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THE IDENTITIES OF SYMMETRIC MATRICES

By
WENXIN MA

A thesis submitted to the School of Graduate Studies
and Research of the University of Ottawa in partial
fulfilment of the requirements for the degree of
Doctor of Philosophy in Mathematics

at the
University of Ottawa



Wenxin Ma, Ottawa, Canada, 1991



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Abstract

Let H_n denote the subspace of symmetric matrices of M_n , the full $n \times n$ matrix algebra with coefficients in a field F . Let

$$T_{2n}(x_1, \dots, x_{2n-1}; x_0) = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^{\sigma+i-1} x_{\sigma(1)} \cdots x_0^{(i)} \cdots x_{\sigma(2n-1)},$$

and $e(n) = n$ if n is even, $n+1$ if n is odd. For all $n \geq 1$, $T_{2n}(x_1, \dots, x_{2n-1}; [x_{2n}, x_{2n+1}])$ is an identity of H_n . If $\text{char} F \nmid e(n)!$, $|F| > 2n$ and $n \neq 3$, then any homogeneous polynomial identity of H_n of degree $2n+1$ is a consequence of

$$T_{2n}(x_1, \dots, x_{2n-1}; x_{2n}), T_{2n}(x_1, \dots, x_{2n-1}; [x_{2n}, x_{2n+1}])$$

by substituting $x_i \circ x_j := x_i x_j + x_j x_i$ or x_k for some of their variables or multiplying them by a variable. For $n = 3$, any identity of H_3 of degree 7 is a consequence of

$$T_6(x_1, \dots, x_5; x_6), T_6(x_1, \dots, x_5; [x_6, x_7]), \\ Q(x_1, \dots, x_6), [S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7],$$

where

$$Q(x_1, \dots, x_6) := \sum_{(123), (456)} \{[x_1, x_2][x_3, x_4][x_5, x_6]\},$$

the commutators are the arguments of the triple product $\{abc\} := abc + cba$, the sum is taken over cyclic permutations of 1 2 3 and 4 5 6, and S_n is the standard polynomial.

To prove the results, a partial ordering on the homogeneous elements of the free associative algebra $F[X]$ over field F with noncommuting generators $X = \{x_1, x_2, \dots\}$ is defined. Let f be an element of $F[X]$ and n the maximum of the degrees of the variables and the multiplicities of the degrees in f . If f is homogeneous and $\text{char} F \nmid n!$ then f can be decomposed into a sum of two polynomials f_0 and f_1 such that for $0 < m \leq n$, f_0 is either symmetric or skew-symmetric in all its arguments of degree m according as m is even or

odd, and f_1 is a consequence of polynomials of lower type than f . Osborn's Theorem about the symmetry of the absolutely irreducible polynomial identities is obtained as a corollary.

The method we set up here is applicable not only to searching for identities of matrices but also to find the identities of arbitrary algebras.

Dedication

**To the memory of my mother
and**

**To my
father,
wife,
and my son
for all your love and support.**

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Chapter 0

Introduction

0.1 The Identities of M_n

Let F be a field, $F[X]$ (respectively $FJ[X]$) denote the free associative (respectively Jordan) algebra over F with generators $X = \{x_1, x_2, \dots\}$, a countable set of elements $x_1, x_2, \dots$. Let $f \in F[X]$. Then $f = f(x_1, \dots, x_n)$ is called a polynomial. Let A be an associative algebra over F and $a_1, a_2, \dots, a_n \in A$. By the universal property of the free algebra $F[X]$, there exists a homomorphism θ from $F[X]$ to A such that $\theta(x_i) = a_i$ for $i = 1, 2, \dots, n$ and $\theta(x_j) = 0$ for $j > n$. Denote $\theta(f)$ by $f(a_1, a_2, \dots, a_n)$. The polynomial f is called an **identity** of A if $f(a_1, a_2, \dots, a_n) = 0$ for any choice of $a_1, a_2, \dots, a_n \in A$.

Let M_n denote the $n \times n$ matrix algebra with coefficients in the field F , H_n (respectively K_n) denote the subspace of symmetric (respectively skew symmetric) matrices of M_n .

Let

$$S_k(x_1, x_2, \dots, x_k) := \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is the symmetric group on k objects and $(-1)^\sigma$ the sign of the permutation σ . $S_k(x_1, x_2, \dots, x_k)$ is a multilinear alternating polynomial of degree k . It is called the **standard polynomial**.

In 1950, Amitsur and Levitzki determined all polynomial identities of M_n of degree $\leq 2n$ in [AL, Theorem 1 and 3].

Theorem 0.1 [Amitsur, Levitzki]

1. M_n has no identity of degree less than $2n$.
2. $S_{2n}(x_1, x_2, \dots, x_{2n})$ is an identity of M_n .
3. If $|F| > 2$ or $n > 2$, any identity of M_n of degree $2n$ is a scalar multiple of the standard identity $S_{2n}(x_1, x_2, \dots, x_{2n})$.

Since $S_{2n}(x_1, \dots, x_{2n})$ is an identity of M_n , obviously the polynomials $x_{2n+1}S_{2n}(x_1, \dots, x_{2n})$, $S_{2n}(x_1, \dots, x_{2n})x_{2n+1}$ and $S_{2n}(x_1, \dots, x_{2n+1}x_i, \dots, x_{2n})$ are identities of M_n of degree $2n + 1$. Naturally one may ask the question: Do all identities of M_n of degree $2n + 1$ come from S_{2n} in this way? It was answered positively by Leron in 1973. He proved the following theorem in [LE]:

Theorem 0.2 [Leron] *If the characteristic of the field is 0 then every identity of M_n of degree $2n + 1$ is a consequence of the standard identity S_{2n} for $n > 2$, and every identity of M_2 of degree 5 is a consequence of the standard identity S_4 and $[[x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2], x_5]$.*

It was also proved by Drensky and Kasparian in [DK] that all identities of M_3 of degree 8 are obtained from S_6 by substituting some variables of S_6 or multiplying S_6 by some variables.

0.2 The identities of H_n

In 1973, Razmyslov [RZ1] introduced the concept of **weak identities**, namely, if V is a subspace of an associative algebra A over a field F . A polynomial $f \in F[X]$ is a weak identity on V if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_1, a_2, \dots, a_n \in V$. Since H_n plays an important role in Jordan theory we are interested in the weak identities of H_n as well as Jordan identities of H_n , a weak identity of H_n which is also in $FJ[X]$.

Let $[a, b]$ denote the commutator $ab - ba$ and $\{abc\}$ the triple product $abc + cba$. Let

$$Q(x_1, \dots, x_6) := \sum_{(123), (456)} \{[x_1, x_2][x_3, x_4][x_5, x_6]\},$$

where the commutators are the arguments of the triple product and the sum is taken over cyclic permutations of 1 2 3 and 4 5 6.

Let

$$T_k^i := \sum_{\substack{\sigma \in S_k \\ \sigma^{-1}(i) \equiv 1, 2 \pmod 4}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}, \quad 1 \leq i \leq k,$$

S_k the symmetric group on k objects and $(-1)^\sigma$ the sign of the permutation σ .

We also define

$$\begin{aligned} T_{2n}(x_1, \dots, x_{2n-1}; x_{2n}) &:= T_{2n}^{2n}(x_1, \dots, x_{2n-1}, x_{2n}), \\ \tilde{T}_{2n}(x_1, \dots, x_{2n-1}; x_{2n}) &:= S_{2n}(x_{2n}, x_1, \dots, x_{2n-1}) - T_{2n}(x_1, \dots, x_{2n-1}; x_{2n}). \end{aligned}$$

In the paper [MR], M. L. Racine and I have determined all the weak identities of minimal degree for the subspace H_n of symmetric matrices of the full matrix algebra M_n . The following theorem was proved:

Theorem 0.3 [MR] *Let F be a field of arbitrary characteristic. For all $n \geq 1$*

1. [Slin'ko] H_n has no weak identity of degree less than $2n$.
2. The polynomials $T_{2n}^i(x_1, x_2, \dots, x_{2n})$ are weak identities of H_n and $Q(x_1, \dots, x_6)$ is a weak identity of H_3 .
3. Let $e(n) := n$ if n is even, $n + 1$ if n is odd. If $\text{char } F \nmid e(n)!$ and $|F| > 2n$, then for $n \neq 3$ all weak identities of H_n of degree $2n$ are consequences of T_{2n}^1 . If $n = 3$, then all identities of degree 6 of H_3 are consequences of T_6^1 and Q .
4. There is no Jordan identity of degree $2n$ of H_n .

The results in [MR] are analogous to Amitsur and Levitzki's results [AL, Theorem 1 and 3]. Since H_n is not closed under the usual matrix product but the circle product $x \circ y := xy + yx$ is closed in H_n , in analogy to Leron's results we may ask the question: Do all weak identities of H_n of degree $2n + 1$ come from T_{2n} by substituting $x \circ y := xy + yx$ for some variables or multiplying them by a variable for $n \neq 3$? For $n = 3$ do all identities of H_3 of degree 7

come from T_6 and Q in this way? This is the topic of this thesis. The answer is no! Because in Chapter 2 we shall get a new identity:

$$T_{2n}(x_1, x_2, \dots, x_{2n-1}; [x_{2n}, x_{2n+1}]),$$

and for H_3 we also have a new identity

$$K(x_1, x_2, \dots, x_7) := \{S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7\}.$$

Nevertheless, the weak identities of H_n of degree $2n + 1$ will be determined. The key step is an induction argument on n to be found in Chapter 6. Since the identities of H_3 do not follow the general scheme, the induction must begin with $n = 4$. This forces us to study the identities of H_2 , H_3 and H_4 very carefully (see Chapter 3, 4 and 5).

Let $f(x_1, \dots, x_m) \in F[X]$ and

$$f = \sum_{\sigma} \alpha_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(m)}.$$

If f is an identity of H_n then we substitute an element of the form $e_{ji} + e_{ji}$ for each x_k in f , $f(x_1, \dots, x_m) = 0$, where e_{ij} is the usual matrix unit. This yields some homogeneous linear equations which have the α_{σ} as unknowns. So, in principle we can do as many substitutions as we need to determine the coefficients α_{σ} . But in practice, it is impossible to get results even using a computer because the number of coefficients α_{σ} is very big. Therefore we have to find ways to cut down the numbers of the unknown. One common way for doing this is following decomposition:

$$f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*)$$

is a sum of a symmetric identity and a skew symmetric identity, where $*$ is the involution of $F[X]$ defined by $x_i^* = x_i$, $\forall x_i \in X$. So we may assume that $f^* = \pm f$. Thus, every monomial $x_{\sigma(1)} \cdots x_{\sigma(m)}$ and its reverse $x_{\sigma(m)} \cdots x_{\sigma(1)}$ have the same coefficient in f up to sign. Hence half of the coefficients are cut off. Usually this is still not enough. In Chapter 1, we follow Marshall Osborn's definition of a partial ordering on the set of homogeneous polynomials of $F[X]$, extend Marshall Osborn's result to set up a new method of searching for identities of matrices. Then we use it to determine all the weak identities of H_n of degree $2n + 1$ (Chapter 3 to Chapter 6).

Throughout the paper F always denotes a field. The polynomials are in the noncommuting free associative algebra $F[X]$, where X is a set of countable noncommuting indeterminates $x_1, x_2, \dots$, and identities mentioned are homogeneous weak identities except when otherwise noted.

Chapter 1

A Decomposition of Elements of the Free Algebra

1.1 Introduction

In [OS], Osborn introduced a partial ordering on the free nonassociative algebra $F\langle X \rangle$ over a field F and proved the following interesting result: if n is a positive integer with $\text{char} F \nmid n!$, and f is an identity of a not necessarily associative algebra A over F such that A has no identity of type lower than f in the partial ordering, then f is either symmetric or skew symmetric in its arguments of degree n , depending on whether n is even or odd. This theorem was used in [MR] to determine the identities of degree $2n$ of the space of $n \times n$ symmetric matrices. However the assumption that A has no identity of lower type limits the use of the result in many cases. In the present chapter we remove the restriction that A has no identities of lower type and consider polynomials at large, not only polynomial identities. Osborn's Theorem follows as a consequence.

1.2 A Partial Order, Δ -operators and Weak T-ideals

Following Osborn [OS, p.78] we introduce a partial ordering on the set of homogeneous polynomials in $F[X]$, the free associative algebra in infinitely many noncommuting variables $X = \{x_1, x_2, \dots\}$. The elements of $F[X]$ are called polynomials. If $p(x_1, \dots, x_m)$ is a homogeneous polynomial of degree n , we say that it is of **type** $[n_1, \dots, n_m]$ if n_j is the

degree of x_j in p and $n_m \neq 0$ but $n_j = 0$ for $j > m$. Define $[[n_1, \dots, n_m]]$ to be $[n_{i_1}, \dots, n_{i_m}]$, where $n_{i_1} \geq n_{i_2} \geq \dots \geq n_{i_m}$ and

$$\{n_{i_1}, \dots, n_{i_m}\} = \{n_1, \dots, n_m\}$$

as multisets. Let p' be another homogeneous polynomial of degree n' and of type $[n'_1, \dots, n'_{m'}]$. If

$$[[n'_1, n'_2, \dots, n'_{m'}]] = [n'_{j_1}, n'_{j_2}, \dots, n'_{j_{m'}}],$$

then p is lower than p' in the **partial ordering** if and only if either (i) $n < n'$ or (ii) $n = n'$ and $n_{i_k} > n'_{j_k}$ for the first integer k such that $n_{i_k} \neq n'_{j_k}$. Otherwise the two polynomials are not comparable. If an integer n_j is repeated k times, we shall denote this by an exponent; for example $[3, 2^2, 1^3]$ means $[3, 2, 2, 1, 1, 1]$.

Let $f(x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_m)$ be a homogeneous polynomial of type $[n_1, n_2, \dots, n_m]$ with coefficients in F . Then we define the polynomial

$$\begin{aligned} f'(x_1, \dots, x_i, \dots, x_j, \dots, x_m) &:= f(x_1, \dots, x_i + x_j, \dots, x_j, \dots, x_m) \\ &= f(x_1, \dots, x_i, \dots, x_j, \dots, x_m) + f_1 + \dots + f_{n_i}, \end{aligned} \quad (1.1)$$

where f_k is the homogeneous component of f' of type $[n_1, \dots, n_i - k, \dots, n_j + k, \dots, n_m]$. Following [ZSSS, p.9] we define

$$\Delta^k(x_i, x_j)f := f_k, \quad k = 1, 2, \dots, n_i, \quad \Delta^0(x_i, x_j)f := f.$$

The mapping Δ^k is called **Δ -operator** (or a **derivation**) and $\Delta^k(x_i, x_j)f$ is called a **partial linearization** of f . If $n_i > 0$ and $n_j = 0$ (which is allowed in the above definition), $\Delta^1(x_i, x_j)f$ is of degree $n_i - 1$ in x_i and 1 in x_j . Therefore if enough variables of degree 0 were present one can linearize f using successive Δ -operators. We denote by Δf the set of all polynomials in $F[X]$ which can be obtained from f by means of repeated Δ -operations:

$$\Delta f = \{g \in F[X] \mid g = \Delta^{j_1}(x_{i_1}, x_{k_1}) \cdots \Delta^{j_s}(x_{i_s}, x_{k_s})f\}.$$

Each f_k is obtained by setting k x_i 's to be x_j in f , and x_i, x_j have degree $n_i - k, n_j + k$ in f_k respectively.

It is well-known that the set $T(A)$ of all polynomial identities of an F -algebra is a T -ideal, i.e. an ideal of $F[X]$ which is invariant under all endomorphisms of $F[X]$. However, this is no longer true for weak identities. So we introduce the concept of weak T -ideals.

Definition 1.2.1 An ideal W of $F[X]$ is called a weak T -ideal if W is invariant under every linear mapping $x_i \mapsto \sum \alpha_{ij} x_j$, for $\alpha_{ij} \in F$.

If S is a subset of $F[X]$ then the smallest weak T -ideal containing S in $F[X]$ is called the weak T -ideal generated by S and denoted by $\langle S \rangle$.

One basic example of weak T -ideal is the ideal $T(A, V)$ consisting of all weak polynomial identities on a subspace V of an algebra A .

Using a standard Vandermonde argument (e.g., [ZSSS, p.12, Theorem 5]) one can prove

Proposition 1.2.1 Let $f(\dots, x_i, \dots, x_j, \dots) \in F[X]$ be a homogeneous polynomial of type $[\dots, n_i, \dots, n_j, \dots]$. If $|F| > n_i$ then $\Delta^k(x_i, x_j)f \in \langle f \rangle$, $k = 0, 1, \dots, n_i$. In particular, if $f \in T(A, V)$ then $\Delta^k(x_i, x_j)f \in T(A, V)$ for $k = 0, 1, \dots, n_i$.

Let $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ be homogeneous polynomials of $F[X]$. We shall say that g is a consequence of (or comes from) the polynomial f if g belongs to the vector space $\text{Span}\{\Delta f\}$.

Since a partial linearization of an identity f on some algebra is not necessarily an identity, the polynomials in Δf need not be identities either. But from Proposition 1.2.1, if F contains enough elements then Δf lies in the weak T -ideal $\langle f \rangle$, and hence if g comes from f then g is a linear combination of identities in $\langle f \rangle$.

1.3 A Decomposition of Elements of $F[X]$

As in [ZSSS], $F[X]$ has a decomposition into a direct sum of subspaces $V^{[n_1, \dots, n_k]}[X]$ consisting of the homogeneous polynomials of type $[n_1, \dots, n_k]$. In this section we shall give a decomposition of the elements of $V^{[n_1, \dots, n_k]}[X]$ which is useful in the study of identities.

Theorem 1.1 Let n be a positive integer and let p be a homogeneous polynomial with coefficients in a field F of characteristic not dividing $n!$. Let x and y be arguments of degree n . Define p_1 by

$$p_1 := p(\dots, y, \dots, x, \dots) - (-1)^n p(\dots, x, \dots, y, \dots).$$

Then the following statements hold:

1. The polynomial p_1 comes from a polynomial of lower type than p .
2. If there exist two variables in p either of even degree in which p is skew symmetric, or of odd degree in which p is symmetric, then $p = \frac{1}{2}p_1$ comes from a polynomial of lower type.
3. If $|F| > 2n - 1$ and p is an identity, then p_1 is a linear combination of identities of type lower than p .
4. If $\Delta(y, x)p = 0$ then p is either symmetric or skew symmetric in x and y depending on whether n is even or odd.

Proof. Define

$$\tilde{p}_1 := \alpha_0 f_n + \alpha_1 f_{n-1} + \dots + \alpha_{n-1} f_1,$$

where $\alpha_k = (-1)^k \frac{k!}{n(n-1)\dots(n-k)}$, $0 \leq k \leq n-1$,

$$\begin{aligned} f_1 &:= \Delta^0(y, x) \Delta^1(x, y) \Delta^1(y, x) p, \\ f_2 &:= \Delta^1(y, x) \Delta^2(x, y) \Delta^1(y, x) p - \binom{n}{1} f_1, \\ f_{m+1} &:= \Delta^m(y, x) \Delta^{m+1}(x, y) \Delta^1(y, x) p - \binom{n}{m} f_1 - \binom{n-1}{m-1} f_2 - \dots \\ &\quad - \binom{n-m+1}{1} f_m. \end{aligned} \tag{1.2}$$

The polynomial $\tilde{p}_1$ comes from $\Delta^1(y, x)p$ which is of lower type than p .

If $|F| > 2n - 1$ and p is an identity, then for each i , $i = 1, \dots, n$, $f_i \in \langle \Delta^1(y, x)p \rangle$ by Proposition 1.2.1 and hence $\tilde{p}_1$ is a linear combination of consequences of lower type identities f_i . Thus, to prove 1) and 3) of the theorem it suffices to show that

$$p(\dots, y, \dots, x \dots) = (-1)^n p(\dots, x, \dots, y, \dots) + \tilde{p}_1 \tag{1.3}$$

Since $\Delta^k(x, y)$ and $\Delta^m(y, x)$ change only the variables x and y , the polynomials $p(\dots, y, \dots, x, \dots)$, $(-1)^n p(\dots, x, \dots, y, \dots)$ and $\tilde{p}_1$ are of the same type. So, they are linear combinations of the same monomials. Thus, to show (1.3) it suffices to show that for

each monomial M , the coefficient of M in $p(\dots, y, \dots, x, \dots)$ and the coefficient of M in $(-1)^n p(\dots, x, \dots, y, \dots) + \bar{p}_1$ are the same.

First, we define an equivalence relation on the set of monomials occurring in p : $M_1 \sim M_2$ if M_2 is obtainable from M_1 by permuting some x 's and y 's and leaving the other variables fixed. We denote by $[M]$ the equivalence class of M . Let I be the set of all associative words in x and y involving exactly n x 's and n y 's. Setting the variables, other than x and y , equal to 1 in an element of $[M]$ yields a bijection from $[M]$ to I ; so we may write $[M] = \{M_i | i \in I\}$ and in particular, $M = M_w$ for some $w \in I$. Let $I^{(k)}, k = -1, 0, 1, \dots, n$, be the set of all associative words which have $n + k$ y 's and $n - k$ x 's. Then $I = I^{(0)}$ and $I' := I^{(-1)}$ is the set of all associative words in x and y involving exactly $n + 1$ x 's and $n - 1$ y 's. For $i, j \in I' \cup I \cup I^{(1)} \cup \dots \cup I^{(n)}$, we define $i \cdot j$ to be the number of positions in which both i and j have a y .

Let c_i be the coefficient of M_i in p and r_j be the coefficient of M_j in $p' := \Delta^1(y, x)p$. Then for each $j \in I'$ the total coefficient of M_j in p' will be the sum of the $n + 1$ coefficients c_i where i runs over all those elements of I such that $i \cdot j = n - 1$, i.e.

$$r_j = \sum_{i: j \cdot i = n-1} c_i, \quad \forall j \in I'. \quad (1.4)$$

If $k = j \cdot w$ then every i occurring in (1.4) satisfies either $i \cdot w = k$ or $i \cdot w = k + 1$. Indeed since j has $n - 1$ elements y 's and $i \cdot j = n - 1$, i has a y in a given position whenever j does. So, in the positions where both j and w have a y , i has a y also. Therefore $j \cdot w = k$ implies $i \cdot w$ is at least k . But i has only one more y than j , so $i \cdot w \leq k + 1$. Each $i \in I$ satisfying $i \cdot w = k$ occurs in exactly $n - k$ equations of type (1.4) for which $j \cdot w = k$. Indeed i occurs in (1.4) iff M_i becomes M_j when replacing one y by an x in M_i ; since $i \cdot w = k$, if we put one y in M_i which occurs in the same position in w to be x to get M_j then $j \cdot w = k - 1$ not k . This can't happen so we have only $n - k$ choices and each one produces distinct M_j 's, so c_i occurs $n - k$ times in the set of equations of type (1.4) satisfying $j \cdot w = k$. Each $i \in I$ satisfying $i \cdot w = k + 1$ occurs in exactly $k + 1$ equations of type (1.4) satisfying $j \cdot w = k$, for, this time, we can only choose those y 's which occur in the same position in i and w in order to get $j \cdot w = k$.

Now, we add up all equations of type (1.4) satisfying $j \cdot w = k$ to get

$$\sum_{j \cdot w = k} r_j = (n - k)C_k + (k + 1)C_{k+1}, \quad k = 0, 1, 2, \dots, n - 1, \quad (1.5)$$

where, for $0 \leq m \leq n$, $C_m := \sum_{i \cdot w = m} c_i$, with c_i occuring in an equation of type (1.4) satisfying $j \cdot w = k$. Substituting the n equations of (1.5) into one another yields

$$\begin{aligned} C_0 &= (-1)^n C_n + (-1)^{n-1} \frac{(n-1)!}{n(n-1) \dots 1} \sum_{j \cdot w = n-1} r_j + (-1)^{n-2} \frac{(n-2)!}{n(n-1) \dots 2} \sum_{j \cdot w = n-2} r_j + \dots \\ &\quad - \frac{1}{n(n-1)} \sum_{j \cdot w = 1} r_j + \frac{1}{n} \sum_{j \cdot w = 0} r_j. \end{aligned}$$

We know that $C_n = c_w$ and $C_0 = c_{w'}$, where w' is the word obtained from w by interchanging all the x 's and y 's. Thus we have that

$$c_{w'} = (-1)^n c_w + \alpha_{n-1} \beta_{n-1} + \dots + \alpha_0 \beta_0,$$

where $\alpha_k = (-1)^k \frac{k!}{n \dots (n-k)}$ and $\beta_k = \sum_{j \cdot w = k} r_j$, $k = 0, 1, \dots, n - 1$. Next we show that $\sum_{i=0}^{n-1} \alpha_i \beta_i$ is the coefficient of the monomial M_w in $\bar{p}_1$, and therefore $\bar{p}_1$ comes from p' . By the definition of $\bar{p}_1$, it suffices to show that each β_{n-k} is the coefficient of M_w in f_k .

For $v, u, u' \in I \cup I' \cup I^{(1)} \cup \dots \cup I^{(n)}$ we define $\#(v) :=$ the degree of y in v , and we say that $u \leq u'$ if whenever u has a y in a given position, so does u' . Define $u \cup u'$ to be the word which has a y in a given position iff u or u' does (e.g. if $u = xxyxyy$ and $u' = yxyxxx$ then $u \cup u' = yxyxyy$). From the definition we have that if $u \leq t$ and $u' \leq t$ then $u \cup u' \leq t$ and $\#(u \cup u') \leq \#(t)$.

Next let us consider β_{n-1} . Since f_1 is obtained by setting one x equal to y in p' , the coefficient of M_w in f_1 is

$$\gamma_w = \sum_{j \cdot w = n-1} r_j = \beta_{n-1}.$$

Suppose $\beta_{n-1}, \beta_{n-2}, \dots, \beta_{n-m}$ are the coefficients of M_w in $f_1, f_2, \dots, f_m$ respectively. Now for β_{n-m-1} putting $m+1$ elements x 's to be y in $\Delta^1(y, x)p$ yields the polynomial $\Delta^{m+1}(x, y)\Delta^1(y, x)p$, then putting m elements y 's to be x in $\Delta^{m+1}(x, y)\Delta^1(y, x)p$ we get the polynomial $\Delta^m(y, x)\Delta^{m+1}(x, y)\Delta^1(y, x)p$. In $\Delta^{m+1}(x, y)\Delta^1(y, x)p$, the monomial M_t has coefficient $\lambda_t = \sum_{j \cdot t = n-1} r_j$, since $\#(j) = n - 1$ and $\#(t) = n + m = (n - 1) + (m + 1)$.

(If $j \cdot t < n - 1$ then putting $m + 1$ x 's to be y in M_j cannot yield M_t which has $n + m$ y 's). Similarly, in $\Delta^m(y, x)\Delta^{m+1}(x, y)\Delta^1(y, x)p$, M_w has coefficient $\mu_w = \sum_{t \in I^{(m)}, t \cdot w = n} \lambda_t$, since $\#(t) = n + m$ and $\#(w) = n$. Thus

$$\mu_w = \sum_{t \in I^{(m)}, t \cdot w = n} \left(\sum_{j \in I', j \cdot t = n-1} r_j \right) \quad (1.6)$$

We claim that

$$(*) \quad \{j \in I' \mid j \cdot t = n - 1 \text{ for some } t \in I^{(m)} \text{ and } t \cdot w = n\} \\ = \{j \in I' \mid n - m - 1 \leq j \cdot w \leq n - 1\}.$$

For if $j \in I'$ with $j \cdot t = n - 1$, $\#(t) = n + m$ and $t \cdot w = n$ then $j \leq t$, $w \leq t$ and $j \cup w \leq t$. Since $\#(j) = n - 1$, $j \cdot w \leq n - 1$. If $j \cdot w \leq n - m - 2$ then $j \cup w \leq t$ implies that $\#(j \cup w) \leq n + m$. But

$$\#(j \cup w) = n - 1 + (n - j \cdot w) \geq n - 1 + n - n + m + 2 = n + m + 1,$$

a contradiction. On the other hand, if $j \cdot w = n - k$, $k = 1, 2, \dots, m + 1$ then

$$\#(j \cup w) = n - 1 + n - (n - k) = n - 1 + k \leq n + m,$$

therefore there exists a word $t \in I^{(m)}$ such that $j \cup w \leq t$. Since $j \leq t$ and $w \leq t$, $j \cdot t = \#(j) = n - 1$ and $w \cdot t = \#(w) = n$. This establishes $(*)$. It also means that r_j occurs in (1.6) iff $j \cdot w = n - k$ for some k with $1 \leq k \leq n - m - 1$.

Each r_j may occur several times in (1.6). We group the r_j 's in (1.6) according as $j \cdot w = n - k$ for each k . We claim that

$$(**) \quad \text{each } r_j \text{ with } j \in I' \text{ satisfying } j \cdot w = n - k \text{ occurs } \binom{n - k + 1}{m - k + 1} \text{ times in (1.6).}$$

Indeed, the number of the occurrences of r_j in (1.6) is exactly the number of words $t \in I^{(m)}$ such that $t \cdot w = n$ and $j \cdot t = n - 1$. That is $t \in I^{(m)}$ such that $w \cup j \leq t$ since $\#(j) = n - 1$. Now, if $w \cdot j = n - k$ then

$$\#(w \cup j) = n + n - 1 - n + k = n + k - 1.$$

So, the word t obtained by changing $m - k + 1$ x 's in $w \cup j$ to be y 's belongs to $I^{(m)}$ and $w \cup j \leq t$. However, the number of choices for such a word t is $2n - (n + k - 1)$ choose $m - k + 1$, that is $\binom{n - k + 1}{m - k + 1}$. Hence the $(**)$ holds and

$$\begin{aligned} \mu_w = & \sum_{t: w \leq t} \sum_{j: t = n-1} r_j = \binom{n}{m} \sum_{j: w \leq n-1} r_j + \binom{n-1}{m-1} \sum_{j: w \leq n-2} r_j + \dots + \\ & \binom{n-m+1}{1} \sum_{j: w \leq n-m} r_j + \sum_{j: w \leq n-m-1} r_j. \end{aligned}$$

Since $\beta_{n-k} = \sum_{j: w \leq n-k} r_j$,

$$\beta_{n-m-1} = \mu_w - \binom{n}{m} \beta_{n-1} - \binom{n-1}{m-1} \beta_{n-2} - \dots - \binom{n-m-1}{1} \beta_{n-m}.$$

Thus, by the induction hypothesis and the definition of f_{m+1} , β_{n-m-1} is the coefficient of M_w in f_{m+1} .

From above we know that $\sum \alpha_k \beta_k$ is the coefficient of M_w in $\tilde{p}_1$. We also know that c_w and $(-1)^n c_w$ are the coefficients of M_w in $p(\dots, y, \dots, x, \dots)$ and $(-1)^n p(\dots, x, \dots, y, \dots)$ respectively. Thus (1.3) holds. So 1) and 3) are proved. The second statement of the theorem follows easily from (1.3).

If $\Delta(y, x)p = 0$ then $\tilde{p}_1 = 0$ by the definition of $\tilde{p}_1$. Hence $p_1 = 0$, which implies that p is either symmetric or skew symmetric in x and y depending on whether n is even or odd. ■

Theorem 1.1 is also true for the free nonassociative algebra. The proof is exactly the same except the definition of the mapping from $[M]$ to I . In this case, erasing the brackets and the variables, other than x and y , in an element of $[M]$ yields a bijection from $[M]$ to I since elements in $[M]$ have the same distribution of the brackets. Thus from Theorem 1.1 we have

Corollary 1.3.1 [Osborn's Theorem]. *Let p be a homogeneous identity in a nonassociative algebra A over a field F of characteristic not dividing $n!$. If A has no identity of type lower than p then p is either symmetric or skew symmetric in its arguments of degree n , depending on whether n is even or odd.*

Theorem 1.1 says that for each pair of arguments x, y of degree n there exists a polynomial $p' = \frac{1}{2}(p(\dots, x, \dots, y, \dots) - (-1)^n p(\dots, y, \dots, x, \dots))$ which comes from polynomials of lower type such that $p - p'$ is symmetric or skew symmetric in x, y according as n is even or odd. In fact, we can find a polynomial p_1 which comes from some polynomials of type lower than that of p such that $p - p_1$ is symmetric or skew symmetric in all its arguments of degree n , according to n is even or odd.

Theorem 1.2 *Let r be an integer and F be a field with $\text{char } F \nmid r!$. Let V be an S_r -module then*

1. $V = \{v \in V | \pi v = v, \forall \pi \in S_r\} + (\sum_{1 \leq i < j \leq r} \{v \in V | (i, j)v = -v\});$
and
2. $V = \{v \in V | \pi v = -v, \forall \pi \in S_r\} + (\sum_{1 \leq i < j \leq r} \{v \in V | (i, j)v = v\}).$

Proof. To prove the theorem we use induction on r . In what follows, we agree that the product of two permutations is performed from right to left. For $r = 1$, the theorem holds obviously. For $r = 2$, every $v \in V$ has a unique decomposition

$$v = \frac{1}{2}(v + (12)v) + \frac{1}{2}(v - (12)v).$$

Suppose the result is true for $r - 1$. We imbed S_{r-1} into S_r as $\{\pi \in S_r | \pi(r) = r\}$. Thus a given S_r -module is also an S_{r-1} -module. Hence by induction,

$$V = \{v \in V | \pi v = v, \forall \pi \in S_{r-1}\} + (\sum_{1 \leq i < j \leq r-1} \{v \in V | (i, j)v = -v\}).$$

Let $v \in V^{S_{r-1}} := \{v \in V | \pi v = v, \forall \pi \in S_{r-1}\}$, and define

$$w := v - \frac{1}{r} \sum_{1 \leq i \leq r-1} (v - (ir)v).$$

It suffices to show $w \in V^{S_r}$. We have

$$w = \frac{1}{r}(v + \sum_{1 \leq i \leq r-1} (ir)v),$$

hence, for $1 \leq j \leq r-1$,

$$\begin{aligned}(jr)w &= \frac{1}{r}[(jr)v + v + \sum_{1 \leq i \leq r-1, i \neq j} (jr)(ir)v] \\ &= w\end{aligned}$$

since $(jr)(ir) = (ir)(ij)$ and thus

$$(jr)(ir)v = (ir)(ij)v = (ir)v$$

using $v \in V^{S_{r-1}}$. Because $\{(jr) | 1 \leq j \leq r-1\}$ generates S_r , we shown that $w \in V^{S_r}$, completing the proof of part 1 of Theorem 1.2.

Symmetrically, to prove part 2 of the theorem we define

$$w' := v' - \frac{1}{r} \sum_{1 \leq i \leq r-1} (v' + (ir)v'),$$

where $v' \in \{v \in V | \pi v = -v, \forall \pi \in S_{r-1}\}$. Now we can repeat the proof of part 1 of the theorem step by step to show that $w' \in \{v \in V | \pi v = -v, \forall \pi \in S_r\}$. This completes the proof of Theorem 1.2. ■

Let $V = V^{[m^r, n^s, \dots, u^t]}[X]$, and

$$\begin{aligned}W_1 &= \text{Span}\{f(x_1, \dots, x_r, y_1, \dots, y_s, \dots) \in V | \\ &\quad f \text{ is skewsymmetric in } x_i \text{ and } x_j \text{ for some } i \neq j\}.\end{aligned}$$

Let U be the set of the elements $f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t) \in V$ which are either symmetric or skew symmetric in some w_i, w_j where $i \neq j$ and $w \in \{x, y, \dots, z\}$, according as the degree of w_i in f is odd or even. Let $V_1 = \text{Span}U$ then we have

Theorem 1.3 *Let $m_0 = \max\{m, n, \dots, u, r, s, \dots, t\}$. If $\text{char}F \nmid m_0!$ then every element f in V has a decomposition into a sum of two polynomials of the same type as f , $f = f_0 + f_1$, where, for each k with $0 < k \leq m_0$, f_0 is either symmetric or skew symmetric in all variables of degree k depending on whether k is even or odd and $f_1 \in V_1$. Moreover V_1 is spanned by elements which come from lower type polynomials: if $g \in V_1$ then $g = \sum g_k$, where $g_k \in U$ and g_k is given by formula (1.2).*

Proof. Let $f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t)$ in V . First we show that $f = f_0 + f_1$, where $f_1 \in V_1$ and f_0 is either symmetric or skew symmetric in all x_i 's according as m is even or odd. For an arbitrary $\sigma \in S_r$ we define

$$\sigma(f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t)) := f(x_{\sigma(1)}, \dots, x_{\sigma(r)}, y_1, \dots, y_s, \dots, z_1, \dots, z_t).$$

Then V is a S_r -module. If m is even then by 1) of Theorem 1.2, $f = f_0 + f_1$, where $f_1 \in W_1$ and f_0 is symmetric in all x_i 's. Since $W_1 \subseteq V_1$ by the definitions of W_1 and V_1 , we get $f_1 \in V_1$. If m is odd then we use 2) of Theorem 1.2.

In any case, we have that $f = h_0 + h_1$, where $h_1 \in V_1$ and h_0 is either symmetric or skew symmetric in all x_i 's according as m is even or odd. Next, we consider h_0 and variables $y_1, \dots, y_s$ as above. The above process will not destroy the symmetry of h_0 in the x_i 's. Thus after a finite number of steps, we have that $f = f_0 + f_1$, where $f_1 \in V_1$ and f_0 is as in the theorem.

The last statement of the Theorem follows from Theorem 1.1 and the the definition of U . ■

To summarize, if $f \in T(A, W) \cap V$ then $f = f_0 + f_1$ with f_i an identity, for $i = 0, 1$ by Theorem 1.3 and the fact that $T(A, W) \cap V$ is an S_r -module. We know in all interesting cases f_1 comes from some lower type identities. Using Proposition 1.2.1 We have

Corollary 1.3.2 *Let $m_0 = \max\{m, n, \dots, u, r, s, \dots, t\}$. If $\text{char} F \nmid m_0!$ and $|F| > 2m_0 - 1$ then every identity f in V has a decomposition into a sum of two identities of the same type as f , $f = f_0 + f_1$ where, for each k with $0 < k \leq m_0$, f_0 is either symmetric or skew symmetric in all variables of degree k depending on whether k is even or odd and f_1 comes from lower type identities.*

Chapter 2

The General Results About The Identities of H_n

2.1 Some Formulas About The Identities of H_n of Degree $2n + 1$

To find polynomial identities we must determine the coefficients of monomials in a polynomial identity. One basic method to do this is substitution. In this section we obtain some important lemmas using specific substitutions. From now on $e[ij]$ denotes the matrix $e_{ij} + e_{ji}$ for $i \neq j$ and $e[ii]$ denotes e_{ii} , where e_{rs} is the usual matrix unit.

In what follows, we agree that the product of two permutations is performed from right to left. Let $\sigma \in S_r$. Then, if

$$\sigma = \begin{pmatrix} 1, \dots, r \\ \sigma(1), \dots, \sigma(r) \end{pmatrix}$$

we denote this σ in the following simply by $\sigma(1) \cdots \sigma(r)$. The permutation $\sigma(r)\sigma(r-1) \cdots \sigma(2)\sigma(1)$ is called the **reverse permutation** of σ and denoted by $\rho\sigma$.

Lemma 2.1.1 *Let $p(x_1, \dots, x_{2n+1})$ be a multilinear identity of H_n . If the monomial $x_{\sigma(1)} \cdots x_{\sigma(2n+1)}$ has coefficient α_σ in p for some $\sigma \in S_{2n+1}$, the symmetric group on $2n+1$ objects, then the following statements hold:*

1. For i, j with $1 \leq i, j \leq n-1$ and $i \neq j$,

$$\alpha_\sigma + \alpha_{(2i-1, 2i)\sigma} + \alpha_{(2i+2j, 2i+2j+1)\sigma} + \alpha_{(2i-1, 2i)(2i+2j, 2i+2j+1)\sigma} = 0.$$

2. $\sum_{\tau \in S_3} \alpha_{\tau\sigma} = 0$, where S_3 is the symmetric group on $2i-1$, $2i$ and $2i+1$.

3. For i, j with $1 \leq i, j \leq n$ and $i \neq j$,

$$\alpha_{\sigma} + \alpha_{(2i-2, 2i-1)\sigma} + \alpha_{(2i+2j+1, 2i+2j+2)\sigma} + \alpha_{(2i+2j+1, 2i+2j+2)(2i-2, 2i-1)\sigma} + \\ \alpha_{\rho\sigma} + \alpha_{\rho(2i-2, 2i-1)\sigma} + \alpha_{\rho(2i+2j+1, 2i+2j+2)\sigma} + \alpha_{\rho(2i+2j+1, 2i+2j+2)(2i-2, 2i-1)\sigma} = 0,$$

where $\rho\tau$ denotes the reverse permutation of the permutation τ .

