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Affine Reflection Groups and Bruhat-tits Buildings

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AFFINE REFLECTION GROUPS AND BRUHAT-TITS BUILDINGS

By

Michel Parent, B.Sc.

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A Thesis

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Abstract

This thesis contains a summary of material about finite and affine reflection groups as well as about Tits' construction of buildings, aiming ultimately at creating buildings associated to split classical algebraic groups over a p-adic field. To this end, this thesis contains a review of p-adic numbers as well as several examples of classical algebraic groups. It also contains numerous examples worked out in detail.

Acknowledgements

I would like to thank my supervisor Monica Nevins for her infinite support and patience. I would also like to thank my wife Carrie for always standing by my side and giving me all the support in the world, as well my parents Suzanne and Gaëtan for doing everything it is that parents are loved for. Finally, I would like to thank my Nan, Marion, for sheltering me for two weeks from the stress and distractions of the world at her home in Norman's Cove, Newfoundland when I needed it most.

Dedication

I dedicate this work to my brand new nephew and godson, Anthony Lawrence Wheeler.

Contents

Chapter 1

Introduction

The role of finite, and later affine, reflection groups in the study and classification of Lie groups and Lie algebras has been known for over a hundred years. The usual classification of simple Lie algebras over the complex numbers, for example, involves showing that each simple Lie algebra gives rise to the crystallographic root system of a finite reflection group, and then showing that the simple Lie algebra is completely determined by this root system. Similar methods are used to classify compact Lie groups and simple algebraic groups.

In the 1960s and 1970s, Bruhat and Tits together developed a way to "recreate" an algebraic group from its reflection group, or more precisely, from the associated Coxeter system (Definition 2.24). That is, from a Coxeter system they produced an object which is now called a Bruhat-Tits building. When the Coxeter system is one coming from an algebraic group, then the automorphisms of the building are an algebraic group whose associated root system corresponds to that Coxeter system.

In fact, they produced much more. According to [Y], the original writings on Bruhat-Tits theory and the theory of buildings ([BT1], [BT2], [BT3], [BT4], [BT5]) spans over 20 years, ranging from 1965 until 1987 (during which time the formulation was still evolving), and covers more than 550 pages. For many people who would like to have this theory as a foundation for study it is a daunting task to attempt to read the writings, because they are so long and make every attempt to give all results in complete generality.

In this thesis, we present the tools for defining a building, and then its associated group of automorphisms, starting from a finite or affine reflection group. We also show the converse, that is, how one can create a building associated to a group which has a BN-pair (Definition 5.34). We conclude with several explicit constructions of buildings and examine several equivalent ways of describing them, arising from different constructions.

In this thesis, we follow mainly [Br] and [G], referring also as necessary to [H3]. In order to quickly reach the most important geometric results about finite and affine reflection groups (found in §2.4 and §3.4), we have omitted a great many technical details about the generating set S of a reflection group W. As a result, we have omitted several proofs (or parts of proofs) in the fifth chapter, where buildings are defined.

The goal of this thesis is to illuminate the topic of Bruhat-Tits buildings and to provide the necessary background to understand their construction in the basic cases. To this end, four examples of low-rank buildings (or rather, of one apartment in each building) were explicitly computed (see Chapters 6, 7 and Appendix A). To our knowledge, these examples would be well-known to experts in the field but are not readily available in the literature.

We have also developed numerous illustrative examples of key points in the text, and provide original proofs of some of the results we cite. These contributions are indicated at the beginning of each chapter, as are the key sources used for the rest of the results.

The thesis is organized as follows. In Chapter 2 we present finite reflection groups, as groups generated by reflections in linear hyperplanes of a Euclidean space, and associated to root systems. (Among these are the crystallographic root systems, which later arise as root systems for affine reflection groups in Chapter 3.) We present some examples, and prove some key algebraic properties of these groups and their associated root systems, before going on to explore in detail the geometric interpretation of the action of finite reflection groups, and introducing the terminology of *chambers* and *walls*.

In Chapter 3, we expand upon the ideas presented in Chapter 2 by considering now also the case of affine reflection groups, that is, infinite groups generated by reflections in affine hyperplanes. We begin by exploring the structure of these groups as subgroups of the affine group on a vector space. We present several examples and then go on to analyse in general how an affine reflection group acts on the set of hyperplanes associated to it. We show that an affine reflection group contains a finite reflection group as a subgroup, and analyse the interplay between the two groups. In particular, we have greatly expanded on the proof in Brown that an affine reflection group is the semidirect product of a finite reflection group and a lattice (Proposition 3.19). We conclude the chapter by describing and proving several geometric properties of affine reflection groups, giving a geometric interpretation analogous to the one given for finite reflection groups in Chapter 2.

In Chapter 4, we provide some background on the fields of p-adic numbers $\mathbb{Q}_p$ which will be required for the development of the examples in Chapter 7. We also present some particular linear algebraic groups which arise in Chapters 6 and 7. This preview is intended to give the reader an example to bear in mind as they read Chapter 5, where the groups involved are presented in abstract generality.

Chapter 5 is the key chapter in the presentation of the theory of Bruhat-Tits buildings. In the first part, we establish the notation and theory of Coxeter complexes, and see that the geometric realizations of the finite and affine reflection groups presented in Chapters 2 and 3 are examples of Coxeter complexes. We then define a building as a collection of apartments, where each apartment is exactly such a Coxeter complex. We then prove that each building admits a unique labelling by elements of the associated reflection group; the labelling determines the *type* of each vertex of the building. This allows us to define the group of type-preserving automorphisms of the building. Among its subgroups are B, the group of type-preserving automorphisms which fix a particular chamber, and N, the group of type-preserving automorphisms which fix a particular apartment containing that chamber. We explore some of their properties.

In the second part of Chapter 5, we do the reverse. That is, beginning with a group G admitting a pair of subgroups B and N which have these properties (called a BN-pair), we construct a poset consisting of *special subgroups* and certain cosets called *special cosets*, and show that this poset is a building in the sense of Bruhat and

Tits.

In Chapter 6, we apply this last construction to the groups $GL_3(\mathbb{R})$ and $Sp_4(\mathbb{R})$ respectively. In each case, we determine the poset of special cosets and use this to draw a fundamental apartment of the associated building. We note that by considering the flags of subspaces stabilized by the special subgroups, we can describe the elements of the apartment with flags of subspaces. Finally, by realizing the associated finite reflection groups as groups acting on these flags, we obtain an identification of each of the affine subspaces of the apartment by the elements of the finite reflection group W.

In Chapter 7, we turn to the case of p-adic groups, where the affine reflection groups naturally arise, and we present the examples of $SL_3(\mathbb{Q}_p)$ and $SL_4(\mathbb{Q}_p)$ in detail. Here, rather than explicitly defining the special subgroups and special cosets, we focus directly on flags of lattice chains preserved by special subgroups. We discuss and prove several results for the general case of $SL_n(\mathbb{Q}_p)$, including the identification of the corresponding affine reflection group as a group normalizing a maximal compact diagonal subgroup.

All this helped us to define the poset structure of the apartment, but to determine the corresponding geometry, we needed to make several additional calculations, which have been relegated to Appendix A. The presentation of the apartments and the derivation of their description by lattice flags conclude the thesis.

Chapter 2

Finite Reflection Groups

This chapter will focus on finite reflection groups, which are the precursor to affine reflection groups. Both of these types of reflection groups are fundamentally tied to buildings. The results in this chapter come mainly from [Br, §I, pages 1-32] and [H3, Chapter 1, pages 1-28]. The Lemmas proven for Example 2.10 are original, and the proof of Proposition 2.15 has been signficantly expanded from the literature.

2.1 Basic Definitions

Let V be a Euclidean space with an inner product denoted by $(\, , \,)$. An example is $V = \mathbb{R}^n$, equipped with the dot product.

Definition 2.1. A *hyperplane* $H \subset V$ is a co-dimension one subspace.

Example 2.2. A plane passing through the origin is a hyperplane in $\mathbb{R}^3$.

Definition 2.3. A *reflection* on V is a linear endomorphism with respect to a hyperplane H, denoted s_H, which acts as the identity on H while acting as multiplication by -1 on the one-dimensional orthogonal complement of H.

If α is a non-zero vector orthogonal to H and λ is any vector in V then the reflection $s_H : V \to V$ is given by the formula

$$s_H(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \cdot \alpha. \tag{1}$$

Since any $\lambda \in H$ satisfies $(\lambda, \alpha) = 0$, and the orthogonal complement of H is spanned by α, the reader may easily check that this formula is equivalent to the above description of a reflection. If we have a non-zero vector α orthogonal to H, we may refer to H as H_α. In such a situation, we will refer to s_H as s_α.

Then we have

$$\begin{aligned} H_\alpha &= \{\lambda \in V | (\alpha, \lambda) = 0\} \\ &= \{\lambda \in V | s_\alpha(\lambda) = \lambda\}. \end{aligned}$$

It is important to note that all reflections are of order two. This is clear from Definition 2.3, as well as from Equation (1).

Definition 2.4. A *reflection group* is a group W of linear transformations generated by reflections s_H, where H ranges over a set of hyperplanes.

Let us consider in this chapter the case that W is finite, and let $\mathcal{H}$ be the set of all hyperplanes H such that $s_H \in W$. Since the set $\{s_H | H \in \mathcal{H}\}$ is contained in W, and W is finite, it follows that $\mathcal{H}$ is finite as well.

It is possible that our space V may be too large, containing irrelevant information. Set $V_0 = \bigcap_{H \in \mathcal{H}} H$.

Lemma 2.5. *The set of fixed points of W acting on V is V_0.*

Proof. If $\lambda \in V$ satisfies $w(\lambda) = \lambda$ for all $w \in W$, then in particular $s_H(\lambda) = \lambda$ for all $H \in \mathcal{H}$, so $\lambda \in \bigcap_{H \in \mathcal{H}} H = V_0$. The converse is clear, since W is generated by the reflections it contains. □

Definition 2.6. V_0 is the *inessential* part of V and its orthogonal complement V_1 is the *essential* part of V. If $V = V_1$, then $(W,\, V)$ is an *essential pair* and we say that W acts *essentially* on V.

We would prefer if $V_0 = \{0\}$, since having a fixed-point set containing more than the origin tells us nothing more about W than a space that has only the origin comprising the fixed-point set.

In general, one can replace V with V_1 without losing any information about W. Note that both V_0 and V_1 are W-invariant and that the inessential part of V_1 is $\{0\}$, so we have that $V = V_0 \oplus V_1$ where W acts trivially on V_0 and essentially on V_1. Thus we may replace V with V_1, and the resulting pair (W, V_1) is essential.

Example 2.7. Our first example is the simplest non-trivial finite reflection group. We will also use it to stress the difference between an essential and an inessential pair (W, V).

Suppose that V is two-dimensional and let H be a line passing through the origin; see Figure 1. Let W be generated by a single reflection s_H; it has order two. Reflection through H simply permutes the two half spaces found on either side of it. Note that the fixed-point set V_0 is not simply the origin, but the entire hyperplane H, so W does not act essentially on V.

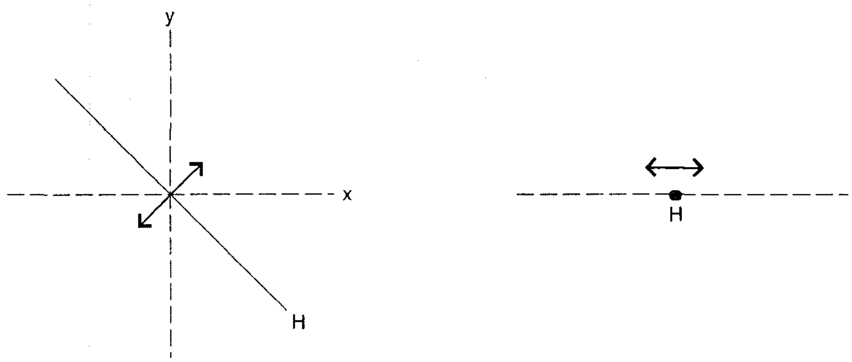

Figure 1: Inessential vs. essential pairs $(W,\ V)$

There is nothing wrong with our choice of V. However, it is bigger than necessary. Here, $V_1 = span\{\alpha\}$ where α is a non-zero vector perpendicular to H. By using the method described above, we can replace V by a one-dimensional Euclidean space, with W acting essentially. Then H becomes a point (the origin) instead of a line, by taking the quotient $V' = V/H$. In this case, $V_0' = (\bigcap_{H \in \mathcal{H}} H)/H = H/H = \{0\}$ and $V_1' = V'$. The finite reflection group W acts on V' by permuting two halves of a line across the origin at its center. This is equivalent to permuting two half spaces across a line, but is obviously simpler. It is more desirable since there has been no loss of information.

From this point on, we will only use essential pairs (W, V).

2.2 Orthogonality

Recall that the orthogonal group $O(V)$ is defined to be the set of all invertible linear transformations t of V which preserve the inner product $(,)$, in the sense that

$$(tx, ty) = (x, y) \quad \forall x, y \in V. \tag{2}$$

This is a group under composition. It is easy to check that a reflection in a hyperplane is an orthogonal map. It follows that any group W which is generated by reflections of V is a subgroup of $O(V)$.

Proposition 2.8. *If $t \in O(V)$ and α is any non-zero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$.*

Proof. Recall that $s_{t\alpha}$ is the reflection in $H_{t\alpha}$, and is thus determined by fixing $H_{t\alpha}$ pointwise and sending $t\alpha$ to its negative. Clearly,

$$ts_\alpha t^{-1}(t\alpha) = ts_\alpha(\alpha) = t(-\alpha) = -t\alpha$$

so we only have to prove that $ts_\alpha t$ fixes $H_{t\alpha}$ pointwise. To do this, note first that since $(\lambda, \alpha) = (t\lambda, t\alpha)$, λ is contained in H_α if and only if $t\lambda$ is contained in $H_{t\alpha}$. Now $\lambda \in H_\alpha$ if and only if $s_\alpha(\lambda) = \lambda$. Therefore, since

$$ts_\alpha t^{-1}(t\lambda) = ts_\alpha(\lambda) = t\lambda,$$

we see that $ts_\alpha t^{-1}$ stabilizes all points $t\lambda$ in $H_{t\alpha}$, as required. □

Corollary 2.9. *Let W be a finite reflection group. If $w \in W$, then $s_{w\alpha}$ belongs to W whenever s_α does. In particular, W permutes the set $\mathcal{H} = \{H | s_H \in W\}$.*

Proof. Since $s_{w\alpha} = ws_\alpha w^{-1}$ by Proposition 2.8, the first part follows. The second part follows from the proof, where it was shown that $H_{w\alpha} = wH_\alpha$. □

We may now characterize all essential pairs (W,V) where $\dim V = 2$.

Example 2.10. Let $\dim(V)=2$ and suppose that W is a finite reflection group acting essentially on V. Recall that since W is finite, so is the set $\mathcal{H}$ of hyperplanes.

We begin with the following lemma.

Lemma 2.11. *Choose H_α and H_β in the finite set $\mathcal{H}$ such that the angle θ between them is minimal. Then $\theta = \pi/m$ for some positive integer m.*

Proof. Without loss of generality, we may suppose that H_β is at an angle θ counterclockwise from H_α. (See Figure 2.) Since the angle θ is minimal, there are no hyperplanes in $\mathcal{H}$ meeting the region from H_α to H_β. Let s_α denote the reflection in H_α and s_β the reflection in H_β. It is readily shown (using matrix algebra, for example) that $s = s_\alpha s_\beta \in W$ is clockwise rotation by 2θ. Since for each $H \in \mathcal{H}$, $sH \in \mathcal{H}$ by Proposition 2.8, we see that applying elements of W to H_α and H_β generates a series of hyperplanes $H_0 = H_\alpha$, $H_1 = H_\beta$, $H_2 = sH_\alpha$, $\cdots$ in $\mathcal{H}$, such that the angle from H_i to H_{i+1} is θ. Let

$$m = \min\{m > 0 | m\theta \leq \pi, (m+1)\theta > \pi\}. \tag{3}$$

We have necessarily that

$$(m+1)\theta - \pi \geq \theta$$

since if it does not hold, then H_{m+1} and H_α make an angle smaller than θ, contradicting the choice of H_β. We subsequently derive:

$$\begin{aligned} (m+1)\theta - \pi &\geq \theta \\ m\theta - \pi &\geq 0 \\ m\theta &\geq \pi \end{aligned} \tag{4}$$

Equations (3) and (4) together force the conclusion

$$\theta = \pi/m.$$

□

We claim that $\mathcal{H} = \{H_0, H_1, \cdots, H_{m-1}\}$. Namely, in the course of the proof we showed that each H_i makes an angle of θ with the hyperplane H_{i+1} and $H_m = H_0$. If $H \in \mathcal{H}$ were not among the H_i, then let $0 < x\theta < \pi$ be the angle it makes with H_0. Set $m = \lfloor x \rfloor$ to be the greatest integer less than or equal to x; then by construction H makes an angle of $(x - \lfloor x \rfloor)\theta < \theta$ with H_m, contradicting the minimality of θ in the statement of the Lemma.

Since the group generated by s_α and s_β includes the reflections in all the hyperplanes H_i (by the proof of the Lemma, and the previous Corollary), and this group is all of W, we deduce that W is generated by s_α and s_β. Moreover, from the proof of the Lemma we deduce that $(s_\alpha s_\beta)^m = 1$.

Let us now use this result to determine the order of W. Let R be the subgroup of W generated by $s = s_\alpha s_\beta$. It has order m.

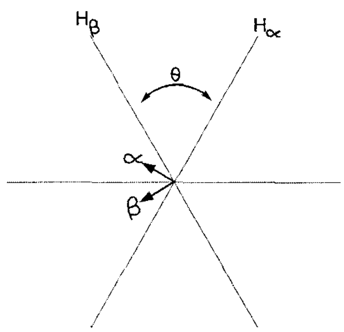

Figure 2: The case when $m = 3$

Lemma 2.12. *We have that $R \lhd W$.*

Proof. Let $s \in R$; then s can be written as $s = (s_\alpha s_\beta)^n$ for some $n \in \mathbb{Z}$. Recall that W is generated by $\{s_\alpha, s_\beta\}$. Since $s_\alpha^2 = 1$ we have that $s_\alpha^{-1} = s_\alpha$. We note that

$$s_\alpha \cdot s \cdot s_\alpha^{-1} = s_\alpha \cdot (s_\alpha s_\beta)^n \cdot s_\alpha = (s_\beta s_\alpha)^n = (s_\alpha s_\beta)^{-n} \in R$$

$$s_\beta \cdot s \cdot s_\beta^{-1} = s_\beta \cdot (s_\alpha s_\beta)^n \cdot s_\beta = (s_\beta s_\alpha)^n = (s_\alpha s_\beta)^{-n} \in R$$

Hence R is preserved under conjugation by the generators of W and therefore $R \lhd W$. □

Since both generators of W are of order two, any element of W must be written as an alternating sequence of the two generators. This, along with Lemma 2.12 implies all elements written using an even number of the generators is equivalent to the identity mod R, and any element written using an odd number of generators is equivalent to either s_α or s_β mod R.

However, $s_\alpha \equiv s_\beta \bmod R$ since $s_\alpha \cdot s_\alpha s_\beta = s_\beta$. Therefore W/R has order two. Knowing the order of both R and W/R allows us to calculate the order of W to be $2m$.

Recall the dihedral group D_{2m}, which is the group of order $2m$ on two generators r and a satisfying the relations

$$r^2 = 1, \quad a^m = 1, \quad rar = a^{-1}.$$

From the above, we deduce that $W \cong D_{2m}$, via the map sending s_α to r and s to a.

2.3 Root Systems

Let us consider the following definition of a root system, from [H3].

Definition 2.13. A *root system* is a finite set of non-zero vectors $\Phi \subset V$ satisfying:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \ \forall\, \alpha \in \Phi$
   The only multiples of any $\alpha \in \Phi$ also contained in Φ are $\pm\alpha$.

2. $s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi$
   Reflecting Φ through a hyperplane orthogonal to any element $\alpha \in \Phi$ leaves Φ unchanged.

   A root system is *crystallographic* if in addition it satisfies:

3. $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \ \forall\, \alpha, \beta \in \Phi$

A vector $\alpha \in \Phi$ is a *root*.

Given a root system Φ, define W to be the group generated by $\{s_\alpha | \alpha \in \Phi\}$. This is a reflection group where $\mathcal{H} = \{H_\alpha | \alpha \in \Phi\}$.

Conversely, given a finite reflection group W with $\mathcal{H} = \{H | s_H \in W\}$, for each $H \in \mathcal{H}$, take the two unit vectors orthogonal to H. Corollary 2.9 implies that W permutes the hyperplanes in $\mathcal{H}$; consequently, W permutes these unit vectors. It follows that this set satisfies the conditions (1) and (2) of Definition 2.13, although it may not be crystallographic. See Example 2.14, below.

Example 2.14. Let $V = \mathbb{R}^2$ and consider the four hyperplanes of the reflection group $W = D_8$ (see Example 2.10) illustrated in Figure 3. It is easy to check that the set

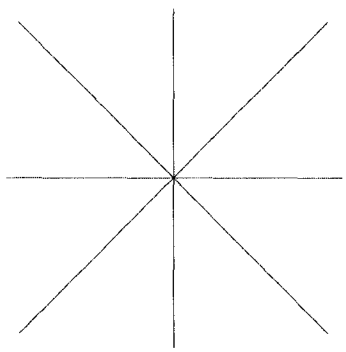

Figure 3: The four hyperplanes of the reflection group $W = D_8$

Φ of eight unit vectors defined by these hyperplanes forms a root system. It is not crystallographic because, for example, taking $\alpha = (1, 0)$ and $\beta = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we have

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \sqrt{2} \notin \mathbb{Z}.$$

On the other hand, the set of eight vectors

$$\Phi' = \{(\pm 1, 0), (\pm 1, \pm 1), (0, \pm 1)\}$$

define the same set of hyperplanes, and one can verify that Φ' is a crystallographic root system. Both Φ and Φ' correspond to the same finite reflection group W, since they give rise to the same set of hyperplanes.

See Table 1 for a summary of notation for finite reflection groups.

V	Euclidean space with an inner product
H	hyperplane contained in V
s_H	reflection with respect to the hyperplane H
W	finite reflection group generated by a finite set of reflections
$\mathcal{H}$	set of hyperplanes used to obtain reflections in a finite reflection group
α	root associated to a hyperplane H
Φ	root system associated to W
H_α	hyperplane with root α
s_α	reflection with respect to the hyperplane H_α

Table 1: Summary of Notation

2.4 Geometric Properties

Except for Proposition 2.15, which is our own, all results presented here come from [Br, §I].

Let us explore the geometric properties of finite reflection groups. We let W be a finite reflection group acting essentially on V and $\mathcal{H} = \{H_1, \ldots, H_k\}$ be its set of reflection hyperplanes.

For each $H_i \in \mathcal{H}$, let $f_i : V \longrightarrow \mathbb{R}$ be a non-zero linear function such that

$$H_i = \{x \in V | f_i(x) = 0\}.$$

Let us use the following shorthand notation:

$$\begin{aligned} f_i^0 &= H_i, \\ f_i^+ &= \{x \in V | f_i(x) > 0\}, \\ f_i^- &= \{x \in V | f_i(x) < 0\}. \end{aligned}$$

So f_i^- and f_i^+ are the two open half spaces of $V \setminus H_i$.

Proposition 2.15. *The function f_i is uniquely determined by its associated H_i up to a non-zero scalar multiple.*

Proof. Let α be a non-zero vector orthogonal to the hyperplane H_i; then every $x \in V$ can be written uniquely as $x = h + a\alpha$ where $h \in H_i$ and $a \in \mathbb{R}$. By definition, $f_i(h) = 0$ for all $h \in H_i$, so by linearity we have

$$f_i(x) = f_i(h + a\alpha) = f_i(h) + af_i(\alpha) = af_i(\alpha).$$

Since f_i is nonzero, $f_i(\alpha) = b \neq 0$. If f_i' is any other nonzero linear function $f_i' : V \to \mathbb{R}$ satisfying $f_i'(h) = 0$ for all $h \in H_i$, then set $f_i'(\alpha) = b' \neq 0$. It follows that

$$f_i' = \frac{b'}{b} f_i.$$

□

Note that if we replace f_i with a negative scalar multiple of f_i it will exchange the open half spaces $f_i^{\pm}$ of $V \setminus H_i$.

