On the Generalized Honeymoon Oberwolfach Problem

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Abstract

The Honeymoon Oberwolfach Problem (HOP), introduced by Šajna, is a recent variant of the classic Oberwolfach Problem. This problem asks whether it is possible to seat $2m_1 + 2m_2 + \cdots + 2m_t = 2n$ participants, consisting of n newlywed couples, at t round tables of sizes $2m_1, 2m_2, \ldots, 2m_t$ for $2n-2$ successive nights, so that each participant sits next to their spouse every night and next to every other participant exactly once. This problem is denoted by $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$. Jerade, Lepine, and Šajna have studied the HOP and resolved several important cases.

In this thesis, we generalize the HOP by allowing tables of size two, relaxing the previous restriction that tables must have a minimum size of four. In the generalized HOP, we aim to seat the $2n$ participants at s tables of size 2 and t round tables of sizes $2m_1, 2m_2, \ldots, 2m_t$, where $2n = 2s + 2m_1 + 2m_2 + \cdots + 2m_t$ and $m_i \geq 2$, while preserving the original adjacency conditions of the HOP. We denote this problem by $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$.

We present a general approach to this problem and provide solutions to several cases. In particular, we present partial results on the generalized HOP with two round tables, showing that a solution to $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2)$ exists when $n \equiv 1 \pmod{2m_1 + 2m_2}$ and when $n \equiv m_1 + m_2 \pmod{2m_1 + 2m_2}$. We also develop solutions for cases with small round tables, showing that $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$ has a solution for all $m = m_1 + \cdots + m_t \leq 10$ whenever $n = s + m$ is odd and $n(n-1) \equiv 0 \pmod{2m}$. Lastly, we establish a complete solution to the generalized HOP with a single round table, demonstrating that the necessary condition for $\mathrm{HOP}(2^{\langle s \rangle}, 2m)$ to have a solution is also sufficient.

Dedications

I dedicate this thesis with love and gratitude to:

- My dearest parents, the true heroes of my life. Though far from home, your love and prayers, day and night, have been my greatest source of strength and purpose. This journey was possible because of you.
- My respected supervisor, Professor Mateja Šajna, whose unwavering support, guidance, and encouragement have shaped every step of this journey. Thank you for believing in me.

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Contents

List of Figures **x**

List of Tables **xi**

1 Introduction **1**

1.1 Basic Terminology 1

1.2 Terminology for the HOP 3

1.3 Terminology for the Generalized HOP 5

1.4 Background and Motivation 5

1.5 Thesis Overview 7

2 Previous Work **9**

2.1 The Oberwolfach Problem 9

2.2 The Honeymoon Oberwolfach Problem 11

2.2.1 The Main Results 12

2.2.2 The Main Approach 13

2.2.3 Additional Tools 13

2.2.4 Constructing HOP 2-factorizations with Specific Starters 15

2.2.5 Summary of the Proof of the Main Results from [36] 19

2.3 Previous Results on Decompositions into 2-regular Subgraphs 23

2.3.1 Decompositions of K_n into 2-regular Subgraphs 23

2.3.2 Decompositions of $2K_n$ into 2-regular Subgraphs 24

3 The Generalized Honeymoon Oberwolfach Problem **25**

3.1 Introduction 25

3.2 The Main Approach 26

3.3 Additional Tools 37

3.3.1 From G to $4G^\bullet$ . . . 38
3.3.2 From $2G^\circ$ to $4G^\bullet$ . . . 41
3.4 Useful Starters . . . 43
3.4.1 From a Decomposition of K_n to an HOP Decomposition of $4K_n^\bullet$ . . . 44
3.4.2 From a Decomposition of $2K_n^\circ$ to an HOP Decomposition of $4K_n^\bullet$ 45
3.4.3 HOP Decompositions of $4K_n^\bullet$ . . . 48

4 The Generalized HOP with Two Round Tables 53
4.1 HOP $(C_2^{\langle\alpha\rangle}, C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition . . . 53
4.2 HOP (C_2, C_m)-decompositions of $4K_n^\bullet$ when $n \equiv 1 \pmod{2(m+2)}$ . . . 55
4.3 HOP (C_2, C_m)-Decompositions of $4K_n^\bullet$ when $n \equiv m+2 \pmod{2(m+2)}$ 66
4.4 HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ . . . 75
4.5 Generalization of Lemmas 4.2.1 and 4.3.1 . . . 76

5 The Generalized HOP with Small Round Tables 78
5.1 HOP $(C_{m_1}, \ldots, C_{m_t})$-decompositions of $4K_n^\bullet$ with Small Cycle Lengths 78
5.1.1 HOP Decompositions for $n \equiv 1 \pmod 8$ . . . 79
5.1.2 HOP Decompositions for $n \equiv 1$ or $9 \pmod{12}$ . . . 79
5.1.3 HOP Decompositions for $n \equiv 1 \pmod{16}$ . . . 83
5.1.4 HOP Decompositions for $n \equiv 1$ or $9 \pmod{18}$ . . . 87
5.1.5 HOP Decompositions for $n \equiv 1$ or $5 \pmod{20}$ . . . 90

6 A Complete Solution to the Generalized HOP with One Round Table 103
6.1 Main Result . . . 104
6.2 Recoloring the edges in a cycle decomposition of $2G^\circ$ . . . 105
6.3 Definitions and Supporting Tools . . . 110
6.4 Extending a (C_m)-decomposition of $2K_n$ to an HOP (C_m)-decomposition of $4K_n^\bullet$ . . . 117
6.4.1 The Case When m is Even and n is Even . . . 118
6.4.2 The Case When m is Even and n is Odd . . . 127
6.4.3 The Case When m is Odd and n is Even . . . 129
6.5 HOP (C_m)-decomposition of $4K_n^\bullet$ . . . 143
6.5.1 The Case When n is Odd . . . 144
6.5.2 The Case When n is Even . . . 167

7 Conclusion and Future Work **196**

7.1 Problems 196

Bibliography **198**

List of Figures

1.1 A 1-factor I of K_8, the multigraph $K_8 + 5I$, and an I-alternating cycle of length 8. . . . 3

1.2 An illustration of a HOP-coloring-orientation. . . . 4

1.3 An illustration of $2K_3^\circ$. . . . 4

3.1 An HOP C_4-factorization of $4K_4^\bullet$ 30

3.2 Part 1: from an HOP C_4-factor to an I-alternating C_8-factor. . . . 31

3.3 Part 2: from an HOP C_4-factor to an I-alternating C_8-factor. . . . 32

3.4 Part 3: from an HOP C_4-factor to an I-alternating C_8-factor. . . . 33

3.5 From an HOP C_4-factorization of $4K_4^\bullet$ to an I-alternating C_8-factorization of $K_8 + 5I$ 34

3.6 An HOP (C_2)-decomposition of $4K_3^\bullet$ 35

3.7 Part 1: from an HOP (C_2)-subgraph to an I-alternating (K_2, C_4)-factor. . . . 36

3.8 Part 2: from an HOP (C_2)-decomposition of $4K_3^\bullet$ to an I-alternating (K_2, C_4)-factorization of $K_6 + 5I$ 37

3.9 A C_5-factorization of K_5 40

3.10 From a C_5-factorization of K_5 to an HOP C_5-factorization of $4K_5^\bullet$ 41

3.11 From a C_4-factorization of $2K_4^\circ$ to an HOP C_4-factorization of $4K_4^\bullet$ 43

4.1 From an $(H \dot\cup H')$-decomposition of K_{11} to an HOP (C_2, C_3)-decomposition of $4K_{11}^\bullet$. . . . 55

4.2 From a (C_3, C_4)-factorization of K_7 to an HOP $(C_2^{\langle 2\rangle}, C_3)$-factorization of $4K_7^\bullet$. 56

4.3 The F_i subgraph in Lemma 4.2.1 for $m \equiv 2 \pmod 4$. . . . 67

4.4 Subgraph F_i in Lemma 4.3.1 for $m \equiv 1 \pmod 4$ and $m \geq 9$. . . . 74

5.1 (C_2, C_4)-subgraphs of $4K_{21}^\bullet$ that generate an HOP (C_2, C_4)-decomposition for $4K_{21}^\bullet$. . . . 84

5.2 (C_3, C_3)-subgraphs of $4K_9^\bullet$ that generate an HOP (C_3, C_3)-decomposition for $4K_9^\bullet$. 85

5.3 (C_2, C_8)-subgraphs of $2K_{25}^\circ$ that generate an HOP (C_2, C_8)-decomposition of $4K_{25}^\bullet$. 96

5.4 (C_2, C_4, C_4)-subgraphs of $2K_{25}^\circ$ that generate an HOP (C_2, C_4, C_4)-decomposition of $4K_{25}^\bullet$. 98

5.5 (C_2, C_3, C_5)-subgraphs of $4K_{25}^\bullet$ that generate an HOP (C_2, C_3, C_5)-decomposition of $4K_{25}^\bullet$. 102

6.1 A (C_4)-decomposition of $2K_8^\circ$ and its decomposition graph. 107

6.2 Recoloring the red vertices green in $\mathscr{D}(\mathcal{D})$ by swapping colors of the parallel edges in their corresponding cycles. 108

6.3 Recoloring the red vertices green in $\mathscr{D}(\mathcal{D})$ by swapping colors of the parallel edges in their corresponding cycles. 109

6.4 Starter (C_m)-subgraphs for an HOP decomposition of $4K_{m+1}^\bullet$ with $m \equiv 2 \pmod 4$. 116

6.5 Lemma 6.4.2 – Starter m-cycles for an HOP (C_m)-decomposition of $4G^\bullet$ for $m \equiv 2 \pmod 4$. 129

6.6 Lemma 6.5.1– The starter peripheral cycle C_0 and the four starter central cycles for $m = 8$ and $n = 13$. 168

6.7 Orientation and coloring of the edges of C_0. 178

6.8 Illustration of the path P_f for $\ell \equiv 3 \pmod 4$. 188

6.9 Illustration of the path P_f for $\ell \equiv 1 \pmod 4$. 190

List of Tables

5.1 Part 1: HOP $(C_{m_1}, \ldots, C_{m_t})$-decompositions with $m_1 + \ldots + m_t \leq 10$ . . . 80
5.2 Part 2: HOP $(C_{m_1}, \ldots, C_{m_t})$-decompositions with $m_1 + \ldots + m_t \leq 10$ . . . 81

Chapter 1

Introduction

1.1 Basic Terminology

In this section, we introduce the definitions and notation used throughout this thesis.

Definition 1.1.1. For a graph G, the symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A graph H is a *subgraph* of G, if it satisfies the following properties:

- The vertex set of H is a subset of the vertex set of G, and
- the edge set of H is a subset of the edge set of G.

If a subgraph H of a graph G has the same vertex set as G, then it is called a *spanning subgraph* of G. The *union* of graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1) \cup V(G_2)$ as the vertex set, and $E(G_1) \cup E(G_2)$ as the edge set. Furthermore, if G_1 and G_2 have no vertex in common, that is, the vertex sets of G_1 and G_2 are disjoint, then $G_1 \cup G_2$ is a *disjoint union*, and is denoted by $G_1 \dot{\cup} G_2$.

Graphs in this thesis will be loopless, but may contain parallel edges or directed edges. For any simple graph G, we use λG to denote the multigraph obtained by replacing each edge of G with λ parallel copies. As usual, K_n denotes the complete graph on n vertices, so λK_n denotes the λ-fold complete graph. We use the symbol $K_{m[k]}$ to denote the complete equipartite graph with m parts of size k, and $K_{m,n}$ to denote the complete bipartite graph with parts of sizes m and n. Furthermore, the graph obtained by replacing each edge of K_n with a pair of opposite arcs is denoted by K_n^*, referred to as the *complete symmetric digraph*.

Definition 1.1.2. Let G be a graph. A collection $\{H_1, H_2, \ldots, H_t\}$ is called a *decomposition* of G if it has the following properties:

- $H_1, H_2, \ldots, H_t$ are subgraphs of G, and
- $\{E(H_1), E(H_2), \ldots, E(H_t)\}$ is a partition of $E(G)$.

Such a decomposition is denoted by $G = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_t$. Moreover, if the subgraphs $H_1, H_2, \ldots, H_t$ are all isomorphic to H, then $\{H_1, H_2, \ldots, H_t\}$ is called an *H-decomposition* of G.

Definition 1.1.3. A graph whose vertices all have the same degree r is called *r-regular*. A spanning subgraph of G which is r-regular is called an *r-factor.* A $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-*factor* of G is a 2-factor of G consisting of t disjoint cycles of lengths $m_1, m_2, \ldots, m_t$. A $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$*-factorization* of a graph G is a decomposition of G into $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors. In the case of uniform cycle lengths, that is, when $m_1 = m_2 = \ldots = m_t = m$, we use the terms *$C_m$-factor* and *$C_m$-factorization* to denote a $(C_m, C_m, \ldots, C_m)$-factor and $(C_m, C_m, \ldots, C_m)$-factorization, respectively.

Definition 1.1.4. A 1-factorization $\{F_1, F_2, \ldots, F_{2n-1}\}$ of K_{2n} is called a *semi-uniform* 1-*factorization of type* $(2m_1, 2m_2, \ldots, 2m_t)$ if, for all $i \geq 2$, the union $F_1 \cup F_i$ is a $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor.

Definition 1.1.5. Let G_1 and G_2 be two vertex-disjoint simple graphs. The *join* of G_1 and G_2, denoted by $G_1 \bowtie G_2$, is a simple graph consisting of the union of graphs G_1 and G_2 along with all possible edges between them.

Definition 1.1.6. The *circulant graph* $\mathrm{Circ}(n; S)$ is a graph with vertex set $\{x_i : i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+d} : i \in \mathbb{Z}_n,\ d \in S \ \text{ or } \ n-d \in S\}$, where $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. An edge of the form $x_i x_{i+d}$ is said to have *difference d.* Since each edge of difference d is also of difference $n-d$, we may assume that all differences lie in the set $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

When n is even, the edge $x_i x_{i+\frac{n}{2}}$ connects a pair of vertices that are diametrically opposite in the cycle. We refer to $d = \frac{n}{2}$ as a *diameter difference.*

The complete graph K_n is a circulant graph of order n with difference set $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$; that is, $K_n = \mathrm{Circ}(n; S)$.

In many of the constructions given in this thesis, the complete graph K_n is viewed as $\mathrm{Circ}(n-1; S) \bowtie K_1$, where $S = \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and the vertex of K_1 is denoted by x_∞. In other words, we have:

- $V(K_n) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$, and
- $E(K_n) = \{x_i x_j : i, j \in \mathbb{Z}_{n-1},\ i \neq j\} \cup \{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Note that an edge of the form $x_i x_\infty$ is said to be of *difference infinity.*

Definition 1.1.7. Let $G = \mathrm{Circ}(n; S)$ be a circulant graph with vertex set $\{x_0, x_1, \ldots, x_{n-1}\}$, where the vertices are cyclically ordered according to increasing subscript modulo n. We define cyclic interval notation as follows: for any $0 \leq i, j < n$,

- $[x_i, x_j]$ denotes the set $\{x_i, x_{i+1}, \ldots, x_j\}$ if $i \leq j$, and the set $\{x_i, x_{i+1}, \ldots, x_{n-1}, x_0, x_1, \ldots, x_j\}$ if $i > j$.
- $[x_i, x_j)$ denotes $[x_i, x_j] \setminus \{x_j\}$,

- $(x_i, x_j]$ denotes $[x_i, x_j] \setminus \{x_i\}$,
- (x_i, x_j) denotes $[x_i, x_j] \setminus \{x_i, x_j\}$.

All indices are interpreted modulo n, and negative indices are used when convenient. For example, $x_{-1} = x_{n-1}$.

The interval $(x_{n-3}, x_2]$ contains the vertices $x_{n-2}, x_{n-1}, x_0, x_1, x_2$. Equivalently, we may write $(x_{-3}, x_2]$, which includes the same vertices: $x_{-2}, x_{-1}, x_0, x_1, x_2$.

Definition 1.1.8. Let G be a graph with $V(G) = \{x_i : i \in \mathbb{Z}_n\}$ and let H be a subgraph of G. Let $\rho = (x_0\ x_1\ \dots\ x_{n-1})$ be a permutation on the vertex set of G. Then an H-decomposition of G of the form $\mathcal{D} = \{\rho^i(H) : i \in \mathbb{Z}_n\}$ is called *cyclic*.

1.2 Terminology for the HOP

In this section, we review the required terminology for the Honeymoon Oberwolfach Problem. This terminology is taken from [36].

Definition 1.2.1. Let I be a 1-factor in K_{2n}. An edge of K_{2n} which belongs to $E(I)$ is called an *I-edge*; all other edges are *non-I-edges*. A graph K_{2n} with deleted I-edges is denoted by $K_{2n} - I$, and a graph K_{2n} with λ additional copies of each I-edge is denoted by $K_{2n} + \lambda I$. Note that additional copies of I-edges are also considered I-edges in the graph $K_{2n} + \lambda I$.

A cycle C in $K_{2n} + \lambda I$ is called an *I-alternating* cycle if the I-edges and non-I-edges alternate along C, as illustrated in Figure 1.1. Notice that an I-alternating cycle is of even length. Let F be a 2-factor of $K_{2n} + \lambda I$. If every cycle in F is I-alternating, then F is said to be *I-alternating*.

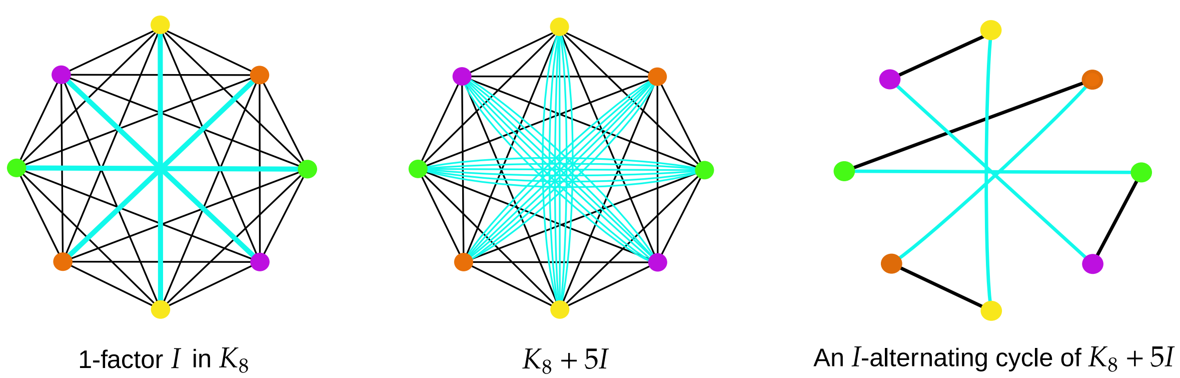


Figure 1.1: A 1-factor I of K_8, the multigraph $K_8 + 5I$, and an I-alternating cycle of length 8.

Next, we introduce *HOP-coloring-orientation*, which will be the main tool used to construct solutions for cases of HOP.

Definition 1.2.2. Let G be a simple graph. An *HOP-coloring-orientation* of $4G$ is a 3-edge-coloring of $4G$ with colors blue, pink, and black, together with an orientation of the black edges such that for any two adjacent vertices in $4G$, among the four parallel edges between them, one edge is blue, one edge is pink, and two of the edges are black with the opposite orientations (as illustrated in Figure 1.2). The multigraph $4G$ with a given HOP-coloring-orientation is denoted by $4G^{\bullet}$.

Figure 1.2: An illustration of a HOP-coloring-orientation.

Definition 1.2.3. Let G be a simple graph and $\mathcal{D}$ a 2-factorization of $4G^{\bullet}$. We say that $\mathcal{D}$ is *HOP* if it satisfies the following condition.

(C1) For every cycle C in $\mathcal{D}$, any two adjacent edges of C satisfy one of the following:

- one is blue and the other pink; or
- one is blue and the other black with an orientation toward the blue edge; or
- one is pink and the other black with an orientation away from the pink edge; or
- both are black and oriented in the same way.

We say that a cycle C in $4G^{\bullet}$ is an *HOP cycle* if it satisfies condition (C1) above. Similarly, a 2-regular subgraph or a 2-factor of $4G^{\bullet}$ is called an *HOP 2-regular subgraph* or an *HOP 2-factor*, respectively, if all of its cycles are HOP.

Definition 1.2.4. Let G be a simple graph. The symbol $2G^{\circ}$ represents the multigraph $2G$ with a 2-edge-coloring with colors pink and black such that for any two vertices in $2G$, the two parallel edges between them have colors pink and black (as illustrated in Figure 1.3).

Figure 1.3: An illustration of $2K_3^{\circ}$.

1.3 Terminology for the Generalized HOP

In this section, we define the required terminology for the generalized Honeymoon Oberwolfach Problem.

Definition 1.3.1. A $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$*-subgraph* of a graph G is a 2-regular subgraph of G consisting of t disjoint cycles of lengths $m_1, m_2, \ldots, m_t$. A $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$*-decomposition* of a graph G is a decomposition of G into $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs. In the case with uniform cycle lengths, that is, $m_1 = m_2 = \ldots = m_t = m$, we use the terms $(C_m^{\langle t \rangle})$*-subgraph* and $(C_m^{\langle t \rangle})$*-decomposition* instead.

Note that a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor of G is a spanning $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph.

Definition 1.3.2. Let F be a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph of K_{2n}, and I a fixed 1-factor of K_{2n}. Analogously to the Definition 1.2.1 , if every cycle in F is I-alternating, then F is said to be I*-alternating*.

Definition 1.3.3. For any graph G, we use the notation $G^{\langle s \rangle}$ to denote the disjoint union of s copies of G.

Definition 1.3.4. A $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$*-factor* of a graph G is a spanning subgraph of G that is a disjoint union of s copies of K_2 and a $(C_{m_1}, \ldots, C_{m_t})$-subgraph of G. A $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$*-factorization* of G is a decomposition of G into $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$-factors.

Let I be a fixed 1-factor of K_{2n}. A $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$-factor of K_{2n} is called I*-alternating* if its $(C_{m_1}, \ldots, C_{m_t})$-subgraph is I-alternating, and all other edges are I-edges. Moreover, a $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$-factorization is I*-alternating* if all of its $(K_2^{\langle s \rangle}, C_{m_1}, \ldots, C_{m_t})$-factors are I-alternating.

Definition 1.3.5. Let G be a simple graph and $\mathcal{D}$ be a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4G^\bullet$. We say that $\mathcal{D}$ is *HOP* if every cycle in $\mathcal{D}$ satisfies Condition **(C1)** of Definition 1.2.3.

1.4 Background and Motivation

The well-known Oberwolfach Problem (OP) is a seating assignment problem posed by Gerhard Ringel at a conference in Oberwolfach. The problem is formulated as follows. We wish to create a seating chart for n participants in a room with t round tables that respectively seat $m_1, m_2, \ldots, m_t$ people, such that all tables are full at each meal; that is, $m_1 + m_2 + \ldots + m_t = n$, and each participant sits beside every other participant exactly once. Since each participant sits next to two others at each meal, every participant must have an even number of neighbors, which implies that n must be odd.

The OP can be expressed precisely in the language of graphs as the question of whether the complete graph on n vertices, K_n, can be decomposed into 2-factors such that each 2-factor is a disjoint union of t cycles of lengths $m_1, m_2, \ldots, m_t$. Clearly, such a decomposition requires all vertex degrees to be even, and thus n must be odd. This problem is denoted by $\mathrm{OP}(m_1, m_2, \ldots, m_t)$.

Since its introduction, the OP has been widely studied and extended. Many cases have been solved, and different variations have been proposed. Although the OP was originally defined for odd values of n, it has also been extended to even values. For even n, two main variations of the OP have been studied. In both versions, the n participants are grouped into $\frac{n}{2}$ couples.

The first variation is called *spouse-avoiding*. Compared to the original OP, it includes the additional requirement that no participant is allowed to sit next to their spouse [30]. In graph theory, such a seating arrangement is equivalent to a decomposition of $K_n - I$ into 2-factors, where $K_n - I$ is the complete graph of even order with the edges of a 1-factor (representing the couples) removed.

The second variation is called *spouse-loving* [12]. In this variation, the same condition as in the OP holds, with the added requirement that each participant must sit next to their spouse exactly twice. Such a seating arrangement is equivalent to a decomposition of $K_n + I$ into 2-factors.

Another variation of the OP involves seating participants so that each person sits to the right of every other participant exactly once. This version corresponds to a decomposition of the complete symmetric digraph K_n^* into spanning subdigraphs consisting of disjoint directed cycles [6].

In general, the OP remains an open problem. However, it is known to have a solution for all sufficiently large values of n [25].

In this thesis, we focus on a variant of the OP called the Honeymoon Oberwolfach Problem (HOP), which is due to Šajna [36]. In this variation, the problem is formulated as the question of whether it is possible to seat $2m_1 + 2m_2 + \ldots + 2m_t = 2n$ participants, consisting of n newlywed couples, at t round tables of sizes $2m_1, 2m_2, \ldots, 2m_t$ for $2n - 2$ successive nights, so that each participant sits next to their spouse every night and next to every other participant exactly once.

In the language of graph theory, a solution to the HOP corresponds to a decomposition of the multigraph $K_{2n} + (2n - 3)I$ into 2-factors, each consisting of disjoint I-alternating cycles of lengths $2m_1, 2m_2, \ldots, 2m_t$. Here, I is a fixed 1-factor of K_{2n} representing the couples. This problem is denoted by $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$, and in the special case where $m_1 = m_2 = \ldots = m_t$ and $n = tm$, it is denoted by $\mathrm{HOP}(2n; 2m)$.

Observe that in each I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor in the decomposition of $K_{2n} + (2n - 3)I$, removing the fixed 1-factor I leaves behind another 1-factor. This observation leads to the notion of a *semi-uniform 1-factorization* of type $(2m_1, 2m_2, \ldots, 2m_t)$, as introduced in [35] and defined in Definition 1.1.4. Hence, there is a direct connection between semi-uniform 1-factorizations and solutions to $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$. In the next chapter, we explore this connection further.

A more general framework for the OP was proposed by Rinaldi, under the name *Oberwolfach Problem with Loving Couples* [43]. In this setting, the n participants are grouped into $\frac{n}{2}$ couples and seated at t round tables of specified sizes over the course of several nights, with the requirement that each person sits next to their partner exactly $r \geq 0$ times, and next to every other participant exactly once. Observe that when $r = 0$, the problem coincides with the *spouse-avoiding* version [30]; when $r = 2$, it corresponds to the *spouse-loving* version [12]; and when $r = 2n - 2$, it matches the *Honeymoon Oberwolfach Problem* [36].

Until now, the HOP has been defined only for tables of size at least 4. In this thesis, we aim to generalize the problem to include tables of size 2, leading to a new version of the problem that has not been previously studied.

The generalized HOP preserves the original conditions, except that the $2n$ participants are seated at s tables of size 2 and t round tables of sizes $2m_1, 2m_2, \ldots, 2m_t$, under the assumption that $n = s + m_1 + \ldots + m_t$ and all $m_i \geq 2$. A solution to the generalized HOP is equivalent to a decomposition of the multigraph $K_{2n} + (\gamma - 1)I$, for an appropriate integer γ which is determined by the other parameters, into subgraphs consisting of disjoint I-alternating cycles of lengths $2m_1, 2m_2, \ldots, 2m_t$, along with s copies of K_2. We denote this generalized problem by $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$.

In the following chapters, we will delve into this problem and establish several new results.

1.5 Thesis Overview

This thesis is structured as follows.

In Chapter 2, we summarize the most important results known to date on the OP and the HOP. These include, for instance, the case with uniform cycle lengths, $\mathrm{HOP}(2n; 2m)$, which has been completely resolved in [36]. We also outline the main techniques used to prove these results and review the proofs of key theorems. At the end of the chapter, in Section 2.3, we present existing results on the decomposition of K_n and $2K_n$ into 2-regular subgraphs. These results will be used in later chapters to address certain questions related to the generalized HOP.

In Chapter 3, we begin by defining the generalized HOP and stating the necessary conditions for the existence of a solution. We then extend the main method used for proving results on the HOP to this generalized setting. The main approach involves converting the problem from the multigraph $K_{2n} + (\gamma - 1)I$ to the multigraph $4K_n$ with an HOP-coloring-orientation. Accordingly, our focus will be on finding an HOP decomposition of $4K_n^{\bullet}$ into appropriate subgraphs. Key techniques include a reduction step, where we first find decompositions of simpler graphs such as K_n and $2K_n$, and then extend those decompositions to obtain HOP decompositions of $4K_n^{\bullet}$.

In Chapter 4, we focus on finding solutions to the generalized HOP involving s tables of size 2 and two round tables of sizes $2m_1$ and $2m_2$. In particular, we show that $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2)$ admits a solution whenever $n \equiv 1 \pmod{2(m_1 + m_2)}$ or $n \equiv m_1 + m_2 \pmod{2(m_1 + m_2)}$.

In Chapter 5, we address the generalized HOP with small round tables. Specifically, we prove that $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution whenever $m = \sum_{i=1}^{t} m_i \leq 10$ and $n(n-1) \equiv 0 \pmod{2m}$. This result follows from the existing work on decompositions of K_n into $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs for $m_1, m_2, \ldots, m_t \geq 3$. However, we also develop new tools to handle the cases where $2 \in \{m_1, m_2, \ldots, m_t\}$.

In Chapter 6, we provide a complete solution to the generalized HOP with one round table of size $2m$. We show that the obvious necessary condition for $\mathrm{HOP}(2^{\langle s\rangle}, 2m)$ to have a solution is indeed sufficient.

In Chapter 7, we present several problems related to the generalized HOP. These problems are formulated in such a way that parts of them can be proven using existing results on the decomposition of K_n or $2K_n$. However, new constructions are required to address the remaining cases. Overall, the generalized HOP is a new problem, and there is still plenty of room for further exploration.

Chapter 2

Previous Work

In this chapter, we introduce the Oberwolfach Problem and the Honeymoon Oberwolfach Problem. We summarize known results on both OP and HOP, and review key proof techniques developed in [36] for the HOP.

2.1 The Oberwolfach Problem

The Oberwolfach Problem (OP) is a well-known problem in graph theory, which was first presented by Gerhard Ringel in 1967. The OP asks whether it is possible to seat the n participants at a conference at t round tables of sizes $m_1, m_2, \ldots, m_t$, where $m_1 + m_2 + \ldots + m_t = n$, for several consecutive nights so that each person sits beside every other person exactly once. We denote this problem by $\mathrm{OP}(m_1, m_2, \ldots, m_t)$. In the case with uniform cycle lengths, that is, $m_1 = m_2 = \ldots = m_t = m$, we use the symbol $\mathrm{OP}(n; m)$.

In graph theory, the OP is equivalent to the existence of a 2-factorization of K_n, in which each 2-factor is a disjoint union of t cycles of lengths $m_1, m_2, \ldots, m_t$. This applies when n is odd. However, for even n, the graph K_n does not admit a 2-factorization. In this case, we are interested in finding a 2-factorization of $K_n - I$. This gives rise to a variation of the OP known as the *spouse-avoiding* variant [30]. In this variation, the n participants form $\frac{n}{2}$ couples, and the question becomes whether it is possible to seat the n participants at t round tables of sizes $m_1, m_2, \ldots, m_t$, where $m_1 + m_2 + \ldots + m_t = n$, for several consecutive nights, so that each person sits next to every other person exactly once, except they never sit next to their spouse.

In general, a solution to $\mathrm{OP}(m_1, m_2, \ldots, m_t)$, for $n = m_1 + m_2 + \ldots + m_t$, is equivalent to a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of K_n when n is odd, or of $K_n - I$ when n is even.

The OP is a generalization of the classical problem of decomposing K_n into triples, which was first proposed by Kirkman [38] in 1850. This problem was solved more than a century later, independently by Jiaxi Lu [40] and by Ray-Chaudhuri and Wilson [41]. The following theorem is the result proved by Ray-Chaudhuri and Wilson.

Theorem 2.1.1. [41] There exists a C_3-factorization of K_n if and only if $n \equiv 3 \pmod 6$.

Although the OP remains open in general, many results are known for specific parameters. The most important ones are stated below in Theorems 2.1.2–2.1.9.

For the OP with tables of equal lengths, it has been shown that, except for the cases $(n,m) \in \{(6,3),(12,3)\}$, the obvious necessary condition is also sufficient. This is stated in the following theorem:

Theorem 2.1.2. [4, 7, 8, 13, 29, 34, 41, 42] Let $3 \leq m \leq n$ be integers. Then $\mathrm{OP}(n;m)$ has a solution if and only if $n \equiv 0 \pmod{m}$ and $(n,m) \notin \{(6,3),(12,3)\}$.

For the OP with tables of varying lengths, it has been shown that the necessary conditions are sufficient for all $n \leq 100$, except for a few specific parameter values. This is stated in the following theorem:

Theorem 2.1.3. [2, 19, 22, 23, 32, 45] Let $m_1, m_2, \ldots, m_t$ be integers such that $3 \leq m_1 \leq m_2 \leq \ldots \leq m_t$ and $m_1 + m_2 + \cdots + m_t \leq 100$. Then $\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution if and only if $(m_1, m_2, \ldots, m_t) \notin \{(3,3),(4,5),(3,3,5),(3,3,3,3)\}$.

Aside from the exceptions listed in the above theorems, it is conjectured that the necessary conditions are sufficient for all values of n.

In 2011, Bryant and Danziger provided a complete solution to the OP in the case where all tables are of even size.

Theorem 2.1.4. [14] Let $m_1, m_2, \ldots, m_t \in \mathbb{Z}$ be even integers, each of size at least 4. Then $\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution.

Gvozdjak solved several cases of the OP with two cycles in his PhD thesis. Later, in 2013, Traetta completed the solution to this case. The result is as follows:

Theorem 2.1.5. [26, 47] Let $3 \leq m_1 \leq m_2$ be integers. Then $\mathrm{OP}(m_1, m_2)$ has a solution if and only if $(m_1, m_2) \notin \{(3,3),(4,5)\}$.

The following is a summary of some of the known results on the OP with three tables. These results have been collected from various sources.

Theorem 2.1.6. [18, 26, 51, 50]

(a) Let $s > 5$ an odd integer, and let $3 < r \leq 13$. Then $OP(r,r,s)$ has a solution.

(b) Let $r \in \{3,4\}$ and $n \geq 2r+3$. Then $OP(r,r,n-2r)$ has a solution.

(c) Let $s \geq 1$ be an integer. Then $OP(2s+1, 2s+1, 2s+2)$ has a solution.

(d) Let $s \geq 1$ be an integer. Then $OP(3, 4s, 4s)$ has a solution.

(e) Let p be a prime number, and $t, r \in \mathbb{Z}^+$ such that $5 \leq t \leq p+1$ and $r \neq 1,2,4$. Then $OP(t, 2p, p^{2r})$ has a solution.

(f) Let $n \geq 10$, and $n \not\equiv 1 (\mathrm{mod}\ 4)$. Then $OP(3,4,n-7)$ has a solution.

(g) Let $n \geq 12$, and $n \not\equiv 1 (\text{mod } 4)$. Then $OP(3, 6, n-9)$ has a solution.

(h) Let $n \equiv 0 \pmod 4$ and $3 \leq a < n-3-a$. Then $OP(3, a, n-3-a)$ has a solution.

In 2021, Glock et al. showed that the necessary conditions for the OP are sufficient when n is large enough; however, their proof relies on probabilistic methods, which are nonconstructive and do not provide an explicit lower bound on the order.

Theorem 2.1.7. [25] There exists $n_0 \in \mathbb{N}$ such that for all odd $n \geq n_0$, if $m_1, m_2, \ldots, m_t$ are integers with $3 \leq m_i \leq n$ and $m_1 + m_2 + \cdots + m_t = n$, then $\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution.

We have the following result due to Burgess, Danziger, and Traetta; they provide an explicit construction of the solutions.

Theorem 2.1.8. [17] Let $3 \leq m_1 \leq m_2 \leq \ldots \leq m_t$ be such that the $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor is a single-flip 2-factor with one sufficiently large cycle. Then $\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution.

Note that a 2-factor is called a *single-flip* if the multiplicity of at most one of its odd cycle lengths is odd.

Moreover, in 2024, Traetta gave a constructive solution to the OP with tables of varying lengths, showing that a solution exists when one of the tables has size greater than an explicit lower bound [48].

Here are some results on the OP for the complete equipartite graph with uniform tables due to Liu.

Theorem 2.1.9. [37] Let α, k, m be positive integers such that $2 \leq \alpha$ and $3 \leq m \leq k\alpha$. Then there exists a C_m-factorization of $K_{\alpha[k]}$ if and only if all of the following conditions are satisfied.

- $\alpha k \equiv 0 \pmod m$,
- $k(\alpha - 1) \equiv 0 \pmod 2$,
- m is even if $\alpha = 2$, and
- $(k, \alpha, m) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}$.

2.2 The Honeymoon Oberwolfach Problem

In this thesis, we focus on a variant of the Oberwolfach Problem called the Honeymoon Oberwolfach Problem (HOP), which was introduced by Šajna in 2019 [36]. In this variation, we have $2n = 2m_1 + 2m_2 + \ldots + 2m_t$ participants consisting of n newlywed couples. We want to see whether it is possible to seat them at t round tables of sizes $2m_1, 2m_2, \ldots, 2m_t$ for $2n-2$

consecutive nights so that each participant sits next to their spouse every time and next to each other participant exactly once. This problem is denoted by $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$. Moreover, $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ with $m_1 = m_2 = \ldots = m_t$ and $n = tm$ is denoted by $\text{HOP}(2n; 2m)$.

A solution to $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ for $2n = 2m_1 + 2m_2 + \ldots + 2m_t$ is equivalent to a decomposition of the multigraph $K_{2n} + (2n-3)I$ into 2-factors, each consisting of disjoint I-alternating cycles of lengths $2m_1, 2m_2, \ldots, 2m_t$. In the case with uniform cycle length, a solution to $\text{HOP}(2n; 2m)$ is equivalent to an I-alternating C_{2m}-factorization of $K_{2n} + (2n-3)I$.

The HOP is not just about arranging couples at tables in a special way. In the following lemma we see that a solution to $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ corresponds to a *semi-uniform* 1-factorization of K_{2n} of type $(2m_1, 2m_2, \ldots, 2m_t)$. Recall that a 1-factorization $\{F_1, F_2, \ldots, F_{2n-1}\}$ of K_{2n} is called *semi-uniform of type* $(2m_1, 2m_2, \ldots, 2m_t)$ if, for all $i \geq 2$, $F_1 \cup F_i$ is a $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor [35].

Lemma 2.2.1. A solution to $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ with $2m_1 + 2m_2 + \ldots + 2m_t = 2n$ is equivalent to a semi-uniform 1-factorization of K_{2n} of type $(2m_1, 2m_2, \ldots, 2m_t)$.

Proof. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_{2n-2}\}$ be a solution to $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$, that is, $\mathcal{D}$ is an I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factorization of $K_{2n} + (2n-3)I$. Let $F_1 = I$, and for all $j \in \{2, \ldots, 2n-1\}$, let $F_j = D_{j-1} - I$. Since each $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor is I-alternating, it follows that each F_j is a 1-factor such that $F_1 \cup F_j$ is a $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor. Notice that $F_2 \cup F_3 \cup \ldots \cup F_{2n-1}$ contains all the non-I-edges of K_{2n}. Therefore, we have $F_1 \bigoplus F_2 \bigoplus \ldots \bigoplus F_{2n-1} = K_{2n}$, and by definition, $\{F_1, F_2, \ldots, F_{2n-1}\}$ is a semi-uniform 1-factorization of K_{2n} of type $(2m_1, 2m_2, \ldots, 2m_t)$.

Conversely, assume $\{F_1, F_2, \ldots, F_{2n-1}\}$ is a semi-uniform 1-factorization of K_{2n} of type $(2m_1, 2m_2, \ldots, 2m_t)$, that is, for all $i \geq 2$, $F_1 \cup F_i$ is a $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor. Let $I = F_1$ so $F_1 \cup F_i$ is an I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factor. Let $I_2, I_3, \ldots, I_{2n-1}$ be pairwise edge-disjoint copies of I. It can be seen that $(I_2 \cup F_2) \bigoplus (I_3 \cup F_3) \bigoplus \ldots \bigoplus (I_{2n-1} \cup F_{2n-1}) = K_{2n} + (2n-3)I$. Thus, $\mathcal{D} = \{I_2 \cup F_2, I_3 \cup F_3, \ldots, I_{2n-1} \cup F_{2n-1}\}$ is an I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-factorization of $K_{2n} + (2n-3)I$, that is, a solution to $\text{HOP}(m_1, m_2, \ldots, m_t)$. ∎

Several cases of the HOP have been previously solved, with results published in [36]. In particular, the case where all tables have uniform length was completely resolved, and solutions to many other cases were established as summarized in Theorem 2.2.2 below.

2.2.1 The Main Results

In this section, we review the results presented in [36], together with recent updates from [31]. The main results are summarized in the following theorem.

Theorem 2.2.2. [36, 31] Let $m_1, m_2, \ldots, m_t$ be integers with $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$. Let $n = m_1 + m_2 + \ldots m_t$. Then $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ has a solution in each of the following cases.

- $m_i \equiv 0 \pmod 4$, for $i = 1, 2 \ldots, t$;
- n is odd and $\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution;
- n is odd and $t = 2$;
- n is odd, $n \leq 100$, and $m_1 \geq 3$;
- $n \leq 20$;
- $m_1 = m_1 = \ldots = m_t = m$.

2.2.2 The Main Approach

The following theorem provides the main tool for proving Theorem 2.2.2. The approach is to convert the problem from the graph $K_{2n} + (2n-3)I$ to a problem in the graph $4K_n^\bullet$.

Theorem 2.2.3. [36] Let $m_1, m_2, \ldots, m_t$ be integers with $m_i \geq 2$, and let $n = m_1 + m_2 + \ldots + m_t$. Then $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ has a solution if and only if $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.

In Chapter 3, we extend Theorem 2.2.3 to Theorem 3.2.1 for the generalized HOP and provide a proof that closely follows the one given for Theorem 2.2.3 in [36].

2.2.3 Additional Tools

Theorem 2.2.3 shows that solving $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ is equivalent to finding an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $4K_n^\bullet$. Thus, we focus on finding HOP 2-factorizations of $4K_n^\bullet$. However, such factorizations are not always easy to find directly. In this section, we present useful tools from [36] that help construct HOP 2-factorizations of $4G^\bullet$ using 2-factorizations of smaller graphs.

2.2.3.1 From G to $4G^\bullet$

Lemma 2.2.4. [36] Let G be a simple graph.

(a) Let $H_1, H_2, \ldots, H_s$ be spanning subgraphs of G such that $G = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_s$. If for each $i \in \{1, \ldots, s\}$, the multigraph $4H_i^\bullet$ admits an HOP 2-factorization $\mathcal{D}_i$, then $4G^\bullet$ admits an HOP 2-factorization $\mathcal{D} = \bigcup_{i=1}^{s} \mathcal{D}_i$.

(b) Assume G is a vertex-disjoint union of $H_1, H_2, \ldots, H_s$, where each H_i for $i \in \{1, \ldots, s\}$ is an r-regular subgraph of G. If for each $i \in \{1, \ldots, s\}$, the multigraph $4H_i^\bullet$ admits an HOP 2-factorization $\mathcal{D}_i = \{F_1^i, \ldots, F_{2r}^i\}$, then $4G^\bullet$ admits an HOP 2-factorization $\mathcal{D} = \{F_1, \ldots, F_{2r}\}$, where each $F_j = \bigcup_{i=1}^{s} F_j^i$ for all $j \in \{1, \ldots, 2r\}$.

Proof.

(a) Notice that $4H_1^\bullet, 4H_2^\bullet, \ldots, 4H_s^\bullet$ are spanning subgraphs of $4G^\bullet$ since $H_1, H_2, \ldots, H_s$ are spanning subgraphs of G. Moreover, since the edge sets of $H_1, H_2, \ldots, H_s$ form a partition of $E(G)$, we have $4G^\bullet = 4H_1^\bullet \bigoplus 4H_2^\bullet \bigoplus \cdots \bigoplus 4H_s^\bullet$. Each $4H_i^\bullet$, for $i \in \{1, \ldots, s\}$, is a spanning subgraph of $4G^\bullet$, so any 2-factor in $4H_i^\bullet$ is a 2-factor in $4G^\bullet$. The multigraphs $4H_1^\bullet, 4H_2^\bullet, \ldots, 4H_s^\bullet$ are edge-disjoint subgraphs, so the union of their 2-factorizations forms a 2-factorization for $4G^\bullet$. In addition, since, for all $i \in \{1, \ldots, s\}$, the 2-factorization D_i of $4H_i^\bullet$ is HOP, the resulting 2-factorization of $4G^\bullet$, that is, $\mathcal{D} = \bigcup_{i=1}^{s} \mathcal{D}_i$ is HOP as well.

(b) Note that G is a vertex-disjoint union of $H_1, H_2, \ldots, H_s$, such that each H_i for $i \in \{1, \ldots, s\}$, is r-regular. Consequently, the multigraph $4G^\bullet$ is a vertex-disjoint union of $4H_1^\bullet, 4H_2^\bullet, \ldots, 4H_s^\bullet$, where each $4H_i^\bullet$, for $i \in \{1, \ldots, s\}$, is a $4r$-regular subgraph of $4G^\bullet$. Moreover, each multigraph $4H_i^\bullet$ admits an HOP 2-factorization $\mathcal{D}_i = \{F_1^i, \ldots, F_{2r}^i\}$. Since $4H_1^\bullet$, $4H_2^\bullet, \ldots, 4H_s^\bullet$ are vertex-disjoint, any 2-factor in $4G^\bullet$ is the union of 2-factors in $4H_1^\bullet, 4H_2^\bullet, \ldots, 4H_s^\bullet$. In particular, $F_j = \bigcup_{i=1}^{s} F_j^i$, for all j, is a 2-factor of $4G^\bullet$. Hence, $\mathcal{D} = \{F_1, \ldots, F_{2r}\}$ is a 2-factorization of $4G^\bullet$. In addition, each $\mathcal{D}_i$ is HOP so any $F_j^i \in \mathcal{D}_i$ is HOP. Thus for all j, F_j is HOP, and consequently the 2-factorization $\mathcal{D}$ is HOP.

∎

Lemma 2.2.5. [36] Let G be a simple graph. If G admits a $(C_{m_1}, \ldots C_{m_t})$-factorization, then $4G^\bullet$ admits an HOP $(C_{m_1}, \ldots C_{m_t})$-factorization.

In Chapter 3, this lemma is extended to Lemma 3.3.3 for the generalized HOP, and a similar proof to the one in [36] is given.

In the following corollary we see how we can find a solution to HOP$(2m_1, 2m_2, \ldots, 2m_t)$ by using a solution to OP$(m_1, m_2, \ldots, m_t)$.

Corollary 2.2.6. [36] Assume that $m_1, m_2, \ldots, m_t$ are integers with $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$. Let $n = m_1 + m_2 + \ldots + m_t$ be odd. Then HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution if OP$(m_1, m_2, \ldots, m_t)$ has a solution.

Proof. Since OP$(m_1, m_2, \ldots, m_t)$ has a solution and n is odd, the complete graph K_n admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization. By Lemma 2.2.5, then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization. Applying Theorem 2.2.3, we conclude that HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution. ∎

2.2.3.2 From $2G^\circ$ to $4G^\bullet$

The following lemma shows how to find a 2-factorization of $4G^\bullet$ using a suitable 2-factorization of $2G^\circ$. Later, in Chapter 3, this lemma will be extended to Lemma 3.3.5 for the generalized HOP, and a proof will be given based on the proof in [36].

Lemma 2.2.7. [36] Assume that $2G^\circ$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization $\mathcal{F}$ with the property that every m-cycle of $\mathcal{F}$, for $m \geq 3$, contains an even number of pink edges. Then $4G^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.

2.2.4 Constructing HOP 2-factorizations with Specific Starters

In this section, we present a method from [36] for constructing HOP 2-factorizations. In this approach, an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization is obtained by applying a rotation with a fixed point to specific $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors of $4K_n^\bullet$, referred to as *starters*.

This method allows us to construct a decomposition of $4K_n^\bullet$ either directly, using starters for $4K_n^\bullet$, or by using starters for a smaller graph and then extending them to a decomposition of $4K_n^\bullet$. The method is especially useful in cases where decompositions of smaller graphs are not available and we need to deal with $4K_n^\bullet$ directly.

2.2.4.1 Construction with a Single Starter

In this case, the rotation is applied to a single $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor of $2K_n^\circ$ to form an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $4K_n^\bullet$.

Proposition 2.2.8. [36] Let n be even, and let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Let F be a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor of $2K_n^\circ$ with the following properties.

(A1) Every cycle of F of length at least 3 contains an even number of pink edges, and

(A2) F contains exactly one edge from each of the orbits of $\langle \rho_\circ \rangle$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.

Proof. By the given labeling for the vertices of $2K_n^\circ$, each edge in $E(2K_n^\circ)$ is either of the form $x_i x_{i+d}$, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, or of the form $x_i x_\infty$ (if it is of difference infinity). Since $\rho_\circ$ is a permutation that preserves the color of the edges, the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$, and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Taken together, the group $\langle \rho_\circ \rangle$ has $2(\frac{n-2}{2}) + 2 = n$ orbits, and the size of each orbit is $n-1$. Moreover, we know $|E(F)| = m_1 + m_2 + \ldots + m_t = n$ and $|E(2K_n^\circ)| = n(n-1)$. Since F contains exactly one edge from each of the orbits of $\langle \rho_\circ \rangle$, we observe that, for all $i \in \mathbb{Z}_{n-1}$, the 2-factors $F_i = \rho_\circ^i(F)$ form a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization $\mathcal{D}$ for $2K_n^\circ$. Since every 2-factor in $\mathcal{D}$ is formed by applying permutation $\rho_\circ$ to the starter F, and $\rho_\circ$ preserves the edge colors, it satisfies condition **(A1)**. Therefore, since every cycle in $\mathcal{D}$ of length at least 3 contains an even number of pink edges, by Lemma 2.2.7, the graph $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization. ∎

The following lemma summarizes a set of results from [36], which were proved using the single starter approach described in Proposition 2.2.8.

Lemma 2.2.9. [36] The following HOP 2-factorizations exist.

(i) an HOP (C_2, C_4)-factorization of $4K_6^\bullet$;

(ii) an HOP (C_3, C_5)-factorization of $4K_8^\bullet$;

(iii) an HOP C_6-factorization of $4K_{12}^\bullet$;

(iv) an HOP C_m-factorization of $4K_{4m}^\bullet$, where m is an odd integer greater than 4;

(v) an HOP C_m-factorization of $4K_m^\bullet$, where $m \equiv 0 \pmod 4$;

(vi) an HOP (C_2, C_m)-factorization of $4K_{m+2}^\bullet$, where $m \equiv 3 \pmod 4$.

2.2.4.2 Construction with Two Starters

In this case, the rotation is applied to two edge-disjoint 2-factors of $4K_n^\bullet$ to form an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $4K_n^\bullet$.

Proposition 2.2.10. [36] Let n be even, and let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Let F_1 and F_2 be two edge-disjoint $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors of $4K_n^\bullet$ with the following properties.

(D1) Every cycle in F_1 and F_2 satisfies Condition **(C1)** of Definition 1.2.3 , and

(D2) F_1 and F_2 together contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i \in \mathbb{Z}_{n-1}\}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $4K_n^\bullet$.

Proof. By the given labeling for the vertices of $4K_n^\bullet$, each edge in $E(4K_n^\bullet)$ is in one of the following forms:

- $x_i x_{i+d}$ with blue color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$;
- $x_i x_{i+d}$ with pink color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$;
- (x_i, x_{i+d}) with black color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \ldots, n-2\}$;
- $x_i x_\infty$ with blue color, for $i \in \mathbb{Z}_{n-1}$;

- $x_i x_\infty$ with pink color, for $i \in \mathbb{Z}_{n-1}$;
- (x_i, x_∞) with black color, for $i \in \mathbb{Z}_{n-1}$;
- (x_∞, x_i) with black color, for $i \in \mathbb{Z}_{n-1}$.

Since $\rho_\bullet$ is a permutation that preserves the color and orientation of the edges, the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$;
- for $d \in \{1, 2, \ldots, n-2\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

Taken together, the group $\langle \rho_\bullet \rangle$ has $2(\frac{n-2}{2}) + 2 + (n-2) + 2 = 2n$ orbits, and the size of each orbit is $n-1$. Moreover, we know $|E(F_1)| + |E(F_2)| = 2n$ and $|E(4K_n^\bullet)| = 2n(n-1)$. Since F_1 and F_2 together contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$, we observe that $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i \in \mathbb{Z}_{n-1}\}$ is a 2-factorization for $4K_n^\bullet$. Since every 2-factor in $\mathcal{D}$ is formed by applying permutation $\rho_\bullet$ to the starter F_1 or F_2, and $\rho_\bullet$ preserves the edge colors, every 2-factor satisfies Condition **(C1)** of Definition 1.2.3. Therefore, $\mathcal{D}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $4K_n^\bullet$. ∎

The following lemma summarizes a set of results from [36], which were proved using two starters construction derived from Proposition 2.2.10.

Lemma 2.2.11. [36] The following HOP 2-factorizations exist.

(i) HOP C_3-factorizations of $4K_6^\bullet$, $4K_{12}^\bullet$, and $4K_{18}^\bullet$;

(ii) HOP (C_2, C_2, C_4)-, (C_2, C_3, C_3)-, and (C_2, C_6)-factorizations of $4K_8^\bullet$;

(iii) an HOP C_m-factorization of $4K_{2m}^\bullet$, for every odd $m \geq 5$;

(iv) an HOP C_m-factorization of $4K_m^\bullet$, for $m \in \mathbb{Z}$ such that $m \geq 6$ and $m \equiv 2 \pmod 4$.

2.2.4.3 Construction with Three Starters

In this case, the rotation is applied to three pairwise edge-disjoint 2-factors of $4K_n^\bullet$ to form an HOP 2-factorization of $4K_n^\bullet$.

Proposition 2.2.12. [36] Let n be odd, and let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Let F_1, F_2, and F_3 be three pairwise edge-disjoint $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors of $4K_n^\bullet$ with the following properties.

(E1) Every cycle in F_1, F_2, and F_3 satisfies Condition **(C1)** of Definition 1.2.3;

(E2) each orbit of $\langle \rho_\bullet \rangle$ has edges either in $F_1 \cup F_2$ or in F_3;

(E3) if e is an edge of $F_1 \cup F_2$, then $\rho_\bullet^{\frac{n-1}{2}}(e)$ is also an edge of $F_1 \cup F_2$;

(E4) $F_1 \cup F_2$ contains a pink and a blue edge of difference $\frac{n-1}{2}$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i = 0, 1, \dots, \frac{n-3}{2}\} \cup \{\rho_\bullet^i(F_3) : i \in \mathbb{Z}_{n-1}\}$ is an HOP $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-factorization of $4K_n^\bullet$.

Proof. By the given labeling for the vertices of $4K_n^\bullet$, each edge in $E(4K_n^\bullet)$ is in one of the following forms:

- $x_i x_{i+d}$ with blue color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \dots, \frac{n-3}{2}\}$;
- $x_i x_{i+d}$ with pink color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \dots, \frac{n-3}{2}\}$;
- $x_i x_{i+\frac{n-1}{2}}$ with blue color, for $i \in \{0, 1, \dots, \frac{n-3}{2}\}$;
- $x_i x_{i+\frac{n-1}{2}}$ with pink color, for $i \in \{0, 1, \dots, \frac{n-3}{2}\}$;
- (x_i, x_{i+d}) with black color, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \dots, n-2\}$;
- $x_i x_\infty$ with blue color, for $i \in \mathbb{Z}_{n-1}$;
- $x_i x_\infty$ with pink color, for $i \in \mathbb{Z}_{n-1}$;
- (x_i, x_∞) with black color, for $i \in \mathbb{Z}_{n-1}$;
- (x_∞, x_i) with black color, for $i \in \mathbb{Z}_{n-1}$.

Since $\rho_\bullet$ is a permutation that preserves the color and orientation of the edges, the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for $d \in \{1, 2, \dots, \frac{n-3}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_{i+\frac{n-1}{2}} : i = 0, 1, \dots, \frac{n-3}{2}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$;
- for $d \in \{1, 2, \dots, n-2\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

Taken together, the group $\langle \rho_\bullet \rangle$ has $2(\frac{n-3}{2}) + 2 + 2 + (n-2) + 2 = 2n + 1$ orbits. Note that size of each orbit is $n-1$, except the pink and the blue orbit consisting of edges of difference $\frac{n-1}{2}$; these two orbits have length $\frac{n-1}{2}$.

Assume O is an orbit of $\langle \rho_\bullet \rangle$; then by **(E2)**, it has edges either in $F_1 \cup F_2$ or in F_3.

Case 1: O has edges in F_3. Then the size of O is $n-1$ since the only orbits of size $\frac{n-1}{2}$ are the pink and the blue orbits consisting of edges of difference $\frac{n-1}{2}$, and by **(E4)** these orbits have edges in $F_1 \cup F_2$. Thus, $\{\rho_\bullet^i(F_3) : i \in \mathbb{Z}_{n-1}\}$ contains all edges of O.

Case 2: O has edges in $F_1 \cup F_2$. Then the size of O is either $n-1$ or $\frac{n-1}{2}$. If the size of O is $\frac{n-1}{2}$, then clearly $\{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i = 0, 1, \dots, \frac{n-3}{2}\}$ contains all edges of O. If the size of O is $n-1$, then by **(E3)**, we have that $e \in E(F_1 \cup F_2)$ implies $\rho_\bullet^{\frac{n-1}{2}}(e) \in E(F_1 \cup F_2)$. Hence, $\{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i = 0, 1, \dots, \frac{n-3}{2}\}$ contains all edges of O.

Note that the collection of orbits forms a partition of $E(4K_n^\bullet)$, and all edges in each orbit are covered by $\{\rho_\bullet^i(F_3) : i \in \mathbb{Z}_{n-1}\}$ or by $\{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2) : i = 0, 1, \dots, \frac{n-3}{2}\}$, and since these rotations cover $2n(n-1)$ edges, which is exactly equal to the number of edges in $4K_n^\bullet$, no edge is covered twice. Thus $\mathcal{D}$ forms a $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-factorization for $4K_n^\bullet$. Since every 2-factor in $\mathcal{D}$ is formed by applying permutation $\rho_\bullet$ to the starters, and every cycle in the starters satisfies Condition **(C1)** of Definition 1.2.3, and $\rho_\bullet$ preserves the edge colors, the 2-factorization D is HOP. Hence, $\mathcal{D}$ is an HOP $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-factorization of $4K_n^\bullet$. ∎

The following lemma summarizes a set of results from [36], which were proved using the three starter approach from Proposition 2.2.12.

Lemma 2.2.13. [36] The following HOP 2-factorizations exist.

(i) An HOP (C_2, C_2, C_3)-factorization of $4K_7^\bullet$;

(ii) HOP (C_2, C_2, C_2, C_3)-, (C_2, C_2, C_5)-, (C_2, C_3, C_4)-, and (C_4, C_5)-factorizations of $4K_9^\bullet$;

(iii) an HOP (C_3, C_3, C_5)-factorization of $4K_{11}^\bullet$;

(iv) an HOP (C_2, C_m)-factorization of $4K_{m+2}^\bullet$, for $m \in \mathbb{Z}$ such that $m \geq 5$ and $m \equiv 1 \pmod 4$.

2.2.5 Summary of the Proof of the Main Results from [36]

With the necessary tools in hand, we now present a summary of the proofs of the main results from [36].

The main result, stated earlier in this chapter as Theorem 2.2.2, is restated below to make the proof easier to follow.

Theorem 2.2.14. Let $m_1, m_2, \dots, m_t$ be integers with $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$. Let $n = m_1 + m_2 + \ldots m_t$. Then HOP$(2m_1, 2m_2, \dots, 2m_t)$ has a solution in each of the following cases.

(i) $m_i \equiv 0 \pmod 4$, for all i;

(ii) n is odd and OP$(m_1, m_2, \dots, m_t)$ has a solution;

(iii) n is odd and $t = 2$;

(iv) n is odd, $n \leq 100$, and $m_1 \geq 3$;

(v) $n \leq 20$;

(vi) $m_1 = m_2 = \ldots = m_t$.

Proof. (i): Assume $m_i \equiv 0 \pmod 4$, for all $i = 1, 2 \ldots, t$. Consider $2K_n^\circ$, and let I be a 1-factor in the pink copy of K_n. Choose a 1-factor I' in the black copy of K_n so that $I \bigoplus I'$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor in $2K_n^\circ$; this is possible since all m_i are even. Since for all $i \in \{1, 2, \ldots, t\}$, we have $m_i \geq 4$ and m_i is even, by Theorem 2.1.4, both $K_n - I$ (pink) and $K_n - I'$ (black) admit $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorizations $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. It can be seen that $\mathcal{D} = \{I \bigoplus I'\} \cup \mathcal{D}_1 \cup \mathcal{D}_2$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $2K_n^\circ$. Notice that any cycle in $\mathcal{D}_1$ is pink with an even length; there is no pink cycle in $\mathcal{D}_2$; and any cycle in $I \bigoplus I'$ has an even number of pink edges since $m_i \equiv 0 \pmod 4$, for all $i = 1, 2 \ldots, t$. Therefore, $\mathcal{D}$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $2K_n^\circ$ with the property that every cycle of $\mathcal{D}$ contains an even number of pink edges. Thus by Lemma 2.2.7, the graph $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization, and by Theorem 2.2.3, HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution.

(ii): Assume n is odd and OP$(m_1, m_2, \ldots, m_t)$ has a solution; then the result follows from Corollary 2.2.6.

(iii): Assume n is odd and $t = 2$. We have the following cases.

Case 1: $3 \leq m_1 \leq m_2$. If $(m_1, m_2) \neq (4, 5)$, then by Theorem 2.1.5, OP(m_1, m_2) has a solution, and by part **(ii)**, HOP$(2m_1, 2m_2)$ has a solution. For $(m_1, m_2) = (4, 5)$, by Lemma 2.2.13, an HOP (C_4, C_5)-factorization of $4K_9^\bullet$ exists.

Case 2: $m_1 = 2$.

If $m_2 \equiv 1 \pmod 4$, then $m_2 \geq 5$, and by Lemma 2.2.13, an HOP (C_2, C_{m_2})-factorization of $4K_{m_2+2}^\bullet$ exists.

If $m_2 \equiv 3 \pmod 4$, then by Lemma 2.2.9, an HOP (C_2, C_{m_2})-factorization of $4K_{m_2+2}^\bullet$ exists.

In all above cases, the graph $4K_n^\bullet$ admits an HOP (C_{m_1}, C_{m_2})-factorization, and by Theorem 2.2.3, HOP$(2m_1, 2m_2)$ has a solution.

(iv): Assume n is odd, $n \leq 100$, and $m_1 \geq 3$. In part **(iii)**, it was proved that, for $t = 2$, HOP$(2m_1, 2m_2)$ has a solution; hence, we may assume $t \geq 3$.

If $(m_1, m_2, m_3) \neq (3, 3, 5)$ and $(m_1, m_2, m_3, m_4) \neq (3, 3, 3, 3)$, then by Theorem 2.1.3, OP$(m_1, m_2, \ldots, m_t)$ has a solution, and by part **(ii)**, HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution.

If $(m_1, m_2, m_3) = (3, 3, 5)$, then by Lemma 2.2.13, an HOP (C_3, C_3, C_5)-factorization of $4K_{11}^\bullet$ exists, and by Theorem 2.2.3, HOP$(6, 6, 10)$ has a solution.

If $(m_1, m_2, m_3, m_4) = (3, 3, 3, 3)$, the result follows from part **(vi)**.

(v): First assume $n \leq 9$. We have the following cases:

- If $m_1 = m_1 = \ldots = m_t$, then the result follows from part **(vi)**.
- If n is odd and $t = 2$, then the result follows from part **(iii)**.
- If n is odd and $3 \leq m_1$, then the result follows from part **(iv)**.
- If $(m_1, m_2, \ldots, m_t) \in \{(2, 4), (3, 5)\}$, then the result follows from Lemma 2.2.9.
- If $(m_1, m_2, \ldots, m_t) \in \{(2, 2, 4), (2, 3, 3), (2, 6)\}$, then the result follows from Lemma 2.2.11.
- If $(m_1, m_2, \ldots, m_t) \in \{(2, 2, 3), (2, 2, 2, 3), (2, 2, 5), (2, 3, 4)\}$, then the result follows from Lemma 2.2.13.

For $10 \leq n \leq 20$, it was verified that $\text{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ has a solution in [31].

(vi): Assume $m_1 = m_2 = \ldots = m_t = m$, that is, $n = tm$. We have the following two cases.

Case 1: m is even. Then n is even as well.

First, let $m = 2$, and let $\mathcal{F}$ be a 1-factorization of K_n. Note that we can construct two C_2-factors of $4K_n$ from each 1-factor F in $\mathcal{F}$ as follows. Each edge $e = xy \in E(F)$ gives rise to a directed black 2-cycle C, and a 2-cycle C' with a pink and a blue edge in $4K_n^\bullet$. Notice that $C \bigoplus C'$ corresponds to a 4-set of parallel edges between x and y with two black arcs in the opposite directions, one blue, and one pink edge. Therefore, the resulting C_2-factors form a C_2-factorization of $4K_n^\bullet$. Moreover, it is easy to verify this C_2-factorization is HOP.

Second, assume $m \geq 4$, but $m \neq 6$.

- If $t = 1$, then $n = m$. We observed in Lemma 2.2.9 that an HOP C_m-factorization of $4K_m^\bullet$ exists for $m \equiv 0 \pmod 4$, and in Lemma 2.2.11 that an HOP C_m-factorization of $4K_m^\bullet$ exists for $m \equiv 2 \pmod 4$. Therefore, the graph $4K_n^\bullet$ admits an HOP C_m-factorization.
- If $t \geq 2$, then $n = tm$. It is easy to verify that K_{tm} can be decomposed into two spanning subgraphs H_1 and H_2, where H_1 is a disjoint union of t copies of K_m, and H_2 is $K_{t[m]}$. In Lemma 2.2.9 and 2.2.11, we observed that $4K_m^\bullet$ admits an HOP C_m-factorization. Furthermore, since $m \neq 6$, by Theorem 2.1.9, the graph $K_{t[m]}$ admits a C_m-factorization, and by Lemma 2.2.5, there exists an HOP C_m-factorization of $4K_{t[m]}^\bullet$. Therefore, each of $4H_1^\bullet$ and $4H_2^\bullet$ admits an HOP C_m-factorization. By Lemma 2.2.4, the graph $4K_n^\bullet$ admits an HOP C_m-factorization.

Third, assume $m = 6$.

- If $t = 1$, then $n = 6$. By Lemma 2.2.11, an HOP C_6-factorization of $4K_6^\bullet$ exists.

- If $t = 2$, then $n = 12$. By Lemma 2.2.9, an HOP C_6-factorization of $4K_{12}^{\bullet}$ exists.
- If $t \geq 3$, then $n = 6t$. Consider a decomposition of K_{6t} into two spanning subgraphs H_1 and H_2, where H_1 is a disjoint union of t copies of K_6, and H_2 is $K_{t[6]}$. By Lemma 2.2.11, the graph $4K_6^{\bullet}$ admits an HOP C_6-factorization. Furthermore, since $t \geq 3$, by Theorem 2.1.9, the graph $K_{t[6]}$ admits a C_6-factorization, and by Lemma 2.2.5, there exists an HOP C_6-factorization of $4K_{t[6]}^{\bullet}$. Therefore, each of $4H_1^{\bullet}$ and $4H_2^{\bullet}$ admits an HOP C_6-factorization. By Lemma 2.2.4, the graph $4K_n^{\bullet}$ admits an HOP C_6-factorization.

Case 2: m is odd. Then n is either odd or even.

– Assume n is odd. By Theorem 2.1.2, $\mathrm{OP}(n; m)$ has a solution, and by part **(ii)**, $\mathrm{HOP}(2n; 2m)$ has a solution.

– Assume n is even. Since m is odd, t must be even.

First, let $m = 3$.

- If $t = 2$, then $n = 6$. By Lemma 2.2.11, an HOP C_3-factorization of $4K_6^{\bullet}$ exists.
- If $t = 4$, then $n = 12$. By Lemma 2.2.11, an HOP C_3-factorization of $4K_{12}^{\bullet}$ exists.
- If $t = 6$, then $n = 18$. By Lemma 2.2.11, an HOP C_3-factorization of $4K_{18}^{\bullet}$ exists.
- If $t \geq 8$, then we can let $n = t'(2 \cdot 3)$, where $t' \geq 4$. Consider a decomposition of $K_{6t'}$ into two spanning subgraphs H_1 and H_2, where H_1 is a disjoint union of t' copies of K_6, and H_2 is $K_{t'[6]}$. By Lemma 2.2.11, the graph $4K_6^{\bullet}$ admits an HOP C_3-factorization. Furthermore, since $t' \geq 4$, by Theorem 2.1.9 the graph $K_{t'[6]}$ admits a C_3-factorization, and by Lemma 2.2.5, there exists an HOP C_3-factorization of $4K_{t'[6]}^{\bullet}$. Therefore, each of $4H_1^{\bullet}$ and $4H_2^{\bullet}$ admits an HOP C_3-factorization. By Lemma 2.2.4, the graph $4K_n^{\bullet}$ admits an HOP C_6-factorization.

Now, let $m \geq 5$.

- If $t = 2$, then $n = 2m$. By Lemma 2.2.11, the graph $4K_{2m}^{\bullet}$ admits an HOP C_m-factorization.
- If $t = 4$, then $n = 4m$. By Lemma 2.2.9, the graph $4K_{4m}^{\bullet}$ admits an HOP C_m-factorization.
- If $t \geq 6$, then we can let $n = t'(2m)$, where $t' \geq 3$. Consider a decomposition of $K_{t'(2m)}$ into two spanning subgraphs H_1 and H_2, where H_1 is a disjoint union of t' copies of K_{2m}, and H_2 is $K_{t'[2m]}$. By Lemma 2.2.11, the graph $4K_{2m}^{\bullet}$ admits an HOP C_m-factorization. Furthermore, since m is odd, $m \geq 5$, and $t' \geq 3$, we have $(m, t') \notin \{(3, 3)\}$; by Theorem 2.1.9, the graph $K_{t'[2m]}$ admits a C_m-factorization, and by Lemma 2.2.5, there exists an HOP C_m-factorization of $4K_{t'[2m]}^{\bullet}$. Therefore, each of $4H_1^{\bullet}$ and $4H_2^{\bullet}$ admits an HOP C_m-factorization. By Lemma 2.2.4, the graph $4K_n^{\bullet}$ admits an HOP C_m-factorization.

In all cases, it is proved that $4K_n^\bullet$ admits an HOP C_m-factorization. Hence, by Theorem 2.2.3, HOP$(2n; 2m)$ has a solution. ∎

2.3 Previous Results on Decompositions into 2-regular Subgraphs

In Chapter 3, we introduce the generalized HOP and demonstrate that the main approach used for proving results on HOP can be extended to the generalized HOP as well. As in HOP, the reduction step involves converting the problem to the multigraph $4K_n^\bullet$. Therefore, the goal is to find HOP decompositions of $4K_n^\bullet$ into 2-regular subgraphs. We use previous results on the decomposition of smaller graphs, such as K_n and $2K_n$, to construct decompositions of $4K_n$. Hence, in this section, we provide up-to-date results on decompositions of K_n and $2K_n$ into 2-regular subgraphs.

2.3.1 Decompositions of K_n into 2-regular Subgraphs

Here, we provide a summary of known results on decompositions of K_n into 2-regular subgraphs, in particular those that will later be used to derive solutions to the generalized HOP.

Theorem 2.3.1. [39] There exists a (C_3)-decomposition of K_n if and only if $n \equiv 1$ or $3 \pmod 6$.

Theorem 2.3.2. [5, 44] Let $3 \leq m \leq n$ be integers, and let n be odd. Then K_n admits a (C_m)-decomposition if and only if m divides the number of edges in K_n.

Theorem 2.3.3. [24] Let $t \geq 1$ and $m \geq 3$ be integers, and let $n = 2tm + 1$. Then there exists a cyclic $(C_m^{\langle t \rangle})$-decomposition of K_n.

Theorem 2.3.4. [1, 3, 11, 15, 20] Let $m_1, m_2 \geq 3$ and $m = m_1 + m_2$. Then there exists a (C_{m_1}, C_{m_2})-decomposition of K_n in each of the following cases.

(1) $n \equiv 1 \pmod{2m}$;

(2) m odd, $n \equiv m \pmod{2m}$, and $(m_1, m_2, n) \neq (4, 5, 9)$.

Theorem 2.3.5. [10, 21] Let $m_1, m_2, \ldots, m_t$ be even integers, and let $m = \sum_{i=1}^t m_i$. Then there exists a $(C_{m_1}, \ldots, C_{m_t})$-decomposition of K_n for all $n \equiv 1 \pmod{2m}$.

Theorem 2.3.6. [33] Let $m_1, m_2, m_3 \geq 3$ be odd integers and $m = m_1 + m_2 + m_3$. There exists a $(C_{m_1}, C_{m_2}, C_{m_3})$-decomposition of each of the following graphs:

- K_{2km+1}, for every positive integer $k \neq 2$;
- $K_{(2k+1)[m]}$, for every positive integer k;
- $K_{k[2m]}$, for any integer $k \geq 3$.

Theorem 2.3.7. [3] Let $m_1, m_2, \ldots, m_t$ be integers such that $m_i \geq 3$ for all i, and $m_1 + \cdots + m_t = m \leq 10$. Then there exists a $(C_{m_1}, \ldots, C_{m_t})$-decomposition of K_n if and only if all of the following conditions hold.

- $m \leq n$;
- n is odd;
- m divides the number of edges in K_n; and
- $n \neq 9$ when $(m_1, m_2, \ldots, m_t) \in \{(3,3),(4,5)\}$.

For any graph G, there are three obvious necessary conditions for the existence of a G-decomposition of K_n:

(i) $|V(G)| \leq n$;

(ii) $n(n-1) \equiv 0 \pmod{2|E(G)|}$; and

(iii) $n-1 \equiv 0 \pmod{d}$ where d is the greatest common divisor of the degrees of the vertices in G.

In [49], Wilson has shown that these conditions are asymptotically sufficient.

Theorem 2.3.8. [49] For any given graph G, there exists an integer $N(G)$ exponentially large in n such that if $n > N(G)$, then conditions (ii) and (iii) above are sufficient for the existence of a G-decomposition of K_n.

Theorem 2.3.9. [46] Let m, r, and s be even positive integers. The complete bipartite graph $K_{r,s}$ has a (C_m)-decomposition if and only if $min\{r,s\} \geq \frac{m}{2}$ and m divides the number of edges in $K_{r,s}$.

2.3.2 Decompositions of $2K_n$ into 2-regular Subgraphs

In this section, we present some known results on decompositions of $2K_n$ into 2-regular subgraphs.

Theorem 2.3.10. [6] Let m and n be integers such that $2 \leq m \leq n$. The multigraph $2K_n$ admits a (C_m)-decomposition if and only if m divides the number of edges in $2K_n$.

Theorem 2.3.11. [16, 27] Let m, n, t be integers such that $2 \leq m \leq n$ and $n = mt + 1$. Then the multigraph $2K_n$ admits a $(C_m^{\langle t \rangle})$-decomposition.

Chapter 3

The Generalized Honeymoon Oberwolfach Problem

In this chapter, we introduce the generalized Honeymoon Oberwolfach Problem. We begin by defining the problem, then adapt the techniques used for the Honeymoon Oberwolfach Problem in [36] to its generalized form. We also discuss the approaches and tools needed to address the generalized HOP.

3.1 Introduction

In [36], the HOP has been defined for tables of size at least 4. We generalize the problem to allow for tables of size 2. As in HOP, we have $2n$ participants consisting of n newlywed couples, but this time we want to see whether it is possible to seat them at s tables of size 2 and t tables of sizes $2m_1, 2m_2, \ldots, 2m_t$, for $m_i \geq 2$, for several consecutive nights so that each participant sits next to their spouse every time, and next to each other participant exactly once. We assume that $2n = 2s + 2m_1 + 2m_2 + \ldots + 2m_t$, where $2 \leq m_1 \leq \ldots \leq m_t$, and $s \geq 0$, and we denote this problem by $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2, \ldots, 2m_t)$. We refer to tables of size at least 4 as *round tables.*

We would like to model the generalized HOP using an appropriate graph derived from K_{2n}. Let I denote the 1-factor of K_{2n} corresponding to the n couples. Then, an arrangement for the first night is equivalent to an I-alternating $(K_2^{\langle s \rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factor of this graph. Moreover, a solution to $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2, \ldots, 2m_t)$ is equivalent to an I-alternating $(K_2^{\langle s \rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factorization of a graph of the form $K_{2n} + (\gamma - 1)I$, for an appropriate γ (to be derived below).

Note that although a solution to $\mathrm{HOP}(2m_1, 2m_2, \ldots, 2m_t)$ is equivalent to a semi-uniform 1-factorization of K_{2n} of type $(2m_1, 2m_2, \ldots, 2m_t)$, as seen in Lemma 2.2.1, in general, a solution to the generalized HOP does not correspond to such a factorization.

The obvious necessary condition for $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2, \ldots, 2m_t)$ to have a solution is stated in the following lemma.

Lemma 3.1.1. Let $s \geq 0$ and $2 \leq m_1 \leq \ldots \leq m_t$ be integers. Let $n = s+m_1+m_2+\ldots+m_t$. Then $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution only if $\left(\sum_{i=1}^{t} m_i\right) \mid 2n(n-1)$.

Proof. Assume $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution. Then $K_{2n} + (\gamma - 1)I$, for an appropriate γ, admits an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factorization. It follows that the number of non-I-edges in each $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factor, that is, $\frac{1}{2}\sum_{i=1}^{t} 2m_i$, must divide the number of non-I-edges in $K_{2n} + (\gamma-1)I$, that is, $\frac{2n(2n-2)}{2}$. In other words, we have $\left(\sum_{i=1}^{t} m_i\right) \mid 2n(n-1)$. ∎

Note that if $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution, then $\frac{2n(n-1)}{\sum_{i=1}^{t} m_i}$ is an integer, and it is exactly the number of $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factors in the required decomposition of $K_{2n} + (\gamma-1)I$; that is to say, $\gamma = \frac{2n(n-1)}{\sum_{i=1}^{t} m_i}$.

3.2 The Main Approach

The following theorem is a generalization of Theorem 2.2.3 [36]. The approach here, just as in [36], is to convert the problem from the multigraph $K_{2n}+(\gamma-1)I$ to the multigraph $4K_n^{\bullet}$.

Theorem 3.2.1. Let $s \geq 0$ and $2 \leq m_1 \leq \ldots \leq m_t$ be integers. Let $n = s + m_1 + m_2 + \ldots + m_t$. Then $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, 2m_2, \ldots, 2m_t)$ has a solution if and only if $4K_n^{\bullet}$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. First, assume $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, 2m_2, \ldots, 2m_t)$ has a solution. By Lemma 3.1.1, we have $\left(\sum_{i=1}^{t} m_i\right) | 2n(n-1)$; let $\gamma = \frac{2n(n-1)}{\sum_{i=1}^{t} m_i}$. Thus we have a factorization of $K_{2n}+(\gamma-1)I$ into I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factors. Let $\mathcal{F}$ be such an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factorization.

