Affine oriented Frobenius Brauer categories and general linear Lie superalgebras

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Abstract

To any Frobenius superalgebra A we associate an *oriented Frobenius Brauer category* and an *affine oriented Frobenius Brauer categeory.* We define natural actions of these categories on categories of supermodules for general linear Lie superalgebras $\mathfrak{gl}_{m|n}(A)$ with entries in A. These actions generalize those on module categories for general linear Lie superalgebras and queer Lie superalgebras, which correspond to the cases where A is the ground field and the two-dimensional Clifford superalgebra, respectively. We include background on monoidal supercategories and Frobenius superalgebras and discuss some possible further directions.

Résumé

Pour chaque superalgèbre A, nous associons une *catégorie orientée de Frobenius–Brauer* et une *catégorie affine orientée de Frobenius–Brauer*. Nous définissons des actions naturelles de ces catégories sur les catégories de supermodules pour les superalgèbres de Lie générales linéaires $\mathfrak{gl}_{m|n}(A)$ avec des entrées dans A. Ces actions généralisent celles sur les catégories de supermodules pour les superalgèbres de Lie générales linéaires et les superalgèbres de Lie queer, qui correspondent aux cas où A est le corps de base et l'algèbre de Clifford bidimensionnelle, respectivement. Nous incluons des informations sur les supercatégories monoïdales et les superalgèbres de Frobenius et discutons de certaines autres directions possibles.

Dedications

For my female ancestors who have been denied the opportunities and access to education that I am afforded today.

Acknowledgement

The uOttawa Women in Math Club has been a constant source of support, friendship, and inspiration for me. I'd like to thank the women who have worked to maintain the club alongside me including Adèle, Behnaz, Jane, Maiko, Maggie, Maribeth, Prof. Monica Nevins, Prof. Tanya Schmah and many others. I'd also like to thank the Association for Women in Math for the motivation and support to create the Women in Math Club at uOttawa.

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Contents

Chapter 1

Introduction

This thesis is an expanded treatment of the material appearing in [18]. We have included additional details and background information for the less advanced reader.

The oriented Brauer category $\mathcal{OB}$ is the free linear rigid symmetric monoidal category generated by a single object $\uparrow$. This universal property immediately implies the existence of a monoidal functor

$$F\colon \mathcal{OB} \to \text{mod-}\mathfrak{gl}_n$$

from $\mathcal{OB}$ to the category of right modules for the general linear Lie algebra $\mathfrak{gl}_n$. (One can also work with left modules, but right modules turn out to be easier for this thesis.) This functor sends the generating object $\uparrow$ and its dual $\downarrow$ to the defining $\mathfrak{gl}_n$-module V and its dual V^*, respectively. For $r \geq 1$, the endomorphism algebra $\operatorname{End}_{\mathcal{OB}}(\uparrow^{\otimes r})$ is the group algebra of the symmetric group on r letters, and the algebra homomorphism

$$\operatorname{End}_{\mathcal{OB}}(\uparrow^{\otimes r}) \to \operatorname{End}_{\mathfrak{gl}_n}(V^{\otimes r})$$

induced by F is the classical one appearing in Schur-Weyl duality. More generally, $\operatorname{End}_{\mathcal{OB}}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s})$ are *walled Brauer algebras* and the induced algebra homomorphisms

$$\operatorname{End}_{\mathcal{OB}}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s}) \to \operatorname{End}_{\mathfrak{gl}_n}(V^{\otimes r} \otimes (V^*)^{\otimes s})$$

were originally defined and studied by Turaev [30] and Koike [16].

The rank n of $\mathfrak{gl}_n$ appears as a parameter in $\mathcal{OB}$. In fact, the definition of $\mathcal{OB}$ makes sense for *any* value of this parameter, i.e. it need not be a positive integer. This observation leads to the definition of Deligne's interpolating category for the general linear Lie groups [9]; this interpolating category is the additive Karoubi envelope of $\mathcal{OB}$.

The functor F yields an action of $\mathcal{OB}$ on mod-$\mathfrak{gl}_n$. More precisely, for a category $\mathcal{C}$, let $\mathcal{E}nd(\mathcal{C})$ denote the corresponding strict monoidal category of endofunctors and natural transformations. Then we have a functor

$$\mathcal{OB} \to \mathcal{E}nd(\text{mod-}\mathfrak{gl}_n),$$

$$X \mapsto F(X) \otimes -,$$
$$f \mapsto F(f) \otimes -,$$

for objects X and morphisms f in $\mathcal{OB}$. In [4], this was extended to an action of the *affine oriented Brauer category* $\mathcal{AOB}$ on mod-$\mathfrak{gl}_n$. The category $\mathcal{AOB}$ is obtained from $\mathcal{OB}$ by adjoining an additional endomorphism of the generating object $\uparrow$, subject to certain natural relations. This additional endomorphism acts by a natural transformation of the functor $V \otimes -$ arising from multiplication by a certain canonical element of $\mathfrak{gl}_n \otimes \mathfrak{gl}_n$. Restricting to endomorphism spaces recovers actions of *affine walled Brauer algebras* studied in [22, 23].

In fact, much of the above picture can be generalized, replacing $\mathfrak{gl}_n$ by the general linear Lie *super*algebra $\mathfrak{gl}_{m|n}$. Remarkably, one does not need to modify $\mathcal{OB}$ or $\mathcal{AOB}$ at all. Here the corresponding Schur–Weyl duality was established by Sergeev [27] and Berele–Regev [1], while the action of the walled Brauer algebras was described in [8, Th. 7.8]. The analogue of the functor F above is described in [10, Th. 4.16]. The extension of the action to $\mathcal{AOB}$ does not seem to have appeared in the literature, but is certainly expected by experts. For example, it is mentioned in the introduction to [4].

The affine oriented Brauer category $\mathcal{AOB}$ is a special case of more general category. The *Heisenberg category* at central charge -1 was first introduced by Khovanov [15] as a tool to study the representation theory of the symmetric group. In [17], it was generalized to arbitrary negative central charge, which corresponds to replacing the symmetric group by more general degenerate cyclotomic Hecke algebras of type A. In [2], Brundan gave a simplified presentation of the Heisenberg category $\mathcal{H}eis_k$ at arbitrary central charge k. When $k = 0$, the Heisenberg category is precisely the affine oriented Brauer category.

The Heisenberg category has been further generalized in [21, 24, 7] to the *Frobenius Heisenberg category* $\mathcal{H}eis_k(A)$ depending on a Frobenius superalgebra A. When $A = \Bbbk$, this construction recovers the Heisenberg category. When $k \neq 0$, the category $\mathcal{H}eis_k(A)$ acts naturally on categories of modules over the cyclotomic wreath product algebras defined in [25]. However, actions in the case $k = 0$ have not yet been studied. In some sense, central charge zero yields the simplest and most interesting case. For example, $\mathcal{H}eis_k(A)$ is symmetric monoidal if and only if $k = 0$.

1.1 Original Contributions of the Thesis

In analogy with the $A = \Bbbk$ case, we call the Frobenius Heisenberg category at central charge zero the *affine oriented Frobenius Brauer category* $\mathcal{AOB}(A)$. It contains a natural Frobenius algebra analogue of the oriented Brauer category, which we call the *oriented Frobenius Brauer category* $\mathcal{OB}(A)$. Note that the Frobenius Heisenberg category has been studied in [21, 24, 7] and as such, this central charge zero case is not original. However, in Chapter 4, we state and prove a number of relations which follow from the defining relations of $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$. These relations are special cases of those appearing in [24], but we are able to greatly simplify many of the proofs for the charge zero case. We define, in Theorems 5.3.4 and 5.3.11, natural functors

$$\mathcal{OB}(A) \to \text{smod-}\mathfrak{gl}_{m|n}(A), \quad \mathcal{AOB}(A) \to \mathcal{E}nd(\text{smod-}\mathfrak{gl}_{m|n}(A)),$$

where smod-$\mathfrak{gl}_{m|n}(A)$ denotes the monoidal supercategory of right supermodules for the general linear Lie superalgebra with entries in the Frobenius superalgebra A.

When $A = \Bbbk$, we recover the functors described above for $\mathfrak{gl}_{m|n} = \mathfrak{gl}_{m|n}(\Bbbk)$. On the other hand, if $A = \mathrm{Cl}$ is the two-dimensional Clifford superalgebra, then $\mathfrak{gl}_{m|n}(\mathrm{Cl})$ is isomorphic to the queer Lie superalgebra $\mathfrak{q}(m+n)$, and our functors recover those defined in [3]. As in the $A = \Bbbk$ case, these functors extend Schur-Weyl duality results for queer Lie superalgebras [28], actions of walled Brauer-Clifford superalgebras [14], and actions of affine walled Brauer-Clifford superalgebras [3, 12]. In fact, the Clifford superalgebra is the main example of interest where the Frobenius superalgebra is not symmetric. Since this case has already been studied in the aforementioned papers, we assume throughout the thesis that A is symmetric, as this simplifies the exposition. We have indicated in Remark 4.1.6 the modification that needs to be made to handle the more general case.

The results of this thesis, and the accompanying paper [18], extend the powerful category theoretic tools that have been used to study the representation theory of general linear Lie superalgebras and queer Lie superalgebras to the setting of general linear Lie superalgebras over Frobenius superalgebras. For example, in Proposition 5.3.12, we see that these functors yield central elements in the universal enveloping algebra generalizing the known generators of this center in the $A = \Bbbk$ and $A = \mathrm{Cl}$ cases. When $A = \Bbbk[x]/(x^l)$, then $\mathfrak{gl}_{m|n}(A)$ is a truncated current superalgebra (also called a Takiff algebra when $m = 0$ or $n = 0$). In this case, Brauer category type methods do not seem to have appeared in the literature before.

The functors defined in Theorems 5.3.4 and 5.3.11 and several of the examples contained in this thesis also appear in the paper, [18]. However, in this thesis we have included additional details in the proofs of Theorems 5.3.4 and 5.3.11 and several of the examples. Additionally, we have included the background knowledge necessary to understand the concepts in this thesis and in [18], as well as proofs for the subsequent relations in $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$. We have also included a review of classical Schur-Weyl duality and some explanation of the connection to the rest of the thesis, which does not appear in [18]. The exact content of the thesis is outlined in more detail below.

1.2 Overview of the Thesis

In Chapters 2 and 3 we cover the preliminaries such as vector superspaces, superalgebras, supercategories, superfunctors, supernatural transformations, and Frobenius superalgebras. We also introduce (symmetric) monoidal supercategories and string diagrams. In Chapter 3 we also discuss in more detail some of the examples mentioned in this introduction such as the two-dimensional Clifford superalgebra (Example 3.2.3) and the supermatrix ring $\mathrm{Mat}_{m|n}(A)$ (Example 3.2.2).

In Chapter 4 we introduce the affine oriented Frobenius Brauer categories and the oriented Frobenius Brauer categories. We give their definition in terms of generators and relations and prove a number of relations that follow. As mentioned, these subsequent relations and their proofs are special cases of those appearing in [24]. We also note which relations in particular correspond to certain properties of $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$ such as rigidity, pivotality, and their symmetric monoidal structure (only in the case of $\mathcal{OB}(A)$). We also recall the Basis Theorem for $\mathcal{AOB}(A)$ from [7, Th. 7.2].

In Chapter 5 we introduce the category of right $\mathfrak{gl}_{m|n}(A)$-supermodules and define the functors mentioned above in Theorems 5.3.4 and 5.3.11.

Finally, in Chapter 6 we discuss the possible connection of the functors defined in Theorems 5.3.4 and 5.3.11 to Schur-Weyl duality which is discussed more below.

1.3 Further Directions

In Chapter 6 we review classical Schur-Weyl duality and note the connection to the functors defined in Theorems 5.3.4 and 5.3.11. We identify some examples of Frobenius superalgebras where an analogue of the proof of classical Schur-Weyl duality goes through as well as some examples where it may not.

Some additional possible further directions which we have not explored are listed below.

1. *Interpolating categories.* The idempotent completion of $\mathcal{OB}(A)$ is a natural candidate for an interpolating category for smod-$\mathfrak{gl}_{m|n}(A)$, which could potentially be used to generalize work of Deligne and others in the case $n = 0$, $A = \Bbbk$.

2. *Frobenius Schur algebras.* One should be able to define Schur algebras depending on a Frobenius superalgebra A such that, when $A = \Bbbk$, one recovers the usual Schur algebras.

3. *Cyclotomic quotients.* In the cases $A = \Bbbk$ and $A = \mathrm{Cl}$, cyclotomic quotients of $\mathcal{AOB}(A)$ have been studied in [3, 4]. We expect that many of these results can be extended to the setting of general Frobenius superalgebras.

4. *Quantum analogues.* The *quantum Frobenius Heisenberg categories*, introduced in [6], are natural quantum analogues of Frobenius Heisenberg categories. The special case of central charge zero yields a natural *quantum affine oriented Frobenius Brauer category*. When $A = \Bbbk$, this is the affine HOMFLY-PT skein category. Then quantum affine oriented Frobenius Brauer categories should act on as-yet-to-be-defined quantum enveloping algebras of $\mathfrak{gl}_{m|n}(A)$ and yield Frobenius analogues of the HOMFLY-PT link invariant.

Chapter 2

Monoidal Supercategories

In this chapter we introduce some basic category theoretic concepts and work up towards symmetric monoidal supercategories. We give a brief overview of string diagrams in the super setting, which will be needed in Chapter 4 and beyond.

2.1 Introduction to Supercategories

Throughout the thesis we will work over a ground field $\Bbbk$. Unadorned tensor products will be over $\Bbbk$.

A $\Bbbk$-vector superspace is a $\Bbbk$-vector space, V, with a $\mathbb{Z}_2$-grading with decomposition

$$V = V_0 \oplus V_1, \quad 0, 1 \in \mathbb{Z}_2.$$

Note that $\Bbbk$ itself is a purely even vector superspace.

A nonzero homogeneous element $v \in V_i$, has *parity* given by $\bar{v} = i$. If $\bar{v} = 0$, v is said to be even and if $\bar{v} = 1$, v is said to be odd. We can consider the *category of vector superspaces*, $\mathcal{SV}ec$, where objects are vector superspaces and morphisms are parity preserving linear maps between them. We say a linear map is even if it is parity preserving and odd if it is parity reversing.

Definition 2.1.1. A *supercategory*, $\mathcal{C}$, is a category enriched in $\mathcal{SV}ec$. This means for each pair of objects, $A, B \in \mathcal{C}$, their morphism space, $\mathrm{Hom}_{\mathcal{C}}(A, B)$, is a $\Bbbk$-vector superspace and composition is parity preserving, meaning $\overline{f \circ g} = \bar{f} + \bar{g}$ for $f \in \mathrm{Hom}_{\mathcal{C}}(B, C), g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 2.1.2. It is immediate that $\mathcal{SV}ec$ is a supercategory since composition of even morphisms is again even.

Remark 2.1.3. *Every category is a supercategory with purely even morphism spaces.*

Definition 2.1.4. For supercategories $\mathcal{C}, \mathcal{D}$ a superfunctor $F : \mathcal{C} \to \mathcal{D}$ is a pair of maps on objects and morphisms of $\mathcal{C}$ such that

- $F1_X = 1_{FX}$ for $X \in \mathcal{C}$,
- $F(g \circ f) = Fg \circ Ff$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and
- $\overline{Ff} = \bar{f}$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Note that the first two conditions above say that F is a functor from $\mathcal{C}$ to $\mathcal{D}$ and it is the third condition that elevates F from a functor to a superfunctor.

Definition 2.1.5. For superfunctors $F, G\colon \mathcal{A} \to \mathcal{B}$, a *supernatural transformation* $\alpha\colon F \Rightarrow G$ *of parity* $r \in \mathbb{Z}_2$ is the data of morphisms $\alpha_X \in \mathrm{Hom}_{\mathcal{B}}(FX, GX)_r$ for each $X \in \mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{(-1)^{r\bar{f}}Ff} & FY \\ {\scriptstyle \alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_Y} \\ GX & \xrightarrow[Gf]{} & GY \end{array} \quad , \tag{2.1.1}$$

that is $Gf \circ \alpha_X = (-1)^{r\bar{f}} \alpha_Y \circ Ff$ for each homogeneous $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$.

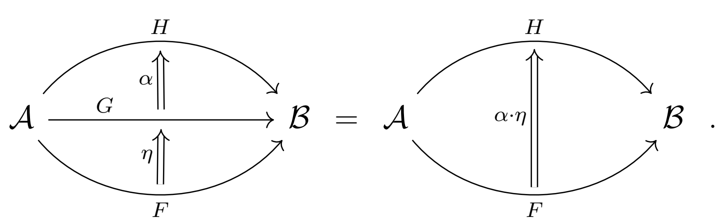
A *supernatural transformation* $\alpha\colon F \Rightarrow G$ is $\alpha = \alpha_0 + \alpha_1$ with each α_r being a supernatural transformation of parity r.

A *supernatural isomorphism* is a supernatural transformation α where each component α_X is an isomorphism.