4. Let $\sigma(1)\tau\sigma(2n+1) := \sigma(1) \cdots \sigma(2n+1) \in S_{2n+1}$ then

$$\alpha_{\sigma(1)\tau\sigma(2n+1)} + \alpha_{\sigma(1)(\rho\tau)\sigma(2n+1)} + \alpha_{\tau\sigma(1)\sigma(2n+1)} + \alpha_{(\rho\tau)\sigma(1)\sigma(2n+1)} + \\ \alpha_{\sigma(1)\sigma(2n+1)\tau} + \alpha_{\sigma(1)\sigma(2n+1)\rho\tau} + \alpha_{\rho(\sigma(1)\tau\sigma(2n+1))} + \alpha_{\rho(\sigma(1)(\rho\tau)\sigma(2n+1))} + \\ \alpha_{\rho(\tau\sigma(1)\sigma(2n+1))} + \alpha_{\rho(\rho\tau\sigma(1)\sigma(2n+1))} + \alpha_{\rho(\sigma(1)\sigma(2n+1)\tau)} + \alpha_{\rho(\sigma(1)\sigma(2n+1)\rho\tau)} = 0.$$

Let f be a homogeneous identity of H_n . If $\text{char} F \neq 2, 3$ then the words having the same indeterminate in positions $2i-1$ and $2i$ and the same indeterminate in positions $2i+2j$ and $2i+2j+1$, or having the same indeterminate in positions $2i-1, 2i$ and $2i+1$, have coefficients 0 in f .

Proof. Let $w := x_{\sigma(1)} \cdots x_{\sigma(2n+1)}$. The operator (i, j) acts on w by permuting $x_{\sigma(i)}$ and $x_{\sigma(j)}$. Substituting

$$e[11], e[12], e[22], e[23], \dots, e[i-1, i-1], e[i-1, i], e[ii], e[ii], e[i, i+1], \\ e[i+1, i+1], e[i+1, i+2], \dots, e[i+j, i+j], e[i+j, i+j], e[i+j, i+j+1], \dots, e[nn]$$

for

$$x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(2n+1)}$$

in p there are only 4 monomials

$$w, (2i-1, 2i)w, (2i+2j, 2i+2j+1)w, (2i-1, 2i)(2i+2j, 2i+2j+1)w$$

which produce e_{1n} . This implies 1.

Similarly, substituting

$$e[11], e[12], e[22], \dots, e[i-1, i], e[ii], e[ii], e[ii], e[i, i+1], \dots, e[nn]$$

for $x_{\sigma(1)}, \dots, x_{\sigma(2n+1)}$ in p and considering the coefficient of c_{1n} yields 2.

Substituting

$$e[12], e[22], e[23], \dots, e[i-1, i], e[ii], e[ii], e[i, i+1], \dots, \\ e[i+j, i+j], e[i+j, i+j], e[i+j, i+j+1], \dots, e[nn], e[n1]$$

for $x_{\sigma(1)}, \dots, x_{\sigma(2n+1)}$ in p and considering the coefficient of e_{11} yields 3.

Substituting

$$e[11], e[12], e[22], e[23], \dots, e[nn], e[n1]$$

for $x_{\sigma(1)}, \dots, x_{\sigma(2n+1)}$ in p and considering the coefficient of e_{11} yields 4.

The last statement is obtained by linearizing f and then applying the results 1 and 2 respectively. ■

As in Lemma 2.2 of [MR], we have

Lemma 2.1.2 *Let $f(x, y, z, \dots)$ be an identity of H_n , n_1 and n_2 be the degree of x and y in f respectively. If the characteristic of F does not divide $(n_1 - 1)!$ and $(n_2 - 1)!$ then the following statements hold:*

1. [MR, Lemma 2.2] *If $f = f_1 yx + \sim$, where $\sim$ is the sum of the terms in f which do not end in yx . Then f_1 is an identity of H_{n-1} .*
2. *If $f = f_1 yx^2 + \sim$, where $\sim$ is the sum of the terms in f which do not end in yx^2 . Then f_1 is an identity of H_{n-1} .*

Proof. We prove 2. Linearizing x and y in f completely we get an identity denoted by $\text{lin}_{x,y}f$. Then from $f = f_1 yx^2 + g$,

$$\text{lin}_{x,y}f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z, \dots) = \\ \text{lin}_{x,y}f_1(x_3, \dots, x_{n_1}, y_2, \dots, y_{n_2}, z, \dots)y_1x_1x_2 + \dots + \text{lin}_{x,y}g.$$

We claim that $\text{lin}_{x,y}f_1$ is an identity of H_{n-1} . For, if it is not then there exist $a_3, \dots, a_{n_1}, b_2, \dots, b_{n_2}, c, \dots \in H_{n-1}$ such that

$$\text{lin}_{x,y}f_1(a_3, \dots, a_{n_1}, b_2, \dots, b_{n_2}, c, \dots) = \alpha e_{ij} + A,$$

with $\alpha \neq 0$, e_{ij} does not occur in the matrix A and $A \in H_{n-1}$. Letting $y_1 = e[jn]$ and $x_1 = x_2 = e_{nn}$ then $\text{lin}_{x,y}f(x_1, x_2, a_3, \dots, a_{n_1}, y_1, b_2, \dots, b_{n_2}, c, \dots) = 0$ implies that

$$\text{lin}_{x,y}f_1(a_3, \dots, a_{n_1}, b_2, \dots, b_{n_2}, c, \dots)y_1(x_1 \circ x_2) + 0 = 0.$$

But

$$\text{lin}_{x,y}f_1(a_3, \dots, a_{n_1}, b_2, \dots, b_{n_2}, c, \dots)y_1(x_1 \circ x_2) = 2\alpha e_{in} + B,$$

with $2\alpha \neq 0$ and e_{in} does not occur in the matrix B , a contradiction. Thus $\text{lin}_{x,y}f_1$ is an identity of H_{n-1} .

Since

$$\text{lin}_{x,y}f_1(x, \dots, x, y, \dots, y, z, \dots) = (n_1 - 2)!(n_2 - 1)!f_1,$$

and the characteristic of the field F does not divide $(n_1 - 2)!$ and $(n_2 - 1)!$, f_1 is an identity of H_{n-1} . ■

Let V be the set of the homogeneous identities of H_n of degree $2n + 1$ and

$$V[n_1, \dots, n_r] = \{f \in V \mid f \text{ is of type } [n_1, \dots, n_r]\}.$$

Then $V[n_1, \dots, n_r]$ is a subspace of the vector space V . Our aim is to find a basis of V . Since $V = \bigoplus V[n_1, \dots, n_r]$, it suffices to find a basis for each $V[n_1, \dots, n_r]$. Although the proof is simple, the lemma below gives a way to reduce the calculations when one searches for a basis of $V[n_1, \dots, n_r]$.

Lemma 2.1.3 *Let W be a subspace of $F[X]$. If $Q_1, \dots, Q_m \in W$ are linearly independent and there exist polynomials $p_1, \dots, p_m \in F[X]$ such that $W \subseteq \text{Span}\{p_1, \dots, p_m\}$, then $\dim W = m$ and $\{Q_1, \dots, Q_m\}$ is a basis of W .*

2.2 The Razmyslov Transformation

In this section we introduce the Razmyslov transformation of polynomials. In the following all identities mentioned are weak identities.

Definition 2.2.1 Let $f(x, y, \dots; z)$ be a polynomial of degree 1 in z . If

$$f = \sum_i g_i z h_i,$$

then following [FM] we define

$$f^\#(z) := \sum_i h_i z g_i,$$

which is called the Razmyslov transform of f with respect to z .

Obviously, the Razmyslov transform of an identity need not be an identity. However it is still a kind of zero operator. So, we introduce the concept of mixed identity.

Definition 2.2.2 Let $f(x, y, \dots, z; w)$ be a polynomial of degree one in w . If $f(x, y, \dots, z; w) = 0$ for $x, y, \dots, z \in H_n$ and $w \in K_n$ then f is called a mixed identity of H_n .

Similar to Lemma 2.1.2 and the Lemma 2.2 of [MR] we have

Proposition 2.2.1 Let $f(x_1, x_2, \dots; x_n)$ be a multilinear mixed identity of H_n and $f = f_1 x_n x_{n-1} + f_2 x_i x_j + \sim$, where $\sim$ denotes the sum of the terms of f which do not end in $x_n x_{n-1}$ or $x_i x_j$ for $i, j \neq n$. Then f_2 is a mixed identity and f_1 is an identity.

If $f(x_1, x_2, \dots; y)$ is a mixed identity over a field F of characteristic not dividing the maximum of the degrees of the variables in f , and $f = f_1 y x_i^2 + f_2 x_i x_j^2 + \sim$, then f_1 is an identity and f_2 is a mixed identity.

In following, $*$ is the involution on $F[X]$ defined by $x_i^* := x_i$, for $i = 1, 2, \dots$. From the definition of the Razmyslov transformation one can show

Proposition 2.2.2 In a polynomial f , if $f^* = \varepsilon f$ then $(f^\#(z))^* = \varepsilon f^\#(z)$, where $\varepsilon = \pm 1$. If $f^* = f$ then f is an identity (mixed identity) on H_n iff $f^\#(z)$ is an identity (mixed identity) on H_n . If $f^* = -f$ then f is an identity on H_n iff $f^\#(z)$ is a mixed identity on H_n .

Proof. Let $f = \sum_i g_i z h_i$ then $f^* = \sum_i h_i^* z g_i^*$ and $f^\#(z) = \sum_i h_i z g_i$. Thus $f^* = \pm f$ implies that $(f^*)^\#(z) = \pm f^\#(z)$, that is, $\sum_i g_i^* z h_i^* = \pm \sum_i h_i z g_i$. However $(f^\#(z))^* = \sum_i g_i^* z h_i^* = (f^*)^\#(z)$. So $(f^\#(z))^* = \pm f^\#(z)$.

To prove the other part of the proposition we use the fact that if A is an $n \times n$ symmetric (skew symmetric) matrix then $A = 0$ iff the trace of AB is zero for every $B \in H_n$ ($B \in K_n$). Let $f(x_1, \dots, x_m, z)$ be $*$ -symmetric and

$$f = \sum_i g_i z h_i$$

then

$$f^\#(z) = \sum_i h_i z g_i$$

and

$$f^\#(x_1, \dots, x_m, z) \in K_n, \forall x_1, \dots, x_m \in H_n, \forall z \in K_n.$$

Thus, denoting by T the trace of a matrix, for $\forall x_1, \dots, x_m \in H_n$, and $\forall z \in K_n$

$$\begin{aligned} f^\#(x_1, \dots, x_m, z) = 0 &\iff T(f^\#(x_1, \dots, x_m, z)B) = 0, \forall B, z \in K_n \\ &\iff \sum_i T(h_i z g_i B) = 0, \forall B, z \in K_n \\ &\iff \sum_i T(g_i B h_i z) = 0, \forall B, z \in K_n \\ &\iff T((\sum_i g_i B h_i)z) = 0, \forall B, z \in K_n \\ &\iff f(x_1, \dots, x_m, B) = 0, \forall x_1, \dots, x_m \in H_n, \forall B \in K_n. \end{aligned}$$

The rest of the proposition can be proved by using the same argument. ■

2.3 A New Identity of H_n

In this section we use the Razmyslov transformation to show that T_{2n} is a zero operator on M_n . Namely $T_{2n}(x_1, \dots, x_{2n-1}; y) = 0$ for all $x_i \in H_n$ and $y \in M_n$. This sharpens the results in [MR], especially the graph-theoretic interpretation of the results. From this result we get a new identity of H_n : $T_{2n}(x_1, \dots, x_{2n-1}; [x_{2n}, x_{2n+1}]) = 0$.

Let us recall the definition of T_{2n} . We define

$$T_{2n}^{2n}(x_1, \dots, x_{2n}) := \sum_{\substack{\sigma \in S_{2n} \\ \sigma^{-1}(2n) \equiv 1, 2 \pmod{4}}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)},$$

S_{2n} the symmetric group on $2n$ objects and $(-1)^\sigma$ the sign of the permutation σ .

$$T_{2n}(x_1, \dots, x_{2n-1}; x_{2n}) := T_{2n}^{2n}(x_1, \dots, x_{2n-1}, x_{2n}).$$

So,

$$\begin{aligned} T_{2n}(x_1, \dots, x_{2n-1}; x_{2n}) \\ = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(i-1)} \overset{(i)}{x_{2n}} x_{\sigma(i)} \cdots x_{\sigma(2n-1)}. \end{aligned}$$

Theorem 2.1 $T_{2n}(x_1, \dots, x_{2n-1}; y) = 0$ for all $x_i \in H_n$ and $y \in M_n$.

Proof. Since T_{2n} is linear in y and an identity on H_n by $[MR]$, it suffices to show that T_{2n} is a mixed identity. If n is odd then T_{2n} is $*$ -skew symmetric. Indeed,

$$\begin{aligned} T_{2n}(x_1, \dots, x_{2n-1}; y)^* \\ = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} (x_{\sigma(1)} \cdots x_{\sigma(i-1)} \overset{(i)}{y} x_{\sigma(i)} \cdots x_{\sigma(2n-1)})^* \\ = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} x_{\sigma(2n-1)} \cdots x_{\sigma(i)} \overset{(2n-i+1)}{y} x_{\sigma(i-1)} \cdots x_{\sigma(1)} \\ = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} (-1)^{n-1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} \overset{(2n-i+1)}{y} x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)} \\ = \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} \overset{(2n-i+1)}{y} x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)}, \end{aligned}$$

since n is odd, $n-1$ is even. On the other hand, $n = 2m+1$ implies that $2n+1-i = 4m-i+3$ and

$$2n+1-i \equiv \begin{cases} 2 \pmod{4} & \text{if } i \equiv 1 \pmod{4} \\ 1 \pmod{4} & \text{if } i \equiv 2 \pmod{4}. \end{cases} \quad (2.1)$$

Hence

$$x_{\sigma(1)} \cdots x_{\sigma(2n-i)} \overset{(2n-i+1)}{y} x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)}$$

is a term of T_{2n} and has different sign by the definition of T_{2n} . So T_{2n} is $*$ -skew symmetric. Since T_{2n} is an identity, $T_{2n}^\#$ is a mixed identity by Proposition 2.2.2. However,

$$\begin{aligned} T_{2n}(x_1, \dots, x_{2n-1}; y)^\#(y) &= \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} x_{\sigma(i)} \cdots x_{\sigma(2n-1)} \overset{(2n-i+1)}{y} x_{\sigma(1)} \cdots x_{\sigma(i-1)} \\ &= \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1+(2n-i)(i-1)} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} \overset{(2n-i+1)}{y} x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)} \\ &= \sum_{\substack{\sigma \in S_{2n-1} \\ i \equiv 1, 2 \pmod{4}}} (-1)^\sigma (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} \overset{(2n-i+1)}{y} x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)} \\ &= -T_{2n}(x_1, \dots, x_{2n-1}; y) \end{aligned}$$

from the last equation. Therefore T_{2n} is also a mixed identity if n is odd.

If n is even then $n+1$ is odd, so $T_{2(n+1)}$ is a mixed identity. However

$$T_{2(n+1)}(x_1, \dots, x_{2n+1}; y) = T_{2n}(x_1, \dots, x_{2n-1}; y) x_{2n} x_{2n+1} + \sim$$

from [MR]. Hence T_{2n} is a mixed identity by Proposition 2.2.1 proving the Theorem. ■

Let (V, E) be a finite graph with vertices $V = \{v_1, \dots, v_n\}$ and (undirected) edges $E = \{w_1, \dots, w_r\}$, $r \geq 2n$, in an arbitrary but fixed ordering. Then unicursal paths correspond to permutations $\sigma \in S_r$, where the path is $w_{\sigma(1)} \cdots w_{\sigma(r)} = p_\sigma$. Define $\varepsilon(p_\sigma)$ to be $(-1)^\sigma$. Choose a distinguished edge w_i . Then for any two fixed vertices v_j, v_k let q_σ be the paths from v_j to v_k with w_i in the positions which are congruent to 1 or 2 modulo 4. Theorem 5 in [MR] said that the number of the paths q_σ with $\varepsilon(q_\sigma) = 1$ is equal to the number of paths q_σ with $\varepsilon(q_\sigma) = -1$. In fact, if we direct w_i then the paths q_σ with $\varepsilon(q_\sigma) = 1$ are divided into two classes depending on the direction of w_i (that is the edge w_i is one way) as are the paths q_σ with $\varepsilon(q_\sigma) = -1$ also. We have

Theorem 2.2 *The number of the paths q_σ with $\varepsilon(q_\sigma) = 1$ and the same direction of w_i is equal to the number of the paths q_σ with $\varepsilon(q_\sigma) = -1$ and the same direction of the w_i . ■*

2.4 Another Proof of The Amitsur Levitzki Theorem

In this section we give another proof of the Amitsur and Levitzki Theorem by using Proposition 1.1 in [MR] and the Razmyslov transformation.

First we note that it can be proved that $\tilde{T}_{2n}$ is also an identity of H_n by using the proof that T_{2n} is an identity (see [MR]), where $\tilde{T}_{2n}$ is the polynomial such that $T_{2n} + \tilde{T}_{2n} = S_{2n}$, the standard identity.

Corollary 2.4.1 $S_{2n}(x_1, \dots, x_{2n-1}, x_{2n})$ is an identity on M_n .

Proof. Since S_{2n} is multilinear alternating, it suffices to show that

$$S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) = 0, \forall x_i \in H_n, \forall y_j \in K_n, \text{ where } r + s = 2n. \quad (2.2)$$

We show (2.2) by induction on s . If $s = 0$ then (2.2) follows from the fact that T_{2n} and $\tilde{T}_{2n}$ are identities on H_n . Assume that (2.2) holds for all $n = 1, 2, 3, \dots$, and $s - 1$. Then for s , we show (2.2) holds by distinguishing two cases.

If s is odd then for all odd n , S_{2n} is $\ast$ -skew symmetric. Hence $S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) \in H_n$ for $\forall x_i \in H_n, \forall y_j \in K_n$. Therefore it suffices to show that

$$T(S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s)z) = 0, \forall z \in H_n.$$

We claim that

$$S_{2n}^\# = -S_{2n}, n = 1, 2, 3, \dots$$

Let $(x_1, \dots, x_r, y_1, \dots, y_s) = (z_1, \dots, z_{2n-1}, y_s)$. Then

$$\begin{aligned} S_{2n}(z_1, \dots, z_{2n-1}, y_s) &= \\ &= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^{2n-i} z_{\sigma(1)} \cdots z_{\sigma(i-1)} \overset{(i)}{y_s} z_{\sigma(i)} \cdots z_{\sigma(2n-1)} \\ &= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^i z_{\sigma(1)} \cdots z_{\sigma(i-1)} \overset{(i)}{y_s} z_{\sigma(i)} \cdots z_{\sigma(2n-1)}. \end{aligned}$$

Thus

$$\begin{aligned}
S_{2n}^\#(y_s) &= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^i z_{\sigma(i)} \cdots z_{\sigma(2n-1)} \frac{(2n-i+1)}{y_s} z_{\sigma(1)} \cdots z_{\sigma(i-1)} \\
&= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^{i+(2n-1)(i-1)} z_{\sigma(1)} \cdots z_{\sigma(2n-i)} \frac{(2n-i+1)}{y_s} z_{\sigma(2n-i+1)} \cdots z_{\sigma(2n-1)} \\
&= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^i z_{\sigma(1)} \cdots z_{\sigma(2n-i)} \frac{(2n-i+1)}{y_s} z_{\sigma(2n-i+1)} \cdots z_{\sigma(2n-1)} \\
&= -S_{2n}(z_1, \dots, z_{2n-1}, y_s).
\end{aligned}$$

Let

$$S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) = \sum_i g_i y_s h_i.$$

Then from the claim we have for all $z \in H_n$

$$\begin{aligned}
T(S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s)z) &= \sum_i T(g_i y_s h_i z) \\
&= \sum_i T(h_i z g_i y_s) \\
&= T(S_{2n}^\#(x_1, \dots, x_r, y_1, \dots, y_{s-1}, z) y_s) \\
&= T(-S_{2n} y_s) = 0
\end{aligned}$$

by the induction hypothesis. For n even, $n+1$ is odd. Thus

$$S_{2n+2}(x_1, \dots, x_{r+1}, y_1, \dots, y_s) = 0.$$

However

$$S_{2n+2}(x_1, \dots, x_{r+2}, y_1, \dots, y_s) = \pm S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) x_{r+1} x_{r+2} + \sim.$$

So

$$S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) = 0$$

by Lemma 2.1.2.

If s is even then for $n = 2, 4, 6, \dots$, S_{2n} is $*$ symmetric and $S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s) \in H_n$ for $\forall x_i \in H_n, \forall y_j \in K_n$. Therefore it suffices to show that

$$T(S_{2n}(x_1, \dots, x_r, y_1, \dots, y_s)z) = 0, \forall z \in H_n, n = 2, 4, 6, \dots$$

This can be shown by using the same argument and the claim above. Then using the same argument as above to show the case when n is odd completes the proof. ■

Chapter 3

The Identities of H_2 of Degree 5

In this chapter we are going to show that every identity of H_2 is a consequence of T_4 . From Lemma 2.1.1 we know immediately that H_2 has no identity of type $[5]$ or $[4,1]$. Thus identities which we need to determine are of type $[3,2]$ or higher. In following, $\mathbf{VI}_n[n_1, \dots, n_r]$ denotes the set of the identities of H_n of type $[n_1, \dots, n_r]$ and $\mathbf{VIN}_n[n_1, \dots, n_r]$ denotes the set of the identities of H_n of type $[n_1, \dots, n_r]$ which are also symmetric or skew symmetric in all their variables of degree m according as m is even or odd, where $m \in \{n_1, \dots, n_r\}$.

Proposition 3.1.1 *If the char $F \nmid 3!$ then every identity of type $[3,2]$ is a scalar multiple of $T_4(x, y, x^2; y)$.*

Proof. Let $f(x, y) \in \mathbf{VI}_2[3, 2]$ then from Lemma 2.1.1

$$\begin{aligned} f = & \alpha_1 yxyxx + \alpha_2 yxyyx + \alpha_3 yxxx y + \alpha_4 xy yxx + \\ & \alpha_5 xyxyx + \alpha_6 xyxxy + \alpha_7 xxyyx + \alpha_8 xxyxy, \end{aligned}$$

and

$$\begin{aligned} \alpha_1 + \alpha_4 &= 0, \quad \alpha_7 + \alpha_8 = 0, \quad \alpha_2 + \alpha_5 + \alpha_7 = 0, \\ \alpha_3 + \alpha_6 + \alpha_8 &= 0, \quad \alpha_4 + \alpha_5 + \alpha_6 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \end{aligned}$$

Substituting $e[12], e[11]$ for x, y in f and considering the coefficient of e_{12} yield $\alpha_2 + \alpha_7 = 0$. Substituting $e[12], e[22]$ for y, x and considering the coefficient e_{11} yield $\alpha_3 = 0$.

The solution of the equations above is $\alpha_1 = -\alpha_2 = -\alpha_4 = \alpha_6 = \alpha_7 = \alpha_8$. So

$$f = \alpha_1[[y, x], [y, x^2]] = \alpha_1 T_4(x, y, x^2; y).$$

completing the proof of the proposition. ■

Proposition 3.1.2 *If the char $F \nmid 3!$ then identities*

$$T_4(x, x^2, y; z), T_4(x, x^2, z; y), T_4(x^2, y, z; x), T_4(x, y, z; x)x$$

form a basis of the vector space $VIN_2[3, 1, 1]$.

Proof. From Theorem 1.3 it suffices to find a basis of $VIN_2[3, 1, 1]$. Let $f(x, y, z) \in VIN_2[3, 1, 1]$ then from Lemma 2.1.1

$$\begin{aligned} f = (Id - (y, z))\{ & \alpha_1 yxzx + \alpha_2 yxxzx + \alpha_3 yxxxz + \alpha_4 xyxxx + \\ & \alpha_5 xyxxz + \alpha_6 xyxzx + \alpha_7 xxyxz + \alpha_8 xxyxz \}, \end{aligned}$$

where Id is the identity mapping and the operator (y, z) acts on the monomials by permuting y and z . Using the same arguments as in the proof of Proposition 3.1.1 we have

$$\begin{aligned} \alpha_1 + \alpha_4 &= 0, \quad \alpha_7 + \alpha_8 = 0, \quad \alpha_2 + \alpha_5 + \alpha_7 = 0, \\ \alpha_3 + \alpha_6 + \alpha_8 &= 0, \quad \alpha_4 + \alpha_5 + \alpha_6 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \end{aligned}$$

The solution of this system is

$$\alpha_1 = -\alpha_3 + \alpha_5 - \alpha_8, \quad \alpha_2 = -\alpha_5 + \alpha_8, \quad \alpha_4 = \alpha_3 - \alpha_5 + \alpha_8, \quad \alpha_6 = -\alpha_3 - \alpha_8, \quad \alpha_7 = -\alpha_8.$$

Thus $f = \alpha_3 f_1 + \alpha_5 f_2 + \alpha_8 f_3$, where f_i , for $i = 1, 2, 3$ are independent from f . So $VIN_2[3, 1, 1]$ is contained in $Span\{f_1, f_2, f_3\}$.

On the other hand, the identities

$$v_1 := T_4(y, z, x^2; x), \quad v_2 := T_4(x, y, z; x)x, \quad v_3 := T_4(x, y, x^2; z) - T_4(x, z, x^2; y)$$

belong to $VIN_2[3, 1, 1]$ and are linearly independent. For, if there exist scalars $\beta_1, \beta_2, \beta_3 \in F$ such that $\sum_{i=1}^3 \beta_i v_i = 0$ then the coefficients of the monomials $yxxxx$, $xxxyx$ and $xxzyx$ are 0, which yields $\beta_1 = \beta_2 = \beta_3 = 0$. Thus v_1, v_2, v_3 form a basis of $VIN_2[3, 1, 1]$.

Since $VI_2[3, 2]$ is spanned by $T_4(x, y, x^2; y)$ by Proposition 3.1.1, it follows from theorem 1.3, $T_4(x, y, x^2; z) + T_4(x, z, x^2; y)$, v_1 , v_2 , v_3 form a basis of $VI_2[3, 1, 1]$. Obviously the identities

$$h_1 := T_4(x, x^2, y; z), \quad h_2 := T_4(x, x^2, z; y),$$

$$h_3 := T_4(y, z, x^2; x), \quad h_4 := T_4(x, y, z; x)x$$

form a basis of $VI_2[3, 1, 1]$. ■

Note. $xT_4(x, y, z; x) = h_1 - h_2 + h_4$ and $T_4(x, y, z; x^2) = -h_1 + h_2 + h_3 - h_4$.

Let

$$k_1 := T_4(y, x \circ y, x; z), \quad k_2 := T_4(y^2, x, z; x), \quad k_3 := T_4(x^2, y, z; y),$$

$$k_4 := T_4(x, y, x \circ z; y), \quad k_5 := T_4(y, x, z \circ y; x), \quad k_6 := T_4(x, y, z; y)x,$$

$$k_7 := xT_4(x, y, z; y) + yT_4(y, x, z; x), \quad k_8 := T_4(y, x, z; x)y.$$

Then we have

Proposition 3.1.3 *If $\text{char} F \nmid 3!$ then $\{k_i | i = 1, \dots, 8\}$ is a basis of the vector space $VI_2[2, 2, 1]$.*

Proof. Let $A := yyzx$, $B := yxyx$, $C := yxxy$. Let X_i be the monomial which is obtained by putting z in the i -th position of X , where $X \in \{A, B, C\}$. For instance, $A_2 = yzyxx$. Thus if $f(x, y, z) \in VI_2[2, 2, 1]$ then the monomials of f are X_i for $i = 1, \dots, 5$ and $X \in \{A, B, C\}$, as well as the monomials which come from the X_i 's by exchanging all x 's and y 's. Let α_i (resp. β_i, γ_i) denote the coefficient of the monomial A_i (resp. B_i, C_i) in f . Since $f(x, y, 1) \in VI_2[2, 2] = \{0\}$ by Theorem 1.2 in [MR], $f(x, y, 1) = 0$. This yields

$$\sum_{i=1}^5 \alpha_i = 0, \quad \sum_{i=1}^5 \beta_i = 0, \quad \sum_{i=1}^5 \gamma_i = 0. \quad (3.1)$$

From Lemma 2.1.1 we have

$$\begin{aligned} \alpha_1 + \beta_1 + \gamma_1 &= 0, \quad \beta_3 + \beta_4 + \beta_5 + \gamma_3 + \gamma_4 + \gamma_5 = 0, \\ \alpha_2 + \beta_2 + \gamma_2 &= 0, \quad \alpha_4 + \beta_4 + \gamma_4' = 0, \quad \alpha_5 + \beta_5 + \gamma_5' = 0, \end{aligned} \quad (3.2)$$

where γ'_4 and γ'_5 are the coefficients of the monomials $xyyzx$, $xyyzx$ respectively.

Since $f(x, y, x) \in VI_2[3, 2]$, from Proposition 3.1.1 there exists a scalar $\alpha \in F$ such that

$$f(x, y, x) = \alpha T_4(x, x^2, y; y). \quad (3.3)$$

In the following we prove that α , α_5 , γ_4 , γ_5 are the parameters of the coefficients of f if f is symmetric or skew symmetric in the variables of degree 1. Since $T_4(x, x^2, y; y)$ is a fixed polynomial, the linear equations obtained from (3.3) have the coefficients of the monomials in f as unknown and $\lambda_i \alpha$ for some $\lambda_i \in F$ as constants. These λ_i 's are independent from f . So we may assume that $\alpha = 0$. Hence $f(x, y, x) = 0$ yields

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \alpha_3 + \alpha_4 + \alpha_5 = 0, \alpha_4 + \beta_1 + \beta_2 = 0, \\ \alpha_5 + \gamma_1 + \gamma_2 &= 0, \beta_3 + \beta_4 + \gamma'_1 = 0, \beta_5 + \beta'_1 + \gamma_3 = 0, \end{aligned} \quad (3.4)$$

where γ'_1 , β'_1 are the coefficients of the monomials $zxyyx$ and $zxyxy$.

Now let us argue in two cases. If f is symmetric in the variables y, z then $\gamma'_i = \gamma_i$ and $\beta'_i = \beta_i$. Solving systems (3.1), (3.2) and (3.4) yields

$$\begin{aligned} \alpha_1 &= -\gamma_5 - \gamma_4 = -\alpha_2, \alpha_3 = 0, \alpha_4 = \gamma_1 = -\alpha_5, \beta_1 = 2\gamma_5 + \gamma_4, \\ \beta_2 &= -2\gamma_5 + \alpha_5 - \gamma_4, \beta_3 = -\alpha_5 + \gamma_4 + \gamma_5, \beta_4 = \alpha_5 - \gamma_4, \\ \beta_5 &= -\alpha_5 - \gamma_5, \gamma_2 = -\alpha_5 + \gamma_5, \gamma_3 = \alpha_5 - \gamma_4 - \gamma_5. \end{aligned}$$

If f is skew symmetric in the variables y, z then $\gamma'_i = -\gamma_i$ and $\beta'_i = -\beta_i$. Solving systems (3.1), (3.2) and (3.4) yields

$$\begin{aligned} \alpha_1 &= \gamma_5 + \gamma_4 = -\alpha_2, \alpha_3 = 0, \alpha_4 = -\alpha_5, \gamma_1 = -\gamma_5, \beta_1 = -\gamma_4, \\ \beta_2 &= \alpha_5 + \gamma_4 = \beta_4, \beta_3 = -\alpha_5 - \gamma_4 - \gamma_5, \beta_5 = -\alpha_5 + \gamma_5 = \gamma_2, \gamma_3 = \alpha_5 - \gamma_4 - \gamma_5. \end{aligned}$$

Therefore in every case the coefficients are the linear combination of α_5 , γ_4 , γ_5 and α . So every $f \in VI_2[2, 2, 1]$ is a linear combination of 8 polynomials which are independent from f . Thus, from Lemma 2.1.3 to show the proposition it suffices to show that $k_1, \dots, k_8$ are linearly independent.

Suppose there exist scalars $\delta_i \in F$ such that $\sum_{i=1}^8 \delta_i k_i = 0$. Then considering the coefficients of the monomials

$$xxxyz, yxxyz, yxxyz, yxyxz, xyxzy, zxxxy, yzxyx, zxyxy$$

yields

$$\begin{aligned} \delta_1 - \delta_2 + \delta_6 - \delta_8 &= 0, \quad 2\delta_2 = 0, \quad \delta_1 + \delta_3 + \delta_4 + \delta_5 - \delta_6 - \delta_7 = 0, \\ -\delta_3 + \delta_4 - \delta_5 - \delta_6 &= 0, \quad -\delta_3 + 2\delta_6 - \delta_8 = 0, \quad \delta_2 - \delta_3 - \delta_4 + \delta_6 = 0, \\ \delta_3 + \delta_4 + \delta_5 + \delta_6 &= 0, \quad -\delta_2 + \delta_3 - \delta_6 - \delta_7 = 0. \end{aligned}$$

The only solution is trivial. Thus $k_1, \dots, k_8$ are linearly independent. ■

Note:

1. We give the basis of $VI_2[2, 2, 1]$ here for later use.
2. $T_4(y, z, x; x \circ y) = -k_1 - k_2 + k_3$, $(T_4(y, z, x; x \circ y))^* = -k_1 - k_4 + k_5$
 $xT_4(x, y, z; y) - yT_4(y, x, z; x) = -2k_1 - k_2 + k_3 - k_4 + k_5 + k_6 - k_8$,
 $T_4(x, z, x \circ y; y) = -k_1 - k_2 - k_4 + k_5$.

Let

$$\begin{aligned} L_1 &:= xT_4(y_1, y_2, y_3; x), \quad L_2 := x(T_4(y_1, y_2, y_3; x))^*, \quad L_3 := T_4(y_1, y_2, y_3; x)x, \\ L_4 &:= (T_4(y_1, y_2, y_3; x))^*x, \quad L_5 := T_4(y_1, y_2, y_3; x^2), \quad L_6 := (T_4(y_1, y_2, y_3; x^2))^*, \\ L_7 &:= \sum_{(123)} T_4(x \circ y_1, y_2, y_3; x), \end{aligned}$$

where the sum is taken over all cyclic permutations of 1 2 3. Then we have

Proposition 3.1.4 *If $\text{char} F \nmid 3!$ then $\{L_i | i = 1, \dots, 7\}$ is a basis of the vector space $VIN_2[2, 1^3]$.*

Proof. Let $f(x, y_1, y_2, y_3) \in VIN_2[2, 1^3]$ then from the symmetry

$$f = \sum_{1 \leq i < j \leq 3} \alpha_{ij} f_{ij}(x, y_1, y_2, y_3),$$

where

$$f_{ij} = \sum_{\sigma \in S_3} (-1)^\sigma y_{\sigma(1)} \cdots \overset{(i)}{x} \cdots \overset{(j)}{x} \cdots y_{\sigma(3)}.$$

Let

$$\alpha_1 = \alpha_{12}, \alpha_2 = \alpha_{13}, \alpha_3 = \alpha_{14}, \alpha_4 = \alpha_{15}, \alpha_5 = \alpha_{23}, \dots, \alpha_{10} = \alpha_{45};$$

and

$$f_1 = f_{12}, f_2 = f_{13}, f_3 = f_{14}, f_4 = f_{15}, f_5 = f_{23}, \dots, f_{10} = f_{45}.$$

From Lemma 2.1.1 we know that the sum of the coefficients of the monomials which have form $y_1 y_2 \dots$ is 0. So $\alpha_8 + \alpha_9 + \alpha_{10} = 0$. Similarly we have

$$\alpha_1 + \alpha_2 + \alpha_5 = 0, \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 = 0.$$

Solving these equations yields

$$\alpha_1 = -\alpha_2 - \alpha_5, \alpha_8 = -\alpha_9 - \alpha_{10}, \alpha_3 = -\alpha_4 - \alpha_6 - \alpha_7.$$

Thus

$$\begin{aligned} f = & \alpha_2(f_2 - f_1) + \alpha_4(f_4 - f_3) + \alpha_5(f_5 - f_1) + \alpha_6(f_6 - f_3) + \\ & \alpha_7(f_7 - f_3) + \alpha_9(f_9 - f_8) + \alpha_{10}(f_{10} - f_8) \end{aligned}$$

It suffices now to show that $L_1, \dots, L_7$ are linearly independent. If there exist scalars $\delta_i \in F$ such that $\sum_{i=1}^7 \delta_i L_i(x, y, z, w) = 0$ then considering the coefficients of the monomials

$$xzwxy, yxzwx, zwxyx, xyxwz, yxxzw, zwpxy, yxxzw$$

yields

$$\delta_2 + \delta_7 = 0, \delta_3 + 2\delta_7 = 0, \delta_4 = 0, \delta_1 - \delta_7 = 0, \delta_5 + 2\delta_7 = 0, \delta_6 = \delta_7 = 0.$$

Obviously the solution is trivial. Thus $L_1, \dots, L_7$ form a basis of $VIN_2[2, 1^3]$. ■

Finally, $VIN_2[1^5]$ has $S_5(x_1, \dots, x_5)$ as a basis. From the definition of standard polynomial and Lemma 1.1 of [MR],

$$S_5 = \sum_{i=1}^5 (-1)^{i-1} x_i S_4(x_1, \dots, \hat{x}_i, \dots, x_5)$$

and

$$S_4(x_1, \dots, \widehat{x}_i, \dots, x_5) = \frac{1}{2} \sum_{j \neq i} (-1)^{j-1} T_4(x_1, \dots, \widehat{x}_j, \dots, \widehat{x}_i, \dots, x_5; x_j).$$

Thus $VIN_2[1^5]$ has a basis which is a consequence of T_4 .

Chapter 4

The Identities of H_3 of Degree 7

In this chapter we shall determine all identities of H_3 of degree 7. In the following, $VI_3[n_1, \dots, n_r]$ denotes the set of the identities of H_3 of type $[n_1, \dots, n_r]$ and $VIN_3[n_1, \dots, n_r]$ denotes the set of the identities of H_3 of type $[n_1, \dots, n_r]$ which are also symmetric or skew symmetric in all their variables of degree $m \in \{n_1, \dots, n_r\}$ according as m is even or odd.

4.1 The Identities of Type $[4, 1^3]$ or Lower

Proposition 4.1.1 *If $\text{char} F \nmid 3!$ then there is no identity of H_3 of type $[4, 3]$ or lower.*

Proof. Let $f(x, y, z)$ be an identity of type $[5, 1, 1]$ then

$$f = f_1xy + f_2yx + f_3xz + f_4zx + f_5yz + f_6zy + f_7yx^2 + f_8zx^2.$$

So f_i is an identity of H_2 from Lemma 2.1.2, for $i = 1, \dots, 8$. However H_2 has no identity of degree 5 which has type lower than $[3, 2]$ and no identity of degree 4 which has type $[3, 1]$. Each $f_i = 0$. Thus $f = 0$.

Now let $f(x, y)$ be an identity of type $[4, 3]$. Since H_2 has no identity type $[3, 1]$ or $[2, 2]$, $f = f_1xy + f_2yx$. Then $\Delta(y, x)f = 0$, because $\Delta(y, x)f \in VI_3[5, 2]$ and, as we have shown above, $VI_3[5, 1, 1] = \{0\}$. But $\Delta(y, x)f = (f_1 + f_2)x^2 + \sim$, where $\sim$ denotes the sum of the terms which do not end in x^2 . So $f_1 + f_2 = 0$.

Since f_1 is an identity by Lemma 2.1.2, $f_1 = \alpha[[y, x], [y, x^2]]$ for some $\alpha \in F$, by Proposition 3.1.1. So

$$f = \alpha[[y, x], [y, x^2]][x, y].$$

Substituting $e[12]$, $e[23]$ for x, y yields $\alpha = 0$, so $f = 0$. ■

Proposition 4.1.2 *If $\text{char} F \nmid 3!$ then $Q(x^2, y, x, z, y, x)$ forms a basis of the vector space $VI_3[4, 2, 1]$.*

Proof. Let $f(x, y, z) \in VI_3[4, 2, 1]$. Then

$$f = f_1xz + f_2zx + f_3xy + f_4yx + f_5yx^2,$$

since H_2 has no identity of degree 4 of type $[2, 2]$, $[4]$ or $[3, 1]$. Since f_i is an identity for $i = 1, \dots, 5$ by Lemma 2.1.2, $f_5 = \alpha_5[[x, y], [x, z]]$ follows from [MR], $f_i = \alpha_i[[y, x], [y, x^2]]$, for $i = 1, 2$, $f_3 = \sum_{i=1}^4 \beta_i h_i$ and $f_4 = \sum_{i=1}^4 \gamma_i h_i$, where $h_1, \dots, h_4$ are the basis of the space $VI_2[3, 1, 1]$ given in Chapter 3, namely

$$\begin{aligned} h_1 &= T_4(x, x^2, y; z) = [z, x][x^2, y] + [z, x^2][y, x], \\ h_2 &= T_4(x, x^2, z; y) = [y, x][x^2, z] + [y, x^2][z, x], \\ h_3 &= T_4(y, z, x; x^2) = [x, y][z, x^2] + [x, z][x^2, y], \\ h_4 &= T_4(x, y, z; x)x = [[x, y], [z, x]]x. \end{aligned}$$

Since $f(x, y, 1) \in VI_2[4, 2] = \{0\}$, $f(x, y, 1) = 0$ implies that $\alpha_1 + \alpha_2 = 0$, because $h_i(x, y, 1) = 0$ for $i = 1, \dots, 4$ and $f_5(x, y, 1) = 0$. So

$$f = \alpha_1[[y, x], [y, x^2]][x, z] + \alpha_5[[x, y], [x, z]]yx^2 + f_3xy + f_4yx.$$

Since $\Delta(y, x)f$ is an identity of type $[5, 1, 1]$, it is 0. Therefore, considering the sum of the terms of $\Delta(y, x)f$ which end in x^2 , yields $f_3 + f_4 + f_5x = 0$. Since $f_5x = -\alpha_5T_4(y, z, x; x)x = -\alpha_5h_4$ and h_1, h_2, h_3, h_4 are linear independent, $\beta_i + \gamma_i = 0$ for $i = 1, 2, 3$ and $\beta_4 + \gamma_4 - \alpha_5 = 0$. Thus

$$f = \alpha_1[[y, x], [y, x^2]][x, z] + \alpha_5(h_4yx - T_4(y, z, x; x)yx) + \sum_{i=1}^4 \beta_i h_i[x, y]. \quad (4.1)$$

Similarly, $\Delta(z, y)f = 0$ implies that $-\alpha_1 + \beta_1 + \beta_2 = 0$. If we write $f = xxzg + \sim$, where $\sim$ is the sum of the terms of f which do not start with xxz , then g is an identity of type $[2, 2]$ of H_2 by Lemma 2.1.2. Since H_2 has no identity of type $[2, 2]$, $g = 0$. Thus f has no terms which start with xxz . So considering the coefficient of the monomial $xxzyxy$ yields $\beta_1 = 0$. So $\alpha_1 = \beta_2$ from the equation above.

