

Lawvere Theories and Definable Operations

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Abstract

We introduce the inner theory or, more verbosely, isotropy Lawvere theory functor, which generalizes the isotropy group/monoid by assigning a Lawvere theory of coherently extendable arrows to each object of a category with finite powers. Then, we characterize the inner theory for categories of models of an algebraic (or, more generally, quasi-equational) theory, and note its relationship with a notion of definability for morphisms. Finally, we explore a variety of examples.

Dedications

I would like to dedicate this thesis to the late Pieter Hofstra, who kindly accepted to supervise this thesis. Though I have only known him for the two short years I spent as his student, I am very grateful for his help and his encouragement, and he will be sorely missed.

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Chapter 1

Introduction

In [1], Bergman shows that conjugation by a group element admits a categorical characterization: inner automorphisms are precisely those automorphisms of a group G which can be coherently extended along any homomorphism $h : G \rightarrow H$. We say that τ_h is the extension of an automorphism $\tau_G : G \rightarrow G$ along h if we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\tau_G} & G \\ h \downarrow & & \downarrow h \\ H & \xrightarrow{\tau_h} & H \end{array}$$

The coherence condition is then that given another homomorphism of groups $k : H \rightarrow K$, the following square commutes:

$$\begin{array}{ccc} H & \xrightarrow{\tau_h} & H \\ k \downarrow & & \downarrow k \\ K & \xrightarrow{\tau_{k \circ h}} & K \end{array}$$

Indeed, given $g \in G$, the family of functions defined by $\tau_h x = (hg)x(hg)^{-1}$ satisfies this condition and extends the map $x \mapsto gxg^{-1} : G \rightarrow G$. The data of a system of coherent extensions of an automorphism can be conveniently formulated as a natural automorphism of the projection functor $P_G : (G \downarrow \mathbf{Grp}) \rightarrow \mathbf{Grp}$. The group $\text{Aut}(P_G)$ is called the extended inner automorphism group of G . Conversely, given an extended inner automorphism of G , it is shown that it is induced by a unique $g \in G$, resulting in a natural isomorphism $G \cong \text{Aut}(P_G)$. Then Bergman goes on to characterize the coherently extendable automorphisms of associative unital algebras over a field, and of Lie algebras. More generally, we might want to characterize extended inner automorphisms in categories of models of an algebraic (or even quasi-equational) theory. This is precisely what is done in [4], and expanded upon in Jason Parker's PhD thesis [13]. Parker characterizes in logical terms, for \mathcal{C} a category of models

of a quasi-equational theory, the functor $\mathcal{C} \rightarrow \mathbf{Grp}$ called the *isotropy group*, which assigns to an object X the extended inner automorphism group of X . This functor is the covariant version of a contravariant isotropy group functor introduced in [2] as an invariant of toposes. The contravariant version of isotropy is more appropriate in geometric settings, while the covariant version is better suited to algebraic settings; we will come back to this idea briefly in the conclusion. Also see [5] for an application of the isotropy group to the category of strict monoidal categories and its relation to the Picard group of such a category.

The isotropy group is interesting in its own right, but already in [1], Bergman considers the more general notion of inner *endomorphisms* of a group, which leads to what we might call the isotropy monoid of the category of groups. In this thesis, not only do we drop the condition of invertibility, but we also go beyond the isotropy monoid by considering coherent systems of extensions of arrows of type $C^n \rightarrow C^m$, for an object C of a category \mathcal{C} with finite powers. All such systems are then assembled into a Lawvere theory for each object $C \in \mathcal{C}$, and these Lawvere theories are bundled into what we might call the *isotropy Lawvere theory* functor $\mathfrak{Z}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Law}$, by analogy with the isotropy group and the isotropy monoid. However, we will prefer the shorter name inner theory for this functor. By considering arrows $C^n \rightarrow C^m$ rather than only arrows $C \rightarrow C$, we can access phenomena invisible to the isotropy group/monoid. The case of the category of commutative rings offers a striking example, which we treat in detail in Chapter 3. Our main technical result is a characterization of the inner theory associated with models of algebraic theories (or even quasi-equational theories), building on ideas from [1] and [13].

The plan of the thesis is as follows. In the remainder of this introduction, we go over preliminary material on category theory, algebraic theories, and quasi-equational theories. In Chapter 2, we motivate and define the inner theory (functor) associated with a category with finite powers. In Section 2.2, the inner theory of the category of models of an algebraic theory is characterized, and in Section 2.3, we do the same for quasi-equational theories. Here, we build on and generalize the approach for the syntactic description of the isotropy group of a quasi-equational theory as developed in [13]. We chose to treat the algebraic case first and the quasi-equational case second, despite the fact that quasi-equational theories subsume algebraic theories, because going from the more specific to the more abstract seemed like the most pedagogically sound approach. Indeed, the ideas and concepts, which are quite clear in the algebraic case, tend to be obscured by concerns of partiality and multi-sortedness in the quasi-equational setting. Once we have our main results in hand, we apply them to a series of examples in Chapter 3. Then, finally, we conclude with some open questions and potential avenues for further research in chapter 4.

A part of the material of this thesis is also collected in the preprint [8].

1.1 Preliminaries

Let us make a note on the use of these preliminaries. The section on category theory is bare bones but meant to make the thesis self-contained; it may safely be skipped by anyone familiar with basic category theory, except maybe its end where Lawvere theories are defined. The reader that wants a little more depth in the treatment of category theory can consult any textbook on the matter, such as [9]. The next section on algebraic theories is worth looking at, even if the reader is familiar with them, since it sets up notations and singles out concepts used throughout the rest of the thesis. Finally, the section on quasi-equational theories is similar to the previous one on algebraic theories, but with slight nuances and notational differences, owing in part to a multi-sorted setting.

1.1.1 Category Theory

Definition 1.1.1. A *category* \mathcal{C} consists of a collection of objects and a collection of arrows such that:

- Each arrow f has associated to it two objects, its source $\text{src } f = C$ and its target $\text{tar } f = D$, and we write $f : C \rightarrow D$ or $C \xrightarrow{f} D$.
- Given arrows $C \xrightarrow{f} D \xrightarrow{g} E$, there is a composite arrow $g \circ f : C \rightarrow E$.
- Composition of arrows is associative (when it is defined).
- For each object $C \in \mathcal{C}$, there exists an arrow $1_C : C \rightarrow C$ which acts as an identity for composition in the sense that, given $f : D \rightarrow C$ and $g : C \rightarrow D$, we have $1_C \circ f = f$ and $g \circ 1_C = g$.

Arrows are sometimes called morphisms, a terminology reminiscent of the homomorphisms of algebra. We also call the collection of all arrows from C to D a *hom-set* (short for homomorphism set, though it may be a proper class), and write it as $\mathcal{C}(C, D)$, or just $\text{Hom}(C, D)$ if there is no risk of confusion.

If the collection of arrows of a category is a set, we say the category is *small*. If it is too big to be a set, hence a proper class (like the collection of all sets), then we say the category is *large*. A further distinction is of interest: the category is *locally small* if all of its hom-sets are sets (even if the collection of *all* arrows is proper class).

Definition 1.1.2. The *coslice category* $(C \downarrow \mathcal{C})$ has as objects the arrows $f : C \rightarrow X$ from C in \mathcal{C} . An arrow from $C \xrightarrow{f} X$ to $C \xrightarrow{g} Y$ is an arrow $h : X \rightarrow Y$ in \mathcal{C} which

makes the following diagram commute:

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{h} & Y \end{array}$$

Definition 1.1.3. Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a mapping which associates to each object $C \in \mathcal{C}$ an object $FC \in \mathcal{D}$, and to each arrow $f : C \rightarrow C'$ in \mathcal{C} an arrow $Ff : FC \rightarrow FC'$ in \mathcal{D} (thus F preserves sources and targets). Moreover, we require that identities are preserved: $F1_C = 1_{FC}$; and that composition is preserved: $F(f' \circ f) = Ff' \circ Ff$ given $C \xrightarrow{f} C' \xrightarrow{f'} C''$.

Definition 1.1.4. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A *natural transformation* $\tau : F \Rightarrow G$ is a family of maps $\tau_A : FA \rightarrow GA$ for $A \in \mathcal{C}$, such that given an arrow $f : A \rightarrow B$ in \mathcal{C} , we obtain a commutative diagram:

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ Ff \downarrow & & \downarrow Gf \\ FB & \xrightarrow{\tau_B} & GB \end{array}$$

If each τ_A is an isomorphism, we say that τ is a *natural isomorphism*.

We now prove the Yoneda lemma for covariant hom-functors.

Lemma 1.1.5 (Yoneda). *Let $C \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathbf{Set}$. There is a bijection*

$$\text{Nat}(\text{Hom}(C, -), F) \cong FC.$$

Proof: We will define maps both ways and show they are inverses.

Let $\tau : \text{Hom}(C, -) \Rightarrow F$. Define $a : \text{Nat}(\text{Hom}(C, -), F) \rightarrow FC$ by $a\tau = \tau_C 1_C$.

Let $x \in FC$. Define $b : FC \rightarrow \text{Nat}(\text{Hom}(C, -), F)$ by

$$(bx)_D(C \xrightarrow{f} D) = Ffx.$$

We check that bx is indeed a natural transformation, i.e. that given $g : D \rightarrow E$ we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}(C, D) & \xrightarrow{(bx)_D} & FD \\ \text{Hom}(C, g) \downarrow & & \downarrow Fg \\ \text{Hom}(C, E) & \xrightarrow{(bx)_E} & FE \end{array}$$

And indeed for $f \in \text{Hom}(C, D)$, we have

$$(bx)_E(\text{Hom}(C, g)f) = (bx)_E(g \circ f) = F(g \circ f)x = Fg(Ffx) = Fg((bx)_Df).$$

Now we check that a and b are inverses. On the one hand,

$$a(bx) = (bx)_C 1_C = F 1_C x = 1_{FC} x = x.$$

On the other hand, we have a naturality square

$$\begin{array}{ccc} \text{Hom}(C, C) & \xrightarrow{\tau_C} & FC \\ \text{Hom}(C, f) \downarrow & & \downarrow Ff \\ \text{Hom}(C, D) & \xrightarrow{\tau_D} & FD \end{array}$$

The commutativity of this diagram yields

$$(b(a\tau))_D f = Ff(a\tau) = Ff(\tau_C 1_C) = \tau_D(\text{Hom}(C, f) 1_C) = \tau_D(f \circ 1_C) = \tau_D f.$$

Thus we have $b(a\tau) = \tau$. ■

Note that the bijection exhibited in the proof of Yoneda's lemma is natural in both C and F once both sides of $\text{Nat}(\text{Hom}(C, -), F) \cong FC$ are considered as functors $\mathcal{C} \times \mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}$. Also note that there is a Yoneda lemma for contravariant hom-functors as well.

Definition 1.1.6. Functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are part of an *adjunction*, written $F \dashv G$, where F is the *left adjoint* and G is the *right adjoint*, if and only if we have an isomorphism

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, GD),$$

natural in both C and D .

A typical example of an adjunction, which we will come back to in the next section, is the so-called free-forgetful adjunction of the category of models of an algebraic theory.

Proposition 1.1.7. *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism of categories if and only if it is bijective on objects and on hom-sets.*

Proof: We have to construct an inverse $G : \mathcal{D} \rightarrow \mathcal{C}$. Given that F is bijective on objects, let GD be the unique $C \in \mathcal{C}$ such that $FC = D$. Also let $G(D \xrightarrow{f} D')$ be the unique $g : GD \rightarrow GD'$ such that $Fg = f$, since F is bijective on hom-sets. Clearly F

and G are inverses, but it remains to show G is a functor. Indeed, $F1_{GD} = 1_{FGD} = 1_D$ implies $G1_D = 1_{GD}$, and, given $D \xrightarrow{f} D' \xrightarrow{f'} D''$, we have

$$F(Gf' \circ Gf) = FGf' \circ FGf = f' \circ f = FG(f' \circ f),$$

which implies that $Gf' \circ Gf = G(f' \circ f)$. Thus G is a functor. ■

The notion of isomorphism will mostly be important for Lawvere theories, which are a special type of category. However, for categories, the more relaxed notion of equivalence is usually more appropriate.

Definition 1.1.8. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence of categories* if and only if it is fully faithful, i.e. bijective on hom-sets, and essentially surjective, which means that for all $D \in \mathcal{D}$, there exists some $C \in \mathcal{C}$ such that $FC \cong D$.

Before we discuss Lawvere theories, we need to define products, which is done via a universal property.

Definition 1.1.9. Let \mathcal{C} be a category. A family of arrows $p_i : D \rightarrow C_i$, $i \in I$, for some index set I , is called a *product diagram* if and only if for all families of arrows $f_i : E \rightarrow C_i$ in \mathcal{C} , there exists a unique arrow $u = (f_i) : E \rightarrow D$ which makes the following diagram commute:

$$\begin{array}{ccc} E & & \\ u \downarrow & \searrow f_i & \\ D & \xrightarrow{p_i} & C_i \end{array}$$

Then, the object D is called the *product* of the objects C_i , and we write $D = \prod_{i \in I} C_i$.

When the index set in the definition above is finite, say of cardinality n , and all the objects C_i are actually the same object C , the product is called a finite power C^n .

Definition 1.1.10. Dual to the notion of product is the notion of *coproduct*. We say that a family $j_i : C_i \rightarrow D$ is a *coproduct diagram* if and only if for all families of arrows $f_i : C_i \rightarrow E$, there exists a unique arrow $u : D \rightarrow E$ which makes the following diagram commute:

$$\begin{array}{ccc} E & & \\ u \uparrow & \swarrow f_i & \\ D & \xleftarrow{j_i} & C_i \end{array}$$

Then we write $D = \coprod_{i \in I} C_i$.

Given an adjunction $F \dashv G$, note that left adjoints preserve coproducts in the sense that

$$F\left(\coprod_{i \in I} C_i\right) \cong \coprod_{i \in I} FC_i,$$

and right adjoints preserve products in the sense that

$$G\left(\prod_{i \in I} C_i\right) \cong \prod_{i \in I} GC_i$$

Of course, these isomorphisms are natural.

Now, let \mathcal{C} be a category with finite powers and assume we have specified a product diagram $p_{C,i} : C^n \rightarrow C$, $1 \leq i \leq n$, for each $C \in \mathcal{C}$ and each $n \in \mathbb{N}$. Further assume that the product diagram for $C^1 = C$ is just $(C^1 \xrightarrow{p_{C,1}} C) = (C \xrightarrow{1_C} C)$ (this benign assumption makes some developments more straightforward later on).

There is an endofunctor $(-)^n : \mathcal{C} \rightarrow \mathcal{C}$ defined on arrows $f : C \rightarrow D$ by $f^n = (f \circ p_{C,1}, \dots, f \circ p_{C,n})$, so that f^n is the unique arrow which makes the following diagram commute:

$$\begin{array}{ccc} C^n & \xrightarrow{p_{C,i}} & C \\ f^n \downarrow & & \downarrow f \\ D^n & \xrightarrow{p_{D,i}} & D \end{array}$$

To see that this indeed defines a functor, consider the following commutative diagram which shows $1_C^n = 1_{C^n}$:

$$\begin{array}{ccc} C^n & \xrightarrow{p_{C,i}} & C \\ 1_{C^n} \downarrow & & \downarrow 1_C \\ C^n & \xrightarrow{p_{C,i}} & C \end{array}$$

Moreover, given $C \xrightarrow{f} D \xrightarrow{g} E$, we have a commutative diagram showing that $(g \circ f)^n = g^n \circ f^n$:

$$\begin{array}{ccc} C^n & \xrightarrow{p_{C,i}} & C \\ \left. \begin{array}{c} f^n \downarrow \\ g^n \downarrow \end{array} \right\} g^n \circ f^n & & \left. \begin{array}{c} \downarrow f \\ \downarrow g \end{array} \right\} g \circ f \\ D^n & \xrightarrow{p_{D,i}} & D \\ E^n & \xrightarrow{p_{E,i}} & E \end{array}$$

We recall the definition of a Lawvere theory. The formulation we use here differs slightly from the original [7], but is equivalent to it.

Definition 1.1.11. A *Lawvere theory* \mathcal{C} is a category together with a choice of an object $C \in \mathcal{C}$ and specified product diagrams $p_{C,i} : C^n \rightarrow C$, $1 \leq i \leq n$, for each

$n \in \mathbb{N}$, such that each object of \mathcal{C} is isomorphic to a power of C . We will always assume that the product diagram for C^1 is just $(C^1 \xrightarrow{p_{C^1}} C) = (C \xrightarrow{1_C} C)$.

Definition 1.1.12. Let \mathcal{C} and \mathcal{D} be Lawvere theories with chosen objects C and D , respectively. A *morphism of Lawvere theories* is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ which strictly preserves the chosen object and the specified product diagrams, in the sense that $FC = D$, and the image under F of $p_{C,i} : C^n \rightarrow C$ is

$$(FC^n \xrightarrow{Fp_{C,i}} FC) = (D^n \xrightarrow{p_{D,i}} D).$$

Lawvere theories and their morphisms form a category **Law**.

1.1.2 Algebraic Theories

Definition 1.1.13. A *signature* Σ is a set of operation symbols, each with their own specified arity $n \in \mathbb{N}$. Given a signature Σ , we define Σ -terms inductively as follows. Let $V = \{x_1, x_2, \dots\}$ be a countably infinite set of variables.

- Each $x_i \in V$ is a Σ -term.
- If $f \in \Sigma$ is an operation symbol of arity n and t_1, \dots, t_n are Σ -terms, then $f(t_1, \dots, t_n)$ is also a Σ -term.

The set of Σ -terms is the smallest set satisfying both of these conditions.

Note that we call a nullary operation c a constant, and write c instead of $c()$ for the corresponding Σ -term.

Definition 1.1.14. An *algebraic theory* $T = (\Sigma, E)$ consists of a signature Σ and a set E of equations (more formally, pairs) between Σ -terms.

Definition 1.1.15. Let $T = (\Sigma, E)$ be a theory. A Σ -structure is a set A equipped with an operation $f^A : A^n \rightarrow A$ for each operation symbol f of arity n in Σ . Then any Σ -term t containing at most the variables $y_1, \dots, y_n \in V$ defines a function $t^A : A^n \rightarrow A$ which we call its *interpretation* in A .

- If $t = y_i$ is a variable, then $t^A = ((y_1, \dots, y_n) \mapsto y_i)$ is a projection.
- If $t = f(t_1, \dots, t_m)$ then $t^A = f^A \circ (t_1^A, \dots, t_m^A)$.

We say that A is a *model* of T if and only if $s^A = t^A$ for each equation $(s = t) \in E$. Intuitively, a T -model A is a Σ -structure such that each equation in E holds true when interpreted in A .

Definition 1.1.16. A *homomorphism* of T -models $h : A \rightarrow B$ is a function which commutes with all operations f of arity n in T :

$$h \circ f^A = f^B \circ h^n.$$

Of course, there is a category $T\text{-Mod}$ whose objects are all T -models and whose arrows are the T -homomorphisms, and this category has products and coproducts.

There is a free-forgetful adjunction $F \dashv U$ for $T\text{-Mod}$. Given a set X , the free model $FX = \langle X \rangle$ is built as follows. Consider all the terms built from the operations in Σ and the elements of X , treated as formal symbols, and impose all equations in E to this set of terms. Crucially, since operations are finitary and Σ is a set, there is only a set's worth (rather than a proper class's worth) of these terms, even before imposing equations. Also note that we write the free model functor as $F = \langle - \rangle$.

We will generally confuse elements of $\langle x_1, \dots, x_n \rangle$ with their Σ -term representatives, and we will call them terms as well. The interpretation of a term $t \in \langle \vec{x} \rangle$ in a T -model A , written t^A , is defined as you would expect: it is the interpretation of any of its representatives in the Σ -structure A (one can check this is well-defined). This function $t^A : A^n \rightarrow A$ is by no means guaranteed to be a T -homomorphism. In fact, we will devote some space later on in the thesis to determining when t^A is a T -homomorphism.

Next, we discuss what we might call models with parameters in a given model A .

Definition 1.1.17. Let $T = (\Sigma, E)$ be a theory and let A be a T -model. We can form a new algebraic theory T_A by adding a constant symbol a to Σ for each $a \in A$, and adding all equations between elements of A to E . We think of T_A -models as *T -models with parameters in A* .

Proposition 1.1.18. A T_A -model B is equivalent to a T -model B with a T -homomorphism $h : A \rightarrow B$. This correspondence gives an isomorphism of categories:

$$T_A\text{-Mod} \cong (A \downarrow T\text{-Mod}).$$

Proof: Given a T_A -model B , define $h : A \rightarrow B$ by $ha = a^B$ (the interpretation of the constant a in B). Let f be an operation symbol of arity n in T . Then, for all $a_1, \dots, a_n \in A$, we have

$$f^B(ha_1, \dots, ha_n) = f^B(a_1^B, \dots, a_n^B) = (f^A(a_1, \dots, a_n))^B = h(f^A(a_1, \dots, a_n)).$$

The second equality follows because the theory T_A incorporates all the equations between elements of A . Thus h commutes with all the operations of T ; it is a T -homomorphism.

Conversely, given a T -homomorphism $h : A \rightarrow B$, define $a^B = ha$ for each $a \in A$. Since h preserves equations involving elements of A , the interpretations a^B of the

constants $a \in A$ of T_A satisfy all the equations of this theory. ■

The isomorphism $T_A\text{-Mod} \cong (A \downarrow T\text{-Mod})$ fits into a commutative triangle with the obvious forgetful functors:

$$\begin{array}{ccc}
 T_A\text{-Mod} & \xrightarrow{\sim} & (A \downarrow T\text{-Mod}) \\
 & \searrow & \swarrow P_A \\
 & T\text{-Mod} &
 \end{array}$$

This justifies our habit of calling elements of the coslice $(A \downarrow T\text{-Mod})$ models with parameters in A , and generally confusing the two categories $T_A\text{-Mod}$ and $(A \downarrow T\text{-Mod})$.