We can define composition of supernatural transformations in two ways. Let $F, G, H\colon \mathcal{A} \to \mathcal{B}$ be superfunctors. Let $\eta\colon F \Rightarrow G, \alpha\colon G \Rightarrow H$ be supernatural transformations. We can define a supernatural transformation $\alpha \cdot \eta\colon F \Rightarrow H$ with components

$$(\alpha \cdot \eta)_X := \alpha_X \circ \eta_X\colon FX \to HX,$$

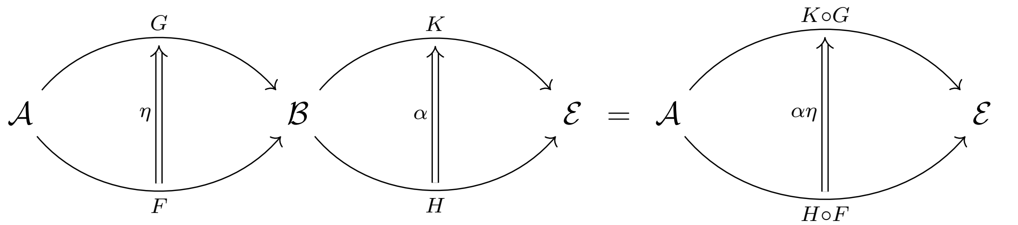
for $X \in \mathcal{A}$ where $\circ$ is regular morphism composition in the category $\mathcal{B}$. This is known as *vertical composition*:



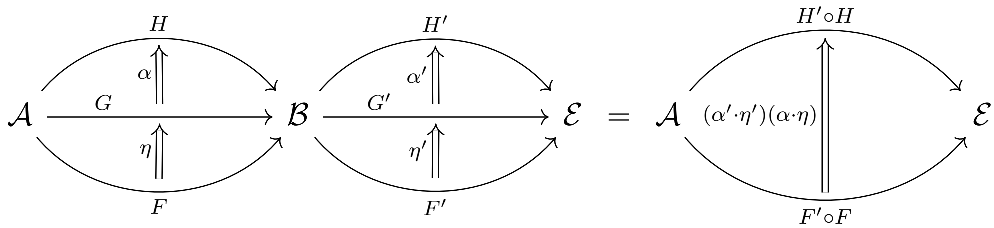
Now let $F, G\colon \mathcal{A} \to \mathcal{B}$, $H, K\colon \mathcal{B} \to \mathcal{E}$ be superfunctors. Let $\eta\colon F \Rightarrow G$, $\alpha\colon H \Rightarrow K$ be supernatural transformations. We can define another composition $\alpha\eta\colon H \circ F \Rightarrow K \circ G$ with components

$$(\alpha\eta)_X := \alpha_{GX} \circ H(\eta_X) : HFX \to KGX,$$

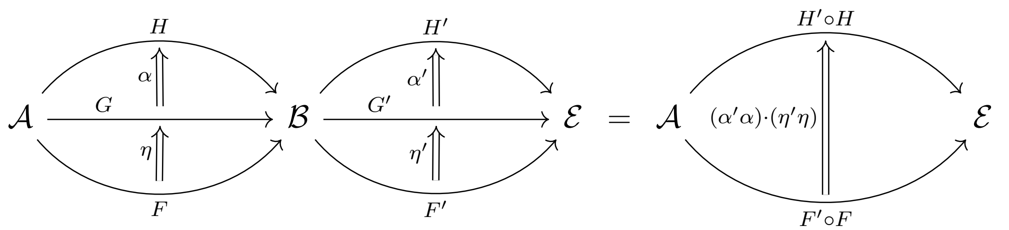
for $X \in \mathcal{A}$, which is called *horizontal composition*:



A natural question to ask is do these compositions commute. If we first compose vertically and then horizontally we have



while if we compose hortizontally and then vertically we have



It turns out that the compositions commute up to a sign such that

$$(\alpha'\alpha) \cdot (\eta'\eta) = (-1)^{\bar{\alpha}\overline{\eta'}}(\alpha' \cdot \eta')(\alpha \cdot \eta).$$

For simplicity, we assume that each supernatural transformation, $\alpha, \alpha', \eta, \eta'$ is homogeneous and then we can extend by linearity. Recall that by supernaturality of η', for $X \in \mathcal{A}$ we have $G'(\alpha_X) \circ \eta'_{GX} = (-1)^{r\overline{\alpha_X}}\eta'_{HX} \circ F'\alpha_X$ where $r = \overline{\eta'}$. Note also that by definition $\overline{\alpha_X} = \bar{\alpha}$ so that $(-1)^{r\overline{\alpha_X}} = (-1)^{\bar{\alpha}\overline{\eta'}}$. We have

$$\begin{aligned} ((\alpha'\alpha) \cdot (\eta'\eta))_X &= (\alpha'\alpha)_X \circ (\eta'\eta)_X \\ &= \alpha'_{HX} \circ G'(\alpha_X) \circ \eta'_{GX} \circ F'(\eta_X) \\ &= (-1)^{\bar{\alpha}\overline{\eta'}}\alpha' HX \circ \eta'_{HX} \circ F'(\alpha_X) \circ F'(\eta_X) \qquad (2.1.2) \\ &= (-1)^{\bar{\alpha}\overline{\eta'}}(\alpha' \cdot \eta')_{HX} \circ F'((\alpha \cdot \eta)_X) \\ &= (-1)^{\bar{\alpha}\overline{\eta'}}\left((\alpha' \cdot \eta')(\alpha \cdot \eta)\right)_X . \end{aligned}$$

Definition 2.1.6. A *monoidal category* is a category $\mathcal{C}$ equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$ such that for all $A, B, C, D \in \mathcal{C}$ there are natural isomorphisms with components $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda_A : \mathbb{1} \otimes A \to A$, and $\rho_A : A \otimes \mathbb{1} \to A$ such that the following diagrams commute:

$$\begin{array}{ccccc} & \overset{\alpha_{A,B,C\otimes D}}{\nearrow} & (A \otimes B) \otimes (C \otimes D) & \overset{\alpha_{A\otimes B,C,D}}{\searrow} & \\ A \otimes (B \otimes (C \otimes D)) & & & & ((A \otimes B) \otimes C) \otimes D \\ 1_A \otimes \alpha_{B,C,D} \downarrow & & & & \uparrow \alpha_{A,B,C} \otimes 1_D \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A \otimes (B \otimes C)) \otimes D \end{array}, \tag{2.1.3}$$

$$\begin{array}{ccc} A \otimes (\mathbb{1} \otimes B) & \xrightarrow{\alpha_{A,\mathbb{1},B}} & (A \otimes \mathbb{1}) \otimes B \\ & {\scriptstyle 1_A \otimes \lambda_B} \searrow \quad \swarrow {\scriptstyle \rho_A \otimes 1_B} & \\ & A \otimes B & \end{array}. \tag{2.1.4}$$

These commutative diagrams are called the *coherence conditions*.

If α, λ, ρ are identities then $\mathcal{C}$ is called a *strict* monoidal category. To show α, λ, ρ are identities, it suffices to show that for all objects $A, B, C \in \mathcal{C}$, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$, $A \otimes \mathbb{1} = A = \mathbb{1} \otimes A$ and that for all morphisms $f, g, h \in \operatorname{Mor} \mathcal{C}$, $f \otimes (g \otimes h) = (f \otimes g) \otimes h$ and $f \otimes \mathrm{id}_1 = f = \mathrm{id}_{\mathbb{1}} \otimes f$.

A *monoidal supercategory* is a supercategory with a superbifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $\mathbb{1}$ subject to the same coherence conditions as above.

In a *strict monoidal supercategory* the coherence maps are again identities. Moreover, in a strict monoidal supercategory composition of morphisms is given by

$$(f \otimes g) \circ (h \otimes k) = (-1)^{\bar{g}\bar{h}} (f \circ h) \otimes (g \circ k). \tag{2.1.5}$$

Equation (2.1.5) is known as the *superinterchange law*. For more on monoidal supercategories, see [5]. Also note that (2.1.5) agrees with (2.1.2).

Example 2.1.7. The category of vector superspaces is a monoidal supercategory with tensor product of vector superspaces and unit object $\Bbbk$, the even one-dimensional vector superspace.

Definition 2.1.8. A monoidal category $\mathcal{C}$ is called *symmetric* if there is a natural isomorphism called a symmetric braiding with components $s_{A,B} : A \otimes B \to B \otimes A$ for $A, B \in \mathcal{C}$ such that the following diagrams commute

$$\begin{array}{ccc} A \otimes \mathbb{1} & \xrightarrow{s_{a,\mathbb{1}}} & \mathbb{1} \otimes A \\ & {\scriptstyle \rho_A} \searrow \quad \swarrow {\scriptstyle \lambda_A} & \\ & A & \end{array} \quad , \tag{2.1.6}$$

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{s_{A,B} \otimes 1_C} & (B \otimes A) \otimes C \\ {\scriptstyle \alpha_{A,B,C}} \downarrow & & \downarrow {\scriptstyle \alpha_{B,A,C}} \\ A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\ {\scriptstyle s_{A,B\otimes C}} \downarrow & & \downarrow {\scriptstyle 1_B \otimes s_{A,C}} \\ (B \otimes C) \otimes A & \xrightarrow[\alpha_{B,C,A}]{} & B \otimes (C \otimes A) \end{array} \quad , \tag{2.1.7}$$

$$\begin{array}{ccc} A \otimes B & \xrightarrow{1_{A\otimes B}} & A \otimes B \\ & {\scriptstyle s_{A,B}} \searrow \quad \nearrow {\scriptstyle s_{B,A}} & \\ & B \otimes A & \end{array} \quad . \tag{2.1.8}$$

If we exclude the last diagram, so that we do not require $s_{B,A} \circ s_{A,B} = 1_{A\otimes B}$ but still have that $A \otimes B$ is naturally isomorphic to $B \otimes A$, we have a *braided monoidal category* instead.

Example 2.1.9. The category of vector superspaces is a symmetric monoidal supercategory with symmetric braiding given on homogenous elements by

$$\begin{aligned} V \otimes W &\to W \otimes V, \\ v \otimes w &\mapsto (-1)^{\bar{v}\bar{w}} w \otimes v, \end{aligned}$$

and extended by linearity.

Definition 2.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal supercategories with tensor products $\otimes_{\mathcal{C}}, \otimes_{\mathcal{D}}$ and unit objects $\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{D}}$ respectively. A monoidal superfunctor is a superfunctor $F : \mathcal{C} \to \mathcal{D}$ with an even supernatural isomorphism

$$\phi_{X,Y} \colon FX \otimes_{\mathcal{D}} FY \to F(X \otimes_{\mathcal{C}} Y)$$

for $X, Y \in \mathcal{C}$ and an even invertible morphism

$$\iota : \mathbb{1}_{\mathcal{D}} \to F\mathbb{1}_{\mathcal{C}}$$

such that the following diagrams commute:

$$\begin{array}{ccc} (FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} FZ & \xrightarrow{\alpha_{FX,FY,FZ}} & FX \otimes_{\mathcal{D}} (FY \otimes_{\mathcal{D}} FZ) \\ \phi_{X,Y} \otimes_{\mathcal{D}} 1_{FZ} \downarrow & & \downarrow 1_{FX} \otimes_{\mathcal{D}} \phi_{Y,Z} \\ F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} FZ & & FX \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\ \phi_{X \otimes_{\mathcal{C}} Y, Z} \downarrow & & \downarrow \phi_{X, Y \otimes_{\mathcal{C}} Z} \\ F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \end{array} \quad , \tag{2.1.9}$$

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} FX & \xrightarrow{\iota \otimes_{\mathcal{D}} 1_{FX}} & F\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{D}} FX \\ \lambda_{FX} \downarrow & & \downarrow \phi_{\mathbb{1}_{\mathcal{C}}, X} \\ FX & \xleftarrow[F\lambda_X]{} & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X) \end{array} \quad , \tag{2.1.10}$$

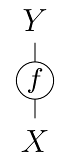
$$\begin{array}{ccc} FX \otimes_{\mathcal{D}} \mathbb{1}_{\mathcal{D}} & \xrightarrow{1_{FX} \otimes_{\mathcal{D}} \iota} & FX \otimes_{\mathcal{D}} F\mathbb{1}_{\mathcal{C}} \\ \rho_{FX} \downarrow & & \downarrow \phi_{X, \mathbb{1}_{\mathcal{C}}} \\ FX & \xleftarrow[F\rho_X]{} & F(X \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}}) \end{array} \quad , \tag{2.1.11}$$

where λ, ρ, α are as in Definition 2.1.6. A monoidal superfunctor is called *strict* if $\phi_{X,Y}$ and ι are identities.

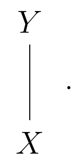
2.2 String Diagrams

We will give a brief overview of string diagrams in both the classical and super setting. For another overview see [26], for more details see [29, Ch. 1, 2], and see [5] for the super setting.

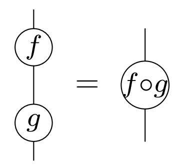
String diagrams are a way to visually represent morphisms in a monoidal supercategory. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we can represent it as



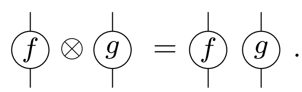
so that we read diagrams from bottom to top. When it is obvious or unimportant, we omit the labels at the top and bottom of the diagram. We represent the identity morphism by an empty strand



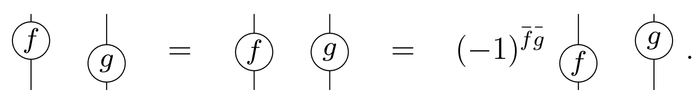
We represent composition by vertically stacking



and we represent the tensor product by horizontal juxtaposition

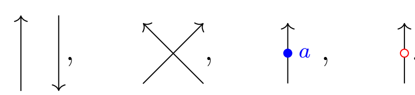


Due to the super interchange law, (2.1.5), we need to be careful about the height of morphisms in our diagrams:



(2.2.1)

Instead of representing a morphism by a coupon, we may occasionally represent it in other ways. For example, by orientation of a strand, a crossing, a token or dot:



In this thesis our morphisms will be represented by orientation and various decorations on strands as pictured above.

Chapter 3

Frobenius Superalgebras

In this chapter we introduce the concepts of superalgebras as well as Frobenius algebras. We provide some motivating examples such as the superalgebra of supermatrices and the two-dimensional Clifford superalgebra.

3.1 Superalgebras

Recall the definition of a vector superspace given in Chapter 2. We can consider applying a similar grading to an algebra to obtain a superalgebra.

Definition 3.1.1. An *associative superalgebra*, A, over a field $\Bbbk$ is a $\Bbbk$-vector superspace with decomposition

$$A = A_0 \oplus A_1$$

such that $A_i A_j \subseteq A_{i+j}$. For homogeneous elements $a, b \in A$, $\overline{ab} = \bar{a} + \bar{b}$.

A superalgebra is said to be symmetric if for all homogeneous $a, b \in A$, $ab = (-1)^{\bar{a}\bar{b}} ba$.

In the remainder of the document, whenever we discuss the parity of an element, $\bar{a}$, we assume $a \in A$ is homogeneous. We also give definitions on homogeneous elements and extend by linearity.

For superalgebras $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$, multiplication in the superalgebra $A \otimes B$ is defined by

$$(a' \otimes b)(a \otimes b') = (-1)^{\bar{a}\bar{b}} a'a \otimes bb' \tag{3.1.1}$$

for $a, a' \in A$, $b, b' \in B$.

Definition 3.1.2. The *center* $Z(A)$ is the subalgebra of A generated by all homogeneous $a \in A$ such that

$$ab = (-1)^{\bar{a}\bar{b}} ba, \qquad \text{for all homogeneous } b \in A \tag{3.1.2}$$

Definition 3.1.3. The *cocenter* $C(A)$ is the quotient of A by the subspace spanned by $ab - (-1)^{\bar{a}\bar{b}} ba$ for all homogeneous $a, b \in A$. Note that in general $C(A)$ is a vector superspace, and not a superalgebra. For $a \in A$, we let $\dot{a}$ denote its canonical image in $C(A)$.

Example 3.1.4. Fix $m, n \in \mathbb{N}$. For $1 \leq i \leq m+n$, define $p(i) \in \mathbb{Z}_2$ by

$$p(i) = \begin{cases} 0 & \text{if } 1 \leq i \leq m, \\ 1 & \text{if } m+1 \leq i \leq m+n. \end{cases} \tag{3.1.3}$$

For a superalgebra A, let $\mathrm{Mat}_{m|n}(A)$ be the ring consisting of $(m+n) \times (m+n)$ matrices with entries in A, where multiplication is given by matrix multiplication. We can equip $\mathrm{Mat}_{m|n}(A)$ with a $\mathbb{Z}_2$-grading which is defined as follows. For $a \in A$ and $1 \leq i, j \leq m+n$, let $a_{(i,j)} \in \mathrm{Mat}_{m|n}(A)$ denote the matrix with a in the (i, j) position and 0 in all other positions. Then, for homogeneous $a \in A$, we define

$$\overline{a_{(i,j)}} = \bar{a} + p(i) + p(j).$$

Since it is straightforward to see $\mathrm{Mat}_{m|n}(A)$ is a vector superspace over $\Bbbk$, to see it is a superalgebra we only need to verify the multiplication respects the $\mathbb{Z}_2$ grading. Recall for $1 \leq i, j, r, s \leq m+n$ and homogeneous $a, b \in A$, we have

$$a_{(i,j)} b_{(r,s)} = \delta_{j,r} (ab)_{(i,s)}.$$

Since the zero matrix is both even and odd, we only need to consider the case when $j = r$. Then we have

$$\begin{aligned} \overline{a_{(i,j)} b_{(j,s)}} &= \overline{(ab)_{(i,s)}} \\ &= \bar{a} + \bar{b} + p(i) + p(s) \\ &= \bar{a} + p(i) + p(j) + \bar{b} + p(j) + p(s) \\ &= \overline{a_{(i,j)}} + \overline{b_{(j,s)}}. \end{aligned}$$

Example 3.1.5. The two-dimensional Clifford superalgebra $\mathrm{Cl} := \langle c : c^2 = 1 \rangle$ is a superalgebra with decomposition given by $\Bbbk \oplus \Bbbk c$ with $\bar{c} = 1$. It is straightforward to see the multiplication respects the $\mathbb{Z}_2$-grading since $c^2 = 1$.

3.2 Introduction to Frobenius Superalgebras

Definition 3.2.1. Let $\mathrm{tr} : A \to \Bbbk$ be a linear map. Let $\mathbf{B}$ be a basis for A. Then A is said to be a *Frobenius superalgebra* if there exists a dual basis, $\mathbf{B}^\vee := \{b^\vee : b \in \mathbf{B}\}$ with respect to tr. That is:

$$\mathrm{tr}(b^\vee c) = \delta_{b,c}, \quad b, c \in \mathbf{B}.$$

Moreover, A is a *symmetric Frobenius superalgebra* if

$$\operatorname{tr}(ab) = (-1)^{\bar{a}\bar{b}} \operatorname{tr}(ba)$$

for all homogeneous $a, b \in A$. For a (not necessarily symmetric) Frobenius superalgebra, one can show that there is an automorphism ψ of A, called the *Nakayama automorphism* (see [19, Sec. 2]), such that $\operatorname{tr}(ab) = (-1)^{\bar{a}\bar{b}} \operatorname{tr}(b\psi(a))$.

Note that traces are not unique (see Example 3.2.3) and in the non-super case, any two traces are related by multiplying by an invertible element. For the super setting see [20, Prop. 4.7].

The supertrace, str, of a matrix, $M = (m_{i,j}) \in \operatorname{Mat}_{m|n}(A)$, is given by

$$\operatorname{str}(M) = \sum_{i=1}^{m+n} (-1)^{p(i)} m_{i,i}.$$

If we write M as a block matrix instead, $M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix} \in \operatorname{Mat}_{m|n}(A)$, where M_{00} is $m \times m$, M_{01} is $m \times n$, M_{10} is $n \times m$ and M_{11} is $n \times n$, then

$$\operatorname{str}(M) = \operatorname{Tr}(M_{00}) - \operatorname{Tr}(M_{11})$$

where Tr is the usual trace of a matrix.

Example 3.2.2. For a symmetric Frobenius superalgebra A with trace map tr, the superalgebra $\operatorname{Mat}_{m|n}(A)$ is a symmetric Frobenius superalgebra with trace

$$\begin{aligned} \operatorname{tr}_{m|n} := \operatorname{tr} \circ \operatorname{str} \colon \operatorname{Mat}_{m|n}(A) &\to \Bbbk, \\ a_{(i,j)} &\mapsto \delta_{i,j} (-1)^{p(i)} \operatorname{tr}(a), \end{aligned}$$

for $1 \leq i, j \leq m+n$ and $a \in A$.