Let the vertex set of $K_{2n} + (\gamma-1)I$ be

$$V(K_{2n} + (\gamma-1)I) = \{x_i^{\alpha} : i \in \mathbb{Z}_n,\ \alpha \in \mathbb{Z}_2\},$$

where $x_i^0 x_i^1 \in E(I)$, for all $i \in \mathbb{Z}_n$. Let the vertex set of $4K_n^{\bullet}$ be

$$V(4K_n^{\bullet}) = \{x_i : i \in \mathbb{Z}_n\}.$$

We know there are four edges between any two vertices of $4K_n^{\bullet}$, one pink, one blue, and two black arcs of opposite directions. Assign to each non-I-edge of $K_{2n} + (\gamma-1)I$ an edge of $4K_n^{\bullet}$ as follows. For $i, j \in \mathbb{Z}_n$, and $i \neq j$,

- Assign the blue copy of $x_i x_j$ to the edge $x_i^0 x_j^0$.
- Assign the pink copy of $x_i x_j$ to the edge $x_i^1 x_j^1$.
- Assign the black arc (x_i, x_j) to the edge $x_i^0 x_j^1$.

- Assign the black arc (x_j, x_i) to the edge $x_i^1 x_j^0$.

This mapping, call it ϕ, is a bijection from the set of non-I-edges of $K_{2n} + (\gamma - 1)I$ to the edge set of $4K_n^\bullet$. Let $C = x_{i_0}^{\alpha} x_{i_0}^{1-\alpha} x_{i_1}^{\beta} x_{i_1}^{1-\beta} \dots x_{i_{m-1}}^{\delta} x_{i_{m-1}}^{1-\delta} x_{i_0}^{\alpha}$ be a $2m$-cycle in $\mathcal{F}$. By contracting the I-edges in the sequence C, and applying the bijection ϕ, we obtain a sequence C' of length m that is a cycle in $4K_n^\bullet$. Thus, ϕ can be viewed as mapping each I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \dots, C_{2m_t})$-factor in $\mathcal{F}$ to a $(C_{m_1}, \dots, C_{m_t})$-subgraph in $4K_n^\bullet$; and hence the I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \dots, C_{2m_t})$-factorization $\mathcal{F}$ to a $(C_{m_1}, \dots, C_{m_t})$-decomposition $\mathcal{D}$ of $4K_n^\bullet$. Notice that the copies of K_2 in the $(K_2^{\langle s\rangle}, C_{2m_1}, \dots, C_{2m_t})$-factors "disappear" in this mapping; in other words, they are irrelevant.

It remains to prove C' satisfies Condition **(C1)** of Definition 1.2.3. Take any two adjacent edges in C', say $x_i x_j$ and $x_j x_k$; they correspond to the path $P = x_i^{\alpha} x_j^{\beta} x_j^{1-\beta} x_k^{\delta} x_k^{1-\delta}$ in C.

- If $x_i x_j$ is blue, then $\alpha = \beta = 0$, and we have $P = x_i^0 x_j^0 x_j^1 x_k^{\delta} x_k^{1-\delta}$.
  - If $\delta = 1$, then $x_j x_k$ is pink.
  - If $\delta = 0$, then $x_j x_k$ is black and oriented from x_k to x_j.
- If $x_i x_j$ is pink, then $\alpha = \beta = 1$ and we have $P = x_i^1 x_j^1 x_j^0 x_k^{\delta} x_k^{1-\delta}$.
  - If $\delta = 1$, then $x_j x_k$ is black and oriented from x_j to x_k.
  - If $\delta = 0$, then $x_j x_k$ is blue.
- If $x_i x_j$ is black with orientation from x_i to x_j, then $\alpha = 0, \beta = 1$ and we have $P = x_i^0 x_j^1 x_j^0 x_k^{\delta} x_k^{1-\delta}$.
  - If $\delta = 1$, then $x_j x_k$ is black and oriented from x_j to x_k.
  - If $\delta = 0$, then $x_j x_k$ is blue.
- If $x_i x_j$ is black with orientation from x_j to x_i, then $\alpha = 1, \beta = 0$ and we have $P = x_i^1 x_j^0 x_j^1 x_k^{\delta} x_k^{1-\delta}$.
  - If $\delta = 1$, then $x_j x_k$ is pink.
  - If $\delta = 0$, then $x_j x_k$ is black and oriented from x_k to x_j.

We observe that any two adjacent edges in C' satisfy Condition **(C1)** of Definition 1.2.3; hence, $\mathcal{D}$ is an HOP $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Conversely, assume $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-decomposition $\mathcal{D}$. Thus, the number of edges of each $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-subgraph in $\mathcal{D}$, that is, $\sum_{i=1}^{t} m_i$, must divide the number of edges of $4K_n^\bullet$, that is, $2n(n-1)$. In other words, we have $\left(\sum_{i=1}^{t} m_i\right) | 2n(n-1)$, and $\gamma = \frac{2n(n-1)}{\sum_{i=1}^{t} m_i}$ is an integer. Label the vertices of $K_{2n} + (\gamma - 1)I$ and $4K_n^\bullet$ as before. Let $C' = x_{i_0} x_{i_1} \dots x_{i_{m-1}} x_{i_0}$ be an arbitrary m-cycle in $\mathcal{D}$. Create a sequence C'' by duplicating each vertex in the sequence of C' except the last vertex. Thus, we have

$C'' = x_{i_0}x_{i_0}x_{i_1}x_{i_1}x_{i_2}x_{i_2}\ldots x_{i_{m-1}}x_{i_{m-1}}x_{i_0}$. Notice that this sequence, viewed cyclically, decomposes into subsequences of length two of the form x_ix_j for $i \neq j$; that is, $C'' = x_{i_0}x_{i_0}x_{i_1}x_{i_1}x_{i_2}x_{i_2}\ldots x_{i_{m-1}}x_{i_{m-1}}x_{i_0}$. For each such subsequence $e' = x_ix_j$, replace e' with an edge e of $K_{2n} + (\gamma - 1)I$ as follows:

- If e' is blue, then let $e = x_i^0x_j^0$.
- If e' is pink, then let $e = x_i^1x_j^1$.
- If e' is black and oriented from x_i to x_j, then let $e = x_i^0x_j^1$.
- If e' is black and oriented from x_j to x_i, then let $e = x_i^1x_j^0$.

Denote the resulting sequence by C. Notice that the cycle C' has property **(C1)**; that is, any two adjacent edges, say x_ix_j and x_jx_k, which correspond to the path $P = x_i^\alpha x_j^\beta x_j^{1-\beta}x_k^\delta x_k^{1-\delta}$ in C, satisfy one of the following:

- One is blue and the other pink.
  - If x_ix_j is blue and x_jx_k is pink, then $P = x_i^0x_j^0x_j^1x_k^1x_k^0$.
  - If x_ix_j is pink and x_jx_k is blue, then $P = x_i^1x_j^1x_j^0x_k^0x_k^1$.
- Both are black and have the same directions.
  - If x_ix_j is oriented from x_i to x_j, and x_jx_k is oriented from x_j to x_k, then $P = x_i^0x_j^1x_j^0x_k^1x_k^0$.
  - If x_ix_j is oriented from x_j to x_i, and x_jx_k is oriented from x_k to x_j, then $P = x_i^1x_j^0x_j^1x_k^0x_k^1$.
- If x_ix_j is blue and x_jx_k is black with the orientation from x_k to x_j, then $P = x_i^0x_j^0x_j^1x_k^0x_k^1$.
- If x_ix_j is pink and x_jx_k is black with the orientation from x_j to x_k, then $P = x_i^1x_j^1x_j^0x_k^1x_k^0$.

Note that in all cases, any consecutive vertices with the same subscript have distinct superscripts in P. Therefore, C is an I-alternating $2m$-cycle in $K_{2n} + (\gamma - 1)I$. Thus, we see that a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph of $4K_n^\bullet$ "lifts" to an I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-subgraph of $K_{2n}+(\gamma-1)I$. We extend this subgraph to an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factor of $K_{2n} + (\gamma - 1)I$ as follows. We know there are $2s$ vertices in the vertex set of $K_{2n} + (\gamma-1)I$ that are not in the cycles $C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t}$, since $2n = 2s+2m_1+2m_2+\ldots+2m_t$. Moreover, since the t cycles are I-alternating, we can find s pairs among the remaining $2s$ vertices such that each pair forms an I-edge. Thus, any I-alternating $(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_t})$-subgraph extends uniquely to an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factor. Therefore, any $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph in $\mathcal{D}$ "lifts" to an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factor in $K_{2n} + (\gamma - 1)I$. Since $|\mathcal{D}| = \gamma$, we have γ I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factors which form an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factorization of $K_{2n} + (\gamma - 1)I$. In other words, an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$ gives rise to an I-alternating $(K_2^{\langle s\rangle}, C_{2m_1}, \ldots, C_{2m_t})$-factorization of $K_{2n} + (\gamma - 1)I$, and this is equivalent to a solution for HOP$(2^{\langle s\rangle}, 2m_1, 2m_2, \ldots, 2m_t)$. ∎

In the following examples, we apply the construction from Theorem 3.2.1 to build solutions to specific cases of the HOP.

In Example 3.2.2, we show that an HOP C_4-factorization of $4K_4^\bullet$ gives rise to a solution to HOP$(8; 8)$, and in Example 3.2.3, we show that an HOP (C_2)-decomposition of $4K_3^\bullet$ gives rise to a solution to HOP$(2, 4)$.

Example 3.2.2. Let $\mathcal{D} = \{C_1', C_2', C_3', C_4', C_5', C_6'\}$ be the HOP C_4-factorization of $4K_4^\bullet$ shown in Figure 3.1 (see Example 3.3.6 for how this 2-factorization was obtained).

Using the construction from the proof of Theorem 3.2.1, we build an I-alternating C_8-factorization of $K_8 + 5I$, which gives a solution to HOP$(8; 8)$ (see Figures 3.2–3.5). Note that here $n = 4$, $s = 0$, and $\gamma = 6$.

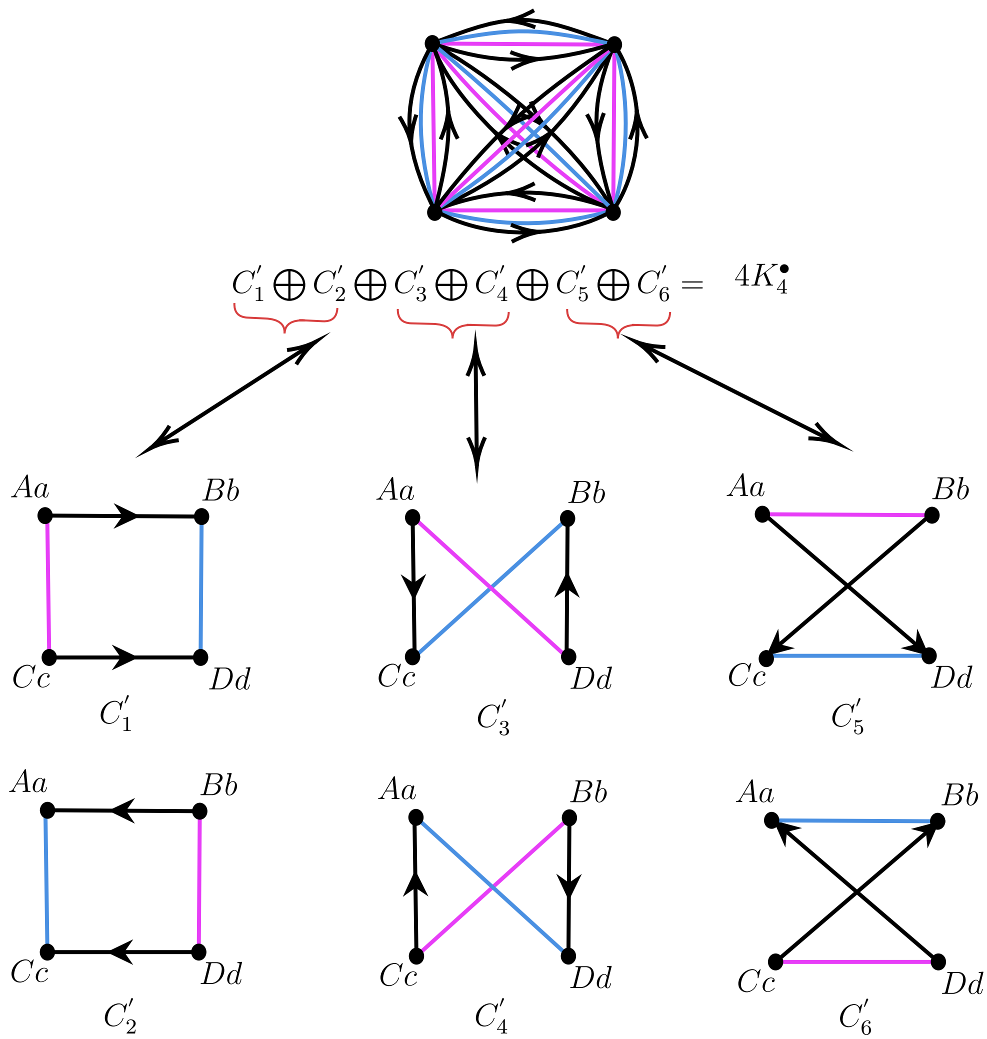


Figure 3.1: An HOP C_4-factorization of $4K_4^\bullet$

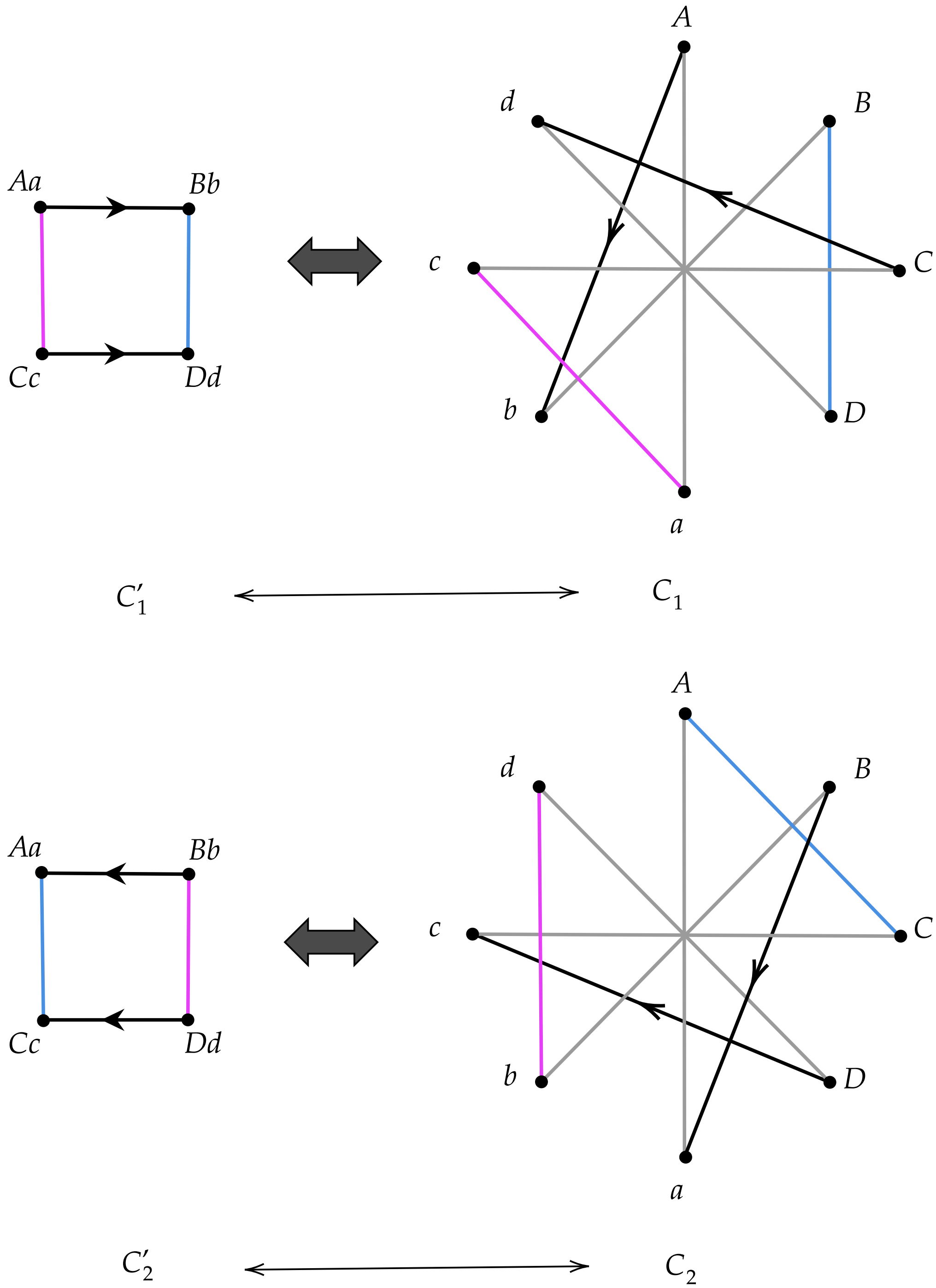


Figure 3.2: Part 1: from an HOP C_4-factor to an I-alternating C_8-factor.

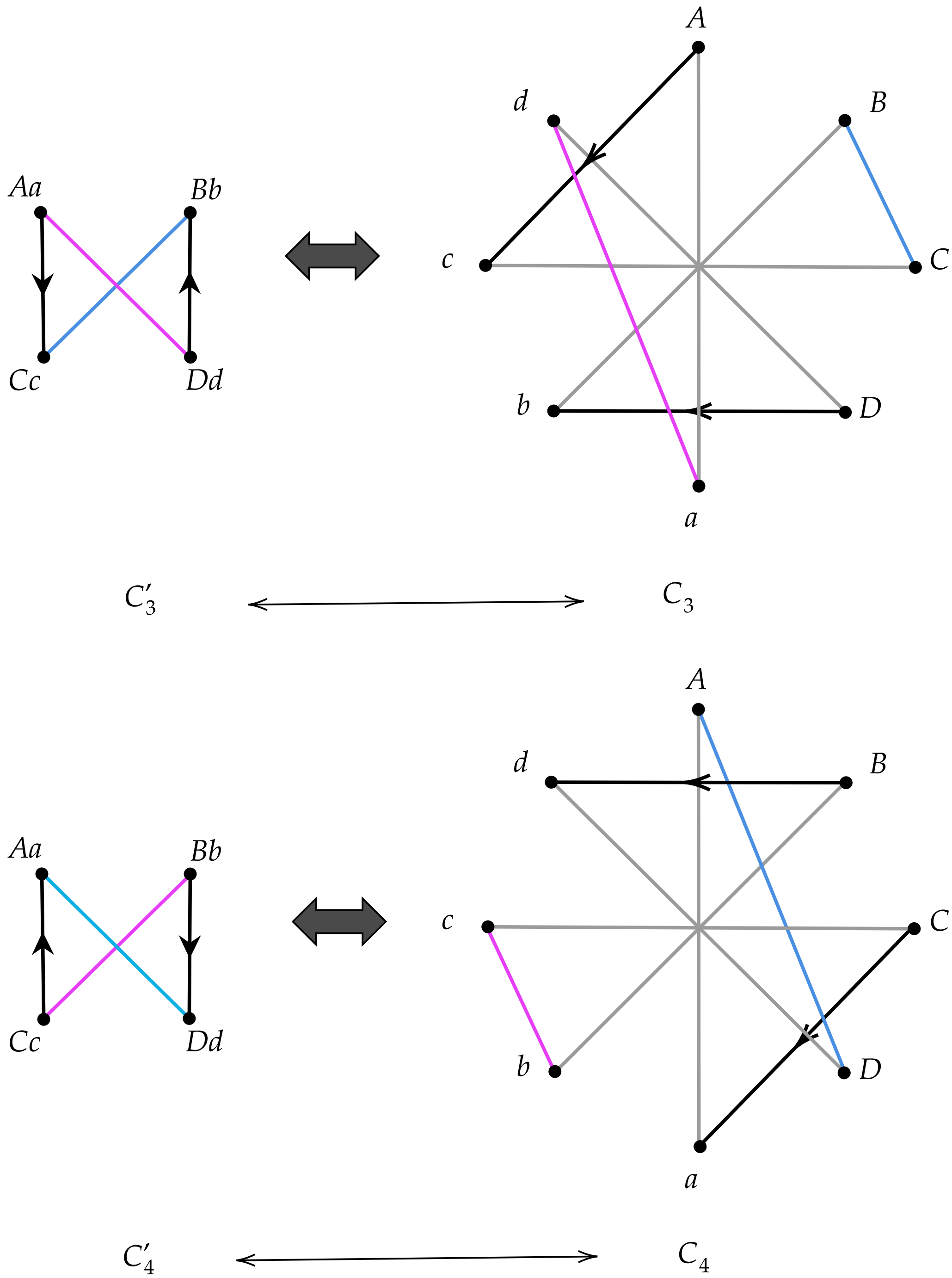


Figure 3.3: Part 2: from an HOP C_4-factor to an I-alternating C_8-factor.

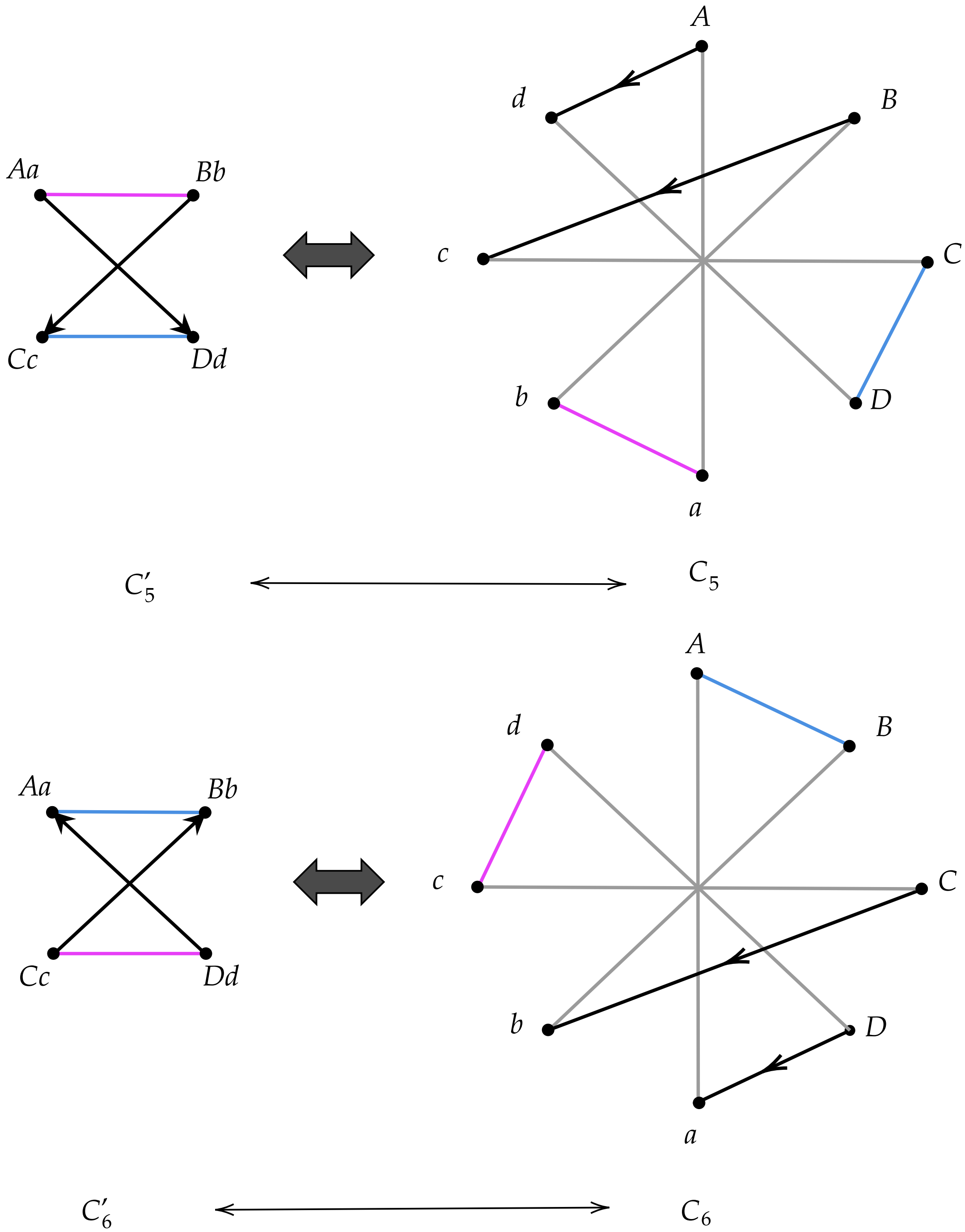
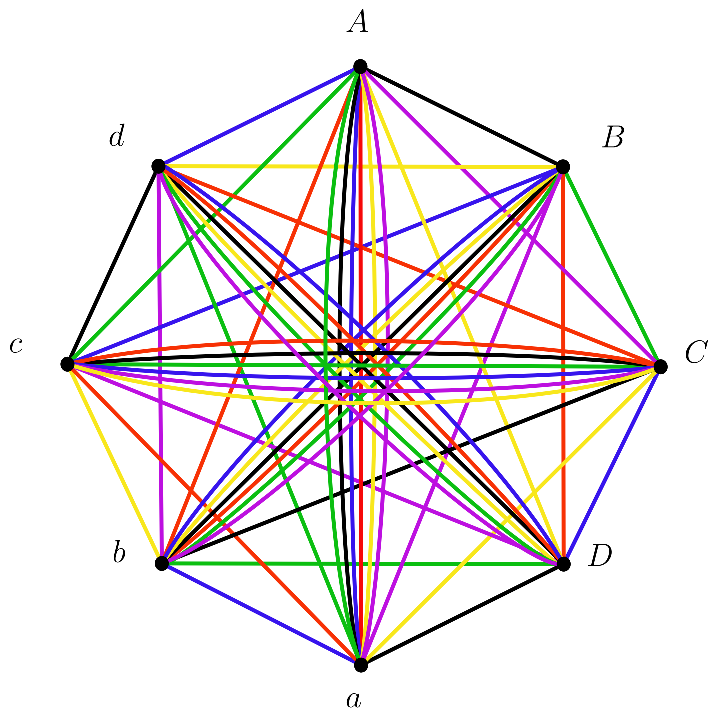


Figure 3.4: Part 3: from an HOP C_4-factor to an I-alternating C_8-factor.



$$C_1 \bigoplus C_2 \bigoplus C_3 \bigoplus C_4 \bigoplus C_5 \bigoplus C_6 = K_8 + 5I$$

Figure 3.5: From an HOP C_4-factorization of $4K_4^{\bullet}$ to an I-alternating C_8-factorization of $K_8 + 5I$

Example 3.2.3. Let $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5, F_6\}$ be the HOP (C_2)-decomposition of $4K_3^{\bullet}$ shown in Figure 3.6. Note that here $n = 3$, $s = 1$, and $\gamma = 6$.

Using the construction from the proof of Theorem 3.2.1, we obtain an I-alternating (K_2, C_4)-factorization of $K_6 + 5I$, which gives a solution to HOP$(2, 4)$ (see Figures 3.7 and 3.8).

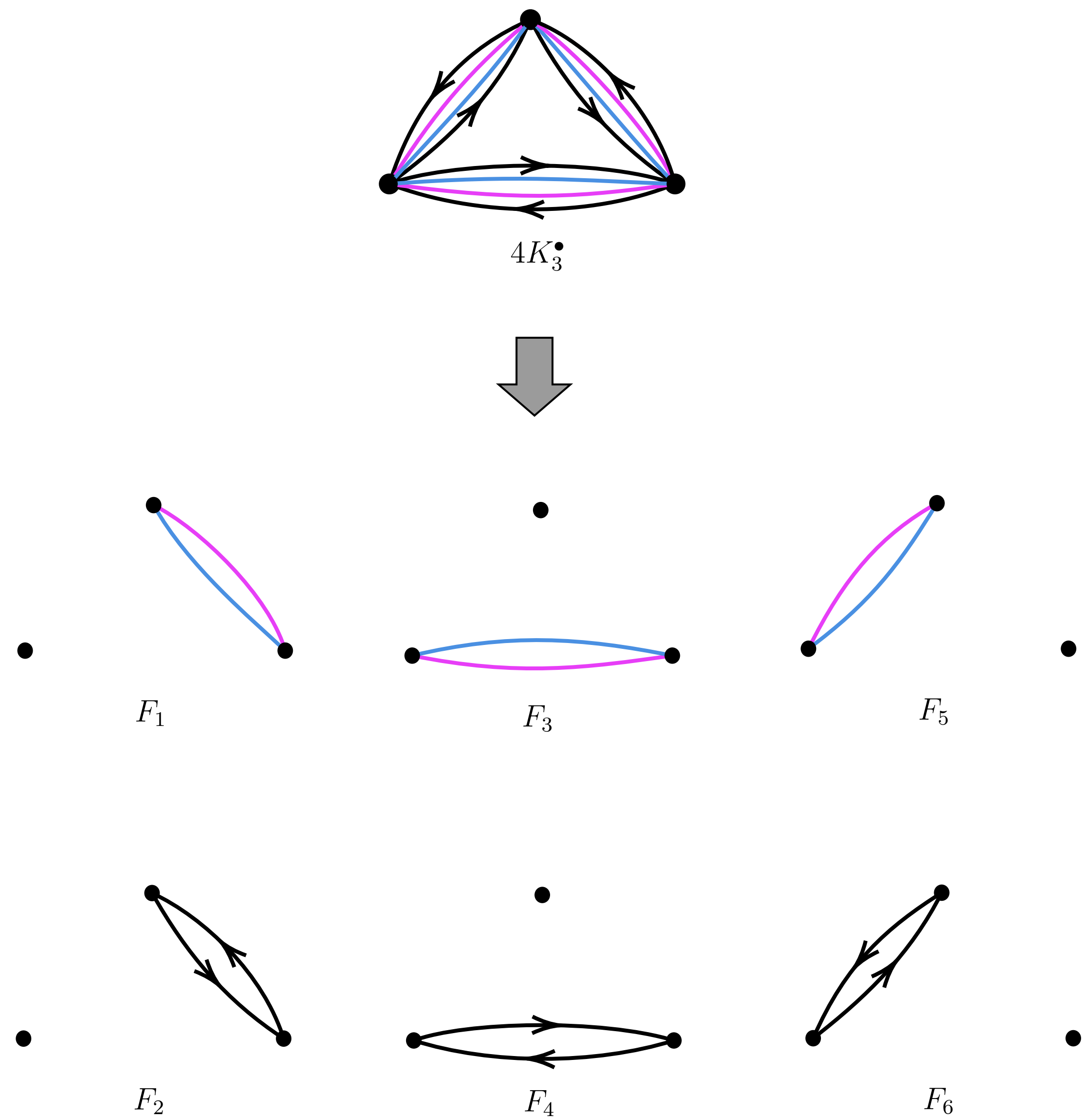


Figure 3.6: An HOP (C_2)-decomposition of $4K_3^\bullet$

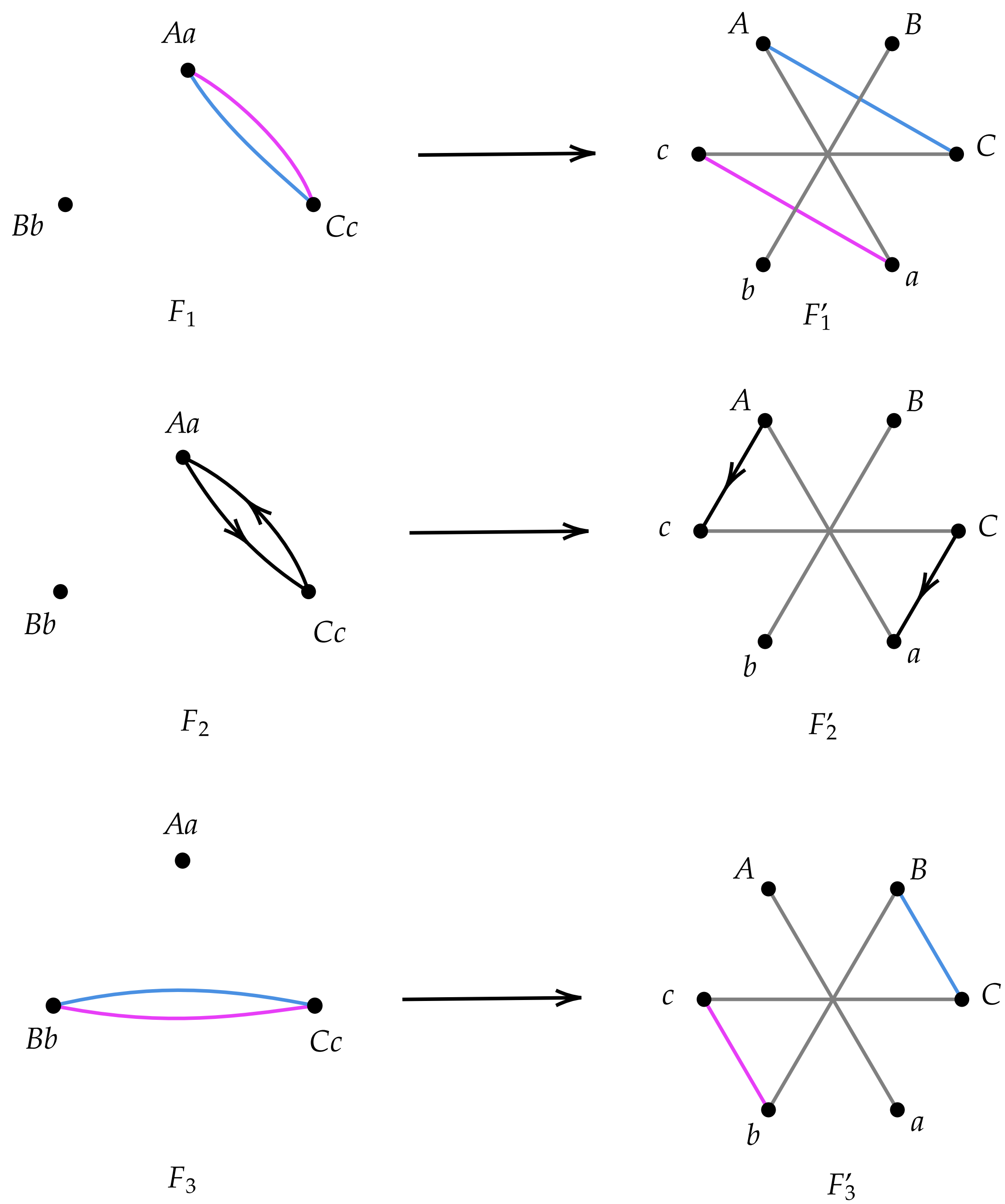
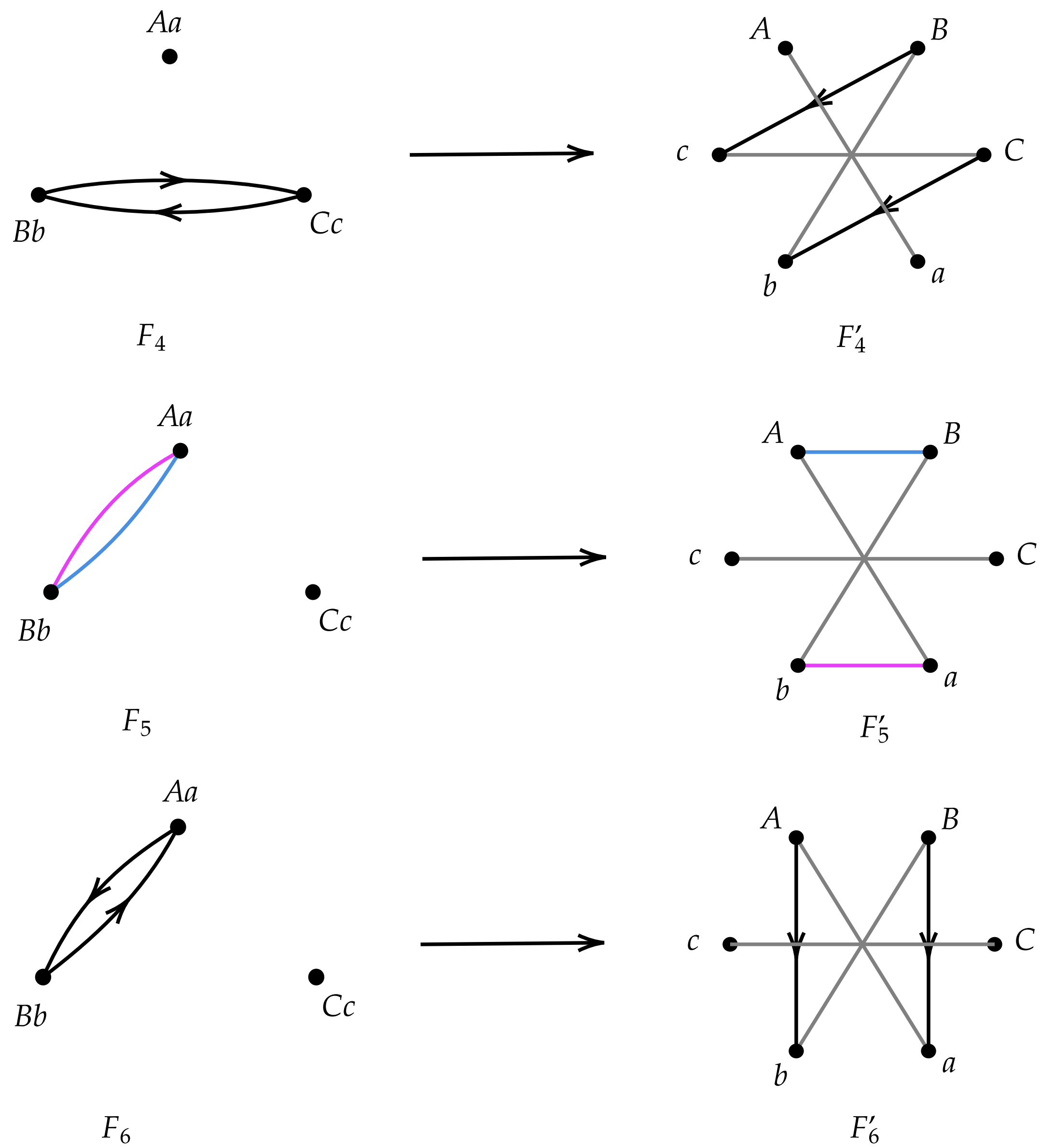


Figure 3.7: Part 1: from an HOP (C_2)-subgraph to an I-alternating (K_2, C_4)-factor.



$$F'_1 \bigoplus F'_2 \bigoplus F'_3 \bigoplus F'_4 \bigoplus F'_5 \bigoplus F'_6 = K_6 + 5I$$

Figure 3.8: Part 2: from an HOP (C_2)-decomposition of $4K_3^\bullet$ to an I-alternating (K_2, C_4)-factorization of $K_6 + 5I$.

3.3 Additional Tools

We observed that HOP$(2^{\langle s\rangle}, 2m_1, 2m_2, \ldots, 2m_t)$ admits a solution if and only if $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. We henceforth focus on finding HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decompositions of $4K_n^\bullet$.

However, finding such decompositions is not straightforward. We introduce some reduction tools that allow us to use decompositions of K_n or $2K_n$, and extend them to obtain HOP decompositions of $4K_n^\bullet$. These tools are discussed below.

3.3.1 From G to $4G^\bullet$

The following lemma is a generalization of Lemma 2.2.4(a) [36].

Lemma 3.3.1. Let G be a simple graph. Let $H_1, H_2, \ldots, H_s$ be subgraphs of G such that $G = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_s$. If, for all $i \in \{1, \ldots, s\}$, the multigraph $4H_i^\bullet$ admits an HOP decomposition $\mathcal{D}_i$ into $(C_{m_1}, \ldots, C_{m_t})$-subgraphs, then $\mathcal{D} = \bigcup_{i=1}^s \mathcal{D}_i$ is an HOP decomposition of $4G^\bullet$ into $(C_{m_1}, \ldots, C_{m_t})$-subgraphs.

Proof. We know that $H_1, H_2, \ldots, H_s$ are subgraphs of G, so $4H_1^\bullet, 4H_2^\bullet, \ldots, 4H_s^\bullet$ are subgraphs of $4G^\bullet$. Therefore, any $(C_{m_1}, \ldots, C_{m_t})$-subgraph of $4H_i^\bullet$, for $i \in \{1, \ldots, s\}$, is also a $(C_{m_1}, \ldots, C_{m_t})$-subgraph of $4G^\bullet$. Moreover, since the edge sets of $H_1, H_2, \ldots, H_s$ form a partition of $E(G)$, we have $4G^\bullet = 4H_1^\bullet \bigoplus 4H_2^\bullet \bigoplus \cdots \bigoplus 4H_s^\bullet$. For each $i \in \{1, \ldots, s\}$, we know $4H_i^\bullet$ admits a decomposition $\mathcal{D}_i$ into $(C_{m_1}, \ldots, C_{m_t})$-subgraphs. Then the union $\mathcal{D} = \bigcup_{i=1}^s \mathcal{D}_i$ is a decomposition of $4G^\bullet$ into $(C_{m_1}, \ldots, C_{m_t})$-subgraphs. Finally, since each $\mathcal{D}_i$ is an HOP decomposition, the resulting decomposition $\mathcal{D}$ is also HOP. ∎

The following lemma gives a sufficient condition for the multigraph $4G^\bullet$ to admit an HOP $(C_2^{\langle t \rangle})$-decomposition.

Lemma 3.3.2. Let G be a simple graph. If G admits a decomposition into 1-regular subgraphs of order $2t$, then $4G^\bullet$ admits an HOP $(C_2^{\langle t \rangle})$-decomposition.

Proof. Let $\mathcal{F} = \{H_1, H_2, \ldots, H_s\}$ be a decomposition of G where each H_i is a 1-regular subgraph of order $2t$; that is to say, each H_i is a disjoint union of t copies of K_2. It is easy to see that each copy of K_2 in each H_i gives rise to a directed black 2-cycle, and a 2-cycle with a pink and a blue edge in $4G^\bullet$. Therefore, each copy of K_2 gives rise to two HOP C_2-subgraphs, and consequently, each H_i gives rise to two HOP $(C_2^{\langle t \rangle})$-subgraphs. Notice that the two HOP $(C_2^{\langle t \rangle})$-subgraphs derived from H_i form an HOP $(C_2^{\langle t \rangle})$-decomposition of $4H_i^\bullet$. Therefore, for all $i \in \{1, \ldots, s\}$, the multigraph $4H_i^\bullet$ admits an HOP $(C_2^{\langle t \rangle})$-decomposition, and by Lemma 3.3.1, the multigraph $4G^\bullet$ admits an HOP $(C_2^{\langle t \rangle})$-decomposition.

∎

The following lemma is a generalization of Lemma 2.2.5 [36], and the proof is very similar.

Lemma 3.3.3. Let G be a simple graph. If G admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition, then $4G^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Let $\mathcal{D}_G = \{H_1, H_2, \ldots, H_s\}$ be a decomposition of G, where each H_i is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph. If we prove that for all $i \in \{1, 2, \ldots, s\}$, the multigraph $4H_i^\bullet$ admits an HOP decomposition into $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs, then by Lemma 3.3.1, the multigraph $4G^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Note that any 2-regular subgraph H_i, for all $i \in \{1, 2, \ldots, s\}$, consists of disjoint cycles of lengths $m_1, m_2, \ldots, m_t$. To show that $4H_i^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition by applying Lemma 3.3.1, it suffices to show that for each cycle C in H_i, the multigraph $4C^\bullet$ admits an HOP C_m-factorization.

To show that $4C^\bullet$ admits an HOP C_m-factorization, we consider the following two cases.

Case 1: Assume m is even. Let C_1, C_2, C_3, C_4 be four copies of C. Color the edges of C_1 alternately pink and blue. Color the edges of C_2 alternately pink and blue, but in the opposite way of C_1. Color all the edges of C_3 and C_4 black, and orient the two cycles in the opposite directions.

Case 2: Assume m is odd. First, fix two adjacent edges in C, say x_0x_1 and x_1x_2. Let C_1, C_2, C_3, C_4 be four copies of C. In C_1, color the edge x_0x_1 black and x_1x_2 blue. Color all the remaining edges alternately pink and blue, starting with pink from the edge x_2x_3. In C_2, color the edge x_0x_1 pink and x_1x_2 black. Color all the remaining edges alternately blue and pink, starting with blue from the edge x_2x_3. In C_3, color the edge x_0x_1 blue and x_1x_2 pink. Color all the remaining edges black. Notice that in these three cycles, each run of blue and pink edges alternates between the two colors, and each run of black edges is bounded by a blue and a pink edge. Orient the black edges from pink to blue. Thus, the three cycles satisfy Condition **(C1)** of Definition 1.2.3. In the fourth copy of C, that is, C_4, color all the edges black and orient them in the opposite direction from the black edges in C_1, C_2, C_3.

We observe that in both cases the resulting cycles satisfy property **(C1)**, and form an HOP C_m-factorization for $4C^\bullet$. ∎

In the following example, we apply Lemma 3.3.3 to construct a C_5-factorization of $4K_5^\bullet$ from an existing C_5-factorization of K_5 (see Figure 3.9).

Example 3.3.4. Let $\mathcal{D} = \{C_1, C_2\}$ be a C_5-factorization of K_5 in Figure 3.9. Figure 3.10 describes the construction of an HOP C_5-factorization of $4K_5^\bullet$ using the procedure from the proof of Lemma 3.3.3.

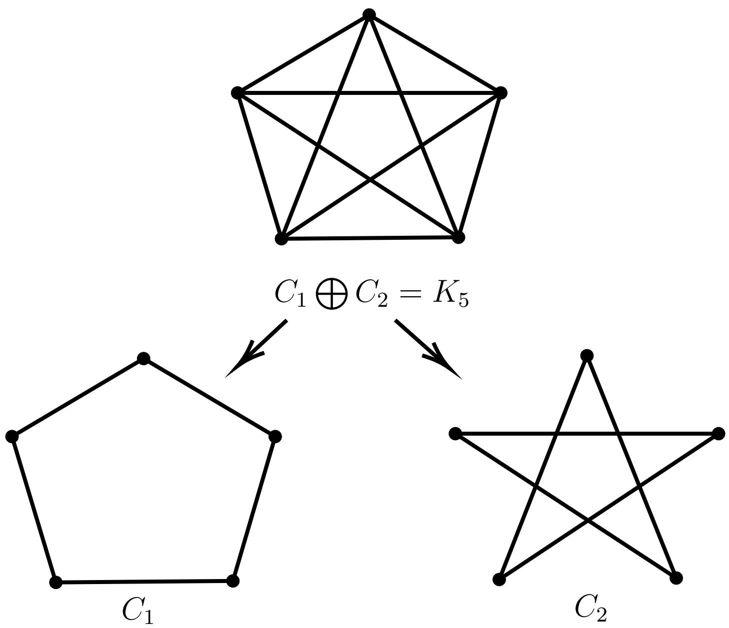
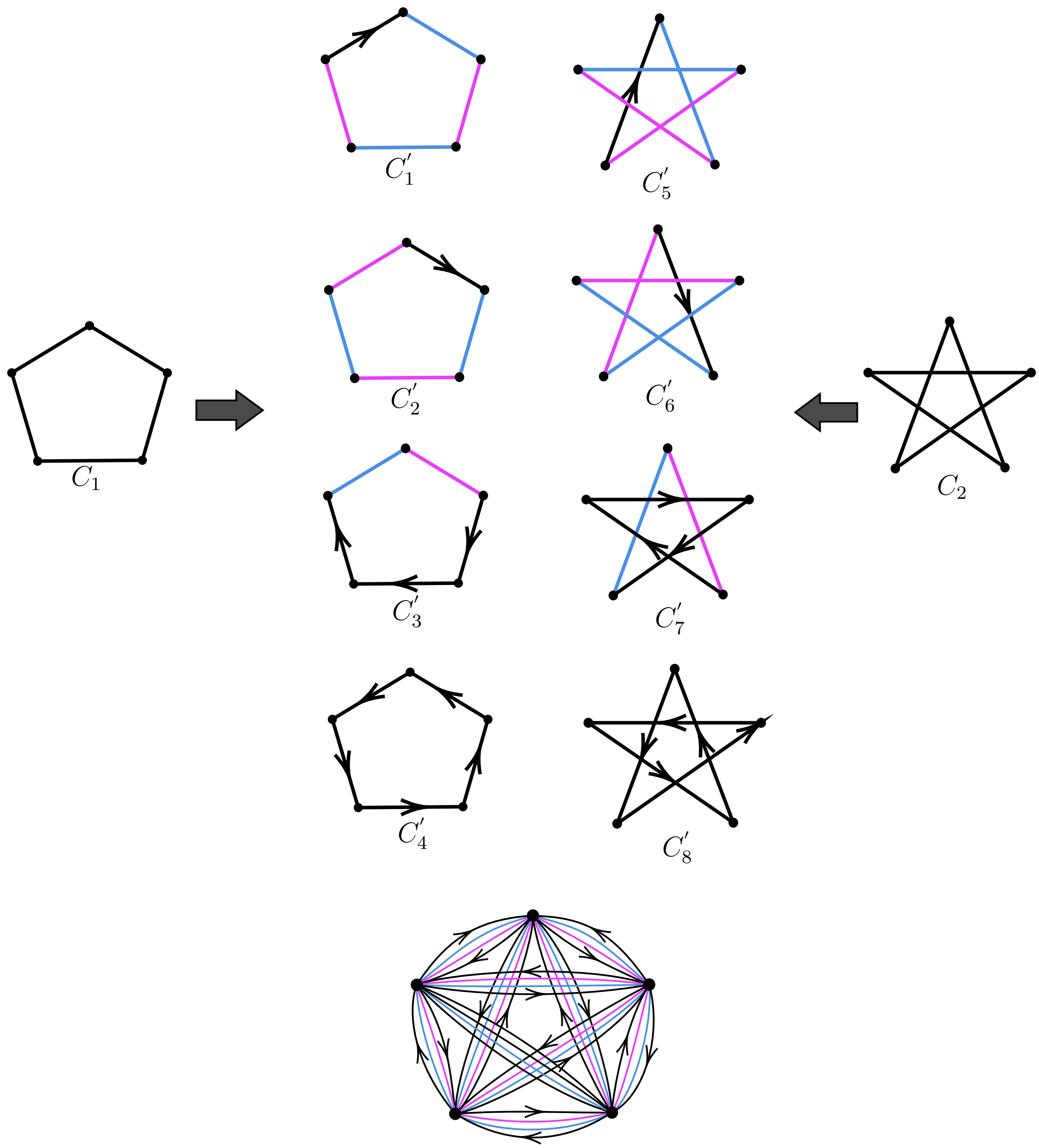


Figure 3.9: A C_5-factorization of K_5



$$C_1' \bigoplus C_2' \bigoplus C_3' \bigoplus C_4' \bigoplus C_5' \bigoplus C_6' \bigoplus C_7' \bigoplus C_8' = 4K_5^\bullet$$

Figure 3.10: From a C_5-factorization of K_5 to an HOP C_5-factorization of $4K_5^\bullet$

3.3.2 From $2G^\circ$ to $4G^\bullet$

The following lemma generalizes Lemma 2.2.7 [36]. As in [36], the idea is to use a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $2G^\circ$ to construct an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4G^\bullet$.

Lemma 3.3.5. Assume that $2G^\circ$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition $\mathcal{F}$ with the property that every m-cycle of $\mathcal{F}$, for $m \geq 3$, contains an even number of pink edges. Then $4G^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Let C be any m-cycle in $\mathcal{F}$. Consider the following two cases.

Case 1: $m \geq 3$. Then C contains an even number of pink edges. Construct C_1 from C as follows: Recolor every second pink edge blue. Since the number of pink edges is even, after recoloring, in each run of blue and pink edges, the two colors alternate, and each run of black edges is bounded by a blue and a pink edge. Orient the black edges from pink to blue. Now, construct C_2 from C_1 by swapping pink edges with blue edges and changing the orientation of the black edges. Notice that when the number of pink edges is zero, C_1 and C_2 are directed black cycles with opposite directions.

Case 2: $m = 2$. Then C contains two parallel edges with colors pink and black. Obtain C_1 from C by recoloring the black edge blue, and obtain C_2 by replacing the pink edge of C with a black edge, and orient the two black edges in opposite directions.

Therefore, each m-cycle in $\mathcal{F}$ gives rise to two HOP m-cycles in $4G^\bullet$. More generally, any $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph of $2G^\circ$ gives rise to two HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs in $4G^\bullet$.

We still need to show that the union of all these HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs is equal to $4G^\bullet$. Let x and y be two arbitrary adjacent vertices of $2G^\circ$. There are two parallel edges between x and y, one colored pink and the other black. Since $\mathcal{F}$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $2G^\circ$, there exist cycles C^p and C^b in $\mathcal{F}$ that contain the pink and black copies of the edge xy, respectively. By the construction described above, the cycle C^p gives rise to two cycles: one containing a pink copy of xy and the other containing a blue copy. Similarly, the cycle C^b gives rise to two cycles: one containing a black copy of xy oriented from x to y, and the other containing a black copy oriented from y to x. Hence, the union of the resulting HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs covers all edges of $4G^\bullet$. Therefore, $4G^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

∎

The following example illustrates Lemma 3.3.5 for $G = K_4$.

Example 3.3.6. The 2-factorization $\mathcal{D} = \{F_1, F_2, F_3\}$ shown in Figure 3.11 is a C_4-factorization of $2K_4^\circ$ that satisfies the condition in Lemma 3.3.5. The figure also illustrates how this factorization can be used to construct an HOP C_4-factorization of $4K_4^\bullet$, following the method described in the proof of Lemma 3.3.5.

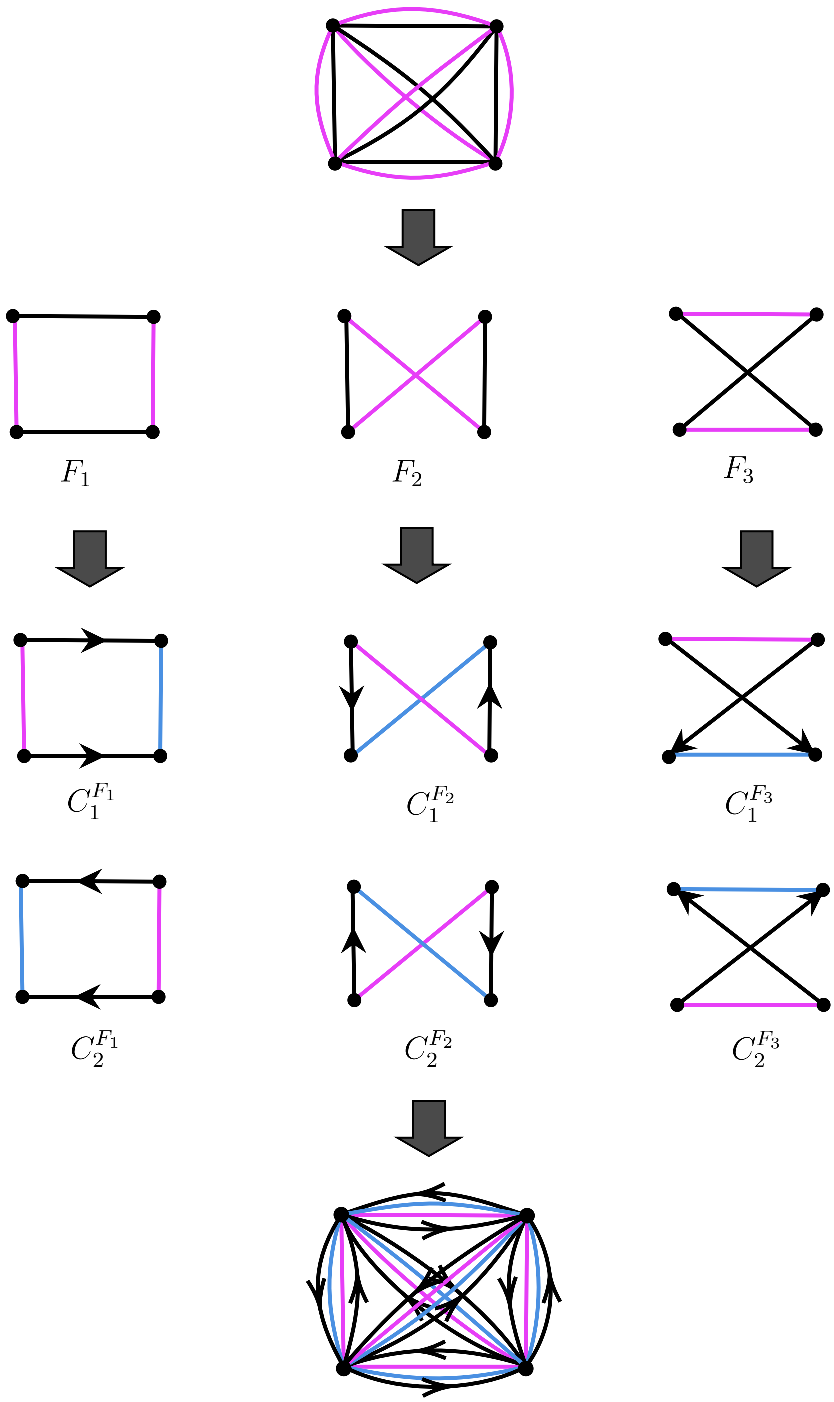


Figure 3.11: From a C_4-factorization of $2K_4^\circ$ to an HOP C_4-factorization of $4K_4^\bullet$

3.4 Useful Starters

In this section, we develop several useful lemmas that provide methods for constructing HOP decompositions of $4K_n^\bullet$. These constructions are obtained either by deriving them from decompositions of smaller graphs, such as K_n and $2K_n^\circ$, or through direct constructions. In each case, the focus is on generating starter cycles that satisfy specific symmetry and HOP coloring conditions, followed by applying a suitable permutation to obtain a cyclic decomposition of $4K_n^\bullet$.

Although not all of the lemmas are used later in the thesis, they are included for completeness and may be useful in related contexts.

3.4.1 From a Decomposition of K_n to an HOP Decomposition of $4K_n^\bullet$

Lemmas 3.4.1 and 3.4.2 are classical results on cyclic decompositions of K_n, which we restate in a form suited to our context.

Lemma 3.4.1. Let $3 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer such that $\frac{n-1}{2} = ms$ for some $s \in \mathbb{Z}^+$. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ x_2\ \ldots\ x_{n-1})$ be a permutation on $V(K_n)$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of K_n that jointly contain exactly one edge from each of the orbits of $\langle\rho\rangle$ in its action on $E(K_n)$. Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. By the given labeling of the vertices of K_n, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, for $i \in \mathbb{Z}_n$ and $d \in \{1, 2, \ldots, \frac{n-1}{2}\}$. Therefore, the group $\langle\rho\rangle$ acting on $E(K_n)$ has $\frac{n-1}{2}$ orbits, each of size n. Since $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle\rho\rangle$, observe $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_s) : i \in \mathbb{Z}_n\}$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of K_n. Applying Lemma 3.3.3, we see that $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. ∎

Lemma 3.4.2. Let $3 \le m_1 \le m_2 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer such that $n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$ be a permutation on $V(K_n)$. Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of K_n with the following properties.

(A1) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges from each orbit of $\langle\rho\rangle$ of length $n-1$, namely, a pair of edges of the form $(e, \rho^{\frac{n-1}{2}}(e))$; and

(A2) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each orbit of $\langle\rho\rangle$ of length $\frac{n-1}{2}$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Notice that the group $\langle\rho\rangle$ has the following orbits on the edge set of K_n:

- for each $d \in \{1, 2, \ldots, \frac{n-3}{2}\}$, we have one orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- one orbit $\{x_i x_{i+\frac{n-1}{2}} : i = 0, 1, \ldots, \frac{n-3}{2}\}$; and

- one orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Hence, the group $\langle\rho\rangle$ has $\frac{n-3}{2}+1+1=\frac{n+1}{2}$ orbits on $E(K_n)$. Note that the size of each orbit is $n-1$, except for the orbit consisting of edges of difference $\frac{n-1}{2}$; this orbit has length $\frac{n-1}{2}$.

Assume O is an orbit of $\langle\rho\rangle$; then either it has length $\frac{n-1}{2}$ or $n-1$. If the length of O is $\frac{n-1}{2}$, then by **(A2)**, clearly $\{\rho^i(F_1),\ldots,\rho^i(F_s) : i=0,1,\ldots,\frac{n-3}{2}\}$ contains all edges of O. If the length of O is $n-1$, then for any $e \in O$, by **(A1)**, we have that $e \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$ implies $\rho^{\frac{n-1}{2}}(e) \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$. Hence, $\{\rho^i(F_1),\ldots,\rho^i(F_s) : i=0,1,\ldots,\frac{n-3}{2}\}$ contains all edges of O.

Note that $F_1, F_2, \ldots, F_s$ jointly contain $ms=n$ edges, and all edges in each orbit are covered by $\{\rho^i(F_1),\ldots,\rho^i(F_s) : i=0,1,\ldots,\frac{n-3}{2}\}$. Since this set covers $n\cdot\frac{n-1}{2}$ edges, which is exactly equal to the number of edges in K_n, each edge is covered precisely once. Thus $\mathcal{D}=\{\rho^i(F_1),\rho^i(F_2),\ldots,\rho^i(F_s) : i=0,1,\ldots,\frac{n-3}{2}\}$ forms a $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-decomposition for K_n. Applying Lemma 3.3.3, the multigraph $4K_n^\bullet$ admits an HOP $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-decomposition. ∎

3.4.2 From a Decomposition of $2K_n^\circ$ to an HOP Decomposition of $4K_n^\bullet$

Lemma 3.4.3. Let $2 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1+\ldots+m_t$. Assume n is an odd integer such that $n-1=ms$ for some $s \in \mathbb{Z}^+$. Let $V(2K_n^\circ)=\{x_i : i \in \mathbb{Z}_n\}$, and let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho=(x_0\ x_1\ x_2\ \ldots\ x_{n-1})$.

Let $F_1,F_2,\ldots,F_s$ be $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-subgraphs of $2K_n^\circ$ with the following properties.

(B1) Every cycle in $F_1,F_2,\ldots,F_s$ of length at least 3 contains an even number of pink edges; and

(B2) $F_1,F_2,\ldots,F_s$ jointly contain exactly one edge from each of the orbits of $\langle\rho_\circ\rangle$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-decomposition.

Proof. Each edge in $E(2K_n^\circ)$ is of the form $x_i x_{i+d}$, for $i \in \mathbb{Z}_n$ and $d \in \{1,2,\ldots,\frac{n-1}{2}\}$. The group $\langle\rho_\circ\rangle$ has a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_n\}$, for each $d \in \{1,2,\ldots,\frac{n-1}{2}\}$. Therefore, we have $2\cdot\frac{n-1}{2}=n-1$ orbits, and the size of each orbit is n. Since $F_1,F_2,\ldots,F_s$ jointly contain exactly one edge from each of the orbits of $\langle\rho_\circ\rangle$, observe $\mathcal{D}=\{\rho_\circ^i(F_1),\rho_\circ^i(F_2),\ldots,\rho_\circ^i(F_s) : i \in \mathbb{Z}_n\}$ is a $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-decomposition of $2K_n^\circ$. By Condition **(B1)** and Lemma 3.3.5, we see that $4K_n^\bullet$ admits an HOP $(C_{m_1},C_{m_2},\ldots,C_{m_t})$-decomposition. ∎

Lemma 3.4.4. Let $2 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1+\ldots+m_t$. Assume n is an even integer such that $2(n-1)=ms$ for some $s \in \mathbb{Z}^+$. Let $V(2K_n^\circ)=\{x_i : i \in \mathbb{Z}_n\}$, and let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho=(x_0\ x_1\ x_2\ \ldots\ x_{n-1})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $2K_n^\circ$ with the following properties.

(B′1) Every cycle in $F_1, F_2, \ldots, F_s$ of length at least 3 contains an even number of pink edges;

(B′2) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges from each orbit of $\langle \rho_\circ \rangle$ of length n, namely, a pair of edges of the form $(e, \rho_\circ^{\frac{n}{2}}(e))$; and

(B′3) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each orbit of $\langle \rho_\circ \rangle$ of length $\frac{n}{2}$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Observe that the group $\langle \rho_\circ \rangle$ has a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_n\}$, for each $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, and a pink and a black orbit $\{x_i x_{i+\frac{n}{2}} : i = 0, 1, \ldots, \frac{n-2}{2}\}$. Therefore, we have $2 \cdot \frac{n-2}{2} + 2 = n$ orbits. Note that the size of each orbit is n, except for the pink and the black orbit consisting of edges of difference $\frac{n}{2}$; these two orbits have length $\frac{n}{2}$.

Assume O is an orbit of $\langle \rho_\circ \rangle$; then either it has length $\frac{n}{2}$ or n. If the length of O is $\frac{n}{2}$, then by **(B′3)** clearly $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ contains all edges of O. If the length of O is n, then for any $e \in O$, by **(B′2)**, we have that $e \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$ implies $\rho_\circ^{\frac{n}{2}}(e) \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$. Hence, $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ contains all edges of O.

We see that $F_1, F_2, \ldots, F_s$ jointly contain $2(n-1)$ edges, and all edges in each orbit are covered by $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$. Since this set covers $2(n-1) \cdot \frac{n}{2} = n(n-1)$ edges which is exactly equal to the number of edges in $2K_n^\circ$, no edge is covered twice. Thus $\mathcal{D} = \{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ forms a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition for $2K_n^\circ$. By Condition **(B′1)** and Lemma 3.3.5, we see that $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. ∎

Lemma 3.4.5. Let $2 \le m_1 \le m_2 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an even integer such that $n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $2K_n^\circ$ with the following properties.

(D1) Every cycle of $F_1, F_2, \ldots, F_s$ of length at least 3 contains an even number of pink edges, and

(D2) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\circ \rangle$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Each edge in $E(2K_n^\circ)$ is either of the form $x_i x_{i+d}$, for $i \in \mathbb{Z}_{n-1}$ and $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, or of the form $x_i x_\infty$ (if it is of difference infinity). Since $\rho_\circ$ is a permutation that preserves the color of the edges, the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for each $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$, and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Hence, the group $\langle \rho_\circ \rangle$ has $2 \cdot \frac{n-2}{2} + 2 = n$ orbits, and the size of each orbit is $n-1$. Since $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\circ \rangle$, we observe that $\mathcal{D} = \{\rho_\circ^i(F_1), \rho_\circ^i(F_2), \ldots, \rho_\circ^i(F_s) : i \in \mathbb{Z}_{n-1}\}$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $2K_n^\circ$. Moreover, since every cycle in $\mathcal{D}$ of length at least 3 contains an even number of pink edges, by Lemma 3.3.5, the multigraph $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. ∎

Lemma 3.4.6. Let $2 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer with $2n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $2K_n^\circ$ with the following properties.

(D′1) Every cycle in $F_1, F_2, \ldots, F_s$ of length at least 3 contains an even number of pink edges;

(D′2) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges from each orbit of $\langle \rho_\circ \rangle$ of length $n-1$, namely, a pair of edges of the form $(e, \rho_\circ^{\frac{n-1}{2}}(e))$; and

(D′3) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each orbit of $\langle \rho_\circ \rangle$ of length $\frac{n-1}{2}$.

Then $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Observe that the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for each $d \in \{1, 2, \ldots, \frac{n-3}{2}\}$, we have a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a black orbit $\{x_i x_{i+\frac{n-1}{2}} : i = 0, 1, \ldots, \frac{n-3}{2}\}$; and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Therefore, we have $2 \cdot \frac{n-3}{2} + 2 + 2 = n + 1$ orbits. Note that the size of each orbit is $n-1$, except for the pink and the black orbit consisting of edges of difference $\frac{n-1}{2}$; these two orbits have length $\frac{n-1}{2}$.

Assume O is an orbit of $\langle \rho_\circ \rangle$; then either it has length $\frac{n-1}{2}$ or $n-1$. If the length of O is $\frac{n-1}{2}$, then by **(D′3)** clearly $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ contains all edges of O. If the length of O is $n-1$, then for any $e \in O$, by **(D′2)**, we have that $e \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$ implies $\rho_\circ^{\frac{n-1}{2}}(e) \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$. Hence, $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ contains all edges of O.

Note that $F_1, F_2, \ldots, F_s$ jointly contain $ms = 2n$ edges, and all edges in each orbit are covered by the set $\{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$. Since this set covers $2n \cdot \frac{n-1}{2} =$

$n(n-1)$ edges, which matches exactly the number of edges in $2K_n^\circ$, no edge is covered more than once.

Thus, $\mathcal{D} = \{\rho_\circ^i(F_1), \ldots, \rho_\circ^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $2K_n^\circ$. By Condition **(D′1)** and Lemma 3.3.5, we conclude that $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. ∎

3.4.3 HOP Decompositions of $4K_n^\bullet$

Lemma 3.4.7. Let $2 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer with $2(n-1) = ms$ for some $s \in \mathbb{Z}^+$. Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_0\ x_1\ x_2\ \ldots\ x_{n-1})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $4K_n^\bullet$ with the following properties.

(E1) Every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3;

(E2) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \ldots, \rho_\bullet^i(F_s) : i \in \mathbb{Z}_n\}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Proof. Since $\rho_\bullet$ is a permutation that preserves the color and orientation of the edges, the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \ldots, \frac{n-1}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_n\}$;
- for each $d \in \{1, 2, \ldots, n-1\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_n\}$.

Altogether, the group $\langle \rho_\bullet \rangle$ has $2 \cdot \frac{n-1}{2} + (n-1) = 2(n-1)$ orbits, and the size of each orbit is n. Since $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$, we see that $\mathcal{D} = \{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i \in \mathbb{Z}_n\}$ is a $(C_{m_1}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. Moreover, since every 2-regular subgraph in $\mathcal{D}$ is formed by applying the permutation $\rho_\bullet$ to the starters in $\{F_1, F_2, \ldots, F_s\}$, and $\rho_\bullet$ preserves the edge colors (and orientation), the decomposition $\mathcal{D}$ is HOP. ∎

Lemma 3.4.8. Let $2 \le m_1 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an even integer with $4(n-1) = ms$ for some $s \in \mathbb{Z}^+$. Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_0\ x_1\ x_2\ \ldots\ x_{n-1})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $4K_n^\bullet$ with the following properties.

(E′1) Every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3;

(E′2) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges from each orbit of $\langle\rho_\bullet\rangle$ of lenght n, namely, a pair of edges of the form $(e, \rho_\bullet^{\frac{n}{2}}(e))$; and

(E′3) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each orbit of $\langle\rho_\bullet\rangle$ of length $\frac{n}{2}$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Proof. Observe that the group $\langle\rho_\bullet\rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_n\}$;
- for each $d \in \{1, 2, \ldots, n-1\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_n\}$; and
- a pink and a blue orbit $\{x_i x_{i+\frac{n}{2}} : i = 0, 1, \ldots, \frac{n-2}{2}\}$.

Hence, the group $\langle\rho_\bullet\rangle$ has $2 \cdot \frac{n-2}{2} + (n-1) + 2 = 2n - 1$ orbits. Note that the size of each orbit is n, except for the pink and the blue orbit consisting of edges of difference $\frac{n}{2}$; these two orbits have length $\frac{n}{2}$.

Assume O is an orbit of $\langle\rho_\bullet\rangle$; then either it has length $\frac{n}{2}$ or n. If the length of O is $\frac{n}{2}$, then by **(E′3)** clearly $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ contains all edges of O. If the length of O is n, then for any $e \in O$, by **(E′2)**, we have that $e \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$ implies $\rho_\bullet^{\frac{n}{2}}(e) \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$. Hence, $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$ contains all edges of O.

Note that $F_1, F_2, \ldots, F_s$ jointly contain $4(n-1)$ edges, and all edges in each orbit are covered by the set $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-2}{2}\}$. Since this set covers $4(n-1) \cdot \frac{n}{2} = 2n(n-1)$ edges, which is exactly the total number of edges in $4K_n^\bullet$, each edge is covered precisely once. Therefore,

$$\mathcal{D} = \{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \tfrac{n-2}{2}\}$$

is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. By Condition **(E′1)** and the fact that $\rho_\bullet$ preserves both edge colors and orientation, we conclude that $\mathcal{D}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. ∎

Lemma 3.4.9. Let $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an even integer with $2n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $4K_n^\bullet$ with the following properties.

(L1) Every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3; and

(L2) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle\rho_\bullet\rangle$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \ldots, \rho_\bullet^i(F_s) : i \in \mathbb{Z}_{n-1}\}$ is an HOP $(C_{m_1}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Proof. Notice that the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- for each $d \in \{1, 2, \ldots, n-2\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

Hence, the group $\langle \rho_\bullet \rangle$ has $2 \cdot \frac{n-2}{2} + (n-2) + 2 + 2 = 2n$ orbits, each of size $n-1$. Since $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$, we see that $\mathcal{D} = \{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i \in \mathbb{Z}_{n-1}\}$ is a $(C_{m_1}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. Moreover, since every 2-regular subgraph in $\mathcal{D}$ is formed by applying the permutation $\rho_\bullet$ to the starters in $\{F_1, F_2, \ldots, F_s\}$, and $\rho_\bullet$ preserves the edge colors (and orientation), the decomposition $\mathcal{D}$ is HOP. ∎

Lemma 3.4.10. Let $2 = m_1 \leq m_2 \leq \ldots \leq m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer with $2n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_2, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $4K_n^\bullet$ with the following properties.

(L′1) Every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3;

(L′2) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges of difference $\frac{n-1}{2}$, which appear in a 2-cycle; and

(L′3) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits of $\langle \rho_\bullet \rangle$ corresponding to differences $1, 2, \ldots, \frac{n-3}{2}, \infty$.

Then there exists an HOP $(C_2, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Proof. Since $\rho_\bullet$ is a permutation that preserves the color and orientation of the edges, the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \ldots, \frac{n-3}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- for each $d \in \{1, 2, \ldots, n-2\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_{i+\frac{n-1}{2}} : i = 0, 1, \ldots, \frac{n-3}{2}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

Hence, the group $\langle\rho_\bullet\rangle$ has $2 \cdot \frac{n-3}{2} + (n-2) + 2 + 2 + 2 = 2n+1$ orbits. Note that the size of each orbit is $n-1$, except for the pink and the blue orbit consisting of edges of difference $\frac{n-1}{2}$; these two orbits have length $\frac{n-1}{2}$.

Let e be an edge of $4K_n^\bullet$ of difference $\frac{n-1}{2}$. Then by **(L′2)**, it appears only in a 2-cycle, say C. We may assume that F_s is the $(C_2, C_{m_2}, \ldots, C_{m_t})$-subgraph that contains C. Recolor the edges of C pink and blue. It is clear that $\{\rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ covers all pink and a blue edges of difference $\frac{n-1}{2}$. Now, form F_s' from F_s by recoloring the edges of C black (as opposite arcs). Observe that $\{\rho_\bullet^{\frac{n-1}{2}+i}(F_s') : i = 0, 1, \ldots, \frac{n-3}{2}\}$ covers all black arcs of difference $\frac{n-1}{2}$. Hence, $\{\rho_\bullet^i(F_s), \rho_\bullet^{\frac{n-1}{2}+i}(F_s') : i = 0, 1, \ldots, \frac{n-3}{2}\}$ covers all four copies of all edges of difference $\frac{n-1}{2}$. Moreover, since $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each of the orbits, except for the orbits that correspond to difference $\frac{n-1}{2}$, we see that

$$\mathcal{D} = \left\{\rho_\bullet^i(F_s), \rho_\bullet^{\frac{n-1}{2}+i}(F_s') : i = 0, 1, \ldots, \tfrac{n-3}{2}\right\} \cup \left\{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \ldots, \rho_\bullet^i(F_{s-1}) : i \in \mathbb{Z}_{n-1}\right\}$$

is a $(C_2, C_{m_2}, \ldots, C_{m_t})$-decomposition for $4K_n^\bullet$. Since every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3, and $\rho_\bullet$ preserves the edge colors (and orientations), decomposition $\mathcal{D}$ is an HOP $(C_2, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. ∎

Lemma 3.4.11. Let $2 \le m_1 \le m_2 \le \ldots \le m_t$ be integers, and $m = m_1 + \ldots + m_t$. Assume n is an odd integer with $4n = ms$ for some $s \in \mathbb{Z}^+$. Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$.

Let $F_1, F_2, \ldots, F_s$ be $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraphs of $4K_n^\bullet$ with the following properties.

(L″1) Every cycle in $F_1, F_2, \ldots, F_s$ satisfies Condition **(C1)** of Definition 1.2.3;

(L″2) $F_1, F_2, \ldots, F_s$ jointly contain exactly two edges from each orbit of $\langle\rho_\bullet\rangle$ of length $n-1$, namely, a pair of edges of the form $(e, \rho_\bullet^{\frac{n-1}{2}}(e))$; and

(L″3) $F_1, F_2, \ldots, F_s$ jointly contain exactly one edge from each orbit of $\langle\rho_\bullet\rangle$ of length $\frac{n-1}{2}$.

Then $\mathcal{D} = \{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$.

Proof. With the same orbit setup as in Lemma 3.4.10, assume O is an orbit of $\langle\rho_\bullet\rangle$; then either it has length $\frac{n-1}{2}$ or $n-1$. If the length of O is $\frac{n-1}{2}$, then by **(L″3)** clearly $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ contains all edges of O. If the length of O is $n-1$, then for any $e \in O$, by **(L″2)**, we have that $e \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$ implies $\rho_\bullet^{\frac{n-1}{2}}(e) \in E(F_1 \cup F_2 \cup \ldots \cup F_s)$. Hence, $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$ contains all edges of O.

Note that $F_1, F_2, \ldots, F_s$ jointly cover $4n$ edges, and all edges in each orbit are covered by $\{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$. Since this set covers $4n \cdot \frac{n-1}{2} = 2n(n-1)$ edges which is exactly equal to the number of edges in $4K_n^\bullet$, no edge is covered twice. Thus,

$$\mathcal{D} = \{\rho_\bullet^i(F_1), \ldots, \rho_\bullet^i(F_s) : i = 0, 1, \ldots, \frac{n-3}{2}\}$$

is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of $4K_n^\bullet$. By Condition **(L″1)**, we see this decomposition is an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. ∎

Chapter 4

The Generalized HOP with Two Round Tables

In this chapter, we prove a partial result on the generalized HOP with s tables of sizes 2 and two round tables of sizes $2m_1$ and $2m_2$, that is, HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$. In particular, we prove in Theorem 4.4.1 that HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution whenever $n \equiv 1 \pmod{2m_1+2m_2}$, and in Theorem 4.4.2, we show that HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution whenever $n \equiv m_1 + m_2 \pmod{2m_1 + 2m_2}$. Later in Section 4.5, a solution to HOP$(2^{\langle s\rangle}, 4^{\langle r\rangle}, 2m)$ under similar conditions is established.

4.1 HOP $(C_2^{\langle\alpha\rangle}, C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition

We start with the following lemma, which describes how a particular decomposition of K_n gives rise to an HOP decomposition of $4K_n^\bullet$. This lemma will later be used to prove the results in Lemmas 4.2.1 and 4.3.1.

Lemma 4.1.1. Assume G is a simple graph. Let H be a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-subgraph of G, and let H' be an edge-disjoint union of two 1-regular subgraphs of G of order 2α.