Definition 2.16. Choose a sign $\sigma_i \in \{+, -, 0\}$ for each i. A *cell* in V, with respect to $\mathcal{H}$, is a non-empty set $A \subseteq V$ defined by

$$A = \bigcap_{i=1\ldots k} f_i^{\sigma_i}.$$

In other terms, $A = \bigcap_i U_i$ where each U_i is either H_i or one of the open half-spaces of its complement.

Example 2.17. Figure 4 shows a set of choices for the f_i's for the finite reflection group shown in Figure 2, as well as the cells that the three hyperplanes define. One can see here that there are 13 cells. Six of the cells are open sections of V, each of which is shaped like a cone, six of the cells are open half lines extending from the origin, and the last cell is the origin itself.

Remark. There are twenty-seven choices in total for the triples $(\sigma_1, \sigma_2, \sigma_3)$. However, many of these choices are inconsistent, that is, they define the empty set. For example, in Figure 4, if we choose f_3^+, f_2^+, and f_1^-, we have the empty set since the third condition contradicts the first two, and there are no points satisfying all three conditions.

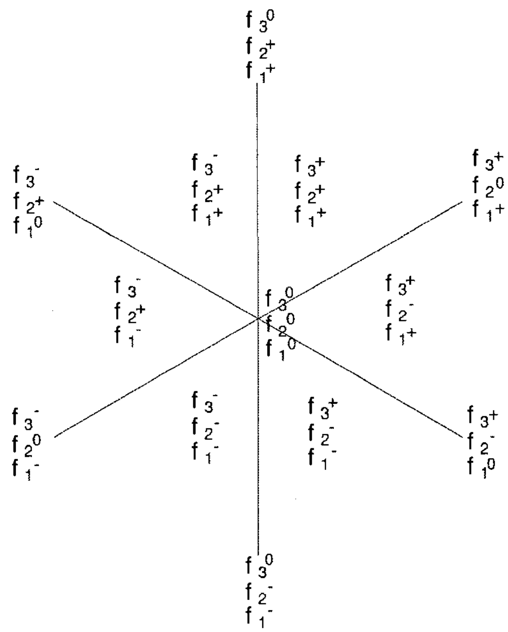

Figure 4: Linear functions defining cells in $\mathbb{R}^2$

Definition 2.18. Let A and B be cells. Supppose A is defined by the sign choices σ_i and B is defined by the sign choices δ_i. Then B is a *face* of A if for all i:

$$\begin{aligned} \sigma_i \in \{+,-\} \;&\Rightarrow\; \delta_i \in \{\sigma_i, 0\} \\ \sigma_i = 0 \;&\Rightarrow\; \delta_i = 0. \end{aligned}$$

This is to say that B is a face of A if its σ_i's are obtained from A's by replacing any number (including none) of them with 0's.

Note that by this definition, all cells are faces of themselves. A less trivial example would be to say that the open half line defined by f_3^+, f_2^0, f_1^+ is a face of the open sector defined by f_3^+, f_2^+, f_1^+. The origin, being defined by $\sigma_3 = \sigma_2 = \sigma_1 = 0$, is a face of every cell.

Recall that a poset, or partially ordered set, is a set A together with a binary relation, which we'll denote by $\leq$, such that this relation is reflexive ($x \leq x$ for all

$x \in A$), antisymmetric (if $x \leq y$ and $y \leq x$ then $y = x$) and transitive (if $x \leq y$ and $y \leq z$ then $x \leq z$). A poset differs from a totally ordered set in that not all pairs of elements $x, y \in A$ satisfy either $x \leq y$ or $y \leq x$.

Let us define a poset of cells as follows. Let Σ be the set of cells and define a binary relation on Σ using the face relation given in Definition 2.18. That is to say, if $A, B \in \Sigma$ then $B \leq A$ if B is a face of A. This relation is clearly reflexive, antisymmetric and transitive, so Σ is a poset.

Definition 2.19. The nonempty cells defined by sign choices σ_i such that $\sigma_i \neq 0$ for all i are called *chambers.*

Since the chambers are intersections of open half-planes, they are open.

Recall that a set U is *convex* if, for any $x, y \in U$, the line segment $[x, y] := \{tx + (1-t)y \colon 0 \leq t \leq 1\}$ is entirely contained in U. If $U = \{z \in V \colon f(z) > 0\}$ is an open half-plane then U is convex by the linearity of f and from the definition it follows that the intersection of convex sets is convex. Hence each chamber is a convex set.

Note that a convex set is connected, since it is path-connected. In fact, by construction the chambers are the connected components of $V \setminus \bigcup_{H \in \mathcal{H}} H$.

Let C denote a fixed chamber. Consider a face A of C which is obtained from C by changing exactly one of the σ_i's to 0. That is to say, A is a face of C such that $A \subset H_i$ but A is not contained in any other H_j, $j \neq i$. Since the codimension of H_i in V is 1, we say A is a face of C of *codimension 1.*

Note that if A is a face of codimension 1, then $A \lneq C$ and there are no faces B such that $A \lneq B \lneq C$.

Definition 2.20. Let C be a chamber. We will say that $\mathcal{H}' \subseteq \mathcal{H}$ *defines* C if there exist choices $\sigma_i \in \{+, -\}$ so that

$$C = \bigcap_{H_i \in \mathcal{H}'} f_i^{\sigma_i}.$$

By definition, the subset $\mathcal{H}$ defines C, for any chamber C. However, some $H_i \in \mathcal{H}$ may be redundant in the sense that there exist $j, k \neq i$ for which $f_j^{\sigma_j} \cap f_k^{\sigma_k} \subset f_i^{\sigma_i}$; in this case the set $\mathcal{H} \setminus H_i$ would also define C.

Proposition 2.21. *There is a unique minimal subset $\mathcal{H}' \subseteq \mathcal{H}$ which defines C.*

Proof. Since $\mathcal{H}$ is finite, we can always obtain a minimal $\mathcal{H}' \subseteq \mathcal{H}$ which defines C by simply picking any $\mathcal{H}' \subseteq \mathcal{H}$ which defines C and removing the redundant $H \in \mathcal{H}'$ as above. To show that $\mathcal{H}'$ is unique, we show that each codimension-1 face of C is contained in a hyperplane in $\mathcal{H}'$, and that conversely each hyperplane of $\mathcal{H}'$ contains a codimension-1 face of C.

Renumber the hyperplanes so that we have $\mathcal{H}' = \{H_1, \ldots, H_r\}$ for some $r \leq k$ and replace each function f_i by $-f_i$ if necessary so that $f_i(x) > 0$ for all $x \in C$, for $i = 1, \ldots, k$.

We will now show that each H_i, $i \leq r$, contains a codimension-1 face. Let

$$A_i = \bigcap_{j \neq i, 1 \leq j \leq k} f_j^+ \cap f_i^0 \subset H_i$$

This is a face of C if it is non-empty. By the minimality of $\mathcal{H}'$, $\mathcal{H}' \setminus H_i$ cannot define C; that is to say,

$$C' = \bigcap_{j \neq i, 1 \leq j \leq r} f_j^+$$

is a strictly larger set than C. Consider $y \in C' \setminus C$ and $x \in C$. By construction, $f_i(x) > 0$ and $f_i(y) \leq 0$, thus there is a point $z \in (x, y]$ such that $f_i(z) = 0$. Since C' is convex, $z \in C'$ as well, and so $z \in C' \cap f_i^0 = A_i$. So A_i is non-empty and is indeed a face of C contained in H_i.

Conversely, suppose A is a codimension-1 face of C. Let i denote the index of the hyperplane such that $A \subset H_i$. If $i \leq r$, we are done. If not, then we have

$$\begin{aligned} A &= \bigcap_{1 \leq j \leq k, j \neq i} f_j^+ \cap f_i^0 \\ &= C \cap f_i^0 \\ &= \emptyset \end{aligned}$$

where the second equality follows from the definition of $\mathcal{H}'$ and the last by $C \subset V \setminus \mathcal{H}$. Hence A is empty, contradicting the hypothesis that A is a face of C. The uniqueness of $\mathcal{H}'$ follows.

□

Corollary 2.22. *The minimal set $\mathcal{H}'$ defining C is the set of hyperplanes in correspondence with the codimension-1 faces of C. The elements of $\mathcal{H}'$ will be referred to as the walls of C.*

Proposition 2.23. *Any cell $A \in \Sigma$ is a face of a chamber. If A is a codimension-1 face of a chamber then it is a face of exactly two chambers.*

Proof. Let the cell A be defined by the sign choices τ_i where i indexes the hyperplanes in $\mathcal{H}$. Choose a point $x \in A$ and a point $y \in V \setminus \bigcup_{H \in \mathcal{H}} H$. Let $\sigma_i = \tau_i$ whenever $\tau_i \in \{+, -\}$ and let σ_i be the sign of $f_i(y)$ otherwise. Since $\mathcal{H}$ is finite, the line segment $[x, y]$ meets at most finitely many hyperplanes, so there is now a neighbourhood U around x such that for all $z \in U \cap (x, y]$ we have $f_i(z) \neq 0$ and the sign of $f_i(z) = \sigma_i$ for all i. Hence the choice of signs σ_i defines a (nonempty) chamber C, and we also have $A \leq C$ (that is, A is a face of C). This proves the first assertion.

Assume A is a codimension-1 face. Then there is exactly one index i such that $\tau_i = 0$, which we can relabel so that we have $\tau_1 = 0$. We thus have a maximum of two possible chambers with A as a face since there are only two possible sign choices for σ_1. We get one chamber by following the proof above and choosing for σ_1 the sign of $f_1(y)$. However, if we consider the intersection of the entire line L from y to x with the neighbourhood U, and noting that f_1 changes sign at x, there is a point $z \in L \cap U$ such that $f_1(z) < 0$. Hence we obtain the second chamber by defining it with σ_1 as the sign of $-f_1(y)$. We thus have that if A is a codimension-1 face, it is the face of exactly two chambers. □

In the case that two chambers C and C' share a codimension-1 face, we say that C and C' are *adjacent*. Thus if C and C' are adjacent, we have $C = C'$ or we have a codimension-1 cell A such that A is a face of the two chambers C and C'. In the latter scenario, we have a wall separating the two chambers, which we took to be H_1 in the proof above.

Note that this is the unique wall shared by both chambers, since by definition of a codimension-1 face, every σ_i with $i > 1$ is unchanged in passing from C to A to C'. Hence C and C' are on the same side of every other hyperplane H_j, $j > 1$.

We conclude this chapter with an important definition taken from [BT1, §1.2.1].

Definition 2.24. A *Coxeter system* is a pair (W, S) where W is a group and S is a subset of W satisfying the following properties:

1. W is generated by S and the elements of S are of order two.

2. Let $m(s, s')$ be the order of the element $ss' \in W$ (where $s, s' \in S$) and let I be the set of pairs (s, s') of elements of S such that $m(s, s')$ is finite; the generating set S and the relations $(ss')^{m(s,s')} = 1$ for $(s, s') \in I$ give a presentation of W.

The group W of Example 2.10, together with $S = \{s_\alpha, s_\beta\}$, satisfies this definition of a Coxeter system. More generally, given any finite reflection group W, let S be the set of reflections in the walls of $\mathcal{H}'$ (Proposition 2.21). Then (W, S) is a Coxeter system ([H3, §1.9], [Br, §II.4]).

Remark. The poset Σ defined here is a geometric realization of a Coxeter system's associated *Coxeter complex*, which we define in Chapter 5.

Chapter 3

Affine Reflection Groups

In this chapter, we explore reflection groups generated by reflections not only through linear hyperplanes, but more generally through affine hyperplanes. The results in Sections 3.1 through 3.3 come mainly from [H3, §4]; those from Section 3.4 come mainly from [Br, §VI]. All the lemmas proven in the first three sections are either original, or their proofs have been significantly expanded upon from the literature. The proof of Proposition 3.19 has also been significantly expanded.

3.1 Basic Definitions

Let V be a Euclidean space as in §2.1.

Definition 3.1. Let U be a hyperplane in V. Then for any $v \in V$, $v + U$ is an *affine hyperplane.*

From this point on, we will refer to affine hyperplanes simply as hyperplanes unless we wish to stress their affine nature.

At this point, we introduce the set $Aff(V)$, which is the set of affine maps on V. Recall that a map is affine if it can be written as a composition of an invertible linear transformation with a translation. Let $g \in GL(V)$, and let t_λ be translation by a vector $\lambda \in V$. Then their composition $t_\lambda g$ is evaluated as

$$(t_\lambda g)(\mu) = g(\mu) + \lambda$$

for any $\mu \in V$.

Lemma 3.2. *If $\mu \in V$ and $g \in GL(V)$, then there exists $\lambda \in V$ such that $t_\mu g = gt_\lambda$.*

Proof. Let $v \in V$,

$$\begin{aligned} t_\mu g(v) &= g(v) + \mu \\ &= g(v + g^{-1}(\mu)) \\ &= g(t_{g^{-1}(\mu)}(v)) \\ &= gt_{g^{-1}(\mu)}(v). \end{aligned}$$

So we may take $\lambda = g^{-1}\mu$. □

Lemma 3.3. *$Aff(V)$ is a group, namely a subgroup of bijections of V.*

Proof. We first note that $Aff(V)$ is closed under composition by Lemma 3.2. That is to say, if gt_λ, $g't_{\lambda'} \in Aff(V)$, then

$$gt_\lambda g't_{\lambda'} = gg't_\mu t_{\lambda'} = (gg')(t_\mu t_{\lambda'}) = ht_{\mu+\lambda'} \in Aff(V)$$

for some $\mu \in V$ and some $h \in GL(V)$.

The identity map is $i = t_0 I$, which is clearly in $Aff(V)$.

Let $t_\lambda g \in Aff(V)$. Cleary, $g^{-1}t_{-\lambda} = (t_\lambda g)^{-1}$, which can also be expressed as $t_{g^{-1}(-\lambda)}g^{-1}$ by Lemma 3.2. This completes the proof.

□

Lemma 3.4. *$Aff(V) = GL(V) \ltimes V$.*

Proof. Let us begin by identifying the set of all translations with V via the correspondence $t_\lambda \leftrightarrow \lambda$. Clearly, $GL(V)$ and V are subgroups of $Aff(V)$ and $Aff(V)$ is generated by $GL(V)$ and V. We also have that $V \cap GL(V) = \{i\}$, where i represents the identity transformation in $Aff(V)$. It is left to show that $V \lhd Aff(V)$. It suffices to show that V is preserved under conjugation by $GL(V)$. Let v, $\lambda \in V$, and let $g \in GL(V)$. Then by the proof of Lemma 3.2 we have that

$$gt_\lambda g^{-1} = t_{g\lambda}. \tag{5}$$

Since $g\lambda \in V$, this shows that $V \lhd Aff(V)$ and thus $Aff(V) = GL(V) \ltimes V$. □

Lemma 3.5. *If $T \in Aff(V)$, H, H' are two hyperplanes, and $T(H) \subseteq H'$, then $T(H) = H'$.*

Proof. Since $T \in Aff(V)$, write $T = t_\lambda g$ for some $\lambda \in V$ and some $g \in GL(V)$. Also write $H = v + U$ and $H' = v' + U'$ for some $v, v' \in V$ and some linear hyperplanes U, U'.

We have that $T(H) = t_\lambda g(H) = t_\lambda g(v + U) = g(v) + \lambda + g(U)$, which is itself an affine hyperplane since $g(U)$ is a linear hyperplane and $g(v) + \lambda \in V$. We thus have that

$$g(v) + \lambda + g(U) \subseteq v' + U', \tag{6}$$

which implies that we have one affine hyperplane contained in another. Rewrite Equation (6) as

$$g(U) \subseteq -g(v) - \lambda + v' + U'.$$

We have that $\vec{0} \in g(U)$ and thus $\vec{0} \in -g(v) - \lambda + v' + U'$, which implies that $-g(v) - \lambda + v' \in U'$. This gives us

$$g(U) \subseteq U',$$

and so we now have a linear hyperplane contained in another linear hyperplane and that they both have the same dimension. By drawing on standard results from linear algebra we can conclude that both hyperplanes are equal. This concludes the proof. □

Definition 3.6. Let H be an affine hyperplane and let α be a non-zero normal vector to H. Write $\lambda \in V$ as $\mu + c\alpha$ for some $c \in \mathbb{R}$ and $\mu \in H$. The *affine reflection* $s_H \in Aff(V)$ is the unique map that acts as the identity on H while acting as multiplication by -1 on the one-dimensional subspace of V orthogonal to H. That is to say,

$$s_H(\lambda) = s_H(\mu + c\alpha) = \mu - c\alpha.$$

One should note that in the above definition of an affine reflection, the decomposition $\lambda = \mu + c\alpha$ is unique. Namely, since for each $\mu \in H$, $(\alpha, \mu) = k$ is a constant, we deduce that $c = \frac{(\alpha,\lambda)-k}{(\alpha,\alpha)}$ and $\mu = \lambda - c\alpha$.

Let $\alpha \in V \setminus \{0\}$. Define, for each $k \in \mathbb{Z}$, the hyperplane

$$H_{\alpha,k} = \{\lambda \in V | (\alpha, \lambda) = k\}. \tag{7}$$

Recall that $H_\alpha = \{\mu \in V | (\alpha, \mu) = 0\}$. Thus $H_\alpha = H_{\alpha,0}$.

Lemma 3.7. $H_{\alpha,k} = \frac{k\alpha}{(\alpha,\alpha)} + H_\alpha$.

Proof. Let $\gamma \in H_{\alpha,k}$. Consider the element $\gamma - \frac{k\alpha}{(\alpha,\alpha)}$.

$$\begin{aligned}
(\alpha, \gamma - \frac{k\alpha}{(\alpha,\alpha)}) &= (\alpha, \gamma) - (\alpha, \frac{k\alpha}{(\alpha,\alpha)}) \\
&= k - k\frac{(\alpha,\alpha)}{(\alpha,\alpha)} \\
&= k - k \\
&= 0.
\end{aligned}$$

Thus we have that $\gamma - \frac{k\alpha}{(\alpha,\alpha)} \in H_\alpha$, which gives us

$$\gamma = (\frac{k\alpha}{(\alpha,\alpha)}) + (\gamma - \frac{k\alpha}{(\alpha,\alpha)}) \in \frac{k\alpha}{(\alpha,\alpha)} + H_\alpha,$$

implying that $H_{\alpha,k} \subseteq \frac{k\alpha}{(\alpha,\alpha)} + H_\alpha$.

By Lemma 3.5 we get equality, which concludes the proof. □

Let $H = H_{\alpha,k}$ and let us derive a formula for $s_H = s_{\alpha,k}$.

Lemma 3.8. *Let $\lambda \in V$. Then $s_{\alpha,k}(\lambda) = \lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\alpha$.*

Proof. Write λ as $\mu + c\alpha$ for some $c \in \mathbb{R}$ and $\mu \in H_{\alpha,k}$. By definition,

$$s_{\alpha,k}(\lambda) = s_{\alpha,k}(\mu + c\alpha) = \mu - c\alpha.$$

Let $R(\lambda) = \lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\alpha$. We have that

$$\begin{aligned}
R(\mu + c\alpha) &= \mu + c\alpha - \frac{2((\mu + c\alpha, \alpha) - k)}{(\alpha,\alpha)}\alpha \\
&= \mu + c\alpha - 2\frac{(\mu,\alpha) + c(\alpha,\alpha) - k}{(\alpha,\alpha)}\alpha
\end{aligned}$$

$$\begin{aligned}
&= \mu + c\alpha - 2\frac{k + c(\alpha,\alpha) - k}{(\alpha,\alpha)}\alpha \\
&= \mu + c\alpha - 2\frac{c(\alpha,\alpha)}{(\alpha,\alpha)}\alpha \\
&= \mu + c\alpha - 2c\alpha \\
&= \mu - c\alpha \\
&= s_{\alpha,k}(\lambda)
\end{aligned}$$

□

Corollary 3.9. $s_{\alpha,k} = t_{\frac{k\alpha}{(\alpha,\alpha)}} s_\alpha t_{\frac{-k\alpha}{(\alpha,\alpha)}}$.

Proof.

$$\begin{aligned}
t_{\frac{k\alpha}{(\alpha,\alpha)}} s_\alpha t_{\frac{-k\alpha}{(\alpha,\alpha)}}(\lambda) &= t_{\frac{k\alpha}{(\alpha,\alpha)}} s_\alpha \left(\lambda - \frac{k\alpha}{(\alpha,\alpha)}\right) \\
&= t_{\frac{k\alpha}{(\alpha,\alpha)}} \left(\lambda - \frac{k\alpha}{(\alpha,\alpha)} - \frac{2\left(\lambda - \frac{k\alpha}{(\alpha,\alpha)}, \alpha\right)}{(\alpha,\alpha)} \cdot \alpha\right) \\
&= t_{\frac{k\alpha}{(\alpha,\alpha)}} \left(\lambda - \frac{k\alpha}{(\alpha,\alpha)} - \frac{2\left((\lambda,\alpha) - \frac{k(\alpha,\alpha)}{(\alpha,\alpha)}\right)}{(\alpha,\alpha)} \cdot \alpha\right) \\
&= t_{\frac{k\alpha}{(\alpha,\alpha)}} \left(\lambda - \frac{k\alpha}{(\alpha,\alpha)} - \frac{2((\lambda,\alpha) - k)}{(\alpha,\alpha)} \cdot \alpha\right) \\
&= \lambda - \frac{k\alpha}{(\alpha,\alpha)} - \frac{2((\lambda,\alpha) - k)}{(\alpha,\alpha)} \cdot \alpha + \frac{k\alpha}{(\alpha,\alpha)} \\
&= \lambda - \frac{2((\lambda,\alpha) - k)}{(\alpha,\alpha)}\alpha \\
&= s_{\alpha,k}(\lambda).
\end{aligned}$$

The second and final equalities come from Lemma 3.8. □

Definition 3.10. A set of hyperplanes $\mathcal{H}$ is *locally finite* if every point in V has a neighbourhood meeting only finitely many $H \in \mathcal{H}$.

Definition 3.11. [Br, §VI.1B page 141] An *affine reflection group* is a group $W \subseteq Aff(V)$ generated by a set of affine reflections such that the set $\mathcal{H} = \{H | s_H \in W\}$ is locally finite.

See Tables 1 and 2 for a summary of notation for this chapter. If an entry appears in both tables, the entry from Table 2 will overide the entry from Table 1.

H	affine hyperplane contained in V
s_H	reflection with respect to the affine hyperplane H
$Aff(V)$	the group of affine transformations on V
t_λ	translation in V by the vector λ
$H_{\alpha,k}$	hyperplane parallel to H_α defined by Equation (7)
$s_{\alpha,k}$	reflection with respect to the hyperplane $H_{\alpha,k}$
W	affine reflection group generated by a set of reflections
$\mathcal{H}$	set of hyperplanes used to obtain reflections in an affine reflection group

Table 2: Summary of Notation

Example 3.12. The simplest example of an affine reflection group is a finite reflection group, since if $\mathcal{H}$ is finite it is certainly locally finite. We need only take the set of generators S from Definition 3.11 to be $S = \{s_{\alpha,k} | \alpha \in \Phi,\ k = 0\}$.

More generally, it can be shown ([H3, §4.2]) that if Φ is a *crystallographic* root system, then the group W generated by $\{s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ is an affine reflection group. In fact, it is true, but non-trivial to show ([H3, §4.10], [Br, §VI.1E, Remark 2]), that every affine reflection group arises from a crystallographic root system.

3.2 An Example

We begin with two important results.

Lemma 3.13. *The distance between any two consecutive hyperplanes $H_{\alpha,k}$ and $H_{\alpha,k+1}$ is $\frac{1}{(\alpha,\alpha)}\alpha$.*

Proof. By Lemma 3.7, $H_{\alpha,k} = \frac{k\alpha}{(\alpha,\alpha)} + H_\alpha$ and $H_{\alpha,k+1} = \frac{(k+1)\alpha}{(\alpha,\alpha)} + H_\alpha$. Therefore we have that $H_{\alpha,k+1} = \frac{1}{(\alpha,\alpha)}\alpha + H_{\alpha,k}$. Since α is orthogonal to $H_{\alpha,k}$ and $H_{\alpha,k+1}$, the distance betwen these hyperplanes is

$$\left\| \frac{1}{(\alpha,\alpha)}\alpha \right\| = \frac{1}{\|\alpha\|}.$$

□

Lemma 3.14. $s_{\alpha,k+1}s_{\alpha,k} = t_{\frac{2}{(\alpha,\alpha)}\alpha}$.