First note that if $\mathbf{B}_A$ is a basis for A then $\operatorname{Mat}_{m|n}(A)$ has basis

$$\mathbf{B}_{m|n} := \{ b_{(i,j)} : b \in \mathbf{B}_A,\ 1 \leq i, j \leq m+n \} \tag{3.2.1}$$

and we claim that

$$(b_{(i,j)})^\vee = (-1)^{p(j)} b^\vee_{(j,i)}. \tag{3.2.2}$$

is a left dual basis. Note that, here and in what follows, we adopt the convention that we apply the symbol $\vee$ before considering subscripts. Thus, for example, $b^\vee_{(i,j)} = (b^\vee)_{(i,j)}$. We have

$$\begin{aligned} \operatorname{tr}_{m|n}((b_{(i,j)})^\vee c_{(r,s)}) &= \operatorname{tr}_{m|n}((-1)^{p(j)} b^\vee_{(j,i)} c_{(r,s)}) \\ &= \operatorname{tr}_{m|n}((-1)^{p(j)} \delta_{i,r} (b^\vee c)_{(j,s)}) \end{aligned}$$

$$= (-1)^{p(j)}\delta_{i,r}\delta_{j,s}(-1)^{p(j)}\operatorname{tr}(b^\vee c)$$
$$= \delta_{i,r}\delta_{j,s}\operatorname{tr}(b^\vee c) = \delta_{i,r}\delta_{j,s}\delta_{b,c}$$

as desired. The fact that $\operatorname{tr}_{m|n}$ is symmetric follows from the fact that tr is symmetric. Thus $\operatorname{Mat}_{m|n}(A)$ is a symmetric Frobenius superalgebra with trace map, $\operatorname{tr}_{m|n}$. Note that $\operatorname{Mat}_{m|n}(A)$ is also a Frobenius superalgebra with trace map $\operatorname{tr}\circ\operatorname{Tr}$, where Tr is the ordinary matrix trace, with dual basis given by

$$(b_{(i,j)})^\vee = b^\vee_{j,i}.$$

To see this simply remove all the signs in the previous calculation and replace str with Tr. However, it is not symmetric since in particular for $1 \le i, j \le m+n$ with $p(i) = 1$ and $p(j) = 0$ we have

$$\operatorname{tr}\circ\operatorname{Tr}(1_{(i,j)}1_{(j,i)}) = \operatorname{tr}\circ\operatorname{Tr}((1)_{(i,i)})$$
$$= \operatorname{tr}(1)$$

while

$$(-1)^{\overline{1_{(i,j)}}\,\overline{1_{(j,i)}}}\operatorname{tr}\circ\operatorname{Tr}(1_{(j,i)}1_{(i,j)}) = (-1)^{p(i)}\operatorname{tr}(1)$$
$$= -\operatorname{tr}(1).$$

Example 3.2.3. The two-dimensional Clifford superalgebra, Cl, is a Frobenius superalgebra. Here, up to scalar multiple, we have two choices for homogeneous $\operatorname{tr} : \mathrm{Cl} \to \Bbbk$ which are defined on the basis and extended by linearity. We have parity preserving:

$$\operatorname{tr}_0(1) = 1 \text{ and } \operatorname{tr}_0(c) = 0,$$

or parity reversing:

$$\operatorname{tr}_1(1) = 0 \text{ and } \operatorname{tr}_1(c) = 1.$$

Note that if we choose tr_1, Cl is a symmetric Frobenius superalgebra since we have

$$\operatorname{tr}_1(c\cdot 1) = (-1)^{\bar{c}\bar{1}}\operatorname{tr}_1(1\cdot c),$$
$$\operatorname{tr}_1(1\cdot 1) = (-1)^{\bar{1}\bar{1}}\operatorname{tr}_1(1\cdot 1)$$
$$\operatorname{tr}_1(c\cdot c) = (-1)^{\bar{c}\bar{c}}\operatorname{tr}_1(c\cdot c).$$

However, if we choose tr_0, this is not the case since in particular

$$\operatorname{tr}_0(c\cdot c) = \operatorname{tr}_0(1) = 1 \neq -1 = (-1)^{\bar{c}}\operatorname{tr}_0(c\cdot c).$$

With tr_0, we see that Cl has nontrivial Nakayama automorphism given by $\psi(c) = -c$ so that

$$\operatorname{tr}_0(c\cdot c) = \operatorname{tr}_0(1) = 1 = (-1)^{\bar{c}}\operatorname{tr}_0(c\cdot\psi(c)).$$

In the remainder of the document A will denote a symmetric Frobenius superalgebra with parity preserving trace map $\operatorname{tr}\colon A \to \Bbbk$. Thus

$$\operatorname{tr}(ab) = (-1)^{\bar{a}\bar{b}} \operatorname{tr}(ba) = (-1)^{\bar{a}} \operatorname{tr}(ba) = (-1)^{\bar{b}} \operatorname{tr}(ba), \quad a, b \in A, \tag{3.2.3}$$

where the second and third equalities follow from the fact that $\operatorname{tr}(ab) = 0$ unless $\bar{a} = \bar{b}$. The definition of a graded Frobenius superalgebra gives that A possesses a homogeneous basis $\mathbf{B}_A$ and a left dual basis $\{b^\vee : b \in \mathbf{B}_A\}$ such that

$$\operatorname{tr}(b^\vee c) = \delta_{b,c}, \quad b, c \in \mathbf{B}_A. \tag{3.2.4}$$

It follows that, for all $a \in A$, we have

$$a = \sum_{b \in \mathbf{B}_A} \operatorname{tr}(b^\vee a) b = \sum_{b \in \mathbf{B}_A} \operatorname{tr}(ab) b^\vee. \tag{3.2.5}$$

Note that $\bar{b} = \overline{b^\vee}$, and that the left dual basis to $\{b^\vee : b \in \mathbf{B}_A\}$ is given by

$$(b^\vee)^\vee = (-1)^{\bar{b}} b. \tag{3.2.6}$$

Lemma 3.2.4. *For all homogeneous $a, c \in A$, we have*

$$\sum_{b \in \mathbf{B}_A} (-1)^{\bar{b}\bar{c}} abc \otimes b^\vee = (-1)^{\bar{a}\bar{c}} \sum_{b \in \mathbf{B}_A} (-1)^{\bar{b}\bar{c}} b \otimes cb^\vee a. \tag{3.2.7}$$

Proof: We have

$$\sum_{b \in \mathbf{B}_A} (-1)^{\bar{b}\bar{c}} abc \otimes b^\vee \overset{(3.2.5)}{=} \sum_{b,e \in \mathbf{B}_A} (-1)^{\bar{b}\bar{c}} \operatorname{tr}(e^\vee abc) e \otimes b^\vee = \sum_{b,e \in \mathbf{B}_A} (-1)^{\bar{c}\bar{e}+\bar{c}\bar{a}} e \otimes \operatorname{tr}(ce^\vee ab) b^\vee$$

$$\overset{(3.2.5)}{=} (-1)^{\bar{a}\bar{c}} \sum_{e \in \mathbf{B}_A} (-1)^{\bar{c}\bar{e}} e \otimes ce^\vee a,$$

where, in the second equality, we used the fact that $\operatorname{tr}(e^\vee abc) = 0$ unless $\bar{e}+\bar{a}+\bar{b}+\bar{c} = 0$ to simply the exponent of -1. ∎

Remark 3.2.5. *Taking $c = 1$ in Lemma 3.2.4 we have*

$$\sum_{b \in B} ab \otimes b^\vee = \sum_{b \in B} b \otimes b^\vee a. \tag{3.2.8}$$

Similarly, taking $a = 1$ we have

$$\sum_{b \in B} (-1)^{\bar{b}\bar{c}} bc \otimes b^\vee = \sum_{b \in B} (-1)^{\bar{b}\bar{c}} b \otimes cb^\vee. \tag{3.2.9}$$

Chapter 4

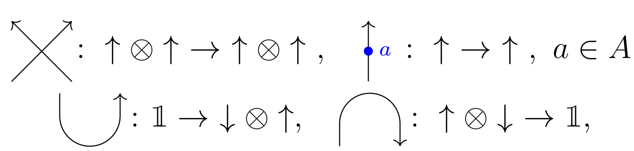
Affine Oriented Frobenius Brauer Categories

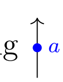
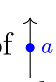
In this chapter we introduce our main object of study: the affine oriented Frobenius Brauer category and the oriented Frobenius category contained within it. As mentioned in Chapter 1, $\mathcal{AOB}(A)$ is the charge 0 case of $\mathcal{Heis}_k(A)$ studied in [21, 24, 7]. However, in our $k = 0$ case the category is symmetric monoidal. In this chapter we define $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$ in terms of generators and relations and prove some additional relations. We also discuss some of the properties of $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$ such as rigidity, pivotality and the symmetric monoidal structure. We also recall the basis theorem for $\mathcal{AOB}(A)$ from [7].

4.1 Definition

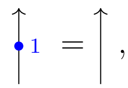
The affine oriented Frobenius Brauer category is the central charge $k = 0$ case of the Frobenius Heisenberg category $\mathcal{Heis}_k(A)$ introduced in [24] and further studied in [7].

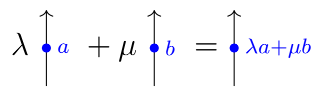
Definition 4.1.1. The *oriented Frobenius Brauer category* $\mathcal{OB}(A)$ associated to the graded Frobenius superalgebra A is the strict graded monoidal supercategory generated by objects $\uparrow$ and $\downarrow$ and morphisms

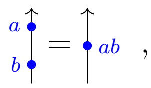


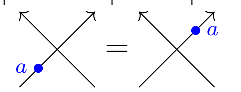
subject to certain relations. We refer to the decorations representing  , $a \in A$, as *tokens*. The parity of  is $\bar{a}$, and all the other generating morphisms are even. We impose the following relations:

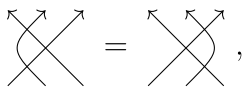
1. *Affine wreath product algebra relations*:

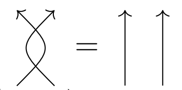

(4.1.1)


(4.1.2)

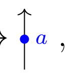

(4.1.3)


(4.1.4)

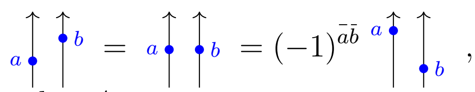

(4.1.5)

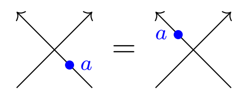

(4.1.6)

for all $a, b \in A$, $\lambda, \mu \in \Bbbk$. It follows from (4.1.1) to (4.1.3) that the map

$$A \to \mathrm{End}_{\mathcal{AOB}(A)}(\uparrow), \quad a \mapsto$$  ,

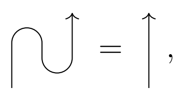
is a superalgebra homomorphism and, also using the super interchange law, that

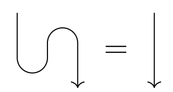

(4.1.7)


(4.1.8)

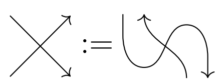
for $a, b \in A$.

2. *Right adjunction relations*: We impose the following relations:

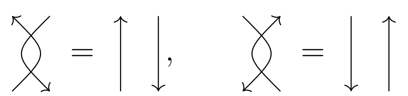

(4.1.9)


(4.1.10)

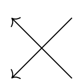
3. *Inversion relation*: The morphism


(4.1.11)

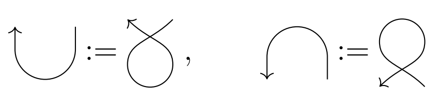
is invertible so that


(4.1.12)

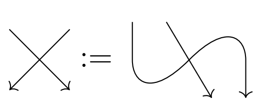
The inversion relation means that there is another generating morphism that is inverse to the right crossing:



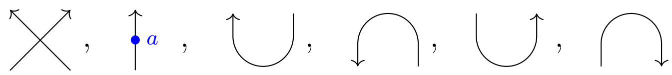
We also define the left cups and caps


(4.1.13)

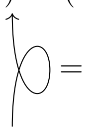
and the downward crossing:

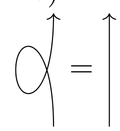

(4.1.14)

Remark 4.1.2. *The oriented Frobenius Brauer category can equivalently be described as the strict* $\Bbbk$*-linear monoidal category generated by* $\uparrow, \downarrow$*, and morphisms*

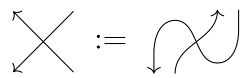


subject to (4.1.1) *to* (4.1.3)*,* (4.1.5)*,* (4.1.6)*,* (4.1.8) *to* (4.1.10) *and* (4.1.12) *as well as*

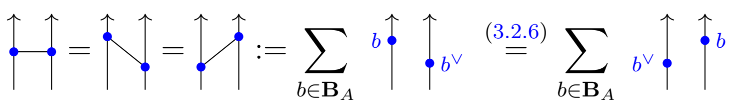

(4.1.15)


(4.1.16)

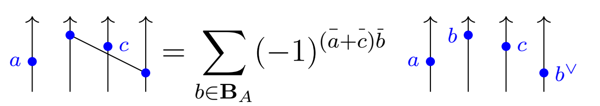
In this case, the left crossing is defined as


(4.1.17)

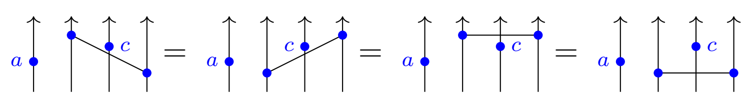
From the imposed relations above we are able to deduce a number of others, which are detailed in Section 4.2. Define the *teleporter*


(4.1.18)

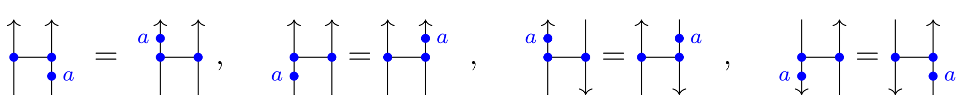
We do not insist that the tokens in a teleporter (4.1.18) are drawn at the same horizontal level. The convention when this is not the case is that b is on the higher of the tokens and $b^\vee$ is on the lower one. We will also draw teleporters in larger diagrams. When doing so, we add a sign of $(-1)^{y\bar{b}}$ in front of the b summand in (4.1.18), where y is the sum of the parities of all morphisms in the diagram vertically between the tokens labeled b and $b^\vee$. For example,



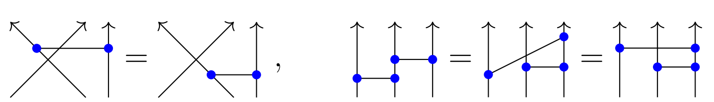
This convention ensures that one can slide the endpoints of teleporters along strands:

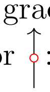


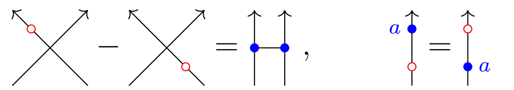
We are able to set many of these conventions as a consequence of the trace map, tr. For example, it follows from (3.2.7) that tokens can "teleport" across teleporters in the sense that, for $a \in A$, we have


(4.1.19)

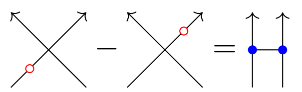
where the strings can occur anywhere in a diagram (i.e. they do not need to be adjacent). The endpoints of teleporters slide through crossings and they can teleport too. For example we have


(4.1.20)

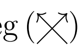
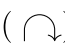
Definition 4.1.3. The *affine oriented Frobenius Brauer category* $\mathcal{AOB}(A)$ associated to the graded Frobenius superalgebra A is the graded monoidal supercategory obtained from $\mathcal{OB}(A)$ by adjoining an even generator : $\uparrow \to \uparrow$, which we call a *dot*, subject to the relations

 $\quad a \in A. \qquad (4.1.21)$

It follows that we also have the relation

 $\qquad (4.1.22)$

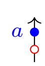
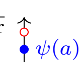
Remark 4.1.4. *If A is $\mathbb{Z}$-graded where* $\deg(\mathrm{tr}) = -d_A$, *the categories $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$ are also naturally $\mathbb{Z}$-graded where the degrees of the generating morphisms are as follows:*

deg () = deg(a), deg () = deg () = deg () = 0.

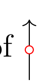
All of the results of this thesis hold in the $\mathbb{Z}$-graded setting.

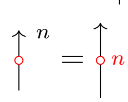
Example 4.1.5. As noted in the introduction, when $A = \Bbbk$, the categories $\mathcal{OB}(\Bbbk)$ and $\mathcal{AOB}(\Bbbk)$ are the *oriented Brauer* and *affine oriented Brauer categories*, respectively; see [4]. The endomorphism algebras of $\mathcal{OB}(\Bbbk)$ are *oriented Brauer algebras*, which are isomorphic to *walled Brauer algebras*.

Remark 4.1.6. *The definitions of $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$ can be generalized to allow for A to be a (not necessarily symmetric) Frobenius superalgebra with trace map of arbitrary parity. The only change to the relations is that the parity of the dot is equal to the parity $\overline{\mathrm{tr}}$ of the trace map (i.e. the dot is odd if the trace map is parity reversing) and the second relation in (4.1.21) becomes*

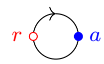
 $= (-1)^{\bar{a}\overline{\mathrm{tr}}}$  , $\quad a \in A,$

where ψ is the Nakayama automorphism. (This level of generality was considered in [24].) With this modification, we can take A to be the two-dimensional Clifford superalgebra Cl*; see Example 3.2.3. Then $\mathcal{OB}(\mathrm{Cl})$ and $\mathcal{AOB}(\mathrm{Cl})$ are the oriented Brauer–Clifford and degenerate affine oriented Brauer–Clifford supercategories, respectively, introduced in [3].*

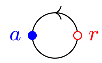
For $n \geq 1$, we denote the n-th power of  by labelling the dot with the exponent n:

 .

We adopt the following conventions for bubbles with a negative number of dots:



$$= -\delta_{r,-1}\,\mathrm{tr}(a), \tag{4.1.23}$$



$$= \delta_{r,-1}\,\mathrm{tr}(a), \tag{4.1.24}$$

for $r < 0$ and $a \in A$.

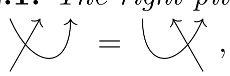
4.2 Subsequent Relations

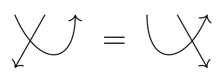
The following relations and their consequences are a result of the above imposed relations on $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$. Relations not involving dots hold in both $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$ while the dotted relations hold only in $\mathcal{AOB}(A)$.

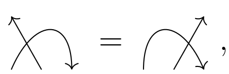
4.2.1 Rigidity

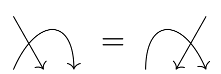
A *rigid category* is a monodial category where every object has a left and right dual. The right adjunction relations, (4.1.9) and (4.1.10), tell us that $\downarrow$ is right dual to $\uparrow$. We also have left adjunction relations and so $\downarrow$ is left dual to $\uparrow$ and $\mathcal{OB}(A)$ is a rigid category. Before showing the left adjunction relations we will need a number of other relations which all follow from the defining relations.

Lemma 4.2.1. *The right pitchfork relations hold:*

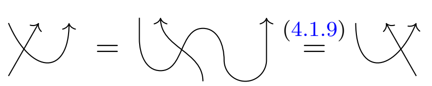
 (4.2.1)

 (4.2.2)

 (4.2.3)

 (4.2.4)

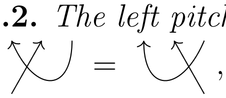
Proof: To see (4.2.1) we have

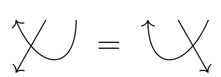


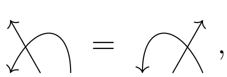
The remaining right pitchfork relations can be verified similarly. ∎

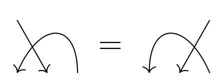
We also have left pitchfork relations.

Lemma 4.2.2. *The left pitchfork relations hold:*

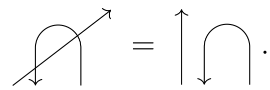
 (4.2.5)

 (4.2.6)

 (4.2.7)

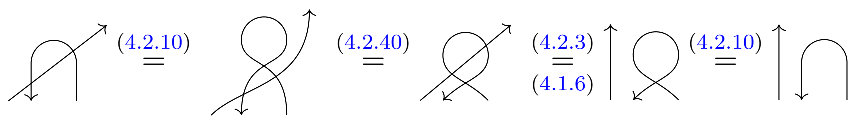
 (4.2.8)

Proof: To verify (4.2.7) we first claim that

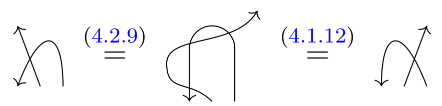


(4.2.9)

We have



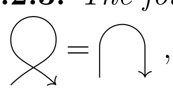
This completes the proof of (4.2.9). Now we have



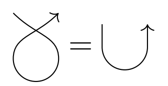
The proof of (4.2.8) is similar and (4.2.5) and (4.2.6) follow from Lemma 4.2.16. ∎

In Definitions 4.1.1 and 4.1.3, we define the left cups and caps as right cups and caps with crossings, in fact, we can view the right cups and caps in a similar way.

