KALMAN ESTIMATION AND CONTROL OF THE ATTITUDE OF THE DUAL-SPIN GEOSTATIONARY SATELLITE

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ABSTRACT

For satellite communication purposes of systems using dual-spin geostationary satellites with non tracking ground antennas, it is necessary to maintain the spin axis of the satellite to within 0.1 degree of the orbit normal by periodic attitude corrections. Normally the data for attitude estimation are obtained electronically at the ground station in the form of digital times. These are determined from the analogue sensor waveforms telemetered to the ground station. This information is supplied to the Attitude Determination programme, which processes the data and outputs the right ascension and declination of the spin axis.

This thesis presents first an application of the Extended Kalman Filter in estimating the attitude of dual-spin geostationary satellites. The precession of the angular momentum vector by the solar radiation torque is considered to be the only natural attitude perturbation. The orbital dynamics are considered to be known and decoupled from the attitude dynamics. A periodic attitude control policy is derived.

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Thanks are also due to my parents for their support and love.

CHAPTER I

INTRODUCTION

The successful launching and orbiting of Telstar and Relay in 1962, with their low, non-synchronous orbit and brief 30 minute transmission periods assured the future of active satellite repeaters. Under a contract from the National Aeronautics and Space Administration (NASA) the synchronous, geostationary satellite series termed SYNCOM was developed. SYNCOM I, intended for a synchronous equatorial orbit 22,300 miles above the earth early in 1963, failed to operate. SYNCOM II was launched later that year. SYNCOM III was placed in a stationary, equatorial orbit over the Pacific Ocean in mid 1964. The SYNCOM series dispelled any final doubts about the feasibility of a geostationary satellite communication network.

It is an interesting historical note that the concept of a geostationary satellite was first published in October 1945 in an article entitled "Extra Terrestrial Relays" by British scientist, A.C. Clarke [1]. Although Mr. Clarke's article seemed pure fiction to many at the time, it now stands as a prophecy to the events which have come to pass.

The world's first commercial communication satellite was orbited in April, 1965. This synchronous satellite, named Early Bird, was placed in a geostationary equatorial orbit over the Atlantic Ocean. From its vantage point 22,300 miles above the Atlantic, Early Bird linked North America and Europe with 240 high quality voice circuits and made live television commercially available across the Atlantic for the first time. In the four short years since Early Bird was launched, six larger and more powerfull satellites had successfully been placed in operation over the equatorial region of the Atlantic, Pacific and Indian Ocean.

Early Bird, designated INTELSAT I was the first of the INTELSAT series. By September 1967 four INTELSAT II satellites had been launched; the first of these failed to achieve orbit due to a malfunction of the apogee motor. The second was successfully parked in orbit over the Pacific Ocean, while the third was positioned, as planned, 6 degree west longitude over the Atlantic Ocean. The fourth of the INTELSAT II satellites was emplaced over the Pacific Ocean at 176 degrees East longitude.

By May 1969, three INTELSAT III satellites had been placed in orbit, one each over the Atlantic, Pacific and Indian Ocean. The INTELSAT IV satellites appeared in 1971. They have more channel capacity and also the ability to switch the transponders in orbit to cover the precise earth area desired.

On November 9, 1972, Telesat Canada successfully launched Anik I into an earth-synchronous orbit. Two months later,. domestic communications via satellite became a reality in Canada. Anik I's beamwidth covers all Canada from coast to coast and parts of the U.S. as well. On April 20, 1973 Anik II, the world's second domestic communications satellite, was launched and subsequently placed into an earth-synchronous orbit.

The possibilities confirmed by the success of the synchronous satellite were revolutionary. With a minimum of three satellites in geostationary orbit, global coverage can be achieved. This provides telephone, radio, television services, a vast communication and navigation service for aircraft, shipping and many more applications.

The immediate implication of geostationary satellites greatly affected earth station technology. The huge tracking antenna requires only very limited movement to follow a "fixed" satellite. This permits tremendous savings in the cost of materials and maintenance. Simplifying earth station technology, particularly the complex tracking requirements, not only allowed more attention to be focused on the development of new and better radio equipment. It realistically meant that many more earth stations could be erected in less time at less expense.

The orientation of the vehicle with respect to a reference frame is usually called attitude. Attitude control is required for many purposes. Observational satellites must be so oriented that the sensors are pointed towards the object to be observed. Where orbit control is executed, the thrust must be applied in the proper direction. For communication purposes, the orientation of the vehicle can be essential, since, if directional antennas are employed on it, they must be pointed toward the ground receiving stations. Some early satellites and space probes had no attitude control; consequently, their communications capability was limited because of the omnidirectional antennas required on the vehicle. Since there is no truly omnidirectional antenna, and certain nulls and peaks will exist in the pattern of a vehicle antenna, uncontrolled tumbling could give rise to undesired signal modulation and fading, which may cause dropouts of signals. For reliable communications, therefore, some form of attitude control seemed necessary.

Generally speaking the attitude of a vehicle is controlled by controling its motion in a specified reference frame. A second reference frame, namely, the body frame is also specified. This frame is fixed with respect to the vehicle. When both reference frames coincide, or are in some specific relation, the vehicle is then at its nominal attitude.

This thesis deals with the periodic attitude control of the Dual-Spin Geostationary Spacecraft. The orbital parameters are assumed to be known. Single axial jets in pulsed mode are activ ated periodically within a cycle for attitude corrections. The attitude corrections are assumed to precede the inclination corrections such that the spin-axis would be normal to the equatorial plane at the time of the inclination maneuvre [2].

An Extended Kalman Filter is used for estimating the spacecraft attitude by processing the sensor data. A model for the precession of the satellite spin vector by considering the solar radiation torque as the only natural disturbing torque is developed. By selecting appropriate initial conditions, a periodic attitude control model for the Dual-Spin Geostationary Satellite is formed.

The Kalman Filter derived in our model could be used for providing the attitude information of the spacecraft and also estimating the new starting positions in each periodic control cycle. The results obtained in this thesis would be helpful in the design of an on-board attitude control system for the dual-spin geostationary satellites utilizing orbital information transmitted to the spacecraft from the ground stations (semi-active control system).

CHAPTER 2

PRELIMINARY MATERIAL

The minimum altitude for a satellite to remain in orbit is 100 miles with a period of 90 minutes. As the orbit altitude increases, the period also increases. The moon is a natural satellite of the earth and its altitude is about a quarter of a million miles. Its orbital period is about 29 days. Between the extremes of 100 miles and a quarter of a million miles, there is an altitude 22,300 miles, at which the orbital period is 24 hours. A satellite placed in this 24 hour orbit is known as a Geostationary Satellite because its movement is synchronized with the rotation of the earth and thus it appears to stand still and hover always in the same position.

This 24 hour or "Geostationary" orbit is a very desirable one due mainly to the following reasons:-

1) Since it is "stationary", only one main earth station antenna is needed. Small earth station antennas with a fairly broad beam do not require tracking.

2) One satellite can cover a vast area of ± 81 degrees. Three satellites located over the Pacific, Indian and Atlantic oceans can cover essentially all the populated surface of the world.

3) The satellite is above the Van Allen Radiation Belt and is therefore subject to less damage.

4) Use of highly stabilized narrow spot beams is possible, giving high antenna gain or high EIRP (Effective Isotropic Radiated Power).

The two commonly used stabilization methods in the geostationary orbit are those of spin-stabilization and three-axis stabilization. The spin-stabilized geostationary communications satellite was first proposed by the

Hughes Aircraft Company in the fall of 1959. The spin-stabilization method was preferred because it is a much simplified one, whereby the entire satellite spins and thus has an angular momentum like a gyroscope. Also, like a gyroscope, the satellite is reluctant to change its attitude and will resist doing so.

A body can experience a pure spin only about two axes - the axis of maximum moment of inertia (major axis), and the axis of minimum moment of inertia (minor axis) [5]. The distribution of the body's mass elements determines which is the major axis. Pure spin about the major axis has the minimum energy state - a state which is characterized by the lowest requirement of rotational energy to keep it in motion. Inversely, pure spin about the minor axis has the highest energy state. Generally, nature will seek a lower energy state.

The cigar-shaped Explorer I, was the first spin-stabilized spacecraft. It was launched in 1958. It spun about its minor axis and hence was inherently unstable. Early on in orbit, Explorer I began to wobble because of energy dissipated by the flexing of the whip-antennas and ultimately tumbled out of control.

With this lesson learned, designers adopted the "drum shaped" vehicle as the basic shape for spin stabilized satellites, and a long series of successful single-body major-axis spinners followed, leading eventually to the sophiscated dual-spin configuration, such as in the Canadian Satellites, Anik I, II and III. In the sequel when we mention dual-spin satellites or spacecraft , we will be referring to the ones of the Anik type (Figs 2.1 and 2.2).

2.1 Orbital Parameters

The motion of a satellite around the earth can be considered as a two-body problem by neglecting the air drag and oblateness terms. In