If G admits a decomposition into subgraphs isomorphic to $H \,\dot\cup\, H'$, then $4G^\bullet$ admits an HOP $(C_2^{\langle\alpha\rangle}, C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Proof. Let $\mathcal{D} = \{F_1, F_2, \ldots, F_k\}$ be a decomposition of G such that each F_i, for $i \in \{1, 2, \ldots, k\}$, is isomorphic to $H \,\dot\cup\, H'$.

Fix $i \in \{1, 2, \ldots, k\}$; then $F_i = H_i \,\dot\cup\, H_i'$, where H_i is isomorphic to H and H_i' is isomorphic to H'. Now, let $H_i^{(1)}, H_i^{(2)}, H_i^{(3)}, H_i^{(4)}$ be four copies of the 2-regular subgraph H_i.

Next, for $j \in \{1, 2, \ldots, 2\alpha\}$, let E_j be the j^{th} subgraph isomorphic to K_2 in H_i'. We may assume $E_1, \ldots, E_\alpha$ as well as $E_{\alpha+1}, \ldots, E_{2\alpha}$ are pairwise disjoint, and $E_1 \cup \ldots \cup E_\alpha$ is edge-disjoint from $E_{\alpha+1} \cup \ldots \cup E_{2\alpha}$. For each $j \in \{1, 2, \ldots, 2\alpha\}$, let $E_j^{(1)}$ and $E_j^{(2)}$ be 2-cycles that jointly contain four (parallel) copies of the edge of E_j. Now, apply the proof of Lemma

3.3.3 to color the edges in $H_i^{(1)}, H_i^{(2)}, H_i^{(3)}, H_i^{(4)}$ and $E_j^{(1)}, E_j^{(2)}$, for all j, so that they satisfy Condition **(C1)** of Definition 1.2.3, and so that

- $H_i^{(1)} \bigoplus H_i^{(2)} \bigoplus H_i^{(3)} \bigoplus H_i^{(4)} = 4H_i^{\bullet}$, and
- $E_j^{(1)} \bigoplus E_j^{(2)} = 4E_j^{\bullet}$.

Now, let $T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}$ be the following subgraphs:

$$\begin{aligned} T_{i_1} &= H_i^{(1)} \cup E_1^{(1)} \cup E_2^{(1)} \cup \ldots \cup E_\alpha^{(1)}; \\ T_{i_2} &= H_i^{(2)} \cup E_{\alpha+1}^{(1)} \cup E_{\alpha+2}^{(1)} \cup \ldots \cup E_{2\alpha}^{(1)}; \\ T_{i_3} &= H_i^{(3)} \cup E_1^{(2)} \cup E_2^{(2)} \cup \ldots \cup E_\alpha^{(2)}; \\ T_{i_4} &= H_i^{(4)} \cup E_{\alpha+1}^{(2)} \cup E_{\alpha+2}^{(2)} \cup \ldots \cup E_{2\alpha}^{(2)}. \end{aligned}$$

It is easy to see that $T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}$ are all $(C_2^{\langle\alpha\rangle}, C_{m_1}, \ldots, C_{m_t})$-subgraphs of $4G^{\bullet}$ that satisfy Condition **(C1)** of Definition 1.2.3, and

$$T_{i_1} \bigoplus T_{i_2} \bigoplus T_{i_3} \bigoplus T_{i_4} = 4(H_i \dot\cup H_i')$$

We apply the above procedure for each F_i, for $i \in \{1, 2, \ldots, k\}$, so each F_i gives rise to four $(C_2^{\langle\alpha\rangle}, C_{m_1}, \ldots, C_{m_t})$-subgraphs. Since $\mathcal{D} = \{F_1, F_2, \ldots, F_k\}$ is a decomposition of G, the resulting $(C_2^{\langle\alpha\rangle}, C_{m_1}, \ldots, C_{m_t})$-subgraphs form an HOP $(C_2^{\langle\alpha\rangle}, C_{m_1}, \ldots, C_{m_t})$-decomposition of $4G^{\bullet}$ (see Example 4.1.2). ∎

Example 4.1.2. Let H be a (C_3)-subgraph of K_{11}, and let H' be an edge-disjoint union of two 1-regular subgraphs of order 2 in K_{11}. Then K_{11} admits an $(H \dot\cup H')$-decomposition, as shown in Figure 4.1, each color class represents an $(H \dot\cup H')$-subgraph.

In Figure 4.1, we use the construction from the proof of Lemma 4.1.1 to show that such an $(H \dot\cup H')$-decomposition of K_{11} gives rise to an HOP (C_2, C_3)-decomposition of $4K_{11}^{\bullet}$.

The following lemma is a direct consequence of Lemma 4.1.1. We will later use this result to prove generalized forms of Lemmas 4.2.1 and 4.3.1.

Lemma 4.1.3. Let $m_1, m_2, \ldots, m_t \geq 3$ be integers. Assume that G is a simple graph that admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition, where $m_1, m_2, \ldots, m_s$ are even for some $s \leq t$.

Then $4G^{\bullet}$ admits an HOP $(C_2^{\langle \frac{m_1+m_2+\ldots+m_s}{2} \rangle}, C_{m_{s+1}}, \ldots, C_{m_t})$-decomposition.

Proof. We observe that, for $i \in \{1, 2, \ldots, s\}$, each even cycle, C_{m_i}, is an edge-disjoint union of two 1-regular subgraphs of order m_i. Thus, $C_{m_1} \dot\cup \ldots \dot\cup C_{m_s}$ is an edge-disjoint union of two 1-regular subgraphs of order $m_1 + m_2 + \ldots + m_s$. Therefore, the $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition of G gives rise to a decomposition of G into subgraphs consisting of a disjoint union of a $(C_{m_{s+1}}, \ldots, C_{m_t})$-subgraph and two edge-disjoint 1-regular subgraphs of order $m_1 + m_2 + \ldots + m_s$. Now, applying Lemma 4.1.1, we see $4G^{\bullet}$ admits an HOP $(C_2^{\langle \frac{m_1+m_2+\ldots+m_s}{2} \rangle}, C_{m_{s+1}}, \ldots, C_{m_t})$-decomposition (see Example 4.1.4). ∎

Example 4.1.4. In Figure 4.2, we see that a (C_3, C_4)-factorization of K_7 gives rise to an HOP $(C_2^{\langle 2\rangle}, C_3)$-factorization of $4K_7^{\bullet}$.

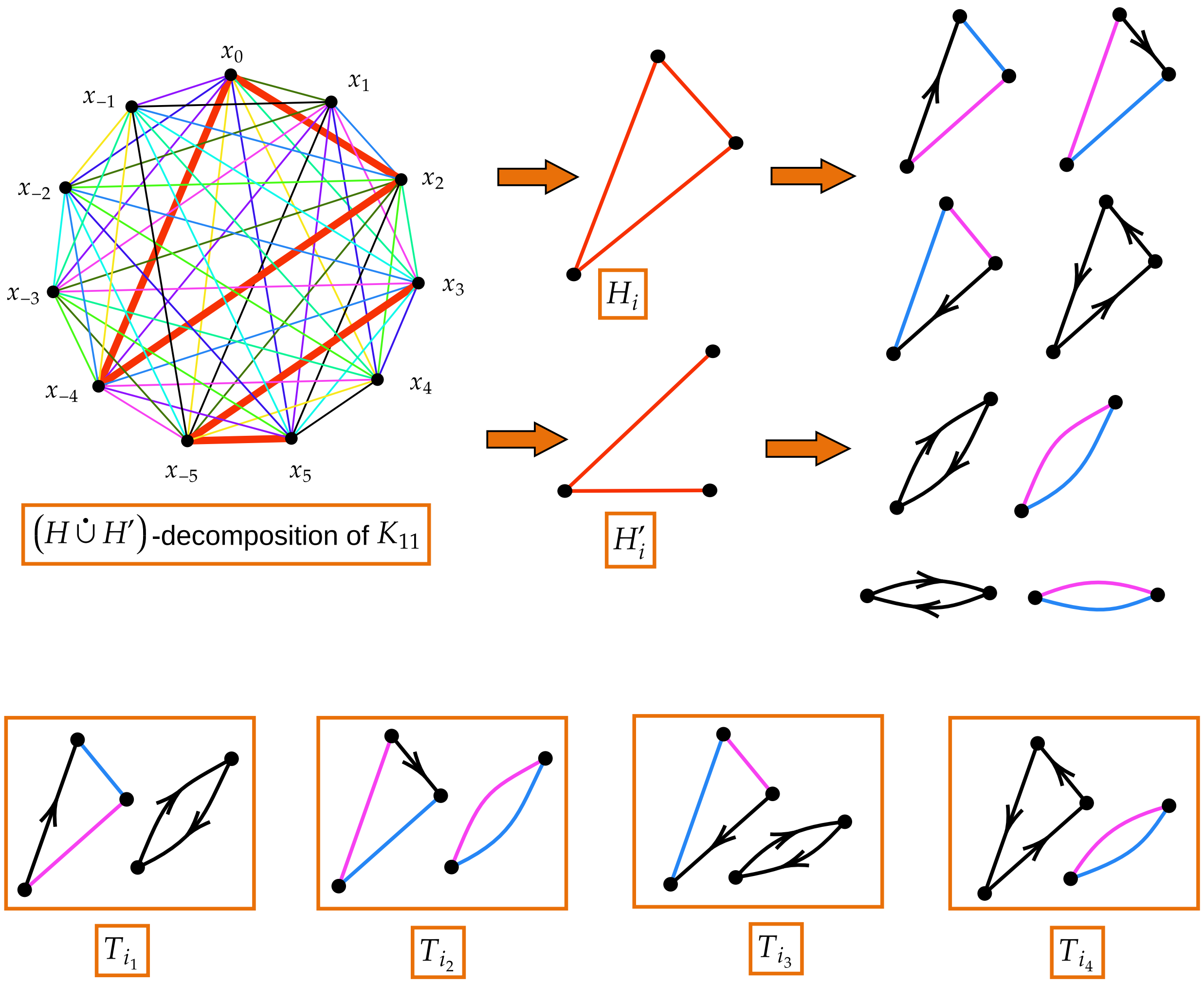


Figure 4.1: From an $(H \dot\cup H')$-decomposition of K_{11} to an HOP (C_2, C_3)-decomposition of $4K_{11}^\bullet$.

4.2 HOP (C_2, C_m)-decompositions of $4K_n^\bullet$ when $n \equiv 1 \pmod{2(m+2)}$

In the following lemma, we show that for $n \equiv 1 \pmod{2(m+2)}$, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_m)-decomposition.

Lemma 4.2.1. Let $m \geq 3$, $k \geq 1$ be integers, and let $n = 2(m+2)k + 1$. Then there exists an HOP (C_2, C_m)-decomposition of $4K_n^\bullet$.

Proof. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_n)$. By the given labeling, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, where $d \in \{1, 2, \ldots, (m+2)k\}$. Therefore, the group $\langle\rho\rangle$ acting on $E(K_n)$ has $(m+2)k$ orbits, each of size n.

Our goal is to construct k subgraphs of K_n, namely $F_1, F_2, \ldots, F_k$, where each F_i is a

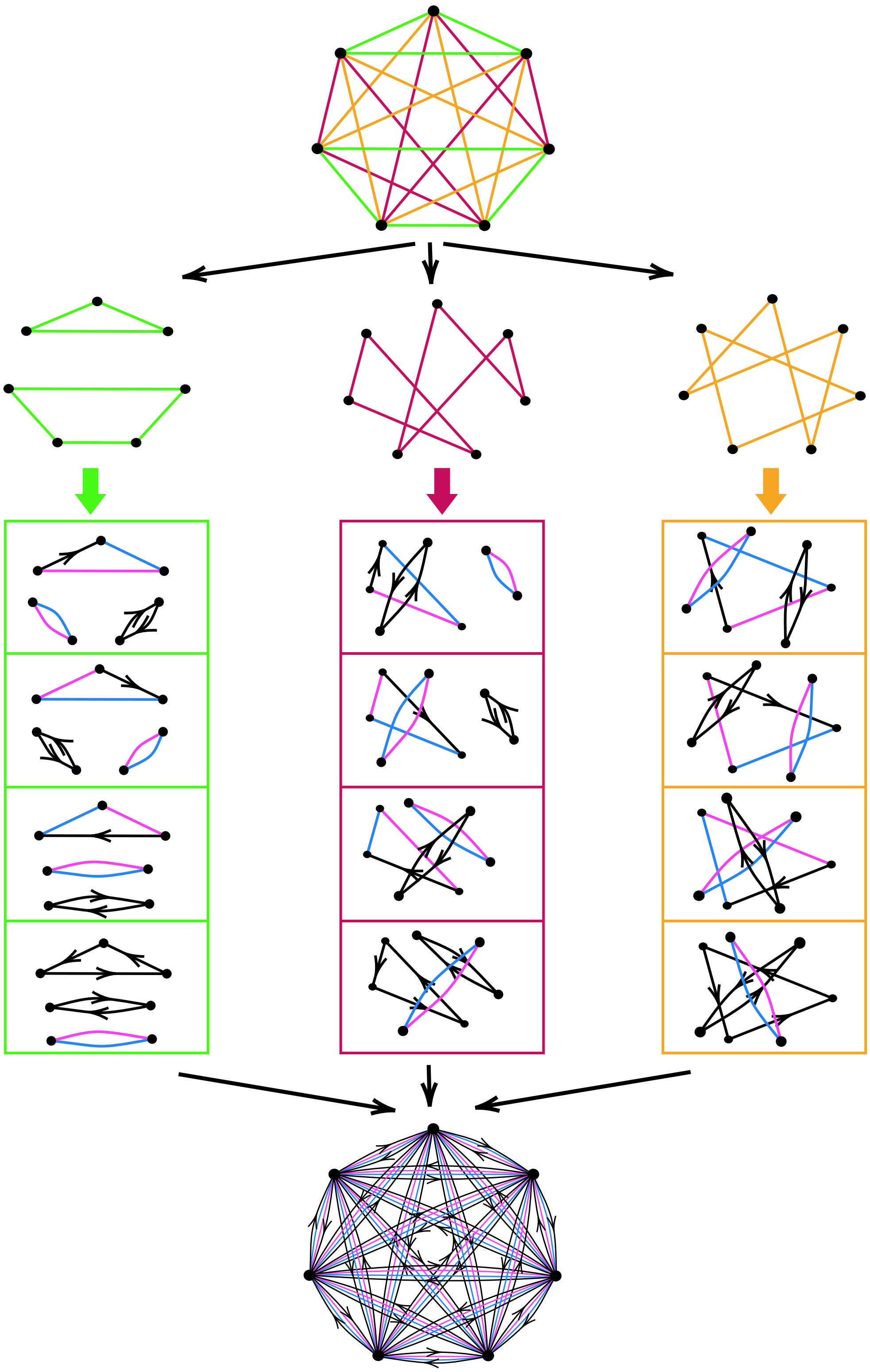


Figure 4.2: From a (C_3, C_4)-factorization of K_7 to an HOP $(C_2^{\langle 2\rangle}, C_3)$-factorization of $4K_7^\bullet$.

disjoint union of a C_m-subgraph and two edge-disjoint 1-regular subgraphs of order 2, and where $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, (m{+}2)k\}$. Therefore, $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$ forms a decomposition of K_n that satisfies the condition in Lemma 4.1.1 (for $t = 1$ and $\alpha = 1$); hence, by Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_m)-decomposition.

We start by constructing m-cycles. First, we build two paths, P and Q, and then form the cycle C by concatenating these paths appropriately. The construction depends on m mod 4, so we consider four cases: $m \equiv 0, 1, 2,$ or $3 \pmod 4$.

Case 1: $m \equiv 0 \pmod 4$.

For $i \in \{1, 2, \ldots, k\}$, define a walk P_i as follows:

$$\begin{aligned} P_i \;\; = \;\; & x_{1-i}\; x_i\; x_{-i}\; x_{2k+i}\; x_{-(2k+i)}\; x_{4k+i}\; x_{-(4k+i)} \cdots \\ & \cdots\; x_{\frac{m-12}{2}k+i}\; x_{-(\frac{m-12}{2}k+i)}\; x_{\frac{m-8}{2}k+i}\; x_{-(\frac{m-8}{2}k+i)}\; x_{\frac{m-4}{2}k+i}. \end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m-2}{2}$, which yields the following sequence of differences:

$$\begin{gathered} 2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \ldots, \\ \ldots, (m-14)k+2i, (m-12)k+2i, (m-10)k+2i, (m-8)k+2i, (m-6)k+2i. \end{gathered}$$

The explicit differences in the sequences $P_1, P_2, \ldots, P_k$ are listed below. We see that the paths $P_1, P_2, \ldots, P_k$, jointly contain the consecutive odd differences from 1 to $2k-1$ (shown in red), and the consecutive even differences from 2 to $(m-4)k$.

$$\begin{array}{ll} P_1: & 1, 2, 2k+2, 4k+2, 6k+2, \ldots, (m-10)k+2, (m-8)k+2, (m-6)k+2 \\ P_2: & 3, 4, 2k+4, 4k+4, 6k+4, \ldots, (m-10)k+4, (m-8)k+4, (m-6)k+4 \\ P_3: & 5, 6, 2k+6, 4k+6, 6k+6, \ldots, (m-10)k+6, (m-8)k+6, (m-6)k+6 \\ & \vdots \\ P_{k-2}: & 2k-5, 2k-4, 4k-4, 6k-4, 8k-4, \ldots, (m-8)k-4, (m-6)k-4, (m-4)k-4 \\ P_{k-1}: & 2k-3, 2k-2, 4k-2, 6k-2, 8k-2, \ldots, (m-8)k-2, (m-6)k-2, (m-4)k-2 \\ P_k: & 2k-1, 2k, 4k, 6k, 8k, \ldots, (m-8)k, (m-6)k, (m-4)k \end{array}$$

Next, for $i \in \{1, 2, \ldots, k\}$, define a walk Q_i as follows:

$$\begin{aligned} Q_i \;\; = \;\; & x_{-(\frac{m+4}{2}k+i)}\;\; x_{\frac{m+4}{2}k+i}\;\; x_{-(\frac{m+8}{2}k+i)}\; x_{\frac{m+8}{2}k+i}\; \cdots \\ & \cdots\; x_{-((m-2)k+i)}\;\; x_{(m-2)k+i}\;\; x_{-(mk+i)}\;\; x_{mk+i}. \end{aligned}$$

It is easy to verify that Q_i, for $i \in \{1, 2, \ldots, k\}$, is a path of length $\frac{m-2}{2}$ and produces the following sequence of differences:

$$mk-2i+1, (m-2)k-2i+1, (m-4)k-2i+1, \ldots, 8k-2i+1, 6k-2i+1, 4k-2i+1.$$

In what follows, the differences in the sequence of Q_i, for $i \in \{1, 2, \dots, k\}$, are explicitly listed. Observe that $Q_1, Q_2, \dots, Q_k$ jointly contain the consecutive odd differences from $2k+1$ to $mk - 1$.

$$\begin{aligned}
Q_1 :& \quad mk-1, (m-2)k-1, (m-4)k-1, \dots, 8k-1, 6k-1, 4k-1 \\
Q_2 :& \quad mk-3, (m-2)k-3, (m-4)k-3, \dots, 8k-3, 6k-3, 4k-3 \\
Q_3 :& \quad mk-5, (m-2)k-5, (m-4)k-5, \dots, 8k-5, 6k-5, 4k-5 \\
&\quad \vdots \\
Q_{k-2} :& \quad (m-2)k+5, (m-4)k+5, (m-6)k+5, \dots, 6k+5, 4k+5, 2k+5 \\
Q_{k-1} :& \quad (m-2)k+3, (m-4)k+3, (m-6)k+3, \dots, 6k+3, 4k+3, 2k+3 \\
Q_k :& \quad (m-2)k+1, (m-4)k+1, (m-6)k+1, \dots, 6k+1, 4k+1, 2k+1
\end{aligned}$$

Now, for $i \in \{1, 2, \dots, k\}$, we construct the following closed walk using paths P_i and Q_i:

$$C_i = P_i \; x_{\frac{m-4}{2}k+i} \; x_{-(\frac{m+4}{2}k+i)} \; Q_i \; x_{mk+i} \; x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$\begin{gathered}
2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \dots, \\
(m-14)k+2i, (m-12)k+2i, (m-10)k+2i, (m-8)k+2i, (m-6)k+2i, \\
mk+2i, mk-2i+1, (m-2)k-2i+1, (m-4)k-2i+1, \dots, \\
8k-2i+1, 6k-2i+1, 4k-2i+1, mk+2i-1.
\end{gathered}$$

Below, the differences in the sequences of $C_1, C_2, \dots, C_k$ are listed explicitly.

$$\begin{aligned}
C_1 :& \quad 1, \dots, (m-6)k+2, mk+2, mk-1, (m-2)k-1, \dots, 4k-1, mk+1 \\
C_2 :& \quad 3, \dots, (m-6)k+4, mk+4, mk-3, (m-2)k-3, \dots, 4k-3, mk+3 \\
C_3 :& \quad 5, \dots, (m-6)k+6, mk+6, mk-5, (m-2)k-5, \dots, 4k-5, mk+5 \\
&\quad \vdots \\
C_{k-2} :& \quad 2k-5, \dots, (m-4)k-4, (m+2)k-4, (m-2)k+5, (m-4)k+5, \dots, 2k+5, (m+2)k-5 \\
C_{k-1} :& \quad 2k-3, \dots, (m-4)k-2, (m+2)k-2, (m-2)k+3, (m-4)k+3, \dots, 2k+3, (m+2)k-3 \\
C_k :& \quad 2k-1, \dots, (m-4)k, (m+2)k, (m-2)k+1, (m-4)k+1, \dots, 2k+1, (m+2)k-1
\end{aligned}$$

Observe that $C_1, C_2, \dots, C_k$ jointly contain exactly one edge of each difference in $\{1, 2, 3, \dots, (m+2)k\}$ except for

$$(m-4)k + 2j, \text{ for } j \in \{1, 2, \dots, 2k\}.$$

Note that we have $2k$ differences from $\{1, 2, 3, \dots, (m+2)k\}$ that are not covered by any C_i. Next, we define subgraphs that cover the leftover differences.

First, assume $m \geq 8$. Define the following subgraphs of K_n (isomorphic to K_2) that cover the leftover differences:

- $E_j^{(1)} = x_{-\left(\frac{m-6}{2}k+j\right)}\ x_{\frac{m-2}{2}k+j}$, for $j \in \{1,2,\dots,k\}$;
- $E_j^{(2)} = x_{-\left(\frac{m-4}{2}k+j\right)}\ x_{\frac{m}{2}k+j}$, for $j \in \{1,2,\dots,k\}$.

Observe that none of the vertices in the intervals $[x_{-(\frac{m-6}{2}k+1)}, x_{-(\frac{m+4}{2}k)}]$ and $[x_{\frac{m-2}{2}k+1}, x_{\frac{m+4}{2}k}]$ are used in any C_i, and that only vertices in these intervals are used to construct the subgraphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from $E_j^{(1)}$ and $E_j^{(2)}$, for all $j \in \{1,2,\dots,k\}$.

For $m = 4$, the missing differences are $2j$, for $j \in \{1,2,\dots,2k\}$, which are covered by the following subgraphs (isomorphic to K_2):

- $E_j^{(1)} = x_{-(6k-j+1)}\ x_{6k-j}$, for $j \in \{1,2,\dots,k-1\}$;
- $E_k^{(1)} = x_{-k}\ x_{-3k}$; and
- $E_j^{(2)} = x_{-(k+j)}\ x_{k+j}$, for $j \in \{1,2,\dots,k\}$.

Case 2: $m \equiv 2 \pmod 4$.

For $i \in \{1,2,\dots,k\}$, define a walk P_i as follows:

$$\begin{aligned} P_i &= x_{1-i}\ x_i\ x_{-i}\ x_{2k+i}\ x_{-(2k+i)}\ x_{4k+i}\ x_{-(4k+i)} \cdots \\ &\cdots\ x_{\frac{m-14}{2}k+i}\ x_{-(\frac{m-14}{2}k+i)}\ x_{\frac{m-10}{2}k+i}\ x_{-(\frac{m-10}{2}k+i)}\ x_{\frac{m-6}{2}k+i}\ x_{-(\frac{m-6}{2}k+i)}. \end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m-2}{2}$, and yields the same sequence of differences as P_i in Case 1:

$$\begin{gathered} 2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \dots, \\ \dots, (m-14)k+2i, (m-12)k+2i, (m-10)k+2i, (m-8)k+2i, (m-6)k+2i. \end{gathered}$$

Next for $i \in \{1,2,\dots,k\}$, define a walk Q_i as follows:

$$\begin{aligned} Q_i &= x_{\frac{m+2}{2}k+i}\ \ x_{-(\frac{m+6}{2}k+i)}\ \ x_{\frac{m+6}{2}k+i}\ \cdots \\ &\quad \cdots\ x_{-((m-2)k+i)}\ \ x_{(m-2)k+i}\ \ x_{-(mk+i)}\ \ x_{mk+i}. \end{aligned}$$

It is easy to verify that Q_i, for $i \in \{1,2,\dots,k\}$, is a path of length $\frac{m-2}{2}$, and yields the same sequence of differences as Q_i in Case 1:

$$mk-2i+1, (m-2)k-2i+1, (m-4)k-2i+1, \dots, 8k-2i+1, 6k-2i+1, 4k-2i+1.$$

Now, for $i \in \{1,2,\dots,k\}$, we construct the following closed walk using paths P_i and Q_i (see Figure 4.3):

$$C_i = P_i\ x_{-(\frac{m-6}{2}k+i)}\ x_{\frac{m+2}{2}k+i}\ Q_i\ x_{mk+i}\ x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \dots,$$

$$(m-14)k+2i, (m-12)k+2i, (m-10)k+2i, (m-8)k+2i, (m-6)k+2i,$$
$$(m-2)k+2i, mk-2i+1, (m-2)k-2i+1, (m-4)k-2i+1, \ldots,$$
$$8k-2i+1, 6k-2i+1, 4k-2i+1, mk+2i-1$$

The differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$\begin{array}{rl}
C_1: & 1, 2, \ldots, (m-6)k+2, (m-2)k+2, mk-1, (m-2)k-1, \ldots, 4k-1, mk+1 \\
C_2: & 3, 4, \ldots, (m-6)k+4, (m-2)k+4, mk-3, (m-2)k-3, \ldots, 4k-3, mk+3 \\
C_3: & 5, 6, \ldots, (m-6)k+6, (m-2)k+6, mk-5, (m-2)k-5, \ldots, 4k-5, mk+5 \\
 & \vdots \\
C_{k-2}: & 2k-5, 2k-4, \ldots, (m-4)k-4, mk-4, (m-2)k+5, (m-4)k+5, \ldots, 2k+5, (m+2)k-5 \\
C_{k-1}: & 2k-3, 2k-2, \ldots, (m-4)k-2, mk-2, (m-2)k+3, (m-4)k+3, \ldots, 2k+3, (m+2)k-3 \\
C_k: & 2k-1, 2k, \ldots, (m-4)k, mk, (m-2)k+1, (m-4)k+1, \ldots, 2k+1, (m+2)k-1
\end{array}$$

Observe that $C_1, C_2, \ldots, C_k$ jointly contain exactly one edge of each difference in $\{1, 2, 3, \ldots, (m+2)k\}$ except for the following differences:

(i) $(m-4)k+2j$, for $j \in \{1, 2, \ldots, k\}$; and

(ii) $mk+2j$, for $j \in \{1, 2, \ldots, k\}$.

Note that we have $2k$ differences from the set $\{1, 2, 3, \ldots, (m+2)k\}$ that are not covered by any C_i. Next, we define the subgraphs that cover the remaining differences.

Define the following subgraphs of K_n (isomorphic to K_2) that cover the differences listed in (i) and (ii):

- $E_j^{(1)} = x_{-\left(\frac{m-4}{2}k+j\right)}\ x_{\frac{m-4}{2}k+j}$, for $j \in \{1, 2, \ldots, k\}$;
- $E_j^{(2)} = x_{-\left(\frac{m}{2}k+j\right)}\ x_{\frac{m}{2}k+j}$, for $j \in \{1, 2, \ldots, k\}$.

Observe that none of the vertices in the intervals $[x_{-(\frac{m-4}{2}k+1)}, x_{-(\frac{m+6}{2}k)}]$ and $[x_{\frac{m-4}{2}k+1}, x_{\frac{m+2}{2}k}]$ are used in any C_i, and that only vertices from these intervals are used to construct the subgraphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$, for $j \in \{1, 2, \ldots, k\}$.

Case 3: $m \equiv 1 \pmod 4$.

For $i \in \{1, 2, \ldots, k\}$, define a walk P_i as follows:

$$\begin{aligned}
P_i \;=\;& x_{1-i}\ x_i\ x_{-i}\ x_{2k+i}\ x_{-(2k+i)}\ x_{4k+i}\ x_{-(4k+i)} \cdots \\
& \cdots\ x_{\frac{m-9}{2}k+i}\ x_{-(\frac{m-9}{2}k+i)}\ x_{\frac{m-5}{2}k+i}\ x_{-(\frac{m-5}{2}k+i)}\ x_{\frac{m-1}{2}k+i}.
\end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m+1}{2}$, and yields the following sequence of differences:

$$2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \ldots,$$
$$\ldots, (m-9)k+2i, (m-7)k+2i, (m-5)k+2i, (m-3)k+2i.$$

Note that the differences in $P_1, P_2, \ldots, P_k$ are listed explicitly below. We see that the paths $P_1, P_2, \ldots, P_k$ jointly contain the consecutive odd differences from 1 to $2k-1$ (shown in red), as well as the consecutive even differences from 2 to $(m-1)k$.

$$\begin{array}{rl}
P_1: & 1, 2, 2k+2, 4k+2, 6k+2, \ldots, (m-7)k+2, (m-5)k+2, (m-3)k+2 \\
P_2: & 3, 4, 2k+4, 4k+4, 6k+4, \ldots, (m-7)k+4, (m-5)k+4, (m-3)k+4 \\
P_3: & 5, 6, 2k+6, 4k+6, 6k+6, \ldots, (m-7)k+6, (m-5)k+6, (m-3)k+6 \\
 & \vdots \\
P_{k-2}: & 2k-5, 2k-4, 4k-4, 6k-4, 8k-4, \ldots, (m-5)k-4, (m-3)k-4, (m-1)k-4 \\
P_{k-1}: & 2k-3, 2k-2, 4k-2, 6k-2, 8k-2, \ldots, (m-5)k-2, (m-3)k-2, (m-1)k-2 \\
P_k: & 2k-1, 2k, 4k, 6k, 8k, \ldots, (m-5)k, (m-3)k, (m-1)k
\end{array}$$

For $m = 5$, let $Q_i = x_{4k+3i-1}$. Next, assume $m \geq 9$, and for $i \in \{1, 2, \ldots, k\}$, define a walk Q_i as follows:

$$\begin{aligned}
Q_i \;\; = \;\; & x_{-(\frac{m+11}{2}k+i)} \;\; x_{\frac{m+3}{2}k+i} \;\; x_{-(\frac{m+15}{2}k+i)} \;\; x_{\frac{m+7}{2}k+i} \;\; \cdots \\
& \cdots \;\; x_{-((m-1)k+i)} \;\; x_{(m-5)k+i} \;\; x_{-((m+1)k+i)} \;\; x_{(m-3)k+i} \;\; x_{(m-1)k+3i-1}.
\end{aligned}$$

It is easy to verify that Q_i is a path of length $\frac{m-5}{2}$, and yields the following sequence of differences:

$$(m-3)k-2i+1, (m-5)k-2i+1, (m-7)k-2i+1, \ldots, 8k-2i+1, 6k-2i+1, 2k+2i-1$$

In what follows, we see that the paths $Q_1, Q_2, \ldots, Q_k$ jointly contain the consecutive odd differences from $2k+1$ to $(m-3)k-1$.

$$\begin{array}{rl}
Q_1: & (m-3)k-1, (m-5)k-1, (m-7)k-1, \ldots, 8k-1, 6k-1, 2k+1 \\
Q_2: & (m-3)k-3, (m-5)k-3, (m-7)k-3, \ldots, 8k-3, 6k-3, 2k+3 \\
Q_3: & (m-3)k-5, (m-5)k-5, (m-7)k-5, \ldots, 8k-5, 6k-5, 2k+5 \\
 & \vdots \\
Q_{k-2}: & (m-5)k+5, (m-7)k+5, (m-9)k+5, \ldots, 6k+5, 4k+5, 4k-5 \\
Q_{k-1}: & (m-5)k+3, (m-7)k+3, (m-9)k+3 \ldots, 6k+3, 4k+3, 4k-3 \\
Q_k: & (m-5)k+1, (m-7)k+1, (m-9)k+1 \ldots, 6k+1, 4k+1, 4k-1
\end{array}$$

Now, for $i \in \{1, 2, \ldots, k\}$, we construct the following closed walk using paths P_i and Q_i:

$$C_i = P_i \; x_{\frac{m-1}{2}k+i} \; x_{-(\frac{m+11}{2}k+i)} \; Q_i \; x_{(m-1)k+3i-1} \; x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$\begin{gathered} 2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \ldots, \\ (m-9)k+2i, (m-7)k+2i, (m-5)k+2i, (m-3)k+2i, \\ (m-1)k-2i+1, (m-3)k-2i+1, (m-5)k-2i+1, (m-7)k-2i+1, \ldots, \\ 10k-2i+1, 8k-2i+1, 6k-2i+1, 2k+2i-1, d. \end{gathered}$$

Note that for $i \in \{1, 2, \ldots, \lfloor \frac{3k+2}{4} \rfloor\}$, the difference $d = (m-1)k+4i-2$, and it is even; for $i \in \{\lfloor \frac{3k+2}{4} \rfloor + 1, \ldots, k\}$, the difference $d = (m+1)k+4(k-i)+3$, and it is odd. Moreover, for all $i \in \{1, 2, \ldots, k\}$, the difference $(m-1)k-2i+1$ is odd.

The explicit differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$\begin{array}{rc} C_1: & 1, 2, \ldots, (m-3)k+2, (m-1)k-1, (m-3)k-1, \ldots, 2k+1, (m-1)k+2 \\ C_2: & 3, 4, \ldots, (m-3)k+4, (m-1)k-3, (m-3)k-3, \ldots, 2k+3, (m-1)k+6 \\ C_3: & 5, 6, \ldots, (m-3)k+6, (m-1)k-5, (m-3)k-5, \ldots, 2k+5, (m-1)k+10 \\ & \vdots \\ C_{k-2}: & 2k-5, 2k-4, \ldots, (m-1)k-4, (m-3)k+5, (m-5)k+5, \ldots, 4k-5, (m+1)k+11 \\ C_{k-1}: & 2k-3, 2k-2, \ldots, (m-1)k-2, (m-3)k+3, (m-5)k+3, \ldots, 4k-3, (m+1)k+7 \\ C_k: & 2k-1, 2k, \ldots, (m-1)k, (m-3)k+1, (m-5)k+1, \ldots, 4k-1, (m+1)k+3 \end{array}$$

Observe that $C_1, C_2, \ldots, C_k$ jointly contain exactly one edge of each difference in $\{1, 2, 3, \ldots, (m+2)k\}$ except for the following differences:

(i) $(m-1)k+2j-1$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $(m-1)k+4j$, for $j \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$; and

(iii) $(m+1)k+4(k-i)+1$, for $j \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$.

Note that there are $2k$ differences from the set $\{1, 2, 3, \ldots, (m+2)k\}$ that are not covered by any C_i. Define the following subgraphs of K_n (isomorphic to K_2) which cover the differences listed in (i)–(iii):

- $E_j^{(1)} = x_{-\left(\frac{m+1}{2}k+j-1\right)} \; x_{\left(\frac{m-3}{2}k+j\right)}$, for $j \in \{1, 2, \ldots, k\}$;
- $E_j^{(2)} = x_{-\left(\frac{m-3}{2}k+3j\right)} \; x_{\left(\frac{m+1}{2}k+j\right)}$, for $j \in \{1, 2, \ldots, k\}$.

Note that the edge in $E_j^{(1)}$ is of difference $(m-1)k+2j-1$. The edge in $E_j^{(2)}$, for $j \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$, is of difference $(m-1)k+4j$, and for $j \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$, it is of difference $(m+1)k+4(k-i)+1$.

Observe that none of the vertices in the intervals $[x_{-(\frac{m-3}{2}k+1)}, x_{-(\frac{m+11}{2}k)}]$, $[x_{\frac{m-3}{2}k+1}, x_{\frac{m-1}{2}k}]$, and $[x_{\frac{m+1}{2}k+1}, x_{\frac{m+3}{2}k}]$ are used in any C_i, and only vertices from these intervals are used to construct the subgraphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$, for $j \in \{1, 2, \ldots, k\}$.

Case 4: $m \equiv 3 \pmod 4$.

For $i \in \{1, 2, \ldots, k\}$, define a walk P_i as follows:

$$\begin{aligned} P_i \;&=\; x_{1-i}\; x_i\; x_{-i}\; x_{2k+i}\; x_{-(2k+i)}\; x_{4k+i}\; x_{-(4k+i)} \cdots \\ &\cdots\; x_{\frac{m-11}{2}k+i}\; x_{-(\frac{m-11}{2}k+i)}\; x_{\frac{m-7}{2}k+i}\; x_{-(\frac{m-7}{2}k+i)}\; x_{\frac{m-3}{2}k+i}. \end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m-1}{2}$, and yields the following sequence of differences:

$$\begin{gathered} 2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \ldots, \\ \ldots, (m-11)k+2i, (m-9)k+2i, (m-7)k+2i, (m-5)k+2i. \end{gathered}$$

The explicit differences in the paths $P_1, P_2, \ldots, P_k$ are listed below. We observe that the paths $P_1, P_2, \ldots, P_k$ jointly contain the consecutive odd differences from 1 to $2k-1$ (shown in red), as well as the consecutive even differences from 2 to $(m-3)k$.

$$\begin{array}{rc} P_1: & 1, 2, 2k+2, 4k+2, 6k+2, \ldots, (m-9)k+2, (m-7)k+2, (m-5)k+2 \\ P_2: & 3, 4, 2k+4, 4k+4, 6k+4, \ldots, (m-9)k+4, (m-7)k+4, (m-5)k+4 \\ P_3: & 5, 6, 2k+6, 4k+6, 6k+6, \ldots, (m-9)k+6, (m-7)k+6, (m-5)k+6 \\ & \vdots \\ P_{k-2}: & 2k-5, 2k-4, 4k-4, 6k-4, 8k-4, \ldots, (m-7)k-4, (m-5)k-4, (m-3)k-4 \\ P_{k-1}: & 2k-3, 2k-2, 4k-2, 6k-2, 8k-2, \ldots, (m-7)k-2, (m-5)k-2, (m-3)k-2 \\ P_k: & 2k-1, 2k, 4k, 6k, 8k, \ldots, (m-7)k, (m-5)k, (m-3)k \end{array}$$

Next, we construct paths Q_i. For $m = 3$, let $Q_i = x_{2k+3i-1}$. Now, assume $m \geq 7$, and for $i \in \{1, 2, \ldots, k\}$, define a path Q_i as follows:

$$\begin{aligned} Q_i \;&=\; x_{-(\frac{m+9}{2}k+i)}\; x_{\frac{m+1}{2}k+i}\; x_{-(\frac{m+13}{2}k+i)}\; x_{\frac{m+5}{2}k+i} \cdots \\ &\cdots\; x_{-((m-1)k+i)}\; x_{(m-5)k+i}\; x_{-((m+1)k+i)}\; x_{(m-3)k+i}\; x_{(m-1)k+3i-1}. \end{aligned}$$

It is easy to verify that Q_i is a path of length $\frac{m-3}{2}$, and yields the following sequence of differences:

$$(m-1)k-2i+1, (m-3)k-2i+1, (m-5)k-2i+1, \ldots, 8k-2i+1, 6k-2i+1, 2k+2i-1$$

Below, we see that the paths $Q_1, Q_2, \ldots, Q_k$ jointly contain the consecutive odd differences from $2k+1$ to $(m-1)k-1$.

$$\begin{aligned}
Q_1 :&\quad (m-1)k-1, (m-3)k-1, (m-5)k-1, \ldots, 8k-1, 6k-1, 2k+1\\
Q_2 :&\quad (m-1)k-3, (m-3)k-3, (m-5)k-3, \ldots, 8k-3, 6k-3, 2k+3\\
Q_3 :&\quad (m-1)k-5, (m-3)k-5, (m-5)k-5, \ldots, 8k-5, 6k-5, 2k+5\\
&\qquad\vdots\\
Q_{k-2} :&\quad (m-3)k+5, (m-5)k+5, (m-7)k+5, \ldots, 6k+5, 4k+5, 4k-5\\
Q_{k-1} :&\quad (m-3)k+3, (m-5)k+3, (m-7)k+3, \ldots, 6k+3, 4k+3, 4k-3\\
Q_k :&\quad (m-3)k+1, (m-5)k+1, (m-7)k+1, \ldots, 6k+1, 4k+1, 4k-1
\end{aligned}$$

Now, for $i \in \{1, 2, \ldots, k\}$, we construct the following closed walk using paths P_i and Q_i:

$$C_i = P_i \; x_{\frac{m-3}{2}k+i} \; x_{-(\frac{m+9}{2}k+i)} \; Q_i \; x_{(m-1)k+3i-1} \; x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$\begin{gathered}
2i-1, 2i, 2k+2i, 4k+2i, 6k+2i, 8k+2i, \ldots,\\
(m-11)k+2i, (m-9)k+2i, (m-7)k+2i, (m-5)k+2i,\\
(m+1)k-2i+1, (m-1)k-2i+1, (m-3)k-2i+1, (m-5)k-2i+1, \ldots,\\
10k-2i+1, 8k-2i+1, 6k-2i+1, 2k+2i-1, d
\end{gathered}$$

Note that for $i \in \{1, 2, \ldots, \lfloor \frac{3k+2}{4} \rfloor\}$, the difference $d = (m-1)k+4i-2$, and it is even; for $i \in \{\lfloor \frac{3k+2}{4} \rfloor + 1, \ldots, k\}$, the difference $d = (m+1)k+4(k-i)+3$, and it is odd. Moreover, for all $i \in \{1, 2, \ldots, k\}$, the difference $(m+1)k-2i+1$ is odd.

Below, the differences in $C_1, C_2, \ldots, C_k$ are listed explicitly.

$$\begin{aligned}
C_1 :&\quad 1, 2, \ldots, (m-5)k+2, (m+1)k-1, (m-1)k-1, \ldots, 6k-1, 2k+1, (m-1)k+2\\
C_2 :&\quad 3, 4, \ldots, (m-5)k+4, (m+1)k-3, (m-1)k-3, \ldots, 6k-3, 2k+3, (m-1)k+6\\
C_3 :&\quad 5, 6, \ldots, (m-5)k+6, (m+1)k-5, (m-1)k-5, \ldots, 6k-5, 2k+5, (m-1)k+10\\
&\qquad\vdots\\
C_{k-2} :&\quad 2k-5, 2k-4, \ldots, (m-3)k-4, (m-1)k+5, (m-3)k+5, \ldots, 4k+5, 4k-5, (m+1)k+11\\
C_{k-1} :&\quad 2k-3, 2k-2, \ldots, (m-3)k-2, (m-1)k+3, (m-3)k+3, \ldots, 4k+3, 4k-3, (m+1)k+7\\
C_k :&\quad 2k-1, 2k, \ldots, (m-3)k, (m-1)k+1, (m-3)k+1, \ldots, 4k+1, 4k-1, (m+1)k+3
\end{aligned}$$

Observe that $C_1, C_2, \ldots, C_k$ jointly contain exactly one edge of each difference in $\{1, 2, 3, \ldots, (m+2)k\}$ except for the following differences:

(i) $(m-3)k+2j$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $(m-1)k+4j$, for $j \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$;

(iii) $(m+1)k+4(k-i)+1$, for $j \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$.

Note that we have $2k$ differences from the set $\{1, 2, 3, \ldots, (m+2)k\}$ that are not covered by any C_i. Next, we define subgraphs that cover the leftover differences.

First assume $m \geq 7$, and for all $j \in \{1, 2, \ldots, k\}$, let $E_j^{(1)}$ and $E_j^{(2)}$ denote the following subgraphs of K_n (isomorphic to K_2):

- $E_j^{(1)} = x_{-(\frac{m-5}{2}k+j)} \;\; x_{\frac{m-1}{2}k+j}$; and
- $E_j^{(2)} = x_{-(\frac{m+3}{2}k+3j)} \;\; x_{\frac{m-5}{2}k+j}$.

Note that the edge in $E_j^{(1)}$ has difference $(m-3)k + 2j$. The edge in $E_j^{(2)}$, for $j \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$, has difference $(m-1)k + 4j$, and for $j \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$, it has difference $(m+1)k + 4(k-j) + 1$.

Observe that none of the vertices in the intervals $[x_{-(\frac{m-5}{2}k+1)}, x_{-(\frac{m+9}{2}k)}]$, $[x_{\frac{m-1}{2}k+1}, x_{\frac{m+1}{2}k}]$, and $[x_{\frac{m-5}{2}k+1}, x_{\frac{m-3}{2}k}]$ are used in any C_i, and only vertices from these intervals are used to construct the graphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$, for $j \in \{1, 2, \ldots, k\}$.

For $m = 3$, the missing differences are as follows:

(i) $2j$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $2k + 4j$, for $j \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$;

(iii) $8k - 4j + 1$, for $j \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$.

The following subgraphs cover the differences listed in (i)-(iii):

- $E_j^{(1)} = x_{-(2k+j)}\, x_{-(2k-j)}$, with difference $2j$, for $j \in \{1, 2, \ldots, k\}$;
- $E_j^{(2)} = x_{-(k+3j)}\, x_{k+j}$, which has difference $2k+4j$ for $j \in \left\{1, 2, \ldots, \left\lfloor \frac{3k}{4} \right\rfloor\right\}$, and difference $8k - 4j + 1$ for $j \in \left\{\left\lfloor \frac{3k}{4} \right\rfloor + 1, \ldots, k\right\}$.

In all cases above, we have shown how to construct k m-cycles, $C_1, C_2, \ldots, C_k$, and $2k$ subgraphs $E_1^{(1)}, E_2^{(1)}, \ldots, E_k^{(1)}$ and $E_1^{(2)}, E_2^{(2)}, \ldots, E_k^{(2)}$, each isomorphic to K_2 such that for all $i, j \in \{1, 2, \ldots, k\}$:

- each C_i is disjoint from each $E_j^{(1)}$ and each $E_j^{(2)}$;
- the $E_j^{(1)}$, as well as the $E_j^{(2)}$, are pairwise disjoint;
- all $E_j^{(1)}$ are edge-disjoint from all $E_j^{(2)}$; and
- all C_i, together with all $E_j^{(1)}$ and $E_j^{(2)}$, jointly contain exactly one edge of each difference in $\{1, 2, \ldots, (m+2)k\}$.

Now, for $i \in \{1, 2, \ldots, k\}$, let $F_i = C_i \dot\cup (E_i^{(1)} \cup E_i^{(2)})$. Since $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, (m+2)k\}$, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition of K_n that satisfies the conditions of Lemma 4.1.1. Applying Lemma 4.1.1, we conclude that the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_m)-decomposition. In Figure 4.3, we illustrate the subgraph F_i for the case $m \equiv 2 \pmod 4$.

∎

4.3 HOP (C_2, C_m)-Decompositions of $4K_n^\bullet$ when $n \equiv m + 2 \pmod{2(m+2)}$

In the following lemma, we prove that if m is odd and $n \equiv m+2 \pmod{2(m+2)}$, then the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_m)-decomposition.

Lemma 4.3.1. Let $m \geq 3$ be an odd integer, and let $k \geq 0$ be an integer. Let $n = (2k+1)(m+2)$. Then there exists an HOP (C_2, C_m)-decomposition of $4K_n^\bullet$.

Proof. Notice that $K_{(2k+1)(m+2)} = H_1 \bigoplus H_2$, where H_1 is the disjoint union of $2k+1$ copies of K_{m+2}, and H_2 is the complete equipartite graph with $2k+1$ parts of size $m+2$, that is, $K_{(2k+1)[m+2]}$. In [36], it is shown that $4K_{m+2}^\bullet$ admits an HOP (C_2, C_m)-factorization. Consequently, by Lemma 3.3.1, $4H_1^\bullet$ admits an HOP (C_2, C_m)-factorization.

If we can construct an HOP (C_2, C_m)-decomposition of $4K_{(2k+1)[m+2]}^\bullet$, then both $4H_1^\bullet$ and $4H_2^\bullet$ admit HOP (C_2, C_m)-decompositions. Applying Lemma 3.3.1 again, it follows that the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_m)-decomposition. Thus, our goal is to construct an HOP (C_2, C_m)-decomposition of $4K_{(2k+1)[m+2]}^\bullet$.

For simplicity, let $\ell = 2k+1$. The complete equipartite graph $K_{\ell[m+2]}$ is a circulant graph, $\mathrm{Circ}(n; S)$, with vertex set $\{x_i : i \in \mathbb{Z}_n\}$, where $n = (m+2)\ell$, and edge set $\{x_i x_{i+d} : i \in \mathbb{Z}_n, d \in S\}$, where

$$S = \left\{1, 2, \ldots, \ell - 1, \ell + 1, \ldots, 2\ell - 1, 2\ell + 1, \ldots, \frac{(m+2)\ell - 1}{2}\right\}.$$

Let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_{\ell[m+2]})$. Observe that S contains all differences from 1 to $\frac{(m+2)\ell-1}{2}$, except those in the set $\{\ell, 2\ell, \ldots, \frac{(m+1)}{2}\ell\}$. Therefore, the group $\langle \rho \rangle$ acting on $E(K_{\ell[m+2]})$ has

$$\frac{(m+2)\ell - 1}{2} - \frac{m+1}{2} = \frac{1}{2}\left((m+2)(2k+1) - 1 - (m+1)\right) = (m+2)k$$

orbits, each of size $n = (m+2)\ell$.

We aim to construct k subgraphs of $K_{\ell[m+2]}$, namely $F_1, F_2, \ldots, F_k$, where each F_i is a disjoint union of a (C_m)-subgraph and two edge-disjoint 1-regular subgraphs of order

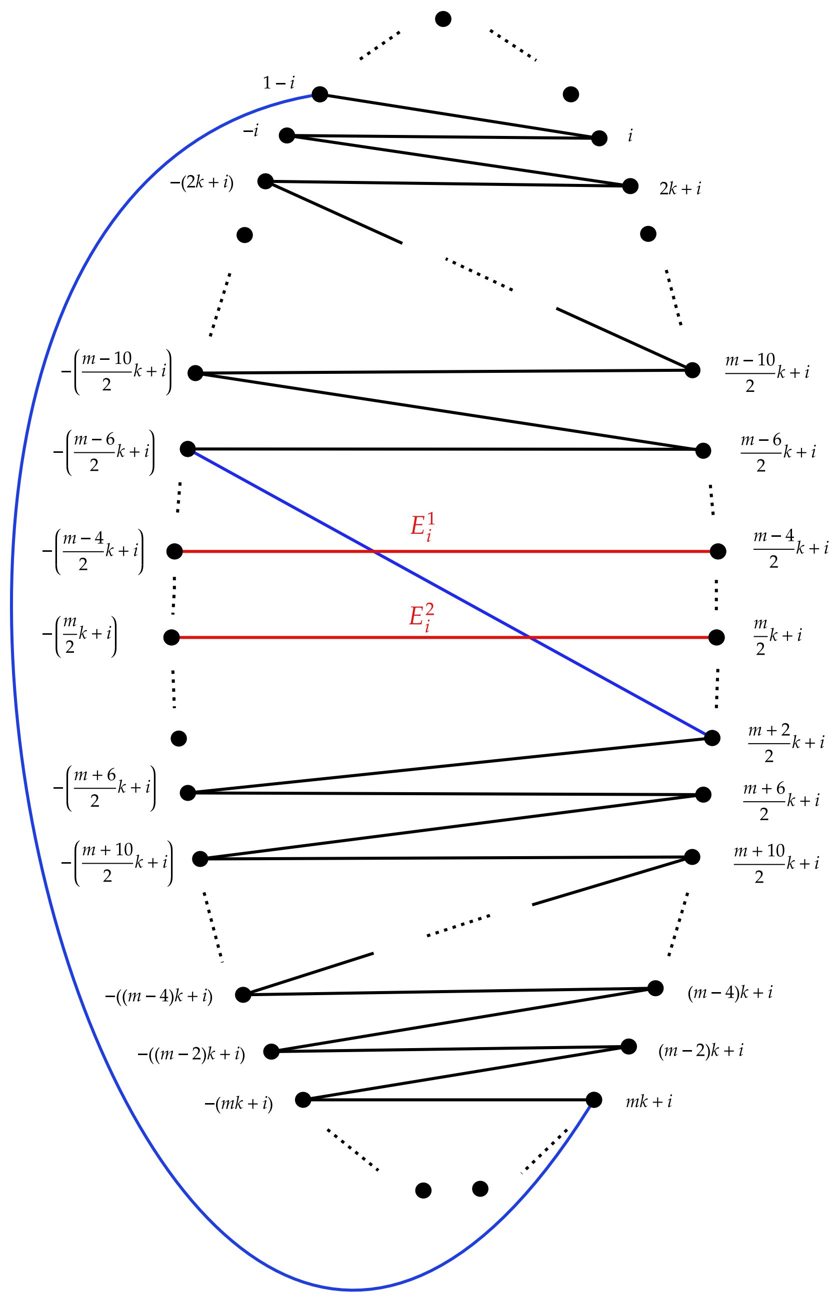


Figure 4.3: The F_i subgraph in Lemma 4.2.1 for $m \equiv 2 \pmod 4$.

2, and $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in S. Therefore, $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$ forms a decomposition of $K_{\ell[m+2]}$ that satisfies the condition of Lemma 4.1.1 (for $t = 1$); hence, by Lemma 4.1.1, the multigraph $4K^\bullet_{\ell[m+2]}$ admits an HOP (C_2, C_m)-decomposition.

We begin by constructing a suitable (C_m)-subgraph. First, we build two paths, P and Q, and then show how to form the cycle C_m by properly joining these paths. Since m is odd, it suffices to consider the following two cases: $m \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$.

Case 1: $m \equiv 1$ (mod 4).

For $i \in \{1, 2, \ldots, k\}$, define a walk P_i as follows:

$$\begin{aligned} P_i \;=\; & x_{1-i}\; x_i\; x_{-i}\; x_{(\ell-1)+i}\; x_{-((\ell+1)+i)}\; x_{2(\ell-1)+i}\; x_{-(2(\ell+1)+i)} \cdots \\ & \cdots\; x_{\frac{m-9}{4}(\ell-1)+i}\; x_{-(\frac{m-9}{4}(\ell+1)+i)}\; x_{\frac{m-5}{4}(\ell-1)+i}\; x_{-(\frac{m-5}{4}(\ell+1)+i)}\; x_{\frac{m-1}{4}(\ell-1)+i}. \end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m+1}{2}$, and yields the following sequence of differences:

$$\begin{gathered} 2i-1, 2i, \ell+2i-1, 2\ell+2i, 3\ell+2i-1, 4\ell+2i, 5\ell+2i-1, 6\ell+2i, \ldots \\ \ldots, \tfrac{m-9}{2}\ell+2i, \tfrac{m-7}{2}\ell+2i-1, \tfrac{m-5}{2}\ell+2i, \tfrac{m-3}{2}\ell+2i-1. \end{gathered}$$

Note that the explicit differences in $P_1, P_2, \ldots, P_k$ are listed below. As can be seen, these paths jointly contain the consecutive odd differences from 1 to $2k-1$ (shown in red), as well as the consecutive even differences from 2 to $\frac{m-1}{2}\ell - 2$.

$$\begin{array}{rl} P_1: & 1, 2, \ell+1, 2\ell+2, 3\ell+1 \ldots, \frac{m-7}{2}\ell+1, \frac{m-5}{2}\ell+2, \frac{m-3}{2}\ell+1 \\ P_2: & 3, 4, \ell+3, 2\ell+4, 3\ell+3 \ldots, \frac{m-7}{2}\ell+3, \frac{m-5}{2}\ell+4, \frac{m-3}{2}\ell+3 \\ P_3: & 5, 6, \ell+5, 2\ell+6, 3\ell+5 \ldots, \frac{m-7}{2}\ell+5, \frac{m-5}{2}\ell+6, \frac{m-3}{2}\ell+5 \\ & \vdots \\ P_{k-2}: & 2k-5, 2k-4, 2\ell-6, 3\ell-5, 4\ell-6, \ldots, \frac{m-5}{2}\ell-6, \frac{m-3}{2}\ell-5, \frac{m-1}{2}\ell-6 \\ P_{k-1}: & 2k-3, 2k-2, 2\ell-4, 3\ell-3, 4\ell-4, \ldots, \frac{m-5}{2}\ell-4, \frac{m-3}{2}\ell-3, \frac{m-1}{2}\ell-4 \\ P_k: & 2k-1, 2k, 2\ell-2, 3\ell-1, 4\ell-2, \ldots, \frac{m-5}{2}\ell-2, \frac{m-3}{2}\ell-1, \frac{m-1}{2}\ell-2 \end{array}$$

Now, we construct the paths Q_i that contain the consecutive odd differences of S, excluding the odd differences already present in P_i.

For $m = 5$, let $Q_i = x_{2\ell+3i+1}$. Now assume that $m \geq 9$, and for each $i \in \{1, 2, \ldots, k\}$, define the path Q_i as follows:

$$\begin{aligned} Q_i \;=\; & x_{-(\frac{m+7}{4}(\ell-1)+\frac{m-1}{2}+3i)}\; x_{\frac{m+7}{4}(\ell+1)-\frac{m-1}{2}-i}\; x_{-(\frac{m+11}{4}(\ell-1)+\frac{m-1}{2}+3i)}\; x_{\frac{m+11}{4}(\ell+1)-\frac{m-1}{2}-i} \cdots \\ & \cdots\; x_{-(\frac{m-3}{2}(\ell-1)+\frac{m-1}{2}+3i)}\; x_{\frac{m-3}{2}(\ell+1)-\frac{m-1}{2}-i}\; x_{-(\frac{m-1}{2}(\ell-1)+\frac{m-1}{2}+3i)}\; x_{\frac{m-1}{2}(\ell+1)-\frac{m-1}{2}-i}\; x_{\frac{m+1}{2}\ell+i}. \end{aligned}$$

It is easy to verify that Q_i is a path of length $\frac{m-5}{2}$, and yields the following sequence of differences:

$$\tfrac{m-3}{2}\ell-2i, \tfrac{m-5}{2}\ell-2i+1, \tfrac{m-7}{2}\ell-2i, \ldots, 5\ell-2i, 4\ell-2i+1, 3\ell-2i, \ell+2i$$

In what follows, we see that the paths $Q_1, Q_2, \ldots, Q_k$ jointly contain the consecutive odd differences from $\ell+2$ to $\frac{m-3}{2}\ell-2$.

$$\begin{array}{rl} Q_1: & \frac{m-3}{2}\ell-2, \frac{m-5}{2}\ell-1, \frac{m-7}{2}\ell-2, \ldots, 5\ell-2, 4\ell-1, 3\ell-2, \ell+2 \\ Q_2: & \frac{m-3}{2}\ell-4, \frac{m-5}{2}\ell-3, \frac{m-7}{2}\ell-4, \ldots, 5\ell-4, 4\ell-3, 3\ell-4, \ell+4 \end{array}$$

$$
\begin{array}{rl}
Q_3: & \frac{m-3}{2}\ell-6, \frac{m-5}{2}\ell-5, \frac{m-7}{2}\ell-6, \ldots, 5\ell-6, 4\ell-5, 3\ell-6, \ell+6 \\
& \vdots \\
Q_{k-2}: & \frac{m-5}{2}\ell+5, \frac{m-7}{2}\ell+6, \frac{m-9}{2}\ell+5, \ldots, 4\ell+5, 3\ell+6, 2\ell+5, 2\ell-5 \\
Q_{k-1}: & \frac{m-5}{2}\ell+3, \frac{m-7}{2}\ell+4, \frac{m-9}{2}\ell+3 \ldots, 4\ell+3, 3\ell+4, 2\ell+3, 2\ell-3 \\
Q_k: & \frac{m-5}{2}\ell+1, \frac{m-7}{2}\ell+2, \frac{m-9}{2}\ell+1 \ldots, 4\ell+1, 3\ell+2, 2\ell+1, 2\ell-1
\end{array}
$$

Now, for $i \in \{1, 2, \ldots, k\}$, we construct the following closed walk using paths P_i and Q_i (see Figure 4.4).

$$C_i = P_i \; x_{\frac{m-1}{4}(\ell-1)+i} \; x_{-(\frac{m+7}{4}(\ell-1)+\frac{m-1}{2}+3i)} \; Q_i \; x_{\frac{m+1}{2}\ell+i} \; x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$
\begin{gathered}
2i-1, 2i, \ell+2i-1, 2\ell+2i, 3\ell+2i-1, 4\ell+2i, 5\ell+2i-1, \ldots, \\
\tfrac{m-9}{2}\ell+2i, \tfrac{m-7}{2}\ell+2i-1, \tfrac{m-5}{2}\ell+2i, \tfrac{m-3}{2}\ell+2i-1, \\
\tfrac{m+1}{2}\ell-4i+2, \tfrac{m-3}{2}\ell-2i, \tfrac{m-5}{2}\ell-2i+1, \tfrac{m-7}{2}\ell-2i, \ldots, \\
5\ell-2i, 4\ell-2i+1, 3\ell-2i, \ell+2i, d
\end{gathered}
$$

Note that for $i \in \{1, 2, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$, the difference $d = \frac{m+1}{2}\ell + 2i - 1$, and it is even; for $i \in \{\lfloor \frac{k+1}{2} \rfloor + 1, \ldots, k\}$, the difference $d = \frac{m+3}{2}\ell - 2i + 1$, and it is odd. Moreover, for all $i \in \{1, 2, \ldots, k\}$, the difference $\frac{m+1}{2}\ell - 4i + 2$ is odd.

The explicit differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$
\begin{array}{rl}
C_1: & 1, 2, \ell+1, \ldots, \frac{m-3}{2}\ell+1, \frac{m+1}{2}\ell-2, \frac{m-3}{2}\ell-2, \ldots, 3\ell-2, \ell+2, \frac{m+1}{2}\ell+1 \\
C_2: & 3, 4, \ell+3, \ldots, \frac{m-3}{2}\ell+3, \frac{m+1}{2}\ell-6, \frac{m-3}{2}\ell-4, \ldots, 3\ell-4, \ell+4, \frac{m+1}{2}\ell+3 \\
C_3: & 5, 6, \ell+5, \ldots, \frac{m-3}{2}\ell+5, \frac{m+1}{2}\ell-10, \frac{m-3}{2}\ell-6, \ldots, 3\ell-6, \ell+6, \frac{m+1}{2}\ell+5 \\
& \vdots \\
C_{k-2}: & 2k-5, 2k-4, 2\ell-6, \ldots, \frac{m-1}{2}\ell-6, \frac{m-3}{2}\ell+12, \frac{m-5}{2}\ell+5, \ldots, 2\ell+5, 2\ell-5, \frac{m+1}{2}\ell+6 \\
C_{k-1}: & 2k-3, 2k-2, 2\ell-4, \ldots, \frac{m-1}{2}\ell-4, \frac{m-3}{2}\ell+8, \frac{m-5}{2}\ell+3, \ldots, 2\ell+3, 2\ell-3, \frac{m+1}{2}\ell+4 \\
C_k: & 2k-1, 2k, 2\ell-2, \ldots, \frac{m-1}{2}\ell-2, \frac{m-3}{2}\ell+4, \frac{m-5}{2}\ell+1, \ldots, 2\ell+1, 2\ell-1, \frac{m+1}{2}\ell+2
\end{array}
$$

Observe that $C_1, C_2, \ldots, C_k$ jointly contain exactly one edge of each difference in S except for the following differences:

(i) $\frac{m-1}{2}\ell + 2j$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $\frac{m-3}{2}\ell + 4j - 2$, for $j \in \{1, 2, \ldots, k\}$.

Note that we have $2k$ differences from S that are not covered by any C_i. Next, we define subgraphs that cover the leftover differences. For all $j \in \{1, 2, \ldots, k\}$, let $E_j^{(1)}$ and $E_j^{(2)}$ denote the follwoing subgraphs of K_n (isomorphic to K_2):

- $E_j^{(1)} = x_{-(\frac{m-3}{4}(\ell+1)+j)} \; x_{\frac{m+1}{4}(\ell-1)+j+1}$; and

- $E_j^{(2)} = x_{-(\frac{m-3}{4}(\ell+1)+3j-2)}\ \ x_{\frac{m-3}{4}(\ell-1)+j}$.

Note that for $j \in \{1, 2, \ldots, k\}$, the edge in $E_j^{(1)}$ is of difference $\frac{m-1}{2}\ell + 2j$, and the edge in $E_j^{(2)}$ is of difference $\frac{m-3}{2}\ell + 4j - 2$.

Moreover, observe that none of the vertices in the intervals $[x_{-(\frac{m-3}{4}(\ell+1))},\ x_{-(\frac{m+7}{4}(\ell+1)-2)}]$, $[x_{\frac{m+1}{4}(\ell-1)+1},\ x_{\frac{m+5}{4}(\ell-1)+3}]$, and $[x_{\frac{m-3}{4}(\ell-1)+1},\ x_{\frac{m-1}{4}(\ell-1)}]$ are used in any C_i, and only vertices in these intervals are used to construct the subgraphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$.

The cycle C_i, for $m = 5$, is as follows:

$$C_i = P_i\ x_{(\ell-1)+i}\ x_{2\ell+3i+1}\ x_{1-i}.$$

It can be seen that C_i traverses edges of the following differences (in order):

$$2i-1, 2i, \ell+2i-1, \ell+2i+2, d$$

Note that for $i \in \{1, 2, \ldots, \lfloor\frac{3k+1}{4}\rfloor\}$, the difference $d = 2\ell + 4i$, and for $i \in \{\lfloor\frac{3k+1}{4}\rfloor + 1, \ldots, k\}$, the difference $d = 5\ell - 4i$.

$$\begin{array}{rc}
C_1: & 1, 2, \ell+1, \ell+4, 2\ell+4 \\
C_2: & 3, 4, \ell+3, \ell+6, 2\ell+8 \\
C_3: & 5, 6, \ell+5, \ell+8, 2\ell+12 \\
 & \vdots \\
C_{k-2}: & 2k-5, 2k-4, 2\ell-6, 2\ell-3, 3\ell+10 \\
C_{k-1}: & 2k-3, 2k-2, 2\ell-4, 2\ell-1, 3\ell+6 \\
C_k: & 2k-1, 2k, 2\ell-2, 2\ell+1, 3\ell+2
\end{array}$$

For $m = 5$, the missing differences are as follows:

(i) $2\ell + 4j - 2 = 4k + 4j$, for $j \in \{1, 2, \ldots, \lfloor\frac{3}{4}(k+1)\rfloor\}$;

(ii) $5\ell - 4j + 2 = 10k - 4j + 7$, for $j \in \{\lfloor\frac{3}{4}(k+1)\rfloor + 1, \ldots, k\}$;

(iii) $2\ell + 2j + 1 = 4k + 2j + 3$, for $j \in \{1, 2, \ldots, k-1\}$; and

(iv) $\ell + 2 = 2k + 3$.

The following subgraphs cover the differences listed in (i)-(iv):

- Let $E_j^{(1)} = x_{-(k+3j)}\, x_{3k+j}$; it is of difference $2\ell + 4j - 2$ for $j \in \left\{1, 2, \ldots, \left\lfloor\frac{3}{4}(k+1)\right\rfloor\right\}$, and of difference $5\ell - 4j + 2$ for $j \in \left\{\left\lfloor\frac{3}{4}(k+1)\right\rfloor + 1, \ldots, k\right\}$;
- Let $E_j^{(2)} = x_{-(3k+j+3)}\, x_{k+j}$; it is of difference $2\ell + 2j + 1$ for $j \in \{1, 2, \ldots, k-1\}$; and

- Let $E_k^{(2)} = x_{-(7k+3)}\, x_{-5k}$; it is of difference $\ell+2$.

Observe that none of the vertices in the intervals $[x_{-(k+1)},\ x_{-(7k+3)}]$, $[x_{k+1},\ x_{2k}]$, and $[x_{3k+1},\ x_{4k+5}]$ are used in any C_i. Only vertices from these intervals are used to construct the graphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$.

Case 2: $m \equiv 3 \pmod 4$.

For $i \in \{1, 2, \ldots, k\}$, define a walk P_i as follows:

$$\begin{aligned} P_i \;&=\; x_{1-i}\, x_i\, x_{-i}\, x_{(\ell-1)+i}\, x_{-((\ell+1)+i)}\, x_{2(\ell-1)+i}\, x_{-(2(\ell+1)+i)} \cdots \\ &\cdots\; x_{\frac{m-11}{4}(\ell-1)+i}\, x_{-(\frac{m-11}{4}(\ell+1)+i)}\, x_{\frac{m-7}{4}(\ell-1)+i}\, x_{-(\frac{m-7}{4}(\ell+1)+i)}\, x_{\frac{m-3}{4}(\ell-1)+i}. \end{aligned}$$

It is easy to verify that P_i is a path of length $\frac{m-1}{2}$, and yields the following sequence of differences:

$$\begin{gathered} 2i-1, 2i, \ell+2i-1, 2\ell+2i, 3\ell+2i-1, 4\ell+2i, 5\ell+2i-1, 6\ell+2i, \ldots \\ \ldots, \tfrac{m-11}{2}\ell+2i, \tfrac{m-9}{2}\ell+2i-1, \tfrac{m-7}{2}\ell+2i, \tfrac{m-5}{2}\ell+2i-1. \end{gathered}$$

Note that the explicit differences in $P_1, P_2, \ldots, P_k$ are listed below. We see that paths $P_1, P_2, \ldots, P_k$ jointly contain the consecutive odd differences from 1 to $2k-1$ (shown in red), and the consecutive even differences of S from 2 to $\frac{m-3}{2}\ell - 2$.

$$\begin{aligned} P_1 :&\quad 1, 2, \ell+1, 2\ell+2, 3\ell+1 \ldots, \tfrac{m-9}{2}\ell+1, \tfrac{m-7}{2}\ell+2, \tfrac{m-5}{2}\ell+1 \\ P_2 :&\quad 3, 4, \ell+3, 2\ell+4, 3\ell+3 \ldots, \tfrac{m-9}{2}\ell+3, \tfrac{m-7}{2}\ell+4, \tfrac{m-5}{2}\ell+3 \\ P_3 :&\quad 5, 6, \ell+5, 2\ell+6, 3\ell+5 \ldots, \tfrac{m-9}{2}\ell+5, \tfrac{m-7}{2}\ell+6, \tfrac{m-5}{2}\ell+5 \\ &\quad\vdots \\ P_{k-2} :&\quad 2k-5, 2k-4, 2\ell-6, 3\ell-5, 4\ell-6, \ldots, \tfrac{m-7}{2}\ell-6, \tfrac{m-5}{2}\ell-5, \tfrac{m-3}{2}\ell-6 \\ P_{k-1} :&\quad 2k-3, 2k-2, 2\ell-4, 3\ell-3, 4\ell-4, \ldots, \tfrac{m-7}{2}\ell-4, \tfrac{m-5}{2}\ell-3, \tfrac{m-3}{2}\ell-4 \\ P_k :&\quad 2k-1, 2k, 2\ell-2, 3\ell-1, 4\ell-2, \ldots, \tfrac{m-7}{2}\ell-2, \tfrac{m-5}{2}\ell-1, \tfrac{m-3}{2}\ell-2 \end{aligned}$$

Now, we construct paths Q_i that contain consecutive odd differences of S excluding the odd differences used in P. For $i \in \{1, 2, \ldots, k\}$, define a walk Q_i as follows:

$$\begin{aligned} Q_i \;=\;& x_{-(\frac{m+5}{4}(\ell-1)+\frac{m+3}{2}-3i)}\, x_{\frac{m+5}{4}(\ell+1)-\frac{m+3}{2}+5i}\, x_{-(\frac{m+9}{4}(\ell-1)+\frac{m+3}{2}-3i)} \\ & x_{\frac{m+9}{4}(\ell+1)-\frac{m+3}{2}+5i}\, x_{-(\frac{m+13}{4}(\ell-1)+\frac{m+3}{2}-3i)}\, x_{\frac{m+13}{4}(\ell+1)-\frac{m+3}{2}+5i} \cdots \\ & \cdots\, x_{-(\frac{m-5}{2}(\ell-1)+\frac{m+3}{2}-3i)}\, x_{\frac{m-5}{2}(\ell+1)-\frac{m+3}{2}+5i}\, x_{-(\frac{m-3}{2}(\ell-1)+\frac{m+3}{2}-3i)} \\ & x_{\frac{m-3}{2}(\ell+1)-\frac{m+3}{2}+5i}\, x_{-(\frac{m-1}{2}(\ell-1)+\frac{m+3}{2}-3i)}\, x_{\frac{m-1}{2}(\ell+1)-\frac{m+3}{2}+5i}\, x_{-(\frac{m+1}{2}(\ell-1)+\frac{m+3}{2}-3i)}. \end{aligned}$$

It is easy to verify that Q_i is a path of length $\frac{m-3}{2}$, and yields the following sequence of differences:

$$\tfrac{m-1}{2}\ell-2i, \tfrac{m-3}{2}\ell-2i+1, \tfrac{m-5}{2}\ell-2i, \tfrac{m-7}{2}\ell-2i+1, \ldots, 5\ell-2i, 4\ell-2i+1, 3\ell-2i, 2\ell-2i+1$$

Below, we see that paths $Q_1, Q_2, \ldots, Q_k$ jointly contain the consecutive odd differences of S from $\ell+2$ to $\frac{m-1}{2}\ell - 2$.

$$\begin{array}{rl}
Q_1: & \frac{m-1}{2}\ell-2, \frac{m-3}{2}\ell-1, \frac{m-5}{2}\ell-2, \ldots, 5\ell-2, 4\ell-1, 3\ell-2, 2\ell-1 \\
Q_2: & \frac{m-1}{2}\ell-4, \frac{m-3}{2}\ell-3, \frac{m-5}{2}\ell-4, \ldots, 5\ell-4, 4\ell-3, 3\ell-4, 2\ell-3 \\
Q_3: & \frac{m-1}{2}\ell-6, \frac{m-3}{2}\ell-5, \frac{m-5}{2}\ell-6, \ldots, 5\ell-6, 4\ell-5, 3\ell-6, 2\ell-5 \\
& \vdots \\
Q_{k-2}: & \frac{m-3}{2}\ell+5, \frac{m-5}{2}\ell+6, \frac{m-7}{2}\ell+5, \ldots, 4\ell+5, 3\ell+6, 2\ell+5, \ell+6 \\
Q_{k-1}: & \frac{m-3}{2}\ell+3, \frac{m-5}{2}\ell+4, \frac{m-7}{2}\ell+3 \ldots, 4\ell+3, 3\ell+4, 2\ell+3, \ell+4 \\
Q_k: & \frac{m-3}{2}\ell+1, \frac{m-5}{2}\ell+2, \frac{m-7}{2}\ell+1 \ldots, 4\ell+1, 3\ell+2, 2\ell+1, \ell+2
\end{array}$$

Now, for $i \in \{1, 2, \ldots, k\}$, we construct the following closed walk using paths P_i and Q_i.

$$C_i = P_i \; x_{\frac{m-3}{4}(\ell-1)+i} \; x_{-(\frac{m+5}{4}(\ell-1)+\frac{m+3}{2}-3i)} \; Q_i \; x_{-(\frac{m+1}{2}(\ell-1)+\frac{m+3}{2}-3i)} \; x_{1-i}.$$

It can be seen that C_i is a cycle of length m, and traverses edges of the following differences (in order):

$$\begin{gathered}
2i-1, 2i, \ell+2i-1, 2\ell+2i, 3\ell+2i-1, 4\ell+2i, 5\ell+2i-1, \ldots, \\
\tfrac{m-11}{2}\ell+2i, \tfrac{m-9}{2}\ell+2i-1, \tfrac{m-7}{2}\ell+2i, \tfrac{m-5}{2}\ell+2i-1, \\
\tfrac{m+1}{2}\ell-2i+1, \tfrac{m-1}{2}\ell-2i, \tfrac{m-3}{2}\ell-2i+1, \tfrac{m-5}{2}\ell-2i, \ldots, \\
5\ell-2i, 4\ell-2i+1, 3\ell-2i, 2\ell-2i+1, \tfrac{m+1}{2}\ell-4i+2
\end{gathered}$$

The explicit differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$\begin{array}{rl}
C_1: & 1, 2, \ell+1, \ldots, \frac{m-5}{2}\ell+1, \frac{m+1}{2}\ell-1, \frac{m-1}{2}\ell-2, \ldots, 3\ell-2, 2\ell-1, \frac{m+1}{2}\ell-2 \\
C_2: & 3, 4, \ell+3, \ldots, \frac{m-5}{2}\ell+3, \frac{m+1}{2}\ell-3, \frac{m-1}{2}\ell-4, \ldots, 3\ell-4, 2\ell-3, \frac{m+1}{2}\ell-6 \\
C_3: & 5, 6, \ell+5, \ldots, \frac{m-5}{2}\ell+5, \frac{m+1}{2}\ell-5, \frac{m-1}{2}\ell-6, \ldots, 3\ell-6, 2\ell-5, \frac{m+1}{2}\ell-10 \\
& \vdots \\
C_{k-2}: & 2k-5, 2k-4, 2\ell-6, \ldots, \frac{m-3}{2}\ell-6, \frac{m-1}{2}\ell+6, \frac{m-3}{2}\ell+5, \ldots, 2\ell+5, \ell+6, \frac{m-3}{2}\ell+12 \\
C_{k-1}: & 2k-3, 2k-2, 2\ell-4, \ldots, \frac{m-3}{2}\ell-4, \frac{m-1}{2}\ell+4, \frac{m-3}{2}\ell+3, \ldots, 2\ell+3, \ell+4, \frac{m-3}{2}\ell+8 \\
C_k: & 2k-1, 2k, 2\ell-2, \ldots, \frac{m-3}{2}\ell-2, \frac{m-1}{2}\ell+2, \frac{m-3}{2}\ell+1, \ldots, 2\ell+1, \ell+2, \frac{m-3}{2}\ell+4
\end{array}$$

Observe that $C_1, C_2, \ldots, C_k$ jointly contain exactly one edge of each difference in S except for the following differences:

(i) $\frac{m-3}{2}\ell + 4j - 2$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $\frac{m+1}{2}\ell + j$, for $j \in \{1, 2, \ldots, k\}$.

Note that we have $2k$ differences from S that are not covered by any C_i. Next, we define subgraphs that cover the leftover differences. For all $j \in \{1, 2, \ldots, k\}$ and $m \geq 7$, let $E_j^{(1)}$ and $E_j^{(2)}$ denote the following subgraphs of K_n (isomorphic to K_2):

- $E_j^{(1)} = x_{-(\frac{m-5}{4}(\ell+1)+2j-1)} \; x_{\frac{m-1}{4}(\ell-1)+2j}$; and

- $E_j^{(2)} = x_{-(\frac{m-1}{4}(\ell+1)+1)} \; x_{\frac{m+3}{4}(\ell-1)+j}$.