Note that A is itself a T_A -model (corresponding to $(A \xrightarrow{1_A} A) \in (A \downarrow T\text{-Mod})$). Moreover, the projection functor $P_A : (A \downarrow T\text{-Mod}) \rightarrow T\text{-Mod}$ is just the forgetful functor $T_A\text{-Mod} \rightarrow T\text{-Mod}$.

Since T_A is an algebraic theory, it has its own free-forgetful adjunction $F_A \dashv U_A = U \circ P_A$, where $U : T\text{-Mod} \rightarrow \mathbf{Set}$ is the forgetful functor. The free T_A -model on X is $A \amalg \langle X \rangle$, with the coproduct taken in $T\text{-Mod}$, but we will shorten it to $A\langle X \rangle$. Its elements are referred to as terms with parameters in A , or simply terms, though they are more properly equivalence classes of terms (up to provable equality); however, this ambiguity will not trouble us. Also note that, given a T -homomorphism $h : A \rightarrow B$, we obtain a T_A -homomorphism $h\langle X \rangle = h \amalg 1_{\langle X \rangle} : A\langle X \rangle \rightarrow B\langle X \rangle$, which acts on a term $t \in A\langle X \rangle$ by replacing each parameter $a \in A$ by $ha \in B$.

1.1.3 Quasi-Equational Theories

Definition 1.1.19. A *signature* Σ consists of a set of sorts \mathcal{S} and a set of operation symbols, each with their own specified type $S_1 \times \cdots \times S_n \rightarrow S$, where S_1, \dots, S_n, S are sorts. Given a signature Σ , we define Σ -terms and their types inductively as follows. For each $S \in \mathcal{S}$, let x_1^S, x_2^S, \dots be an infinite list of variables.

- Each x_i^S is a Σ -term of type S ; we write $x_i^S : S$.
- If f is an operation symbol of type $S_1 \times \cdots \times S_n \rightarrow S$, and $t_i : S_i$, $1 \leq i \leq n$, are Σ -terms, then $f(t_1, \dots, t_n) : S$ is also a Σ -term.

The set of Σ -terms is the smallest set satisfying both of these conditions.

Definition 1.1.20. Recall that a *sequent* is an expression of the form $\phi \vdash \psi$, where ϕ and ψ are formulas, and the turnstile represents entailment. In particular, given a signature Σ , a *horn sequent* is a sequent of the form

$$t_1 = s_1 \wedge \cdots \wedge t_n = s_n \vdash r_1 = q_1 \wedge \cdots \wedge r_m = q_m,$$

where the Σ -terms t_i and s_i have the same type for $i = 1, \dots, n$, and similarly for r_j and q_j . A *quasi-equational theory* $T = (\Sigma, E)$ consists of a signature Σ and a set E of horn sequents called the axioms of the theory.

Note that quasi-equational theories are strictly more general than algebraic theories. Indeed, algebraic theories are just those quasi-equational theories which are single-sorted and whose every axiom has an empty antecedent, so that it merely expresses a conjunction of equalities between terms.

Also note that we use partial equational logic to manipulate the axioms and make deductions. This means that substitution is restricted to defined terms. For more information, consult [12]. We should remember that the equality $t = t$ expresses the fact that t is defined, written $t \downarrow$; conversely, if t is not defined, we cannot assert $t = t$. Models of quasi-equational theories are allowed to have partially defined operations. One can think of composition in categories, which is only defined for pairs of arrows with compatible sources and targets.

Definition 1.1.21. Let $T = (\Sigma, E)$ be a theory. A Σ -*structure* is a family of sets $A = (A_S)_{S \in \mathcal{S}}$ equipped with a partial operation $f^A : A_{S_1} \times \cdots \times A_{S_n} \rightarrow A_S$ for each operation symbol $f : S_1 \times \cdots \times S_n \rightarrow S$ in the signature Σ . Then more generally any Σ -term $t : S$ containing at most the variables $y_1 : S_1, \dots, y_n : S_n$ defines a partial function $t^A : A_{S_1} \times \cdots \times A_{S_n} \rightarrow A_S$ which we call its *interpretation* in A .

- If $t = y_i$ is a variable, then $t^A = ((y_1, \dots, y_n) \mapsto y_i)$ is a (total) projection.
- If $t = f(t_1, \dots, t_m)$ then $t^A = f^A \circ (t_1^A, \dots, t_m^A)$, and its domain of definition consists of those tuples $\vec{a} \in A_{S_1} \times \cdots \times A_{S_n}$ such that $t_j^A \vec{a}$ is defined for each j , and moreover $f(t_1^A \vec{a}, \dots, t_m^A \vec{a})$ is defined.

We say that A is a *model* of T if and only if each axiom in E holds true when interpreted in A . More precisely, if E contains the axiom

$$t_1 = s_1 \wedge \cdots \wedge t_n = s_n \vdash r_1 = q_1 \wedge \cdots \wedge r_m = q_m,$$

then we must have that

$$t_1^A = s_1^A \wedge \cdots \wedge t_n^A = s_n^A \quad \text{implies} \quad r_1^A = q_1^A \wedge \cdots \wedge r_m^A = q_m^A.$$

Definition 1.1.22. A *homomorphism* of T -models $h : A \rightarrow B$ is a family of *total* functions $h_S : A_S \rightarrow B_S$ which commutes with all operations $f : S_1 \times \cdots \times S_n \rightarrow S$ of the theory in the sense that, whenever $f^A(a_1, \dots, a_n) \downarrow$, we have

$$h_S(f^A(a_1, \dots, a_n)) = f^B(h_{S_1}a_1, \dots, h_{S_n}a_n).$$

(In particular, the right-hand side of this equation must be defined.)

Of course, there is a category $T\text{-Mod}$ whose objects are the T -models and whose arrows are the T -homomorphisms. It has a free-forgetful adjunction $F \dashv U$ where $U : T\text{-Mod} \rightarrow \mathbf{Set}^{\mathcal{S}}$. Given a family of sets $X = (X_S)_{S \in \mathcal{S}}$, we write $FX = \langle X \rangle$ for the free model on X .

Definition 1.1.23. Let $T = (\Sigma, E)$ be a theory and let A be a T -model. We can form a new algebraic theory T_A by adding a constant symbol $a : S$ to Σ for each $a \in A_S$ and all $S \in \mathcal{S}$, and adding all equations between elements of A to E . We think of T_A -models as *T -models with parameters in A* .

Proposition 1.1.24. *A T_A -model B is equivalent to a T -model B with a T -homomorphism $h : A \rightarrow B$. This correspondence gives an isomorphism of categories:*

$$T_A\text{-Mod} \cong (A \downarrow T\text{-Mod}).$$

Proof: Similar to the proof of proposition 1.1.18. ■

Once again, this proposition justifies our habit of calling elements of the coslice $(A \downarrow T\text{-Mod})$ models with parameters in A , and generally confusing the two categories $T_A\text{-Mod}$ and $(A \downarrow T\text{-Mod})$.

Note that A is itself a T_A -model (corresponding to $(A \xrightarrow{1_A} A) \in (A \downarrow T\text{-Mod})$). Moreover, the projection functor $P_A : (A \downarrow T\text{-Mod}) \rightarrow T\text{-Mod}$ is just the forgetful functor $T_A\text{-Mod} \rightarrow T\text{-Mod}$.

The set of sorts \mathcal{S} can be considered as a discrete category, so that $\mathbf{Set}^{\mathcal{S}} \cong \prod_{S \in \mathcal{S}} \mathbf{Set}$. This category has as objects \mathcal{S} -indexed families of sets, and as arrows it has families of functions. Hence the data of a functor $F : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{S}}$ is the same as the data of a family of functors $F_S : \mathcal{C} \rightarrow \mathbf{Set}$ for each $S \in \mathcal{S}$, and given F , the corresponding family has F_S defined by $F_S C = (FC)_S$ for objects C and by $F_S f = (Ff)_S$ for arrows $f : C \rightarrow D$.

Furthermore, if we have two functors $F, G : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{S}}$, a natural transformation $\tau : F \Rightarrow G$ corresponds to a family of natural transformations $\tau_S : F_S \rightarrow G_S$ defined by $(\tau_S)_C = (\tau_C)_S$. In fact, this gives us a natural bijection $\text{Nat}(F, G) \cong \prod_{S \in \mathcal{S}} \text{Nat}(F_S, G_S)$.

Since $(A \downarrow T\text{-Mod}) \cong T_A\text{-Mod}$, we have a free-forgetful adjunction

$$(A \downarrow T\text{-Mod}) \begin{array}{c} \xrightarrow{U_A} \\ \xleftarrow{F_A} \end{array} \mathbf{Set}^{\mathcal{S}} .$$

Looking at things sort-wise, we can build a left adjoint to the component $(U_A)_S : T\text{-Mod} \rightarrow \mathbf{Set}$. For a set X , define the family $X^S = (X_R^S)_{R \in \mathcal{S}}$ as follows:

$$X_R^S = \begin{cases} X & \text{if } R = S \\ \emptyset & \text{otherwise} \end{cases} .$$

Then, we have a functor $F_{A,S} : \mathbf{Set} \rightarrow T\text{-Mod}$ defined by $F_{A,S}X = F_A X^S$, and $F_{A,S} \dashv (U_A)_S$.

A family of terms $t = (t_S) \in \prod_{S \in \mathcal{S}} A\langle \vec{x}^S \rangle_S$ induces a family of total functions $t^A : A^n \rightarrow A$, whose component $t_S^A : A_S^n \rightarrow A_S$ is defined at $(a_1, \dots, a_n) \in A_S^n$ by

$$t_S^A(a_1, \dots, a_n) = t_S[a_i/x_i^S].$$

Note that t_S^A is just the function induced by t_S , as per Definition 1.1.21. Moreover, it really is always total, since any operation defined between freely adjoined variables must be again defined when those variables are replaced by arbitrary elements of A .

Given variables $x_1^S, \dots, x_n^S : S$ and a homomorphism $h : A \rightarrow B$, we have a homomorphism $h\langle \vec{x}^S \rangle : A\langle \vec{x}^S \rangle \rightarrow B\langle \vec{x}^S \rangle$ which satisfies $h\langle \vec{x}^S \rangle_R a = h_R a$ for all $a \in A_R$ and all $R \in \mathcal{S}$, and $h\langle \vec{x}^S \rangle_S x_i^S = x_i^S$ for $i = 1, \dots, n$. In fact, $h\langle \vec{x}^S \rangle = h \amalg 1_{\langle \vec{x}^S \rangle}$. Moreover, given $t = (t_S) \in \prod_{S \in \mathcal{S}} A\langle \vec{x}^S \rangle_S$, we can define $h\langle \vec{x} \rangle t \in \prod_{S \in \mathcal{S}} B\langle \vec{x}^S \rangle_S$ by $(h\langle \vec{x} \rangle t)_S = h\langle \vec{x}^S \rangle_S t_S$. It applies (the relevant component of) h to each parameter, while leaving the variables untouched.

Chapter 2

The Inner Theory

2.1 Definition of the Inner Theory

Let \mathcal{C} be a category with finite powers. In the remainder of this section, we assume that \mathcal{C} comes equipped with a choice of product diagrams $p_{C,i} : C^n \rightarrow C$, $1 \leq i \leq n$, for each $C \in \mathcal{C}$ and each $n \in \mathbb{N}$, where C^1 is just C . In particular, we then have, for each n , an endofunctor $(-)^n : \mathcal{C} \rightarrow \mathcal{C}$.

A *system of extensions* of an arrow $\tau_C : C^n \rightarrow C^m$ is a family of arrows $\tau_f : X^n \rightarrow X^m$ for all arrows $f : C \rightarrow X$, such that the following diagram commutes:

$$\begin{array}{ccc} C^n & \xrightarrow{\tau_C} & C^m \\ f^n \downarrow & & \downarrow f^m \\ X^n & \xrightarrow{\tau_f} & X^m \end{array}$$

In particular, extending τ_C along $1_C : C \rightarrow C$, we find that $\tau_C = 1_C^m \circ \tau_C = \tau_{1_C} \circ 1_C^n = \tau_{1_C}$. We also want these extensions to be coherent. Consider the following diagram:

$$\begin{array}{ccc} C^n & \xrightarrow{\tau_C} & C^m \\ f^n \downarrow & & \downarrow f^m \\ X^n & \xrightarrow{\tau_f} & X^m \\ k^n \downarrow & & \downarrow k^m \\ Y^n & \xrightarrow{\tau_{k \circ f}} & Y^m \end{array}$$

The top square and the outer rectangle both commute, because τ_f and $\tau_{k \circ f}$ are extensions of τ_C along f and $k \circ f$, respectively. The coherence condition we impose is the commutativity of the bottom square.

Thus, a coherent system of extensions of an arrow $C^n \rightarrow C^m$ is just a family of arrows $\tau_f : X^n \rightarrow X^m$ for each $f : C \rightarrow X$ in \mathcal{C} , such that whenever $k \circ f = g$, we

have a commutative square

$$\begin{array}{ccc} X^n & \xrightarrow{\tau_f} & X^m \\ k^n \downarrow & & \downarrow k^m \\ Y^n & \xrightarrow{\tau_g} & Y^m \end{array}$$

This requirement can be packaged neatly into the statement that τ is a natural transformation $P_C^n \Rightarrow P_C^m$, where P_C is the canonical projection from the coslice $(C \downarrow \mathcal{C})$ to \mathcal{C} .

Note that the functor category $\mathcal{C}^{(C \downarrow \mathcal{C})}$ is itself a category with products induced by those of \mathcal{C} , so that in particular $P_C^n = (-)^n \circ P_C$. The relevant product diagrams $\pi_i : P_C^n \Rightarrow P_C$ involve natural transformations defined by $\pi_i(C \xrightarrow{f} X) = p_{X,i} : X^n \rightarrow X$.

Definition 2.1.1. Let \mathcal{C} be a category with finite powers. The *inner theory functor* $\mathfrak{Z} = \mathfrak{Z}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Law}$ (we omit the subscript if the category is clear from context) is defined as follows. For an object $C \in \mathcal{C}$, the Lawvere theory $\mathfrak{Z}C$ has objects the natural numbers and hom-sets $\mathfrak{Z}C(n, m) = \text{Nat}(P_C^n, P_C^m)$. Given $h : C \rightarrow D$, we define a morphism of Lawvere theories $\mathfrak{Z}h : \mathfrak{Z}C \rightarrow \mathfrak{Z}D$. On objects, we let $\mathfrak{Z}hn = n$. To define $\mathfrak{Z}h$ on arrows, let $\tau : P_C^n \Rightarrow P_C^m$ and $(D \xrightarrow{f} X) \in (D \downarrow \mathcal{C})$. We define

$$\mathfrak{Z}h\tau : P_D^n \Rightarrow P_D^m, \quad (\mathfrak{Z}h\tau)_f = \tau_{f \circ h}.$$

Note that, in particular, $\mathfrak{Z}C(1, 1)$ is the isotropy monoid, and those of its elements which are invertible form the isotropy group.

Given an object C , the arrows of $\mathfrak{Z}C$ are called *inner morphisms*¹. This terminology is consistent with the name inner theory for the functor \mathfrak{Z} . It comes from the observation that elements of the isotropy group of the category of groups are just the inner automorphisms (see [1] or [4]), and our inner theory generalizes this notion to more general morphisms in arbitrary categories with finite products.

Proposition 2.1.2. \mathfrak{Z} is a well-defined functor.

Proof: First, we check that $\mathfrak{Z}h\tau$ is a natural transformation $P_D^n \Rightarrow P_D^m$. Let $(D \xrightarrow{f} X), (D \xrightarrow{g} Y) \in (D \downarrow \mathcal{C})$, and let $k : f \rightarrow g$ be an arrow between them. Then $k \circ f = g$ implies $k \circ (f \circ h) = g \circ h$, which means k is an arrow from $f \circ h$ to $g \circ h$ in $(C \downarrow \mathcal{C})$. Since $\tau : P_C^n \Rightarrow P_C^m$ is a natural transformation, we have a commutative diagram:

$$\begin{array}{ccc} X^n & \xrightarrow{\tau_{f \circ h} = (\mathfrak{Z}h\tau)_f} & X^m \\ k^n \downarrow & & \downarrow k^m \\ Y^n & \xrightarrow{\tau_{g \circ h} = (\mathfrak{Z}h\tau)_g} & Y^m \end{array}$$

¹In the previous literature on the isotropy group/monoid, inner morphisms were sometimes called elements of isotropy or isotropy elements, but on the suggestion of Prof. Hofstra we have decided to introduce this shorter and neater terminology.

Next, we check that $\mathfrak{3}h : \mathfrak{3}C \rightarrow \mathfrak{3}D$ is functorial. We have

$$(\mathfrak{3}h1_{P_C^n})_f = (1_{P_C^n})_{f \circ h} = 1_{X^n} = (1_{P_D^n})_f = (1_{\mathfrak{3}hP_C^n})_f.$$

Moreover, given $P_C^n \xrightarrow{\tau} P_C^m \xrightarrow{\sigma} P_C^l$, we have

$$(\mathfrak{3}h\sigma \circ \mathfrak{3}h\tau)_f = (\mathfrak{3}h\sigma)_f \circ (\mathfrak{3}h\tau)_f = \sigma_{f \circ h} \circ \tau_{f \circ h} = (\sigma \circ \tau)_{f \circ h} = (\mathfrak{3}h(\sigma \circ \tau))_f.$$

To see that $\mathfrak{3}h$ preserves the product diagrams $\pi_i : P_C^n \Rightarrow P_C$, notice $\mathfrak{3}hn = n$, and

$$(\mathfrak{3}h\pi_i)_f = (\pi_i)_{f \circ h} = p_{X,i} = (\pi_i)_f,$$

where the rightmost π_i is the projection of P_D^n rather than P_C^n . We conclude that $\mathfrak{3}h$ is a morphism of Lawvere theories.

Finally, we check that $\mathfrak{3}$ is functorial. We have $\mathfrak{3}1_C : \mathfrak{3}C \rightarrow \mathfrak{3}C$, with $\mathfrak{3}1_C n = n = 1_{\mathfrak{3}C} n$ and

$$(\mathfrak{3}1_C \tau)_f = \tau_{f \circ 1_C} = \tau_f = (1_{\mathfrak{3}C} \tau)_f.$$

Moreover, given $C \xrightarrow{h} D \xrightarrow{k} E$, we have

$$((\mathfrak{3}k \circ \mathfrak{3}h)\tau)_g = (\mathfrak{3}k(\mathfrak{3}h\tau))_g = ((k \circ h)\tau)_g.$$

This completes the verification. ■

We prove a theorem that shows that the inner theory does not depend on the particular choice of product diagrams for powers of objects of \mathcal{C} .

Theorem 2.1.3. *Let \mathcal{C} be a category with finite powers, specified by product diagrams $p_{C,i} : C^n \rightarrow C$ and $p'_{C,i} : (C^n)' \rightarrow C$. As usual, we assume*

$$(C^1 \xrightarrow{p_{C,1}} C) = (C \xrightarrow{1_C} C) = ((C^1)' \xrightarrow{p'_{C,1}} C).$$

Let $\mathfrak{3}, \mathfrak{3}' : \mathcal{C} \rightarrow \mathbf{Law}$ be the corresponding inner theory functors. Then there is a natural isomorphism $\mathfrak{3} \cong \mathfrak{3}'$.

Proof: We must construct a natural isomorphism $\gamma : \mathfrak{3} \Rightarrow \mathfrak{3}'$. For $C \in \mathcal{C}$, we must define an isomorphism of Lawvere theories $\gamma_C : \mathfrak{3}C \rightarrow \mathfrak{3}'C$. On objects, let $\gamma_C n = n$. Towards a definition of $\gamma_C \tau$ for an arrow $\tau : n \rightarrow m$ in $\mathfrak{3}C$, we make the observation that the universal property of products ensures they are unique up to a unique isomorphism. Therefore, we have, for any object $X \in \mathcal{C}$ and any $n \in \mathbb{N}$, an isomorphism $f_{X,n} : X^n \rightarrow (X^n)'$ which is the unique arrow making the following diagram commutative for $i = 1, \dots, n$:

$$\begin{array}{ccc} X^n & \xrightarrow{f_{X,n}} & (X^n)' \\ & \searrow p_{X,i} & \downarrow p'_{X,i} \\ & & X \end{array}$$

Of course, its inverse is the unique arrow $f_{X,n}^{-1} : (X^n)' \rightarrow X^n$ making the following triangle commutative for $i = 1, \dots, n$:

$$\begin{array}{ccc} (X^n)' & \xrightarrow{f_{X,n}^{-1}} & X^n \\ & \searrow p'_{X,i} & \downarrow p_{X,i} \\ & & X \end{array}$$

Thus, for $h : C \rightarrow X$, we should define

$$(\gamma_{C\tau})_h = f_{X,m} \circ \tau_h \circ f_{X,n}^{-1}.$$

Let us show that $\gamma_{C\tau}$ is an arrow $n \rightarrow m$ in $\mathfrak{3}'C$, i.e. a natural transformation $(P_C^n)' \Rightarrow (P_C^m)'$. We need to prove that a commutative triangle

$$\begin{array}{ccc} & C & \\ h \swarrow & & \searrow g \\ X & \xrightarrow{k} & Y \end{array}$$

induces a commutative square

$$\begin{array}{ccc} (X^n)' & \xrightarrow{(\gamma_{C\tau})_h} & (X^m)' \\ (k^n)' \downarrow & & \downarrow (k^m)' \\ (Y^n)' & \xrightarrow{(\gamma_{C\tau})_g} & (Y^m)' \end{array}$$

First, note that for $i = 1, \dots, n$, we have that

$$p_{Y,i} \circ k^n \circ f_{X,n}^{-1} = k \circ p_{X,i} \circ f_{X,n}^{-1} = k \circ p'_{X,i} = p'_{Y,i} \circ (k^n)' = p_{Y,i} \circ f_{Y,n}^{-1} \circ (k^n)',$$

hence $k^n \circ f_{X,n}^{-1} = f_{Y,n}^{-1} \circ (k^n)'$. From this equality, and the fact that τ is a natural transformation $P_C^n \Rightarrow P_C^m$, we calculate

$$\begin{aligned} p'_{Y,j} \circ (k^m)' \circ (\gamma_{C\tau})_h &= k \circ p'_{X,j} \circ f_{X,m} \circ \tau_h \circ f_{X,n}^{-1} \\ &= k \circ p_{X,j} \circ \tau_h \circ f_{X,n}^{-1} \\ &= p_{Y,j} \circ k^m \circ \tau_h \circ f_{X,n}^{-1} \\ &= p_{Y,j} \circ \tau_g \circ k^n \circ f_{X,n}^{-1} \\ &= p'_{Y,j} \circ f_{Y,m} \circ \tau_g \circ f_{Y,n}^{-1} \circ (k^n)' \\ &= p'_{Y,j} \circ (\gamma_{C\tau})_g \circ (k^n)' \end{aligned}$$

for $j = 1, \dots, m$, hence $(k^m)' \circ (\gamma_{C\tau})_h = (\gamma_{C\tau})_g \circ (k^n)'$ as desired.