Lemma 4.2.3. *The following relations hold:*

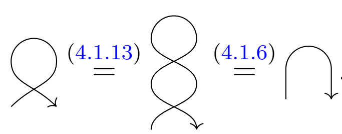


(4.2.10)



(4.2.11)

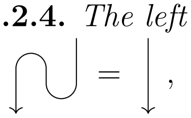
Proof: To see (4.2.10) we have



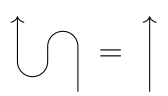
(4.2.11) follows from Lemma 4.2.16. ∎

Now we can verify the left adjunction relations.

Lemma 4.2.4. *The left adjunction relations hold:*

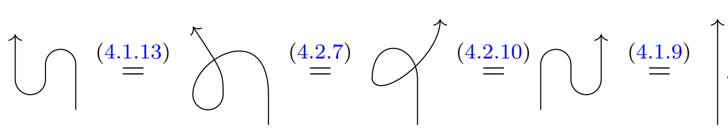


(4.2.12)



(4.2.13)

Proof: To see (4.2.12) we have

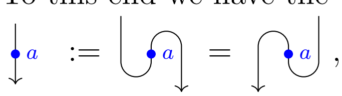


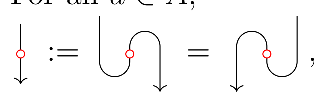
(4.2.12) follows by Lemma 4.2.16. ∎

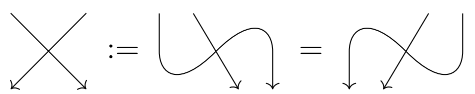
4.2.2 Pivotality

A *pivotal category* is a rigid category with a natural isomorphism ${}^*(-) \Rightarrow (-)^*$, where ${}^*(-)$ is the left dual functor sending each object and morphism to its left dual and $(-)^*$ is the right. In a *strict pivotal category* the above natural isomorphism is in fact the identity. In Section 4.2.1 we saw that $\downarrow$ is left and right dual to $\uparrow$, so to see that $\mathcal{OB}(A)$ is strictly pivotal, we need to verify that the left and right dual of each generating morphism are equal.

To this end we have the *rotation relations*. For all $a \in A$,

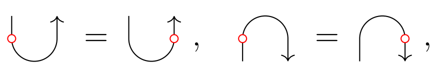

(4.2.14)

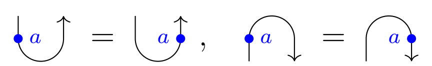

(4.2.15)


(4.2.16)

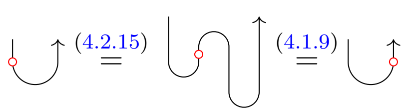
Note that in (4.2.14) to (4.2.16) we take the first equalities as definitions and prove the second equalities in Lemma 4.2.11. These definitions give us tokens and dots, in addition to crossings, on downward strands.

Lemma 4.2.5. *For all $a \in A$,*

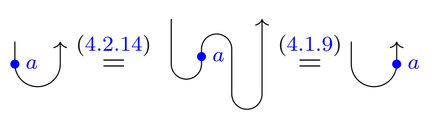

(4.2.17)


(4.2.18)

Proof: The first relation in (4.2.17) can be verified as follows

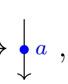


To see the first relation in (4.2.18) we have



The remaining relations follow from Lemma 4.2.16. ∎

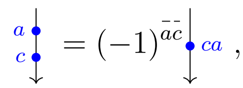
Lemma 4.2.6. *There is an anti-homomorphism of superalgebras*

$$A \to \operatorname{End}\downarrow, \quad a \mapsto$$  (4.2.19)

that is, a homomorphism

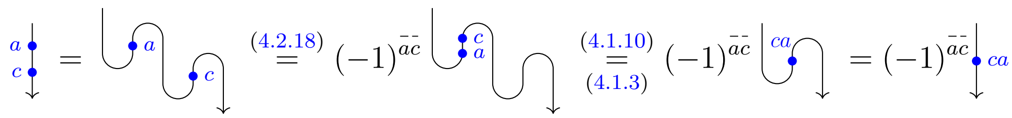
$$A \to (\operatorname{End}\downarrow)^{\mathrm{op}}.$$

Proof: It suffices to show



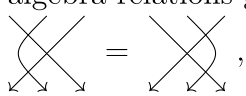
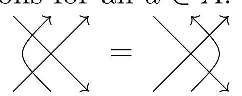
$a, c \in A,$

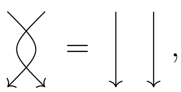
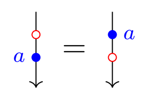
We have

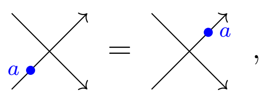
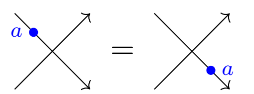


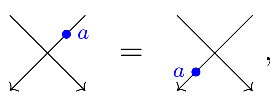
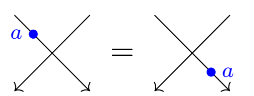
∎

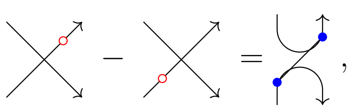
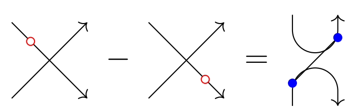
Attaching right caps to the top and right cups to the bottom of the affine wreath product algebra relations gives the following relations for all $a \in A$:

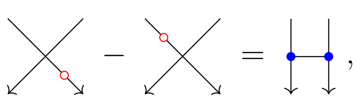
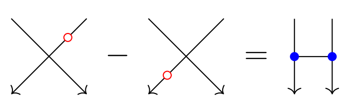
 (4.2.20)  (4.2.21)

 (4.2.22)  (4.2.23)

 (4.2.24)  (4.2.25)

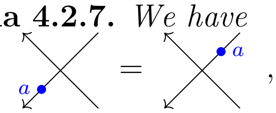
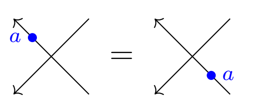
 (4.2.26)  (4.2.27)

 (4.2.28)  (4.2.29)

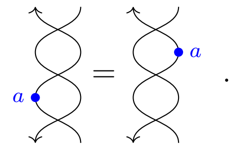
 (4.2.30)  (4.2.31)

In addition to the above downwards and right crossing relations, we have some similar ones for left crossings.

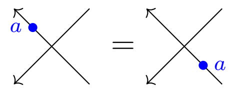
Lemma 4.2.7. *We have*

 (4.2.32)  (4.2.33)

Proof: To prove (4.2.32), first compose (4.2.24) on the top and bottom with the left crossing as follows



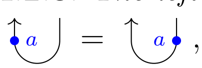
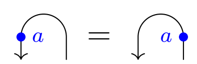
Then using (4.1.12) on the top left and bottom right we have



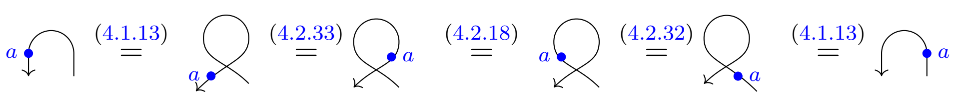
as desired. The proof for (4.2.33) is similar beginning instead with (4.2.25). ∎

Now we can show tokens can slide over left caps and cups.

Lemma 4.2.8. *The left cup and cap token slide relations hold:*

 (4.2.34)  (4.2.35)

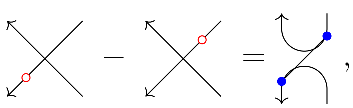
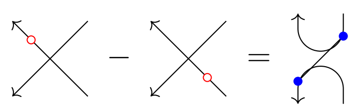
Proof: We prove the cap version and the cup version is similar. We have



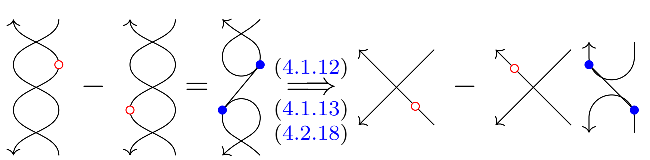
∎

We have similar results involving dots.

Lemma 4.2.9. *The left dot slide relations hold:*

 (4.2.36)  (4.2.37)

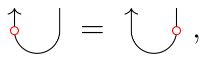
Proof: We prove (4.2.37), since (4.2.36) is similar. Composing (4.2.28) on the top and bottom with we have

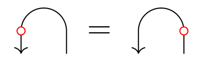


∎

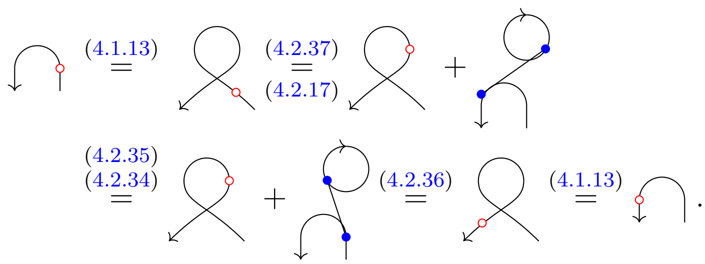
Now we can show that dots can slide over left cups and caps:

Lemma 4.2.10. *The left cup and cap dot slide relations hold:*


(4.2.38)


(4.2.39)

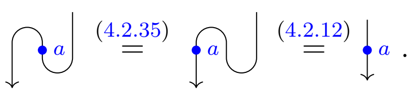
Proof: We prove (4.2.39), since (4.2.38) is similar. We have



∎

Lemma 4.2.11. *The rotation relations,* (4.2.14) *to* (4.2.16)*, hold.*

Proof: We have

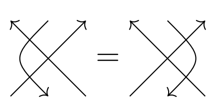


Similarly, (4.2.15) follows from (4.2.12) and (4.2.39) and (4.2.16) follows from (4.2.8) and (4.2.12). ∎

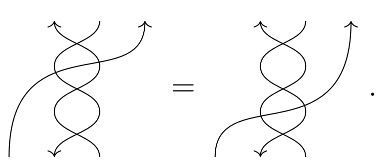
4.2.3 Symmetric Monoidal Structure

We have seen that for a number of our defining relations, they actually hold for all orientations. This is also the case for the braid relations.

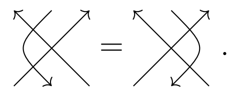
Lemma 4.2.12. *We have the alternating braid relation:*


(4.2.40)

Proof: We start with (4.2.21) and attach crossings to the top left and bottom right pairs of strands to obtain

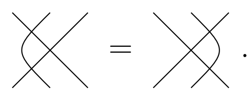

(4.2.41)

Using (4.1.12) on the bottom left and the top right, (4.2.41) becomes



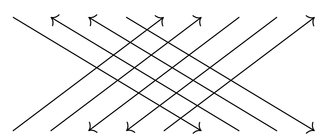
∎

Remark 4.2.13. *We now have the braid relations for all orientations*



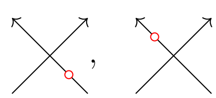
(4.2.42)

Remark 4.2.14. *The oriented Frobenius Brauer category, $\mathcal{OB}(A)$, is a symmetric monoidal supercategory with symmetric braiding, s, given by the obvious diagram crossing. For example, for $A =\uparrow\uparrow\downarrow\downarrow\uparrow$ and $B =\downarrow\uparrow\uparrow\downarrow$, $s_{A\otimes B}$ is given by*



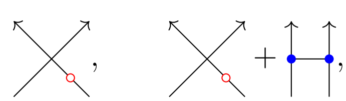
To see $\mathcal{OB}(A)$ is symmetric, note that diagram (2.1.6) follows immediately and (2.1.8) follows from (4.1.6), (4.1.12) and (4.2.22). Diagram (2.1.7) follows from associativity of the tensor product and supernaturality of the symmetric braiding follows from (4.1.4), (4.1.8), (4.2.24) to (4.2.27), (4.2.32), (4.2.33) and (4.2.42).

Remark 4.2.15. *The affine oriented Frobenius Brauer category, $\mathcal{AOB}(A)$ is not symmetric monoidal. In particular, rewriting*



(4.2.43)

using the basis $D_0(\uparrow\uparrow,\uparrow\uparrow)$ from Theorem 4.3.1, as well as (4.1.21), we have

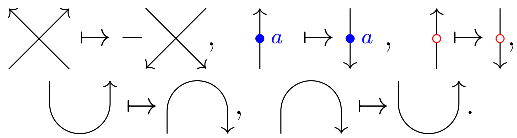


respectively. Then, we can conclude the two diagrams in (4.2.43) are not equal since the coefficients are distinct when written in the basis and so $\mathcal{AOB}(A)$ is not symmetric monoidal.

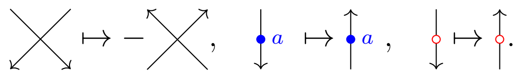
Lemma 4.2.16. *There is an isomorphism of monoidal categories*

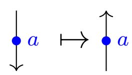
$$\omega\colon \mathcal{AOB}(A) \xrightarrow{\cong} \mathcal{AOB}^{\mathrm{op}}(A^{\mathrm{op}}) \tag{4.2.44}$$

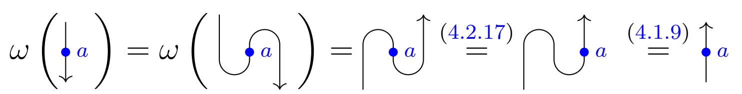
interchanging the objects ↑ and ↓ and defined on the generating morphisms by



Proof: The functor ω preserves the affine wreath product algebra relations, the right adjunction relations, and the inversion relations by (4.1.12), (4.2.12), (4.2.13), (4.2.19), (4.2.22), (4.2.23), (4.2.27), (4.2.31) and (4.2.40). By (4.1.9) and (4.1.10) we have



For example, to see  note that



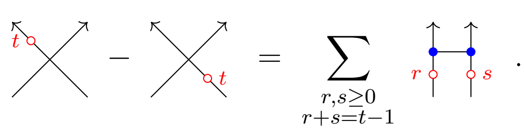
Thus $\omega^2 = \mathrm{id}$. Hence ω is an isomorphism. ∎

Diagrammatically, ω reflects diagrams in the horizontal axis, and then multiplies by $(-1)^{\binom{n}{2}+k}$, where k is the total number of crossings appearing in the diagram and n is the number of odd tokens. Note that (4.1.23) and (4.1.24) are compatible with the action of ω.

4.2.4 Bubble Slide and Infinite Grassmanian Relations

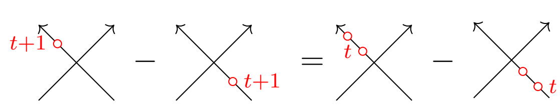
We can generalize each of the dot slide relations, (4.1.22), (4.2.28) to (4.2.31), (4.2.36) and (4.2.37) for an arbitrary number of dots. We give the proof for (4.1.21) and the rest are similar. These general relations are needed in the proof of the bubble slide relations, (4.2.49) and (4.2.50).

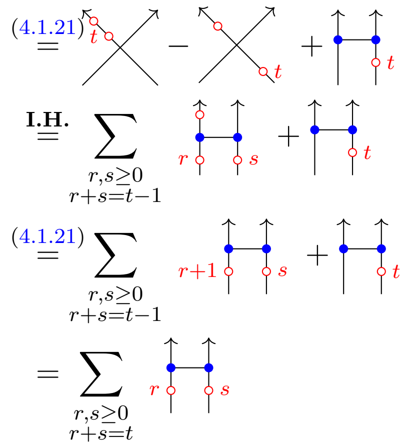
Lemma 4.2.17. *The following generalization of* (4.1.21) *holds for $a \in A$ and $t \geq 1$:*



(4.2.45)

Proof: We proceed by induction. In the case $t = 1$ we arrive at (4.1.21). Suppose the result holds for $t \geq 1$. We have





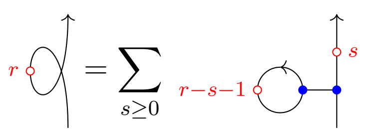
where the third equality comes from our induction hypothesis. ∎

For any homogeneous $a \in A$, we define

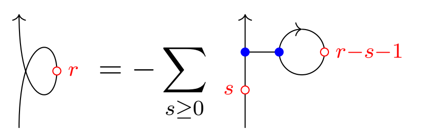
$$a^{\dagger} := \sum_{b \in \mathbf{B}_A} (-1)^{\bar{a}\bar{b}} bab^{\vee}, \tag{4.2.46}$$

which is well-defined independent of the choice of the basis $\mathbf{B}_A$. First we prove the curl relations, which are needed in the proof of (4.2.49) and (4.2.50).

Proposition 4.2.18. *For all $r \geq 0$, we have the curl relations:*



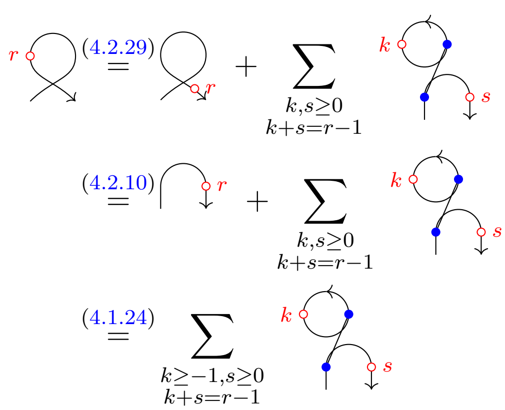
(4.2.47)

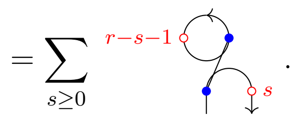


(4.2.48)

Proof: First note that the sums in these relations are indeed finite since by (4.1.23) and (4.1.24), the bubbles will be 0 whenever $s > r$.

We have

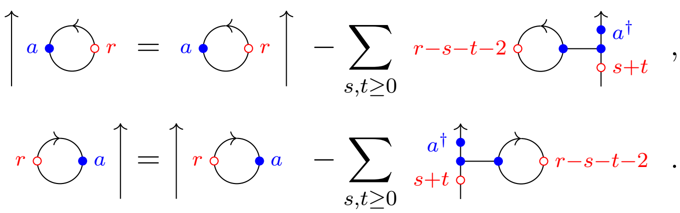




Placing an upward strand on the right, joining the bottom of the two rightmost strands with a right cup, and using (4.1.9) gives (4.2.47). Relation (4.2.48) is obtained similarly. ∎

Note that taking $r = 0$ in (4.2.47) and (4.2.48), using (4.1.23) and (4.1.24), we arrive at (4.1.15) and (4.1.16) from the alternate definition of $\mathcal{OB}(A)$ in Remark 4.1.2.