It can be proved symmetrically that

$$f = \alpha'_1[x, z][[y, x], [y, x^2]] + \alpha'_5(yxh_4 - xxyT_4(y, z, x; x)) + \sum_{i=1}^4 \beta'_i[x, y]h_i. \quad (4.2)$$

From (4.1) and (4.2), considering the coefficients of the monomials $xxxyyxx$ and $xxxyxy$ yields $\alpha'_1 = \alpha_5$ and $-\alpha'_1 = \beta_3$ since $\beta_1 = 0$. So $\alpha_5 = -\beta_3$. From (4.2) we know the coefficient of the monomial $xxxyxyx$ is 0. But this monomial has coefficient $-\beta_4 + \alpha_5$. So $\beta_4 = \alpha_5$.

Let $g(x_1, x_2, x_3, x_4, y_1, y_2, z)$ be the complete linearization of f , then g is also an identity. Substituting

$$e[11], e[12], e[12], e[22], e[23], e[33]$$

for z, x_1, x_2, x_3, y, x_4 , considering the coefficient of e_{11} in $g(x_1, x_2, x_3, x_4, y, y, z)$ yields $\alpha_1 + \alpha_5 = 0$. For, the monomials which evaluate to e_{11} are

$$zx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}yx_{\sigma(4)}, \quad zx_{\sigma(1)}yx_{\sigma(2)}yx_{\sigma(3)}x_{\sigma(4)},$$

$$x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}yx_{\sigma(4)}z, \quad x_{\sigma(1)}yx_{\sigma(2)}yx_{\sigma(3)}x_{\sigma(4)}z,$$

$\forall \sigma \in S_4$. In fact, the coefficient of e_{11} is $4!$ times the sum of the coefficients of the following monomials in f since g is the linearization of f :

$$xxxyxyx, \quad zxyxyxx, \quad xxyxyxz, \quad xyxyxxz.$$

This yields $\alpha_1 + \alpha_5 = 0$ because $\beta_1 = 0$ and $\text{char} F \nmid 3!$. So $f = \alpha_5 g$, where g is independent from f . Thus from Lemma 2.1.3, $Q(x^2, y, x, y, x, z)$ is a basis of $VI_3[4, 2, 1]$ since it is a nonzero polynomial. ■

Proposition 4.1.3 *If $\text{char} F \nmid 3!$ then*

$$T_6(y_1, y_2, y_3, x, x^2; x), \quad K := [S_3([y_1, x], [y_2, x], [y_3, x]), x]$$

form a basis of the vector space $VIN_3[4, 1^3]$.

Proof. First K is an identity since $S_3(z_1, z_2, z_3)$ is in the center of $M_3 \forall z_i \in K_3$. Indeed, K_3 has a basis $e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32}$. Substituting these elements for z_1, z_2, z_3 yields $S_3(z_1, z_2, z_3) = -2I$, where I is the unit matrix of M_3 . So K is an identity of H_3 , which was already observed in Remark 2.5.15 of [RW1].

Next, we show that $T_6(y_1, y_2, y_3, x, x^2; x)$ and K are linearly independent. If there exist scalars $\alpha_i \in F$ such that $\alpha_1 T_6(y_1, y_2, y_3, x, x^2; x) + \alpha_2 K = 0$ then considering the coefficient of the monomials $y_1 x y_2 x y_3 x x$, $x y_1 x y_2 x y_3 x$ respectively yields $\alpha_1 + \alpha_2 = 0$, $-2\alpha_2 = 0$. So they are linearly independent.

Let $f(x, y_1, y_2, y_3) \in VIN_3[4, 1^3]$ then

$$f = \sum_{(123)} \{f_1(x, y_1, y_2) y_3 x^2 + f_2(x, y_1, y_2) y_3 x + f_3(x, y_1, y_2) x y_3\}$$

since H_2 has no identity of type $[4, 1]$. Moreover

$$f_1(x, y_1, y_2) = \alpha[[x, y_1], [x, y_2]], f_2 = \beta_1(h_1 - h_2) + \beta_3 h_3 + \beta_4 h_4, f_3 = \gamma_1(h_1 - h_2) + \gamma_3 h_3 + \gamma_4 h_4,$$

where $h_1 - h_2, h_3, h_4$ are the basis of the space $VIN_2[3, 1, 1]$ given in Proposition 3.1.2.

Since $f(x, y_1, y_2, 1)$ is an identity of type $[4, 1, 1]$ it is 0 identically. But

$$f(x, y_1, y_2, 1) = \{(\beta_1 + \gamma_1)(h_1 - h_2) + (\beta_3 + \gamma_3)h_3 + (\beta_4 + \gamma_4 - \alpha)h_4\}x$$

and $h_1 - h_2, h_3, h_4$ are linearly independent, thus $\beta_i + \gamma_i = 0$ for $i = 1, 3$ and $\beta_4 + \gamma_4 = \alpha$.

By Lemma 2.1.1 the sum of the coefficients of the following monomials is 0:

$$x y_2 x y_3 y_1 x x, y_1 y_3 x y_2 x x x, x x x y_2 x y_3 y_1,$$

$$x x y_2 x y_3 y_1 x, x x y_1 y_3 x y_2 x, x y_1 y_3 x y_2 x x.$$

This yields $\alpha - \beta_1 = 0$.

Let $f'(x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ be the complete linearization of f . f' is also an identity of H_3 . Substituting

$$e[12], e[22], e[23], e[33], e[31]$$

for y_1, x, y_2, y_3, x_4 and considering the coefficient of e_{11} in $f'(x, x, x, x_4, y_1, y_2, y_3)$ yields $6(\beta_3 - \beta_1 + \beta_3 + \beta_4) = 0$ since only $y_1 x x x y_2 y_3 x_4, x_4 y_3 y_2 x x x y_1$ evaluate to e_{11} . Since

$\beta_1 = \alpha = \beta_4 + \gamma_4, \gamma_4 = 2\beta_3$ from $-\beta_1 + 2\beta_3 + \beta_4 = 0$. Thus every coefficient of the monomials in f is a linear combination of β_3, β_4 . So from Lemma 2.1.3 the proposition holds. ■

4.2 The Identities of Type $[3, \dots]$

Proposition 4.2.1 *If $\text{char } F \nmid 3!$ then the vector space $VIN_3[3, 3, 1] = \{0\}$.*

Proof. Let $f(x, y, z) \in VIN_3[3, 3, 1]$. Then

$$\begin{aligned} f = & f_1(x, y, z)xy - f_1(y, x, z)yx + f_2(x, y)yz + f'_2(x, y)zy - \\ & f_2(y, x)zx - f'_2(y, x)zx + f_3(x, y, z)yx^2 - f_3(y, x, z)xy^2. \end{aligned} \quad (4.3)$$

Since $\Delta(z, y)f = 0$, considering the sum of the terms of $\Delta(z, y)f$ which end in yy yields $f_2 + f'_2 = 0$ because f_3 is of type $[2, 1, 1]$, $f_3 = \gamma[[y, x], [y, z]]$ for some $\gamma \in F$ and hence $\Delta(z, y)f_3(y, x, z) = 0$. We have $f_1 = \sum_{i=1}^8 \alpha_i k_i$ and $f'_1 = \sum_{i=1}^8 \alpha'_i k_i$, where $k_1, \dots, k_8$ are the basis of $V I_2[2, 2, 1]$ given in Proposition 3.1.3. Also, $f_2 = \alpha[[y, x], [y, x^2]]$ since f_2 is an identity of type $[3, 2]$ of H_2 . Symmetrically, $f = [y, z]\tilde{f}_2 + \sim$ and $\tilde{f}_2 = \beta[[y, x], [y, x^2]]$. Since f has degree 1 in z ,

$$\begin{aligned} f = & \alpha\{[[y, x], [y, x^2]][y, z] - [[x, y], [x, y^2]][x, z]\} + \\ & \beta\{[y, z][[y, x], [y, x^2]] - [x, z][[x, y], [x, y^2]] + \sim\} \end{aligned} \quad (4.4)$$

From (4.3) the monomial $yzxyx^2$ has coefficient $-\gamma$ but by (4.4) it has coefficient β . So $\beta = -\gamma$. If

$$f = xxyg(y, x, z) - yyxg(x, y, z) + \sim \quad (4.5)$$

then $g(y, x, z) = \delta[[y, x], [y, z]]$ for some $\delta \in F$. Similarly we have $\alpha = \delta$ by considering the coefficient of the monomial $xyxyzy$ in f in the two different expressions.

The monomials zy^2x^3y and zy^2x^2yx have coefficients $\alpha_1 + \alpha_3 + \alpha_6$ and $\alpha_1 - \alpha_2 - \alpha_8$ respectively from (4.3). But, from (4.4) we know that they have coefficients 0 and β respectively. So

$$\alpha_1 + \alpha_3 + \alpha_6 = 0, \alpha_1 - \alpha_2 - \alpha_8 = \beta. \quad (4.6)$$

Using the same argument and considering the monomials

$$zxyxyxy, zxyxyyx, zxyyxxy, zxyyxxy, yyxxzxy, yyxxzyx$$

yields

$$\begin{aligned} \alpha_1 - \alpha_4 - \alpha_8 &= 0, \alpha_1 + \alpha_5 + \alpha_6 = \beta, -\alpha_2 + \alpha_4 = \beta, \\ \alpha_3 - \alpha_5 &= -\beta, \alpha_3 + \alpha_7 = -\alpha, -\alpha_2 - \alpha_7 = \alpha. \end{aligned} \quad (4.7)$$

Since $\Delta(y, x)f \in VI_3[4, 2, 1]$, there exists a scalar in F such that $\Delta(y, x)f$ is a scalar multiple of $Q(x^2, x, y, z, x, y)$. So the monomial $xyxzyx$ has coefficient 0 in $\Delta(y, x)f$. Since this monomial comes from the monomials

$$yyxzyx, xyxzyx, xyxzyx, xyxzyx,$$

which have coefficients in f

$$-\alpha_3 - \alpha_7, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8, -\alpha_4 - \alpha_5 - \alpha_6 + \alpha_7, -\gamma.$$

respectively and since $-\gamma = \beta$ from above, we have

$$\beta - \alpha_6 + \alpha_8 = 0. \quad (4.8)$$

Since $\Delta(z, y)f = 0$, considering the sum of its terms which end in xy and yx respectively yield $f_1(x, y, y) - f_2(y, x) = 0$ and $f_1(y, x, y) + f_2(y, x) = 0$. From these two equations we have

$$\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 - \alpha = 0, \alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_4 + \alpha = 0. \quad (4.9)$$

Now we may assume that $f^* = \pm f$. Then considering the coefficients of the monomials $xyx^2yz, xyxzyx$ and their reverse yields

$$\alpha = s\beta, \alpha_4 + \alpha_5 - \alpha_7 + \alpha_8 = -s(\alpha_4 + \alpha_5 - \alpha_7 + \alpha_6), \quad (4.10)$$

where $s = \pm 1$ when $f^* = \pm f$. The solution of system (4.6), ..., (4.10) is trivial if $s = \pm 1$. So $f = 0$. ■

Let

$$\begin{aligned} Q_1(x, y, z) &:= Q(x \circ z, y, x, z, y, x), \quad Q_2(x, y, z) := Q(x \circ y, z, x, y, z, x), \\ Q_3(x, y, z) &:= Q(x^2, y, z, x, y, z), \quad K(x, y, z) := [S_3([x, y], [x, z], [y, z]), x]. \end{aligned}$$

They are all identities in $VI_3[3, 2, 2]$. Moreover, $Q_1 + Q_2, Q_3, K$ are all symmetric in y, z . In following we are going to show

Proposition 4.2.2 *If $\text{char } F \nmid 3!$ then $Q_1 + Q_2, Q_3, K$ form a basis of the vector space $VIN_3[3, 2, 2]$.*

Proof. First we show that $Q_1 + Q_2, Q_3, K$ are linearly independent. It suffices to show that Q_1, Q_2, Q_3, K are linearly independent. From the definition of Q ,

$$\begin{aligned} Q_1 &= [[[x \circ z, y], [x, y]], [z, x]] + [[[y, x], [z, x^2]], [y, z]] + \{[x \circ z, y][x, z][x, y]\} + \\ &\quad \{[y, x][x, z^2][x, y]\} + \{[x^2, z][y, z][x, y]\}, \\ Q_2 &= Q_1(x, z, y), \\ Q_3 &= [[[y, z], [z, x^2]], [y, x]] + [[[y, x^2], [z, y]], [x, z]] + \{[y, x^2][z, x][z, y]\} + \\ &\quad \{[z, x^2][y, x][y, z]\}. \end{aligned}$$

If $\sum_{i=1}^3 \alpha_i Q_i + \alpha_4 K = 0$ for some $\alpha_i \in F$ then considering the coefficient of the monomial $xyxzyzx$ yields $\alpha_1 = 0$. Considering the coefficient of the monomial $yzxxzyx$ in $\alpha_2 Q_2 + \alpha_3 Q_3 + \alpha_4 K$ yields $\alpha_3 = 0$. Considering the coefficient of the monomial $xxzyyzz$ in $\alpha_2 Q_2 + \alpha_4 K$ yields $\alpha_4 = 0$. Since $Q_2 \neq 0$, $\alpha_2 = 0$. So Q_1, Q_2, Q_3, K are linearly independent.

Now let $f(x, y, z) \in VIN_3[3, 2, 2]$. Since H_2 has no identity of type $[2, 2]$ or $[3, 1]$ by [MR], f has no terms which end in y^2 or z^2 by Lemma 2.1.2. So

$$\begin{aligned} f &= f_1(x, y, z)xz + f'_1(x, y, z)zx + f_1(x, z, y)xy + f'_1(x, z, y)yx + \\ &\quad f_2(x, y, z)yz + f_2(x, z, y)zy + f_3(x, y, z)zx^2 + f_3(x, z, y)yx^2. \end{aligned} \quad (4.11)$$

Moreover,

$$f_1 = \sum_{i=1}^8 \alpha_i k_i, \quad f'_1 = \sum_{i=1}^8 \alpha'_i k_i, \quad f_2 = \sum_{i=1}^4 \gamma_i h_i, \quad f_3 = \lambda[[y, x], [y, z]],$$

where $k_1, \dots, k_8$ and $h_1, \dots, h_4$ are the basis of the vector spaces $VI_2[2, 2, 1]$ and $VI_2[3, 1, 1]$ given in Chapter 3.

From Lemma 2.1.1 the sum of the coefficients of the monomials $yxxyzxz, yxxyxzz, yxxyzxz$ is 0. So from (4.11) we have

$$\alpha_3 + \alpha'_3 - \alpha_5 - \alpha'_5 = 0. \quad (4.12)$$

Similarly for each $w = xyxx, yxyx, xyxy$ considering the sum of the coefficients of the monomials of form $w---$ with z, z, x in the last three positions yield

$$\alpha_2 + \alpha'_2 - \alpha_4 - \alpha'_4 = 0, \alpha_4 + \alpha'_4 + \alpha_7 + \alpha'_7 = 0, \alpha_5 + \alpha'_5 + \alpha_7 + \alpha'_7 = 0. \quad (4.13)$$

Because for each $w = zyyx, yzxy, yxzy, xyxz$ the sum of the coefficients of the monomials of form $w---$ with x, x, z in the last three positions is 0 we have

$$\begin{aligned} \alpha_1 + \alpha'_1 + \alpha_3 + \alpha'_3 + \alpha_6 + \alpha'_6 + \lambda &= 0, \\ \alpha_4 + \alpha'_4 + \alpha_5 + \alpha'_5 - \alpha_6 - \alpha'_6 + \alpha_7 + \alpha'_7 - \lambda &= 0, \\ \alpha_1 + \alpha'_1 + \alpha_5 + \alpha'_5 - \alpha_6 - \alpha'_6 + \alpha_7 + \alpha'_7 - \lambda &= 0, \\ \alpha_2 + \alpha'_2 + \alpha_4 + \alpha'_4 + \alpha_5 + \alpha'_5 + \alpha_6 + \alpha'_6 + \lambda &= 0. \end{aligned} \quad (4.14)$$

Solving (4.12), (4.13) and (4.14) yields

$$\alpha_i + \alpha'_i = 0, i \neq 6, 8, \alpha_6 + \alpha'_6 = -\lambda. \quad (4.15)$$

Since $f(x, y, y)$ is of type $[4, 3]$, it is 0 identically. So from (4.11) and

$$\begin{aligned} k_i(x, y, y) &= 0, \quad i = 3, 6, 7, 8, \quad h_j(x, y, y) = 0, \quad j = 1, 2, \\ k_1(x, y, y) &= -k_2(x, y, y) = k_4(x, y, y) = \frac{1}{2}k_5(x, y, y) = [[y, x], [y^2, x]], \\ h_3(x, y, y) &= h_4(x, y, y) = [[y, x], [x^2, y]], \end{aligned}$$

we have

$$\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 = 0, \beta_3 + \beta_4 = 0. \quad (4.16)$$

Using Lemma 2.1.1 and considering the monomials

$$\begin{aligned} & -yyxxx-, -yxyxx-, -xyxyx-, -yyxzz-, -zyxyz-, \\ & -yzxyx-, yyyzz-, -yyzxx-, -yxzyx-, \end{aligned}$$

where the empty positions of the first three words are z, z while others are x, x yields

$$\begin{aligned} \alpha_1 + \alpha_3 + \alpha_6 - \beta_1 &= 0, \alpha_1 + \alpha_5 + \alpha_6 + \beta_2 - \beta_3 = 0, \alpha_1 - \alpha_4 - \alpha_8 = 0, \\ -\alpha_2 + \alpha_4 &= 0, \alpha_4 + \alpha_5 + \alpha_7 + \alpha'_8 - \beta_3 - \lambda = 0, \\ \alpha_1 - \alpha_4 + \alpha_5 - \alpha_7 - \alpha'_8 - \beta_4 + \lambda &= 0, \alpha_3 + \alpha_7 - \lambda = 0, \\ \alpha_2 - \alpha_3 + \alpha_4 - \alpha'_6 - 2\alpha_7 + \lambda &= 0, \\ -\alpha_1 + \alpha_3 - \alpha_4 - \alpha_5 + \alpha'_6 + 3\alpha_7 + \alpha'_8 + \beta_4 - 2\lambda &= 0. \end{aligned} \quad (4.17)$$

For instance, from Lemma 2.1.1 the sum of the coefficients of the monomials

$$zyyxxxx, xxxxyyz, zzyyxxx, zzzxyy, yyxxxxx, xxxyyzz$$

is 0. Since f has no terms which end in zz and symmetrically it has no terms which start with zz from (4.11), the sum of the coefficients of the monomials $zyyxxxx, xxxxyyz$ is 0. So $\alpha_1 + \alpha_3 + \alpha_6 - \beta_1 = 0$. This is the first equation in system (4.17). The others in (4.17) can be gotten in the same way.

Next, substituting $e[12], e[22], e[23]$ for x, y, z in $f(x, y, z)$ yield

$$-\alpha_1 - \alpha_6 + \alpha_7 - 2e + \beta_1 - \beta_3 = 0, \alpha_2 + \alpha_5 + \alpha'_6 + 4\alpha_7 + \alpha'_8 = 0. \quad (4.18)$$

The solution of the systems of (4.15), (4.16), (4.17) and (4.18) is

$$\begin{aligned} \alpha_8 &= -5\alpha_7 - 2\beta_2, \beta_1 = -4\alpha_7 - 2\beta_2, \beta_4 = \alpha_3 + \alpha_7, \alpha'_8 = -3\alpha_7 - \beta_2, \\ \alpha'_6 &= -3\alpha_7, \alpha_6 = 2\alpha_7 - \alpha_3, \alpha_4 = -\alpha_7, \alpha_1 = -6\alpha_7 - 2\beta_2, \\ \beta_3 &= -\alpha_3 - \alpha_7, \alpha_2 = -\alpha_7, \alpha_5 = 3\alpha_7 + \beta_2, \lambda = \alpha_3 + \alpha_7. \end{aligned}$$

Thus $f = \alpha_3 g_1 + \alpha_7 g_2 + \beta_3 g_3$, where g_1, g_2, g_3 are independent from f . So $Q_1 + Q_2, Q_3, K$ are the basis of $VIN_3[3, 2, 2]$ by Lemma 2.1.3. ■

Let

$$\begin{aligned}
Q_1 &:= Q(x \circ y, y, x, z_1, z_1), \quad Q_2 := (Id - (z_1, z_2))Q(x \circ y, z_1, x, x, y, z_2), \\
Q_3 &:= (Id - (z_1, z_2))Q(x^2, z_1, y, y, x, z_2), \quad Q_4 := (Id - (z_1, z_2))Q(x \circ z_1, y, x, y, x, z_2), \\
Q_5 &:= K(x, y, x, z_1, x, z_2, y), \quad Q_6 := (Id - (z_1, z_2))K(x, y, x, z_1, y, z_2, x), \\
Q_7 &:= T_6(x \circ y, y, x, z_1, z_2; x), \quad Q_8 := T_6(x^2, z_1, z_2, x, y; y), \\
Q_9 &:= xQ(x, y, z_1, x, y, z_2), \quad Q_{10} := Q(x, y, z_1, x, y, z_2)x,
\end{aligned}$$

where $K(x_1, x_2, x_3, x_4, x_5, x_6, x_7) := [S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7]$. Then we have

Proposition 4.2.3 *The set $\{Q_1, \dots, Q_{10}\}$ is a basis of the vector space $VIN_3[3, 2, 1, 1]$.*

Proof. First we show that $Q_1, \dots, Q_{10}$ are linearly independent. Let $\alpha_i \in F$ such that $\sum_{i=1}^{10} \alpha_i Q_i = 0$. Then considering the coefficients of the following monomials

$$\begin{aligned}
&yxxyz_1z_2x, \quad yxz_1xyx_2x, \quad yxyx_1xz_2x, \quad yxxz_1yz_2x, \quad z_1xxyyz_2x, \\
&yx_1xyxz_2x, \quad z_1xyxyx_2x, \quad xxz_1z_2yyx, \quad yxxyxz_1z_2, \quad yxz_1xxz_2y,
\end{aligned}$$

yields

$$\begin{aligned}
\alpha_3 + 2\alpha_2 - \alpha_8 &= 0, \quad -\alpha_1 - \alpha_4 - \alpha_5 + \alpha_6 + \alpha_7 - \alpha_{10} = 0, \\
\alpha_4 - \alpha_5 - \alpha_6 - \alpha_{10} &= 0, \quad 2\alpha_6 - \alpha_9 + \alpha_{10} = 0, \quad -\alpha_4 + \alpha_6 - \alpha_{10} = 0, \\
\alpha_4 + \alpha_5 + \alpha_6 + \alpha_{10} &= 0, \quad -\alpha_2 - \alpha_6 - \alpha_7 + \alpha_{10} = 0, \\
\alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 &= 0, \quad \alpha_1 + \alpha_7 + \alpha_8 = 0, \quad \alpha_3 + 2\alpha_4 + \alpha_5 = 0.
\end{aligned} \tag{4.19}$$

The solution of this system is trivial. So $Q_1, \dots, Q_{10}$ are linearly independent.

Let $f(x, y, z_1, z_2) \in VIN_3[3, 2, 1, 1]$ then

$$\begin{aligned}
f &= (Id + (z_1, z_2))\{f_1xz_2 + f'_1z_2x + f_2yz_2 + f'_2z_2y + f_3z_2xx\} + f_4[z_1, z_2] + \\
&\quad f_5yxx + f_6xyy + f_7xy + f_8yx,
\end{aligned} \tag{4.20}$$

since H_2 has no identity of type $[3, 1]$. Moreover

$$\begin{aligned}
f_1 &= \sum_{i=1}^8 \alpha_i k_i, \quad f'_1 = \sum_{i=1}^8 \alpha'_i k_i, \quad f_2 = \sum_{i=1}^4 \beta_i h_i, \quad f'_2 = \sum_{i=1}^4 \beta'_i h_i, \\
f_3 &= \alpha[[y, x], [y, z_1]], \quad f_4 = \beta[[y, x], [y, x^2]], \quad f_6 = \gamma[[x, z_1], [x, z_2]],
\end{aligned}$$

where $k_1, \dots, k_8$ and $h_1, \dots, h_4$ are the basis of the spaces $VI_2[2, 2, 1]$ and $VI_2[3, 1, 1]$ given in proposition 3.1.3 and 3.1.2 respectively.

Since $f(x, y, z_1, 1)$ is an identity of H_3 of type $[3, 2, 1]$, it is 0 identically. From (4.20), f_5 and f_6 are of type $[1^4]$ and $[2, 1, 1]$ respectively. Thus they are linear combinations of the products of two commutators by Corollary 3.2 of [MR]. So $f_5(x, y, z_1, 1) = f_6(x, y, z_1, 1) = 0$. From the definitions of k_i and h_i , $k_i(x, y, z_1, 1) = h_i(x, y, z_1, 1) = 0$. So

$$f(x, y, z_1, 1) = f_1x + f'_1x + f_2y + f'_2y + f_3xx + f_7xy + f_8yx.$$

Hence the coefficients of the following monomials are 0:

$$xyyxz_1x, yxxyz_1x, yxyxz_1x, z_1yyxx, yxyz_1xx, yxz_1yxx, xyz_1yxx, xyxyz_1x.$$

Since the last two variables of the monomials above are not xy and yx , these monomials occur in $f_1x + f'_1x + f_3xx$. Thus we have

$$\begin{aligned} \alpha_2 + \alpha'_2 - \alpha_4 - \alpha'_4 &= 0, \quad \alpha_3 + \alpha'_3 - \alpha_5 - \alpha'_5 = 0, \\ \alpha_5 + \alpha'_5 + \alpha_7 + \alpha'_7 &= 0, \quad \alpha_4 + \alpha'_4 + \alpha_7 + \alpha'_7 = 0, \\ \alpha_1 + \alpha'_1 + \alpha_6 + \alpha'_6 - \alpha_7 - \alpha'_7 + \alpha &= 0, \\ \alpha_1 + \alpha'_1 + \alpha_3 + \alpha'_3 + \alpha_6 + \alpha'_6 + \alpha &= 0, \\ \alpha_4 + \alpha'_4 + \alpha_5 + \alpha'_5 + \alpha_6 + \alpha'_6 - \alpha_7 - \alpha'_7 + \alpha &= 0, \\ \alpha_1 + \alpha'_1 - \alpha_4 - \alpha'_4 + \alpha_5 + \alpha'_5 - \alpha_6 - \alpha'_6 + \alpha_7 + \alpha'_7 - \alpha &= 0. \end{aligned}$$

Similarly considering the coefficients of the monomials

$$xyxxz_1y, xz_1xyy, z_1xyy$$

in $f(x, y, z_1, 1)$ yields

$$-\beta_1 - \beta'_1 - \beta_4 - \beta'_4 = 0, \quad \beta_1 + \beta'_1 - \beta_3 - \beta'_3 = 0, \quad \beta_1 + \beta'_1 = 0.$$

Solving the equations above we have

$$\alpha_i = -\alpha'_i, \quad i \neq 6, 8, \quad \beta_j = -\beta'_j, \quad j \neq 2. \quad (4.21)$$

Now we assume that $f^* = \pm f$. Let g be the sum of the terms of f in which the first two positions and the last two positions are xy , yx or xx . Since f is skew symmetric in z_1, z_2 , we need only consider the coefficients of the following monomials in g when $f^* = \pm f$.

$$\begin{aligned} &xyz_1z_2xxy, xyz_1z_2xyx, yxz_1z_2xxy, yxz_1z_2xyx, xyz_1xz_2xy, xyz_1xz_2yx, yxz_1xz_2xy, \\ &yxz_1xz_2yx, xyz_1z_2yxx, xxz_1z_2yyx, xxz_1z_2yxy, yxz_1yz_2xx, xyz_1yz_2xx. \end{aligned}$$

Let γ_i denote the coefficients of the first 7 monomials above in the polynomial g and let δ_i denote the coefficients of the last 6 monomials above. Then using Lemma 2.1.1 we have

$$\delta_1 + \delta_2 = 0, \delta_3 + \delta_4 = 0, \delta_5 + \delta_6 = 0, \gamma_1 + \gamma_2 + \delta_2 = 0, \gamma_3 + \gamma_4 + \delta_1 = 0. \quad (4.22)$$

By Lemma 2.1.1, the sum of the coefficients of the following monomials is 0:

$$xywxxy, xywxyx, yxwxxy, yxwxyx,$$

where $w = z_1xz_2$. That is,

$$\gamma_5 \pm (-\gamma_5) + \gamma_6 + \gamma_7 = 0 \quad (4.23)$$

since $f^* = \pm f$,

Next we distinguish two cases. First assume that $f^* = f$. Since $f(x, y, z_1, y)$ and $f(x, y, z_1, x)$ are identities of type [3,3,1] and [4,2,1] respectively, there exist scalars $\lambda_i \in F$ such that

$$f(x, y, z_1, y) = \lambda_1 Q(x \circ y, x, y, z_1, x, y), \quad f(x, y, z_1, x) = \lambda_2 Q(x^2, x, y, z_1, x, y).$$

But $f^* = f$ and $Q^* = -Q$, so $\lambda_i = 0$ for $i = 1, 2$. Hence $f(x, y, z_1, y) = f(x, y, z_1, x) = 0$. Thus considering the coefficients of monomials

$$\begin{aligned} &xyz_1yxxy, xyz_1yxxy, yxz_1yxxy, yxz_1yxxy, xyz_1xyxy, \\ &xyyxz_1y, z_1yyxxxy, yxyxz_1xy, xyxzy_1xy, xxyyz_1xy, xyxz_1xxy, \end{aligned}$$

in $f(x, y, z_1, y)$ yields

$$\gamma_1 + \alpha_1 - \alpha_3 + \alpha_5 + \alpha_6 - \alpha_7 = 0, \quad \gamma_2 - \alpha_1 - \alpha_5 + \alpha'_6 = 0,$$

$$\begin{aligned}
\gamma_3 + \alpha_1 + \alpha_3 - \alpha_4 - \alpha_6 + \alpha_7 &= 0, \quad \gamma_4 - \alpha_1 + \alpha_4 - \alpha'_6 = 0, \\
\gamma_5 - \alpha_1 - \alpha_5 - \alpha_8 &= 0, \quad \alpha_1 + \alpha_2 - \alpha_4 + \alpha_7 + \alpha'_8 - \gamma_5 = 0, \\
\alpha_1 + \alpha_3 + \alpha_6 - \beta_1 &= 0, \quad -\alpha_1 + \alpha_4 + \alpha_8 = 0, \\
\alpha_3 - \alpha_5 + \beta - \beta_1 - \beta_2 - \beta_4 &= 0, \\
-\alpha_3 - \alpha_7 + \delta_1 + \delta_5 &= 0, \\
-\alpha_2 + \alpha_3 - \alpha_6 + \alpha_8 - \gamma_1 &= 0.
\end{aligned} \tag{4.24}$$

For instance, the monomial xyz_1yxxxy comes from the monomials xz_2z_1yxxxy , xyz_1z_2xxxy and xyz_1yxxz_2 of f which have coefficients $-\alpha_3 + \alpha_5$, γ_1 , $\alpha_1 + \alpha_6 - \alpha_7$ respectively by (4.20), the expression of g and $f^* = f$. So we get the first equation of (4.24). The others were obtained in the same way.

Since $f(x, y, z_1, y) = -f_2(x, y, y)yz_1 + f_4[z_1, y] + \sim$, considering the sum of the terms of $f(x, y, z_1, y)$ which end in yz_1 yields

$$\beta - \beta_3 - \beta_4 = 0. \tag{4.25}$$

It can be proved also that

$$\gamma + \beta_2 + \beta'_2 = 0, \tag{4.26}$$

by considering the sum of the terms of $f(x, y, z_1, y)$ which end in xyy . Indeed, from (4.20) this kind of terms occur in

$$\begin{aligned}
&f_2(x, y, z_1)yy + f'_2(x, y, z_1)yy + f_7(x, z_1, y)xyy \\
&= (\beta_2 + \beta'_2 + \gamma)[[x, z_1], [x, y]]xyy.
\end{aligned}$$

Thus (4.26) holds.

By $f(x, y, z_1, 1) = 0$, considering the coefficients of the monomials xyz_1xyx , yxz_1xxy and xyz_1xxy yields

$$\gamma_6 = \gamma_7 = 0, \quad \gamma_5 - \alpha_8 - \alpha'_8 + \beta_2 + \beta'_2 = 0. \tag{4.27}$$

From $f^* = f$, considering the coefficient of the monomial yz_2yxxz_1x in f yields

$$-\alpha_1 - \alpha_3 + \alpha'_6 = -\beta_1 + \beta_3. \tag{4.28}$$

From $f(x, y, z_1, x) = 0$, considering the coefficients of the monomials

$$\begin{aligned} & yxz_1xyxx, \quad xzx_1xyyx, \quad xzx_1yxx, \quad xyz_1yxxx, \\ & yxyxxz_1x, \quad xyz_1xxxy, \quad xyz_1xxx, \end{aligned}$$

yields

$$\begin{aligned} \delta_1 - \beta_4 + \alpha_8 + \alpha'_8 &= 0, \quad \delta_3 + \beta_3 = 0, \quad \beta_3 - \delta_5 = 0, \quad \delta_6 + \alpha_6 + \alpha'_6 = 0, \\ \beta - \alpha_1 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 0, \quad \beta_2 + \gamma_2 = 0, \quad \beta'_2 - \beta_4 + \gamma_1 + \gamma_5 = 0. \end{aligned} \quad (4.29)$$

[Note. When both $---z_2x$ and $---xz_2$ occur in calculating the coefficients of the monomials above we need only consider α_i and α'_i for $i = 6, 8$ by (4.21)].

Next we use Lemma 2.1.1 to produce more equations. Let $w = z_1z_2xyy$ then the sum of the coefficients of the following monomials is 0:

$$xwx, \quad xw^*x, \quad xxw, \quad xxw^*, \quad wxx, \quad w^*xx.$$

This yields

$$-\beta + \alpha_2 - \alpha_4 = 0. \quad (4.30)$$

Similarly considering $w = z_1xz_2yy$, xz_1z_2yy , z_1z_2yxy yields

$$-\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 + \alpha'_6 + \alpha_7 = 0, \quad \delta_3 - \alpha_3 - \alpha_7 = 0, \quad \beta + \delta_4 + \alpha_5 + \alpha_7 = 0. \quad (4.31)$$

Solving systems (4.22), ..., (4.31) we have

$$\begin{aligned} \alpha &= \delta_6, \quad \beta = \delta_6, \quad \gamma = \gamma_5, \quad \alpha_1 = \gamma_5 - \frac{1}{2}\delta_6, \quad \alpha_2 = \frac{1}{2}\gamma_5 + \frac{3}{4}\delta_6, \quad \alpha_3 = -\frac{1}{2}\gamma_5 + \frac{7}{4}\delta_6, \\ \alpha_4 &= \frac{1}{2}\gamma_5 - \frac{1}{4}\delta_6, \quad \alpha_5 = -\frac{1}{2}\gamma_5 + \frac{3}{4}\delta_6, \quad \alpha_6 = -\frac{1}{2}\gamma_5 - \frac{5}{4}\delta_6, \quad \alpha_7 = \frac{1}{2}\gamma_5 - \frac{3}{4}\delta_6, \\ \alpha'_6 &= \frac{1}{2}\gamma_5 + \frac{1}{4}\delta_6, \quad \alpha_8 = \frac{1}{2}\gamma_5 - \frac{1}{4}\delta_6, \quad \alpha'_8 = -\frac{1}{2}\gamma_5 + \frac{1}{4}\delta_6, \quad \beta_1 = 0, \quad \beta_2 = 0, \\ \beta_3 &= -\delta_6, \quad \beta_4 = 2\delta_6, \\ \beta'_2 &= -\gamma_5, \quad \gamma_1 = 2\delta_6, \quad \gamma_2 = \gamma_6 = \gamma_7 = 0, \\ \gamma_3 &= -\gamma_5 - 2\delta_6, \quad \gamma_4 = \gamma_5, \quad \delta_1 = 2\delta_6 = -\delta_2, \quad \delta_3 = -\delta_4 = -\delta_5 = \delta_6. \end{aligned} \quad (4.32)$$

Let

$$x = e[22] + e[12] + e[13], \quad y = e[23], \quad z_1 = e[11], \quad z_2 = e[33]$$

then $f(x, y, z_1, z_2) = 0$ implies that $\gamma_5 + \delta_6 = 0$ by considering the coefficients of e_{23} . Indeed, only the following 11 monomials in f evaluate to e_{23} :

$$\begin{aligned} &xyyxz_1xz_2, \quad xyxz_1xyz_2, \quad xxz_1xyyz_2, \quad xyz_2xz_1xy, \quad xyz_2yxz_1x, \quad xz_1xyxyz_2, \\ &yxz_1xxyz_2, \quad yz_2yxxz_1x, \quad yz_2xz_1xxy, \quad yyxzx_1xz_2, \quad xz_1xz_2yxy. \end{aligned}$$

Their coefficients in f are

$$\begin{aligned} &\alpha_2 - \alpha_4, \quad -\beta_2, \quad \beta_3, \quad -\gamma_5, \quad \alpha_1 - \alpha'_6 - \alpha_7, \quad \beta_2, \quad -\beta_1 - \beta_2 - \beta_4, \\ &\alpha_1 + \alpha'_6 - \alpha_3, \quad \beta'_2 - \beta_4, \quad -\alpha_2 - \alpha_7, \quad \alpha_4 + \alpha_5 - \alpha'_6 - \alpha_7. \end{aligned}$$

Thus from (4.32) and adding them up we get $-4\gamma_5 - 4\delta_6 = 0$.

So far we have determined the coefficients of $f_1, f'_1, f_2, f'_2, f_3, f_4$ and f_6 . The coefficients of the monomials in f_5yxx, f_7xy and f_8yx have also been determined since the monomials in these polynomials either start with $z_i x, x z_i, y z_i, z_i y$ or the monomials in g . Since $f^* = f$, the coefficients of the first kind are the linear combination of $\alpha_i, \alpha'_i, \beta_i, \beta'_i$. While the second kind have coefficients γ_j or δ_j . So $f = \delta_6 h$, where h is independent from f . On the other hand, $Q_9 - Q_{10}$ is $\star$ -symmetric and skew in z_1, z_2 . From Lemma 2.1.3 the vector space $W := \{f \in VIN_3[3, 2, 1, 1] | f^* = f\}$ has a basis $Q_9 - Q_{10}$.