Next, we must show that γ_C is a morphism of Lawvere theories. Consider the identity arrow $1_n : n \rightarrow n$ in $\mathfrak{3}C$. We find that

$$(\gamma_C 1_n)_h = f_{X,n} \circ (1_n)_h \circ f_{X,n}^{-1} = f_{X,n} \circ 1_{X^n} \circ f_{X,n}^{-1} = 1_{(X^n)'} = (1_n)_h,$$

hence $\gamma_C 1_n = 1_n = 1_{\gamma_C n}$. Now we consider a composite $n \xrightarrow{\tau} m \xrightarrow{\sigma} p$ in $\mathfrak{3}C$. We calculate

$$\begin{aligned} (\gamma_C(\sigma \circ \tau))_h &= f_{X,p} \circ (\sigma \circ \tau)_h \circ f_{X,n}^{-1} = f_{X,p} \circ \sigma_h \circ \tau_h \circ f_{X,n}^{-1} \\ &= f_{X,p} \circ \sigma_h \circ f_{X,m}^{-1} \circ f_{X,m} \circ \tau_h \circ f_{X,n}^{-1} = (\gamma_C \sigma)_h \circ (\gamma_C \tau)_h = (\gamma_C \sigma \circ \gamma_C \tau)_h, \end{aligned}$$

hence $\gamma_C(\sigma \circ \tau) = \gamma_C \sigma \circ \gamma_C \tau$. This shows that γ_C is a functor. Finally, consider the projections $\pi_i : P_C^n \Rightarrow P_C$. Since we have assumed $(X^1 \xrightarrow{P_{X,1}} X) = (X \xrightarrow{1_X} X)$, we have that $f_{X,1} = 1_X$ and we calculate

$$(\gamma_C \pi_i)_h = f_{X,1} \circ (\pi_i)_h \circ f_{X,n}^{-1} = 1_X \circ p_{X,i} \circ f_{X,n}^{-1} = p'_{X,i} = (\pi'_i)_h.$$

This implies that $\gamma_C \pi_i = \pi'_i$, which is to say that γ_C preserves product diagrams. Therefore, γ_C is a morphism of Lawvere theories. But more than that, it is an isomorphism of Lawvere theories. Indeed, it is obviously bijective on objects, and we can also verify that it is bijective on hom-sets. The inverse of $\gamma_C : \mathfrak{3}C(n, m) \rightarrow \mathfrak{3}'C(n, m)$ is defined on $\rho : (P_C^n)' \Rightarrow (P_C^m)'$ by

$$(\gamma_C^{-1} \rho)_h = f_{X,m}^{-1} \circ \rho_h \circ f_{X,n}.$$

Similarly to what was done earlier for $\gamma_C \tau$, we can show that $\gamma_C^{-1} \rho$ is a natural transformation $P_C^n \Rightarrow P_C^m$.

All that is left is to show that γ is a natural transformation. Given $h : C \rightarrow D$ in \mathcal{C} , we need to prove that we have a commutative square

$$\begin{array}{ccc} \mathfrak{3}C & \xrightarrow{\gamma_C} & \mathfrak{3}'C \\ \mathfrak{3}h \downarrow & & \downarrow \mathfrak{3}'h \\ \mathfrak{3}D & \xrightarrow{\gamma_D} & \mathfrak{3}'D \end{array}$$

On objects, $(\gamma_D \circ \mathfrak{3}h)n = n = (\mathfrak{3}'h \circ \gamma_C)n$. On arrows $\tau : n \rightarrow m$, we find that, for $g : D \rightarrow X$, we have

$$\begin{aligned} ((\gamma_D \circ \mathfrak{3}h)\tau)_g &= (\gamma_D(\mathfrak{3}h\tau))_g = f_{X,m} \circ (\mathfrak{3}h\tau)_g \circ f_{X,n}^{-1} \\ &= f_{X,m} \circ \tau_{g \circ h} \circ f_{X,n}^{-1} = (\gamma_C \tau)_{g \circ h} = (\mathfrak{3}'h(\gamma_C \tau))_g = ((\mathfrak{3}'h \circ \gamma_C)\tau)_g, \end{aligned}$$

hence $\gamma_D \circ \mathfrak{3}h = \mathfrak{3}'h \circ \gamma_C$ as desired. ■

Proposition 2.1.4. *If $f : C \xrightarrow{\sim} D$ is an isomorphism, then we have an isomorphism of Lawvere theories $F : \mathfrak{3}C \xrightarrow{\sim} \mathfrak{3}D$.*

Proof: Define $Fn = n$ and, for $\tau : n \rightarrow m$ and $h : D \rightarrow X$, let $(F\tau)_h = \tau_{h \circ f}$.

First, we have to show that $F\tau$ is a natural transformation $P_D^n \Rightarrow P_D^m$. Given a commutative triangle

$$\begin{array}{ccc} & D & \\ h \swarrow & & \searrow g \\ X & \xrightarrow{k} & Y \end{array}$$

we must show we have a commutative square

$$\begin{array}{ccc} X^n & \xrightarrow{(F\tau)_h} & X^m \\ k^n \downarrow & & \downarrow k^m \\ Y^n & \xrightarrow{(F\tau)_g} & Y^m \end{array}$$

But $k \circ (h \circ f) = g \circ f$, and $\tau : P_C^n \Rightarrow P_C^m$ is a natural transformation, so we calculate

$$k^m \circ (F\tau)_h = k^m \circ \tau_{h \circ f} = \tau_{g \circ f} \circ k^n = (F\tau)_g \circ k^n.$$

Now let us show that F is functorial. First,

$$(F1_n)_h = (1_n)_{h \circ f} = 1_{X^n} = (1_{Fn})_h.$$

Then, given $n \xrightarrow{\tau} m \xrightarrow{\sigma} p$, we have that

$$(F(\sigma \circ \tau))_h = (\sigma \circ \tau)_{h \circ f} = \sigma_{h \circ f} \circ \tau_{h \circ f} = (F\sigma)_h \circ (F\tau)_h = (F\sigma \circ F\tau)_h.$$

Also F is a morphism of Lawvere theories, since

$$(F\pi_i)_h = (\pi_i)_{h \circ f} = p_{X,i} = (\pi_i)_h.$$

Finally, F is an isomorphism of Lawvere theories. Clearly, it is bijective on objects. It is also bijective on hom-sets, as we can check that $F : \mathfrak{3}C(n, m) \rightarrow \mathfrak{3}D(n, m)$ has inverse defined for $h : C \rightarrow X$ by $(F^{-1}\rho)_h = \rho_{h \circ f^{-1}}$; we can also check that this is well-defined, i.e. that $F^{-1}\rho$ is a natural transformation. \blacksquare

Lemma 2.1.5. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories and $p_{C,i} : C^n \rightarrow C$ is a product diagram in \mathcal{C} , then $Fp_{C,i} : FC^n \rightarrow FC$ is also a product diagram.*

Proof: Suppose we have arrows $f_i : Y \rightarrow FC$. Since F is essentially surjective, there exists $X \in \mathcal{C}$ with an isomorphism $g : FX \rightarrow Y$. Since $f_i \circ g : FX \rightarrow FC$, and F is fully faithful, there exists a unique arrow $\overline{f_i \circ g} : X \rightarrow C$ such that $F\overline{f_i \circ g} = f_i \circ g$. By the universal property of the product C^n , there exists a unique arrow $u : X \rightarrow C^n$ such that $p_{C,i} \circ u = \overline{f_i \circ g}$ for $i = 1, \dots, n$, hence $Fp_{C,i} \circ Fu = F\overline{f_i \circ g} = f_i \circ g$. This means that $Fp_{C,i} \circ (Fu \circ g^{-1}) = f_i$, showing existence.

Now, assume there exists some other arrow $v : Y \rightarrow FC^n$ such that $Fp_{C,i} \circ v = f_i$ for $i = 1, \dots, n$. Since $v \circ g : FX \rightarrow FC$, there exists a unique arrow $\overline{v \circ g} : X \rightarrow C$ such that $F\overline{v \circ g} = v \circ g$. We find that

$$F(p_{C,i} \circ \overline{v \circ g}) = Fp_{C,i} \circ F\overline{v \circ g} = Fp_{C,i} \circ v \circ g = f_i \circ g = F\overline{f_i \circ g}.$$

Since F is faithful, we obtain that $p_{C,i} \circ \overline{v \circ g} = \overline{f_i \circ g}$, hence $\overline{v \circ g} = u$ and $v = Fu \circ g^{-1}$, showing unicity. \blacksquare

In particular, if \mathcal{C} and \mathcal{D} have specified product diagrams, we have a canonical isomorphism $(FC)^n \rightarrow FC^n$, which makes the following diagram commutative for $i = 1, \dots, n$:

$$\begin{array}{ccc} (FC)^n & \longrightarrow & FC^n \\ & \searrow p_{FC,i} & \downarrow Fp_{C,i} \\ & & FC \end{array}$$

Furthermore, given $f : X \rightarrow X'$, we have a commutative diagram as follows:

$$\begin{array}{ccccc} & & \text{---} & & \\ & & \text{---} & & \\ & & \text{---} & & \\ (FX)^n & \xrightarrow{p_{FX,i}} & FX & \xleftarrow{Fp_{X,i}} & FX^n \\ (Ff)^n \downarrow & & Ff \downarrow & & \downarrow Ff^n \\ (FX')^n & \xrightarrow{p_{FX',i}} & FX' & \xleftarrow{Fp_{X',i}} & F(X')^n \\ & & \text{---} & & \\ & & \text{---} & & \end{array}$$

Theorem 2.1.6. *If there is an equivalence of categories $F : \mathcal{C} \rightarrow \mathcal{D}$, then there is a natural isomorphism $\gamma : \mathfrak{Z}_{\mathcal{C}} \cong \mathfrak{Z}_{\mathcal{D}} \circ F$.*

Proof: For each $C \in \mathcal{C}$, we need to define an isomorphism of Lawvere theories $\gamma_C : \mathfrak{Z}_{\mathcal{C}}C \rightarrow \mathfrak{Z}_{\mathcal{D}}FC$. On objects, obviously $\gamma_C n = n$. Now, let $\tau : n \rightarrow m$ in $\mathfrak{Z}_{\mathcal{C}}C$. We must define a natural transformation $\gamma_C \tau : P_{FC}^n \Rightarrow P_{FC}^m$. Let $h : FC \rightarrow Y$. Since F is essentially surjective, there exists $X \in \mathcal{C}$ with an isomorphism $g : Y \rightarrow FX$. Since F is fully faithful, there exists a unique arrow $\overline{g \circ h} : C \rightarrow X$ such that

$F\overline{g \circ h} = g \circ h : FC \rightarrow FX$. Define $(\gamma_C \tau)_h$ to be the unique arrow which makes the following diagram commute (we have isomorphisms along the sides):

$$\begin{array}{ccc}
 Y^n & \xrightarrow{(\gamma_C \tau)_h} & Y^m \\
 g^n \downarrow & & \downarrow g^m \\
 (FX)^n & & (FX)^m \\
 \downarrow & & \downarrow \\
 FX^n & \xrightarrow{F\tau_{\overline{g \circ h}}} & FX^m
 \end{array}$$

First, we show that $\gamma_C \tau$ is indeed a natural transformation. Suppose we have a commutative triangle:

$$\begin{array}{ccc}
 & FC & \\
 h \swarrow & & \searrow h' \\
 Y & \xrightarrow{k} & Y'
 \end{array}$$

As before, let $g : Y \rightarrow FX$ be an isomorphism. Also let $g' : Y' \rightarrow FX'$ be an isomorphism. Then we have that

$$\begin{aligned}
 F(\overline{g' \circ k \circ g^{-1} \circ g \circ h}) &= F\overline{g' \circ k \circ g^{-1}} \circ F\overline{g \circ h} \\
 &= g' \circ k \circ g^{-1} \circ g \circ h = g' \circ h' = F\overline{g' \circ h'}.
 \end{aligned}$$

By the faithfulness of F , we have a commutative triangle:

$$\begin{array}{ccc}
 & C & \\
 \overline{g \circ h} \swarrow & & \searrow \overline{g' \circ h'} \\
 X & \xrightarrow{\overline{g' \circ k \circ g^{-1}}} & X'
 \end{array}$$

Applying the naturality of τ gives a commutative square:

$$\begin{array}{ccc}
 X^n & \xrightarrow{\tau_{\overline{g \circ h}}} & X^m \\
 \overline{g' \circ k \circ g^{-1}}^n \downarrow & & \downarrow \overline{g' \circ k \circ g^{-1}}^m \\
 (X')^n & \xrightarrow{\tau_{\overline{g' \circ h'}}} & (X')^m
 \end{array}$$

Putting everything together yields a commutative diagram as follows:

$$\begin{array}{ccc}
 Y^n & \xrightarrow{(\gamma_C \tau)_h} & Y^m \\
 g^n \downarrow & & \downarrow g^m \\
 (FX)^n & & (FX)^m \\
 \downarrow & & \downarrow \\
 FX^n & \xrightarrow{F\tau_{g \circ h}} & FX^m \\
 \downarrow \overline{Fg' \circ k \circ g^{-1}{}^n} & & \downarrow \overline{Fg' \circ k \circ g^{-1}{}^m} \\
 F(X')^n & \xrightarrow{F\tau_{g' \circ h'}} & F(X')^m \\
 \downarrow & & \downarrow \\
 (FX')^n & & (FX')^m \\
 \downarrow ((g')^{-1})^n & & \downarrow ((g')^{-1})^m \\
 (Y')^n & \xrightarrow{(\gamma_C \tau)'_h} & (Y')^m
 \end{array}
 \quad (2.1.1)$$

Note that the sides are indeed k^n and k^m since, for example,

$$\begin{aligned}
 ((FX)^n \rightarrow FX^n \xrightarrow{\overline{Fg' \circ k \circ g^{-1}{}^n}} F(X')^n \rightarrow (FX')^n) &= (\overline{Fg' \circ k \circ g^{-1}{}^n})^n \\
 &= (g' \circ k \circ g^{-1})^n = (g')^n \circ k^n \circ (g^n)^{-1}.
 \end{aligned}$$

This suffices to show that $\gamma_C \tau$ is a natural transformation.

At this point, we can also conclude that $(\gamma_C \tau)_h$ does not depend on our choice of isomorphism g . Indeed, if we let $h' = h$ and $k = 1_Y$ in (2.1.1), it reduces to the following commutative square

$$\begin{array}{ccc}
 Y^n & \xrightarrow{(\gamma_C \tau)_h} & Y^m \\
 1_Y^n = 1_{Y^n} \downarrow & & \downarrow 1_Y^m = 1_{Y^m} \\
 Y^n & \xrightarrow{(\gamma_C \tau)_h} & Y^m
 \end{array}$$

where $(\gamma_C \tau)_h$ is defined in terms of $g : Y \rightarrow FX$ on the top, and in terms of $g' : Y \rightarrow FX'$ on the bottom.

Now, we show that $\gamma_C : \mathfrak{Z}_C C \rightarrow \mathfrak{Z}_D FC$ is an isomorphism of Lawvere theories for each $C \in \mathcal{C}$. Once again we take $h : FC \rightarrow Y$ and an isomorphism $g : Y \rightarrow FX$.

$$\begin{aligned}
 (\gamma_C 1_n)_h &= (Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F(1_n)_{g \circ h}} FX^n \rightarrow (FX)^n \xrightarrow{(g^{-1})^n} Y^n) \\
 &= 1_{Y^n} = (1_n)_h,
 \end{aligned}$$

since $F(1_n)_{\overline{goh}} = F1_{X^n} = 1_{FX^n}$. Moreover, given that $n \xrightarrow{\tau} m \xrightarrow{\sigma} p$, we have that

$$\begin{aligned} (\gamma_C \sigma \circ \gamma_C \tau)_h &= (\gamma_C \sigma)_h \circ (\gamma_C \tau)_h \\ &= \left(Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F\tau_{\overline{goh}}} FX^m \xrightarrow{F\sigma_{\overline{goh}}} (FX)^p \xrightarrow{(g^{-1})^p} Y^p \right) \\ &= (\gamma_C(\sigma \circ \tau))_h, \end{aligned}$$

since $F\sigma_{\overline{goh}} \circ F\tau_{\overline{goh}} = F(\sigma_{\overline{goh}} \circ \tau_{\overline{goh}}) = F(\sigma \circ \tau)_{\overline{goh}}$. Thus γ_C is a functor.

$$\begin{aligned} (\gamma_C(n \xrightarrow{\pi_i} 1))_h &= \left(Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F(\pi_i)_{\overline{goh}}} FX \xrightarrow{g^{-1}} Y \right) \\ &= \left(Y^n \xrightarrow{g^n} (FX)^n \xrightarrow{p_{FX,i}} FX \xrightarrow{g^{-1}} Y \right) \\ &= p_{Y,i}. \end{aligned}$$

Thus γ_C is a morphism of Lawvere theories. Clearly γ_C is bijective on objects.

We ask the question: Given $\rho : P_{FC}^n \Rightarrow P_{FC}^m$, does there exist $\tau : P_C^n \Rightarrow P_C^m$ such that $\gamma_C \tau = \rho$? Let us construct such a τ . Since F is fully faithful, for $f : C \rightarrow X$, let τ_f be the unique arrow such that

$$F\tau_f = \left(FX^n \rightarrow (FX)^n \xrightarrow{\rho_{Ff}} (FX)^m \rightarrow FX^m \right).$$

We check that τ is a natural transformation $P_C^n \Rightarrow P_C^m$. Consider a commutative triangle

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{k} & Y \end{array}$$

We then have that $Fk \circ Ff = F(k \circ f) = Fg$. Thus, by naturality of ρ , we obtain a commutative diagram:

$$\begin{array}{ccccccc} & & & F\tau_f & & & \\ & & & \curvearrowright & & & \\ FX^n & \longrightarrow & (FX)^n & \xrightarrow{\rho_{Ff}} & (FX)^m & \longrightarrow & FX^m \\ Fk^n \downarrow & & (Fk)^n \downarrow & & \downarrow (Fk)^m & & \downarrow Fk^m \\ FY^n & \longrightarrow & (FY)^n & \xrightarrow{\rho_{Fg}} & (FY)^m & \longrightarrow & FY^m \\ & & & \curvearrowleft & & & \\ & & & F\tau_g & & & \end{array}$$

Given that F is faithful we find that $k^m \circ \tau_f = \tau_g \circ k^n$, as desired. Now, let $h : FC \rightarrow Y$, and let $g : Y \rightarrow FX$ be an isomorphism.

$$(\gamma_C \tau)_h = \left(Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F\tau_{\overline{goh}}} FX^m \rightarrow (FX)^m \xrightarrow{(g^{-1})^m} Y^m \right)$$

and

$$F\tau\overline{g \circ h} = (FX^n \rightarrow (FX)^n \xrightarrow{\rho_{F\overline{g \circ h}} = \rho_{g \circ h}} (FX)^m \rightarrow FX^m),$$

hence

$$(\gamma_C\tau)_h = (Y^n \xrightarrow{g^n} (FX)^n \xrightarrow{\rho_{g \circ h}} (FX)^m \xrightarrow{(g^{-1})^m} Y^m) = \rho_h,$$

since ρ is a natural transformation $P_{FC}^n \Rightarrow P_{FC}^m$. Therefore $\gamma_C\tau = \rho$. It follows that γ_C is full, i.e. surjective on hom-sets.

We show that γ_C is injective on hom-sets. Suppose that $\gamma_C\tau = \gamma_C\sigma$, and let $f : C \rightarrow X$. We have that

$$\begin{aligned} ((FX)^n \rightarrow FX^n \xrightarrow{F\tau_f} FX^m \rightarrow (FX)^m) &= (\gamma_C\tau)_{Ff} \\ &= (\gamma_C\sigma)_{Ff} = ((FX)^n \rightarrow FX^n \xrightarrow{F\sigma_f} FX^m \rightarrow (FX)^m). \end{aligned}$$

Hence $F\tau_f = F\sigma_f$, and faithfulness of F implies $\tau_f = \sigma_f$. Since f was arbitrary, $\tau = \sigma$. It follows that γ_C is faithful.