Proposition 4.2.19. *We have the bubble slide relations:*

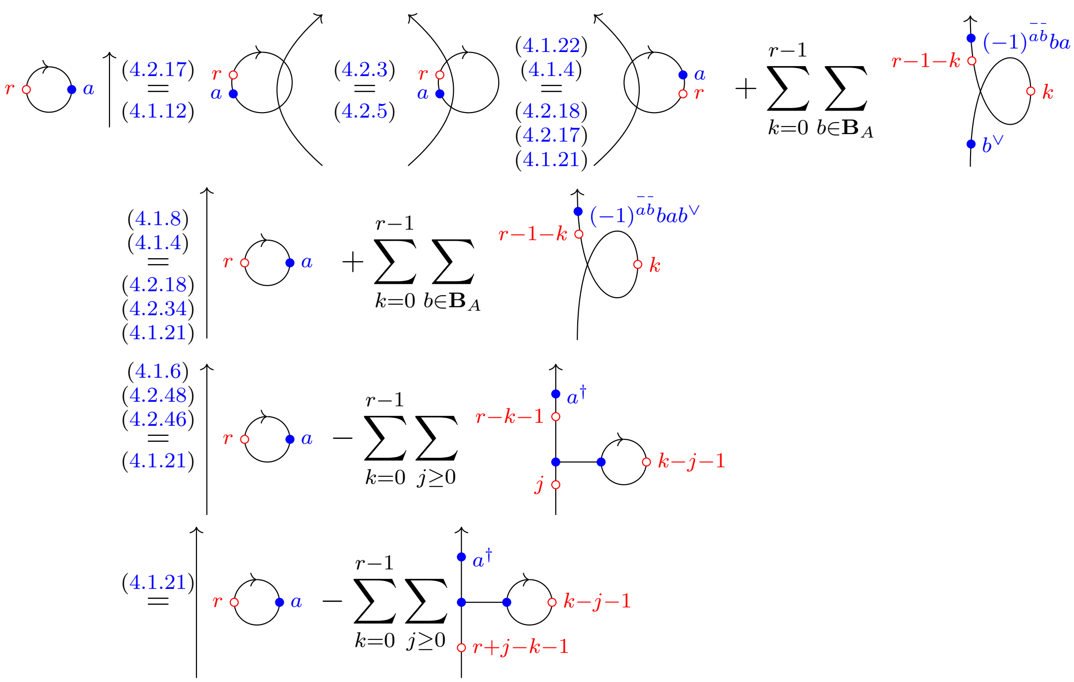


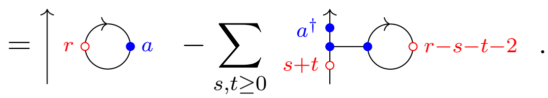
(4.2.49)

(4.2.50)

Proof:

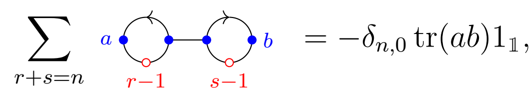
It suffices to prove (4.2.50), since then (4.2.49) follows by applying ω from Lemma 4.2.16, placing upward strands to the left and right of all diagrams, connecting the tops of the two leftmost straight strands with a right cap, connecting the bottoms of the rightmost strands with a right cup, and using (4.1.9), (4.2.17) and (4.2.18). We have





where, in the last equality, we changed the summations over k and j to summations over s and t where $s+t = r+j-k-1$ (using (4.1.23) in the process). ∎

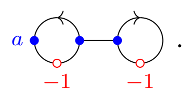
Proposition 4.2.20. *We have the infinite Grassmannian relation:*



(4.2.51)

Proof: First note that the sum is finite since by (4.1.23) and (4.1.24), whenever $r < 0$ or $s < 0$, the left and right dotted bubbles will be 0, respectively. Thus we are only summing over non negative r and s and are bounded above by n.

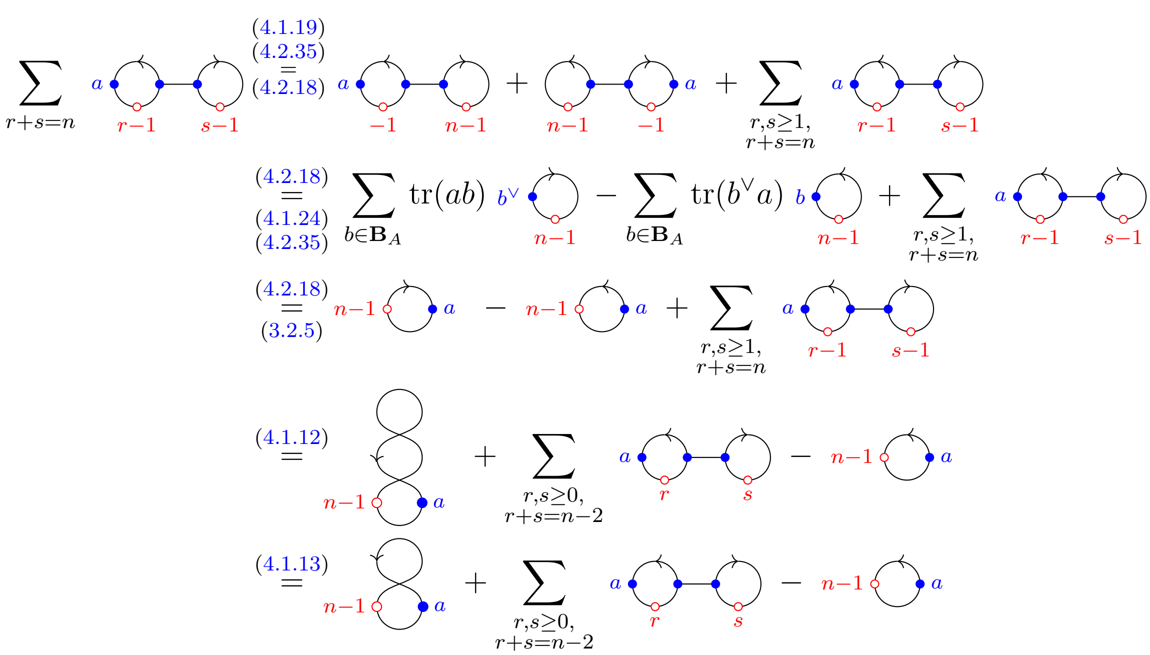
By (3.2.7), we can consider the case when $b = 1$. When $n = 0$, the left of (4.2.51) becomes

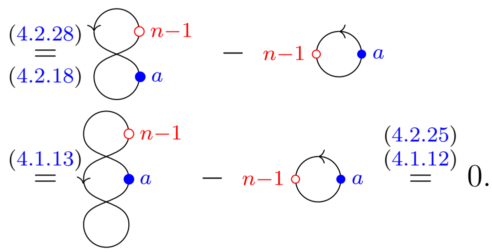


By (4.1.23) and (4.1.24), this becomes

$$-\sum_{b\in B}\mathrm{tr}(ab)\,\mathrm{tr}(b^\vee) = -\,\mathrm{tr}\left(\sum_{b\in B}\mathrm{tr}(ab)b^\vee\right) \overset{(3.2.5)}{=} -\,\mathrm{tr}(a)$$

as desired. When $n > 0$, we have





■

4.3 The Basis Theorem

We next recall the basis theorem for $\mathcal{AOB}(A)$.

We define $\mathrm{Sym}(A)$ to be the symmetric superalgebra generated by the vector superspace $C(A)[x]$, where x here is an even indeterminate. Recall that for $a \in A$, we let $\dot{a}$ denote its canonical image in $C(A)$. For $n \in \mathbb{Z}$ and $a \in A$, let $e_n(a) \in \mathrm{Sym}(A)$ denote

$$e_n(a) := \begin{cases} 0 & \text{if } n < 0, \\ \mathrm{tr}(a) & \text{if } n = 0, \\ \dot{a}x^{n-1} & \text{if } n > 0. \end{cases} \tag{4.3.1}$$

Since tr is parity-preserving, this defines a parity-preserving linear map $e_n \colon A \to \mathrm{Sym}(A)$. By [7, Lem. 7.1], for each $n \in \mathbb{Z}$ there is a unique parity-preserving linear map $h_n \colon A \to \mathrm{Sym}(A)$ such that

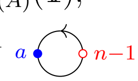
$$\sum_{r+s=n} \sum_{c \in \mathbf{B}_A} (-1)^r e_r(ac) h_{r-s}(c^\vee b) = \delta_{n,0}\, \mathrm{tr}(ab), \qquad \text{for all } a, b \in A \tag{4.3.2}$$

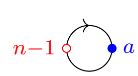
and $h_n(a) = 0$ for $a \in A$ and $n < 0$.

In the special case that $A = \Bbbk$, $\mathrm{Sym}(A)$ may be identified with the algebra of symmetric functions so that $e_n(1)$ corresponds to the n-th elementary symmetric function and $h_n(1)$ corresponds to the n-th complete symmetric function.

We have a homomorphism of superalgebras

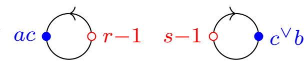
$$\beta \colon \mathrm{Sym}(A) \to \mathrm{End}_{\mathcal{AOB}(A)}(\mathbb{1}),$$

$$e_n(a) \mapsto (-1)^{n-1}$$  , (4.3.3)

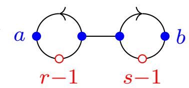
$$h_n(a) \mapsto$$  , $n \geq 1$.

To see this is a superalgebra homomorphism, since the $e_n(a), h_n(a)$ generate $\mathrm{Sym}(A)$ as a superalgebra and (4.3.2) allows us to write the h_n in terms of the e_n, it suffices to show (4.3.2) holds in $\mathrm{End}_{\mathcal{AOB}(A)}(\mathbb{1})$. Under β, we have that

$$\sum_{r+s=n} \sum_{c\in \mathbf{B}_A} (-1)^r e_r(ac) h_s(c^\vee b) \mapsto \sum_{r+s=n} \sum_{c\in \mathbf{B}_A} (-1)^r (-1)^{r-1}$$



$$\overset{(4.2.18)}{\underset{(4.1.3)\,(4.2.35)}{=}} \sum_{r+s=n} (-1)^{2r-1}$$



$$\overset{(4.2.51)}{=} -\delta_{n,0}\,\mathrm{tr}(ab) 1_{\mathbb{1}}$$

as desired.

Let $X = X_n \otimes \cdots \otimes X_1$ and $Y = Y_m \otimes \cdots \otimes Y_1$ be objects of $\mathcal{AOB}(A)$ for $X_i, Y_j \in \{\uparrow, \downarrow\}$. An (X,Y)-*matching* is a bijection between the sets

$$\{i : X_i = \uparrow\} \sqcup \{j : Y_j = \downarrow\} \quad \text{and} \quad \{i : X_i = \downarrow\} \sqcup \{j : Y_j = \uparrow\}.$$

A *reduced lift* of an (X,Y)-matching is a string diagram representing a morphism $X \to Y$ such that

- the endpoints of each string are points which correspond under the given matching;
- there are no floating bubbles and no dots or tokens on any string;
- there are no self-intersections of strings and no two strings cross each other more than once.

For each (X,Y) matching, fix a set $D(X,Y)$ consisting of a choice of reduced lift for each (X,Y)-matching. Then let $D_\circ(X,Y)$ denote the set of all morphisms that can be obtained from the elements of $D(X,Y)$ by adding a nonnegative number of dots and one element of $\mathbf{B}_A$ near to the terminus of each string (i.e. such that there are no crossings between the terminus and the dots and elements of $\mathbf{B}_A$).

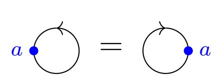
Using the homomorphism β from (4.3.3), we have that, for $X, Y \in \mathcal{AOB}(A)$, $\mathrm{Hom}_{\mathcal{AOB}(A)}(X,Y)$ is a right $\mathrm{Sym}(A)$-supermodule under the action

$$\phi\theta := \phi \otimes \beta(\theta), \quad \phi \in \mathrm{Hom}_{\mathcal{AOB}(A)}(X,Y),\ \theta \in \mathrm{Sym}(A).$$

Theorem 4.3.1 ([7, Th. 7.2])**.** *For $X, Y \in \mathcal{AOB}(A)$, the morphism space $\mathrm{Hom}_{\mathcal{AOB}(A)}(X,Y)$ is a free right $\mathrm{Sym}(A)$-supermodule with basis $D_\circ(X,Y)$.*

It follows from Theorem 4.3.1 that the map (4.3.3) is an isomorphism of superalgebras.

Taking $n = 1$ in (4.2.51) and using (4.1.23), (3.2.5) and (4.1.24) gives

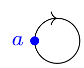


for all $a \in A$. (4.3.4)

It follows from the bubble slide relations (4.2.49) and (4.2.50) these bubbles, which have the same degree and parity as a, are strictly central:

For further discussion of the above bubbles see [24, Sec. 1].

For any linear map $\theta \colon C(A) \to \Bbbk$, we can define the *specialized oriented Frobenius Brauer category* $\mathcal{OB}(A, \theta)$ by imposing on $\mathcal{OB}(A)$ the additional relation



$$= \theta(a), \quad a \in A. \tag{4.3.5}$$

Similarly, we can define the *specialized affine oriented Frobenius Brauer category* $\mathcal{AOB}(A, \theta)$ by imposing on $\mathcal{AOB}(A)$ the relation (4.3.5). We will see in (5.3.7) that, under the categorical action to be defined in Chapter 5, the bubbles (4.3.4) act by multiplication by the supertrace of the map $V_+ \to V_+$, $v \mapsto av$. Hence these actions factor through the corresponding specialized categories.

Chapter 5

Categorical Action

In this chapter we work towards defining the categorical action of $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$ on the general linear Lie superalgebra. When $k \neq 0$, the category $\mathcal{H}eis_k(A)$ acts naturally on categories of modules over the cyclotomic wreath product algebras defined in [25] and so in this chapter, we fill in the gap and consider the natural action in the $k = 0$ case.

5.1 General Linear Lie Superalgebra

Let $\mathfrak{g}$ denote the Lie superalgebra associated to $\mathrm{Mat}_{m|n}(A)$. Precisely, $\mathfrak{g}$ is equal to $\mathrm{Mat}_{m|n}(A)$ as a $\Bbbk$-supermodule and the Lie superbracket is defined by

$$[M, N] = MN - (-1)^{\overline{M}\,\overline{N}} NM.$$

for homogeneous $M, N \in \mathfrak{g}$ and extended by linearity.

Example 5.1.1. When $A = \Bbbk$, $\mathfrak{g}$ is the usual general linear superalgebra over $\Bbbk$.

Example 5.1.2. When $A = \Bbbk[t]/(t^\ell)$, $\ell \geq 2$, $\mathfrak{g} = \mathfrak{gl}_{m|n}(A)$ is a *truncated current superalgebra*. When $n = 0$, $\mathfrak{g}$ is a *Takiff algebra*.

Example 5.1.3. When $A = \mathrm{Cl}$ (see Example 3.2.3), $\mathfrak{g}$ is the *queer Lie superalgebra*. The queer Lie superalgebra, $\mathfrak{q}(n)$, has even and odd parts

$$\mathfrak{q}(n)_{\bar{0}} = \left\{ \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} : M \in \mathrm{Mat}_n(\Bbbk) \right\}, \quad \mathfrak{q}(n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} : M \in \mathrm{Mat}_n(\Bbbk) \right\},$$

where $\mathrm{Mat}_n(\Bbbk)$ denotes the set of $n \times n$ matrices with entries in $\Bbbk$. Then we have an isomorphism of Lie superalgebras $\mathfrak{q}(m+n) \xrightarrow{\cong} \mathfrak{g}_{m|n}(\mathrm{Cl})$ given by

$$\begin{pmatrix} 1_{(i,j)} & 0 \\ 0 & 1_{(i,j)} \end{pmatrix} \mapsto c^{p(i)+p(j)} 1_{(i,j)}, \quad \begin{pmatrix} 0 & 1_{(i,j)} \\ 1_{(i,j)} & 0 \end{pmatrix} \mapsto c^{p(i)+p(j)+1} 1_{(i,j)}, \quad 1 \leq i, j \leq m+n.$$

Let $\vartheta\colon \mathfrak{q}(m+n) \xrightarrow{\cong} \mathfrak{g}_{m|n}(\mathrm{Cl})$ denote the given map. Define

$$E_{i,j} := \begin{pmatrix} 1_{(i,j)} & 0 \\ 0 & 1_{(i,j)} \end{pmatrix}, \quad E'_{i,j} := \begin{pmatrix} 0 & 1_{(i,j)} \\ 1_{(i,j)} & 0 \end{pmatrix}.$$

The map ϑ is clearly an isomorphism of vector superspaces. For $1 \le i, j \le n$, we have

$$\begin{aligned} [\vartheta(E_{i,j}), \vartheta(E_{k,l})] &= [c^{p(i)+p(j)} 1_{(i,j)}, c^{p(k)+p(l)} 1_{(k,l)}] \\ &= \delta_{j,k} c^{p(i)+p(l)} 1_{(i,l)} - \delta_{i,l} c^{p(j)+p(k)} 1_{(k,j)} \\ &= \vartheta\left(\delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j}\right) \\ &= \vartheta\left([E_{i,j}, E_{k,l}]\right), \end{aligned}$$

$$\begin{aligned} [\vartheta(E'_{i,j}), \vartheta(E'_{k,l})] &= [c^{p(i)+p(j)+1} 1_{(i,j)}, c^{p(k)+p(l)+1} 1_{(k,l)}] \\ &= \delta_{j,k} c^{p(i)+p(l)} 1_{(i,l)} + \delta_{i,l} c^{p(j)+p(k)} 1_{(k,j)} \\ &= \vartheta\left(\delta_{j,k} E_{i,l} + \delta_{i,l} E_{k,j}\right) \\ &= \vartheta\left([E'_{i,j}, E'_{k,l}]\right), \end{aligned}$$

and

$$\begin{aligned} [\vartheta(E_{i,j}), \vartheta(E'_{k,l})] &= [c^{p(i)+p(j)} 1_{(i,j)}, c^{p(k)+p(l)+1} 1_{(k,l)}] \\ &= \delta_{j,k} c^{p(i)+p(l)+1} 1_{(i,l)} - \delta_{i,l} c^{p(j)+p(k)+1} 1_{(k,j)} \\ &= \vartheta\left(\delta_{j,k} E'_{i,l} - \delta_{i,l} E'_{k,j}\right) \\ &= \vartheta\left([E_{i,j}, E'_{k,l}]\right). \end{aligned}$$

5.2 Supercategory of Right $\mathfrak{g}$-Supermodules

Let $A^{m|n}$ denote the $\Bbbk$-supermodule equal to A^{m+n} as a $\Bbbk$-module, with $\mathbb{Z}_2$-grading determined by

$$\overline{ae_i} = \bar{a} + p(i),$$

where e_i, $1 \le i \le m+n$, denotes the element of $A^{m|n}$ with a 1 in the i-th entry and all other entries equal to 0. We will also consider $A^{m|n}$ as a left A-module with action

$$a(a_1, \dots, a_{m+n}) = (aa_1, \dots, aa_{m+n}).$$

Definition 5.2.1. A right supermodule, W, over a Lie superalgebra, L, is a $\Bbbk$-vector space with a $\Bbbk$-bilinear map $W \times L \to W$, $(w, X) \mapsto wX$ such that

$$W_i L_j \subseteq W_{i+j}, \quad i, j \in \mathbb{Z}_2,$$

and

$$w[X,Y] = (wX)Y - (-1)^{\overline{X}\,\overline{Y}}(wY)X, \quad X, Y \in L, w \in W.$$

Definition 5.2.2. For a Lie superalgebra L and right L-supermodules, V and W, a right L-supermodule homomorphism is a linear map $f : V \to W$ such that $f(vX) = f(v)X$ for $v \in V, X \in L$.

Let smod-$\mathfrak{g}$ denote the category of right $\mathfrak{g}$-supermodules where objects are right $\mathfrak{g}$-supermodules and morphisms are linear maps between them.