Note that, for $j \in \{1, 2, \ldots, k\}$, the edge in $E_j^{(1)}$ is of difference $\frac{m-3}{2}\ell + 4j - 2$, and the edge in $E_j^{(2)}$ is of difference $\frac{m+1}{2}\ell + j$. Moreover, none of the vertices in the intervals $[x_{-(\frac{m-5}{4}(\ell+1))}, x_{-(\frac{m-1}{4}(\ell+1)+1)}]$ and $[x_{\frac{m-1}{4}(\ell-1)+1}, x_{\frac{m+5}{4}(\ell-1)+5}]$ are used in any C_i, and only vertices from these intervals are used to construct the subgraphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$.

For $m = 3$, the missing differences are as follows:

(i) $4j - 2$, for $j \in \{1, 2, \ldots, k\}$;

(ii) $2\ell + j = 4k + j + 2$, for $j \in \{1, 2, \ldots, k\}$.

The following subgraphs cover the differences listed in (i) and (ii):

- the graph $E_j^{(1)} = x_{3k+2j} \, x_{3k-2j+2}$, which is of difference $4j - 2$; and
- the graph $E_j^{(2)} = x_{-(k+2)} \; x_{3k+j}$, which is of difference $4k + j + 2$.

Observe that the vertex $x_{-(k+2)}$, as well as all vertices in the interval $[x_{k+1}, \; x_{5k+2}]$, are not used in any C_i. Moreover, only vertices from this interval are used to construct the graphs $E_j^{(1)}$ and $E_j^{(2)}$. Hence, each C_i is disjoint from all subgraphs $E_j^{(1)}$ and $E_j^{(2)}$.

In all the above cases, we have shown how to construct k m-cycles $C_1, C_2, \ldots, C_k$, and $2k$ subgraphs $E_1^{(1)}, E_2^{(1)}, \ldots, E_k^{(1)}$ and $E_1^{(2)}, E_2^{(2)}, \ldots, E_k^{(2)}$, each isomorphic to K_2, such that for all $i, j \in \{1, 2, \ldots, k\}$:

- each C_i is disjoint from each $E_j^{(1)}$ and each $E_j^{(2)}$;
- all C_i, together with all $E_j^{(1)}$ and $E_j^{(2)}$, jointly contain exactly one edge of each difference in S; and
- all $E_j^{(1)}$ and $E_j^{(2)}$ are edge-disjoint.

Now, for $i \in \{1, 2, \ldots, k\}$, let $F_i = C_i \,\dot\cup\, \left(E_i^{(1)} \cup E_i^{(2)}\right)$. Since $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in S, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition of $K_{\ell[m+2]}$ that satisfies the conditions of Lemma 4.1.1. Applying Lemma 4.1.1, we conclude that the multigraph $4K_{\ell[m+2]}^\bullet$ admits an HOP (C_2, C_m)-decomposition.

∎

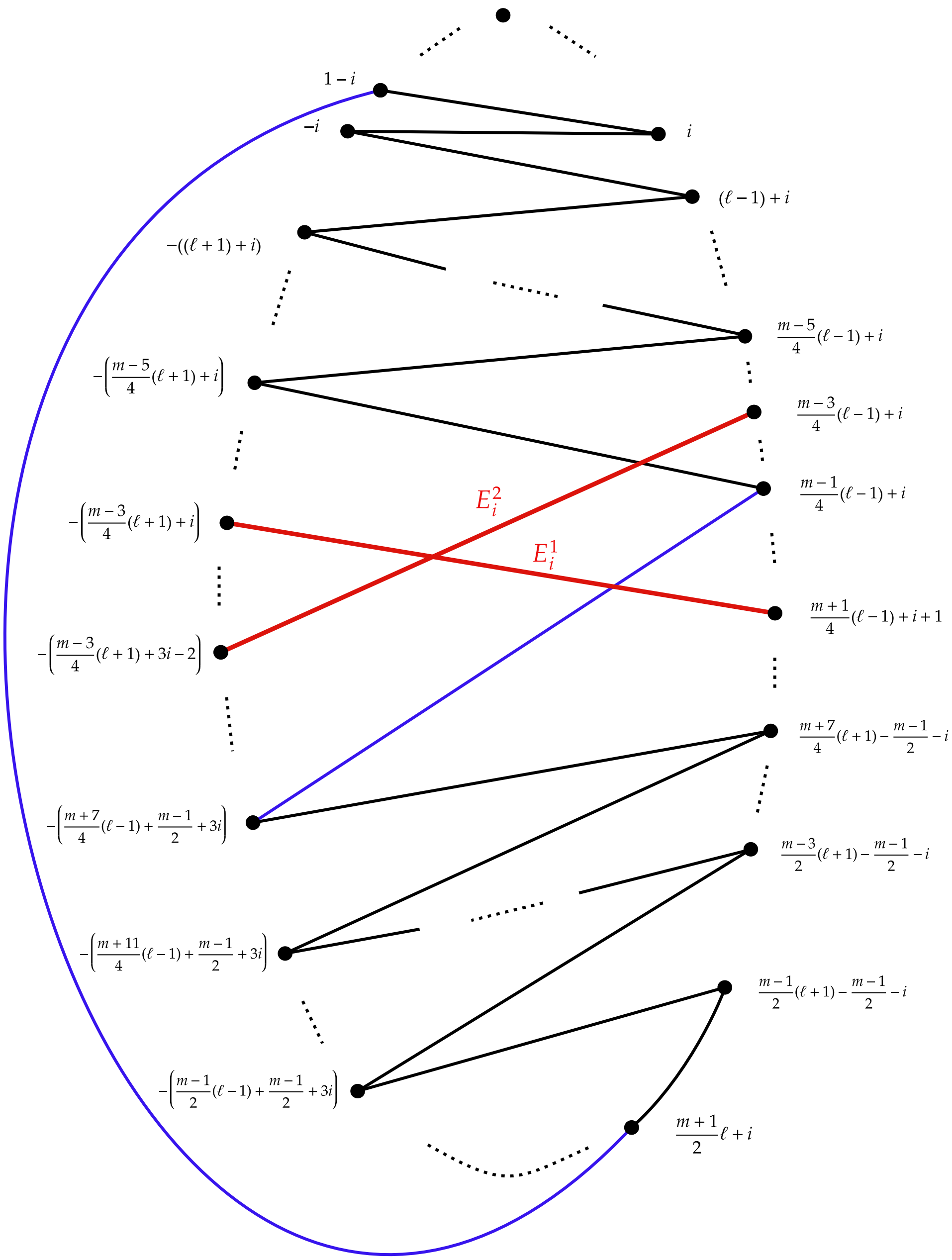


Figure 4.4: Subgraph F_i in Lemma 4.3.1 for $m \equiv 1 \pmod 4$ and $m \geq 9$.

4.4 HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$

Combining the results of Lemmas 4.2.1 and 4.3.1, together with the earlier results in Theorem 2.3.4, we obtain the following theorems.

Theorem 4.4.1. Let s, m_1, m_2 be integers such that $s \geq 0$ and $2 \leq m_1 \leq m_2$. Let $m = m_1 + m_2$, and $n = s + m$. Assume $n \equiv 1 \pmod{2m}$. Then HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution.

Proof. Since $n \equiv 1 \pmod{2m}$, we have $n = 2mk + 1$ for some integer $k \geq 1$. There are three cases to consider.

- $m_1 = m_2 = 2$. Then $n = 8k + 1$. By Theorem 2.3.2, there exists a (C_4)-decomposition of K_n. Applying Lemma 4.1.3, there exists an HOP (C_2, C_2)-decomposition of $4K_n^\bullet$.
- $m_1 = 2$ and $m_2 \geq 3$. By Lemma 4.2.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_{m_2})-decomposition.
- $m_1 \geq 3$. Then by Theorem 2.3.4**(1)**, there exists a (C_{m_1}, C_{m_2})-decomposition of K_n. Applying Lemma 3.3.3, the multigraph $4K_n^\bullet$ admits an HOP (C_{m_1}, C_{m_2})-decomposition.

In each of the above cases, we see that $4K_n^\bullet$ admits an HOP (C_{m_1}, C_{m_2})-decomposition. Therefore, by Theorem 3.2.1, HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution.

∎

Theorem 4.4.2. Let s, m_1, m_2 be integers such that $s \geq 0$ and $2 \leq m_1 \leq m_2$. Let $m = m_1 + m_2$ be odd, and $n = s + m$. Assume $n \equiv m \pmod{2m}$. Then HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution.

Proof. We have the following two cases to consider.

- $m_1 = 2$ and $m_2 \geq 3$. Since $m = m_1 + m_2$ is odd, m_2 must be odd. Then, by Lemma 4.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_{m_2})-decomposition.
- $m_1 \geq 3$. If $(m_1, m_2, n) \neq (4, 5, 9)$, then by Theorem 2.3.4(2), there exists a (C_{m_1}, C_{m_2})-decomposition of K_n. Applying Lemma 3.3.3, the multigraph $4K_n^\bullet$ admits an HOP (C_{m_1}, C_{m_2})-decomposition. If $(m_1, m_2, n) = (4, 5, 9)$, then by Lemma 2.2.13(ii) [36], an HOP (C_4, C_5)-factorization of $4K_9^\bullet$ exists.

In each of the above cases, we see that $4K_n^\bullet$ admits an HOP (C_{m_1}, C_{m_2})-decomposition. Now, applying Theorem 3.2.1, we conclude that HOP$(2^{\langle s\rangle}, 2m_1, 2m_2)$ has a solution.

∎

4.5 Generalization of Lemmas 4.2.1 and 4.3.1

We now generalize the (C_2, C_m)-decompositions constructed in Lemmas 4.2.1 and 4.3.1 by introducing an additional parameter r, which allows for multiple copies of C_2 in the decomposition.

The generalized forms are given in Lemmas 4.5.1 and 4.5.4, and they establish the existence of HOP $(C_2^{\langle r\rangle}, C_m)$-decompositions under the same conditions as before.

Lemma 4.5.1. Let $m \geq 3$, and $k, r \geq 1$ be integers. Let $n = 2(m+2r)k+1$. Then there exists an HOP $(C_2^{\langle r\rangle}, C_m)$-decomposition of $4K_n^\bullet$.

Proof. We have the following two cases to consider.

- $r = 1$. Then $n = 2(m+2)k+1$. By Lemma 4.2.1, we see that there exists an HOP (C_2, C_m)-decomposition of $4K_n^\bullet$.
- $r \geq 2$. Since $n \equiv 1 \pmod{2(m+2r)}$, by Theorem 2.3.4(1) there exists a (C_{2r}, C_m)-decomposition of K_n. Now, applying Lemma 4.1.3, we obtain an HOP$(C_2^{\langle r\rangle}, C_m)$-decomposition of $4K_n^\bullet$.

∎

The following corollary is a direct consequence of Lemma 4.5.1 and Theorem 3.2.1.

Corollary 4.5.2. Let $s \geq 0$, $r \geq 1$, and $m \geq 3$ be integers, and let $n = s+2r+m$. If $n \equiv 1 \pmod{2(2r+m)}$, then HOP$(2^{\langle s\rangle}, 4^{\langle r\rangle}, 2m)$ has a solution.

To prove Lemma 4.5.4, we first show the existence of an HOP $(C_2^{\langle r\rangle}, C_m)$-factorization of $4K_{m+2r}^\bullet$ in the following lemma.

Lemma 4.5.3. Let $m \geq 3$ be an odd integer, and let $r \geq 1$ be an integer. Then there exists an HOP $(C_2^{\langle r\rangle}, C_m)$-factorization of $4K_{m+2r}^\bullet$.

Proof. We have the following three cases to consider.

- $r = 1$. By Lemmas 2.2.9(vi) and 2.2.13(iv), the multigraph $4K_{m+2}^\bullet$ admits an HOP (C_2, C_m)-factorization.
- $r = 2$ and $m = 5$. By Lemma 2.2.13(ii), the multigraph $4K_9^\bullet$ admits an HOP $(C_2^{\langle 2\rangle}, C_5)$-factorization.
- $r \geq 2$ and $m \neq 5$. By Theorem 2.1.5, we know that there exists a 2-factorization of K_{m+2r} into (C_{2r}, C_m)-factors. Applying Lemma 4.1.3, we see the multigraph $4K_{m+2r}^\bullet$ admits an HOP $(C_2^{\langle r\rangle}, C_m)$-factorization.

∎

Lemma 4.5.4. Let $m \geq 3$ be an odd integer, $k \geq 0$, and $r \geq 1$ be integers. Let $n = (2k+1)(m+2r)$. Then there exists an HOP $(C_2^{\langle r \rangle}, C_m)$-decomposition of $4K_n^\bullet$.

Proof. For $k = 0$, the result follows by Lemma 4.5.3. For $k \geq 1$, we have the following three cases to consider.

- $r = 1$. Then $n = (2k+1)(m+2)$. By Lemma 4.3.1, we see that there exists an HOP (C_2, C_m)-decomposition of $4K_n^\bullet$.
- $r = 2$, and $(m, n) = (5, 9)$. By Lemma 2.2.13(ii), the multigraph $4K_9^\bullet$ admits an HOP $(C_2^{\langle 2 \rangle}, C_5)$-factorization.
- $r \geq 2$ and $(m, n) \neq (5, 9)$. By Theorem 2.3.4(2), the complete graph K_n admits a (C_{2r}, C_m)-decomposition. Now, by Lemma 4.1.3, there exists an HOP $(C_2^{\langle r \rangle}, C_m)$-decomposition of $4K_n^\bullet$.

∎

The following corollary is a direct consequence of Lemma 4.5.4 and Theorem 3.2.1.

Corollary 4.5.5. Let $s \geq 0$, $r \geq 1$, and $m \geq 3$ be integers with m odd, and let $n = s+2r+m$. If $n \equiv 2r + m \pmod{2(2r+m)}$, then HOP$(2^{\langle s \rangle}, 4^{\langle r \rangle}, 2m)$ has a solution.

Chapter 5

The Generalized HOP with Small Round Tables

In this chapter, we study $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ under the condition that $m = m_1 + \cdots + m_t \leq 10$. Our goal is to show that $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution whenever $n = s+m$ is odd and $n(n-1) \equiv 0 \pmod{2m}$.

We apply the tools developed in Chapter 3 to prove our main result. We proceed case by case, depending on the value of $m = m_1 + \cdots + m_t$, where $2 \leq m_1 \leq \cdots \leq m_t$ and $m \leq 10$. For each case, we either use known theorems on $(C_{m_1}, \ldots, C_{m_t})$-decompositions of K_n and $2K_n$, and extend them to HOP decompositions of $4K_n^\bullet$, or we explicitly construct such decompositions. We summarize all cases in Tables 5.1 and 5.2.

5.1 HOP $(C_{m_1}, \ldots, C_{m_t})$-decompositions of $4K_n^\bullet$ with Small Cycle Lengths

The following theorem presents the main results of this chapter. It establishes several results on the generalized HOP with small round tables.

Theorem 5.1.1. Let $s \geq 0$, and $2 \leq m_1 \leq \ldots \leq m_t$ be integers. Let $m = m_1 + \ldots + m_t \leq 10$, and let $n = s + m$ be an odd integer such that $n(n-1) \equiv 0 \pmod{2m}$. Then $\mathrm{HOP}(2^{\langle s\rangle}, 2m_1, \ldots, 2m_t)$ has a solution.

Proof. The proof of this theorem is summarized in Table 5.2. Note that for the cases highlighted in pink rows, Theorems 2.3.7, 2.3.2, and 2.3.1 are used to show that K_n admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition. For these cases, we then apply Theorem 3.3.3 to show that $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

For the remaining cases, we combine the results of Lemmas 4.5.1, 4.5.4, and Lemmas 5.1.2–5.1.17 to show that $4K_n^\bullet$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decomposition.

Tables 5.1 and 5.2 list all possible cases where $n(n-1) \equiv 0 \pmod{2m}$, with $m \leq 10$ and n odd. Observe that in each case presented in the table, the multigraph $4K_n^\bullet$ admits an

HOP $(C_{m_1}, C_{m_2}, \dots, C_{m_t})$-decomposition. Applying Theorem 3.2.1, we conclude that HOP $(2^{\langle s\rangle}, 2m_1, \dots, 2m_t)$ has a solution.

Note that in the tables, when a row of the 3rd and 4th columns straddles two rows of column 2, then these decompositions may fall under both cases.

∎

5.1.1 HOP Decompositions for $n \equiv 1 \pmod 8$

Lemma 5.1.2. Let $n \equiv 1 \pmod 8$. Then $4K_n^\bullet$ admits an HOP (C_2, C_2)-decomposition.

Proof. By Theorem 2.3.2, there exists a (C_4)-decomposition of K_n. Applying Lemma 4.1.3, there exists an HOP (C_2, C_2)-decomposition of $4K_n^\bullet$. ∎

5.1.2 HOP Decompositions for $n \equiv 1$ or $9 \pmod{12}$

Lemma 5.1.3. Let $n \equiv 1$ or $9 \pmod{12}$. Then $4K_n^\bullet$ admits an HOP (C_2, C_2, C_2)-decomposition.

Proof. By Theorem 2.3.2, there exists a (C_6)-decomposition of K_n. Applying Lemma 4.1.3, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_2, C_2)-decomposition. ∎

Lemma 5.1.4. Let $n = 12k + 9$ for any integer $k \geq 0$. There exists an HOP (C_2, C_4)-decomposition of $4K_n^\bullet$.

Proof. Observe that $2n = 6(4k + 3)$; let $s = 4k + 3$. By Lemma 3.4.10 with $m_1 = 2$ and $m_2 = 4$, it suffices to find (C_2, C_4)-subgraphs $F_1, F_2, \dots, F_s$ in $4K_n^\bullet$ that satisfy Conditions (L′1) to (L′3).

As in Lemma 3.4.10, let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \dots\ x_{n-2})$. Observe that the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \dots, 6k + 3\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- for each $d \in \{1, 2, \dots, 12k + 7\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_{i+(6k+4)} : i = 0, 1, \dots, \frac{n-3}{2}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

m	n	$(C_{m_1},\ldots,C_{m_t})$-decomposition	Theorem or Lemma
2	$4k+1,\ k\geq 1$	(C_2)-decomposition	Lemma 3.3.2 with $t=1$
3	$6k+1,\ k\geq 1$	(C_3)-decomposition	Theorem 2.3.1
	$6k+3,\ k\geq 0$		
4	$8k+1,\ k\geq 1$	(C_2,C_2)-decomposition	Lemma 5.1.2
		(C_4)-decomposition	Theorem 2.3.2
5	$10k+1,\ k\geq 1$	(C_2,C_3)-decomposition	Lemma 4.5.1 with $m=3,\ r=1$
		(C_5)-decomposition	Theorem 2.3.2
	$10k+5,\ k\geq 0$		
		(C_2,C_3)-decomposition	Lemma 4.5.4 with $m=3,\ r=1$
6	$12k+1,\ k\geq 1$	(C_2,C_2,C_2)-decomposition	Lemma 5.1.3
		(C_2,C_4)-decomposition	Lemma 4.5.1 with $m=4,\ r=1$
		(C_3,C_3)-decomposition	Theorem 2.3.7 if $n\neq 9$
	$12k+9,\ k\geq 0$	(C_6)-decomposition	Theorem 2.3.2
		(C_3,C_3)-decomposition	Lemma 5.1.6 if $n=9$
		(C_2,C_2,C_2)-decomposition	Lemma 5.1.3
		(C_2,C_4)-decomposition	Lemma 5.1.4
7	$14k+1,\ k\geq 1$	(C_2,C_2,C_3)-decomposition	Lemma 4.5.1 with $m=3,\ r=2$
		(C_2,C_5)-decomposition	Lemma 4.5.1 with $m=5,\ r=1$
		(C_3,C_4)-decomposition	Theorem 2.3.7
	$14k+7,\ k\geq 0$	(C_7)-decomposition	Theorem 2.3.2
		(C_2,C_2,C_3)-decomposition	Lemma 4.5.4 with $m=3,\ r=2$
		(C_2,C_5)-decomposition	Lemma 4.5.4 with $m=5,\ r=1$
8	$16k+1,\ k\geq 1$	(C_2,C_2,C_2,C_2)-decomposition	Lemma 5.1.7
		(C_2,C_2,C_4)-decomposition	Lemma 4.5.1 with $m=4,\ r=2$
		(C_2,C_3,C_3)-decomposition	Lemma 5.1.8
		(C_2,C_6)-decomposition	Lemma 4.5.1 with $m=6,\ r=1$
		(C_3,C_5)-, (C_4,C_4)-decompositions	Theorem 2.3.7
		(C_8)-decompositions	Theorem 2.3.2

Table 5.1: Part 1: HOP $(C_{m_1},\ldots,C_{m_t})$-decompositions with $m_1+\ldots+m_t\leq 10$

m	n	$(C_{m_1}, \ldots, C_{m_t})$-decomposition	Theorem or Lemma
9	$18k+1,\ k \geq 1$	(C_2, C_2, C_2, C_3)-decomposition	Lemma 4.5.1 with $m = 3$, $r = 3$
		(C_2, C_2, C_5)-decomposition	Lemma 4.5.1 with $m = 5$, $r = 2$
		(C_2, C_3, C_4)-decomposition	Lemma 5.1.9
		(C_2, C_7)-decomposition	Lemma 4.5.1 with $m = 7$, $r = 1$
		(C_3, C_3, C_3)-, (C_3, C_6)-, (C_4, C_5)-,	Theorem 2.3.7 (except for (C_4, C_5)-decomp when $n = 9$)
	$18k+9,\ k \geq 0$	(C_9)-decompositions	Theorem 2.3.2
		(C_4, C_5)-decomposition	Lemma 2.2.13(ii) if $n = 9$
		(C_2, C_2, C_2, C_3)-decomposition	Lemma 4.5.4 with $m = 3$, $r = 3$
		(C_2, C_2, C_5)-decomposition	Lemma 4.5.4 with $m = 5$, $r = 2$
		(C_2, C_3, C_4)-decomposition	Lemma 5.1.10
		(C_2, C_7)-decomposition	Lemma 4.5.4 with $m = 7$, $r = 1$
10	$20k+1,\ k \geq 1$	$(C_2, C_2, C_2, C_2, C_2)$-decomposition	Lemma 5.1.11(i)
		(C_2, C_2, C_2, C_4)-decomposition	Lemma 4.5.1 with $m = 4$, $r = 3$
		(C_2, C_2, C_3, C_3)-decomposition	Lemma 5.1.11(iii)
		(C_2, C_2, C_6)-decomposition	Lemma 4.5.1 with $m = 6$, $r = 2$
		(C_2, C_3, C_5)-decomposition	Lemma 5.1.13
		(C_2, C_4, C_4)-decomposition	Lemma 5.1.12
		(C_2, C_8)-decomposition	Lemma 4.5.1 with $m = 8$, $r = 1$
		(C_3, C_3, C_4)-, (C_3, C_7)-, (C_4, C_6)-, (C_5, C_5)-decompositions	Theorem 2.3.7
	$20k+5,\ k \geq 1$	(C_{10})-decompositions	Theorem 2.3.2
		$(C_2, C_2, C_2, C_2, C_2)$-, (C_2, C_2, C_2, C_4)-, (C_2, C_2, C_3, C_3)-, (C_2, C_2, C_6)-decomposition	Lemma 5.1.11
		(C_2, C_3, C_5)-decomposition	Lemma 5.1.17
		(C_2, C_4, C_4)-decomposition	Lemma 5.1.16
		(C_2, C_8)-decomposition	Lemma 5.1.14

Table 5.2: Part 2: HOP $(C_{m_1}, \ldots, C_{m_t})$-decompositions with $m_1 + \ldots + m_t \leq 10$

For convenience, let $S=\{1,2,\dots,6k+4,\infty\}$ be the set of all differences.

We start by constructing subgraphs $F_1,F_2,\dots,F_{s-1}$, where each F_i is a disjoint union of a 4-cycle and a 2-cycle, and the subgraphs $F_1,F_2,\dots,F_{s-1}$ jointly cover exactly four edges corresponding to the following differences from S:

$$1,2,\dots,3k+1,3k+3,\dots,6k+3,\infty.$$

First, we construct the 4-cycles. For $i\in\{1,2,\dots,2k+1\}$, let C_i be the following closed walk (see Figure 5.1 in Example 5.1.5, where $k\geq 1$):

$$C_i=x_0\ x_i\ x_{6k+4}\ x_{-(6k+4-i)}\ x_0.$$

It can be seen that C_i is a cycle of length four and traverses edges of differences i and $6k+4-i$ twice. This means that for $i\in\{1,2,\dots,2k+1\}$, the cycles C_i jointly cover each of the following differences exactly twice:

$$1,2,\dots,2k+1,\ 4k+3,4k+4,\dots,6k+2,6k+3.$$

Next, construct cycles of length two, denoted by E_i, as follows:

- for $i\in\{1,2,\dots,2k+1\}$ and $i\neq k+1$, let $E_i=x_{3k+2}\ x_{5k+3+i}\ x_{3k+2}$; and
- for $i=k+1$, let $E_i=x_{3k+2}\ x_\infty\ x_{3k+2}$.

Observe that the 2-cycles E_i, for $i\in\{1,2,\dots,2k+1\}$, jointly cover each of the following differences exactly twice:

$$2k+2,2k+3,\dots,3k+1,\ \infty,\ 3k+3,\dots,4k+1,4k+2.$$

It can be easily checked that, for $i\in\{1,2,\dots,2k+1\}$, each C_i is disjoint from each E_i. Now, for $i\in\{1,2,\dots,2k+1\}$, let $G_i=C_i\,\dot\cup\,E_i$. We see that $G_1,G_2,\dots,G_{2k+1}$ jointly contain exactly two edges of each of the differences in S, except for differences $3k+2$ and $6k+4$.

Consider two copies of each G_i, for all $i\in\{1,2,\dots,2k+1\}$. We want to color each copy so that it satisfies Condition **(C1)** of Definition 1.2.3. In the first copy of G_i, color the edges of E_i pink and blue; and in C_i, color the edge $x_0\ x_i$ pink, $x_{6k+4}\ x_{-(6k+4-i)}$ blue, and color the other two edges black and orient them away from the pink edge and towards the blue edge. In the second copy of G_i, color the edges of E_i black and orient them in opposite directions; and in C_i, color $x_i\ x_{6k+4}$ pink, $x_{-(6k+4-i)}\ x_0$ blue, and color the other two edges black and orient them away from the pink edge and towards the blue edge. Considering two copies of each G_i, for all $i\in\{1,2,\dots,2k+1\}$, we have $4k+2$ (C_2,C_4)-subgraphs, which we relabel as $F_1,F_2,\dots,F_{s-1}$. It is easy to verify that except for differences $3k+2$ and $6k+4$, all other differences occur exactly four times (exactly one pink, one blue, and two opposite black with opposite directions) in $F_1,F_2,\dots,F_{s-1}$.

Next, we construct a (C_2,C_4)-subgraph F_s that covers differences $3k+2$ and $6k+4$. Let $F_s=C_s\,\dot\cup\,E_s$, where:

- $C_s = x_0\ x_{3k+2}\ x_{6k+4}\ x_{-(3k+2)}\ x_0$; and
- $E_s = x_1\ x_{-(6k+3)}\ x_1$.

Color the edges of C_s as follows: $x_{3k+2}\ x_{6k+4}$ blue, $x_{-(3k+2)}\ x_0$ pink, and the other two edges black and orient them away from the pink edge and towards the blue edge. We see that C_s covers the difference $3k+2$ exactly four times (one pink, one blue, and two black arcs), and E_s covers $6k+4$ twice. Observe that the subgraphs $F_1, F_2, \ldots, F_s$ satisfy Conditions (L′1) to (L′3) of Lemma 3.4.10. Hence, $4K_n^\bullet$ admits an HOP (C_2, C_4)-decomposition.

∎

Example 5.1.5. Let $F_1, F_2, \ldots, F_7$ be the (C_2, C_4)-subgraphs of $4K_{21}^\bullet$ shown in Figure 5.1, constructed using the method from the proof of Lemma 5.1.4. Observe that $F_1, F_2, \ldots, F_7$ satisfy Conditions (L′1) to (L′3) of Lemma 3.4.10. Therefore, $4K_{21}^\bullet$ admits an HOP (C_2, C_4)-decomposition.

Lemma 5.1.6. There exists an HOP (C_3, C_3)-decomposition of $4K_9^\bullet$.

Proof. Notice that $4 \cdot 9 = 6 \cdot 6$; let $s = 6$. By Lemma 3.4.11 with $m_1 = 3$ and $m_2 = 3$, it suffices to find (C_3, C_3)-subgraphs $F_1, F_2, \ldots, F_6$ in $4K_n^\bullet$ that satisfy Conditions (L″1) to (L″3). Observe that such $F_1, F_2, \ldots, F_6$ subgraphs are shown in Figure 5.2. Therefore, $4K_9^\bullet$ admits an HOP (C_3, C_3)-decomposition. ∎

5.1.3 HOP Decompositions for $n \equiv 1 \pmod{16}$

Lemma 5.1.7. Let $n \equiv 1 \pmod{16}$. Then $4K_n^\bullet$ admits an HOP (C_2, C_2, C_2, C_2)-decomposition.

Proof. By Theorem 2.3.2, there exists a (C_8)-decomposition of K_n. Applying Lemma 4.1.3, we see that $4K_n^\bullet$ admits an HOP (C_2, C_2, C_2, C_2)-decomposition. ∎

Lemma 5.1.8. Let $n = 16k + 1$ for any integer $k \geq 1$. There exists an HOP (C_2, C_3, C_3)-decomposition of $4K_n^\bullet$.

Proof. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_n)$. By the given labeling, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, where $d \in \{1, 2, \ldots, 8k\}$. Therefore, the group $\langle\rho\rangle$ acting on $E(K_n)$ has $8k$ orbits, each of size n.

Our goal is to construct k subgraphs of K_n, namely $F_1, F_2, \ldots, F_k$, where each F_i is a disjoint union of a (C_3, C_3)-subgraph and two edge-disjoint 1-regular subgraphs of order 2 such that $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 8k\}$. Therefore, $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$ is a decomposition for K_n that satisfies the condition of Lemma 4.1.1 (for $\alpha = 1$ and $m_1 = m_2 = 3$); hence, by Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_3, C_3)-decomposition.

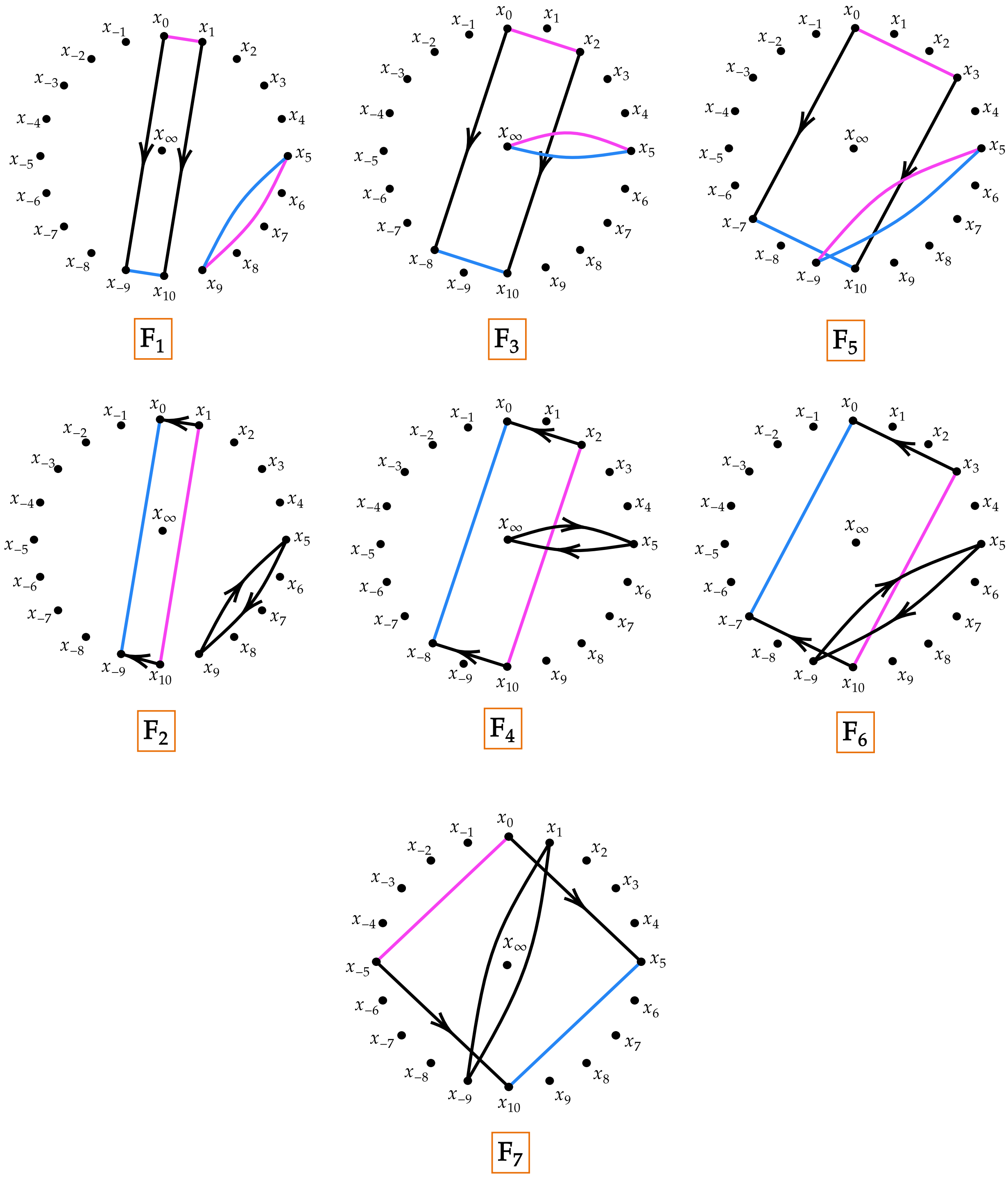


Figure 5.1: (C_2, C_4)-subgraphs of $4K_{21}^{\bullet}$ that generate an HOP (C_2, C_4)-decomposition for $4K_{21}^{\bullet}$.

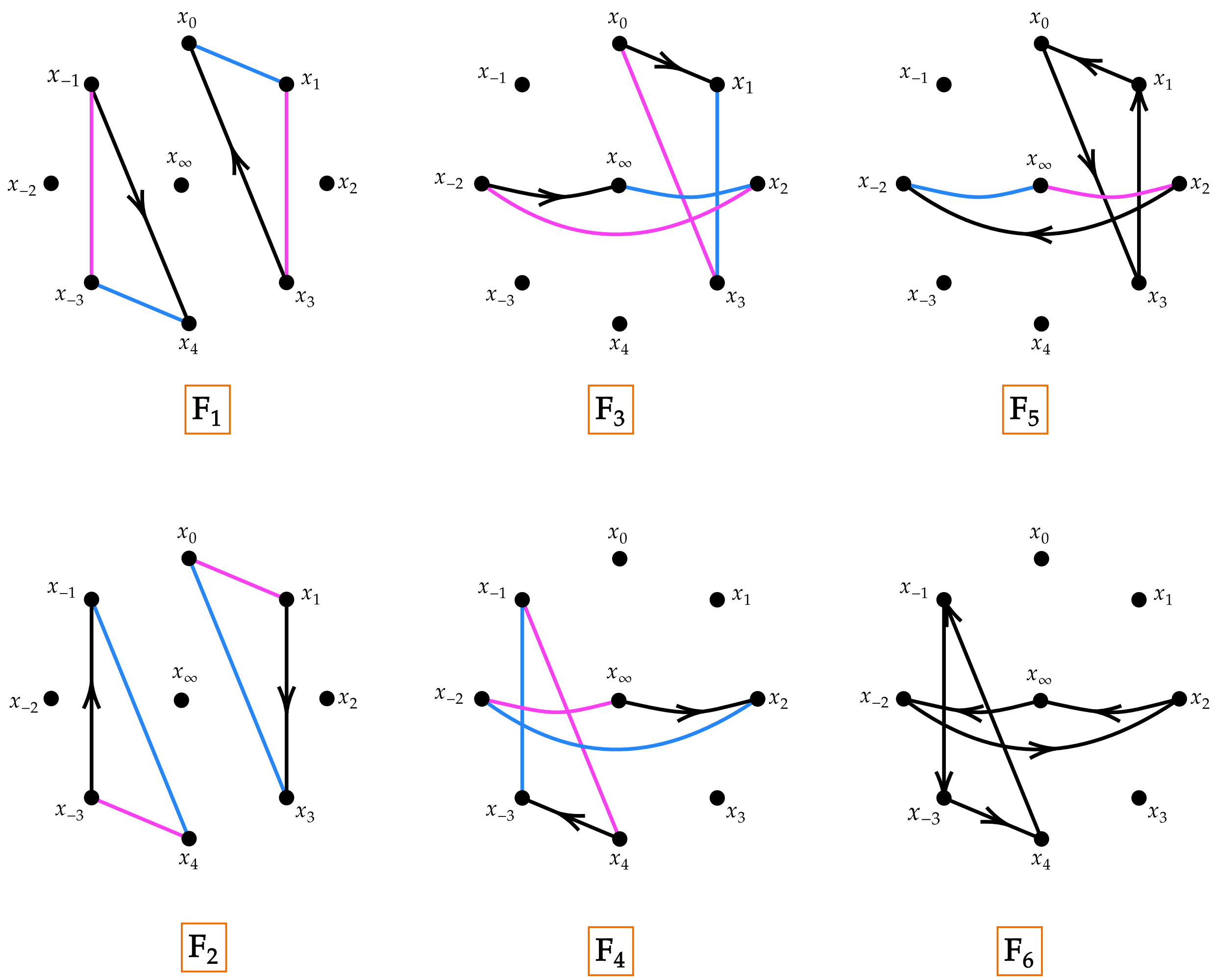


Figure 5.2: (C_3, C_3)-subgraphs of $4K_9^{\bullet}$ that generate an HOP (C_3, C_3)-decomposition for $4K_9^{\bullet}$.

Now, for $i \in \{1, 2, \dots, 2k\}$, construct the following closed walk:

$$C_i = x_{1-i}\ x_{-(8k-3i+1)}\ x_i\ x_{1-i}.$$

It can be seen that C_i is a cycle of length three, and traverses edges of differences

$$8k-4i+2, 8k-2i+1, 2i-1.$$

The explicit differences in $C_1, C_2, \dots, C_{2k}$ are listed below.

$$\begin{array}{rc}
C_1: & 8k-2, 8k-1, 1 \\
C_2: & 8k-6, 8k-3, 3 \\
C_3: & 8k-10, 8k-5, 5 \\
 & \vdots \\
C_{2k-2}: & 10, 4k+5, 4k-5 \\
C_{2k-1}: & 6, 4k+3, 4k-3 \\
C_{2k}: & 2, 4k+1, 4k-1
\end{array}$$

Observe that $C_1, C_2, \dots, C_{2k}$ jointly contain exactly one edge of each difference in $\{1, 2, \dots, 8k\}$ except for the following differences:

(i) $4i$, for $i \in \{1, 2, \dots, 2k\}$.

The following subgraphs of K_n (isomorphic to K_2) cover the differences in (i) exactly once:

- $E_i = x_{2k+1}\ \ x_{2k+4i+1}$, for $i \in \{1, 2, \dots, k\}$;
- $E_i = x_{2k+1}\ \ x_{-(14k-4i)}$, for $i \in \{k+1, \dots, 2k\}$.

Observe that none of the vertices in the interval $[x_{2k+1}, x_{8k+2}]$ are used in any C_i for $i \in \{1, 2, \dots, k\}$, and only vertices from this interval are used to construct E_i for $i \in \{1, 2, \dots, k\}$. Furthermore, none of the vertices in the interval $[x_{2k+1}, x_{11k+2}]$ are used in any C_i for $i \in \{k+1, \dots, 2k\}$, and only vertices from this interval are used to construct E_i for $i \in \{k+1, \dots, 2k\}$. Hence, for $i \in \{1, 2, \dots, 2k\}$, the subgraph C_i is disjoint from the subgraph E_i. Moreover, we have the following:

- C_i and C_j are disjoint for $i \neq j$ and $i, j \in \{1, 2, \dots, 2k\}$;
- E_{2i-1} and C_{2i} are disjoint for $i \in \{1, 2, \dots, k\}$; and
- E_{2i} and C_{2i-1} are disjoint for $i \in \{1, 2, \dots, k\}$.

Now, for $i \in \{1, 2, \dots, k\}$, let $F_i = C_{2i-1} \,\dot\cup\, C_{2i} \,\dot\cup\, (E_{2i-1} \cup E_{2i})$. Since $F_1, F_2, \dots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \dots, 8k\}$, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \dots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition for K_n that satisfies the condition of Lemma 4.1.1. Applying Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_3, C_3)-decomposition. ∎

5.1.4 HOP Decompositions for $n \equiv 1$ or 9 (mod 18)

Lemma 5.1.9. Let $n = 18k + 1$ for any integer $k \geq 1$. There exists an HOP (C_2, C_3, C_4)-decomposition of $4K_n^\bullet$.

Proof. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ \dots\ x_{n-1})$ be a permutation on $V(K_n)$. By the given labeling, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, where $d \in \{1, 2, \dots, 9k\}$. Therefore, the group $\langle \rho \rangle$ acting on $E(K_n)$ has $9k$ orbits, each of size n.

We construct subgraphs $F_1, F_2, \dots, F_k$, where each F_i is a disjoint union of a (C_3, C_4)-subgraph and two edge-disjoint 1-regular subgraphs of order 2 such that $F_1, F_2, \dots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \dots, 9k\}$. Therefore,

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \dots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

forms a decomposition for K_n that satisfies the condition of Lemma 4.1.1 (for $\alpha = 1$, $m_1 = 3$, and $m_2 = 4$); hence, by Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_3, C_4)-decomposition.

Now, for $i \in \{1, 2, \dots, k\}$, construct the following closed walks:

$$C_i = x_0\ x_{4i-3}\ x_{-1}\ x_{4i-1}\ x_0;$$

$$C'_i = x_{-2}\ x_{4k+2i-2}\ x_{-(4k+2i+1)}\ x_{-2}.$$

It can be seen that C_i and C'_i are two disjoint cycles of lengths four and three, respectively, and traverse edges of differences

$$C_i : 4i-3, 4i-2, 4i, 4i-1;$$

$$C'_i : 4k+2i, d, 4k+2i-1.$$

In C'_i, we have $d = 8k + 4i - 1$ for $i \in \{1, 2, \dots, \lfloor \frac{k+1}{4} \rfloor\}$, and $d = 10k - 4i + 2$ for $i \in \{\lceil \frac{k+1}{4} \rceil, \dots, k\}$.

The explicit differences in $C_1, C_2, \dots, C_k$ are listed below.

$$\begin{array}{rc}
C_1 : & 1, 2, 4, 3 \\
C_2 : & 5, 6, 8, 7 \\
C_3 : & 9, 10, 12, 11 \\
 & \vdots \\
C_{k-2} : & 4k-11, 4k-10, 4k-8, 4k-9 \\
C_{k-1} : & 4k-7, 4k-6, 4k-4, 4k-5 \\
C_k : & 4k-3, 4k-2, 4k, 4k-1
\end{array}$$

The explicit differences in $C'_1, C'_2, \dots, C'_k$ are listed below.

$$C'_1 : \quad 4k+2, 8k+3, 4k+1$$

$$
\begin{aligned}
C_2' &: \quad 4k+4, 8k+7, 4k+3 \\
C_3' &: \quad 4k+6, 8k+11, 4k+5 \\
&\quad\vdots \\
C_{k-2}' &: \quad 6k-4, 6k+10, 6k-5 \\
C_{k-1}' &: \quad 6k-2, 6k+6, 6k-3 \\
C_k' &: \quad 6k, 6k+2, 6k-1
\end{aligned}
$$

Observe that, for $i \in \{1, 2, \ldots, k\}$, C_i and C_i' jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 9k\}$ except for the following differences:

(i) $6k + 2i - 1$, for $i \in \{1, 2, \ldots, k\}$;

(ii) $6k + 4i$, for $i \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$;

(iii) $12k - 4i + 1$, for $i \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$.

The following subgraphs of K_n (isomorphic to K_2) cover the differences listed in (i)-(iii) exactly once:

(i) $E_i = x_{-3} \;\; x_{-(6k+2i+2)}$, for $i \in \{1, 2, \ldots, k\}$;

(ii) $E_i' = x_{-3} \;\; x_{-(6k+4i+3)}$, for $i \in \{1, 2, \ldots, \lfloor \frac{3k}{4} \rfloor\}$;

(iii) $E_i' = x_{-3} \;\; x_{12k-4i-2}$, for $i \in \{\lfloor \frac{3k}{4} \rfloor + 1, \ldots, k\}$.

Notice that the vertex x_{-3}, as well as all vertices in the interval $[x_{6k-1}, x_{12k-1}]$, are not used in any of the cycles C_i or C_i', and that only vertices from this interval are used to construct the graphs E_i and E_i'. Hence, for $i \in \{1, 2, \ldots, k\}$, the cycles C_i and C_i' are disjoint from the edge-disjoint subgraphs E_i and E_i'.

Now, for $i \in \{1, 2, \ldots, k\}$, let $F_i = C_i \,\dot\cup\, C_i' \,\dot\cup\, (E_i \cup E_i')$. Since $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 9k\}$, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition of K_n that satisfies the condition of Lemma 4.1.1. By applying Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_3, C_4)-decomposition. ∎

Lemma 5.1.10. Let $n = 18k + 9$ for any integer $k \geq 0$. There exists an HOP (C_2, C_3, C_4)-decomposition of $4K_n^\bullet$.

Proof. Notice that $n = 18k + 9 = 9(2k + 1)$, and we have $K_{9(2k+1)} = H_1 \bigoplus H_2$, where H_1 is a disjoint union of $2k + 1$ copies of K_9, and H_2 is a complete equipartite graph with $2k + 1$ parts of size 9. In Lemma 2.2.13(ii) [36], it is proved that $4K_9^\bullet$ admits an HOP (C_2, C_3, C_4)-factorization. Consequently, by Lemma 3.3.1, $4H_1^\bullet$ admits an HOP (C_2, C_3, C_4)-decomposition. Hence, we may assume $k \geq 1$ and it suffices to show that $4K_{(2k+1)[9]}^\bullet$ admits an HOP (C_2, C_3, C_4)-decomposition.

The complete equipartite graph $K_{(2k+1)[9]}$ can be viewed as a circulant graph, $\mathrm{Circ}(n; S)$, with the vertex set $\{x_i : i \in \mathbb{Z}_n\}$, where $n = 18k+9$, and the edge set $\{x_i x_{i+d} : i \in \mathbb{Z}_n, d \in S\}$, where

$$S = \{1, 2, \ldots, 2k, 2k+2, \ldots, 4k+1, 4k+3, \ldots, 6k+2, 6k+4, \ldots, 8k+3, 8k+5, \ldots, 9k+4\}.$$

Let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_{(2k+1)[9]})$. Observe that S contains all differences from 1 to $9k+4$, except for the differences $2k+1, 4k+2, 6k+3, 8k+4$. Therefore, the group $\langle\rho\rangle$ acting on $E(K_{(2k+1)[9]})$ has $(9k+4) - 4 = 9k$ orbits, each of size $n = 18k+9$.

First, we construct subgraphs $F_1, F_2, \ldots, F_k$, where each F_i is a disjoint union of a (C_3, C_4)-subgraph and two edge-disjoint 1-regular subgraphs of order 2, such that $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in S.

Now, for $i \in \{1, 2, \ldots, k\}$, construct the following closed walks:

$$C_i = x_0\ x_{2i-1}\ x_{-1}\ x_{4k+2i+1}\ x_0;$$

$$C_i' = x_{-2}\ x_{2k+i-1}\ x_{-(6k+i+5)}\ x_{-2}.$$

It can be seen that C_i and C_i' are disjoint cycles of lengths four and three, respectively. The 4-cycle C_i traverses edges of differences

$$C_i : 2i-1, 2i, 4k+2i+2, 4k+2i+1.$$

The explicit differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$\begin{array}{rc}
C_1 : & 1, 2, 4k+4, 4k+3 \\
C_2 : & 3, 4, 4k+6, 4k+5 \\
C_3 : & 5, 6, 4k+8, 4k+7 \\
 & \vdots \\
C_{k-2} : & 2k-5, 2k-4, 6k-2, 6k-3 \\
C_{k-1} : & 2k-3, 2k-2, 6k, 6k-1 \\
C_k : & 2k-1, 2k, 6k+2, 6k+1
\end{array}$$

The 3-cycle C_i' traverses edges of differences

$$C_i' : 2k+i+1, d, 6k+i+3.$$

Notice that, for $i \in \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$, the difference $d = 8k+2i+4$, and for $i \in \{\lfloor \frac{k}{2} \rfloor + 1 \ldots, k\}$, the difference $d = 10k - 2i + 5$. The explicit differences in $C_1', C_2', \ldots, C_k'$ are listed below.

$$\begin{array}{rc}
C_1' : & 2k+2, 8k+6, 6k+4 \\
C_2' : & 2k+3, 8k+8, 6k+5 \\
C_3' : & 2k+4, 8k+10, 6k+6 \\
 & \vdots \\
C_{k-2}' : & 3k-1, 8k+9, 7k+1
\end{array}$$

$$\begin{aligned} C'_{k-1}: &\quad 3k, 8k+7, 7k+2 \\ C'_k: &\quad 3k+1, 8k+5, 7k+3 \end{aligned}$$

Observe that the subgraphs C_i and C'_i, for $i \in \{1, 2, \dots, k\}$, jointly contain exactly one edge of each difference in S except for the following differences:

(i) $4k - i + 2$, for $i \in \{1, 2, \dots, k\}$; and

(ii) $8k - i + 4$, for $i \in \{1, 2, \dots, k\}$.

Note that there are $2k$ differences from S that are not covered by any C_i or C'_i. The following subgraphs of K_n (isomorphic to K_2) cover the differences in (i) and (ii), respectively, exactly once:

- $E_i = x_{-3} \;\; x_{-(4k-i+5)}$, for $i \in \{1, 2, \dots, k\}$; and
- $E'_i = x_{-(4k+4)} \;\; x_{4k-i}$, for $i \in \{1, 2, \dots, k\}$.

Notice that none of the vertices in the intervals $[x_{3k}, x_{4k+2}]$ and $[x_{-(6k+5)}, x_{-3}]$ are used in any C_i or C'_i, and only vertices in these intervals are used to construct the subgraphs E_i and E'_i. Hence, C_i and C'_i are disjoint from E_i and E'_i.

For $i \in \{1, 2, \dots, k\}$, let $F_i = C_i \,\dot\cup\, C'_i \,\dot\cup\, (E_i \cup E'_i)$. Since $F_1, F_2, \dots, F_k$ jointly contain exactly one edge of each difference in S, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \dots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition for $K_{(2k+1)[9]}$ that satisfies the condition of Lemma 4.1.1. Applying Lemma 4.1.1, the multigraph $4K^\bullet_{(2k+1)[9]}$ admits an HOP (C_2, C_3, C_4)-decomposition.

∎

5.1.5 HOP Decompositions for $n \equiv 1$ or $5 \pmod{20}$

Lemma 5.1.11. Let $n \equiv 1$ or $5 \pmod{20}$. Then the following HOP 2-regular-subgraph decompositions of $4K_n^\bullet$ exist:

(i) HOP $(C_2, C_2, C_2, C_2, C_2)$-decomposition;

(ii) HOP (C_2, C_2, C_2, C_4)-decomposition;

(iii) HOP (C_2, C_2, C_3, C_3)-decomposition; and

(iv) HOP (C_2, C_2, C_6)-decomposition.

Proof. By Theorems 2.3.2 and 2.3.7, we know there exist a (C_{10})-decomposition, a (C_6, C_4)-decomposition, a (C_4, C_3, C_3)-decomposition, and a (C_4, C_6)-decomposition of K_n, and by Lemma 4.1.3, these decompositions give rise to HOP decompositions listed in (i)-(iv), respectively. ∎

Lemma 5.1.12. Let $n = 20k + 1$ for any integer $k \geq 1$. There exists an HOP (C_2, C_4, C_4)-decomposition of $4K_n^\bullet$.

Proof. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_n)$. By the given labeling, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, where $d \in \{1, 2, \ldots, 10k\}$. Therefore, the group $\langle \rho \rangle$, acting on $E(K_n)$, has $10k$ orbits, each of size n.

We construct subgraphs $F_1, F_2, \ldots, F_k$, where each F_i is a disjoint union of a (C_4, C_4)-subgraph and two edge-disjoint 1-regular subgraphs of order 2, such that $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 10k\}$. Therefore, $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$ forms a decomposition of K_n that satisfies the condition of Lemma 4.1.1 (for $\alpha = 1$, $m_1 = 4$, and $m_2 = 4$); hence, by Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_4, C_4)-decomposition.

Now, for $i \in \{1, 2, \ldots, 2k\}$, construct the following closed walk:

$$C_i = x_{1-i}\ x_i\ x_{-(2k+i)}\ x_{4k+i}\ x_{1-i}.$$

It can be seen that C_i is a cycle of length four, and traverses edges of differences

$$C_i : 2i - 1, 2k + 2i, 6k + 2i, 4k + 2i - 1.$$

The explicit differences in $C_1, C_2, \ldots, C_{2k}$ are listed below.

$$\begin{array}{rc}
C_1 : & 1, 2k+2, 6k+2, 4k+1 \\
C_2 : & 3, 2k+4, 6k+4, 4k+3 \\
C_3 : & 5, 2k+6, 6k+6, 4k+5 \\
 & \vdots \\
C_{2k-2} : & 4k-5, 6k-4, 10k-4, 8k-5 \\
C_{2k-1} : & 4k-3, 6k-2, 10k-2, 8k-3 \\
C_{2k} : & 4k-1, 6k, 10k, 8k-1
\end{array}$$

Observe that the C_i, for $i \in \{1, 2, \ldots, 2k\}$, jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 10k\}$ except for the following differences:

(i) $2i$, for $i \in \{1, 2, \ldots, k\}$;

(ii) $8k + 2i - 1$, for $i \in \{1, 2, \ldots, k\}$.

The following subgraphs of K_n (isomorphic to K_2) cover the differences listed in (i) and (ii) exactly once:

(i) $E_i = x_{6k+1}\ \ x_{6k+2i+1}$, for $i \in \{1, 2, \ldots, k\}$;

(ii) $E_i' = x_{6k+1}\ \ x_{-(6k-2i+1)}$, for $i \in \{1, 2, \ldots, k\}$.

Notice that none of the vertices in the interval $[x_{6k+1}, x_{16k}]$ are used in any C_i, and only vertices in this interval are used to construct the subgraphs E_i and E_i'. Hence, for $i \in \{1, 2, \ldots, k\}$, cycles C_{2i-1} and C_{2i} are disjoint from the subgraphs E_i and E_i'.

Now, for $i \in \{1, 2, \ldots, k\}$, let $F_i = C_{2i-1} \dot\cup C_{2i} \dot\cup (E_i \cup E_i')$. Since $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 10k\}$, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition for K_n that satisfies the condition of Lemma 4.1.1. Applying Lemma 4.1.1, the multigraph $4K_n^{\bullet}$ admits an HOP (C_2, C_4, C_4)-decomposition. ∎

Lemma 5.1.13. Let $n = 20k + 1$ for any integer $k \geq 1$. There exists an HOP (C_2, C_3, C_5)-decomposition of $4K_n^{\bullet}$.

Proof. Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$, and let $\rho = (x_0\ x_1\ \ldots\ x_{n-1})$ be a permutation on $V(K_n)$. By the given labeling, each edge in $E(K_n)$ is of the form $x_i x_{i+d}$, where $d \in \{1, 2, \ldots, 10k\}$. Therefore, the group $\langle \rho \rangle$ acting on $E(K_n)$ has $10k$ orbits, each of size n.

We construct subgraphs $F_1, F_2, \ldots, F_k$, where each F_i is a disjoint union of a (C_3, C_5)-subgraph and two edge-disjoint 1-regular subgraphs of order 2, and $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1, 2, \ldots, 10k\}$. Therefore, $\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$ forms a decomposition for K_n that satisfies the condition of Lemma 4.1.1 (for $\alpha = 1$, $m_1 = 3$, and $m_2 = 5$); hence, by Lemma 4.1.1, the multigraph $4K_n^{\bullet}$ admits an HOP (C_2, C_3, C_5)-decomposition.

Now, for $i \in \{1, 2, \ldots, k\}$, construct the following closed walks:

$$C_i = x_0\ x_{2i-1}\ x_{-1}\ x_{2k+i-1}\ x_{-(3k+i)}\ x_0;$$

$$C_i' = x_{-(k+i)}\ x_{3k}\ x_{-(8k-i+2)}\ x_{-(k+i)}.$$

It can be seen that C_i and C_i' are disjoint cycles of length five and three, respectively. The 5-cycle C_i traverses edges of differences:

$$C_i : 2i-1, 2i, 2k+i, 5k+2i-1, 3k+i.$$

The explicit differences in $C_1, C_2, \ldots, C_k$ are listed below.

$$\begin{array}{rc}
C_1: & 1, 2, 2k+1, 5k+1, 3k+1 \\
C_2: & 3, 4, 2k+2, 5k+3, 3k+2 \\
C_3: & 5, 6, 2k+3, 5k+5, 3k+3 \\
 & \vdots \\
C_{2k-2}: & 2k-5, 2k-4, 3k-2, 7k-5, 4k-2 \\
C_{2k-1}: & 2k-3, 2k-2, 3k-1, 7k-3, 4k-1 \\
C_{2k}: & 2k-1, 2k, 3k, 7k-1, 4k
\end{array}$$

The 3-cycle C_i' traverses edges of differences:

$$C_i' : 4k+i, 9k+i-1, 7k-2i+2.$$

The explicit differences in $C_1', C_2', \ldots, C_k'$ are listed below.

$$\begin{array}{rc}
C_1': & 4k+1, 9k, 7k \\
C_2': & 4k+2, 9k+1, 7k-2 \\
C_3': & 4k+3, 9k+2, 7k-4 \\
 & \vdots \\
C_{2k-2}': & 5k-2, 10k-3, 5k+6 \\
C_{2k-1}': & 5k-1, 10k-2, 5k+4 \\
C_{2k}': & 5k, 10k-1, 5k+2
\end{array}$$

Observe that, for $i \in \{1,2,\ldots,k\}$, C_i and C_i' jointly contain exactly one edge of each difference $\{1,2,\ldots,10k\}$ except for the following differences:

(i) $7k+i$, for $i \in \{1,2,\ldots,2k-1\}$;

(ii) $10k$.

The following subgraphs of K_n (isomorphic to K_2) cover the differences listed in (i) and (ii) exactly once:

- $E_i = x_{-5k}\ \ x_{8k-i+1}$, for $i \in \{1,2,\ldots,2k-1\}$, and
- $E_{2k} = x_{-5k}\ \ x_{5k}$.

Notice that the vertex x_{-5k}, as well as none of the vertices in the interval $[x_{3k+1}, x_{12k-1}]$, are used in any C_i or C_i', and only vertices in this interval are used to construct the subgraphs E_i. Hence, for $i \in \{1,2,\ldots,k\}$, cycles C_i and C_i' are disjoint from the subgraphs E_{2i-1} and E_{2i}.

Now, for $i \in \{1,2,\ldots,k\}$, let $F_i = C_i \,\dot\cup\, C_i' \,\dot\cup\, (E_{2i-1} \cup E_{2i})$. Since $F_1, F_2, \ldots, F_k$ jointly contain exactly one edge of each difference in $\{1,2,\ldots,10k\}$, we see that

$$\mathcal{D} = \{\rho^i(F_1), \rho^i(F_2), \ldots, \rho^i(F_k) : i \in \mathbb{Z}_n\}$$

is a decomposition of K_n that satisfies the condition of Lemma 4.1.1. Applying Lemma 4.1.1, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_3, C_5)-decomposition.

∎

Lemma 5.1.14. Let $n = 20k+5$ for any integer $k \geq 1$. There exists an HOP (C_2, C_8)-decomposition of $4K_n^\bullet$.

Proof. Observe that $2n = 10(4k+1)$; let $s = 4k+1$. By Lemma 3.4.6 with $m_1 = 2$ and $m_2 = 8$, it suffices to find (C_2, C_8)-subgraphs $F_1, F_2, \ldots, F_s$ in $2K_n^\circ$ that satisfy Conditions (D′1) to (D′3).

As in Lemma 3.4.6, let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for each $d \in \{1, 2, \dots, 10k+1\}$, we have a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a black orbit $\{x_i x_{i+(10k+2)} : i = 0, 1, \dots, \frac{n-3}{2}\}$; and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

Let $S = \{1, 2, \dots, 10k+2\}$ be the set of differences. Note that $10k+2$ is the diameter difference.

First, for $i \in \{1, 2, \dots, 4k+1\}$, construct the following closed walk (see Figure 5.3 in Example 5.1.15):

$$C_i = x_0\ x_i\ x_{-(5k+1)}\ x_{10k+2-i}\ x_{10k+2}\ x_{-(10k+2-i)}\ x_{5k+1}\ x_{-i}\ x_0.$$

It can be seen that C_i is a cycle of length eight, and traverses edges of differences i and $5k+i+1$ four times. Since these differences come from an orbit of size $n-1$, the four edges occur as two pairs of the form $\{e, \rho_\circ^{\frac{n-1}{2}}(e)\}$.

The 8-cycles $C_1, C_2, \dots, C_{4k+1}$ jointly cover each of the following differences exactly four times:

$$1, 2, \dots, 4k+1, 5k+2, 5k+3, \dots, 9k+1, 9k+2.$$

The differences that are not covered by any C_i are the following:

(i) $9k+2+i$, for $i \in \{1, 2, \dots, k\}$;

(ii) ∞; and

(iii) $4k+1+i$, for $i \in \{1, 2, \dots, k\}$.

Next, we construct cycles of length two that jointly cover each of the leftover differences exactly four times, except for the diameter difference $10k+2$.

- The following 2-cycles cover the differences $9k+2+j$, for $j \in \{1, 2, \dots, k-1\}$, and the difference ∞ exactly four times.
  - $E_j = x_{-(4k+1)}\ x_{5k+1+j}\ x_{-(4k+1)}$, for $j \in \{1, 2, \dots, k-1\}$, and $E_k = x_{-(3k+2)}\ x_\infty\ x_{-(3k+2)}$.
  - $E'_j = x_{6k+1}\ x_{-(5k+1-j)}\ x_{6k+1}$, for $j \in \{1, 2, \dots, k-1\}$, and $E'_k = x_{7k}\ x_\infty\ x_{7k}$.

  Notice that $E'_j = \rho_\circ^{\frac{n-1}{2}}(E_j)$. Moreover, none of the vertices in the intervals $[x_{2k+1}, x_{5k}]$, $[x_{5k+2}, x_{8k+1}]$, $[x_{-(8k+1)}, x_{-(5k+2)}]$, and $[x_{-5k}, x_{-(2k+1)}]$ are used in any C_i, for $i \in \{1, 2, \dots, 2k\}$, and only vertices in these intervals are used to construct the subgraphs E_j and E'_j. Hence, for $j \in \{1, 2, \dots, k\}$ and $i \in \{1, 2, \dots, 2k\}$, the subgraphs E_j, E'_j, and C_i are pairwise disjoint.

  Relabel E_j and E'_j jointly as E_i, for $i \in \{1, 2, \dots, 2k\}$. Now, for $i \in \{1, 2, \dots, 2k\}$, let $F_i = C_i \dot\cup E_i$. We see that $F_1, F_2, \dots, F_{2k}$ are (C_2, C_8)-subgraphs of $2K_n$.

- The following 2-cycles cover difference the $4k+1+j$, for $j \in \{1, 2, \ldots, k\}$, exactly four times.
  - $E''_j = x_{-(4k+1)}\ x_j\ x_{-(4k+1)}$, for $j \in \{1, 2, \ldots, k\}$.
  - $E'''_j = x_{6k+1}\ x_{-(10k+2-j)}\ x_{6k+1}$, for $j \in \{1, 2, \ldots, k\}$.

  First, notice that $E'''_j = \rho_\circ^{\frac{n-1}{2}}(E''_j)$. Moreover, none of the vertices in the intervals $[x_1, x_{2k}]$, $[x_{5k+2}, x_{6k+1}]$, $[x_{-(10k+1)}, x_{-(8k+2)}]$, and $[x_{-5k}, x_{-(4k+1)}]$ are used in any C_i, for $i \in \{2k+1, 2k+2, \ldots, 4k\}$, and only vertices in these intervals are used to construct the subgraphs E''_j and E'''_j. Hence, the subgraphs E''_j, E'''_j, and C_i are pairwise disjoint.

  Next, relabel E''_j and E'''_j jointly as E_i, for $i \in \{2k+1, 2k+2, \ldots, 4k\}$. Now, for $i \in \{2k+1, 2k+2, \ldots, 4k\}$, let $F_i = C_i \dot\cup E_i$. We see that $F_{2k+1}, F_{2k+2}, \ldots, F_{4k}$ are (C_2, C_8)-subgraphs of $2K_n$.

- The only difference that is not covered yet is the diameter difference $10k+2$. Let $E_{4k+1} = x_1\ x_{-(10k+1)}\ x_1$, and observe that E_{4k+1} is disjoint from C_{4k+1}. Now, let $F_{4k+1} = C_{4k+1} \dot\cup E_{4k+1}$. We see that F_{4k+1} is a (C_2, C_8)-subgraph of $2K_n$.

Next, color the edges of the cycles in the (C_2, C_8)-subgraphs alternately pink and black. Observe that the subgraphs $F_1, F_2, \ldots, F_s$ satisfy Conditions (D′1) to (D′3) of Lemma 3.4.6. Therefore, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_8)-decomposition.

∎

Example 5.1.15. Let $F_1, F_2, \ldots, F_5$ be the (C_2, C_8)-subgraphs of $2K_{25}^\circ$ shown in Figure 5.3, constructed following the method used in the proof of Lemma 5.1.14.

Observe that $F_1, F_2, \ldots, F_5$ satisfy Conditions (D′1) to (D′3) of Lemma 3.4.6. Therefore, $4K_{25}^\bullet$ admits an HOP (C_2, C_8)-decomposition.

Lemma 5.1.16. Let $n = 20k+5$ for any integer $k \geq 1$. There exists an HOP (C_2, C_4, C_4)-decomposition of $4K_n^\bullet$.

Proof. Observe that $2n = 10(4k+1)$; let $s = 4k+1$. By Lemma 3.4.6 with $m_1 = 2, m_2 = m_3 = 4$, it suffices to find (C_2, C_4, C_4)-subgraphs $F_1, F_2, \ldots, F_s$ in $2K_n^\circ$ that satisfy Conditions (D′1) to (D′3).

Let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for each $d \in \{1, 2, \ldots, 10k+1\}$, we have a pink and a black orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a black orbit $\{x_i x_{i+(10k+2)} : i = 0, 1, \ldots, \frac{n-3}{2}\}$; and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

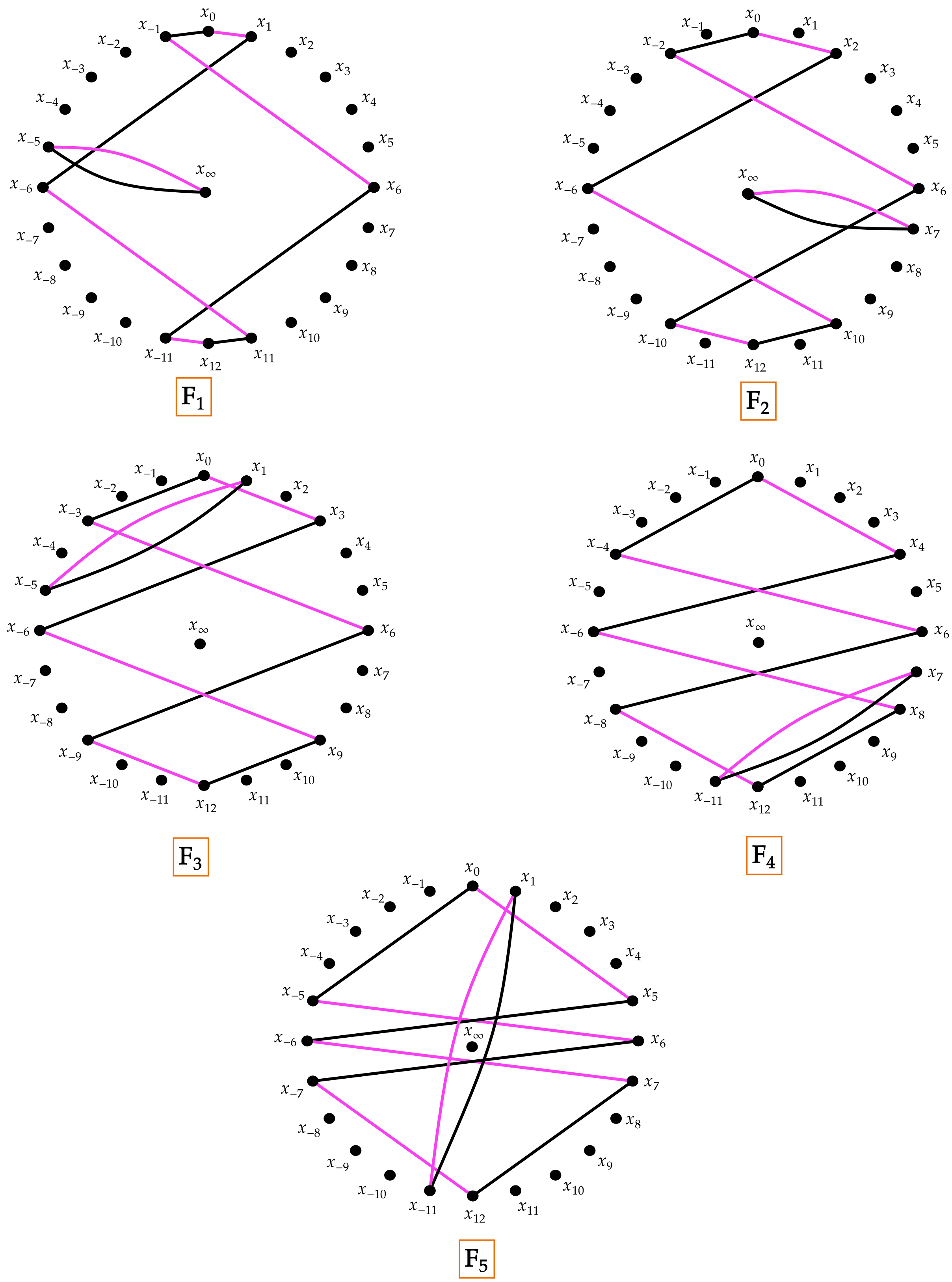


Figure 5.3: (C_2, C_8)-subgraphs of $2K_{25}^{\circ}$ that generate an HOP (C_2, C_8)-decomposition of $4K_{25}^{\bullet}$.

For $k = 1$, see Figure 5.4. For $k \geq 2$ and $i \in \{1, 2, \ldots, 4k+1\}$, construct the following two closed walks.

$$C_i = x_1\ x_{i+1}\ x_{-(10k+1)}\ x_{-(10k+1-i)}\ x_1;$$
$$C'_i = x_{-1}\ x_{-(i+1)}\ x_{10k+1}\ x_{10k+1-i}\ x_{-1}.$$

Notice that for $i \in \{1, 2, \ldots, 4k+1\}$, C_i and C'_i are disjoint cycles of length four and traverse edges of differences $i, 10k-i+2, i, 10k-i+2$. Observe that edges of the same difference occur in diametrical pairs. Hence, C_i and C'_i, for $i \in \{1, 2, \ldots, 4k+1\}$, jointly cover each of the following differences exactly four times:

$$1, 2, \ldots, 4k+1, 6k+1, 6k+2, \ldots, 10k, 10k+1.$$

The differences not covered by any C_i or C'_i are the following:

(i) $4k+1+i$, for $i \in \{1, 2, \ldots, 2k-1\}$;

(ii) $10k+2$; and

(iii) ∞.

Next, we construct cycles of length two that cover the leftover differences exactly four times, except for the diameter difference $10k+2$, which should be covered exactly twice.

- The following 2-cycles cover the differences $4k+1+j$, for $j \in \{1, 2, \ldots, 2k-1\}$, as well as the difference ∞, exactly four times.

  - $E_j = x_0\ x_{-(4k+1+j)}\ x_0$, for $j \in \{1, 2, \ldots, 2k-1\}$, and $E_{2k} = x_0\ x_\infty\ x_0$.
  - $E'_j = x_{10k+2}\ x_{6k+1-j}\ x_{10k+2}$, for $j \in \{1, 2, \ldots, 2k-1\}$, and $E'_{2k} = x_{10k+2}\ x_\infty\ x_{10k+2}$.

  Notice that $E'_j = \rho_\circ^{\frac{n-1}{2}}(E_j)$. Moreover, the vertices x_0, x_{10k+2}, and the vertices in the intervals $[x_{4k+2}, x_{6k}]$ and $[x_{-(6k)}, x_{-(4k+2)}]$ are not used in any C_i or C'_i, for $i \in \{1, 2, \ldots, 4k\}$, and only vertices in these intervals are used to construct graphs E_j and E'_j. Hence, the subgraphs E_j, E'_j, C_i, and C'_i are pairwise disjoint.

  Next, relabel E_j and E'_j jointly as E_i, for $i \in \{1, 2, \ldots, 4k\}$. Now, let $F_i = C_i \dot\cup C'_i \dot\cup E_i$. We see that $F_1, F_2, \ldots, F_{4k}$ are (C_2, C_4, C_4)-subgraphs of $2K_n$.

- The only difference that is not covered yet is $10k+2$. Let $E_{4k+1} = x_0\ x_{10k+2}\ x_0$, and observe E_{4k+1} is disjoint from C_{4k+1} and C'_{4k+1}. Now, let $F_{4k+1} = C_{4k+1} \dot\cup C'_{4k+1} \dot\cup E_{4k+1}$. We see that F_{4k+1} is a (C_2, C_4, C_4)-subgraph of $2K_n$.

Next, in each (C_2, C_4, C_4)-subgraph, color the edges of the 2-cycle pink and black. Color the edges of one of the 4-cycles, say C_i, alternately pink and black. Note that edges of the same length receive the same color. Then, color the edges of the other 4-cycle, C'_i, in the opposite way of C_i (see Figure 5.4).

The (C_2, C_4, C_4)-subgraphs $F_1, F_2, \ldots, F_s$ satisfy Conditions (D′1) to (D′3) of Lemma 3.4.6. Therefore, the multigraph $4K_n^\bullet$ admits an HOP (C_2, C_4, C_4)-decomposition.

∎

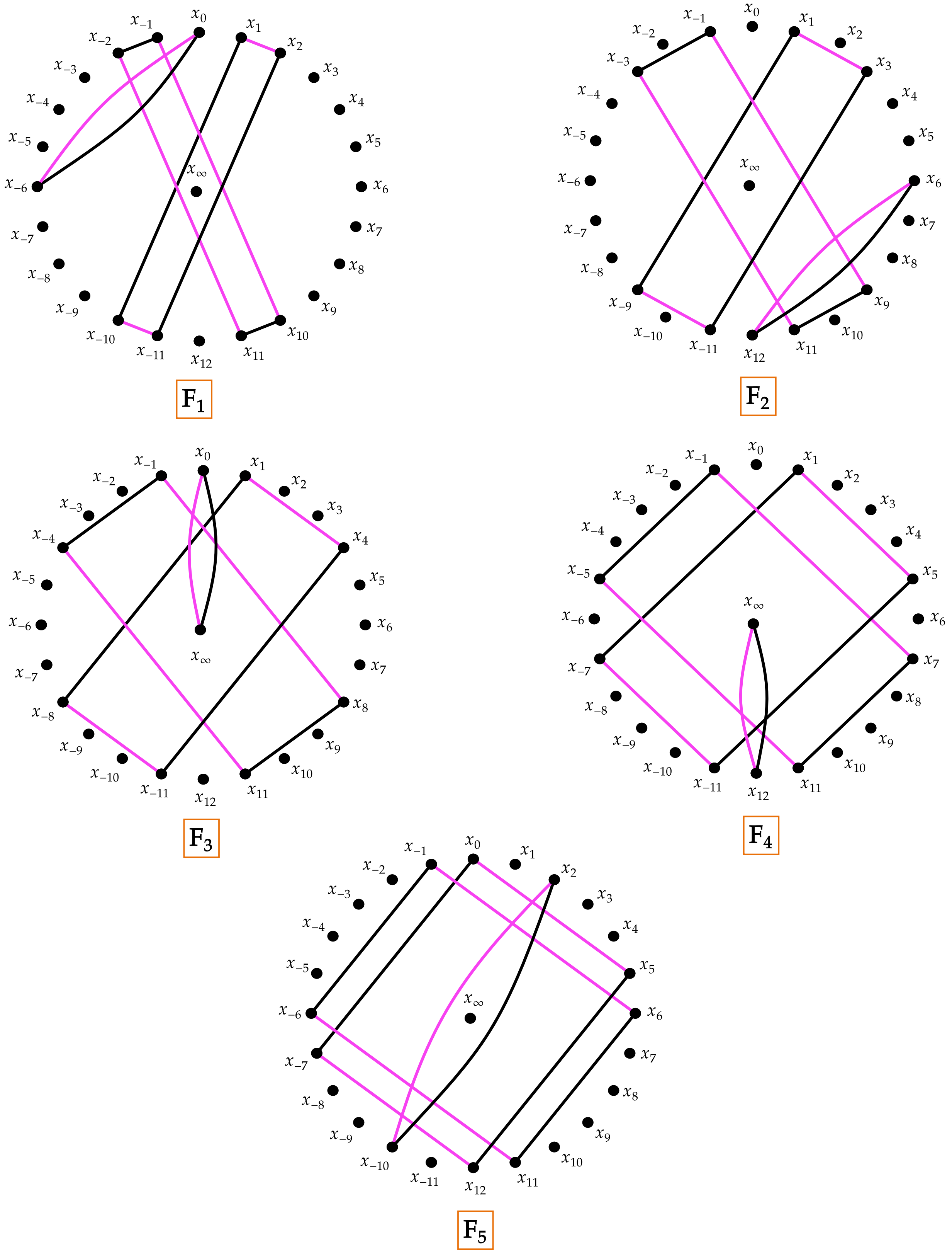


Figure 5.4: (C_2, C_4, C_4)-subgraphs of $2K_{25}^{\circ}$ that generate an HOP (C_2, C_4, C_4)-decomposition of $4K_{25}^{\bullet}$.

Lemma 5.1.17. Let $n = 20k + 5$ for any integer $k \geq 1$. There exists an HOP (C_2, C_3, C_5)-decomposition of $4K_n^\bullet$.

Proof. Observe that $2n = 10(4k + 1)$; let $s = 4k + 1$. By Lemma 3.4.10 with $m_1 = 2$, $m_2 = 3$, and $m_3 = 5$, it suffices to find (C_2, C_3, C_5)-subgraphs $F_1, F_2, \dots, F_s$ in $4K_n^\bullet$ that satisfy Conditions (L′1) to (L′3).

As in Lemma 3.4.10, let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \dots\ x_{n-2})$. Observe that the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $d \in \{1, 2, \dots, 10k + 1\}$, we have a pink and a blue orbit $\{x_i x_{i+d} : i \in \mathbb{Z}_{n-1}\}$;
- for each $d \in \{1, 2, \dots, 20k + 3\}$, we have a black orbit $\{(x_i, x_{i+d}) : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_{i+(10k+2)} : i = 0, 1, \dots, \frac{n-3}{2}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

First, we construct the 5-cycles and the 3-cycles of the (C_2, C_3, C_5)-subgraphs. For $k = 1$ see Figure 5.5. For $k \geq 2$ and $i \in \{1, 2, \dots, k\}$, construct the following closed walks:

$$C_i = x_0\ x_{2i-1}\ x_{-3}\ x_{2k+i-1}\ x_{-(3k+i+2)}\ x_0$$

$$C_i' = x_{-(k+i+2)}\ x_{3k}\ x_{-(8k-i+6)}\ x_{-(k+i+2)}$$

It can be seen that C_i and C_i' are disjoint cycles of length five and three, respectively.