We have proved that γ_C is bijective on objects and fully faithful. Therefore, it is an isomorphism of Lawvere theories. It only remains to show that γ is a natural transformation. The question is as follows: If $f : C \rightarrow D$, do we have a commutative square as below?

$$\begin{array}{ccc} \mathfrak{Z}_C C & \xrightarrow{\gamma_C} & \mathfrak{Z}_D FC \\ \mathfrak{Z}_C f \downarrow & & \downarrow \mathfrak{Z}_D Ff \\ \mathfrak{Z}_C X & \xrightarrow{\gamma_D} & \mathfrak{Z}_D FD \end{array}$$

Let $\tau : P_C^n \Rightarrow P_C^m$, $h : FD \rightarrow Y$ in \mathcal{D} , and $g : Y \rightarrow FX$ an isomorphism. Then $F(\overline{h \circ f}) = F\overline{h} \circ Ff = h \circ Ff$, hence $\overline{h \circ f} = \overline{h \circ Ff}$, and thus

$$F(\mathfrak{Z}_C f\tau)_{\overline{h}} = F\tau_{\overline{h \circ f}} = F\tau_{\overline{h \circ Ff}}.$$

From this we calculate

$$\begin{aligned} ((\gamma_D \circ \mathfrak{Z}_C f)\tau)_h &= (\gamma_D(\mathfrak{Z}_C f\tau))_h \\ &= (Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F(\mathfrak{Z}_C f\tau)_{\overline{h}}} FX^m \rightarrow (FX)^m \xrightarrow{(g^{-1})^m} Y^m) \\ &= (Y^n \xrightarrow{g^n} (FX)^n \rightarrow FX^n \xrightarrow{F\tau_{\overline{h \circ Ff}}} FX^m \rightarrow (FX)^m \xrightarrow{(g^{-1})^m} Y^m) \\ &= (\gamma_C\tau)_{h \circ Ff} = (\mathfrak{Z}_D Ff(\gamma_C\tau))_h = ((\mathfrak{Z}_D Ff \circ \gamma_C)\tau)_h. \end{aligned}$$

This completes the proof and shows that $\gamma : \mathfrak{Z}_C \cong \mathfrak{Z}_D \circ F$. ■

2.2 The Inner Theory for Algebraic Theories

Now that the inner theory functor has been properly defined, we turn to the specific case of $\mathcal{C} = T\text{-Mod}$ for an algebraic theory T , equipped with its canonical product diagrams. Let $A \in T\text{-Mod}$. Since $\mathfrak{Z}A$ is a Lawvere theory, it is entirely determined by the hom-sets of the form $\mathfrak{Z}A(n, 1)$.

Owing to the isomorphism of categories $T_A\text{-Mod} \cong (A \downarrow T\text{-Mod})$, we can interpret the projection functor $P_A : (A \downarrow T\text{-Mod}) \rightarrow T\text{-Mod}$ simply as the forgetful functor $T_A\text{-Mod} \rightarrow T\text{-Mod}$, which forgets that the elements of A are constants in the theory T_A . Now, let $U : T\text{-Mod} \rightarrow \mathbf{Set}$ be the forgetful functor. Then $U_A = U \circ P_A$ is just the forgetful functor $T_A\text{-Mod} \rightarrow \mathbf{Set}$, and we have an inclusion

$$\mathfrak{Z}A(n, 1) = \text{Nat}(P_A^n, P_A) \subseteq \text{Nat}(U_A^n, U_A).$$

Indeed, the only difference between an element $\tau \in \text{Nat}(P_A^n, P_A)$ and an element $\tau \in \text{Nat}(U_A^n, U_A)$ is that, in the former case, all components of τ must be not only functions, but also T -homomorphisms. So let us investigate the elements of $\text{Nat}(U_A^n, U_A)$ first and impose suitable restrictions on them later.

If the functor U_A^n was representable, we could apply Yoneda's lemma to simplify $\text{Nat}(U_A^n, U_A)$; and indeed it is. Recall the free-forgetful adjunction $F_A \dashv U_A$. Given a set X , we have $F_A X = (A \xrightarrow{\iota} A\langle X \rangle)$ (thinking in terms of the coslice category $(A \downarrow T\text{-Mod})$). We will write $\vec{x} = (x_1, \dots, x_n)$ when n is clear from context. Then, by the definition of an adjunction, and by the universal properties of coproducts and products, we have

$$\begin{aligned} B^n &= U_A^n(A \xrightarrow{h} B) \cong \mathbf{Set}(\{x\}, U_A^n h) \\ &= \mathbf{Set}\left(\{x\}, \prod_{i=1}^n U_A h\right) \\ &\cong \prod_{i=1}^n \mathbf{Set}(\{x\}, U_A h) \\ &\cong \mathbf{Set}\left(\prod_{i=1}^n \{x\}, U_A h\right) \\ &\cong (A \downarrow T\text{-Mod})\left(F_A\left(\prod_{i=1}^n \{x\}\right), h\right) \\ &\cong (A \downarrow T\text{-Mod})(A \xrightarrow{\iota} A\langle \vec{x} \rangle, h) \end{aligned}$$

naturally in h , which shows that $U_A^n \cong (A \downarrow T\text{-Mod})(\iota, -)$ is a representable functor. Intuitively, a T_A -homomorphism from $A\langle x_1, \dots, x_n \rangle$ to B corresponds to the n -tuple of elements of B which are the images of x_1, \dots, x_n under this homomorphism.

Now, applying Yoneda's lemma,

$$\text{Nat}(U_A^n, U_A) \cong \text{Nat}((A \downarrow T\text{-Mod})(\iota, -), U_A) \cong U_A \iota = A\langle x_1, \dots, x_n \rangle.$$

Let us make this natural correspondence explicit. Consider a natural transformation $\tau \in \text{Nat}(U_A^n, U_A)$; we get another natural transformation $\sigma : (A \downarrow T\text{-Mod})(\iota, -) \Rightarrow U_A$ from the natural bijection $(A \downarrow T\text{-Mod})(\iota, -) \cong U_A^n$. Under this bijection, $f \in (A \downarrow T\text{-Mod})(\iota, A \xrightarrow{h} B)$ corresponds to $(fx_1, \dots, fx_n) \in B^n = U_A^n h$, so we have $\sigma_h f = \tau_h(fx_1, \dots, fx_n)$. Now, the natural bijection of Yoneda's lemma,

$$\text{Nat}((A \downarrow T\text{-Mod})(\iota, -), U_A) \cong U_A \iota = A\langle \vec{x} \rangle,$$

acts on σ as follows:

$$\sigma \mapsto \sigma_\iota 1_\iota = \tau_\iota(1_\iota x_1, \dots, 1_\iota x_n) = \tau_\iota \vec{x}.$$

Let us look at things the other way around, starting with a term $t \in A\langle \vec{x} \rangle$. Once again we consider the natural bijection given by Yoneda's lemma:

$$A\langle \vec{x} \rangle = U_A \iota \cong \text{Nat}((A \downarrow T\text{-Mod})(\iota, -), U_A).$$

It sends t to a natural transformation σ defined by

$$\sigma_h f = U_A f t = f t$$

for $(A \xrightarrow{h} B) \in (A \downarrow T\text{-Mod})$ and $f \in (A \downarrow T\text{-Mod})(\iota, h)$. We get a natural transformation $\tau : U_A^n \Rightarrow U_A$ using the natural bijection $U_A^n \cong (A \downarrow T\text{-Mod})(\iota, -)$, which sends a tuple $(b_1, \dots, b_n) \in B^n = U_A^n h$ to the unique T_A -homomorphism $g : \iota \rightarrow h$ satisfying $g x_i = b_i, 1 \leq i \leq n$. Then, we have

$$\tau_h(b_1, \dots, b_n) = \sigma_h g = g t = (h\langle \vec{x} \rangle t)^B(b_1, \dots, b_n).$$

This τ is a natural transformation $P_A^n \Rightarrow P_A$ if and only if each component $\tau_h = (h\langle \vec{x} \rangle t)^B$ is a T -homomorphism. This prompts the following definition.

Definition 2.2.1. We say that a term $t \in A\langle \vec{x} \rangle$ with parameters in A *generically induces a T -homomorphism* if and only if $(h\langle \vec{x} \rangle t)^B \in T\text{-Mod}(B^n, B)$ for all T_A -models $h : A \rightarrow B$. We say that a tuple of terms (t_1, \dots, t_m) generically induces a T -homomorphism if and only if each t_j generically induces a T -homomorphism.

This definition leads into the next one, which describes a useful functor \mathfrak{G} .

Definition 2.2.2. We define a functor $\mathfrak{G} : T\text{-Mod} \rightarrow \mathbf{Law}$ as follows. Consider the category \mathcal{T}_A whose objects are the natural numbers and whose arrows $n \rightarrow m$ are m -tuples of terms in $A\langle x_1, \dots, x_n \rangle$. Composition is defined by substitution, and identities are given as $1_n = (x_1, \dots, x_n)$. Moreover, this category is a Lawvere theory, since we have canonical product diagrams $x_i : n \rightarrow 1, 1 \leq i \leq n$. (In fact, it is the Lawvere theory corresponding to the algebraic theory T_A .) We define $\mathfrak{G}A$ to be the

subcategory of \mathcal{T}_A with the same objects and with arrows only those tuples of terms $(t_1, \dots, t_m) : n \rightarrow m$ which generically induce T -homomorphisms. Then the projections $x_i : n \rightarrow 1$ are in $\mathfrak{G}A$, since $(h\langle \vec{x} \rangle x_i)^B$ is the i -th projection $B^n \rightarrow B$, which is a T -homomorphism; therefore, $\mathfrak{G}A$ is a Lawvere theory with the same specified product diagrams as \mathcal{T}_A .

It remains to define a morphism of Lawvere theories $\mathfrak{G}f : \mathfrak{G}A \rightarrow \mathfrak{G}C$ given a T -homomorphism $f : A \rightarrow C$. On objects, we let $\mathfrak{G}fn = n$. Given $(t_1, \dots, t_m) : n \rightarrow m$, we define

$$\mathfrak{G}f(t_1, \dots, t_m) = f\langle x_1, \dots, x_n \rangle(t_1, \dots, t_m).$$

Proposition 2.2.3. $\mathfrak{G} : T\text{-Mod} \rightarrow \mathbf{Law}$ is a well-defined functor.

Proof: First, we check that $\mathfrak{G}(A \xrightarrow{f} C)$ is well-defined, that is we make sure that $\mathfrak{G}f\vec{t}$ is an arrow of $\mathfrak{G}C$ when \vec{t} is an arrow of $\mathfrak{G}A$. Indeed, we have

$$(h\langle \vec{x} \rangle(\mathfrak{G}f\vec{t}))^B = (h\langle \vec{x} \rangle(f\langle \vec{x} \rangle\vec{t}))^B = ((h \circ f)\langle \vec{x} \rangle\vec{t})^B$$

for $\vec{t} : n \rightarrow m$ in $\mathfrak{G}A$, so that $(h\langle \vec{x} \rangle(\mathfrak{G}f\vec{t}))^B$ is a T -homomorphism for all $(A \xrightarrow{h} B) \in (A \downarrow T\text{-Mod})$.

We check that $\mathfrak{G}f$ is functorial. It preserves identities:

$$\mathfrak{G}f(x_1, \dots, x_n) = f\langle \vec{x} \rangle(x_1, \dots, x_n) = (x_1, \dots, x_n);$$

and composition:

$$\begin{aligned} \mathfrak{G}f((s_1, \dots, s_l) \circ (t_1, \dots, t_m)) &= \mathfrak{G}f(s_1[t_i/x_i], \dots, s_l[t_i/x_i]) \\ &= f\langle \vec{x} \rangle(s_1[t_i/x_i], \dots, s_l[t_i/x_i]) \\ &= ((f\langle \vec{x} \rangle s_1)[f\langle \vec{x} \rangle t_i/x_i], \dots, (f\langle \vec{x} \rangle s_l)[f\langle \vec{x} \rangle t_i/x_i]) \\ &= (f\langle \vec{x} \rangle s_1, \dots, f\langle \vec{x} \rangle s_l) \circ (f\langle \vec{x} \rangle t_1, \dots, f\langle \vec{x} \rangle t_m) \\ &= \mathfrak{G}f(s_1, \dots, s_l) \circ \mathfrak{G}f(t_1, \dots, t_m). \end{aligned}$$

In the above, note that, assuming from context that i ranges from 1 to n , $s[t_i/x_i]$ is shorthand for $s[t_1/x_1, \dots, t_n/x_n]$, which means s with x_i substituted by t_i for each i , simultaneously.

In fact, $\mathfrak{G}f$ is a morphism of Lawvere theories, because it preserves the product diagrams:

$$\mathfrak{G}f x_i = f\langle \vec{x} \rangle x_i = x_i : n \rightarrow 1.$$

Finally, we check that \mathfrak{G} is functorial. We have

$$\mathfrak{G}1_A \vec{t} = 1_A \langle \vec{x} \rangle \vec{t} = \vec{t} = 1_{\mathfrak{G}A} \vec{t}.$$

Given $A \xrightarrow{f} C \xrightarrow{g} D$, we also have

$$\mathfrak{G}(g \circ f) \vec{t} = (g \circ f) \langle \vec{x} \rangle \vec{t} = g \langle \vec{x} \rangle (f \langle \vec{x} \rangle \vec{t}) = \mathfrak{G}g(\mathfrak{G}f \vec{t}).$$

This completes the verification. ■

Now, since we have defined \mathfrak{G} in such a way that $\mathfrak{G}A(n, m) \cong \mathfrak{Z}A(n, m)$, we expect these bijections to extend to a natural isomorphism of the functors \mathfrak{G} and \mathfrak{Z} .

Theorem 2.2.4. *There is a natural isomorphism $\mathfrak{G} \cong \mathfrak{Z}$.*

Proof: We define $\delta : \mathfrak{G} \Rightarrow \mathfrak{Z}$. Given $A \in T\text{-Mod}$, we have a component $\delta_A : \mathfrak{G}A \rightarrow \mathfrak{Z}A$, which is a morphism of Lawvere theories. On objects, we let $\delta_A n = n$. Given an arrow $\vec{t} : n \rightarrow m$, we define $\delta_A \vec{t} : P_A^n \Rightarrow P_A^m$, for any $h \in (A \downarrow T\text{-Mod})$, by $(\delta_A \vec{t})_h = (h \langle \vec{x} \rangle \vec{t})^B$. Up to now, there have been no surprises. Our definition merely reflects the bijection that led us to the definition of \mathfrak{G} . In particular, $\delta_A \vec{t}$ really is a natural transformation, and δ_A is clearly bijective on both objects and hom-sets. It remains to show that δ_A is a morphism of Lawvere theories.

Consider the projections $x_i : n \rightarrow 1$ in $\mathfrak{G}A$, and let $p_{B,i} : B^n \rightarrow B$ be projections in $T\text{-Mod}$. If, furthermore, we let $\pi_i : n \rightarrow 1$ be the projections in $\mathfrak{Z}A$, and $h : A \rightarrow B$, then

$$(\delta_A x_i)_h = (h \langle \vec{x} \rangle x_i)^B = p_{B,i} = (\pi_i)_h,$$

which implies $\delta_A x_i = \pi_i$. Therefore, δ_A preserves products. Now, we check that it is a functor. Note that $1_n = (x_1, \dots, x_n)$ in $\mathfrak{G}A$. We have that

$$(\delta_A 1_n)_h = (h \langle \vec{x} \rangle (x_1, \dots, x_n))^B = 1_{B^n} = 1_{P_A^n h} = (1_n)_h,$$

hence $\delta_A 1_n = 1_n$ (note that 1_n is an arrow of $\mathfrak{G}A$ on the left-hand side, but of $\mathfrak{Z}A$ on the right-hand side). Moreover, given arrows

$$n \xrightarrow{\vec{t}} m \xrightarrow{\vec{s}} l,$$

we calculate:

$$\begin{aligned} (\delta_A(\vec{s} \circ \vec{t}))_h &= \left(\delta_A(s_1[t_j/x_j], \dots, s_l[t_j/x_j]) \right)_h \\ &= \left(h \langle \vec{x} \rangle (s_1[t_j/x_j], \dots, s_l[t_j/x_j]) \right)^B \\ &= \left((h \langle \vec{x} \rangle s_1)[h \langle \vec{x} \rangle t_j/x_j], \dots, (h \langle \vec{x} \rangle s_l)[h \langle \vec{x} \rangle t_j/x_j] \right)^B \\ &= (h \langle \vec{x} \rangle \vec{s} \circ h \langle \vec{x} \rangle \vec{t})^B = (h \langle \vec{x} \rangle \vec{s})^B \circ (h \langle \vec{x} \rangle \vec{t})^B \\ &= (\delta_A \vec{s})_h \circ (\delta_A \vec{t})_h = (\delta_A \vec{s} \circ \delta_A \vec{t})_h. \end{aligned}$$

We conclude that $\delta_A(\vec{s} \circ \vec{t}) = \delta_A \vec{s} \circ \delta_A \vec{t}$.

Finally, we must show that $\delta : \mathfrak{G} \Rightarrow \mathfrak{Z}$ is indeed a natural transformation. Let $f : A \rightarrow C$ in $T\text{-Mod}$. We have to establish the commutativity of the following diagram:

$$\begin{array}{ccc} \mathfrak{G}A & \xrightarrow{\delta_A} & \mathfrak{Z}A \\ \mathfrak{G}f \downarrow & & \downarrow \mathfrak{Z}f \\ \mathfrak{G}C & \xrightarrow{\delta_C} & \mathfrak{Z}C \end{array}$$

First, we verify that, on objects, $\mathfrak{Z}f(\delta_A n) = \mathfrak{Z}fn = n = \delta_C n = \delta_C(\mathfrak{G}fn)$. Next, let $\vec{t} : n \rightarrow m$ and $(C \xrightarrow{h} B) \in (C \downarrow T\text{-Mod})$. We have that

$$\begin{aligned} (\delta_C(\mathfrak{G}f\vec{t}))_h &= (\delta_C(f\langle \vec{x} \rangle \vec{t}))_h = (h\langle \vec{x} \rangle (f\langle \vec{x} \rangle \vec{t}))^B \\ &= ((h \circ f)\langle \vec{x} \rangle \vec{t})^B = (\delta_A \vec{t})_{h \circ f} = (\mathfrak{Z}f(\delta_A \vec{t}))_h. \end{aligned}$$

This completes the proof. ■

In light of the previous theorem, we shall say that the arrows of $\mathfrak{G}A$ are inner morphisms, like those of $\mathfrak{Z}A$.

Now, we give a definition and a theorem which help to further characterize the elements of $\mathfrak{G}A(n, 1)$. These are terms which induce T -homomorphisms on all T -models with parameters in A (in prior terminology, they generically induce such a homomorphism). If we were to check by hand that a term induces a homomorphism on a particular model B , we would verify that the function it induces commutes with all the operations on T . If we want this to happen for all models with parameters in A , we should not use any particular property of B in our calculations. Equivalently, we would be checking that the function induced by the term commutes with all operations of T in a model without any extraneous relations, i.e. a free model. This is the idea behind what follows.

Definition 2.2.5. Let f be an operation symbol of arity l , and let $t \in A\langle \vec{x} \rangle$. Also let $\vec{y} = (y_{11}, \dots, y_{ln})$ be a list of $l \cdot n$ variables. Then t *commutes generically* with f if and only if

$$f^{A\langle \vec{y} \rangle}(t[y_{1i}/x_i], \dots, t[y_{li}/x_i]) = t[f^{A\langle \vec{y} \rangle}(y_{1i}, \dots, y_{li})/x_i].$$

Let us give a few examples of generic commutation. If f is a constant symbol c , then generic commutation of t with c means that

$$c = t[c/x_i],$$

where we let $c^{A\langle \vec{y} \rangle} = c$ for simplicity of notation. If instead f is a binary operation \cdot , written in infix notation, then generic commutation of t with \cdot means that

$$t[y_{1i}/x_i] \cdot t[y_{2i}/x_i] = t[y_{1i} \cdot y_{2i}/x_i],$$

where we let $\cdot^{A\langle\vec{y}\rangle} = \cdot$ for simplicity.

And now the theorem hinted at when motivating generic commutation:

Theorem 2.2.6. *Let $t \in A\langle\vec{x}\rangle$. Then $t \in \mathfrak{G}A(n, 1)$ if and only if t commutes generically with all operation symbols of T .*

Proof: We prove the forward direction first. If t generically induces a T -homomorphism, then in particular $t^{A\langle\vec{y}\rangle} : (A\langle\vec{y}\rangle)^n \rightarrow A\langle\vec{y}\rangle$ is a T -homomorphism. For $k = 1, \dots, l$, let $\vec{y}_k = (y_{k1}, \dots, y_{kn})$. We calculate:

$$\begin{aligned} t[f^{A\langle\vec{y}\rangle}(y_{1i}, \dots, y_{li})/x_i] &= t^{A\langle\vec{y}\rangle}(f^{A\langle\vec{y}\rangle}(y_{11}, \dots, y_{l1}), \dots, f^{A\langle\vec{y}\rangle}(y_{1n}, \dots, y_{ln})) \\ &= t^{A\langle\vec{y}\rangle}(f^{A\langle\vec{y}\rangle^n}(\vec{y}_1, \dots, \vec{y}_l)) \\ &= f^{A\langle\vec{y}\rangle}(t^{A\langle\vec{y}\rangle}\vec{y}_1, \dots, t^{A\langle\vec{y}\rangle}\vec{y}_l) \\ &= f^{A\langle\vec{y}\rangle}(t[y_{1i}/x_i], \dots, t[y_{li}/x_i]). \end{aligned}$$

Conversely, suppose that t commutes generically with all operation symbols f of T . Let $h : A \rightarrow B$ be a T -homomorphism. We show that $(h\langle\vec{x}\rangle t)^B \in T\text{-Mod}(B^n, B)$, which is to say that $(h\langle\vec{x}\rangle t)^B$ commutes with f for all operation symbols f of T . Let $\vec{b}_k = (b_{k1}, \dots, b_{kn}) \in B^n$ for $k = 1, \dots, l$, and collect all these vectors into a single vector \vec{b} . Since t commutes generically with f , we have that

$$t[f^{A\langle\vec{y}\rangle}(y_{1i}, \dots, y_{li})/x_i] = f^{A\langle\vec{y}\rangle}(t[y_{1i}/x_i], \dots, t[y_{li}/x_i]). \quad (2.2.1)$$

Moreover, since $A\langle\vec{y}\rangle = A \amalg \langle\vec{y}\rangle$, we have a homomorphism $h^{\vec{b}} : A\langle\vec{y}\rangle \rightarrow B$ such that $h^{\vec{b}}a = ha$ for all $a \in A$ and $h^{\vec{b}}y_{ki} = b_{ki}$, $1 \leq k \leq l$, $1 \leq i \leq n$. Applying it to both sides of (2.2.1), we find that

$$\begin{aligned} (h\langle\vec{x}\rangle t)^B(f^{B^n}(\vec{b}_1, \dots, \vec{b}_l)) &= (h\langle\vec{x}\rangle t)^B(f^B(b_{11}, \dots, b_{l1}), \dots, f^B(b_{1n}, \dots, b_{ln})) \\ &= h^{\vec{b}}(t[f^{A\langle\vec{y}\rangle}(y_{1i}, \dots, y_{li})/x_i]) \\ &= h^{\vec{b}}(f^{A\langle\vec{y}\rangle}(t[y_{1i}/x_i], \dots, t[y_{li}/x_i])) \\ &= f^B(h^{\vec{b}}(t[y_{1i}/x_i]), \dots, h^{\vec{b}}(t[y_{li}/x_i])) \\ &= f^B((h\langle\vec{x}\rangle t)^B\vec{b}_1, \dots, (h\langle\vec{x}\rangle t)^B\vec{b}_l), \end{aligned}$$

as desired. ■

2.3 The Inner Theory for Quasi-Equational Theories

This section covers many of the same ideas as the previous section, so we will not provide detailed proofs when they are very similar to the proofs of previous results, instead providing hints and discussing the differences from the algebraic case, while trusting the reader to fill in the gaps.