Remark 5.2.3. *We can also define the smaller category of right $\mathfrak{g}$-supermodules where objects are right $\mathfrak{g}$-supermodules and morphisms are purely even maps between them. This category is a monoidal category with tensor product of supermodules.*

Lemma 5.2.4. *The category of right $\mathfrak{g}$-supermodules, smod-$\mathfrak{g}$, is a monoidal supercategory with tensor product of supermodules such that*

$$(m \otimes n) \cdot X = (-1)^{\bar{n}\overline{X}} m \cdot X \otimes n + m \otimes n \cdot X$$

for $m \otimes n \in M \otimes N$, $M, N \in \text{smod-}\mathfrak{g}$, $X \in \mathfrak{g}$ and trivial object $\Bbbk$ where $k \cdot X = 0$ for $X \in \mathfrak{g}$ and $k \in \Bbbk$.

Proof: First we need to verify that for right $\mathfrak{g}$-supermodules, M, N, their tensor product, $M \otimes N$ is once again a right $\mathfrak{g}$-supermodule.

It is straightforward to see $M \otimes N$ is a $\Bbbk$-vector superspace, so it remains to verify

$$(m \otimes n)[X, Y] = ((m \otimes n)X)Y - (-1)^{\overline{X}\overline{Y}}((m \otimes n)Y)X$$

for $m \otimes n \in M \otimes N$, $M, N \in \text{smod-}\mathfrak{g}$, $X, Y \in \mathfrak{g}$. We have

$$\begin{aligned}
&(m \otimes n)[X, Y] = (-1)^{\bar{n}(\overline{X}+\overline{Y})} m[X, Y] \otimes n + m \otimes n[X, Y] \\
&= (-1)^{\bar{n}(\overline{X}+\overline{Y})}(mYX \otimes n - (-1)^{\overline{X}\overline{Y}} mXY \otimes n) + m \otimes nYX - (-1)^{\overline{X}\overline{Y}} m \otimes nXY \\
&= (-1)^{\bar{n}(\overline{X}+\overline{Y})} mYX \otimes n + m \otimes nYX - (-1)^{\overline{X}\overline{Y}}((-1)^{\bar{n}(\overline{X}+\overline{Y})} mXY \otimes n + m \otimes nXY) \\
&= ((-1)^{\bar{n}\overline{Y}} mY \otimes n + m \otimes nY)X - (-1)^{\overline{X}\overline{Y}}((-1)^{\bar{n}\overline{X}} mX \otimes n + m \otimes nX)Y \\
&= ((m \otimes n)X)Y - (-1)^{\overline{X}\overline{Y}}((m \otimes n)Y)X.
\end{aligned}$$

Any morphism $\phi : M \to N$ can be decomposed into parity preserving and parity reversing morphisms $\phi = \phi_0 + \phi_1$ where

$$\phi_0(m) = \begin{cases} \phi(m), & \overline{\phi(m)} = \overline{m}, \\ 0, & \overline{\phi(m)} = \overline{m} + 1 \end{cases}$$

and

$$\phi_1(m) = \begin{cases} \phi(m), & \overline{\phi(m)} = \overline{m} + 1, \\ 0, & \overline{\phi(m)} = \overline{m} \end{cases}$$

for homogeneous $m \in M$. To see ϕ_0 is indeed a morphism note that for $m \in M, X \in \mathfrak{g}$ we have

$$\begin{aligned}\phi_0(mX) &= \begin{cases}\phi(mX), & \overline{\phi(mX)} = \overline{m} + \overline{X}, \\ 0, & \overline{\phi(mX)} = \overline{m} + \overline{X} + 1\end{cases} \\ &= \begin{cases}\phi(m)X, & \overline{\phi(m)X} = \overline{m} + \overline{X}, \\ 0, & \overline{\phi(m)X} = \overline{m} + \overline{X} + 1\end{cases} \\ &= \begin{cases}\phi(m)X, & \overline{\phi(m)} + \overline{X} = \overline{m} + \overline{X}, \\ 0, & \overline{\phi(m)} + \overline{X} = \overline{m} + \overline{X} + 1\end{cases} \\ &= \begin{cases}\phi(m)X, & \overline{\phi(m)} = \overline{m}, \\ 0, & \overline{\phi(m)} = \overline{m} + 1\end{cases} = \phi_0(m)X.\end{aligned}$$

To see ϕ_1 is a morphism note that $\phi_1 = \phi - \phi_0$. So the morphism spaces are vector superspaces and smod-$\mathfrak{g}$ is a supercategory.

Associativity of $\otimes$, or the fact that diagram (2.1.3) commutes, follows from associativity of the tensor product of supermodules. The fact that $\otimes$ is unital, or the fact that diagram (2.1.4) commutes, follows from the natural isomorphisms

$$\begin{aligned}\lambda_M : \Bbbk \otimes M &\to M, \\ k \otimes m &\mapsto mk, \\ \rho_M : M \otimes \Bbbk &\to M, \\ m \otimes k &\mapsto mk.\end{aligned} \tag{5.2.1}$$

We will verify λ is a supernatural isomorphism and the proof for ρ is similar. We need to verify that λ is natural in M and each component, λ_M, is indeed a right $\mathfrak{g}$-supermodule isomorphism. It is immediate λ_M is bijective. We have that

$$\lambda_M((k\otimes m)X) = \lambda((-1)^{\overline{m}\overline{X}}kX\otimes m + k\otimes mX) = \lambda_M(k\otimes mX) = mX = (\lambda_M(k\otimes M))X$$

for $X \in \mathfrak{g}$ and

$$\overline{\lambda_M(k \otimes m)} = \overline{km} = \overline{k} + \overline{m} = \overline{k \otimes m}.$$

Moreover, for $M, N \in$ smod-$\mathfrak{g}$ and $f : M \to N$ we have

$$\begin{array}{ccc} M \otimes \Bbbk & \xrightarrow{(-1)^{r\overline{f}} f \otimes 1_{\Bbbk}} & N \otimes \Bbbk \\ {\scriptstyle \lambda_M}\downarrow & & \downarrow{\scriptstyle \lambda_N} \\ M & \xrightarrow[f]{} & \mathbb{N} \end{array}$$

since $f(\lambda_M(m\otimes k)) = f(km) = kf(m) = \lambda_N(f(m)\otimes k) = (-1)^{r\bar{f}}\lambda_N \circ (f\otimes 1_{\Bbbk})(m\otimes k)$ since $r = 0$. Thus λ_M is indeed natural in M. So λ_M is a right $\mathfrak{g}$-supermodule homomorphism. ∎

Remark 5.2.5. *When working with commuting actions of superalgebras, it is most natural to work with one left action and one right action, as this avoids signs arising from the actions of the two algebras commuting past one another. Since we will want the category introduced in Chapter 4 to act on the left, we choose to work with* right *$\mathfrak{g}$-supermodules. The reader who wishes to work with left $\mathfrak{g}$-supermodules instead can use the standard equivalence between the categories of right and left supermodules. Precisely, if V is a right $\mathfrak{g}$-supermodule, then it becomes a left $\mathfrak{g}$-module with action given by $X \cdot v := -(-1)^{\overline{X}\bar{v}} vX$, $X \in \mathfrak{g}$, $v \in V$. For example, see Lemma 5.3.1.*

5.3 Categorical Action of $\mathcal{AOB}(A)$ and $\mathcal{OB}(A)$

Let $V_+ = A^{m|n}$, written as row matrices, and let $V_- = A^{m|n}$, written as column matrices. Then V_+ is naturally a right $\mathfrak{g}$-supermodule with action given by right matrix multiplication.

Lemma 5.3.1. *We have that V_- is a right $\mathfrak{g}$-supermodule with action*

$$v \cdot M := -(-1)^{\bar{v}\overline{M}} Mv, \quad v \in V_-. \tag{5.3.1}$$

Proof: For $v \in V_-$ and $M, N \in \mathfrak{g}$, we have

$$\begin{aligned} v \cdot [M,N] &= (v\cdot M)\cdot N - (-1)^{\overline{M}\,\overline{N}}(v\cdot N)\cdot M \\ &= (-1)^{\bar{v}(\overline{M}+\overline{N})+\overline{M}\,\overline{N}} NMv - (-1)^{\bar{v}(\overline{M}+\overline{N})} MNv \\ &= -(-1)^{\bar{v}(\overline{M}+\overline{N})}[M,N]v. \end{aligned}$$

∎

Lemma 5.3.2. *The $\Bbbk$-bilinear form*

$$B\colon V_- \otimes V_+ \to \Bbbk, \quad B(v\otimes w) := (-1)^{\bar{v}\bar{w}}\operatorname{tr}(wv),$$

is a homomorphism of right $\mathfrak{g}$-supermodules.

Proof: For $v, w \in V$ and $M \in \mathfrak{g}$, we have

$$\begin{aligned} B((v \otimes w)M) &= B\left(v \otimes wM - (-1)^{\overline{M}(\bar{v}+\bar{w})} Mv \otimes w\right) \\ &= (-1)^{\bar{v}(\bar{w}+\overline{M})}(\operatorname{tr}(wMv) - \operatorname{tr}(wMv)) \\ &= 0. \end{aligned}$$

∎

For $a \in A$ and $1 \le i \le m+n$, let $a_{i,\pm}$ denote the element of $V_\pm$ with a in the i-th position and 0 in every other position. Then

$$\mathbf{B}_+ := \{b_{i,+} : b \in \mathbf{B}_A,\ 1 \le i \le m+n\} \tag{5.3.2}$$

and

$$\mathbf{B}_- := \{b_{i,-}^\vee : b \in \mathbf{B}_A,\ 1 \le i \le m+n\} \tag{5.3.3}$$

are dual bases of V_+ and V_- with respect to the bilinear form B. For $v = b_{i,+} \in \mathbf{B}_+$, let $v^\vee = (-1)^{p(i)} b_{i,-}^\vee \in \mathbf{B}_-$. Note that $\overline{b_{i,+}} = \overline{b_{i,-}^\vee} = \bar{b} + p(i)$. Thus, we have

$$B(v^\vee \otimes w) = \delta_{v,w}, \quad v, w \in \mathbf{B}_+. \tag{5.3.4}$$

To see this let $1 \le i, j \le m+n$ and $a, b \in \mathbf{B}_A$. Then we have

$$\begin{aligned} B((-1)^{p(i)} b_{i,-}^\vee \otimes a_{j,+}) &= (-1)^{p(i)+(\bar{b}+p(i))(\bar{a}+p(j))} \operatorname{tr}(\delta_{i,j} a b^\vee) \\ &\overset{(3.2.3)}{=} (-1)^{p(i)+(\bar{b}+p(i))(\bar{a}+p(j))+\bar{a}\bar{b}} \delta_{i,j} \operatorname{tr}(b^\vee a) \\ &\overset{(3.2.4)}{=} (-1)^{p(i)+(\bar{b}+p(i))(\bar{a}+p(j))+\bar{a}\bar{b}} \delta_{i,j}\delta_{a,b} \\ &= \delta_{i,j}\delta_{a,b} \end{aligned}$$

where the last equality follows since $(-1)^{p(i)+(\bar{b}+p(i))(\bar{a}+p(j))+\bar{a}\bar{b}} \delta_{i,j}\delta_{a,b} = 0$ unless $a = b$ and $i = j$, in which case we have that $p(i) + (\bar{b}+p(i))(\bar{a}+p(j)) + \bar{a}\bar{b} = p(i) + (\bar{a} + p(i))^2 + \bar{a}^2 = 0$.

Define

$$\Omega := \sum_{M \in \mathbf{B}_{m|n}} M \otimes M^\vee \in \mathfrak{g} \otimes \mathfrak{g}, \qquad \tau := \sum_{b \in \mathbf{B}_A} b \otimes b^\vee \in A \otimes A. \tag{5.3.5}$$

Lemma 5.3.3. *For all $u, v \in V_+$, we have*

$$\tau(u \otimes v) = (-1)^{\bar{u}\bar{v}} (v \otimes u)\Omega. \tag{5.3.6}$$

Proof: It suffices to prove the result for $u = a_{k,+}$ and $v = c_{l,+}$, where $a, c \in A$ and $1 \le k, l \le m+n$. We have

$$
\begin{aligned}
(a_{k,+} \otimes c_{l,+})\Omega &= \sum_{i,j=1}^{m+n} \sum_{b \in \mathbf{B}_A} (-1)^{(\bar{b}+p(i)+p(j))(\bar{c}+p(l))+p(j)} a_{k,+} b_{(i,j)} \otimes c_{l,+} b^\vee_{(j,i)} \\
&= \sum_{b \in \mathbf{B}_{m|n}} (-1)^{(\bar{b}+p(k))(\bar{c}+p(l))+\bar{c}p(l)} (ab)_{l,+} \otimes (cb^\vee)_{k,+} \\
&\overset{(3.2.5)}{=} \sum_{b,e \in \mathbf{B}_A} (-1)^{(\bar{b}+p(k))(\bar{c}+p(l))+\bar{c}p(l)} \operatorname{tr}(e^\vee ab) e_{l,+} \otimes (cb^\vee)_{k,+} \\
&\overset{(3.2.5)}{=} (-1)^{(\bar{a}+p(k))(\bar{c}+p(l))} \sum_{e \in \mathbf{B}_A} (-1)^{\bar{e}(\bar{c}+p(l))+\bar{c}p(l)} e_{l,+} \otimes (ce^\vee a)_{k,+} \\
&\overset{(3.2.5)}{=} (-1)^{(\bar{a}+p(k))(\bar{c}+p(l))} \sum_{b,e \in \mathbf{B}_A} (-1)^{\bar{e}(\bar{c}+p(l))+\bar{c}p(l)} e_{l,+} \otimes \operatorname{tr}(ce^\vee b)(b^\vee a)_{k,+} \\
&\overset{(3.2.3)}{=} (-1)^{(\bar{a}+p(k))(\bar{c}+p(l))} \sum_{b,e \in \mathbf{B}_A} (-1)^{(\bar{e}+\bar{c})(\bar{c}+p(l))} \operatorname{tr}(e^\vee bc) e_{l,+} \otimes (b^\vee a)_{k,+} \\
&\overset{(3.2.5)}{=} (-1)^{(\bar{a}+p(k))(\bar{c}+p(l))} \sum_{b \in \mathbf{B}_A} (-1)^{\bar{b}(\bar{c}+p(l))} (bc)_{l,+} \otimes (b^\vee a)_{k,+} \\
&= (-1)^{(\bar{a}+p(k))(\bar{c}+p(l))} \tau(c_{l,+} \otimes a_{k,+}).
\end{aligned}
$$

∎

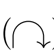
Theorem 5.3.4. *There is a monoidal superfunctor $\psi\colon \mathcal{OB}(A) \to$ smod-$\mathfrak{g}$ that sends the objects of $\mathcal{OB}(A)$ to objects of smod-$\mathfrak{g}$ as follows*

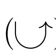
$$\uparrow \;\mapsto V_+ \;, \qquad \downarrow \;\mapsto V_-$$

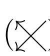
and the generating morphisms as follows

ψ () $\colon V_+ \otimes V_+ \to V_+ \otimes V_+, \qquad u \otimes v \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u,$

ψ () $\colon V_+ \to V_+\,, \qquad v \mapsto av,$

ψ () $\colon V_+ \otimes V_- \to \Bbbk, \qquad u \otimes v \mapsto (-1)^{\bar{u}\bar{v}} B(v \otimes u)$

ψ () $\colon \Bbbk \to V_- \otimes V_+, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} v^\vee \otimes v,$

ψ () $\colon V_- \otimes V_+ \to V_+ \otimes V_-, \qquad u \otimes v \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u.$

Proof: We need to verify that the image of each generating morphism above is a right $\mathfrak{g}$-supermodule homomorphism and that relations (4.1.4) to (4.1.6), (4.1.9), (4.1.10) and (4.1.12) hold in smod-$\mathfrak{g}$.

We begin with showing the images of the generating morphisms are right $\mathfrak{g}$-supermodule homomorphisms. For $\psi(\nwarrow\!\!\!\!\nearrow)$, note that linearity follows from linearity of the tensor product and for $u, v \in V_+, M \in \mathfrak{g}$ we have

$$\begin{aligned}
\psi(\nwarrow\!\!\!\!\nearrow)(u \otimes v)M &= ((-1)^{\bar{u}\bar{v}} v \otimes u)M \\
&= (-1)^{\bar{u}\bar{v}}((-1)^{\bar{u}\overline{M}} vM \otimes u + v \otimes uM) \\
&= (-1)^{\bar{u}\bar{v}+\bar{u}\overline{M}} vM \otimes u + (-1)^{\bar{v}\overline{M}+\bar{v}\bar{u}+\bar{v}\overline{M}} v \otimes uM \\
&= (-1)^{\bar{u}\overline{vM}} vM \otimes u + (-1)^{\bar{v}\overline{M}}(-1)^{\overline{uM}\bar{v}} v \otimes uM \\
&= \psi(\nwarrow\!\!\!\!\nearrow)\left(u \otimes vM + (-1)^{\bar{v}\overline{M}} uM \otimes v\right) \\
&= \psi(\nwarrow\!\!\!\!\nearrow)((u \otimes v)M)
\end{aligned}$$

so $\psi(\nwarrow\!\!\!\!\nearrow)$ is a right $\mathfrak{g}$-supermodule homomorphism. The case is similar for $\psi(\nwarrow\!\!\!\!\swarrow)$. For $\psi(\uparrow\!\bullet\, a)$, linearity is immediate and we have that

$$\psi(\uparrow\!\bullet\, a)(v)M = (av)M = avM = \psi(\uparrow\!\bullet\, a)(vM).$$

The fact that $\psi(\curvearrowright)$ is a right $\mathfrak{g}$-supermodule homomorphism follows from Lemma 5.3.2. Let $a \in A$ and $1 \le r, s \le m+n$. To see $\psi(\cup\!\!\uparrow)$ is a right $\mathfrak{g}$-supermodule we have

$$\begin{aligned}
\psi(\cup\!\!\uparrow)(1)a_{(r,s)} &= \left(\sum_{b\in\mathbf{B}_A}\sum_{i=1}^{m+n}(-1)^{\bar{b}_i+p(i)} b_i^\vee \otimes b_i\right) a_{(r,s)} \\
&= \sum_{b\in\mathbf{B}_A}\sum_{i=1}^{m+n}(-1)^{\bar{b}+\bar{b}_i\overline{a_{(r,s)}}} b_i^\vee \cdot a_{(r,s)} \otimes b_i + (-1)^{\bar{b}} b_i^\vee \otimes b_i a_{(r,s)} \\
&\overset{(5.3.1)}{=} \sum_{b\in\mathbf{B}_A}\sum_{i=1}^{m+n} -(-1)^{\bar{b}} a_{(r,s)} b_i^\vee \otimes b_i + (-1)^{\bar{b}} b_i^\vee \otimes b_i a_{(r,s)} \\
&= \sum_{b\in\mathbf{B}_A}\sum_{i=1}^{m+n} -(-1)^{\bar{b}} \delta_{i,s}(ab^\vee)_r \otimes b_i + (-1)^{\bar{b}} b_i^\vee \otimes \delta_{i,r}(ba)_s \\
&= \sum_{b\in\mathbf{B}_A} -(-1)^{\bar{b}}(ab^\vee)_r \otimes b_s + (-1)^{\bar{b}} b_r^\vee \otimes (ba)_s \\
&= \sum_{b\in\mathbf{B}_A} -(-1)^{\bar{b}}(a(b^\vee)^\vee)_r \otimes b_s^\vee + (-1)^{\bar{b}}(b^\vee)_r^\vee \otimes (b^\vee a)_s \\
&\overset{(3.2.6)}{=} \sum_{b\in\mathbf{B}_A} -(ab)_r \otimes b_s^\vee + b_r \otimes (b^\vee a)_s \\
&\overset{(3.2.7)}{=} \sum_{b\in\mathbf{B}_A} -b_r \otimes (b^\vee a)_s + b_r \otimes (b^\vee a)_s
\end{aligned}$$

$$= 0 = \psi(\circlearrowleft)(0) = \psi(\circlearrowleft)(1 \cdot a_{(r,s)}).$$

where in the sixth equality we use the fact that $\sum_{b \in \mathbf{B}_A} b^\vee \otimes b$ is independent of basis and so we are free to instead sum over the dual basis.