Proof.

$$\begin{aligned}
s_{\alpha,k+1}s_{\alpha,k}(\lambda) &= s_{\alpha,k+1}(\lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\alpha) \\
&= \lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\alpha - \frac{2((\lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\alpha,\alpha)-(k+1))}{(\alpha,\alpha)}\alpha \\
&= \lambda + (-\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} + \frac{2k}{(\alpha,\alpha)} - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} + \frac{4(((\lambda,\alpha)-k)\alpha,\alpha)}{(\alpha,\alpha)^2} + \frac{2k+2}{(\alpha,\alpha)})\alpha \\
&= \lambda + (-\frac{4(\lambda,\alpha)}{(\alpha,\alpha)} + \frac{4k+2}{(\alpha,\alpha)} + \frac{4(\lambda,\alpha)}{(\alpha,\alpha)} + \frac{-4k}{(\alpha,\alpha)})\alpha \\
&= \lambda + \frac{2}{(\alpha,\alpha)}\alpha.
\end{aligned}$$

□

Now, let $V = \mathbb{R}$, and let $\Phi = \{\alpha, -\alpha\} \subseteq V$, $\alpha \neq 0$. Let $\mathcal{H} = \{H_{\alpha,k} | k \in \mathbb{Z}\}$. All of these hyperplanes are parallel (they all share the same normal vector and never intersect), and consecutive hyperplanes are always the same distance apart (see Figure 5).

Let W be the affine reflection group generated by $S = \{s_H | H \in \mathcal{H}\}$. Let us show that the entire group can be generated by $s_{\alpha,0}$ and $s_{\alpha,1}$. Let $s = s_{\alpha,1}s_{\alpha,0}$. By Lemma 3.14 we have that s is a translation by $\frac{2}{(\alpha,\alpha)}\alpha$ in the direction of $H_{\alpha,0} \to H_{\alpha,1}$.

Note that $s = t_{\frac{2}{(\alpha,\alpha)}\alpha}$ takes 0 to $\frac{2}{(\alpha,\alpha)}\alpha$ and s^{-1} does the opposite. By Corollary 3.9,

$$s_{\alpha,2} = t_{\frac{2\alpha}{(\alpha,\alpha)}} s_{\alpha,0} t_{\frac{-2\alpha}{(\alpha,\alpha)}} = s s_{\alpha,0} s^{-1} = s_{\alpha,1}s_{\alpha,0}s_{\alpha,0}s_{\alpha,0}s_{\alpha,1} = s_{\alpha,1}s_{\alpha,0}s_{\alpha,1}.$$

More generally, if $m \in \mathbb{Z}$ then

$$s_{\alpha,2m} = t_{\frac{2m\alpha}{(\alpha,\alpha)}} s_{\alpha,0} t_{\frac{-2m\alpha}{(\alpha,\alpha)}} = (t_{\frac{2\alpha}{(\alpha,\alpha)}})^m s_{\alpha,0} (t_{\frac{2\alpha}{(\alpha,\alpha)}})^{-m} = s^m s_{\alpha,0} s^{-m}$$

which is obtained solely using $s_{\alpha,0}$ and $s_{\alpha,1}$. Also,

$$s_{\alpha,2m+1} = t_{\frac{(2m+1)\alpha}{(\alpha,\alpha)}} s_{\alpha,0} t_{\frac{-(2m+1)\alpha}{(\alpha,\alpha)}} = (t_{\frac{2\alpha}{(\alpha,\alpha)}})^m t_{\frac{\alpha}{(\alpha,\alpha)}} s_{\alpha,0} t_{\frac{-\alpha}{(\alpha,\alpha)}} (t_{\frac{2\alpha}{(\alpha,\alpha)}})^{-m} = s^m s_{\alpha,1} s^{-m}$$

which is also obtained solely using $s_{\alpha,0}$ and $s_{\alpha,1}$, where the last equality comes directly from Corollary 3.9.

We now have that W is generated by $s_{\alpha,0}$ and $s_{\alpha,1}$.

Both generators have order two. However, the element s, being a translation on V, has infinite order, implying that there are no relations besides ${s_{\alpha,0}}^2 = {s_{\alpha,1}}^2 = 1$. This tells us that our group is isomorphic to D_∞, the Dihedral group of infinite order, which has the following presentation: $\langle x, y | x^2 = y^2 = 1\rangle$. One can view this example in Figure 5. In this figure, our V is one-dimensional, and the points represent our hyperplanes.

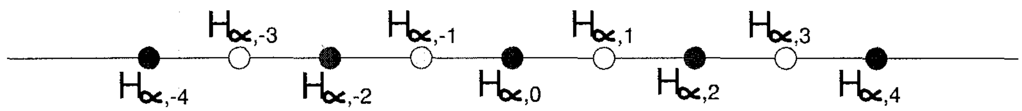

Figure 5: The hyperplanes $H_{\alpha,k}$ of the example from §3.2. See also Example 3.31.

Note that with $S = \{s_{\alpha,0}, s_{\alpha,1}\}$, the pair (W, S) satisfies Definition 2.24 of a Coxeter system.

Lemma 3.15. *The hyperplanes $H_{\alpha,k} \in \mathcal{H}_{even} = \{H_{\alpha,k} \in \mathcal{H} | k \text{ is even}\}$, and the hyperplanes $H_{\alpha,k} \in \mathcal{H}_{odd} = \{H_{\alpha,k} \in \mathcal{H} | k \text{ is odd}\}$, in the above example, form two distinct orbits under action by W.*

Proof. Recall that $s = s_{\alpha,1}s_{\alpha,0}$. Let $\lambda \in H_{\alpha,2m}$ for some $m \in \mathbb{Z}$. Then $s\lambda \in H_{\alpha,2m+2}$ and $s_{\alpha,0}\lambda \in H_{\alpha,-2m}$, by which we can also conclude that $s_{\alpha,1}\lambda$ will be contained in a hyperplane in $\mathcal{H}_{even}$. We thus have that W preserves $\mathcal{H}_{even}$, and we can argue similarly for $\mathcal{H}_{odd}$. □

These orbits are denoted by the black circles (even) and white circles (odd) in Figure 5.

3.3 Algebraic Properties

Let W be an affine reflection group, with associated crystallographic root system Φ and hyperplanes $\mathcal{H}$. From now on we assume that we are in the infinite affine case, that is $\mathcal{H} = \{H_{\alpha,k} | \alpha \in \Phi, k \in \mathbb{Z}\}$. We introduce several more definitions.

Definition 3.16. Let W_0 be the subgroup of W generated by all the reflections $s_{\alpha,0}$ in W.

Since $\mathcal{H}$ is locally finite, there are finitely many linear hyperplanes $H_{\alpha,0}$ in $\mathcal{H}$. Thus W_0 is a finite reflection group.

Definition 3.17. If $\alpha \in \Phi$ is a root, then we will refer to $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ as a *coroot.* We will refer to the set $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ as the *coroot system.* The *root lattice* $L(\Phi)$ is the $\mathbb{Z}$-span of Φ. The *coroot lattice* $L(\Phi^\vee)$ is the $\mathbb{Z}$-span of $\Phi^\vee$.

We are now ready to give several results. The first proposition shows that the elements of $\mathcal{H}$ are permuted by elements of W.

Proposition 3.18. *[H3, Proposition 4.1]*

1. *If $w \in W_0$ then $wH_{\alpha,k} = H_{w\alpha,k}$ and $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$.*
2. *If $\lambda \in V$ satisfies $(\lambda, \alpha) \in \mathbb{Z}$ for all roots $\alpha \in \Phi$ then $t_\lambda H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha)} \in \mathcal{H}$ and $t_\lambda s_{\alpha,k} t_{-\lambda} = s_{\alpha,k+(\lambda,\alpha)}$.*

Proof. Let $w \in W_0$.
First, let $\lambda \in H_{\alpha,k}$. Then

$$(w(\lambda), w(\alpha)) = (\lambda, \alpha) = k$$

which gives us that $w(\lambda) \in H_{w(\alpha),k}$, implying that $wH_{\alpha,k} \subseteq H_{w(\alpha),k}$. By Lemma 3.5 we thus have that $wH_{\alpha,k} = H_{w(\alpha),k}$.

Let $\gamma \in V$. Since $w\alpha$ is a non-zero vector orthogonal to $H_{w\alpha,k}$, we may write γ as a sum $\gamma = \lambda + c(w\alpha)$ where $\lambda \in H_{w\alpha,k}$ and $c \in \mathbb{R}$. Then

$$s_{w\alpha,k}(\lambda + c(w\alpha)) = \lambda - c(w\alpha).$$

On the other hand,

$$ws_{\alpha,k}w^{-1}(\gamma) = ws_{\alpha,k}(w^{-1}\lambda + w^{-1}cw\alpha) = ws_{\alpha,k}(w^{-1}\lambda + c\alpha).$$

By the first part of the proof above, $w^{-1}\lambda \in H_{\alpha,k}$, so we have

$$s_{\alpha,k}(w^{-1}\lambda + c\alpha) = w^{-1}\lambda - c\alpha.$$

Hence

$$ws_{\alpha,k}w^{-1}(\lambda + c(w\alpha)) = w(w^{-1}\lambda - c\alpha) = \lambda - c(w\alpha).$$

Thus

$$s_{w\alpha,k}(\gamma) = ws_{\alpha,k}w^{-1}(\gamma).$$

To prove the second item, let $\lambda \in V$ be as in the hypotheses and let $\mu \in H_{\alpha,k}$. Then

$$\begin{aligned}
(t_\lambda(\mu), \alpha) &= (\mu + \lambda, \alpha) \\
&= (\mu, \alpha) + (\lambda, \alpha) \\
&= k + (\lambda, \alpha),
\end{aligned}$$

which gives us $t_\lambda(H_{\alpha,k}) \subseteq H_{\alpha,k+(\lambda,\alpha)}$. By Lemma 3.5 we have $t_\lambda(H_{\alpha,k}) = H_{\alpha,k+(\lambda,\alpha)}$.

Finally, let $\eta \in V$.

$$\begin{aligned}
t_\lambda s_{\alpha,k} t_{-\lambda}(\eta) &= t_\lambda s_{\alpha,k}(\eta - \lambda) \\
&= t_\lambda(\eta - \lambda - \frac{2((\eta - \lambda, \alpha) - k)}{(\alpha, \alpha)}\alpha) \\
&= \eta - \lambda - \frac{2((\eta - \lambda, \alpha) - k)}{(\alpha, \alpha)}\alpha + \lambda \\
&= \eta - \frac{2((\eta, \alpha) - (\lambda, \alpha) - k)}{(\alpha, \alpha)}\alpha.
\end{aligned}$$

On the other hand

$$\begin{aligned}
s_{\alpha,k+(\lambda,\alpha)}(\eta) &= \eta - \frac{2((\eta, \alpha) - (k + (\lambda, \alpha)))}{(\alpha, \alpha)}\alpha \\
&= \eta - \frac{2((\eta, \alpha) - (\lambda, \alpha) - k)}{(\alpha, \alpha)}\alpha \\
&= t_\lambda s_{\alpha,k} t_{-\lambda}(\eta).
\end{aligned}$$

□

In the next proposition we will describe the structure of W. Corollary 3.21 will show how W acts on the set $\mathcal{H}$, as well as on the reflections associated to its elements.

Proposition 3.19. *[H3, Proposition 4.2] W is the semi-direct product of W_0 and the translation group corresponding to the coroot lattice $L = L(\Phi^\vee)$.*

We will use the following lemma in the proof.

Lemma 3.20. $s_{\alpha,k} = t_{(k\alpha^\vee)}s_\alpha$.

Proof. Let $\lambda \in V$ and consider the element $t_{(k\alpha^\vee)}s_\alpha$,

$$\begin{aligned} t_{(k\alpha^\vee)}s_\alpha(\lambda) &= t_{(k\alpha^\vee)}(\lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\cdot\alpha) \\ &= \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\cdot\alpha + k\alpha^\vee \\ &= \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\cdot\alpha + k\frac{2\alpha}{(\alpha,\alpha)} \\ &= \lambda - \frac{2((\lambda,\alpha)-k)}{(\alpha,\alpha)}\cdot\alpha \\ &= s_{\alpha,k}(\lambda) \end{aligned}$$

by Lemma 3.8, which gives us

$$s_{\alpha,k} = t_{(k\alpha^\vee)}s_\alpha. \tag{8}$$

□

Proof of Proposition 3.19. We will beging by showing that W_0 normalizes the group of translations $\{t_\lambda | \lambda \in L\}$ (which we will identify with L). We know that any element $w \in W_0$ is generated by reflections, and thus is also an element of $GL(V)$, therefore by Equation (5) we know that $wt_\lambda w^{-1} = t_{w\lambda}$. It suffices to show that $t_{w\lambda} \in \{t_\lambda | \lambda \in L\}$, or more to the point, that $w\lambda \in L$. Since w is generated by reflections $s_{\alpha,0}$, $\alpha \in \Phi$, it suffices to show that $s_{\alpha,0}(\lambda) \in L$. Recall that $\lambda \in L$ and therefore $\lambda = \sum_{\beta\in\Phi}\frac{2r_\beta}{(\beta,\beta)}\beta$ for some $r_\beta \in \mathbb{Z}$. Then

$$s_{\alpha,0}(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$$

$$\begin{aligned} &= \lambda - \sum_{\beta\in\Phi} \frac{2(\frac{2r_\beta}{(\beta,\beta)}\beta, \alpha)}{(\alpha,\alpha)}\alpha \\ &= \lambda - \sum_{\beta\in\Phi} \frac{4r_\beta(\beta,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\alpha. \end{aligned}$$

Recall that Φ is crystallographic, which gives us $\frac{2(\beta,\alpha)}{(\beta,\beta)} = m_\beta$ for some $m_\beta \in \mathbb{Z}$. Therefore

$$\begin{aligned} s_{\alpha,0}(\lambda) &= \sum_{\beta\in\Phi} \lambda - \frac{4r_\beta(\beta,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\alpha \\ &= \lambda - \sum_{\beta\in\Phi} m_\beta r_\beta \frac{2}{(\alpha,\alpha)}\alpha \\ &= \lambda - \sum_{\beta\in\Phi} m_\beta r_\beta \alpha^\vee \in L, \end{aligned}$$

which shows us that W_0 normalizes L.

Also, since W_0 consists only of linear transformations, the intersection $W_0 \cap L$ is trivial. Therefore, W_0 and L form a semi-direct product. Denote this product by W'.

Lemma 3.20 implies that all generators of W lie in W'. Thus, $W \subseteq W'$. We already know that $W_0 \subseteq W$, but rearranging Equation (8) gives us

$$t_{k\alpha^\vee} = s_{\alpha,k}s_\alpha \in W$$

which implies that $L \subseteq W$. Therefore $W' \subseteq W$, so

$$W = W' = L \rtimes W_0.$$

□

Corollary 3.21. *[H3, Corollary 4.2] If $w \in W$ and $H_{\alpha,k} \in \mathcal{H}$ then $wH_{\alpha,k} = H_{\beta,l}$ for some $\beta \in \Phi$, $l \in \mathbb{Z}$, and thus $ws_{\alpha,k}w^{-1} = s_{\beta,l}$.*

Proof. By Proposition 3.19 we have that $W = W_0 \ltimes L$, thus we can write $w \in W$ as $w = w_0 t_l$ for some $w_0 \in W_0$ and some $l \in L$. Furthermore, if $l \in L$, then $l = \sum_{\beta\in\Phi} \frac{2r_\beta}{(\beta,\beta)}\beta$ for some $r_\beta \in \mathbb{Z}$, and thus we have that $(l,\alpha) = \sum_{\beta\in\Phi}(\frac{2r_\beta}{(\beta,\beta)}\beta,\alpha) = \sum_{\beta\in\Phi} r_\beta \frac{2(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha \in \Phi$ since Φ is crystallographic.

We may thus apply Proposition 3.18 to deduce

$$wH_{\alpha,k} = w_0 t_l H_{\alpha,k} = w_0 H_{\alpha,k+(l,\alpha)} = H_{w_0(\alpha),k+(l,\alpha)},$$

and

$$ws_{\alpha,k}w^{-1} = w_0 t_l s_{\alpha,k} t_{-l} w_0^{-1} = w_0 s_{\alpha,k+(l,\alpha)} w_0^{-1} = s_{w_0(\alpha),k+(l,\alpha)}.$$

□

3.4 Geometric Properties

Let us explore the geometric properties of affine reflection groups. For each $H_{\alpha,k} \in \mathcal{H}$, let $f_\alpha : V \to \mathbb{R}$ be a non-zero linear function such that

$$H_{\alpha,k} = \{x \in V | f_\alpha(x) = k\}.$$

Let us use the following shorthand notation:

$$\begin{aligned} f_\alpha^k &= H_{\alpha,k} \\ f_\alpha^{k+} &= \{x \in V | f_\alpha(x) > k\} \\ f_\alpha^{k-} &= \{x \in V | f_\alpha(x) < k\}. \end{aligned}$$

This is analogous to the f_i's we had in §2.4. In this case f_α^{k-} and f_α^{k+} are the two open half spaces of $V \setminus H_{\alpha,k}$. We now have a partition of V into cells just as we did in the finite reflection group case.

Definition 3.22. For each $\alpha \in \Phi$ choose a sign $\sigma_\alpha \in \{k, k+, k- | k \in \mathbb{Z}\}$. A *cell* in V, with respect to $\mathcal{H}$, is a non-empty set $A \subseteq V$ defined by

$$A = \bigcap_{\alpha \in \Phi} f_\alpha^{\sigma_\alpha}.$$

Remark. Of course, just as with the finite reflection groups, many of the possible choices leave us with the empty set.

Definition 3.23. Let A and B be cells. Suppose A is defined by the sign choices σ_α and B is defined by the sign choices δ_α, as α runs over Φ. Then B is a *face* of A if for all $\alpha \in \Phi$:

$$\begin{aligned} \sigma_\alpha \in \{k+, k-\} &\Rightarrow \delta_\alpha \in \{\sigma_\alpha, k\} \\ \sigma_\alpha = k &\Rightarrow \delta_\alpha = k. \end{aligned}$$

Definition 3.24. The non-empty cells defined by sign choices σ_α such that $\sigma_\alpha \in \{k+, k-\}$ for all $\alpha \in \Phi$ are called *chambers*.

As in Chapter 2, we deduce that the chambers are the non-empty open convex connected components of $V \setminus \bigcup_{H \in \mathcal{H}} H$.

Definition 3.25. Let C be a chamber. A *wall* of C is a hyperplane $H \in \mathcal{H}$ such that there exists a face A of C such that H is the minimal affine subspace of V containing A.

Choose a chamber C, and call it the *fundamental chamber*. Let $S = \{s_H | H \text{ is a wall of } C\}$. It is a useful fact that for any proper subset S' of S, the normal vectors of the corresponding hyperplanes H are linearly independent [H3, §4.3]. It follows that any face of C is determined as an intersection of co-dimension one faces in exactly one way.

Definition 3.26. Let G be a group acting on a set A. The action of G on A is *transitive* if for any a, $b \in A$ there exists a $g \in G$ such that $ga = b$. The action is *simply-transitive* if there is a unique such g for all $a, b \in A$.

Proposition 3.27. *[H3, Proposition 4.3]*

1. *W is generated by S.*

2. *W acts transitively on the set $\mathcal{C}$ of chambers.*

3. *C has only finitely many walls, hence S is finite.*

Proof. Let W' be the subgroup of W generated by S. We will begin by showing that W' acts transitively on $\mathcal{C}$. It is enough to show that for any $A \in \mathcal{C}$ there exists a $w \in W'$ such that $wA = C$. Let $\gamma \in C$ and $\delta \in A$. The orbit of δ under the action of the translation group L is a discrete subset of V, and since W is an extension of this lattice group by a finite group (see Proposition 3.19) we have that the orbit under the action of W (and thus also by W') is also a discrete subset of V. Choose an element $\tau = w\delta$ from the orbit of δ of smallest possible distance from γ with $w \in W'$. At this point, if we can show that $\tau \in C$, we will have that $wA \cap C \neq \emptyset$, which implies that $wA = C$ since both are chambers.

Let us proceed by proof by contradiction. Suppose $\tau \notin C$. We must then have that γ and δ must lie in different half-spaces relative to some wall H of C. Let s_H be the corresponding reflection. Note that $s_H \in W'$. Consider the trapezoid having vertices $\tau, s_H\tau, s_H\gamma, \gamma$ which is bisected by H. The lines segments $s_H\tau$ to τ and $s_H\gamma$ to γ are parallel and $\|s_H\tau - \gamma\| = \|\tau - s_H\gamma\|$ since s_H is an affine map and $s_H{}^2 = 1$. It can now be shown, using the law of cosines, that

$$\|s_H\tau - \gamma\| < \|\tau - \gamma\| .$$

However, we have that $s_H\tau$ is in the W' orbit of δ which contracts our choice of τ.

This gives us that W' permutes $\mathcal{C}$ transitively. This implies furthermore that the walls of each chamber will be the images of the walls of C. We would like to show next that $W = W'$, and to do this, it suffices to show that all the $s_{\alpha,k}$ lie in W'. Let A be any chamber with $H_{\alpha,k}$ as a wall, and let $w \in W'$ such that $wA = C$. We then have that $wH_{\alpha,k}$ coincides with one of the walls of H of C, whose corresponding reflection s_H lies in W'. By Corollary 3.21 we get $ws_{\alpha,k}w^{-1} = s_H$, which implies

$$s_{\alpha,k} = w^{-1}s_H w \in W'.$$

This completes the proof of parts (1) and (2) of the proposition. Since we assume that every hyperplane in H is parallel to a linear hyperplane in H, the local finiteness of H implies that there are at most finitely many hyperplanes bounding H, that is, finitely many walls. This completes the proof.

□

We also have that this Proposition holds for finite reflection groups. The arguments follow almost exactly as they do in this proof. See [H3, §1.8] and [H3, §1.9]. Property (3) will of course follow naturally since W itself is finite in this case.

Furthermore, we also have that the action of W on C is simply-transitive on the chambers. We will need a few more results to show this. Since W is generated by S, each element of W can be written as a product of elements s_i. Call an expression $w = s_1 \cdots s_r$ *reduced* if r is minimal among all such expressions.

Lemma 3.28. *[H3, Lemma 4.5] Suppose $w \in W$ is not the identity, and let $w = s_1 \cdots s_r$ be a reduced expression. Then the hyperplanes in the set*

$$L(w) = \{H_{s_1}, s_1 H_{s_2}, s_1 s_2 H_{s_3}, \ldots, s_1 \cdots s_{r-1} H_{s_r}\}$$

are all distinct.

Proof. If not, then there is some $1 \leq m < n \leq r$ such that

$$s_1 \cdots s_{m-1} H_{s_m} = s_1 \cdots s_{n-1} H_{s_n},$$

or, after multiplying both sides on the left by $s_{m-1} \cdots s_1$,

$$H_{s_m} = s_m \cdots s_{n-1} H_{s_n}.$$

Applying Corollary 3.21, we deduce $s_m = (s_m \cdots s_{n-1}) s_n (s_{n-1} \cdots s_m)$. Rearranging terms, we deduce $s_{m+1} \cdots s_{n-1} = s_m \cdots s_n$. But this would imply that

$$w = s_1 \cdots (s_m \cdots s_n) \cdots s_r = s_1 \cdots (s_{m+1} \cdots s_{n-1}) \cdots s_r$$

which is a shorter expression, contradicting that $s_1 \cdots s_r$ was a reduced expression for w. □

Theorem 3.29. *[H3, Theorem 4.5] Let C be the fundamental chamber. The hyperplanes separating C from wC are exactly those in $L(w)$.*

Proof. We proceed by induction on the length of a reduced expression for w. If $w = s$, then $L(w) = \{H_s\}$ and this is the unique hyperplane separating C from sC by construction.