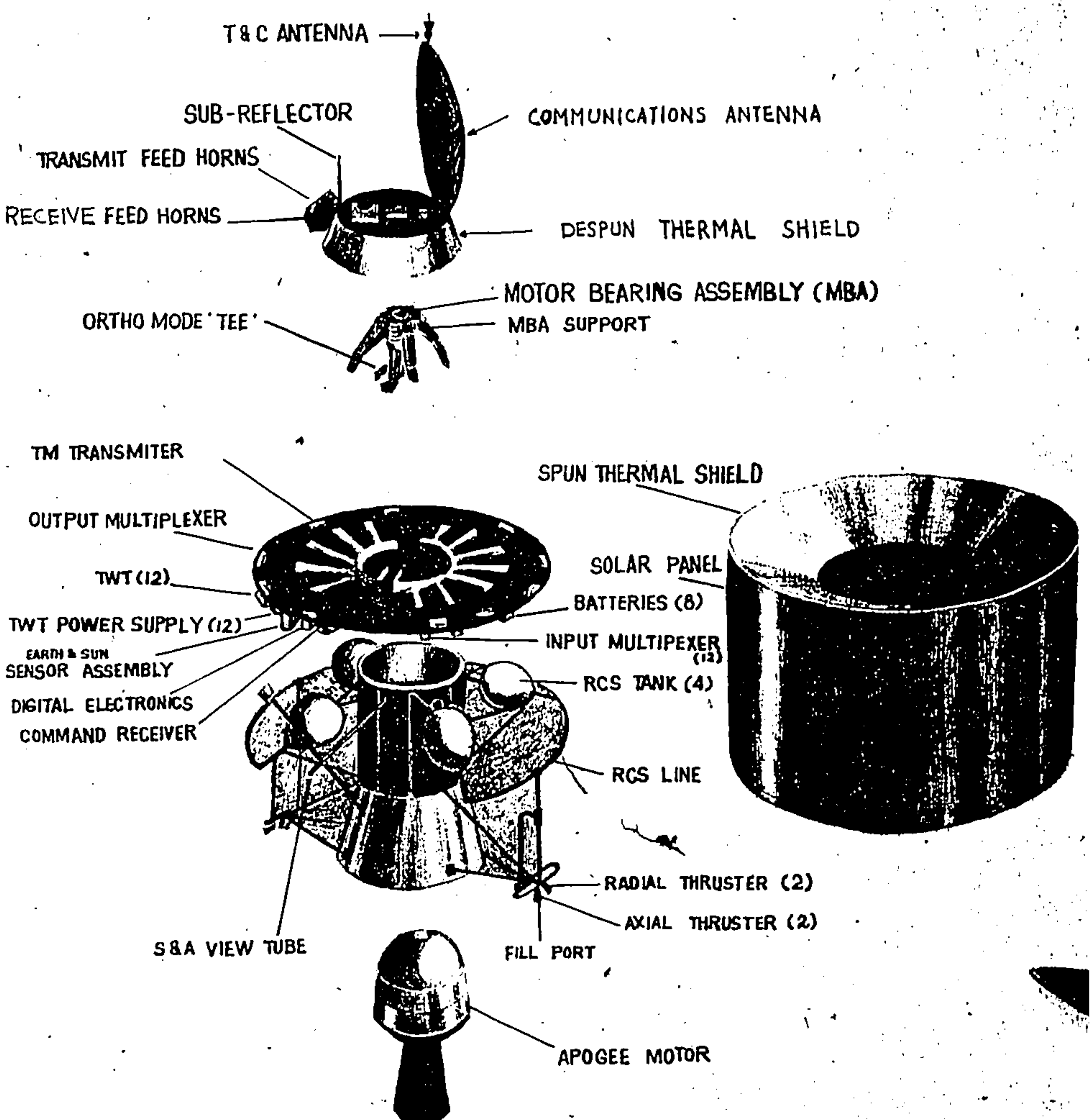


FIG. 2.1 SPACECRAFT ELEMENTS

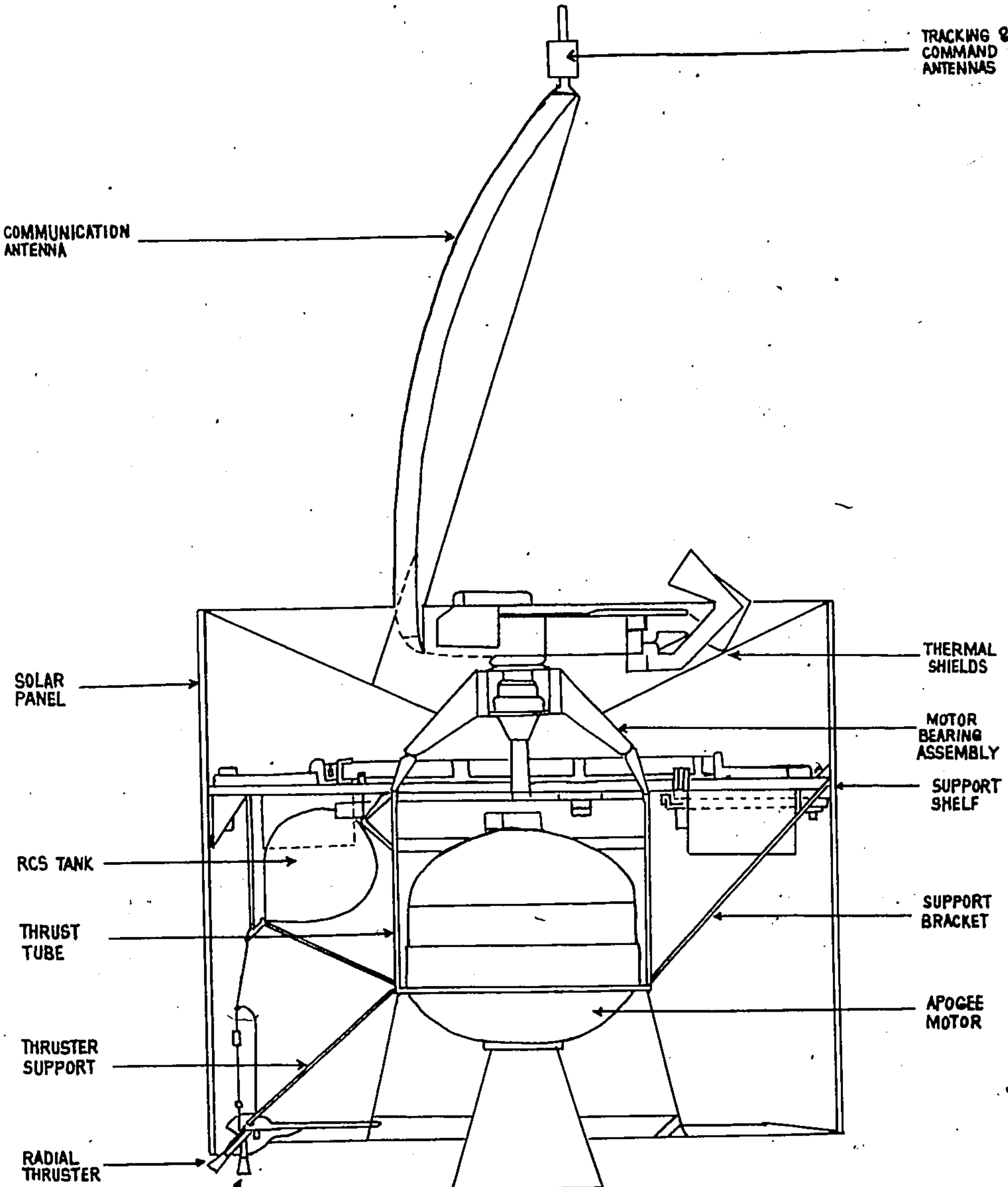


FIG. 2.2
DUAL SPIN STABILIZED SATELLITE - GENERAL CONFIGURATION

essence, this is the motion of a particle in a central force field, the earth gravitational field in this case. Newton's law of gravitation can thus be written as:

$$\ddot{\vec{r}} = -(M_e + M_s)\, G \frac{\vec{r}}{r^3} \qquad (2.1)$$

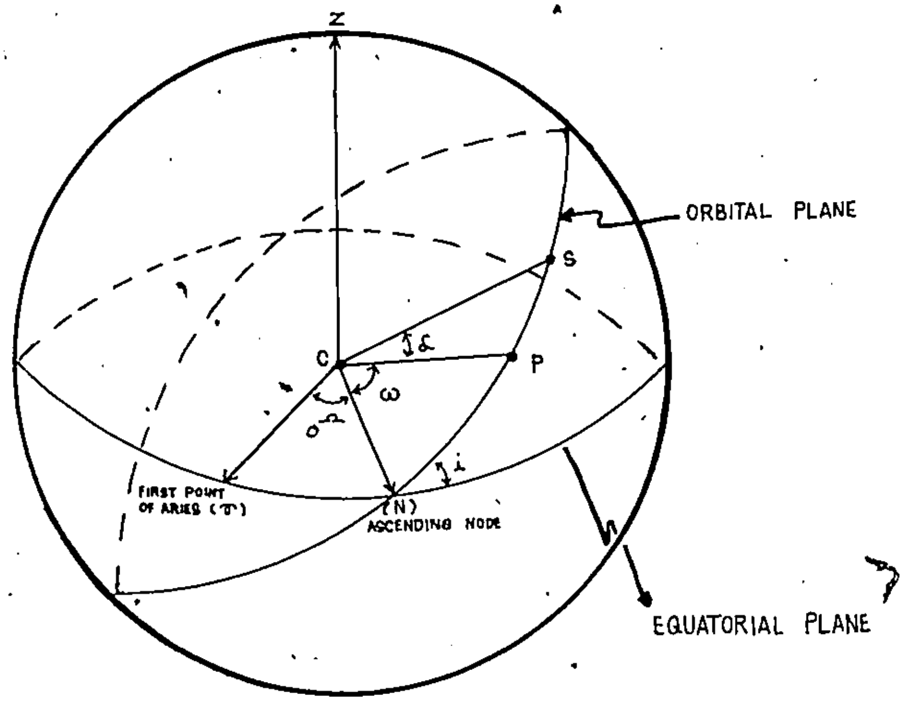
where $\vec{r}$ is the position vector, M_e the mass of the earth, M_s the mass of the satellite, and G the gravitational constant ($G = 6.67 \times 10^{-11} M^3\, kg^{-1} sec^{-2}$).

For all practical purposes, $M_s < M_e$; hence M_s can be neglected. The three independent equations for each component are:

$$\ddot{x} = -\mu \frac{x}{r^3}$$
$$\ddot{y} = -\mu \frac{y}{r^3} \qquad (2.2)$$
$$\ddot{z} = -\mu \frac{z}{r^3}$$

where $\mu = GM_e$ and $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. A solution of this set of equations, representing the path motion of the orbit, is a conic section which is determined by six integration constants. These constants are referred to as: semi-major axis, a; the numerical eccentricity, ϵ ; the epoch of the perifocal passage, τ ; the inclination, i; the right ascension of the ascending node, Ω; and the argument of the perigee, ω ; (Fig. 2.3).

At this point, we would like to emphasize again that we assumed that the spacecraft in our model has achieved its desired orbit and the orbital parameters are known at each instant of time.



C ≙ CENTER OF EARTH
S ≙ SATALLITE POSITION
P ≙ PERIGEE

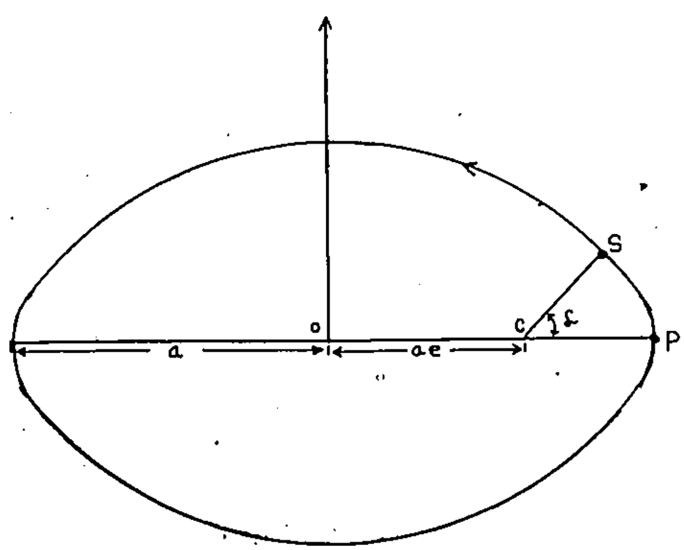


FIG. 2.3 SATELLITE PATH AND CONIC SECTION

2.2 Dual-Spin Spacecraft

The dual-spin stabilized satellites have two spinning bodies (Fig. 2.1), the platform or top body is "despun" - spinning at the same rate but counter to its stabilizing rotor. In essence, then, the platform is standing still. Therefore, with the platform seemingly motionless in a despun position, unidirectional payload items can be mounted so that they always point toward the earth. Additionally, as the rotor spins about the major axis, the satellite is nutationally stable and this provides a stable platform.

The control of the dual-spin spacecrafts (Anik I, II & III) is achieved through a cyclic sequence of orbit and attitude corrections using four 1 lb. hydrazine thrusters [2]. Two of the thrusters are mounted axially so that their thrust is very nearly parallel to the spin-axis but displaced from the centre of mass by 2.8 feet. These thrusters are separated in azimuth by 180 degrees. The remaining two jets are canted from the spin axis by 45 degrees.

The orbital elements are determined from range information which is obtained by looping through the spacecraft a modulated signal originating in the ground tracking station [3]. The attitude information of the spacecraft is provided by the earth and sun sensors.

2.3 Earth and Sun Sensors

The earth sensors are essentially spin infrared (15 micron wavelength) telescopes having a bolometer detector and a video amplifier. One sensor points 5 degrees North of the satellite equatorial plane, and the other 5 degrees South (Fig. 2.4). As the satellite spins, the earth sensors generate output pulses with leading and trailing edges which coincide with the horizon crossings of their optical axes. The sensor video is telemetered and transformed on the ground to "t-times" or times of earth-space horizon crossings with respect to a reference time.