Consider a quasi-equational theory T , and let $A \in T\text{-Mod}$. Since $\mathfrak{Z}A$ is a Lawvere theory, it is entirely determined by the hom-sets of the form $\mathfrak{Z}A(n, 1)$.

Owing to the isomorphism of categories $T_A\text{-Mod} \cong (A \downarrow T\text{-Mod})$, we can interpret the projection functor $P_A : (A \downarrow T\text{-Mod}) \rightarrow T\text{-Mod}$ simply as the forgetful functor $T_A\text{-Mod} \rightarrow T\text{-Mod}$. Now, let $U : T\text{-Mod} \rightarrow \mathbf{Set}^S$ be the forgetful functor. Then $U_A = U \circ P_A$ is just the forgetful functor $T_A\text{-Mod} \rightarrow \mathbf{Set}^S$, and we have an inclusion

$$\mathfrak{Z}A(n, 1) = \text{Nat}(P_A^n, P_A) \subseteq \text{Nat}(U_A^n, U_A).$$

Indeed, the only difference between an element $\tau \in \text{Nat}(P_A^n, P_A)$ and an element $\tau \in \text{Nat}(U_A^n, U_A)$ is that, in the former case, all components of τ must be not arbitrary families of functions, but T -homomorphisms. So let us investigate the elements of $\text{Nat}(U_A^n, U_A)$ first and impose suitable restrictions on them later.

Given that $\text{Nat}(U_A^n, U_A) \cong \prod_{S \in \mathcal{S}} \text{Nat}((U_A^n)_S, (U_A)_S)$, we will proceed sort-wise. Note that for any $h : A \rightarrow B$, we have

$$(U_A^n)_S h = (U_A^n h)_S = (B^n)_S = (B_S)^n = ((U_A)_S h)^n = \prod_{i=1}^n (U_A)_S h.$$

Also recall the free-forgetful adjunction $F_{A,S} \dashv (U_A)_S$. Therefore, we have natural isomorphisms as follows:

$$\begin{aligned} (U_A^n)_S h &\cong \mathbf{Set}(\{x\}, (U_A^n)_S h) \\ &= \mathbf{Set}\left(\{x\}, \prod_{i=1}^n (U_A)_S h\right) \\ &\cong \prod_{i=1}^n \mathbf{Set}(\{x\}, (U_A)_S h) \\ &\cong \mathbf{Set}\left(\prod_{i=1}^n \{x\}, (U_A)_S h\right) \\ &\cong (A \downarrow T\text{-Mod})\left(F_{A,S}\left(\prod_{i=1}^n \{x\}\right), h\right) \\ &\cong (A \downarrow T\text{-Mod})(A \xrightarrow{\iota^S} A\langle \bar{x}^S \rangle, h). \end{aligned}$$

This shows that $(U_A^n)_S \cong (A \downarrow T\text{-Mod})(\iota^S, -)$ is a representable functor. Now, applying Yoneda's lemma,

$$\text{Nat}((U_A^n)^S, (U_A)_S) \cong \text{Nat}((A \downarrow T\text{-Mod})(\iota^S, -), (U_A)_S) \cong (U_A)_S \iota^S = A\langle \vec{x}^S \rangle_S.$$

Under this correspondence, a term $t_S \in A\langle \vec{x}^S \rangle_S$ corresponds to the natural transformation $\tau_S : (U_A^n)_S \Rightarrow (U_A)_S$ defined by $(\tau_S)_h = (h\langle \vec{x}^S \rangle_S t_S)^B$. Therefore families of terms

$$t = (t_S)_{S \in \mathcal{S}} \in A\langle \vec{x}^S \rangle_S$$

are in bijective correspondence with elements of $\text{Nat}(U_A^n, U_A)$, in such a way that t corresponds to $\tau : U_A^n \Rightarrow U_A$ with $\tau_h = (h\langle \vec{x} \rangle t)^B$. If, moreover, we require that $\tau : P_A^n \Rightarrow P_A$, then each component $\tau_h = (h\langle \vec{x} \rangle t)^B$ must be a homomorphism. This motivates the following definition.

Definition 2.3.1. The functor $\mathfrak{G} : T\text{-Mod} \rightarrow \mathbf{Law}$ is defined as follows. The Lawvere theory $\mathfrak{G}A$ has objects the natural numbers, and its hom-sets are determined by the fact that $\mathfrak{G}A(n, 1)$ contains those families of terms $t = (t_S) \in \prod_{S \in \mathcal{S}} A\langle \vec{x}^S \rangle_S$ such that $(h\langle \vec{x} \rangle t)^B$ is a homomorphism $B^n \rightarrow B$ for all $h : A \rightarrow B$. It remains to define a morphism of Lawvere theories $\mathfrak{G}f : \mathfrak{G}A \rightarrow \mathfrak{G}C$ given a T -homomorphism $f : A \rightarrow C$. On objects, we let $\mathfrak{G}fn = n$. Given $(t_1, \dots, t_m) : n \rightarrow m$, we define

$$\mathfrak{G}f(t_1, \dots, t_m) = f\langle \vec{x} \rangle(t_1, \dots, t_m) = (f\langle \vec{x} \rangle t_1, \dots, f\langle \vec{x} \rangle t_m).$$

Much as in the case of algebraic theories, the functor \mathfrak{G} is well-defined.

Theorem 2.3.2. *There is a natural isomorphism $\mathfrak{G} \cong \mathfrak{Z}$.*

Proof: Similar to the proof of Theorem 2.2.4. ■

To formulate generic commutation for quasi-equational theories, a technical tool is needed. Indeed, the situation is complicated by the fact that operations may only be partially defined, and thus it is possible that an operation is defined for some tuple of elements in some model, but undefined when applied to variables in a free model. To remedy this defect, given an operation $f : S_1 \times \dots \times S_l \rightarrow S$, we introduce a new quasi-equational theory $T_{A\langle \vec{y} \rangle}^f$, where \vec{y} is composed of variables $y_{ki} : S_k$, $1 \leq k \leq l$, $1 \leq i \leq n$. Actually, to be entirely precise, \vec{y} should be replaced by the family of sets $Y = (Y_R)_{R \in \mathcal{S}}$ defined by

$$Y_R = \begin{cases} \{y_{k1}, \dots, y_{kn}\} & \text{if } R = S_k \\ \emptyset & \text{otherwise} \end{cases}.$$

Now, let us consider $T_{A\langle \vec{y} \rangle}$, the theory obtained from T by adjoining a (totally defined) constant for each element of $A\langle \vec{y} \rangle$, together with all the Horn sequents which hold in $A\langle \vec{y} \rangle$. (It can also be obtained by adjoining totally defined constants $y_{ki} : S_k$ to T_A .) Then, we obtain $T_{A\langle \vec{y} \rangle}^f$ by adding the axiom $\vdash f(y_{1i}, \dots, y_{li}) \downarrow$ to $T_{A\langle \vec{y} \rangle}$ for $i = 1, \dots, n$.

Definition 2.3.3. A family of terms $t = (t_S) \in \prod_{S \in \mathcal{S}} A\langle x_1^S, \dots, x_n^S \rangle_S$ is said to *commute generically* with an operation $f : S_1 \times \dots \times S_l \rightarrow S$ if and only if the following sequent is provable in $T_{A\langle \vec{y} \rangle}^f$:

$$\vdash t_S[f(y_{1i}, \dots, y_{li})/x_i^S] = f(t_{S_1}[y_{1i}/x_i^{S_1}], \dots, t_{S_l}[y_{li}/x_i^{S_l}]).$$

(The appropriate deductive logic here is partial Horn logic, which accounts for partially defined operations. See [12] for details.)

Note that $T_{A\langle \vec{y} \rangle}^f$ has an initial model, which we will write as $A\langle \vec{y} \rangle^f$. Its usefulness lies in the fact that whenever we have a T -homomorphism $h : A \rightarrow B$, and a vector \vec{b} of elements $b_{ki} \in B_{S_k}$, $1 \leq k \leq l$, $1 \leq i \leq n$, such that $f^{B^n}(\vec{b}_1, \dots, \vec{b}_l) \downarrow$ (equivalently $f^B(b_{1i}, \dots, b_{li}) \downarrow$ for $i = 1, \dots, n$), then there exists a T -homomorphism $h^{\vec{b}} : A\langle \vec{y} \rangle^f \rightarrow B$ satisfying $h_R^{\vec{b}} a = h_R a$ for all $a \in A_R$ and all $R \in \mathcal{S}$, and $h_{S_k}^{\vec{b}} y_{ki} = b_{ki}$.

Lemma 2.3.4. *A family of terms $t = (t_S)$ commutes generically with an operation $f : S_1 \times \dots \times S_l \rightarrow S$ if and only if we have the following equality in $A\langle \vec{y} \rangle^f$:*

$$t_S[f^{A\langle \vec{y} \rangle^f}(y_{1i}, \dots, y_{li})/x_i^S] = f^{A\langle \vec{y} \rangle^f}(t_{S_1}[y_{1i}/x_i^{S_1}], \dots, t_{S_l}[y_{li}/x_i^{S_l}]).$$

Proof: This follows immediately from the fact that $A\langle \vec{y} \rangle^f$ is the initial model of the theory $T_{A\langle \vec{y} \rangle}^f$. ■

Theorem 2.3.5. *A family of terms $t = (t_S) \in \prod_{S \in \mathcal{S}} A\langle x_1^S, \dots, x_n^S \rangle_S$ is an element of $\mathfrak{G}A(n, 1)$ if and only if it commutes generically with all the operations of T .*

Proof: Making use of the previous lemma, the proof is similar to the proof of Theorem 2.2.6. ■

Chapter 3

Examples

3.1 Algebraic Theories

First, we explore a variety of examples from algebra. Then, we turn to examples from order theory which still admit axiomatizations as algebraic theories. Note that our main theorem can be readily adapted to algebraic theories with infinitary operations, as long as sufficiently many free models exist. It only means an infinity of variables may be necessary to express some generic commutation equations. The problem is that not all theories with infinitary operations have free models, even on finitely many generators. The reason is simple: when we build free models for algebraic theories, we start by constructing the set of terms, and then apply an equivalence relation to ensure that all the equations of the theory hold; but in the presence of operations of unbounded arities, the set of terms is a proper class and not a set at all. It should then be regarded as exceptional that in some cases, this proper class becomes a set after taking a quotient by the appropriate equivalence relation.

For each example, we pick an algebraic theory T and proceed as follows. We let $A \in T\text{-Mod}$ and we characterize all elements $t \in \mathfrak{G}A(n, 1)$, for each $n \in \mathbb{N}$. Since \mathfrak{G} is naturally isomorphic to the inner theory \mathfrak{Z} , in particular we have a bijection $\mathfrak{G}A(n, 1) \cong \mathfrak{Z}A(n, 1)$, which makes t correspond to the family of homomorphisms $(h\langle \vec{x} \rangle t)^B$ indexed by morphisms $h : A \rightarrow B$ with source A . This can be seen as a family of coherent extensions of the morphism $t^A = (1_A\langle \vec{x} \rangle t)^A$. Therefore, it can be helpful to think of this particular morphism as representative of the whole family. Now that we have determined $\mathfrak{Z}A(n, 1)$ for all $n \in \mathbb{N}$, we effectively have a full picture of $\mathfrak{Z}A$, since as a Lawvere theory it is determined by those hom-sets. In this way, we obtain a description of the inner theory \mathfrak{Z} . Also recall that the morphisms of $\mathfrak{Z}A$ (or $\mathfrak{G}A$) are called inner morphisms. For reasons outlined above, we will mostly concentrate on the elements of $\mathfrak{G}A(n, 1)$, which may also be called inner morphisms.

3.1.1 Sets

Proposition 3.1.1. *Let T be the theory of sets, which has no operations and no equations. Let $A \in \mathbf{Set}$. Then $\mathfrak{G}A(n, 1) = A \amalg \{x_1, \dots, x_n\}$ (a disjoint union of sets). Therefore, inner morphisms correspond to constants or projections.*

Proof: We have that $A\langle x_1, \dots, x_n \rangle = A \amalg \{x_1, \dots, x_n\}$ is just a disjoint union, and all of its elements represent inner morphisms, since there are no operations with which they would have to commute generically. ■

Note that $\mathfrak{G}A$ is actually the Lawvere theory corresponding to the theory of sets with parameters in A , or sets under A , which was called \mathcal{T}_A previously.

3.1.2 Sets with an Action

Let M be a monoid (in particular, it could be a group). An M -set is a set S together with an action $(m, x) \mapsto m \cdot x : M \times S \rightarrow S$, such that $e \cdot x = x$ for all $x \in S$, and $m \cdot (n \cdot x) = (mn) \cdot x$ for all $m, n \in M$ and all $x \in S$. We can construe M -sets as the models of an algebraic theory much in the same way as we do modules, by making $x \mapsto m \cdot x$ an operation for each $m \in M$. Thus, we let T be that algebraic theory.

Proposition 3.1.2. *Let $A \in T\text{-Mod}$ be an M -set. Then an element of $\mathfrak{G}A(n, 1)$ is either an element $a \in A$ which is M -invariant (i.e. $p \cdot a = a$ for all $p \in M$), or of the form $m \cdot x_i$ where m is in the center of M .*

Proof: Let $t \in A\langle x_1, \dots, x_n \rangle$. Then $t \in \mathfrak{G}A(n, 1)$ if and only if it commutes generically with the action of each $m \in M$. Two cases present themselves: either $t = a \in A$, or $t = m \cdot x_i$ for some $i = 1, \dots, n$.

In the first case, $t = a$ commutes generically with the action of p for all $p \in M$, so that $p \cdot a = a$ for all p . This means that a must be M -invariant.

In the second case, $t = m \cdot x_i$, so generic commutation with the action $x \mapsto p \cdot x$ leads us to the equation

$$(pm) \cdot y_{1i} = p \cdot (m \cdot y_{1i}) = m \cdot (p \cdot y_{1i}) = (mp) \cdot y_{1i}.$$

Since we are working over a free model with parameters in A , this equation can only hold if $pm = mp$, and this must be true for all $p \in M$. Thus $m \in CM$, the center of the monoid M . ■

Note that $\mathfrak{G}A$ is actually the Lawvere theory corresponding to the theory of CM -sets with parameters in the CM -set of M -invariant elements of A .

3.1.3 Semigroups

Proposition 3.1.3. *Let T be the theory of semigroups, and let $A \in T\text{-Mod}$. An element of $\mathfrak{G}A(n, 1)$ is either an idempotent of A or a projection x_i .*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. Then t must commute generically with the product:

$$t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i].$$

In the right-hand side, all variables of type y_{1i} appear to the left of all variables of type y_{2i} . For that to be the case on the left-hand side, t can contain at most one variable (and no power of this variable higher than one). Thus either $t = a$, or $t = ax_i b$, or $t = ax_i$, or $t = x_i b$, or $t = x_i$, with $a, b \in A$. Let us once again write the equation for generic commutation with the product. In case $t = a$, we obtain $a = a^2$, which is to say that a is idempotent. In case $t = ax_i b$, we obtain $ay_{1i}y_{2i}b = ay_{1i}bay_{2i}b$, which is impossible. Similarly for $t = ax_i$ and $t = x_i b$. Projections $t = x_i$ are obviously inner morphisms. Therefore, the idempotents of A and the projections x_i exhaust all inner morphisms in $\mathfrak{G}A(n, 1)$. \blacksquare

Note that $\mathfrak{G}A$ is actually the Lawvere theory corresponding to the theory of sets with parameters in the set of idempotents of A .

3.1.4 Monoids

Proposition 3.1.4. *Let $T = (\Sigma, E)$ be the theory of monoids, and let $A \in T\text{-Mod}$. The hom-set $\mathfrak{G}A(n, 1)$ contains e and those terms of the form $ax_i a^{-1}$ for $i = 1, \dots, n$, and $a \in A$ an invertible element.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle \vec{x} \rangle$. Considering t properly as a Σ -term, that is as a representative of an element of $A\langle \vec{x} \rangle$, we can reduce it to a normal form. The reduction process is as follows. If the string t has length 1, it is reduced. Otherwise, we remove any instance of the neutral element e , and replace any two consecutive elements of A by their product. This is discussed in [13, Definition 3.2.1] and [13, Lemma 3.2.2]. Now, the term t must commute generically with the product, so

$$t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i].$$

Assuming t was written in reduced form, the left-hand side is also reduced, since the variables y_{1i} and y_{2i} in $A\langle \vec{y} \rangle$ are not linked by any relations that would permit a simplification. When we write the right-hand side in reduced form, all variables y_{1i} appear to the left of all variables y_{2i} , and these two sets of variables cannot be moved past each other. But if t contains more than a single x_i , then the left-hand side will have variables of the form y_{2i} to the left of variables of the form y_{1i} , despite

being in reduced form. This contradiction shows that either $t = e$ or $t = ax_i b$ for some $i = 1, \dots, n$ and some $a, b \in A$. In the first case, e commutes generically with all monoid operations, so $e \in \mathfrak{G}A(n, 1)$. In the second case, we have

$$ay_{1i}y_{2i}b = t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i] = ay_{1i}bay_{2i}b,$$

which implies $ba = e$. Moreover, t commutes generically with the neutral element e , so

$$ab = aeb = e,$$

which together with $ba = e$ implies $b = a^{-1}$. ■

3.1.5 Groups

Proposition 3.1.5. *Let $T = (\Sigma, E)$ be the theory of groups, and let $A \in T\text{-Mod}$. The hom-set $\mathfrak{G}A(n, 1)$ contains e and conjugations $ax_i a^{-1}$ for $i = 1, \dots, n$, and $a \in A$.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle \vec{x} \rangle$. Considering t properly as a Σ -term, that is as a representative of an element of $A\langle \vec{x} \rangle$, we can reduce it to a normal form. This is the same reduction as we performed for monoids, but with the additional step of removing products $x_i x_i^{-1}$ or $x_i^{-1} x_i$. As is noted in Section 3.2 of [13], only minor adjustments to [13, Definition 3.2.1] and [13, Lemma 3.2.2] are necessary to explain reduction to a normal form for groups. Now, the term t must commute generically with the product, so

$$t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i].$$

Assuming t was written in reduced form, the left-hand side is also reduced, since the variables y_{1i} and y_{2i} in $A\langle \vec{y} \rangle$ are not linked by any relations that would permit a simplification. When we write the right-hand side in reduced form, all variables y_{1i} appear to the left of all variables y_{2i} , and these two sets of variables cannot be moved past each other. But if t contains the inverse x_i^{-1} of a variable, or more than a single variable x_i (including powers of said variable higher than one), then the left-hand side will have variables of the form y_{2i} to the left of variables of the form y_{1i} , and reduction will not change this fact. We have a contradiction which shows that either $t = e$ or $t = ax_i b$ for some $i = 1, \dots, n$ and some $a, b \in A$. In the first case, e commutes generically with all group operations, so $e \in \mathfrak{G}A(n, 1)$. In the second case, we have

$$ay_{1i}y_{2i}b = t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i] = ay_{1i}bay_{2i}b,$$

which implies $ba = e$, i.e. $b = a^{-1}$. We check that $ax_i a^{-1}$ commutes generically with

- the product:

$$t[y_{1i}y_{2i}/x_i] = ay_{1i}y_{2i}a^{-1} = ay_{1i}aa^{-1}y_{2i}a^{-1} = t[y_{1i}/x_i]t[y_{2i}/x_i].$$

- inverses : $t[y_{1i}^{-1}/x_i] = ay_{1i}^{-1}a^{-1} = (ay_{1i}a^{-1})^{-1} = (t[y_{1i}/x_i])^{-1}$.
- the neutral element: $t[e/x_i] = aea^{-1} = e$.

This completes the verification. ■

It is interesting to compare this to the results previously obtained by Bergman [1]. He showed that the isotropy group contains only conjugation by group elements, and that e is added to this group to obtain the isotropy monoid. Thus, in this case, the inner theory does not add much to the isotropy monoid; in fact, the inner theory of groups is generated, as a Lawvere theory, by the isotropy monoid $\mathfrak{G}A(1, 1)$.

3.1.6 Abelian Groups and Commutative Monoids

Proposition 3.1.6. *Let T be the theory of abelian groups. Let $A \in \mathbf{Ab} = T\text{-Mod}$. We use additive notation. We find that*

$$\mathfrak{G}A(n, 1) = \{ k_1x_1 + \cdots + k_nx_n \mid k_1, \dots, k_n \in \mathbb{Z} \}.$$

Proof: Let $t \in \mathfrak{G}A(n, 1)$. We can write t as

$$t = a + k_1x_1 + \cdots + k_nx_n, \quad a \in A, \quad k_i \in \mathbb{Z}.$$

This term must commute generically with the neutral element 0, so

$$0 = t[0/x_i] = a.$$

We show that $t = k_1x_1 + \cdots + k_nx_n$ is an inner morphism. It commutes generically with

- the sum: $k_1(y_{11} + y_{21}) + \cdots + k_n(y_{1n} + y_{2n}) = (k_1y_{11} + \cdots + k_ny_{1n}) + (k_1y_{21} + \cdots + k_ny_{2n})$.
- inverses: $k_1(-y_{11}) + \cdots + k_n(-y_{1n}) = -(k_1y_{11} + \cdots + k_ny_{1n})$.
- the neutral element: $k_10 + \cdots + k_n0 = 0$.