Suppose that the result is true for any w' with a reduced expression of length $r-1$, and suppose $w = s_1 \cdots s_r$ is a reduced expresssion for w. Set $w' = s_2 \cdots s_r$. By induction, the hyperplanes separating C and $w'C$ are exactly those in $L(w')$. Note that from the definition of $L(w')$ we have that $s_1 L(w') \subset L(w) \setminus \{H_{s_1}\}$. So if $H_{s_1} \in L(w')$, we would have $s_1 H_{s_1} \in s_1 L(w')$, but $s_1 H_{s_1} = H_{s_1}$, so this is impossible. Thus C and $w'C$ lie on the same side of H_{s_1}. Now $s_1 w'C = wC$ must lie on the opposite side of H_{s_1}, so H_{s_1} separates C and wC. On the other hand, given any $H \in L(w)$, H separates C from $w'C$ so $s_1 H$ separates $s_1 C$ from $s_1 w'C = wC$. Since C and $s_1 C$ lie on the same side of every hyperplane except H_{s_1}, it follows that all the hyperplanes in $s_1 L(w')$ also separate C from wC, as required. □

Corollary 3.30. *W acts simply transitively on the set $\mathcal{C}$ of chambers.*

Proof. If $wC = C$ for some w, then since by the theorem the number of hyperplanes separating C and wC is equal to the length of a reduced expression for w, we deduce w cannot be nontrivial. □

Example 3.31. Recall the example from §3.2 and Figure 5. Choose the chamber whose faces are contained in $H_{\alpha,0}$ and $H_{\alpha,1}$ as the fundamental chamber C. The reflections in these two hyperplanes are $s_{\alpha,0}$ and $s_{\alpha,1}$ respectively. In §3.2 we proved that $s = s_{\alpha,1} s_{\alpha,0} = t_{\frac{2\alpha}{(\alpha,\alpha)}}$ and that our affine reflection group W is generated by $s_{\alpha,1}$ and $s_{\alpha,0}$.

In Figure 6 we denote the fundamental chamber by C. By Corollary 3.30, W acts simply transitively on the set of all chambers and so is in bijection with the set of all chambers by the map $w \mapsto wC$. To illustrate this, we identify each chamber wC with the unique element $w \in W$ (above the line).

We also identify the hyperplanes (below the line) as per §3.2, that is, by writing down the corresponding reflection in terms of the generators. By the proof of Proposition 3.27, each hyperplane maps, by some element of W, to a face of the fundamental chamber C (which is one of either $H_{\alpha,0}$ or $H_{\alpha,1}$). By our discussion of orbits in §3.2, this face is unique. The element of W is not unique because each hyperplane is the face of two chambers. Using Corollary 3.21 to identify the reflection in wH with the reflection $ws_H w^{-1} \in W$, we see that the different choices correspond to different ways of writing the same element in W. Thus the labels below the diagram in Figure 6 represent one way of writing the reflections in the associated hyperplanes.

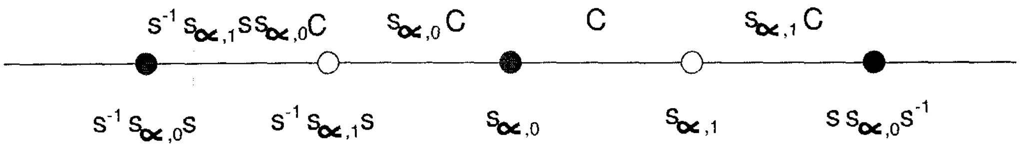

Figure 6: Geometric representation of an affine reflection group derived from a finite reflection group

Remark. As in the finite group case, the pair (W, S) from this chapter is another example of a Coxeter system, and the poset Σ of cells, ordered by the face relation, is again a geometric realization of its associated Coxeter complex (which we will see more of in Chapter 5).

Chapter 4

p-adic Numbers and Algebraic Groups

This chapter is aimed mainly at giving a review of p-adic numbers in §4.1 and some examples of algebraic groups in §4.2 that we will use later. The section on p-adic numbers will construct the field of p-adic numbers $\mathbb{Q}_p$ as well as the ring of p-adic integers $\mathbb{Z}_p$ following [K, §1 pages 1-20, §3 pages 52-65]. The section on algebraic groups will give several classical examples paying special care to define them in a field-independent way. See [Bo, §1 pages 46, §23 pages 253-264] and [H2, §2.7 pages 51-53] for the reference material used.

4.1 p-adic Numbers

We begin this section with a series of definitions.

Definition 4.1. A *metric* on a set X is a function d from pairs of elements (x, y) of X to the non-negative real numbers satisfying

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y) \;\; \forall\, z \in X$.

Definition 4.2. A *norm* on a field F, denoted by $\| \; \|$, is a map from F to the non-negative real numbers satisfying

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|x \cdot y\| = \|x\| \cdot \|y\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4.3. A metric d is *induced* by a norm $\| \; \|$ if it is defined by $d(x, y) = \|x - y\|$.

We of course know many metrics and norms already. For instance, the regular distance function on the real numbers is a metric, and the absolute value function is a norm. Let us take this time to explore another norm, this one defined on the rational numbers.

Let p be any prime number and let a be any non-zero integer. Define $ord_p a$ to be the highest power of p which divides a. That is to say, $ord_p a$ is the greatest integer m such that $a \equiv 0 \pmod{p^m}$. We will define $ord_p 0 = \infty$.

Remark. One should note that $ord_p(ab) = ord_p a + ord_p b$.

We would now like to extend our map to allow it to work on rational numbers as well as integers. Let $x = \frac{a}{b}$ (with $a, b \in \mathbb{Z},\ b \neq 0$) be a rational number. Define $ord_p x = ord_p a - ord_p b$. This is well defined by the remark above. Now, use this to further define a map $| \; |_p$ on $\mathbb{Q}$:

$$|x|_p = \begin{cases} p^{-ord_p x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Proposition 4.4. *$| \; |_p$ is a norm on $\mathbb{Q}$*

Proof. Our proof of part (3) follows that in [K, I §2]. It suffices to verify that $| \; |_p$ satisfies the properties given in Definition 4.2.

1. This first property is obtained by definition of $| \; |_p$.

2. Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. If x or y is 0, the second property is obvious, so assume both a and c are non-zero. Let us begin by exploring some properties of ord_p.

$$\begin{aligned}
-ord_p\left(\frac{ac}{bd}\right) &= -ord_p(ac) + ord_p(bd) \\
&= -ord_p(a) - ord_p(c) + ord_p(b) + ord_p(d) \\
&= -ord_p(a) + ord_p(b) - ord_p(c) + ord_p(d) \\
&= -ord_p\left(\frac{a}{b}\right) - ord_p\left(\frac{c}{d}\right)
\end{aligned}$$

This gives us

$$\begin{aligned}
|x \cdot y|_p &= \left|\frac{ac}{bd}\right|_p \\
&= p^{-ord_p(\frac{ac}{bd})} \\
&= p^{-ord_p(\frac{a}{b}) - ord_p(\frac{c}{d})} \\
&= p^{-ord_p(\frac{a}{b})} \cdot p^{-ord_p(\frac{c}{d})} \\
&= p^{-ord_p x} \cdot p^{-ord_p y} \\
&= |x|_p \cdot |y|_p.
\end{aligned}$$

3. If x, y or $x+y$ is 0 the third property is obvious, so assume they are all non-zero. Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. We now have

$$x + y = \frac{ad + bc}{bd}$$

and

$$ord_p(x + y) = ord_p(ad + bc) - ord_p b - ord_p d.$$

The highest power of p dividing the sum of two numbers is at least the minimum of the highest power dividing either of the two numbers. Therefore

$$\begin{aligned}
ord_p(x + y) &\geq \min\{ord_p(ad), ord_p(bc)\} - ord_p b - ord_p d \\
&= \min\{ord_p a + ord_p d, ord_p b + ord_p c\} - ord_p b - ord_p d \\
&= \min\{ord_p a - ord_p b, ord_p c - ord_p d\} \\
&= \min\{ord_p x, ord_p y\}. \qquad (9)
\end{aligned}$$

Finally, since p^{-t} is a decreasing function of t, we have

$$\begin{aligned} |x+y|_p = p^{-ord_p(x+y)} &\leq \max\{p^{-ord_p x}, p^{-ord_p y}\} \\ &= \max\{|x|_p, |y|_p\} \\ &\leq |x|_p + |y|_p. \end{aligned}$$

□

Definition 4.5. The norm $|\ |_p$ is the *p-adic norm.*

Remark. By combining Definition 4.3 and Proposition 4.4 we can define the *p-adic metric* on $\mathbb{Q}$.

Definition 4.6. A norm on F is called *non-Archimedean* if $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ holds for all $x, y \in F$.

We have not only proven that $|\ |_p$ is a norm, but also that it is a non-Archimedean norm.

Definition 4.7. A *discrete valuation* on a field F is a surjective homomorphism $v : F^\times \to \mathbb{Z}$ such that

$$v(x+y) \geq \min\{v(x), v(y)\}$$

for all $x, y \in F^\times$ with $x + y \neq 0$.

We have from Equation (9) that ord_p is a discrete valuation.

We are now ready to give a definition of p-adic numbers. One may define the field $\mathbb{Q}_p$ of *p-adic numbers* as the completion of $\mathbb{Q}$ with respect to the p-adic norm. That is, each element of $\mathbb{Q}_p$ can be viewed as an equivalence class of sequences of rational numbers which are Cauchy with respect to $|\ |_p$, where two sequences are equivalent if their difference converges to 0. The sum and product of such sequences are also Cauchy sequences, and hence define elements of $\mathbb{Q}_p$. In fact $\mathbb{Q}_p$ is a field [K, I §4].

It can be shown that each equivalence class of sequences contains a unique representative of a particularly convenient form, as follows [K, I §4].

The non-zero elements of $\mathbb{Q}_p$ can be realized as formal sums of the form

$$a = \frac{b_{-m}}{p^m} + \frac{b_{-m+1}}{p^{m-1}} + \cdots + \frac{b_{-1}}{p} + b_0 + b_1 p + b_2 p^2 + \cdots$$

where $b_i \in \{0, 1, \ldots, p-1\}$ and $m \in \mathbb{N}$. That is, each element of $\mathbb{Q}_p$ is expressible as a Laurent series in p, with coefficients in $\{0, 1, \ldots, p-1\}$. Such sums converge in the p-adic norm. Addition and multiplication are computed by carrying higher powers of p to the right, and taking the limit.

This realization is particularly convenient to use, because it mimics the real numbers. That is to say, each p-adic number can be thought of as a base p "decimal" number, with the b_i's as the "digits", and only finitely many non-zero digits (if any) "to the right of the decimal point", which are realized as the digits with negative powers of p and written starting from the left.

Example 4.8. One could write the rational number

$$13.56 \in \mathbb{Q} \subseteq \mathbb{Q}_5$$

as

$$\frac{4}{5^2} + \frac{2}{5^1} + 3 + 2 \cdot 5^1.$$

This is its 5-adic expansion. A more non-trivial example would be

$$-1 = 4 + 4 \cdot 5^1 + 4 \cdot 5^2 + 4 \cdot 5^3 \cdots.$$

Definition 4.9. The *p-adic integer ring* is $\mathbb{Z}_p = \{a \in \mathbb{Q}_p | \, |a|_p \leq 1\}$.

Note that 0 and 1 are in $\mathbb{Z}_p$, and by the proof of Proposition 4.4 we have that $|xy|_p = |x|_p|y|_p$ and $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. Thus $\mathbb{Z}_p$ is indeed a subring of $\mathbb{Q}_p$ comprised of all numbers of $\mathbb{Q}_p$ with no negative powers of p in their expansion.

Definition 4.10. A *uniformizing element* is any $\pi \in \mathbb{Z}_p$ such that $ord_p(\pi) = 1$.

We will very often choose $\pi = p$.

4.2 Algebraic Groups

The theory of algebraic groups is quite complex, and we will not present a general definition of an algebraic group here. The algebraic groups of interest to us here are principally some of the simple split classical groups over the real and p-adic numbers. The interested reader may note that over the real numbers, our groups are Lie groups, endowed with a smooth manifold structure, but we will be interested in them only from an algebraic perspective. Our goal is to present several examples of split classical groups and some particular important subgroups.

We start with a few definitions. Let K be a field.

Definition 4.11. The *diagonal group* $D_n(K)$ is the group of $n \times n$ invertible diagonal matrices.

Definition 4.12. For any n, the *monomial group* is the group of $n \times n$ invertible matrices with exactly one non-zero entry in each row and column.

Definition 4.13. A group G is *solvable* if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for $i = 0, 1, \ldots, s-1$.

Let us now present the classical examples, all of which can be found at [Bo, §I.1.6]. In the following, if $v_1, \cdots, v_k \in K^n$, we write $[v_1, \cdots, v_k]$ for the span over K of $\{v_1, \cdots, v_k\}$. In particular, if $v \neq 0$ then $[v]$ denotes a line in K^n.

Example 4.14. Define $G = GL_n(K)$ to be the set of all $n \times n$ invertible matrices with entries in K. This is the multiplicative group known as the *general linear group*.

The upper triangular group $B \subset GL_n(K)$ is a *Borel subgroup*, that is, B is a maximal group among the solvable subgroups of G. Let $N \subset GL_n(K)$ be the monomial group. The diagonal group $T = D_n(K) \subseteq G$ is a maximal *torus*, that is, a maximal diagonalizable subgroup of G.

The group N permutes the set of lines $\{[e_1], \ldots, [e_n]\}$ where the e_i's form the standard basis of K^n. We obtain a surjection of N onto the symmetric group on n

letters $\Omega : N \to S_n$ by mapping $g \in N$ to the element of S_n that permutes the labels of the lines $\{[e_1], \ldots, [e_n]\}$ in the same way that g permutes the lines. The preimage of the identity in S_n is exactly T. Thus we have that T is the kernel of Ω and that N normalizes T. In fact, we have that $N = N_G(T)$ is the normalizer of T in G and that $T = C_G(T)$ is self-centralizing in G. The quotient $W = N_G(T)/C_G(T) = N/T$, called the *Weyl group*, is thus isomorphic to S_n via our map Ω above.

Example 4.15. Define $SL_n(K)$ to be the set of all matrices of determinant 1 in $GL_n(K)$. This is the group known as the *special linear group*.

The subgroups B, T, $N \subset GL_n(K)$ intersected with $SL_n(K)$ give a Borel, torus, and normalizer of the torus in $SL_n(K)$ respectively. We also have that $W = (N \cap SL_n(K))/(T \cap SL_n(K)) \cong S_n$ the symmetric group on n letters as we did in the case of $GL_n(K)$.

Example 4.16. [Br, §v.6] Set

$$J = \begin{bmatrix} 0 & \cdots & & 1 \\ & & \cdot & \\ \vdots & \cdot & & \vdots \\ & \cdot & & \\ 1 & \cdots & & 0 \end{bmatrix}.$$

Let us equip the space K^{2n} with a symplectic (skew-symmetric, non-degenerate, bilinear) form $B(\ ,\ )$, which we may represent by the matrix $\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$. It is proven in [A, §III.5] that any symplectic vector space is the orthogonal direct sum of hyperbolic planes, that is, 2-dimensional subspaces in which there exists a basis $\{e, f\}$ such that $B(e,e) = B(f,f) = 0$ and $B(e,f) = -B(f,e) = 1$. Thus there exists a basis $\{e_1, \ldots, e_n, f_n, \ldots, f_1\}$ such that each $\{e_i, f_i\}$ is a hyperbolic pair and, for $i \neq j$, span$\{e_i, f_i\}$ is orthogonal to span$\{e_j, f_j\}$. Given any linear transformation $x \in GL_{2n}(K)$ which preserves B, that is, for which $B(xu, xv) = B(u, v)$ for all $u, v \in K^{2n}$, its matrix must satisfy

$$x^t \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix} x = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}.$$

The collection of all such x is denoted by $Sp_{2n}(K)$ and is known as the *symplectic group*.

The subgroup $B \subset Sp_{2n}(K)$ of upper triangular symplectic matrices is a Borel subgroup. Let $N \subset Sp_{2n}(K)$ be the group of monomial matrices which lie in $Sp_{2n}(K)$. A maximal torus in $Sp_{2n}(K)$ is $T = B \cap N$, the group of diagonal symplectic matrices. A quick calculation shows that

$$T = \left\{ \begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & a_n^{-1} & & \\ & & & & \ddots & \\ & & & & & a_1^{-1} \end{bmatrix} : a_1, \ldots, a_n \in K^* \right\}.$$

Again, N normalizes T, and we get a Weyl group $W = N/T$. However, it is slightly more complicated than our two previous examples. In this case, we get that W is the group of permutations of the basis vectors $\{e_1, \ldots, e_n, f_n, \ldots, f_1\}$ with the proviso that every pair $\{e_i, f_i\}$ is mapped to another such pair. In fact, W is generated by the set $S = \{s_1, \ldots, s_n\}$ of permutations where s_i for $i < n$ is the product of the two transpositions $e_i \leftrightarrow e_{i+1}$ and $f_i \leftrightarrow f_{i+1}$, and s_n is the transposition $e_n \leftrightarrow f_n$.

Chapter 5

Buildings

The chapter begins by giving the details of simplicial complexes that we will need. We begin §5.2 with the definition of a building, and follow it by giving the definition of a labelling and then proving that a building (and each apartment) admits a labelling. We define a group G of certain automorphisms of the building. We continue in sections §5.3 and §5.4 by defining a BN-pair and showing that every building admits a BN-pair, and then showing how to construct a building from a BN-pair via the idea of parabolic subgroups. Please see [Br, §IV pages 76-78, §V pages 99-114] and [G, §4 pages 51-53, §5 pages 63-84] for the reference material used in these sections. Although most proofs given in this section are adapted from one or more of these sources, the following contributions are either original or expanded upon from the literature: Lemma 5.28, Lemma 5.29, Lemma 5.30, Lemma 5.38, and Lemma 5.39.

5.1 Simplicial Complexes

Let us begin with some definitions.

Definition 5.1. A *simplicial complex* Δ, with associated *vertex set* $\mathcal{V}$, is a collection of finite subsets of $\mathcal{V}$, *simplices*, such that every singleton $\{v\} \subset \mathcal{V}$ is a simplex, and every subset of a simplex is a simplex.

Note that the empty set is also a simplex. If A and B are simplices such that

$A \subseteq B$, then A is a *face* of B. We now have two definitions of what a face is. We shall see shortly that both definitions will coincide under a suitable identification.

Definition 5.2. A *simplicial homomorphism* is a map $\gamma : \mathcal{V} \to \mathcal{V}'$ from a vertex set $\mathcal{V}$, associated with a simplicial complex Δ, to a vertex set $\mathcal{V}'$, associated to a simplicial complex Δ', such that if A is a simplex in Δ then $\gamma(A)$ is a simplex in $\mathcal{V}'$.

Definition 5.3. A *subcomplex* of Δ is a subset $\Delta' \subset \Delta$ such that if $A \in \Delta'$ then Δ' also contains all the faces of A.

We thus have that a subcomplex $\Delta' \subset \Delta$ is itself a simplicial complex with its own associated vertex set $\mathcal{V}' \subset \mathcal{V}$.

Posets

A simplicial complex Δ is a poset, ordered by the face relation.

Lemma 5.4. *Any poset Δ which satisfies the following two properties can be identified with the poset of simplices of a simplicial complex.*

1. *If $A, B \in \Delta$ then there exists a greatest lower bound in Δ.*

2. *If $A \in \Delta$ then $\Delta_{\leq A} = \{B \in \Delta | B \leq A\}$ is isomorphic to the poset of subsets of $\{1, \ldots, k\}$ for some $k \geq 0$.*

Sketch of proof. First note that the poset of a simplicial complex satisfies (1) and (2), where we take the greatest lower bound as $A \cap B$. Conversely, suppose Δ satisfies (1) and (2), then take the vertex set $\mathcal{V}$ to be the set of rank 1 elements of Δ where the rank of $A \in \Delta$ is the unique integer k such that property (2) above holds. In this instance we can identify each $A \in \Delta$ with $A' = \{v \in \mathcal{V} | \{v\} \leq A\}$ where $\leq$ denotes the face relation. Thus, the map $A \mapsto A'$ defines a poset isomorphism from Δ onto a simplicial complex with vertex set $\mathcal{V}$. $\square$

Coxeter Systems

Let (W, S) be a Coxeter system, as in Definition 2.24.

Definition 5.5. A *special subgroup* $\langle S' \rangle$ of W is a subgroup generated by a subset S' of S. A *special coset* is a coset $w \langle S' \rangle$ with $w \in W$ and $S' \subseteq S$.

Lemma 5.6. *[Br, Lemma §III.1] The function $S' \mapsto \langle S' \rangle$ is a poset isomorphism from the set of subsets of S to the set of special subgroups of W.*

The proof of Lemma 5.6 relies heavily on the deletion condition, which we have not covered here, but can found at [Br, Condition (D) §II.1].

Let $\Sigma = \Sigma(W, S)$ be the poset of special cosets ordered by inverse inclusion. This means that if A, $B \in \Sigma$ then $B \leq A$ (B is a face of A) if and only if $A \subseteq B$ as subsets of W.

Theorem 5.7. *[Br, Theorem 1 §III page 58] The poset $\Sigma = \Sigma(W, S)$ is a simplicial complex.*

Sketch of proof. The vertices of Σ are special cosets of the form $w\langle S' \rangle$ where $w \in W$ and S' is a maximal proper subset of S.

By Lemma 5.4 we have two things to verify.

1. Any two elements of Σ have a greatest lower bound.

   Suppose that $w \langle S' \rangle$ and $w' \langle S'' \rangle$ are two elements of Σ. Since $\tilde{w} \left\langle \tilde{S} \right\rangle$ is a greatest lower bound of $w \langle S' \rangle$ and $w' \langle S'' \rangle$ if and only if $w^{-1}\tilde{w} \left\langle \tilde{S} \right\rangle$ is a greatest lower bound of 1 and $w^{-1}w'$, we may without loss of generality assume that $w = 1$. We must then prove that a special subgroup $\langle S' \rangle$ and a special coset $w' \langle S'' \rangle$ have a least upper bound (since we ordered Σ by inverse inclusion). Any special coset containing the two above contains the identity and is thus a special subgroup. It also contains w' and thus also contains $\langle S'' \rangle = w'^{-1}w' \langle S'' \rangle$. So any upper bound of the two given special cosets is a special subgroup containing S', S'', and w'. The least upper bound is the intersection of all the special subgroups containing S', S'', and w. Specifically, there exists a smallest $T \subseteq S$ such that $w \in \langle T \rangle$ and $S', S'' \subseteq T$.

2. For any $A \in \Sigma$ the poset $\Sigma_{\leq A}$ of all faces of A is isomorphic to the set of subsets of some finite set.

   It suffices to prove this for A maximal with respect to the ordering on the poset Σ, which means a minimal special coset. These are exactly the elements of W, and it suffices to prove the result for $A = \{e\}$, the identity element. Then the special subgroups containing A are exactly all special subgroups, so we have $\Sigma_{\leq A} = \{B \in \Sigma | B \leq A\} = \{B \text{ special coset} | B \supseteq A\} = \{\text{all special subgroups}\}$. The set of all special subgroups, ordered by inverse inclusion, is isomorphic to the poset of all subsets of $\{1, 2, \ldots, |S|\}$ by Lemma 5.6 where $\{1, 2, \ldots, |S|\} \cong \{s_1, s_2, \ldots, s_{|S|}\}$, so the result follows.

$\square$

Definition 5.8. Let (W, S) be a Coxeter system. The poset $\Sigma = \Sigma(W, S)$ of special cosets ordered by inverse inclusion is called a *Coxeter complex.*

In Chapters 2 and 3, we described certain Coxeter systems (W, S) via their geometric actions on vector spaces V, and consequently produced a poset, call it Σ', associated to (W, S) where the objects were cells in V, ordered by the face relation. From this point on we will only consider Coxeter systems that arise from either a finite or affine reflection group.

Lemma 5.9. *The posets Σ' and $\Sigma(W, S)$ are isomorphic, and hence Σ' is a Coxeter complex.*

Proof. We define a map $\phi : \Sigma' \to \Sigma(W, S)$ as follows. Choose a fundamental chamber $C \in \Sigma'$. The walls of C are in one-to-one correspondence with the elements of S, and each vertex v of C is contained in all but one wall, say the one corresponding to s. So we define $\phi(v) = \langle S \setminus \{s\} \rangle$, which is a vertex of Σ'. More generally, any face A of C may be uniquely identified by a subset $S' \subset S$ of reflections stabilizing it, and so we may define $\phi(A) = \langle S' \rangle$.