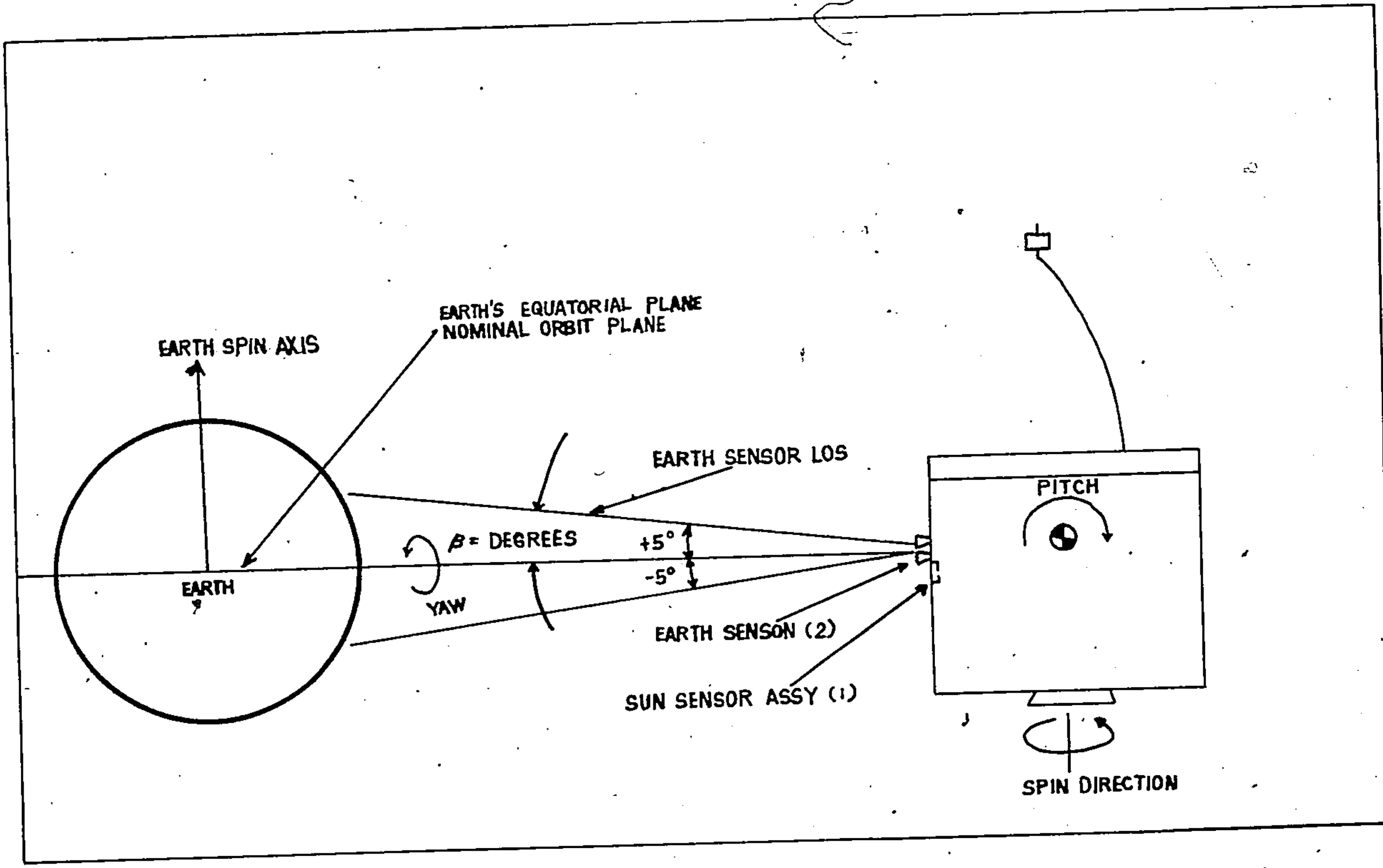


FIG. 2.4 EARTH & SUN SENSOR LOCATION

A reference time-pulse is generated once per spin period by the sun sensor when its ψ_1 slit scans the sun (Fig. 2.5). The earth pulses are converted to t-times by the attitude pulse digitizer and the Telemetry, Tracking and Command Computer (TTAC). The t-times are later input to the attitude determination program, which processes the data and outputs the right ascension and declination of the spin axis. The sun sensor also generates a second pulse as its inclined ψ_2 slit scans the sun. This pulse is converted to a t-time and is used as a measure of the angle between the spin axis and the sun.

2.4 Reaction Control Subsystem (RCS)

The reaction control subsystem is used to provide thrust for orienting the satellite, for acquiring station, and for controlling the attitude and the geostationary orbit for the life-span of the spacecraft. The fuel used is monopropellant hydrazine stored in four tanks (Fig. 2.2). The fuel is divided equally between two independent systems, either of which can be used to perform attitude and orbit control. This provides some means of redundancy, although performance would be degraded by the loss of one system, and useful life decreased.

The hydrazine is initially pressurized to 360 psia (pounds per square inch absolute) with gaseous nitrogen. The pressure of the nitrogen, and the centrifugal force due to the spacecraft spin, forces the fuel through the propellant manifold to the thrusters. On command from the ground, a valve in the thruster can be opened, allowing the fuel to flow into the thrust chamber . The hydrazine is rapidly decomposed in the presence of a catalyst (Shell 405). The reaction is exothermic and the expanding gases provide the thrust which is 1 lb nominally.

To provide a velocity increment to the spacecraft in the spin axis direction, the axial jet is fired continuously. Normally, in this mode both axial jets are fired simultaneously to eliminate torque - induced

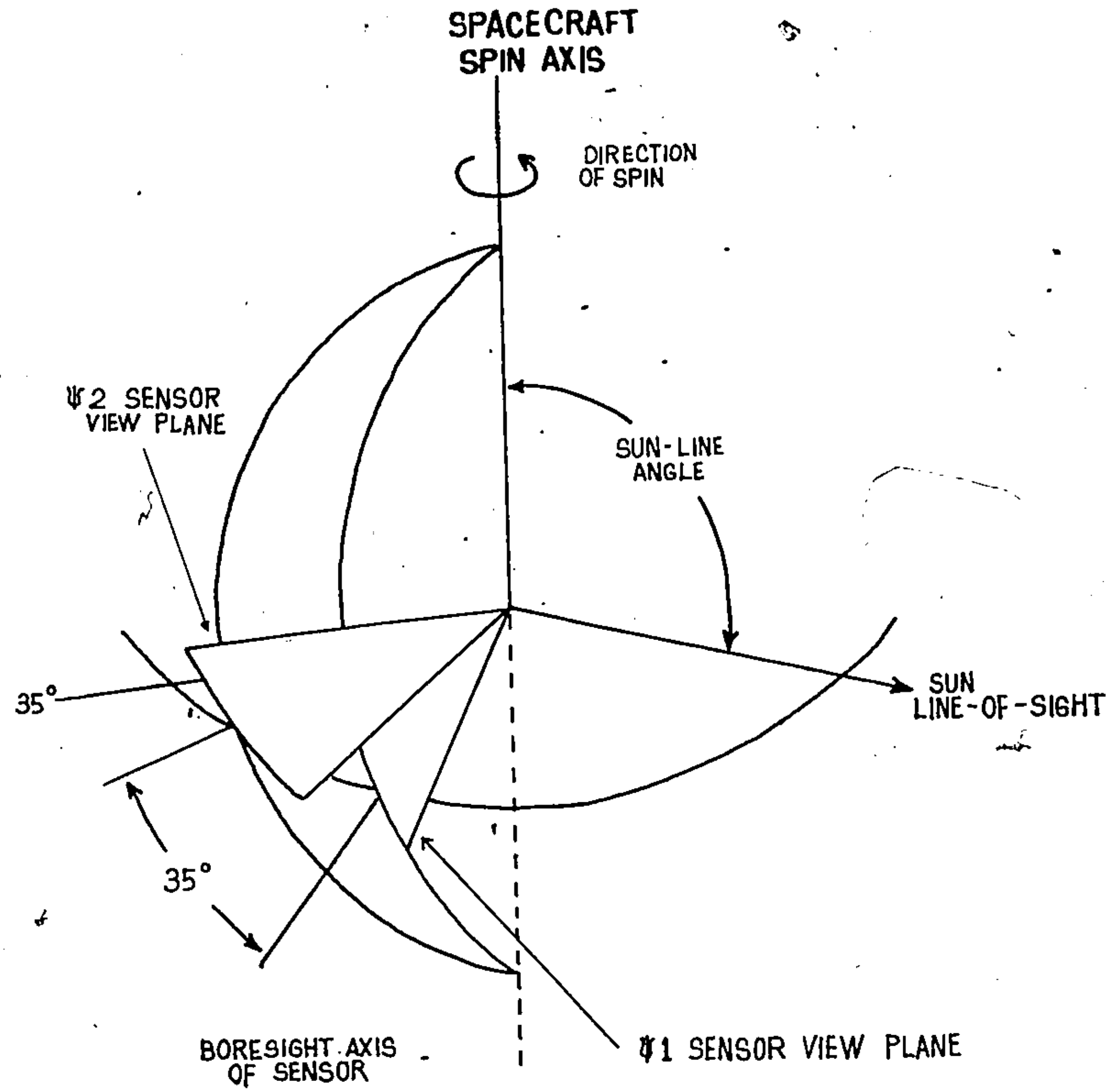


FIG 2.5 SUN SENSOR GEOMETRY

nutations. To provide a velocity increment normal to the spin axis, the canted jet is fired in pulsed mode at spin frequency. By varying the phasing in the spin cycle, the radial thrust component can be shifted through 360 degrees in the plane normal to the spin axis.

The spin axis attitude is changed by the application of precession torques. The satellite's axial thrusters are aligned parallel to the spin axis. By pulsing one of the axial jets during a particular portion of each revolution of the satellite, the required precession torque is produced.

CHAPTER 3

DESIGN OF AN EXTENDED KALMAN FILTER FOR ATTITUDE ESTIMATION

The Extended Kalman Filter equations will be derived in this chapter. The filter will be used to process the sensor data and output the attitude of the spacecraft. Before deriving the filter equations, it is first necessary to describe the coordinate systems used and formulate the state equations.

3.1 Coordinate Systems and State Equations

In order to describe the motion of a space vehicle, a coordinate system is required. This coordinate system is chosen to fit best the type of motion desired. Three reference coordinate frames are used to define the geometrical relationships involved in the system model. These frames shall be referred to as basic inertial (I), orbital (0), and body-fixed (B). Transformations between the frames are derived below and their geometrical relations are depicted in Figs. 3.1 and 3.2.

The basic inertial is derived from the more general Equatorial Coordinate System (ECS). The origin coincides with the center of earth; the xy plane coincides with the equatorial plane with the x axis pointing to the direction of the First Point of Aries and the z axis pointing to the North pole. By neglecting the precession of the first point of Aries, this frame is considered to be inertial.

Derivation of the Transformation Matrix T_{OI}

The orbital frame differs from the basic inertial frame by two Euler angles, the right ascension Ω, and the inclination i, which define the orbital plane of the satellite. Consequently, only two rotations are sufficient

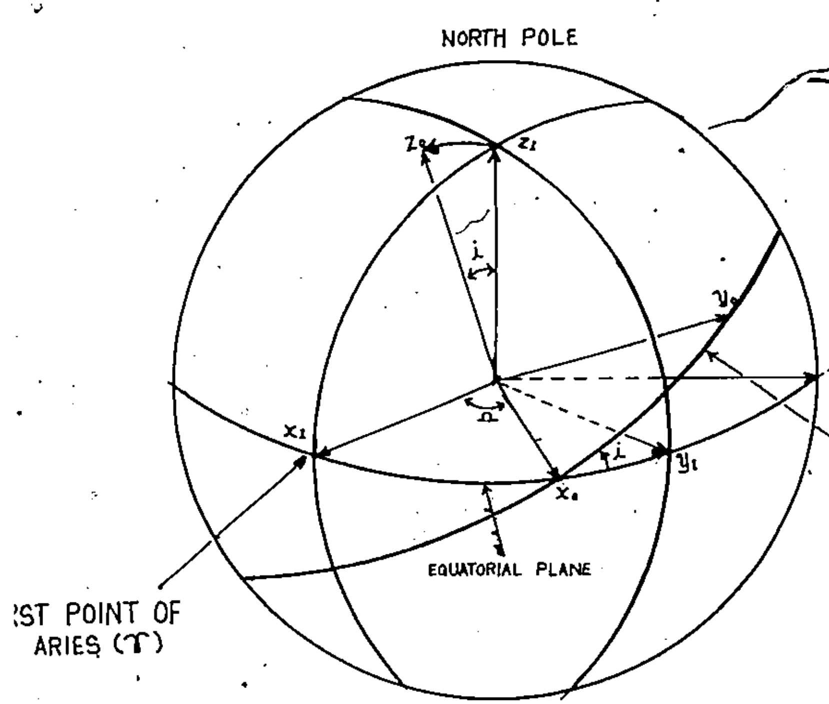


FIG. 3.1 BASIC INERTIAL AND ORBITAL COORDINATE SYSTEMS

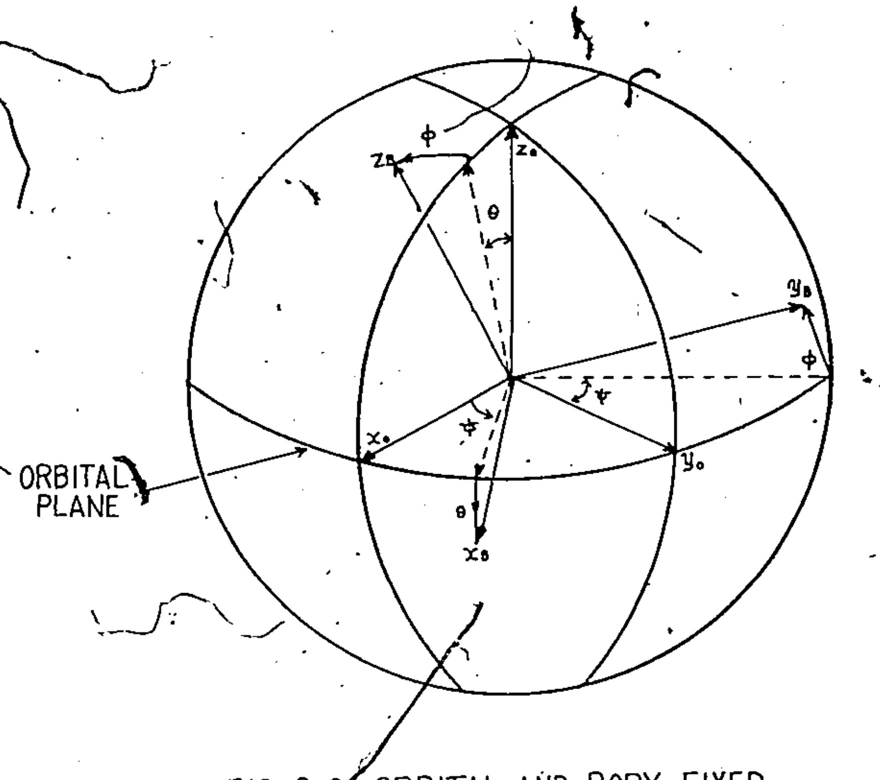


FIG. 3.2 ORBITAL AND BODY-FIXED COORDINATE SYSTEMS

to bring the two frames into coincidence. The transformation boxes [7] for the two rotations are shown in Fig. 3.3 and the transformation matrix T_{OI} can easily be derived as:

$$T_{OI} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\cos i \sin\Omega & \cos i \cos\Omega & \sin i \\ \sin i \sin\Omega & -\sin i \cos\Omega & \cos i \end{bmatrix} \qquad (3.1)$$

Since we are considering a known orbit, the matrix T_{OI} will thus be a constant matrix.