Now let $f^* = -f$. We also have

$$f(x, y, z_1, y) = \lambda_1 Q(x \circ y, x, y, z_1, x, y), \quad f(x, y, z_1, x) = \lambda_2 Q(x^2, x, y, z_1, x, y)$$

for some $\lambda_i \in F$. We may assume that $\lambda_i = 0, i = 1, 2$. Repeating the proof of the case $f^* = f$, we also have (4.23), (4.25) and (4.26). Similarly to the first 7 equations in (4.24) we have

$$\begin{aligned} &\gamma_1 + \alpha_1 - \alpha_3 + \alpha_5 + \alpha_6 - \alpha_7 = 0, \quad \gamma_2 - \alpha_1 + \alpha_5 + \alpha'_6 + 2\alpha_7 = 0, \\ &\gamma_3 + \alpha_1 + \alpha_3 - \alpha_4 + 2\alpha_5 - \alpha_6 + \alpha_7 = 0, \\ &\gamma_4 - \alpha_1 + \alpha_4 - 2\alpha_5 - \alpha'_6 - 2\alpha_7 = 0, \quad \gamma_5 - \alpha_1 + 2\alpha_4 - \alpha_5 + 2\alpha_7 - \alpha_8 = 0, \\ &-\gamma_5 - \alpha_1 + \alpha_2 - \alpha_4 - \alpha_7 - \alpha'_8 = 0, \quad \alpha_1 + \alpha_3 + \alpha_6 - \beta_1 = 0. \end{aligned} \quad (4.33)$$

Similarly to (4.29) we have

$$\begin{aligned}\delta_1 + \beta_4 + \alpha_8 + \alpha'_8 &= 0, \delta_3 + \beta_3 = 0, \beta_3 + \delta_5 = 0, \\ \delta_6 + \alpha_6 + \alpha'_6 &= 0, \beta - \alpha_1 - \alpha_3 + 2\alpha_4 + \alpha_5 = 0, \\ 2(\beta_1 + \beta_4) + \beta_2 + \gamma_2 + \gamma_6 &= 0, -2\beta_1 + \beta'_2 - \beta_4 + \gamma_1 + \gamma_5 = 0.\end{aligned}\quad (4.34)$$

From Lemma 2.1.1 if $w = z_2 xxy$ then the sum of the coefficients of the monomials

$$xyz_1w, yz_1xw, z_1xyw, z_1yxw, yxz_1w, xz_1yw$$

is 0. Similarly if $w = yxz_1x$ then the sum of the coefficients of the monomials of the form $w---$ is 0, where empty positions are x, y, z_2 . These yield

$$\beta_2 + \beta'_2 + \gamma_1 + \gamma_3 - \alpha_8 - \alpha'_8 = 0, -\beta_2 - \beta'_2 + \gamma_5 + \gamma_7 + \alpha_8 + \alpha'_8 = 0. \quad (4.35)$$

From $f(x, y, z_1, 1) = 0$, considering the coefficients of the monomials $xyz_1xxy, xyz_1xyx, yxz_1xxy$ in it respectively yields

$$\gamma_5 + \alpha_8 + \alpha'_8 + \beta_2 + \beta'_2 = 0, \gamma_6 - 2(\alpha_8 + \alpha'_8) = 0, \gamma_7 - 2(\beta_2 + \beta'_2) = 0. \quad (4.36)$$

Solving the systems (4.22), (4.23), (4.25), (4.26), (4.33), (4.34), (4.35) and (4.36) we know that

$$\alpha'_8, \beta'_2, \gamma_4, \gamma_6, \gamma_7, \delta_2, \delta_6$$

are the parameters of the coefficients, i.e. every coefficient is a linear combination of

$$\alpha'_8, \beta'_2, \gamma_4, \gamma_6, \gamma_7, \delta_2, \delta_6.$$

Since we assume that $\lambda_i = 0, i = 1, 2$,

$$f = \alpha'_8 g_1 + \beta'_2 g_2 + \gamma_4 g_3 + \gamma_6 g_4 + \gamma_7 g_5 + \delta_2 g_6 + \delta_6 g_7 + \lambda_1 g_8 + \lambda_2 g_9,$$

where $g_1, \dots, g_9$ are independent from f . Since $Q_1, \dots, Q_8, Q_9 + Q_{10}$ are $\ast$ -skew symmetric and linearly independent, by Lemma 2.1.3, $\{Q_1, \dots, Q_8, Q_9 - Q_{10}\}$ is a basis of the vector space $U := \{f \in VIN_3[3, 2, 1, 1] | f^\ast = -f\}$. Hence $\{Q_1, \dots, Q_{10}\}$ is a basis of the vector space $VIN_3[3, 2, 1, 1]$. ■

Let

$$\begin{aligned}
Q_1 &:= T_6(x, y_1, y_2, y_3, y_4; x^2), \quad Q_2 := T_6(x^2, y_1, y_2, y_3, y_4; x), \\
Q_3 &:= xT_6(x, y_1, y_2, y_3, y_4; x), \quad Q_4 := T_6(x, y_1, y_2, y_3, y_4; x)x, \\
Q_5 &:= \sum_{(1234)} (-1)^\sigma K(x, y_1, x, y_2, x, y_3, y_4), \\
Q_6 &:= \sum_{(1234)} (-1)^\sigma T_6(x, x \circ y_1, y_2, y_3, y_4; x), \\
Q_7 &:= \sum_{(1234)} (-1)^\sigma T_6(x, x^2, y_1, y_2, y_3; y_4),
\end{aligned}$$

where the sum is taken over all cyclic permutations of 1234 and $(-1)^\sigma$ is the sign of the cyclic permutation σ . Then

Proposition 4.2.4 *The set $\{Q_1, \dots, Q_7\}$ is a basis of the vector space $VIN_3[3, 1^4]$.*

Proof. Let $f(x, y_1, y_2, y_3, y_4) \in VIN_3[3, 1^4]$. Then

$$\begin{aligned}
f &= \sum_{\sigma(i) < \sigma(i+1), i=1,3} (-1)^\sigma f_1(x, y_{\sigma(1)}, y_{\sigma(2)})(y_{\sigma(3)}, y_{\sigma(4)}) + \\
&= \sum_{\sigma(1) < \sigma(2) < \sigma(3)} (-1)^\sigma \{f_2(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)})x y_{\sigma(4)} + \\
&\quad f'_2(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)})y_{\sigma(4)}x + f_3(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)})y_{\sigma(4)}xx\}. \quad (4.37)
\end{aligned}$$

Moreover,

$$f_1 = \sum_{i=1}^4 \alpha_i h_i, \quad f'_1 = \sum_{i=1}^4 \alpha'_i h_i, \quad f_2 = \sum_{i=1}^7 \beta_i L_i, \quad f'_2 = \sum_{i=1}^7 \beta'_i L_i,$$

where $h_1, \dots, h_4$ and $L_1, \dots, L_7$ are the bases of $VI_2[3, 1, 1]$ and $VIN_2[2, 1, 1, 1]$ given in Chapter 3. f_3 is an identity of H_2 of type $[1^4]$. So it is a linear combination of the products of two commutators by Corollary 3.2 of [MR]. Since $f_3(x, y_1, y_2, y_3)$ is skew in the y_i 's,

$$f_3 = \sum_{(123)} \{\gamma_1[x, y_1][y_2, y_3] + \gamma_2[y_1, y_2][x, y_3]\}.$$

Since f_1 is skew in the variables of degree 1, from the definition of h_i , we have $\alpha_3 = -\alpha_4$. Also from the definitions of h_i and L_i if we let one of degree 1 variables in L_i or in h_j

be equal to 1 then $L_i = h_j = 0$, for $i = 1, \dots, 6$ and $j = 1, \dots, 4$. Thus from (4.37), $f(x, y_1, y_2, y_3, 1) = 0$ implies that

$$\gamma_1 + \beta_7 + \beta'_7 = 0, \gamma_2 = 0, 3\beta'_7 + \beta_7 = 0, \quad (4.38)$$

by considering the coefficients of the monomials $xy_1y_2y_3xx$, $y_1y_2y_3xxx$, $xy_2xy_1y_3x$ in $f(x, y_1, y_2, y_3, 1)$ respectively.

Since $f(x, y_1, y_2, y_3, x) \in VIN_3[4, 1, 1, 1]$, there exist $\alpha, \beta \in F$ such that $f(x, y_1, y_2, y_3, x) = \alpha T_6 + \beta K$, where T_6, K are the basis of $VIN_3[4, 1^3]$ given in Proposition 4.1.3. As before we assume that $\alpha = \beta = 0$. Then the sum of the terms in $f(x, y_1, y_2, y_3, x)$ which end in xx is 0 implies that

$$(*) \quad f_3(x, y_1, y_2, y_3)x + f_2 + f'_2 = 0.$$

Since $f_3(x, y_1, y_2, y_3)x = \gamma_1 L_3(x, y_1, y_2, y_3)$ and $L_1, \dots, L_7$ are linearly independent, the equation $(*)$ implies that

$$\gamma_1 + \beta_3 + \beta'_3 = 0, \beta_i + \beta'_i = 0, i \neq 3. \quad (4.39)$$

The sum of the terms of $f(x, y_1, y_2, y_3, x)$ which end in y_3x is 0. This implies that

$$(**) \quad f_1(x, y_1, y_2)y_3x - f'_2(x, y_1, y_2, x)y_3x = 0.$$

This implies that

$$\alpha_1 h_1 + \alpha_2 h_2 + \alpha_3(h_3 - h_4) + \sum_{i=1}^6 \beta_i L_i(x, y_1, y_2, x) = 0$$

since $\alpha_7 = 0$ from (4.38) and (4.39). Since

$$\begin{aligned} L_1(x, y_1, y_2, x) &= h_2 + h_3 - h_4, \quad L_2(x, y_1, y_2, x) = -h_2 - h_3 + h_4, \\ L_3(x, y_1, y_2, x) &= h_2, \quad L_4(x, y_1, y_2, x) = -h_2, \\ L_5(x, y_1, y_2, x) &= h_1 + h_3 + h_4, \quad L_6(x, y_1, y_2, x) = -h_1, \end{aligned}$$

and $h_1, \dots, h_4$ are linearly independent,

$$\alpha_1 + \beta_5 - \beta_6 = 0, \alpha_2 + \beta_1 - \beta_2 + \beta_3 - \beta_4 = 0, \alpha_3 + \beta_1 - \beta_2 - \beta_5 = 0. \quad (4.40)$$

Let $w = y_1 x x y_2 y_3$, then from Lemma 2.1.1 the sum of the coefficients of the monomials

$$x w y_4, x w^* y_4, x y_4 w, x y_4 w^*, w x y_4, w^* x y_4,$$

$$y_4 w x, y_4 w^* x, y_4 x w, y_4 x w^*, w y_4 x, w^* y_4 x$$

is 0. Since $\beta_i = -\beta_i^*$ for $i = 1, \dots, 6$ from (4.38) and (4.39), the sum of the coefficients of the above 12 monomials is

$$-2\alpha_1 - \alpha_2 - 2\alpha_4 - \beta_2 + \beta_4 + 2\beta_6 = 0. \quad (4.41)$$

Solving the systems (4.38), ..., (4.41) we know that $\beta_2, \dots, \beta_6$ are parameters of the coefficients. Since we assume that $\alpha = \beta = 0$,

$$f = \alpha g_0 + \beta g_1 + \sum_{i=2}^6 \beta_i g_i,$$

where $g_0, \dots, g_6$ are independent from f .

Next we show that $Q_1, \dots, Q_7$ are linearly independent. If there exist scalars $\alpha_i \in F$ such that $\sum_{i=1}^7 \alpha_i Q_i = 0$ then considering the coefficients of the monomials

$$x y_1 y_2 y_3 y_4 x x, y_1 y_2 x x y_3 y_4 x, x y_1 x y_2 x y_3 y_4, y_1 y_2 x y_3 x y_4 x,$$

$$x y_1 x y_2 y_3 y_4 x, x y_1 y_2 y_3 x y_4 x, x y_1 y_2 x y_3 x y_4, y_1 x x x y_2 y_3 y_4,$$

yields

$$-\alpha_1 + \alpha_2 + 2\alpha_7 = 0, -\alpha_2 + \alpha_5 - 2\alpha_6 - 3\alpha_7 = 0, -\alpha_3 + \alpha_5 - \alpha_6 = 0,$$

$$\alpha_4 - \alpha_5 + \alpha_6 = 0, -\alpha_4 - \alpha_6 = 0, \alpha_3 + \alpha_6 = 0, \alpha_3 + \alpha_5 + 3\alpha_6 = 0, \alpha_1 + \alpha_2 + 2\alpha_6 = 0.$$

The solution is trivial. So $Q_1, \dots, Q_7$ are linearly independent, completing the proof. $\blacksquare$

Note. Let

$$Q_8 := \sum_{\sigma(i) < \sigma(i+1), i=1,3} (-1)^\sigma K(x, y_{\sigma(1)}, x, y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}, x),$$

$$Q_9 := \sum_{\sigma(i) < \sigma(i+1), i=1,3} (-1)^\sigma Q(x^2, x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}),$$

$$Q_{10} := \sum_{\sigma(3) < \sigma(4)} (-1)^\sigma Q(x, x \circ y_{\sigma(1)}, y_{\sigma(2)}, x, y_{\sigma(3)}, y_{\sigma(4)}).$$

Then

$$\begin{aligned} -34Q_1 - 10Q_2 + 8Q_3 + 8Q_4 + 6Q_5 + 9Q_6 + 2Q_8 + 2Q_9 + 13Q_{10} &= 0, \\ 7Q_1 + Q_2 - 2Q_6 + Q_7 - 2Q_{10} &= 0, \\ \frac{21}{4}Q_1 - \frac{5}{4}Q_2 + 4Q_3 + 3Q_5 - 2Q_6 + \frac{11}{4}Q_7 + Q_8 &= 0. \end{aligned}$$

4.3 The Identities of H_3 of Type $[2, \dots]$

Let

$$\begin{aligned} Q_1 &:= \sum_{\sigma \in S_3} Q(x_{\sigma(1)} \circ x_{\sigma(2)}, x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(2)}, x_{\sigma(3)}, y), \\ Q_2 &:= \sum_{(123)} Q(x_1 \circ y, x_2, x_3, x_1, x_2, x_3), \\ Q_3 &:= \sum_{(123)} Q(x_1, x_2, x_3, y, x_2, x_3)x_1, \quad Q_4 := \sum_{(123)} x_1 Q(x_1, x_2, x_3, y, x_2, x_3), \\ Q_5 &:= \sum_{(123)} Q(x_1^2, x_2, x_3, y, x_2, x_3), \quad Q_6 := K(x_1, x_2, x_1, x_3, x_2, x_3, y), \\ Q_7 &:= \sum_{\sigma \in S_3} K(x_{\sigma(1)}, y, x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(2)}). \end{aligned}$$

Then

Proposition 4.3.1 *The set $\{Q_i | i = 1, \dots, 7\}$ is a basis of the vector space $VIN_3[2, 2, 2, 1]$.*

Proof. Let $f(x, y, z, w) \in VIN_3[2^3, 1]$ then $f(x, x, z, w)$ and $f(x, y, z, x)$ are identities of type $[4, 2, 1]$ and $[3, 2, 2]$ respectively, and $f(x, y, z, x)$ is symmetric in the variables of degree

2. Thus there exist $\alpha, \lambda_1, \lambda_2, \lambda_3 \in F$ such that

$$f(x, x, z, w) = \alpha Q(x^2, x, z, w, x, z), \quad f(x, y, z, x) = \sum_{i=1}^3 \lambda_i g_i,$$

where g_1, g_2, g_3 are the basis of $VIN_3[3, 2, 2]$ given in Proposition 4.2.2.

Since $g_1, g_2, g_3, Q(x^2, x, z, w, x, z)$ are independent from f we assume that $\alpha = \lambda_1 = \lambda_2 = \lambda_3 = 0$. Then $f(x, x, z, w) = f(x, y, z, x) = 0$.

The words involving exactly 2 x's and 2 y's are

$$xxyy, xyxy, xyxx, yyyx, yxyx, yxxy.$$

Let $A := xxyy$, $B := xyxy$, $C := xyxx$. Putting w in the i -th position of A we get a new word denoted by A_i (e.g. $A_1 = wxxyy$). Similarly we define B_i and C_i . Let A_{ij} (resp. A_{ijk}) denote the word which is obtained by inserting z in j -th (resp. k -th) position of A_i (resp. A_{ij}). (e.g. $A_{13} = wxzxyy$, $A_{135} = wxzxyzxyy$). Similarly we can define B_{ijk} , C_{ijk} . Since $f(x, y, z, w)$ is symmetric in x, y, z , it suffices to determine the coefficients of the monomials A_{ijk} , B_{ijk} , C_{ijk} . Let α_{ijk} (resp. β_{ijk} , γ_{ijk}) denote the coefficient of the monomial A_{ijk} (resp. B_{ijk} , C_{ijk}).

For an arbitrary but fixed pair of j, k , if we denote α_{ijk} by α_i then $f(x, x, z, w) = 0$ implies that

$$\alpha_i + \beta_i + \gamma_i = 0, \quad i = 1, 2, \dots, 5. \quad (4.42)$$

Now $f(x, y, z, x) = 0$ implies that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad (4.43)$$

since the monomial $xxxyy$ (because z is fixed at positions j, k we can forget it) has coefficient 0 in $f(x, y, z, x)$ and this monomial comes from the monomials $wxxyy$, $xwxyy$, $xxwyy$ (the variable z occur in positions j, k we forget it). Similarly considering the monomials

$$xyxyx, xyxx, xyxyx, xxyyx$$

yields

$$\begin{aligned} \beta_1 + \beta_2 + \alpha_4 &= 0, \quad \gamma_1 + \beta_3 + \beta_4 = 0, \quad \alpha_1 + \gamma_4 + \gamma_5 = 0, \\ \beta_1 + \gamma_3 + \beta_5 &= 0, \quad \gamma_1 + \gamma_2 + \alpha_5 = 0. \end{aligned} \quad (4.44)$$

[The symmetry of f in the variables x, y was used here].

Since f is symmetric in x, y, z , $f(x, y, z, x) = 0$ implies that $f(x, y, z, y) = 0$. Thus, for an arbitrary but fixed pair of j and k , the monomials $xyxyy$, $xyxyy$, $xyyyx$ have coefficients 0 in $f(x, y, z, y)$. This implies that

$$\alpha_3 + \alpha_4 + \alpha_5 = 0, \quad \beta_2 + \beta_3 + \gamma_5 = 0, \quad \alpha_2 + \beta_4 + \beta_5 = 0, \quad \gamma_2 + \gamma_3 + \gamma_4 = 0. \quad (4.45)$$

The solution of the systems (4.42)...(4.45) is

$$\begin{aligned}\alpha_1 &= -\gamma_4 - \gamma_5, \alpha_2 = -\beta_4 - \beta_5, \alpha_3 = \beta_4 + \beta_5 + \gamma_4 + \gamma_5, \alpha_4 = -\beta_4 - \gamma_4, \\ \alpha_5 &= -\beta_5 - \gamma_5, \beta_1 = -\beta_5 - \gamma_3, \beta_2 = \beta_4 + \beta_5 + \gamma_3 + \gamma_4, \\ \beta_3 &= -\beta_4 - \beta_5 - \gamma_3 - \gamma_4 - \gamma_5, \gamma_1 = \gamma_3 + \gamma_4, \gamma_2 = -\gamma_3 - \gamma_4.\end{aligned}\quad (4.46)$$

From (4.46) we know that it suffices to determine $\beta_4, \beta_5, \gamma_3, \gamma_4, \gamma_5$ for each pair of j and k . But if in B_{4jk} one z occurs in the position 1 or 2, that is j or $k \in \{1, 2\}$ then B_{4jk} must have x or y occurring in the positions m_1, m_2 in B_{4jk} with $m_r \geq 3$ for $r = 1, 2$. So interchanging this variable with z in B_{4jk} yields a word $X_{sj_0k_0}$ with $s \leq 4$ and $3 \leq j_0 < k_0 \leq 7$. By the symmetry of f , B_{4jk} and $X_{sj_0k_0}$ have the same coefficients in f . That is, $\beta_{4jk} = d_{sj_0k_0}$, where $d_{sj_0k_0}$ is the coefficient of $X_{sj_0k_0}$. Therefore from (4.46) we need only to determine $\beta_{4j_0k_0}$ with $3 \leq j_0 < k_0 \leq 7$. (e.g. if $B_{426} = xzyxwzy$ then interchanging y, z yields the word $xyzxwyz$ which is B_{437} . So $\beta_{426} = \beta_{437}$.) Similarly, we need only to determine $\beta_{5jk}, \gamma_{rjk}$ for $r = 3, 4, 5$ and $3 \leq j < k \leq 7$. So we have shown that

Lemma 4.3.1 *The coefficients of f which we need to determine are $\beta_{rjk}, \gamma_{sjk}$ for $r = 4, 5, s = 3, 4, 5$ and $3 \leq j < k \leq 7$. ■*

Since H_2 has no identity of type $[2, 2]$, the symmetry of f implies that

$$\begin{aligned}f &= \sum_{(x,y,z)} (f_1(y, z, x)xw + f'_1(y, z, x)wx) + \\ &\quad f_2(x, y, w)yz^2 + \cdots + \\ &\quad f_3(x, y, z, w)yz + \cdots.\end{aligned}\quad (4.47)$$

Moreover

$$f_1 = \sum_{i=1}^8 \alpha_i k_i, \quad f'_1 = \sum_{i=1}^8 \alpha'_i k_i, \quad f_2 = \beta[[x, y], [x, w]].$$

Since $f_3(x, y, z, w)$ is an identity of H_2 of type $[2, 1^3]$ from Theorem 1.2, $f_3 = q_0 + q_1 + q_2 + q_3$, where q_0 is skew in y, z, w and q_1 (resp. q_2, q_3) is symmetric in the variables y, z (resp. y, w ; and z, w). Thus from Theorem 1.1

$$\begin{aligned}q_1(x, y, z, w) &= \sum_{i=1}^8 \gamma_i k'_i(x, y, z, w), \quad q_2(x, y, z, w) = \sum_{i=1}^8 \delta_i k'_i(x, y, w, z), \\ q_3(x, y, z, w) &= \sum_{i=1}^8 \mu_i k'_i(x, z, w, y),\end{aligned}$$

where $k'_i(x, y, z, w) := \Delta(y, z)k_i(x, y, w)$. $q_0 \in VIN_2[2, 1^3]$, so $q_0 = \sum_{i=1}^7 \beta_i L_i$.

Now $f(x, y, z, 1) \in VIN_3[2^3] = \{0\}$, so $f(x, y, z, 1) = 0$. Since

$$\begin{aligned} L_i(z, x, y, 1) &= 0, \quad i = 1, 2, \dots, 6, \quad L_7(z, x, y, 1) = 2T_4(x, y, z; z), \\ k'_i(z, x, y, 1) &= 0, \quad i = 1, \dots, 8, \quad k'_i(z, x, 1, y) = 0, \quad k'_i(z, y, 1, x) = 0, \quad i = 1, 3, 4, 6, \\ k'_i(z, x, 1, y) &= 2T_4(x, z, y; z), \quad i = 2, 5, \quad k'_i(z, x, 1, y) = T_4(x, z, y; z), \quad i = 7, 8, \\ k'_i(z, y, 1, x) &= 2T_4(y, z, x; z), \quad i = 2, 5, \quad k'_i(z, x, 1, y) = T_4(y, z, x; z), \quad i = 7, 8. \end{aligned}$$

Thus from (4.47) the sum of the terms of $f(x, y, z, 1)$ which end in x is 0, which implies that

$$f_1(y, z, x) + f'_1(y, z, x) + f_3(z, y, x, 1)y + f_3(y, z, x, 1)z = 0.$$

Hence

$$\sum_{i=1}^8 (\alpha_i + \alpha'_i) k_i(y, z, x) + \lambda (T_4(y, x, z; z)y + T_4(z, x, y; y)z) = 0,$$

where

$$\lambda := 2\beta_7 - 2\delta_2 - 2\delta_5 - \delta_7 - \delta_8 + 2\mu_2 + 2\mu_5 + \mu_7 + \mu_8.$$

Since $T_4(y, x, z; z)y = -k_6(y, z, x)$, $T_4(z, x, y; y)z = -k_8(y, z, x)$ and $k_1, \dots, k_8$ are linearly independent,

$$\alpha_i + \alpha'_i = 0, \quad i \neq 6, 8, \quad \alpha_j + \alpha'_j - \lambda = 0, \quad j = 6, 8. \quad (4.48)$$

Since f_1 is symmetric in the variables of degree 2, from the definition of k_i we have

$$\alpha_1 = 0, \quad \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_5, \quad \alpha_6 = \alpha_8, \quad \alpha'_6 = \alpha'_8. \quad (4.49)$$

Next we show that following equations hold:

$$\alpha_3 = 3\alpha_4, \quad \gamma_{456} = 0, \quad \gamma_{457} - \beta = 0. \quad (4.50)$$

Proof. Since $f(x, x, z, w) = 0$ by assumption, by (4.47) the sum of the terms of $f(x, x, z, w)$ which end in xw is 0, which implies that $2f_1(x, z, x) = 0$. But we have

$$\begin{aligned} f_1(x, z, x) &= \alpha_3[[z, x^2], [z, x]] + 2\alpha_4[[z, x], [z, x^2]] + \alpha_5[[x, z], [x^2, z]] + \\ &\quad (\alpha_3 - 3\alpha_4)[[z, x^2], [z, x]] \end{aligned}$$

because $\alpha_1 = 0$ and $\alpha_4 = \alpha_5$. Thus $\alpha_3 = 3\alpha_4$.

$C_{456} = xywzxx$. Substituting

$$e[12], e[22], e[23], e[33], e[31]$$

for x_1, y, w, z, x_2 in the partial linearization $g(x_1, x_2, y, z, w) := \text{lin}_x f$, the coefficient of e_{11} is the sum of the coefficients of the monomials $x_1 y w z z x_2$ and $x_2 z z w y y x_1$ in g , that is, the sum of the coefficients of the monomials $xywzxx$ and $xzzwyx$. By the symmetry of f in y, z we have $\gamma_{456} = 0$.

From Lemma 2.1.1 the sum of the coefficients of the monomials $xywzxx, xywzxx, xywzxx$ is 0. This implies that $\gamma_{456} + \gamma_{457} - \beta = 0$. Since $\gamma_{456} = 0, \gamma_{457} = \beta$.

Let $\tilde{f}(x, y, z, z_1, w) := \Delta(z, z_1)f(x, y, z, w)$. Then $\tilde{f}(x, y, x, y, w)$ is an identity of type $[3, 3, 1]$. So there exists a scalar $\tilde{\beta} \in F$ such that $\tilde{f}(x, y, x, y, w) = \tilde{\beta}Q(x \circ y, x, y, w, x, y)$. Then $\tilde{\beta} = 0$. Indeed, from (4.47) the sum of the terms of $\tilde{f}(x, y, x, y, w)$ which end in xw is

$$\{\tilde{f}_1(y, x, y, x) + f_1(x, y, y)\}xw = \left\{ \sum_{i=2}^8 \alpha_i \{k'_i(y, x, y, x) + k_i(x, y, y)\} \right\}xw,$$

where $\tilde{f}_1(y, z, z_1, x) := \Delta(z, z_1)f_1(y, z, x)$ and $k'_i(x, y, y_1, z) := \Delta(y, y_1)k_i(x, y, z)$. If we denote $k_i(x, y, y)$ (resp. $k'_i(y, x, y, x)$) by k_i (resp. k'_i), then

$$\begin{aligned} k_2 &= -k_4 = -\frac{1}{2}k_5 = [[x, y^2], [x, y]], \quad k_i = 0, \quad i = 3, 6, 7, 8, \\ k'_2 &= -k'_3 = k'_4 = -k'_5 = [[x, y^2], [x, y]], \quad k'_i = 0, \quad i = 6, 7, 8. \end{aligned}$$

In $\tilde{\beta}Q(x \circ y, x, y, w, x, y)$ the sum of the terms which end in xw is $\tilde{\beta}[[x, y^2], [x, y]]xw$. So $\tilde{f}(x, y, x, y, w) = \tilde{\beta}Q(x \circ y, x, y, w, x, y)$, which implies that $2\alpha_2 - \alpha_3 - 3\alpha_5 = \tilde{\beta}$. Since $\alpha_2 = \alpha_3, \alpha_4 = \alpha_5$ and $\alpha_3 = 3\alpha_4$ from (4.50), $\tilde{\beta} = 0$.

Next we show that

$$\beta = 0, \alpha_6 + \alpha'_6 = 0, \alpha_8 + \alpha'_8 = 0. \quad (4.51)$$

Since $f(x, y, z, x) = 0$ the sum of the terms of $f(x, y, z, x)$ which end in xy, yx, xx , or yz are 0. This implies that

$$\begin{aligned} f'_1(x, z, y) + f_3(z, x, y, x) &= 0, \quad f_1(x, z, y) + f_3(z, y, x, x) = 0, \\ f_1(y, z, x) + f'_1(y, z, x) + f_2(z, y, x)y + f_2(y, z, x)z &= 0, \quad f_3(x, y, z, x) = 0. \end{aligned} \quad (4.52)$$

By (4.47), the equations in (4.52) imply

$$\begin{aligned}
-\gamma_1 - \mu_1 + 2\delta_1 &= 0, \quad -\alpha_3 - \gamma_1 - \mu_1 - \gamma_4 - \mu_4 - 2\mu_5 - 2\gamma_5 + 2\delta_2 = 0, \\
-\alpha_2 - \gamma_3 - \mu_3 + 2\delta_3 &= 0, \quad -\alpha_5 - \gamma_4 - \mu_4 + 2\delta_4 = 0, \\
-\alpha_4 + \gamma_1 + \mu_1 - \gamma_2 - \mu_2 + \gamma_4 + \mu_4 + \gamma_5 + \mu_5 + 2\delta_5 &= 0, \\
\alpha'_8 - \gamma_6 - \mu_6 + 2\delta_6 &= 0, \quad -\alpha_7 - \gamma_7 - \mu_7 + 2\delta_7 = 0, \quad \alpha'_6 - \gamma_8 - \mu_8 + 2\delta_8 = 0. \quad (4.53)
\end{aligned}$$

$$\begin{aligned}
-\gamma_1 - \delta_1 + 2\mu_1 &= 0, \quad \alpha_3 - \gamma_1 - \delta_1 - \gamma_4 - \delta_4 - 2\gamma_5 - 2\delta_5 + 2\mu_2 = 0, \\
\alpha_2 - \gamma_3 - \delta_3 + 2\mu_3 &= 0, \quad \alpha_5 - \gamma_4 - \delta_4 + 2\mu_4 = 0, \\
\alpha_4 + \gamma_1 + \delta_1 - \gamma_2 - \delta_2 + \gamma_4 + \delta_4 + \gamma_5 + \delta_5 + 2\mu_5 &= 0, \\
\alpha_8 - \gamma_6 - \delta_6 + 2\mu_6 &= 0, \quad \alpha_7 - \gamma_7 - \delta_7 + 2\mu_7 = 0, \quad \alpha_6 - \gamma_8 - \delta_8 + 2\mu_8 = 0. \quad (4.54)
\end{aligned}$$

$$\beta + \alpha_6 + \alpha'_6 = 0, \quad \beta + \alpha_8 + \alpha'_8 = 0. \quad (4.55)$$

$$\begin{aligned}
\beta_5 - \beta_6 + 4\beta_7 - \delta_2 + \delta_3 + \delta_4 - \delta_5 + \mu_2 - \mu_3 - \mu_4 + \mu_5 &= 0, \\
\beta_1 - \beta_2 + \beta_3 - \beta_4 + \delta_6 - \delta_8 - \mu_6 + \mu_8 &= 0, \\
\beta_1 - \beta_2 + \beta_5 + \beta_7 + \gamma_1 + \gamma_3 - 2\gamma_4 - \gamma_5 - 2\delta_1 + \delta_4 - \delta_5 + \mu_2 - \mu_3 &= 0, \\
-\beta_1 + \beta_2 - \beta_5 - \beta_7 + \gamma_1 + \gamma_3 - 2\gamma_4 - \gamma_5 + \delta_2 - \delta_3 - 2\mu_1 + \mu_4 - \mu_5 &= 0. \quad (4.56)
\end{aligned}$$

Since $\tilde{f}(x, y, x, y, w) = 0$, the sum of the terms of $\tilde{f}(x, y, x, y, w)$ which end in xy is 0, which implies that

$$f_2(y, x, w)x + f_2(x, y, w)y + \tilde{f}_3(x, y, x, y, w) + f_3(y, x, x, w) + f_3(x, y, y, w) = 0,$$

where $\tilde{f}_3(z, z_1, x, y, w) := \Delta(z, z_1)f_3(z, x, y, w)$. Computing coefficients, this equation implies that

$$\begin{aligned}
2\beta_2 - 2\beta_1 - \beta_5 - \beta_6 + 2\beta_7 - \delta_1 + \mu_1 &= 0, \\
\beta_2 - \beta_1 - \beta_5 - \beta_7 + 2\gamma_1 + \gamma_2 - \gamma_3 + 2\gamma_4 - \delta_1 + 2\delta_4 - \delta_5 + 2\mu_1 + \mu_3 - \mu_4 - \mu_5 &= 0, \\
\beta_1 - \beta_2 + \beta_5 + \beta_7 + 2\gamma_1 + \gamma_2 - \gamma_3 + 2\gamma_4 + 2\delta_1 + \delta_3 - \delta_4 - \delta_5 - \mu_1 + 2\mu_4 - \mu_5 &= 0, \\
\beta_2 - \beta_1 - \beta_6 + 3\beta_7 - 2\gamma_1 + \gamma_4 + \gamma_5 - 2\delta_1 - \delta_3 + \delta_4 + \delta_5 + \mu_1 - \mu_2 + \mu_3 - \mu_4 &= 0, \\
\beta_1 - \beta_2 + \beta_6 - 3\beta_7 - 2\gamma_1 + \gamma_4 + \gamma_5 + \delta_1 - \delta_2 + \delta_3 - \delta_4 - 2\mu_1 - \mu_3 + \mu_4 + \mu_5 &= 0, \\
\beta_1 - \beta_2 + \beta_3 - \beta_4 + \gamma_6 + \gamma_8 - \delta_6 - \mu_8 &= 0, \\
\beta_2 - \beta_1 - \beta_3 + \beta_4 + \gamma_6 + \gamma_8 - \delta_8 - \mu_6 &= 0. \quad (4.57)
\end{aligned}$$

From (4.50), $\gamma_{456} = 0$, $\gamma_{457} = \beta$. But calculating the coefficients of the monomials C_{456} , C_{457} from (4.47) yields

$$\begin{aligned} -\beta_5 - 2\beta_7 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_8 - \delta_1 + \delta_2 + \delta_8 - \mu_1 + \mu_2 + \mu_8 &= 0, \\ \beta_5 + 2\beta_7 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_8 + \delta_3 - \delta_5 + \mu_3 - \mu_8 - \beta &= 0. \end{aligned} \quad (4.58)$$

Solving the systems (4.49), (4.50), (4.53), (4.54), (4.55), (4.56), (4.57) and (4.58) yields $\beta = 0$. Hence $\alpha_6 + \alpha'_6 = \alpha_8 + \alpha'_8 = 0$ by (4.55).

We also have

$$\alpha_2 = -\alpha_7. \quad (4.59)$$

Since we have shown that $\beta = 0$ by assuming that $\alpha = \lambda_i = 0$, for $i = 1, 2, 3$, the coefficients of the monomials which end in xx , yy or zz are 0. More precisely, the coefficients are linear combinations of α , $\lambda_1, \lambda_2, \lambda_3$. f^* is also in $VIN_3[2, 2, 2, 1]$. We claim that

$$f^*(x, x, z, w) = -\alpha Q(x^2, x, z, w, x, z), \quad f^*(x, y, z, x) = -\sum_{i=1}^3 \lambda_i Q_i,$$

where Q_1, Q_2, Q_3 is the basis of $VIN_3[3, 2, 2]$. For, the operators $*$ and $\Delta(z, y)$ commute on each monomial, and so commute on every polynomial. Thus

$$\begin{aligned} f^*(x, x, z, w) &= \frac{1}{2} \Delta(y, x) \Delta(y, x) f^*(x, y, z, w) \\ &= \left(\frac{1}{2} \Delta(y, x) \Delta(y, x) f(x, y, z, w) \right)^* \\ &= (f(x, x, z, w))^* \\ &= \alpha (Q(x^2, x, z, w, x, z))^* \\ &= -\alpha Q(x^2, x, z, w, x, z). \end{aligned}$$

Similarly we can show that the second equation of the claim holds. Thus we also have $f^*(x, x, z, w) = f^*(x, y, z, x) = 0$ and $\tilde{f}^*(x, y, x, y, w) = 0$, where $\tilde{f}^*(x, y, z, z_1, w) := \Delta(z, z_1) f^*(x, y, z, w)$ if we assume that $\alpha = \lambda_i = 0$, for $i = 1, 2, 3$. Repeating the proof of $\beta = 0$ we get the result that $f^*(x, y, z, w)$ has no terms which end in xx , yy or zz . This implies that f has no term $yyxxzzw$. However from (4.47), this monomial has coefficient $-\alpha_2 - \alpha_7$. So $\alpha_2 = -\alpha_7$.

Next we show

Lemma 4.3.2 *The coefficients α_{ijk} , β_{ijk} , γ_{ijk} are linear combinations of α_2 , α_6 , γ_{334} .*

Proof. In the following if a coefficient is a linear combination of α_i 's then we say it is congruent 0 (modulo the α_i 's) for short. By Lemma 4.3.1, we need only consider the coefficients

$$\beta_{4jk}, \beta_{5jk}, \gamma_{3jk}, \gamma_{4jk}, \gamma_{5jk},$$

where $3 \leq j < k \leq 7$. Since $B_4 = xyxwy$, the monomials B_{434} , B_{435} , B_{445} occur in $f'_1(x, y, z)wy$. So their coefficients are all linear combination of α_i 's. Since we have shown that

$$\alpha'_i = -\alpha_i, i = 1, \dots, 8, \alpha_1 = 0, \alpha_3 = \alpha_2, \alpha_5 = \alpha_4 = \frac{1}{3}\alpha_3, \alpha_7 = -\alpha_2, \alpha_8 = \alpha_6,$$

the coefficients β_{434} , β_{435} , β_{445} are all linear combinations of α_2 , α_6 . So they are congruent 0 modulo the α_i 's. So are

$$\beta_{434}, \beta_{435}, \beta_{445}, \gamma_{424}, \gamma_{425}, \gamma_{434}, \gamma_{435}, \gamma_{445}.$$

(4.60)

We also know from (4.47) and (4.50) that

$$\beta_{5jk}, \gamma_{5jk}, \gamma_{456} = 0, \beta_{467} = \gamma_{467} = \gamma_{457} = \beta = 0,$$

where $i \leq j < k \leq 7$. Thus the coefficients which we need to determine are

$$\gamma_{334}, \gamma_{335}, \gamma_{336}, \gamma_{337}, \gamma_{345}, \gamma_{346}, \gamma_{347}, \gamma_{356}, \gamma_{357}, \gamma_{436},$$

$$\gamma_{437}, \gamma_{446}, \gamma_{447}, \beta_{436}, \beta_{437}, \beta_{446}, \beta_{447}, \beta_{456}, \beta_{457}.$$

By Lemma 2.1.1, given a word w the sum of the coefficients of monomials of the form $w---$ is 0, which implies that $\gamma_{356} + \gamma_{357} \equiv 0$ and $\gamma_{436} + \gamma_{437} \equiv 0$.

Substituting

$$e[12], e[22], e[22], e[23], e[33], e[33], e[31]$$

for x, y, y, z, w, z_1, x_1 in $\Delta(x, x_1)\Delta(z, z_1)f$ and considering the coefficient of e_{11} yields that the sum of the coefficients of the monomials

$$xyyzwzx, xyzzwx, xzwzyyx, xwzzyyx$$

in f is 0. So $\gamma_{446} + \gamma_{445} + \gamma_{224} + \gamma_{234} = 0$. Since f is symmetric in z, y , $\gamma_{224} = \gamma_{356}$, $\gamma_{234} = \gamma_{256}$. So $\gamma_{446} + \gamma_{356} + \gamma_{256} \equiv 0$ by $\gamma_{445} \equiv 0$ in (4.60). But $\gamma_{256} = -\gamma_{356} - \gamma_{456}$ by (4.46) and (4.50), so

$$\gamma_{446} \equiv 0.$$

Thus $\gamma_{224} + \gamma_{234} \equiv 0$, $\gamma_{2jk} \equiv -\gamma_{3jk} - \gamma_{4jk}$ in (4.46) and $\gamma_{445} \equiv \gamma_{434} \equiv \gamma_{424} \equiv 0$ in (4.60) imply that $\gamma_{324} + \gamma_{334} \equiv 0$. However $\gamma_{324} = \gamma_{436}$ by the symmetry of f in y, z , so

$$\gamma_{436} + \gamma_{334} \equiv 0.$$

By the symmetry of f in y, z and the formulas in (4.46),

$$\gamma_{346} = \gamma_{225} = -\gamma_{325} - \gamma_{425} \equiv \gamma_{336}$$

since $\gamma_{425} \equiv 0$.