The 5-cycles C_i traverse edges of difference

$$2i - 1, 2i + 2, 2k + i + 2, 5k + 2i + 1, 3k + i + 2.$$

The explicit differences in $C_1, C_2, \dots, C_k$ are listed below.

$$\begin{array}{rc}
C_1: & 1, 4, 2k+3, 5k+3, 3k+3 \\
C_2: & 3, 6, 2k+4, 5k+5, 3k+4 \\
C_3: & 5, 8, 2k+5, 5k+7, 3k+5 \\
 & \vdots \\
C_{k-2}: & 2k-5, 2k-2, 3k, 7k-3, 4k \\
C_{k-1}: & 2k-3, 2k, 3k+1, 7k-1, 4k+1 \\
C_k: & 2k-1, 2k+2, 3k+2, 7k+1, 4k+2
\end{array}$$

The 3-cycles C_i' traverse edges of difference

$$C_i' : 4k + i + 2, 9k + i - 2, 7k - 2i + 4.$$

The explicit differences in $C'_1, C'_2, \ldots, C'_k$ are listed below.

$$\begin{aligned}
C'_1:&\quad 4k+3, 9k-1, 7k+2\\
C'_2:&\quad 4k+4, 9k, 7k\\
C'_3:&\quad 4k+5, 9k+1, 7k-2\\
&\quad\vdots\\
C'_{k-2}:&\quad 5k, 10k-4, 5k+8\\
C'_{k-1}:&\quad 5k+1, 10k-3, 5k+6\\
C'_k:&\quad 5k+2, 10k-2, 5k+4
\end{aligned}$$

The differences that are not covered by any C_i or C'_i are the following:

(i) $7k+i+2$, for $i \in \{1,2,\ldots,2k-4\}$;

(ii) $2k+1$;

(iii) $10k-1$;

(iv) $10k$;

(v) ∞; and

(vi) $2, 10k+1, 10k+2$

The following C_2-subgraphs of K_n cover the differences in (i)-(v), respectively, exactly once:

(i) $E_i = x_{-(3k+2)}\ x_{4k+i}\ x_{-(3k+2)}$, for $i \in \{1,2,\ldots,2k-4\}$;

(ii) $E_{2k-3} = x_{3k+1}\ x_{5k+2}\ x_{3k+1}$;

(iii) $E_{2k-2} = x_{-(3k+2)}\ x_{7k-3}\ x_{-(3k+2)}$;

(iv) $E_{2k-1} = x_{-(3k+2)}\ x_{7k-2}\ x_{-(3k+2)}$; and

(v) $E_{2k} = x_{-(3k+2)}\ x_{\infty}\ x_{-(3k+2)}$.

Note that neither the vertex $x_{-(3k+2)}$ nor any of the vertices in the interval $[x_{3k+1}, x_{12k-2}]$ are used by any C_i or C'_i, and only vertices in this interval are used to construct the 2-cycles E_i. Hence, all C_i and C'_i are disjoint from the E_i.

For each $i \in \{1,2,\ldots,k\}$, let $C_i^{(1)}, C_i^{(2)}, C_i^{(3)}, C_i^{(4)}$ be four copies of C_i, and let $C_i'^{(1)}, C_i'^{(2)}$, $C_i'^{(3)}, C_i'^{(4)}$ be four copies of C'_i. For each $i \in \{1,2,\ldots,2k\}$, let $E_i^{(1)}$ and $E_i^{(2)}$ be two copies of E_i. Now, apply the proof of Lemma 3.3.3 to color the copies of each cycle so that they satisfy Condition **(C1)** of Definition 1.2.3, and so that

- $C_i^{(1)} \bigoplus C_i^{(2)} \bigoplus C_i^{(3)} \bigoplus C_i^{(4)} = 4C_i^\bullet$,

- $C_i'^{(1)} \bigoplus C_i'^{(2)} \bigoplus C_i'^{(3)} \bigoplus C_i'^{(4)} = 4C_i'^\bullet$, and
- $E_i^{(1)} \bigoplus E_i^{(2)} = 4E_i^\bullet$.

Now, let $T_i^{(1)}, T_i^{(2)}, T_i^{(3)}, T_i^{(4)}$ be the following subgraphs:

$$T_i^{(1)} = C_i^{(1)} \cup C_i'^{(1)} \cup E_i^{(1)};$$

$$T_i^{(2)} = C_i^{(2)} \cup C_i'^{(2)} \cup E_{k+i}^{(1)};$$

$$T_i^{(3)} = C_i^{(3)} \cup C_i'^{(3)} \cup E_i^{(2)};$$

$$T_i^{(4)} = C_i^{(4)} \cup C_i'^{(4)} \cup E_{k+i}^{(2)}.$$

It is easy to see that $T_i^{(1)}, T_i^{(2)}, T_i^{(3)}, T_i^{(4)}$ are all (C_2, C_3, C_5)-subgraphs of $4K_n^\bullet$ that satisfy Condition **(C1)** of Definition 1.2.3. We apply the above procedure for each $i \in \{1, 2, \ldots, k\}$, so we have $4k$ (C_2, C_3, C_5)-subgraphs, which we relabel as $F_1, F_2, \ldots, F_{s-1}$.

Observe that except for differences 2, $10k+1$, and $10k+2$, all other differences occur exactly four times (exactly one pink, one blue, and two black with opposite directions) in $F_1, F_2, \ldots, F_{s-1}$.

Next, we construct a (C_2, C_3, C_5)-subgraph F_s that covers differences 2, $10k+1$, and $10k+2$. Let $F_s = C_s \,\dot\cup\, C_s' \,\dot\cup\, E_s$, where:

- $C_s = x_1\ x_{-1}\ x_{10k}\ x_{10k+2}\ x_{-10k}\ x_1$;
- $C_s' = x_0\ x_{10k+1}\ x_{-(10k+1)}\ x_0$; and
- $E_s = x_{-(5k+1)}\ x_{5k+1}\ x_{-(5k+1)}$.

Color the edges of C_s as follows: $x_{10k+2}\ x_{-10k}$ blue, $x_{-10k}\ x_1$ pink, and $x_1\ x_{-1}$, $x_{-1}\ x_{10k}$, $x_{10k}\ x_{10k+2}$ black with orientation away from the pink edge and towards the blue edge. Color the edges of C_s' as follows: $x_{10k+1}\ x_{-(10k+1)}$ pink, $x_{-(10k+1)}\ x_0$ blue, and $x_0\ x_{10k+1}$ black with orientation away from the pink edge and towards the blue edge (see Figure 5.5). Notice that C_s and C_s' jointly cover differences $10k+1$ and 2 exactly four times (one pink edge, one blue edge, and two opposite black arcs), and E_s covers difference $10k+2$ twice. Observe that the subgraphs $F_1, F_2, \ldots, F_s$ satisfy Conditions (L$'$1) to (L$'$3) of Lemma 3.4.10. Hence, $4K_n^\bullet$ admits an HOP (C_2, C_3, C_5)-decomposition. ∎

Example 5.1.18. Let $F_1, F_2, \ldots, F_5$ be the (C_2, C_3, C_5)-subgraphs of $4K_{25}^\bullet$ in Figure 5.5. Observe that $F_1, F_2, \ldots, F_5$ satisfy Conditions (L$'$1) to (L$'$3) of Lemma 3.4.10. Therefore, $4K_{25}^\bullet$ admits an HOP (C_2, C_3, C_5)-decomposition.

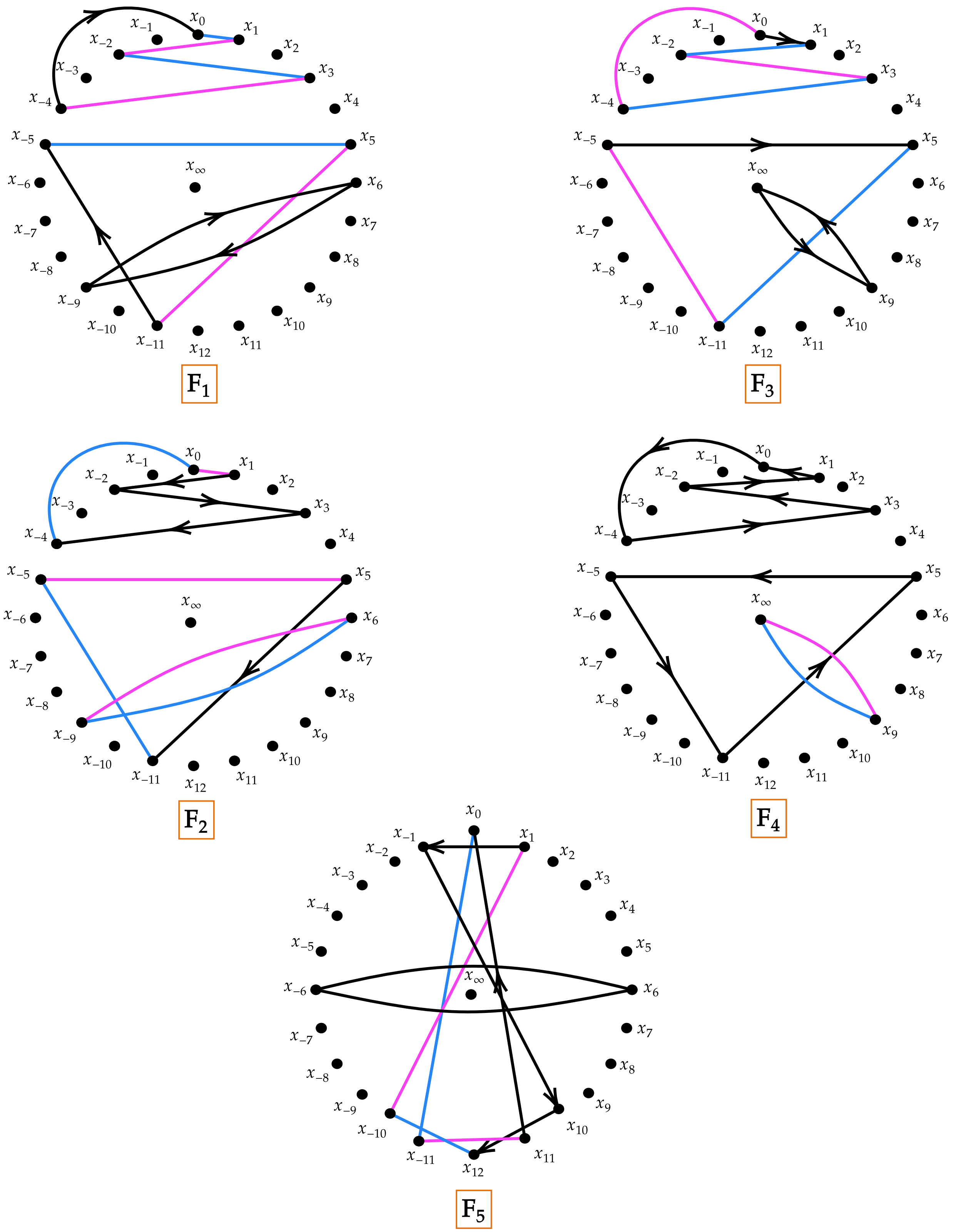


Figure 5.5: (C_2, C_3, C_5)-subgraphs of $4K_{25}^\bullet$ that generate an HOP (C_2, C_3, C_5)-decomposition of $4K_{25}^\bullet$.

Chapter 6

A Complete Solution to the Generalized HOP with One Round Table

In this chapter, we aim to provide a complete solution to the generalized HOP with one round table, showing that the necessary condition for HOP$(2^{\langle s\rangle}, 2m)$ to have a solution is indeed sufficient. This is equivalent to proving that the multigraph $4K_n^\bullet$ admits an HOP(C_m)-decomposition if and only if m divides the number of edges in $4K_n^\bullet$. Thus, the obvious necessary condition is $m \mid 2n(n-1)$, and this may occur in one of the following three cases:

- **Case 1:** m divides the number of edges in K_n.
- **Case 2:** m does not divide the number of edges in K_n but divides the number of edges in $2K_n$.
- **Case 3:** m does not divide the number of edges in $2K_n$ but divides the number of edges in $4K_n$.

This division simplifies the problem. In the first case, we use the existing results on (C_m)-decompositions of K_n and extend them to HOP (C_m)-decompositions for $4K_n^\bullet$. For the second case, we either use the results on $2K_n$ from [6] or directly construct a (C_m)-decomposition of $2K_n$, which we then extend to HOP (C_m)-decompositions for $4K_n^\bullet$. The third case, which is the most challenging, requires us to work directly with $4K_n^\bullet$ to construct an HOP (C_m)-decomposition.

This chapter is organized as follows. In Section 6.1, we present the main result. In Section 6.2, we introduce a method for coloring edges in a cycle decomposition of $2G^\circ$ so that each cycle has an even number of pink edges. In Section 6.3, we introduce definitions and tools that are commonly used and relevant to both Sections 6.4 and 6.5. In Section 6.4, we extend a (C_m)-decomposition of $2K_n$ to an HOP (C_m)-decomposition of $4K_n^\bullet$. Finally, in Section 6.5, we directly construct an HOP (C_m)-decomposition of $4K_n^\bullet$.

Throughout this chapter, we assume $m \geq 3$, as an HOP (C_m)-decomposition of $4G^\bullet$ can be easily obtained for $m = 2$.

6.1 Main Result

Here, we present the main result of this chapter. The terminology, tools, and lemmas used in its proof will be introduced in the subsequent sections.

Theorem 6.1.1. Let $2 \leq m \leq n$ be integers. Then there exists an HOP (C_m)-decomposition of $4K_n^\bullet$ if and only if $2n(n-1) \equiv 0 \pmod{m}$.

Proof. It is clear that if $4K_n^\bullet$ has an HOP(C_m)-decomposition, then $2n(n-1) \equiv 0 \pmod{m}$. Conversely, assume $2n(n-1) \equiv 0 \pmod{m}$. We want to show $4K_n^\bullet$ admits an HOP(C_m)-decomposition.

First, let $m = 2$. Then it follows from Lemma 3.3.2 that there exists an HOP(C_m)-decomposition of $4K_n^\bullet$.

From now on, let $m \geq 3$. Then, there are three cases to consider.

Case 1: $m \mid \frac{n(n-1)}{2}$.

First, assume n is odd. By Theorem 2.3.2, there exists a (C_m)-decomposition of K_n, and by Lemma 3.3.3, we can extend this decomposition to an HOP (C_m)-decomposition of $4K_n^\bullet$.

Second, assume n is even. Depending on the parity of m, the following two subcases arise:

- m is even. By Theorem 2.3.10, $2K_n$ admits a (C_m)-decomposition. Consequently, $2K_n^\circ$ admits a (C_m)-decomposition. If $m \equiv 0 \pmod{4}$, then by Corollary 6.2.5, we can recolor each m-cycle in this decomposition so that it contains an even number of pink edges, and by Lemma 3.3.5 such a decomposition can be extended to an HOP (C_m)-decomposition of $4K_n^\bullet$.

  If $m \equiv 2 \pmod{4}$, then since $m \mid \frac{n(n-1)}{2}$, it follows from Lemma 6.4.1 (Case 1) that $4K_n^\bullet$ admits an HOP (C_m)-decomposition.
- m is odd. By Lemma 6.4.11, there exists an HOP (C_m)-decomposition of $4K_n^\bullet$.

Case 2: $m \nmid \frac{n(n-1)}{2}$, but $m \mid n(n-1)$.

This implies that m must be even. Moreover, if $n(n-1)$ is an even multiple of m, it follows that m divides $\frac{n(n-1)}{2}$, which is a contradiction. Hence, $n(n-1)$ must be an odd multiple of m, that is, $n(n-1) \equiv m \pmod{2m}$.

By Theorem 2.3.10, $2K_n$ admits a (C_m)-decomposition. Consequently, $2K_n^\circ$ also admits a (C_m)-decomposition. Since m is even, first assume $m \equiv 0 \pmod{4}$. By Corollary 6.2.5, each m-cycle in this decomposition can be recolored so that it has an even number of pink edges. We can then apply Lemma 3.3.5 to extend this to an HOP (C_m)-decomposition of $4K_n^\bullet$.

Second, assume $m \equiv 2 \pmod{4}$. Depending on the parity of n, two subcases arise.

- n is odd. Then, by Lemma 6.4.3, there exists an HOP (C_m)-decomposition of $4K_n^\bullet$.
- n is even. Since m does not divide $\frac{n(n-1)}{2}$ by assumption, Lemma 6.4.1 (Case 2) implies that the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Case 3: $m \nmid n(n-1)$, but $m \mid 2n(n-1)$.

This implies that m must be a multiple of four. Moreover, if $2n(n-1)$ is an even multiple of m, it follows that m divides $n(n-1)$, which is a contradiction. Hence, $2n(n-1)$ must be an odd multiple of m, that is, $2n(n-1) \equiv m \pmod{2m}$. Depending on the parity of n, we have the following two cases:

- n is odd. Then, by Lemma 6.5.1, there exists an HOP (C_m)-decomposition of $4K_n^\bullet$.
- n is even. Then, by Lemma 6.5.2, there exists an HOP (C_m)-decomposition of $4K_n^\bullet$.

∎

The following corollary is a direct consequence of Theorem 6.1.1.

Corollary 6.1.2. Let $s \geq 0$, $m \geq 2$, and $n = s + m$. Then $\mathrm{HOP}(2^{\langle s \rangle}, 2m)$ has a solution if and only if $m \mid 2n(n-1)$.

6.2 Recoloring the edges in a cycle decomposition of $2G^\circ$

In this section, we demonstrate how to recolor a given (C_m)-decomposition of $2K_n^\circ$ so that every cycle in the decomposition contains an even number of pink edges. For this approach, it suffices to know that a (C_m)-decomposition of $2K_n^\circ$ exists; the explicit construction of the decomposition is not required. However, this method is applicable only for certain values of m and n. The cases requiring a detailed analysis of the (C_m)-decomposition of $2K_n$ are addressed in Section 6.4.

We begin this section with the definition of the *decomposition graph*.

Definition 6.2.1. Assume G is a simple graph. Let $\mathcal{D}$ be a (C_m)-decomposition of $2G^\circ$. The *decomposition graph* of $\mathcal{D}$, denoted as $\mathscr{D}(\mathcal{D})$, is defined as a graph with vertex set $\mathcal{D}$, and two vertices in $\mathscr{D}(\mathcal{D})$ are adjacent via k edges if and only if the corresponding cycles in $\mathcal{D}$ share exactly k pairs of parallel edges in $2G^\circ$.

Observe that $\mathscr{D}(\mathcal{D})$ is an m-regular graph of order $|\mathcal{D}|$, and hence has $\frac{|\mathcal{D}|\cdot m}{2}$ edges.

Lemma 6.2.2. Assume G is a simple graph. Let $\mathcal{D}$ be a (C_m)-decomposition of $2G^\circ$. Let H be a connected component in $\mathscr{D}(\mathcal{D})$ and D its corresponding set of cycles in $\mathcal{D}$. If $|E(H)| \equiv 0 \pmod 2$, then the number of cycles in D with an odd number of pink edges is even.

Proof. Note that for any cycle $C \in D$ and any edge $e \in E(C)$, there exists a cycle $C' \in D$ and $e' \in E(C')$ such that e and e' are parallel edges of $2G^\circ$. Also, observe that the number of edges of H is equal to the number of pairs of parallel edges in D. Since H has an even number of edges, and each edge of H corresponds to a pair of parallel edges in D, the number of pink and the number of black edges in D are equal and even. If there is an odd number of cycles with an odd number of pink edges in D, then the total number of pink edges is odd, which is a contradiction. Hence, the number of cycles with an odd number of pink edges in D is even. ∎

Lemma 6.2.3. Let $m \geq 3$ be a positive integer, and let $\mathcal{D} = \{F_1, F_2, \ldots, F_t\}$ be a (C_m)-decomposition of $2G^\circ$. If all connected components in $\mathscr{D}(\mathcal{D})$ have an even number of edges, then there exists a recoloring of the edges in $\mathcal{D}$ such that every cycle in $\mathcal{D}$ has an even number of pink edges.

Proof. Label the vertices of $\mathscr{D}(\mathcal{D})$ as x_i for $i \in \{1, 2, \ldots, t\}$. For any cycle F_i in $\mathcal{D}$, if the cycle has an even number of pink edges, color the corresponding vertex x_i in $\mathscr{D}(\mathcal{D})$ green; otherwise color it red. If all vertices in $\mathscr{D}(\mathcal{D})$ are green, then we are done. Otherwise, by Lemma 6.2.2, there must be an even number of red vertices in each connected component.

Let H be one of the connected components in $\mathscr{D}(\mathcal{D})$ with some red vertices. Since the number of red vertices is even, we can pair them. Without loss of generality, assume vertices x_i and x_j in H are red. As H is connected, there exists a path from x_i to x_j, denoted as $P = x_i x_\ell \ldots x_k x_j$. This path P represents a set of cycles, namely $D = \{F_i, F_\ell, \ldots, F_k, F_j\}$ in $\mathcal{D}$. Observe that if two vertices are adjacent in P, then the corresponding cycles in D share at least one pair of parallel edges.

Now, consider each two adjacent vertices in P and swap the colors in one pair of parallel edges in their corresponding cycles (see Example 6.2.4). Note that the color of exactly one edge in each of F_i and F_j changes, while in each of the other cycles in D, the colors of two edges change. Therefore, the parity of the number of pink edges in F_i and F_j changes and become even, while the parity of the number of pink edges in each of the other cycles of D remains the same. Consequently, the two red vertices x_i and x_j are recolored green and no other vertices are recolored. We can repeat this process for any two red vertices in H to fully color all the vertices green. Repeating this process for each connected component of $\mathscr{D}(\mathcal{D})$ completes the proof. ∎

In the following example, we apply the proof of Lemma 6.2.3 to demonstrate how to recolor the 4-cycles in a given (C_4)-decomposition of $2K_8^\circ$ so that each cycle contains an even number of pink edges.

Example 6.2.4. Let $\mathcal{D} = \{F_1, F_2, \ldots, F_{14}\}$ be the (C_4)-decomposition of $2K_8^\circ$ in Figure 6.1. The decomposition graph $\mathscr{D}(\mathcal{D})$ is also shown at the bottom of Figure 6.1. Observe that all connected components in $\mathscr{D}(\mathcal{D})$ have an even number of edges, and there are two connected components with red vertices. In Figure 6.2 and 6.3, we use the proof of Lemma 6.2.3 to show that it is possible to recolor the cycles in $\mathcal{D}$ so that each cycle has an even number of pink edges.

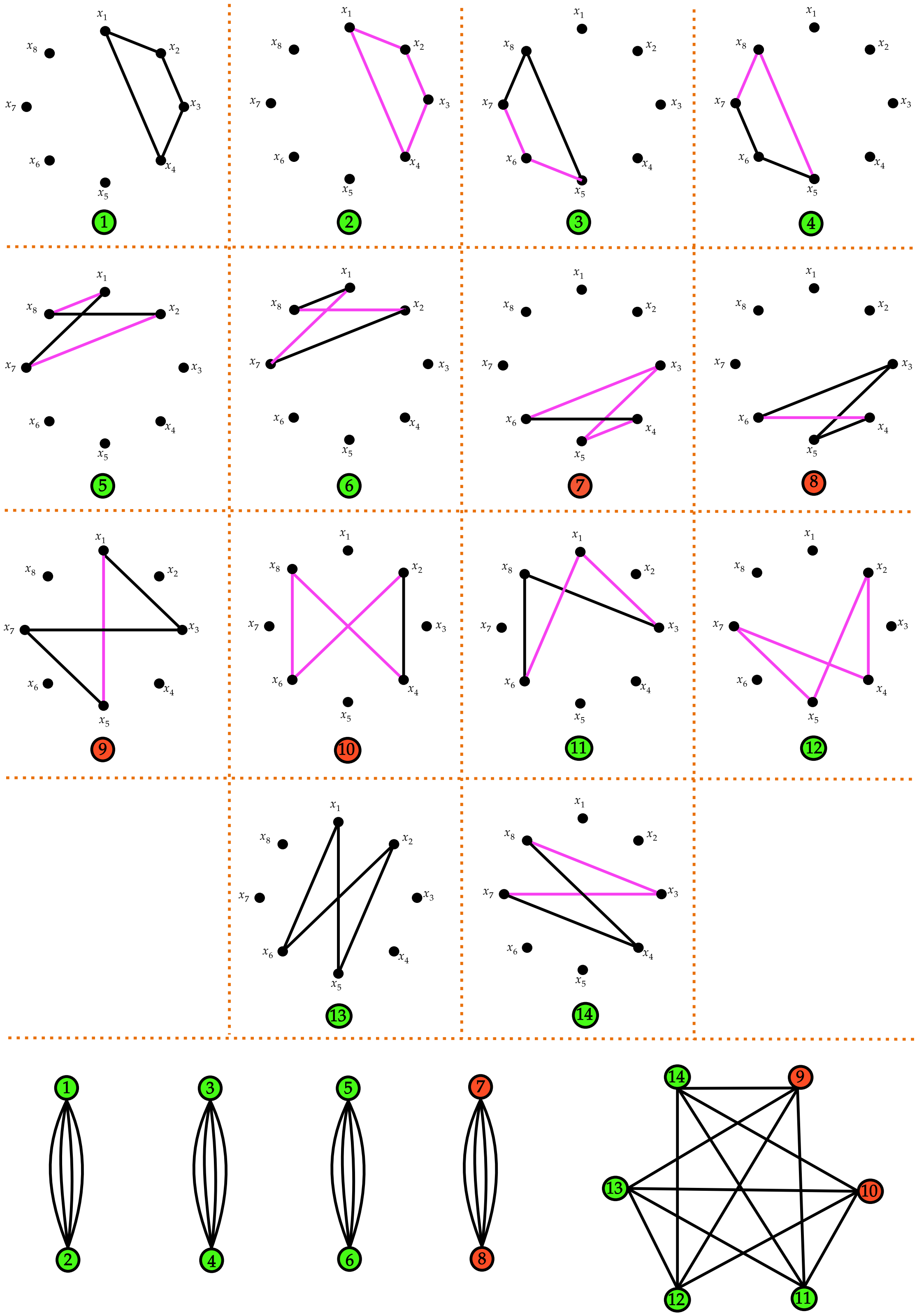


Figure 6.1: A (C_4)-decomposition of $2K_8^\circ$ and its decomposition graph.

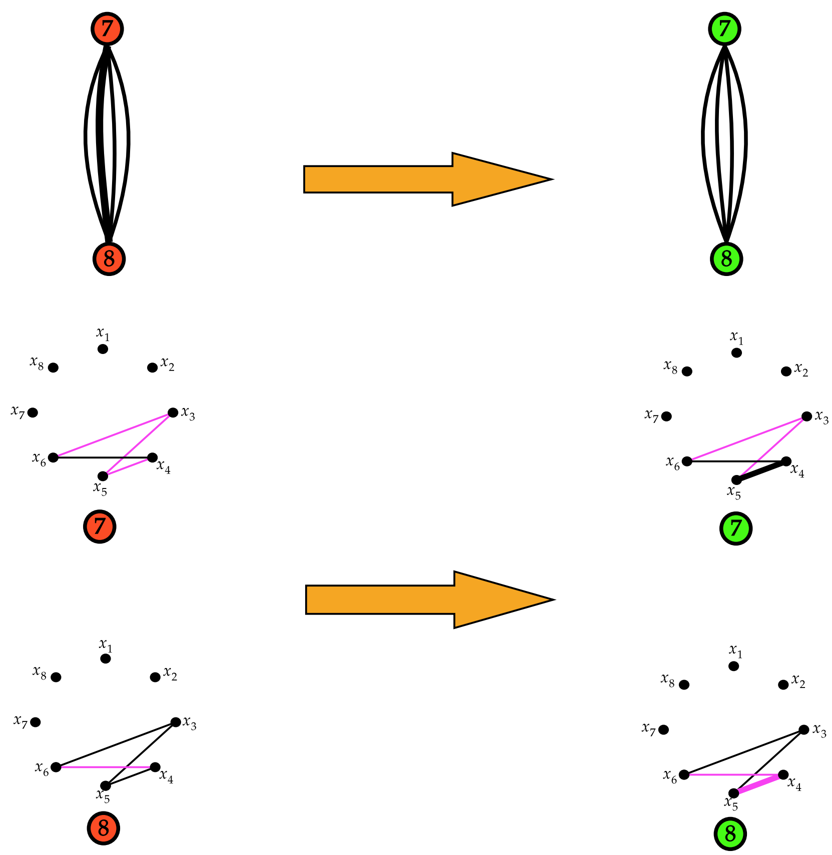


Figure 6.2: Recoloring the red vertices green in $\mathscr{D}(\mathcal{D})$ by swapping colors of the parallel edges in their corresponding cycles.

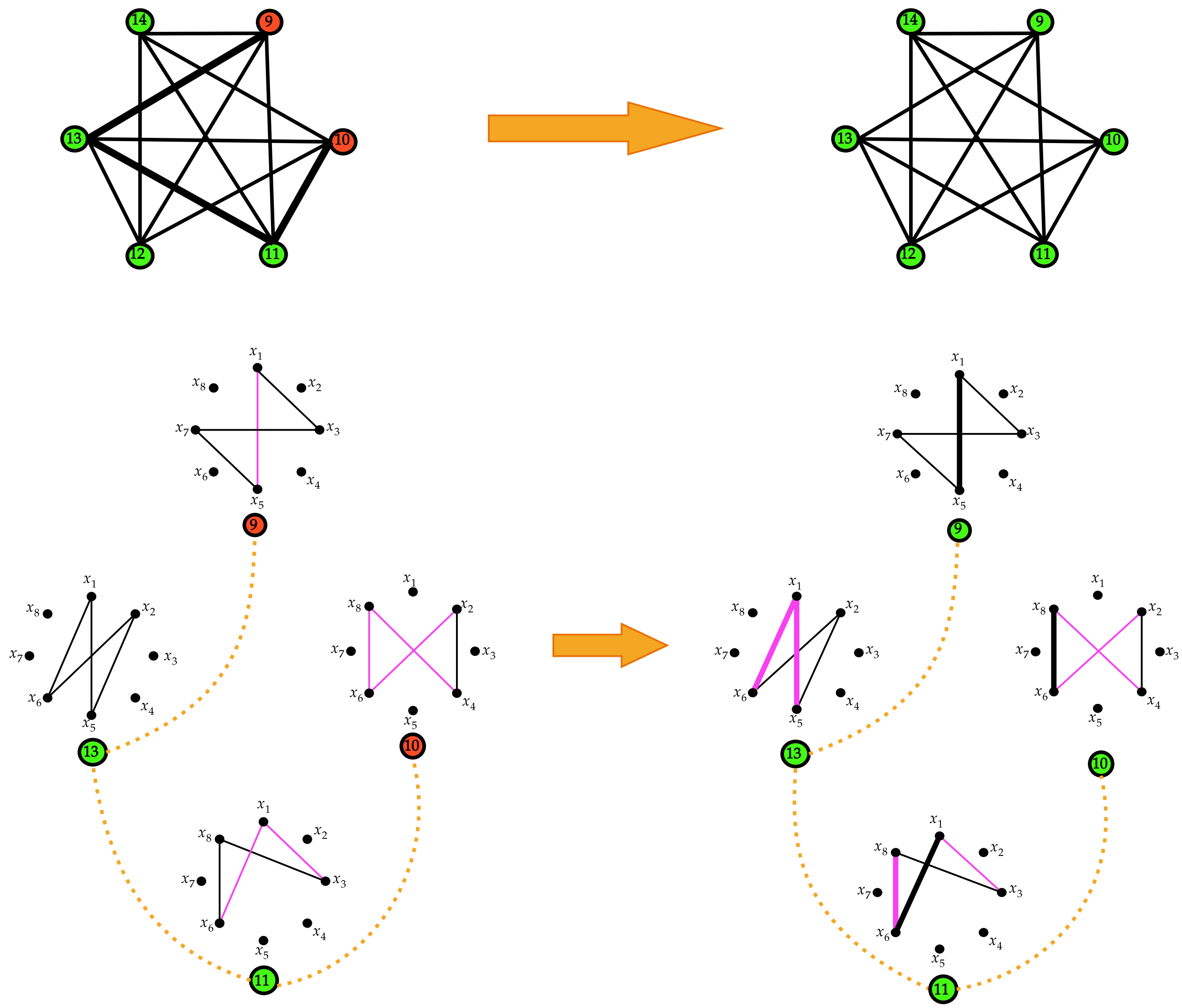


Figure 6.3: Recoloring the red vertices green in $\mathscr{D}(\mathcal{D})$ by swapping colors of the parallel edges in their corresponding cycles.

The following corollary is an immediate result of Lemma 6.2.3.

Corollary 6.2.5. Let $m \equiv 0 \pmod 4$, and let n be a positive integer. If $2K_n^\circ$ admits a (C_m)-decomposition, then every m-cycle in the decomposition can be recolored so that it contains an even number of pink edges.

Proof. Let $\mathcal{D}$ be a (C_m)-decomposition of $2K_n^\circ$. Construct the decomposition graph $\mathscr{D}(\mathcal{D})$. In $\mathscr{D}(\mathcal{D})$, the number of edges in any connected component H is $\frac{|V(H)|\cdot m}{2}$. Since $m \equiv 0 \pmod 4$, the connected component H has an even number of edges. Thus, by Lemma 6.2.3, we can recolor the decomposition $\mathcal{D}$ such that each cycle in it contains an even number of pink edges. ∎

6.3 Definitions and Supporting Tools

In this section, we introduce the definitions, lemmas, and tools that will be used in both Sections 6.4 and 6.5.

We first define the terms *peripheral cycle* and *central cycle*, which are the two types of cycles used in the construction of (C_m)-decompositions of $2K_n$ and $4K_n^\bullet$.

Definition 6.3.1. Let $G = Circ(n-1; S) \bowtie K_1$ be a circulant with the vertex set $\{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$ and the difference set S.

- A *central cycle* C is a cycle of G that passes through the central vertex x_∞ and is constructed as $C = x_\infty P x_\infty$, where P is a path.
- A cycle C in G that is not a central cycle is referred to as a *peripheral cycle*.

Let $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$ be a permutation on the vertex set of G. If C is a central (peripheral) cycle, then $\rho(C)$ is also a central (peripheral) cycle.

Definition 6.3.2. Let $P = x_0x_1\ldots x_{n-1}x_n$ be a path. The *reversal path* of P, denoted as $\overleftarrow{P}$, is obtained by reversing the order of the vertices in P, that is, $\overleftarrow{P}= x_nx_{n-1}\ldots x_1x_0$.

The following lemma is about the existence of a path in a circulant graph that covers a given set of differences in a specified order. It also shows that we can replace any one of the differences in the set with a difference 1, and still find a path that covers the differences in the given order.

Lemma 6.3.3. Let n be a positive integer, and let $A = \{a_1, a_2, \ldots, a_t\}$ be a set of integers such that $1 \le a_1 < a_2 < \cdots < a_t \le n-2$. Then, the circulant graph $G = \mathrm{Circ}(n-1; A \cup \{1\})$ contains a path of length t, denoted P, that starts at x_0 and covers differences in the following order:

$$a_1, a_2, \ldots, a_{t-1}, a_t.$$

Moreover, for any $i \in \{1, \ldots, t\}$, the difference a_i can be replaced with difference 1; that is, there exists a path of length t, denoted Q, covering differences in the following order:

$$a_1, a_2, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_{t-1}, a_t.$$

Additionally, the path Q can be constructed so that it coincides with P up to the initial of the edge corresponding to difference a_{i-1}.

Proof. Let

$$\begin{aligned}
v_0 &= 0, \\
v_1 &= a_1, \\
v_2 &= a_1 - a_2, \\
v_3 &= a_1 - a_2 + a_3, \\
&\vdots \\
v_t &= a_1 - a_2 + \ldots + (-1)^t a_{t-1} + (-1)^{t+1} a_t.
\end{aligned}$$

Let P be the following walk:

$$P = x_{v_0}\ x_{v_1}\ x_{v_2}\ \ldots\ x_{v_{t-1}}\ x_{v_t}.$$

Here, we call vertices with indices $v_0, v_2, \ldots$, even vertices, and those with indices $v_1, v_3, \ldots$, odd vertices. Observe that the subscripts of the even vertices are decreasing, while the subscripts of the odd vertices are increasing.

We want to show that the even vertices do not "cross over" the odd vertices; that is, for even vertex x_{v_k} and odd vertex $x_{v_{k'}}$, we have $v_k + (n-1) > v_{k'}$. Suppose to the contrary that the even vertex x_{v_j} is the first even vertex that "crosses" the odd vertex $x_{v_{j-1}}$; that is, $v_j + (n-1) < v_{j-1}$. Since $v_j = v_{j-1} - a_j$, we have $(v_{j-1} - a_j) + (n-1) < v_{j-1}$. This simplifies to $a_j > n-1$, which is a contradiction.

Now, suppose x_{v_j} is the first odd vertex that "crosses" the even vertex $x_{v_{j-1}}$; that is, $v_j > v_{j-1} + (n-1)$. Since $v_j = v_{j-1} + a_j$, we have $(v_{j-1} + a_j) > v_{j-1} + (n-1)$. This simplifies to $a_j > n-1$, which is a contradiction.

Therefore, the vertices of the walk P are pairwise distinct, which means P is a path.

We next show there exists a path Q covering such a list of differences in the given order:

$$a_1, a_2, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_{t-1}, a_t.$$

We know that

$$1 \leq a_1 < \ldots < a_{i-2} < a_{i-1} < a_i < a_{i+1} < \ldots < a_t \leq n-2.$$

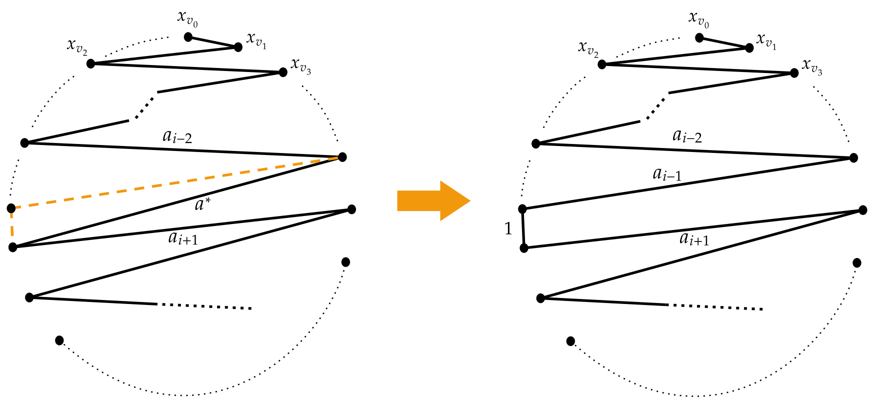
Let $a^* = a_{i-1} + 1$. It is clear that $a^* - a_{i-2} \geq 2$ and that $a_{i-2} < a^* < a_{i+1}$. Thus, $A' = \{a_1, a_2, \ldots, a_{i-2}, a^*, a_{i+1}, \ldots, a_t\}$ is a set of size $t-1$ such that

$$a_1 < a_2 < \ldots < a_{i-2} < a^* < a_{i+1} < \ldots, a_t.$$

By the first part of this lemma, there exists a path of length $t-1$ that starts at x_0 and covers differences in the following order

$$a_1, a_2, \ldots, a_{i-2}, a^*, a_{i+1}, \ldots, a_t.$$

Moreover, since $a^* = a_{i-1} + 1$ and $a_{i-2} < a_{i-1} = a^* - 1$, we can easily replace the edge with difference a^* with two edges of differences a_{i-1} and 1, respectively, as shown below.



Thus, the path Q is constructed that covers the differences in the following order:

$$a_1, a_2, \ldots, a_{i-2}, a_{i-1}, 1, a_{i+1}, \ldots, a_t.$$

Note that if we follow the same initial steps used to construct the path P, the resulting path Q coincides with P up to the initial of the edge corresponding to the difference a_{i-1}.

∎

Corollary 6.3.4. Let n be an even positive integer. Consider the following sets of differences:

- $A = \{a_1, a_2, \ldots, a_t\}$ with $1 \leq a_1 < a_2 < \cdots < a_t \leq \frac{n-2}{2}$, and
- $B = \{b_1, b_2, \ldots, b_s\}$ with $1 \leq b_1 < b_2 < \cdots < b_s \leq \frac{n-2}{2}$.

Then, the circulant graph $G = \mathrm{Circ}(n-1; A \cup B)$ contains a path of length $t+s$ that covers the differences in the following order:

$$a_1, a_2, \ldots, a_t, b_s, b_{s-1}, \ldots, b_2, b_1.$$

Proof. Let $b_i' = (n-1) - b_i$ for $i = 1, 2, \ldots, s$. Observe that $b_i' \equiv -b_i \pmod{n-1}$, and we have

$$\frac{n}{2} \leq b_s' < b_{s-1}' < \cdots < b_1' \leq n-2,$$

and thus:

$$1 \leq a_1 < \ldots < a_t < b_s' < b_{s-1}' < \cdots < b_1' \leq n-2.$$

Using Lemma 6.3.3, we see that there exists a path of length $t+s$ that covers the differences mentioned above, in the specified order.

∎

Corollary 6.3.5. Let n be an odd positive integer. Consider the following sets of differences:

- $A = \{a_1, a_2, \ldots, a_t\}$ with $1 \leq a_1 < a_2 < \cdots < a_t \leq \frac{n-1}{2}$, and
- $B = \{b_1, b_2, \ldots, b_s\}$ with $1 \leq b_1 < b_2 < \cdots < b_s \leq \frac{n-3}{2}$.

Then, the circulant graph $G = \mathrm{Circ}(n-1; A \cup B)$ contains a path of length $t+s$ that covers the differences in the following order:

$$a_1, a_2, \ldots, a_t, b_s, b_{s-1}, \ldots, b_2, b_1.$$

Proof. Let $b_i' = (n-1) - b_i$ for $i = 1, 2, \ldots, s$. Observe that $b_i' \equiv -b_i \pmod{n-1}$, and we have

$$\frac{n+1}{2} \leq b_s' < b_{s-1}' < \cdots < b_1' \leq n-2,$$

and thus:

$$1 \leq a_1 < \ldots < a_t < b_s' < b_{s-1}' < \cdots < b_1' \leq n-2.$$

Using Lemma 6.3.3, we see that there exists a path of length $t+s$ that covers the differences mentioned above, in the specified order.

∎

The following lemma serves as a reduction step, showing that to find an HOP (C_m)-decomposition of $4K_n^\bullet$ for even m and all even $n \geq m$, it suffices to find such a decomposition for even values of n in the interval $m \leq n < 2m$. This lemma extends the approach from [6], originally developed for $2K_n$, to the multigraph $4K_n^\bullet$.

Lemma 6.3.6. Let m be an even integer, and let $\alpha \in \mathbb{Z}^+$. If $4K_n^\bullet$ admits an HOP (C_m)-decomposition for all even n such that $m \leq n < 2m$ and $m \mid \alpha n(n-1)$, then $4K_n^\bullet$ also admits an HOP (C_m)-decomposition for all even $n \geq m$ such that $m \mid \alpha n(n-1)$.

Proof. Take any even $n' \geq m$ such that $m \mid \alpha n'(n'-1)$. We can write $n' = n + qm$, for integers $q \geq 0$ and $m \leq n < 2m$. Moreover, since $m \mid \alpha n'(n'-1)$, it follows that $m \mid \alpha n(n-1)$.

Notice that we can partition the vertex set of $4K_{n'}^\bullet$ into q sets of m vertices and one set of n vertices. Each set of m vertices induces a subgraph isomorphic to $4K_m^\bullet$. The edges between any two sets of m vertices induce a subgraph isomorphic to the bipartite graph $4K_{m,m}^\bullet$. The set of n vertices induces a subgraph isomorphic to $4K_n^\bullet$. The edges between the set of n vertices and any of the sets of m vertices induce a subgraph isomorphic to the bipartite graph $4K_{m,n}^\bullet$. Therefore, $4K_{n'}^\bullet$ is an edge-disjoint union of:

- q copies of $4K_m^\bullet$,
- $\binom{q}{2}$ copies of $4K_{m,m}^\bullet$,
- one copy of $4K_n^\bullet$, and

- q copies of $4K^{\bullet}_{m,n}$.

Since $m \leq n < 2m$, by supposition, $4K^{\bullet}_n$ admits an HOP (C_m)-decomposition. In Lemma 2.2.11 (iv) and Lemma 2.2.9 (v), it is proved that $4K^{\bullet}_m$ admits an HOP (C_m)-decomposition. Furthermore, since m and n are even, $\min\{m,n\} \geq \frac{m}{2}$, and m divides the number of edges in both $K_{m,m}$ and $K_{m,n}$, by Theorem 2.3.9, both $K_{m,m}$ and $K_{m,n}$ have (C_m)-decompositions. Now, by applying Lemma 3.3.3, we see that both $4K^{\bullet}_{m,m}$ and $4K^{\bullet}_{m,n}$ admit an HOP (C_m)-decomposition. Finally, by applying Lemma 3.3.1, the multigraph $4K^{\bullet}_{n'}$ admits an HOP (C_m)-decomposition. ∎

In Lemma 6.3.7, we show that for any even m, the multigraph $4K^{\bullet}_{m+1}$ admits an HOP (C_m)-decomposition. To construct this decomposition, we use the (C_m)-decomposition of $2K_{m+1}$ provided in [6]. Specifically, we double each cycle in the decomposition and color the two copies appropriately to extend the decomposition to an HOP (C_m)-decomposition for $4K^{\bullet}_{m+1}$. Later, in Lemma 6.3.8, we use this result to establish the reduction step for odd n.

Lemma 6.3.7. Let m be an even positive integer. Then there exists an HOP (C_m)-decomposition of $4K^{\bullet}_{m+1}$.

Proof. Let $V(4K^{\bullet}_{m+1}) = \{x_i : i \in \mathbb{Z}_m\} \cup \{x_\infty\}$, and let $\rho_\bullet$ be the permutation on $E(4K^{\bullet}_{m+1})$ that preserves the edge colors (and orientations), and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{m-1})$. In [6], Alspach et al. showed that $\mathcal{W} = \{C', C, \rho(C), \rho^2(C), \ldots, \rho^{m-1}(C)\}$, where

$$C = x_\infty\ x_0\ x_{-1}\ x_1\ x_{-2}\ x_2\ \ldots\ x_{-\frac{m-2}{2}}\ x_{\frac{m-2}{2}}\ x_\infty, \text{ and}$$

$$C' = x_0\ x_1\ x_2\ \ldots\ x_{-2}\ x_{-1}\ x_0,$$

is a (C_m)-decomposition of $2K_{m+1}$.

For $m \equiv 2 \pmod 4$, let P be the path

$$P = x_{-1}\ x_1\ x_{-2}\ x_2\ \ldots\ x_{-\frac{m-2}{4}}\ x_{\frac{m-2}{4}},$$

and represent the cycle C as

$$C = x_\infty\ x_0\ x_{-1}\ P\ x_{\frac{m-2}{4}}\ x_{-\frac{m+2}{4}}\ \rho^{\frac{m}{2}}(\overleftarrow{P})\ x_{\frac{m-2}{2}}\ x_\infty.$$

For $m \equiv 0 \pmod 4$, let P be the path

$$P = x_{-1}\ x_1\ x_{-2}\ x_2\ \ldots\ x_{\frac{m-4}{4}}\ x_{-\frac{m}{4}},$$

and represent the cycle C as

$$C = x_\infty\ x_0\ x_{-1}\ P\ x_{-\frac{m}{4}}\ x_{\frac{m}{4}}\ \rho^{\frac{m}{2}}(\overleftarrow{P})\ x_{\frac{m-2}{2}}\ x_\infty.$$

We explain the coloring for the case $m \equiv 2 \pmod 4$. The case $m \equiv 0 \pmod 4$ is analogous.

Now, let F_1, F_2, F_3, F_4 be four copies of C that are colored as follows (see Figure 6.4).

- In F_1, color the edges of P black and orient them forward, color the edge $x_{\frac{m-2}{4}}x_{-\frac{m+2}{4}}$ blue, color the edges of $\rho^{\frac{m}{2}}(\overleftarrow{P})$ alternately pink and blue, beginning with pink, and since $\overleftarrow{P}$ has odd length, it ends with pink. Now color the edge $x_{\frac{m-2}{2}}x_\infty$ blue, $x_\infty x_0$ pink, and color the arc (x_0, x_{-1}) black.

- In F_2, color the edges of P black and orient them backward, color the edge $x_{\frac{m-2}{4}}x_{-\frac{m+2}{4}}$ pink, color the edges of $\rho^{\frac{m}{2}}(\overleftarrow{P})$ alternately blue and pink, beginning with blue, and since $\overleftarrow{P}$ has odd length, it ends with blue. Now color the arcs (x_{-1}, x_0), (x_0, x_∞), and $(x_\infty, x_{\frac{m-2}{2}})$ black.

- In F_3, color the edges of P black and orient them backward, color the arc $(x_{-\frac{m+2}{4}}, x_{\frac{m-2}{4}})$ black, color the edges of $\rho^{\frac{m}{2}}(\overleftarrow{P})$ alternately pink and blue, beginning with pink, and since $\overleftarrow{P}$ has odd length, it ends with pink. Now color the edge $x_{\frac{m-2}{2}}x_\infty$ blue and the arcs (x_{-1}, x_0) and (x_0, x_∞) black.

- In F_4, color the edges of P black and orient them forward, color the arc $(x_{\frac{m-2}{4}}, x_{-\frac{m+2}{4}})$ black, color the edges of $\rho^{\frac{m}{2}}(\overleftarrow{P})$ alternately blue and pink, beginning with blue, and since $\overleftarrow{P}$ has odd length, it ends with blue. Then color the arcs $(x_\infty, x_{\frac{m-2}{2}})$, (x_0, x_{-1}) black and the edge $x_\infty x_0$ pink.

Let F_1', F_2' be two copies of C'. In F_1', color the edges alternately pink and blue, begin with pink. In F_2', color the edges alternately blue and pink, begin with blue.

We claim that

$$\mathcal{W}' = \left\{\rho_\bullet^i(F_1), \rho_\bullet^i(F_2), \rho_\bullet^{\frac{m}{2}+i}(F_3), \rho_\bullet^{\frac{m}{2}+i}(F_4) : i = 0, 1, \dots, \tfrac{m-2}{2}\right\} \cup \left\{F_1', F_2'\right\}$$

is a (C_m)-decomposition for $4K^\bullet_{m+1}$. Notice that F_1, F_2, F_3, F_4 jointly contain exactly one edge from each orbit of $\langle\rho_\bullet\rangle$ corresponding to difference $\frac{m}{2}$, and two edges from each orbit of $\langle\rho_\bullet\rangle$ corresponding to each difference $d \in \{2, 3, \dots, \frac{m-2}{2}\}$, namely, a pair of edges of the form $(e, \rho_\bullet^{\frac{m}{2}}(e))$. The black orbits corresponding to difference 1 are covered by F_1, F_2, F_3, F_4, while the pink and the blue orbits are completely covered by F_1', F_2'. Since $F_1, F_2, F_3, F_4, F_1', F_2'$ satisfy Condition **(C1)** of Definition 1.2.3, and $\rho_\bullet$ preserves the edge colors (and orientations), it can be verified that $\mathcal{W}'$ is an HOP (C_m)-decomposition of $4K^\bullet_{m+1}$. ∎

The following lemma serves as a reduction step, showing that to find an HOP (C_m)-decomposition of $4K^\bullet_n$ for even m and all odd $n > m$, it suffices to find such a decomposition for values of n in the interval $m < n < 2m$. This lemma generalizes the approach outlined in [6] for $2K_n$.

Lemma 6.3.8. Let m be an even integer, and let $\alpha \in \mathbb{Z}^+$. If $4K^\bullet_n$ admits an HOP (C_m)-decomposition for all odd n such that $m < n < 2m$ and $m \mid \alpha n(n-1)$, then $4K^\bullet_n$ also admits an HOP (C_m)-decomposition for all odd $n > m$ such that $m \mid \alpha n(n-1)$.

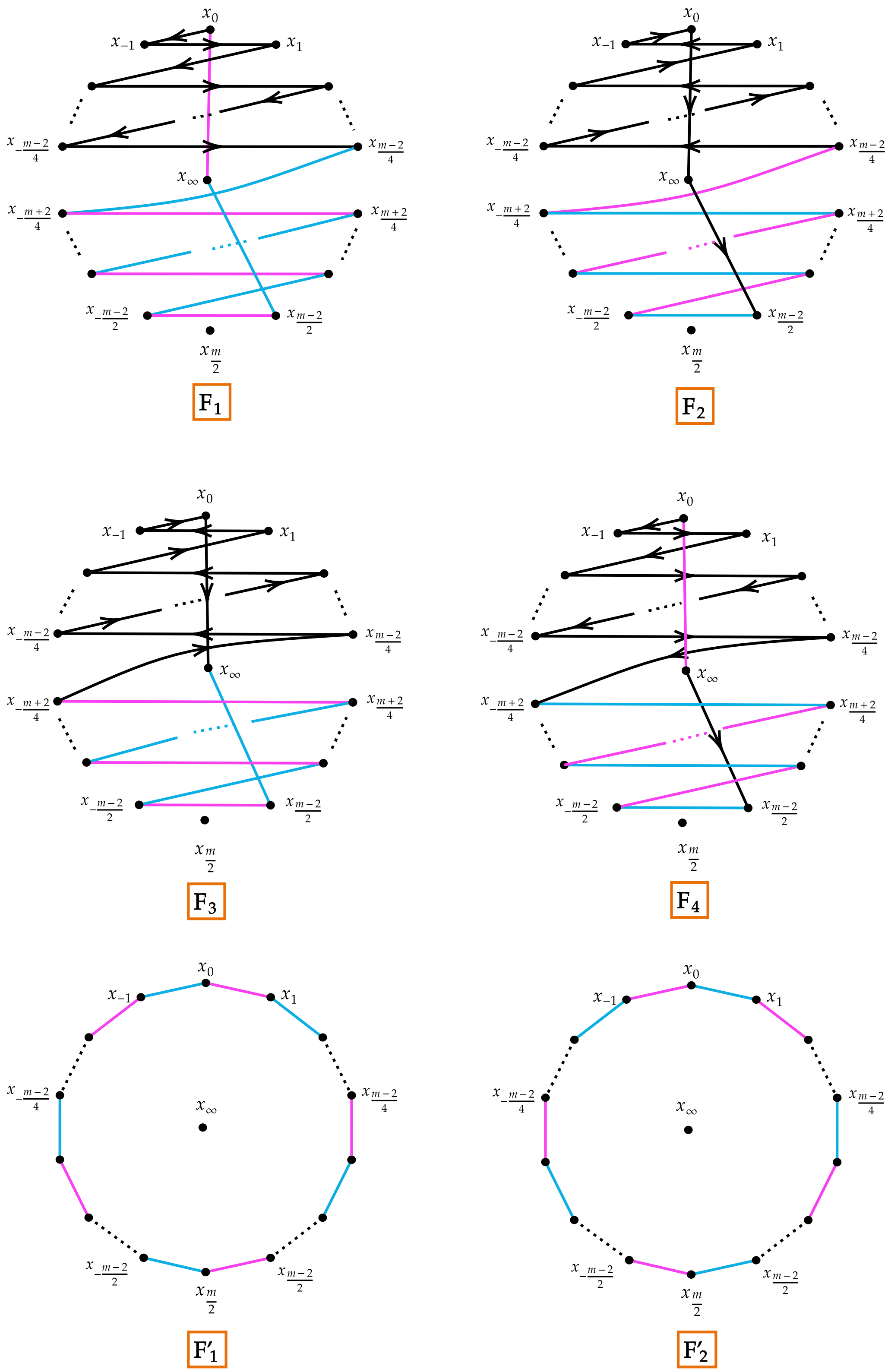


Figure 6.4: Starter (C_m)-subgraphs for an HOP decomposition of $4K^{\bullet}_{m+1}$ with $m \equiv 2 \pmod 4$.

Proof. Take any odd $n > m$ such that $m \mid \alpha n(n-1)$. We can write $n = qm + r + 1$, for integers $q \geq 1$ and even $0 \leq r < m - 1$. We can further rewrite n as $(q-1)m + (m+r) + 1$. Label one vertex of $4K_n^\bullet$ as x_∞ and partition the remaining vertices into $q - 1$ sets of m vertices and one set of $m + r$ vertices. Notice that:

- Each set of m vertices together with vertex x_∞ induces a subgraph isomorphic to $4K_{m+1}^\bullet$.
- The edges between any two sets of m vertices induce a subgraph isomorphic to the bipartite graph $4K_{m,m}^\bullet$.
- The set of $m + r$ vertices, along with vertex x_∞, induces a subgraph isomorphic to $4K_{m+r+1}^\bullet$.
- The edges between the set of $m + r$ vertices and any of the sets of m vertices induce a subgraph isomorphic to the bipartite graph $4K_{m,m+r}^\bullet$.

Therefore, $4K_n^\bullet$ is an edge-disjoint union of:

- $q - 1$ copies of $4K_{m+1}^\bullet$,
- $\binom{q-1}{2}$ copies of $4K_{m,m}^\bullet$,
- one copy of $4K_{m+r+1}^\bullet$, and
- $q - 1$ copies of $4K_{m,m+r}^\bullet$.

Since m divides $\alpha n(n-1)$ and $n = qm + r + 1$, it follows that m divides $\alpha r(r+1)$. Consequently, m divides $\alpha(m+r+1)(m+r)$. Additionally, considering that $0 \leq r < m-1$, we have $m < m+1 \leq m+r+1 < 2m$. Therefore, by the given hypothesis, $4K_{m+r+1}^\bullet$ admits an HOP (C_m)-decomposition.

Also, in Lemma 6.3.7, we proved that $4K_{m+1}^\bullet$ admits an HOP (C_m)-decomposition. Since m and r are even, $\min\{m, m+r\} \geq \frac{m}{2}$, and m divides the number of edges in both $K_{m,m}$ and $K_{m,m+r}$, by Theorem 2.3.9, both $K_{m,m}$ and $K_{m,m+r}$ have a (C_m)-decomposition. Now, by applying Lemma 3.3.3, we see that both $4K_{m,m}^\bullet$ and $4K_{m,m+r}^\bullet$ admit an HOP (C_m)-decomposition. Therefore, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition. ∎

6.4 Extending a (C_m)-decomposition of $2K_n$ to an HOP (C_m)-decomposition of $4K_n^\bullet$

In this section, we focus on constructing a (C_m)-decomposition of $2K_n$ and then extending it to an HOP (C_m)-decomposition of $4K_n^\bullet$. To find a decomposition for $2K_n$, we use different methods depending on the parity of m and n. The following subsections present these constructions, beginning with the case where both m and n are even.

6.4.1 The Case When m is Even and n is Even

Here, we consider the case where both m and n are even. Since the case $m \equiv 0 \pmod 4$ can be handled using Corollary 6.2.5, we focus on the case where $m \equiv 2 \pmod 4$.

Lemma 6.4.1. Let $m \equiv 2 \pmod 4$, and let n be an even positive integer with $6 \leq m \leq n$. If $n(n-1) \equiv 0 \pmod m$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Proof. By Lemma 6.3.6, it suffices to prove this result for even n in the range $m \leq n < 2m$. Note that if $n = m$, then by Lemma 2.2.11 (iv), there exists an HOP (C_m)-decomposition of $4K_m^\bullet$. Hence, we assume that $m < n < 2m$.

The approach is to construct starter central and starter peripheral cycles and to color them appropriately. These starter cycles are designed to generate all cycles in the decomposition through suitable permutations.

We begin by defining the necessary parameters and determining how many starter central and starter peripheral cycles are needed.

Let $V(2K_n^\circ) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$, and let $\rho_\circ$ be the permutation on $E(2K_n^\circ)$ that preserves the color of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho_\circ \rangle$ has the following orbits on the edge set of $2K_n^\circ$:

- for each $s \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a black orbit $\{x_i x_{i+s} : i \in \mathbb{Z}_{n-1}\}$; and
- a pink and a black orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

For convenience, let $S = \{1, 2, \ldots, \frac{n-2}{2}, \infty\}$ be the set of all differences.

Let $n = 2^e a$, where a is odd, and let $m = 2a'b'$, where a' and b' are both odd, with $a' \mid a$ and $b' \mid (n-1)$.

To construct the peripheral cycles, we partition the $n-1$ vertices in $\{x_i : i \in \mathbb{Z}_{n-1}\}$ into b' segments, each containing $\ell = \frac{n-1}{b'}$ consecutive vertices. Note that each segment will contribute $2a'$ edges toward an m-cycle.

Now, we explore the required number of central and peripheral cycles for the decomposition. If $2K_n^\circ$ is (C_m)-decomposable, then the number of m-cycles in the decomposition is

$$\begin{aligned}
\frac{n(n-1)}{m} &= \frac{(n-1)(m+n-m)}{m} \\
&= \frac{(n-1)m + (n-1)(n-m)}{m} \\
&= (n-1) + \frac{(n-1)(n-m)}{2a'b'} \\
&= (n-1) + \frac{n-1}{b'} \cdot \frac{n-m}{2a'}
\end{aligned}$$

$$= (n-1) + \ell \cdot \frac{n-m}{2a'}$$

Observe that

$$\frac{n-m}{2a'} = \frac{2\left(2^{e-1}a - a'b'\right)}{2a'} = \frac{2^{e-1}a - a'b'}{a'}.$$

Since $a' \mid a$ and $e \geq 1$, it follows that $\frac{2^{e-1}a-a'b'}{a'}$ is an integer. Let $F = \frac{2^{e-1}a-a'b'}{a'}$. The number of m-cycles can be expressed in the following form:

$$\frac{n(n-1)}{m} = (n-1) + \ell F$$

This suggests constructing one starter central cycle and F starter peripheral cycles. The starter central cycle is rotated through all $n-1$ positions, while each of the F starter peripheral cycles is rotated through ℓ positions to generate all the cycles in the decomposition.

Next, we establish some inequalities that will be useful later in the proof.

We know that $n < 2m$; that is, $2^e a < 2^2 a'b'$. Dividing both sides by $2a'$ gives $\frac{2^{e-1}a}{a'} < 2b'$. Therefore, $\frac{2^{e-1}a-a'b'}{a'} < b'$, which means $F < b'$. Therefore, we have

$$F \leq b' - 1. \tag{6.4.1}$$

Additionally, we have $m < n-1 < 2m$; that is, $2a'b' < b'\ell < 2^2 a'b'$. Dividing each side by b', we get

$$2a' < \ell < 4a'. \tag{6.4.2}$$

Now, we show that if $\ell - a' \leq 2a' - 1$, then $F \leq \frac{b'}{2}$:

$$\begin{aligned}
\ell - a' &\leq 2a' - 1\\
\ell &\leq 3a' - 1\\
b'\ell &\leq 3a'b' - b'\\
(\ b'\ell = n-1) \qquad n &\leq 3a'b' - b' + 1\\
(\ n = 2^e a) \qquad 2^e a &\leq 3a'b' - b' + 1\\
\frac{2^{e-1}a}{a'} &\leq \frac{3a'b' - b' + 1}{2a'} = b' + \frac{a'b' - b' + 1}{2a'}\\
\frac{2^{e-1}a - a'b'}{a'} &\leq \frac{a'b' - b' + 1}{2a'} = \frac{b'}{2} - \frac{b'-1}{2a'}\\
(\text{Since } b' \geq 1) \qquad F &\leq \frac{b'}{2}
\end{aligned}$$

Therefore,

$$\text{if } \ell - a' \leq 2a' - 1, \text{ then } F \leq \frac{b'}{2}. \tag{6.4.3}$$

Now that we have established the necessary inequalities, we proceed with the main proof by dividing it into two cases: one where m divides $\frac{n(n-1)}{2}$, and one where m does not divide $\frac{n(n-1)}{2}$.

Case 1: Assume $m \mid \frac{n(n-1)}{2}$. We know that $n = 2^e a$, and since $m \mid \frac{n(n-1)}{2}$, it follows that $e \geq 2$, and thus $n \equiv 0 \pmod 4$. Moreover, since a', b', and a are all odd, it follows that $F = \frac{2^{e-1}a - a'b'}{a'}$ is odd.

The idea is to first construct $\frac{F-1}{2} + 1$ starter peripheral cycles for K_n. Then, for each of the $\frac{F-1}{2}$ cycles, we take two copies: we color the edges of one copy pink and the edges of the other copy black. This gives us $F-1$ starter peripheral cycles for $2K_n^\circ$.

One starter peripheral cycle remains, which we color pink. The differences covered by this cycle will also be covered by the starter central cycle. We make sure that the corresponding edges in the starter central cycle are colored black.

We start by constructing the starter peripheral cycles. For $i = 0, 1, \ldots, \frac{F-1}{2}$, define the path P_i as follows:

$$P_i = x_0 \; x_{i\ell+1} \; x_{-1} \; x_{i\ell+2} \; x_{-2} \ldots x_{i\ell+a'-1} \; x_{-(a'-1)} \; x_{i\ell+a'} \; x_{\frac{b'+1}{2}\ell}.$$

The differences covered by P_0 are:

$$1,\; 2,\; 3,\; 4,\; \ldots,\; 2a'-2,\; 2a'-1,\; \frac{(b'-1)\ell}{2} + a', \tag{6.4.4}$$

and the differences covered by P_i, for $i = 1, \ldots, \frac{F-1}{2}$, are:

$$i\ell+1,\; i\ell+2,\; i\ell+3,\; \ldots,\; i\ell+2a'-2,\; i\ell+2a'-1,\; \frac{(b'-2i+1)\ell}{2} - a'. \tag{6.4.5}$$

Since $2a' < \ell$ by (6.4.2), the paths $\rho^j(P_i)$, for $j = 0, \ell, 2\ell, \ldots, (b'-1)\ell$, are pairwise vertex-disjoint except at the endpoints. Also, since b' is odd, we have $\gcd\left(b', \frac{b'+1}{2}\right) = 1$. Thus,

$$C_i = P_i \cup \rho^\ell(P_i) \cup \ldots \cup \rho^{(b'-1)\ell}(P_i)$$

is an m-cycle.

Next, we show that the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-1}{2}}$, as listed in (6.4.4) and (6.4.5), are pairwise distinct.

First, we show that the differences $i\ell+1, i\ell+2, \ldots, i\ell+2a'-1$ are pairwise distinct for $i = 0, 1, \ldots, \frac{F-1}{2}$. It suffices to show that the largest difference does not exceed $\frac{n-2}{2}$; that is,

$$\frac{F-1}{2}\ell + 2a' - 1 \leq \frac{n-2}{2}.$$

By (6.4.1), we have $F \leq b'-1$, and since F and b' are odd, it follows that $F \leq b'-2$. Therefore, $\frac{F-1}{2}\ell \leq \frac{b'-3}{2}\ell$. Additionally, in (6.4.2), we proved that $2a' < \ell$. Combining these results, we obtain

$$\frac{F-1}{2}\ell + 2a' - 1 \leq \frac{b'-3}{2}\ell + 2a' - 1$$

$$\begin{aligned}
&= \frac{b'-3}{2}\ell + \ell - 1 \quad (\text{since } 2a' < \ell)\\
&= \frac{b'-1}{2}\ell - 1\\
&< \frac{(b'-1)\ell}{2} + a'\\
&< \frac{b'\ell}{2} = \frac{n-1}{2}
\end{aligned}$$

In the above inequality, observe that

$$\frac{F-1}{2}\ell + 2a' - 1 < \frac{(b'-1)\ell}{2} + a' < \frac{n-1}{2}.$$

This guarantees that for all $j = 0, 1, \ldots, \frac{F-1}{2}$,

$$\frac{(b'-1)\ell}{2} + a' \notin \big\{j\ell + 1, j\ell + 2, \ldots, j\ell + 2a' - 1\big\}.$$

Next, we show that for all $i = 1, \ldots, \frac{F-1}{2}$ and $j = 0, 1, \ldots, \frac{F-1}{2}$,

$$\frac{(b'-2i+1)\ell}{2} - a' \notin \big\{j\ell + 1, j\ell + 2, \ldots, j\ell + 2a' - 1\big\}.$$

Observe that

$$0 < \tfrac{(b'-2(\frac{F-1}{2})+1)\ell}{2} - a' < \tfrac{(b'-2(\frac{F-3}{2})+1)\ell}{2} - a' < \ldots < \tfrac{(b'-2\cdot 1+1)\ell}{2} - a' < \tfrac{(b'-1)\ell}{2} + a'.$$

Thus, it suffices to show that for any $j = 0, 1, \ldots, \frac{F-1}{2}$ and any $\alpha = 1, \ldots, 2a' - 1$,

$$\frac{(b' - 2\left(\frac{F-1}{2}\right) + 1)\ell}{2} - a' \neq j\ell + \alpha.$$

We prove this by contradiction. Suppose to the contrary that

$$\begin{aligned}
\frac{(b' - 2\left(\frac{F-1}{2}\right) + 1)\ell}{2} - a' &= j\ell + \alpha,\\
\frac{b'-F}{2}\ell + \ell - a' &= j\ell + \alpha.
\end{aligned}$$

Since F and b' are odd, the expression $\frac{b'-F}{2}$ is an integer, and since $2a' < \ell$ we have

$$\frac{b'-F}{2} = j \;\text{ and }\; \ell - a' = \alpha.$$

Given that $j \leq \frac{F-1}{2}$, it follows that $F \geq \frac{b'+1}{2}$.

On the other hand, knowing that $1 \leq \alpha \leq 2a' - 1$ and $\ell - a' = \alpha$, it follows that $\ell - a' \leq 2a' - 1$. In (6.4.3), we showed that $F \leq \frac{b'}{2}$ under this condition. Therefore, we have $\frac{b'+1}{2} \leq F \leq \frac{b'}{2}$, which is a contradiction.

We conclude that the differences listed in (6.4.4) and (6.4.5) are pairwise distinct.

Let B be the set of differences covered by the m-cycles $C_1, C_2, \ldots, C_{\frac{F-1}{2}}$, and let $G_1 = \mathrm{Circ}(n-1; B)$. Then,

$$\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 1, \ldots, \tfrac{F-1}{2}\}$$

is a (C_m)-decomposition of G_1. Take two copies of each C_i, for $i = 1, \ldots, \frac{F-1}{2}$, color one copy pink and the other black. By doing so, we generate $F-1$ starter peripheral cycles, which we denote $C'_1, \ldots, C'_{F-1}$.

It is now easy to see that

$$\{C'_i, \rho_\circ(C'_i), \ldots, \rho_\circ^{\ell-1}(C'_i) \mid i = 1, \ldots, F-1\}$$

is a (C_m)-decomposition of $2G_1^\circ$. Moreover, since m is even, each cycle contains an even number of pink edges. Thus, by Lemma 3.3.5, this decomposition can be extended to an HOP (C_m)-decomposition for $4G_1^\bullet$.

Notice that $A = \{1, 2, \ldots, 2a'-2, 2a'-1, \frac{(b'-1)\ell}{2} + a'\}$ is the set of differences covered by C_0. Color the edges of C_0 pink. Observe that the family of peripheral cycles generated by C_0; that is, $\{C_0, \rho_\circ(C_0), \ldots, \rho_\circ^{\ell-1}(C_0)\}$, covers the pink orbits corresponding to the differences in A.

The differences in the set A will also appear in the central cycle C, to be constructed below. There, we need to ensure that the black orbits corresponding to the differences listed in A are covered.

Now, we focus on constructing the starter central cycle C. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus (A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 = 2a' < a_2 < \ldots < a_t \leq \frac{n-2}{2}$ are positive integers and $t = \frac{n-2}{2} - (F+1)a'$.

Since $m < n$, it follows that $b' \geq 3$. Also, having $A = \{1, 2, \ldots, 2a'-2, 2a'-1, \frac{(b'-1)\ell}{2} + a'\}$, we observe that

$$2a' - 1 < a_1 < \ell < \tfrac{(b'-1)\ell}{2} + a' \leq \tfrac{n-2}{2}.$$

Consider the following sets of differences:

- $W_1 = \{1, 2, \ldots, 2a'-1, a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + a', \ldots, a_{t-1}, a_t\}$
- $W_2 = \{a_1, a_2, \ldots, a_{t-1}, a_t\}$

Note that $\frac{(b'-1)\ell}{2} + a' \notin W_2$, and in the sequence $a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + a', \ldots, a_{t-1}, a_t$, we assume

$$a_1 < a_2 < \ldots < \tfrac{(b'-1)\ell}{2} + a' < \ldots < a_{t-1} < a_t,$$

with the understanding that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t < \tfrac{(b'-1)\ell}{2} + a'$$

is also possible.

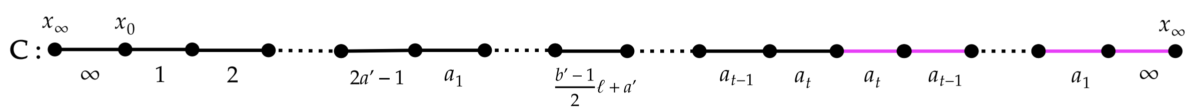
Here, W_1 contains all the differences in A and $\mathscr{L}$ except for ∞, and has size $|W_1| = t+2a'$. The set W_2 contains all the differences in $\mathscr{L}$ except for ∞, and has size $|W_2| = t$. By Corollary 6.3.4, there exists a path, call it T, of length $2t + 2a'$ that starts at x_0 and covers the differences in the following order:

$$T : 1, 2, \ldots, 2a' - 1, a_1, a_2, \ldots, \tfrac{(b'-1)\ell}{2} + a', \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, a_2, a_1.$$

Notice that the length of the path T is $2t + 2a'$, and

$$\begin{aligned} 2t + 2a' &= 2\left(\frac{n-2}{2} - (F+1)a'\right) + 2a' \\ &= n - 2 - 2Fa' \\ \text{(substitute } F = \frac{n-m}{2a'}\text{)} \quad &= n - 2 - n + m \\ &= m - 2. \end{aligned}$$

Let $C = x_\infty T x_\infty$. We color the edges corresponding to the second occurrences of the differences $\infty, a_1, a_2, \ldots, a_{t-1}, a_t$ pink, and color all remaining edges black, as shown below.



Given $t = \frac{n-2}{2} - (F+1)a'$, where $n \equiv 0 \pmod 4$ and both F and a' are odd, we see that t is odd. Hence, $t+1$, the number of pink edges in C, is even.

Observe that C contains exactly one pink edge and one black edge from each orbit of $\langle \rho_\circ \rangle$ corresponding to the differences in $\mathscr{L}$. Additionally, it contains exactly one black edge from each orbit of $\langle \rho_\circ \rangle$ corresponding to the differences in A. Thus,

$$\{C, \rho_\circ(C), \ldots, \rho_\circ^{n-2}(C)\} \cup \{C_0, \rho_\circ(C_0), \ldots, \rho_\circ^{\ell-1}(C_0)\}$$

is a (C_m)-decomposition for $2G_2^\circ$, where

$$G_2 = \mathrm{Circ}(n-1; A \cup (\mathscr{L} - \{\infty\})) \bowtie K_1.$$

Notice that in the above decomposition, each cycle has an even number of pink edges. Thus, by Lemma 3.3.5, this decomposition can be extended to an HOP (C_m)-decomposition for $4G_2^\bullet$.

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Case 2: Assume $m \nmid \frac{n(n-1)}{2}$, meaning that $2a'b' \nmid 2^{e-1}a(n-1)$. Since $b' \mid (n-1)$ and $a' \mid a$, it follows that 2 cannot divide 2^{e-1}. Consequently, we must have $e = 1$, allowing us to write $n = 2a$, which implies $n \equiv 2 \pmod 4$.

Since $e = 1$ and a', b', and a are all odd, it follows that $F = \frac{2^{e-1}a - a'b'}{a'}$ is even. We begin by constructing $\frac{F}{2}$ starter peripheral cycles for K_n. Next, we take two copies of each of these $\frac{F}{2}$ cycles. We color one copy pink and the other black in each pair. This process gives us F starter peripheral cycles for $2K_n^{\circ}$.

As in Case 1, for $i = 0, 1, \ldots, \frac{F-2}{2}$, let

$$P_i = x_0\ x_{i\ell+1}\ x_{-1}\ x_{i\ell+2}\ x_{-2} \ldots x_{i\ell+a'-1}\ x_{-(a'-1)}\ x_{i\ell+a'}\ x_{\frac{b'+1}{2}\ell}.$$

The differences covered by P_0 are:

$$1,\ 2,\ 3,\ \ldots,\ 2a'-2,\ 2a'-1,\ \tfrac{(b'-1)\ell}{2} + a', \tag{6.4.6}$$

and for $i = 1, \ldots, \frac{F-2}{2}$, the differences covered by P_i are:

$$i\ell+1,\ i\ell+2,\ i\ell+3,\ \ldots,\ i\ell+2a'-2,\ i\ell+2a'-1,\ \tfrac{(b'-2i+1)\ell}{2} - a'. \tag{6.4.7}$$

Since $2a' < \ell$ by (6.4.2), the paths $\rho^j(P_i)$, for $j = 0, \ell, 2\ell, \ldots, (b'-1)\ell$, are pairwise vertex-disjoint except at the endpoints. Also, since b' is odd, we have $\gcd\left(b', \frac{b'+1}{2}\right) = 1$. Thus,

$$C_i = P_i \cup \rho^{\ell}(P_i) \cup \ldots \cup \rho^{(b'-1)\ell}(P_i)$$

is an m-cycle.

Next, we show that all the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-2}{2}}$, as listed in (6.4.6) and (6.4.7), are pairwise distinct.

First, we show that the differences $i\ell+1, i\ell+2, \ldots, i\ell+2a'-1$ are all pairwise distinct for $i = 0, 1, \ldots, \frac{F-2}{2}$. It suffices to show that

$$\frac{F-2}{2}\ell + 2a' - 1 \le \frac{n-2}{2}.$$

By (6.4.1), we have $F \le b'-1$, which implies that $\frac{F-2}{2}\ell \le \frac{b'-3}{2}\ell$. Additionally, in (6.4.2), we proved that $2a' < \ell$. Combining these results, we have

$$\begin{aligned} \frac{F-2}{2}\ell + 2a' - 1 &\le \frac{b'-3}{2}\ell + 2a' - 1 \\ &< \frac{b'-1}{2}\ell \quad (\text{since } 2a' < \ell) \\ &< \frac{(b'-1)\ell}{2} + a' \\ &< \frac{n-1}{2}. \end{aligned}$$

In the above inequality, observe that

$$\frac{F-2}{2}\ell + 2a' - 1 < \frac{(b'-1)\ell}{2} + a' < \frac{n-1}{2}.$$

This guarantees that for all $j = 0, 1, \dots, \frac{F-2}{2}$,

$$\frac{(b'-1)\ell}{2} + a' \notin \{j\ell+1, j\ell+2, \dots, j\ell+2a'-1\}.$$

Next, we show that for all $i = 1, \dots, \frac{F-2}{2}$ and $j = 0, 1, \dots, \frac{F-2}{2}$,

$$\frac{(b'-2i+1)\ell}{2} - a' \notin \{j\ell+1, j\ell+2, \dots, j\ell+2a'-1\}.$$

Observe that

$$0 < \tfrac{(b'-2\cdot\frac{F-2}{2}+1)\ell}{2} - a' < \tfrac{(b'-2\cdot\frac{F-4}{2}+1)\ell}{2} - a' < \dots < \tfrac{(b'-2\cdot 1+1)\ell}{2} - a' < \tfrac{(b'-1)\ell}{2} + a'.$$

Thus, it suffices to show that for any $j = 0, 1, \dots, \frac{F-2}{2}$ and any $\alpha = 1, \dots, 2a'-1$,

$$\frac{(b' - 2\left(\frac{F-1}{2}\right) + 1)\ell}{2} - a' \neq j\ell + \alpha.$$

We prove this by contradiction. Suppose to the contrary that

$$\begin{aligned}\frac{(b' - 2\left(\frac{F-2}{2}\right) + 1)\ell}{2} - a' &= j\ell + \alpha\\ \frac{b'-F+1}{2}\ell + \ell - a' &= j\ell + \alpha.\end{aligned}$$

Since F is even and b' is odd, the expression $\frac{b'-F+1}{2}$ is an integer. Moreover, since $2a' < \ell$, we have

$$\frac{b'-F+1}{2} = j \quad \text{and} \quad \ell - a' = \alpha.$$

Given that $j \leq \frac{F-2}{2}$, it follows that $F \geq \frac{b'+3}{2}$.