Derivation of the Transformation Matrix T_{BO}

The body frame differs from the orbital frame by three Euler angles ψ, θ, ϕ. Under nominal attitude conditions, the orientation of the body-fixed frame is such that X_B is in the orbital plane, Y_B is along the local zenith, and Z_B completes the right-handed triad. There are six possible ways of transforming the orbital frame to the body frame. The sequence of rotations and their transformation boxes selected is shown in Fig. 3.4 and the matrix T_{BO} can be derived as:

$$T_{BO} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \qquad (3.2)$$

State Equations

As mentioned in the introduction, the attitude of a spacecraft is defined as the orientation of the body axes with respect to reference axes (orbital frame). Thus, the state X of the model is chosen as the three

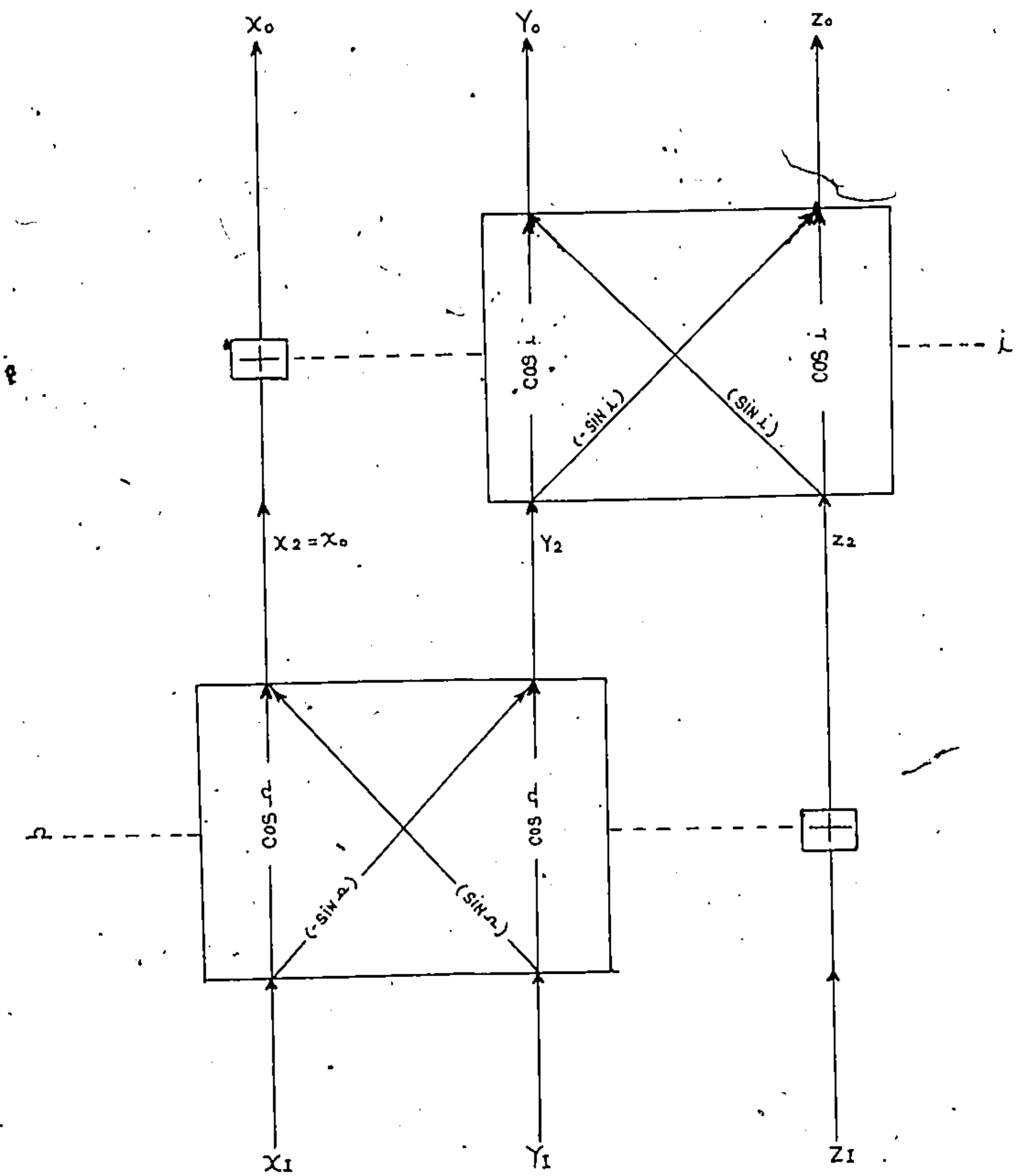


FIG. 3.3 TRANSFORMATION BOXES FOR RELATIONSHIPS BETWEEN I AND O COORDINATE SYSTEMS

component vector

$$(X)^T = (\psi, \theta, \phi) \tag{3.3}$$

where ψ, θ, and ϕ are Euler angles defining the body attitude as shown in Fig. 3.2.

The differential equations relating the state rates ($\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$) to the body rates (ω_x, ω_y, ω_z) as measured about the x, y, z body axes can be derived from the transformation boxes (Fig. 3.4) as:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 1 & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{3.4}$$

The general state equation can thus be written as

$$\dot{X} = G(X)\,\omega \tag{3.5}$$

where $(\omega)^T = (\omega_x, \omega_y, \omega_z)$ is the vector of the body rates.

3.2 Measurement Equations

To formulate the Kalman Filter, it is essential to know the measurement equations. There are two different measuring devices, namely, the sun sensors and the earth sensors (Chapter 2). From them, the declination, the right ascension of the spin axis and the angle between the spin axis and the sun are obtained. Thus three measurement equations are required.

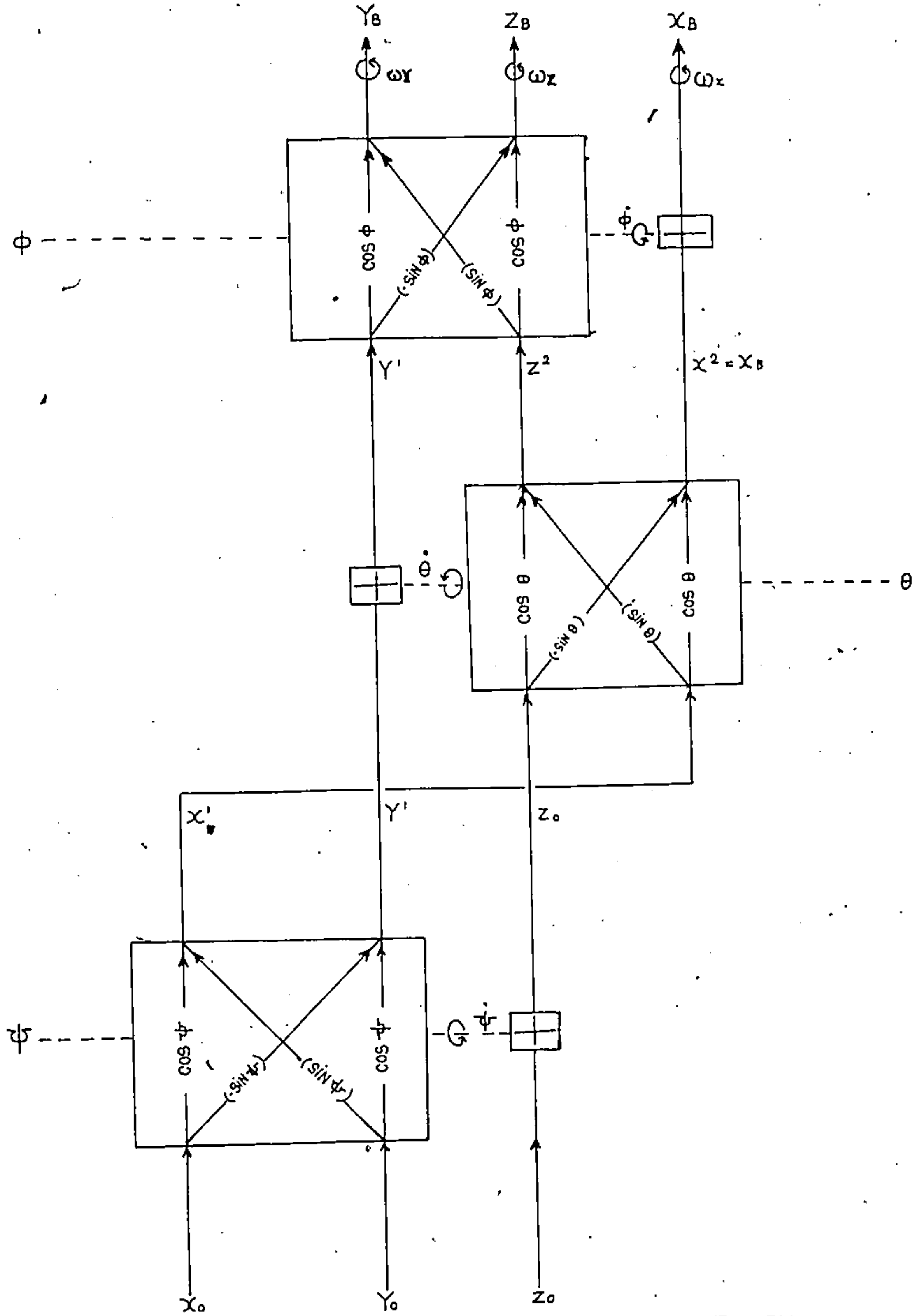


FIG. 3.4 TRANSFORMATION BOXES FOR RELATIONSHIPS BETWEEN O AND B COORDINATE SYSTEMS

Earth Sensor Measurement Equations

The earth sensor measurement equations can be derived by transforming the unit spin vector from the orbital to the body frame. Consider the unit spin vector, A, in the orbital frame as

$$A_o = \begin{bmatrix} \sin\delta \ \cos\xi \\ \sin\delta \ \sin\xi \\ \cos\delta \end{bmatrix} \tag{3.6}$$

where δ and ξ are the declination and right ascension of the spin vector respectively (Fig. 3.5).