From (4.46), $\gamma_{156} = \gamma_{356} + \gamma_{456} \equiv \gamma_{356}$ since $\gamma_{456} \equiv 0$ by (4.50). Similarly we have $\gamma_{156} \equiv \gamma_{134} \equiv \gamma_{334}$. Thus

$$\gamma_{356} \equiv \gamma_{334}.$$

If we substitute

$$e[12], e[22], e[23], e[33]$$

for x, y, z, w in f then the monomials which produce e_{11} are $xyyzwzx, xyzwzyx, xzwyzyx$. So the sum of their coefficients is 0. Since $\gamma_{446} \equiv 0$ and since f is symmetric, $\gamma_{335} + \gamma_{356} \equiv 0$. So

$$\gamma_{335} \equiv -\gamma_{356} \equiv -\gamma_{334}.$$

By Lemma 2.1.1, $\gamma_{335} + \gamma_{336} = 0$. So

$$\gamma_{336} \equiv -\gamma_{335} \equiv \gamma_{334}.$$

From (4.46) and $\gamma_{445} \equiv 0$, $\gamma_{145} \equiv \gamma_{345}$. But from the symmetry of f

$$\gamma_{145} = \gamma_{136} = \gamma_{336} + \gamma_{436} \equiv \gamma_{334} + \gamma_{436}.$$

However $\gamma_{436} \equiv -\gamma_{334}$ from above, thus

$$\gamma_{345} \equiv \gamma_{145} \equiv 0.$$

$\beta_{5jk} \equiv 0$ implies that $\beta_{147} \equiv -\gamma_{347}$ by (4.46). From the symmetry, $\beta_{147} \equiv \beta_{136} \equiv -\gamma_{336} \equiv \gamma_{334}$. So

$$\gamma_{347} \equiv -\gamma_{334}.$$

Similarly we have

$$\gamma_{337} \equiv \beta_{325} \equiv -\beta_{425} - \gamma_{325} - \gamma_{425} \equiv \beta_{537} - \gamma_{336} - \gamma_{434} \equiv -\gamma_{336} \equiv -\gamma_{334}.$$

From Lemma 2.1.1 the sum of the coefficients of the monomials of form $v---$ is 0. Since $C_{447} \equiv xy y z w x z$, $C_{446} \equiv xy y z w z x$ and since the coefficients of the monomials

$$xy y z x z w, xy y z z w x, xy y z z x w, xy y z x w z$$

are linear combinations of α_i 's, $\gamma_{446} + \gamma_{447} \equiv 0$. So $\gamma_{447} \equiv -\gamma_{446} \equiv 0$. Similarly, we have $\gamma_{436} + \gamma_{437} \equiv 0$, $\beta_{437} + \beta_{436} \equiv 0$ and $\beta_{446} + \beta_{447} \equiv 0$.

From Lemma 2.1.1 the sum of the coefficients of the monomials

$$xxywzzzy, yzzwyyxx, xywzzzyx, xyzzwyx, ywzzzyxx, xxyzzzyy$$

is 0. This implies that $\alpha_{456} + \gamma_{345} + \gamma_{334} - \beta = 0$. Since $\beta = 0$ and $\alpha_{456} = -\beta_{456} - \gamma_{456}$ by (4.51) and (4.46) respectively, $\beta_{456} \equiv \gamma_{345} + \gamma_{334} - \gamma_{456} \equiv \gamma_{334}$ because $\gamma_{345} \equiv \gamma_{456} \equiv 0$. By Lemma 2.1.1 the sum of the coefficients of the monomials $xyxwzzzy$, $xyxwzyz$, $xyxwyzz$ is 0. So $\beta_{456} + \beta_{457} \equiv 0$ since $\beta = 0$. Thus $\beta_{457} \equiv -\beta_{456} \equiv -\gamma_{334}$.

By the symmetry of f in y, z the monomials $xyxwzzzy$ and $xxwyyzzz$ have the same coefficients in f . So $\beta_{456} \equiv \alpha_{327} \equiv \beta_{427} + \gamma_{427}$ by (4.46) and $\beta_{5jk} \equiv \gamma_{5jk} \equiv 0$. From the symmetry of f , we have $\beta_{427} \equiv \beta_{436}$, $\gamma_{427} \equiv \beta_{334}$. Since $\beta_{434} \equiv \gamma_{434} \equiv \beta_{5jk} \equiv \gamma_{5jk} \equiv 0$, from (4.46) $\beta_{334} \equiv -\gamma_{334}$. So $\beta_{456} \equiv \beta_{436} - \gamma_{334}$ and $\beta_{436} \equiv -\beta_{456} + \gamma_{334} \equiv 0$.

By the symmetry of f in y, z , $\beta_{446} \equiv \alpha_{427}$. $\alpha_{427} \equiv -\beta_{427} - \gamma_{427}$ by (4.46). On the other hand from the symmetry of f $\beta_{427} \equiv \beta_{436}$, $\gamma_{427} \equiv \beta_{334}$. So $\beta_{446} \equiv -\beta_{436} - \beta_{334} \equiv -\beta_{436} + \gamma_{334}$ because $\beta_{334} \equiv -\beta_{434} - \gamma_{334} - \gamma_{434}$ by (4.46) and $\beta_{5jk} \equiv \gamma_{5jk} \equiv \beta_{434} \equiv \gamma_{434} \equiv 0$. Thus $\beta_{446} \equiv \gamma_{334}$ completing the proof of the lemma. ■

To prove Proposition 4.3.1 we need now only to show that $Q_1, \dots, Q_7$ are linearly independent. Suppose there exist scalars $\alpha_i \in F$ such that $\sum_{i=1}^7 \alpha_i Q_i = 0$. Then the

coefficient of the monomial $xyzwxzy$ is 0, which implies that $\alpha_1 + \alpha_3 - \alpha_4 = 0$. Since $(\sum_{i=1}^7 \alpha_i Q_i)^* = 0$, the monomial $xyzwxzy$ has coefficient 0 in it, which implies that $-\alpha_1 + \alpha_3 - \alpha_4 = 0$. So $\alpha_1 = 0$, $\alpha_3 = \alpha_4$. Since $\alpha_1 = 0$, the monomial $xwyxzy$ has coefficient 0 in $\sum_{i=1}^7 \alpha_i Q_i$, which implies that $-2\alpha_2 + \alpha_7 = 0$. Similarly considering the coefficients of the monomials $yxzxzyw$, $zxyxzyw$ yields

$$-\alpha_2 - \alpha_5 - \alpha_6 = 0, \quad -\alpha_2 - \alpha_6 = 0.$$

So $\alpha_5 = 0$. Since $\alpha_1 = \alpha_5 = 0$, considering the coefficients of the monomials $xwyxzy$, $xyxwzy$ yields

$$\alpha_4 + \alpha_7 = 0, \quad 2\alpha_2 + \alpha_3 - \alpha_4 + 2\alpha_7 = 0.$$

So $\alpha_i = 0$ for $i = 1, \dots, 7$ and $Q_1, \dots, Q_7$ are linearly independent. ■

We will now determine a basis for the vector space $VIN_3[2, 2, 1^3]$. So let $f(y, z, x_1, x_2, x_3) \in VIN_3[2, 2, 1^3]$. Then

$$\begin{aligned} f = & \sum_{(123)} \{f_1(y, z, x_1)[x_2, x_3] + (Id + (y, z))(f_3(z, x_1, x_2, x_3)zy^2 + f_5(y, z, x_1, x_2, x_3)yz + \\ & \sum_{(123)} \{f_2(y, x_1, x_2)x_3z^2 + f_4(y, z, x_1, x_2)x_3z + f'_4(y, z, x_1, x_2)zx_3\}. \end{aligned} \quad (4.61)$$

Moreover, since $f_2 \in VIN_2[2, 1, 1]$, $f_3 \in VI_2[1^4]$ which are skew in x_i 's, they are linear combinations of the products of two commutators by Corollary 3.2 of [MR]. So there exist scalars $\alpha, \beta, \gamma \in F$ such that

$$f_2 = \alpha[[y, x_1], [y, x_2]], \quad f_3 = \beta \sum_{(123)} [y, x_1][x_2, x_3].$$

$f_5(y, z, x_1, x_2, x_3) \in VI_2[1^5]$ which is skew in x_i 's. We need to determine the coefficients of following 20 monomials:

$$yzx_1x_2x_3, yx_1zx_2x_3, \dots, x_1x_2x_3yz, zyx_1x_2x_3, zx_1yx_2x_3, \dots, x_1x_2x_3zy.$$

Let $\beta_i, \beta'_i, i = 1, \dots, 10$ denote their coefficients.

Since $f(y, z, x_1, x_2, y) \in VIN_3[3, 2, 1, 1]$, there exist scalars $\alpha_1, \dots, \alpha_{10} \in F$ such that $f(y, z, x_1, x_2, y) = \sum_{i=1}^{10} \alpha_i Q_i$, where $\{Q_1, \dots, Q_{10}\}$ is the basis of $VIN_3[3, 2, 1, 1]$ given in

Proposition 4.2.3. Since f is skew in the variables of degree 1, $f(y, z, x_1, y, y) = 0$. This implies that $\sum_{i=1}^{10} \alpha_i Q_i(y, z, x_1, y) = 0$. So by calculation we have $\alpha_2 = 2\alpha_4$.

From now on we assume that $f(y, z, x_1, x_2, y) = 0$. Then the sum of the terms of $f(y, z, x_1, x_2, y)$ which end in $x_2 y$ (resp. in yx_2, yz, zy, yzz) is 0, which implies

$$\begin{aligned} f_1(y, z, x_1) + f_4(z, y, y, x_1) &= 0, \quad f_1(y, z, x_1) + f'_4(z, y, y, x_1) = 0, \\ f_4(y, z, x_1, x_2) + f_5(y, z, x_1, x_2, y) &= 0, \quad f'_4(y, z, x_1, x_2) + f_5(z, y, x_1, x_2, y) = 0, \\ f_2(y, x_1, x_2) + f_3(y, x_1, x_2, y) &= 0. \end{aligned} \quad (4.62)$$

Thus it suffices to determine β_i, β'_i and α, β, γ . From the last equation of (4.62), we have

$$\alpha - \beta + \gamma = 0. \quad (4.63)$$

By Lemma 2.1.1, the sum of the coefficients of the monomials

$$yzx_1x_2x_3yz, zyx_1x_2x_3yz, zyx_1x_2x_3zy, yzx_1x_2x_3zy,$$

is 0. This implies that

$$\beta_1 + \beta'_1 = 0. \quad (4.64)$$

The sum of the coefficients of the monomials

$$x_1x_2yzx_3yz, x_1x_2yzx_3zy, x_1x_2zyx_3yz, x_1x_2zyx_3zy$$

is 0, which implies that

$$\beta_8 + \beta'_8 = 0. \quad (4.65)$$

The sum of the coefficients of the monomials $wzzy, wzyz, wyzz$ is 0 for each

$$w = yx_1x_2x_3, x_1yx_2x_3, x_1x_2yx_3, x_1x_2x_3y,$$

which implies that

$$\beta_4 + \beta'_4 + \beta = 0, \quad \beta_7 + \beta'_7 - \beta = 0, \quad \beta_8 + \beta'_8 + \gamma = 0, \quad \beta_{10} + \beta'_{10} - \gamma = 0. \quad (4.66)$$

Similarly considering the monomials $---x_2x_3yz$ and $---x_2zx_3---$, respectively, we have

$$\beta_2 + \beta'_2 + \beta_5 + \beta'_5 = 0, \quad \beta_3 + \beta'_3 + \beta_6 + \beta'_6 = 0. \quad (4.67)$$

Since f is symmetric in y, z , $f_1(y, z, x_1) = f_1(z, y, x_1)$ by (4.61). Therefore from (4.62)

$$f_5(z, y, y, x_1, z) - f_5(y, z, z, x_1, y) = 0.$$

So the coefficients of the monomials

$$yzzyx_1, yzx_1zy, x_1zzyy, yx_1zyz$$

are 0 in

$$f_5(z, y, y, x_1, z) - f_5(y, z, z, x_1, y)$$

which implies that

$$\begin{aligned} \beta_1 + \beta_2 - \beta_6 - \beta_8 + \beta'_1 + \beta'_2 - \beta'_6 - \beta'_8 &= 0, \\ \beta_1 - \beta_3 + \beta_7 - \beta_{10} + \beta'_1 - \beta'_3 + \beta'_7 - \beta'_{10} &= 0, \\ \beta_6 + \beta_7 + \beta_8 + \beta_8 + \beta'_6 + \beta'_7 + \beta'_8 + \beta'_8 &= 0, \\ -\beta_2 + \beta_4 + \beta_8 + \beta_{10} - \beta'_2 + \beta'_4 + \beta'_8 + \beta'_{10} &= 0. \end{aligned} \quad (4.68)$$

By Lemma 2.1.1 the sum of the coefficients of the monomials $ywy, yw^*y, yyw, yyw^*, wyy, w^*yy$ is 0 for $w = x_1zx_2x_3z, x_1x_2x_3zz, x_1x_2zx_3z$. This implies that

$$\begin{aligned} -\beta_3 - \beta_6 + \beta'_1 + 2\beta'_2 + \beta'_4 + \beta'_5 - \beta + \alpha &= 0, \\ \beta'_1 + 2\beta'_2 + \beta'_3 + \beta'_4 + \beta'_5 + \beta'_6 - \gamma &= 0, \\ -\beta_4 - \beta_7 - \beta'_1 - \beta'_2 + 2\beta'_3 + \beta'_4 - \beta'_5 + \beta'_7 + \gamma - \alpha &= 0. \end{aligned} \quad (4.69)$$

Next let us do some substitutions. Substituting

$$e[12], e[13], e[11], e[22], e[23]$$

for y, z, x_1, x_2, x_3 and considering the coefficient e_{23} we have

$$-\beta'_1 + \beta'_3 + \beta'_5 + \beta'_6 - \beta'_8 + \beta'_8 + \beta_1 + \beta_{10} - \beta_2 - \beta_4 - 2\beta_5 - \beta_6 + \beta_8 = 0. \quad (4.70)$$

Indeed if a substitution of the matrix $e[ij]$ denotes the path from i to j and if a substitution of matrix eii denotes the loop at point i , then the monomials which evaluate to e_{23} are the paths from vertex 2 to vertex 3 which pass through all edges once. These paths are

$$\begin{aligned} & yx_1zx_3x_2yz, yzx_3x_2yx_1z, x_2yx_1zx_3yz, x_2yzx_3yx_1z, \\ & yyx_2x_3zx_1z, yx_1yx_2x_3zz, x_2yyx_3zx_1z, x_2yx_1yx_3zz, \\ & yzzx_1yx_2x_3, yx_1zzyx_2x_3, x_2yzzx_1yx_3, x_2yx_1zzyx_3, \\ & x_3zyx_2yx_1z, x_3zx_1yx_2yz, x_2x_3zyyx_1z, x_2x_3zx_1yyz. \end{aligned}$$

The sum of the coefficients of these monomials in f is the coefficient of e_{23} in this substitution. It yields (4.70).

Similarly substituting

$$e[13], e[33], e[11], e[12], e[22]$$

for y, z, x_1, x_2, x_3 and considering the coefficients of e_{12}, e_{21} respectively yields

$$\begin{aligned} \beta'_1 + \beta'_2 + \beta'_5 + \beta'_6 - \beta_6 - \beta_8 - \beta_8 - \beta_{10} &= 0, \\ \beta_8 + \beta_8 - \beta'_{10} &= 0, \end{aligned} \quad (4.71)$$

since the monomials which produce e_{12} (resp. e_{21}) in the substitution are $yzzyx_1x_2x_3$ and $x_1yzzyx_2x_3$ (resp. $x_3x_2yzzyx_1$ and $x_3x_2x_1yzzy$).

Next we assume that $f^* = \pm f$. If $f^* = -f$ then by symmetry and considering the coefficients of monomials

$$\begin{aligned} & yx_1zx_2x_3yz, yx_1x_2zx_3yz, yx_1x_2x_3zyz, x_1yzx_2x_3yz, x_1yx_2zx_3yz, \\ & x_1x_2yzx_3yz, x_1x_2x_3yzzy, zx_1yx_2x_3yz, zx_1x_2yx_3yz, x_1x_2zyx_3yz, x_1x_2x_3zyyz, \end{aligned}$$

we have

$$\begin{aligned} \beta_1 + \beta_2 + \beta'_7 &= 0, -\beta_1 + \beta_3 + \beta'_6 = 0, \beta_1 + \beta_4 + \beta'_5 = 0, \beta'_1 + \beta_5 + \beta_7 = 0, \\ 2\beta_8 - \beta'_1 &= 0, -\beta_1 - \beta_4 + \beta_8 - \beta_{10} + \beta'_6 = 0, -\beta_1 + \beta_3 - \beta_8 + \beta_{10} - \beta'_5 = 0, \\ \beta'_1 + \beta'_2 + \beta'_4 &= 0, 2\beta'_3 - \beta'_1 = 0, \beta_1 - \beta_3 + \beta'_7 + \beta'_8 - \beta'_{10} = 0, \\ \beta_1 + \beta_2 - \beta'_6 - \beta'_8 + \beta'_{10} &= 0. \end{aligned} \quad (4.72)$$

The solution of the systems (4.64)... (4.72) is

$$\begin{aligned}
\beta_1 &= 2\beta'_4 + 2\beta'_8 + 4\beta'_{10} + 2\gamma, \quad \beta_2 = \beta'_4 - \beta'_8 - 3\beta'_{10} + \gamma, \\
\beta_3 &= -\beta'_4 + \beta'_{10} - 2\gamma, \quad \beta_4 = \beta'_4 + \beta'_8 + \beta'_{10} + 2\gamma, \\
\beta_5 &= \beta'_4 + 2\beta'_8 + 4\beta'_{10} + \gamma, \quad \beta_6 = -\beta'_4 - \beta'_8 - 2\beta'_{10} - \gamma, \\
\beta_7 &= \beta'_4 + \gamma, \quad \beta_8 = \beta'_8 + \beta'_{10} + \gamma, \quad \beta_9 = -\beta'_8 - \gamma, \quad \beta_{10} = \gamma - \beta'_{10}, \\
\beta'_1 &= -\beta_1, \quad \beta'_2 = \beta'_4 + 2\beta'_8 + 4\beta'_{10} - 2\gamma, \quad \beta'_3 = -\beta'_4 - \beta'_8 - 2\beta'_{10} - \gamma, \\
\beta'_5 &= -3\beta'_4 - 3\beta'_8 - 5\beta'_{10} - 4\gamma, \quad \beta'_6 = 3\beta'_4 + 2\beta'_8 + 3\beta'_{10} + 4\gamma, \\
\beta'_7 &= -3\beta'_4 - \beta'_8 - \beta'_{10} - 3\gamma, \quad \beta'_8 = -\beta'_8 - \beta'_{10} - \gamma, \\
\alpha &= -2\beta'_4 - \beta'_8 - \beta'_{10} - 3\gamma, \quad \beta = -2\beta'_4 - \beta'_8 - \beta'_{10} - 2\gamma.
\end{aligned} \tag{4.73}$$

Thus $\beta'_4, \beta'_8, \beta'_{10}, \gamma, \alpha_i, i \neq 4, 10$ are 12 parameters of the coefficients of f when f is $\ast$ -skew symmetric. Since $f^\ast = -f$ implies that $f(y, z, x_1, x_2, y)^\ast = -f(y, z, x_1, x_2, y)$. So $\sum_{i=1}^{10} \alpha_i Q_i$ is $\ast$ -skew symmetric. This implies that $\alpha_{10} = \alpha_9$. We have $\alpha_3 = 2\alpha_4$ already.

If $f^\ast = f$ then corresponding to (4.72) we have

$$\begin{aligned}
\beta_1 &= 0, \quad \beta_2 - \beta'_7 = 0, \quad \beta_1 + \beta_3 - \beta'_6 = 0, \quad \beta_4 - \beta_1 - \beta'_5 = 0, \\
\beta_5 - \beta_7 - \beta'_1 &= 0, \quad \beta'_1 = 0, \quad \beta_1 + \beta_4 + \beta_8 + \beta_{10} - \beta'_6 = 0, \\
2\beta_8 - \beta'_5 - \beta_1 - \beta_4 &= 0, \quad \beta_1 + \beta_8 - \beta_3 + \beta_{10} - \beta'_5 = 0, \\
\beta'_2 - \beta'_4 - \beta'_1 &= 0, \quad \beta'_6 + \beta_1 - \beta_3 = 0, \quad \beta'_{10} + \beta'_8 - \beta'_7 - \beta_1 + \beta_3 = 0, \\
\beta'_{10} + \beta'_6 + \beta'_8 - \beta_1 - \beta_2 &= 0,
\end{aligned} \tag{4.74}$$

The solution of the systems (4.64)... (4.71) and (4.74) is

$$\begin{aligned}
\beta_1 &= \beta'_1 = 0, \quad \beta_3 = \beta'_2 = \beta'_4 = \beta'_6 = \gamma, \\
\beta_2 &= \beta_6 = -\frac{1}{2}\beta_5 = -\frac{1}{2}\beta_7 = \beta'_7 = -\beta'_8 = \beta_8 + \gamma, \\
\beta_4 &= \beta_5 = \beta_8, \quad \beta_8 = -\beta'_8 = \beta_{10} - \beta_8, \quad \beta_{10} = c - \beta'_{10}, \quad \beta'_3 = -\beta_8 - 3\gamma.
\end{aligned}$$

Thus there are 4 parameters $\gamma, \alpha_9, \beta_8, \beta'_{10}$ when $f^\ast = f$. Since if $f^\ast = f$ then $\sum_{i=1}^{10} \alpha_i Q_i$ is symmetric. This implies that $\alpha_i = 0$ for $i \neq 9, 10$ and $\alpha_9 = -\alpha_{10}$.

Next we shall find 12 linear independent identities of $VIN_3[2, 2, 1, 1, 1]$ which are $*$ -skew symmetric, and 4 linear independent identities of $VIN_3[2, 2, 1, 1, 1]$ which are $*$ -symmetric.

Let

$$\begin{aligned}
Q_1 &:= \sum_{(123)} (Id + (x, y)) Q(z_1 \circ y, y, x, x, z_2, z_3), \\
Q_2 &:= \sum_{(123)} (Id + (x, y)) Q(x^2, z_1, y, y, z_2, z_3), \\
Q_3 &:= \sum_{\sigma \in S_3} (Id + (x, y)) (-1)^\sigma Q(y \circ z_{\sigma(1)}, z_{\sigma(2)}, x, x, y, z_{\sigma(3)}), \\
Q_4 &:= \sum_{(123)} (Id + (x, y)) T_6(z_1 \circ y, y, x, z_2, z_3; x), \\
Q_5 &:= (Id + (x, y)) T_6(x \circ y, y, z_1, z_2, z_3; x), \\
Q_6 &:= (Id + (x, y)) T_6(x^2, z_1, z_2, z_3, y; y), \\
Q_7 &:= (Id + (x, y)) K(x, z_1, x, z_2, y, z_3, y), \\
Q_8 &:= \sum_{(123)} (Id + (x, y)) K(x, y, z_2, z_1, x, z_3, y), \\
Q_9 &:= \sum_{\sigma \in S_3} (-1)^\sigma K(x, y, x, z_{\sigma(1)}, y, z_{\sigma(2)}, z_{\sigma(3)}), \\
Q_{10} &:= \sum_{(123)} (Id + (x, y)) Q(z_1, y, z_2, x, y, z_3) x, \\
Q_{11} &:= \sum_{(123)} (Id + (x, y)) Q(x, y, z_1, x, y, z_2) z_3, \\
Q_{12} &:= \sum_{(123)} (Id + (x, y)) x Q(z_1, y, z_2, x, y, z_3), \\
Q_{13} &:= \sum_{(123)} (Id + (x, y)) z_3 Q(x, y, z_1, x, y, z_2), \\
Q_{14} &:= (Id + (x, y)) T_6(x, y, z_1, z_2, z_3; x) y, \\
Q_{15} &:= (Id + (x, y)) y T_6(x, y, z_1, z_2, z_3; x), \\
Q_{16} &:= T_6(x, y, z_1, z_2, z_3; [x, y]).
\end{aligned}$$

Then Q_{16} is an identity of H_3 by Theorem 3.1 and Q_i for $i \neq 16$ is an identity by $[MR]$. Next we show that $Q_1, \dots, Q_{16}$ are linearly independent. If there exist scalars $\alpha_i \in F$ such that $\sum_{i=1}^{16} \alpha_i Q_i = 0$ then

$$\sum_{i=1}^{16} \alpha_i Q_i(x, y, z_1, z_2, x) = 0.$$

This implies that

$$\begin{aligned} 3\alpha_1 + \alpha_2 &= 0, \quad -2\alpha_1 - \alpha_2 + \alpha_3 = 0, \quad \alpha_1 + \alpha_3 = 0, \quad 2\alpha_{12} + \alpha_{13} = 0, \\ -\alpha_7 - 2\alpha_8 &= 0, \quad -\alpha_7 + \alpha_8 + \alpha_9 = 0, \quad 2\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 = 0, \\ -2\alpha_1 - \alpha_2 + 2\alpha_4 + \alpha_6 &= 0, \quad 2\alpha_{11} + \alpha_{12} = 0, \end{aligned} \quad (4.75)$$

since if we denote $Q_i(x, y, z_1, z_2, x)$ by g_i then

$$\begin{aligned} g_1 &= Q'_1 + 2Q'_0 + Q'_4, \quad g_2 = Q'_0 - Q'_3, \quad g_3 = Q'_2 + 2Q'_3 + Q'_4, \quad g_4 = Q'_7 + 2Q'_8, \quad g_5 = Q'_7, \\ g_6 &= Q'_8, \quad g_7 = -Q'_5 - Q'_6, \quad g_8 = -2Q'_5 + Q'_6, \quad g_9 = Q'_6, \quad g_{10} = 2Q'_{10}, \\ g_{11} &= 2Q'_{10}, \quad g_{12} = 2Q'_9, \quad g_{13} = Q'_9, \quad g_{14} = g_{15} = g_{16} = 0, \end{aligned}$$

where $Q'_0 = Q'_1 - Q'_2 + Q'_3 + Q'_7 - Q'_8$ and $\{Q'_1, \dots, Q'_{10}\}$ is the basis of $VIN_3[3, 2, 1, 1]$ given in Proposition 4.2.3.

Since $\sum_{i=1}^{16} \alpha_i Q_i(x, y, z_1, z_2, x) = 0$, $(\sum_{i=1}^{16} \alpha_i Q_i(x, y, z_1, z_2, x))^* = 0$. This implies that

$$2\alpha_{13} + \alpha_{10} = 0, \quad 2\alpha_{10} + \alpha_{11} = 0.$$

because $Q_i^* = -Q_i$, for $i \neq 10, 11, 12, 13, 14, 15, 16$, $Q_{10}^* = -Q_{12}$, $Q_{11}^* = -Q_{13}$, $Q_{14}^* = -Q_{15}$ and $Q_{16}^* = Q_{16}$.

From $\sum_{i=1}^{16} Q_i(x, x, z_1, z_2, z_3) = 0$ we have

$$\alpha_7 = 0, \quad -4\alpha_2 + 4\alpha_5 - 2\alpha_6 = 0.$$

Considering the coefficients of the monomials

$$yz_1yz_2xz_3x, \quad z_1z_2z_3xyxy, \quad yxyxz_1z_2z_3, \quad yxz_1z_2z_3yx, \quad z_1z_2xyyxz_3, \quad z_1z_2xxxyyz_3,$$

in $\sum_{i=1}^{16} \alpha_i Q_i$ yields

$$\begin{aligned} -\alpha_{16} - \alpha_{15} - \alpha_{13} - \alpha_{11} + 2\alpha_8 - 2\alpha_5 + 2\alpha_4 &= 0, \quad -\alpha_{15} + \alpha_{11} - \alpha_9 + \alpha_5 + \alpha_1 = 0, \\ -\alpha_{16} + \alpha_{13} - \alpha_9 + \alpha_5 + \alpha_1 &= 0, \quad -2\alpha_{13} - 2\alpha_{11} + 2\alpha_9 + 4\alpha_4 = 0, \\ \alpha_5 + \alpha_2 + \alpha_3 + \alpha_1 + \alpha_6 + \alpha_7 &= 0, \quad -\alpha_{14} - 2\alpha_{10} - \alpha_7 - \alpha_6 + \alpha_5 + \alpha_4 - \alpha_3 - \alpha_2 = 0. \end{aligned}$$

The solution of these systems is trivial. So $Q_1, \dots, Q_{16}$ are linearly independent. Thus we have shown the following

Proposition 4.3.2 *If $\text{char} F \nmid 3!$ then the set $\{Q_1, \dots, Q_{16}\}$ is a basis of the vector space $VIN_3[2, 2, 1, 1, 1]$. ■*

Let

$$\begin{aligned}
Q_1 &:= \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^\sigma Q(y^2, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}), \\
Q_2 &:= \sum_{\sigma(i) < \sigma(i+1), i=1,2} (-1)^\sigma Q(y, x_{\sigma(1)}, x_{\sigma(2)}, y \circ x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}), \\
Q_3 &:= \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^\sigma y Q(y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}), \\
Q_4 &:= \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^\sigma Q(y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}) y, \\
Q_5 &:= T_6(x_1, x_2, x_3, x_4, x_5; y^2), \\
Q_6 &:= \sum_{(12345)} T_6(y \circ x_1, x_2, x_3, x_4, x_5; y), \\
Q_7 &:= y T_6(x_1, x_2, x_3, x_4, x_5; y), \quad Q_8 := T_6(x_1, x_2, x_3, x_4, x_5; y) y, \\
Q_9 &:= \sum_{(12345)} x_1 T_6(y, x_2, x_3, x_4, x_5; y), \\
Q_{10} &:= \sum_{(12345)} T_6(y, x_2, x_3, x_4, x_5; y) x_1.
\end{aligned}$$

Then we have

Proposition 4.3.3 *The set $\{Q_1, \dots, Q_{10}\}$ is a basis of the vector space $VIN_3[2, 1^5]$.*

Proof. Let $f(y, x_1, x_2, x_3, x_4, x_5) \in VIN_3[2, 1^5]$. Then we need only to determine the coefficients of the following monomials in f :

$$yyx_1x_2 \cdots x_5, yx_1yx_2 \cdots x_5, \dots, yx_1 \cdots x_5, x_1yyx_2 \cdots x_5, \dots, x_1 \cdots x_5yy.$$

Let $\alpha_1, \dots, \alpha_{21}$ be their coefficients in f respectively. From Lemma 2.1.1 the sum of the coefficients of the monomials yyx_1w , yx_1yw , x_1yyw is 0 for $w = x_2x_3x_4x_5$. This yields

$$\alpha_1 + \alpha_2 + \alpha_7 = 0. \quad (4.76)$$

Similarly we have

$$\alpha_{12} + \alpha_{13} + \alpha_{16} = 0, \quad \alpha_{19} + \alpha_{20} + \alpha_{21} = 0. \quad (4.77)$$

By Lemma 2.1.1, the sum of the coefficients of the monomials yx_1wx_5y , x_1ywx_5y , x_1ywyx_5 , yx_1wyx_5 for $w = x_2x_3x_4$ is also 0. So

$$\alpha_5 + \alpha_6 + \alpha_{10} + \alpha_{11} = 0. \quad (4.78)$$

Since the sum of the coefficients of the monomials

$$x_1x_2yx_3x_4x_5y, x_1x_2yx_3x_4yx_5, x_1x_2x_3yx_4x_5y, x_1x_2x_3yx_4yx_5,$$

and the sum of the coefficients of the monomials

$$yx_1x_2yx_3x_4x_5, yx_1x_2x_3yx_4x_5, x_1yx_2yx_3x_4x_5, x_1yx_2x_3yx_4x_5,$$

are 0, we have that

$$\alpha_{14} + \alpha_{15} + \alpha_{17} + \alpha_{18} = 0, \alpha_3 + \alpha_4 + \alpha_8 + \alpha_9 = 0. \quad (4.79)$$

Let $w = x_1x_2x_3x_4x_5$ then the sum of the coefficients of the monomials yyw , yyw^* , wyy , w^*yy , ywy , yw^*y is 0, which implies that

$$\alpha_1 + \alpha_6 + \alpha_{21} = 0. \quad (4.80)$$

Similarly, the sum of the coefficients of the monomials of the form $-w-$, $-w^*-$, $--w$, $--w^*$, $w--$, w^*-- is 0 for

$$w = x_1x_2yx_3x_4, x_1x_2x_3yx_4, yx_2x_3x_4x_5,$$

which implies that

$$\begin{aligned} \alpha_3 + \alpha_4 + \alpha_9 + \alpha_{14} + \alpha_{15} + \alpha_{18} &= 0, \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8 + 2\alpha_{10} + \alpha_{11} + \alpha_{15} + \alpha_{17} + \alpha_{18} + \alpha_{20} &= 0, \\ \alpha_1 + \alpha_2 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_{11} + \alpha_{19} + \alpha_{20} + \alpha_{21} &= 0. \end{aligned} \quad (4.81)$$

Substituting

$$e[11], e[12], e[22], e[23], e[33], e[13], e[13]$$

for $x_1, x_2, x_3, x_4, x_5, y, y$, and considering the coefficient of e_{31} yield

$$\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7 - \alpha_8 - \alpha_{10} - \alpha_{17} + \alpha_{19} + \alpha_{20} + \alpha_{21} = 0. \quad (4.82)$$

Indeed if a substitution of the matrix $e[ij]$ denotes the path from i to j and if a substitution of matrix eii denotes the loop at point i , then the monomials which evaluate to e_{31} are the paths from vertex 3 to vertex 1 which pass each edge only once. They are

$$yx_1x_2x_3x_4x_5y, yx_2x_3x_4x_5yx_1, x_3yx_1x_2x_4x_5y, x_3yx_2x_4x_5yx_1,$$

$$x_4x_3x_2x_1yx_5y, x_4x_3x_2yx_5yx_1, x_5x_4x_3x_2x_1yy, x_5x_4x_3x_2yyx_1,$$

$$yx_1yx_5x_4x_3x_2, yyx_5x_4x_3x_2x_1, x_5yx_1yx_4x_3x_2, x_5yyx_4x_3x_2x_1.$$

Thus the sum of these monomials is 0 yields (4.82).

Similarly substituting $e[11], e[12], e[22], e[23], e[31], e[33]$ for $x_1, \dots, x_5, y$ and considering the coefficient of e_{11} we have

$$\alpha_7 + \alpha_{12} + \alpha_{16} + \alpha_{19} = 0. \quad (4.83)$$

The solution of the systems (4.76), ..., (4.83) is

$$\alpha_7 = -\alpha_8 - \alpha_9 + \alpha_{15} + \alpha_{18} + \alpha_{20} + \alpha_{21},$$

$$\alpha_2 = \alpha_8 + \alpha_9 - \alpha_{15} - \alpha_{18} - \alpha_{20} + \alpha_6,$$

$$\alpha_{12} = \alpha_8 + \alpha_9 - \alpha_{15} - \alpha_{18} - \alpha_{16},$$

$$\alpha_5 = -\alpha_6 - \alpha_{11}, \alpha_1 = -\alpha_6 - \alpha_{21},$$

$$\alpha_{14} = -\alpha_{15} + \alpha_8 - \alpha_{18}, \alpha_{13} = -\alpha_8 - \alpha_9 + \alpha_{15} + \alpha_{18},$$

$$\alpha_3 = -\alpha_4 - \alpha_8 - \alpha_9, \alpha_{17} = -\alpha_8, \alpha_{10} = 0, \alpha_{19} = -\alpha_{20} - \alpha_{21}.$$

So there are 10 parameters.

Next, we show that $Q_1, \dots, Q_{10}$ are linearly independent. If there exist scalars $\alpha_i \in F$ such that $\sum_{i=1}^{10} \alpha_i Q_i = 0$ then $(\sum_{i=1}^{10} \alpha_i Q_i)^* = 0$. Adding up these 2 equations yields

$$(\alpha_4 - \alpha_3)(Q_4 - Q_3) + (\alpha_8 - \alpha_7)(Q_8 - Q_7) + (\alpha_{10} - \alpha_9)(Q_{10} - Q_9) = 0. \quad (4.84)$$

Since

$$\begin{aligned} Q_i(y, x_1, x_2, x_3, x_4, y) &= 0, \quad i = 3, 4, \\ Q_j(y, x_1, x_2, x_3, x_4, y) &= Q'_1, \quad j = 7, 9, \\ Q_k(y, x_1, x_2, x_3, x_4, y) &= Q'_2, \quad k = 8, 10, \end{aligned}$$

and Q'_3, Q'_4 are linearly independent, (4.84) implies that

$$\alpha_8 - \alpha_7 + \alpha_{10} - \alpha_9 = 0.$$

Considering the coefficients of the monomials

$$\begin{aligned} yx_1x_2yx_3x_4x_5, \quad yx_1x_2x_3yx_4x_5, \quad yx_1x_2x_3x_4yx_5, \quad yx_1x_2x_3x_4x_5y, \quad x_1yyx_2x_3x_4x_5, \\ yyx_1x_2x_3x_4x_5, \quad yx_1yyx_2x_3x_4x_5, \quad x_1yx_2x_3yx_4x_5, \quad x_1x_2yyx_3x_4x_5, \end{aligned}$$

yields

$$\begin{aligned} 3\alpha_2 + 2\alpha_3 + 2\alpha_6 + \alpha_{10} &= 0, \quad 2\alpha_2 - 2\alpha_3 + 2\alpha_6 = 0, \quad 6\alpha_2 + 2\alpha_3 + 3\alpha_6 + \alpha_7 = 0, \\ -2\alpha_3 + 2\alpha_4 - \alpha_7 + \alpha_8 &= 0, \quad -2\alpha_1 - \alpha_2 - \alpha_5 - 2\alpha_6 + \alpha_{10} = 0, \\ 2\alpha_1 + 2\alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 &= 0, \quad \alpha_2 - 2\alpha_3 + \alpha_6 - \alpha_7 - \alpha_9 = 0, \\ -5\alpha_2 - 2\alpha_6 + \alpha_9 &= 0, \quad 2\alpha_1 + 3\alpha_2 + \alpha_9 = 0. \end{aligned}$$

Since $(\sum_{i=1}^{10} \alpha_i Q_i)^* = 0$ we have

$$\begin{aligned} -\alpha_1 Q_1 - \alpha_2 Q_2 - \alpha_3 Q_4 - \alpha_4 Q_3 - \alpha_5 Q_5 - \alpha_6 Q_6 - \\ \alpha_7 Q_8 - \alpha_8 Q_7 - \alpha_9 Q_{10} - \alpha_{10} Q_9 &= 0. \end{aligned} \tag{4.85}$$

Thus considering the coefficients of the monomials

$$yx_1x_2yx_3x_4x_5, \quad yx_1x_2x_3yx_4x_5, \quad yyx_1x_2x_3x_4x_5, \quad yx_1yyx_2x_3x_4x_5, \quad x_1yyx_2x_3yx_4x_5,$$

in (4.85) yields

$$\begin{aligned} 3\alpha_2 + 2\alpha_4 + 2\alpha_6 + \alpha_{10} &= 0, \quad 2\alpha_2 - 2\alpha_4 + 2\alpha_6 = 0, \quad 2\alpha_1 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_8 = 0, \\ \alpha_2 - 2\alpha_4 + \alpha_6 - \alpha_8 - \alpha_9 &= 0, \quad -5\alpha_2 - 2\alpha_6 + \alpha_{10} = 0. \end{aligned}$$

The solution of these systems is trivial. So $Q_1, \dots, Q_{10}$ are linearly independent. ■

Chapter 5

The Identities of Degree 9 on H_4

5.1 Introduction

In Chapter 4, we have determined all identities of degree 7 of H_3 . A basis of the vector space $VIN_3[t]$ was given for each type t . As in Chapter 4 let $VI_4[n_1, \dots, n_r]$ denote the set of all identities of degree 9 on H_4 which have type $[n_1, \dots, n_r]$, and $VIN_4[n_1, \dots, n_r]$ denote the set of all identities of degree 9 on H_4 which have type $[n_1, \dots, n_r]$ and are symmetric or skew symmetric in all their variables of degree m according as m is even or odd for any $m \in \{n_1, \dots, n_r\}$.

We also recall

$$\begin{aligned}\tilde{T}_k(x_1, \dots, x_{k-1}; x_k) &:= \\ S_k(x_k, x_1, \dots, x_{k-1}) - T_k(x_1, \dots, x_{k-1}; x_k)\end{aligned}$$

In this chapter we shall find a basis of the vector space $VIN_4[t]$ for each type t . The main results are Theorem 5.1 and Theorem 5.2.

Theorem 5.1 *If the characteristic of the field F does not divide $3!$ and $|F| > 8$ then:*

1. *For any type $[n_1, \dots, n_r]$ with $\sum_{i=1}^r n_i = 9$ and*

$$\begin{aligned}[n_1, \dots, n_r] \neq & [4, 1^5], [3, 2, 1^4], [3, 1^6], [2^3, 1^3] \\ & [2^2, 1^5], [2, 1^7] \text{ or } [1^9]\end{aligned}$$

we have $VI_4[n_1, \dots, n_r] = \{0\}$.