Now if C' is any other chamber in Σ', then the simple transitivity of the action of W on the set of chambers in Σ' from Corollary 3.30 implies there exists a unique $w \in W$ so that $wC = C'$. Given a face A of C', set $\phi(A) = w(\phi(w^{-1}A))$, where $w^{-1}A$

is a face of C and so $\phi(w^{-1}A)$ is a special subgroup, and thus $\phi(A)$ is a special coset. In particular, we see that ϕ maps vertices to vertices and simplices to simplices, and so is a simplicial homomorphism.

Note that ϕ is bijective by the simple transitivity of the action of W on Σ'. Namely, for any $w\langle S'\rangle$, let A be the face of C contained in the intersection of the hyperplanes corresponding to S'; then $\phi(wA) = w\langle S'\rangle$. If $\phi(wA) = \phi(w'B)$ for $w.w' \in W$ and A, B faces of C, then $w\langle S'\rangle = w'\langle S''\rangle$ for subsets S' and S'' of S corresponding to A and B. Hence $w'^{-1}w\langle S'\rangle = \langle S''\rangle$ is a subgroup, which implies that $w'^{-1}w \in \langle S'\rangle$ and $\langle S'\rangle = \langle S''\rangle$. Thus the stabilizers of A and B are equal, implying $A = B$ since both are faces of C. Finally, note that this implies $wA = w\langle S'\rangle A = w'\langle S''\rangle B = w'B$ so ϕ is injective.

□

The advantage to the poset Σ' is that it encodes geometric information about the action of W, which lets us picture the Coxeter complex in a very nice way.

Definition 5.10. Let Λ, Γ be simplicial complexes and let A be a simplex such that $A \in \Lambda$ and $A \in \Gamma$. Let $\mathcal{V}_\Lambda$ and $\mathcal{V}_\Gamma$ be the vertex sets of Λ and Γ respectively. Set $\mathcal{V} = \mathcal{V}_\Lambda \cap \mathcal{V}_\Gamma$. A simplicial homomorphism $f : \Lambda \to \Gamma$ fixes A *pointwise* if $f(v) = v$ for all $v \in \mathcal{V}$ such that $\{v\} \leq A$.

5.2 Buildings from Coxeter Complexes

Definition 5.11. [Br, §IV.1] A *building*, related to a Coxeter system (W, S), is a simplicial complex Δ which can be expressed as the union of subcomplexes Σ, henceforth called *apartments*, satisfying:

1. Each apartment Σ is isomorphic to the Coxeter complex of (W, S).

2. For any two simplices $A, B \in \Delta$ there is an apartment Σ containing both of them.

3. If Σ and Σ' are two apartments containing A and B then there is a simplicial isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Definition 5.12. If Σ is an apartment of a building Δ, then a *chamber* $C \in \Sigma$ is a maximal element of the poset.

Labellings

Definition 5.13. Let Δ be a simplicial complex with associated vertex set $\mathcal{V}$. Let $A = \{v_1, \ldots, v_n\}$ be a simplex. The *rank* of A is equal to n, the number of vertices used to define A.

Definition 5.14. A simplicial complex Δ with all simplices having finite rank is a *chamber complex* if all maximal simplices have the same rank and any two can be connected by a sequence of maximal simplices in which any two consecutive ones are *adjacent* (have a common co-dimension one face). In this case, the *rank* of Δ is equal to the rank of one of its maximal simplices.

Suppose now that Δ is a building. Since every chamber C of Δ lies in an apartment isomorphic to the Coxeter complex of (W, S), it follows that the rank of C is equal to $|S|$. Hence Δ satisfies the first condition for being a chamber complex. The chambers of a building Δ all have the same rank (namely $|S|$). See Theorem 5.7 combined with [Br, Theorem §III.1] for a proof that the second condition is also fulfilled so that a Coxeter complex is a chamber complex.

Proposition 5.15. *[Br, Remark §III.4A] If ϕ and ψ are two simplicial isomorphisms from a chamber complex Σ to a chamber complex Σ' such that $\phi(v) = \psi(v)$ for all vertices $v \in C$ where C is a chamber in Σ, then $\phi = \psi$.*

Proof. Let ϕ and ψ be two isomorphisms $\Sigma \to \Sigma'$ such that they both agree on C pointwise. Let D be a chamber in Σ adjacent to, and distinct from, C. That is, C and D share a common codimension-1 face A.

We have that $C_1 = \phi(C) = \psi(C)$. Since ϕ and ψ are simplicial homomorphisms which agree on C pointwise they must necessarily agree on all vertices of C pointwise, and hence all faces of C as well. Let $A_1 = \phi(A) = \psi(A)$, and let D_1 be the unique chamber in Σ' distinct from C_1 with A_1 as a face. We then have necessarily that $\phi(D) = D_1 = \psi(D)$ since the only other possibility is that either ϕ or ψ maps D to

C_1 contradicting the injectivity of ϕ and ψ. We also have that ϕ and ψ must agree pointwise on D since they already agree on all but one vertex of D. That is to say, both ϕ and ψ need to send the only vertex of D not in A to the only vertex of D_1 not in A_1.

The above gives us that if two isomorphisms agree pointwise on a chamber C then they agree pointwise on any chamber D adjacent to C. Since every two chambers are connected by a sequence of adjacent chambers, we deduce by iterating the above argument that these two isomorphisms agree pointwise on all chambers of Σ □

Definition 5.16. A simplicial homomorphism γ is *non-degenerate* if $\gamma(A)$ has the same rank as A for all simplices A.

A non-degenerate simplicial homomorphism is the same as a poset map $\gamma : \Delta \to \Delta'$ such that γ will map $\Delta_{\leq A}$ to $\Delta'_{\leq \gamma(A)}$ isomorphically for every $A \in \Delta$. Also, if Δ and Δ' have the same rank then a simplicial homomorphism γ being non-degenerate is equivalent to it taking chambers to chambers. This is true since if $C \in \Delta$ is a chamber, then $\gamma(C) \in \Delta'$ has the same rank as C since γ is non-degenerate, and since Δ' has the same rank as Δ, than $\gamma(C)$ must be a chamber.

Definition 5.17. A non-degenerate simplicial map γ between two chamber complexes Δ and Δ' of equal rank is a *chamber map*. Furthermore, if Δ' is a subcomplex of Δ and $\gamma : \Delta \to \Delta'$ acts as the identity on Δ' then γ is a *retraction*.

Definition 5.18. A *labelling* of an apartment Σ by a set I is a function which assigns to each vertex of Σ an element of I in such a way that the vertices of every chamber are mapped bijectively onto I.

Let Δ be a chamber complex. Let I be a set with cardinality n where n is the rank of Δ, and let $\mathcal{I}$ be the simplicial complex consisting of all subsets of I. Then a labelling of Δ by I is the same as a chamber map $\lambda : \Delta \to \mathcal{I}$.

Definition 5.19. If $A \in \Delta$ and $\lambda : \Delta \to \mathcal{I}$ is a labelling of Δ, then the simplex $\lambda(A)$ is called the *type* of A. A simplicial automorphism $f : \Delta \to \Delta$ is *type preserving* if $\lambda(A) = \lambda(f(A))$ for all $A \in \Delta$.

Lemma 5.20. *[Br, B2″ §IV.1] If Σ and Σ' are two apartments with a common chamber, then there is an isomorphism $\Sigma \to \Sigma'$ fixing every simplex in $\Sigma \cap \Sigma'$. In fact, this condition is equivalent to property (3) of Definition 5.11.*

Proof. Let Σ and Σ' be apartments with a common chamber C. In light of property (3) of Definition 5.11 we have, for each $A \in \Sigma \cap \Sigma'$, an isomorphism $\phi_A : \Sigma \to \Sigma'$ fixing A and C pointwise. However, Proposition 5.15 shows that there is at most one isomorphism from Σ to Σ' fixing C pointwise and thus all the ϕ_A are equal to a single isomorphism ϕ which fixes the entire intersection $\Sigma \cap \Sigma'$.

Conversely, suppose first that Σ and Σ' contain a chamber C and a simplex A in their intersection. Then by hypothesis there is a simplicial isomorphism from Σ to Σ' fixing A and C pointwise. More generally, let A and B be two simplices contained in two apartments Σ and Σ'. Choose chambers C and D with $A \leq C \in \Sigma$ and $B \leq D \in \Sigma'$, and choose an apartment Σ'' containing both C and D. In view of our weaker property, we have isomorphisms $\Sigma \to \Sigma'' \to \Sigma'$, where the first isomorphism fixes C and B pointwise and the second fixes A and D pointwise. Their composition is then an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise. □

Corollary 5.21. *[Br, Proposition 1 §IV.1] Suppose Σ, Σ' are two apartments containing a common chamber C. If Σ and Σ' admit labellings by I which agree on $\Sigma \cap \Sigma'$ then the map from Lemma 5.20 is label preserving.*

Sketch of proof. Let $\lambda : \Sigma \to I$ and $\lambda' : \Sigma' \to I$ be the two labellings. Let C be a chamber in $\Sigma \cap \Sigma'$ and let $\phi : \Sigma \to \Sigma'$ be a map from Lemma 5.20. We have that $I = \lambda(C) = \lambda' \circ \phi(C)$. A similar argument as in Proposition 5.15 (with $C_1 = D_1 = I$, for example) shows that $\lambda = \lambda' \circ \phi$. □

We now give an example of a retraction, which will become important again later.

Example 5.22. Let Δ be a building with associated fundamental apartment and chamber (Σ, C). We construct a retraction $\Delta \to \Sigma$ fixing C pointwise as follows. If we have an apartment Σ' which also contains C we know from Lemma 5.20 that there is an isomorphism $\Sigma' \to \Sigma$ fixing C pointwise. Let Σ'' be an apartment which does not contain C. Let B be a chamber in Σ''. We then have that, by property (2) of

Definition 5.11, that there exists an apartment Σ' containing both B and C. By applying the above argument to this situation twice, we can construct an isomorphism $\Sigma'' \to \Sigma' \to \Sigma$ such that the first fixes B pointwise and the second fixes C pointwise. By Corollary 5.21 we also know that these maps are all label preserving.

Proposition 5.23. *[Br, §I.Appendix C page 31] A chamber complex Δ admits a labelling if and only if it admits a retraction onto $\Delta_{\leq C}$ where C is a chamber of Δ.*

Proof. Let C be a chamber in Δ. Assume Δ admits a labelling $\lambda : \Delta \to \mathcal{I}$. We thus have that λ maps the subcomplex $\Delta' = \Delta_{\leq C}$ isomorphically onto $\mathcal{I}$. Thus, we get a retraction $\gamma = \lambda|_{\Delta'}{}^{-1} \circ \lambda : \Delta \to \Delta'$. Conversely, we can view a retraction onto Δ' as a labelling where the set I of labels is the set of vertices of C. □

Theorem 5.24. *The Coxeter complex Σ of a Coxeter system (W, S) admits a labelling.*

The following proof has been greatly expanded upon from [Br, Proposition §I.Appendix C]. In particular, it applies to affine, not just finite, reflection groups.

Proof of Theorem 5.24. Choose a chamber C. By Corollary 3.30 we have that W acts simply transitively on the set of chambers. Therefore, for any other chamber C' there is exactly one element $w \in W$ such that $wC = C'$. Furthermore, if A is a face of C, there is only one face of C' that A can be mapped to via the W-action. Define $\gamma : \Sigma \to \Sigma_{\leq C}$ by setting $\gamma(A)$ to be the unique face of C which is W-equivalent to A. We have that γ is a non-degenerate chamber map since the W-action is bijective, and we also have that it acts as the identity on the $\Sigma_{\leq C}$. By Proposition 5.23 we have a labelling. □

Remark 5.25. Any two labellings of a chamber complex Σ are equivalent. Let γ and λ be two labellings on Σ with label sets I_γ and I_λ respectively. We have that the cardinality of both of these sets must be equal, call it n. We can thus write $I_\gamma = \{a_1, \ldots, a_n\}$ and $I_\lambda = \{b_1, \ldots, b_n\}$. Let C be a chamber of Σ and let $\mathcal{C}$ be the set of its vertices. Then since γ and λ are labellings, there are bijective maps $\tau_\gamma: I_\gamma \to \mathcal{C}$ and $\tau_\lambda: I_\lambda \to \mathcal{C}$. Then $\tau_\lambda^{-1}\tau_\gamma$ defines a bijection from I_γ to I_λ, which extends to a map $\tau: \mathcal{I}_\gamma \to \mathcal{I}_\lambda$. We then have $\lambda = \tau\gamma$.

Corollary 5.26. *[Br, Proposition 1 §IV.1] A building Δ admits a labelling.*

Proof. Let Δ be a building. Fix a chamber C and label its vertices by some set I. Let Σ be any apartment containing C. We then have, by Theorem 5.24 and Remark 5.25, that there is a unique labelling λ_Σ of Σ which agrees with the labelling of C. Let Σ and Σ' be two apartments containing C. We then have that the labellings λ_Σ, $\lambda_{\Sigma'}$ agree on $\Sigma \cap \Sigma'$, which follows from the fact that $\lambda_{\Sigma'}$ can be constructed as $\lambda_\Sigma \circ \phi$ where $\phi : \Sigma' \to \Sigma$ is the unique isomorphism fixing $\Sigma \cap \Sigma'$ from Lemma 5.20. We thus have that the labellings λ_Σ all fit together to give a labelling λ defined on the union of all of the apartments containing C. Let A be any simplex in Δ, then by property (2) of Definition 5.11 we have that there exists an apartment Σ containing both A and C, which means that all simplices in Δ are contained in at least one apartment which contains C. Therefore, our labelling λ is a labelling of the entire building Δ. □

Buildings and Coxeter Systems

Let Δ be a building and $\mathcal{A}$ the collection of all its apartments. Suppose G is a group that acts as type-preserving simplicial automorphisms on Δ, stabilizes $\mathcal{A}$ (that is, if Σ is an apartment then so is $g\Sigma$) and also acts transitively on the set of pairs (Σ, C) where $\Sigma \in \mathcal{A}$ and C is a chamber in Σ. Fix an arbitrary such pair (Σ, C) and refer to these specific Σ and C as the "fundamental" apartment and chamber respectively.

Following Remark 5.25 we have that any two labellings on Σ are equivalent, so let us choose the one given in the proof of Theorem 5.24. By construction, this labelling gives us the fact that W is a group of type-preserving automorphisms of Σ. We also have that for any chamber C' there exists a unique element of W which maps C to C' in a type preserving way. This implies that if $\gamma \in G$ is a type-preserving automorphism on Σ we must have that its action on Σ is equal to an element of W. This gives us the fact that W is the group of all type preserving automorphisms on Σ.

Remark 5.27. We will impose a further restriction on our buildings by insisting they be *thick*. This means that any co-dimension one face of C is a face of at least three chambers.

We take this opportunity to introduce several subgroups of G, following [Br, §V.1A].

$$\begin{aligned} B &= \{g \in G | gC = C\} \\ N &= \{g \in G | g\Sigma = \Sigma\} \\ T &= \{g \in G | g \text{ fixes } \Sigma \text{ pointwise}\} \end{aligned}$$

The choice of names for these subgroups is meant to be suggestive of the **B**orel, **T**orus, and **N**ormalizer of the torus of an algebraic group.

Lemma 5.28. *$W \cong N/T$.*

Proof. Recall that N is the set of all elements of G, which are type-preserving automorphisms on Δ, such that $g\Sigma = \Sigma$ and that W is the set of all type-preserving automorphisms of Σ. Since G acts transitively on all pairs (Σ, C) it contains all type preserving automorphisms of Σ. We therefore get a surjective homomorphism $h : N \to W$ by restricting the action of N from Δ to Σ. We also find that T is the kernel of h and thus we have that T is normal in N and $W \cong N/T$. □

Lemma 5.29. *If $g \in G$ fixes C then it fixes C pointwise.*

Proof. Since g fixes C in a type preserving way, and there is only one way to map C to C in a type preserving way, we must have that g fixes C pointwise. □

Lemma 5.30. *$T = B \cap N$.*

Proof. Let $g \in B \cap N$. Since $g \in N$, we have by the proof of Lemma 5.28 that g acts on Σ as an element of W via the surjective homomorphism h. But we also have that $g \in B$, so it will fix C. By the simple transitivity of the W-action, this implies that $h(g) = 1 \in W$, and so by Lemma 5.28 we have $g \in T$. Thus $B \cap N \subseteq T$; but the converse is clear since each of B and N contains T by definition. Thus $T = B \cap N$. □

Let us now make a small digression which will ultimately lead us to a useful result; namely that G is generated by B and N. Recall Lemma 5.9. Let A be the face of C whose stabilizer in W (identified with N/T) is the special subgroup $W' = \langle S' \rangle$. Let $P_{S'}$ be the stabilizer of A in G.

Proposition 5.31. *[Br, § V.1D]* $P_{S'} = \bigcup_{w \in W'} BwB$.

Proof. For some $g \in P_{S'}$ choose an apartment Σ' containing the two chambers C and gC. We may choose an element $b \in B \subset G$ such that $b\Sigma' = \Sigma$, since G is transitive on the pairs (Σ, C) and the fact that any element which fixes C is contained in B ($bC = C \ \ \forall\, b \in B$). Since bgC is a chamber in Σ and W acts transitively on the set of chambers in Σ (Proposition 3.27) we have that

$$bgC = wC \tag{10}$$

for some $w \in W$. Equation (10) implies that $bg \in wB$ and therefore $g \in BwB$. To show that $w \in W'$ write $g = b_1 w b_2$. Then

$$A = gA = b_1 w b_2 A = b_1 w A \qquad \Rightarrow \qquad wA = b_1^{-1} A = A.$$

Therefore $wA = A$ and $w \in W'$. The same argument, in reverse, shows that $BW'B \subseteq P_{S'}$. Finally, this gives us $P_{S'} = \bigcup_{w \in W'} BwB$. $\square$

We will be using the shorthand notations

$$\bigcup_{w \in W'} BwB = BW'B = B\langle S' \rangle B = B \langle s_1, \ldots, s_n \rangle B$$

where

$$W' = \langle S' \rangle = \langle s_1, \ldots, s_n \rangle$$

interchangeably.

From this, we derive two interesting results. First, when $S' = \{s\}$, we have that

$$P_s = B \cup BsB. \tag{11}$$

Second, when $S' = S$, we have that $W' = W$ and $P_{S'} = P_S = G$. We conclude this because the face A of C stabilized by $W = \langle S \rangle$ is either the origin if W is a finite reflection group or the empty set if W is an infinite affine reflection group, and the corresponding stabilizer in G in both cases is G itself. This tells us that

$$G = BWB.$$

Recall that we identified W with N/T. Therefore, we can identify every element of W with an element of N (any element of $n \in N$ such that $w = nt$ for some $t \in T$). This gives us the fact that G is generated by B and N.

Lemma 5.32. *[Br, § V.1E] $G = \coprod_{w \in W} BwB$ is a disjoint union.*

This is called the *Bruhat decomposition* of G.

Proof. We have already established that $G = \bigcup_{w \in W} BwB$; at issue is to show that if $BwB = Bw'B$ then $w = w'$.

Let $g \in G$ and choose an apartment Σ' containing both C and gC. Also, choose $b \in B$ such that $b\Sigma' = \Sigma$. Since b is type preserving, the action of b induces the unique isomorphism $\Sigma' \to \Sigma$ fixing C pointwise. We have

$$bgC = \rho(gC)$$

where ρ is the retraction $\rho_{\Sigma,C} : \Delta \to \Sigma$ fixing C pointwise constructed in Example 5.22. What we have now then, as in the proof of Proposition 5.31, is that $g \in BwB$ where w is the unique element of W such that $\rho(gC) = wC$.

To complete the proof, it suffices to show that for any $w \in W$ and $g \in BwB$, that $\rho(gC) = wC$. Choose $\tilde{w} \in N$ representing w, and consider an element $g = b\tilde{w}b' \in BwB$, $b, b' \in B$. We then have that

$$gC = b\tilde{w}b'C = b\tilde{w}C = bwC.$$

Hence, since $bC = C$, we have that both gC and C are in the same apartment $b\Sigma$. Furthermore, b^{-1} maps this apartment back to Σ, from which we derive

$$\rho(gC) = b^{-1}gC = wC.$$

It now follows from the simply transitive action of W that these double cosets are all distinct. $\square$

Proposition 5.31 gives us that $P_{S'} = BW'B$, however, Lemma 5.32 lets us know that all of the BwB are disjoint as w ranges over W. The set $\mathcal{P} = \{P_{S'} | S' \subset S\}$ is a poset with relation given by inclusion. Let $\mathcal{S}$ be the collection of all subsets of S. We can now create a function $\lambda : \mathcal{S} \to \mathcal{P}$ which maps the subset $S' \subseteq S$ to $P_{S'}$.

Proposition 5.33. *The function $S' \mapsto P_{S'}$ is a poset isomorphism.*

Proof. It is clear that λ is a poset homomorphism. We claim its inverse is the map which sends $P \in \mathcal{P}$ to the set

$$S' = \{s \in S | BsB \subseteq P\}.$$

Namely, write $P = B\tilde{W}B$ for some $\tilde{W} = \langle \tilde{S} \rangle \subset W$ as in Proposition 5.31. If $BsB \subseteq P$ then we must have that $BsB = BwB$ for some $w \in \tilde{W}$, since double cosets are either equal or disjoint. By Lemma 5.32, however, this in turn implies $w = s$ since both are elements of W. By Lemma 5.6, we deduce that in fact $s \in \tilde{S}$. Hence $S' \subseteq \tilde{S}$. That $\tilde{S} \subset S'$ follows from our choice of $\tilde{S}$, so in fact $S' = \tilde{S}$, as required. □

5.3 BN-pairs

We are now ready to define a BN-pair.

Definition 5.34. [BT1, §1.2.6] A pair of subgroups (B, N) of a group G will be called a *BN-pair* if:

1. The subgroups B and N generate G.
2. The intersection $T = B \cap N$ is normal in N.
3. The quotient $W = N/T$ has a set of generators S, all of which have order 2, such that
   (a) $BsB \cdot BwB \subseteq BwB \cup BswB$
   (b) $sBs \not\subseteq B$

   hold for all $s \in S$ and $w \in W$.

Note that in [BT1] a BN-pair is called a *Tits system.*

The following Lemma will prove useful, the proof of which can be found at [Br, Lemma 1 §V.2A]. The proof of Lemma 5.35 is done by induction on the length of the reduced expression for w, using properties of the length function we have not given here.

Lemma 5.35. *$G = \bigcup_{w \in W} BwB$.*

Proposition 5.36. *The subgroups B and N from §5.2 are a BN-pair.*

We proved property (1) at the end of §5.2. property (2) is given by Lemmas 5.28 and 5.30. See [Br, §V.1F] for a proof of property (3a) which uses an idea called s-adjacency, which we have not covered here.

Proof of property (3b). By Remark 5.27 we have that any co-dimension one face of C is a face of at least three chambers. Also, there exists an element $s \in S$ which stabilizes this face. Two of the (at least) three chambers are C and sC, and thus we must have a distinct third chamber hC for some $h \in P_s$. The condition $hC \neq C$ implies that $h \notin B$, so we must have $h \in BsB$ by Equation (11). The condition $hC \neq sC$ implies that $h \notin sB$. Therefore $BsB \not\subseteq sB$. This implies that $Bs \not\subseteq sB$, which we rewrite as $sBs \not\subseteq B$. This proves property (3b). □

We now have that every building which admits a group G (as in the section Buildings and Coxeter Systems in §5.2) gives rise to a BN-pair. We would like to be able to deduce the converse as well, namely, that every BN-pair gives rise to a building. We will continue in that vein.

From Properties (3a) and (3b) in Definition 5.34 one can deduce several other key properties of BN-pairs, as in [Br, Theorem §V.2A]. We will need the following one now, along with one more further on.

Lemma 5.37. *$B \cup BsB$ is a subgroup of G for every $s \in S$.*

Proof. We know that B itself is a group, and thus $BB = B$, $BBsB = BsB$, and $BsBB = BsB$ are all contained in $B \cup BsB$. Finally, we also get $BsBBsB \subseteq BsB \cup BssB = B \cup BsB$ from property (3a) and the fact that $s^2 = 1$. □

The other property we will need is that $BW'B$ is a subgroup of G for any special subgroup $W' \subseteq W$ and is proven in Lemma 5.39.

