Uncertainty Principle and the Hermite Functions

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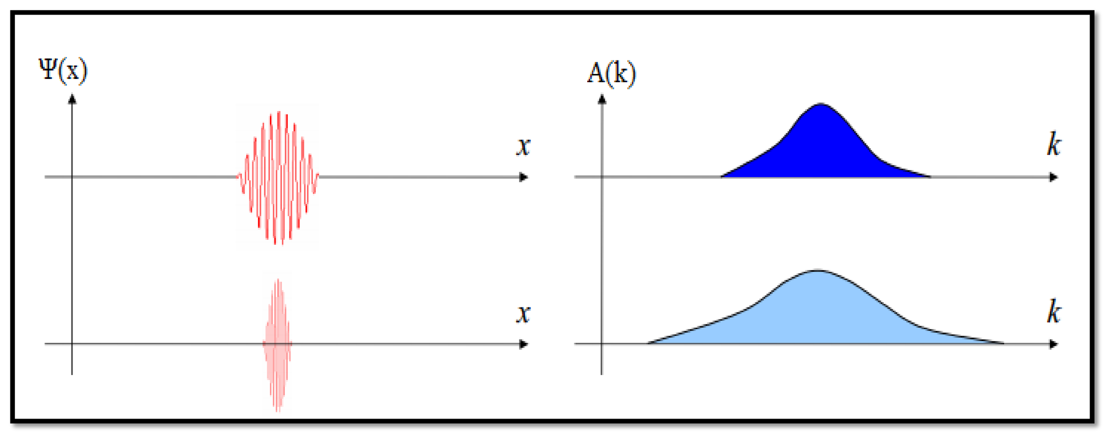
ABSTRACT

The Heisenberg uncertainty principle tells us that it is impossible to determine simultaneously with unlimited precision the position and momentum of a particle. In this research project, we studied various mathematical formulations of the uncertainty principle and their strength as well as learning about Fourier analysis on the real line. Moreover, we explored the Hermite functions, a set of functions that form an orthogonal basis of the bounded, continuous and square integrable functions, in order to formulate a variation of the uncertainty principle based on them. We also discuss the quantum harmonic oscillator and its relation to the Hermite functions.



From left to right: graphs of wave functions and corresponding graphs of amplitude in function of the wave number showing the relationship between uncertainty in position and in momentum, Werner Heisenberg (1901-1976)

UNCERTAINTY PRINCIPLE

The mathematical uncertainty principle states that if f is a real valued function that belongs to $L^2(\mathbb{R})$,i.e the set of measurable functions that are square integrable, then

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\hat{f}(\xi)|^2d\xi\right) \geqslant \frac{\| f \|_2^4}{16\pi^2}$$

As mentioned before there's a lot of generalizations and modifications of this inequality. For example, one can consider the following variation:

Suppose $p, q \in [1, \infty]$ and $a, b > 0$. There is a constant K such that

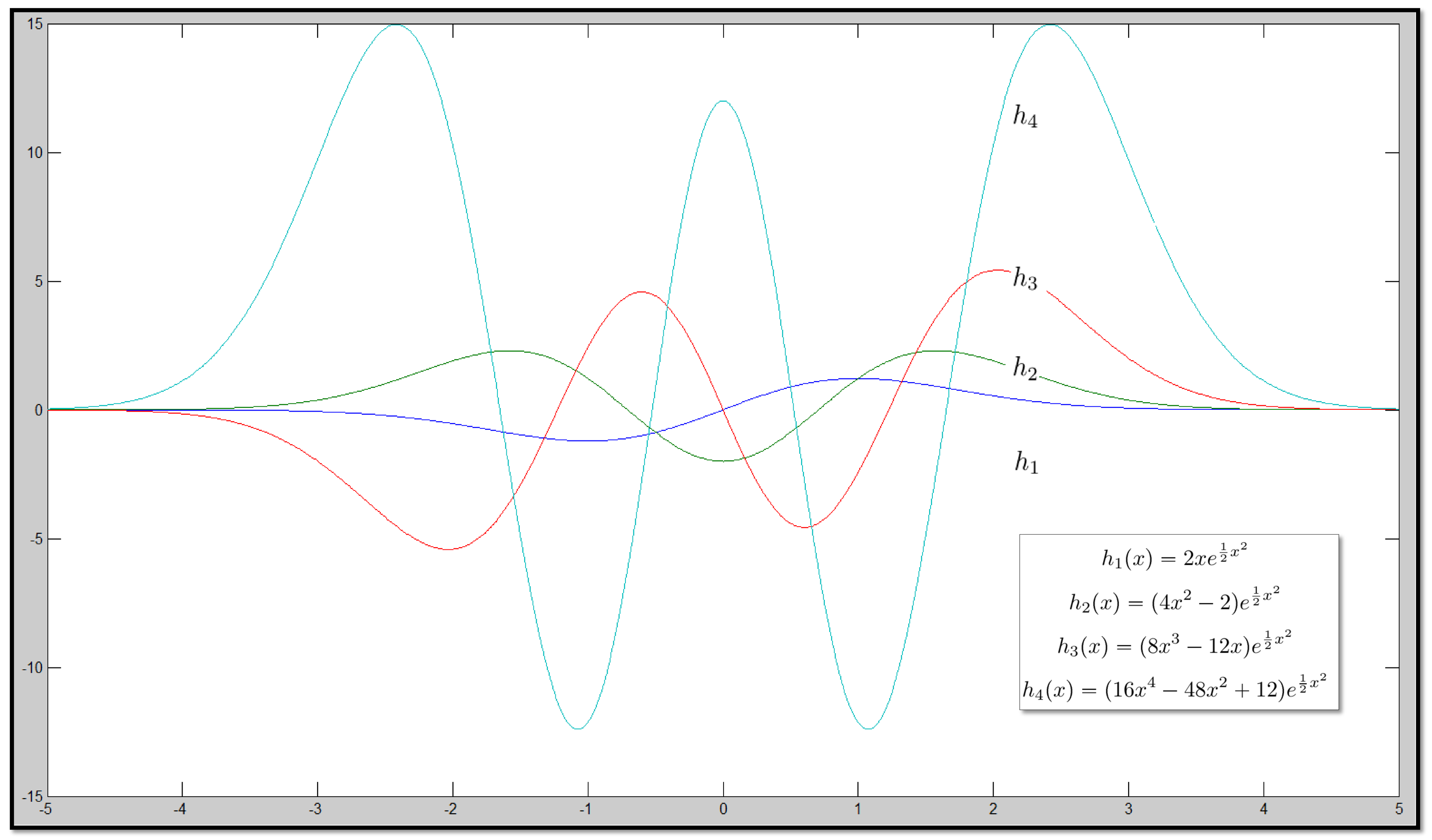
$$\| \, |x|^a f \, \|_p + \| \, |\xi|^b \hat{f} \, \|_q \geqslant K \| f \|_2$$

for all functions that belongs to $L^2(\mathbb{R})$, if and only if

$$a > \frac{1}{2} - \frac{1}{p} \qquad \text{and} \qquad b > \frac{1}{2} - \frac{1}{q}.$$

HERMITE FUNCTIONS

The Hermite functions are defined by $h_k(x) = H_k(x)e^{-\frac{1}{2}x^2}$ where H_k are called the Hermite polynomials and are defined by $(-1)^k e^{x^2}\frac{d^k}{dx^k}(e^{-x^2})$.



Graph of the first four Hermite functions

These functions belong to $L^2(\mathbb{R})$ thus they satisfy the uncertainty principle. Moreover, one of the things we've done is we've found the coefficients C_k that multiply the right hand side of the inequality in question to be an equality and we've obtained that $C_k = 2\pi(2k+1)$. Furthermore, the hermite functions not only belong to $L^2(\mathbb{R})$ but they form an orthogonal basis of this set. In particular, for any function f that belong to $L^2(\mathbb{R})$ can be expressed as a linear combination of these functions such that $f(x) = \sum_k c_k h_k(x)$.

From this fact,we've derived in this research project a new statement which implies the uncertainty principle which states that

$$\boxed{\sqrt{\pi}\sum_k c_k^2\, 2^{k-1}k!(2k+1) + \sqrt{\pi}\sum_k c_k c_{k+2}\, 2^{k+1}(k+1)! \geqslant \frac{1}{\sqrt{\pi}}\sum_k c_k^2 2^{k-2}k!}$$