The unit spin vector in the body frame is A_B, (Fig. 3.6), where

$$A_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.7}$$

Since the transformation matrix T_{BO} is known, we have

$$A_B = T_{BO} \cdot A_o$$

After carrying out the matrix multiplication, simplification and equalization, the following equations are obtained:

$$\sin\delta \cos\xi = \frac{-\sin^2\psi \sin\theta \cos\phi + \sin\psi \sin\phi \cos\psi - \sin\theta \cos\psi}{\cos\phi}$$

$$\sin\delta \sin\xi = \sin\theta \cos\phi \sin\psi - \sin\phi \cos\psi$$

$$\cos\delta = \cos\theta \cos\phi$$

For simplicity of the linearization procedure, the two earth sensor measurement equations are related as

$$Y_1 \triangleq \sin\delta \sin\xi = \sin\theta \cos\phi \sin\psi - \sin\phi \cos\psi \qquad (3.8)$$

$$Y_2 \triangleq \cos\delta = \cos\theta \cos\phi \qquad (3.9)$$

Sun Sensor Measurement Equation

The sun sensor measurement equation can be similarly derived by considering the unit sun vector in the inertial coordinate system and transforming it to the body frame. The unit sun vector in the inertial frame is, S_I, (Fig. 3.7);

$$S_I = \begin{bmatrix} \sin Q \quad \cos P \\ \sin Q \quad \sin P \\ \cos Q \end{bmatrix} \qquad (3.10)$$

where Q is given by (3.11) describing the annual movement of the sun with time zero as the vernal equinox (21st March).

$$Q = 25 \cdot \sin(2\pi \cdot \omega_s t / 365) \qquad (3.11)$$

and $P = \omega_s t \quad ; \quad \omega_s = 1$ degree /day

$t = 0, 1, 2, \ldots\ldots, 365.$

The sun vector in the body frame can be represented by the vector, S_B, (Fig. 3.8):

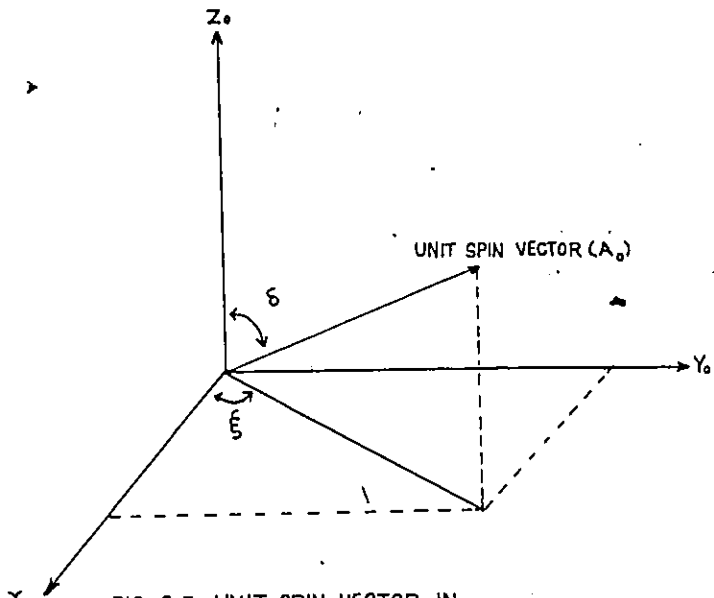


FIG. 3.5 UNIT SPIN VECTOR IN ORBITAL FRAME

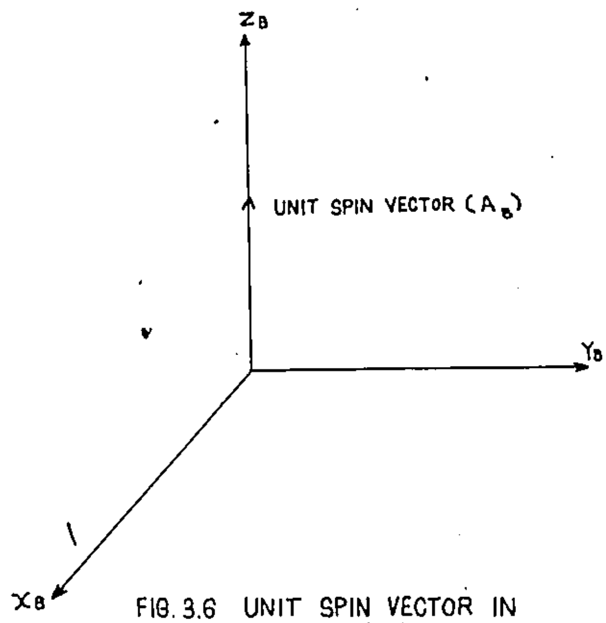


FIG. 3.6 UNIT SPIN VECTOR IN BODY FRAME

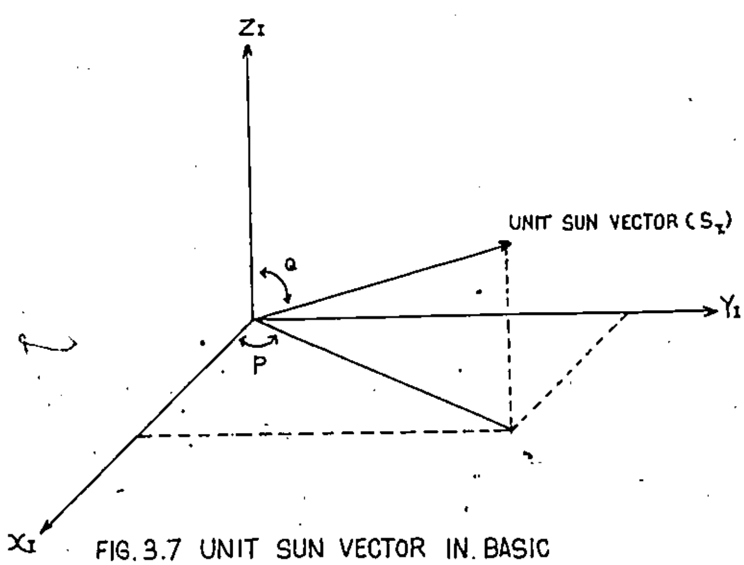


FIG. 3.7 UNIT SUN VECTOR IN BASIC INERTIAL FRAME

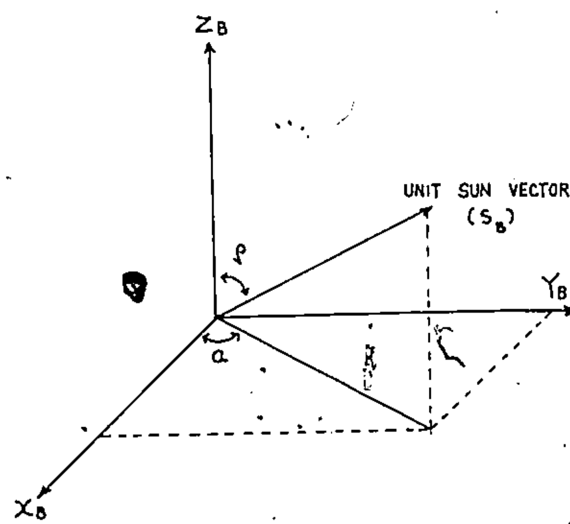


FIG. 3.8 UNIT SUN VECTOR IN BODY FRAME

$$S_B = \begin{bmatrix} \sin\rho & \cos\alpha \\ \sin\rho & \sin\alpha \\ \cos\rho \end{bmatrix} \qquad (3.12)$$

The angle ρ is the angle between the sun vector and the spin axis and is provided by the sun sensor measurements. Equating the two vectors, we obtain:

$$S_B = T_{BO} \cdot T_{OI} \cdot S_I$$

Since we are considering a known orbit, the values of Ω and P can be precalculated. Thus $T_{OI} \cdot S_I$ is assumed to be known.

Let us define

$$T_{OI} \cdot S_I \triangleq \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where V_1, V_2, and V_3 are constants. After carrying out the matrix multiplication as given in equation 3.12, we select the sun sensor measurement equation as

$$Y_3 \triangleq \cos\rho = V_1 (\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi) + V_2 (\cos\phi \sin\theta \sin\psi - \sin\phi \sin\psi) +$$
$$V_3 (\cos\phi \cos\theta) \qquad (3.13)$$

The general measurement equation for the system can be written as:-

$$Y = h(X) \qquad (3.14)$$

where, $Y = [Y_1, Y_2, Y_3]^T$

3.3 Kalman Estimation

The state and measurement equations derived earlier appear in terms of quantities that have a nonlinear algebric relation with the state variables. The non-linearity of the dynamical and output equations must be removed for the application of Kalman Estimation techniques. To achieve this, the following fundamental assumption is used. [6, pp 265] Fundamental Assumption. A nominal solution of the nonlinear differential equations must exist. This solution must provide a "good" approximation of the actual behaviour of the system. The approximation is "good" if the difference between the nominal and actual solutions can be described by a system of linear differential equations. These equations shall be called "linear perturbation euqations".

As described in earlier section, the dynamical system evolves according to the vector differential equation

$$\dot{X} = G(X)\omega \tag{3.5}$$

where X is the three dimensional state vector. The nominal values of our model are selected by considering the dual-spin geostationary satellite in an ideal attitude position. This is the situation when the Euler angles θ and ϕ are both equal to zero, the angle ψ is moving at the orbital rate, ω_o, and the spacecraft is spinning about its body Z axis at the nominal rate, ω_z^*.

Thus, the nominal values of X are defined as X^*

where, $(X^*)^T = (\psi^*, \theta^*, \phi^*) = (\omega_o t, o, o)$.

and $(\omega^*)^T = (\omega_x^*, \omega_y^*, \omega_z^*) = (o, o, \omega_z^*)$

Linearization of state and Measurement equations

To linearize the state equations, equation (3. 5) is expanded in a Taylor series about the nominal values. We neglect all terms except those of first order. Then we obtain

$$\delta \dot{X} = F \cdot \delta X \tag{3.15}$$

where, $$(\delta X)^T = (X - X^*)^T = (\delta X_1, \delta X_2, \delta X_3) \tag{3.16}$$

and

$$F = \frac{\partial}{\partial X} \left[G(X)\, \omega \right] \Bigg|_{\omega = \omega^*, \; X = X^*}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_z^* \\ 0 & \omega_z^* & 0 \end{bmatrix}$$

Equation (3.15) represents the state linear perturbation equation. This equation describes the motion of the actual system relative to the nominal.

The measurement equations must also be linearized. The measurement equations are related to the state by

$$Y = h(X) \tag{3.14}$$

Using the same technique as that of the state linearization, we obtained the linearized measurement equations as

$$\delta Y = H \cdot \delta X \tag{3.17}$$

where, $(\delta Y)^T = (Y - Y^*)^T = (\delta Y_1, \delta Y_2, \delta Y_3)$ and

$$Y^* = h(X^*)$$

$$H = \left.\frac{\partial h}{\partial X}\right|_{X = X^*} = \begin{bmatrix} 0 & \sin\omega_o t & -\cos\omega_o t \\ 0 & 0 & 0 \\ 0 & V_1\cos w_o t + V_2 \sin\omega_o t & V_1 \sin\omega_o t - V_2\cos\omega_o t \end{bmatrix}$$

The measurement noise, $\{V(t), t \geq t_o\}$ is considered to be an additive zero mean Gaussian white noise with covariance matrix

$$E\{V(t)\, V^T(t)\} = R(t) \cdot \delta(t-t)$$

where, $R(t)$ is a known positive definite matrix and $\delta(t)$ is the impulse function. Thus the linearized measurement equation becomes:

$$\delta Y = H \cdot \delta X + V \tag{3.18}$$

3.4 Kalman Filter Equations

From the linearized state and measurement equations (3.15) and (3.18) respectively we obtained the filter euqations [8] :

$$\begin{aligned} \delta\dot{\hat{X}}(t) &= F(t)\delta\hat{X}(t) + K(t)\left[\delta Y(t) - H(t)\,\delta\hat{X}(t)\right]; \ \forall\ t \geq t_o \\ \text{with} \quad & \delta\hat{X}(t_o) = 0 \\ K(t) &= F(t)\,H^T(t)\,R^{-1}(t) \\ \dot{P}(t) &= F(t)P(t) + P(t)F^T(t) - P(t)\,H^T(t)\,R^{-1}(t)\,H(t)P(t) \end{aligned} \tag{3.19}$$

where $\delta\hat{X}(t)$ is the optimal estimate of $\delta X(t)$, $K(t)$ is the Kalman Gain Matrix, and $P(t)$ is the error covariance matrix with initial condition given by:

$$P(t_o) = E\{\delta X(t_o)\ \delta X^T(t_o)\}$$

which is assumed to be known.

Thus, from equations (3.15), (3.16), (3.18), (3.19) and knowing the values of ω_o and ω_z^* (As an example, in the Anik satellites ω_o = 1 degree/day, ω_z^* = 100 RPM), the attitude of the spacecraft can be determined at each instant of time. With this attitude information, it is then possible to decide the attitude control cycle.

CHAPTER 4

A PERIODIC ATTITUDE CONTROL POLICY

For communication purposes with non-tracking ground antennas, it is necessary to maintain the declination of the spin-axis of the dual-spin geostationary satellite from the orbit normal to within 0.1 degree. The periodic attitude control model derived in this chapter is such that the spin-axis declination is kept within this constraint. Before deriving the attitude control policy, it is essential to model the natural precession of the spacecraft under the natural disturbing torque.

4.1 Natural Precession of Spacecraft

The precession of the spacecraft angular momentum vector H by the solar radiation torque is considered to be the only natural attitude perturbation of the dual-spin geostationary satellite.

$$H = I \cdot \omega \tag{4.1}$$

where, H $\triangleq$ angular momentum of satellite,

I $\triangleq$ rotational inertia of satellite,

ω $\triangleq$ angular velocity of satellite.

A torque is produced by the radiation force of the Sun acting through an effective centre of mass displaced from the centre of mass of the satellite. The torque direction is perpendicular to the angular momentum vector resulting in precession of this vector.

Prediction of the natural precession of the satellite can be achieved by solving the following equations [2]:

$$\dot{x}_1 = - \frac{\tau}{H} \cos \omega_s t \tag{4.2}$$

$$\dot{x}_2 = - \frac{\tau}{H} \sin \omega_s t \tag{4.3}$$

where x_1, x_2 are orthogonal components of the spin vector projected onto a plane passing from the center of the satellite and parallel to the inertial x, y plane, ω_s is the angular rate of projection of the Sun vector on the same plane, H is the angular momentum of the satellite and τ is the solar radiation torque magnitude.