Now we verify that each of the generating relations of $\mathcal{OB}(A)$ hold in smod-$\mathfrak{g}$. The image of the left of (4.1.4) is given by

$$u \otimes v \mapsto au \otimes v \mapsto (-1)^{\overline{au}\bar{v}} v \otimes au = (-1)^{(\bar{a}+\bar{u})\bar{v}} v \otimes au$$

and the right by

$$u \otimes v \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u \mapsto (-1)^{\bar{u}\bar{v}+\bar{a}\bar{v}} v \otimes au.$$

So we have (4.1.4). For (4.1.5) on the left we have

$$u \otimes v \otimes w \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u \otimes w \mapsto (-1)^{\bar{u}\bar{v}+\bar{u}\bar{w}} v \otimes w \otimes u \mapsto (-1)^{\bar{u}\bar{v}+\bar{u}\bar{w}+\bar{v}\bar{w}} w \otimes v \otimes u$$

and on the right

$$u \otimes v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} u \otimes w \otimes v \mapsto (-1)^{\bar{u}\bar{w}+\bar{v}\bar{w}} w \otimes u \otimes v \mapsto (-1)^{\bar{u}\bar{v}+\bar{u}\bar{w}+\bar{v}\bar{w}} w \otimes v \otimes u.$$

Therefore the braid relation holds. The image under ψ of the right of relation (4.1.6) is given by

$$u \otimes v \mapsto (-1)^{\bar{v}\bar{u}} v \otimes u \mapsto u \otimes v$$

so relation (4.1.6) holds. Applying ψ to the left of relation (4.1.9) we have

$$u \otimes 1 \mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} u \otimes v^\vee \otimes v \mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}+\bar{u}\bar{v}} B(v^\vee \otimes u) \otimes v$$

and we have that

$$\begin{aligned} \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}+\bar{u}\bar{v}} B(v^\vee \otimes u) \otimes v &= \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} \operatorname{tr}(uv^\vee) \otimes v \\ &\overset{(3.2.3)}{=} \sum_{v \in \mathbf{B}_+} \operatorname{tr}(v^\vee u) \otimes v \\ &\overset{(3.2.5)}{=} u. \end{aligned}$$

Therefore relation (4.1.9) holds. The proof for relation (4.1.10) is similar. To verify (4.1.12) first note that applying ψ to (4.1.11) we have

$$V_+ \otimes V_- \to V_- \otimes V_+, \quad u \otimes w \mapsto (-1)^{\bar{u}\bar{w}} w \otimes u.$$

So (4.1.12) is immediate. So we have verified the defining relations and so smod-$\mathfrak{g}$ is a monoidal supercategory. ∎

Remark 5.3.5. *Note that the image of each generating morphism, except $\psi(\uparrow\!\!\bullet a)$, is an even $\mathfrak{g}$-supermodule homomorphism since*

$$\overline{u \otimes v} = \bar{u} + \bar{v} = \overline{v \otimes u} = \overline{(-1)^{\bar{u}\bar{v}} v \otimes u} = \overline{\psi(\nwarrow\!\!\nearrow)(u \otimes v)} = \overline{\psi(\nwarrow\!\!\swarrow)(u \otimes v)},$$

$$\bar{1} = 0 = \sum_{v \in \mathbf{B}_+} \bar{v} + \bar{v} = \overline{\sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} v^\vee \otimes v} = \overline{\psi(\cup\!\!\uparrow)(1)},$$

and it is immediate $\psi(\cap\!\!\downarrow)$ is even since tr *is parity preserving. However*

$$\overline{\psi(\uparrow\!\!\bullet a)(v)} = \overline{av} = \bar{a} + \bar{v} \neq \bar{v}, \quad \textit{for } \bar{a} = 1.$$

Thus, since $\psi(\uparrow\!\!\bullet a)$ is not always purely even, we need the larger supercategory smod-$\mathfrak{g}$, *instead of the category of right $\mathfrak{g}$-supermodules where morphisms are purely even homomorphisms.*

Remark 5.3.6. *To use the alternate definition of $\mathcal{OB}(A)$ in Remark 4.1.2 we maintain the above definition of ψ, remove the definition on $\nwarrow\!\!\swarrow$ and add in that*

$$\psi(\uparrow\!\!\cup)\colon \Bbbk \mapsto V_+ \otimes V_-, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_+} v \otimes v^\vee,$$

$$\psi(\downarrow\!\!\cap)\colon V_- \otimes V_+ \to \Bbbk, \qquad v \otimes w \mapsto B(v \otimes w).$$

Then, in addition to the relations we verified above, we also check (4.1.15) *and* (4.1.16). *Moreover, we note that the image of the definition of the left crossing,* (4.1.17) *under ψ, which is given by $u \otimes w \mapsto (-1)^{\bar{u}\bar{w}} w \otimes v$, is the same as the definition of ψ on $\nwarrow\!\!\swarrow$. For* (4.1.15) *we have*

$$u \mapsto \sum_{v \in \mathbf{B}_+} u \otimes v \otimes v^\vee \mapsto \sum_{v \in B_+} (-1)^{\bar{u}\bar{v}} v \otimes u \otimes v^\vee \mapsto \sum_{v \in \mathbf{B}_+} v \otimes B(v^\vee \otimes u)$$
$$\overset{(5.3.4)}{=} \sum_{v \in \mathbf{B}_+} v \otimes \delta_{v,u} \overset{(3.2.5)}{=} u.$$

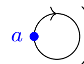
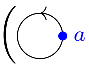
The proof for relation (4.1.16) *is similar. Also note that the image of* (4.1.13) *under ψ is consistent with our definition of ψ on $\uparrow\!\!\cup, \downarrow\!\!\cap$ above. For example, for the left cup definition in* (4.1.13) *we have*

$$1 \mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} v^\vee \otimes v \mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v} + \bar{v}\bar{v}} v \otimes v^\vee = \sum_{v \in \mathbf{B}_+} v \otimes v^\vee$$

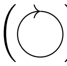
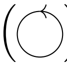
as desired. The proof for the left cap definition is similar.

This version of Theorem 5.3.4 is stated in [18, Thm. 5.1].

Note that, for $a \in A$,

$\psi\Big($  $\Big) = \psi\Big($  $\Big) = \sum_{b \in \mathbf{B}_+} (-1)^{\bar{v}} B(v^\vee, av) 1_{\mathbb{1}}$ (5.3.7)

is multiplication by the supertrace of the map $V_+ \to V_+$, $v \mapsto av$. In particular,

$\psi\Big($  $\Big) = \psi\Big($  $\Big) = \mathrm{sdim}(A) 1_{\mathbb{1}},$ (5.3.8)

where $\mathrm{sdim}(A) = m\dim(A_{\bar{0}}) + n\dim(A_{\bar{1}}) - m\dim(A_{\bar{1}}) - n\dim(A_{\bar{0}})$ is the super dimension of V_+.

Definition 5.3.7. Let $\mathcal{C}$ be a supercategory. The *supercategory of endofunctors* of $\mathcal{C}$, denoted $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$, consists of super endofunctors $F\colon \mathcal{C} \to \mathcal{C}$, and morphisms which are supernatural transformations between super endofunctors.

Lemma 5.3.8. *The supercategory of endofunctors $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ is a strict monoidal supercategory for any supercategory $\mathcal{C}$ with tensor product denoted by $\cdot$ and unit object $\mathrm{id}_{\mathcal{C}}$. This tensor product, $\cdot\colon \mathcal{E}nd_{\Bbbk}(\mathcal{C}) \times \mathcal{E}nd_{\Bbbk}(\mathcal{C}) \to \mathcal{E}nd_{\Bbbk}(\mathcal{C})$, is defined as regular composition on objects and as horizontal composition on morphisms.*

Proof: First we show that $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ is indeed a supercategory. It is straightforward to see the morphism spaces are vector superspaces because supernatural transformations are defined to be the sum of an odd and even supernatural transformation as in Definition 2.1.5. It is also straightforward to see composition of supernatural transformations preserves parity since for homogenous supernatural transformations $\alpha : F \Rightarrow G$, $\beta : H \Rightarrow K$ we have

$$\overline{(\alpha\beta)_X} = \overline{\alpha_{KX} \circ F(\beta_X)} = \overline{\alpha_{KX}} + \overline{F(\beta_X)} = \bar{\alpha} + \bar{\beta},$$

for $X \in \mathcal{C}$.

Now, to verify $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ is strict monoidal, we need to verify that the supernatural isomorphisms α, λ, ρ from Definition 2.1.6 are identities. First we show that composition of objects of $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ is associative and unital. This follows immediately since composition of functors is associative and unital. Then, we show that composition of morphisms is associative and unital. For all supernatural transformations $\alpha\colon F \Rightarrow G, \beta\colon H \Rightarrow K, \gamma\colon J \Rightarrow L$ of $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ and $X \in \mathcal{C}$ we have

$$\begin{aligned}((\alpha\beta)\gamma)_X &= (\alpha\beta)_{LX} \circ F(H(\gamma_X)) \\ &= \alpha_{KLX} \circ F(\beta_{LX}) \circ F(H(\gamma_X)) \\ &= \alpha_{KLX} \circ F(\beta_{LX} \circ H(\gamma_X)) \\ &= \alpha_{KLX} \circ F((\beta\gamma)_X) \\ &= \alpha(\beta\gamma).\end{aligned}$$

So we have associativity. The morphism identity is the identity supernatural transformation, $\mathrm{id}\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$. For $\alpha\colon F \to G$ and $X \in \mathcal{C}$ we have

$$\begin{aligned}(\alpha\,\mathrm{id})_X &= \alpha_{\mathrm{id}_{\mathcal{C}}(X)} \circ F(\mathrm{id}_X)\\ &= \alpha_X \circ \mathrm{id}_{FX}\\ &= \alpha_X.\end{aligned}$$

The proof that $(\mathrm{id}\,\alpha)_X = \alpha_X$ is similar. Thus $\mathcal{E}nd_{\Bbbk}(\mathcal{C})$ is a strict monoidal supercategory for any supercategory $\mathcal{C}$. ∎

Theorem 5.3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal supercategories. For a monoidal superfunctor $\psi\colon \mathcal{C} \to \mathcal{D}$ there is an induced monoidal superfunctor*

$$\begin{aligned}\Psi\colon \mathcal{C} &\to \mathcal{E}nd_{\Bbbk}(\mathcal{D})\\ X &\mapsto \psi(X) \otimes_{\mathcal{D}} -,\\ f &\mapsto \psi(f) \otimes_{\mathcal{D}} -,\end{aligned} \tag{5.3.9}$$

where

$$\Psi(X)(Y) = \psi(X) \otimes_{\mathcal{D}} Y, \quad \Psi(X)(g) = 1_{\psi(X)} \otimes_{\mathcal{D}} g$$

for $Y \in \mathcal{D}$, $g \in \mathrm{Mor}\,\mathcal{D}$ and for $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, $\Psi(f) : \Psi A \to \Psi B$ has components given by $\Psi(f)_Y : \Psi(A)(Y) \to \Psi(B)(Y)$, $\Psi(f)_Y = \psi(f) \otimes_{\mathcal{D}} 1_Y$.

Proof: First we show Ψ is a superfunctor. That is, Ψ preserves composition, $\overline{\Psi(f)} = \overline{f}$ and that $\Psi(1_X) = 1_{\Psi(X)}$. For $f \in \mathrm{Hom}_{\mathcal{C}}(A,B), g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$ and $Y \in \mathcal{D}$ we have

$$\begin{aligned}\Psi(g \circ f)_Y &= \psi(g \circ f) \otimes_{\mathcal{D}} 1_Y\\ &= (\psi(g) \circ \psi(f)) \otimes_{\mathcal{D}} (1_Y \circ 1_Y)\\ &= (\psi(g) \otimes_{\mathcal{D}} 1_Y) \circ (\psi(f) \otimes_{\mathcal{D}} 1_Y)\\ &= \Psi(g)_Y \circ \Psi(f)_Y,\end{aligned}$$

where the second line follows from the fact that ψ is a superfunctor and the third line follows from the superinterchange law (2.1.5) and the fact that $\overline{1_Y} = 0$. We have

$$\Psi(1_X)_Y = \psi(1_X) \otimes_{\mathcal{D}} 1_Y = 1_{\psi(X)} \otimes_{\mathcal{D}} 1_Y = 1_{\psi(X) \otimes_{\mathcal{D}} Y}.$$

Finally, we have that

$$\overline{\Psi(f)_Y} = \overline{\psi(f) \otimes_{\mathcal{D}} 1_Y} = \overline{\psi(f)} + \overline{1_Y} = \overline{f} + 0 = \overline{f}.$$

So Ψ is a superfunctor.

To see Ψ is a monoidal superfunctor, recall the coherence maps, ϕ and ι, from Definition 2.1.10. We need to show $\phi_{X,Y} : \Psi(X) \otimes_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})} \Psi(Y) \to \Psi(X \otimes_{\mathcal{C}} Y)$ is an even supernatural isomorphism and $\iota : \mathbb{1}_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})} \to \Psi(\mathbb{1}_{\mathcal{C}})$ is an even invertible morphism, such that the appropriate diagrams commute. For ϕ, it suffices to show $\Psi(X) \otimes_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})} \Psi(Y) \simeq \Psi(X \otimes_{\mathcal{C}} Y)$, and for ι, that $\mathbb{1}_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})} \simeq \Psi(\mathbb{1}_{\mathcal{C}})$ for all $X, Y \in \mathcal{C}$. It is immediate that parity is preserved across these isomorphisms since Ψ and ψ are superfunctors and thus preserve parity.

We have

$$
\begin{aligned}
(\Psi(X) \otimes_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})} \Psi(Y))(Z) &= \Psi(X)(\Psi(Y)(Z)) \\
&= \Psi(X)(\psi(Y) \otimes_{\mathcal{D}} Z) \\
&= \psi(X) \otimes_{\mathcal{D}} (\psi(Y) \otimes_{\mathcal{D}} Z) \\
&\simeq (\psi(X) \otimes_{\mathcal{D}} \psi(Y)) \otimes_{\mathcal{D}} Z \\
&\simeq \psi(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} Z \\
&= \Psi(X \otimes_{\mathcal{C}} Y)(Z)
\end{aligned}
$$

where the first isomorphism follows from the fact that $\mathcal{D}$ is a monoidal category, and the second from the fact that ψ is a monoidal superfunctor. We also have

$$
\Psi(\mathbb{1}_{\mathcal{C}})(Z) = \psi(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} Z \simeq \mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} Z \simeq Z = \mathbb{1}_{\mathcal{E}nd_{\Bbbk}(\mathcal{D})}(Z)
$$

where the first isomorphism follows from the fact that ψ is a monoidal functor, and the second from the fact that $\mathcal{D}$ is a monoidal category. From these isomorphisms and the fact that $\mathcal{D}$ and ψ are monoidal, it is straightforward to check that the appropriate diagrams in Definition 2.1.10 commute. So Ψ is a monoidal superfunctor. ∎

Remark 5.3.10. *Using the notation in Theorem 5.3.9, if ψ is a* strict *monoidal superfunctor (and $\mathcal{C}, \mathcal{D}$ are strict monoidal categories), the induced functor, Ψ is also strict monoidal. In this case, the isomorphisms in the proof above are replaced by equalities.*

Theorem 5.3.11. *We have a strict monoidal superfunctor $\Psi \colon \mathcal{AOB}(A) \to \mathcal{E}nd_{\Bbbk}(\text{smod-}\mathfrak{g})$ given on objects by $\uparrow \mapsto V_+ \otimes -$, $\downarrow \mapsto V_- \otimes -$ and on morphisms by*

$$
\Psi(f) = \psi(f) \otimes -, \quad f \in \{\times, \uparrow_{\bullet a}, \cup, \cap, \times : a \in A\},
$$

and $\Psi(\uparrow_{\circ}) \colon V_+ \otimes - \to V_+ \otimes -$ is the functor with components

$$
\Psi(\uparrow_{\circ})_W \colon V_+ \otimes W \to V_+ \otimes W, \quad v \otimes w \mapsto (v \otimes w)\Omega,
$$

for $W \in \text{smod-}\mathfrak{g}$, where Ω is the element defined in (5.3.5).

Proof: In light of Theorem 5.3.4 and Theorem 5.3.9, it suffices to check that $\Psi(\uparrow)$ is a supernatural transformation and that Ψ respects the relations (4.1.21).

First we show $\Psi(\uparrow)$ is a supernatural transformation. Note that $\uparrow$ is purely even, so $\Psi(\uparrow)$ is also even. This means there are no signs appearing in the supernaturality diagram (2.1.1) for $\Psi(\uparrow)$. Let $W, Y \in$ smod-$\mathfrak{g}$, $f\colon W \to Y$, $v \in V_+$, and $w \in W$. We have

$$\begin{aligned}
(\mathrm{id}_{V_+} \otimes f)(\Psi(\uparrow))_W(v \otimes w)) &= (\mathrm{id}_{V_+} \otimes f)((v \otimes w)\Omega) \\
&= (\mathrm{id}_{V_+} \otimes f)\Big(\sum_{M \in \mathbf{B}_{m|n}} (-1)^{\overline{M}\bar{w}} vM \otimes wM^\vee\Big) \\
&= \sum_{M \in \mathbf{B}_{m|n}} (-1)^{\overline{M}\bar{w} + \overline{vM}\bar{f}} vM \otimes f(wM^\vee) \\
&= \sum_{M \in \mathbf{B}_{m|n}} (-1)^{\overline{M}\bar{w} + \bar{v}\bar{f} + \overline{M}\bar{f}} vM \otimes f(w)M^\vee \\
&= ((-1)^{\bar{v}\bar{f}} v \otimes f(w))\Omega \\
&= \Psi(\uparrow)\big((\mathrm{id}_{V_+} \otimes f)(v \otimes w)\big)
\end{aligned}$$

where the fourth equality follows since f is a $\mathfrak{g}$-supermodule homomorphism. So $\Psi(\uparrow)$ is a supernatural transformation.