QUANTUM HARMONIC OSCILLATOR

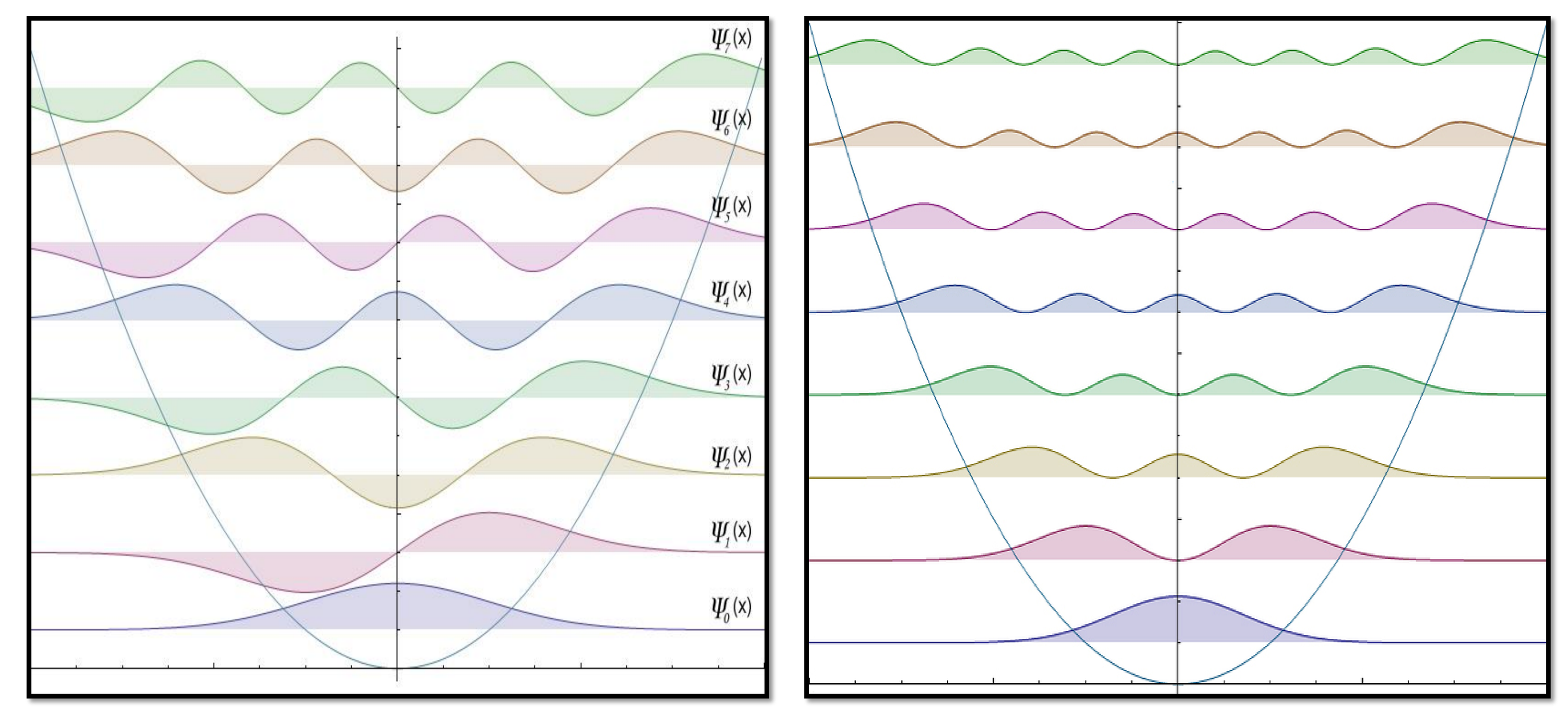
The harmonic oscillator is a very important model in quantum mechanics; it's a particle which is subject to a restoring force that is proportional to its displacement. In the classical case, the angular frequency of such a system is defined as $\omega = \sqrt{\frac{k}{m}}$ and so its potential is $U(x) = \frac{1}{2}m\omega^2x^2$. Furthermore, the study of a quantum mechanical system's evolution in time begins with the specifications of the Schrödinger equation :

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

In this case, we obtain the following equation

$$\frac{d^2\psi(x)}{dt^2} = \frac{2m}{\hbar^2}\left(\frac{1}{2}m\omega^2x^2 - E\right)\psi(x)$$

The solutions to this partial differential equation are in fact the Hermite functions which highlights their importance to study them as we've done.



From left to right: graphs of wavefunctions for the first eight bound eigenstates , corresponding probability densities.

REMARKS

Recently, Terry Tao published an article on an uncertainty principle for various abelian and nilpotent finite groups. We hope in the future to have the opportunity to extend his work and prove a new proposition on this subject. Furthermore, it would be a very engaging experience to study other orthogonal family of functions to derive another statement which entails the uncertainty principle and investigate their physical representation as we've done with the Hermite functions.

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