This completes the verification. ■

We recover the isotropy monoid $\mathfrak{G}A(1, 1) = \{ kx \mid k \in \mathbb{Z} \}$, which is larger than the isotropy group $\{ x, -x \}$ plus the constant 0. Moreover, the form of elements of $\mathfrak{G}A(n, 1)$ is more general than that of elements of $\mathfrak{G}A(1, 1)$, unlike in the case of groups.

Note that $\mathfrak{G}A$ is actually the Lawvere theory corresponding to the theory of \mathbb{Z} -modules.

If instead we let T be the theory of commutative monoids, since there is no inverse operation, we find that

$$\mathfrak{G}A(n, 1) = \{ k_1x_1 + \cdots + k_nx_n \mid k_1, \dots, k_n \in \mathbb{N} \}.$$

3.1.7 Modules Over a Ring

Proposition 3.1.7. *To generalize the case of abelian groups, let R be a ring, let T be the theory of R -modules, and let $A \in T\text{-Mod}$. Then $\mathfrak{G}A$ is isomorphic to the category of matrices with entries in the center of R . In particular, if R is a field, then $\mathfrak{G}A$ is equivalent to the category of finite-dimensional R -vector spaces, and all the homomorphisms are inner morphisms.*

Proof: Consider $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. There are unique $a \in A$ and $r_1, \dots, r_n \in R$ such that $t = a + r_1x_1 + \cdots + r_nx_n$. Since t commutes generically with zero, we must have $a = 0$. Moreover, t must commute with scalar multiplication by any $r \in R$:

$$r_1(ry_{11}) + \cdots + r_n(ry_{1n}) = r(r_1y_{11} + \cdots + r_ny_{1n}) = rr_1y_{11} + \cdots + rr_ny_{1n}$$

if and only if $r_i r = r r_i$ for $i = 1, \dots, n$. This means that for each i , r_i must be in the center of R :

$$r_i \in CR = \{ s \in R \mid rs = sr \text{ for all } r \in R \}.$$

We show that, given $r_i \in CR$, we have that $t = r_1x_1 + \cdots + r_nx_n$ is an inner morphism. It commutes generically with

- the sum: $r_1(y_{11} + y_{21}) + \cdots + r_n(y_{1n} + y_{2n}) = (r_1y_{11} + \cdots + r_ny_{1n}) + (r_1y_{21} + \cdots + r_ny_{2n})$.
- inverses: $r_1(-y_{11}) + \cdots + r_n(-y_{1n}) = -(r_1y_{11} + \cdots + r_ny_{1n})$.
- zero: $r_10 + \cdots + r_n0 = 0$.
- multiplication by the scalar $r \in R$:

$$r(r_1y_{11} + \cdots + r_ny_{1n}) = rr_1y_{11} + \cdots + rr_ny_{1n} = r_1(ry_{11}) + \cdots + r_n(ry_{1n}).$$

Thus

$$\mathfrak{G}A(n, 1) = \{ r_1x_1 + \cdots + r_nx_n \mid r_1, \dots, r_n \in CR \}.$$

Now, CR is a subring of R , which is equal to R if and only if R is a commutative ring. We have a correspondence

$$\mathfrak{G}A(n, m) \leftrightarrow (CR)^{m \times n}$$

$$\left(\sum_i r_{1i} x_i, \dots, \sum_i r_{mi} x_i \right) \leftrightarrow (r_{ji})$$

So consider the category $\mathbf{Mat}(CR)$ of matrices with coefficients in the center of R . Its objects are the natural numbers, and its hom-sets are given as follows:

$$\mathbf{Mat}(CR)(n, m) = (CR)^{m \times n}.$$

Composition is by matrix multiplication. This is the same composition as in \mathfrak{GA} under the correspondence above. Indeed, given

$$n \xrightarrow{\vec{r}} m \xrightarrow{\vec{s}} l,$$

we have

$$\begin{aligned} \vec{s} \circ \vec{r} &= \left(\sum_j s_{1j} x_j, \dots, \sum_j s_{lj} x_j \right) \circ \left(\sum_i r_{1i} x_i, \dots, \sum_i r_{mi} x_i \right) \\ &= \left(\sum_j s_{1j} \sum_i r_{ji} x_i, \dots, \sum_j s_{lj} \sum_i r_{ji} x_i \right) \\ &= \left(\sum_i \left(\sum_j s_{1j} r_{ji} \right) x_i, \dots, \sum_i \left(\sum_j s_{lj} r_{ji} \right) x_i \right). \end{aligned}$$

Thus we have an isomorphism of categories $\mathfrak{GA} \cong \mathbf{Mat}(CR)$. ■

Note that \mathfrak{GA} is actually the Lawvere theory corresponding to the theory of CR -modules.

3.1.8 Rings and Rigs

Proposition 3.1.8. *Let T be the theory of rings (which we always take to be unital), and let $A \in T\text{-Mod}$. The elements of $\mathfrak{GA}(n, 1)$ take the form $t = \sum_{j=1}^n a_j x_j b_j$, where the coefficients $a_j, b_j \in A$ satisfy a few conditions. First, the term t has to be reduced in the sense that $i_j = i_k$ implies $a_j \neq a_k$ and $b_j \neq b_k$ (this can always be achieved using distributivity). Second, we assume without loss of generality that $a_j \neq 0$ and $b_j \neq 0$ for $j = 1, \dots, n$. Third, we require that $\sum_{j=1}^n a_j b_j = 1$ and*

$$b_j a_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Proof: Let $t \in A\langle x_1, \dots, x_n \rangle$. Then $t \in \mathfrak{GA}(n, 1)$ if and only if it commutes generically with all ring operations. We can write $t = t_1 + \dots + t_m$ where each

summand is a product of elements of A and variables among x_1, \dots, x_n . Since t commutes generically with multiplication, we have

$$\begin{aligned} \sum_{j=1}^m t_j [y_{1i} y_{2i} / x_i] &= \left(\sum_{j=1}^m t_j [y_{1i} / x_i] \right) \left(\sum_{j=1}^m t_j [y_{2i} / x_i] \right) \\ &= \sum_{j=1}^m \sum_{k=1}^m t_j [y_{1i} / x_i] t_k [y_{2i} / x_i]. \end{aligned}$$

In the right-hand side, variables of type y_{1i} always appear before those of type y_{2i} in each summand, but in the left-hand side, this condition is violated as soon as some nonzero t_j contains more than one variable (including powers of a single variable which are greater than one). Since we are working in a free model where multiplication is non-commutative, variables of type y_{1i} and those of type y_{2i} cannot be moved past each other in a summand, hence both sides of the equation can only be equal if each summand contains at most a single variable. Moreover, t commutes generically with zero, so it has no constant summand (an element of A not multiplied by any variables). It follows that

$$t = \sum_{j=1}^m a_j x_{i_j} b_j$$

for some elements $a_j, b_j \in A$, $j = 1, \dots, m$. We can assume without loss of generality that t is reduced in the sense that $i_j = i_k$ implies $a_j \neq a_k$ and $b_j \neq b_k$ (this can always be achieved using distributivity). We can also assume that $a_j \neq 0$ and $b_j \neq 0$ for $j = 1, \dots, m$ (else $a_j x_{i_j} b_j = 0$ and can be removed from the summation entirely). The equation for generic commutation with multiplication then becomes

$$\sum_{j=1}^m a_j y_{1i_j} y_{2i_j} b_j = \sum_{j=1}^m \sum_{k=1}^m a_j y_{1i_j} b_j a_k y_{2i_k} b_k.$$

For these two sums to be equal, whenever $j \neq k$, we must have $b_j a_k = 0$ to nullify the summand $a_j y_{1i_j} b_j a_k y_{2i_k} b_k$ on the right-hand side, since this summand cannot appear on the left-hand side. Moreover, for each $j = 1, \dots, m$, we must have $b_j a_j = 1$, to ensure $a_j y_{1i_j} y_{2i_j} b_j = a_j y_{1i_j} b_j a_j y_{2i_j} b_j$. Clearly, t already commutes generically with addition and additive inverses. It remains to impose generic commutation with unity, which amounts to the requirement that $\sum_{j=1}^m a_j b_j = 1$. \blacksquare

The above also applies to rigs¹, since additive inverses were not used in our reasoning.

¹Rigs are rings without negatives, and are also sometimes called semirings. See [11] for more details.

3.1.9 Commutative Rings and Commutative Rigs

Proposition 3.1.9. *Let T be the theory of commutative rings, let $A \in T\text{-Mod}$. Then an element of $\mathfrak{G}A(n, 1)$ has the form $b_1x_1 + \cdots + b_nx_n$ where $b_1, \dots, b_n \in A$ are orthogonal idempotents summing to 1.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle \vec{x} \rangle$. Note that $A\langle \vec{x} \rangle = A[x_1, \dots, x_n]$ is a ring of polynomials in n variables. To simplify notation, given an n -tuple $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, we shall write $x^I = x_1^{i_1} \cdots x_n^{i_n}$. Thus $t = \sum_{I \in \mathbb{N}^n} a_I x^I$, where only finitely many of the coefficients $a_I \in A$ are nonzero. Since t commutes generically with sums, we have

$$\sum_{I \in \mathbb{N}^n} a_I y_1^I + \sum_{I \in \mathbb{N}^n} a_I y_2^I = \sum_{I \in \mathbb{N}^n} a_I (y_{11} + y_{21})^{i_1} \cdots (y_{1n} + y_{2n})^{i_n}. \quad (3.1.1)$$

Expand out the right-hand side into a polynomial in $2n$ variables, namely $y_{11}, \dots, y_{1n}, y_{21}, \dots, y_{2n}$. Either $t = 0$ or some $a_I \neq 0$ for some I . In the former case, t commutes with unity if and only if $0 = 1$, that is if and only if A is the trivial ring, and otherwise $t = 0$ is not an inner morphism. In the latter case, if $i_1 \geq 1$, once the right-hand side of (3.1.1) is expanded out, the term $a_I y_{11}^{i_1-1} y_{21} y_{12}^{i_2} \cdots y_{1n}^{i_n}$ appears (by n applications of the binomial theorem). If $i_1 > 1$, this term contains both variables of type y_1 and of type y_2 , something which does not occur on the left-hand side of (3.1.1). This contradiction shows that $i_1 = 1$. Moreover, if say $i_2 > 0$, then we run into the same contradiction. Thus $i_2 = 0$ and, by similar arguments, i_3, \dots, i_n are all equal to 0. We can repeat this argument to show that $i_2 \leq 1$, and that $i_2 = 1$ implies all other exponents i_1, i_3, \dots, i_n are zero, and so on. Therefore, provided $t \neq 0$, we have that t must be of degree 1. Hence it has the form $t = b + \sum_{i=1}^n b_i x_i$ with $b, b_1, \dots, b_n \in A$. Given that t commutes generically with zero, we have that $b = 0$, so that $t = \sum_{i=1}^n b_i x_i$. Such a t commutes generically with sums:

$$\sum_{i=1}^n b_i y_{1i} + \sum_{i=1}^n b_i y_{2i} = \sum_{i=1}^n b_i (y_{1i} + y_{2i}).$$

To ensure that t commutes generically with unity, the equation $\sum_{i=1}^n b_i = 1$ must hold. To make t commute generically with the product means that

$$\sum_{i=1}^n b_i y_{1i} y_{2i} = \left(\sum_{i=1}^n b_i y_{1i} \right) \left(\sum_{i=1}^n b_i y_{2i} \right) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j y_{1i} y_{2j}.$$

This equation only holds if the corresponding coefficients are equal, that is if

$$b_i b_j = \begin{cases} b_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This expresses the fact that b_1, \dots, b_n is a system of orthogonal idempotents. This exhausts all ring operations, hence $t \in \mathfrak{G}A(n, 1)$ if and only if $t = \sum_{i=1}^n b_i x_i$ where

b_1, \dots, b_n is an orthogonal system of idempotents summing to 1 (note this also includes the degenerate case $t = 0$ for the trivial ring, since in that ring $0 = 1$). ■

This is interesting, because of the following well-known fact:

Proposition 3.1.10. *Let A be a ring. There is a bijective correspondence between systems of orthogonal idempotents summing to 1 and decompositions of A as a direct sum of left ideals (equivalently, A -submodules).*

Proof: We sketch a proof by explaining how the correspondence works. Given a system b_1, \dots, b_n of orthogonal idempotents summing to 1, we obtain a decomposition of A as

$$A = Ab_1 \oplus \cdots \oplus Ab_n.$$

Conversely, if $A = I_1 \oplus \cdots \oplus I_n$, then there exist unique elements $b_i \in I_i$ such that $1 = b_1 + \cdots + b_n$, and they form a system of orthogonal idempotents.

A more complete proof of this fact can be found, for example, in exercise 1.7 of the first chapter of [6]. ■

This means that elements of $\mathfrak{S}A(n, 1)$ represent decompositions of A seen as an A -module. Also note that in the special case that A is an integral domain, if a certain b_i in a system of orthogonal idempotents summing to 1 is nonzero, then $b_j = 0$ for all $j \neq i$. Hence by the condition of summing to unity, $b_i = 1$, and $t = x_i$. So in the particular case of an integral domain, $\mathfrak{S}A(n, 1) = \{x_1, \dots, x_n\}$.

Inner morphisms for commutative rigs (rings without additive inverses) also correspond to systems of orthogonal idempotents in the same way, since additive inverses did not appear in our reasoning above.

3.1.10 Boolean Rings

A Boolean ring is a ring in which each element is idempotent. This forces it to have characteristic 2 and to be commutative. Indeed, in such a ring, we have

$$a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b,$$

which implies that $0 = ab + ba$ for all elements a and b . Upon setting $b = 1$, we find that $2a = 0$ for all elements a . It then follows that

$$ba = 0 + ba = (ab + ba) + ba = ab + 2(ba) = ab$$

for all elements a and b .

The reasoning in this section will resemble that of the case of commutative rings.

Proposition 3.1.11. *Let T be the algebraic theory of Boolean rings, and let $A \in T\text{-Mod}$. An element of $\mathfrak{G}A(n, 1)$ has the form $b_1x_1 + \cdots + b_nx_n$ where the coefficients $b_i \in A$, $1 \leq i \leq n$, satisfy $\sum_{i=1}^n b_i = 1$, and $b_ib_j = 0$ whenever $i \neq j$.*

Proof: Let $t \in A\langle x_1, \dots, x_n \rangle$. Then, since all the elements of $A\langle \vec{x} \rangle$ are idempotent, and since multiplication is commutative, we can write $t = \sum_I a_I x^I$ where $x^{(i_1, \dots, i_n)} = x_1^{i_1} \cdots x_n^{i_n}$, and I ranges over all of $\{0, 1\}^n$ in the sum.

Since t commutes generically with addition, we have

$$\sum_I a_I y_1^I + \sum_I a_I y_2^I = \sum_I a_I (y_{11} + y_{21})^{i_1} \cdots (y_{1n} + y_{2n})^{i_n}. \quad (3.1.2)$$

Expand out the right-hand side into a polynomial in $2n$ variables, namely $y_{11}, \dots, y_{1n}, y_{21}, \dots, y_{2n}$. Either $t = 0$ or some summand $a_I x^I \neq 0$ for some I (it is not enough to assume $a_I \neq 0$, since Boolean rings have characteristic 2, and hence, for example, $2x^I = 0$). In the former case, t commutes with unity if and only if $0 = 1$, that is if and only if A is the trivial ring, and otherwise $t = 0$ is not an inner morphism. In the latter case, if $i_1 = i_2 = 1$, once the right-hand side of equation 3.1.1 is expanded out, the term $a_I y_{21} y_{12} y_{13}^3 \cdots y_{1n}^{i_n}$ appears (by n applications of the binomial theorem). This term contains both variables of type y_{1i} and of type y_{2i} , something which does not occur on the left-hand side of equation 3.1.1. This contradiction shows that $i_1 = 0$ or $i_2 = 0$. More generally, this argument can be used to show that only one of the exponents i_1, \dots, i_n may be equal to 1; all others must be 0. Moreover, this applies to all I such that $a_I x^I \neq 0$. Therefore, provided $t \neq 0$, it must be of degree 1. Hence it has the form $t = b + \sum_{i=1}^n b_i x_i$ with $b, b_1, \dots, b_n \in A$. Given that t commutes generically with zero, we have that $b = 0$, so that $t = \sum_{i=1}^n b_i x_i$. Such a t commutes generically with addition:

$$\sum_{i=1}^n b_i y_{1i} + \sum_{i=1}^n b_i y_{2i} = \sum_{i=1}^n b_i (y_{1i} + y_{2i}).$$

To ensure that t commutes generically with unity, the equation $\sum_{i=1}^n b_i = 1$ must hold. To make t commute generically with the product means that

$$\sum_{i=1}^n b_i y_{1i} y_{2i} = \left(\sum_{i=1}^n b_i y_{1i} \right) \left(\sum_{i=1}^n b_i y_{2i} \right) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j y_{1i} y_{2j}.$$

This equation only holds if the corresponding coefficients are equal, that is if

$$b_i b_j = \begin{cases} b_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Finally, note that the condition $b_i^2 = b_i$ is redundant, since all elements of a Boolean ring are already idempotent. Thus, the orthogonality of the coefficients b_i is what is

really important. ■

Note that the category of Boolean rings is isomorphic to the category of Boolean algebras, so this result also allows us to determine the inner theory of the category of Boolean algebras, which is order-theoretic in nature. We will work out the details of this remark, as well as other interesting cases, in the next sections.

3.1.11 Join Semi-Lattices

A join semi-lattice is a set A together with an associative, commutative, and idempotent binary operation \vee , called join, and a nullary operation \perp , called bottom, which is a neutral element for the join operation.

Proposition 3.1.12. *Let T be the theory of join semi-lattices, and let $A \in T\text{-Mod}$. Then, inner morphisms in $\mathfrak{G}A(n, 1)$ are exactly joins of variables among x_1, \dots, x_n .*

Proof: Let $t \in A\langle x_1, \dots, x_n \rangle$. Then $t \in \mathfrak{G}A(n, 1)$ if and only if it commutes generically with bottom and join. We can write $t = a \vee \bigvee_{j \in J} x_j$ where $a \in A$ and $J \subseteq \{1, \dots, n\}$, using commutativity and idempotence. It commutes generically with bottom if and only if

$$\perp = a \vee \bigvee_{j \in J} \perp = a.$$

Thus we need to have $t = \bigvee_{j \in J} x_j$. Using commutativity, we check that t already commutes with join:

$$\left(\bigvee_{j \in J} y_{1j} \right) \vee \left(\bigvee_{j \in J} y_{2j} \right) = \bigvee_{j \in J} y_{1j} \vee y_{2j}.$$

It follows that inner morphisms in $\mathfrak{G}A(n, 1)$ are exactly joins of variables among x_1, \dots, x_n . ■

Note that $\mathfrak{G}A$ is actually the Lawvere theory corresponding to the theory of join semi-lattices.

Also note that meet semi-lattices are the same as join semi-lattices, from an algebraic point of view.

3.1.12 Suplattices

A suplattice is a set A together with arbitrary joins as operations (these joins are still associative, commutative, and idempotent, and the join of no elements at all is the bottom). Thus suplattices have infinitary operations, and their theory is not

algebraic. However, given a set S , we can build the free suplattice on S as the power set $\mathcal{P}S$, together with unions for joins. Since we have a free-forgetful adjunction for the category **SupLat**, we can still prove our main theorem for the almost algebraic theory of suplattices.

Proposition 3.1.13. *Let T be the theory of suplattices, and let $A \in T\text{-Mod}$. Then,*

$$\mathfrak{G}A(n, 1) = \left\{ \bigvee_{j \in J} x_j \mid J \subseteq \{1, \dots, n\} \right\}.$$

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. Reasoning as in the case of join semi-lattices and forcing generic commutation with the bottom, we can write $t = \bigvee_{j \in J} x_j$ where $J \subseteq \{1, \dots, n\}$. We can then check that t commutes generically with arbitrary joins. Let I be a set; we have that:

$$\bigvee_{i \in I} \bigvee_{j \in J} y_{ij} = \bigvee_{j \in J} \bigvee_{i \in I} y_{ij},$$

as desired. ■

Note that a suplattice also has arbitrary meets, but they are not considered part of the structure, so need not be preserved by suplattice homomorphisms. This is important, as the category of complete lattices, i.e. the category whose objects are lattices with arbitrary joins and arbitrary meets, and whose morphisms are functions preserving arbitrary joins and arbitrary meets, does not admit a free-forgetful adjunction to **Set**. Indeed, if $|S| > 2$, then the free complete lattice on S is a proper class (see [3]).

Also note that inflattices are the same as suplattices, from an algebraic point of view.

3.1.13 Distributive Lattices

A (bounded) lattice is a set A equipped with binary operations of meet \wedge and join \vee , as well as nullary operations top \top and bottom \perp . We require that meet and join are associative, commutative, and idempotent, and that they satisfy the absorption laws:

$$a \vee (a \wedge b) = a \quad \text{and} \quad a \wedge (a \vee b) = a.$$

Furthermore, we require that \top and \perp are neutral elements for meet and join, respectively. The lattice A is distributive if meet distributes over join (and vice-versa):

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

(Note that only one of the two distributivity laws must be assumed; the other follows as a consequence.)

Proposition 3.1.14. *Let T be the theory of distributive lattices, and let $A \in T\text{-Mod}$. Inner morphisms in $\mathfrak{G}A(n, 1)$ are of the form $(b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \cdots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. Then, we may write $t = \bigvee_I a_I \wedge \bigwedge_{i \in I} x_i$, where I ranges over all subsets of $\{1, \dots, n\}$, and $a_I \in A$ for each I .