To solve equations (4.2) and (4.3), it is necessary to know the solar radiation torque magnitude. The data for solar radiation force and the natural precession rate of a typical dual-spin geostationary satellite (Anik I) with an angular momentum of 900 ft lb. sec. are obtained from reference [2]. Since

$$\text{Precession Rate} = \frac{F \times LA}{H} = \frac{\tau}{H} \tag{4.4}$$

where, F $\triangleq$ solar radiation force

LA $\triangleq$ lever or moment arm length

and knowing the value of H, we can find the solar radiation torque magnitude. The natural precession rate curve, the solar radiation torque curve and the variation of moment arm curve are plotted in Figs. 4.1, 4.2 and 4.3 respectively as functions of time. Time zero is considered to be the time of Earth - Sun - Aries coincidence i.e. 21st March. At this time the Sun is in the equatorial plane and moreover in the inertial x - axis.

FIG 4.1 NATURAL SPIN AXIS PRECESSION RATE

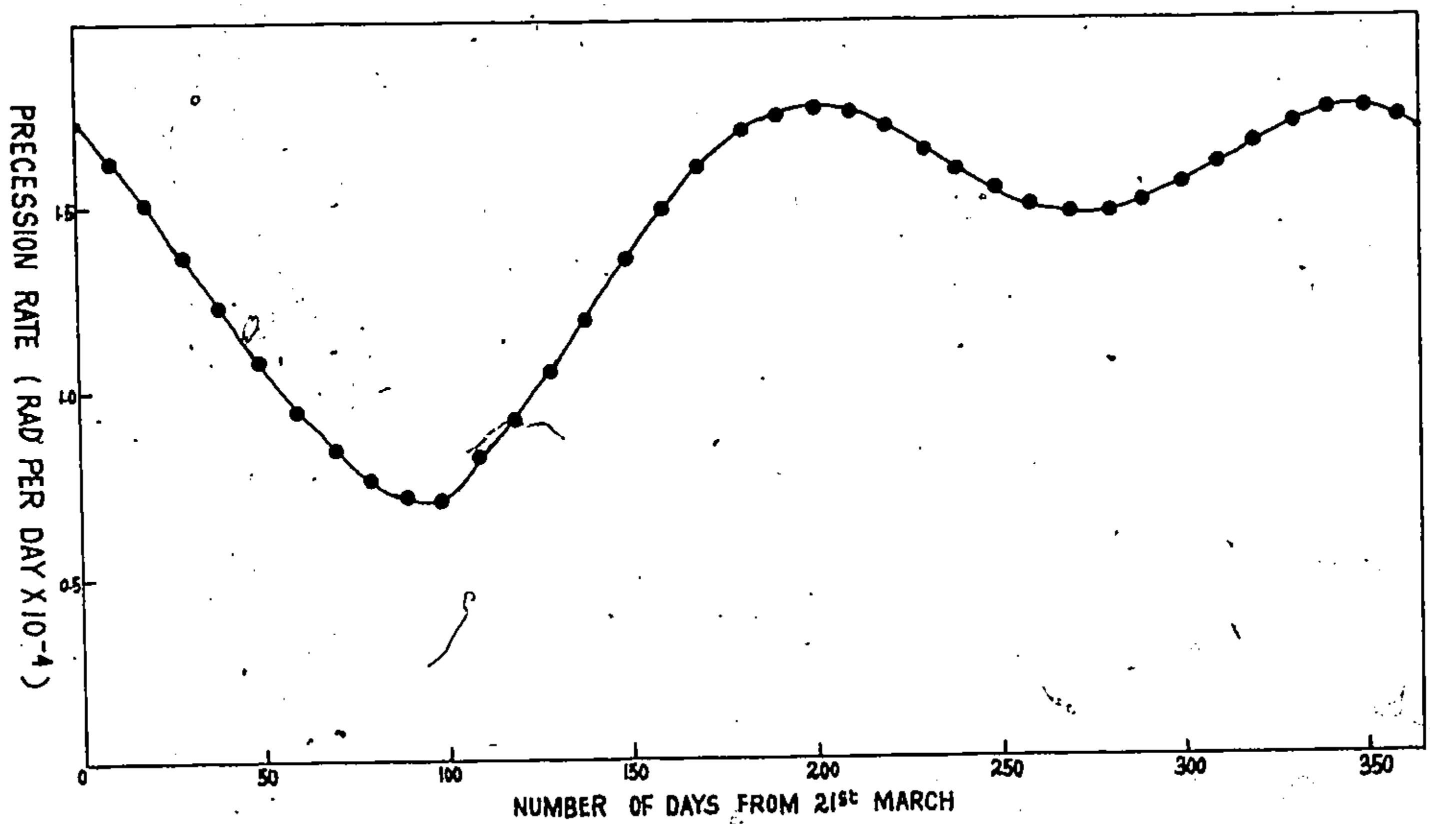


FIG 4.2 VARIATION OF SOLAR RADIATION TORQUE

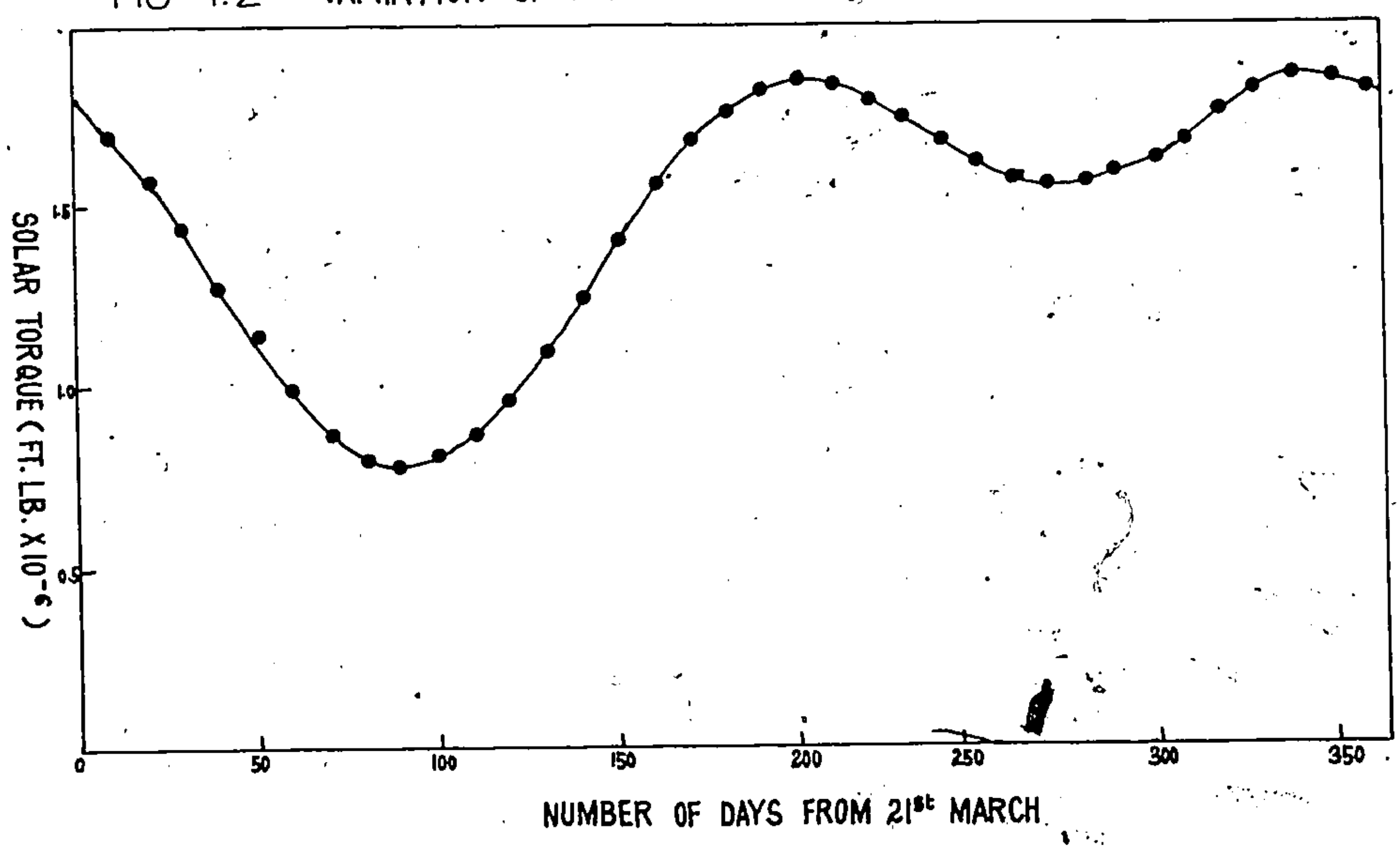
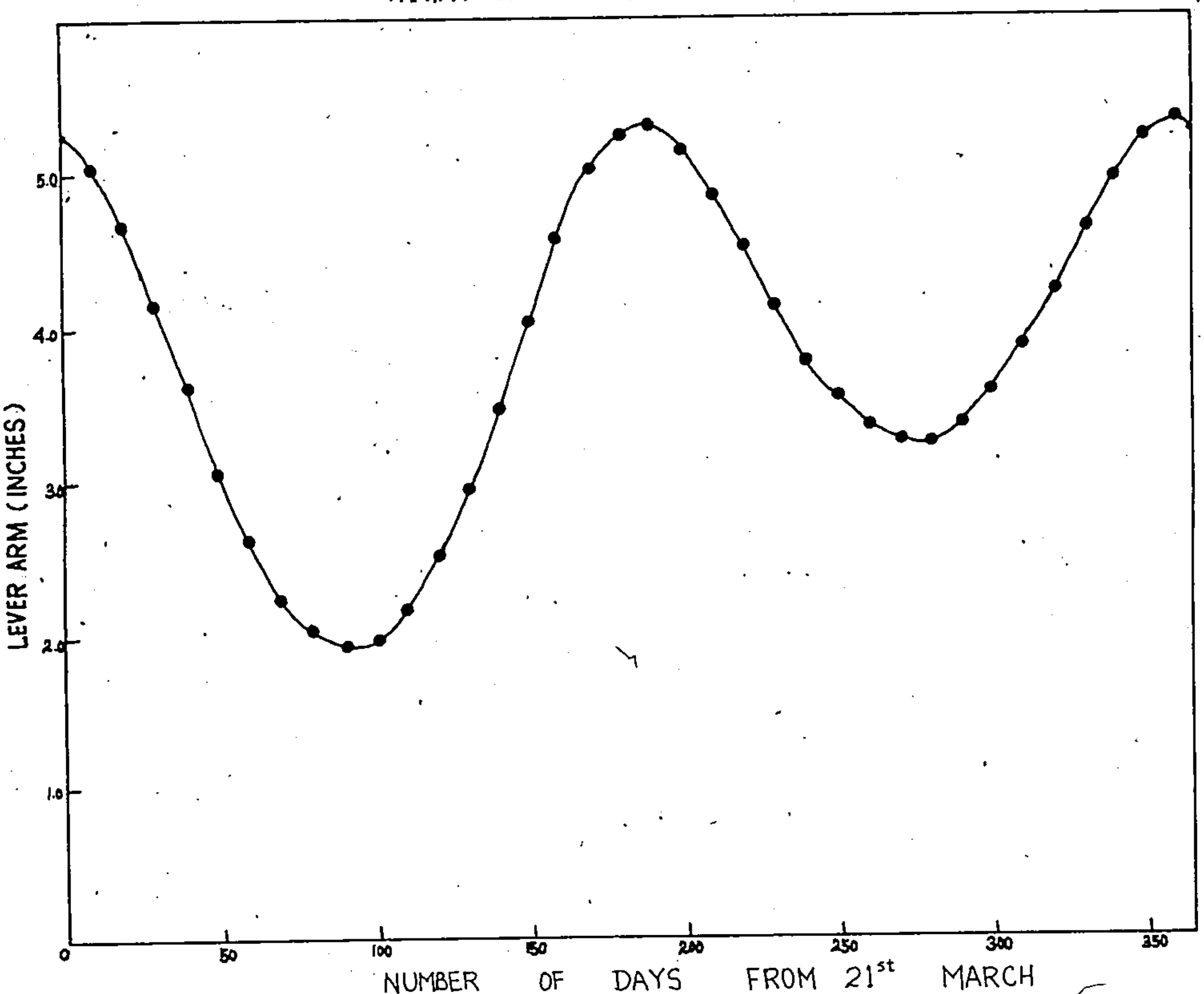


FIG 4.3
VARIATION OF LEVER ARM



4.2 Attitude Control Policy

As mentioned earlier, the attitude constraint of this model is that of limiting the spin axis declination to within 0.1 degree. That is, the projection of the unit-spin vector should not be permitted to move outside the limit circle of radius equal to 1.74533×10^{-3} ($= \sin 0.1°$) units in the x_1, x_2 plane. No control is necessary as long as the spin-vector projection is within the limit circle.