On the other hand, knowing that $1 \leq \alpha \leq 2a'-1$ and $\ell - a' = \alpha$, it follows that $\ell - a' \leq 2a' - 1$. In (6.4.3), we showed that $F \leq \frac{b'}{2}$ under this condition. Therefore, we have $\frac{b'+3}{2} \leq F \leq \frac{b'}{2}$, which is a contradiction.

We conclude that the differences listed in (6.4.6) and (6.4.7) are pairwise distinct.

Let B be the set of differences covered by the m-cycles $C_0, C_1, \dots, C_{\frac{F-2}{2}}$, and let $G_1 = \mathrm{Circ}(n-1; B)$. Then,

$$\{C_i, \rho(C_i), \dots, \rho^{\ell-1}(C_i) \mid i = 0, 1, \dots, \tfrac{F-2}{2}\}$$

is a (C_m)-decomposition of G_1. Take two copies of each C_i, for $i = 0, \dots, \frac{F-2}{2}$, color one copy pink and the other black. By doing so, we generate F starter peripheral cycles, which we denote $C_1', \dots, C_F'$.

It is now easy to see that

$$\{C_i', \rho_\circ(C_i'), \dots, \rho_\circ^{\ell-1}(C_i') \mid i = 1, \dots, F\}$$

is a (C_m)-decomposition of $2G_1^\circ$. Moreover, since m is even, each cycle contains an even number of pink edges. Thus, by Lemma 3.3.5, this decomposition can be extended to an HOP (C_m)-decomposition for $4G_1^\bullet$.

Now, we focus on constructing the starter central cycle C. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus B$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 < a_2 < \ldots < a_t \le \frac{n-2}{2}$ are positive integers and $t = \frac{n-2}{2} - Fa'$. Consider the following sets of differences:

- $W_1 = W_2 = \{a_1, a_2, \ldots, a_{t-1}, a_t\}$

Note that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t \le \tfrac{n-2}{2}.$$

Here, $|W_1| = |W_2| = t$. By Corollary 6.3.4, there exists a path, call it T, of length $2t$ that starts at x_0 and covers the differences in the following order:

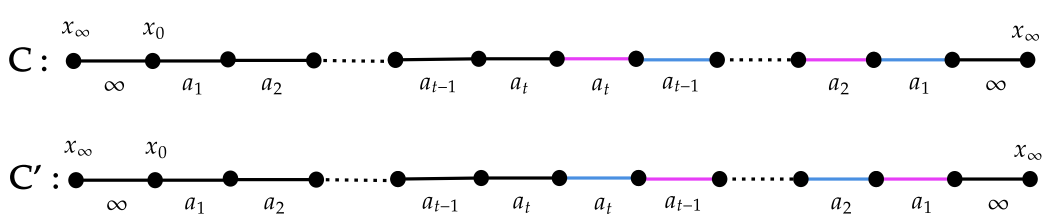
$$T : a_1, a_2, \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, a_2, a_1.$$

Notice that the length of the path T is $2t$, and

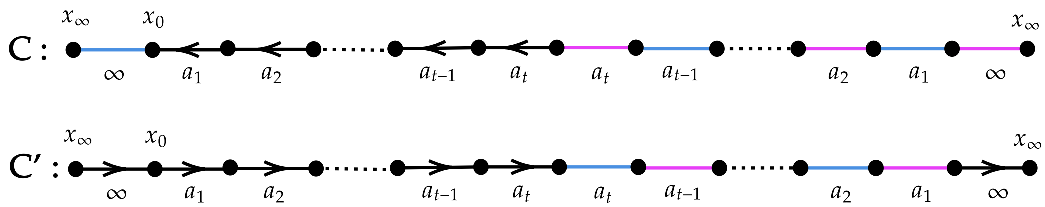
$$\begin{aligned} 2t &= 2\left(\frac{n-2}{2} - Fa'\right) \\ &= n - 2 - 2Fa' \\ \text{(substitute } F = \frac{n-m}{2a'}\text{)} \quad &= n - 2 - n + m \\ &= m - 2. \end{aligned}$$

Let $C = x_\infty T x_\infty$. Observe that C covers each difference in $\mathscr{L}$ exactly twice. Let C' be another copy of C.

In C, color the edges corresponding to the second occurrences of the differences $a_1, a_2, \ldots,$ a_t alternately pink and blue, starting with a_t as pink. Since $n \equiv 2 \pmod 4$, F is even, and a' is odd, it follows that $t = \frac{n-2}{2} - Fa'$ is even, ensuring that the edge corresponding to the difference a_1 is colored blue. In C', color the same edges alternately pink and blue, but in the opposite way of C, as shown below.



Finally, we complete the coloring of the cycles C and C' as shown below.



Observe that C and C' jointly contain exactly one blue edge, one pink edge, and two black arcs corresponding to the differences in $\mathscr{L}$. Additionally, the two cycles satisfy Condition **(C1)** of Definition 1.2.3. Therefore,

$$\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, n-2\}$$

is an HOP (C_m)-decomposition of $4G_2^\bullet$, where

$$G_2 = \mathrm{Circ}(n-1; \mathscr{L} - \{\infty\}) \bowtie K_1.$$

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

∎

6.4.2 The Case When m is Even and n is Odd

In this section, we delve into the construction of (C_m)-decompositions for $2K_n$ as given in [6], and we aim to extend them to HOP (C_m)-decompositions of $4K_n^\bullet$. In particular, we focus on the case where m is even and n is odd.

Since the case where $m \mid \frac{n(n-1)}{2}$ can be addressed using (C_m)-decompositions of K_n, we may assume $m \nmid \frac{n(n-1)}{2}$; that is, $n(n-1) \equiv m \pmod{2m}$. Moreover, when $m \equiv 0 \pmod 4$, the problem can be addressed by Corollary 6.2.5, so we restrict our attention to the case where $m \equiv 2 \pmod 4$.

Proceeding step by step, we concentrate on the starter m-cycles. We show that by taking copies of these starters and applying appropriate coloring, an HOP (C_m)-decomposition for $4K_n^\bullet$ can be obtained. We first prove the following lemma and later use it to prove Lemma 6.4.3.

Lemma 6.4.2. Let $m = 2k$ with k being an odd positive integer. Let n be an odd integer with $6 \le m < n < 2m$, and $n(n-1) \equiv m \pmod{2m}$. Let $G = Circ(n; S)$, where $S = \{a_1, a_2, \ldots, a_k\}$ and $a_1, a_2, \ldots, a_k$ are positive integers with

$$a_1 < a_2 < \ldots < a_k < \frac{n}{2}.$$

Then $4G^\bullet$ admits an HOP (C_m)-decomposition.

Proof. Let $V(4G^\bullet) = \{x_i : i \in \mathbb{Z}_n\}$. Let $\rho_\bullet$ be the permutation on $E(4G^\bullet)$ that preserves the edge colors (and orientations), and is induced by the permutation $\rho = (x_0\ x_1\ x_2\ \dots\ x_{n-1})$.

Let $\ell_i = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{i-1} a_i$, for $i \in \{1, 2, \dots, k-1\}$. Then $P = x_0\ x_{\ell_1}\ x_{\ell_2}\ x_{\ell_3}\ \dots\ x_{\ell_{k-1}}$ is a path of length $k-1$ in G. Now let

$$C = P\ x_{\ell_{k-1}} x_{\ell_{k-1}-a_k}\ \rho^{n-a_k}(\overleftarrow{P})\ x_{-a_k} x_0.$$

Observe that C is an m-cycle that traverses the following differences (in order):

$$a_1, a_2, a_3, \dots, a_{k-1}, a_k, a_{k-1}, a_{k-2}, \dots, a_3, a_2, a_1, a_k.$$

In [6], Alspach et al. showed that $\mathcal{W} = \{C, \rho(C), \rho^2(C), \dots, \rho^{n-1}(C)\}$ forms a (C_m)-decomposition for $2G$.

Let C' be a copy of C. Color C and C' as follows. (see Figure 6.5)

- In C, color the edge $x_{\ell_{k-1}} x_{\ell_{k-1}-a_k}$ blue, and color the first edge in $\rho^{n-a_k}(\overleftarrow{P})$, which is an edge of difference a_{k-1}, pink. Color the rest of the edges black and orient them away from the pink edge and towards the blue edge.
- In C', color the edges in path P alternately pink and blue, starting with pink. Since P is a path of length $k-1$, and $k-1$ is even, it ends with blue. Now, color edge $x_{\ell_{k-1}} x_{\ell_{k-1}-a_k}$ pink, and color the first edge in $\rho^{n-a_k}(\overleftarrow{P})$, which is an edge of difference a_{k-1}, black and orient it away from the pink edge. Color the rest of the edges in $\rho^{n-a_k}(\overleftarrow{P})$ alternately blue and pink, start with blue, and since $k-2$ is odd, it ends with blue. Finally, color the remaining edge $x_{-a_k} x_0$ black and orient it away from the pink edge and towards the blue edge.

Observe that C and C' satisfy Condition **(C1)** of Definition 1.2.3, and they jointly contain exactly one edge from each orbit of $\langle \rho_\bullet \rangle$ corresponding to differences $a_1, a_2, \dots, a_k$. Thus, $\mathcal{W}' = \left\{\rho_\bullet^i(C), \rho_\bullet^i(C') : i = 0, 1, \dots, n-1\right\}$ forms an HOP (C_m)-decomposition for $4G^\bullet$. ∎

Lemma 6.4.3. Let $m \equiv 2 \pmod 4$, and let n be an odd integer such that $6 \le m < n$. If $n(n-1) \equiv m \pmod{2m}$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Proof. By Lemma 6.3.8, it is enough to prove the result for odd n satisfying $m < n < 2m$. Write $n = m + r + 1$, where $0 \le r < m - 1$ is even.

If $r = 0$, then $n = m + 1$, and by Lemma 6.3.7, the multigraph $4K_{m+1}^\bullet$ admits an HOP (C_m)-decomposition. Thus, we assume $0 < r < m - 1$.

Let $V(K_n) = \{x_i : i \in \mathbb{Z}_n\}$. Then, $K_n = \mathrm{Circ}(n; A)$, where $A = \{1, 2, \dots, \frac{n-1}{2}\}$. Define $G_1 = \mathrm{Circ}(n; B)$, where $|B| = \frac{r}{2}$, $B \subseteq A$, and B is carefully selected as in the proof of [6, Theorem 3.1] to ensure that $2G_1$ admits a (C_m)-decomposition. Let $G_2 = \mathrm{Circ}(n; A \setminus B)$. Note that:

$$K_n = \mathrm{Circ}(n; A) = \mathrm{Circ}(n; B) \oplus \mathrm{Circ}(n; A \setminus B) = G_1 \oplus G_2.$$

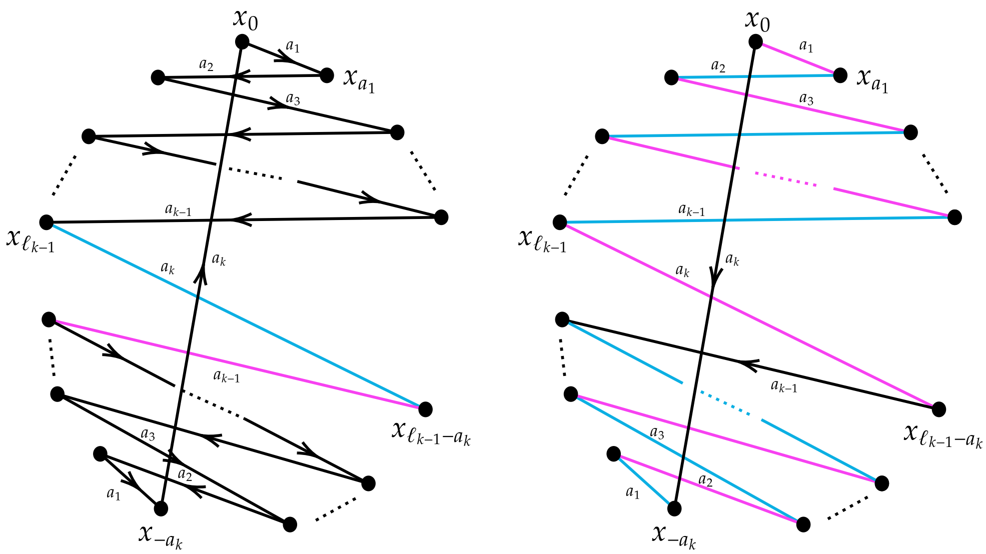


Figure 6.5: Lemma 6.4.2 – Starter m-cycles for an HOP (C_m)-decomposition of $4G^{\bullet}$ for $m \equiv 2 \pmod 4$.

Since $|A \setminus B| = \frac{m}{2}$, Lemma 6.4.2 implies that $4G_2^{\bullet}$ admits an HOP (C_m)-decomposition. Furthermore, the (C_m)-decomposition $\mathcal{D}$ of $2G_1$ from [6] has the property that for any m-cycle $C \in \mathcal{D}$, there exists an m-cycle C' such that $C \oplus C' = 2C$. In other words, the authors essentially constructed a (C_m)-decomposition of G_1. Thus, by Lemma 3.3.3, there exists an HOP (C_m)-decomposition of $4G_1^{\bullet}$.

Observe that $4K_n^{\bullet} = 4G_1^{\bullet} \bigoplus 4G_2^{\bullet}$, and hence, by Lemma 3.3.1, the multigraph $4K_n^{\bullet}$ admits an HOP (C_m)-decomposition. ∎

6.4.3 The Case When m is Odd and n is Even

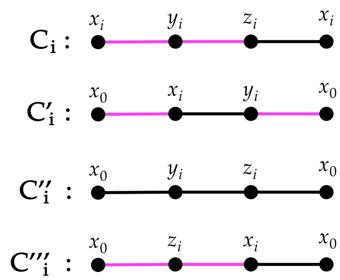
In this subsection, we aim to construct an HOP (C_m)-decomposition of $4K_n^{\bullet}$ from a (C_m)-decomposition of $2K_n$. Specifically, we focus on the case where m is odd and n is even.

We first address the case when $m = 3$, then we assume $m \geq 5$ for the remainder of this subsection.

Lemma 6.4.4. Let n be an even integer. There exists an HOP (C_3)-decomposition of $4K_n^{\bullet}$ whenever $3|n(n-1)$.

Proof. Since n is even and $3|n(n-1)$, we have $n = 3k$ for an even k, or $n = 3k+1$ for an odd k. First, assume $n = 3k$. In [36], it is shown that there exists an HOP C_3-factorization of $4K_{3k}^{\bullet}$. Observe that each C_3-factor consists of k disjoint (C_3)-subgraphs. Thus, $4K_{3k}^{\bullet}$ admits an HOP (C_3)-decomposition.

For $n = 3k+1$ with an odd k, we have $n \equiv 4 \pmod 6$. We use the (C_3)-decomposition of $2K_n$ provided in [9] by Bermond. Let $n = 6t+4$, and let x_0 be an arbitrary vertex of K_n. Obtain $K_n - x_0$ from K_n by deleting x_0 and all edges having x_0 as an endpoint. The graph $K_n - x_0$ has $6t+3$ vertices. Ray-Chaudhuri and Wilson in [41, Theorem 2.1.1], proved that there exists a decomposition of $K_n - x_0$ into $3t+1$ subgraphs (parallel classes), each consisting of $2t+1$ pairwise vertex-disjoint 3-cycles. Let $\mathcal{P}$ be one of these subgraphs, and label its 3-cycles as $C_i = x_i\ y_i\ z_i\ x_i$, for $1 \le i \le 2t+1$. Now, for each $i = 1, \dots, 2t+1$, construct 3-cycles $C_i' = x_0\ x_i\ y_i\ x_0$, $C_i'' = x_0\ y_i\ z_i\ x_0$, and $C_i''' = x_0\ z_i\ x_i\ x_0$ of K_n. Observe that C_i, C_i', C_i'', C_i''' jointly cover each of the edges x_0x_i, x_0y_i, x_0z_i, x_iy_i, x_iz_i, and y_iz_i exactly twice. We color the four cycles as follows, ensuring they cover exactly one pink and one black copy of the mentioned edges, and each cycle has an even number of pink edges.



Take two copies of each of the cycles C_i, C_i', C_i'', C_i''', and apply Lemma 3.3.5 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. We thus have an HOP (C_3)-decomposition of $4\mathcal{P}^{\bullet}$.

Take four copies of each of the 3-cycles in the (C_3)-decomposition of $K_n - \mathcal{P}$, and apply Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. Hence, we have an HOP (C_3)-decomposition of $4(K_n - \mathcal{P})^{\bullet}$. Furthermore, we have $4K_n^{\bullet} = 4(K_n - \mathcal{P})^{\bullet} \bigoplus 4\mathcal{P}^{\bullet}$. Therefore, by Lemma 3.3.1, $4K_n^{\bullet}$ admits an HOP (C_3)-decomposition. ∎

We now define the necessary terminology and tools from [6] to establish a key reduction step. Specifically, these reductions prepare us for the proof of Lemma 6.4.11, allowing us to reduce the problem of finding an HOP (C_m)-decomposition of $4K_n^{\bullet}$, for odd m and even $n > m$, to the case where n lies in the range $m < n < 3m$.

Definition 6.4.5. Let $X = \{x_0, x_1, \dots, x_{m-1}\}$ and $Y = \{y_0, y_1, \dots, y_{m-1}\}$ be two sets of size m that partition the vertex set of K_{2m}, and let $S, S' \subseteq \{1, 2 \dots, \frac{m-1}{2}\}$ and $T \subseteq \{0, 1, 2 \dots, m-1\}$. Then $K_{2m}\langle S, T, S' \rangle$ denotes a subgraph G of K_{2m} such that $G[X] \cong \mathrm{Circ}(m; S)$, and $G[Y] \cong \mathrm{Circ}(m; S')$, and $x_iy_{i+d} \in E(G)$ if and only if $d \in T$.

The following proposition is proved in [28] and [6].

Proposition 6.4.6. For odd $m \ge 3$, each of the following graphs has a (C_m)-decomposition.

(a) $K_{2m}\big\langle \emptyset, \{0, 1, \dots, m-2s-1\}, \emptyset \big\rangle \bowtie \overline{K_s}$ for odd s with $1 \le s \le \frac{m-1}{2}$;

(b) $K_{2m}\left\langle\{\frac{m-1}{2}\}, \{0, 1, \ldots, m-2s-2\}, \emptyset\right\rangle \bowtie \overline{K_s}$ for even s with $0 \leq s \leq \frac{m-3}{2}$;

(c) $K_{2m}\left\langle\{\frac{m-1}{2}\}, S \cup (S+1), \emptyset\right\rangle$ where $S \subseteq \{1, 3, 5 \ldots, m-2\}$; and

(d) $K_{2m}\langle A, \emptyset, B\rangle$ where each of A and B is either $\{1, 2, \ldots, \frac{m-1}{2}\}$ or $\{1, 2, \ldots, \frac{m-3}{2}\}$.

Lemma 6.4.7. Let $G_1 = K_{2m}\langle\emptyset, \{0\}, \emptyset\rangle \bowtie \overline{K_{\frac{m-1}{2}}}$ and $G_2 = K_{2m}\langle\{1\}, \emptyset, \emptyset\rangle \bowtie \overline{K_{\frac{m-1}{2}}}$. Then for odd $m \geq 3$, the multigraphs $4G_1^{\bullet}$ and $4G_2^{\bullet}$ admit an HOP (C_m)-decomposition.

Proof. Let $m = 2k+1$ for $k \in \mathbb{Z}$, and let $V(\overline{K_k}) = \{z_0, z_1, \ldots, z_{k-1}\}$. Let

$$\omega = (x_0\ x_1\ \ldots\ x_{m-1})(y_0\ y_1\ \ldots\ y_{m-1})$$

be a permutation that fixes all vertices in $\overline{K_k}$.

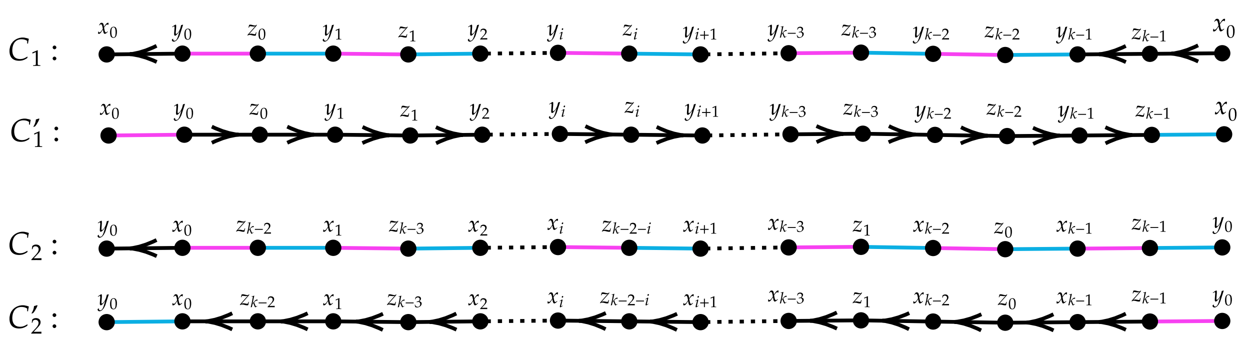
Consider the following m-cycles in G_1:

$$C_1 = x_0\ y_0\ z_0\ y_1\ z_1\ y_2\ z_2\ \ldots\ z_{k-2}\ y_{k-1}\ z_{k-1}\ x_0$$

$$C_2 = y_0\ x_0\ z_{k-2}\ x_1\ z_{k-3}\ x_2\ \ldots\ z_1\ x_{k-2}\ z_0\ x_{k-1}\ z_{k-1}\ y_0$$

The authors in [6, Proposition 5.4] showed that $\{\omega^i(C_1), \omega^i(C_2) : i = 0, 1, \ldots, m-1\}$ is a (C_m)-decomposition of $2G_1$.

Let $\omega_{\bullet}$ be the permutation induced by ω that preserves the color (and orientation) of the edges of $4G_1^{\bullet}$. Take two copies of each of C_1 and C_2 and color them as follows:



In C_1 and C_1', the path $y_i z_i y_{i+1}$, for $i = \{0, 1, \ldots, k-2\}$, consists of two edges from the same orbit of $\langle\omega\rangle$. Notice that the pink and the blue copies of these edges are covered by C_1 and the opposite black arcs are covered by C_1'. Similarly, in C_2 and C_2', the path $x_i z_{k-2-i} x_{i+1}$, for $i = \{0, 1, \ldots, k-2\}$, consists of two edges from the same orbit of $\langle\omega\rangle$. Observe that the pink and the blue copies of these edges are covered by C_2, and the opposite black arcs are covered by C_2'. The edge $x_0 y_0$ is covered by the four cycles, C_1, C_1', C_2, C_2'. The cycles C_1 and C_2 cover the opposite black arcs and C_1' and C_2' cover the pink and the blue copies of $x_0 y_0$, respectively. The edge $z_{k-1} x_0$ in C_1, C_1' and the edge $x_{k-1} z_{k-1}$ in C_2, C_2' are edges from the same orbit of $\langle\omega\rangle$. Similarly, the edge $y_{k-1} z_{k-1}$ in C_1, C_1' and the edge $z_{k-1} y_0$ in C_2, C_2' are edges from the same orbit of $\langle\omega\rangle$. Observe that C_1, C_1', C_2, C_2' jointly contain exactly one blue, one pink, and two black arcs of these edges. Additionally, all four cycles satisfy Condition **(C1)** of Definition 1.2.3. Hence,

$$\{\omega_\bullet^i(C_1), \omega_\bullet^i(C_1'), \omega_\bullet^i(C_2), \omega_\bullet^i(C_2') : i = 0, 1, \dots, m-1\}$$

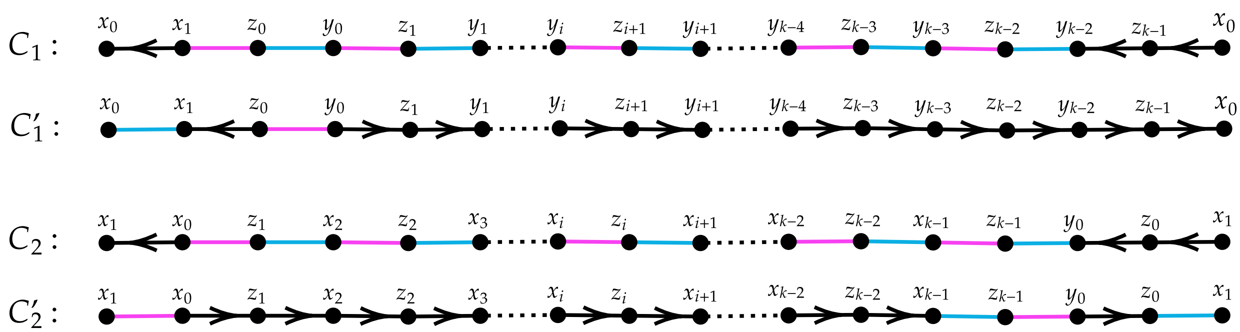
is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Now, consider the following m-cycles in G_2:

$$C_1 = x_0\ x_1\ z_0\ y_0\ z_1\ y_1\ z_2\ \dots\ z_{k-2}\ y_{k-2}\ z_{k-1}\ x_0$$

$$C_2 = x_1\ x_0\ z_1\ x_2\ z_2\ x_3\ \dots\ z_{k-3}\ x_{k-2}\ z_{k-2}\ x_{k-1}\ z_{k-1}\ y_0\ z_0\ x_1$$

In [6, Proposition 5.4], it is proved that $\{\omega^i(C_1), \omega^i(C_2) : i = 0, 1, \dots, m-1\}$ is a (C_m)-decomposition of $2G_2$. Take two copies of each cycle and color them as follows:



In C_1, C_1', the edges of the path $y_i z_{i+1} y_{i+1}$, for $i = \{0, 1, \dots, k-3\}$, are from the same orbit of $\langle\omega\rangle$. Cycle C_1 covers the pink and the blue copies of these edges, and C_1' covers the opposite black arcs. For $i = \{2, 3, \dots, k-2\}$, the edges of the path $x_i z_i x_{i+1}$ in C_2, C_2' are from the same orbit of $\langle\omega\rangle$. C_2 covers the pink and the blue copies of these edges and C_2' covers the opposite black arcs. Furthermore, the following edges are also from the same orbits of $\langle\omega\rangle$:

- the edge x_0x_1 in C_1, C_1' and the edge x_1x_0 in C_2, C_2';
- the edge x_1z_0 in C_1, C_1' and the edge z_0x_1 in C_2, C_2';
- the edge z_0y_0 in C_1, C_1' and the edge y_0z_0 in C_2, C_2';
- the edge $z_{k-1}x_0$ in C_1, C_1' and the edge $x_{k-1}z_{k-1}$ in C_2, C_2';
- the edge $y_{k-2}z_{k-1}$ in C_1, C_1' and the edge $z_{k-1}y_0$ in C_2, C_2'; and
- the edge x_0z_1 and the edge z_1x_2 in C_2, C_2'.

Observe that in each of the above cases, C_1, C_1', C_2, and C_2' jointly contain exactly one blue, one pink, and two opposite black arcs corresponding to these edges. Additionally, these cycles satisfy Condition **(C1)** of Definition 1.2.3. Therefore,

$$\{\omega_\bullet^i(C_1), \omega_\bullet^i(C_1'), \omega_\bullet^i(C_2), \omega_\bullet^i(C_2') : i = 0, 1, \dots, m-1\}$$

is an HOP (C_m)-decomposition of $4G_2^\bullet$.

∎

The next proposition is based on [6, Proposition 5.5].

Proposition 6.4.8. Let $m = 2k+1$, and let $t = qk + r$ for $0 \le r \le k-1$ and $1 \le q \le m + 2r - 1$. Let $G = K_{2m} \bowtie \overline{K_t}$. Then $4G^\bullet$ admits an HOP (C_m)-decomposition. In particular, $4G^\bullet$ admits an HOP (C_m)-decomposition whenever $\frac{m-1}{2} \le t \le \frac{(m-1)^2}{2}$.

Proof. We have $t = qk + r$. Based on the parity of r, we have the following two cases.

Case 1: r is odd. We partition the vertices in $\overline{K_t}$ into $B_0, B_1, \ldots, B_q$, where $|B_i| = k$, for $i \in \{1, 2, \ldots, q\}$ and $|B_0| = r$. First, consider $K_{2m} \bowtie \overline{K_r}$. We can decompose this graph into the following subgraphs:

(i) $G_1 = K_{2m}\big\langle \emptyset, \{0, 1, \ldots, m-2r-1\}, \emptyset \big\rangle \bowtie \overline{K_r}$,

(ii) $G_2 = K_{2m}\big\langle \emptyset, \{m-2r, m-2r+1 \ldots, m-1\}, \{k\} \big\rangle$, and

(iii) $G_3 = K_{2m}\big\langle \{1, 2, \ldots, k\}, \emptyset, \{1, 2, \ldots, k-1\} \big\rangle$.

By Proposition 6.4.6 **(c)** and **(d)**, we know that G_2 and G_3 admit a (C_m)-decomposition, respectively. The number of m-cycles in the (C_m)-decomposition of G_2 is $\frac{|E(G_2)|}{m} = \frac{2rm+m}{m} = 2r+1$, and the number of m-cycles in the (C_m)-decomposition of G_3 is $\frac{|E(G_3)|}{m} = \frac{km+(k-1)m}{m} = 2k-1$. Therefore, G_2 and G_3 jointly have $(2r+1)+(2k-1) = m+2r-1$ m-cycles in their (C_m)-decompositions. By assumption, this means there are at least q m-cycles. Let q of these m-cycles be $R_1, R_2, \ldots, R_q$, and if $q < m+2r-1$, then we have extra cycles $R_{q+1}, \ldots, R_{m+2r-1}$. Consider four copies of the extra cycles and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3.

Notice that none of the vertices in $B_1, \ldots, B_q$ are used. For $i \in \{1, 2, \ldots, q\}$, let R_i' be obtained from R_i by adjoining m isolated vertices and let $H_i = R_i' \bowtie \overline{K_{B_i}}$, where $\overline{K_{B_i}}$ is the empty graph with vertex set B_i. Observe that H_i is isomorphic to $K_{2m}\big\langle \{1\}, \emptyset, \emptyset \big\rangle \bowtie \overline{K_k}$. Applying Lemma 6.4.7, the subgraph $4H_i^\bullet$, for all $i \in \{1, 2, \ldots, q\}$, admits an HOP (C_m)-decomposition. Moreover, by Proposition 6.4.6 **(a)**, since r is odd, the subgraph G_1 admits a (C_m)-decomposition, and using Lemma 3.3.3, $4G_1^\bullet$ admits an HOP (C_m)-decomposition. Thus,

$$4G^\bullet = 4G_1^\bullet \bigoplus 4H_1^\bullet \bigoplus \cdots \bigoplus 4H_q^\bullet \bigoplus 4R_{q+1}^\bullet \bigoplus \cdots \bigoplus 4R_{m+2r-1}^\bullet.$$

By Lemma 3.3.1, the multigraph $4G^\bullet$ admits an HOP (C_m)-decomposition.

Case 2: r is even. Again we partition the vertex set of $\overline{K_t}$ as in the previous case. The graph $K_{2m} \bowtie \overline{K_r}$ can be decomposed into the following four subgraphs.

(i) $G_1 = K_{2m}\big\langle \{k\}, \{0, 1, \ldots, m-2r-2\}, \emptyset \big\rangle \bowtie \overline{K_r}$,

(ii) $G_2 = K_{2m}\big\langle \emptyset, \{m-2r, m-2r+1 \ldots, m-1\}, \{k\} \big\rangle$,

(iii) $G_3 = K_{2m}\big\langle\{1,2,\ldots,k-1\},\emptyset,\{1,2,\ldots,k-1\}\big\rangle$, and

(iv) $G_4 = K_{2m}\big\langle\emptyset,\{m-2r-1\},\emptyset\big\rangle$

Notice that since $q \geq 1$, there exists at least one part of cardinality k, namely B_1. Let H_1 be the union of the empty graph on the vertex set B_1 and G_4 with all possible edges between them. Observe that H_1 is isomorphic to $K_{2m}\big\langle\emptyset,\{m-2r-1\},\emptyset\big\rangle \bowtie \overline{K_k}$.

Furthermore, parts **(c)** and **(d)** of Proposition 6.4.6 show that G_2 and G_3 each admits a (C_m)-decomposition, respectively. The total number of m-cycles in the (C_m)-decomposition of G_2 and G_3 is $m+2r-2$. That means there are at least $q-1$ m-cycles. Let $q-1$ of these m-cycles be $R_2, R_3, \ldots, R_q$, and if $q < m+2r-1$, then we have extra cycles $R_{q+1}, \ldots, R_{m+2r-1}$. Consider four copies of the extra cycles and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3..

For $i \in \{2,3,\ldots,q\}$, let R_i' be obtained from R_i by adjoining m isolated vertices and let $H_i = R_i' \bowtie \overline{K_{B_i}}$, where $\overline{K_{B_i}}$ is the empty graph with vertex set B_i. Observe that H_i is isomorphic to $K_{2m}\big\langle\{1\},\emptyset,\emptyset\big\rangle \bowtie \overline{K_k}$. Applying Lemma 6.4.7, the subgraph $4H_1^\bullet$ and all subgraphs $4H_i^\bullet$, for $i \in \{2,\ldots,q\}$ admit an HOP (C_m)-decomposition. Moreover, by Proposition 6.4.6 **(b)**, since $r \leq \frac{m-3}{2}$, the subgraph G_1 admits a (C_m)-decomposition, and using Lemma 3.3.3, $4G_1^\bullet$ admits an HOP (C_m)-decomposition. Therefore,

$$4G^\bullet = 4G_1^\bullet \bigoplus 4H_1^\bullet \bigoplus \cdots \bigoplus 4H_q^\bullet \bigoplus 4R_{q+1}^\bullet \bigoplus \cdots \bigoplus 4R_{m+2r-1}^\bullet.$$

admits an HOP (C_m)-decomposition by Lemma 3.3.1.

We now prove that $4G^\bullet$ admits an HOP (C_m)-decomposition whenever $\frac{m-1}{2} \leq t \leq \frac{(m-1)^2}{2}$. Since $m = 2k+1$, we have $\frac{m-1}{2} = k$ and $\frac{(m-1)^2}{2} = 2k^2$, thus $k \leq t \leq 2k^2$. We show that if $k \leq t \leq 2k^2$, then $1 \leq q \leq m+2r-1$. We have $t = qk+r$ for $0 \leq r \leq k-1$. Since $t \geq k$ and $r \geq 0$, we have $q \geq 1$.

On the other hand, we have $t = qk+r \leq 2k^2$ and $r \geq 0$, thus $q \leq 2k = m-1 \leq m+2r-1$. Therefore, for t in the range $k \leq t \leq 2k^2$, with $t = qk+r$ and $0 \leq r \leq k-1$, we have $1 \leq q \leq m+2r-1$, as required.

∎

Having established the necessary terminology and tools analogous to those from [6], we are now ready to state the key reduction step.

Lemma 6.4.9. Let $m \geq 5$ be an odd integer. If $4K_n^\bullet$ admits an HOP (C_m)-decomposition for all even n such that $m < n < 3m$ and $m|n(n-1)$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition for all even $n > m$ such that $m|n(n-1)$.

Proof. Take any even $n' > m$ such that $m|n'(n'-1)$. We can write $n' = n + 2qm$, for integers $q \geq 0$ and $m < n < 3m$. Moreover, since $m|n'(n'-1)$, it follows that $m|n(n-1)$. Notice that we can partition the vertex set of $4K_{n'}^\bullet$ into q sets of $2m$ vertices and one set of n vertices. Depending on the value of q, there are three cases:

- $q = 0$. Then $n' = n$ and we obtain the multigraph $4K_n^\bullet$.
- $q = 1$. The multigraph $4K_{n'}^\bullet$ is an edge-disjoint union of $4K_{2m}^\bullet \bowtie \overline{4K_n^\bullet}$ and $4K_n^\bullet$.
- $q \geq 3$. The multigraph $4K_{n'}^\bullet$ is an edge-disjoint union of q copies of $4K_{2m}^\bullet \bowtie \overline{4K_n^\bullet}$, one copy of $4K_{q[2m]}^\bullet$, and one copy of $4K_n^\bullet$.
- $q = 2$. The multigraph $4K_{n'}^\bullet$ is an edge-disjoint union of one copy of $4K_{2m}^\bullet \bowtie \overline{4K_n^\bullet}$, one copy of $4K_n^\bullet$, and one copy of $4K_{2m}^\bullet \bowtie \overline{4K_{2m+n}^\bullet}$.

By [28], the complete multipartite graph with $q \geq 3$ parts of cardinality $2m$, that is, $K_{q[2m]}$, has a (C_m)-decomposition; applying Lemma 3.3.3, we can extend this decomposition to an HOP (C_m)-decomposition of $4K_{q[2m]}^\bullet$. Moreover, by supposition $4K_n^\bullet$ has an HOP (C_m)-decomposition. It remains to show that $4K_{2m}^\bullet \bowtie \overline{4K_n^\bullet}$ and $4K_{2m}^\bullet \bowtie \overline{4K_{2m+n}^\bullet}$ admit HOP (C_m)-decompositions.

- For $m \geq 13$, we have $n < 2m + n < 5m \leq \frac{(m-1)^2}{2}$; hence, by Proposition 6.4.8, each of $4K_{2m}^\bullet \bowtie \overline{4K_n^\bullet}$ and $4K_{2m}^\bullet \bowtie \overline{4K_{2m+n}^\bullet}$ admits an HOP (C_m)-decomposition.
- For $m \leq 11$, given that $m \mid n(n-1)$ and since m is odd and $\gcd(n, n-1) = 1$, it follows that $m \mid n$ or $m \mid (n-1)$. Since n is even and $m < n < 3m$, it follows that n must be either $m+1$ or $2m$. Consequently, we need an HOP (C_m)-decomposition for $4K_{2m}^\bullet \bowtie \overline{4K_{m+1}^\bullet}$, $4K_{2m}^\bullet \bowtie \overline{4K_{2m}^\bullet}$, $4K_{2m}^\bullet \bowtie \overline{4K_{3m+1}^\bullet}$, and $4K_{2m}^\bullet \bowtie \overline{4K_{4m}^\bullet}$. First, observe that the case $4K_{2m}^\bullet \bowtie \overline{4K_{4m}^\bullet}$arises from $n = 2m$ and $q = 2$. Thus, we actually seek an HOP (C_m)-decomposition of $4K_{6m}^\bullet$. The multigraph $4K_{6m}^\bullet$ is an edge-disjoint union of three copies of $4K_{2m}^\bullet$ and one copy of $4K_{3[2m]}^\bullet$, each of which admits an HOP (C_m)-decomposition. For the remaining cases, if $m \geq 9$, then $m+1 < 2m < 3m+1 \leq \frac{(m-1)^2}{2}$. Hence, each of these multigraphs admits an HOP (C_m)-decomposition by Proposition 6.4.8. Thus, we only need to verify the cases where $m \leq 7$.
- For $m = 7$, we need to find an HOP (C_m)-decomposition for $4K_{14}^\bullet \bowtie \overline{4K_8^\bullet}$, $4K_{14}^\bullet \bowtie \overline{4K_{14}^\bullet}$, and $4K_{14}^\bullet \bowtie \overline{4K_{22}^\bullet}$. It is clear that $m + 1 = 8 < 2m = 14 \leq \frac{(m-1)^2}{2} = 18$. Therefore, by Proposition 6.4.8, we see that $4K_{14}^\bullet \bowtie \overline{4K_8^\bullet}$ and $4K_{14}^\bullet \bowtie \overline{4K_{14}^\bullet}$ admit an HOP (C_7)-decomposition. For $4K_{14}^\bullet \bowtie \overline{4K_{22}^\bullet}$, using Proposition 6.4.8, we have that $m = 2 \cdot 3 + 1$, $k = 3$, and $t = 7 \cdot 3 + 1$; hence, $r = 1$ and $q = 7$. Thus, $q = 7 \leq 8 = m + 2r - 1$, and the conditions in Proposition 6.4.8 are satisfied. Thus $4K_{14}^\bullet \bowtie \overline{4K_{22}^\bullet}$ admits an HOP (C_7)-decomposition.
- For $m = 5$, we need to find an HOP (C_5)-decomposition for $4K_{10}^\bullet \bowtie \overline{4K_6^\bullet}$, $4K_{10}^\bullet \bowtie \overline{4K_{10}^\bullet}$, and $4K_{10}^\bullet \bowtie \overline{4K_{16}^\bullet}$.

  For $4K_{10}^\bullet \bowtie \overline{4K_6^\bullet}$, we have $6 \leq \frac{(m-1)^2}{2}$. By Proposition 6.4.8, it admits an HOP (C_5)-decomposition.

  For $4K_{10}^\bullet \bowtie \overline{4K_{10}^\bullet}$, partition the vertices in $\overline{4K_{10}^\bullet}$ into five sets of two vertices. Then, $4K_{10}^\bullet \bowtie \overline{4K_{10}^\bullet}$ can be viewed as an edge-disjoint union of five isomorphic copies of $4G^\bullet$ where $G = K_{10}\langle \emptyset, \{0\}, \emptyset \rangle \bowtie \overline{K_2}$ and two copies of $4K_5^\bullet$. Clearly, $4K_5^\bullet$ admits an HOP

(C_5)-decomposition, and by Lemma 6.4.7, there exists an HOP (C_5)-decomposition of $4G^\bullet$. Hence, $4K_{10}^\bullet \bowtie \overline{4K_{10}^\bullet}$ admits an HOP (C_5)-decomposition.

For $4K_{10}^\bullet \bowtie \overline{4K_{16}^\bullet}$, partition the vertices in $\overline{4K_{16}^\bullet}$ into eight sets of two vertices. Then, $4K_{10}^\bullet \bowtie \overline{4K_{16}^\bullet}$ can be viewed as an edge-disjoint union of five isomorphic copies of $4G_1^\bullet$ where $G_1 = K_{10}\langle \emptyset, \{0\}, \emptyset \rangle \bowtie \overline{K_2}$, three isomorphic copies of $4G_2^\bullet$ where $G_2 = K_{10}\langle \{1\}, \emptyset, \emptyset \rangle \bowtie \overline{K_2}$, and one $4C_5^\bullet$. By Lemma 6.4.7, there exist HOP (C_5)-decompositions of $4G_1^\bullet$ and $4G_2^\bullet$. Thus, $4K_{10}^\bullet \bowtie \overline{4K_{16}^\bullet}$ admits an HOP (C_5)-decomposition.

Therefore, in all cases, $4K_{n'}^\bullet$ admits an HOP (C_m)-decomposition. ∎

We now address the special cases $n = m+1$ and $n = 2m$.

Lemma 6.4.10. For any odd integer $m \geq 5$, each of the following multigraphs admits an HOP (C_m)-decomposition.

(i) $4K_{m+1}^\bullet$

(ii) $4K_{2m}^\bullet$

Proof. **(i)** Let $V(4K_{m+1}^\bullet) = \{x_i : i \in \mathbb{Z}_m\} \cup \{x_\infty\}$. Let $\rho_\bullet$ be the permutation on $E(4K_{m+1}^\bullet)$ that preserves the edge colors (and orientations), and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{m-1})$.

In [6, Lemma 5.7], Alspach et al. showed that $\mathcal{W} = \{C_1, \rho(C_1), \ldots, \rho^{m-1}(C_1), C_2\}$ is a (C_m)-decomposition of $2K_{m+1}$, where

$$C_1 = x_0\ x_1\ x_{-1}\ x_2\ \ldots\ x_{\frac{m-1}{4}}\ x_{-\frac{m-1}{4}}\ x_{\frac{m+7}{4}}\ x_{-\frac{m+3}{4}}\ \ldots\ x_{\frac{m+3}{2}}\ x_{\frac{m+1}{2}}\ x_\infty\ x_0,$$

if $m \equiv 1 \pmod 4$, and

$$C_1 = x_0\ x_1\ x_{-1}\ x_2\ \ldots\ x_{\frac{m-3}{4}}\ x_{-\frac{m-3}{4}}\ x_{\frac{m+5}{4}}\ x_{-\frac{m+1}{4}}\ x_{\frac{m+9}{4}}\ \ldots\ x_{\frac{m+3}{2}}\ x_{\frac{m+1}{2}}\ x_\infty\ x_0$$

if $m \equiv 3 \pmod 4$, and C_2 is

$$C_2 = x_0\ x_{\frac{m-1}{2}}\ x_{m-1}\ x_{\frac{m-3}{2}}\ \ldots\ x_{\frac{m+1}{2}}\ x_0.$$

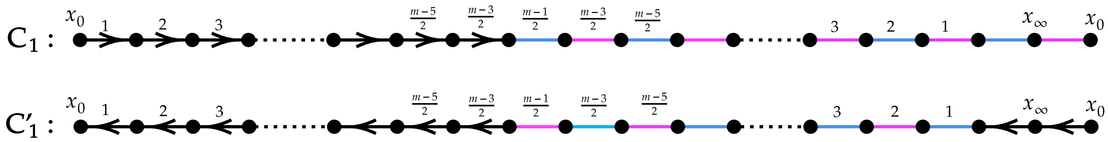
Observe that C_2 only covers difference $\frac{m-1}{2}$. Let C_2' be another copy of C_2. Color C_2 and C_2' black and orient them in opposite directions. By doing this, we cover the opposite black arcs corresponding to the difference $\frac{m-1}{2}$. This difference also occurs in C_1. When coloring C_1, it's important to ensure that pink and blue copies of the edges corresponding to the difference $\frac{m-1}{2}$ are covered.

Notice that C_1 covers each difference in $\{1, 2\ldots, \frac{m-3}{2}\}$ exacatly twice and difference $\frac{m-1}{2}$ exactly once in the following order:

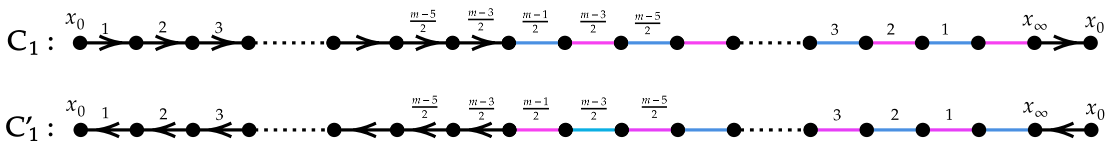
$$1, 2, 3, \ldots, \tfrac{m-3}{2}, \tfrac{m-1}{2}, \tfrac{m-3}{2}, \ldots, 3, 2, 1, \infty, \infty.$$

Let C_1' be another copy of C_1 and color them as follows:

- Assume $m \equiv 1 \pmod 4$.



- Assume $m \equiv 3 \pmod 4$.



Observe that C_1, C_1' and C_2, C_2' satisfy Condition **(C1)** of Definition 1.2.3. Also, C_1 and C_1' jointly contain exactly one edge from each orbit of $\langle \rho_\bullet \rangle$ corresponding to differences in $\{1, 2, \ldots, \frac{m-3}{2}, \infty\}$, as well as one edge from the pink and the blue orbits of $\langle \rho_\bullet \rangle$ corresponding to the difference $\frac{m-1}{2}$. The black orbit corresponding to the difference $\frac{m-1}{2}$ is covered by C_2 and C_2'. Thus

$$\mathcal{W}' = \left\{\rho_\bullet^i(C_1), \rho_\bullet^i(C_1') : i = 0, 1, \ldots, m-1\right\} \cup \{C_2, C_2'\}$$

is an HOP (C_m)-decomposition for $4K_{m+1}^\bullet$.

(ii) By Lemma 2.2.11(iii), we know that there exists an HOP C_m-factorization of $4K_{2m}^\bullet$ for any odd $m \geq 5$. Each C_m-factor consists of two (C_m)-subgraphs. Thus, $4K_{2m}^\bullet$ admits an HOP (C_m)-decomposition for any odd $m \geq 5$. ∎

We are now ready to prove the main lemma.

Lemma 6.4.11. Let m be odd, and let n be an even integer with $3 \leq m \leq n$. If $m|n(n-1)$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Proof. The case $m = 3$ is handled in Lemma 6.4.4. For $m \geq 5$, by Lemma 6.4.9, it is enough to prove the result for even n in the range $m < n < 3m$.

We begin by defining the required parameters and determining the number of starter central and starter peripheral cycles needed for the construction. For the peripheral cycles, we use the construction provided for $2K_n$ in [6] and extend it to starter peripheral cycles for $4K_n^\bullet$. However, for the central cycles, we introduce a new construction that differs from the one in [6].

The authors in [6, Theorem 5.1] split the proof into two cases: $m < n \leq 2m$ and $2m < n < 3m$. By Lemma 6.4.10, we know that $4K_{m+1}^\bullet$ and $4K_{2m}^\bullet$ admit HOP (C_m)-decompositions. Therefore, we may assume $m + 2 < n < 2m$ or $2m < n < 3m$.

Let $V(2K_n) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$, and let $\rho = (x_\infty)(x_0\ x_1\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho \rangle$ has the following orbits on the edge set of K_n:

- for each $s \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have an orbit $\{x_i x_{i+s} : i \in \mathbb{Z}_{n-1}\}$; and
- $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$.

For convenience, let $S = \{1, 2, \ldots, \frac{n-2}{2}, \infty\}$ be the set of all differences.

Let $n = 2^e a$, where a is odd, and let $m = a'b'$, where a' and b' both are odd with $a'|a$ and $b'|(n-1)$. Note that $b' \geq 3$ since $m \nmid n$.

To construct the peripheral cycles, we partition the $n-1$ vertices in $\{x_i : i \in \mathbb{Z}_{n-1}\}$ into b' segments, each containing $\ell = \frac{n-1}{b'}$ consecutive vertices. Note that each segment will contribute a' edges toward a peripheral m-cycle.

Now, we determine the required number of central and peripheral cycles for the decomposition. If $2K_n$ is (C_m)-decomposable, then the number of m-cycles in the decomposition is

$$\begin{aligned}
\frac{n(n-1)}{m} &= \frac{(n-1)(m+n-m)}{m} \\
&= \frac{(n-1)m + (n-1)(n-m)}{m} \\
&= (n-1) + \frac{(n-1)(n-m)}{a'b'} \\
&= (n-1) + \frac{n-1}{b'} \cdot \frac{n-m}{a'} \\
&= (n-1) + \ell \cdot \frac{n-m}{a'}.
\end{aligned}$$

Let $F = \frac{n-m}{a'} = \frac{2^e a - a'b'}{a'}$. Since $a' \mid a$, and a', b', a are all odd, and $e \geq 1$, it follows that F is an odd integer. The number of m-cycles can be expressed in the following form:

$$\frac{n(n-1)}{m} = (n-1) + \ell F.$$

This suggests constructing one starter central cycle and F starter peripheral cycles. The starter central cycle is rotated through all $n-1$ positions, while each of the F starter peripheral cycles is rotated through ℓ positions to generate all the cycles in the decomposition.

As mentioned earlier, the authors of [6] divided the proof into two cases: one for $m+2 < n < 2m$ and the other for $2m < n < 3m$. In the latter case, the subcases $(m, n) \in \{(15, 36), (15, 40)\}$ are treated separately in [6]. We address these special subcases at the end of the proof. Except for these special subcases, the construction of the peripheral cycles follows the same approach in both cases.

The authors of [6] first construct a starter peripheral cycle C_0, which covers a set of differences $A = \{s_1, s_2, \ldots, s_{a'}\} \subset S$, where $s_1 < s_2 < \ldots < s_{a'}$. If $F = 1$, the remaining differences can be covered by the central cycles. However, if $F \geq 3$, additional families of peripheral cycles are needed.

For $F \geq 3$, let $c = \frac{F-1}{2}$. The authors of [6] construct c starter peripheral cycles, called $C_1, C_2, \ldots, C_c$ such that the families generated by these cycles, that is, $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 1, 2, \ldots, c\}$, jointly cover a set of differences denoted as:

$$B = \{s_1^i, s_2^i, \ldots, s_{a'}^i \mid i = 1, 2, \ldots, c\} \subset S \setminus A$$

where $s_1^i < \ldots < s_{a'}^i$, for $i = 1, 2, \ldots, c$.

Later, they duplicate these cycles to obtain $2c$ m-cycles. The families generated by these $2c$ cycles form a (C_m)-decomposition of $2G_1$, where $G_1 = \mathrm{Circ}(n-1; B)$.

For our purpose, we take four copies of each C_i, for $i = 1, 2, \ldots, c$, and apply Lemma 3.3.3 to color them so that they satisfy the HOP property. By doing so, we generate $4c$ starter peripheral cycles for $4G_1^\bullet$, which we denote by $C_1', C_2', \ldots, C_{4c}'$.

It is now easy to see that

$$\{C_i', \rho_\bullet(C_i'), \ldots, \rho_\bullet^{\ell-1}(C_i') \mid i = 1, \ldots, 4c\}$$

is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Notice that the cycle C_0 covers the differences in $A = \{s_1, s_2, \ldots, s_{a'}\}$. Let C_0' be another copy of C_0. Color both C_0 and C_0' black and orient them in opposite directions. The families of cycles generated by these two cycles cover the black orbits corresponding to the differences in A. The differences in A will also appear in the starter central cycle C, to be constructed below, where we need to ensure that the pink and blue orbits corresponding to the differences in A are covered.

Now, we focus on constructing the starter central cycle C. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus (A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 < a_2 < \ldots < a_t \leq \frac{n-2}{2}$ are positive integers and $t = \frac{n-2}{2} - \frac{F+1}{2}a'$.

Consider the following sets of differences:

- $W_1 = \{s_1, s_2, \ldots, s_{a'}, a_1, a_2, \ldots, a_{t-1}, a_t\}$
- $W_2 = \{a_1, a_2, \ldots, a_{t-1}, a_t\}$

Here, W_1 contains all the differences from the sets A and $\mathscr{L}$ except for ∞, and its size is $|W_1| = a' + t$. Since the differences in W_1 are pairwise distinct, we may relabel them as

$$W_1 = \{b_1, b_2, \ldots, b_{a'+t}\} \quad \text{with} \quad b_1 < b_2 < \cdots < b_{a'+t}.$$

By Corollary 6.3.4, there exists a path, call it T, of length $2t + a'$ that starts at x_0 and covers the differences in the following order:

$$T : b_1, b_2, \ldots, b_{a'+t}, a_t, a_{t-1}, \ldots, a_2, a_1.$$

Notice that the length of the path T is $2t + a'$, and

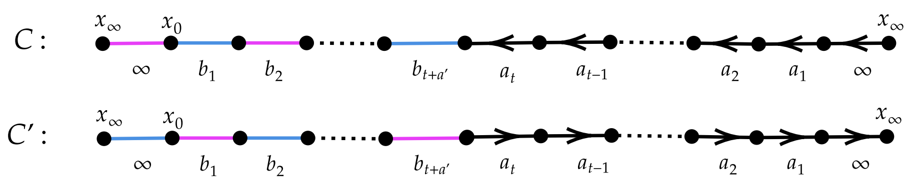
$$2t + a' = 2\left(\frac{n-2}{2} - \frac{F+1}{2}a'\right) + a'$$

$$
\begin{aligned}
&= n-2-Fa' \\
\text{(substitute } F=\frac{n-m}{a'}\text{)} \quad &= n-2-n+m \\
&= m-2.
\end{aligned}
$$

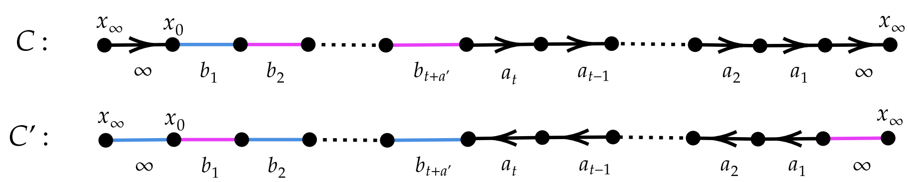
Let $C = x_\infty T x_\infty$. Observe that C covers each difference in $\mathscr{L}$ exactly twice and each difference in A exactly once. Let C' be another copy of C.

In C, color the first $t+a'$ edges with differences $b_1, b_2, \ldots, b_{a'+t}$ alternately pink and blue, starting with b_1 in blue. The remaining coloring depends on whether t is even or odd:

- Assume t is even. Since a' is odd, $t+a'$ is also odd, so the edge with difference $b_{a'+t}$ is colored blue. In C', color the same $t+a'$ edges alternately pink and blue, but in the opposite way of C. Then, color the remaining edges as shown below.



- Assume t is odd. Since a' is odd, $t+a'$ is even, so the edge with difference $b_{a'+t}$ is colored pink. In C', color the same $t+a'$ edges alternately pink and blue, but in the opposite way of C. Then, color the remaining edges as shown below.



Observe that the opposite black arcs corresponding to each difference in A are covered by C_0 and C_0', while the pink and the blue edges corresponding to each difference in A are covered by the starter central cycles C and C'.

Moreover, the starter central cycles jointly contain exactly one blue edge, one pink edge, and two black arcs corresponding to each difference in $\mathscr{L}$.

Therefore,

$$\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, n-2\} \cup \{\rho_\bullet^i(C_0), \rho_\bullet^i(C_0') \mid i = 0, 1, \ldots, \ell-1\}$$

is an HOP (C_m)-decomposition of $4G_2^{\bullet}$, where

$$G_2 = \mathrm{Circ}(n-1; A \cup (\mathscr{L} - \{\infty\}) \bowtie K_1.$$

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

To complete the proof of Lemma 6.4.11, we prove that HOP (C_{15})-decompositions of $4K_{36}^\bullet$ and $4K_{40}^\bullet$ exist.

As before, we construct starter peripheral and starter central cycles. We begin with the starter peripheral cycles for $2K_{36}$ and $2K_{40}$, as provided in [6], and extend them to obtain the starter peripheral cycles for $4K_{36}^\bullet$ and $4K_{40}^\bullet$. Finally, we describe the construction of the starter central cycles.

We start with $n = 36$. Since $n = 2^e{\cdot}a = 2^2{\cdot}9$, we have $a = 9$. Also, $m = a'b' = 3{\cdot}5$, where $a' \mid a$ and $b' \mid (n-1)$. Thus, $a' = 3$ and $b' = 5$. The set of differences is $S = \{1, 2, \ldots, 17, \infty\}$.

We partition the $n-1 = 35$ vertices into $b' = 5$ segments, where each segment contains $\ell = \frac{n-1}{b'} = 7$ consecutive vertices. Each segment contributes $a' = 3$ edges toward a 15-cycle.

The number of families of peripheral cycles is $F = \frac{n-m}{a'} = 7$. Consider the following paths from [6]:

- $P_0 = x_0\ x_5\ x_{-1}\ x_{14}$;
- $P_1 = x_0\ x_1\ x_{-1}\ x_7$;
- $P_2 = x_0\ x_3\ x_{-1}\ x_{-14}$;
- $P_3 = x_0\ x_9\ x_{-2}\ x_{-14}$.

For each $i = 0, 1, 2, 3$, we have

$$C_i = P_i \cup \rho^\ell(P_i) \cup \rho^{2\ell}(P_i) \cup \rho^{3\ell}(P_i) \cup \rho^{4\ell}(P_i),$$

which is a cycle of length 15.

The families generated by the peripheral cycles C_1, C_2, C_3 jointly cover differences in $B = \{1, 2, 3, 4, 8, 9, 11, 12, 13\}$. Let $G_1 = \mathrm{Circ}(35; B)$. Taking four copies of each of C_1, C_2, C_3, we obtain 12 cycles. Applying Lemma 3.3.3, we color these cycles to satisfy Condition **(C1)** from Definition 1.2.3, and denote them $C_1', C_2', \ldots, C_{12}'$. Observe that

$$\{\rho_\bullet^i(C_1'), \ldots, \rho_\bullet^i(C_{12}') \mid i = 0, 1, \ldots, 6\}$$

is an HOP (C_{15})-decomposition of $4G_1^\bullet$.

Note that the family generated by the starter peripheral cycle C_0 covers the differences in the set $A = \{5, 6, 15\}$.

Next, we construct the starter central cycle. Let $\mathscr{L}$ be the set of unused differences:

$$\mathscr{L} = S \setminus (A \cup B) = \{7, 10, 14, 16, 17, \infty\}.$$

Consider the following sets of differences:

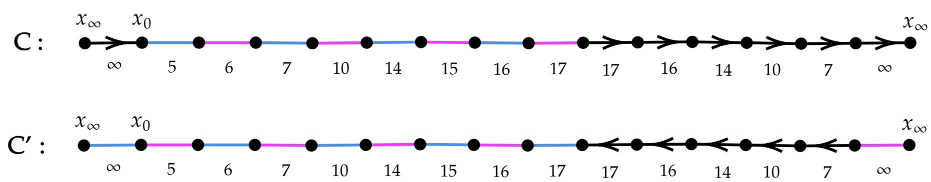
- $W_1 = \{5, 6, 7, 10, 14, 15, 16, 17\}$

- $W_2 = \{7, 10, 14, 16, 17\}$

Using Corollary 6.3.4, we construct a path T starting from x_0, covering differences in the following order:

$$5, 6, 7, 10, 14, 15, 16, 17, 17, 16, 14, 10, 7.$$

Let $C = x_\infty T x_\infty$. Notice that C is a cycle of length 15. Take another copy of C, call it C'. We color them as follows:



Now, let C_0' be another copy of C_0, color both C_0 and C_0' black, and orient them in the opposite directions. We see that

$$\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, 34\} \cup \{\rho_\bullet^i(C_0), \rho_\bullet^i(C_0') \mid i = 0, 1, \ldots, 6\},$$

is an HOP (C_{15})-decomposition of $2G_2^\bullet$, where $G_2 = \mathrm{Circ}(35; A \cup (\mathscr{L} - \{\infty\}) \bowtie K_1$.

Since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_{15})-decomposition, by Lemma 3.3.1, it follows that $4K_{36}^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$ also admits an HOP (C_{15})-decomposition.

For $n = 40$, we have $a = 5$, $a' = 5$, $b' = 3$, and $\ell = 13$. The set of differences is $S = \{1, 2, \ldots, 19, \infty\}$.

Consider the following paths from [6]:

- $P_0 = x_0\ x_9\ x_{-1}\ x_{10}\ x_{-2}\ x_{13}$;
- $P_1 = x_0\ x_1\ x_{-3}\ x_2\ x_{-5}\ x_{13}$;
- $P_2 = x_0\ x_2\ x_{-1}\ x_5\ x_{-3}\ x_{13}$.

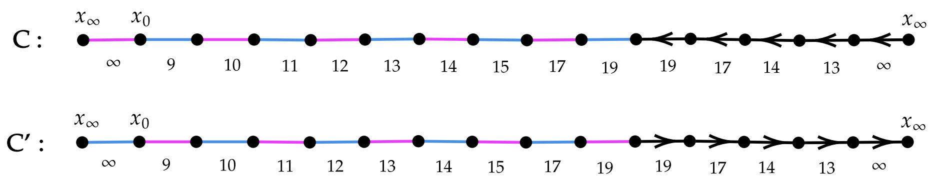
For $i = 0, 1, 2$, we see that $C_i = P_i \cup \rho^\ell(P_i) \cup \rho^{2\ell}(P_i)$ is a cycle of length 15. The families generated by the peripheral cycles C_1 and C_2 jointly cover the differences in $B = \{1, 2, 3, 4, 5, 6, 7, 8, 16, 18\}$. Following the previous approach, we take four copies of each cycle and use Lemma 3.3.3 to color them appropriately, resulting in an HOP (C_{15})-decomposition of $4G_1^\bullet$, where $G_1 = \mathrm{Circ}(39; B)$.

Note that the family generated by starter peripheral cycle C_0 covers the differences in the set $A = \{9, 10, 11, 12, 15\}$.

The starter central cycle uses the remaining differences, namely $\{13, 14, 17, 19, \infty\}$. Let $C = x_\infty T x_\infty$, where T is a path of length 13 constructed using Corollary 6.3.4. The path T starts at x_0 and covers the differences in the following order:

$$9, 10, 11, 12, 13, 14, 15, 17, 19, 19, 17, 14, 13.$$

Let C' be another copy of the starter central cycle C. We color them as follows:



Now, let C_0' be another copy of C_0, color them black, and orient them in the opposite directions. Observe that

$$\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, 38\} \cup \{\rho_\bullet^i(C_0), \rho_\bullet^i(C_0') \mid i = 0, 1, \ldots, 12\}$$

is an HOP (C_{15})-decomposition of $4G_2^\bullet$, where $G_2 = \mathrm{Circ}(39; (\mathscr{L} \cup A) - \{\infty\}) \bowtie K_1$.

Since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_{15})-decomposition, by Lemma 3.3.1, it follows that $4K_{40}^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$ also admits an HOP (C_{15})-decomposition.

∎

6.5 HOP (C_m)-decomposition of $4K_n^\bullet$

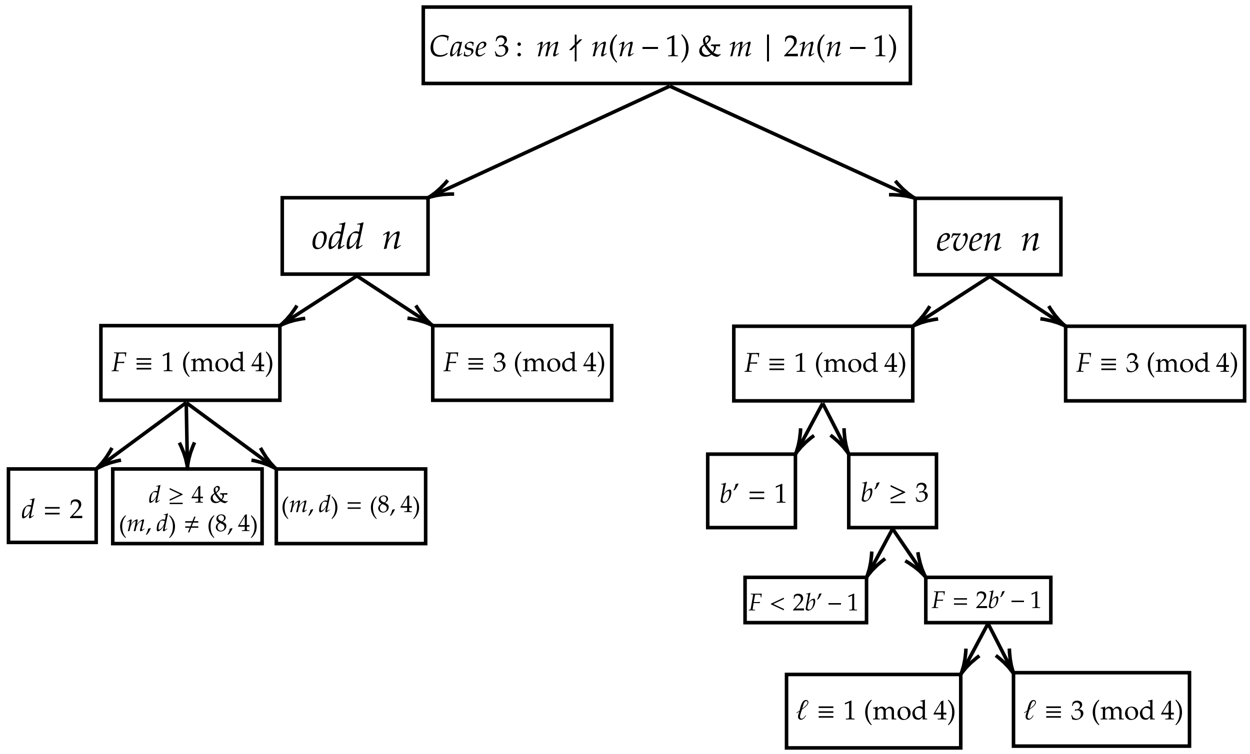
In this section, we address the final case of our main result in Theorem 6.1.1, where $m \mid 2n(n-1)$ but $m \nmid n(n-1)$. The divisibility constraints imply that $m \equiv 0 \pmod 4$ and $2n(n-1) \equiv m \pmod{2m}$. Our goal is to show that $4K_n^\bullet$ admits an HOP (C_m)-decomposition under these conditions. To handle this case, we study the multigraph $4K_n^\bullet$ directly. We split the proof into two cases based on whether n is odd or even.

In both cases, the overall strategy is similar. We construct a set of starter peripheral cycles and starter central cycles, and use appropriate permutations to build the required decomposition. The number of peripheral cycles F will be shown to be odd, which leads to a further division of the proof into two subcases depending on whether $F \equiv 1 \pmod 4$ or $F \equiv 3 \pmod 4$.

When $F \equiv 3 \pmod 4$, we first construct $\frac{F-3}{4} + 1$ starter peripheral cycles for K_n. By taking four copies of each of $\frac{F-3}{4}$ cycles and coloring them such that they satisfy HOP properties, we obtain $F - 3$ peripheral cycles in $4K_n^\bullet$. Then, we take three copies of the one remaining cycle to complete all F starter peripheral cycles for $4K_n^\bullet$. Finally, we construct starter central cycles.

Similarly, when $F \equiv 1 \pmod 4$, we first construct $\frac{F-1}{4} + 1$ starter peripheral cycles for K_n, and use four copies of each of the $\frac{F-1}{4}$ cycles, together with the one remaining cycle, to properly color and generate all F starter peripheral cycles for $4K_n^\bullet$. As in the previous case, starter central cycles are constructed in the final step. In each case, some special cases arise and are handled separately.

The proof is structured into cases and subcases as outlined below.



The odd case, along with all its subcases, is handled in Lemma 6.5.1, while the even case and its subcases are examined in Lemma 6.5.2. Together, these two lemmas establish the final case in the proof of Theorem 6.1.1.

6.5.1 The Case When n is Odd

Lemma 6.5.1. Let $m \equiv 0 \pmod 4$, and let n be an odd integer such that $4 \leq m < n$. If $2n(n-1) \equiv m \pmod{2m}$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Proof. By Lemma 6.3.8, it suffices to prove this result for odd n in the range $m < n < 2m$.

As before, the approach is to construct starter central and starter peripheral cycles. We first outline the parameters, then explain how many of each type of cycle are needed.

Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$, and let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $s \in \{1, 2, \ldots, \frac{n-3}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+s} : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_{i+\frac{n-1}{2}} : i = 0, 1, \ldots, \frac{n-3}{2}\}$;
- for each $s \in \{1, 2, \ldots, n-2\}$, we have a black orbit $\{(x_i, x_{i+s}) : i \in \mathbb{Z}_{n-1}\}$;

- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

For convenience, let $S = \{1, 2, \ldots, \frac{n-3}{2}, \frac{n-1}{2}, \infty\}$ be the set of differences. Notice that $\frac{n-1}{2}$ is the diameter difference.

Let $n-1 = 2^a p$ and $m = 2^b q$, where p and q are both odd. Observe that $b \geq 2$ since $m \equiv 0 \pmod 4$ and $a \geq 1$ since $n-1$ is even. First, we show that $b = a+1$. We know that $2n(n-1) \equiv m \pmod{2m}$. Consequently, $m \mid 2n(n-1)$ but $m \nmid n(n-1)$. Since $m \mid 2n(n-1)$, we have $2^b q \mid 2n \cdot 2^a p = 2^{a+1} np$. Given that n, p, and q are odd, we have $q \mid np$. Thus, 2^b must divide 2^{a+1}, implying that $b \leq a+1$. On the other hand, we have $m \nmid n(n-1)$, that is, $2^b q \nmid n \cdot 2^a p$. It follows that 2^b cannot divide 2^a. Thus, $b > a$, which implies $b \geq a+1$. Therefore, we conclude that $b = a+1$, and write $m = 2^{a+1} q$.

Let $d = \gcd(m, n-1)$. Since $m = 2^{a+1} q$ and $n-1 = 2^a p$, it follows that $2^a \mid d$. Since $a \geq 1$, d is even. Now, let $m = m'd$ and $n-1 = n'd$. We see that $\gcd(n', m') = 1$, n' is odd, and $m' \equiv 2 \pmod 4$.

Let $r = n - m$. Since n is odd and m is even, we have that r is odd. If $r = 1$, then $n = m+1$, which implies that $m \mid n(n-1)$, leading to a contradiction. Thus,

$$r > 1. \tag{6.5.1}$$

Now, partition the $n-1$ vertices in $\{x_i : i \in \mathbb{Z}_{n-1}\}$ into d segments, each containing n' consecutive vertices. Each segment will contribute m' edges toward a peripheral m-cycle.

Now, we explore the required number of central and peripheral cycles for the decomposition. If $4K_n^\bullet$ is (C_m)-decomposable, then the number of m-cycles in the decomposition is

$$\begin{aligned}
\frac{2n(n-1)}{m} = \frac{2(m+r)(n-1)}{m} &= \frac{(2m+2r)(n-1)}{m} \\
&= \frac{2m(n-1)}{m} + \frac{2r(n-1)}{m} \\
&= 2(n-1) + \frac{2rn'd}{m'd} \\
&= 4 \cdot \frac{n-1}{2} + \frac{2r}{m'} \cdot n'
\end{aligned}$$

Since $\gcd(n', m') = 1$, r is odd, and $m' \equiv 2 \pmod 4$, it is easy to see that $\frac{2r}{m'}$ is an odd integer. Let $F = \frac{2r}{m'}$. The number of m-cycles can be expressed in the following form:

$$\frac{2n(n-1)}{m} = 4 \cdot \frac{n-1}{2} + F \cdot n'$$

This suggests constructing four starter central cycles and F starter peripheral cycles. Here, we consider four starter central cycles due to the presence of a diameter difference. The four starter central cycles will be rotated through $\frac{n-1}{2}$ positions, while each of the F starter peripheral cycles will be rotated through n' positions to generate all cycles in the decomposition.

Before proceeding further, we establish some inequalities to be used later. From (6.5.1), we know that $r > 1$; it follows that $m < n - 1$. Additionally, since $n - 1 < 2m$, we have $m < n - 1 < 2m$, and thus $m'd < n'd < 2m'd$. Dividing each part by d, we get:

$$m' < n' < 2m'. \tag{6.5.2}$$

Next, we know that $n < 2m$, which implies $n - m < m$ and thus $2(n - m) < 2m$. Dividing both sides by m' gives $\frac{2r}{m'} < \frac{2m}{m'}$. Therefore, $F < 2d$, which implies

$$F \leq 2d - 1. \tag{6.5.3}$$

Moreover, since d is even, if $F \equiv 1 \pmod 4$, then we have

$$F \leq 2d - 3. \tag{6.5.4}$$

Now, we show that for all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 2$, except for $(m', d) = (6, 2)$, the following inequality holds:

$$\frac{m'+2}{m'-2} < d \tag{6.5.5}$$

Since $m' > 2$ and $m' \equiv 2 \pmod 4$, we have $m' = 4k + 2$, for some integer $k \geq 1$. Therefore,

$$\frac{m'+2}{m'-2} = \frac{4k+4}{4k} = 1 + \frac{1}{k}.$$

It is clear that $1 + \frac{1}{k} < d$ except for $k = 1$ and $d = 2$. Therefore, the inequality in (6.5.5) holds for all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 2$, except for $(m', d) = (6, 2)$.

When $F = 2d-1$, we have $\frac{2(n-m)}{m'} = 2d-1$. Thus, we obtain $n = \frac{2d-1}{2}m' + m = \frac{4d-1}{2}m'$. Using this, we calculate n' as

$$n' = \frac{n-1}{d} = 2m' - \frac{m'+2}{2d}. \tag{6.5.6}$$

Using the inequality in (6.5.5), we have:

$$\begin{aligned} \frac{m'+2}{d} &< m' - 2 \\ 2m' - \frac{m'+2}{2d} &> 2m' - \frac{m'-2}{2} \\ \text{(Using the equation in (6.5.6))} \qquad n' &> \frac{3m'+2}{2}. \end{aligned}$$

Thus, for $F = 2d - 1$, all $m' > 2$ with $m' \equiv 2 \pmod 4$, and even $d \geq 2$, except for $(m', d) = (6, 2)$, we have

$$n' > \frac{3m' + 2}{2} \tag{6.5.7}$$

Next, we show that for all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 4$, except for $\{(m', d)\} \in \{(6, 4), (10, 4)\}$, the following inequality holds:

$$\frac{3m' + 2}{m' - 2} < d. \tag{6.5.8}$$

Since $m' > 2$ and $m' \equiv 2 \pmod 4$, we have $m' = 4k + 2$, for some integer $k \geq 1$. Therefore,

$$\frac{3m' + 2}{m' - 2} = \frac{12k + 8}{4k} = 3 + \frac{2}{k}.$$

It is clear that $3 + \frac{2}{k} < d$ except for $k = 1, 2$ and $d = 4$. Therefore, the inequality in (6.5.8) holds for all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 4$, except for $\{(m', d)\} \in \{(6, 4), (10, 4)\}$.

When $F = 2d - 3$, we have $\frac{2(n-m)}{m'} = 2d - 3$. Thus, we obtain $n = \frac{2d-3}{2}m' + m = \frac{4d-3}{2}m'$. Using this, we calculate n' as

$$n' = \frac{n - 1}{d} = 2m' - \frac{3m' + 2}{2d}. \tag{6.5.9}$$

Using the inequality in (6.5.8), we have:

$$\begin{aligned} \frac{3m' + 2}{d} &< m' - 2, \\ 2m' - \frac{3m' + 2}{2d} &> 2m' - \frac{m' - 2}{2}, \\ \text{(Using the formula for } n' \text{ in (6.5.9))} \qquad n' &> \frac{3m' + 2}{2}. \end{aligned}$$

Thus, for $F = 2d - 3$, all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 4$, except for $\{(m', d)\} \in \{(6, 4), (10, 4)\}$, we have

$$n' > \frac{3m' + 2}{2} \tag{6.5.10}$$

With the necessary inequalities established, we now begin the main part of the proof.

We have seen that F is odd, so we have two cases to consider: one where $F \equiv 1 \pmod 4$ and the other where $F \equiv 3 \pmod 4$. We begin with the simpler case.

Case 1: $F \equiv 3 \pmod 4$.

The approach is to construct $\frac{F-3}{4} + 1$ starter peripheral cycles for K_n. Then, take four copies of each of the $\frac{F-3}{4}$ cycles and color them appropriately using Lemma 3.3.3. This process results in $F - 3$ starter peripheral cycles in $4K_n^\bullet$. We will then be left with one starter peripheral cycle in K_n; we take three copies of it to generate the remaining three starter peripheral cycles in $4K_n^\bullet$.