2. The vector space $VIN_4[2^3, 1^3] = \{0\}$.

Theorem 5.2 *If the characteristic of the field F does not divide $3!$ and $|F| > 8$ then:*

1. $\{T_8(x^2, x, y_1, \dots, y_5; x)\}$ is a basis of the vector space $VI_4[4, 1^5]$.
2. $\{T_8(x^2, x, y, z_1, \dots, z_4; y), T_8(x \circ y, x, y, z_1, \dots, z_4; x)\}$ is a basis of the space $VI_4[3, 2, 1^4]$.
3. The following identities form a basis of the space $VIN_4[3, 1^6]$

$$\begin{aligned} &T_8(x^2, y_1, \dots, y_6; x), \quad T_8(x, y_1, \dots, y_6; x^2), \\ &\sum_{i=1}^6 (-1)^{i-1} T_8(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_6; x), \\ &\sum_{i=1}^6 (-1)^i T_8(x^2, x, y_1, \dots, \hat{y}_i, \dots, y_6; y_i), \\ &T_8(x, y_1, \dots, y_6; x)x, \quad xT_8(x, y_1, \dots, y_6; x). \end{aligned}$$

4. The identities

$$\begin{aligned} &T_8(x^2, y, z_1, \dots, z_5; y) + T_8(y^2, x, z_1, \dots, z_5; x), \\ &T_8(x \circ y, x, z_1, \dots, z_5; y) + T_8(x \circ y, y, z_1, \dots, z_5; x), \\ &\sum_{i=1}^5 (-1)^{i-1} (T_8(x \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_5, x, y; y) + \\ &\quad T_8(y \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_5, y, x; x)), \\ &xT_8(x, y, z_1, \dots, z_5; y) + yT_8(y, x, z_1, \dots, z_5; x), \\ &T_8(x, y, z_1, \dots, z_5; y)x + T_8(y, x, z_1, \dots, z_5; x)y, \\ &T_8(x, y, z_1, \dots, z_5; [x, y]), \quad \bar{T}_8(x, y, z_1, \dots, z_5; [x, y]) \end{aligned}$$

form a basis of the space $VIN_4[2, 2, 1^5]$.

5. The identities

$$\begin{aligned} &T_8(y_1, \dots, y_7; x^2), \quad \bar{T}_8(y_1, \dots, y_7; x^2), \\ &\sum_{i=1}^7 (-1)^{i-1} T_8(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_7; x), \end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^7 (-1)^{i-1} \hat{T}_8(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_7; x), \\
& \sum_{i=1}^7 (-1)^{i-1} T_8(x, y_1, \dots, \hat{y}_i, \dots, y_7; y_i \circ x), \\
& T_8(y_1, \dots, y_7; x)x, \quad xT_8(y_1, \dots, y_7; x), \\
& \tilde{T}_8(y_1, \dots, y_7; x)x, \quad x\tilde{T}_8(y_1, \dots, y_7; x), \\
& \sum_{i=1}^7 (-1)^{i-1} y_i T_8(y_1, \dots, \hat{y}_i, \dots, y_7, x; x), \\
& \sum_{i=1}^7 (-1)^{i-1} \tilde{T}_8(y_1, \dots, \hat{y}_i, \dots, y_7, x; x) y_i, \\
& \sum_{i=1}^7 (-1)^{i-1} T_8(x, y_1, \dots, \hat{y}_i, \dots, y_7; [x, y_i])
\end{aligned}$$

form a basis of the space $VIN_4[2, 1^7]$.

6. $S_9(x_1, \dots, x_9)$ forms a basis of the space $VIN_4[1^9]$.

Thus, all identities of H_4 of degree 9 are determined.

Throughout the chapter, we assume that the characteristic of the field F does not divide $3!$ and $|F| > 8$.

5.2 The Proof of the Theorem 5.1

In this section we shall prove the Theorem 5.1. The proof will be completed in the following Propositions.

Proposition 5.2.1 H_4 has no identity of degree 9 which is of type $[5, 3, 1]$.

Proof. Let $f(x, y, z) \in VI_4[5, 3, 1]$. Then $f = f_1xy + f'_1yx$ because H_3 has no identity of degree 7 which is lower than $[4, 2, 1]$, and no identity of degree 6 which is lower than $[2, 2, 1, 1]$. Since f_1, f'_1 are identities of H_3 of type $[4, 2, 1]$ by Lemma 2.1.2, there exist scalars $\alpha, \beta \in F$ such that

$$f_1 = \alpha Q(x^2, x, y, z, x, y), \quad f'_1 = \beta Q(x^2, x, y, z, x, y).$$

However, from $f = f_1xy + f'_1yx$ we know that f has no terms which end in yx^2 . f^* is also an identity of type $[5, 3, 1]$, thus it has no terms which end in yx^2 . Thus f has no terms which start with x^2y . Since $x^2yxzxxyxy$ has coefficient $-\alpha$ in f , $\alpha = 0$. Similarly we can show that $\beta = 0$. So $f = 0$. ■

Proposition 5.2.2 *The vector space $VI_4[5, 2, 2] = \{0\}$.*

Proof. Let $f(x, y, z) \in VI[5, 2, 2]$ then by Theorem 1.1, f is symmetric in y, z since $\Delta(y, z)f \in VI_4[5, 3, 1] = \{0\}$. Thus

$$f = f_1(x, y, z)xz + f'_1(x, y, z)zx + f_1(x, z, y)xy + f'_1(x, z, y)yx.$$

Since f_1, f'_1 are identities of H_3 of type $[4, 2, 1]$, using the same argument as in the proof of Proposition 5.2.1 we have $f_1 = f'_1 = 0$. So $f = 0$. ■

Proposition 5.2.3 *The vector space $VI_4[5, 2, 1, 1] = \{0\}$.*

Proof. From Proposition 5.2.2, $VI_4[5, 2, 1^2] = VIN_4[5, 2, 1^2]$. Let $f(x, y, z, w) \in VIN_4[5, 2, 1^2]$ then

$$\begin{aligned} f = & f_1xy + f'_1yx + f_2(x, y, w)xz + \\ & f'_2(x, y, w)zx - f_2(x, y, z)xw - f'_2(x, y, z)wx, \end{aligned} \quad (5.1)$$

since H_3 has no identity which has type $[5, 1, 1]$, $[5, 1]$, $[3, 2, 1]$ or $[4, 1, 1]$.

Since f can be decomposed into the sum of a $*$ -symmetric identity in $VIN_4[5, 2, 1, 1]$ and a $*$ -skew symmetric identity in $VFN_4[5, 2, 1, 1]$, we may assume that $f^* = \pm f$. From (5.1) we know that every identity of type $[5, 2, 1, 1]$ has no terms which end in x^2 . So arguing as in the proof of Proposition 5.2.1 yields $f_2 = f'_2 = 0$. Thus $f = f_1xy + f'_1yx$. Since $\Delta(z, y)f \in VI_4[5, 3, 1] = \{0\}$, $\Delta(z, y)f_1 = 0$ as is seen by considering the sum of the terms of $\Delta(z, y)f$ which end in xy . Thus f_1 is skew symmetric in the variables of degree 1. Since $f^* = \pm f$ and f has only terms which end in xy or yx , f_1 must have only terms which start with xy or yx . However f_1 is skew symmetric in the variables y, z , so $f_1 = 0$ and $f = 0$. ■

Proposition 5.2.4 *The vector space $VI_4[5, 1^4] = \{0\}$. Therefore there is no identity of H_4 which is of type $[5, 1^4]$ or of lower type.*

Proof. Since H_4 has no identity which is lower than $[5, 1^4]$, $VI_4[5, 1^4] = VIN_4[5, 1^4]$. Let $f(x, y_1, y_2, y_3, y_4) \in VIN_4[5, 1^4]$ then

$$f = \sum_{i=1}^4 (-1)^i (f_1(x, y_1, \dots, \widehat{y}_i, \dots, y_4)xy_i + f'_1(x, y_1, \dots, \widehat{y}_i, \dots, y_4)y_ix).$$

Since $\Delta(y_4, 1)f \in VI_4[5, 1^3] = \{0\}$, $\Delta(y_4, 1)f_1(\dots, y_4, \dots) \in VI_3[4, 1^2] = \{0\}$ and $\Delta(y_4, 1)f'_1(\dots, y_4, \dots) \in VI_3[4, 1^2] = \{0\}$, from the equation above we have $f_1 + f'_1 = 0$. Because $f_1 \in VIN_3[4, 1^3]$ there exist scalars $\alpha, \beta \in F$ such that

$$f_1 = \alpha T_6(y_1, y_2, y_3, x^2, x; x) + \beta [S_3([y_1, x], [y_2, x], [y_3, x]), x].$$

We may assume that $f^* = \pm f$. Thus f has no terms which end in x^2 . This implies that the monomial $x^2y_1xy_2y_3xy_4$ has coefficient 0 in f . This implies that $\alpha - \beta = 0$. If $f^* = -f$ then the fact that the monomial $xy_1xy_2xy_3xy_4x$ has coefficient 0 in f implies that $\beta = 0$. So $f_1 = 0$ and therefore $f = 0$. If $f^* = f$ then the Razmyslov transform $f^\#(y_4)$ is also an identity of type $[5, 1^4]$. Since there is no identity of H_3 of degree 6 with type $[3, 1^3]$, $f^\#(y_4)$ has no terms which end in x^2 . So the monomial $y_4xy_1xy_2xy_3xx$ has coefficient 0 in $f^\#(y_4)$. This means the monomial $xy_1xy_2xy_3xxxy_4$ has coefficient 0 in f . However it has coefficient $-\beta$, thus $\beta = 0$. So $\alpha = 0$. Hence $f = 0$.

Let f be an identity of degree 9. If f has type lower than $[5, 1^4]$ then by Proposition 1.2.1, one of its partial linearizations is an identity of type $[5, 1^4]$ since $|F| > 8$. Thus this partial linearization is 0 by the result we have proved above. Therefore f is 0 since the partial linearization did not set some variables of f equal and there is no identity of H_4 which is lower than $[5, 1^4]$. ■

Proposition 5.2.5 *The vector space $VI_4[4, 3, 1, 1] = \{0\}$.*

Proof. Let $f(x, y, z_1, z_2) \in VI_4[4, 3, 1, 1]$. Then

$$\begin{aligned} f &= f_1xy + f'_1yz + f_4yx^2 + \\ &\quad \sum_{i=1}^2 (f_{2i}xz_i + f'_{2i}z_ix + f_{3i}yz_i + f'_{3i}z_iy), \end{aligned} \tag{5.2}$$

since H_3 has no identity of degree 7 which is lower than $[4,2,1]$ and no identity of degree 6 which is lower than $[2,2,1,1]$. Moreover

$$f_{21} = \alpha_1 Q(x \circ y, x, y, z_2, x, y), \quad f_4 = \beta Q(x, y, z_1, x, y, z_2)$$

because they are identities of H_3 . We may assume that $f^* = \pm f$. Then from (5.2) f has no terms which start with y^2 . So the monomial $y^2 x z_2 x^2 y x z_1$ has coefficient 0 in f . But it has coefficient $-\alpha_1$, thus $f_{21} = 0$. Since $\Delta(y, x)f \in VIN_4[5, 2, 1, 1] = \{0\}$, $\Delta(y, x)f_{21} + f_{31} = 0$. So $f_{31} = 0$. Using the same argument one can show that $f_{22} = f'_{22} = f_{32} = f'_{32} = 0$. Thus $f = f_1 xy + f'_1 yx + f_4 y^2$. Since $f^* = \pm f$ and f has no terms which end in z_1 , the monomial $z_1 x y x y z_2 y x^2$ must have coefficient 0. But it has coefficient β , so $f_4 = 0$ and $f = f_1 xy + f'_1 yx$. Thus $\Delta(y, x)f = 0$ implies that $f_1 = -f'_1$. Since f_1 is an identity of H_3 of type $[3, 2, 1, 1]$, by the decomposition Theorem 1.1 $f_1 = g + h$, where $g \in VIN_3[3, 2, 1, 1]$ and h is a partial linearization of an identity of type $[3, 2, 2]$. Thus $g = \sum_{i=1}^{10} \alpha_i Q_i$ and $h = \sum_{i=1}^4 \beta_i \tilde{Q}_i$, where $\{Q_i | i = 1, \dots, 10\}$ and $\{\tilde{Q}_i(x, y, z, z) | i = 1, \dots, 4\}$ are the bases of $VIN_3[3, 2, 1, 1]$ and $VI_3[3, 2, 2]$ given in Proposition 4.2.3 and Proposition 4.2.2 respectively. Since f has no terms which end in $z_2 x, z_2 y, z_1 z_2$ or $z_1 x^2$ and $f^* = \pm f$, the monomial $w^* xy$ has coefficient zero, where w is either as in the proof of Proposition 4.2.3 or $xxz_1 z_2 y yx$. Thus by the $*$ -symmetry of Q_i , changing α_i to be $-\alpha_i$, α_9 to be α_{10} and α_{10} to be α_9 in the system (4.19) except equation $\alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 = 0$ will give a new system. Solving it yields $\alpha_i = 0$. Since $f(x, y, z_1, z_2)$ has only terms which end in xy or yx and $f^* = \pm f$, $f(x, y, z, z)$ has only terms which start with xy or yx . So $h(x, y, z, z)$ has only terms which start with xy or yx . Considering the coefficients of the monomials $xzyzxyxyxy$, $yzxxzyxyxy$, $xxxyyzxxxy$ [see the proof of Proposition 4.2.2] yields $\beta_i = 0$. Thus $f = 0$. ■

Proposition 5.2.6 *The vector space $VIN_4[4, 2, 2, 1] = \{0\}$.*

Proof. Let $f(x, y_1, y_2, z) \in VIN_4[4, 2, 2, 1]$. Then

$$\begin{aligned} f = & (Id + (1, 2))\{f_1(x, y_2, y_1, z)xy_1 + f'_1(x, y_2, y_1, z)y_1x + f_3(x, y_1, y_2, z)y_1y_2 + \\ & f_4(x, y_1, y_2)y_2z + f'_4(x, y_1, y_2)zy_2 + f_5(x, y_2, y_1, z)y_1x^2\} + \\ & f_2(x, y_1, y_2)xz + f'_2(x, y_1, y_2)zx. \end{aligned} \tag{5.3}$$

where Id denotes the identity mapping and the operator $(1, 2)$ acts on polynomials to permute y_1 and y_2 . Since f_1 is an identity of H_3 of type $[3, 2, 1, 1]$,

$$f_1 = \sum_{i=1}^{10} \alpha_i Q_i + \sum_{i=1}^4 \beta_i \tilde{Q}_i,$$

where $\{Q_i | i = 1, \dots, 10\}$ and $\{\frac{1}{2}\tilde{Q}_i(x, y_2, z, z) | i = 1, \dots, 4\}$ are respectively the basis of the space $VIN_3[3, 2, 1, 1]$ and $VI_3[3, 2, 2]$ given in Chapter 4. Since $f_4 \in VI_3[4, 2, 1]$ and $f_5 \in VI_3[2, 2, 1, 1]$,

$$f_4 = \gamma Q(x^2, x, y_2, y_1, x, y_2), \quad f_5 = \delta Q(x, y_2, y_1, x, y_2, z).$$

If we let $z = 1$ then $f(x, y_1, y_2, 1) = f_i(x, y_1, y_2, 1) = 0$ for $i = 1, 3, 5$ because

$$VI_4[4, 2, 2] = VI_3[3, 2, 1] = VI_3[4, 1, 1] = 0.$$

Thus from (5.3) we have $f_i + f'_i = 0$ for $i = 2, 4$. Since $\Delta(z, y_1)f$ belongs to $VI_4[4, 3, 2] = \{0\}$ by Proposition 5.2.5, $f_1(x, y_2, y_1, y_1) + f_2 = 0$. So $f_1(x, y_2, y_1, y_1)$ is symmetric in y_1, y_2 by the symmetry of f_2 . Hence $f_1(x, y_2, y_1, y_1) \in VIN_3[3, 2, 2]$ which has dimension 3. From this we get $\beta_1 = \beta_2$. Since $\Delta(z, y_2)f = 0$, considering the sum of the terms of $\Delta(z, y_2)f$ which end in xy_1 yields $f_1(x, y_2, y_1, y_2) = 0$. While $f_1(x, y_2, y_1, y_2) = 0$ and $\beta_1 = \beta_2$ imply $\alpha_1 + \alpha_2 - \alpha_4 = 0$. From $\Delta(z, x)f = 0$ we have $f_1(x, y_2, y_1, x) - f_4(x, y_2, y_1) = 0$. This equation yields $-\alpha_3 + 2\alpha_4 - 2\beta_1 + \beta_3 + \gamma = 0$.

We may assume that $f^* = \pm f$. Thus using the $\star$ -symmetry of f and considering the coefficients of the monomials

$$\begin{aligned} &xxzx y_2 y_1 y_2 x y_1, \quad y_1 y_1 x y_2 x z x x y_2, \quad x y_2 y_1 x y_2 z x x y_1, \\ &y_2 x x y_2 y_1 z x x y_1, \quad y_2 x y_1 x y_2 z x x y_1, \quad y_2 x y_2 y_1 x z x x y_1, \\ &x y_2 x y_1 y_2 z x x y_1, \quad y_2 x x y_2 x y_1 z x y_1, \quad y_2 x y_1 x x z y_2 x y_1, \\ &y_1 x x y_2 y_2 z x x y_1, \quad y_1 x y_2 x y_2 z x x y_1, \quad x x y_1 z y_2 y_2 x x y_1, \\ & \quad \quad \quad x x y_2 y_2 y_1 z x x y_1, \end{aligned}$$

yields a system of linear equations. Solving this system yields $f_1 = f_5 = 0$. Since $\Delta(z, y_2)f = 0$ we have $f_1(x, y_1, y_2, y_2) + f_2 = 0$ and $f_3(x, y_1, y_2, y_2) + f_4(x, y_2, y_1) = 0$. While $\Delta(y_1, x)f = 0$ implies $f_1(x, x, y_2, z) + f_3 = 0$. So $f_i = 0$ for $i = 1, 2, 3, 4, 5$. Thus $f = 0$. ■

Proposition 5.2.7 *The vector space $VIN_4[4, 2, 1^3] = \{0\}$.*

Proof. Let $f(x, y, z_1, z_2, z_3) \in VIN_4[4, 2, 1^3]$ then

$$\begin{aligned} f &= f_1xy + f'_1yx + f_5yx^2 + f_6(x, y, z_1, z_2)z_3x^2 + \\ &\quad \sum_{(1,2,3)} \{f_2(x, y, z_1, z_2)xz_3 + f'_2(x, y, z_1, z_2)z_3x + \\ &\quad f_3(x, y, z_1, z_2)yz_3 + f'_3(x, y, z_1, z_2)z_3y + f_4(x, y, z_1)[z_2, z_3]\}. \end{aligned} \quad (5.4)$$

Since $\Delta(z_3, y)f \in VI_4[4, 3, 1^2] = \{0\}$, from (5.4) we have

$$\begin{aligned} f_2 &= -f_1(x, y, z_1, z_2, y), \quad f'_2 = -f'_1(x, y, z_1, z_2, y), \\ f_4 &= f_3(x, y, y, z_1), \quad f_3 = -f'_3, \quad f_6 = -f_5(x, y, z_1, z_2, y). \end{aligned}$$

Similarly from $\Delta(y, x)f = 0$ we have

$$f_3(x, y, z_1, z_2) = -\Delta(y, x)f_2(x, y, z_1, z_2).$$

Thus f is determined by f_1 , f'_1 and f_5 .

Since $\Delta(y, 1)f \in VI_4[4, 1^4] = \{0\}$ and $\Delta(y, 1)f_i = 0$, from (5.4) we have

$$f_1 + f'_1 + f_5x = 0.$$

f_5 is an identity of H_3 of type $[2, 1^4]$ and it is skew symmetric in the last 3 variables. Because every identity of H_3 of degree 6 comes from T_6 and Q , there exist scalars $\beta, \gamma \in F$ such that

$$\begin{aligned} f_5 &= \beta T_6(y, z_1, z_2, z_3, x, x) + \gamma \{Q(x, y, z_1, x, z_2, z_3) + \\ &\quad Q(x, y, z_2, x, z_3, z_1) + Q(x, y, z_3, x, z_1, z_2)\}. \end{aligned}$$

From the results in Chapter 4 every identity of H_3 of type $[3, 1^4]$ is a linear combination of the identities which come from T_6 , Q and K by substituting variables, especially the identity $f_1(x, y, z_1, z_2, z_3)$ which is skew symmetric in the z_i 's. Let $f_1 = \sum_i \alpha_i Q_i$, where Q_i is an identity which comes from one of Q , T_6 and K by substituting some variables. Let $\sigma \in S_3$ then

$$\sigma(h(x, y, z_1, z_2, z_3)) := h(x, y, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}),$$

where $h \in \{f_1, Q_i\}$. So

$$\begin{aligned} \sum_{\sigma \in S_3} (-1)^\sigma \sigma(f_1(x, y, z_1, z_2, z_3)) &= \sum_{\sigma \in S_3} (-1)^\sigma f_1(x, y, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) \\ &= \sum_i \alpha_i \left\{ \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i) \right\}. \end{aligned}$$

Thus

$$6f_1 = \sum_i \alpha_i \left\{ \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i) \right\}.$$

Since $\text{Char } F \nmid 3!$, we may assume that

$$f_1 = \sum_i \alpha_i g_i,$$

where $g_i = \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i)$. So g_i comes from one of Q , T_6 and K and is skew symmetric in the variables z_1, z_2, z_3 .

By the symmetry of the variables of T_6 , Q and K , there are only 21 identities which come from T_6 , Q and K by substituting some variables and which are skew in z_i 's. Thus $f_1 = \sum_{i=1}^{21} \alpha_i g_i$.

Now we may assume that $f^* = \pm f$. Then, by the $*$ -symmetry of f and since f has no terms which end in y^2 , considering the coefficients of some monomials yields a system (I) of linear equations.

If $f^* = -f$ then the solution of the system (I) implies that $f_1 = f_5 = 0$ which can be checked by computer easily.

If $f^* = f$ then the Razmyslov transform $f^\#(z_1)$ of f is also an identity of H_4 by Proposition 2.2.2. Thus the monomial $z_2 z_3 x y z_1 y x x x$ must have coefficient 0 in $f^\#(z_1)$ by Lemma 2.1.1. So the monomial $y x x x z_1 z_2 z_3 x y$ has coefficient 0 in f . That is

$$-\alpha_{17} + \alpha_{16} + 2\alpha_{13} + \alpha_{11} + \alpha_6 - \alpha_5 + \alpha_4 - \alpha_3 = 0.$$

The solution of this equation and the system (I) implies that $f_1 = f_5 = 0$. Hence $f = 0$. ■

Proposition 5.2.8 *The vector space $VIN_4[3, 3, 3] = \{0\}$.*

Proof. Let $f(x_1, x_2, x_3) \in VIN_4[3, 3, 3]$ then

$$f = \sum_{i \neq j} \pm f_1(x_k, x_i, x_j) x_i x_j$$

where (i, j, k) is a permutation of $1, 2, 3$, since there is no identity of H_3 of type $[3, 2, 1]$. Since f_1 is an identity of H_3 which has type $[3, 2, 2]$,

$$\begin{aligned} f_1 = & \alpha_1 Q(x_1 \circ x_3, x_1, x_2, x_3, x_1, x_2) + \alpha_2 Q(x_1 \circ x_2, x_1, x_3, x_2, x_1, x_3) + \\ & \alpha_3 Q(x_1^2, x_2, x_3, x_1, x_2, x_3) + \alpha_4 [S_3([x_1, x_2], [x_1, x_3], [x_2, x_3]), x_1]. \end{aligned}$$

We may assume that $f^* = \pm f$. Then by the symmetry of f , $\alpha(w) + s\alpha(w^*) = 0$, where $\alpha(w)$ denotes the coefficient of the monomial w in f and $s := \pm 1$ if $f^* = \mp f$. Thus for

$$w = x_1 x_2 x_1 x_3 x_2 x_3 x_1 x_2 x_3, x_2 x_3 x_1 x_1 x_3 x_2 x_1 x_2 x_3, x_1 x_3 x_1 x_2 x_2 x_3 x_1 x_2 x_3,$$

we have

$$\begin{aligned} 2\alpha_1 + s(\alpha_4 + \alpha_2 + 2\alpha_1) &= 0, \\ -\alpha_3 + s(-\alpha_4 - \alpha_2 - \alpha_3) &= 0, \\ -\alpha_4 - 2\alpha_2 - \alpha_1 - s\alpha_3 &= 0. \end{aligned} \tag{5.5}$$

Since f has no terms which end in x_i^2 and $f^* = \pm f$, the monomials $x_1 x_1 x_2 x_3 x_1 x_3 x_2 x_2 x_3$ and $x_1 x_1 x_3 x_1 x_2 x_2 x_3 x_2 x_3$ have coefficient 0. That is,

$$-\alpha_4 - \alpha_2 - \alpha_3 = 0, \quad \alpha_4 + \alpha_1 + \alpha_3 = 0, \tag{5.6}$$

The solutions in both cases are trivial, so $f = 0$. ■

Let $f(x_1, x_2, y, z)$ be an identity of H_4 of type $[3, 3, 2, 1]$ then

$$\begin{aligned} f = & (Id - (x_1, x_2)) \{ f_1(x_1, x_2, y, z) x_1 x_2 + f_2(x_1, x_2, y, z) x_2 y + \\ & f_2'(x_1, x_2, y, z) y x_2 + f_3(x_1, x_2, y) x_2 z + \\ & f_3'(x_1, x_2, y) z x_2 + f_5(x_2, x_2, y, z) x_1 x_2^2 \} + \\ & f_4(x_1, x_2, y) y z + f_4'(x_1, x_2, y) z y, \end{aligned} \tag{5.7}$$

since H_3 has no identities which are lower than $[3, 1, 1, 1]$.

The identity $f(x_1, x_2, y, y)$ is of type $[3, 3, 3]$, and so is zero identically. But

$$\begin{aligned} f(x_1, x_2, y, y) &= f_1(x_1, x_2, y, y)x_1x_2 - f_1(x_2, x_1, y, y)x_2x_1 + \\ &\quad (f_2(x_1, x_2, y, y) + f_3(x_1, x_2, y))x_2y + \\ &\quad (f_2'(x_1, x_2, y, y) + f_3'(x_1, x_2, y))yx_2 + \\ &\quad (f_4(x_1, x_2, y) + f_4'(x_1, x_2, y))y^2. \end{aligned}$$

Thus,

$$\begin{aligned} f_2(x_1, x_2, y, y) + f_3(x_1, x_2, y) &= 0, f_4 + f_4' = 0, \\ f_2'(x_1, x_2, y, y) + f_3'(x_1, x_2, y) &= 0, f_1(x_1, x_2, y, y) = 0. \end{aligned} \quad (5.8)$$

Similarly, by considering $f(x_1, x_2, y, x_1) = 0$ we have

$$\begin{aligned} f_3(x_2, x_1, y) + f_3'(x_2, x_1, y) + f_5(x_2, x_1, y, x_1)x_2 &= 0, \\ f_4(x_1, x_2, y) - f_2'(x_2, x_1, y, x_1) &= 0, f_3(x_1, x_2, y) - f_1(x_1, x_2, y, x_1) = 0, \\ f_2(x_1, x_2, y, x_1) &= 0, f_2'(x_1, x_2, y, x_1) = 0. \end{aligned} \quad (5.9)$$

Since f_5 is an identity of H_3 of type $[2, 2, 1, 1]$, there exists a scalar $\gamma \in F$ such that $f_5(x_1, x_2, y, z) = \gamma Q(x_1, y, x_2, x_1, y, z)$. Hence $\Delta(z, x_1)f_5(x_2, x_1, y, z) = 0$, and

$$f_3(x_1, x_2, y) + f_3'(x_1, x_2, y) = 0. \quad (5.10)$$

The equalities above tell us we need only determine f_1 , f_2 and f_5 . The identity $f_2(x_1, x_2, y, z)$ is of type $[3, 2, 1, 1]$, so $f_2 = g_0 + g_1$, where

$$\begin{aligned} g_0 &= \frac{1}{2}(f_2(x_1, x_2, y, z) - f_2(x_1, x_2, z, y)), \\ g_1 &= \frac{1}{2}(f_2(x_1, x_2, y, z) + f_2(x_1, x_2, z, y)). \end{aligned}$$

Since g_0 (respectively g_1) are skew symmetric (respectively symmetric) in y and z ,

$$g_0 = \sum_{i=1}^{10} \beta_i Q_i, \quad g_1 = \sum_{i=1}^4 \gamma_i \Delta^1(y, z) \tilde{Q}_i(x_1, x_2, y),$$

where $Q_1, \dots, Q_{10}$ are the basis of $VIN_3[3, 2, 1, 1]$, and $\tilde{Q}_1, \dots, \tilde{Q}_4$ are the basis of $VIN[3, 2, 2]$, both of which were given in Chapter 4.

Lemma 5.2.1 $f_5 = 0$, that is in f the coefficient γ is zero.

Proof. The proof will use Razmyslov's transformation. If f is $\star$ symmetric then the Razmyslov transform $f^\#(z)$ of f is also an identity by Proposition 2.2.2. Since the monomial $x_2x_1zy^2x_1$ has coefficient γ in f , $y^2x_1zx_2x_1$ has coefficient γ in $f^\#(z)$ also. But from (5.7), every identity of type $[3, 3, 2, 1]$ on H_4 has no terms which start with y^2x_1 or end in x_1y^2 . Thus $\gamma = 0$, since $f^\#(z)$ is an identity of H_4 of type $[3, 3, 2, 1]$.

If $f^\star = -f$ then $f^\#(z)(x_1, x_2, y, z) = 0$ for $x_1, x_2, y \in H_4$, and $z \in K_4$ by Proposition 2.2.2. Since

$$f^\#(z) = g(x_1, x_2, z)x_2y^2 + \sim,$$

$g(x_1, x_2, z) = 0$ for $x_1, x_2 \in H_3$ and $z \in K_3$ by Proposition 2.2.1. Next, we show that g is identically zero. If g is $\star$ -skew symmetric, then $g^\#(z)$ is an identity of type $[3, 2, 1]$ on H_3 , therefore $g^\#(z) = 0$, hence g is identically zero. Thus we may assume that g is $\star$ -symmetric. Since H_2 has no identities of type lower than $[2, 2]$ and has no identity of degree less than 4 either,

$$g(x, y, z) = g_1xy + g'_1yx + g_2yx^2 + g_3xy^2, \quad (5.11)$$

where $g_2(x, y, z)$ is multilinear and $g_2(x, y, z) = 0$ for $x, y \in H_2$ and $z \in K_2$, and $g_3(x, z)$ is of type $[2, 1]$. Hence

$$g_3(x, z) = \alpha_1zx^2 + \alpha_2xzx + \alpha_3x^2z. \quad (5.12)$$

From (5.11) we know that g has no terms ending in one of the monomials

$$zx, xz, zx^2, zy, yz, zy^2.$$

Since g is $\star$ -symmetric, it has no terms starting with one of the monomials

$$xz, zx, x^2z, y^2z, zy, yz.$$

Thus g_3 is identically zero. Since H_2 has no identity of degree less than 4, we may assume that g_2 is $\star$ -symmetric by Proposition 2.2.2. Thus

$$g_2(x, y, z) = \beta_1(xyz + zyx) + \beta_2(yzx + xzy) + \beta_3(zxy + yxz).$$

The same argument as above yields that g_2 is also identically zero. Thus the terms in g must start with either xy or yx , and end in either xy or yx also. Hence

$$\begin{aligned} g = & \gamma_1(xyzx^2y + yx^2zyx) + \gamma_2(xyxxzy + yxxzyx) + \\ & \gamma_3(xyzyx + yxzyx) + \gamma_4(yxx^2y + yx^2zyx). \end{aligned}$$

Let us consider the complete linearization $g'(x_1, x_2, x_3, y_1, y_2, z)$ of g . Substituting $e[12], e[22], e_{23} - e_{32}, e[33], e[33], e[33]$ for $x_1, y_1, z, x_2, x_3, y_2$ in g' and considering the coefficient of e_{13} yield $\gamma_1 + \gamma_3 = 0$.

Substituting $e[12], e[22], e_{23} - e_{32}, e[33], e[33], e[33]$ for $y_1, x_1, z, x_2, y_2, x_3$ in g' and considering the coefficient of e_{13} yield $\gamma_2 + \gamma_4 = 0$.

Substituting $e[11], e[11], e_{12} - e_{21}, e[22], e[23], e[33]$ for $x_1, y_1, z, x_2, x_3, y_2$ in g' and considering the coefficient of e_{13} yield $\gamma_1 + \gamma_4 = 0$.

Substituting $e[11], e[12], e[22], e_{23} - e_{32}, e[33], e[33]$ for $x_1, y_1, x_2, z, y_2, x_3$ in g' and considering the coefficient of e_{13} yield $\gamma_2 + \gamma_3 = 0$.

So

$$\begin{aligned} g = & \gamma_1(xyzx^2y + yx^2zyx + xyxxzy + yxxzyx \\ & - xyzyx - yxzyx - yxx^2y - yx^2zyx). \end{aligned}$$

Let $x = e[11] + e[12]$, $y = e[23]$ and $z = e_{13} - e_{31}$. Then $g(x, y, z) = 0$ implies $\gamma_1 = 0$. So $g = 0$ identically.

From $g = 0$ we know that $f^\#(z)$ has no terms ending either in x_2y^2 or starting with y^2x_2 . Thus $y^2x_1^2x_2^2zx_2x_1$ has coefficient 0 in $f^\#(z)$. However, it has coefficient γ since the monomial $x_2x_1zy^2x_1^2x_2^2$ has coefficient γ in f . So, $\gamma = 0$. ■

Since $\Delta(y, x_2)f \in VI_4[4, 3, 1, 1] = \{0\}$, from Lemma 5.2.1 and (5.7), considering the sum of the terms of $\Delta(y, x_2)f$ which end in x_2^2 yields $f_2 + f'_2 = 0$. While $\Delta(x_2, x_1)f \in VI_4[4, 2, 2, 1] = \{0\}$ implies that $f_1(x_1, x_2, y, z)$ is symmetric in x_1, x_2 from Lemma 5.2.1 and (5.7).

Lemma 5.2.2 *The identity f_2 is zero identically.*

Proof. It suffices to show that f_2 is zero if f is either $\ast$ -symmetric or $\ast$ -skew symmetric. In both cases, the coefficients of the following monomials are zero by (5.7) and $f_5 = 0$,

$$x_1^2 x_2 y x_1 x_2 z x_2 y, x_1^2 y x_2 z x_1 x_2^2 y, x_2^2 x_1 y x_1 z x_1 x_2 y, x_1^2 x_2 y z x_2 x_1 x_2 y, \\ x_1^2 x_2^2 x_1 y z x_2 y, x_2^2 y^2 x_1 x_2 x_1^2 z, x_1^2 x_2^2 y x_2 x_1 y z.$$

That is,

$$\begin{aligned} \beta_2 + 2\beta_3 + \beta_6 - \beta_7 + \beta_9 + \gamma_2 + 2\gamma_3 + \gamma_4 &= 0, \\ 2\beta_3 + \beta_4 + \beta_6 + 2\beta_8 + \beta_9 + \gamma_1 + 2\gamma_3 + \gamma_4 &= 0, \\ \beta_1 + \beta_5 - \beta_7 &= 0, \quad \beta_2 + 2\beta_3 + \beta_9 = 0, \\ \beta_1 + \beta_7 + 2\beta_8 &= 0, \quad \gamma_1 + 2\gamma_3 + \gamma_4 = 0, \\ \beta_1 + \beta_2 - \beta_4 + \gamma_1 - \gamma_2 &= 0. \end{aligned} \tag{5.13}$$

From $f_2(x_1, x_2, y, x_1) = 0$, we have

$$-\beta_3 + 2\beta_4 - 2\gamma_1 + \gamma_3 = 0. \tag{5.14}$$

If $h(x_1, \dots, x_9)$ is a multilinear identity on H_4 then substituting

$$e[11], e[12], e[22], e[23], e[33], e[33], e[34], e[44], e[42]$$

for $x_1, \dots, x_9$ and considering the coefficient of e_{12} in h yields that the sum of the coefficients of the following monomials is 0,

$$\begin{aligned} x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9, x_1 x_2 x_3 x_4 x_6 x_5 x_7 x_8 x_9, \\ x_1 x_2 x_4 x_5 x_6 x_7 x_8 x_9 x_3, x_1 x_2 x_4 x_6 x_5 x_7 x_8 x_9 x_3, \\ x_1 x_2 x_9 x_8 x_7 x_6 x_5 x_4 x_3, x_1 x_2 x_9 x_8 x_7 x_5 x_6 x_4 x_3, \\ x_1 x_2 x_3 x_9 x_8 x_7 x_6 x_5 x_4, x_1 x_2 x_3 x_9 x_8 x_7 x_5 x_6 x_4. \end{aligned}$$

Using this result and Lemma 2.1.1, considering the monomials $x_1 x_2 x_1 y z x_2 x_1 x_2 y$ and $x_2 x_1 y x_1 z x_1 x_2 x_2 y$ in f (in fact, in the complete linearization of f) respectively yield

$$\beta_9 - \beta_{10} = 0, \quad \beta_2 - 2\beta_3 - 3\beta_4 - 2\beta_5 + \beta_7 + 3\gamma_1 + 3\gamma_2 + 2\gamma_3 + 2\gamma_4 = 0. \tag{5.15}$$

If f is $\ast$ -symmetric then the coefficients of the monomials

$$x_1 y x_1 x_2^2 z x_1 x_2 y, x_2 y x_1 x_2 x_1 z x_1 x_2 y, y x_1 x_2 x_1 x_2 z x_1 x_2 y, x_2 y x_1^2 z x_2 x_1 x_2 y$$

are equal to the coefficients of their image under $\ast$, respectively, in f , which yields

$$\begin{aligned} \beta_{10} + \beta_1 - 2\beta_2 - 2\beta_3 - \beta_4 - \beta_5 - \beta_7 + 2\beta_8 - \gamma_1 - 2\gamma_2 + 2\gamma_3 - \gamma_4 &= 0, \\ -\beta_{10} - \beta_2 - \beta_4 - \beta_5 - 2\beta_6 - \beta_7 - \beta_{10} - \gamma_1 - \gamma_2 + 2\gamma_4 &= 0, \\ \beta_{10} - \beta_2 - \beta_4 - \beta_6 - \beta_7 - \gamma_1 - \gamma_2 - \gamma_4 &= 0, \\ -\beta_{10} - 4\beta_3 - 3\beta_4 - \beta_5 - \beta_6 - 2\beta_8 + \gamma_1 + \gamma_4 &= 0. \end{aligned} \quad (5.16)$$

Solving the equations (5.13), ..., (5.16) yields $f_2 = 0$ if f is $\ast$ -symmetric.