Attitude corrections using single axial jet in pulsed mode are performed periodically within a cycle. There are two cases [2] : (a) attitude corrections that precede the inclination corrections are such that the spin axis would be normal to the equatorial plane at the time of the inclination manoeuvre, and (b) if an inclination manoeuvre does not follow the attitude correction, then the attitude is targeted to coincide with the orbit normal midway between attitude corrections. Thus, if

$$a = \left(\frac{\pi}{2} - \delta \right) \text{ EXP } j \left(\alpha + \frac{\pi}{2} \right) \qquad (4.5)$$

is a representation of the spin vector with δ and α as the declination and right ascension respectively, then corresponding to cases(a) and (b) we have the targeting strategies [2] ;

(a) $a_T + \Delta a_t = 0$

(b) $a_T + \Delta a_t = i_t$.

where a_T is the target spin axis representation in the attitude manoeuvre, Δa_t is the precession (due principally to solar torque) over half the attitude correction cycle and i_t is the inclination vector [2] at the midpoint of attitude correction cycle. The attitude control policy selected in this model corresponds to that of case (a). Since the orbit normal precesses approximately symmetrically about the pole during an inclination manoeuvre, this strategy minimizes the cross-coupling of the inclination velocity increment into the orbit plane. The drift rate and eccentricity are therefore disturbed as little as possible [2].

The natural precession of the satellite is predicted by solving equations (4.2) and (4.3) with an appropriate set of initial conditions. The spacecraft spin is allowed to precesse according to its natural path until it reaches the boundary of the limit circle. Control is then applied by activating one of the axial jets in pulse mode during a particular segment of the satellite spin and thus producing the required torque to bring the spacecraft spin to a new location within the limit circle. The satellite is permitted to move naturally again and the cycle is repeated. The control period is dependent on the initial position of the satellite spin axis.

<u>An Example</u>

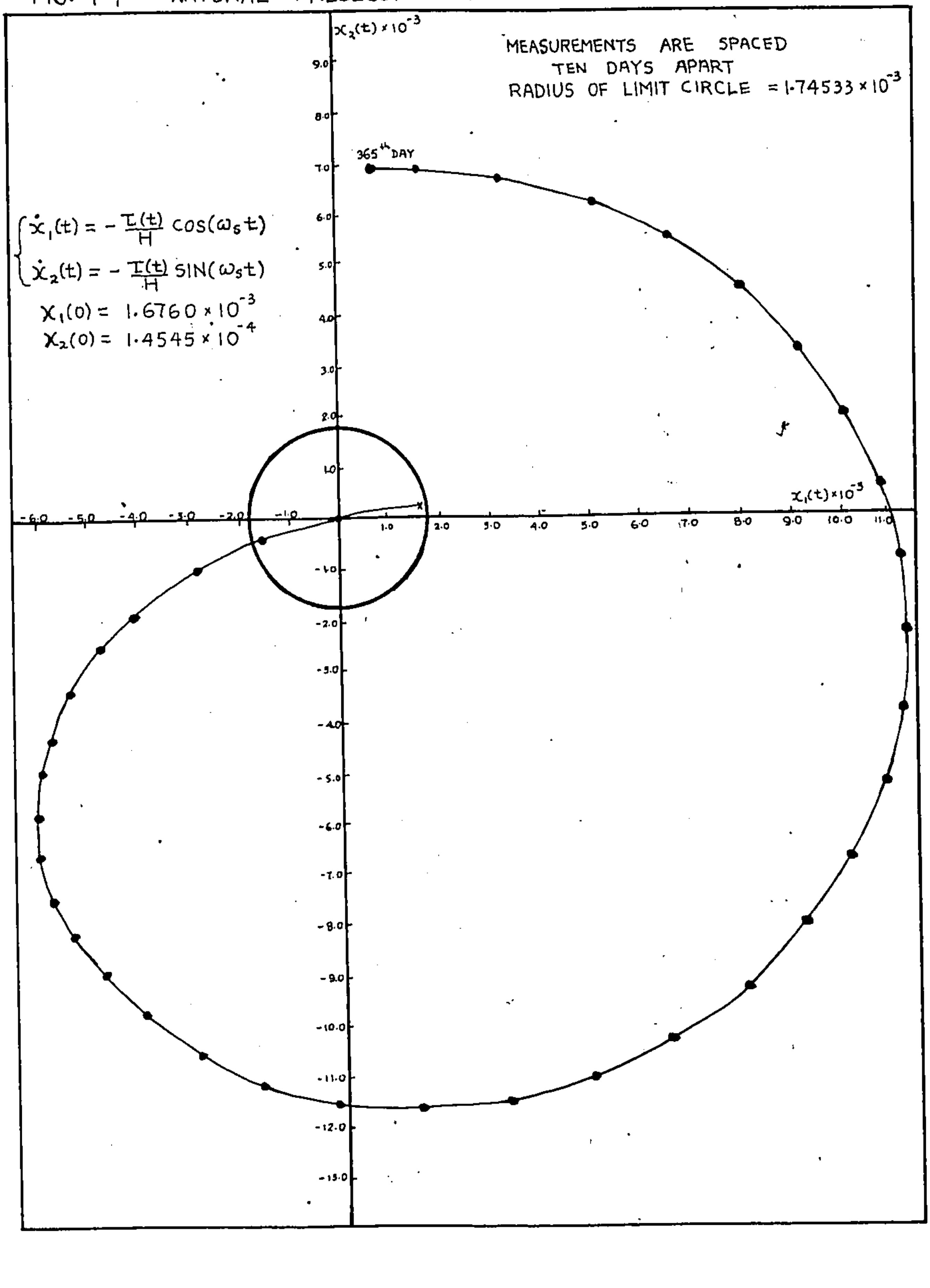
Consider that time zero is the time when the Sun is in the equatorial plane and in particular on the inertial x-axis (21st March). Equations (4.2) and (4.3) are solved with a digital computer with inital conditions selected as:

$$x_1(0) = 1.6760 \times 10^{-3} \qquad (4.6)$$

$$x_2(0) = 1.4545 \times 10^{-4} \qquad (4.7)$$

The initial conditions were selected such that the Sun being initially at the equatorial plane, starts to "push" the spin axis from its initial position causing its projection to move pass the center of x_1, x_2 coordinate system on the tenth day and beyond the limit circle after 21 days. If no control is applied after 21 days, the satellite projections will continue to move in a spiral fashion as shown in Fig. 4.4.

FIG. 4.4 NATURAL PRECESSION OF SATELLITE IN A YEAR



Since the spin axis projection moves outside the constraining circle after 21 days, it is possible to formulate an attitude control model with a period of 21 days. That is, the satellite spin is moved to a new starting position within the limit circle by applying controls. To determine the new starting conditions and the control magnitude, equations (4.2) and (4.3) have to be modified.

The Sun has travelled 21 degrees around the Earth in the course of 21 days and the solar radiation torque is varied accordingly (refer to Fig. 4.2). Thus the modified equations are:

$$\dot{x}_1(t) = \frac{\tau(t)}{H} \left[\sin(\omega_s t) \cdot \sin(21^\circ \cdot i) - \cos(\omega_s t) \cdot \cos(21^\circ \cdot i)\right] \quad (4.8)$$

$$\dot{x}_2(t) = \frac{\tau(t)}{H} \left[-\cos(\omega_s t) \cdot \sin(21^\circ \cdot i) - \sin(\omega_s t) \cdot \cos(21^\circ \cdot i)\right] \quad (4.9)$$

where $t = 0, 1, 2, \ldots\ldots 21.$

The new starting positions after each 21 day cycle are found from the following recursive formulas:

$$x_1^j(0) = x_1^{j-1}(0) \cdot \cos(21^\circ) - x_2^{j-1}(0) \cdot \sin(21^\circ) \quad (4.10)$$

$$x_2^j(0) = x_1^{j-1}(0) \cdot \sin(21^\circ) + x_2^{j-1}(0) \cdot \cos(21^\circ) \quad (4.11)$$

with

$$j = i + 1 \quad (4.12)$$

The periodic attitude control model is found by integrating equations (4.8), (4.9) with initial conditions given by (4.10) and (4.11). The first starting conditions, $x_1^o(0)$ and $x_2^o(0)$ are selected to be equal to (4.6) and (4.7) respectively. With i equal to 0 and j equal to 1 the equations are solved with t running from 0 to 21 days. The solutions provide the natural precession of the satellite spin axis. After each 21 day cycle, the satellite spin is brought to a new starting location as defined by equations (4.10) and (4.11) with i incremented by 1 and the value of j as specified by condition (4.12), by applying a corrective force.

The magnitude of the corrective force can be determined as follows:

$$\tau_{cr} = H \cdot \ell \qquad (4.13)$$

where,

$\tau_{cr} \triangleq$ corrective torque

$\ell \triangleq$ distance between the end position of the spin projection in a cycle and the new starting position

$$\text{and,} \quad \ell = \left[\left(x_1^j(0) - x_1^{j-1}(0) \right)^2 + \left(x_2^j(0) - x_2^{j-1}(0) \right)^2 \right]^{\frac{1}{2}} \qquad (4.14)$$

Also,

$$\tau_{cr} = F_{cr} \cdot D$$

$F_{cr} \triangleq$ corrective force

$D \triangleq$ distance of axial jet from center of mass of the spacecraft

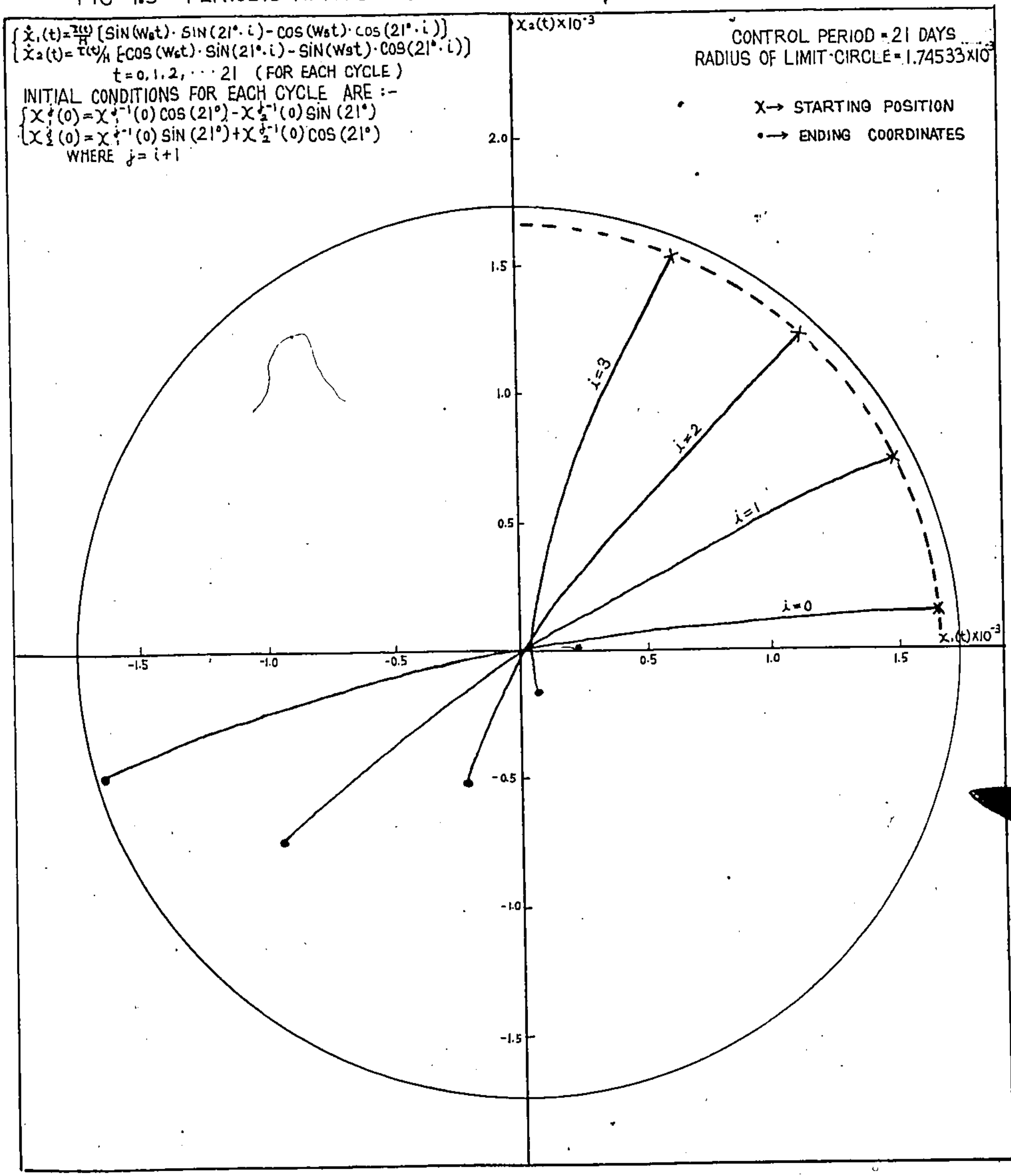
$$\therefore \quad F_{cr} = \frac{\tau_{cr}}{D}$$