We introduce a lemma which will be useful in Chapter 6.

Lemma 5.38. *$B\langle s\rangle B = sB\langle s\rangle Bs$ $\forall s \in S$.*

Proof. We must show that $B \cup BsB = sBs \cup sBsBs$. Let $s \in S$. Clearly,

$$sBs \subseteq BsB \cdot BsB,$$

and Lemma 5.37 gives us

$$BsB \cdot BsB \subseteq B \cup BsB.$$

This implies that

$$sBs \subseteq B \cup BsB. \tag{12}$$

Note that

$$sB \subseteq BsB, \tag{13}$$

and thus

$$\begin{aligned} sBsBs &= sB \cdot sBs \\ &\subseteq BsB \cdot (B \cup BsB) && (14) \\ &= B \cup BsB && (15) \end{aligned}$$

where Equation (14) is obtained using Equations (12) and (13), and Equation (15) follows from Lemma 5.37. Combining Equations (12) and (15) yields

$$sBs \cup sBsBs \subseteq B \cup BsB \tag{16}$$

and we have now shown containment on one side. To get containment on the other side, multiply left and right by s. This gives us $B \cup BsB \subseteq sBs \cup sBsBs$. Finally, we conclude

$$B\langle s \rangle B = sB\langle s \rangle Bs.$$

□

5.4 Parabolic Subgroups

Let G be a group with a BN-pair as in Definition 5.34. We stated after Lemma 5.37 that $P_{S'} = BW'B$ is a subgroup. This follows directly from the following Lemma.

Lemma 5.39. *[Br, Lemma 1 § V.2A] Let $W' = \langle S' \rangle$ be a special subgroup of W. Then for all $w, w' \in W'$, $BwBBw'B \subseteq BW'B$.*

Proof. Write $w = s_1 \cdots s_d$ where each s_i are contained in the set of generators S' of W'. If $d = 1$ the lemma is proven by property (3a). Proceeding by induction on d, we will show that $BwBBw'B \subseteq \bigcup Bw''w'B$ where w'' ranges over the 2^d elements of W' obtained from $w = s_1 \cdots s_d$ by deleting zero or more letters. Assume this holds for $d = k - 1$, and let's prove it for $d = k$.

Write $w = s_k s_{k-1} \cdots s_1$. We then have that

$$\begin{aligned} BwBBw'B &= Bs_k s_{k-1} \cdots s_1 BBw'B \\ &\subseteq Bs_k BBs_{k-1} \cdots s_1 BBw'B \\ &\subseteq Bs_k B \left(\bigcup Bw''w'B \right) \end{aligned}$$

by the induction hypothesis where w'' ranges over the 2^{k-1} elements of W' obtained as in the induction hypothesis. We complete the proof by applying property (3a) 2^{k-1} times to the final equation above to get $BwBBw'B \subseteq \bigcup Bw'''w'B$ where w''' now ranges over the 2^k elements of W' as promised. □

Definition 5.40. A subgroup of G of the form $BW'B$ is a *special subgroup.*

Theorem 5.41. *[Br, Theorem 1 § V.2B] The special subgroups of G are precisely the subgroups of G containing B.*

We will need the following Lemma, whose proof we omit. Note that a *reduced expression* for $w \in W$ is a word in the generating set S of shortest length such that it is equal to w.

Lemma 5.42. *[Br, Lemma § V.2B] Let $w \in W$ admit a reduced expression $w = s_1 \cdots s_n$ where each $s_i \in S$. Then the subgroup of G generated by BwB contains Bs_iB for $i = 1, \ldots, n$. Moreover, this subgroup is generated by B and wBw'.*

The proof of Lemma 5.42 is done by induction on the length of the reduced expression for w, using properties of the length function we have not given here.

Proof of Theorem 5.41. Obviously, all special subgroups contain B. Now, suppose P is a subgroup containing B. Then P is a union of double cosets, namely $P = BW'B$ where

$$W' = \{w \in W | BwB \subseteq P\}.$$

We have that $Bw^{-1}B = (BwB)^{-1}$ and $Bww'B \subseteq BwBBw'B$, thus W' is a subgroup of W. Lemma 5.42 implies that W' contains the generators $s \in S$ which occur in any reduced expression of each of its elements w. Hence, W' is the subgroup of W generated by $W' \cap S$, and P is therefore a special subgroup of G. □

Corollary 5.43. *The set S consists of all non-trivial elements $w \in W$ such that $B \cup BwB$ is a subgroup of G.*

Proof. If $s \in S$, then we know that $B \cup BsB = P_s$ is a subgroup of G from Lemma 5.37. All we need to show then is that if $B \cup BwB$ is a subgroup of G, then $w = s \in S$. Assume that $B \cup BwB$ is a subgroup of G. Obviously, $B \cup BwB$ contains B. Therefore, by Theorem 5.41, we have that $B \cup BwB$ is a special subgroup of G, thus

$$B \dot{\cup} BwB = BW'B = \coprod_{w' \in W'} Bw'B$$

where $W' = \langle S' \rangle$ for some subset $S' \subset S$. We thus have that $W' = \{1, w\}$. Therefore, since $|W'| = 2$, we must have that $|S'| = 1$. Thus, we have that $S' = \{s\}$, and $BW'B = B \cup BsB$. We then have that $BwB = BsB$. Therefore, by Lemma 5.35, $w = s \in S$. □

Definition 5.44. A *special coset* in G is a coset gP where $g \in G$ and P is a special subgroup.

Definition 5.45. A subgroup of $A \leq G$ is *parabolic* if it contains a conjugate of B.

Lemma 5.46. *A subgroup of $A \leq G$ is parabolic if it is conjugate to a special subgroup.*

Proof. Since A is conjugate to a special subgroup, we have that $A = gQg^{-1}$ for some $g \in G$ where Q is a special subgroup. This can be rewritten as $Q = g^{-1}Ag$, and by Theorem 5.41, we have that $B \subseteq Q = g^{-1}Ag$, which again can be rewritten as $A \supseteq gBg^{-1}$, which completes the proof. □

Proposition 5.47. *[Br, Theorem 2 § V.2B] Every special subgroup is self-normalizing and no two distinct special subgroups are conjugate.*

Proof. Let P and P' be special subgroups, and suppose $gP'g^{-1} = P$ for some $g \in G$. Let BwB be the double coset containing g. We then have that $wP'w^{-1} = P$, and so P contains B and wBw^{-1}. Lemma 5.42 now gives us that $BwB \subseteq P$, and so we have $g \in P$ and $P' = P$. □

The following Corollaries are immediate.

Corollary 5.48. *There exists a bijection from the set of special cosets to the set of parabolic subgroups given by $gP \mapsto gPg^{-1}$. This bijection is compatible with the inclusion relation.*

Corollary 5.49. *The poset of special cosets of G is isomorphic to the poset of parabolic subgroups of G.*

Corollary 5.49 gives us one of the most practical ways to construct a building from a group G as it allows us to construct it from parabolic subgroups rather than special cosets. We will use this in Chapters 6 and 7, but first, we need to prove that these posets actually give us our building.

Define Δ to be the poset of special cosets ordered via inverse inclusion. Also, define the subcomplex $\Sigma = \{wP | w \in W, P \text{ is a special subgroup}\}$ and the collection of transforms $g\Sigma$ via the G-action.

Theorem 5.50. *[Br, Theorem § V.3] The simplicial complex Δ of special cosets ordered via inverse inclusion is a building.*

Proof. Recall Definition 5.11. We have three properties to prove.

Let us begin with the map $\iota : \Sigma(W, S) \to \Delta$ given by $\iota(wW') = wP$, where $W' = \langle S' \rangle$ is a special subgroup of W and $P = BW'B$ is the corresponding subgroup of G. Note that the image of ι is Σ as defined above.

We also have a second map $\rho : \Delta \to \Sigma(W, S)$ given by $\rho(gP) = wW'$ if $g \in BwB$ and $P = BW'B$. We will proceed to show that ρ is well defined. Let $g' = gh$

be another representative of the coset gP. We then have that $h \in Bw'B$ for some $w' \in W'$. We then get that

$$g' = gh = BwBBw'B \subseteq \bigcup Bww''B$$

where w'' ranges over the elements of W' obtained from the word w' (as in the proof of Lemma 5.39). We obtain the same special coset $ww''W' = wW'$ if we use g' instead of g, which gives us that ρ is well defined.

Note first that $\rho\iota$ is the identity on $\Sigma(W, S)$. Also, we have that ι maps $\Sigma(W, S)$ isomorphically onto Σ. Therefore Σ is isomorphic to the Coxeter complex of (W, S). It follows that the collection of transforms $g\Sigma$ satisfy the first property of Definition 5.11.

To prove the second property, we may assume without loss of generality that one of the simplices is a special subgroup P. The second simplex must then be gP' for some $g \in G$ and some special subgroup P'. Write $g = bnb'$ for some $n \in N$ and $b, b' \in B$. Since $B \subseteq P$, we have that $P = bP \in b\Sigma$. We also have that $gP' = bnb'P' = bnP' \in b\Sigma$ since $nP' \in \Sigma$ by the definition of Σ. Therefore we have that $b\Sigma$ is an apartment containing both P and gP, which proves the second property.

For property (3), we apply Lemma 5.20. Without loss of generality, we may assume that one of the apartments is Σ and that the common chamber C is the special coset B. Call the second apartment Σ'. Since G is transitive on the collection of apartments and B is the stabilizer of our chamber C, we have an isomorphism $\phi : \Sigma' \to \Sigma$ which is given by the action of some $b \in B$ (that is, we have that $b\Sigma' = \Sigma$).

Note that the map $\bar{\rho} = \iota\rho : \Delta \to \Sigma$ given by $\bar{\rho}(gP) = wP$ if $g \in BwB$ is a retraction since $\bar{\rho}$ acts as the identity on Σ (in much the same way the map $\rho\iota$ was the identity on $\Sigma(W, S)$ which is isomorphic to Σ). We claim that $\phi = \bar{\rho}|_{\Sigma'}$.

Note that every simplex of $\Sigma' = b^{-1}\Sigma$ is of the form $b^{-1}wP$ for some $w \in W$ and some special subgroup P. We therefore have

$$\bar{\rho}(b^{-1}wP) = \bar{\rho}(b^{-1}wbP) = wP = bb^{-1}wP = \phi(b^{-1}wP)$$

as we had claimed.

Since $\bar{\rho}$ acts as the identity on Σ, it acts as the identity on $\Sigma \cap \Sigma'$, and thus so does ϕ, which completes the proof. $\square$

Remark. It is also possible to prove that our building is thick by reversing the calculations in the proof of property 3b in Proposition 5.36.

We have now shown that any building, together with a G-action as in the section Buildings and Coxeter Systems in §5.2, gives rise to a BN-pair and vice-versa.

Chapter 6

Spherical Buildings for Algebraic Groups

In this chapter we will explicitly construct a fundamental apartment associated to specific real algebraic groups. These results are well-known to researchers in the field but are not readily found in the literature.

6.1 $GL_3(\mathbb{R})$

This section deals with the construction of the fundamental apartment for the building of $GL_3(\mathbb{R})$. To begin this construction, we will need a BN-pair for our group.

As in Example 4.14, let $B \subset GL_3(\mathbb{R})$ be the upper triangular subgroup and let $N \subset GL_3(\mathbb{R})$ be the monomial subgroup. We also have that $T = B \cap N$, which is the set of diagonal matrices contained in $GL_3(\mathbb{R})$. It is normalized by N. The group $W = N/T$ is identified with S_3, the symmetric group on 3 letters. This gives us

$$\langle (1\ 2), (2\ 3) \rangle \cong \langle S \rangle = \langle s_1, s_2 \rangle = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

Proposition 6.1. *The subgroups B, $N \in GL_3(\mathbb{R})$ specified above form a BN-pair for the group $GL_3(\mathbb{R})$.*

Proof. We already have that T is normal in N. Since B consists of upper triangular matrices, N contains the permutation matrices, and, setting $w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we have that $w^{-1}Bw$ consists of all lower triangular matrices. The fact that G is generated by B and N follows from the LU-decomposition of Linear Algebra. We have also that W is generated by S which is composed of elements of order two. Property (3b) follows since

$$s_1Bs_1 = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} \not\subseteq B$$

and

$$s_2Bs_2 = \begin{bmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \not\subseteq B.$$

Finally, we prove property (3a). We must show $BsB \cdot BwB \subseteq BwB \cup BswB$. We follow the argument of [Br, §V.5].

The inclusion $BsB \cdot BwB \subseteq BwB \cup BswB$ is equivalent to $sBw \subseteq BwB \cup BswB$ by multiplying on the right and left by suitable elements of B. By multiplying on the right by w^{-1} we can rewrite this as $sB \subseteq BB' \cup BsB'$, where $B' = wBw^{-1}$.

Let us show this inclusion for the case $s = s_2$. The case of $s = s_1$ is entirely analagous. A typical element $x \in sB$ has the form

$$x = \begin{bmatrix} 0 & a & b \\ c & d & e \\ 0 & 0 & f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$ and f is necessarily non-zero. This can be reduced to

$$bx = \begin{bmatrix} 1 & 0 & -bf^{-1} \\ 0 & 1 & -ef^{-1} \\ 0 & 0 & f^{-1} \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $b \in B$.

If $d = 0$ then

$$tbx = \begin{bmatrix} c^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & 0 \\ c & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = s_2$$

where $t \in B$ and $acd \neq 0$ (since our matrix is invertible), so $x \in Bs_2$.

We will now use right multiplication by B'. To avoid ruining the work already accomplished, we will restrict ourselves to $g \in B'$ such that $ge_3 = e_3$ and $g[e_1, e_2] \subseteq [e_1, e_2]$ where $[e_1, e_2] = span\{e_1, e_2\}$. Note that we have that $B' = wBw^{-1}$ stabilizes $[e_{w(1)}]$ as well as $[e_{w(1)}, e_{w(2)}]$ since B stabilizes $[e_1]$ and $[e_1, e_2]$.

There are two possibilities. First, if $w^{-1}(1) < w^{-1}(2)$, then either $w(1) = 1$ and g preserves e_1, or $w^{-1}(1) = 2 < w^{-1}(2) = 3$, so $w(1) = 3, w(2) = 1, w(3) = 2$, and g preserves $[e_3, e_1]$ and we have that g is of the form

$$\begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}.$$

Secondly, if $w^{-1}(2) < w^{-1}(1)$ then we deduce that g is of the form

$$\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}.$$

In the first case,

$$bxb' = \begin{bmatrix} 0 & a & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c^{-1} & -da^{-1}c^{-1} & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = s_2,$$

where $b' \in B'$ and so we have $x \in Bs_2B'$.

In the second case,

$$bxb' = \begin{bmatrix} 0 & a & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -da^{-1}c^{-1} & 0 & 0 \\ a^{-1} & d^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ad^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in B$$

where $b' \in B'$ and so we have $x \in bBb' \in BB'$. □

We will now construct a fundamental apartment of the building for $GL_3(\mathbb{R})$ (determined by this choice of BN-pair) using Corollary 5.49. More precisely we will construct the apartment Σ as

$$\Sigma = \{wPw^{-1} | P \text{ is a special subgroup}, w \in W\}.$$

It is important to note that we will only be using W-conjugates to get the apartment since using all conjugates will give the entire building.

Recall from Lemma 5.46 that $Q \subseteq G$ is parabolic if Q contains a conjugate of B, or equivalently, if Q is conjugate to a special subgroup, which is of the form $BW'B$ where $W' = \langle S' \rangle$ and $S' \subseteq S$. Since $S' \subseteq S$, the only possibilities for S' are $\emptyset, \{s_1\}, \{s_2\}$, and $\{s_1, s_2\}$. A few matrix multiplications give us the following four special subgroups (where $* \in \mathbb{R}$ and each matrix must be an element of $GL_3(\mathbb{R})$) :

$$B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \ B\langle s_1 \rangle B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

$$B\langle s_2 \rangle B = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}, \ B\langle s_1, s_2 \rangle B = BWB = GL_3(\mathbb{R})$$

Next, we compute all these parabolic subgroups, for which we will need to know all the elements of W. We know that S_3 has only six elements. These are the following elements, which can be calculated by matrix multiplications:

$$W = \{I, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}.$$

Since all six of these matrices are different mod T, we must have the entire group W.

To compute the remaining parabolic subgroups, we must calculate $wB\langle s \rangle Bw^{-1}$ $\forall\, w \in W$, $\forall\, s \in S$. Some of these will be redundant. We can reduce our calculations by half by realizing that, for example, $s_2s_1B\langle s_1 \rangle Bs_1s_2 = s_2B\langle s_1 \rangle Bs_2$ (see Lemma 5.38). By conjugating all special subgroups by elements of W we can obtain the parabolic subgroups, which are listed in Table 3 and given explicitly just after.

Conjugates of B:		
$B_1 = B$	$B_2 = s_1 B s_1$	$B_3 = s_2 B s_2$
$B_4 = s_1 s_2 B s_2 s_1$	$B_5 = s_2 s_1 B s_1 s_2$	$B_6 = s_1 s_2 s_1 B s_1 s_2 s_1$
Conjugates of $B\langle s_1\rangle B$:		
$P_1 = B\langle s_1\rangle B$	$P_2 = s_2 B\langle s_1\rangle B s_2$	$P_3 = s_1 s_2 B\langle s_1\rangle B s_2 s_1$
Conjugates of $B\langle s_2\rangle B$:		
$P_4 = B\langle s_2\rangle B$	$P_5 = s_1 B\langle s_2\rangle B s_1$	$P_6 = s_2 s_1 B\langle s_2\rangle B s_1 s_2$

Table 3: Proper parabolic subgroups of $GL_3(\mathbb{R})$

Conjugates of B :

$$B_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \ B_2 = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix}, \ B_3 = \begin{bmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$B_4 = \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix}, \ B_5 = \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix}, \ B_6 = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Conjugates of $B\langle s_1\rangle B$:

$$P_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \ P_2 = \begin{bmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{bmatrix}, \ P_3 = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Conjugates of $B\langle s_2\rangle B$:

$$P_4 = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}, \ P_5 = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{bmatrix}, \ P_6 = \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$$

We can now read the following containment relations directly from these matrices.

$$B_1, B_2 \subset P_1 \qquad B_3, B_5 \subset P_2 \qquad B_4, B_6 \subset P_3$$
$$B_1, B_3 \subset P_4 \qquad B_2, B_4 \subset P_5 \qquad B_5, B_6 \subset P_6$$

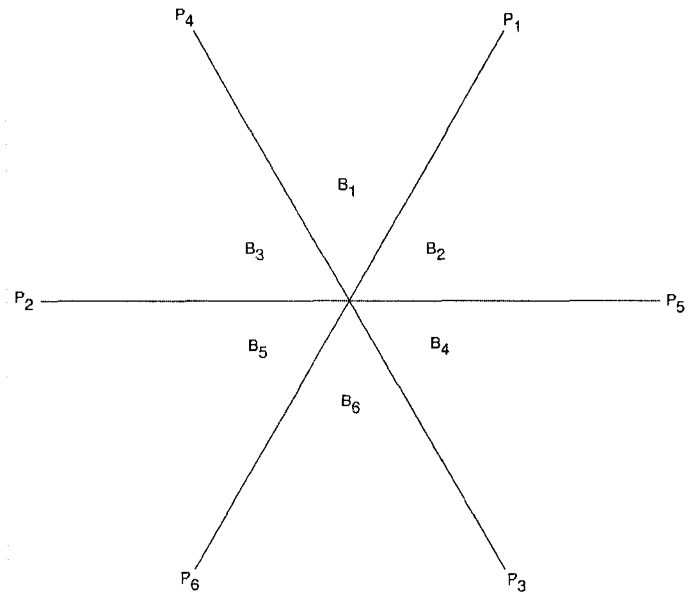

Figure 7: Fundamental apartment for $GL_3(\mathbb{R})$

Finally, by ordering via inverse inclusion, we obtain the fundamental apartment depicted in Figure 7. We see that the maximal elements are B_1 through B_6, the minimal element is G, and all P_i are in between. So G corresponds to a vertex, P_1 through P_6 to 1-simplices, and B_1 through B_6 to 2-simplices.

Bijection of Cells with Flags

As we saw in Chapter 5, there are several equivalent ways to describe the apartments of the building associated to a group G. One way, which is analogous to the method of lattices we choose first in Chapter 7, is as follows.

Definition 6.2. Let R be a set with a binary relation called *incidence* (which will be defined differently for each situation). A *flag* in R is a set of pairwise incident elements of R.

Remark. We can identify the parabolics by the flags they stabilize. If we take the flags to be flags of subspaces, then the incidence relations is given by inclusion. We will see another kind of incidence relation for the p-adic case in Chapter 7.

Example 6.3. Let $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{R}^3$. Consider the flag $[e_1] \subset [e_1, e_2]$. We say that $g \in GL_3(\mathbb{R})$ stabilizes the flag if $g[e_1] = [e_1]$ and $g[e_1, e_2] = [e_1, e_2]$. It follows immediately that the stabilizer of this flag is B. More generally, the stabilizer of a flag of subspaces in $\mathbb{R}^3$ is a parabolic subgroup.

Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. Instead of identifying the cells in the apartment with parabolic subgroups, we could identify them with the flags of subspaces, such that the subspaces are spanned by a subset of the standard basis, stabilized by each parabolic subgroup. We know that W permutes the lines, so given that B stabilizes the full flag, all parabolics stabilize some flag. For example, $B = B_1$ is identified with $[e_1] \subset [e_1, e_2]$, and P_1 is identified with $[e_1]$ since they stabilize those respective flags. The rest of the cells can similarly be identified with flags, as in Figure 8.

The Apartment as a Coxeter Complex

Another way to describe the apartment is by using the elements of the reflection group. In this case, each hyperplane is identified individually (as opposed to each cell). The identification for each hyperplane should be the non-zero (and non-identity) element of the affine reflection group that leaves that hyperplane invariant.

To proceed, we identify W with S_3, and think of it as permuting the standard basis vectors of $\mathbb{R}^3$. The hyperplane identified with P_1 and P_6 stabilizes $[e_1]$ and $[e_2, e_3]$. Therefore, the hyperplane would have to be identified with the element of the affine reflection group which stabilizes this as well. This would be the element (2 3) in S_3, or simply by the element $s_1 \in W$ to be concise. One now views the element (2 3) as the reflection in the hyperplane that it is identified with.

To identify the chambers of the apartment, we start by identifying the fundamental chamber $B = B_1$ with C, and then identifying every other chamber with wC where w is the element of the reflection group, written as products of elements of S, such that

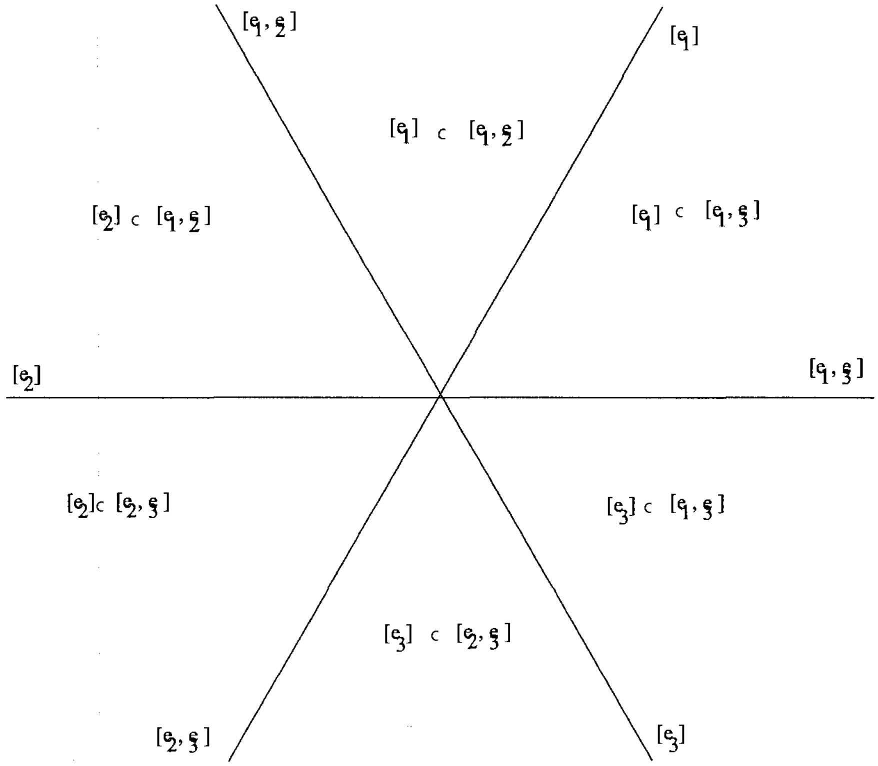

Figure 8: Fundamental apartment for $GL_3(\mathbb{R})$ identified with stabilized flags

the action of w on C results in the chamber wC (see Figure 9). This is possible by Corollary 3.30 which gives us that W acts simply transitively on the set of chambers. This action is the reflection in the hyperplanes denoted by the products of elements of S.