If f is $\ast$ -skew symmetric then (5.16) becomes

$$\begin{aligned} \beta_{10} - \beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 - \beta_6 + \beta_7 - 2\beta_8 - \gamma_1 - 2\gamma_2 - 2\gamma_3 - \gamma_4 &= 0, \\ \beta_{10} + \beta_2 - \beta_4 - \beta_5 + \beta_7 - \beta_{10} - \gamma_1 + \gamma_2 = 0, \gamma_1 - \beta_4 &= 0, \\ \beta_{10} - \beta_4 - \beta_5 + \beta_6 + 2\beta_8 - \gamma_1 - 4\gamma_3 - \gamma_4 &= 0. \end{aligned} \quad (5.17)$$

The solutions of (5.13), (5.14), (5.15), (5.17) are also trivial. Thus in any case $f_2 = 0$. ■

From $f_2 = 0$ and $f'_2 = -f_2$ we know that for every identity of type $[3, 3, 2, 1]$ which is $\ast$ -skew symmetric in the variables of degree 3 the only nonzero terms must end in $x_1 x_2$ or $x_2 x_1$. Since $f^\ast$ is such an identity, its terms end in $x_1 x_2$ or $x_2 x_1$. This implies that the terms of f must start with $x_1 x_2$ or $x_2 x_1$. Thus

$$f = [x_1, x_2]g[x_1, x_2] \quad (5.18)$$

Hence, the polynomial $g(y, x_1, x_2, z)$ is an identity of H_2 of type $[2, 1^3]$ and has the same $\ast$ -symmetry as f . Since f is skew symmetric in x_1, x_2 , g is also skew symmetric in x_1, x_2 from (5.18). By $f_1(x_1, x_2, y, x_1) = 0$ and (5.18), g is skew in x_1, z . Thus g is skew symmetric in all variables of degree 1. Therefore, if $f^\ast = -f$ then

$$g = \lambda_1(L_1 - L_4) + \lambda_2(L_2 - L_3) + \lambda_3(L_5 - L_6) + \lambda_4(L_7 - L_7^\ast),$$

where $\{L_i \mid i = 1, \dots, 7\}$ is the basis of $VIN_3[2, 1^3]$ given in Proposition 3.1.4. If $f^\ast = f$ then

$$g = \lambda_1(L_1 + L_4) + \lambda_2(L_2 + L_3) + \lambda_3(L_5 + L_6) + \lambda_4(L_7 + L_7^\ast).$$

Since $f^\#(z)$ has no terms which end in $x_1 y^2$ if $f^* = \pm f$ (for the case where $f^* = -f$ see the proof of Lemma 5.2), the monomials

$$x_1 x_2 y^2 z x_1 x_2 x_1 x_2, \quad x_1 x_2 x_1 y^2 z x_2 x_1 x_2$$

have coefficients zero in f . Thus $y^2 z x_1 x_2, x_1 y^2 z x_2$ have coefficients zero in g if $g^* = \pm g$. This implies that

$$\lambda_1 + \lambda_3 + \lambda_4 = 0, \quad \lambda_3 + 2\lambda_4 = 0. \quad (5.19)$$

On the other hand, substituting

$$e[12] + e[24], \quad e[23], \quad e[34], \quad e[14]$$

for x_1, x_2, y, z and considering the coefficients of e_{32} and e_{13} in f respectively yields, if f is skew symmetric,

$$\lambda_1 - \lambda_2 + \lambda_3 = 0, \quad -2\lambda_1 - \lambda_3 + 2\lambda_4 = 0, \quad (5.20)$$

and if f is symmetric,

$$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 = 0, \quad 2\lambda_2 - \lambda_3 + 2\lambda_4 = 0. \quad (5.21)$$

In any case, the solution is trivial. Hence $f = 0$ identically. Thus we have shown

Proposition 5.2.9 *There is no identity of type $[3, 3, 2, 1]$ on H_4 . ■*

Let $f(x_1, x_2, y_1, y_2, y_3) \in VIN[3, 3, 1, 1, 1]$. Then

$$\begin{aligned} f(x_1, x_2, y_1, y_2, y_3) = & \sum_{\sigma \in S_2} (-1)^\sigma f_1(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2, y_3) x_{\sigma(1)} x_{\sigma(2)} + \\ & \sum_{\substack{(1,2,3) \\ \sigma \in S_2}} (-1)^\sigma f_2(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2) x_{\sigma(2)} y_3 + \\ & \sum_{\substack{(1,2,3) \\ \sigma \in S_2}} (-1)^\sigma f'_2(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2) y_3 x_{\sigma(2)} + \\ & \sum_{(1,2,3)} f_3(x_1, x_2, y_1, y_2, y_3) [y_2, y_3] + \\ & \sum_{\sigma \in S_2} (-1)^\sigma f_4(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2, y_3) x_{\sigma(1)} x_{\sigma(2)}^2, \end{aligned} \quad (5.22)$$

where $(1, 2, 3)$ denotes the cyclic permutations of 1, 2, 3. Since $VIN[4, 3, 1, 1] = 0$, $\Delta^1(y_3, x_1)f$ is zero identically. It implies that

$$\begin{aligned} f_2 - \Delta(y_3, x_1)f_1 &= 0, \quad f'_2 + \Delta(y_3, x_1)f_1 = 0, \\ f_2 + f'_2 - \Delta(y_3, x_1)f_4 &= 0, \quad f_3 - \Delta(y_3, x_1)f'_2 = 0. \end{aligned} \quad (5.23)$$

From (5.22), $f_4(x_1, x_2, y_1, y_2, y_3) \in VI_3[2, 1, 1, 1]$ and is skew symmetric in the last three variables. Therefore it is a linear combination of the identities g_1 which come from T_6 and Q by $[MR]$ and skew symmetric in y_i 's. So

$$\begin{aligned} f_4 &= \beta_2 T_6(x_2, y_1, y_2, y_3, x_1; x_1) + \\ &\quad \sum_{i=1}^3 (-1)^{i-1} \beta_1 Q(x_1, x_2, y_i, x_1, y_1, \dots, \widehat{y_i}, \dots, y_3). \end{aligned}$$

Hence $\Delta^1(y_3, x_1)f_4 = 0$ and $f_2 + f'_2 = 0$. So, it suffices to determine f_2 and f_4 .

Since $f_2(x_1, x_2, y_1, y_2) \in VIN_3[3, 2, 1, 1]$ there exist scalars $\alpha_i \in F$ such that $f_2 = \sum_{i=1}^{10} \alpha_i Q_i$, where $\{Q_1, \dots, Q_{10}\}$ is the basis of $VIN_4[3, 2, 1, 1]$ given in Chapter 4.

Lemma 5.2.3 *The identity f_2 is zero identically.*

Proof. It is sufficient to show the lemma by assuming that $f^* = \pm f$. Since $VI_3[3, 1, 1, 1] = 0$, there are no terms in f and its Razmyslov transform $f^\#(y_i)$ which start with $x^2 y_i$ or end in $y_i x^2$ by Proposition 2.1.1, Proposition 2.2.2 and the skew symmetry of f in y_i 's. Thus the following monomials have coefficients zero in f :

$$x_1^2 y_1 y_2 x_2 x_1 x_2^2 y_3, \quad y_1 x_1 x_2 x_1^2 y_2 x_2^2 y_3, \quad y_3 x_2^2 y_2 x_1^2 x_2^2 y_1, \quad x_2 x_1^2 y_1 x_1 y_2 x_2^2 y_3.$$

It follows that

$$\begin{aligned} \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 &= 0, \quad -\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 + \alpha_7 + \alpha_8 = 0, \\ \alpha_1 + \alpha_3 + 2\alpha_4 - \alpha_7 + \alpha_8 &= 0, \quad \alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 - \alpha_7 = 0. \end{aligned}$$

By the proof of Lemma 5.2.2 and considering the coefficient of the monomial $x_1 y_1 x_2^2 y_2 x_2 x_1^2 y_3$ give

$$2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 + 2\alpha_8 + \alpha_9 - \alpha_{10} = 0.$$

If $f^* = f$ then every monomial and its $*$ have the same coefficients in f . By considering monomials

$$\begin{aligned} & x_1 y_1 x_2 x_1 x_2 y_2 x_1 x_2 y_3, \quad x_1^2 x_2^2 y_1 y_2 x_1 x_2 y_3, \quad x_2^2 x_1^2 y_1 y_2 x_1 x_2 y_3, \\ & x_2 y_1 x_1^2 x_2 y_2 x_1 x_2 y_3, \quad x_2 y_1 x_1 x_2 x_1 y_2 x_1 x_2 y_3, \quad x_1^2 x_2 y_1 x_2 y_2 x_1 x_2 y_3, \\ & y_1 x_1^2 x_2^2 y_2 x_1 x_2 y_3, \quad x_2 y_1 x_1 x_2 x_1 y_2 x_1 x_2 y_3, \quad y_1 x_1 x_2 x_1 x_2 y_2 x_1 x_2 y_3, \\ & y_1 x_1 x_2 x_1^2 y_2 x_2^2 y_3, \end{aligned}$$

we have

$$\begin{aligned} \alpha_2 + 4\alpha_4 - \alpha_6 + \alpha_7 + \alpha_9 &= 0, \quad \alpha_1 - \alpha_2 - \alpha_3 + \alpha_6 + \alpha_8 - \alpha_9 = 0, \\ 2\beta_1 + \beta_2 - \alpha_1 - \alpha_5 + \alpha_7 &= 0, \quad \alpha_4 + \alpha_5 - \alpha_6 - \alpha_8 - \alpha_{10} = 0, \\ \alpha_2 - \alpha_4 - \alpha_5 + \alpha_7 &= 0, \quad -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_6 - \alpha_8 + \alpha_9 = 0, \\ \alpha_1 - 2\alpha_2 - \alpha_3 - 3\alpha_4 + \alpha_6 - \alpha_7 + \alpha_8 - \alpha_{10} &= 0, \\ \alpha_2 - \alpha_4 - \alpha_5 + \alpha_7 &= 0, \quad \alpha_4 = 0, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 - 2\alpha_7 &= 0. \end{aligned}$$

The solution of the equations above is $\alpha_i = 0, i = 1, \dots, 10$ and $\beta_2 = -2\beta_1$.

If $f^* = -f$ then considering the monomials

$$\begin{aligned} & x_1 y_1 x_2 x_1 x_2 y_2 x_1 x_2 y_3, \quad x_1^2 x_2^2 y_1 y_2 x_1 x_2 y_3, \quad x_2^2 x_1^2 y_1 y_2 x_1 x_2 y_3, \\ & x_2 y_1 x_1^2 x_2 y_2 x_1 x_2 y_3, \quad x_2 y_1 x_1 x_2 x_1 y_2 x_1 x_2 y_3, \quad y_1 x_1^2 x_2^2 y_2 x_1 x_2 y_3, \\ & y_1 x_1 x_2 x_1 x_2 y_2 x_1 x_2 y_3, \quad x_2^2 x_1^2 y_1 x_1 y_2 x_2 y_3, \quad y_1 x_1 x_2 x_1^2 y_2 x_2^2 y_3 \end{aligned}$$

yields

$$\begin{aligned} \alpha_2 + 3\alpha_6 + \alpha_7 - \alpha_9 - 2\alpha_{10} &= 0, \quad \alpha_1 - \alpha_2 - \alpha_3 + \alpha_6 + \alpha_8 - \alpha_9 = 0, \\ 2\beta_1 + \beta_2 + \alpha_1 + \alpha_5 - \alpha_7 &= 0, \\ 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 2\alpha_7 - \alpha_8 + \alpha_{10} &= 0, \\ \alpha_2 + \alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 - 2\alpha_{10} &= 0, \\ -\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 - \alpha_8 - \alpha_{10} &= 0, \\ \alpha_2 + \alpha_4 + \alpha_6 + \alpha_7 - \alpha_{10} &= 0, \quad \alpha_1 + \alpha_5 - \alpha_7 + \beta_2 + 2\beta_1 = 0, \\ \alpha_1 - \alpha_2 + \alpha_4 - \alpha_5 + 2\alpha_8 &= 0, \quad \alpha_4 = 0. \end{aligned}$$

The solution is also $\alpha_i = 0, i = 1, \dots, 10$ and $\beta_2 = -2\beta_1$.

Now we can prove

Proposition 5.2.10 *The vector space $VI_4[3, 3, 1, 1, 1] = \{0\}$.*

Proof. Since $f_2 = 0$, it follows that f has no term which has y_i either in one of the last two positions or in one of the first two positions. But the coefficient of the monomial $y_1 y_2 y_3 x_1 x_2 x_1^2 x_2^2$ in f_4 is $\pm 4\beta_1$, so $\beta_1 = 0$. Therefore

$$f = f_1(x_1, x_2, y_1, y_2, y_3)x_1 x_2 - f_1(x_2, x_1, y_1, y_2, y_3)x_2 x_1.$$

Since $\Delta(x_2, x_1)f \in VI_4[1, 2, 1^3] = \{0\}$, $(\Delta(x_2, x_1)f_1(x_1, x_2, y_1, y_2, y_3))x_1 x_2 = 0$. So $\Delta(x_2, x_1)f_1(x_1, x_2, y_1, y_2, y_3) = 0$. Therefore f_1 is symmetric in x_1, x_2 by Theorem 1.1. So $f = f_1[x_1, x_2]$. Since $f^* = \pm f$, $f = [x_1, x_2]g[x_1, x_2]$. g is multilinear, skew symmetric in the x_i 's, and in the y_i 's by the symmetry of f . In fact g is alternating and therefore is a scalar multiple of S_5 . This follows from

$$[x_1, x_2]g(x_2, x_1, y_1, y_2, x_1) = f_1(x_2, x_1, y_1, y_2, x_1) = f_2 = 0.$$

Hence

$$f = \alpha[x_1, x_2]S_5(x_1, x_2, y_1, y_2, y_3)[x_1, x_2].$$

Since $f(x_1, x_2, y_1, y_2, 1) \in VI_4[3, 3, 1, 1] = \{0\}$ and $S_5(x_1, x_2, y_1, y_2, 1) = S_4(x_1, x_2, y_1, y_2) \neq 0$ by [DR], $\alpha = 0$. So $f = 0$. ■

Let $f(x, y_1, y_2, y_3) \in VIN_4[3, 2, 2, 2]$ then

$$\begin{aligned} f = & \sum_{(1,2,3)} f_1(x, y_1, y_2, y_3)x y_3 + \sum_{(1,2,3)} f'_1(x, y_1, y_2, y_3)y_3 x + \\ & \sum_{(1,2,3)} f_2(x, y_1, y_2, y_3)y_2 y_3 + \sum_{(1,2,3)} f_3(y_1, y_2, y_3, x)y_3 x^2, \end{aligned} \quad (5.24)$$

since $VI_3[3, 2, 1] = VIN_3[2, 2, 2] = 0$.

Since we have shown that $V I_4[1, 2, 2, 1] = 0$, $\Delta(y_3, x)f(x, y_1, y_2, y_3) \in V I_4[4, 2, 2, 1]$ implies that

$$\begin{aligned}\Delta(y_3, x)f_2(x, y_1, y_2, y_3) &= 0, \quad \Delta(y_3, x)f_2(x, y_3, y_1, y_2) = 0, \\ \Delta(y_3, x)f_1(x, y_3, y_1, y_2) + f_2(x, y_1, y_3, y_2) &= 0, \\ f_1(x, y_1, y_2, y_3) + f'_1(x, y_1, y_2, y_3) + \Delta(y_3, x)f_3(y_1, y_2, y_3, x)y_3 + \\ f_3(y_1, y_2, y_3, x)x + \Delta(y_3, x)f_3(y_2, y_3, y_1, x)y_1 + \Delta(y_3, x)f_3(y_2, y_3, y_1, x)y_2 &= 0. \quad (5.25)\end{aligned}$$

by considering the sums of terms of $\Delta(y_3, x)f$ which end in y_2y_3 , y_1y_2 , xy_2 and x^2 respectively. Since $\Delta(y_3, y_1)f \in V I N_4[3, 3, 2, 1] = 0$, the sum of the terms which end in y_1y_3 in $\Delta(y_3, y_1)f$ is zero, which yields

$$\Delta(y_3, y_1)f_2(x, y_2, y_1, y_3) = 0.$$

So, f_2 is skew symmetric in the variables of degree one. Hence, there exist scalars $\alpha_i \in F$ such that

$$f_2 = \sum_{i=1}^{10} \alpha_i Q_i,$$

where $Q_1, \dots, Q_{10}$ is the basis of $V I N_3[3, 2, 1, 1]$ given in Chapter 4. Since $f_3 \in V I_3[2, 2, 1, 1]$, there exists a scalar $\gamma \in F$ such that

$$f_3(y_1, y_2, y_3, x) = \gamma Q(y_1, y_2, y_3, y_1, y_2, x).$$

Next we show

Lemma 5.2.4 *The polynomials $f_2 = f_3 = 0$.*

Proof. From $\Delta(y_3, x)f_2(x, y_1, y_2, y_3) = 0$, $\Delta(y_3, x)f_2(x, y_3, y_1, y_2) = 0$ we have

$$\begin{aligned}2\alpha_4 - \alpha_3 &= 0, \quad 2\alpha_1 + 2\alpha_7 - \alpha_8 = 0, \\ 2\alpha_2 - \alpha_3 - 2\alpha_7 + \alpha_8 &= 0, \quad \alpha_5 + 2\alpha_6 = 0. \quad (5.26)\end{aligned}$$

In the following we may assume that $f^* = \pm f$. From (5.24) the coefficient of the monomial $y_1^2 x y_2 x y_3 x y_2 y_3$ is zero. So we have

$$-\alpha_7 + \alpha_5 + \alpha_1 = 0. \quad (5.27)$$

Let $\alpha(w)$ denote the coefficient of the monomial w in f . Then $\alpha(w) + s\alpha(w^*) = 0$, where $s = 1$ if $f^* = -f$ and $s = -1$ if $f^* = f$. Thus considering the monomials:

$$\begin{aligned} &xx y_2 y_1 y_3 x y_1 y_2 y_3, \quad x x y_1 y_2 y_3 y_1 x y_2 y_3, \quad x x y_1 y_1 x y_2 y_3 y_2 y_3, \\ &y_2 y_3 x x y_1 x y_3 y_2 y_3, \quad y_2 y_3 x y_1 x x y_3 y_2 y_3, \end{aligned}$$

yields

$$\begin{aligned} &-\alpha_9 - \alpha_8 - \alpha_6 - \alpha_4 - \alpha_3 + s\gamma = 0, \\ &-2\alpha_9 - 2\alpha_2 - 2\alpha_3 + 2s\gamma = 0, \quad \alpha_7 + \alpha_8 + \alpha_1 = 0, \\ &\alpha_7 + \alpha_4 + \alpha_3 + \alpha_2 - s(-\alpha_7 + \alpha_2 + \alpha_6 + \alpha_3 + \alpha_{10}) = 0, \\ &-2\alpha_7 + \alpha_3 - s(\alpha_{10} + \alpha_6 - \alpha_4 - \alpha_3) = 0. \end{aligned} \tag{5.28}$$

The solutions of (5.26), (5.27) and (5.28) is $\alpha_i = 0$ for $i \neq 9$ and $\alpha_9 + s\gamma = 0$ when $s = \pm 1$ if $f^* = \mp f$.

Next we show that $\alpha_9 = \gamma = 0$. Since $VI_3[2^3, 1]$ is spanned by the identities which come from Q , K and T_6 by substituting some variables, by the symmetry of Q , K and T_6 the subspace consisting of the elements $g(x, y, z, w) \in VI_3[2^3, 1]$ and symmetric in y, z is spanned by following 17 elements:

$$\begin{aligned} &Q(x^2, y, z, y, z, w), \quad \sigma(Q(y^2, x, z, x, z, w)), \quad \sigma(Q(y \circ x, z, y, z, x, w)), \\ &\sigma(Q(y \circ x, z, x, z, y, w)), \quad \sigma(Q(y \circ x, z, w, z, x, y)), \quad \sigma(Q(y \circ z, x, y, x, w, z)), \\ &Q(x \circ w, y, z, y, z, x), \quad \sigma(Q(y \circ w, x, z, x, z, y)), \quad xQ(x, y, z, y, z, w), \\ &Q(x, y, z, y, z, w)x, \quad \sigma(yQ(y, x, z, x, z, w)), \quad \sigma(Q(y, x, z, x, z, w)y), \\ &\sigma(T_6(y \circ x, y, z, w, x, z)), \quad K(x, y, x, z, y, z, w), \quad \sigma(K(x, y, w, z, y, z, x)), \\ &\sigma(K(x, y, w, z, x, z, y)), \quad \sigma(K(z, y, w, x, x, z, y)), \end{aligned}$$

where $\sigma := Id + (z, y)$. [Note $\sigma(K(y, w, x, z, z, y, x)) - \sigma(K(x, y, w, z, y, z, x)) = 0$]. Let g_i denote these identities. Then $f_1 = \sum_{i=1}^{17} \beta_i g_i$ for some $\beta_i \in F$. By the $\star$ -symmetry of f , considering the coefficients of the monomials

$$y_2 y_1 x y_2 y_1 y_3 x x y_3, \quad y_2 y_1 x y_2 y_3 x y_1 x y_3, \quad y_2 y_3 y_1 x x y_2 y_1 x y_3,$$

$$\begin{aligned}
& y_1 y_3 y_2 y_1 x y_2 x x y_3, x y_1 y_3 x y_1 x y_2 y_2 y_3, y_3 y_2 y_2 x y_1 x y_1 x y_3, \\
& y_1 x y_2 y_3 y_1 x y_2 x y_3, x x y_2 y_2 y_1 y_3 y_1 x y_3, y_2 y_1 x y_3 x y_1 y_2 x y_3, \\
& y_3 y_2 x x y_1 y_2 y_1 x y_3, y_2 y_1 y_2 y_3 x y_1 x x y_3, y_1 x y_2 x y_2 y_1 y_3 x y_3,
\end{aligned}$$

yields

$$\begin{aligned}
& -\beta_{17} - \beta_{15} - \beta_{11} + \beta_{10} - \beta_3 - \beta_7 = 0, \\
& \beta_7 - \beta_{17} + \beta_8 + \beta_{12} - \beta_6 + \beta_3 - 2\beta_{11} + 2\beta_5 + \beta_4 = 0, \\
& \beta_6 - \beta_{16} - \beta_5 - \beta_{13} + \beta_{12} + \beta_1 - \beta_4 - \beta_3 = 0, \\
& \beta_{16} - 2\beta_8 - \beta_{13} + \beta_6 - \beta_{15} + \beta_{11} - \beta_{10} - \beta_3 = 0, \\
& \alpha_9 + s(-\beta_{12} + \beta_{17} + \beta_{13} - \beta_6 - \beta_5 + \beta_4 + \beta_3 - \beta_2) = 0, \\
& \beta_{16} + 2\beta_{13} + \beta_5 - \beta_{12} + \beta_3 = 0, \\
& 2\beta_{12} - 2\beta_{11} + s(\beta_4 + \beta_{17} - 2\beta_5 + \beta_{16} + \beta_{12} - \beta_3 - \beta_{11} + \beta_8 + \beta_7) = 0, \\
& \beta_9 - \beta_3 + \beta_{15} + \beta_{13} + \beta_1 = 0, -\beta_{12} + \beta_{11} = 0, \\
& -\beta_4 + \beta_{14} + \beta_{13} + \beta_8 + \beta_1 - 2\beta_5 = 0, \beta_{17} - \beta_{15} + \beta_{11} - \beta_{10} + \beta_8 - \beta_7 = 0, \\
& -\beta_4 + 2\beta_3 - \beta_{14} + \beta_{13} - \beta_8 + s(-\beta_{12} + \beta_{11}) = 0.
\end{aligned}$$

Since $\Delta(y_3, x)f = 0$ implies that $\Delta(y_3, x)f_1 = 0$, the monomials

$$y_2 y_1 x y_2 x y_1 x, x y_1 x y_2 y_1 y_2 x, y_1 x x y_1 y_2 y_2 x,$$

have coefficients 0 in $f_1(x, y_1, y_2, x)$. Since f has no terms ending in y_i^2 the monomials

$$y_2 y_2 x x y_1 y_1 y_3, y_2 y_2 y_1 x y_3 y_1 x,$$

have coefficients 0 in f_1 . Thus we have

$$\begin{aligned}
& \beta_{14} - \beta_8 - \beta_5 + \beta_3 + 2\beta_{15} = 0, 2\beta_8 + 2\beta_5 - 2\beta_3 = 0, 2\beta_7 + \beta_1 = 0, \\
& -\beta_2 + \beta_{16} + \beta_{13} - \beta_{11} + \beta_4 = 0, \beta_{17} + \beta_{11} + \beta_2 + \beta_6 = 0.
\end{aligned}$$

Since we have $\alpha_9 + s\gamma = 0$ already, solving these 2 systems yields $\alpha_9 = \gamma = 0$ when $s = \pm 1$. Thus $f_2 = f_3 = 0$. ■

Now we can show

Proposition 5.2.11 *The vector space $VIN_4[3, 2, 2, 2] = \{0\}$.*

Proof. From $f_2 = f_3 = 0$ and (5.24) we have $f_1 + f'_1 = 0$ and $\Delta(y_3, x)f_1(x, y_3, y_1, y_2) = 0$. We also have $f_1(x, y_1, y_2, y_3) \in VIN_3[2, 2, 2, 1]$ and is symmetric in the variables y_1, y_2 . The fact that $\Delta(y_3, x)f_1(x, y_3, y_1, y_2) = 0$ implies that f_1 is symmetric in x, y_3 also by Theorem 1.1. Thus $f_1 \in VIN_3[2, 2, 2, 1]$ and

$$f = \sum_{(1,2,3)} f_1(x, y_1, y_2, y_3)[x, y_3]$$

if $f^* = \pm f$. So, f has no terms which start with one of $y_i y_j, x^2 y_i, y_i^2 x, y_i^2 y_j$, by the $\star$ -symmetry of f . Hence f_1 has no such terms either. This means that only terms of the form $xy_i - - - -$ or $y_i x - - - -$ occur in f_1 . But any monomial of the form $y_i y_j - - - -$ has coefficient 0 in f_1 , and f_1 is symmetric in y_i, x . Hence the monomials of the form $y_i x - - - -$ and $xy_i - - - -$ also have coefficients zero in f_1 . Thus $f_1 = 0$, completing the proof. ■

Proposition 5.2.12 *The vector space $VIN_4[3, 2, 2, 1, 1] = \{0\}$.*

Proof. Let $f(x, y_1, y_2, z_1, z_2) \in VIN_4[3, 2, 2, 1, 1]$ then

$$\begin{aligned} f = & (Id + (y_1, y_2))\{f_1(x, y_1, y_2, z_1, z_2)xy_2 + f'_1(x, y_1, y_2, z_1, z_2)y_2x + \\ & f_3(x, y_1, y_2, z_1, z_2)y_1y_2 + f_6(x, y_1, y_2, z_1, z_2)y_2x^2 + f_8(x, y_1, z_1, z_2)xy_2^2\} + \\ & (Id - (z_1, z_2))\{f_2(x, y_1, y_2, z_1)xz_2 + f'_2(x, y_1, y_2, z_1)z_2x + f_7(x, y_1, y_2, z_1)z_2x^2 + \\ & (Id + (y_1, y_2))f_4(x, y_1, y_2, z_1)y_2z_2 + f'_4(x, y_1, y_2, z_1)z_2y_2\} + \\ & f_5(x, y_1, y_2)[z_1, z_2], \end{aligned} \tag{5.29}$$

where Id denotes the identity map, (y_1, y_2) denotes the operator which permutes y_1, y_2 and (z_1, z_2) denotes the operator which permutes z_1, z_2 .

Since there are no identities of H_4 of degree 9 which are lower than $[3, 2, 2, 1, 1]$, $\Delta(z_i, x)f = \Delta(z_i, y_j)f = \Delta(y_i, x)f = 0$ imply that

$$\begin{aligned} f_4(x, y_1, y_2, z_1) &= \Delta(y_2, x)f_2(x, y_1, y_2, z_1), \quad f_5(x, y_1, y_2) = f_4(x, y_1, y_2, y_2), \\ f_7(x, y_1, y_2, z_1) &= f_6(x, y_1, y_2, z_1, y_2), \quad f_4(x, y_1, y_2, z_1) = f_3(x, y_2, y_1, z_1, y_1). \end{aligned}$$

Thus we need only determine f_2, f_3, f_6, f_8 . As before we may assume that $f^* = \pm f$. $f_2(x, y_1, y_2, z_1) \in VI_3[2^3, 1]$ and symmetric in y_1, y_2 . Since $VI_3[2^3, 1]$ is spanned by the identities which come from Q, K and T_6 by substituting some variables, the subspace consisting of the elements $g(x, y, z, w) \in VI_3[2^3, 1]$ and symmetric in y, z , is spanned by g_i , $i = 1, \dots, 17$, which were given in the proof of the Lemma 5.2.4:

$$\begin{aligned} & \sigma(Q(y^2, x, z, x, z, w)), Q(x^2, y, z, y, z, w), \sigma(Q(y \circ x, z, w, z, y, x)), \\ & \sigma(Q(y \circ x, z, y, z, x, w)), \sigma(Q(y \circ x, z, x, z, y, w)), \sigma(Q(y \circ z, x, z, x, y, w)), \\ & \sigma(Q(y \circ w, x, z, x, z, y)), Q(x \circ w, y, z, y, z, x), \sigma(yQ(y, x, z, x, z, w)), \\ & xQ(x, y, z, y, z, w), \sigma(Q(y, x, z, x, z, w)y), Q(x, y, z, y, z, w)x, \\ & \sigma(K(y, x, x, z, z, w, y)), \sigma(K(y, z, x, z, x, w, y)), \sigma(K(y, x, y, z, z, w, x)), \\ & \sigma(T_6(y \circ x, y, z, w, x, z)), K(x, y, x, z, y, z, w), \sigma(K(y, w, x, z, z, y, x)), \end{aligned}$$

where $\sigma := Id + (y, z)$. Since $g_{15} + g_{18} = 0$, we can forget g_{18} and let $f_2 = \sum_{i=1}^{17} \alpha_i g_i$. Similarly, since $f_6(x, y_1, y_2, z_1, z_2)$ is of type $[2, 1^4]$ and is skew symmetric in z_1 and z_2

$$\begin{aligned} f_6(x, y_1, y_2, z_1, z_2) = \\ \beta_1 T_6(x, y_1, y_2, z_1, z_2, y_1) + \beta_2 * Q(y_1, y_2, x, y_1, z_1, z_2) + \\ \beta_3 (Q(y_1, x, z_1, y_1, y_2, z_2) + Q(y_1, x, z_2, y_1, y_2, z_1)). \end{aligned}$$

We also have

$$\begin{aligned} f_8(x, y_1, z_1, z_2) &= \gamma Q(x, y_1, z_1, x, y_1, z_2), \\ f'_4(x, y_1, y_2, z_1) &= -f_4(x, y_1, y_2, z_1) - \gamma Q(x, y_1, z_1, x, y_1, y_2)x. \end{aligned}$$

By $f^* = \pm f$, monomials which start with $y_1^2 y_2$ have coefficients 0 and $\alpha(w) + s\alpha(w^*) = 0$ for any monomial w , where $s = \pm 1$ if $f^* = \mp f$. Considering the 48 monomials we have a system (I) of linear equations.

If $f^* = f$ then its Razmyslov transform $f^\#(z_2)$ is also an identity. So the monomial $z_2 z_1 y_1^2 y_2^2 x x x$ must have coefficient 0 in $f^\#(z_2)$ by Lemma 2.1.1. Thus the monomial $z_1 y_1 y_1 y_2 y_2 x x x z_2$ has coefficient 0 in f . Since $f^\#(z_1)$ is another identity of H_4 of type $[3, 2, 2, 1, 1]$, from (5.29) the monomials of the form $--- y_2 y_1 y_1$ must have coefficients 0. Since $f^* = f$ implies that $(f^\#)^* = f^\#$, the monomial $y_1 y_1 y_2 ---$

has coefficient 0 in $f^\#(z_1)$. This fact implies that the monomial $xz_1y_1y_1y_2y_2xxz_2$ has coefficient 0 in f . We get 2 equations. The solution of (I) and these 2 equations yields $f_6 = f_7 = f_8 = 0$.

Next, since $VI_4[3, 3, 1, 1, 1] = \{0\}$, any identity of type $[3, 2, 2, 1, 1]$ must be symmetric in the variables of degree 2. So $\Delta(y_2, z_1)f(x, y_1, y_2, z_1, z_2, w)$ is symmetric in y_1, z_1 and $\Delta(y_2, z_1)f_2(x, y_1, y_2, z_1)$ is symmetric in y_1z_1 . Considering the monomial $xz_1z_1y_1y_1y_2x$ yields another equation. By solving these 3 equations and (I), $f_2 = 0$. Therefore

$$f_i = 0, \quad i = 2, 4, 5, 6, 7, 8.$$

This implies that $f'_1 = -f_1$ and f_3 is skew symmetric in y_1, y_2, z_1, z_2 . Thus (5.29) becomes

$$f = (Id + (y_1, y_2))(f_1[x, y_2] + f_3(x, y_1, y_2, z_1, z_2)y_1y_2).$$

From this equation we know that f has only terms which end in y_iy_j, xy_i or y_ix . Since f is $\ast$ -symmetric, f_3 has only terms which start with y_iy_j, xy_i or y_ix . But f_3 is skew symmetric in y_1, y_2, z_1, z_2 , so $f_3 = 0$.

If $f^\ast = -f$ then solving (I) yields $f_i = 0, \quad i = 6, 7, 8$. This implies that $f'_i = -f_i$, for $i = 1, 2, 4$ and $f_3(x, y_1, y_2, z_1, z_2)$ is skew symmetric in y_1, y_2 . Since $f_3(x, y_1, y_2, z_1, z_2)$ is of type $[3, 1^4]$ and skew symmetric in y_1, y_2 and in z_1, z_2 , it is a linear combination of 23 identities h_i which come from Q, T_6 and K . Let $f_3 = \sum_{i=1}^{23} \lambda_i h_i$. Then from $f^\ast = -f$, $f_3(x, y_2, y_1, z_1, y_1) + f_4(x, y_1, y_2, z_1) = 0$ and monomials which start with x^2z_i have coefficients 0 we have a system of linear equations (II). Unfortunately, the systems (I) and (II) are still not enough. We have to find more equations.

From the results in Chapter 4, any identity $p(x, y_1, y_2, z_1, z_2)$ of type $[2, 2, 1, 1, 1]$ which is skew symmetric in z_1, z_2 is a linear combination of 57 such identities p_i which come from Q, T_6, K and $T_6(x_1, \dots, x_5; [x_6, x_7])$. So let $f_1 = \sum_{i=1}^{57} \delta_i p_i$. Then from (5.29)

$$f_2(x, y_1, y_2, z_1) = f_1(x, y_1, y_2, z_1, y_2), \quad f_3(x, y_1, y_2, z_1, z_2) = \Delta(y_1, x)f_1(x, y_1, y_2, z_1, z_2),$$

$$f_4(x, y_1, y_2, z_1) = -\Delta(y_2, x)f_2(x, y_1, y_2, z_1), \quad f_5(x, z_1, y_2) = -f_4(x, z_1, y_2, y_2).$$

Using these new definitions of f_i and $f^\ast = -f$ we get a system (III). Solving (I), (II) and (III) yields $f_2 = f_3 = 0$. Thus when $f^\ast = \pm f$ we have

$$f = (Id + (y_1, y_2))f_1(x, y_1, y_2, z_1, z_2)[x, y_2]. \quad (5.30)$$

Since $\Delta(y_1, x)f = 0$, $\Delta(y_1, x)f_1 = 0$. So f_1 is symmetric in x, y_1 . Similarly, it can be shown that f_1 is skew in y_2, z_1 . Thus $f_1 \in VIN_3[2, 2, 1^3]$. Since $f^* = \pm f$, from (5.30) and the symmetry of the variables of f_1 ,

$$f = (Id + (y_1, y_2))[x, y_1]g(x, y_1, y_2, z_1, z_2)[x, y_2],$$

where g is multilinear and skew symmetric in the variables y_2, z_1, z_2 . Again, from $\Delta(y_2, x)f = 0$ and $\Delta(y_2, y_1)f = 0$ we have g is skew symmetric in x, y_2, y_1, y_2 . Thus g is alternating. So, there exists a scalar $\alpha \in F$ such that $g = \alpha S_5(x, y_1, y_2, z_1, z_2)$. So

$$f(x, y_1, y_2, z_1, z_2) = \alpha(Id + (y_1, y_2))[x, y_1]s_5(x, y_1, y_2, z_1, z_2)[x, y_2].$$

Since $S_5(x, y_1, y_2, z_1, 1) = S_4(x, y_1, y_2, z_1)$ by [DR],

$$\begin{aligned} f(x, y_1, y_2, z_1, 1) &= \alpha(Id + (y_1, y_2))[x, y_1]S_5(x, y_1, y_2, z_1, 1)[x, y_2] \\ &= \alpha(Id + (y_1, y_2))[x, y_1]S_4(x, y_1, y_2, z_1)[x, y_2]. \end{aligned}$$

$f(x, y_1, y_2, z_1, 1)$ is an identity of H_4 of degree 8 which has type $[3, 2, 2, 1]$. So $f(x, y_1, y_2, z_1, 1) = 0$. Hence $\alpha = 0$ and $f = 0$. ■

Proposition 5.2.13 *The vector space $VIN_4[2^4, 1] = \{0\}$.*

Proof. Let $f(x_1, x_2, x_3, x_4, y) \in VIN_4[2^4, 1]$ then by the symmetry of the variables x_i 's of f ,

$$\begin{aligned} f &= f_1(x_1, x_2, x_3, x_4, y)x_3x_4 + f_1(x_1, x_2, x_3, x_4, y)x_4x_3 + \dots \\ &\quad + f_2(x_1, x_2, x_3, x_3)x_4y + f_2'(x_1, x_2, x_3, x_4)yx_4 + \dots \\ &\quad + f_3(x_2, x_3, x_4, y)x_4x_1^2 + \dots \end{aligned} \tag{5.31}$$

$f_1(x_1, x_2, x_3, x_4, y)$ is an identity of H_3 of type $[2^2, 1^3]$ and is symmetric in x_1, x_2 . So from the results in Chapter 4, f_1 is a linear combination of $g_i(x_1, x_2, x_3, x_4, y)$ which is symmetric in x_1, x_2 and comes from one of Q, T_6, K and $T(x_1, x_2, x_3, x_3, x_5; [x_6, x_7])$ by substituting some variables or multiplying by a variable. From the symmetry of the variables of Q, T_6 and K , there are only 47 such g_i 's. So $f_1 = \sum_{i=1}^{47} \alpha_i g_i$. $\Delta(y, x_3)f = 0$ implies that

$$f_2(x_1, x_2, x_3, x_4) = f_1(x_1, x_2, x_4, x_3, x_3).$$

Since $f_3 \in VIN_3[2, 2, 1, 1]$, there exists a scalar $\beta \in F$ such that $f_3 = \beta Q(x_2, x_3, x_4, x_2, x_3, y)$.

As before we may assume that $f^* = \pm f$. So from the $\star$ -symmetry we have a system (I). From (5.31) and the symmetry of the variables of f , f_2 is symmetric in its variables of degree 2. This yields a system (II). Finally the system (III) is obtained from

$$\Delta(y, x_1)f_1(x_1, x_2, x_3, x_4, y) = 0,$$

$$\Delta(x_4, x_1)f_1(x_1, x_2, x_3, x_4, y) = 0,$$

$$\Delta(x_3, x_1)f_1(x_1, x_2, x_3, x_4, y) = 0.$$

When $f^* = -f$, the solution of (I), (II) and (III) is trivial. So $f_1 = 0$, hence $f = 0$. If $f^* = f$, let $g(x_1, x_2, x_4, x_3, z, y) := \Delta(x_3, z)f$. Then g is an identity of type $[2^3, 1^3]$ and is $\star$ -symmetric. So the Razmyslov transform $g^\#(z)$ is an identity of H_4 . Hence $g^\#(z)(x_1, x_2, x_4, z, z, y)$ is an identity of type $[2^4, 1]$ and is $\star$ -symmetric. Thus from (5.31) and the fact that every identity of H_4 of type $[2^4, 1]$ has no terms which end in yx_i^2 , $g^\#(z)$ has no terms which start with x_i^2y . Particularly, the monomial $w := x_1x_1yx_2x_4x_2zx_4z$ has coefficient 0 in $g^\#(z)(x_1, x_2, x_4, z, z, y)$. Since w comes from the monomials $x_1x_1yx_2x_4x_2zx_4x_3$ and $x_1x_1yx_2x_4x_2x_3x_4z$, the sum of the coefficients in $g^\#(z)(x_1, x_2, x_4, x_3, z, y)$ of the monomials $x_1x_1yx_2x_4x_2zx_4x_3$ and $x_1x_1yx_2x_4x_2x_3x_4z$ is 0. So by the definition of the Razmyslov transformation, the sum of the coefficients of the monomials $x_4x_3zx_1x_1yx_2x_4x_2$ and $zx_1x_1yx_2x_4x_2x_3x_4$ in $g(x_1, x_2, x_4, x_3, z, y)$ is 0. Since $g(x_1, x_2, x_3, x_3, z, y) = \Delta(x_3, z)f$, this implies that, in f , the sum of the coefficients of the monomials

$$x_4x_3zx_1x_1yx_2x_4x_2, \quad zx_1x_1yx_2x_4x_2x_3x_4,$$

$$x_4zx_3x_1x_1yx_2x_4x_2, \quad x_3x_1x_1yx_2x_4x_2zx_4,$$

is 0. This yields a system of equations (IV). Solving (I) ..., (IV) we get $f_1 = 0$. Hence $f = 0$. ■

Proposition 5.2.14 *The vector space $VIN[2^3, 1^3] = \{0\}$.*

Proof. Let $f(x_1, x_2, x_3, y_1, y_2, y_3) \in VIN_4[2^3, 1^3]$ then

$$f = f_1(x_1, x_2, x_3, y_1, y_2, y_3)x_2x_3 + f_1(x_1, x_3, x_2, y_1, y_2, y_3)x_3x_2 + \dots$$

$$\begin{aligned}
& + f_2(x_1, x_2, x_3, y_1, y_2) y_3 x_3 + f_2'(x_1, x_2, x_3, y_1, y_2) x_3 y_2 + \cdots \\
& + f_3(x_1, x_2, x_3, y_1) [y_2, y_3] + \cdots \\
& + f_4(x_2, x_3, y_1, y_2, y_3) x_3 x_1^2 + \cdots \\
& + f_5(x_2, x_3, y_1, y_2) y_3 x_1^2 + \cdots,
\end{aligned} \tag{5.32}$$

where f_i is an identity of H_3 for $i = 1, \dots, 5$.

Since $\Delta(y_i, x_2)f \in VIN_4[3, 2^2, 1^2] = \{0\}$, by considering the sum of the terms of $\Delta(y_3, x_2)f$ which end in $x_2 x_3$, $x_3 x_2$ and $x_3 x_1^2$ respectively we get

$$\begin{aligned}
f_1(x_1, x_2, x_3, y_1, y_2, x_2) + f_2(x_1, x_2, x_3, y_1, y_2) &= 0, \\
f_1(x_1, x_3, x_2, y_1, y_2, x_2) + f_2'(x_1, x_2, x_3, y_1, y_2) &= 0, \\
f_4(x_2, x_3, y_1, y_2, x_3) + f_5(x_2, x_3, y_1, y_2) &= 0.
\end{aligned} \tag{5.33}$$

Similarly from $\Delta(y_2, x_3)f = 0$ we have

$$f_2(x_1, x_2, x_3, y_1, x_3) - f_3(x_1, x_2, x_3, y_1) = 0. \tag{5.34}$$

Thus f is determined by f_1 and f_4 .