We begin by constructing the starter peripheral cycles. We ensure that the diameter difference $\frac{n-1}{2}$ does not occur in any of the peripheral cycles. We construct the path P_i as follows.

- For $i = 0$, define

$$P_0 = x_0\ x_1\ x_{-1}\ x_2\ x_{-2} \dots x_{\frac{m'-2}{2}}\ x_{-\frac{m'-2}{2}}\ x_{\frac{m'}{2}}\ x_{-n'}.$$

The differences covered by P_0 are:

$$1, 2, 3, 4, \dots, m'-2, m'-1, n' + \frac{m'}{2}. \tag{6.5.11}$$

- For $i = 1, 2, \dots, \lceil \frac{F-3}{8} \rceil$, define the path P_{2i-1} as follows.

$$\begin{aligned} P_{2i-1} = x_0\ x_{(2i-1)n'+1}\ x_{-1}\ x_{(2i-1)n'+2}\ x_{-2}\ \dots\ x_{-\frac{m'-6}{4}}\ x_{(2i-1)n'+\frac{m'-2}{4}}\ x_{-\frac{m'-2}{4}} \\ x_{(2i-1)n'+\frac{m'+6}{4}}\ x_{-\frac{m'+2}{4}}\ \dots\ x_{-\frac{m'-2}{2}}\ x_{(2i-1)n'+\frac{m'+2}{2}}\ x_{-n'}. \end{aligned}$$

The differences covered by P_{2i-1} are:

$$\begin{aligned} &(2i-1)n'+1,\ (2i-1)n'+2,\ (2i-1)n'+3, \dots, (2i-1)n' + \tfrac{m'-4}{2}, \\ &(2i-1)n' + \tfrac{m'-2}{2},\ (2i-1)n' + \tfrac{m'+2}{2},\ (2i-1)n' + \tfrac{m'+4}{2}, \dots, \\ &(2i-1)n' + m',\ 2in' + \tfrac{m'+2}{2}. \end{aligned} \tag{6.5.12}$$

- For $i = 1, 2, \dots, \lfloor \frac{F-3}{8} \rfloor$, define the path P_{2i} as follows.

$$\begin{aligned} P_{2i} = x_0\ x_{2in'+1}\ x_{-1}\ x_{2in'+2}\ x_{-2}\ \dots\ x_{2in'+\frac{m'-2}{4}}\ x_{-\frac{m'-2}{4}} \\ x_{2in'+\frac{m'+2}{4}}\ x_{-\frac{m'+6}{4}}\ x_{2in'+\frac{m'+6}{4}}\ \dots\ x_{-\frac{m'}{2}}\ x_{2in'+\frac{m'}{2}}\ x_{-n'}. \end{aligned}$$

The differences covered by P_{2i} are:

$$\begin{aligned} &2in'+1,\ 2in'+2,\ 2in'+3, \dots, 2in' + \tfrac{m'-2}{2}, 2in' + \tfrac{m'}{2}, \\ &2in' + \tfrac{m'+4}{2}, 2in' + \tfrac{m'+6}{2} \dots, 2in' + m',\ (2i+1)n' + \tfrac{m'}{2}. \end{aligned} \tag{6.5.13}$$

Since $m' < n'$ by (6.5.2), the paths $\rho^j(P_i)$, for $j = 0, n', 2n', \dots, (d-1)n'$, are pairwise vertex-disjoint except at their endpoints. Thus, each path P_i generates an m-cycle via rotation:

$$C_i = P_i \cup \rho^{n'}(P_i) \cup \dots \cup \rho^{(d-1)n'}(P_i).$$

Note that the differences $(2i+1)n' + \frac{m'}{2}$ occur at the ends of the paths P_{2i}, and these differences are carefully avoided in the paths P_{2i-1}. Similarly, the differences $2in' + \frac{m'+2}{2}$ appear at the ends of the paths P_{2i-1} and are carefully avoided in the paths P_{2i}.

First, assume $d = 2$. By (6.5.3), we have $F \leq 2d-1$, which implies that $F = 3$. Thus, there is only one path, P_0, and it covers the following differences:

$$1, 2, 3, \ldots, m'-2, m'-1, n' - \frac{m'}{2}.$$

By (6.5.7), we have $n' > \frac{3m'+2}{2}$, which implies $n' - \frac{m'}{2} > m'+1$. Therefore,

$$m'-1 < m' < n' - \frac{m'}{2} < n' + \frac{m'}{2}. \tag{6.5.14}$$

We know that inequality (6.5.7) holds for all $m' > 2$ with $m' \equiv 2 \pmod 4$, and even $d \geq 2$, except for the case $(m', d) = (6, 2)$.

Note that for $(m', d) = (6, 2)$, we have $n' = 10$ by (6.5.6). However, since n' must be odd, this case does not arise.

For the case $m' = 2$, we use the inequality in (6.5.2), which gives $m' < n' < 2m'$. Substituting $m' = 2$, we obtain $n' = 3$. Therefore, by (6.5.6), we have $d = 2$. In this case, $m = 4$, $n = 7$, and $F = 3$. There is only one path, $P_0 = x_0\, x_1\, x_3$, and the covered differences are 1 and 2, which are distinct.

Now, assume $d \geq 4$. The paths P_i cover the following differences:

- For $i = 0$:
$$1, 2, \ldots, m'-2, m'-1$$
- For $i = 1, 2, \ldots, \frac{F-3}{4}$:
$$in'+1,\ in'+2,\ \ldots,\ in'+m'$$
- One additional difference, depending on the parity of $\frac{F-3}{4}$, which is either
$$\left(\frac{F-3}{4}+1\right)n' + \frac{m'}{2} \quad \text{or} \quad \left(\frac{F-3}{4}+1\right)n' + \frac{m'+2}{2}.$$

To show that the above differences are distinct, it suffices to prove the following inequalities.

(i) $\frac{F-3}{4}n' + m' < \frac{n-1}{2}$

By (6.5.3), we have $F \leq 2d-1$. This implies $\frac{F-3}{4} \leq \frac{d-2}{2}$. Moreover, by (6.5.2), we know that $m' < n'$; thus,

$$\frac{F-3}{4}n' + m' < \frac{d-2}{2}n' + n' = \frac{dn'}{2} = \frac{n-1}{2}.$$

(ii) $\frac{F+1}{4}n' + \frac{m'+2}{2} < \frac{n-1}{2}$

We know $F \equiv 3 \pmod 4$. If $F < 2d-1$, then $F \leq 2d-5$. This implies $\frac{F+1}{4} \leq \frac{d-2}{2}$. Moreover, by (6.5.2), we know that $m' < n'$; consequently, $\frac{m'+2}{2} < n'$. Therefore, we have

$$\frac{F+1}{4}n' + \frac{m'+2}{2} < \frac{d-2}{2}n' + n' = \frac{dn'}{2} = \frac{n-1}{2}.$$

(iii) $\frac{F-3}{4}n' + m' < \frac{F+1}{4}n' - \frac{m'+2}{2}$

If $F = 2d-1$, then $\frac{F+1}{4}n' + \frac{m'+2}{2} = \frac{n-1}{2} + \frac{m'+2}{2}$. Thus, the largest difference covered by $P_{\frac{F-3}{4}}$ is either $\left(\frac{F+1}{4}\right)n' - \frac{m'}{2}$ or $\left(\frac{F+1}{4}\right)n' - \frac{m'+2}{2}$, depending on the parity of $\frac{F-3}{4}$. We show that

$$\frac{F-3}{4}n' + m' < \frac{F+1}{4}n' - \frac{m'+2}{2},$$

which is equivalent to showing that

$$n' > \frac{3m'+2}{2}.$$

Since $F = 2d-1$ and $d \geq 4$, by (6.5.7), we have $n' > \frac{3m'+2}{2}$. Therefore,

$$\frac{F-3}{4}n' + m' < \frac{F+1}{4}n' - \frac{m'+2}{2},$$

Thus, all the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-3}{4}}$ listed in (6.5.11), (6.5.12), and (6.5.13) are pairwise distinct.

Let B be the set of differences listed in (6.5.12) and (6.5.13) that are covered by the m-cycles $C_1, C_2, \ldots, C_{\frac{F-3}{4}}$. Let $G_1 = \mathrm{Circ}(n-1; B)$. Then,

$$\{C_i, \rho(C_i), \ldots, \rho^{n'-1}(C_i) \mid i = 1, \ldots, \tfrac{F-3}{4}\}$$

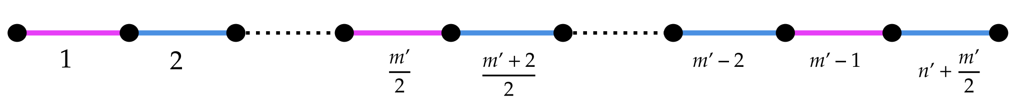
is a (C_m)-decomposition of G_1. Take four copies of each C_i, for $i = 1, \ldots, \frac{F-3}{4}$, and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. By doing so, we generate $F-3$ starter peripheral cycles of $4K_n^\bullet$ with the appropriate HOP coloring, which we label as $C'_3, \ldots, C'_{F-1}$.

It is now easy to see that

$$\{C'_i, \rho_\bullet(C'_i), \ldots, \rho_\bullet^{n'-1}(C'_i) \mid i = 3, \ldots, F-1\}$$

is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Notice that $A = \{1, 2, 3, \ldots, m'-2, m'-1, n' + \frac{m'}{2}\}$ is the set of differences covered by C_0. Take three copies of C_0 and label them as C'_0, C'_1 and C'_2. Notice that these three cycles are the remaining three starter peripheral cycles. Color the edges of the cycles C'_1 and C'_2 black, and orient them in the opposite directions. Color the edges of P_0 in the cycle C'_0 alternately pink and blue (as shown below), starting with pink. Since P_0 has length m', and $m' \equiv 2 \pmod 4$, it ends with blue.



Observe that the families generated by the starter cycles C'_1 and C'_2 jointly cover the black orbits corresponding to the differences in the set A. The family generated by the starter cycle C'_0 covers:

(i) The pink orbits corresponding to the differences

$$1, 3, 5, \ldots, \tfrac{m'}{2}, \tfrac{m'+4}{2}, \ldots, m'-3, m'-1. \tag{6.5.15}$$

(ii) The blue orbits corresponding to the differences

$$2, 4, 6, \ldots, \tfrac{m'-2}{2}, \tfrac{m'+2}{2}, \ldots, m'-2, n'+\tfrac{m'}{2}. \tag{6.5.16}$$

Note that each difference in the set A is already covered by the peripheral cycles C'_0, C'_1, and C'_2, and has been assigned the appropriate coloring. We now need to cover each difference exactly one more time using the central cycles C and C', to be constructed below. The differences in the set A are carefully divided into two parts, with each half covered by one of the central cycles and assigned the proper coloring, such that the blue orbits corresponding to the differences listed in (6.5.15) and the pink orbits corresponding to the differences listed in (6.5.16) are covered.

Now, we focus on constructing the starter central cycles C and C'. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus (A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 = m' < a_2 < \ldots < a_t = \frac{n-1}{2}$ are positive integers and $t = \frac{n-1}{2} - \frac{(F+1)m'}{4}$.

Note that for $d \geq 4$, we have

$$A = \{1, 2, 3, \ldots, m'-2, m'-1, n'+\tfrac{m'}{2}\},$$

and we have seen that $a_1 = m' < n' + \frac{m'}{2} < \frac{n-1}{2}$.

For $d = 2$, we have

$$A = \{1, 2, 3, \ldots, m'-2, m'-1, n'-\tfrac{m'}{2}\},$$

and we have shown in (6.5.14) that $m' < n' - \frac{m'}{2}$; hence,

$$a_1 = m' < n' - \tfrac{m'}{2} < \frac{n-1}{2}.$$

Let $s = n' \pm \frac{m'}{2}$. Thus, $A = \{1, 2, \ldots, m'-2, m'-1, s\}$, and $a_1 < s < a_t$.
Consider the following sets of differences:

- $W_1 = \{a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2 = \{2, 5, 6, \ldots, m'-9, m'-8, m'-5, m'-4, m'-1, a_1, a_2, \ldots, s, \ldots, a_{t-1}\}$

Note that $s \notin W_1$, and in W_2, we have

$$2 < 5 < \ldots < m'-4 < m'-1 < a_1 < a_2 < \ldots < s < \ldots < a_{t-1}.$$

Here, $|W_1| = t$ and $|W_2| = t + \frac{m'-2}{2}$. By Corollary 6.3.5, there exists a path, call it T_1, of length $2t + \frac{m'-2}{2}$ that starts at x_0 and covers the differences in the following order:

$$T_1 : a_1, a_2 \ldots, a_{t-1}, a_t, a_{t-1}, \ldots, s, \ldots, a_1, m'-1, m'-4, m'-5, \ldots, 6, 5, 2.$$

Now, consider the following sets of differences:

- $W_1 = \{a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2' = \{3, 4, \ldots, m'-10, m'-7, m'-6, m'-3, m'-2, a_1, \ldots, s, \ldots, a_{t-1}\}$

Again $s \notin W_1$, and in W_2', we have

$$3 < 4 < \ldots < m'-3 < m'-2 < a_1 < \ldots < s < \ldots < a_{t-1}.$$

Here, $|W_1| = t$ and $|W_2'| = t + \frac{m'-2}{2}$. By Corollary 6.3.5, there exists a path of length $2t + \frac{m'-2}{2}$ that starts at x_0 and covers the differences in the following order:

$$a_1, \ldots, a_{t-1}, a_t, a_{t-1}, \ldots, s, \ldots, a_1, m'-2, m'-3, m'-6, m'-7, \ldots, 4, 3.$$

By Lemma 6.3.3, the difference s in the above path can be replaced with difference 1; that is, there exists a path, call it T_2, that covers the differences in the following order:

$$T_2 : a_1, \ldots, a_{t-1}, a_t, a_{t-1}, \ldots, 1, \ldots, a_1, m'-2, m'-3, m'-6, m'-7, \ldots, 4, 3.$$

Moreover, by Lemma 6.3.3, the paths T_1 and T_2 can be constructed to be identical before the occurrence of the difference s in T_1 and the difference 1 in T_2.

Notice that the lengths of the paths T_1 and T_2 are equal to $2t + \frac{m'-2}{2}$, and

$$\begin{aligned}
2t + \frac{m'-2}{2} &= 2\left(\frac{n-1}{2} - \frac{(F+1)m'}{4}\right) + \frac{m'-2}{2} \\
&= n - 2 - \frac{Fm'}{2} \\
\text{(substitute } F = \frac{2(n-m)}{m'}\text{)} \qquad &= n - 2 - (n-m) \\
&= m - 2
\end{aligned}$$

To simplify the explanation of the coloring process, we express T_1 and T_2 as the following concatenation of subpaths:

- $T_1 = PQ_1R_1$, and
- $T_2 = PQ_2R_2$.

Here, P, Q_1, Q_2, R_1, and R_2 represent subpaths that cover specific sequences of differences:

- $P : a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t$
- $Q_1 : a_{t-1}, a_{t-2}, \ldots, n' + \frac{m'}{2}, \ldots, a_3, a_2, a_1$
- $Q_2 : a_{t-1}, a_{t-2}, \ldots, 1, \ldots, a_3, a_2, a_1$
- $R_1 : m'-1, m'-4, m'-5, m'-8, m'-9, \ldots, 6, 5, 2$

- $R_2 : m' - 2, m' - 3, m' - 6, m' - 7, m' - 10, m' - 11, \ldots, 4, 3$

The difference $a_t = \frac{n-1}{2}$ occurs only in P, and importantly, P is identical in both T_1 and T_2.

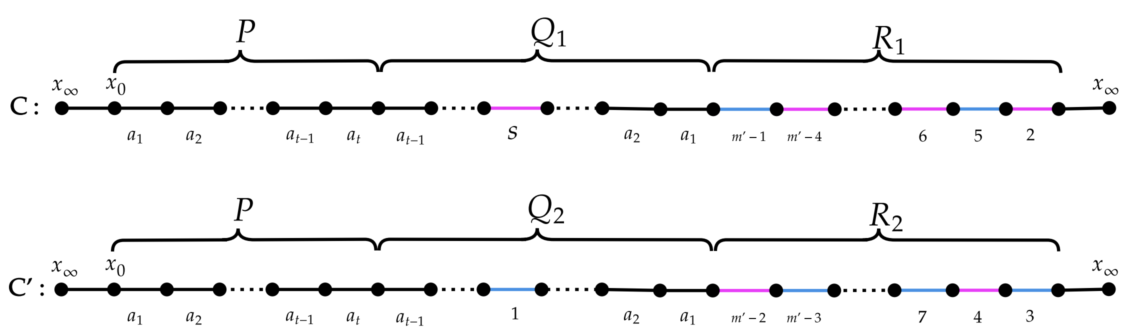
Also, notice that P, Q_1, and Q_2 each have length t. The paths R_1 and R_2 each have length $\frac{m'-2}{2}$. Moreover, since $m' \equiv 2 \pmod 4$:

- In R_1, the differences alternate between $1 \pmod 4$ and $2 \pmod 4$.
- In R_2, the differences alternate between $0 \pmod 4$ and $3 \pmod 4$.

The above property of the paths R_1 and R_2 ensures that they can be colored as required. We now describe their coloring.

Let $C = x_\infty P Q_1 R_1 x_\infty$ and $C' = x_\infty P Q_2 R_2 x_\infty$. First, color the edges in the path R_1 alternately pink and blue, starting with blue. Since its length is $\frac{m'-2}{2}$, which is even, it ends with pink. Similarly, color the edges in the path R_2 alternately pink and blue, starting with pink. Since its length is even, it ends with blue.

Next, in the path Q_1, color the edge with difference s pink, and in the path Q_2, color the edge with difference 1 blue, as shown below. With this coloring, we ensure that the blue orbits corresponding to the differences listed in (6.5.15) and the pink orbits corresponding to the differences listed in (6.5.16) are covered.

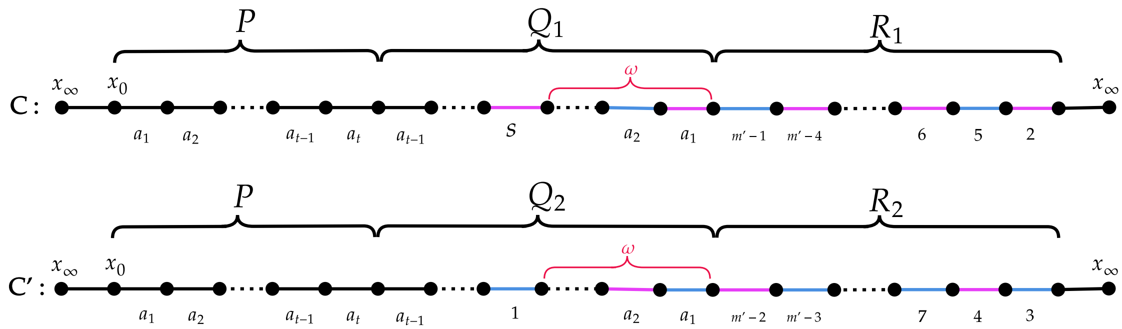


Let ω be the set of differences in Q_1 that fall within the interval $[m', s)$, and recall that $a_1 = m'$, and value of s depends on d.

- If $d \geq 4$, then $s = n' + \frac{m'}{2}$.
  - If $F > 3$, then $\omega = \{m', m'+1, \ldots, n'\}$, and $|\omega| = n' - m' + 1$.
  - If $F = 3$, then $\omega = \{m', m'+1, \ldots, n' + \frac{m'}{2} - 1\}$, and $|\omega| = n' - \frac{m'}{2}$.
- If $d = 2$, then $s = n' - \frac{m'}{2}$, and $\omega = \{m', m'+1, \ldots, n' - \frac{m'}{2} - 1\}$, so

$$|\omega| = n' - \frac{3m'}{2}.$$

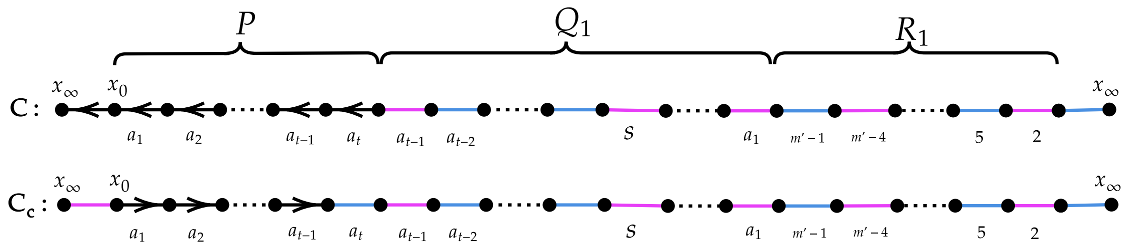
Since n' is odd and $m' \equiv 2 \pmod 4$, in all the above cases, $|\omega|$ is even. This ensures that we can properly color the edges with the differences in ω. We start by coloring the edge with difference a_1 pink in Q_1 and blue in Q_2, then continue coloring the remaining edges in ω alternately pink and blue, as shown below.



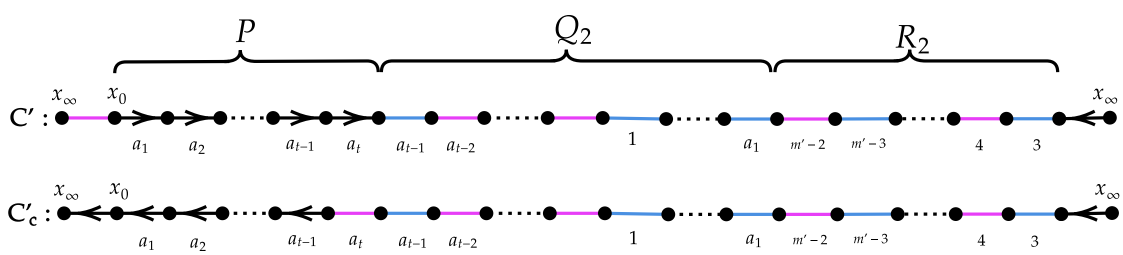
Let C_c be a copy of C, and let C_c' be a copy of C' with the same coloring as described above. Note that the number of uncolored edges in Q_1 and Q_2 is $t-(|\omega|+1)$. Depending on the parity of t, we proceed with the rest of the coloring as follows.

(i) t is odd. Then $t-(|\omega|+1)$ is even.

In C and C_c, start by coloring the uncolored edges in Q_1 alternately pink and blue, beginning with the edge of difference a_{t-1} in pink. Additionally, in C_c, color the edge with difference a_t blue. Then, proceed to color the remaining edges as shown below.

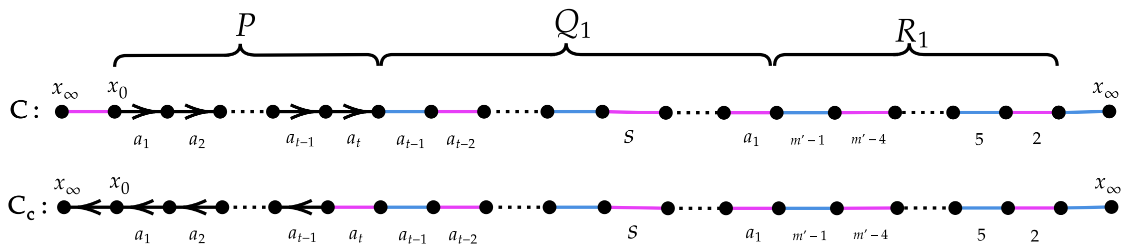


In C' and C_c', start by coloring the uncolored edges in Q_2 alternately pink and blue, beginning with the edge of difference a_{t-1} in blue. Additionally, in C_c', color the edge with difference a_t pink. Then, color the remaining edges as shown below.

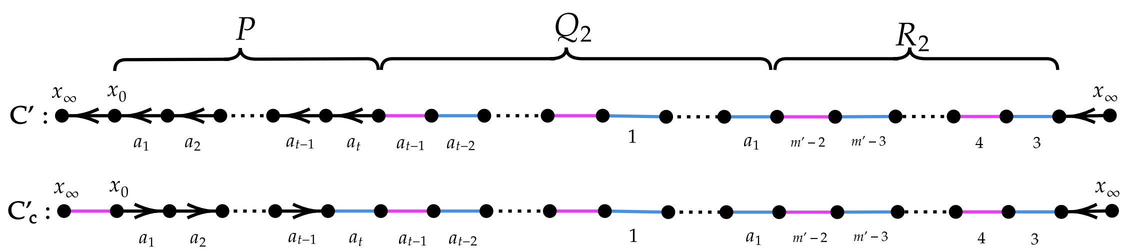


(ii) t is even. Then $t-(|\omega|+1)$ is odd.

In C and C_c, start by coloring the uncolored edges in Q_1 alternately pink and blue, beginning with the edge of difference a_{t-1} in blue. Additionally, in C_c, color the edge with difference a_t pink. Then, proceed to color the remaining edges as shown below.



In C' and C'_c, start by coloring the uncolored edges in Q_2 alternately pink and blue, beginning with the edge of difference a_{t-1} in pink. Additionally, in C'_c, color the edge with difference a_t blue. Then, color the remaining edges as shown below.



We now show that

$$\{\rho_\bullet^i(C), \rho_\bullet^{\frac{n-1}{2}+i}(C_c), \rho_\bullet^i(C'), \rho_\bullet^{\frac{n-1}{2}+i}(C'_c) \mid i=0,1,\ldots,\tfrac{n-3}{2}\}$$
$$\cup\ \{C'_i, \rho_\bullet(C'_i), \ldots, \rho_\bullet^{n'-1}(C'_i) \mid i=0,1,2\}$$

is a (C_m)-decomposition for $4G_2^\bullet$, where

$$G_2 = \mathrm{Circ}(n-1; A\cup(\mathscr{L}-\{\infty\})) \bowtie K_1.$$

Observe that C, C_c, C', C'_c jointly contain exactly one blue edge, one pink edge, and two black arcs from the orbit of $\langle\rho_\bullet\rangle$ corresponding to the diameter difference a_t. Additionally, they jointly contain exactly two blue edges and two pink edges from each orbit of $\langle\rho_\bullet\rangle$ corresponding to each difference $d\in\{a_1,a_2,\ldots,a_{t-1},\infty\}$, namely, a pair of edges of the form $(e,\rho_\bullet^{\frac{n-1}{2}}(e))$.

The four starter central cycles exactly cover four black arcs from each orbit of $\langle\rho_\bullet\rangle$ corresponding to each difference $d\in\{a_1,a_2,\ldots,a_{t-1},\infty\}$. Since the path P is identical in all four starter central cycles, it enables us to color and orient them properly.

The black orbits corresponding to differences in A are covered by C'_1 and C'_2, while the pink and the blue orbits of differences in A are covered by C, C_c, C', C'_c, and C'_0.

Moreover, $C, C_c, C', C'_c, C'_0, C'_1, C'_2$ satisfy Condition **(C1)** of Definition 1.2.3, and $\rho_\bullet$ preserves the edge colors (and orientations), thus we conclude that the above decomposition is an HOP (C_m)-decomposition of $4G_2^\bullet$.

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Case 2: $F \equiv 1 \pmod 4$.

For Case 2, we divide the proof into three subcases: one where $d = 2$; another where $d \geq 4$ and $(m, d) \neq (8, 4)$; and the last one where $(m, d) = (8, 4)$.

Subcase 2.1: $d = 2$. From (6.5.4), we have $F \leq 2d - 3$. Thus, $F = 1$, and we need to construct one family of peripheral cycles. Note that since $d = 2$, we have $n' = \frac{n-1}{2}$.

Define the path P_0 as follows.

$$P_0 = x_0\ x_1\ x_{-1}\ x_2\ x_{-2} \dots x_{\frac{n'+1}{2}-\frac{m'+6}{4}}\ x_{-(\frac{n'+1}{2}-\frac{m'+6}{4})}\ x_{\frac{n'+1}{2}-\frac{m'+2}{4}}$$
$$x_{-(\frac{n'-1}{2}-\frac{m'-2}{4})}\ x_{\frac{n'+1}{2}-\frac{m'-2}{4}}\ \dots\ x_{-\frac{m'-2}{2}}\ x_{\frac{m'-2}{2}}\ x_{-\frac{m'}{2}}\ x_{\frac{m'}{2}}\ x_{n'}.$$

The differences covered by P_0 are:

$$1, 2, 3, 4, \dots, n' - \frac{m'}{2} - 2, n' - \frac{m'}{2} - 1, n' - \frac{m'}{2} + 1, n' - \frac{m'}{2} + 2, \dots, m' - 2, m' - 1, m', n' - \frac{m'}{2}.$$

The difference $n' - \frac{m'}{2}$ appears at the end of the path. Its occurrence was avoided earlier in the path since $n' - \frac{m'}{2} < m'$, as shown below.

Since $F = 1$, we have $2(n - m) = m'$, which implies

$$n = m + \frac{m'}{2}. \tag{6.5.17}$$

Since $d = 2$, we have $m = 2m'$. Thus, we see that $n = \frac{5m'}{2}$. Furthermore, we can write $n' = \frac{n-1}{2} = \frac{5m'-2}{4}$. Therefore, we have

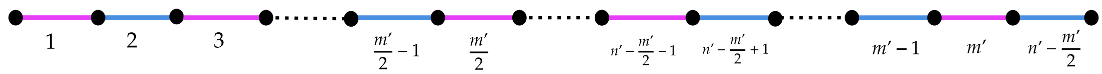
$$n' - \frac{m'}{2} = \frac{3m'}{4} - \frac{1}{2} < m'. \tag{6.5.18}$$

Since $m' < n'$ by (6.5.2), the paths P_0 and $\rho^{n'}(P_0)$ are pairwise vertex-disjoint except at the endpoints. Thus, $C_0 = P_0 \cup \rho^{n'}(P_0)$ is an m-cycle.

By (6.5.2), we know that $m' < n'$, which implies $\frac{m'}{2} < n' - \frac{m'}{2}$. Additionally, using the inequality in (6.5.18) we have

$$\frac{m'}{2} < n' - \frac{m'}{2} < m'.$$

Let $A = \{1, 2, 3, \dots, \frac{m'}{2}, \dots, n' - \frac{m'}{2}, \dots, m' - 1, m'\}$ be the set of differences covered by C_0. Color the edges of P_0 in the cycle C_0 alternately pink and blue (as shown below), starting with pink. Since the length of P_0 is m' and $m' \equiv 2 \pmod 4$, it ends with blue. Note that since n' is odd, $n' - \frac{m'}{2} - 1$ is also odd, and the edge with this difference gets pink color.



Observe that the family generated by the starter cycle C_0 covers:

(i) The pink orbits corresponding to the differences

$$1, 3, 5, \dots, \tfrac{m'}{2}, \dots, n' - \tfrac{m'}{2} - 1, n' - \tfrac{m'}{2} + 2, \dots, m' - 2, m'. \tag{6.5.19}$$

(ii) The blue orbits corresponding to the differences

$$2, 4, 6, \dots, \tfrac{m'}{2} - 1, \dots, n' - \tfrac{m'}{2} - 2, n' - \tfrac{m'}{2} + 1, \dots, m' - 1, n' - \tfrac{m'}{2}. \tag{6.5.20}$$

The differences in the set A will also appear in the central cycles C and C', to be constructed below. There, we need to ensure that the black orbits corresponding to the differences in A, the blue orbits corresponding to the differences of A listed in (6.5.19), and the pink orbits corresponding to the differences of A listed in (6.5.20) are covered.

Now, we focus on constructing the starter central cycles C and C'. Let $\mathscr{L}$ be the set of unused differences in S; that is, $\mathscr{L} = \{m'+1, m'+2, \dots, n'\} \cup \{\infty\}$.

Consider the following sets of differences:

- $W_1 = \{1, 2, 3, \dots, n'-2, n'-1, n'\}$
- $W_2 = \{2, 3, 4, \dots, \frac{m'}{2} - 2, \frac{m'}{2} - 1, \frac{m'}{2}, n' - \frac{m'}{2}, m'+1, m'+2, \dots, n'-1\}$

The set W_1 contains all the differences from the set A and all differences from $\mathscr{L}$, except the difference ∞, and has size $|W_1| = n'$. The set W_2 contains half of the differences from A and all differences from $\mathscr{L}$, excluding differences ∞ and n', and has size $|W_2| = n' - \frac{m'+2}{2}$. By Corollary 6.3.5, there exists a path, call it T_1, of length $2n' - \frac{m'+2}{2}$ that starts at x_0 and covers the differences in the following order:

$$T_1: \ 1, 2, 3, \dots, m'+1, \dots, n'-1, n', n'-1, \dots, m'+1, n' - \tfrac{m'}{2}, \tfrac{m'}{2}, \tfrac{m'}{2} - 1, \dots, 4, 3, 2.$$

Now, consider the following sets of differences:

- $W_1 = \{1, 2, 3, \dots, n'-2, n'-1, n'\}$
- $W_2' = \{1, \frac{m'}{2} + 1, \frac{m'}{2} + 2, \dots, n' - \frac{m'}{2} - 1, n' - \frac{m'}{2} + 1, \dots, m'-2, m'-1, m', m'+1, m'+2, \dots, n'-2, n'-1\}$

Note that $n' - \frac{m'}{2} \notin W_2'$. Here, $|W_1| = n'$ and $|W_2'| = n' - \frac{m'+2}{2}$. By Corollary 6.3.5, there exists a path, call it T_2, of length $2n' - \frac{m'+2}{2}$ that starts at x_0 and covers the differences in the following order:

$$T_2: 1, 2, 3, \dots, n'-1, n', n'-1, \dots, m'+2, m'+1, m', m'-1, \dots,$$

$$n' - \frac{m'}{2} + 1, n' - \frac{m'}{2} - 1, \ldots, \frac{m'}{2} + 2, \frac{m'}{2} + 1, 1.$$

Moreover, by Lemma 6.3.3, the paths T_1 and T_2 can be constructed to be identical up to the occurrence of the difference $m' + 1$ in both T_1 and T_2.

Notice that the lengths of the paths T_1 and T_2 are equal to $2n' - \frac{m'+2}{2}$, and

$$\begin{aligned} 2n' - \frac{m'+2}{2} &= 2(\frac{n-1}{2}) - \frac{m'+2}{2} \\ &= n - 2 - \frac{m'}{2} \\ \text{(Using the equation given in (6.5.17))} \qquad &= m + \frac{m'}{2} - 2 - \frac{m'}{2} \\ &= m - 2 \end{aligned}$$

As before, to simplify the explanation of the coloring process, we express T_1 and T_2 as the following concatenation of subpaths:

- $T_1 = PQR_1$, and
- $T_2 = PQR_2$.

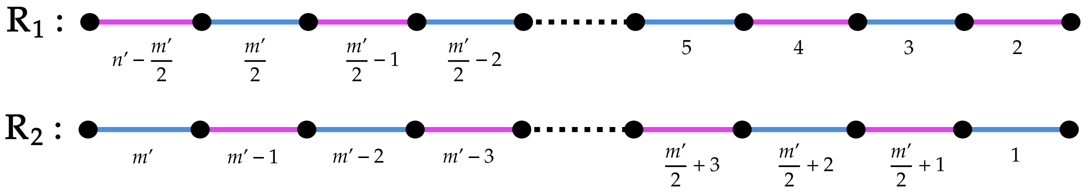
Here, P, Q, R_1, and R_2 represent the subpaths that cover specific sequences of differences:

- $P : 1, 2, 3, \ldots, n' - 2, n' - 1, n'$
- $Q : n' - 1, n' - 2, \ldots, m' + 2, m' + 1$
- $R_1 : n' - \frac{m'}{2}, \frac{m'}{2}, \frac{m'}{2} - 1, \frac{m'}{2} - 2, \ldots, 4, 3, 2$
- $R_2 : m', m' - 1, m' - 2, \ldots, \frac{m'}{2} + 2, \frac{m'}{2} + 1, 1$

The diameter difference $n' = \frac{n-1}{2}$ occurs only in P, and importantly, P and Q are identical in both T_1 and T_2.

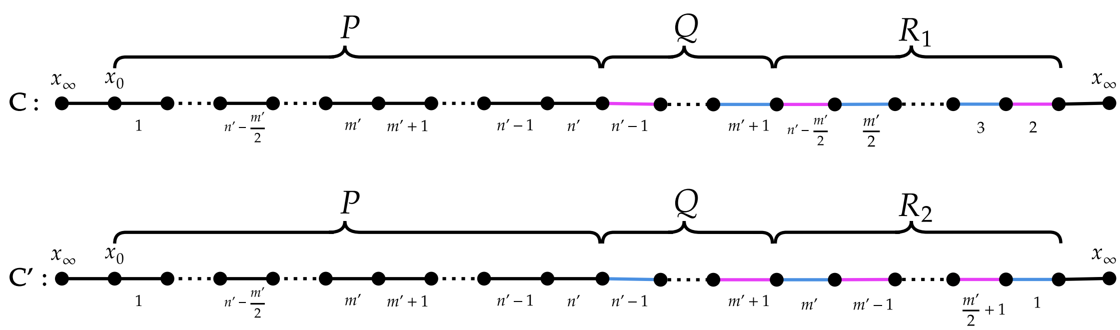
Also, notice that P has length n', and Q has length $n' - m' - 1$. The paths R_1 and R_2 each have length $\frac{m'}{2}$. Moreover, they are constructed so that they can be colored as required. We now describe their coloring.

As shown below, color the edges in the path R_1 alternately pink and blue, starting with pink. Since its length is $\frac{m'}{2}$, which is odd, it ends with pink. Similarly, color the edges in the path R_2 alternately pink and blue, starting with blue. Since its length is odd, it ends with blue. Note that the difference $n' - \frac{m'}{2}$ is covered by R_1 but not R_2.

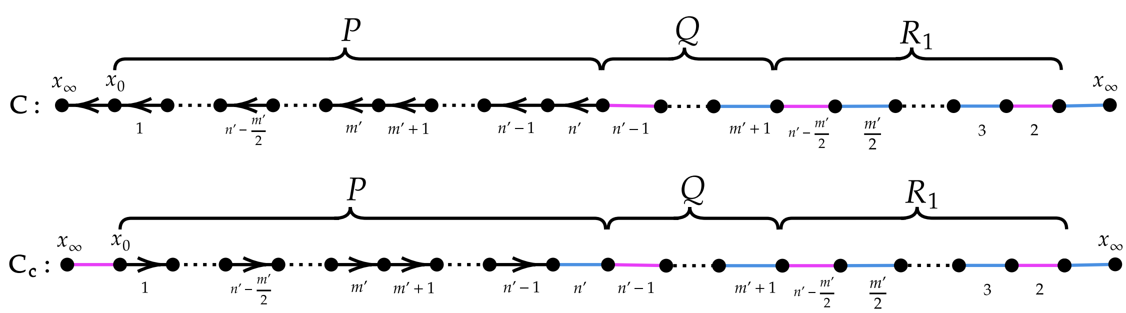


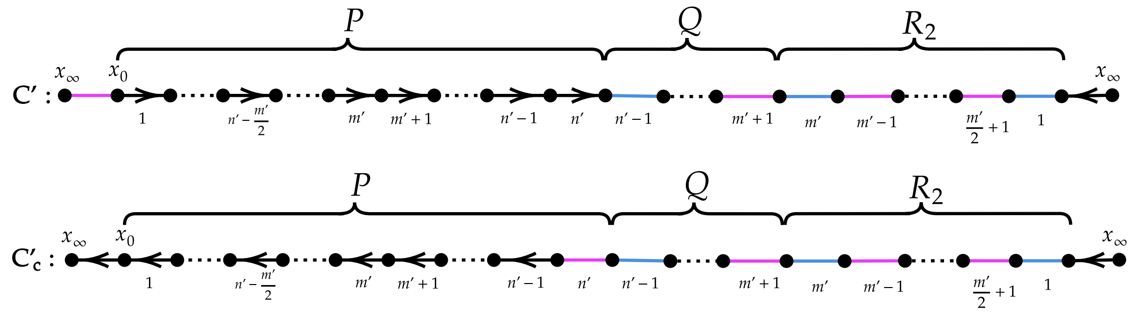
With this coloring, we ensure that the blue orbits corresponding to the differences listed in (6.5.19) and the pink orbits corresponding to the differences listed in (6.5.20) are covered.

Let $C = x_\infty PQR_1x_\infty$ and $C' = x_\infty PQR_2x_\infty$. We know that the length of Q is $n'-m'-1$, and given that n' is odd and m' is even, the length of Q is even. Begin by coloring the edges of Q in C alternately pink and blue, starting with pink; it ends with blue. In C', color the edges of Q alternately pink and blue, starting with blue; it ends with pink, as shown below.



Let C_c be a copy of C, and let C'_c be a copy of C' with the same coloring as described above. In C_c, color the edge with difference n' blue, and in C'_c, color the edge with difference n' pink. Then, color the remaining edges as shown below. Note that, since the path P is identical in all four starter central cycles, it enables us to color and orient them properly.





As in Case 1, it can be verified that

$$\left\{\rho_\bullet^i(C), \rho_\bullet^{\frac{n-1}{2}+i}(C_c), \rho_\bullet^i(C'), \rho_\bullet^{\frac{n-1}{2}+i}(C'_c) \mid i = 0, 1, \ldots, \tfrac{n-3}{2}\right\}$$
$$\cup\ \{C_0, \rho_\bullet(C_0), \ldots, \rho_\bullet^{n'-1}(C_0)\}$$

is an HOP (C_m)-decomposition for $4K_n^{\bullet}$.

Subcase 2.2: $d \geq 4$ and $(m, d) \neq (8, 4)$. Here, the approach is to construct $\frac{F-1}{4} + 1$ starter peripheral cycles for K_n. Then, take four copies of each of the $\frac{F-1}{4}$ cycles and color them appropriately using Lemma 3.3.3 to generate $F - 1$ starter peripheral cycles in $4K_n^{\bullet}$. The $F-1$ families of peripheral cycles cover a set of differences B, for some $B \subset S$. We will then be left with one remaining starter peripheral cycle, which together with the two central cycles will cover the remaining differences; that is, the differences in the set $S \setminus B$.

We begin by constructing the starter peripheral cycles. As in Case 1, define the path P_i as follows:

- For $i = 0$, define

$$P_0 = x_0\ x_1\ x_{-1}\ x_2\ x_{-2} \ldots x_{\frac{m'-2}{2}}\ x_{-\frac{m'-2}{2}}\ x_{\frac{m'}{2}}\ x_{-n'}.$$

  The differences covered by P_0 are:

$$1, 2, 3, 4, \ldots, m'-2, m'-1, n' + \frac{m'}{2}. \tag{6.5.21}$$

- For $i = 1, 2, \ldots, \lceil \frac{F-1}{8} \rceil$, define the path P_{2i-1} as follows.

$$P_{2i-1} = x_0\ x_{(2i-1)n'+1}\ x_{-1}\ x_{(2i-1)n'+2}\ x_{-2}\ \ldots\ x_{-\frac{m'-6}{4}}\ x_{(2i-1)n'+\frac{m'-2}{4}}\ x_{-\frac{m'-2}{4}}$$
$$x_{(2i-1)n'+\frac{m'+6}{4}}\ x_{-\frac{m'+2}{4}}\ \ldots\ x_{-\frac{m'-2}{2}}\ x_{(2i-1)n'+\frac{m'+2}{2}}\ x_{-n'}.$$

  The differences covered by P_{2i-1} are:

$$\begin{aligned}&(2i-1)n'+1,\ (2i-1)n'+2,\ (2i-1)n'+3, \ldots, (2i-1)n' + \tfrac{m'-4}{2},\\ &(2i-1)n' + \tfrac{m'-2}{2},\ (2i-1)n' + \tfrac{m'+2}{2},\ (2i-1)n' + \tfrac{m'+4}{2}, \ldots,\\ &(2i-1)n' + m',\ 2in' + \tfrac{m'+2}{2}.\end{aligned} \tag{6.5.22}$$

- For $i = 1, 2, \ldots, \lfloor \frac{F-1}{8} \rfloor$, define the path P_{2i} as follows.

$$P_{2i} = x_0\ x_{2in'+1}\ x_{-1}\ x_{2in'+2}\ x_{-2}\ \ldots\ x_{2in'+\frac{m'-2}{4}}\ x_{-\frac{m'-2}{4}}$$
$$x_{2in'+\frac{m'+2}{4}}\ x_{-\frac{m'+6}{4}}\ x_{2in'+\frac{m'+6}{4}}\ \ldots\ x_{-\frac{m'}{2}}\ x_{2in'+\frac{m'}{2}}\ x_{-n'}.$$

The differences covered by P_{2i} are:

$$\begin{aligned}&2in'+1,\ 2in'+2,\ 2in'+3, \ldots, 2in'+\tfrac{m'-2}{2}, 2in'+\tfrac{m'}{2},\\ &2in'+\tfrac{m'+4}{2}, 2in'+\tfrac{m'+6}{2} \ldots, 2in'+m',\ (2i+1)n'+\tfrac{m'}{2}.\end{aligned} \tag{6.5.23}$$

Since $m' < n'$ by (6.5.2), the paths $\rho^j(P_i)$, for $j = 0, n', 2n', \ldots, (d-1)n'$, are pairwise vertex-disjoint, except at their endpoints. Thus, for $i = 0, 1, \ldots, \frac{F-1}{2}$

$$C_i = P_i \cup \rho^{n'}(P_i) \cup \ldots \cup \rho^{(d-1)n'}(P_i)$$

is an m-cycle.

As in Case 1, the differences $(2i+1)n' + \frac{m'}{2}$ occur at the ends of the paths P_{2i}, and these differences are carefully avoided in the paths P_{2i-1}. Similarly, the differences $2in' + \frac{m'+2}{2}$ appear at the ends of the paths P_{2i-1} and are carefully avoided in the paths P_{2i}.

The paths P_i cover the following differences:

- For $i = 0$:
$$1, 2, \ldots, m'-2, m'-1$$
- For $i = 1, 2, \ldots, \frac{F-1}{4}$:
$$in'+1,\ in'+2,\ \ldots,\ in'+m'$$
- One additional difference, depending on the parity of $\frac{F-1}{4}$, which is either
$$\left(\frac{F-1}{4}+1\right)n' + \frac{m'}{2} \quad \text{or} \quad \left(\frac{F-1}{4}+1\right)n' + \frac{m'+2}{2}.$$

To show that the above differences are distinct, it suffices to prove the following inequalities.

(i) $\frac{F-1}{4}n' + m' < \frac{n-1}{2}$
As shown in (6.5.4), we have $F \leq 2d-3$. This implies $\frac{F-1}{4} \leq \frac{d-2}{2}$. Moreover, by (6.5.2), we know that $m' < n'$; thus,
$$\frac{F-1}{4}n' + m' < \frac{d-2}{2}n' + n' = \frac{dn'}{2} = \frac{n-1}{2}.$$

(ii) $\frac{F+3}{4}n' + \frac{m'+2}{2} < \frac{n-1}{2}$
Since $F \equiv 1 \pmod 4$, if $F < 2d-3$, then we have $F \leq 2d-7$. This implies $\frac{F+3}{4} \leq \frac{d-2}{2}$. Moreover, by (6.5.2), we know that $m' < n'$; consequently, $\frac{m'+2}{2} < n'$. Therefore, we have
$$\frac{F+3}{4}n' + \frac{m'+2}{2} < \frac{d-2}{2}n' + n' = \frac{dn'}{2} = \frac{n-1}{2}.$$

(iii) $\frac{F-1}{4}n' + m' < \frac{F+3}{4}n' - \frac{m'+2}{2}$,

If $F = 2d-3$, then $\frac{F+3}{4}n' + \frac{m'+2}{2} = \frac{n-1}{2} + \frac{m'+2}{2}$. We show that

$$\frac{F-1}{4}n' + m' < \frac{n-1}{2} - \frac{m'+2}{2} = \frac{F+3}{4}n' - \frac{m'+2}{2},$$

which is equivalent to showing that

$$n' > \frac{3m'+2}{2}.$$

In (6.5.10), for $F = 2d-3$, all $m' > 2$, $m' \equiv 2 \pmod 4$, and even $d \geq 4$, except for $(m', d) \in \{(6,4), (10,4)\}$, we showed that $n' > \frac{3m'+2}{2}$.

For the case $(m', d) = (6, 4)$, by (6.5.9), we have $n' = 9.5$, and for the case $(m', d) = (10, 4)$, we have $n' = 16$. However, since n' must be an odd integer, these cases do not arise.

For the case $m' = 2$, we have $n' = 3$, since $m' < n' < 2m'$ by (6.5.2). Given that $F = 2d-3$, it follows that $d = 4$. In this case, $m = 8$ and $n = 13$. The given construction does not work for this case, and we address it separately in Subcase 2.3.

Thus, we conclude that all the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-1}{4}}$ listed in (6.5.21), (6.5.22), and (6.5.23) are pairwise distinct.

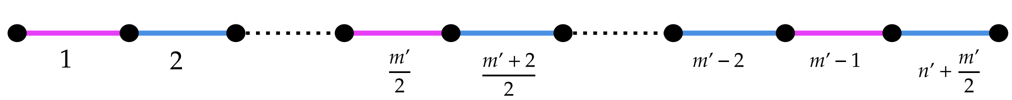
Let B be the set of differences covered by the m-cycles $C_1, C_2 \ldots, C_{\frac{F-1}{4}}$ listed in (6.5.22), and (6.5.23). Let $G_1 = \mathrm{Circ}(n-1; B)$. Then, $\{C_i, \rho(C_i), \ldots, \rho^{n'-1}(C_i) \mid i = 1, \ldots, \frac{F-1}{4}\}$ is a (C_m)-decomposition of G_1. Take four copies of each C_i, for $i = 1, \ldots, \frac{F-1}{4}$, and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. By doing so, we generate $F-1$ starter peripheral cycles of $4K_n^\bullet$ with the appropriate HOP coloring, which we label as $C'_1, \ldots, C'_{F-1}$.

It is now easy to see that

$$\{C'_i, \rho_\bullet(C'_i), \ldots, \rho_\bullet^{n'-1}(C'_i) \mid i = 1, \ldots, F-1\}$$

is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Notice that $A = \{1, 2, 3, \ldots, m'-2, m'-1, n' + \frac{m'}{2}\}$ is the set of differences covered by C_0. Color the edges of P_0 in the cycle C_0 alternately pink and blue (as shown below), starting with pink. Since P_0 has length m', and $m' \equiv 2 \pmod 4$, it ends with blue.



Observe that the family generated by the starter cycle C_0 covers:

(i) The pink orbits corresponding to the differences

$$1, 3, 5, \ldots, \tfrac{m'}{2}, \tfrac{m'+4}{2}, \ldots, m'-3, m'-1. \tag{6.5.24}$$

(ii) The blue orbits corresponding to the differences

$$2, 4, 6, \ldots, \tfrac{m'-2}{2}, \tfrac{m'+2}{2}, \ldots, m'-2, n'+\tfrac{m'}{2}. \tag{6.5.25}$$

Note that each difference in the set A is already covered once by the peripheral cycle C_0 and has been assigned the appropriate coloring, as mentioned above. We now need to cover each difference three more times using the starter central cycles C and C', to be constructed below. All differences in A will appear in both C and C', colored black and properly oriented, accounting for two additional occurrences and ensuring coverage of the black orbits. To cover the blue orbits corresponding to the differences listed in (6.5.24), and the pink orbits corresponding to the differences listed in (6.5.25), the set A is carefully divided into two parts, with each half covered by one of the starter central cycles and assigned the appropriate coloring.

Now, we focus on constructing the starter central cycles C and C'. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus (A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $m' = a_1 < a_2 < \ldots < a_t = \frac{n-1}{2}$ are positive integers and $t = \frac{n-1}{2} - \frac{(F+3)m'}{4}$.

We have $m' < n'$ by (6.5.2), and given that $A = \{1, 2, 3, \ldots, m'-2, m'-1, n'+\frac{m'}{2}\}$ and $d \geq 4$, we see that

$$m'-1 < m' = a_1 < n' + \frac{m'}{2} < a_t.$$

Recall that $m' \equiv 2 \pmod 4$. Now, consider the following sets of differences:

- $W_1 = \{1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n'+\frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2 = \{2, 5, 6, \ldots, m'-8, m'-5, m'-4, m'-1, a_1, a_2, \ldots, n'+\frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}\}$

The set W_1 contains all the differences from the set A and all the differences from $\mathscr{L}$, except the difference ∞, and has size $|W_1| = t + m'$. The set W_2 contains half of the differences from A and all differences from $\mathscr{L}$, excluding differences ∞ and a_t, and has size $|W_2| = t - 1 + \frac{m'}{2}$. By Corollary 6.3.5, there exists a path, call it T_1, of length $2t + \frac{3m'-2}{2}$ that starts at x_0 and covers the differences in the following order:

$$\begin{aligned} T_1 : {} & 1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n'+\frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t, \\ & a_{t-1}, a_{t-2}, \ldots, n'+\frac{m'}{2}, \ldots, a_2, a_1, m'-1, m'-4, m'-5, m'-8, \ldots, 6, 5, 2. \end{aligned}$$

Now, consider the following sets of differences:

- $W_1 = \{1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n'+\frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t\}$

- $W_2' = \{3, 4, \ldots, m'-7, m'-6, m'-3, m'-2, a_1, a_2, \ldots, n' + \frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}\}$

Here, $|W_1| = t + m'$ and $|W_2'| = t + \frac{m'-2}{2}$. By Corollary 6.3.5, there exists a path of length $2t + \frac{3m'-2}{2}$ that starts at x_0 and covers the differences in the following order:

$$1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n' + \frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t,$$
$$a_{t-1}, a_{t-2}, \ldots, n' + \frac{m'}{2}, \ldots, a_2, a_1, m'-2, m'-3, m'-6, m'-7, \ldots, 4, 3.$$

By Lemma 6.3.3, the second occurrence of difference $n' + \frac{m'}{2}$ in the above path can be replaced with difference 1; that is, there exists a path, call it T_2, that covers the differences in the following order:

$$\begin{aligned} T_2 : &1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n' + \frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t, \\ &a_{t-1}, a_{t-2}, \ldots, 1, \ldots, a_2, a_1, m'-2, m'-3, m'-6, m'-7, \ldots, 4, 3. \end{aligned}$$

Notice that the lengths of the paths T_1 and T_2 are equal to $2t + \frac{3m'-2}{2}$, and

$$\begin{aligned} 2t + \frac{3m'-2}{2} &= 2\left(\frac{n-1}{2} - \frac{(F+3)m'}{4}\right) + \frac{3m'-2}{2} \\ &= n - 2 - \frac{Fm'}{2} \\ \text{(substitute } F = \frac{2(n-m)}{m'}\text{)} \quad &= n - 2 - (n-m) \\ &= m - 2 \end{aligned}$$

As in Case 1, to simplify the explanation of the coloring process, we express T_1 and T_2 as the following concatenation of subpaths:

- $T_1 = PQ_1R_1$, and
- $T_2 = PQ_2R_2$.

Here, P, Q_1, Q_2, R_1, and R_2 represent subpaths that cover specific sequences of differences:

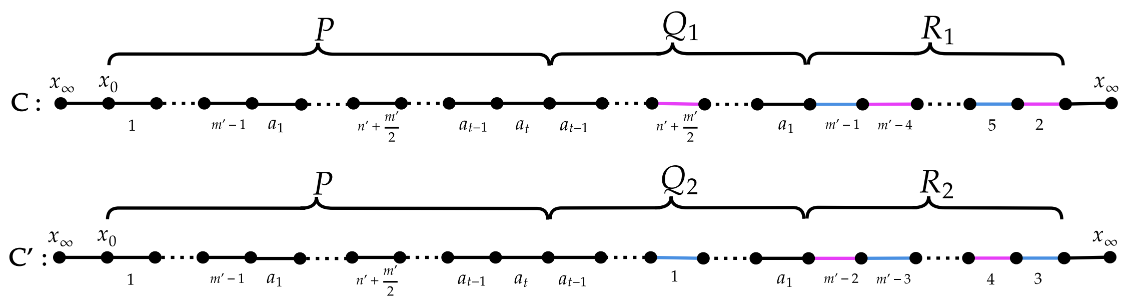
- $P : 1, 2, \ldots, m'-2, m'-1, a_1, a_2, \ldots, n' + \frac{m'}{2}, \ldots, a_{t-2}, a_{t-1}, a_t$
- $Q_1 : a_{t-1}, a_{t-2}, \ldots, n' + \frac{m'}{2}, \ldots, a_3, a_2, a_1$
- $Q_2 : a_{t-1}, a_{t-2}, \ldots, 1, \ldots, a_3, a_2, a_1$
- $R_1 : m'-1, m'-4, m'-5, m'-8, m'-9, \ldots, 6, 5, 2$
- $R_2 : m'-2, m'-3, m'-6, m'-7, m'-10, m'-11, \ldots, 4, 3$

The difference $a_t = \frac{n-1}{2}$ occurs only in P, and this path has length $t + m'$. Importantly, P is identical in both T_1 and T_2. The paths Q_1 and Q_2 each have length t, and the paths R_1 and R_2 each have length $\frac{m'-2}{2}$. Moreover, since $m' \equiv 2 \pmod 4$:

- In R_1, the differences alternate between $1 \pmod 4$ and $2 \pmod 4$.
- In R_2, the differences alternate between $0 \pmod 4$ and $3 \pmod 4$.

Observe that, except for the subpath P, all other subpaths are identical to those in Case 1 (see page 152) in terms of the sequence of differences they cover. The only distinction is that P is longer and includes differences from the set A. The coloring process remains exactly the same as in Case 1.

Let $C = x_\infty P Q_1 R_1 x_\infty$ and $C' = x_\infty P Q_2 R_2 x_\infty$. As in Case 1, we first fix the colors of the edges in paths R_1 and R_2, as well as the colors of the edge with difference $n' + \frac{m'}{2}$ in Q_1 and the edge with difference 1 in Q_2, as shown below.



With this coloring, we ensure that the blue orbits corresponding to the differences listed in (6.5.24) and the pink orbits corresponding to the differences listed in (6.5.25) are covered.

Let ω be the set of differences in Q_1 that falls within the interval $\left[m', n' + \frac{m'}{2}\right)$, and recall $a_1 = m'$.

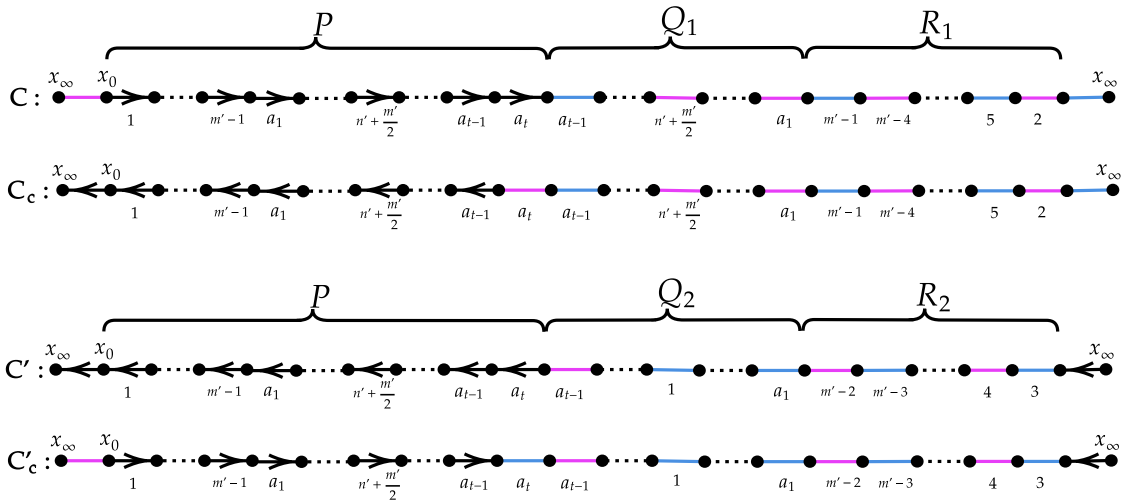
- For $F > 1$, we have $\omega = \{m', m'+1, \ldots, n'\}$ and $|\omega| = n' - m' + 1$.
- For $F = 1$, since $d \geq 4$, we have $\omega = \{m', m'+1, \ldots, n' + \frac{m'}{2} - 1\}$, and $|\omega| = n' - \frac{m'}{2}$.

Since n' is odd and $m' \equiv 2 \pmod 4$, in both cases, $|\omega|$ is even. As in Case 1, this ensures that we can properly color the edges with the differences in ω.

Let C_c be a copy of C, and let C'_c be a copy of C' with the same coloring of Q_1, Q_2, R_1, R_2, as described above. Following the same steps as in Case 1, we can color the uncolored edges in C_c, C, C'_c, and C' so that they satisfy Condition (**C1**) of Definition 1.2.3.

As in Case 1, depending on the parity of t, we will have two cases. For odd t, see (i) on page 154; for even t, see (ii) on page 155.

Here is an illustration of the four starter central cycles when t is even.



Note that, in this case, the black orbits corresponding to the differences in the set A are covered by the families generated by the starter central cycles C_c, C, C'_c, and C'.

Analogous to Case 1, it can be verified that

$$\{\rho_\bullet^i(C), \rho_\bullet^{\frac{n-1}{2}+i}(C_c), \rho_\bullet^i(C'), \rho_\bullet^{\frac{n-1}{2}+i}(C'_c) \mid i = 0, 1, \ldots, \tfrac{n-3}{2}\}$$
$$\cup \ \{C_0, \rho_\bullet(C_0), \ldots, \rho_\bullet^{n'-1}(C_0)\}$$

is an HOP (C_m)-decomposition for $4G_2^\bullet$, where

$$G_2 = \text{Circ}(n-1; A \cup (\mathscr{L} - \{\infty\})) \bowtie K_1.$$

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Subcase 2.3: $(m, d) = (8, 4)$. In this case, $n = 13$, $F = 5$, $n' = 3$, $m' = 2$, and the set of differences is $S = \{1, 2, 3, 4, 5, 6, \infty\}$.

Let $P_0 = x_0\ x_1\ x_3$, and $P_1 = x_0\ x_4\ x_{-3}$. The path P_0 covers differences 1 and 2, and the path P_1 covers differences 4 and 5.

Observe that, for $i = 0, 1$,

$$C_i = P_i \cup \rho^{n'}(P_i) \cup \rho^{2n'}(P_i) \cup \rho^{3n'}(P_i),$$

is a cycle of length 8.

Take four copies of C_1, and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. By doing so, we generate four starter peripheral cycles of $4K_{13}^\bullet$ with the appropriate HOP coloring, which we label as C'_1, C'_2, C'_3, C'_4.

Note that $\{1, 2\}$ is the set of differences covered by C_0. In P_0, color the edge with difference 1 pink, and the edge with difference 2 blue. The black copies of the edges with differences 1 and 2, as well as the blue copy of the edge with difference 1 and the pink copy of the edge with difference 2, will be covered by the central cycles, which will be constructed below.

Notice that the set of uncovered differences is $\{3, 6, \infty\}$. Let C, C_c, C', C_c' be the four central cycles with the HOP coloring, as shown in Figure 6.6.

It can be easily verified that

$$\{\rho_\bullet^i(C), \rho_\bullet^{6+i}(C_c), \rho_\bullet^i(C'), \rho_\bullet^{6+i}(C_c') \mid i = 0, 1, \ldots, 5\} \cup \{C_0, \rho_\bullet(C_0), \rho_\bullet^2(C_0)\}$$
$$\cup \{C_i', \rho_\bullet(C_i'), \rho_\bullet^2(C_i') \mid i = 1, 2, 3, 4\}$$

is an HOP (C_8)-decomposition for $4K_{13}^\bullet$

∎

6.5.2 The Case When n is Even

Lemma 6.5.2. Let $m \equiv 0 \pmod 4$, and let n be an even positive integer such that $4 \leq m \leq n$. If $2n(n-1) \equiv m \pmod{2m}$, then $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Proof. By Lemma 6.3.6, it suffices to prove this result for even n in the range $m \leq n < 2m$. Moreover, since $2n(n-1) \equiv m \pmod{2m}$, we have $m \neq n$, so we restrict our attention to the range $m < n < 2m$.

As before, the approach is to first construct starter central and starter peripheral cycles, then rotate them using appropriate permutations to generate all the cycles.

We begin by outlining the parameters and then explain how many starter central and starter peripheral cycles are needed.

Let $V(4K_n^\bullet) = \{x_i : i \in \mathbb{Z}_{n-1}\} \cup \{x_\infty\}$, and let $\rho_\bullet$ be the permutation on $E(4K_n^\bullet)$ that preserves the color (and orientation) of the edges, and is induced by the permutation $\rho = (x_\infty)(x_0\ x_1\ x_2\ \ldots\ x_{n-2})$. Observe that the group $\langle \rho_\bullet \rangle$ has the following orbits on the edge set of $4K_n^\bullet$:

- for each $s \in \{1, 2, \ldots, \frac{n-2}{2}\}$, we have a pink and a blue orbit $\{x_i x_{i+s} : i \in \mathbb{Z}_{n-1}\}$;
- for each $s \in \{1, 2, \ldots, n-2\}$, we have a black orbit $\{(x_i, x_{i+s}) : i \in \mathbb{Z}_{n-1}\}$;
- a pink and a blue orbit $\{x_i x_\infty : i \in \mathbb{Z}_{n-1}\}$; and
- black orbits $\{(x_\infty, x_i) : i \in \mathbb{Z}_{n-1}\}$ and $\{(x_i, x_\infty) : i \in \mathbb{Z}_{n-1}\}$.

For convenience, let $S = \{1, 2, \ldots, \frac{n-2}{2}, \infty\}$ be the set of all differences.

Let $n = 2^e a$, where a is odd, and let $m = 2^d a' b'$, where a' and b' are both odd, with $a' \mid a$ and $b' \mid (n-1)$. Since $m \equiv 0 \pmod 4$, we have that $d \geq 2$.

First, we show that $d = e + 1$. We know that $2n(n-1) \equiv m \pmod{2m}$. Consequently, $m \mid 2n(n-1)$ but $m \nmid n(n-1)$. Since $m \mid 2n(n-1)$, we have $2^d a' b' \mid 2 \cdot 2^e a \cdot (n-1) =$

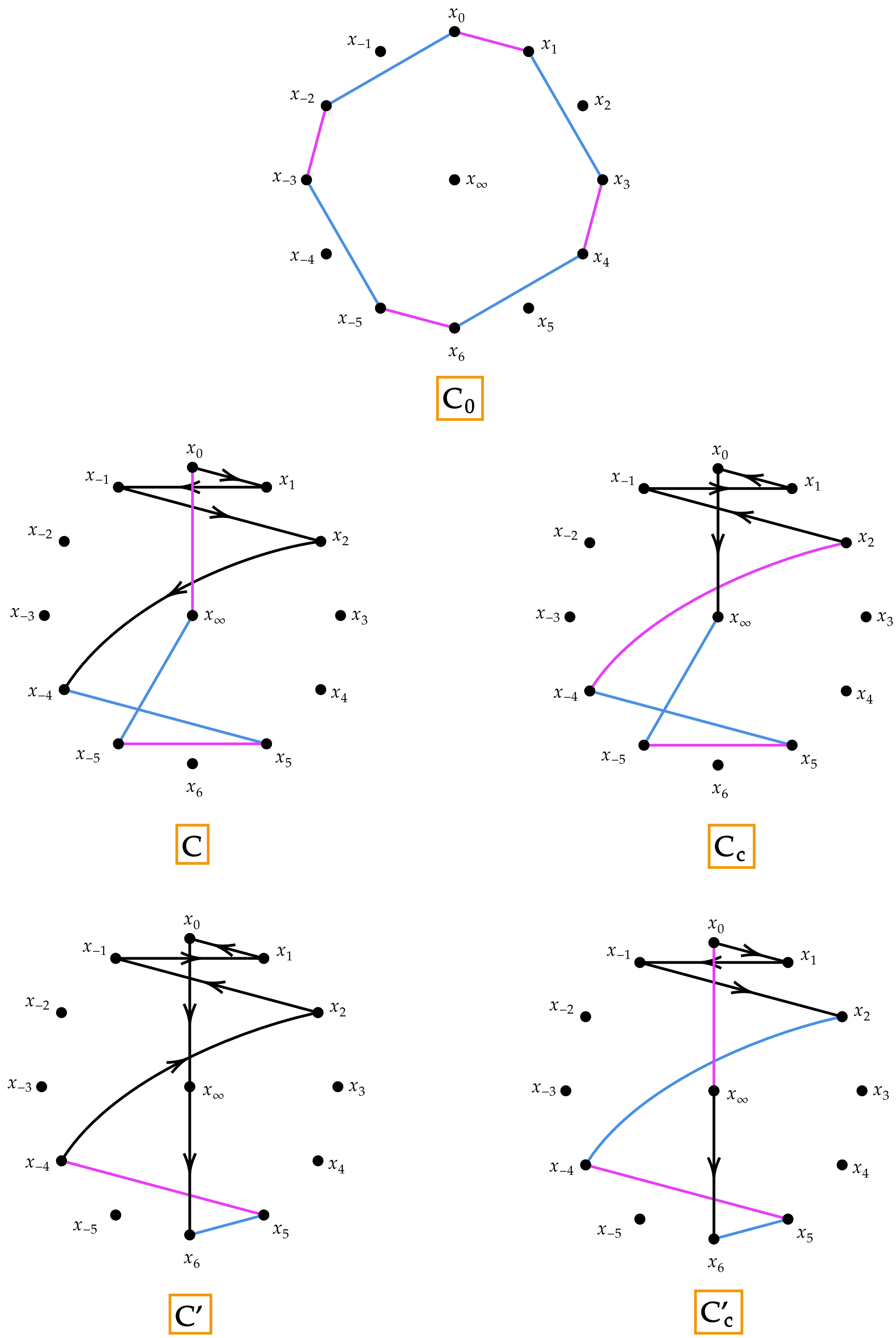


Figure 6.6: Lemma 6.5.1– The starter peripheral cycle C_0 and the four starter central cycles for $m = 8$ and $n = 13$.

$2^{e+1}a(n-1)$. Given that $b' \mid (n-1)$, $a' \mid a$, and a and $n-1$ are both odd, it follows that 2^d must divide 2^{e+1}. Therefore, $d \leq e+1$. On the other hand, we have $m \nmid n(n-1)$; that is, $2^d a'b' \nmid 2^e a(n-1)$. Again, knowing that $b' \mid (n-1)$ and $a' \mid a$, it follows that 2^d cannot divide 2^e. Thus, $d > e$, which implies $d \geq e+1$. Thus, we conclude that $d = e+1$, and we can write $n = 2^{d-1}a$.

To construct peripheral cycles, we partition the $n-1$ vertices in $\{x_i : i \in \mathbb{Z}_{n-1}\}$ into b' segments, each containing $\ell = \frac{n-1}{b'}$ consecutive vertices. Notice that each segment will contribute $2^d a'$ edges toward a peripheral m-cycle.

Now, we explore the required number of central and peripheral cycles for the decomposition. If $4K_n^\bullet$ is (C_m)-decomposable, then the number of m-cycles in the decomposition is

$$\begin{aligned}
\frac{2n(n-1)}{m} &= \frac{2(n-1)(m+n-m)}{m} \\
&= \frac{2(n-1)m + 2(n-1)(n-m)}{m} \\
&= 2(n-1) + \frac{2(n-1)(n-m)}{2^d a'b'} \\
&= 2(n-1) + \frac{(n-1)}{b'} \cdot \frac{2(n-m)}{2^d a'} \\
&= 2(n-1) + \ell \cdot \frac{2(n-m)}{2^d a'}
\end{aligned}$$

Observe that

$$\frac{2(n-m)}{2^d a'} = \frac{2\left(2^{d-1}a - 2^d a'b'\right)}{2^d a'} = \frac{2^d(a-2a'b')}{2^d a'} = \frac{a-2a'b'}{a'}.$$

Since $a' \mid a$ and both a and a' are odd, it follows that $\frac{a-2a'b'}{a'}$ is an odd integer. Let $F = \frac{a-2a'b'}{a'}$. The number of m-cycles can be expressed in the following form:

$$\frac{2n(n-1)}{m} = 2(n-1) + \ell F$$

This suggests constructing two starter central cycles and F starter peripheral cycles. The two starter central cycles are rotated through all $n-1$ positions, while each of the F starter peripheral cycles is rotated through ℓ positions to generate all the cycles in the decomposition.

We showed that F is odd, so we have two cases to consider: one where $F \equiv 1 \pmod 4$ and the other where $F \equiv 3 \pmod 4$.

Before starting the proof for each case, we will first prove some inequalities that will be useful in both cases.

We know that $n < 2m$; that is, $2^{d-1}a < 2^{d+1}a'b'$. Dividing both sides by $2^{d-1}a'$ gives $\frac{a}{a'} < 4b'$. Therefore, $\frac{a-2a'b'}{a'} < 2b'$, which means $F < 2b'$ and thus

$$F \leq 2b' - 1. \tag{6.5.26}$$

Since $F \leq 2b' - 1$, it follows that $b' \geq 3$ whenever $F \equiv 3 \pmod 4$, and $b' \geq 1$ whenever $F \equiv 1 \pmod 4$. In the latter case, when $b' = 1$, we handle it separately.

Additionally, we have $m < n - 1 < 2m$; that is, $2^d a'b' < b'\ell < 2^{d+1}a'b'$. Dividing each side by b', we get

$$2^d a' < \ell < 2^{d+1} a'. \tag{6.5.27}$$

Next, we show that if $\ell - 2^{d-1}a' \leq 2^d a' - 1$, then $F \leq b'$.

$$\begin{aligned}
& & \ell - 2^{d-1}a' &\leq 2^d a' - 1 \\
& & \ell &\leq 3 \cdot 2^{d-1}a' - 1 \\
& & b'\ell &\leq 3 \cdot 2^{d-1}a'b' - b' \\
&(\, b'\ell = n-1) & n &\leq 3 \cdot 2^{d-1}a'b' - b' + 1 \\
& & 2n &\leq 3 \cdot 2^d a'b' - 2b' + 2 \\
&(\, -2b' \leq -2 \text{ since } b' \geq 1) & 2n &\leq 3 \cdot 2^d a'b' \\
&(\, n = 2^{d-1}a) & 2^d a &\leq 3 \cdot 2^d a'b' \\
&(\text{Dividing by } 2^d a') & \frac{a}{a'} &\leq 3b' \\
& & \frac{a - 2a'b'}{a'} &\leq b' \\
& & F &\leq b'
\end{aligned}$$

Therefore,

$$\text{if } \ell - 2^{d-1}a' \leq 2^d a' - 1, \text{ then } F \leq b'. \tag{6.5.28}$$

Now that we have established the necessary inequalities, we proceed with the main proof. We begin with the simpler case.

Case 1: $F \equiv 3 \pmod 4$.

The approach is to construct $\frac{F-3}{4} + 1$ starter peripheral cycles for K_n. Then, take four copies of each of the $\frac{F-3}{4}$ cycles and color them appropriately using Lemma 3.3.3. This process results in $F - 3$ starter cycles in $4K_n^\bullet$. We will then be left with one starter cycle in K_n; we take three copies of it to generate the remaining three starter peripheral cycles in $4K_n^\bullet$.

We start by constructing the starter peripheral cycles.

For $i = 0, 1, \ldots, \frac{F-3}{4}$, define the path P_i as follows:

$$P_i = x_0 \ x_{i\ell+1} \ x_{-1} \ x_{i\ell+2} \ x_{-2} \ldots x_{i\ell+2^{d-1}a'-1} \ x_{-(2^{d-1}a'-1)} \ x_{i\ell+2^{d-1}a'} \ x_{\frac{b'+1}{2}\ell}.$$

The differences covered by P_0 are:

$$1, \ 2, \ 3, \ \ldots, \ 2^d a' - 2, \ 2^d a' - 1, \ \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \tag{6.5.29}$$

and the differences covered by P_i, for $i = 1, \ldots, \frac{F-3}{4}$, are:

$$i\ell+1,\ i\ell+2,\ i\ell+3,\ \ldots,\ i\ell+2^d a'-2,\ i\ell+2^d a'-1,\ \tfrac{(b'-2i+1)\ell}{2} - 2^{d-1}a'. \tag{6.5.30}$$

Since $\ell > 2^d a'$ by (6.5.27), the paths $\rho^j(P_i)$, for $j = 0, \ell, 2\ell, \ldots, (b'-1)\ell$, are pairwise vertex-disjoint except at the endpoints. Also, since b' is odd, we have $\gcd\left(b', \frac{b'+1}{2}\right) = 1$. Thus,

$$C_i = P_i \cup \rho^\ell(P_i) \cup \ldots \cup \rho^{(b'-1)\ell}(P_i)$$

is an m-cycle.

Next, we show that all the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-3}{4}}$, as listed in (6.5.29) and (6.5.30), are distinct.

First, we show that the differences $i\ell+1, i\ell+2, \ldots, i\ell+2^d a'-1$ are all pairwise distinct for $i = 0, 1, \ldots, \frac{F-3}{4}$. It suffices to show that the largest number on this list does not exceed $\frac{n-2}{2}$; that is, to show

$$\frac{F-3}{4}\ell + 2^d a' - 1 \le \frac{n-2}{2}.$$

By (6.5.26), we have $F \le 2b'-1$, which implies that $\frac{F-3}{4}\ell \le \frac{b'-2}{2}\ell$. Additionally, in (6.5.27), we proved that $2^d a' < \ell$. Combining these results, we have

$$\begin{aligned}
\frac{F-3}{4}\ell + 2^d a' - 1 &\le \frac{b'-2}{2}\ell + 2^d a' - 1 \\
&= \frac{b'-1}{2}\ell - \frac{\ell}{2} + 2^d a' - 1 \\
&< \frac{b'-1}{2}\ell + 2^{d-1}a' \qquad (\text{since } 2^d a' < \ell) \\
&< \frac{n-1}{2}.
\end{aligned}$$

In the above inequality, we see that

$$\frac{F-3}{4}\ell + 2^d a' - 1 < \frac{b'-1}{2}\ell + 2^{d-1}a' < \frac{n-1}{2}.$$

This guarantees that for all $j = 0, 1, \ldots, \frac{F-3}{4}$, we have

$$\frac{b'-1}{2}\ell + 2^{d-1}a' \notin \{j\ell+1, j\ell+2, \ldots, j\ell+2^d a'-1\}.$$

Next, we show that for all $i = 1, \ldots, \frac{F-3}{4}$ and $j = 0, 1, \ldots, \frac{F-3}{4}$, the difference $\frac{(b'-2i+1)\ell}{2} - 2^{d-1}a'$ does not appear among the differences

$$j\ell+1, j\ell+2, \ldots, j\ell+2^d a'-1.$$

Since

$$0 < \tfrac{(b'-2\cdot\frac{F-3}{4}+1)\ell}{2} - 2^{d-1}a' < \tfrac{(b'-2\cdot\frac{F-7}{4}+1)\ell}{2} - 2^{d-1}a' < \ldots < \tfrac{(b'-2\cdot 1+1)\ell}{2} - 2^{d-1}a' < \tfrac{(b'-1)\ell}{2} + 2^{d-1}a'.$$

It suffices to show that for all $j = 0, 1, \ldots, \frac{F-3}{4}$ and $1 \leq \alpha \leq 2^d a' - 1$

$$\frac{(b' - 2(\frac{F-3}{4}) + 1)\ell}{2} - 2^{d-1}a' \neq j\ell + \alpha.$$

We prove this by contradiction. Suppose to the contrary that

$$\frac{(b' - 2\left(\frac{F-3}{4}\right) + 1)\ell}{2} - 2^{d-1}a' = j\ell + \alpha,$$
$$\frac{2b' - F + 1}{4}\ell + \ell - 2^{d-1}a' = j\ell + \alpha.$$

Since $F \equiv 3 \pmod 4$, we can write $F = 4k + 3$ for some integer k. Since b' is odd, we can write $b' = 2k' + 1$ for some integer k'. We have $2b' - F + 1 = 4(k' - k)$, which is a multiple of 4. Therefore, $\frac{2b'-F+1}{4}\ell$ is an integer, and since $\ell > 2^d a'$ by (6.5.27), we have

$$\frac{2b' - F + 1}{4} = j \quad \text{and} \quad \ell - 2^{d-1}a' = \alpha.$$

Given that $j \leq \frac{F-3}{4}$, it follows that $F \geq b' + 2$.