To verify the first relation in (4.1.21), we compute that

$$\Psi(\times)_W : V_+ \otimes V_+ \otimes W \to V_+ \otimes V_+ \otimes W$$

is the map given by

$$\begin{aligned}
u \otimes v \otimes w &\mapsto (-1)^{\bar{u}\bar{v}} v \otimes u \otimes w \\
&\mapsto \sum_{M \in \mathbf{B}_{m|n}} (-1)^{\bar{u}\bar{v}} \left((-1)^{\bar{u}\overline{M}} vM \otimes uM^\vee \otimes w + (-1)^{(\bar{u}+\bar{w})\overline{M}} vM \otimes u \otimes wM^\vee\right).
\end{aligned}$$

Similarly,

$$\Psi(\times)_W : V_+ \otimes V_+ \otimes W \to V_+ \otimes V_+ \otimes W$$

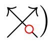
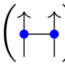
is the map given by

$$u \otimes v \otimes w \mapsto \sum_{M \in \mathbf{B}_{m|n}} (-1)^{\bar{w}\overline{M}} u \otimes vM \otimes wM^\vee \mapsto \sum_{M \in \mathbf{B}_{m|n}} (-1)^{\bar{u}(\bar{v}+\overline{M}) + \bar{w}\overline{M}} vM \otimes u \otimes wM^\vee.$$

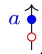
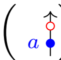
Then, note that

$$\Psi\left(\uparrow\!\!\!-\!\!\!\uparrow\right)(u \otimes v \otimes w) = \tau \otimes \mathrm{id}_W(u \otimes v \otimes w).$$

Thus,

$\Psi\,($  $-$  $)_W\,(u\otimes v\otimes w) = (-1)^{\bar{u}\bar{v}}(v\otimes u)\Omega\otimes w \overset{(5.3.6)}{=} \Psi\Big($  $\Big)\,(u\otimes v\otimes w).$

To verify the second relation in (4.1.21) we compute that, for $a\in A$, we have

$\Psi\Big($  $\Big)_W\,(v\otimes w) = \sum_{M\in\mathbf{B}_{m|n}} (-1)^{\bar{w}\bar{M}} avM\otimes wM^\vee = \Psi\Big($  $\Big)_W\,(v\otimes w).$

∎

When $A=\Bbbk$, Theorems 5.3.4 and 5.3.11 recover known results for $\mathfrak{gl}_{m|n}=\mathfrak{gl}_{m|n}(\Bbbk)$. Furthermore, as noted in Remark 4.1.6, the definitions of $\mathcal{OB}(A)$ and $\mathcal{AOB}(A)$ can be generalized to allow A to be the two-dimensional Clifford superalgebra. In this case, the actions described in Theorems 5.3.4 and 5.3.11 correspond to those described in [3, §4.2 and Th. 4.4] on supermodules for the queer Lie superalgebra (see Example 3.2.3).

The center $Z(\mathcal{E}nd(\text{smod-}\mathfrak{g})) := \mathrm{End}_{\mathcal{E}nd(\text{smod-}\mathfrak{g})}(\mathbb{1})$ of the category $\mathcal{E}nd(\text{smod-}\mathfrak{g})$ can be naturally identified with $Z(U(\mathfrak{g}))$ via the map

$$\rho\colon Z(U(\mathfrak{g})) \xrightarrow{\cong} Z(\mathcal{E}nd(\text{smod-}\mathfrak{g})),\quad u\mapsto \rho_u, \tag{5.3.10}$$

where ρ_u is the natural transformation whose W-component for $W\in\text{smod-}\mathfrak{g}$ is

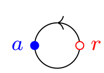
$$(\rho_u)_W\colon W\to W,\quad w\mapsto (-1)^{\bar{u}\bar{w}}wu.$$

Then it follows from Theorem 5.3.11 and (4.3.3) that we have a homomorphism of superalgebras

$$\rho^{-1}\circ\Psi\circ\beta\colon\ \mathrm{Sym}(A)\to Z(U(\mathfrak{g})).$$

The following proposition describes this map explicitly.

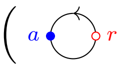
Proposition 5.3.12. *The element*

$\rho^{-1}\circ\Psi\Big($  $\Big) = (-1)^r\rho^{-1}\circ\Psi\circ\beta(e_{r+1}(a))\in Z(U(\mathfrak{g}))$

is given by

$$\sum_{\substack{1\le i_1,\dots,i_r\le d\\ b_1,\dots,b_r\in\mathbf{B}_A}} (-1)^{a\bar{b}_r+\sum_{k=1}^r \bar{b}_k\bar{b}_{k+1}}(b_2b_1)_{i_2,i_1}(b_3b_2^\vee)_{i_3,i_2}\cdots(b_rb_{r-1}^\vee)_{i_r,i_{r-1}}(b_{r+1}^\vee ab_r^\vee)_{i_{r+1},i_r},$$

where we adopt the convention that $i_{r+1}=i_1$ and $b_{r+1}=b_1$.

Proof: For $W \in \text{smod-}\mathfrak{gl}_d(A)$, we compute that $\Psi\Big($  $\Big)_W$ is the map

$$
\begin{aligned}
w &\mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}} v^\vee \otimes v \otimes w \\
&\mapsto \sum_{v \in \mathbf{B}_+} (-1)^{\bar{v}+\bar{a}\bar{v}} v^\vee \otimes (av \otimes w)\Omega^r \\
&= \sum_{\substack{c \in \mathbf{B}_A \\ 1 \le k \le d}} \sum_{\substack{1 \le i_1,\dots,i_r \le d \\ 1 \le j_1,\dots,j_r \le d \\ b_1,\dots,b_r \in \mathbf{B}_A}} (-1)^{\bar{a}\bar{c}+\bar{w}\sum_p \bar{b}_p + \sum_{p<q} \bar{b}_p \bar{b}_q} c_{k,-} \otimes a c_{k,+}^\vee (b_1)_{i_1,j_1} \cdots \\
&\qquad\qquad (b_r)_{i_r,j_r} \otimes w (b_1^\vee)_{j_1,i_1} \cdots (b_r^\vee)_{j_r,i_r} \\
&= \sum_{\substack{1 \le i_1,\dots,i_r,j_r \le d \\ b_1,\dots,b_r,c \in \mathbf{B}_A}} (-1)^{\bar{a}\bar{c}+\bar{w}\sum_p \bar{b}_p + \sum_{p<q} \bar{b}_p \bar{b}_q} c_{i_1,-} \otimes (a c^\vee b_1 \cdots b_r)_{j_r,+} \\
&\qquad\qquad \otimes w (b_1^\vee)_{i_2,i_1} \cdots (b_{r-1}^\vee)_{i_r,i_{r-1}} (b_r^\vee)_{j_r,i_r} \\
&\mapsto \sum_{\substack{1 \le i_1,\dots,i_r \le d \\ b_1,\dots,b_r,c \in \mathbf{B}_A}} (-1)^{\bar{a}\bar{c}+\bar{w}\sum_p \bar{b}_p + \sum_{p<q} \bar{b}_p \bar{b}_q} \operatorname{tr}(c a c^\vee b_1 \cdots b_r) w (b_1^\vee)_{i_2,i_1} \cdots \\
&\qquad\qquad (b_{r-1}^\vee)_{i_r,i_{r-1}} (b_r^\vee)_{i_1,i_r} \\
&= \sum_{\substack{1 \le i_1,\dots,i_r \le d \\ b_1,\dots,b_{r-1},c \in \mathbf{B}_A}} (-1)^{\bar{a}\bar{c}+\bar{a}\bar{w}+\sum_{p<q<r} \bar{b}_p \bar{b}_q + \bar{a}\sum_{p=1}^{r-1} \bar{b}_p + \sum_{p=1}^{r-1} \bar{b}_p} w (b_1^\vee)_{i_2,i_1} \cdots \\
&\qquad\qquad (b_{r-1}^\vee)_{i_r,i_{r-1}} (c a c^\vee b_1 \cdots b_{r-1})_{i_1,i_r} \\
&= (-1)^{\bar{a}\bar{w}} w \sum_{\substack{1 \le i_1,\dots,i_r \le d \\ b_1,\dots,b_r \in \mathbf{B}_A}} (-1)^{\sum_{p<q<r} \bar{b}_p \bar{b}_q + \bar{a}\sum_p \bar{b}_p} (b_1)_{i_2,i_1} \cdots \\
&\qquad\qquad (b_{r-1})_{i_r,i_{r-1}} (b_r a b_r^\vee b_1^\vee \cdots b_{r-1}^\vee)_{i_1,i_r} \\
&\overset{(3.2.7)}{=} (-1)^{\bar{a}\bar{w}} w \sum_{\substack{1 \le i_1,\dots,i_r \le d \\ b_1,\dots,b_r \in \mathbf{B}_A}} (-1)^{\sum_{k=1}^r \bar{b}_k \bar{b}_{k+1} + \bar{a}\bar{b}_{r-1}} (b_1 b_r)_{i_2,i_1} (b_2 b_1^\vee)_{i_3,i_2} \cdots \\
&\qquad\qquad (b_{r-1} b_{r-2}^\vee)_{i_r,i_{r-1}} (b_r^\vee a b_{r-1}^\vee)_{i_1,i_r}.
\end{aligned}
$$

The result then follows by shifting the indices of the b_i by 1. ∎

When $A = \Bbbk$, Proposition 5.3.12 recovers the elements described in [4, Rem. 1.4]. For $A = \mathrm{Cl}$ (see Remark 4.1.6), these central elements were computed in [3, Th. 4.5].

Chapter 6

Further Directions: Schur-Weyl Duality

In this chapter we review classical Schur-Weyl duality and show that we have commuting actions of $U(\mathfrak{g})$ and the wreath product superalgebra on $V^{\otimes k}$ where $V = A^{m|n}$. A super analogue of Schur-Weyl duality was estabilished by Berele-Regev and Sergeev in [1] and [27], respectively, for queer Lie superalgebras and Sergeev algebras. We see that in the Frobenius superalgebra analogue, the action of the wreath product superalgebra, which replaces the symmetric group in the classical version, is contained in the endomorphism algebra $\mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes k})$. The functors defined in Theorems 5.3.4 and 5.3.11 extend Schur–Weyl duality results for queer Lie superalgebras [28], actions of walled Brauer–Clifford superalgebras [14], and actions of affine walled Brauer–Clifford superalgebras [3, 12] although the case for Frobenius superalgebras is left open.

6.1 Classical Schur-Weyl Duality

Let $V = \Bbbk^n$, where $\Bbbk$ is an algebraically closed field of characteristic zero. In classical Schur-Weyl duality, the symmetric group $\mathfrak{S}_k$ acts on $V^{\otimes k}$ on the left as follows

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \quad \sigma \in \mathfrak{S}_k, v_1, \ldots, v_k \in V,$$

and the general linear Lie algebra, $\mathfrak{gl}_n(V)$, acts on $V^{\otimes k}$ on the right as

$$(v_1 \otimes \cdots \otimes v_k) \cdot X = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes v_i X \otimes \cdots \otimes v_k, \quad X \in \mathfrak{gl}_n(V), v_1, \ldots, v_k \in V.$$

It is straightforward to verify these actions commute. Schur-Weyl duality states that $\Bbbk\mathfrak{S}_k$ and $\mathfrak{gl}_n(V)$ generate each other's commutants in the endomorphism algebra, $\mathrm{End}_{\Bbbk}(V^{\otimes k})$. That is $\Bbbk\mathfrak{S}_k$ generates $\mathrm{End}_{\mathfrak{gl}_n(V)}(V^{\otimes k})$ and $\mathfrak{gl}_n(V)$ generates

$\mathrm{End}_{\Bbbk\mathfrak{S}_k}(V^{\otimes k})$. Equivalently, we may apply the following theorem to attain another statement of Schur-Weyl duality.

Theorem 6.1.1 (Double Centralizer Theorem, [11, Thm. 4.54]). *Let W be a finite dimensional vector space over $\Bbbk$, A be a semisimple subalgebra of $\mathrm{End}(W)$ such that W is a finite direct sum of simple A-modules and $B = \mathrm{End}_A(W)$. Then*

1. *B is semisimple,*
2. *$A = \mathrm{End}_B(W)$,*
3. *as an $A \otimes B$-module we have the decomposition*

$$W \simeq \bigoplus U_i \otimes W_i \tag{6.1.1}$$

where U_i are pairwise non-isomorphic simple modules of A and $W_i := \mathrm{Hom}_A(U_i, W)$ are simple modules of B.

Using the notation of Theorem 6.1.1, let $W = V^{\otimes k}$, $A = \Bbbk\mathfrak{S}_k$ and $B = \mathrm{End}_{\mathfrak{S}_k}(V^{\otimes k})$. Then by Item 3, $V^{\otimes k}$ decomposes as a direct sum of representations of $\mathfrak{S}_k \times \mathfrak{gl}_n(V)$,

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} V_\lambda \otimes L_\lambda,$$

where $\lambda \vdash k$ means that λ is an integer partition of k, the V_λ are irreducible representations of $\mathfrak{S}_k$ and the $L_\lambda = \mathrm{Hom}_{\Bbbk\mathfrak{S}_k}(V_\lambda, V)$ are irreducible representations of $\mathfrak{gl}_n(V)$ or zero when the number of parts in the partition λ is greater than n. For further details see [11, Thm. 4,57].

6.2 The Frobenius Superalgebra Analogue

Now, let $V = A^{m|n}$. Recall the Lie superalgebra $\mathfrak{g}$ from Section 5.1 and that $V = A^{m|n}$ is a right $\mathfrak{g}$-supermodule. Since smod-$\mathfrak{g}$ is a monoidal category, it follows that $V^{\otimes k}$ is also a right $\mathfrak{g}$-supermodule with action given by

$$(v_k \otimes \cdots \otimes v_1) \cdot X = \sum_{i=1}^{k} (-1)^{\sum_{j<i} \overline{v_j}\,\overline{X}} v_k \otimes \cdots \otimes v_i \cdot X \otimes \cdots \otimes v_1 \tag{6.2.1}$$

for homogenous $v_k, \ldots, v_1 \in V$ and $X \in \mathfrak{g}$.

Definition 6.2.1. The wreath superalgebra $A^{\otimes k} \rtimes \mathfrak{S}_k$ is the vector superspace $A^{\otimes k} \otimes_\Bbbk \Bbbk\mathfrak{S}_k$ with multiplication given by

$$(\mathbf{a} \otimes \sigma)(\mathbf{b} \otimes \tau) = \mathbf{a}\sigma(\mathbf{b}) \otimes \sigma\tau \quad \mathbf{a}, \mathbf{b} \in A^{\otimes k}, \sigma, \tau \in \mathfrak{S}_k,$$

and extended by linearity.

Note that the wreath product superalgebra is generated as a superalgebra by $A^{\otimes k}$ and $\Bbbk\mathfrak{S}_k$. Let $a_i := 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1 \in A^{\otimes k}$ where $a \in A$ is in the ith position. It follows from Theorem 4.3.1 that the following superalgebra homomorphism is injective

$$\begin{aligned} \Phi : A^{\otimes k} \rtimes \mathfrak{S}_k &\to \mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes k}), \\ a_i \otimes \mathrm{id} &\mapsto \underbrace{\uparrow \cdots \uparrow}_{k-i} \underbrace{\overset{\bullet a}{\uparrow} \ \uparrow \cdots \uparrow}_{i}, \\ 1 \otimes s_i &\mapsto \underbrace{\uparrow \cdots \uparrow}_{k-i-1} \underbrace{\times \uparrow \cdots \uparrow}_{i+1}. \end{aligned} \tag{6.2.2}$$

Composing (6.2.2) with ψ from Theorem 5.3.4 we obtain an action map

$$A^{\otimes k} \rtimes \mathfrak{S}_k \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes k}) \tag{6.2.3}$$

given by

$$\begin{aligned} a_i \otimes \mathrm{id} &\mapsto (v_k \otimes \cdots \otimes v_1 \mapsto (-1)^{\sum_{j=0}^{k-i-1} \overline{v_{k-j}}\,\bar{a}} v_k \otimes \cdots \otimes v_{i+1} \otimes a v_i \otimes v_{i-1} \otimes \cdots \otimes v_1), \\ 1 \otimes s_i &\mapsto (v_k \otimes \cdots \otimes v_1 \mapsto (-1)^{\overline{v_i}\,\overline{v_{i+1}}} v_k \otimes \cdots \otimes v_{i+2} \otimes v_i \otimes v_{i+1} \otimes v_{i-1} \otimes \cdots \otimes v_1) \end{aligned}$$

on homogenous elements and extended by linearity. Thus we have a left $A^{\otimes k} \rtimes \mathfrak{S}_k$-action on the superspace $V^{\otimes k}$ and note that it commutes with the $\mathfrak{g}$-action. Thus we have

$$A^{\otimes k} \rtimes \mathfrak{S}_k \xrightarrow{(6.2.3)} \mathrm{End}_{\Bbbk}(V^{\otimes k}) \xleftarrow{(6.2.1)} U(\mathfrak{g})$$

To show that the image of $\mathfrak{g}$ in $\mathrm{End}_{\Bbbk}(V^{\otimes k})$ under the action map corresponding to (6.2.1) is $\mathrm{End}_{A^{\otimes k} \rtimes \mathfrak{S}_k}(V^{\otimes k})$, we can follow a modified version of the usual proof for classical Schur-Weyl duality. Then, in the case that A is a semisimple superalgebra, by [13, Claim 4.22], $A^{\otimes k} \rtimes \mathfrak{S}_k$ is also semisimple, so we can apply Theorem 6.1.1, to see that (6.2.3) is an isomorphism and obtain Schur-Weyl duality. There are several examples where A is semisimple and we believe a Schur-Weyl duality result holds.

Example 6.2.2. When $A = \Bbbk$ we recover classical Schur-Weyl duality where $A^{\otimes k} \rtimes \mathfrak{S}_k = \Bbbk\mathfrak{S}_k$, which is semisimple by Maschke's Theorem when $\mathrm{char}(\Bbbk)$ does not divide k.

Example 6.2.3. In fact, for any finite group G with $|G|$ not divisible by $\mathrm{char}(\Bbbk)$, $A = \Bbbk G$ is a semisimple algebra by Maschke's Theorem and so $A^{\otimes k} \rtimes \mathfrak{S}_k$ is semisimple.

Example 6.2.4. The two-dimensional Clifford superalgebra, Cl, from Example 3.2.3, is a central simple superalgebra. So when $A = \mathrm{Cl}$, $A^{\otimes k} \rtimes \mathfrak{S}_k$ is the Sergeev algebra of [27] and is semisimple.

However, there are many interesting cases where $A^{\otimes k} \rtimes \mathfrak{S}_k$ is not semisimple.

Example 6.2.5. Let $A = \Bbbk[t]/(t^\ell)$ for $\ell \geq 2$ as in Example 5.1.2. Note that the ideal $I = (t)$ is nonzero but $I^\ell = 0$, so I is nilpotent. Therefore A is not semisimple and so $A^{\otimes k} \rtimes \mathfrak{S}_k$ is not semisimple.

In the case where we do not assume A is semisimple, whether or not Schur-Weyl duality holds is open. To show directly that the image of (6.2.3) in $\mathrm{End}_\Bbbk(V^{\otimes k})$ is $\mathrm{End}_\mathfrak{g}(V^{\otimes k})$ is somewhat complex. Equivalently, we might wonder for which Frobenius superalgebras, A, is $\psi \colon \mathcal{OB}(A) \to \text{smod-}\mathfrak{g}$ from Theorem 5.3.4 full? The functor ψ is said to be full if for any objects $X \in \{\uparrow, \downarrow\}^{\otimes k_1}, Y \in \{\uparrow, \downarrow\}^{\otimes k_2}$, the map

$$\mathrm{Hom}_{\mathcal{OB}(A)}(X, Y) \to \mathrm{Hom}_{\text{smod-}\mathfrak{g}}(\psi(X), \psi(Y)) \tag{6.2.4}$$

is surjective.

A particular case of (6.2.4) is showing the superalgebra homomorphism

$$\mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes k}) \to \mathrm{End}_{\text{smod-}\mathfrak{g}}(V_+^{\otimes k}),$$

is surjective. Recalling that $A^{\otimes k} \rtimes \mathfrak{S}_k \hookrightarrow \mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes k})$ by (6.2.2), we can more clearly see the resemblance of (6.2.4) to classical Schur-Weyl duality.

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