6.2 $Sp_4(\mathbb{R})$

The construction of a fundamental apartment for $Sp_4(\mathbb{R})$ will proceed in basically the same manner as the construction of a fundamental apartment of $GL_3(\mathbb{R})$. It is important to note that since we are working with $Sp_4(\mathbb{R})$, we are working with a

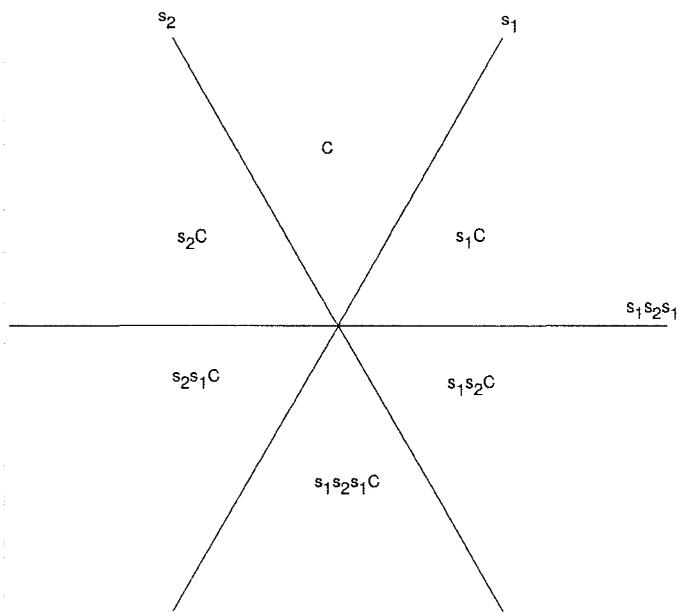

Figure 9: Fundamental apartment for $GL_3(\mathbb{R})$ identified with the affine reflection group

symplectic space. This means that we have a skew symmetric non-degenerate bilinear form on our space. As in §4.2, we choose an ordered symplectic basis $\{e_1, e_2, f_2, f_1\}$ so that it has the following properties:

$$\langle e_i, f_i \rangle = 1 = -\langle f_i, e_i \rangle$$

$$\langle e_i, f_j \rangle = 0 = \langle f_j, e_i \rangle \text{ if } i \neq j.$$

The merit of this choice will be seen in the relative simplicity of the subgroups we will use.

To be able to evaluate the inner product of any two vectors in the space, we will express them in coordinates with respect to the ordered basis above, then use the following equation:

$$\langle (a,b,c,d),(w,x,y,z) \rangle \;=\; \langle (a,0,0,0),(0,0,0,z) \rangle + \langle (0,b,0,0),(0,0,y,0) \rangle$$

$$\begin{aligned} & +\langle(0,0,c,0),(0,x,0,0)\rangle+\langle(0,0,0,d),(w,0,0,0)\rangle \\ = \; & az+by-cx-dw \end{aligned}$$

where $(a,b,c,d),(w,x,y,z)\in\mathbb{R}^4$.

We now introduce a BN-pair. As in Example 4.16 let

$$B=\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\bigcap Sp_4(\mathbb{R})$$

where $*\in\mathbb{R}$, and let N be all monomial matrices contained in $Sp_4(\mathbb{R})$.

Using B and N, we can now describe T and W. $T=B\cap N$, which is the set of all diagonal symplectic matrices. That is to say, T is the set of all matrices of the form $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2^{-1},\lambda_1^{-1})$ where $\lambda_i\in\mathbb{R}$. We define W as N/T. Let $L_i=[e_i]$ and $L_i'=[f_i]$ be the lines spanned by e_i and f_i respectively. Then $W\cong\langle s_1,s_2\rangle$ where s_1 is the product of the two transpositions $L_1\leftrightarrow L_2$ and $L_1'\leftrightarrow L_2'$ and s_2 is the transposition $L_2\leftrightarrow L_2'$. That is to say,

$$s_1=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad s_2=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 6.4. *The subgroups B and N given in this section constitute a BN-pair of $Sp_4(\mathbb{R})$.*

The proof flows much the same as that of Proposition 6.1.

Note that we must keep the fact that we are working $\mathrm{mod}(T)$ in mind, since

$$s_1s_2s_1s_2=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

squares to $-I_4$, but $-I_4 \cong I_4 \bmod(T)$. A few quick calculations $\bmod(T)$ show that W is non-abelian and of order 8,

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_2s_1s_2s_1\}.$$

This means W must be isomorphic to either the quaternions, or the dihedral group of order 8, which are the only two non-abelian groups of order 8 (see [DF, Table §5.3]). Since the group of quaternions has only 1 element of order 2 and our group is generated by two elements of order 2 our group cannot be the group of quaternions. Thus W is the dihedral group of order 8.

As in §6.1 we now calculate the parabolic subgroups of $Sp_4(\mathbb{R})$. However, to facilitate the flow of this example, the calculations in Table 4 are left to the reader, who should at this point recall Lemma 5.38 to simplify these calculations.

$B_1 = B$	$B_2 = s_1Bs_1$
$B_3 = s_2Bs_2$	$B_4 = s_1s_2Bs_2s_1$
$B_5 = s_2s_1Bs_1s_2$	$B_6 = s_1s_2s_1Bs_1s_2s_1$
$B_7 = s_2s_1s_2Bs_2s_1s_2$	$B_8 = s_2s_1s_2s_1Bs_1s_2s_1s_2$
$P_1 = B\langle s_1\rangle B$	$P_2 = s_2B\langle s_1\rangle Bs_2$
$P_3 = s_1s_2B\langle s_1\rangle Bs_2s_1$	$P_4 = s_2s_1s_2B\langle s_1\rangle Bs_2s_1s_2$
$P_5 = B\langle s_2\rangle B$	$P_6 = s_1B\langle s_2\rangle Bs_1$
$P_7 = s_2s_1B\langle s_2\rangle Bs_1s_2$	$P_8 = s_1s_2s_1B\langle s_2\rangle Bs_1s_2s_1$

Table 4: Parabolic Subgroups of $Sp_4(\mathbb{R})$

The containment relations are found in Table 5.

$B_1, B_2 \subset P_1$	$B_3, B_5 \subset P_2$
$B_4, B_6 \subset P_3$	$B_7, B_8 \subset P_4$
$B_1, B_3 \subset P_5$	$B_2, B_4 \subset P_6$
$B_5, B_7 \subset P_7$	$B_6, B_8 \subset P_8$

Table 5: Containment of the Parabolic Subgroups of $Sp_4(\mathbb{R})$

By ordering via inverse inclusion, we obtain the fundamental apartment depicted in Figure 10.

As in §6.1, we can identify this apartment with either the flags of symplectic subspaces spanned by standard basis vectors stabilized by each parabolic subgroup which is a W-conjugate of a special subgroup, or with the reflection group and its action on the fundamental chamber.

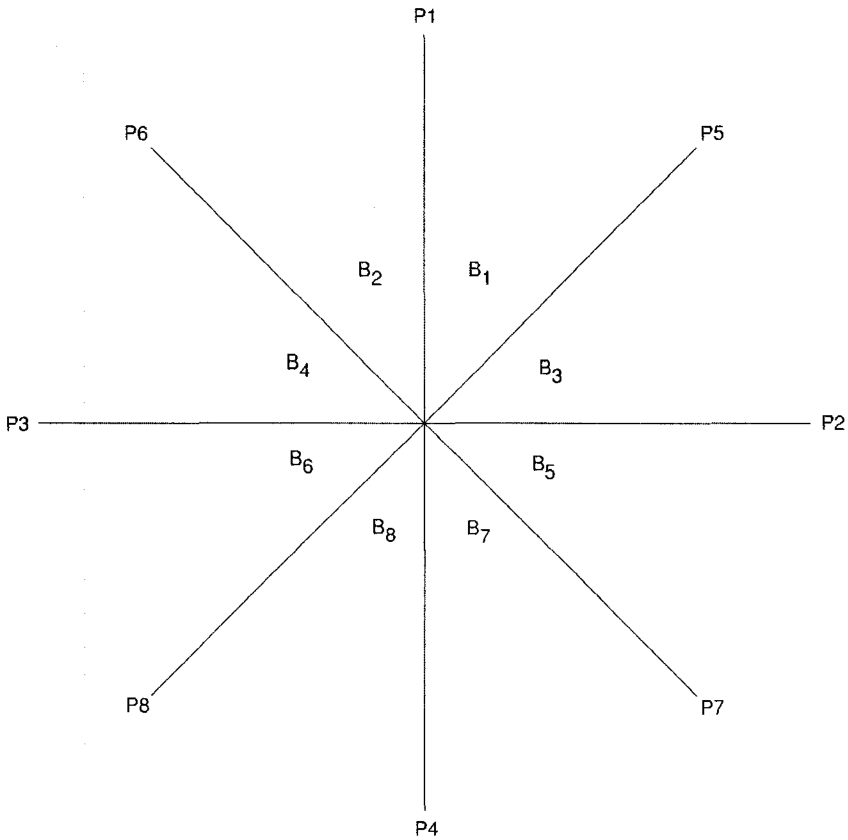

Figure 10: Fundamental apartment for $Sp_4(\mathbb{R})$

Chapter 7

Affine Buildings for p-adic Algebraic Groups

In this chapter we begin in §7.1 by giving a treatment of lattices with the goal being to simplify the construction of a building associated to a p-adic linear group. We generalize this to $SL_n(\mathbb{Q}_p)$ in §7.2. These sections follow [Br, §V.8] closely. In the remaining sections, we will explicitly construct the fundamental apartment associated to specific p-adic algebraic groups. Again, the results of these last sections are well-known to experts but not readily available in the literature.

7.1 Lattice Constructions of Affine Buildings

The nicest way to understand the parabolic subgroups in the case of a classical group over $\mathbb{Q}_p$ with a natural representation on a $\mathbb{Q}_p$-vector space $V = \mathbb{Q}_p^n$ is by understanding which lattices they preserve. The notation used here will follow that used in §4.1.

Lemma 7.1. *$\mathbb{Z}_p$ is a principal ideal domain.*

Proof. Let I be a non-zero ideal of $\mathbb{Z}_p$ and let $n = min\{ord(a)|a \in I\}$. We then have that I contains π^n and every element of I is divisible by π^n. □

As a result of this, we can apply some basic facts about modules over a principal ideal domain. For instance, we will need the fact that a submodule of a free module over a PID is a free module.

Definition 7.2. A *lattice* in V is a $\mathbb{Z}_p$-submodule $L \subset V$ of the form $L = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$ for some basis $\{e_1, \ldots, e_n\}$ of V. If $\{e_1, \ldots, e_n\}$ is the standard basis of V then the resulting lattice is $\mathbb{Z}_p^n$, which will be called the *standard lattice.*

We now continue in an effort to describe our building. Call two lattices L and L' equivalent if $L = \lambda L'$ for some $\lambda \in \mathbb{Q}_p^\times$. In fact, this is an equivalence relation. One should note that the scalar λ can then be taken to be a power of π. Let $[L]$ denote the equivalence class of a lattice L.

Let L be a lattice. Let $\Lambda = [L]$ be the equivalence class of L. If $L' \in \Lambda$, then we have that that $\Lambda = [L] = [L']$ and we call L (or L') a *representative* of the class Λ.

Definition 7.3. Two lattice classes Λ and Λ' are *incident* if they have respective representatives L and L' such that

$$\pi L \subset L' \subset L.$$

Remark. If we have representatives L and L' as in the definition, then the representatives πL and L' satisfy $\pi L' \subset \pi L \subset L'$, which gives us the fact that the incidence relation is symmetric.

7.2 $SL_n(\mathbb{Q}_p)$

One BN-pair of $SL_n(K)$, where K is an arbitrary field, is obtained from Example 4.15. That is, $B \subset SL_n(K)$ is the upper triangular group of $GL_n(K)$ intersected with $SL_n(K)$ and $N \subset SL_n(K)$ is the monomial group of $GL_n(K)$ intersected with $SL_n(K)$. See [Br, §V.5] for a proof of that this choice of B and N is a BN-pair.

Proceeding with this choice of BN-pair gives a finite reflection group W, and consequently a spherical building (as in Chapter 6). However, we can make a finer construction for $SL_n(\mathbb{Q}_p)$.

Consider the following diagram, where $\mathbb{F}_p$ is the finite field of p elements (and is congruent to $\mathbb{Z}_p/p\mathbb{Z}_p$):

$$\begin{array}{ccc} SL_n(\mathbb{Z}_p) & \hookrightarrow & SL_n(\mathbb{Q}_p) \\ \downarrow & & \\ SL_n(\mathbb{F}_p) & & \end{array} .$$

The map $SL_n(\mathbb{Z}_p) \hookrightarrow SL_n(\mathbb{Q}_p)$ is the inclusion map. To understand the map $SL_n(\mathbb{Z}_p) \rightarrow SL_n(\mathbb{F}_p)$, first note also that there is a surjection of $\mathbb{Z}_p$ onto $\mathbb{F}_p$ and the kernel of this map is precisely the non-invertible elements of $\mathbb{Z}_p$. The map $SL_n(\mathbb{Z}_p) \rightarrow SL_n(\mathbb{F}_p)$ is defined by taking the quotient of each matrix entry. We claim that this map is a group homomorphism. First, note that the image is indeed contained in $SL_n(\mathbb{F}_p)$. Also, note that products of elements are mapped to products in the image. This is well defined since multiplying two matrices in $SL_n(\mathbb{Z}_p)$ reduces to multiplying and adding elements of $\mathbb{Z}_p$, and by the rules of valuation, the terms of valuation greater than 0 do not contribute to the leading term of the sum or product.

Lift the subgroup of upper triangular matrices in $SL_n(\mathbb{F}_p)$ to a subgroup, call it B again, of $SL_n(\mathbb{Z}_p)$. This subgroup is not the Borel subgroup of $SL_n(\mathbb{Q}_p)$ considered previously. Its elements are the matrices of determinant 1 in the set

$$\begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \cdots & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times & \cdots & \mathbb{Z}_p \\ \vdots & \vdots & \ddots & \vdots \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \cdots & \mathbb{Z}_p^\times \end{bmatrix}.$$

This subgroup is called an Iwahori subgroup of $SL_n(\mathbb{Q}_p)$, as in [G, §14.1].

Were we to construct the N for our BN-pair in the same way, as an inverse image of a subgroup of $SL_n(\mathbb{F}_p)$, then both B and N would be subgroups of $SL_n(\mathbb{Z}_p)$ and could not possibly generate $SL_n(\mathbb{Q}_p)$. Instead, note the following useful result.

Lemma 7.4. *$SL_n(\mathbb{Q}_p)$ is generated by its monomial subgroup together with $SL_n(\mathbb{Z}_p)$.*

Proof. Let $A \in SL_n(\mathbb{Q}_p)$. Choose a_{ij} to be the matrix coefficient of least valuation in A, in row i and column j. Suppose $d = ord(a_{ij})$. Then, using elementary row and column operations, we see that there are elementary matrices l_1 and r_1 so that l_1Ar_1 is a matrix with zeroes in its ith row and jth column, and element π^d in the (i,j) position. We claim that l_1 and r_1 lie in $SL_n(\mathbb{Z}_p)$.

Namely, for each $k \neq i$, we zero out the (k,j) entry by multiplying on the left by the elementary matrix which has 1s on the diagonal and the element $-a_{kj}a_{ij}^{-1}$ in position (k,i). Since $ord(a_{kj}) \geq ord(a_{ij})$ by hypothesis, each such matrix lies in $SL_n(\mathbb{Z}_p)$. Hence $l_1 \in SL_n(\mathbb{Z}_p)$. The same reasoning applies to the elementary matrices whose product is r_1.

Now repeat this process on the resulting matrix $A_1 = l_1Ar_1$, choosing the element of highest valuation not already used. Since each successive element will lie in a row and a column not previously zeroed out by row and column operations, this process can continue inductively and will terminate after n steps. We arrive at a matrix $A_n = l_nl_{n-1}\cdots l_1Ar_1r_2\cdots r_n = gAg'$ such that $g, g' \in SL_n(\mathbb{Z}_p)$ and A_n is a monomial matrix. Rewriting this yields $A = g'^{-1}A_ng^{-1}$, which shows the result. □

As a consequence of this Lemma, we set N to be the monomial subgroup of $SL_n(\mathbb{Q}_p)$. Our subgroup B contains the upper triangular subgroup of $SL_n(\mathbb{Z}_p)$. Hence, the subgroup of $SL_n(\mathbb{Q}_p)$ generated by B and N contains both the upper and lower triangular subgroups of $SL_n(\mathbb{Z}_p)$. Thus, it contains all elementary matrices of $SL_n(\mathbb{Z}_p)$. By the proof of Lemma 7.4, this subgroup is all of $SL_n(\mathbb{Q}_p)$.

The intersection $T = B \cap N$ consists of all monomial matrices in B. Since each such matrix has determinant equal to 1, and $ord(1) = 0$, it cannot contain a lower triangular entry of B. Hence T is the subgroup of $SL_n(\mathbb{Z}_p)$ composed of diagonal matrices. This is normalized by N.

Let $N(R)$ and $T(R)$ denote the monomial and diagonal subgroups of $SL_n(R)$ respectively where R is any commutative ring. Our N and T above are $N(\mathbb{Q}_p)$ and $T(\mathbb{Z}_p)$, and thus we have that $W = N/T = N(\mathbb{Q}_p)/T(\mathbb{Z}_p) = N(\mathbb{Q}_p)/T(\mathbb{Q}_p) \cdot T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$. The quotient $N(\mathbb{Q}_p)/T(\mathbb{Q}_p)$ is just the symmetric group, which is the Weyl group W_0 of $SL(n)$. Since $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \simeq \mathbb{Z}$ by the valuation map (since the invertible elements of $\mathbb{Z}_p$ are exactly those with valuation equal to 1), it follows that the second quotient

$T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ may be identified with

$$\left\{ \begin{bmatrix} \pi^{x_1} & & & & \\ & \pi^{x_2} & & & \\ & & \ddots & & \\ & & & \pi^{x_{n-1}} & \\ & & & & \pi^{x_n} \end{bmatrix} \middle| \sum_{i=1}^{n} x_i = 0 \right\},$$

or equivalently with $F = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n | \sum_{i=1}^{n} x_i = 0\}$. We note that $W = F \rtimes W_0$ is an affine reflection group, as follows.

The finite reflection group W_0 acts on F (or more generally, on $T(\mathbb{Q}_p)$) by permuting its coordinates; the transposition $(i\ i+1)$ can be thought of as the reflection in the hyperplane orthogonal to $\alpha_i = e_i - e_{i+1}$ since it fixes every element of this hyperplane and sends α_i to its negative [H3, §1.1]. Then we compute that $\alpha_i^\vee = \alpha_i$, and furthermore these vectors form a basis for the lattice F, for $i \in \{1, 2, \cdots, n-1\}$. That W is an affine reflection group now follows from Proposition 3.19.

What we would like next is to identify the set S of generators of W such that (W, S) is a Coxeter system. Let the first $n-1$ of these be the following generators of the symmetric group W_0: s_i for $i < n$ is the permutation matrix in the embedded $SL_2(\mathbb{Z}_p) \subset SL_2(\mathbb{Q}_p) \hookrightarrow SL_n(\mathbb{Q}_p)$ acting on the subspace $[e_i, e_{i+1}]$. As for s_n, take the element of W which is represented by

$$s_n = \begin{bmatrix} 0 & & & & -\pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & 0 \end{bmatrix}. \tag{17}$$

This element is in the embedded $SL_2(\mathbb{Q}_p)$ and acts on the subspace $[e_n, e_1]$. Of course, we need to show that these s_i's constitute a generating set for W. First, note that the subgroup $\langle S \rangle$ contains the symmetric group W_0. Hence it contains an element representing the transposition $(1, n)$. Multiplying s_n by this element gives the element $e_1 - e_n \in F$. Acting on this element by the symmetric group gives the lattice basis

$\{e_1 - e_2, e_3 - e_3, \ldots e_{n-1} - e_n\}$ for F (and hence a generating set). We conclude that we have that $\langle S \rangle = W$.

Proposition 7.5. *The B and N above are a BN-pair.*

Proof. We have already shown Properties (1) and (2) and the beginning of property (3) of a BN-pair above.

Property (3b) can be shown by the simplified property

$$sBw \subseteq BwB \cup BswB.$$

However, we will show the equivalent property

$$wBs \subseteq BwB \cup BwsB.$$

Let $\tilde{s} \in N$ be a representative of s and let $g = wb\tilde{s} \in wBs$. We now have that the chambers wC, wsC, and gC have a common codimension one face. Both of wC and wsC lie in Σ since w stabilizes Σ. Therefore, we have that $\rho(gC) = wC$ or wsC where $\rho = \rho_{\Sigma,C}$ is the retraction of Δ onto the apartment Σ with associated fundamental chamber C. We also know from the proof of Lemma 5.32 that $\rho(gC) = b'gC$ for some $b' \in B$. We thus have that $gC = b'^{-1}wC$ or $b'^{-1}wsC$. This gives us that $g \in BwB \cup BwsB$ as desired, which proves property (3b).

The proof of property (3a) has several cases involved which can be shown technically using row operations to reduce to a 2×2 matrix computation. An example of how to do so may be found in Proposition 6.1. $\square$

7.3 $SL_3(\mathbb{Q}_p)$

Let $G = SL_3(\mathbb{Q}_p)$. To construct the fundamental apartment for G, given the BN-pair of §7.2, we could proceed as in Chapter 6, and explicitly calculate the special subgroups of G. It seems easier, however, to consider instead the flags of lattice classes preserved by the special subgroups, using the ideas developed in §7.1. It is important to note that B preserves flags and that W permutes basis vectors, which will be key in the following arguments.

Remark. For convenience, these lattice classes will be represented by matrices constructed by taking the basis elements of the lattice, representative of the class, as the columns of the matrix, in order. For example, if the basis vectors for a lattice class representative L are $\{\pi e_1, e_2, \pi e_3\}$ (where e_i has a 1 in the i'th position and 0's elsewhere), we will represent it as

$$\begin{bmatrix} \pi e_1 & e_2 & \pi e_3 \end{bmatrix} = (\pi, 1, \pi).$$

The vertices of our apartment are to be identified with lattice classes, with faces corresponding to the incidence geometry on these lattice classes. We also have that the fundamental chamber of our building has vertices $\begin{bmatrix} e_1 & \cdots & e_i & \pi e_{i+1} & \cdots & \pi e_3 \end{bmatrix}$, $i = 1, \ldots, 3$, where e_i denotes the ith standard basis vector written as a column vector. This comes from the fact that these vertices form the full flag

$$\begin{aligned} \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} &\subseteq \begin{bmatrix} e_1 & e_2 & \pi e_3 \end{bmatrix} \\ &\subseteq \begin{bmatrix} e_1 & \pi e_2 & \pi e_3 \end{bmatrix}, \end{aligned}$$

where each lattice class is preserved by B.

The linear part of our affine reflection group, W_0, is isomorphic to S_3 and acts on the standard basis vectors of $\mathbb{Q}_p^3$. These group elements correspond to matrices acting on the matrix forms of the lattices. For example, the element (1 2) will be written as the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in §7.2 there is only one generator of our affine reflection group which is not an element of S_3, which we derive using Equation (17):

$$s_3 = \begin{bmatrix} 0 & 0 & -\pi^{-1} \\ 0 & 1 & 0 \\ \pi & 0 & 0 \end{bmatrix}.$$

Please see Appendix A.1 for an explanation as to how we obtain the equilateral triangle as the shape of our fundamental chamber.