The term t commutes generically with joins if and only if

$$\left(\bigvee_I a_I \wedge \bigwedge_{i \in I} y_{1i} \right) \vee \left(\bigvee_I a_I \wedge \bigwedge_{i \in I} y_{2i} \right) = \bigvee_I a_I \wedge \bigwedge_{i \in I} y_{1i} \vee y_{2i}.$$

Writing $I = \{i_1, \dots, i_{|I|}\}$ and $\mathbf{k} = (k_1, \dots, k_{|I|})$, we calculate

$$\begin{aligned} \bigvee_I a_I \wedge \bigwedge_{i \in I} y_{1i} \vee y_{2i} &= \bigvee_I a_I \wedge (y_{1i_1} \vee y_{2i_1}) \wedge \cdots \wedge (y_{1i_{|I|}} \vee y_{2i_{|I|}}) \\ &= \bigvee_I a_I \wedge \bigvee_{\mathbf{k} \in \{1,2\}^{|I|}} y_{k_1 i_1} \wedge \cdots \wedge y_{k_{|I|} i_{|I|}}. \end{aligned}$$

Thus, applying distributivity once more, we arrive at the equation

$$\left(\bigvee_I a_I \wedge \bigwedge_{i \in I} y_{1i} \right) \vee \left(\bigvee_I a_I \wedge \bigwedge_{i \in I} y_{2i} \right) = \bigvee_I \bigvee_{\mathbf{k} \in \{1,2\}^{|I|}} a_I \wedge y_{k_1 i_1} \wedge \cdots \wedge y_{k_{|I|} i_{|I|}}.$$

All the disjuncts on the left-hand side also appear on the right-hand side. To have equality, the remaining disjuncts on the right-hand side must be absorbed. For example, suppose for some I we have $a_I \neq \perp$ and $|I| \geq 2$. Then the disjunct $a_I \wedge y_{1i_1} \wedge y_{2i_2} \wedge \cdots \wedge y_{2i_{|I|}}$ appears on the right-hand side but not on the left-hand side. For it to be absorbed by a disjunct appearing on both sides, there must exist $J \subset I$ such that $\{k_p \mid i_p \in J\} = \{1\}$ or $\{k_p \mid i_p \in J\} = \{2\}$, and $a_I = a_J \wedge b$ for some $b \in A$. Then one of the terms

$$a_J \wedge \bigwedge_{j \in J} y_{1j}, \quad a_J \wedge \bigwedge_{j \in J} y_{2j},$$

which appear on both sides of the equation, will absorb the disjunct $a_I \wedge y_{1i_1} \wedge y_{2i_2} \wedge \cdots \wedge y_{2i_{|I|}}$. But then notice that $a_I \wedge \bigwedge_{i \in I} x_i$ is already absorbed by $a_J \wedge \bigwedge_{j \in J} x_j$ in the expression of t , and $|J| < |I|$. Since this reasoning applies whenever $|I| > 1$ and $a_I \neq \perp$, we can simplify t to the form

$$t = b_0 \vee (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n),$$

where $b_0, b_1, \dots, b_n \in A$.

The term t commutes generically with \perp if and only if

$$\perp = b_0 \vee (b_1 \wedge \perp) \vee \cdots \vee (b_n \wedge \perp) = b_0.$$

Hence $t = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$.

The term t commutes generically with \top if and only if

$$\top = (b_1 \wedge \top) \vee \cdots \vee (b_n \wedge \top) = b_1 \vee \cdots \vee b_n.$$

Finally, t commutes generically with meet if and only if

$$\begin{aligned} & (b_1 \wedge y_{11} \wedge y_{21}) \vee \cdots \vee (b_n \wedge y_{1n} \wedge y_{2n}) \\ &= \left((b_1 \wedge y_{11}) \vee \cdots \vee (b_n \wedge y_{1n}) \right) \wedge \left((b_1 \wedge y_{21}) \vee \cdots \vee (b_n \wedge y_{2n}) \right) \\ &= \bigvee_{i=1}^n \bigvee_{j=1}^n b_i \wedge b_j \wedge y_{1i} \wedge y_{2j}. \end{aligned}$$

To have equality, whenever $i \neq j$, the disjunct $b_i \wedge b_j \wedge y_{1i} \wedge y_{2j}$ on the right-hand side must be absorbed by a disjunct of the form $b_k \wedge y_{1k} \wedge y_{2k}$, which is only possible if $b_i \wedge b_j = \perp$. \blacksquare

Notice the similarities between this example and that of commutative rings. When we have two associative and commutative binary operations, one of which distributes over the other, we can expect similar results.

3.1.14 Frames

A frame is a distributive lattice, but with arbitrary joins as operations. Accordingly, we require infinite distributive laws:

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i.$$

Note that the theory of frames is equationally defined but not strictly algebraic, since it involves infinitary operations. However, we have a free-forgetful adjunction from **Frm** to **Set**. Indeed, let S be a set, and let M be the free meet semi-lattice on S . Recall that B becomes a poset by defining $a \leq b$ if and only if $a \wedge b = a$. A subset $B \subseteq M$ is called a lower set if $b \in B$ and $a \leq b$ imply $a \in B$, for all $a, b \in M$. Then the set L of all lower sets of M , equipped with unions as joins and intersection as meet, is the free frame on S .

Proposition 3.1.15. *Let T be the theory of frames, and let $A \in T\text{-Mod} = \mathbf{Frm}$. Inner morphisms in $\mathfrak{G}A(n, 1)$ are of the form $(b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \cdots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. Then, reasoning as in the case of distributive lattices, we may write $t = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters

$b_1, \dots, b_n \in A$ satisfy $b_1 \vee \dots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$. Such a term has been verified to commute generically with all frame operations except infinitary joins. Let us check that t commutes with arbitrary joins:

$$\bigvee_{j \in J} \bigvee_{i=1}^n b_i \wedge y_{ji} = \bigvee_{i=1}^n \bigvee_{j \in J} b_i \wedge y_{ji} = \bigvee_{i=1}^n b_i \wedge \bigvee_{j \in J} y_{ji}.$$

Therefore, the inner morphisms for frames are the same as those for distributive lattices. \blacksquare

3.1.15 Boolean Algebras

A Boolean algebra is a complemented distributive lattice. Accordingly, it has all the same operations and equations as distributive lattices, as well as a unary complement \neg , satisfying the equations

$$a \wedge \neg a = \perp, \quad a \vee \neg a = \top.$$

Note that complements are automatically preserved by lattice homomorphisms, so that the category of Boolean algebras is a full subcategory of both **Dist** and **Lat**.

A Boolean algebra can be considered as a Boolean ring, and inversely. These constructions induce an isomorphism (not just an equivalence) between the category of Boolean algebras and the category of Boolean rings, for which we have already calculated the inner theory.

Proposition 3.1.16. *Let T be the theory of Boolean algebras, and let $A \in T\text{-Mod}$. Inner morphisms in $\mathfrak{G}A(n, 1)$ are of the form $(b_1 \wedge x_1) \vee \dots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \dots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$.*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. If we consider A as a Boolean ring, we find that t must be of the form $t = b_1 x_1 + \dots + b_n x_n$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 + \dots + b_n = 1$, and $b_i b_j = 0$ when $i \neq j$. Translating back into the language of Boolean algebra, products are meets and sums are exclusive disjunctions, hence $ab = a \wedge b$ and $a + b + ab = a \vee b$. We have that $b_i \wedge b_j = b_i b_j = 0 = \perp$ when $i \neq j$, so we can show that

$$b_1 \vee \dots \vee b_n = b_1 + \dots + b_n = 1 = \top.$$

Moreover, we also find that $(b_i \wedge x_i) \wedge (b_j \wedge x_j) = \perp$ when $i \neq j$, so the same reasoning as above reveals that

$$t = b_1 x_1 + \dots + b_n x_n = (b_1 \wedge x_1) \vee \dots \vee (b_n \wedge x_n).$$

We happen upon the same inner morphisms as in the cases of distributive lattices and frames. ■

3.1.16 Quantales

A (unital) quantale² is a suplattice with an associative unital multiplication which distributes over joins:

$$x * \bigvee_{i \in I} y_i = \bigvee_{i \in I} x * y_i, \quad \text{and} \quad \left(\bigvee_{i \in I} y_i \right) * x = \bigvee_{i \in I} y_i * x.$$

In particular, for $I = \emptyset$, we find that $x * \perp = \perp = \perp * x$.

The theory of quantales has free models despite its infinitary join operations. Indeed, given a set X , let M be the free monoid on X ; then $\mathcal{P}M$ is the free quantale on X , with unions as joins, and multiplication defined by

$$A * B = \{ ab \mid a \in A \text{ and } b \in B \}.$$

Proposition 3.1.17. *Let T be the theory of quantales, and let A be a quantale. Then an element of $\mathfrak{G}A(n, 1)$ has the form $\bigvee_{j=1}^m a_j * x_{i_j} * b_j$, where $m \in \mathbb{N}$, the integers i_j are taken among $1, \dots, n$, and the coefficients $a_j, b_j \in A$ satisfy the following conditions. First, the term t has to be reduced in the sense that $i_j = i_k$ implies $a_j \neq a_k$ and $b_j \neq b_k$ (this can always be achieved using distributivity). Second, we assume without loss of generality that $a_j \neq \perp$ and $b_j \neq \perp$ for $j = 1, \dots, m$. Third, we require that $\bigvee_{j=1}^m a_j * b_j = e$ and*

$$b_j * a_k = \begin{cases} e & \text{if } j = k \\ \perp & \text{if } j \neq k \end{cases}$$

where e is the unit of the multiplication.

Proof: Let $t \in \mathfrak{G}A(n, 1)$. Using distributivity, we may write $t = \bigvee_{j \in J} t_j$ where each t_j is a product of variables and parameters in A . We have that t commutes generically with products, hence

$$\begin{aligned} \bigvee_{j \in J} t_j[y_{1i} * y_{2i}/x_i] &= \left(\bigvee_{j \in J} t_j[y_{1i}/x_i] \right) * \left(\bigvee_{j \in J} t_j[y_{2i}/x_i] \right) \\ &= \bigvee_{j \in J} \bigvee_{k \in J} t_j[y_{1i}/x_i] * t_k[y_{2i}/x_i]. \end{aligned}$$

In the products $t_j[y_{1i}/x_i] * t_k[y_{2i}/x_i]$ of the right-hand side, all variables of type y_{1i} occur to the left of those of type y_{2i} . However, if any t_j contains more than a single

²The reader may consult [14] for more information on quantales.

variable, this is not the case in $t_j[y_{1i} * y_{2i}/x_i]$ on the left-hand side. Thus each t_j contains at most a single variable (in fact, exactly one variable, since t commutes generically with \perp), and we can write

$$t = \bigvee_{j=1}^m a_j * x_{i_j} * b_j.$$

Reasoning exactly as in the case of rigs, we find that the coefficients $a_j, b_j \in A$ satisfy the following conditions. First, the term t has to be reduced in the sense that $i_j = i_k$ implies $a_j \neq a_k$ and $b_j \neq b_k$ (this can always be achieved using distributivity). Second, we assume without loss of generality that $a_j \neq \perp$ and $b_j \neq \perp$ for $j = 1, \dots, m$. Third, we require that $\bigvee_{j=1}^m a_j * b_j = e$ and

$$b_j * a_k = \begin{cases} e & \text{if } j = k \\ \perp & \text{if } j \neq k \end{cases}$$

where e is the unit of the multiplication. ■

3.1.17 Commutative Quantales

Proposition 3.1.18. *Let T be the theory of commutative quantales (unital quantales with a commutative multiplication), and let A be a commutative quantale. Elements of $\mathfrak{G}A(n, 1)$ have the form $(b_1 * x_1) \vee \dots \vee (b_n * x_n)$ for some $b_1, \dots, b_n \in A$ such that*

$$b_i * b_j = \begin{cases} b_i & \text{if } i = j \\ \perp & \text{if } i \neq j \end{cases}$$

and $b_1 \vee \dots \vee b_n = e$.

Proof: Let $t \in \mathfrak{G}A(n, 1)$. We may write $t = \bigvee_{j \in J} t_j$ where each t_j is a product of elements of A and variables.

If $t = \perp$, then $e = \perp$, since t commutes generically with the multiplicative unit e . Then, for all $a \in A$, we have that $a = a * e = a * \perp = \perp$, and A is the trivial commutative quantale.

For the rest, we assume A is non-trivial. We know that t commutes with binary joins and the bottom element. These correspond to the addition and zero of commutative rigs, so, reasoning as we did for commutative rigs, we find that $t = (b_1 * x_1) \vee \dots \vee (b_n * x_n)$ for some $b_1, \dots, b_n \in A$ such that

$$b_i * b_j = \begin{cases} b_i & \text{if } i = j \\ \perp & \text{if } i \neq j \end{cases}$$

Moreover we must have that

$$b_1 \vee \cdots \vee b_n = e$$

since t commutes generically with e . ■

3.2 Contravariant Isotropy

The inner theory functor is covariant and admits a nice characterization for more algebraic categories, but a contravariant functor would be more appropriate for geometric categories, since it is a well-known principle that algebra and geometry are dual to each other. Thus, for a category \mathcal{C} of geometric objects and their morphisms, which we require to have all finite coproducts, we let the contravariant inner theory of \mathcal{C} be just the covariant inner theory $\mathfrak{Z} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Law}$.

Examples of interest include the category of affine schemes, which is the opposite of the category of commutative rings, and the category of locales, which is opposite to the category of frames.

3.2.1 Affine Schemes

The category \mathbf{Aff} of affine schemes is the opposite of the category of commutative rings. Therefore the contravariant inner theory of affine schemes is the covariant inner theory of commutative rings, which has already been calculated.

Let A be a commutative ring. Then an inner morphism in $\mathfrak{GA}(n, 1)$ corresponds to a family b_1, \dots, b_n of orthogonal idempotents summing to 1. Let us think of this geometrically. We can roughly think of A as a ring of functions on the topological space $\text{Spec } A$ of its prime ideals, endowed with the Zariski topology. Then a point $a \in A$ corresponds to the open subset $D_a = \{P \in \text{Spec } A \mid a \notin P\}$, and all open subsets of this form taken together form a basis of the Zariski topology. The fact that $b_i b_j = 0$ when $i \neq j$ implies that $D_{b_i} \cap D_{b_j} = \emptyset$, since a prime ideal contains $0 = b_i b_j$, and hence also contains b_i or b_j . The fact that $b_1 + \cdots + b_n = 1$ corresponds to the fact that D_{b_1}, \dots, D_{b_n} form an open cover of $\text{Spec } A$ (a prime ideal P must be different from A , hence cannot contain $1 = b_1 + \cdots + b_n$, hence cannot contain all the b_i , hence must be in one of the D_{b_i}). Therefore, D_{b_1}, \dots, D_{b_n} form a partition of $\text{Spec } A$ into mutually disjoint open components (which we may call a separation of the space $\text{Spec } A$), and each D_{b_i} must be clopen (its complement being a union of open subsets). Inversely, since clopen subsets of $\text{Spec } A$ correspond exactly to idempotents of A , all partitions of $\text{Spec } A$ into mutually disjoint open components correspond bijectively to families of orthogonal idempotents summing to 1. We conclude that an element of $\mathfrak{GA}(n, 1)$, in the geometrical setting, is a separation of $\text{Spec } A$ into n open subsets.

See [15, Tag 04PN] for more information on the correspondence between idempotents and clopen subsets.

3.2.2 Stone Duality

Let $\mathbf{Loc} = \mathbf{Frm}^{\text{op}}$ be the category of locales. By definition, of course, this category is the opposite of the category of frames. Let us give a geometric interpretation of its contravariant inner theory. We take $A \in \mathbf{Loc}$. We can think of elements of A as open subspaces. Then, an element of $\mathfrak{G}A(n, 1)$ is of the form $t = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \cdots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$. Translating to the geometrical context, t corresponds to an n -tuple of opens b_1, \dots, b_n , which cover the whole locale A , and which are mutually disjoint. One might call this a partition of A into n opens.

Next, we will look at distributive lattices. But first, we need to define coherent spaces.

Let R be a commutative ring. Its spectrum $\text{Spec } R$ is the set of its prime ideals, equipped with the Zariski topology. Open subsets are of the form

$$\{ P \in \text{Spec } R \mid S \not\subseteq P \}$$

for some $S \subseteq R$. Thus the Zariski topology has a basis of open sets

$$D_a = \{ P \in \text{Spec } R \mid a \notin P \}$$

for $a \in R$.

A coherent space is a topological space which is homeomorphic to the spectrum of a commutative ring [10]. Equivalently, it is a compact sober topological space whose collection of compact open subsets is closed under finite intersections and forms a topological base. Morphisms of coherent spaces are continuous maps such that preimages of compact open subsets are compact.

Let \mathbf{CohSp} be the category of coherent spaces and coherent maps. (Note that we may define coherent locales, and the category of coherent locales is equivalent to that of coherent spaces.) We have that $\mathbf{CohSp}^{\text{op}} \simeq \mathbf{Dist}$, and this contravariant equivalence sends a coherent space A to the (bounded) distributive lattice of its compact open subsets. Then, an element of $\mathfrak{G}A(n, 1)$ is of the form $t = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \cdots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$. Translating to the geometrical context, t corresponds to an n -tuple of compact open subsets b_1, \dots, b_n which cover the whole space A , and which are mutually disjoint.

Finally, let us discuss the classical Stone duality between Boolean algebras and Stone spaces, i.e. compact Hausdorff totally disconnected topological spaces. Let \mathbf{Stone} be the category of Stone spaces, and let \mathbf{Bool} be the category of Boolean algebras. We have that $\mathbf{Stone}^{\text{op}} \simeq \mathbf{Bool}$, and this contravariant equivalence works

by sending a Stone space to its Boolean algebra of clopen subsets. Let $A \in \mathbf{Stone}$. Passing to the corresponding Boolean algebra, we see that an element of $\mathfrak{G}A(n, 1)$ is of the form $t = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n)$, where the parameters $b_1, \dots, b_n \in A$ satisfy $b_1 \vee \cdots \vee b_n = \top$, and $b_i \wedge b_j = \perp$ when $i \neq j$. Translating back to the geometrical context, t corresponds to an n -tuple of clopen subsets b_1, \dots, b_n which cover the whole space A , and which are mutually disjoint. Furthermore, note that actually the open sets b_1, \dots, b_n are automatically closed, since they are mutually disjoint, hence $A \setminus b_i = \bigcup_{j \neq i} b_j$ is open for $i = 1, \dots, n$. Therefore, inner morphisms actually correspond to n -tuples of open subsets b_1, \dots, b_n which cover the whole space A , and which are mutually disjoint.

Broadly speaking, it seems that the inner theory specifies finite partitions of a space into open (and possibly compact) subsets.

3.3 Quasi-Equational Theories

3.3.1 Inverse Semigroups

An inverse semigroup is a set A together with a product $A \times A \rightarrow A$ and an inverse operation $(-)^{-1} : A \rightarrow A$, which satisfy the following axioms:

- $a(bc) = (ab)c$
- $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- $a^2 = a, b^2 = b \vdash ab = ba$.

Thus the theory of semigroups is single-sorted, but not strictly algebraic, as the axiom expressing that idempotents commute is a sequent with nonempty antecedent.

Suppose A is an inverse semigroup and $a \in A$. Then aa^{-1} and $a^{-1}a$ are idempotents. This can be readily checked:

$$(aa^{-1})^2 = (aa^{-1}a)a^{-1} = aa^{-1}$$

and

$$(a^{-1}a)^2 = (a^{-1}aa^{-1})a = a^{-1}a.$$

If, in addition, a is idempotent, then $a^{-1} = a$. Indeed,

$$\begin{aligned} a^{-1} &= a^{-1}aa^{-1} = a^{-1}a^2a^{-1} = (a^{-1}a)(aa^{-1}) \\ &= (aa^{-1})(a^{-1}a) = a^2a^{-1}a^{-1}a^2 = a(aa^{-1})(a^{-1}a)a \\ &= a(a^{-1}a)(aa^{-1})a = (aa^{-1}a)(aa^{-1}a) = aa = a. \end{aligned}$$

(More generally, this line of reasoning can be adapted to show that inverses are unique in an inverse semigroup.)

Proposition 3.3.1. *Let T be the single-sorted quasi-equational theory of inverse semigroups, and let $A \in T\text{-Mod}$. Inner morphisms in $\mathfrak{G}A(n, 1)$ are just the idempotents of A and the projections x_i .*

Proof: Let $t \in \mathfrak{G}A(n, 1) \subseteq A\langle x_1, \dots, x_n \rangle$. Further suppose that t is written as a product of elements of A , variables, and inverses of variables, in a form which is suitably reduced (multiply together adjacent elements of A , and use the axioms to reduce the length of the word as much as possible). Then t must commute generically with the product:

$$t[y_{1i}y_{2i}/x_i] = t[y_{1i}/x_i]t[y_{2i}/x_i].$$

On the right-hand side, all variables of type y_{1i} are to the left of all variables of type y_{2i} . However, this does not occur on the left if t contains an inverse x_i^{-1} of a variable, or a power of a variable which is higher than one. Therefore, $t = a \in A$, or $t = ax_i b$ where $a, b \in A$, or $t = x_i$ (this is obviously an inner morphism). In the first case, forcing generic commutation with the product and the inverse yields the equations $a^2 = a$ and $a^{-1} = a$, which mean precisely that a must be idempotent. In the second case, the equation for generic commutation with the product becomes $ay_{1i}y_{2i}b = ay_{1i}bay_{2i}b$, which is plainly impossible. Hence the idempotents of A and the projections x_i exhaust all the inner morphisms, much as in the case of semigroups. ■

3.3.2 Categories

Let T be the quasi-equational theory of categories. It has a sort O of objects and a sort A of arrows, along with operations as follows:

$$1 : O \rightarrow A, \quad \text{src} : A \rightarrow O, \quad \text{tar} : A \rightarrow O, \quad \circ : A \times A \rightarrow A.$$

These operations are called, respectively, identity, source, target, and composition. The three first operations are total:

$$\vdash 1_C \downarrow, \quad \vdash (\text{src } f) \downarrow, \quad \vdash (\text{tar } f) \downarrow,$$

while composition is partial. Furthermore, they are subject to the following axioms:

$$\begin{aligned} \text{src } f = \text{tar } g \vdash (f \circ g) \downarrow, \quad (f \circ g) \vdash \text{src } f = \text{tar } g, \\ (f \circ g) \downarrow \vdash \text{src}(f \circ g) = \text{src } g, \quad (f \circ g) \downarrow \vdash \text{tar}(f \circ g) = \text{tar } f, \\ \text{src } f = \text{tar } g \wedge \text{src } g = \text{tar } h \vdash (f \circ g) \circ h = f \circ (g \circ h), \\ \vdash \text{src } 1_C = C, \quad \vdash \text{tar } 1_C = C, \\ \vdash f \circ 1_{\text{src } f} = f, \quad \vdash 1_{\text{tar } f} \circ f = f. \end{aligned}$$

Then of course $T\text{-Mod} = \mathbf{Cat}$ is the category of categories and functors between them.