The computer results for the modified equations are tabulated in Table I and some of the trajectories are shown in Fig. 4.5. From the results, it is observed that the distance travelled by the spin axis is not uniform for each of the 21 day cycle. This is due to the fact that the solar radiation torque magnitude is not constant (Fig. 4.2). Thus the applied corrective force will also vary. However, the 21 day control cycle is selected for

TABLE 1. ATTITUDE CONTROL PERIOD OF 21 DAYS

CYCLE NO. (i)	TIME IN NO. OF DAYS	POSITION OF SPIN AXIS AT THE BEGINNING AND THE END OF THE CONTROL CYCLE, (x_1, x_2).	RADIUS OF LIMIT CYCLE $R \leq 1.74533 \times 10^{-3}$
0	0 → 21	$(1.6760 \times 10^{-3}, 1.4545 \times 10^{-4})$	1.6823×10^{-3}
		$(-1.6749 \times 10^{-3}, -4.6223 \times 10^{-4})$	1.7375×10^{-3}
1	21 → 42	$(1.5126 \times 10^{-3}, 7.3641 \times 10^{-4})$	1.6823×10^{-3}
		$(-9.3923 \times 10^{-4}, -7.4344 \times 10^{-4})$	1.1979×10^{-3}
2	42 → 63	$(1.1482 \times 10^{-3}, 1.2296 \times 10^{-3})$	1.6823×10^{-3}
		$(-2.2438 \times 10^{-4}, -5.2890 \times 10^{-4})$	5.7452×10^{-4}
3	63 → 84	$(6.3129 \times 10^{-4}, 1.5594 \times 10^{-3})$	1.6823×10^{-3}
		$(1.2531 \times 10^{-4}, -1.1235 \times 10^{-4})$	1.6830×10^{-4}
4	84 → 105	$(3.0538 \times 10^{-4}, 1.6820 \times 10^{-3})$	1.6823×10^{-3}
		$(1.5853 \times 10^{-4}, 8.1072 \times 10^{-5})$	1.7806×10^{-4}
5	105 → 126	$(-5.7427 \times 10^{-4}, 1.5812 \times 10^{-3})$	1.6823×10^{-3}
		$(2.4155 \times 10^{-4}, -9.7798 \times 10^{-3})$	2.6060×10^{-4}
6	126 → 147	$(-1.1028 \times 10^{-3}, 1.2704 \times 10^{-3})$	1.6823×10^{-3}
		$(6.6751 \times 10^{-4}, -3.8261 \times 10^{-4})$	7.6939×10^{-4}
7	147 → 168	$(-1.4848 \times 10^{-3}, 7.9083 \times 10^{-4})$	1.6823×10^{-4}
		$(1.3413 \times 10^{-3}, -3.6032 \times 10^{-4})$	1.3889×10^{-3}
8	168 → 189	$(-1.6696 \times 10^{-3}, 2.0619 \times 10^{-4})$	1.6823×10^{-3}
		$(1.8534 \times 10^{-3}, 1.2426 \times 10^{-4})$	1.8575×10^{-3}
9	189 → 210	$(-1.6326 \times 10^{-3}, -4.0584 \times 10^{-4})$	1.6823×10^{-3}
		$(1.8564 \times 10^{-3}, 8.3034 \times 10^{-4})$	2.0337×10^{-3}
10	210 → 231	$(-1.3787 \times 10^{-3}, -9.6396 \times 10^{-4})$	1.6823×10^{-3}
		$(1.3606 \times 10^{-3}, 1.3672 \times 10^{-3})$	1.9289×10^{-3}
11	231 → 252	$(-9.4171 \times 10^{-3}, -1.3940 \times 10^{-3})$	1.6823×10^{-3}
		$(6.6510 \times 10^{-4}, 1.5492 \times 10^{-3})$	1.6859×10^{-3}

FIG 4.5 PERIODIC ATTITUDE CONTROL OF DUAL SPIN GEOSTATIONARY SATELLITE



the practical reason that the controllers will perform the attitude manoeuvre on the same day after every three weeks. In addition, this choice might also help the personnel manager in scheduling the shift and vacation periods for the controllers !

CONCLUSIONS

In this thesis, a mathematical model for the estimation of the attitude of the dual-spin geostationary satellite using the Extended Kalman Filter has been formulated. In addition, a periodic attitude control model has also been established.

The state for the Kalman Filter has been selected to be a vector of the three Euler angles ψ, θ, and ϕ defining the attitude of the spacecraft. This choice of state variables arose from the definition of the attitude of a spacecraft. The measurement equations were derived by transforming appropriate unit vectors from one reference frame to another. Due to the inherent nonlinear characteristics of the state and measurement equations, linearization by means of Taylor Series Expansion had to be introduced. The nominal values were chosen by considering an ideal attitude situation.

In deriving the periodic attitude control model, the assumption that the spacecraft had achieved an almost perfect orbit and also that the orbital parameters were known at each instant of time, were made. The solar radiation torque magnitude and the moment arm length were found by combining the precession rate data and the solar radiation force data given in reference [2].

The targeting strategy selected was that the attitude corrections preceded the inclination manoeuvre. The inclination corrections were done in the middle of the attitude corrections cycle when the spin axis is normal to the equatorial plane. This strategy minimized the cross-coupling of the inclination velocity increment into the orbit plane and also did not perturb the drift rate and eccentricity.

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APPENDIX A

MASS PROPERTIES OF ANIK I (F - 1)

	WEIGHT (POUNDS)	MOMENT OF INERTIA (SLUG - FEET SQ) ROLL IZ	PITCH IX	YAW IY	MOMENT OF INERTIA RATIO IR / IP
TOTAL SPACECRAFT-(S/C)	1240.59	108.5	98.8	100.0	1.091
ORIENTATION PROPELLANT	5.0	0.5	0.2	0.2	
S/C AT IGNITION	1235.59	108.0	98.6	99.8	1.088
APOGEE MOTOR EXPENDABLES	582.79	10.9	9.4	9.4	
S/C AFTER INJECTION	652.80	97.0	82.0	83.2	1.157
HYDRAZINE REQUIRED	110.52	14.2	7.1	7.1	
S/C AFTER 7 YEARS	542.28	82.8	74.8	76.0	1.079

APPENDIX B

CONTINUOUS SYSTEM MODELING PROGRAMS (CSMP)

```
                    ****CONTINUOUS SYSTEM MODELING PROGRAM****

           ***PROBLEM INPUT STATEMENTS***

TITLE      NATURAL PRECESSION OF SATELLITE. NO CONTROL APPLIED
RENAME TIME=T
*          WS=1.0 DEGREE PER DAY CONVERTED IN TO RADIAN
*          H=ANGULAR MOMENTUM OF SATELLITE IN FT.LB.DAY.
*          R=ALLOWABLE TOLERANCE ,SHOULD .LE. 1.7453283E-3
*          Q=SUN DECLIATION ,BETWEEN -25.0 TO 25.0 DEGREE
*          T=NO. OF DAYS FROM 21ST MARCH
CONST  H=1.041667E-2,PY=3.141593,WS=1.745329E-2
INITIAL
        X1Z=1.6760E-3
       X2Z=1.4545E-4
DYNAMIC
       Q=25.0*SIN((2.0*PY*T)/365.0)
       FORCE=(4.065128-0.01768768*Q+0.00203274*(Q**2))*(10.0**(-6))
       P=(9.86549-0.08436*Q-0.005459*(Q**2))*(10.0**(-3))
       PREC=P*WS
       LA=((PREC*H)/FORCE)*12.0
       TORQ=(FORCE*LA)/12.0
        X1DT=-PREC*COS(WS*T)
        X2DT=-PREC*SIN(WS*T)
       X1=INTGRL(X1Z,X1DT)
       X2=INTGRL(X2Z,X2DT)
       R=SQRT(X1**2+X2**2)
TIMER  FINTIM=365.0,DELT=1.0,OUTDEL=1.0
METHOD RKSFX
PRTPLT X1,X2,R
CRSPLT X1,X2
END
STOP

OUTPUT VARIABLE SEQUENCE
X1Z     X2Z     Q       P       PREC    X1DT    X1      X2DT    X2      FORCE
LA      TORQ    R

 OUTPUTS    INPUTS     PARAMS   INTEGS + MEM BLKS   FORTRAN   DATA CDS
 17(500)   43(1400)    6(400)     2+   0=  2(300)   14(600)       7

       ENDJOB
```

Comment:

This program solves equations (4.2) and (4.3) with initial conditions as XIZ and X2Z for a period of 365 days.

```
****CONTINUOUS SYSTEM MODELING PROGRAM****

     ***PROBLEM INPUT STATEMENTS***

TITLE   ATTITUDE CONTROL OF DUAL SPIN SATELLITE
RENAME  TIME=T
*       WS=1.0 DEGREE PER DAY CONVERTED IN TO RADIAN
*       H=ANGULAR MOMENTUM OF SATELLITE IN FT.LB.DAY.
*       R=ALLOWABLE TOLERANCE ,SHOULD .LE. 1.7453283E-3
*       Q=SUN DECLIATION ,BETWEEN -25.0 TO 25.0 DEGREE
*       T=NO. OF DAYS FROM 21ST MARCH
*       LA=LEVER ARM IN INCHES
*       TORQ=SOLAR RADIATION TORQUE.
*       A=21.0 DEG CONVERTED INTO RAD
*       TRC=CORRECTIVE TORQUE
*       FCR=CORRECTIVE FORCE
CONST   H=1.041667E-2,PY=3.141593,WS=1.745329E-2,.A=3.665191E-1,...
        X1ZO=1.676CE-3,X2ZO=1.4545E-4,E=1.0,S=0.0 ,D=0.0
INITIAL
        X1Z=X1ZO
        X2Z=X2ZO
DYNAMIC
        Q=25.0*SIN((2.0*PY*V)/365.0)
        P=(9.86549-0.08436*Q-0.00545*(Q**2))*(10.0**(-3))
        PREC=P*WS
        V=T+D
         X1DT=PREC*(SIN(WS*T)*SIN(A*S)-COS(WS*T)*COS(A*S))
         X2DT=PREC*(-COS(WS*T)*SIN(A*S)-SIN(WS*T)*COS(A*S))
        X1=INTGRL(X1Z,X1DT)
        X2=INTGRL(X2Z,X2DT)
        R=SQRT(X1**2+X2**2)
          U1=X1ZO*COS(A)-X2ZO*SIN(A)
          U2=X1ZO*SIN(A)+X2ZO*COS(A)
TERMINAL
        X1ZO=U1
        X2ZO=U2
        FCR=(H*SQRT((X1ZO-X1)**2+(X2ZO-X2)**2))/2.8
        WRITE (3,5) FCR
  5     FORMAT ('0',F15.12)
        D=D+21.0
        S=S+1.0
        E=E+1.0
        IF (E .GT. 17.0) GO TO 11
        CALL RERUN
 11     CONTINUE
TIMER  FINTIM=21.0,DELT=0.1,OUTDEL=1.0
METHOD RKSFX
PRTPLT P
PRINT  V,X1,X2,X1ZO,X2ZO,R
END
STOP

OUTPUT VARIABLE SEQUENCE
X1Z      X2Z      V        Q        P        PREC     X1DT     X1       X2DT     X2
R        U1       U2       X1ZO     X2ZO     FCR      ZZ0003   D        S        E

OUTPUTS    INPUTS    PARAMS  INTEGS + MEM BLKS  FORTRAN  DATA CDS
24(500)   58(1400)  12(400)    2+  0=  2(300)   25(600)      8

ENDJOB
```

Comment:

This program solves equations (4.8) and (4.9) with starting coordinates given by conditions (4.10) and (4.11).