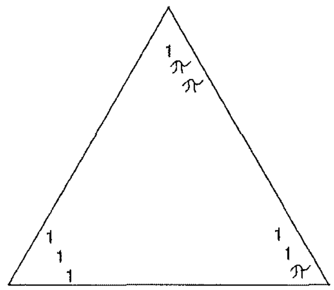

Figure 11: Fundamental chamber for $SL_3(\mathbb{Q}_p)$

At this point, we know one chamber in our apartment, namely, the one depicted in Figure 11. It would be simple enough to tessalate 2-space with this chamber. However, we do not yet know how to identify it, which is what we are interested in. Since we have the generators for our affine reflection group, let us apply them to our fundamental chamber. One sees that each of the three generators is a reflection in one of the three sides of the triangle (which is as expected from Proposition 3.27) as depicted in Figure 12. We thus observe that the three sides of the triangle are contained in hyperplanes which define these reflections.

One may continue to apply the elements of the affine reflection group to obtain the identification of the entire tessalation. However, since there are infinitely many vertices, we will not do so. See Figure 13 for an unidentified diagram of the apartment. We must content ourselves with the ability to identify any chamber in question as we need it by determining the affine reflection group element that moves the fundamental chamber to the chamber in question, and then applying this element to the vertices of the fundamental chamber. However, one thing we shall do is explicitly compute the non-affine part of the apartment, by which we mean the set of chambers obtained from the fundamental chamber via an element in the finite reflection group W_0.

We will now instead identify the vertex previously identified with the identity matrix (corresponding to the lattice class $\left[\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}\right]$) with the origin. One may compute all the affine reflection group elements needed to move the fundamental

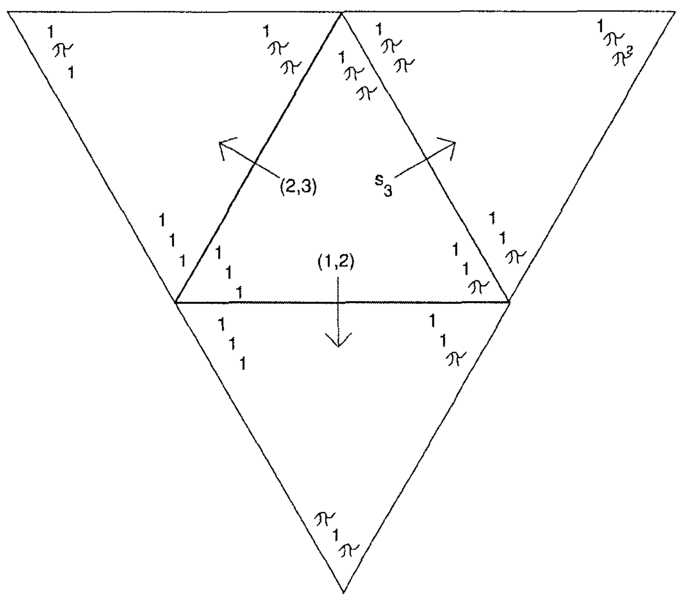

Figure 12: Generators of the affine reflection group for $SL_3(\mathbb{Q}_p)$ acting on the fundamental chamber

chamber to another chamber which has one vertex at the origin and then apply these matrices to the vertices of the fundamental chamber to obtain the non-affine part of the apartment. However, we shall use a different approach.

We know that each chamber in the non-affine part of the apartment has one vertex at the origin, and we also know that the identifications of all three vertices must be incident. One can compile a list of possible chambers with these two properties. One should note that since we have one vertex at the origin, and the origin is not incident with any lattice class represented using π^2's or π^{-2}'s or any higher exponents we are forced to use only 1's and π's (this follows directly from Definition 7.3 as seen in Appendix A.1). Table 6 gives a list of all eligible lattice classes.

We now compile a list of all flags of vertices (where we can only choose lattice classes from Table 6) that are all incident one to the other. Note that each chamber will have a vertex at the origin, a vertex with two 1's and one π, and a vertex with

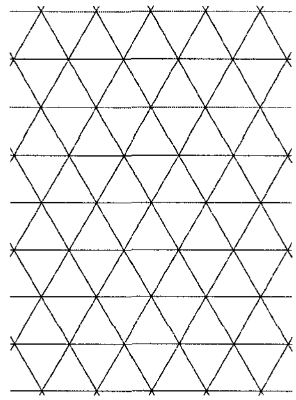

Figure 13: Unlabeled apartment for $SL_3(\mathbb{Q}_p)$

$(1,1,1)$	$(1,1,\pi)$	$(1,\pi,1)$	$(1,\pi,\pi)$
$(\pi,1,1)$	$(\pi,1,\pi)$	$(\pi,\pi,1)$	

Table 6: Eligible Lattice Classes for Lables of Vertices of Chambers Having One Vertex at the Origin

one 1 and two π's. After writing down every such flag one has compiled a list of chambers, of which we will have only 6. A list of these flags is seen in Table 7. We can now fit each of these chambers together since any adjacent chambers will share two vertices in common. The resulting picture can be seen in Figure 14.

7.4 $SL_4(\mathbb{Q}_p)$

We proceed with $SL_4(\mathbb{Q}_p)$ in much the same manner as we did for $SL_3(\mathbb{Q}_p)$. There are some differences we should note. First of all, our lattices are in four dimensions instead of three, and we thus have four basis vectors to consider instead of three. Other changes are that our fundamental chamber has vertices $\begin{bmatrix} e_1 & \cdots & e_i & \pi e_{i+1} & \cdots & \pi e_4 \end{bmatrix}$,

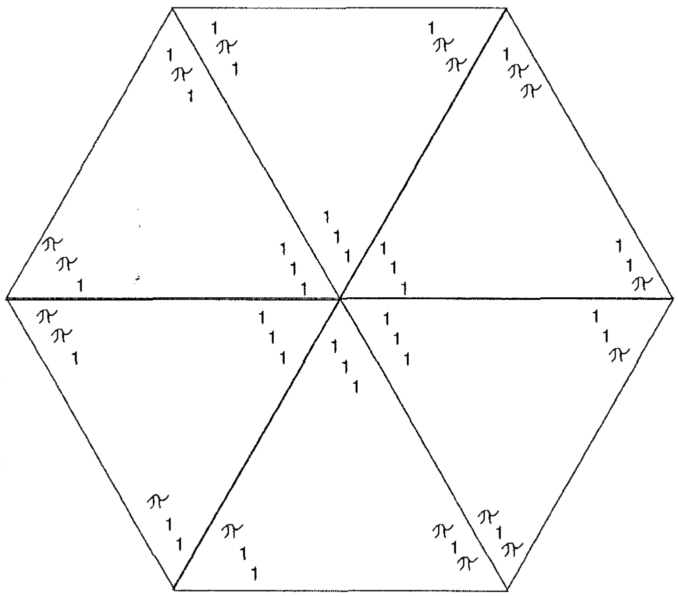

Figure 14: Non-affine portion of an apartment of the building of $SL_3(\mathbb{Q}_p)$

$i = 1, \ldots, 4$ and we now have S_4 as the linear part of our affine reflection group instead of S_3. As with $SL_3(\mathbb{Q}_p)$ we will have one generator of our affine reflection group which does not lie in the linear portion of it, and is in this case denoted by

$$s_4 = \begin{bmatrix} 0 & 0 & 0 & -\pi^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pi & 0 & 0 & 0 \end{bmatrix}.$$

Our next job is to contruct our fundamental chamber. After reading §7.3 one would probably picture our fundamental chamber as an equilateral pyramid since the fundamental chamber for $SL_3(\mathbb{Q}_p)$ was an equilateral triangle. As it turns out, our fundamental chamber isn't as straightforward a shape as an equilateral pyramid, yet possesses a great deal of symmetry. Please see Appendix A.2 for the construction of our fundamental chamber.

$$\begin{array}{ll} (1,1,1) \subseteq (1,1,\pi) \subseteq (1,\pi,\pi) & (1,1,1) \subseteq (1,1,\pi) \subseteq (\pi,1,\pi) \\ (1,1,1) \subseteq (1,\pi,1) \subseteq (1,\pi,\pi) & (1,1,1) \subseteq (1,\pi,1) \subseteq (\pi,\pi,1) \\ (1,1,1) \subseteq (\pi,1,1) \subseteq (\pi,1,\pi) & (1,1,1) \subseteq (\pi,1,1) \subseteq (\pi,\pi,1) \end{array}$$

Table 7: List of full flags starting with $(1,1,1)$

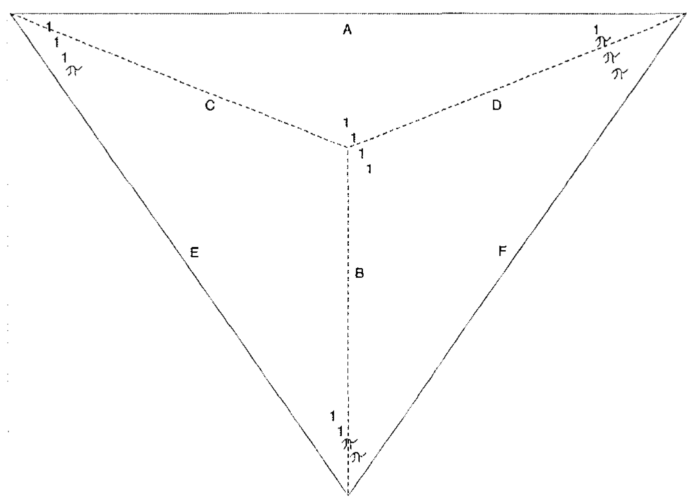

Figure 15: Fundamental chamber for $SL_4(\mathbb{Q}_p)$

We now see how the generators of our affine reflection group affect our fundamental chamber. Please use Figure 15 as a reference for the following discussion as it is a 2-dimensional projection of the 3-dimensional fundamental chamber. One sees that each of the generators reflects the fundamental chamber in one of its faces. The permutation (1 2) reflects in the face with edges labelled as B, C, and E. The permutation (2 3) reflects in the face with edges labelled as A, C, and D. the permutation (3 4) reflects in the face with edges labelled as B, D, and F. Finally, the generator s_4 reflects in the face with edges labelled as A, E, and F.

As in §7.3 the apartment is a tessalation of $\mathbb{R}^3$ by copies of this fundamental chamber. We will proceed by identifying the portion of the apartment generated by the linear part of the affine reflection group with lattice classes. As in §7.3 we only have lattice classes composed of 1's and π's. Using the methods we established in §7.3 one can deduce that there are twenty-four chambers in this portion of the apartment, and we will know all of the lattice flags to which they correspond.

The only difficulty will be in visualizing the non-affine part of the apartment, which consists of the fundamental chamber and all of its images under the action of the finite reflection group W_0. One should recall from Appendix A.2 that the two faces meeting at edge A meet at a 90° angle, and thus the chamber obtained by reflecting in the face with edges A, C, and D will have one face in the same hyperplane as the face with edges A, E, and F and will share the edge A. Thus, if we look at these two faces combined, they form one facet of the geometric shape which is the non-affine part of the apartment, and realize that this same simplification can be made for every pair of chambers sharing a face as these two did, we can simply view the non-affine part as a shape with twelve of these facets instead of a shape with twenty-four of the smaller faces. This shape is a Rhombic Dodecahedron. See Figure 16 for a fully realized non-affine part of the fundamental apartment.

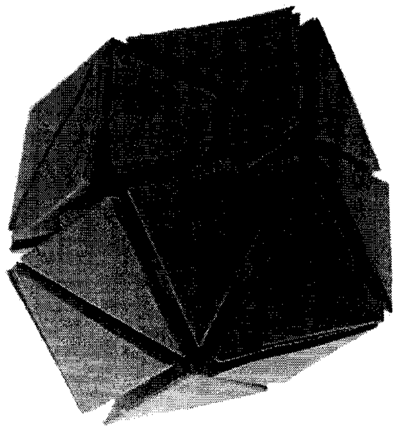

Figure 16: Non-affine part of the fundamental apartment for $SL_4(\mathbb{Q}_p)$

Appendix A

Construction of a Chamber for a Building

A.1 Determination of the Geometry of the Fundamental Chamber of $SL_3(\mathbb{Q}_p)$

We already know that our fundamental chamber has three vertices identified with $[e_1\ e_2\ e_3]$, $[e_1\ e_2\ \pi e_3]$, and $[e_1\ \pi e_2\ \pi e_3]$. Therefore, our chamber is a triangle. We will now attempt to measure the angles of our triangle. We will do this by picking a vertex at which we wish to know the angle, and finding the number of chambers that share that vertex. For example, if only three chambers share the vertex $[e_1\ e_2\ \pi e_3]$, then we know by symmetry that that vertex has angle $360° \div 3 = 120°$.

From now on, we will use the shorthand $[ae_1\ be_2\ ce_3] = (a, b, c)$. Let's start by isolating vertex $(1,\ 1,\ \pi)$. Each chamber is defined by a 3-tuple of vertices. Our fundamental chamber is $(1,\ 1,\ 1)$, $(1,\ 1,\ \pi)$, $(1,\ \pi,\ \pi)$. What we are looking for is all chambers that contain $(1,\ 1,\ \pi)$ as one of its three vertices.

Lemma A.1. *The following lattice classes are all of the lattice classes incident to $(1,\ 1,\ \pi)$:*

$$\begin{array}{ccc} (\pi, 1, \pi) & (1, \pi, \pi) & (1, 1, \pi^2) \\ (\pi, \pi, \pi) & (\pi, 1, \pi^2) & (1, \pi, \pi^2). \end{array}$$

Proof. The manner in which we will do this is by finding all lattice classes which are incident to (1, 1, π) and completing them to full flags.

Recall that the definition of incidence states that two lattice classes are incident if they have representatives L_1 and L_2 which satisfy $\pi L_1 \subset L_2 \subset L_1$. One should note that if one of the components of L_1 and L_2 differ by a power of π greater than 1 and another component is equal, then it is impossible for them to be incident. Therefore, we can change each component of (1, 1, π) by at most one power of π. Also, if we use both π and π^{-1}, that is equivalent (by definition of lattice classes) to using π^2 and 1, which changes by more than 1 power, so we can't use them both. Lastly, if we use π^{-1} and 1 to change (1, 1, π), this is equivalent to using 1 and π, which we will prefer. In conclusion, we can obtain representatives of all lattice classes incident with (1, 1, π) by multiplying each component by either 1 or π. Keep in mind that multiplying all three components by 1, or all three components by π produces the same lattice class.

We thus obtain the following list of lattice classes incident to (1, 1, π),

$$\begin{array}{ccc} (\pi, 1, \pi) & (1, \pi, \pi) & (1, 1, \pi^2) \\ (\pi, \pi, \pi) & (\pi, 1, \pi^2) & (1, \pi, \pi^2), \end{array}$$

which completes the proof. □

We now compare these six lattice classes to each other to determine incidence relations and discover that we have six pairs of two each incident to each other (this computation is left to the reader). Each of these six pairs, together with (1, 1, π), form a chamber containing (1, 1, π). We therefore have six chambers sharing this vertex, and thus the angle at this vertex is $360° \div 6 = 60°$.

We will proceed in exactly the same manner with vertex (1, π, π). We discover the following incident lattice classes:

$$\begin{array}{ccc} (\pi, \pi, \pi) & (1, \pi^2, \pi) & (1, \pi, \pi^2) \\ (\pi, \pi^2, \pi) & (\pi, \pi, \pi^2) & (1, \pi^2, \pi^2). \end{array}$$

The reader can then verify that among these lattice classes we again find six pairs that are incident one to the other, and thus we have a total of six chambers that share the vertex (1, π, π). Therefore we have that the angle at this vertex is again 60°.

Knowing the first two angles allows us to calculate the third angle to be 60°, and we thus have an equilateral triangle.

A.2 Determination of the Geometry of the Fundamental Chamber of $SL_4(\mathbb{Q}_p)$

We know that our fundamental chamber has four vertices identified with $(1,1,1,1)$, $(1, 1, 1, \pi)$, $(1, 1, \pi, \pi)$, and $(1, \pi, \pi, \pi)$. We deduce from this that our shape has four facets, all of which are triangles. We will now measure the angles between the facets of our shape. This will be done almost exactly as in Appendix A.1, but this time we will hold pairs of vertices stationary instead of simply one vertex. Since we have four facets, we will have to measure the angle of six facet intersections.

Let us start by holding $(1, 1, 1, 1)$ and $(1, 1, 1, \pi)$ stationary. Since we're holding two vertices stationary instead of one, we will first find all lattice classes incident with $(1, 1, 1, 1)$ and then all lattice classes incident with $(1, 1, 1, \pi)$, and then intersect these two sets. This will leave us with the following lattice classes incident to both $(1, 1, 1, 1)$ and $(1, 1, 1, \pi)$:

$$\begin{array}{ccc} (1,1,\pi,\pi) & (1,\pi,\pi,\pi) & (1,\pi,1,\pi) \\ (\pi,\pi,1,\pi) & (\pi,1,1,\pi) & (\pi,1,\pi,\pi). \end{array}$$

We need a total of four vertices to realize a chamber, and we've started with two. Therefore, among these six new vertices, we have to find pairs of which that are incident to both of our current vertices (already done) as well as each other. Comparing these lattice classes one to the other, we discover that we have six pairs of vertices incident to each other, and thus we have six chambers that share this particular edge of our chamber. These pairs are

$$(1,1,\pi,\pi) \leftrightarrow (1,\pi,\pi,\pi), \quad (1,1,\pi,\pi) \leftrightarrow (\pi,1,\pi,\pi),$$

$$(1,\pi,1,\pi) \leftrightarrow (1,\pi,\pi,\pi), \quad (1,\pi,1,\pi) \leftrightarrow (\pi,\pi,1,\pi),$$

$$(\pi,1,1,\pi) \leftrightarrow (\pi,\pi,1,\pi), \quad (\pi,1,1,\pi) \leftrightarrow (\pi,1,\pi,\pi).$$

Thus, the two facets of our chamber meeting at (1, 1, 1, 1) and (1, 1, 1, π) meet at a 60° angle.

Using these exact same techniques for each edge of our construct, we discover that the edge with vertices (1, 1, 1, 1) and (1, 1, π, π) yields a 90° angle, the edge with vertices (1, 1, 1, 1) and (1, π, π, π) yields a 60° angle, the edge with vertices (1, 1, 1, π) and (1, π, π, π) yields a 90° angle, the edge with vertices (1, 1, π, π) and (1, π, π, π) yields a 60° angle, and finally, the edge with vertices (1, 1, 1, π) and (1, 1, π, π) yields a 60° angle. It is clearly evident at this point that our shape is not the equilateral pyramid one may have assumed it should have been.

Visualization of the Fundamental Chamber

To visualize our shape, we will assign cartesian co-ordinates to the vertices in $\mathbb{R}^3$. Without loss of generality, let the vertex $A = (1, 1, 1, 1)$ be at the origin. That is to say, let it have the co-ordinate (0, 0, 0). Since we know that the edge with vertices A and $C = (1, 1, \pi, \pi)$ has a 90° angle, we will place C on the positive z-axis, and thus we will have one facet entirely in the xz-plane, and one facet entirely in the yz-plane. Because of this, our two remaining vertices will be contained within the above mentioned planes.

We will place $D = (1, \pi, \pi, \pi)$ in the xz-plane and $B = (1, 1, 1, \pi)$ in the yz-plane. Knowing that the facets ABD and ADC meeting at the edge with vertices A and D have a 60° angle, we know that the normal vectors of the planes containing ABD and ADC have a 60° difference between them. Similarly, both facets DBC and ABD have a 60° angle with both of the facets ADC and ABC.

For the sake of scaling, we will place the vertex B at a height of 1 on the z-axis. Let the equation of the plane containing ABD be

$$P_1 : ax + by + cz = 0$$

which is justifiable since we want this facet to contain the vertex A. That is to say, we want P_1 to contain the origin.

We will be using the dot product between the normal vectors of three planes in play as needed along with the following formula:

$$X \cdot Y = \|X\|\|Y\|cos(\theta).$$

At this moment it is unclear whether we should express the angle between our planes as $60°$ or $-60°$, but we can take comfort in the fact that the cosine of both of these values is $\frac{1}{2}$ and thus we will not concern ourselves with it. Note first that

$$a = (a, b, c) \cdot (1, 0, 0) = \sqrt{a^2 + b^2 + c^2}\frac{1}{2},$$

but since (a, b, c) is a normal vector, it has magnitude 1, and thus

$$a = \frac{1}{2}.$$

Next we have

$$b = (a, b, c) \cdot (0, 1, 0) = \sqrt{a^2 + b^2 + c^2}\frac{1}{2} = \frac{1}{2}.$$

Now that we know both a and b, and that $\sqrt{a^2 + b^2 + c^2} = 1$, we can compute

$$c^2 = \frac{1}{2}.$$

Our only debate then is whether or not c should be positive or negative. If c is positive, then it is impossible to have any points in P_1 that have all three co-ordinates positive and non-zero. However, we have constructed our plane such that the points we are interested in have exactly such co-ordinates, and thus we conclude that c should be negative:

$$c = \frac{-1}{\sqrt{2}}.$$

We now have the exact equation for P_1:

$$P_1 : \frac{x}{2} + \frac{y}{2} + \frac{-z}{\sqrt{2}} = 0.$$

Knowing that D has a y value of 0, and that B has an x value of 0 and a z value of 1, we can compute that $D = (\sqrt{2}z, 0, z)$ and $B = (0, \sqrt{2}, 1)$.

We now have co-ordinates of two of the four vertices. To get the final co-ordinates, we will create another plane, call it P_2. The plane P_2 also has $60°$ angles with both the xz and yz planes, as well as a $90°$ angle with the plane P_1. Furthermore, we know it does not pass through the origin, but it does pass through both B and D, as well as C, which is located somewhere on the positive z-axis.

Let the plane P_2 have the formula

$$P_2 : ax + by + cz = d.$$

By using the same formulas as above, we obtain that

$$a = b = \frac{1}{2}.$$

By using the fact that P_2 has a $90°$ angle with P_1, we obtain

$$\begin{aligned} 0 &= (\frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}}) \cdot (\frac{1}{2}, \frac{1}{2}, c) \\ 0 &= \frac{1}{4} + \frac{1}{4} + \frac{-c}{\sqrt{2}}. \end{aligned}$$

Therefore

$$c = \frac{1}{\sqrt{2}}.$$

By substituting B into the equation for P_2, we obtain

$$d = \sqrt{2},$$

and thus we have obtained the exact equation of P_2,

$$P_2 : \frac{x}{2} + \frac{y}{2} + \frac{z}{\sqrt{2}} = \sqrt{2}.$$

We know C is on the positive z-axis, and thus has x and y co-ordinates both equal to 0. This gives us the fact that its z co-ordinate must be 2, and we have that $C = (0, 0, 2)$.

The equation of P_2 also gives us that $D = (2\sqrt{2} - \sqrt{2}z, 0, z)$. In conjunction with the fact that $D = (\sqrt{2}z, 0, z)$, we have that

$$\begin{aligned} \sqrt{2}z &= 2\sqrt{2} - \sqrt{2}z \\ 2\sqrt{2}z &= 2\sqrt{2} \\ z &= 1, \end{aligned}$$

from which we conclude that $D = (\sqrt{2}, 0, 1)$.

Now that we have the four co-ordinate points of the vertices of our construct, we can use linear algebra to determine that each of the four facets is an isoceles triangle with a hypotenuse length of 2 and two sides of length $\sqrt{3}$ (the height of such a triangle being $\sqrt{2}$).

See Figure 17 for a picture of a fully realized fundamental chamber of $SL_4(\mathbb{Q}_p)$.

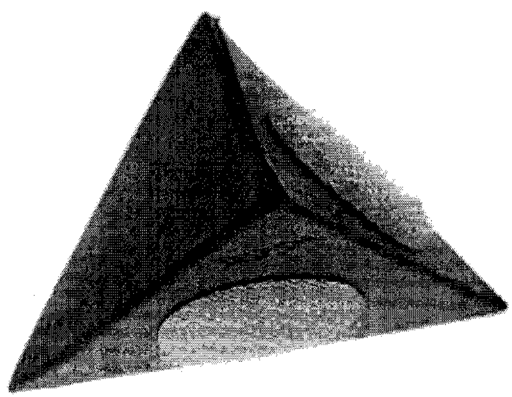

Figure 17: Fundamental chamber for $SL_4(\mathbb{Q}_p)$

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