Proposition 3.3.2. *Let $\mathcal{A} \in \mathbf{Cat}$. An element of $\mathfrak{GA}(n, 1)$ either has the form $(C, 1_C)$, where $C \in \mathcal{A}$, or has the form (x_i^O, x_i^A) , for $i = 1, \dots, n$.*

Proof: Let

$$t = (t_O, t_A) \in \mathfrak{GA}(n, 1) \subseteq \mathcal{A}\langle x_1^O, \dots, x_n^O \rangle_O \times \mathcal{A}\langle x_1^A, \dots, x_n^A \rangle_A.$$

Since in both categories $\mathcal{A}\langle x_1^O, \dots, x_n^O \rangle$ and $\mathcal{A}\langle x_1^A, \dots, x_n^A \rangle$ the new objects or arrows are added as discretely as possible, we only have a few options for t_O and t_A . Either $t_O = C \in \mathcal{A}$ or $t_O = x_i^O$. If $t_O = C$, then the fact that t commutes generically with identity arrows amounts to the equation

$$1_C = 1_{t_O[y_{1i}/x_i^O]} = t_A[1_{y_{1i}/x_i^A}],$$

which implies $t_A = 1_C$. One can then verify that $t = (C, 1_C)$ commutes generically with sources, targets, and composition. Thus it is an inner morphism, and the inner theory of categories is not quite trivial like the isotropy group of categories. The relevant difference is that inner morphisms in the inner theory are not required to be invertible.

In what follows, we assume $t_O = x_i^O$. The possibilities for t_A are: $t_A = f$, where f is an arrow in \mathcal{A} , $t_A = x_j^A$, $t_A = 1_{\text{src } x_j^A}$, or $t_A = 1_{\text{tar } x_j^A}$. Of course, the projections (x_i^O, x_i^A) are in \mathfrak{GA} , since it is a Lawvere theory. Let us show $t = (x_i^O, x_i^A)$ is in fact the only possibility. We know that t commutes generically with sources:

$$\text{src } t_A[y_{1i}/x_i^A] = t_O[\text{src } y_{1i}/x_i^O] = \text{src } y_{1i}.$$

This rules out the options $t_A = f$ and $t_A = 1_{\text{tar } x_j^A}$ (for $j = 1, \dots, n$). It also eliminates the possibility that $t_A = x_j^A$ for some $j \neq i$. Similarly, t commutes generically with targets:

$$\text{tar } t_A[y_{1i}/x_i^A] = t_O[\text{tar } y_{1i}/x_i^O] = \text{tar } y_{1i}.$$

This eliminates the option $t_A = 1_{\text{src } x_j^A}$. ■

3.3.3 Groupoids

Let T be the theory of groupoids. This is a refinement of the theory of categories, with an extra inverse operation $(-)^{-1} : A \rightarrow A$, and the following axioms:

$$\begin{aligned} \vdash f^{-1} \downarrow, \quad \vdash \text{src } f^{-1} = \text{tar } f, \quad \vdash \text{tar } f^{-1} = \text{src } f, \\ \vdash f \circ f^{-1} = 1_{\text{tar } f}, \quad \vdash f^{-1} \circ f = 1_{\text{src } f}. \end{aligned}$$

Proposition 3.3.3. *Let $\mathcal{A} \in T\text{-Mod}$ be a groupoid. As in the case of categories, an element of $\mathfrak{GA}(n, 1)$ either has the form $(C, 1_C)$, where $C \in \mathcal{A}$, or has the form (x_i^O, x_i^A) , for $i = 1, \dots, n$.*

Proof: Let

$$t = (t_O, t_A) \in \mathfrak{GA}(n, 1) \subseteq \mathcal{A}\langle x_1^O, \dots, x_n^O \rangle_O \times \mathcal{A}\langle x_1^A, \dots, x_n^A \rangle_A.$$

Reasoning as in the case of categories, we find that either $t = (C, 1_C)$ for some $C \in \mathcal{A}$, in which case we can verify that t also commutes generically with inverses, or that $t_O = x_i^O$ for some $i = 1, \dots, n$. Like before, we find that $t_A = x_i^A$ yields an inner morphism and we eliminate all other alternatives built solely from sources, targets, identities and composition. However, the inverse operation affords us another possibility: $t_A = (x_j^A)^{-1}$ for some $j = 1, \dots, n$. Suppose $(x_i^O, (x_j^A)^{-1})$ commutes generically with sources:

$$\text{tar } y_{1j} = \text{src } (y_{1j})^{-1} = \text{src } t_A[y_{1i}/x_i^A] = t_O[\text{src } y_{1i}/x_i^O] = \text{src } y_{1i}.$$

This is plainly impossible. ■

3.3.4 Categories with a Terminal Object

Let T be the theory of categories with a chosen terminal object. This is a refinement of the theory of categories, with an extra constant $1 : O$, and an extra unary operation $! : O \rightarrow A$, subject to the following additional axioms:

$$\begin{aligned} \vdash 1 \downarrow, \quad \vdash !_C \downarrow, \quad \vdash \text{src } !_C = C, \\ \vdash \text{tar } !_C = 1, \quad \text{src } f = C \wedge \text{tar } f = 1 \vdash f = !_C. \end{aligned}$$

Proposition 3.3.4. *Let \mathcal{A} be a category with a terminal object. An element of $\mathfrak{GA}(n, 1)$ is either $(1, 1_1)$, or it is a projection (x_i^O, x_i^A) .*

Proof: Let

$$t = (t_O, t_A) \in \mathfrak{GA}(n, 1) \subseteq \mathcal{A}\langle x_1^O, \dots, x_n^O \rangle_O \times \mathcal{A}\langle x_1^A, \dots, x_n^A \rangle_A.$$

Either $t_O = C \in \mathcal{A}$ or $t_O = x_i^O$. In the first case, $t_A = 1_C$ since t commutes generically with identity arrows, but also $t_A = !_C$ because t commutes generically with $!$. To make targets match up, we must have $C = 1$, hence $t = (1, 1_1)$. It is easily checked that this is indeed an inner morphism.

From now on, we assume $t_O = x_i^O$. Then obviously (x_i^O, x_i^A) is an inner morphism, since it is a projection in \mathfrak{GA} . As for categories, we eliminate the $t_A = f$ for an arrow

f of \mathcal{A} , $t_A = x_j^A$ for some $j \neq i$, $t_A = 1_{\text{src } x_j^A}$, and $t_A = 1_{\text{tar } x_j^A}$ (where possibly $j = i$ in the last two options). However, the new operations afford us two new options to consider : $t_A = f \circ !_{\text{src } x_j^A}$ and $t_A = f \circ !_{\text{tar } x_j^A}$. In both cases, since t commutes generically with targets, we find that

$$\text{tar } y_{1i} = t_O[\text{tar } y_{1i}/x_i^O] = \text{tar } t_A[y_{1i}/x_i^A] = \text{tar } f.$$

This is of course a contradiction.

We are left with two possibilities for t . Either $t = (1, 1_1)$, or $t = (x_i^O, x_i^A)$ for $i = 1, \dots, n$. ■

3.3.5 Strict Monoidal Categories

The theory T of strict monoidal categories extends the theory of categories. It adds operations $I : O$, $\otimes_O : O \times O \rightarrow O$, and $\otimes_A : A \times A \rightarrow A$, as well as the following axioms:

- $\vdash I \downarrow$
- $\vdash (A \otimes_O B) \downarrow$
- $\vdash (f \otimes_A g) \downarrow$
- $\vdash \text{src}(f \otimes_A g) = \text{src } f \otimes_O \text{src } g$
- $\vdash \text{tar}(f \otimes_A g) = \text{tar } f \otimes_O \text{tar } g$
- $\vdash 1_{A \otimes_O B} = 1_A \otimes_A 1_B$
- $(f' \circ f) \downarrow \wedge (g' \circ g) \downarrow \vdash (f' \circ f) \otimes_A (g' \circ g) = (f' \otimes_A g') \circ (f \otimes_A g)$
- $\vdash A \otimes_O (B \otimes_O C) = (A \otimes_O B) \otimes_O C$
- $\vdash f \otimes_A (g \otimes_A h) = (f \otimes_A g) \otimes_A h$
- $\vdash A \otimes_O I = A = I \otimes_A A$
- $\vdash f \otimes_A 1_I = f = 1_I \otimes_A f$

In what follows, we will write \otimes for both \otimes_O and \otimes_A , as long as the meaning is clear from context. The axioms guarantee that \otimes is an associative bifunctor with a unit.

Proposition 3.3.5. *Let \mathcal{A} be a strict monoidal category. An element of $\mathfrak{GA}(n, 1)$ is either $(I, 1_I)$, or it is of the form $(A \otimes x_i^O \otimes B, 1_A \otimes x_i^A \otimes 1_B)$ where A and B are inverses of each other in the so-called Picard group of \mathcal{A} .*

Proof: Let

$$t = (t_O, t_A) \in \mathfrak{GA}(n, 1) \subseteq \mathcal{A}\langle x_1^O, \dots, x_n^O \rangle_O \times \mathcal{A}\langle x_1^A, \dots, x_n^A \rangle_A.$$

Any object of $\mathcal{A}\langle x_1^O, \dots, x_n^O \rangle$, such as t_O , can be written in the form

$$t_O = c_0 \otimes x_{i_1}^O \otimes C_1 \otimes x_{i_2}^O \otimes \cdots \otimes C_{m-1} \otimes x_{i_m}^O \otimes C_m,$$

where the objects C_j are in \mathcal{A} , and possibly $C_j = I$ for some j . Assume $m > 1$. Since t commutes generically with \otimes_O , we have that

$$t_O[y_{1i} \otimes y_{2i}/x_i^O] = t_O[y_{1i}/x_i^O] \otimes t_O[y_{2i}/x_i^O].$$

On the left-hand side, some variables of type y_{2i} appear to the left of some variables of type y_{1i} , but this does not happen on the right-hand side, a blatant contradiction. Therefore, $m \leq 1$. Either $m = 0$, and $t_O = C_0 = I$ since t must commute generically with I , or $m = 1$, in which case $t_O = A \otimes x_i^O \otimes B$. Since t commutes generically with \otimes_O , we find that

$$A \otimes y_{1i} \otimes y_{2i} \otimes B = A \otimes y_{1i} \otimes B \otimes A \otimes y_{2i} \otimes B,$$

which is only possible if $B \otimes A = I$. Since t commutes generically with I , we also find that

$$I = A \otimes I \otimes B = A \otimes B.$$

It follows that A and B are inverses of each other in the Picard group of \mathcal{A} (see [5]).

It remains to determine t_A in both cases presented above. First, if $t_O = I$, we must have $\text{src } t_A = I = \text{tar } t_A$, since t commutes generically with sources and targets. But then $t_A : I \rightarrow I$ cannot contain any variables, so it is an arrow of \mathcal{A} . Since furthermore t commutes generically with identity arrows, we have that $t_A = 1_I$. We can check that $t = (I, 1_I)$ commutes generically with all operations.

For the remainder of this section, we assume $t_O = A \otimes x_i^O \otimes B$, where A and B are inverses in the Picard group. Since t commutes generically with both sources and targets, we have that

$$\text{src } t_A[y_{1i}/x_i^A] = t_O[\text{src } y_{1i}/x_i^O] = A \otimes \text{src } y_{1i} \otimes B$$

and

$$\text{tar } t_A[y_{1i}/x_i^A] = t_O[\text{tar } y_{1i}/x_i^O] = A \otimes \text{tar } y_{1i} \otimes B,$$

from which we conclude that $t_A : A \otimes \text{src } x_i^A \otimes B \rightarrow A \otimes \text{tar } x_i^A \otimes B$. The only arrows with this source and target are of the form $f \otimes x_i^A \otimes g$, where $f : A \rightarrow A$ and $g : B \rightarrow B$. Since t commutes generically with identity arrows, we find that

$$f \otimes 1_{y_{1i}} \otimes g = 1_{A \otimes y_{1i} \otimes B} = 1_A \otimes 1_{y_{1i}} \otimes 1_B,$$

which implies $f = 1_A$ and $g = 1_B$. We can verify that in fact $(A \otimes x_i^O \otimes B, 1_A \otimes x_i^A \otimes 1_B)$ commutes generically with all operations. Indeed, we already know that it commutes with sources, targets, and identity arrows, as well as the unit and the tensor product on objects. It also commutes generically with the tensor product on arrows:

$$\begin{aligned} 1_A \otimes y_{1i} \otimes 1_B \otimes 1_A \otimes y_{2i} \otimes 1_B &= 1_A \otimes y_{1i} \otimes 1_{B \otimes A} \otimes y_{2i} \otimes 1_B \\ &= 1_A \otimes y_{1i} \otimes 1_I \otimes y_{2i} \otimes 1_B = 1_A \otimes y_{1i} \otimes y_{2i} \otimes 1_B. \end{aligned}$$

To check that it commutes generically with composition, we must assume that $(y_{1i} \circ y_{2i}) \downarrow$ for $i = 1, \dots, n$ (see Definition 2.3.3). Then

$$\begin{aligned} (1_A \otimes y_{1i} \otimes 1_B) \circ (1_A \otimes y_{2i} \otimes 1_B) &= (1_A \circ 1_A) \otimes (y_{1i} \circ y_{2i}) \otimes (1_B \circ 1_B) \\ &= 1_A \otimes (y_{1i} \circ y_{2i}) \otimes 1_B. \end{aligned}$$

This completes the verification. ■

Note that strict monoidal categories are obtained by a (strictified) categorification of monoids, and their inner theory reflects that fact.

3.3.6 Strict Symmetric Rig Categories

In the algebraic setting, a case of particular interest is that of commutative rings and rigs. Therefore, let us consider the theory T of a (strictified) categorification of commutative rigs, which we shall call a *strict symmetric rig category*. This theory T has all the operations and axioms of strict monoidal categories, as well as operations $0 : O$, $+_O : O \times O \rightarrow O$, and $+_A : A \times A \rightarrow A$, subject to the additional axioms:

- $\vdash A \otimes_O B = B \otimes_O A$
- $\vdash f \otimes_A g = g \otimes_A f$
- $\vdash 0 \downarrow$
- $\vdash (A +_O B) \downarrow$
- $\vdash (f +_A g) \downarrow$
- $\vdash \text{src}(f +_A g) = \text{src } f +_O \text{src } g$
- $\vdash \text{tar}(f +_A g) = \text{tar } f +_O \text{tar } g$
- $\vdash 1_{A+_O B} = 1_A +_A 1_B$
- $\vdash (f' \circ f) \downarrow \wedge (g' \circ g) \downarrow \vdash (f' \circ f) +_A (g' \circ g) = (f' +_A g') \circ (f +_A g)$

- $\vdash A +_O (B +_O C) = (A +_O B) +_O C$
- $\vdash f +_A (g +_A h) = (f +_A g) +_A h$
- $\vdash A +_O B = B +_O A$
- $\vdash f +_A g = g +_A f$
- $\vdash A +_O 0 = A$
- $\vdash f +_A 1_0 = f$
- $\vdash f \otimes_O (g +_O h) = f \otimes_O g +_O f \otimes_O h$
- $\vdash f \otimes_A (g +_A h) = f \otimes_A g +_A f \otimes_A h$
- $\vdash A \otimes_O 0 = 0$
- $\vdash f \otimes_A 1_0 = 1_0$

In what follows, we will write $+$ without subscripts when there is no ambiguity.

Proposition 3.3.6. *Let \mathcal{A} be a strict symmetric rig category. Inner morphisms in $\mathfrak{GA}(n, 1)$ have the form*

$$\left(\sum_{i=1}^n C_i \otimes x_i^O, \sum_{i=1}^n 1_{C_i} \otimes x_i^A \right),$$

where $\sum_{i=1}^n C_i = I$ and

$$C_i \otimes C_j = \begin{cases} C_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Proof: Let

$$t = (t_O, t_A) \in \mathfrak{GA}(n, 1) \subseteq \mathcal{A}\langle x_1^O, \dots, x_n^O \rangle_O \times \mathcal{A}\langle x_1^A, \dots, x_n^A \rangle_A.$$

Note that $\mathcal{A}\langle \vec{x}^O \rangle_O$, equipped with the coproduct $+$ and the tensor product \otimes_O , is the free commutative rig on the variables x_1^O, \dots, x_n^O with parameters in \mathcal{A}_O . Therefore, we can apply the reasoning of Section 3.1.9 to obtain $t_O = \sum_{i=1}^n C_i \otimes x_i^O$ where $\sum_{i=1}^n C_i = I$ and

$$C_i \otimes C_j = \begin{cases} C_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Now, since t commutes generically with sources and targets, we find that

$$t_A : \sum_{i=1}^n C_i \otimes \text{src } x_i^A \rightarrow \sum_{i=1}^n C_i \otimes \text{tar } x_i^A.$$

It follows that $t_A = \sum_{i=1}^n f_i \otimes x_i^A$ where $f_i : C_i \rightarrow C_i$ for $i = 1, \dots, n$. Moreover, since t commutes generically with identity arrows, we have

$$\sum_{i=1}^n f_i \otimes 1_{y_{1i}} = 1_{\sum_{i=1}^n C_i \otimes y_{1i}} = \sum_{i=1}^n 1_{C_i} \otimes 1_{y_{1i}},$$

from which it follows that $f_i = 1_{C_i}$ for each i .

Thus it seems that inner morphisms have the form

$$t = \left(\sum_{i=1}^n C_i \otimes x_i^O, \sum_{i=1}^n 1_{C_i} \otimes x_i^A \right),$$

where $\sum_{i=1}^n C_i = I$ and

$$C_i \otimes C_j = \begin{cases} C_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We only have to check that this commutes generically with the remaining operations. For the tensor product of arrows, we find

$$\begin{aligned} \left(\sum_{i=1}^n 1_{C_i} \otimes y_{1i} \right) \otimes \left(\sum_{i=1}^n 1_{C_i} \otimes y_{2i} \right) &= \sum_{i=1}^n \sum_{j=1}^n 1_{C_i} \otimes 1_{C_j} \otimes y_{1i} \otimes y_{2j} \\ &= \sum_{i=1}^n \sum_{j=1}^n 1_{C_i \otimes C_j} \otimes y_{1i} \otimes y_{2j} \\ &= \sum_{i=1}^n C_i \otimes y_{1i} \otimes y_{2i}. \end{aligned}$$

For the sum of arrows, we find

$$\sum_{i=1}^n 1_{C_i} \otimes (y_{1i} + y_{2i}) = \sum_{i=1}^n 1_{C_i} \otimes y_{1i} + \sum_{i=1}^n 1_{C_i} \otimes y_{2i}.$$

For composition, we assume that $(y_{1i} \circ y_{2i}) \downarrow$ for $i = 1, \dots, n$, and we calculate

$$\begin{aligned} \left(\sum_{i=1}^n 1_{C_i} \otimes y_{1i} \right) \circ \left(\sum_{i=1}^n 1_{C_i} \otimes y_{2i} \right) &= \sum_{i=1}^n (1_{C_i} \otimes y_{1i}) \circ (1_{C_i} \otimes y_{2i}) \\ &= \sum_{i=1}^n (1_{C_i} \circ 1_{C_i}) \otimes (y_{1i} \circ y_{2i}) \\ &= \sum_{i=1}^n 1_{C_i} \otimes (y_{1i} \circ y_{2i}). \end{aligned}$$

This completes the verification. ■

Chapter 4

Conclusion

We have introduced a method for associating a Lawvere theory to each object of a category with finite powers in a functorial manner. Intuitively, the theory $\mathfrak{Z}C$ records the coherent systems of extensions of arrows $C^n \rightarrow C^m$. Then, we characterized the inner theory $\mathfrak{Z}_{T\text{-Mod}}A$ for an algebraic or a quasi-equational theory T , and showed that the morphisms of this inner theory are, first, induced by tuples of terms with parameters in A , and, second, that these terms commute generically with all the operations of T . Finally, we showed that in specific cases the inner theory is related to algebraic concepts already singled out for their interest, such as sets of orthogonal idempotents summing to unity in a commutative ring. It is also interesting to point out that in some cases, such as that of groups, the inner theory is entirely determined by the isotropy monoid, while in other cases, such as that of commutative rings, the inner theory is much richer than the isotropy monoid.

As concerns avenues for further research, note that strict monoidal categories and strict symmetric rig categories are (strictified) categorifications of monoids and commutative rigs, respectively. In both cases, the object part t_O of an inner morphism is determined exactly as in the non-categorified theory, by noting that $\mathcal{A}\langle\vec{x}^O\rangle_O$ is the free model on the generators x_i^O with parameters in \mathcal{A}_O (for the non-categorified theory). Then, forcing t to commute generically with sources, targets, and identity arrows yields the form of the arrow part t_A . This strategy may be more broadly applicable, so it would be interesting to see how the inner theory of the categorification of a structure is related to the inner theory of this structure. Do note, however, that we only dealt with the strict case in this thesis, as it simplifies the syntactic description of the categorified structure a lot. A more general question we can ask is: If a theory is built from another in some specified way, does this imply a relation between the corresponding inner theories?

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