On the other hand, knowing that $1 \leq \alpha \leq 2^d a' - 1$ and $\ell - 2^{d-1}a' = \alpha$, it follows that $\ell - 2^{d-1}a' \leq 2^d a' - 1$. In (6.5.28), we showed that $F \leq b'$ under this condition. Therefore, we have $b' + 2 \leq F \leq b'$, which is a contradiction.

We conclude that the differences listed in (6.5.29) and in (6.5.30) are pairwise distinct. Let B be the set of differences covered by the m-cycles $C_1, C_2 \ldots, C_{\frac{F-3}{4}}$; that is,

$$B = \{i\ell + 1, i\ell + 2, \ldots, i\ell + 2^d a' - 1, \tfrac{(b'-2i+1)\ell}{2} - 2^{d-1}a' \mid i = 1, \ldots, \tfrac{F-3}{4}\}.$$

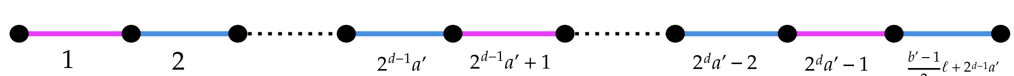
Let $G_1 = \mathrm{Circ}(n-1; B)$. Then, $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 1, \ldots, \frac{F-3}{4}\}$ is a (C_m)-decomposition of G_1. Take four copies of each C_i, for $i = 1, \ldots, \frac{F-3}{4}$, and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. By doing so, we generate $F - 3$ starter peripheral cycles with the appropriate coloring, which we denote $C'_3, \ldots, C'_{F-1}$.

It is now easy to see that

$$\{C'_i, \rho_\bullet(C'_i), \ldots, \rho_\bullet^{\ell-1}(C'_i) \mid i = 3, \ldots, F-1\}$$

is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Notice that $A = \{1, 2, \ldots, 2^d a' - 2, 2^d a' - 1, \frac{(b'-1)\ell}{2} + 2^{d-1}a'\}$ is the set of differences covered by C_0. Take three copies of C_0 and denote them C'_0, C'_1 and C'_2. These are the remaining three peripheral cycles. Color the edges of the cycles C'_1 and C'_2 black, and orient them in the opposite directions. Color the edges of P_0 in the cycle C'_0 alternately pink and blue (as shown below), starting with pink. Since P_0 has length $2^d a'$, which is even, it ends with blue.



The families of cycles generated by C_1' and C_2'; that is, $\{C_i', \rho_\bullet(C_i'), \ldots, \rho_\bullet^{\ell-1}(C_i') \mid i = 1, 2\}$, jointly cover the black orbits corresponding to the differences in the set A. The family of cycles generated by C_0'; that is, $\{C_0', \rho_\bullet(C_0'), \ldots, \rho_\bullet^{\ell-1}(C_0')\}$, covers:

(i) the pink orbits corresponding to the differences

$$1, 3, 5, \ldots, 2^{d-1}a' + 1, 2^{d-1}a' + 3, \ldots, 2^d a' - 3, 2^d a' - 1, \text{ and} \tag{6.5.31}$$

(ii) the blue orbits corresponding to the differences

$$2, 4, 6, \ldots, 2^{d-1}a', 2^{d-1}a' + 2, \ldots, 2^d a' - 2, \frac{(b'-1)\ell}{2} + 2^{d-1}a'. \tag{6.5.32}$$

The differences in the set A will also appear in the central cycles C and C', to be constructed below. There, we need to ensure that the blue orbits corresponding to the differences listed in (6.5.31) and the pink orbits corresponding to the differences listed in (6.5.32) are covered.

Now, we focus on constructing the starter central cycles C and C'. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \backslash (A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 < a_2 < \ldots < a_t \leq \frac{n-2}{2}$ are positive integers and $t = \frac{n-2}{2} - \frac{(F+1)2^d a'}{4}$.

Since $F \leq 2b' - 1$ by (6.5.26), and $F \equiv 3 \pmod 4$, we have $b' \geq 3$. Moreover, since difference ℓ is not covered by any peripheral cycles, we observe that

$$a_1 \leq \ell < \frac{(b'-1)\ell}{2} + 2^{d-1}a' \leq \frac{n-2}{2}.$$

Consider the following sets of differences:

- $W_1 = \{a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2 = \{2^{d-1}a' + 1, 2^{d-1}a' + 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t\}$

Note that $\frac{(b'-1)\ell}{2} + 2^{d-1}a' \notin W_1$, and in the sequence $a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_t$, we assume

$$a_1 < a_2 < \ldots < \frac{(b'-1)\ell}{2} + 2^{d-1}a' < \ldots < a_{t-1} < a_t,$$

with the understanding that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t < \frac{(b'-1)\ell}{2} + 2^{d-1}a'$$

is also possible.

Here, the set W_1 contains all the differences from the set $\mathscr{L}$ except ∞, and has size $|W_1| = t$. The set W_2 contains half of the differences from the set A, along with all the differences from $\mathscr{L}$ except ∞, and has size $|W_2| = t + 2^{d-1}a'$. By Corollary 6.3.4, there exists a path, call it T_1, of length $2t + 2^{d-1}a'$ that starts at x_0 and covers the differences in the following order:

$$T_1 : a_1, \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_1, 2^d a' - 1, \ldots, 2^{d-1}a' + 1.$$

By Lemma 6.3.3, the difference $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ can be replaced with difference 1; that is, there exists a path, call it T_2, that covers the differences in the following order:

$$T_2 : a_1, a_2, \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, 1, \ldots, a_2, a_1, 2^d a' - 1, \ldots, 2^{d-1}a' + 1.$$

Now, consider the following sets of differences:

- $W_1 = \{a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2' = \{2, 3, 4, \ldots, 2^{d-1}a' - 1, 2^{d-1}a', a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t\}$

Again, note that $\frac{(b'-1)\ell}{2} + 2^{d-1}a' \notin W_1$, and in the sequence $a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t$, we assume

$$a_1 < a_2 < \ldots < \frac{(b'-1)\ell}{2} + 2^{d-1}a' < \ldots < a_{t-1} < a_t,$$

with the understanding that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t < \frac{(b'-1)\ell}{2} + 2^{d-1}a'$$

is also possible.

Here, $|W_1| = t$ and $|W_2'| = t + 2^{d-1}a'$. By Corollary 6.3.4, there exists a path, call it T_3, of length $2t + 2^{d-1}a'$ that starts at x_0 and covers the differences in the following order:

$$T_3 : a_1, \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_1, 2^{d-1}a', 2^{d-1}a' - 1, \ldots, 3, 2.$$

By Lemma 6.3.3, the difference $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ can be replaced with difference 1; that is, there exists a path, call it T_4, that covers the differences in the following order:

$$T_4 : a_1, \ldots, a_{t-1}, a_t, a_t, a_{t-1}, \ldots, 1, \ldots, a_1, 2^{d-1}a', 2^{d-1}a' - 1, \ldots, 3, 2.$$

Notice that the lengths of the paths T_1, T_2, T_3, T_4 are all equal to $2t + 2^{d-1}a'$, and

$$\begin{aligned}
2t + 2^{d-1}a' &= 2\left(\frac{n-2}{2} - \frac{(F+1)2^d a'}{4}\right) + 2^{d-1}a' \\
&= n - 2 - (F+1)2^{d-1}a' + 2^{d-1}a' \\
&= n - 2 - F \cdot 2^{d-1}a' \\
\text{(substitute } F = \frac{a - 2a'b'}{a'}\text{)} \quad &= n - 2 - 2^{d-1}a + 2^d a'b'
\end{aligned}$$

$$\begin{aligned}(\text{substitute } n = 2^{d-1}a \ \&\ m = 2^d a'b') \quad &= n - 2 - n + m \\ &= m - 2.\end{aligned}$$

To simplify the explanation of the coloring process, we express T_1, T_2, T_3, and T_4 as the following concatenations of subpaths:

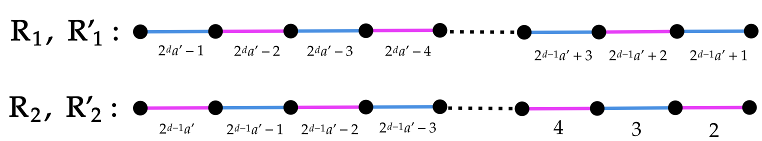
- $T_1 = PQ_1R_1$
- $T_2 = PQ_2R_1'$
- $T_3 = PQ_1R_2$
- $T_4 = PQ_2R_2'$

Here, P, Q_1, Q_2, R_1, R_1', R_2', and R_2 represent subpaths that cover specific sequences of differences:

- $P : a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t$
- $Q_1 : a_t, a_{t-1}, a_{t-2}, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_3, a_2, a_1$
- $Q_2 : a_t, a_{t-1}, a_{t-2}, \ldots, 1, \ldots, a_3, a_2, a_1$
- $R_1, R_1' : 2^d a' - 1, 2^d a' - 2, 2^d a' - 3, \ldots, 2^{d-1}a' + 3, 2^{d-1}a' + 2, 2^{d-1}a' + 1$
- $R_2, R_2' : 2^{d-1}a', 2^{d-1}a' - 1, 2^{d-1}a' - 2, \ldots, 4, 3, 2$

Note that P has length t, while Q_1 and Q_2 each have length $t+1$. The paths R_1 and R_1' cover the same sequence of differences, as do R_2 and R_2', with each path having length $2^{d-1}a' - 1$.

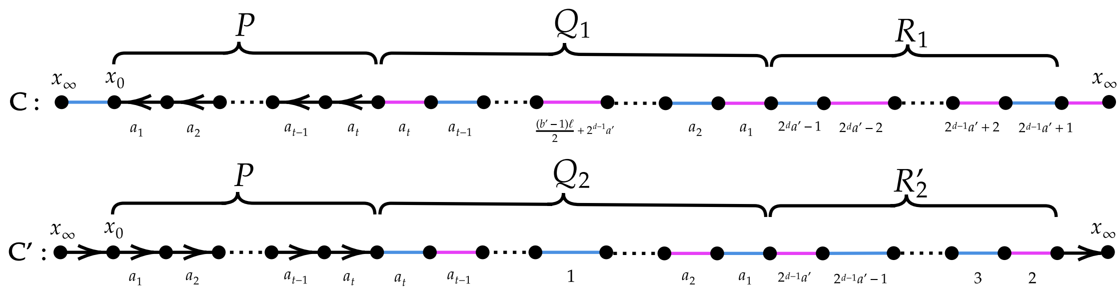
Now, color the edges in the paths R_1 and R_1' alternately pink and blue, starting with blue. Since their length is $2^{d-1}a' - 1$, which is odd, they end with blue. Similarly, color the edges in the paths R_2 and R_2' alternately pink and blue, starting with pink. Since their length is odd, they end with pink.



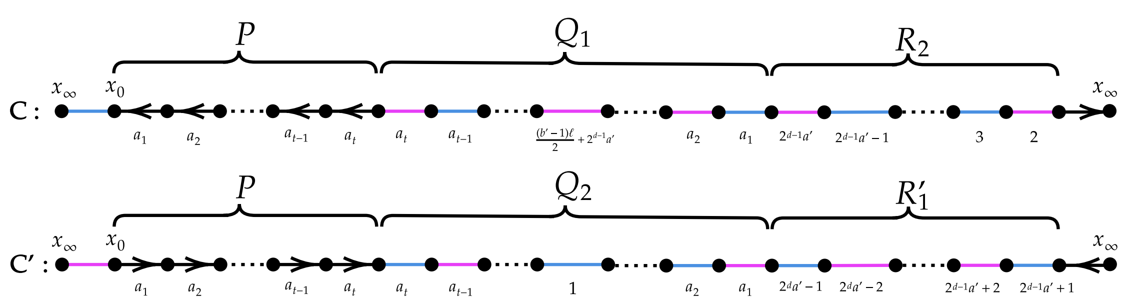
Next, color the edges in the path Q_1 alternately pink and blue, ensuring that the edge with difference $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ is pink. Color the edges in the path Q_2 alternately pink and blue, but in the opposite way of Q_1. Since Q_1 and Q_2 cover the same sequence of differences except for $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ and 1, this ensures that the edge with difference 1 is blue. With this coloring approach, we ensure that the blue copies of the differences listed in (6.5.31) and the pink copies of the differences listed in (6.5.32) are covered. We color the remaining paths as described below and then construct the central cycles based on the coloring.

First, assume that Q_1 starts with pink, meaning that the edge with difference a_t is assigned the pink color. Then Q_2 must start with blue. Depending on the parity of t, there are two cases:

(i) If t is even, since Q_1 and Q_2 have length $t+1$, we know Q_1 ends with pink and Q_2 ends with blue. In this case, let $C = x_\infty\, P\, Q_1\, R_1\, x_\infty$ and $C' = x_\infty\, P\, Q_2\, R_2'\, x_\infty$, and color the rest of the edges as follows.

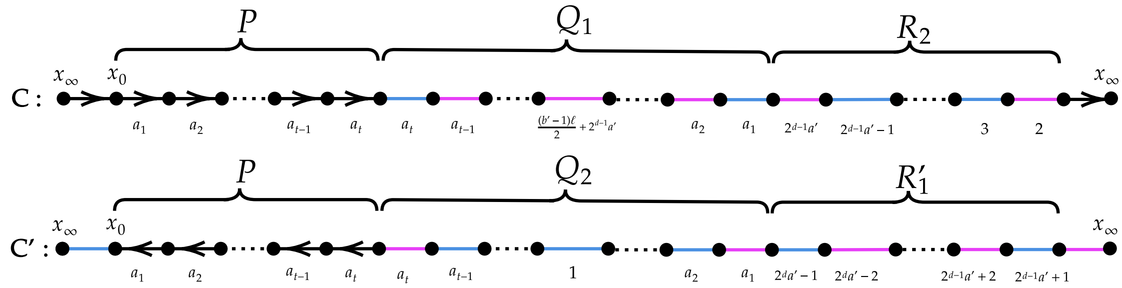


(ii) If t is odd, as Q_1 and Q_2 have length $t+1$, we know Q_1 ends with blue and Q_2 ends with pink. In this case, let $C = x_\infty\, P\, Q_1\, R_2\, x_\infty$ and $C' = x_\infty\, P\, Q_2\, R_1'\, x_\infty$, and color the rest of the edges as follows.

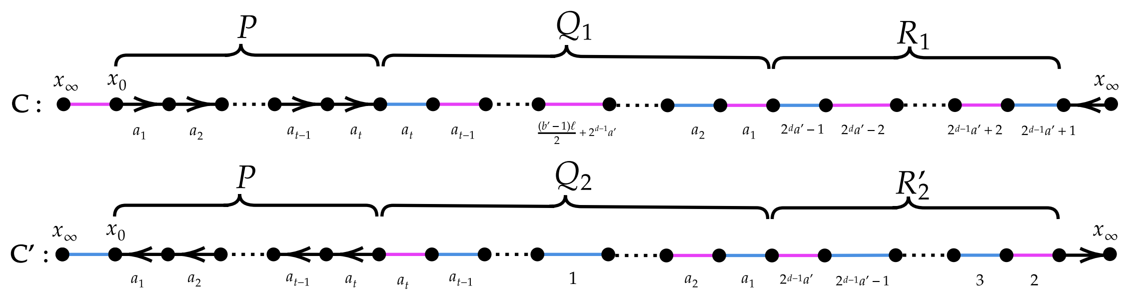


Next, assume that Q_1 starts with blue. Then Q_2 must start with pink. Depending on the parity of t, there are two cases:

(i) If t is even, then Q_1 ends with blue and Q_2 ends with pink. In this case, let $C = x_\infty\, P\, Q_1\, R_2\, x_\infty$ and $C' = x_\infty\, P\, Q_2\, R_1'\, x_\infty$, and color the rest of the edges as follows.



(ii) If t is odd, then Q_1 ends with pink and Q_2 ends with blue. In this case, let $C = x_\infty\, P\, Q_1\, R_1\, x_\infty$ and $C' = x_\infty\, P\, Q_2\, R_2'\, x_\infty$, and color the rest of the edges as follows.



We can easily verify that both C and C' are cycles of length m, and that they satisfy Condition **(C1)** of Definition 1.2.3. Moreover, they jointly contain exactly one blue edge, one pink edge, and two black arcs from each orbit of $\langle \rho_\bullet \rangle$ corresponding to differences in $\mathscr{L}$.

The black orbits corresponding to the differences in A are covered by C_1' and C_2', while the pink and the blue orbits of the differences in A are covered by C, C', and C_0'. Thus,

$$\left\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, n-2\right\} \cup \left\{C_i', \rho_\bullet(C_i'), \ldots, \rho_\bullet^{\ell-1}(C_i') \mid i = 0, 1, 2\right\}$$

is an HOP (C_m)-decomposition for $4G_2^\bullet$, where

$$G_2 = \mathrm{Circ}(n-1; A \cup (\mathscr{L} - \{\infty\})) \bowtie K_1.$$

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

Case 2: $F \equiv 1 \pmod 4$.

For Case 2, we first divide the proof into two subcases: one where $b' = 1$ and the other where $b' \geq 3$.

Subcase 2.1: $b' = 1$. It follows from (6.5.26) that $F = 1$, and thus we only need to construct one family of peripheral cycles.

Since $F = 1$, we have $\frac{a - 2a'b'}{a'} = 1$. Moreover, since $b' = 1$, it follows that $a = 3a'$. Thus, $n = \frac{3}{2}m$. In this case, the set of differences is $S = \{1, 2, \ldots, \frac{3m-4}{4}, \infty\}$.

We define the starter peripheral cycle C_0 as follows:

$$C_0 = x_{-1}\ x_1\ x_2\ x_{-2}\ x_3\ x_{-3}\ x_4\ x_{-4}\ \ldots$$
$$x_{-(\frac{m}{4}-2)}\ x_{\frac{m}{4}-1}\ x_{-(\frac{m}{4}-1)}\ x_{\frac{m}{4}}\ x_{-\frac{m}{4}}\ x_{\frac{m}{2}}\ x_{-(\frac{m}{2}-1)}\ x_{\frac{m}{2}+1}\ x_{-\frac{m}{2}}\ \ldots$$
$$x_{\frac{3m}{4}-3}\ x_{-(\frac{3m}{4}-4)}\ x_{\frac{3m}{4}-2}\ x_{-(\frac{3m}{4}-3)}\ x_{-(\frac{3m}{4}-2)}\ x_{\frac{3m}{4}-1}\ x_{-1}.$$

The cycle C_0 covers the differences in the following order:

$$2, 1, 4, 5, 6, \ldots, \tfrac{m}{2}-3, \tfrac{m}{2}-2, \tfrac{m}{2}-1, \tfrac{m}{2}, \tfrac{3m-4}{4}, \tfrac{m}{2}, \tfrac{m}{2}-1, \tfrac{m}{2}-2, \ldots, 6, 5, 4, 1, 2, \tfrac{3m-4}{4}.$$

Observe that C_0 covers each difference in $A = \{1, 2, 4, 5, \ldots, \frac{m}{2}-1, \frac{m}{2}, \frac{3m-4}{4}\}$ exactly twice. Color the edges of C_0 black and orient them forwards as shown in Figure 6.7.

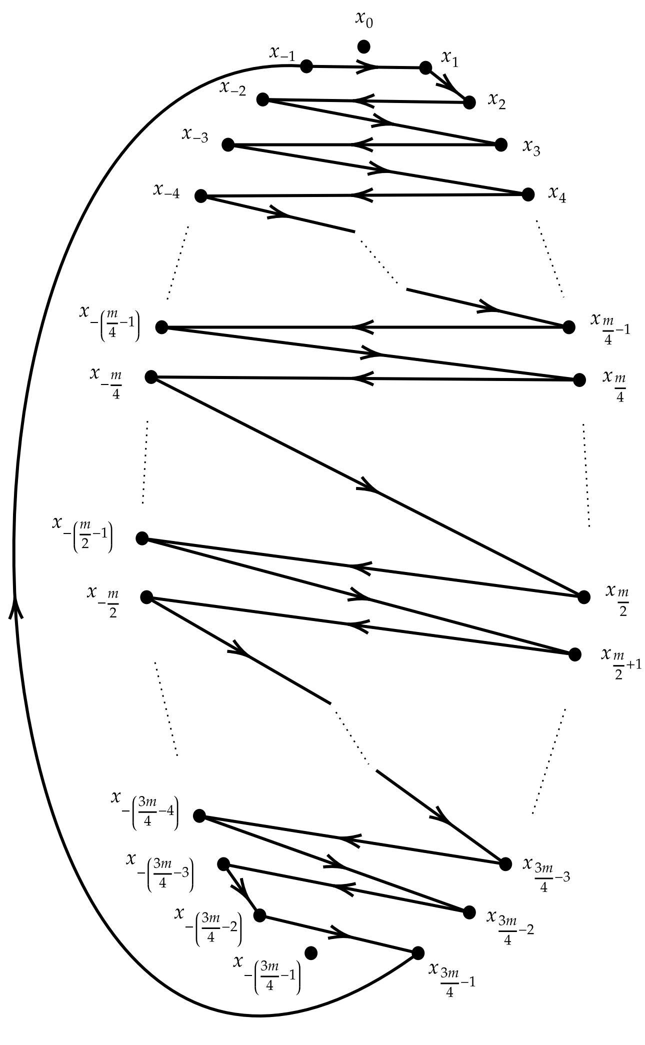


Figure 6.7: Orientation and coloring of the edges of C_0.

Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S \setminus A$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 = 3 < a_2 < \ldots < a_t < \frac{n-2}{2} = \frac{3m-4}{4}$ are integers and $t = \frac{m-4}{4}$. Also, observe that $\frac{m}{2} < a_2$. Consider the following sets of differences:

- $W_1 = \{1, 2, a_1, 4, 5 \ldots, \frac{m}{2} - 1, \frac{m}{2}, a_2, a_3, \ldots, a_{t-1}, a_t, \frac{3m-4}{4}\}$
- $W_2 = \{a_1, a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t\}$

Observe that

$$1 < 2 < a_1 < 4 < 5 < \ldots < \frac{m}{2} < a_2 < a_3 < \ldots < a_{t-1} < a_t < \frac{3m-4}{4}.$$

Here, $|W_1| = \frac{m}{2} + t$ and $|W_2| = t$. By Corollary 6.3.4, there exists a path, call it T, of length $2t + \frac{m}{2} = m - 2$ that starts at x_0 and covers the differences in the following order:

$$T : 1, 2, a_1, 4, 5, \ldots, \tfrac{m}{2} - 1, \tfrac{m}{2}, a_2, a_3, \ldots, a_{t-1}, a_t, \tfrac{3m-4}{4}, a_t, a_{t-1}, \ldots, a_2, a_1.$$

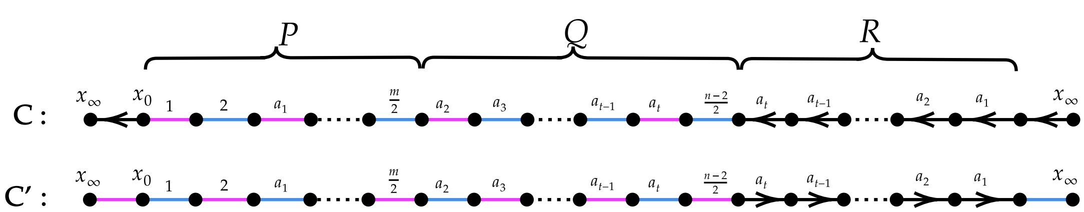
To simplify the explanation of the coloring process, let P, Q, and R be the subpaths of T that cover the following sequences of differences.

- $P : 1, 2, a_1, 4, 5, \ldots, \frac{m}{2} - 1, \frac{m}{2}$
- $Q : a_2, a_3, \ldots, a_{t-2}, a_{t-1}, a_t, \frac{3m-4}{4}$
- $R : a_t, a_{t-1}, a_{t-2}, \ldots, a_3, a_2, a_1$

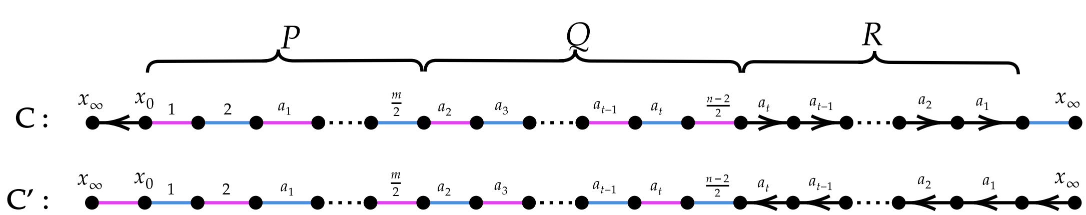
Note that $T = PQR$ and P has length $\frac{m}{2}$, while Q and R each have length t. Let $C = x_\infty PQRx_\infty$; observe that C is a cycle of length m.

Let C' be another copy of C. In C, color the edges of the path P alternatingly pink and blue, starting with pink. Since the length of the path P is even, it ends with blue. In C', color the edges of the path P alternatingly pink and blue, starting with blue; it ends with pink. Depending on the parity of t, we color the rest of the edges as follows:

- If t is even, Q ends with blue in C and with pink in C'.



- If t is odd, Q ends with pink in C and with blue in C'.



The cycle C_0 covers the black copies of the differences in the set A, while the pink and blue copies of these differences are covered by the paths P and Q in the central cycles. Additionally, the central cycles C and C' together cover exactly one blue edge, one pink edge, and two opposite black arcs corresponding to the differences in the set $\mathscr{L}$. Furthermore, the cycles C_0, C, and C' satisfy Condition **(C1)** in Definition 1.2.3. Therefore,

$$\{\rho_\bullet^i(C_0), \rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, n-2\}$$

is an HOP (C_m)-decomposition of $4K_n^\bullet$.

Subcase 2.2: $b' \geq 3$. Here, the approach is to construct $\frac{F-1}{4} + 1$ starter peripheral cycles for K_n. Then, take four copies of each of the $\frac{F-1}{4}$ cycles and color them appropriately using Lemma 3.3.3 to generate $F-1$ starter peripheral cycles in $4K_n^\bullet$. The $F-1$ families of peripheral cycles cover a subset $B \subset S$ of differences. We will then be left with one remaining starter peripheral cycle, which, together with the two central cycles, will cover the remaining differences; that is, the differences in the set $S \setminus B$.

We start by constructing the starter peripheral cycles. We construct paths P_i for $i = 0, 1, 2, \ldots, \frac{F-1}{4}$. From (6.5.26), it follows that $\frac{F-1}{4} \leq \frac{b'-1}{2}$. Note that the construction of P_i differs depending on whether $\frac{F-1}{4} < \frac{b'-1}{2}$ or $\frac{F-1}{4} = \frac{b'-1}{2}$. We address both situations below.

Situation 1: $\frac{F-1}{4} < \frac{b'-1}{2}$.

As in Case 1, for $i = 0, 1, \ldots, \frac{F-1}{4}$, let P_i be:

$$P_i = x_0\ x_{i\ell+1}\ x_{-1}\ x_{i\ell+2}\ x_{-2} \ldots x_{i\ell+2^{d-1}a'-1}\ x_{-(2^{d-1}a'-1)}\ x_{i\ell+2^{d-1}a'}\ x_{\frac{b'+1}{2}\ell}.$$

The differences covered by P_0 are:

$$1,\ 2,\ 3,\ \ldots,\ 2^d a' - 2,\ 2^d a' - 1,\ \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \tag{6.5.33}$$

and the differences covered by P_i, for $i = 1, \ldots, \frac{F-1}{4}$, are:

$$i\ell + 1,\ i\ell + 2,\ i\ell + 3,\ \ldots,\ i\ell + 2^d a' - 2,\ i\ell + 2^d a' - 1,\ \tfrac{(b'-2i+1)\ell}{2} - 2^{d-1}a'. \tag{6.5.34}$$

Since $\ell > 2^d a'$ by (6.5.27), the paths $\rho^j(P_i)$, for $j = 0, \ell, 2\ell, \ldots, (b'-1)\ell$, are pairwise vertex-disjoint except at the endpoints. Also, since b' is odd, we have $\gcd(b', \frac{b'+1}{2}) = 1$. Thus,

$$C_i = P_i \cup \rho^\ell(P_i) \cup \ldots \cup \rho^{(b'-1)\ell}(P_i)$$

is an m-cycle.

Next, we show that all the differences covered by the m-cycles $C_0, C_1, \ldots, C_{\frac{F-1}{4}}$, as listed in 6.5.33 and 6.5.34, are pairwise distinct. First, we show that the differences $i\ell+1, i\ell+2, \ldots, i\ell + 2^d a' - 1$ are all distinct for $i = 0, 1, \ldots, \frac{F-1}{4}$. It suffices to show that

$$\frac{F-1}{4}\ell + 2^d a' - 1 \leq \frac{n-2}{2}.$$

Since $\frac{F-1}{4} < \frac{b'-1}{2}$, we have $\frac{F-1}{4}\ell \leq \frac{b'-3}{2}\ell$. Additionally, in (6.5.27), we proved that $2^d a' < \ell$. Therefore, we have

$$\begin{aligned}\frac{F-1}{4}\ell + 2^d a' - 1 &\leq \frac{b'-3}{2}\ell + 2^d a' - 1\\ &= \frac{b'-1}{2}\ell - \ell + 2^d a' - 1\\ &< \frac{b'-1}{2}\ell + 2^{d-1}a' \qquad (\text{since } 2^d a' < \ell)\\ &< \frac{n-1}{2}.\end{aligned}$$

In the above inequality, we see that

$$\frac{F-1}{4}\ell + 2^d a' - 1 < \frac{b'-1}{2}\ell + 2^{d-1}a' < \frac{n-1}{2}.$$

This guarantees that for all $j = 0, 1, \ldots, \frac{F-1}{4}$, we have

$$\frac{b'-1}{2}\ell + 2^{d-1}a' \notin \{j\ell+1, j\ell+2, \ldots, j\ell + 2^d a' - 1\}.$$

Next, we show that for all $i = 1, \ldots, \frac{F-1}{4}$ and $j = 0, 1, \ldots, \frac{F-1}{4}$

$$\frac{(b'-2i+1)\ell}{2} - 2^{d-1}a' \notin \{j\ell+1, j\ell+2, \ldots, j\ell + 2^d a' - 1\}.$$

Since

$$0 < \tfrac{(b'-2\cdot\frac{F-1}{4}+1)\ell}{2} - 2^{d-1}a' < \tfrac{(b'-2\cdot\frac{F-5}{4}+1)\ell}{2} - 2^{d-1}a' < \ldots < \tfrac{(b'-2\cdot 1+1)\ell}{2} - 2^{d-1}a' < \tfrac{(b'-1)\ell}{2} + 2^{d-1}a',$$

it suffices to show that for all $j = 0, 1, \ldots, \frac{F-1}{4}$ and $1 \leq \alpha \leq 2^d a' - 1$

$$\frac{(b' - 2(\frac{F-1}{4}) + 1)\ell}{2} - 2^{d-1}a' \neq j\ell + \alpha.$$

We prove this by contradiction. Suppose to the contrary that

$$\begin{aligned}\frac{(b' - 2\left(\frac{F-1}{4}\right) + 1)\ell}{2} - 2^{d-1}a' &= j\ell + \alpha\\ \frac{2b'-F-1}{4}\ell + \ell - 2^{d-1}a' &= j\ell + \alpha.\end{aligned}$$

Since $F \equiv 1 \pmod 4$, we can write $F = 4k+1$ for some integer k. Since b' is odd, we can write $b' = 2k'+1$ for some integer k'. We have $2b' - F - 1 = 4(k'-k)$, which is a multiple of 4. Therefore, $\frac{2b'-F-1}{4}\ell$ is an integer. Also, knowing that $2^d a' < \ell$ by (6.5.27), we have

$$\frac{2b'-F-1}{4} = j \quad \text{and} \quad \ell - 2^{d-1}a' = \alpha,$$

Given $j \le \frac{F-1}{4}$, it follows that $F \ge b'$.

On the other hand, we have $1 \le \alpha \le 2^d a' - 1$ and $\ell - 2^{d-1}a' = \alpha$, it follows that $\ell - 2^{d-1}a' \le 2^d a' - 1$. In (6.5.28), we showed that $F \le b'$ under this condition. Therefore, we have $b' \le F \le b'$, which implies $F = b'$. From this, it follows that

$$\frac{a - 2a'b'}{a'} = b'.$$

Simplifying further, we obtain $a = 3a'b'$, and consequently, $n = 2^{d-1} \cdot 3a'b'$. Since $b' \mid (n-1)$, it must hold that $b' = 1$. However, this contradicts the fact that $b' \ge 3$. Thus, all the differences listed in (6.5.33) and (6.5.34) are pairwise distinct.

Situation 2: $\frac{F-1}{4} = \frac{b'-1}{2}$.

First, we explain the reasoning behind separating the case $\frac{F-1}{4} = \frac{b'-1}{2}$. Let $f = \frac{F-1}{4} = \frac{b'-1}{2}$.

In Situation 1, for $i = 0, 1, \ldots, \frac{F-1}{4}$, where $\frac{F-1}{4} < \frac{b'-1}{2}$, we showed that all differences covered by the paths P_i are pairwise distinct. If we were to follow the same pattern for P_f, then path P_f would cover the following differences:

$$\begin{aligned} & f\ell+1,\ f\ell+2,\ \ldots,\ f\ell+2^{d-1}a'-1,\ f\ell+2^{d-1}a',\ f\ell+2^{d-1}a'+1, \\ & \ldots,\ f\ell+\tfrac{\ell-3}{2},\ f\ell+\tfrac{\ell-1}{2},\ f\ell+\tfrac{\ell-1}{2},\ f\ell+\tfrac{\ell-3}{2},\ \ldots,\ \ell-2^{d-1}a'. \end{aligned} \tag{6.5.35}$$

Notice that $f\ell + 2^{d-1}a' = \frac{b'-1}{2}\ell + 2^{d-1}a'$. However, this difference has already occurred in P_0. Continuing further, we encounter $f\ell + \frac{\ell-1}{2} = \frac{n-2}{2}$, after which more repeated differences occur.

To address this issue, we use the same construction up to a certain point but modify it to avoid repeated differences. Specifically:

- We carefully avoid the difference $f\ell + 2^{d-1}a' = \frac{b'-1}{2}\ell + 2^{d-1}a'$.
- We replace differences occurring after $f\ell + \frac{\ell-1}{2} = \frac{n-2}{2}$ with differences from the interval $[2^d a', \ell]$.

Given that the length of P_f must be $2^d a'$, we replace $2^d a' - \frac{\ell-3}{2}$ of the differences listed in (6.5.35).

Next, we explain why we chose differences from the interval $[2^d a', \ell]$.

- The second-to-last difference used in P_0 is $2^d a' - 1$, while the first difference used in P_1 is $\ell + 1$.
- The differences $\frac{(b'-2i+1)\ell}{2} - 2^{d-1}a'$ decrease as i increases. The last path before P_f is P_{f-1}. For $i = f - 1 = \frac{b'-3}{2}$, the corresponding difference is $2\ell - 2^{d-1}a'$. By (6.5.27), we have $2^{d-1}a' < 2^d a' < \ell$. Multiplying by -1, we obtain $-\ell < -2^{d-1}a'$. Adding 2ℓ to both sides, we get:
$$\ell < 2\ell - 2^{d-1}a'.$$

Therefore, replacing the (potentially) repeated differences with differences in the range $[2^d a', \ell]$ is a safe option, as it ensures that the differences covered by P_f are distinct from those covered by P_i for $i = 0, 1, \ldots, f-1$.

Before discussing the path construction, we first prove some inequalities that will be useful later.

For any $d \geq 2$, odd $a' \geq 1$, and odd $b' \geq 3$, except for the cases $(a', d, b') \in \{(1,2,3), (1,2,5), (1,3,3)\}$, we show below that the following inequality holds:

$$2\left(\frac{2^{d-1}a'+1}{2^{d-1}a'-1}\right) < b' \tag{6.5.36}$$

Rewriting the left-hand side, we have:

$$\begin{aligned}2\left(\frac{2^{d-1}a'+1}{2^{d-1}a'-1}\right) &= \frac{2\cdot 2^{d-1}a'+2}{2^{d-1}a'-1},\\ &= 2+\frac{4}{2^{d-1}a'-1}.\end{aligned}$$

Observe that:

- If $(a', d) = (1, 2)$, then we have $2 + \frac{4}{1} = 6 \not< b'$ for $b' \in \{3, 5\}$.
- If $(a', d) = (1, 3)$, then we have $2 + \frac{4}{3} \approx 3.33 \not< b'$ for $b' = 3$.

Except for the above cases, we can see that $2^{d-1}a' \geq 6$, so:

$$\frac{4}{2^{d-1}a'-1} \leq \frac{4}{5} < 1.$$

Thus:

$$2\left(\frac{2^{d-1}a'+1}{2^{d-1}a'-1}\right) < 2+1 = 3.$$

Since $b' \geq 3$, we conclude:

$$2\left(\frac{2^{d-1}a'+1}{2^{d-1}a'-1}\right) < b'.$$

Note that $\frac{F-1}{4} = \frac{b'-1}{2}$, which implies $F = 2b' - 1$. Consequently, we have $\frac{a}{a'} = 4b' - 1$. From this, it follows that $a = 4a'b' - a'$. Substituting this into the expression for n, we obtain

$$n = 2^{d-1}(4a'b' - a').$$

Using this, we calculate ℓ as

$$\ell = \frac{n-1}{b'} = 2^{d+1}a' - \frac{2^{d-1}a'+1}{b'}, \tag{6.5.37}$$

Using the inequality in (6.5.36), we try to determine a lower bound for ℓ.

$$\begin{aligned}
2\left(\frac{2^{d-1}a'+1}{2^{d-1}a'-1}\right) &< b' \\
\frac{2^{d-1}a'+1}{b'} &< \frac{2^{d-1}a'-1}{2} \\
-\frac{2^{d-1}a'+1}{b'} &> -\frac{2^{d-1}a'-1}{2} \\
2^{d+1}a' - \frac{2^{d-1}a'+1}{b'} &> 2^{d+1}a' - \frac{2^{d-1}a'-1}{2} \\
2^{d+1}a' - \frac{2^{d-1}a'+1}{b'} &> 2^{d+1}a' - 2^{d-2}a' + \frac{1}{2} \\
\text{(Using (6.5.37))} \qquad \ell &> 7\cdot 2^{d-2}a' + \frac{1}{2}.
\end{aligned}$$

Therefore, for any $d \geq 2$, odd $a' \geq 1$, and odd $b' \geq 3$, except for the cases $(a', d, b') \in \{(1,2,3),(1,2,5),(1,3,3)\}$, we have the following:

$$\ell > 7\cdot 2^{d-2}a' + \frac{1}{2}. \tag{6.5.38}$$

Now, using the inequality (6.5.38), we prove that the following inequality holds, except for the cases $(a', d, b') \in \{(1,2,3),\ (1,2,5),\ (1,3,3)\}$:

$$-(2^{d-2}a'+1) \geq -\frac{\ell-1}{4}. \tag{6.5.39}$$

The proof is as follows:

$$\begin{aligned}
\text{(By inequality given in (6.5.38))} \qquad \ell &> 7\cdot 2^{d-2}a' + \frac{1}{2} \\
\ell - 1 &> 7\cdot 2^{d-2}a' - \frac{1}{2} \\
\frac{\ell-1}{4} &> \frac{7\cdot 2^{d-2}a'}{4} - \frac{1}{8} \\
\frac{\ell-1}{4} &> 2^{d-2}a' + \frac{3\cdot 2^{d-1}a'-1}{8} \\
(a' \geq 1, d \geq 2, \text{ and } (a', d) \neq (1,2)) \qquad \frac{\ell-1}{4} &> 2^{d-2}a' + 1 \\
-(2^{d-2}a'+1) &> -\frac{\ell-1}{4}.
\end{aligned}$$

Next, using the inequality (6.5.38), we have

$$\frac{3\ell-1}{2} > 21\cdot 2^{d-3}a' + \frac{1}{4} > 16\cdot 2^{d-3}a' = 2^{d+1}a'.$$

Thus, the following inequality holds, except for the cases $(a', d, b') \in \{(1,2,3), (1,2,5), (1,3,3)\}$:

$$\frac{3\ell - 1}{2} > 2^{d+1}a' \tag{6.5.40}$$

We now proceed to show the following inequality:

$$\frac{\ell - 3}{4} < 2^d a' - \frac{\ell + 1}{4} \tag{6.5.41}$$

From (6.5.27), we know that $\frac{\ell}{2} < 2^d a'$. By subtracting $\frac{\ell+1}{4}$ from both sides, we obtain $\frac{\ell-1}{4} < 2^d a' - \frac{\ell+1}{4}$. Thus, it follows that

$$\frac{\ell - 3}{4} < \frac{\ell - 1}{4} < 2^d a' - \frac{\ell + 1}{4}.$$

As mentioned earlier $2^d a' - \frac{\ell-3}{2}$ is the number of differences that will be replaced. Observe below that this number is positive.

Since $n < 2m$, we have

$$\begin{aligned} n - 1 &< 2m \\ b'\ell &< 2^{d+1}a'b' \\ \text{(Divide by } b'\text{)} \qquad \ell &< 2^{d+1}a' \\ \frac{\ell - 3}{2} &< 2^d a'. \end{aligned}$$

Next, we explain how we construct the path P_f. Depending on the value of ℓ, there are two cases to consider: when $\ell \equiv 3 \pmod 4$ and when $\ell \equiv 1 \pmod 4$. Moreover, the construction will be verified using the inequalities proven above. Note that the inequality (6.5.38) is proved for any $d \geq 2$, odd $a' \geq 1$, and odd $b' \geq 3$, except for the cases $(a', d, b') \in \{(1,2,3), (1,2,5), (1,3,3)\}$. When $(a', d, b') = (1, 2, 5)$, by (6.5.37), we have $\ell = 7.4$, and when $(a', d, b') = (1, 3, 3)$, we have $\ell \approx 14.33$. In both cases, ℓ is not an integer. Thus, these cases do not arise.

When $(a', d, b') = (1, 2, 3)$, we have $\ell = 7$ by (6.5.37), $m = 12$, $n = 22$, $f = 1$, and $F = 5$. In this case, $\ell \equiv 3 \pmod 4$, and we verify the construction of the path P_f without using inequality (6.5.38).

- **Situation 2(a):** $\ell \equiv 3 \pmod 4$.

  Define a walk P_f as follows (see Figure 6.8):

$$\begin{aligned} P_f = & x_0\ x_{f\ell+1}\ x_{-1}\ x_{f\ell+2}\ x_{-2} \dots \\ & x_{-(2^{d-2}a'-1)}\ x_{f\ell+2^{d-2}a'}\ x_{-(2^{d-2}a'+1)}\ x_{f\ell+2^{d-2}a'+1}\ x_{-(2^{d-2}a'+2)}\ x_{f\ell+2^{d-2}a'+2} \dots \\ & x_{f\ell+\frac{\ell-3}{4}}\ x_{-\frac{\ell+1}{4}}\ x_{2^d a'-\frac{\ell+1}{4}}\ x_{-\frac{\ell+5}{4}}\ x_{2^d a'-\frac{\ell-3}{4}} \dots \\ & x_{-(2^{d-1}a'-1)}\ x_{3\cdot 2^{d-1}a'-\frac{\ell+3}{2}}\ x_{-(2^{d-1}a')}\ x_{3\cdot 2^{d-1}a'-\frac{\ell+1}{2}}\ x_\ell \end{aligned}$$

First, note that for the case $(a', d, b') = (1, 2, 3)$, we have $P_f = x_0\ x_8\ x_{-2}\ x_2\ x_7$, and the differences covered are $8, 10, 4, 5$. It is clear that P_f is a path and the differences covered are pairwise distinct.

For the rest, to show that P_f is a path, we verify the following three inequalities.

- First, we show that
$$-(2^{d-2}a' + 1) \geq -\frac{\ell+1}{4}.$$
This is proved in (6.5.39).

- Second, we show that:
$$3 \cdot 2^{d-1}a' - \frac{\ell+1}{2} \geq 2^d a' - \frac{\ell+1}{4}, \text{ which is equivalent to } 2^{d-1}a' \geq \frac{\ell+1}{4}.$$
From (6.5.27), we have $\ell < 2^{d+1}a'$. Dividing both sides by 4, we get
$$\frac{\ell}{4} < 2^{d-1}a'.$$
Since $\ell \equiv 3 \pmod 4$, it follows that
$$\frac{\ell+1}{4} \leq 2^{d-1}a'.$$

- Third, we show
$$3 \cdot 2^{d-1}a' - \frac{\ell+1}{2} < \ell, \text{ which is equivalent to } 3 \cdot 2^{d-1}a' < \frac{3\ell+1}{2}.$$
This follows from (6.5.38), as we have $\frac{3\ell}{2} > 21 \cdot 2^{d-3}a' + \frac{3}{4}$, and thus
$$3 \cdot 2^{d-1}a' = 12 \cdot 2^{d-3}a' < 21 \cdot 2^{d-3}a' + \frac{3}{4} < \frac{3\ell}{2} < \frac{3\ell+1}{2}.$$

Therefore, we conclude that P_f is a path.

The differences covered by P_f are:
$$f\ell+1, f\ell+2, f\ell+3, \ldots, f\ell+2^{d-1}a'-1, f\ell+2^{d-1}a'+1, f\ell+2^{d-1}a'+2, \ldots$$
$$\ldots, f\ell+\tfrac{\ell-1}{2},\ 2^d a', 2^d a'+1, \ldots, 2^{d+1}a' - \frac{\ell+3}{2}, 2^{d+1}a' - \frac{\ell+1}{2}, \frac{3\ell+1}{2} - 3 \cdot 2^{d-1}a'.$$

Next, we show that the differences
$$2^d a', 2^d a'+1, \ldots, 2^{d+1}a' - \frac{\ell+3}{2}, 2^{d+1}a' - \frac{\ell+1}{2}, \frac{3\ell+1}{2} - 3 \cdot 2^{d-1}a'$$
in P_f are pairwise distinct and in the interval $[2^d a', \ell]$. We do this by proving the following inequality.
$$\frac{3\ell+1}{2} - 3 \cdot 2^{d-1}a' > 2^{d+1}a' - \frac{\ell+1}{2}, \text{ which is equivalent to } 2\ell+1 > 7 \cdot 2^{d-1}a'.$$

By (6.5.38), we have $\ell > 7 \cdot 2^{d-2}a' + \frac{1}{2}$. Doubling both sides, we get

$$2\ell > 7 \cdot 2^{d-1}a' + 1,$$

and thus,

$$2\ell + 1 > 7 \cdot 2^{d-1}a'.$$

Now that we know $\frac{3\ell+1}{2} - 3 \cdot 2^{d-1}a' > 2^{d+1}a' - \frac{\ell+1}{2}$, in order to show that all differences lie in the interval $[2^d a', \ell]$, it suffices to prove the inequality

$$\frac{3\ell+1}{2} - 3 \cdot 2^{d-1}a' < \ell \quad \Longleftrightarrow \quad \ell + 1 < 6 \cdot 2^{d-1}a'.$$

Since we know $\ell < 2^{d+1}a' = 4 \cdot 2^{d-1}a'$, it is easy to see that

$$\ell + 1 < 4 \cdot 2^{d-1}a' + 1 < 6 \cdot 2^{d-1}a'.$$

You can see an illustration of the path P_f in Figure 6.8. Observe that P_f starts at x_0 and ends at x_ℓ. Moreover, we show that the internal vertices of the path P_f are pairwise distinct modulo ℓ. We do this by proving the following two inequalities.

(i) $\frac{\ell-3}{4} < 2^d a' - \frac{\ell+1}{4}$.
This is proved in (6.5.41).

(ii) $\ell - 2^{d-1}a' > 3 \cdot 2^{d-1}a' - \frac{\ell+1}{2}$, which is equivalent to $\frac{3\ell+1}{2} > 2^{d+1}a'$.
By (6.5.40), we have $\frac{3\ell-1}{2} > 2^{d+1}a'$, and consequently, $\frac{3\ell+1}{2} > 2^{d+1}a'$.

Therefore, the internal vertices of the path P_f are pairwise distinct modulo ℓ.

- **Situation 2(b):** $\ell \equiv 1 \pmod 4$.

  Define a walk P_f as follows (see Figure 6.9):

$$\begin{aligned} P_f = &x_0\ x_{f\ell+1}\ x_{-1}\ x_{f\ell+2}\ x_{-2} \cdots \\ &x_{-(2^{d-2}a'-1)}\ x_{f\ell+2^{d-2}a'}\ x_{-(2^{d-2}a'+1)}\ x_{f\ell+2^{d-2}a'+1}\ x_{-(2^{d-2}a'+2)}\ x_{f\ell+2^{d-2}a'+2} \cdots \\ &x_{f\ell+\frac{\ell-5}{4}}\ x_{-\frac{\ell-1}{4}}\ x_{2^d a'-\frac{\ell-1}{4}}\ x_{-\frac{\ell+3}{4}}\ x_{2^d a'-\frac{\ell-5}{4}} \cdots \\ &x_{-(2^{d-1}a'-1)}\ x_{3\cdot 2^{d-1}a'-\frac{\ell+1}{2}}\ x_{-(2^{d-1}a')}\ x_{3\cdot 2^{d-1}a'-\frac{\ell-1}{2}}\ x_\ell \end{aligned}$$

To show that P_f is a path, we verify the following three inequalities.

- First, we show that

$$-(2^{d-2}a' + 1) \geq -\frac{\ell - 1}{4}.$$

  This inequality was proved in (6.5.39).

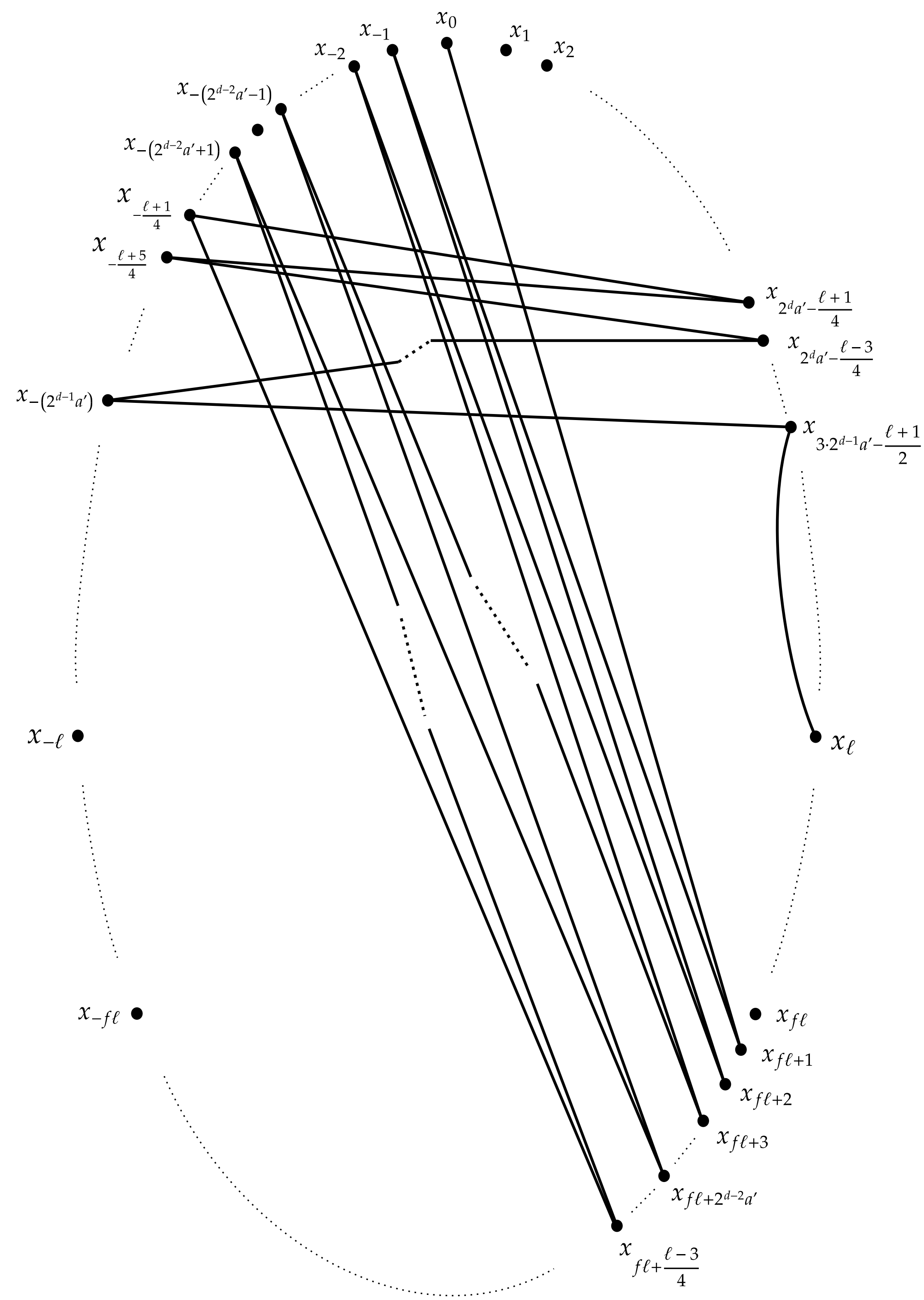


Figure 6.8: Illustration of the path P_f for $\ell \equiv 3 \pmod 4$.

- Second, we show that:

  $$3 \cdot 2^{d-1}a' - \frac{\ell-1}{2} \geq 2^d a' - \frac{\ell-1}{4}, \text{ which is equivalent to } 2^{d-1}a' \geq \frac{\ell-1}{4}.$$

  From (6.5.27), we have $\ell < 2^{d+1}a'$. Dividing both sides by 4, we get $\frac{\ell}{4} < 2^{d-1}a'$. Now, it is easy to see that

  $$\frac{\ell-1}{4} < 2^{d-1}a'.$$

- Third, we show

  $$3 \cdot 2^{d-1}a' - \frac{\ell-1}{2} < \ell, \text{ which is equivalent to } 3 \cdot 2^{d-1}a' < \frac{3\ell-1}{2}.$$

  By (6.5.38), we have $\ell > 7 \cdot 2^{d-2}a' + \frac{1}{2}$. With a simple calculation we get:

  $$\frac{3\ell-1}{2} > 21 \cdot 2^{d-3}a' + \frac{1}{4} > 12 \cdot 2^{d-3}a' = 3 \cdot 2^{d-1}a'.$$

Thus, we conclude that P_f is a path.

The differences covered by P_f are:

$$f\ell+1,\ f\ell+2,\ f\ell+3,\ \ldots,\ f\ell+2^{d-1}a'-1,\ f\ell+2^{d-1}a'+1,\ f\ell+2^{d-1}a'+2,\ \ldots$$
$$f\ell+\tfrac{\ell-3}{2}, 2^d a',\ 2^d a'+1,\ 2^d a'+2,\ \ldots,\ 2^{d+1}a'-\tfrac{\ell+1}{2},\ 2^{d+1}a'-\tfrac{\ell-1}{2}, \tfrac{3\ell-1}{2}-3\cdot 2^{d-1}a'.$$

By proving $\frac{3\ell-1}{2} - 3 \cdot 2^{d-1}a' > 2^{d+1}a' - \frac{\ell-1}{2}$, we show that the differences

$$2^d a', 2^d a'+1, 2^d a'+2, \ldots, 2^{d+1}a' - \frac{\ell+1}{2}, 2^{d+1}a' - \frac{\ell-1}{2}, \frac{3\ell-1}{2} - 3 \cdot 2^{d-1}a'$$

in P_f are pairwise distinct and in the interval $[2^d a', \ell]$.

Observe that $\frac{3\ell-1}{2} - 3 \cdot 2^{d-1}a' > 2^{d+1}a' - \frac{\ell-1}{2}$ is equivalent to $2\ell > 7 \cdot 2^{d-1}a' + 1$, and this follows directly by doubling both sides of the inequality given in (6.5.38).

You can see an illustration of the path P_f in Figure 6.9. Notice that P_f starts at x_0 and ends at x_ℓ. Moreover, the following two inequalities show that the internal vertices of the path P_f are pairwise distinct modulo ℓ.

(i) $\frac{\ell-5}{4} < 2^d a' - \frac{\ell-1}{4}$.

This follows from (6.5.41).

(ii) $\ell - 2^{d-1}a' > 3 \cdot 2^{d-1}a' - \frac{\ell-1}{2}$, which is equivalent to $\frac{3\ell-1}{2} > 2^{d+1}a'$.

This was proved in (6.5.40).

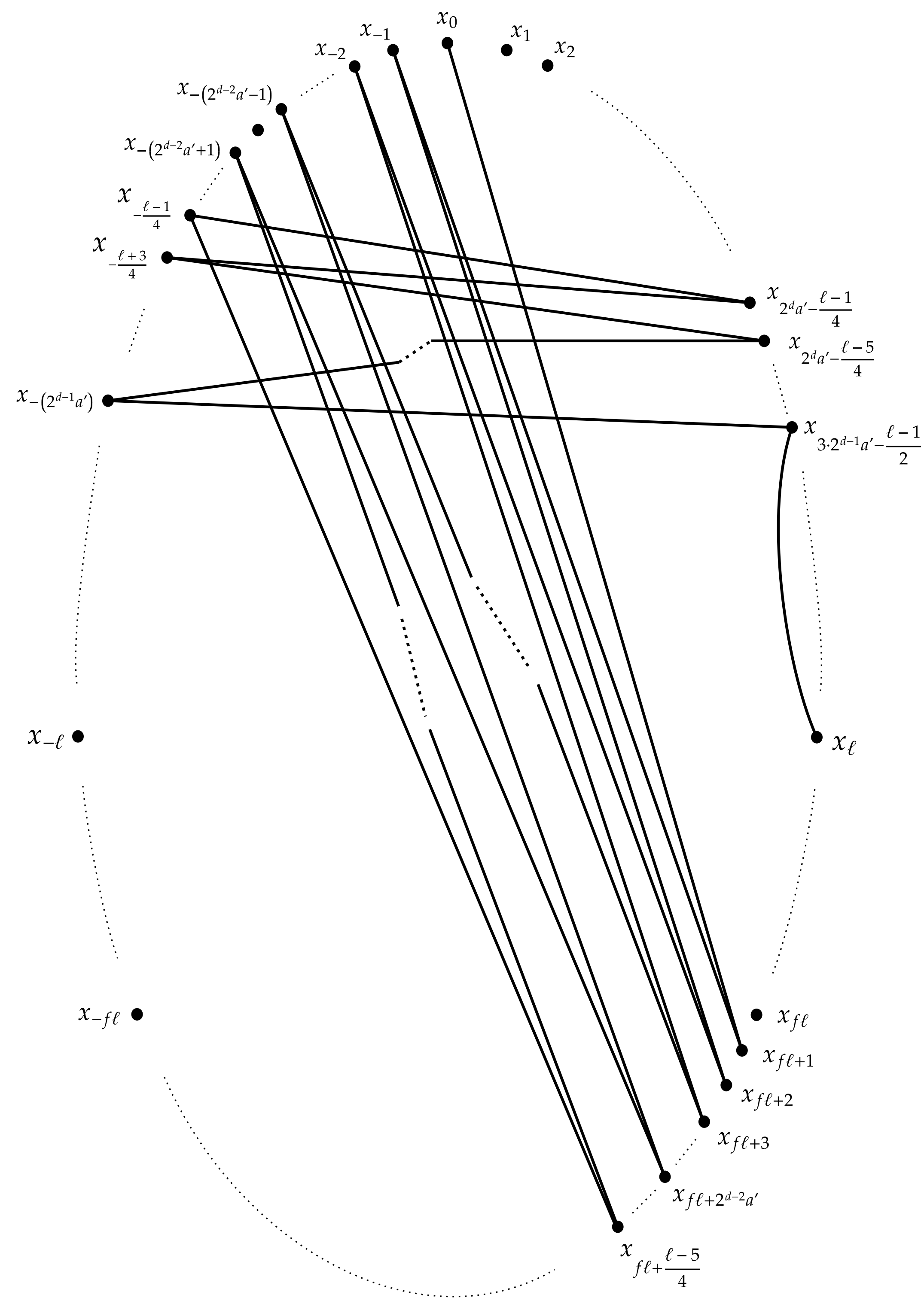


Figure 6.9: Illustration of the path P_f for $\ell \equiv 1 \pmod 4$.

Note that the internal vertices of the path P_f constructed in both Situation 2(a) and Situation 2(b), are pairwise distinct modulo ℓ. That is, for $j = 0, \ell, 2\ell, \dots, (b'-1)\ell$, the paths $\rho^j(P_f)$ are pairwise vertex-disjoint, except at the endpoints. Hence,

$$C_f = P_f \cup \rho^\ell(P_f) \cup \cdots \cup \rho^{(b'-1)\ell}(P_f)$$

is an m-cycle.

Furthermore, the path P_f is constructed so that the differences it covers are distinct from those covered by the paths P_i for $i = 0, 1, \dots, f-1$, which are constructed in Situation 1. Hence, the differences covered by the paths P_i for $i = 0, 1, \dots, \frac{F-1}{4}$ are pairwise distinct.

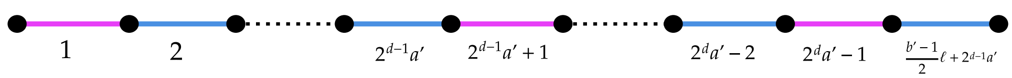
Let B be the set of differences covered by the m-cycles $C_1, C_2 \dots, C_{\frac{F-1}{4}}$. Let $G_1 = \mathrm{Circ}(n-1; B)$. Then, $\{C_i, \rho(C_i), \dots, \rho^{\ell-1}(C_i) \mid i = 1, \dots, \frac{F-1}{4}\}$ is a (C_m)-decomposition of G_1. Take four copies of each C_i, for $i = 1, \dots, \frac{F-1}{4}$, and use Lemma 3.3.3 to color them so that they satisfy Condition **(C1)** of Definition 1.2.3. By doing so, we generate $F-1$ starter peripheral cycles with the appropriate coloring, which we denote $C_1', C_2' \dots, C_{F-1}'$.

It is now easy to see that

$$\{C_i', \rho_\bullet(C_i'), \dots, \rho_\bullet^{\ell-1}(C_i') \mid i = 1, \dots, F-1\}$$

is an HOP (C_m)-decomposition of $4G_1^\bullet$.

Notice that $A = \{1, 2, \dots, 2^d a' - 2, 2^d a' - 1, \frac{(b'-1)\ell}{2} + 2^{d-1}a'\}$ is the set of differences covered by C_0. As in Case 1, color the edges of P_0 alternately pink and blue (as shown below), starting with pink. Since P_0 has length $2^d a'$, which is even, it ends with blue.



The family of starter cycles generated by C_0; namely, $\{C_0, \rho_\bullet(C_0), \dots, \rho_\bullet^{\ell-1}(C_0)\}$, covers:

(i) the pink orbits corresponding to the differences

$$1, 3, 5, \dots, 2^{d-1}a' + 1, 2^{d-1}a' + 3, \dots, 2^d a' - 3, 2^d a' - 1, \text{ and} \tag{6.5.42}$$

(ii) the blue orbits corresponding to the differences

$$2, 4, 6, \dots, 2^{d-1}a', 2^{d-1}a' + 2, \dots, 2^d a' - 2, \frac{(b'-1)\ell}{2} + 2^{d-1}a'. \tag{6.5.43}$$

The differences in the set A will also appear in the central cycles C and C', to be constructed below. There, we need to ensure that the blue orbits corresponding to the differences of A listed in (6.5.42), the pink orbits corresponding to the differences of A listed in (6.5.43), as well as the black orbits of all differences in the set A, are all covered.

Now, we focus on constructing the starter central cycles C and C'. Let $\mathscr{L}$ be the set of unused differences in S, that is, $\mathscr{L} = S\backslash(A \cup B)$. We can express $\mathscr{L} = \{a_1, a_2, \ldots, a_t\} \cup \{\infty\}$, where $a_1 < a_2 < \ldots < a_t \leq \frac{n-2}{2}$ are positive integers and $t = \frac{n-2}{2} - \frac{(F+3)2^d a'}{4}$.

Notice that since $b' \geq 3$, we have $\frac{(b'-1)\ell}{2} + 2^{d-1}a' > \ell \geq 2^d a'$. Also, having $A = \{1, 2, \ldots, 2^d a' - 2, 2^d a' - 1, \frac{(b'-1)\ell}{2} + 2^{d-1}a'\}$, we observe that

$$2^d a' - 1 < a_1 \leq \frac{(b'-1)\ell}{2} + 2^{d-1}a' \leq \frac{n-2}{2}.$$

Consider the following sets of differences:

- $W_1 = \{1, 2, \ldots, 2^d a' - 1, a_1, a_2, a_3, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t\}$
- $W_2 = \{2^{d-1}a' + 1, 2^{d-1}a' + 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t\}$

In the sequence $a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t$, we assume

$$a_1 < a_2 < \ldots < \frac{(b'-1)\ell}{2} + 2^{d-1}a' < \ldots < a_{t-1} < a_t,$$

with the understanding that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t < \frac{(b'-1)\ell}{2} + 2^{d-1}a'$$

is also possible.

Here, the set W_1 contains all the differences from A and all the differences from $\mathscr{L}$, except the difference ∞, and has size $|W_1| = t + 2^d a'$. The set W_2 contains half of the differences from A and all differences from $\mathscr{L}$, except the difference ∞, and has size $|W_2| = t + 2^{d-1}a'$. By Corollary 6.3.4, there exists a path, call it T_1, of length $2t + 3 \cdot 2^{d-1}a'$ that starts at x_0 and covers the differences in the following order:

$$\begin{aligned} T_1 : &1, 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t, \\ &a_t, a_{t-1}, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_2, a_1, 2^d a' - 1, \ldots, 2^{d-1}a' + 2, 2^{d-1}a' + 1. \end{aligned}$$

Similar to Case 1, we use Lemma 6.3.3 to construct a path T_2 by replacing the second occurrence of the difference $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ with difference 1 in the path T_1.

$$\begin{aligned} T_2 : &1, 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t, \\ &a_t, a_{t-1}, \ldots, 1, \ldots, a_2, a_1, 2^d a' - 1, \ldots, 2^{d-1}a' + 2, 2^{d-1}a' + 1. \end{aligned}$$

Now, consider the following sets of differences:

- $W_1 = \{1, 2, \ldots, 2^d a' - 1, a_1, a_2, a_3, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t\}$

- $W_2' = \{2, 3, 4, \ldots, 2^{d-1}a' - 1, 2^{d-1}a', a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t\}$

Again, in the sequence $a_1, a_2, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-1}, a_t$, we assume

$$a_1 < a_2 < \ldots < \frac{(b'-1)\ell}{2} + 2^{d-1}a' < \ldots < a_{t-1} < a_t,$$

with the understanding that

$$a_1 < a_2 < \ldots < a_{t-1} < a_t < \frac{(b'-1)\ell}{2} + 2^{d-1}a'$$

is also possible.

Here, $|W_1| = t + 2^d a'$ and $|W_2'| = t + 2^{d-1}a'$. By Corollary 6.3.4, there exists a path, call it T_3, of length $2t + 3 \cdot 2^{d-1}a'$ that starts at x_0 and covers the differences in the following order:

$$\begin{aligned} T_3 : &1, 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t, \\ &a_t, a_{t-1}, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_2, a_1, 2^{d-1}a', 2^{d-1}a' - 1, \ldots, 3, 2. \end{aligned}$$

By Lemma 6.3.3, the second occurrence of the difference $\frac{(b'-1)\ell}{2} + 2^{d-1}a'$ can be replaced with difference 1; that is, there exists a path, call it T_4, that covers the differences in the following order:

$$\begin{aligned} T_4 : &1, 2, \ldots, 2^d a' - 1, a_1, a_2, \ldots, \tfrac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t, \\ &a_t, a_{t-1}, \ldots, 1, \ldots, a_2, a_1, 2^{d-1}a', 2^{d-1}a' - 1, \ldots, 3, 2. \end{aligned}$$

Notice that the lengths of the paths T_1, T_2, T_3, T_4 are all equal to $2t + 3 \cdot 2^{d-1}a'$, and

$$\begin{aligned} 2t + 3 \cdot 2^{d-1}a' &= 2\left(\frac{n-2}{2} - \frac{(F+3)2^d a'}{4}\right) + 3 \cdot 2^{d-1}a' \\ &= n - 2 - (F+3)2^{d-1}a' + 3 \cdot 2^{d-1}a' \\ &= n - 2 - F \cdot 2^{d-1}a' \\ \text{(substitute } F = \frac{a - 2a'b'}{a'}\text{)} \quad &= n - 2 - 2^{d-1}a + 2^d a'b' \\ \text{(substitute } n = 2^{d-1}a \text{ \& } m = 2^d a'b'\text{)} \quad &= n - 2 - n + m \\ &= m - 2 \end{aligned}$$

As in Case 1, to simplify the explanation of the coloring process, we express T_1, T_2, T_3, and T_4 as the following concatenations of subpaths:

- $T_1 = PQ_1R_1$
- $T_2 = PQ_2R_1'$
- $T_3 = PQ_1R_2$

- $T_4 = PQ_2R_2'$

Here, P, Q_1, Q_2, R_1, R_1', R_2', and R_2 represent subpaths that cover the following sequences of differences:

- $P : 1, 2, \ldots, 2^d a' - 1, a_1, a_2, a_3, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_{t-2}, a_{t-1}, a_t$
- $Q_1 : a_t, a_{t-1}, a_{t-2}, \ldots, \frac{(b'-1)\ell}{2} + 2^{d-1}a', \ldots, a_3, a_2, a_1$
- $Q_2 : a_t, a_{t-1}, a_{t-2}, \ldots, 1, \ldots, a_3, a_2, a_1$
- $R_1, R_1' : 2^d a' - 1, 2^d a' - 2, 2^d a' - 3, \ldots, 2^{d-1}a' + 3, 2^{d-1}a' + 2, 2^{d-1}a' + 1$
- $R_2, R_2' : 2^{d-1}a', 2^{d-1}a' - 1, 2^{d-1}a' - 2, \ldots, 4, 3, 2$

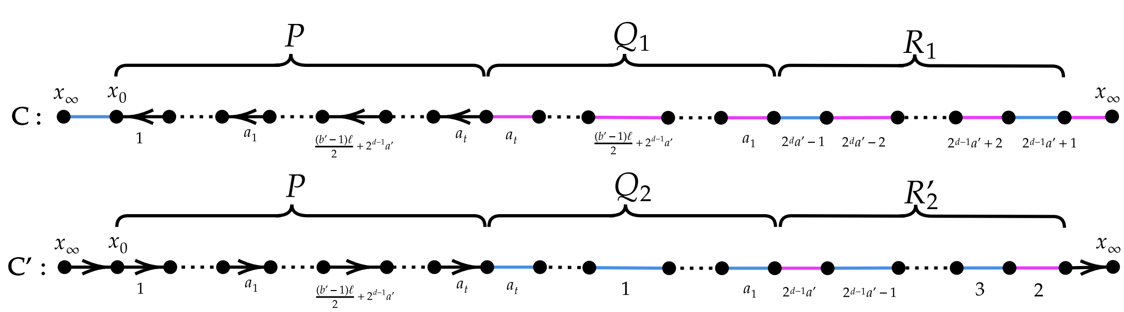
Note that P has length $t + 2^d a'$, while Q_1 and Q_2 each have length $t+1$. The paths R_1 and R_1' cover the same sequences of differences, as do R_2 and R_2', with each path having length $2^{d-1}a' - 1$.

The coloring process is the same as in Case 1 (see page 175). We start by fixing the colors of the edges in R_1, R_1', R_2, and R_2' and then we color the paths Q_1 and Q_2 as described in Case 1. The only difference from Case 1 is that the path P is longer and will covere the black copies of the differences in the set A.

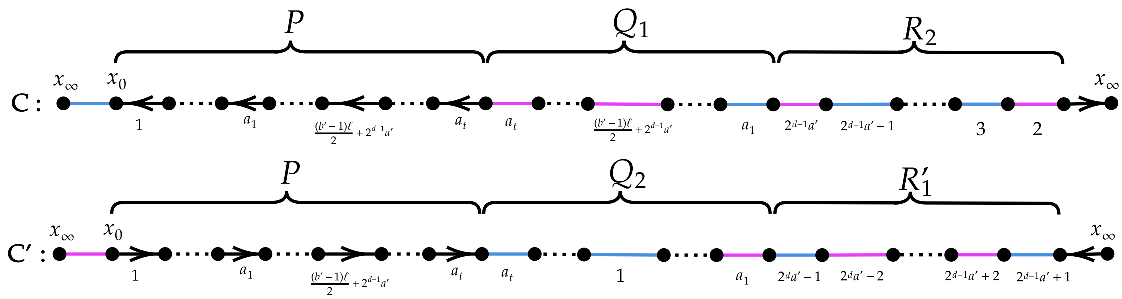
If Q_1 starts with pink, the coloring is similar to that described in Case 1 (for even t, see (i) on page 176; for odd t, see (ii) on page 176). If Q_1 starts with blue, the coloring is similar to that described in Case 1 (for even t, see (i) on page 176; for odd t, see (ii) on page 177).

Here is an illustration of the central cycles C and C', along with their corresponding colorings, when Q_1 starts with pink.

- t is even. Let $C = x_\infty PQ_1R_1x_\infty$ and $C' = x_\infty PQ_2R_2'x_\infty$.



- t is odd. Let $C = x_\infty PQ_1R_2x_\infty$ and $C' = x_\infty PQ_2R_1'x_\infty$.



We can easily verify that both C and C' are cycles of length m, and they satisfy Condition **(C1)** of Definition 1.2.3. Moreover, they jointly contain exactly one blue edge, one pink edge, and two black arcs from each orbit of $\langle \rho_\bullet \rangle$ corresponding to differences in $\mathscr{L}$. Also, the families generated by cycles C_0, C, and C' jointly cover the blue, pink, and black orbits corresponding to the differences in A. Thus,

$$\left\{\rho_\bullet^i(C), \rho_\bullet^i(C') \mid i = 0, 1, \ldots, n-2\right\} \cup \{C_0, \rho_\bullet(C_0), \ldots, \rho_\bullet^{\ell-1}(C_0)\}$$

is an HOP (C_m)-decomposition for $4G_2^\bullet$, where

$$G_2 = \mathrm{Circ}(n-1; A \cup (\mathscr{L} - \{\infty\})) \bowtie K_1.$$

We have $4K_n^\bullet = 4G_1^\bullet \bigoplus 4G_2^\bullet$, and since each of $4G_1^\bullet$ and $4G_2^\bullet$ admits an HOP (C_m)-decomposition, by Lemma 3.3.1, the multigraph $4K_n^\bullet$ admits an HOP (C_m)-decomposition.

∎

Chapter 7

Conclusion and Future Work

As mentioned earlier, the generalized HOP is a distinct problem from the HOP. Although some proof techniques used for the HOP have been extended to the generalized version, no prior results were known for this problem.

In this thesis, several results on the generalized HOP were established. We proved that $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2)$ has a solution whenever $n \equiv 1 \pmod{2m_1 + 2m_2}$ or $n \equiv m_1 + m_2 \pmod{2m_1 + 2m_2}$, and we extended this to $\mathrm{HOP}(2^{\langle s \rangle}, 4^{\langle r \rangle}, 2m)$ under similar conditions. We also considered the case where the sum of the cycle lengths is at most 10, and showed that $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$ has a solution whenever $n = s + m$ is odd and $n(n-1) \equiv 0 \pmod{2m}$. Finally, we gave a complete solution for $\mathrm{HOP}(2^{\langle s \rangle}, 2m)$, proving that the necessary condition $m \mid 2n(n-1)$ is also sufficient.

In this thesis, we resolved several significant cases; however, many remain open. For future work, new constructions may be developed to prove the problems presented below.

In Section 2.3, we gave a summary of known results on decompositions of K_n and $2K_n$ into 2-regular subgraphs. Here, we use those results to propose the following problems.

7.1 Problems

Problem 7.1.1 provides some results on $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2, 2m_3)$ with a constraint on the number of participants. One can apply the previous result on the decomposition of K_n into $(C_{m_1}, C_{m_2}, C_{m_3})$-subgraphs, given in Theorem 2.3.6, to prove this problem. However, to handle the cases not covered by Theorem 2.3.6, new constructions are needed, in particular for the case where $2 \in \{m_1, m_2, m_3\}$.

Problem 7.1.1. Let $s \in \mathbb{Z}^+$, and $2 \le m_1 \le m_2 \le m_3$ be odd integers. Let $m = m_1 + m_2 + m_3$, and $n = s + m$. Then $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, 2m_2, 2m_3)$ has a solution in each of the following cases.

(i) $n \equiv 1 \pmod{2m}$;

(ii) $n \equiv m \pmod{2m}$.

To prove the second part of the above problem, first note that $n = 2km + m$, and the complete graph K_{2km+m} can be written as $H_1 \bigoplus H_2$, where H_1 is a disjoint union of $2k + 1$ copies of K_m, and H_2 is the complete equipartite graph $K_{(2k+1)[m]}$. Therefore, the plan could be to combine the result on the $(C_{m_1}, C_{m_2}, C_{m_3})$-decomposition of $K_{(2k+1)[m]}$ with the partial results given in Theorem 2.1.6 for $\mathrm{OP}(m_1, m_2, m_3)$.

In the following problem, the result given in Theorem 2.3.5 on $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-decompositions of K_n can be applied to tackle $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$. However, new constructions are needed to address the case where $2 \in \{m_1, m_2, \ldots, m_t\}$.

Problem 7.1.2. Let $s \geq 0$, and $2 \leq m_1 \leq \ldots \leq m_t$ be even integers. Let $m = m_1 + \ldots + m_t$, and $n = s + m$. Then $\mathrm{HOP}(2^{\langle s \rangle}, 2m_1, \ldots, 2m_t)$ has a solution if $n \equiv 1 \pmod{2m}$.

Problem 7.1.3 provides a partial result on $\mathrm{HOP}(2^{\langle s \rangle}, (2m)^{\langle t \rangle})$. To prove this problem, one may try to find a suitable coloring for the decomposition of $2K_n$ given in Theorem 2.3.11, and then extend it to an HOP decomposition of $4K_n^{\bullet}$.

Problem 7.1.3. Let $t \geq 1$ and $m \geq 2$ be integers, and let $n = tm+1$. Then $\mathrm{HOP}(2, (2m)^{\langle t \rangle})$ has a solution.

Note that when using results on decompositions of $2K_n$, more work is required compared to using results on decompositions of K_n. Specifically, it is not enough to know that a decomposition exists; one must also find a suitable 2-edge-coloring of the decomposition, which may not exist in general. In other words, Theorem 2.3.11 on its own is not very helpful. One needs to refer to its proof to extract the actual decompositions and provide a proper two-edge-coloring.

Based on the previous results on decompositions of K_n and $2K_n$, we believe that the above problems can be solved, although there may be a few small exceptions. The goal is to identify those exceptions and provide a proof for the remaining cases.

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