It is known by $[MR]$ that every identity of degree 6 of H_3 comes from T_6 and Q . So

$$\begin{aligned}
f_4 &= \gamma_1 T_6(x_2, x_3, y_1, y_2, y_3; x_2) + \\
& \gamma_2 (Q(x_2, x_3, y_1, x_2, y_2, y_3) + Q(x_2, x_3, y_2, x_2, y_3, y_1) + Q(x_2, x_3, y_3, x_2, y_1, y_2))
\end{aligned}$$

by the symmetry of the variables of T_6 and Q .

Since $f_1(x_1, x_2, x_3, y_1, y_2, y_3)$ is skew symmetric in the y_i 's, it suffices to determine the coefficients of the monomials in which the y_i occur in the numerical order. Let $w_1, \dots, w_{20}$ denote the monomials

$$x_1 x_1 y_1 y_2 y_3, x_1 y_1 x_1 y_2 y_3, x_1 y_1 y_2 x_1 y_3, \dots, y_1 y_2 y_3 x_1 x_1,$$

respectively. Thus by inserting one x_2 and one x_3 into each w_i will give all monomials which we need to determine the coefficients. Let w_{ij} (w_{ijk}) denote the monomial obtained from w_i (w_{ij}) by inserting x_2 (x_3) into the position j (k). e.g $w_{11} = x_2 x_1 x_1 y_1 y_2 y_3$ and $w_{114} = x_2 x_1 x_1 x_3 y_1 y_2 y_3$. Let α_{ijk} denote the coefficient of w_{ijk} in f_1 , then we have

Lemma 5.2.5 For an arbitrary but fixed pair of j, k if $\alpha_i := \alpha_{ijk}$ then

$$\begin{aligned}\alpha_1 + \alpha_2 + \alpha_5 &= 0, \quad \alpha_1 - \alpha_3 - \alpha_6 = 0, \\ \alpha_1 + \alpha_4 + \alpha_7 &= 0, \quad -\alpha_2 - \alpha_3 + \alpha_8 = 0, \\ \alpha_2 - \alpha_4 + \alpha_9 &= 0, \quad \alpha_3 + \alpha_4 + \alpha_{10} = 0.\end{aligned}$$

Proof. Since $\Delta(y_r, x_1)f \in VIN_4[3, 2^2, 1^2] = \{0\}$, $\Delta(y_r, x_1)f_1 = 0$ from (5.32). Thus from $\Delta(y_1, x_1)f_1 = 0$, for an arbitrary but fixed pair of j, k the monomial obtained from $x_1x_1x_1y_2y_3$ by inserting x_2, x_3 at positions j, k respectively has coefficient zero in $\Delta(y_1, x_1)f_1$. Since $\Delta(y_1, x_1)$ acts on f_1 without changing x_2, x_3 we can forget x_2, x_3 and simply say $x_1x_1x_1y_2y_3$ has coefficient 0 in $\Delta(y_1, x_1)f_1$. Since $x_1x_1x_1y_2y_3$ has preimage w_1, w_2, w_5 under $\Delta(y_1, x_1)$, $\alpha_1 + \alpha_2 + \alpha_5 = 0$.

Similarly $\Delta(y_1, x_1)f_1 = 0$ implies

$$-\alpha_2 - \alpha_3 + \alpha_8 = 0, \quad \alpha_2 - \alpha_4 + \alpha_9 = 0, \quad \alpha_3 + \alpha_4 + \alpha_{10} = 0.$$

Since $\Delta(y_2, x_1)f_1 = 0$ and $\Delta(y_3, x_1)f_1 = 0$ we have

$$\alpha_1 - \alpha_3 - \alpha_6 = 0, \quad \alpha_1 + \alpha_4 + \alpha_7 = 0,$$

completing the proof of the Lemma. ■

Lemma 5.2.6 For an arbitrary but fixed k if $\alpha_{ij} := \alpha_{ijk}$ then

$$\begin{aligned}\alpha_{42} &= -\alpha_{45} - \alpha_{46} - \alpha_{16} - \alpha_{32} - \alpha_{15}, \\ \alpha_{21} &= -\alpha_{34} - \alpha_{35} + \alpha_{32} + \alpha_{15}, \\ \alpha_{44} &= -\alpha_{36} - \alpha_{34} - \alpha_{35} - \alpha_{45} - \alpha_{46}, \\ \alpha_{13} &= \alpha_{32} + \alpha_{15} + \alpha_{25} + \alpha_{33} - \alpha_{12}, \\ \alpha_{22} &= \alpha_{34} - \alpha_{32} - \alpha_{15} - \alpha_{14} + \alpha_{35}, \\ \alpha_{41} &= \alpha_{45} + \alpha_{46} + \alpha_{32} + \alpha_{15}, \\ \alpha_{31} &= -\alpha_{32} - \alpha_{15}, \\ \alpha_{24} &= -\alpha_{25} - \alpha_{23} - \alpha_{33} - \alpha_{34} - \alpha_{35}, \\ \alpha_{11} &= -\alpha_{32} - \alpha_{15} - \alpha_{25} - \alpha_{33}, \\ \alpha_{43} &= \alpha_{45} - \alpha_{26} + \alpha_{46} - \alpha_{25} - \alpha_{33}.\end{aligned}$$

Proof. Since

$$f_1(x_1, x_2, x_3, y_1, y_2, x_2) = -f_2(x_1, x_2, x_3, y_1, y_2)$$

from (5.33) and since f_2 is symmetric in its variables of degree 2 from (5.32) and f is symmetric in the variables of degree 2, $f_1(x_1, x_2, y_1, y_2, x_2)$ must be symmetric in x_1, x_2 . So $w_{ijk}(x_1, x_2, x_3, y_1, y_2, x_2)$ and $w_{ijk}(x_2, x_1, x_3, y_1, y_2, x_1)$ have the same coefficients in $f_1(x_1, x_2, x_3, y_1, y_2, x_2)$ for arbitrary i, j, k , especially for a fixed k and arbitrary pair of i, j . So for an arbitrary but fixed k if we let

$$p(x_1, x_2, y_1, y_2, y_3) = \sum_{i,j,\sigma \in S_3} (-1)^\sigma a_{ij} w_{ij}(x_1, x_2, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}),$$

then $p(x_1, x_2, y_1, y_2, x_2)$ is symmetric in x_1, x_2 . Hence $w_{ij}(x_1, x_2, y_1, y_2, x_2)$ for $i = 1, \dots, 4$ and $j = 1, \dots, 5$ have coefficients 0 in $p(x_1, x_2, y_1, y_2, x_2) - p(x_2, x_1, y_1, y_2, x_1)$. So using Lemma 5.2.5 to write out p and then using the fact that the coefficients of monomials $w_{ij}(x_1, x_2, y_1, y_2, x_2)$ for $i = 1, \dots, 4$ and $j = 1, \dots, 5$ have coefficients 0 in $p(x_1, x_2, y_1, y_2, x_2) - p(x_2, x_1, y_1, y_2, x_1)$ yields 20 equations. By solving the system we get the equations of the Lemma. ■

Next, we may assume that $f^* = \pm f$. Then from the $*$ -symmetry of f , by considering the coefficients of the monomials

$$w_{ijk}, \quad i = 1, \dots, 4, \quad j = 1, \dots, 6, \quad k = 1, \dots, 7,$$

we obtain a system (I). From the symmetry of f_3 in the variables of degree 2 we have a system (II). Since

$$f_1(x_1, x_2, 1, y_1, y_2, y_3) + 2f_4(x_1, x_2, y_1, y_2, y_3) = 0,$$

and since $\Delta(x_3, 1)f \in VI_4[2^2, 1^4] = \{0\}$ by [MR], we have system (III).

Solving (I), (II), (III) yields $\gamma_1 = \gamma_2 = 0$ when $f^* = -f$ and $\gamma_2 = 0$ when $f^* = f$. This easily implies from (5.33) that $f_5 = 0$ when $f^* = \pm f$. Thus by the decomposition of a polynomial into a symmetric and a skew symmetric part, we have

Lemma 5.2.7 *Every identity $f(x_1, x_2, x_3, y_1, y_2, y_3) \in VIN_4[2^3, 1^3]$ has no term which ends in $y_i x_i^2$.*

In fact we also have $\gamma_1 = 0$ when $f^* = f$. Indeed, since f is $\ast$ -symmetric, its Razmyslov transform $f^\#(y_2)$ is also an identity. Obviously $f^\#(y_2)$ is of type $[2^3, 1^3]$ and symmetric in the variables of degree 2. For, if the monomial $V_1(x_i, x_j)y_2V_2(x_i, x_j)$ has coefficient α in $f^\#(y_2)$ then by the definition of the Razmyslov transformation, $V_2(x_i, x_j)y_2V_1(x_i, x_j)$ has coefficient α in f , where V_1, V_2 are monomials. Thus $V_1(x_j, x_i)y_2V_2(x_j, x_i)$ has coefficient α in f since f is symmetric in the x_i 's. So $V_2(x_j, x_i)y_2V_1(x_j, x_i)$ has coefficient α in $f^\#(y_2)$. That is, $f^\#(y_2)$ is symmetric in x_i, x_j . Since $VI_4[2^4, 1] = \{0\}$, $f^\#(y_2)$ must be skew symmetric in the variables of degree 1 by Theorem 1.1. Thus $f^\#(y_2) \in VIN_4[2^3, 1^3]$ and by Lemma 5.2.7 $x_3y_3x_3x_1x_1y_2y_1x_2x_2$ has coefficient 0 in $f^\#(y_2)$. This means that $y_1x_2x_2y_2x_3y_3x_3x_1x_1$ has coefficient 0 in f . This and $\gamma_2 = 0$ imply that $\gamma_1 = 0$.

Finally, using $\gamma_1 = \gamma_2 = 0$, Lemma 2.1.1 and the following substitutions we get a system (IV): substituting

$$e[12], e[22], e[23], e[33], e[34], e[44], e[41], e[11]$$

for

$$x_1, x_2, x_3, y_3, z_1, z_2, y_2, y_1$$

in $\Delta(x_1, z_1)\Delta(x_2, z_2)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of e_{23} , substituting

$$e[33], e[31], e[12], e[22], e[11], e[14], e[44]$$

for

$$x_1, y_1, x_3, y_3, y_2, x_2, z_1$$

in $\Delta(x_1, z_1)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of e_{31} , substituting

$$e[11], e[12], e[22], e[23], e[33], e[34], e[44]$$

for

$$x_1, y_1, y_2, x_2, y_3, x_3, z_1$$

in $\Delta(x_1, z_1)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of e_{12} , substituting

$$e[12], e[22], e[23], e[33], e[34], e[44]$$

for

$$x_1, y_1, x_2, y_2, x_3, y_3$$

in $f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of e_{11} .

Solving (I), (II), (III) and (IV) we get $f_1 = 0$. Thus $f = 0$. ■

This completes the proof of Theorem 5.1.

5.3 The Proof of The Theorem 5.2

Proposition 5.3.1 *The identity $T_8(x^2, x, y_1, \dots, y_5; x)$ forms a basis of the vector space $VIN_4[4, 1^5]$.*

Proof. Let $f(x, y_1, \dots, y_5) \in VIN_4[4, 1^5]$ then

$$f = \sum_{(1, \dots, 5)} \{f_1(x, y_1, \dots, y_4)xy_5 + f'_1(x, y_1, \dots, y_4)y_5x + f_3(x, y_1, \dots, y_4)y_5x^2\} + \sum_{i < j} (-1)^{i+j-1} f_2(x, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_5)[y_i, y_j].$$

Since $f_2(x, y_1, y_2, y_3) \in VIN_3[4, 1^3]$, $f_2(x, y_1, y_2, 1) \in VIN_3[4, 1^2] = \{0\}$. Similarly $f_3(x, y_1, y_2, y_3, 1) = 0$. Thus $f(x, y_1, \dots, y_4, 1) \in VIN_4[4, 1^4] = \{0\}$ implies that

$$f_1(x, y_1, \dots, y_4) + f'_1(x, y_1, \dots, y_4) + f_3(x, y_1, \dots, y_4)x = 0.$$

Furthermore, $f_3 = \beta T_6(x, y_1, \dots, y_4; x)$ and $f_1 = \sum_{i=1}^7 \alpha_i Q_i$, where $\{Q_1, \dots, Q_7\}$ is the basis of $VIN_3[3, 1^4]$ given in Chapter 4. $f_2 = f_1(x, y_1, y_2, y_3, x)$ because $\Delta(y_4, x)f \in VIN_4[5, 1^3] = \{0\}$. Thus f is determined by f_1 and f_3 .

We may assume that $f^* = \pm f$. Let $\alpha(w)$ denote the coefficient of the monomial w in f . Then $\alpha(w) + s\alpha(w^*) = 0$ for every monomial w , where $s := \pm 1$ if $f^* = \mp f$. Thus considering the coefficients of the monomials

$$\begin{aligned} & y_1 y_2 x x y_3 y_4 x x y_5, y_1 y_2 x y_3 x y_4 x x y_5, x y_1 y_2 y_3 y_4 x x x y_5, \\ & x y_1 x y_2 x y_3 y_4 x y_5, x y_1 x y_2 y_3 y_4 x x y_5, x y_1 y_2 y_3 x y_4 x x y_5, \\ & x y_1 y_2 x y_3 x y_4 x y_5, y_1 x x x y_2 y_3 y_4 x y_5, \end{aligned}$$

yields

$$\begin{aligned}
-3\alpha_7 - 2\alpha_6 + \alpha_5 - \alpha_2 + s(-2\alpha_6 - \alpha_7 - \alpha_5 + \alpha_2) &= 0, \\
\alpha_6 - \alpha_5 + \alpha_4 + s(\alpha_2 - \alpha_7 + \alpha_5 - 2\alpha_6) &= 0, \\
2\alpha_7 + \alpha_2 - \alpha_1 + s(-2\alpha_6 - \alpha_2 - \alpha_1) &= 0, \\
-\alpha_6 + \alpha_5 - \alpha_3 + s(-2\alpha_6 - \alpha_5) &= 0, \\
-\alpha_6 - \alpha_4 + s(-\beta - \alpha_4 - \alpha_6 - 2\alpha_7 - \alpha_2 + \alpha_1) &= 0, \\
\alpha_6 + \alpha_3 + s(-3\alpha_7 + \alpha_1 + \alpha_5) &= 0, \\
3\alpha_6 + \alpha_5 + \alpha_3 &= 0, \\
\alpha_2 + 2\alpha_6 + \alpha_1 + s(-2\alpha_7 + \alpha_1 - \alpha_2) &= 0.
\end{aligned} \tag{5.35}$$

If $f^* = -f$ then the solution of (5.35) is

$$\alpha_i = 0, \quad i \neq 2, 4, \quad \beta = \alpha_2 = -\alpha_4.$$

So $f = \beta g(x, y_1, \dots, y_5)$, where g is independent from f . On the other hand $T_8(x^2, x, y_1, \dots, y_5; x) \in VIN_4[4, 1^5]$ and is $*$ -skew symmetric. Thus $T_8(x^2, x, y_1, \dots, y_5; x)$ forms a basis of the subspace consisting of the $*$ -skew symmetric elements of $VIN_4[4, 1^5]$.

If $f^* = f$ then $f^\#(y_2)$ is also an identity by Proposition 2.2.2. Thus the monomial $y_3y_4xy_5y_2y_1xxx$ must have coefficient 0 in $f^\#(y_2)$ by Lemma 2.1.1. This means the monomial $y_1xxx y_2y_3y_4xy_5$ has coefficient 0 in f . So we have

$$2\alpha_6 + \alpha_2 + \alpha_1 = 0.$$

The solution of (5.35) and this equation is trivial. Thus $f_1 = 0$ and $f = 0$. So every identity in $VIN_4[4, 1^5]$ is $*$ -skew symmetric and $T_8(x^2, x, y_1, \dots, y_5; x)$ is a basis of $VIN_4[4, 1^5]$. ■

Proposition 5.3.2 *The identities $T_6(x^2, x, y, z_1, \dots, z_4; y)$ and $T_6(x \circ y, x, y, z_1, \dots, z_4; x)$ form a basis of the vector space $VIN_4[3, 2, 1^4]$.*

Proof. Let $f(x, y, z_1, \dots, z_4) \in VIN_4[3, 2, 1^4]$ then $\Delta(y, x)f \in VI_4[4, 1^5]$ implies that there exists a scalar $\alpha \in F$ such that $\Delta(y, x)f = \alpha T_4(x^2, x, y, z_1, \dots, z_4; x)$. Since

$T_4(x^2, x, y, z_1, \dots, z_4; y)$ belongs to $VIN_4[3, 2, 1^4]$ and $\Delta(y, x)(f - \alpha T_4(x^2, x, y, z_1, \dots, z_4; y)) = 0$, we may assume that $\Delta(y, x)f = 0$. We show that the identity

$$-2T_6(x^2, x, y, z_1, \dots, z_4; y) + T_6(x \circ y, x, y, z_1, \dots, z_4; x)$$

forms a basis of the subspace

$$W := \{f \in VIN_4[3, 2, 1^4] \mid \Delta(y, x)f = 0\}.$$

Let $f \in W$, then $\Delta(y, x)f = 0$. Since the actions of the operator $\Delta(y, x)$ and $*$ commute, $\Delta(y, x)f^* = 0$. So

$$f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*)$$

is a sum of a $*$ -symmetric identity of W and a $*$ -skew symmetric identity of W . Therefore, we may also assume $f^* = \pm f$.

Since

$$\begin{aligned} f = & f_1xy + f'_1yx + f_5yx^2 + \\ & \sum_{i=1}^4 (-1)^i (f_2(x, y, z_1, \dots, \widehat{z}_i, \dots, z_4)xz_i + f'_2(x, y, z_1, \dots, \widehat{z}_i, \dots, z_4)z_ix + \\ & f_3(x, y, z_1, \dots, \widehat{z}_i, \dots, z_4)yz_i + f'_3(x, y, z_1, \dots, \widehat{z}_i, \dots, z_4)z_iy + \\ & f_6(x, y, z_1, \dots, \widehat{z}_i, \dots, z_4)z_ix^2) + \\ & \sum_{i < j} \pm f_4(x, y, z_1, \dots, \widehat{z}_i, \dots, \widehat{z}_j, z_4)[z_i, z_j] + f_7xy^2, \end{aligned} \quad (5.36)$$

f_1 is an identity of H_3 of type $[2, 1^5]$ and is skew symmetric in the variables $z_1, \dots, z_4$, it is a linear combination of the identities g_i which come from $Q, T_6, T_6(z_1, z_2, x_3, x_4, x_5; [x_6, x_7])$ and K by substituting some variables and are skew symmetric in z_1, z_2, z_3, z_4 . By the symmetry of the variables of Q, T_6 and K , there are only 41 such g_i 's. So, let $f_1 = \sum_{i=1}^{41} \alpha_i g_i$.

Since $\Delta(z_4, x)f = \Delta(z_4, y)f = 0$.

$$\begin{aligned} f_2 &= -f_1(x, y, z_1, z_2, z_3, y), \\ f_3 &= -\Delta(y, x)f_2(x, y, z_1, z_2, z_3) \\ &= f_1(x, x, z_1, z_2, z_3, y) + f_1(x, y, z_1, z_2, z_3, x), \\ f_4 &= -\Delta(z_3, y)f_3 = f_1(x, x, z_1, z_2, y, y) + f_1(x, y, z_1, z_2, y, x), \\ f_6 &= -f_5(x, y, z_1, z_2, z_3, y). \end{aligned} \quad (5.37)$$

Since f_5 is an identity of H_3 of type $[1^4]$, $f_5(x, y, z_1, \dots, z_4) = \sum_i \beta_i p_i$, where p_i is an identity of H_3 coming from T_6 and Q and is skew symmetric in z_i 's. From the symmetry of the variables of Q and T_6 there are only 6 such g_i 's:

$$\begin{aligned} g_1 &= T_6(x, z_1, z_2, z_3, z_4, y), \quad g_2 = T_6(y, z_1, z_2, z_3, z_4, x), \\ g_3 &= \sum_{i=1}^4 (-1)^i T_6(x, y, z_1, \dots, \hat{z}_i, \dots, z_4; z_i), \\ g_4 &= \sum_{\substack{\sigma \in S_4 \\ \sigma(1) < \sigma(2), \sigma(3) < \sigma(4)}} (-1)^\sigma Q(x, z_{\sigma(1)}, z_{\sigma(2)}, y, z_{\sigma(3)}, z_{\sigma(4)}), \\ g_5 &= \sum_{i=1}^4 (-1)^{i-1} Q(x, y, z_i, z_1, \dots, \hat{z}_i, \dots, z_4), \quad g_6 = S_6(x, y, z_1, z_2, z_3, z_4). \end{aligned}$$

It can be shown that $g_6 + 2g_4 + 2g_5 = 0$, so $f_5 = \sum_{i=1}^5 \beta_i g_i$. By the results in $[MR]$, $f_7 = \alpha_6 T_6(x, z_1, z_2, z_3, z_4; x)$.

Next from $f^* = \pm f$ and

$$\begin{aligned} f_2 + f'_2 + \Delta(z_4, x)(f_5 y - f_6 z_1 + f_6 z_2 - f_6 z_3) + f_6 x &= 0, \\ f_3 + f'_3 + \Delta(z_4, y)f_7 x &= 0, \end{aligned}$$

we have (I). If $f^* = f$ then the Razmyslov transform $f^\#(z_1)$ is also an identity of type $[3, 2, 1^4]$. Thus it has no terms which end in $z_2 y y$. This implies that the coefficient of the monomial $x z_2 y y z_1 z_3 x x z_4$ is 0 which yields an equation (II). Solving (I) and (II), and substituting the solution into f_1 yields $f_2 = f_3 = 0$. Thus $f = f_1 x y + f'_1 y x$. Since $\Delta(y, x)f = 0$ and f is $*$ -symmetric,

$$f = [x, y]g(x, z_1, z_2, z_3, z_4)[x, y],$$

where g is multilinear. Furthermore it can be shown easily that g is alternating because $\Delta(z_1, x)f = 0$. So there exists a scalar $\alpha \in F$ such that $g = \alpha S_5(x, z_1, z_2, z_3, z_4)$ and $f = \alpha[x, y]S_5(x, z_1, z_2, z_3, z_4)[x, y]$. However $f(x, y, z_1, z_2, z_3, 1)$ is an identity of H_4 of type $[3, 2, 1, 1, 1]$, so $f(x, y, z_1, z_2, z_3, 1) = 0$. Hence $\alpha = 0$ and $f = 0$.

If $f^* = -f$ then solving (I) and (II), and substituting the solution into f_1 yields $f_2 = \alpha_{23}h_2, f_3 = \alpha_{23}h_5$. Thus $f'_1 = -f_1 + \alpha h_1$ from $\Delta(y, x)f = 0$ and (5.36). So by

$f^* = -f$, (5.36) and (5.37),

$$f = [x, y]g[x, y] + \alpha h_2, \quad (5.38)$$

where g is multilinear. Since $\Delta(y, x)f = 0$, from (5.38) we have $\Delta(z_1, x)g(x, z_1, z_2, z_3, z_4) + \alpha h_3 = 0$. This means modulo α , g is alternating. So there exists a scalar $\beta \in F$ such that $g = \beta S_5(x, z_1, z_2, z_3, z_4) + \alpha h_4$. Thus

$$f = \beta[x, y]S_5(x, z_1, z_2, z_3, z_4)[x, y] + \alpha h_5. \quad (5.39)$$

Since $f(x, y, z_1, z_2, z_3, 1) = 0$ and $S_5(x, z_1, z_2, z_3, 1) = S_4(x, z_1, z_2, z_3)$, from (5.39) we have $\beta = m\alpha$ for some $m \in F$ and $f = \alpha h$. So the dimension of the subspace W is at most 1. Since

$$-2T_6(x^2, x, y, z_1, \dots, z_4; y) + T_6(x \circ y, x, y, z_1, \dots, z_4; x) \in W,$$

the dimension of W is 1. ■

Proposition 5.3.3 *The following identities form a basis of the space $VIN_4[3, 1^6]$.*

$$\begin{aligned} & T_8(x^2, y_1, \dots, y_6; x), \quad T_8(x, y_1, \dots, y_6; x^2), \\ & \sum_{i=1}^6 (-1)^{i-1} T_8(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_6, x; x), \\ & \sum_{i=1}^6 (-1)^i T_8(x^2, x, y_1, \dots, \hat{y}_i, \dots, y_6; y_i), \\ & T_8(x, y_1, \dots, y_6; x)x, \quad xT_8(x, y_1, \dots, y_6; x). \end{aligned}$$

Proof. First, we show that the space $VIN_4[3, 1^6]$ is spanned by 6 elements. Let $f(x, y_1, \dots, y_6) \in VIN_4[3, 1^6]$ then $\Delta(y_5, x)f \in VIN_4[4, 1^5]$. So there exists a scalar $\alpha \in F$ such that

$$\Delta(y_5, x)f = \alpha T_8(x^2, x, y_1, \dots, y_4, y_6; x).$$

Thus we can correct f by subtracting $\alpha T_8(x^2, x, y_1, \dots, y_6; x)$ so that

$$\Delta(y_5, x)(f - \alpha T_8(x^2, x, y_1, \dots, y_6; x)) = 0.$$

Thus it suffices to show that the subspace

$$W := \{f \in VIN_4[3, 1^6] \mid \Delta(y_5, x)f = 0\}$$

is spanned by 5 elements.

Let $f(x, y_1, \dots, y_6) \in W$ then

$$\begin{aligned} f = & \sum_{(1, \dots, 6)} \{f_1(x, y_1, \dots, y_5)x y_6 + f'_1(x, y_1, \dots, y_5)y_6 x + \\ & f_3(x, y_1, \dots, y_5)y_6 x^2\} + \\ & \sum_{i < j} f_2(x, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_6)[y_i, y_j]. \end{aligned} \quad (5.40)$$

Since $f_1 \in VIN_3[2, 1^5]$, there exist scalars $\alpha_i \in F$ such that $f_1 = \sum_{i=1}^{10} \alpha_i g_i$, where $\{g_1, \dots, g_{10}\}$ is the basis of the space $VIN_3[2, 1^5]$ given in Chapter 4. $\Delta(y_5, x)f = 0$ implies that

$$\Delta(y_5, x)f_1 + f_2 = 0,$$

from (5.40). Since $f_3(x, y_1, y_2, y_3, y_4, y_5) \in VIN_3[1^6]$, $f_3 = \sum_i \beta_i g_i$, where

$$\begin{aligned} g_1 &= \sum_{(1, \dots, 5)} T_6(x, y_1, y_2, y_3, y_4; y_5), \quad g_2 = T_6(y_1, \dots, y_5; x), \\ g_3 &= \sum_{i < j} Q(x, y_i, y_j, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_5). \end{aligned}$$

Since $g_2 - g_1 - 2g_3 = 0$, we may assume $f_3 = \sum_{i=1}^2 \beta_i g_i$. Thus from the $\ast$ -symmetry of f we have a system of equations (I). Solving (I) for $f^\ast = f$ and $f^\ast = -f$ respectively we see that there are only 5 parameters in total. So W is spanned by 5 elements.

On the other hand, if we denote by t_i^4 the identities given in the Proposition, then

$$t_i^4 = t_i^3[y_5, y_6], \quad i = 1, 3, 4, 6, \quad t_2^4 = (t_2^3 + t_5^3)[y_5, y_6],$$

where t_i^3 , $i = 1, \dots, 6$ are the elements occurring in the basis of $VIN_3[3, 1^4]$ given in Chapter 4. So they are linear independent. If $\sum_{i=1}^6 \alpha_i t_i^4 = 0$ then

$$\sum_{i=1}^6 \alpha_i t_i^4 = \{\alpha_2(t_2^3 + t_5^3) + \sum_{i \neq 2} \alpha_i t_i^3\}[y_5, y_6] + \sim$$

implies that $\alpha_i = 0$, for $i = 1, \dots, 6$, and we have proved the Proposition. ■

Let

$$\begin{aligned}
t_1 &:= T_8(x^2, y, z_1, \dots, z_5; y) + T_8(y^2, x, z_1, \dots, z_5; x), \\
t_2 &:= T_8(x \circ y, x, z_1, \dots, z_5; y) + T_8(x \circ y, y, z_1, \dots, z_5; x), \\
t_3 &:= \sum_{i=1}^5 (-1)^{i-1} (T_8(x \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_5, x, y; y) + \\
&\quad T_8(y \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_5, y, x; x)), \\
t_4 &:= xT_8(x, y, z_1, \dots, z_5; y) + yT_8(y, x, z_1, \dots, z_5; x), \\
t_5 &:= T_8(x, y, z_1, \dots, z_5; y)x + T_8(y, x, z_1, \dots, z_5; x)y, \\
t_6 &:= T_8(x, y, z_1, \dots, z_5; [x, y]), \\
t_7 &:= \tilde{T}_8(x, y, z_1, \dots, z_5; [x, y]).
\end{aligned}$$

Then

Proposition 5.3.4 *The set $\{t_1, \dots, t_7\}$ is a basis of the vector space $VIN_4[2, 2, 1^5]$.*

Proof. Let f be an identity of type $[2, 2, 1^5]$. Then

$$\begin{aligned}
f &= f_1(x_1, x_2, y_1, \dots, y_5)x_1x_2 + f'_1(x_2, x_1, y_1, \dots, y_5)x_2x_1 + \\
&\quad f_2y_5x_2 + f'_2x_2y_5 + \dots \\
&\quad + f_3[y_4, y_5] + \dots \\
&\quad + f_4(x_1, y_1, \dots, y_5)x_1x_2^2 + f_4(x_2, y_1, \dots, y_5)x_2x_1^2 + \\
&\quad f_5y_5x_2 + \dots.
\end{aligned} \tag{5.41}$$

Since f_4 is an identity of degree 6 on H_3 by Lemma 2.1.2, it comes from T_6 and Q by $[MR]$. More precisely, by the symmetry of T_6 and Q , f_4 is a linear combination of identities of the form $T_6(y_{i_1}, \dots, y_{i_5}; x_1)$, $T_6(x_1, y_{i_1}, \dots, y_{i_4}; y_{i_5})$ and $Q(x_1, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}, y_{i_5})$. But, $f_4(x_1, y_1, \dots, y_5)$ is skew symmetric in the variables y_i 's, from the symmetry of T_6 and Q it is a linear combination of the following identities:

$$\begin{aligned}
g_1 &:= T_6(y_1, \dots, y_5; x_1), \quad g_2 := \sum_{(12345)} T_6(x_1, y_1, \dots, y_4; y_5), \\
g_3 &:= \sum_{i < j} (-1)^{i+j-1} Q(x_1, y_i, y_j, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_5).
\end{aligned}$$

[Note $4S_6(x_1, y_1, \dots, y_5) = g_1 - g_2$ by [MR]] It can be checked easily by computer that $g_3 = 2S_6(x_1, y_1, \dots, y_5)$, thus $-g_1 + g_2 + 2g_3 = 0$ and $f_4 = \beta_1 g_1 + \beta_2 g_2$.

Let $w(i, j)$ denote the monomial $y_1 \cdots \overset{(i)}{x_1} \cdots \overset{(j)}{x_2} \cdots y_5$ and $(1, 2)w(i, j)$ denote the monomial obtained by interchanging x_1 and x_2 in $w(i, j)$. e.g. $w(1, 3) = x_1 y_1 x_2 y_2 y_3 y_4 y_5$ and $(1, 2)w(1, 3) = x_2 y_1 x_1 y_2 y_3 y_4 y_5$. Let $w_i := w_i(x_1, x_2, y_1, \dots, y_5), i = 1, \dots, 21$ denote monomials

$$w(1, j), j = 2, 3, 4, 5; w(2, j), j = 3, 4, 5;$$

$$w(3, j), j = 4, 5; w(4, 5);$$

$$w(i, 6), w(i, 7), i = 1, \dots, 5; w(6, 7),$$

respectively and let $w_i, i = 22, \dots, 42$ denote the monomials $(1, 2)w_i, i = 1, \dots, 21$, respectively. Then

$$f_1 = \sum_{i=1}^{42} \alpha_i \left(\sum_{\sigma \in S_5} (-1)^\sigma \sigma(w_i) \right),$$

where σ acts on w_i to permute y_i 's.

Since $\Delta(y_5, x_1)f \in VIN_4[3, 2, 1^6]$, there exist scalars $\gamma_1, \gamma_2 \in F$ such that

$$\Delta(y_5, x_1)f = \gamma_1 T_8(x_1^2, x_1, x_2, y_1, \dots, y_4; x_2) + \gamma_2 T_8(x_1 \circ x_2, x_1, x_2, y_1, \dots, y_4; x_1)$$

Let $h := \gamma_1 t_1 + \gamma_2 t_2$ then $\Delta(y_5, x_1)h = -\Delta(y_5, x_1)f$. Hence $\Delta(y_5, x_1)(f + h) = 0$. So we may assume that $\Delta(y_5, x_1)f = 0$. Thus from (5.41) we have

$$\begin{aligned} f_2(x_1, x_2, y_1, \dots, y_4) &= -f_1(x_1, x_2, y_1, \dots, y_4, x_1), \\ f'_2(x_1, x_2, y_1, \dots, y_4) &= -f_1(x_2, x_1, y_1, \dots, y_4, x_1), \\ f_3(x_1, x_2, y_1, y_2, y_3) &= -f_1(x_1, x_2, y_1, \dots, y_4, x_1), \\ f_5(x_1, y_1, \dots, y_4) &= -f_4(x_1, y_1, \dots, y_4, x_1). \end{aligned} \tag{5.42}$$

It is easy to see that the operator Δ and involution $*$ commute on every monomial. So they commute on any polynomial. Hence $\Delta(y_5, x_1)f = 0$ implies that $\Delta(y_5, x_1)f^* = 0$. Thus the decomposition $f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*)$ allows us to discuss only these identities in $VIN_4[2, 2, 1^5]$ which are $*$ -skew symmetric or $*$ -symmetric and on which $\Delta(y_5, x_1)$ acts as zero.

So from now on we assume that $f^* = \pm f$ and $\Delta(y_5, x_1)f = 0$.

We also can correct f by subtracting an identity in such a way that

$$\alpha_{31} = \alpha_{33} = \alpha_{42} = 0, \text{ if } f^* = -f,$$

$$\alpha_{31} = \alpha_{33} = 0, \text{ if } f^* = f.$$

Indeed, when $f^* = -f$ let

$$g := \lambda_1(2t_1 + t_3) + \lambda_2(t_4 + t_5) + \lambda_3(t_6 - t_7).$$

Then g is $*$ -skew symmetric and $\Delta(y_5, x)g = 0$. In g , the monomial w_{31} has coefficient $-\lambda_2 - 2\lambda_3$, while w_{42} has coefficient $-\lambda_1$ and w_{33} has coefficient $4\lambda_1 + \lambda_2 + \lambda_3$. Let

$$-\lambda_2 - 2\lambda_3 = \alpha_{31}, \quad -\lambda_1 = \alpha_{42}, \quad 4\lambda_1 + \lambda_2 + \lambda_3 = \alpha_{33}.$$

This system has a solution, therefore there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in F$ such that w_{31}, w_{42}, w_{33} have coefficients zero in $f - g$. When $f^* = f$ let

$$g := y(t_4 - t_5) + z(t_6 + t_7).$$

Then g is $*$ -symmetric and $\Delta(y_5, x)g = 0$. The monomial w_{31} has coefficient $-\lambda_2 - 2\lambda_3$, while w_{33} has coefficient $\lambda_2 - \lambda_3$. Let

$$\lambda_2 + 2\lambda_3 = \alpha_{31}, \quad \lambda_2 - \lambda_3 = \alpha_{33}$$

Solving this system there exist scalars $\lambda_2, \lambda_3 \in F$ such that w_{31}, w_{33} have coefficients 0 in $f - g$.

From $f^* = \pm f$, considering the coefficients of monomials wx_1x_2 and its $*$ for

$$\begin{aligned} w &= w(1,3), \dots, w(1,7), w(2,4), \dots, w(2,7), \\ &w(3,4), \dots, w(3,7), w(4,5), \dots, w(4,7), \\ &w(5,6), w(5,7), w(1,2), w(1,3) \end{aligned}$$

we obtain a system of equations (I). Since f_3 is symmetric in x_1 and x_2 , $f_3(x_1, x_2, y_1, y_2, y_3) - f_3(x_2, x_1, y_1, y_2, y_3) = 0$. Thus considering the coefficients of monomials $w_i(x_1, x_2, y_1, y_2, y_3, x_2, x_1)$ for $i = 1, \dots, 21$ respectively, yields a system (II).

Next we apply Lemma 2.1.1 to produce more equations. Considering the monomials

$$- - y_1 y_2 y_3 y_4 y_5 - -, y_1 y_2 - - y_3 y_4 y_5 - -, y_1 y_2 y_3 y_4 - - y_5 - -,$$

where the blank positions are filled by the variables x_1, x_2, x_1, x_2 , yields three equations. For instance, by Lemma 2.1.1, the sum of the coefficients of the monomials

$$x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_1 x_2, x_2 x_1 y_1 y_2 y_3 y_4 y_5 x_1 x_2,$$

$$x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_2 x_1, x_2 x_1 y_1 y_2 y_3 y_4 y_5 x_2 x_1,$$

is 0. Applying the Lemma 2.1.1 to monomials $w x_2 x_1 x_2$ for

$$w = x_1 y_1 y_2 y_3 y_4 y_5, y_1 x_1 y_2 y_3 y_4 y_5, y_1 y_2 x_1 y_3 y_4 y_5, \\ y_1 y_2 y_3 x_1 y_4 y_5, y_1 y_2 y_3 y_4 x_1 y_5, y_1 y_2 y_3 y_4 y_5 x_1,$$

produces six equations. That is the sum of the coefficients of monomials $w x_1 x_2 x_1, w x_1 x_1 x_2, w x_2 x_1 x_1$ is zero

Applying the Lemma 2.1.1 to the positions 1, 2, 3 of the monomial $x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_1 x_2$, the positions 3, 4, 5 of the monomial $y_1 y_2 x_1 x_2 y_3 y_4 y_5 x_1 x_2$ and the position 5, 6, 7 of the monomial $y_1 y_2 y_3 y_4 x_1 x_2 y_5 x_1 x_2$ yield three equations. Using Lemma 2.1.1, consider monomials $- - w - -$, where the first two blank positions are variables x_1, y_1 and the last two positions are x_1, x_2 for

$$w = x_2 y_2 y_3 y_4 y_5, y_2 x_2 y_3 y_4 y_5, y_2 y_3 x_2 y_4 y_5, \\ y_2 y_3 y_4 x_2 y_5, y_2 y_3 y_4 y_5 x_2.$$

Using Lemma 2.1.1, consider the following monomials and position 3, 4 and 8, 9

$$x_2 y_1 x_1 y_2 y_3 y_4 y_5 x_1 x_2, y_1 x_2 x_1 y_2 y_3 y_4 y_5 x_1 x_2, y_1 y_2 x_1 y_3 x_2 y_4 y_5 x_1 x_2, \\ y_1 y_2 x_1 y_3 y_4 x_2 y_5 x_1 x_2, y_1 y_2 x_1 y_3 y_4 y_5 x_2 x_1 x_2.$$

Using Lemma 2.1.1, consider the following monomials and position 5, 6 and 8, 9

$$x_2 y_1 y_2 y_3 x_1 y_4 y_5 x_1 x_2, y_1 x_2 y_2 y_3 x_1 y_4 y_5 x_1 x_2, y_1 y_2 x_2 y_3 x_1 y_4 y_5 x_1 x_2, \\ y_1 y_2 y_3 x_2 x_1 y_4 y_5 x_1 x_2, y_1 y_2 y_3 y_4 x_1 y_5 x_2 x_1 x_2.$$

All these applications of Lemma 2.1.1 produce a system (III).

Let us do the following substitutions to produce equations (IV).

1. Substituting

$$e[12], e[14], e[22], e[1], e[34], e[23], e[33]$$

for $x_1, x_2, y_1, \dots, y_5$ and considering coefficients of e_{24} .

2. Substituting

$$e[11], e[12], e[22], e[23], e[31], e[34], e[44]$$

for $y_1, x_1, y_2, y_3, x_2, y_4, y_5$ and considering the coefficient of e_{24} [see type $[2, 2, 1^3]$ id. of H_3].

3. Substituting

$$e[11], e[12], e[22], e[13], e[33], e[34], e[44]$$

for $y_1, x_1, x_2, y_2, y_3, y_4, y_5$ and considering the coefficients of e_{14} and e_{41} respectively.

4. Substituting

$$e[11], e[13], e[32], e[22], e[33], e[34], e[44]$$

for $y_1, y_2, x_1, x_2, y_3, y_4, y_5$ and considering the coefficients of e_{14} and e_{41} respectively.

5. Substituting

$$e[11], e[12], e[22], e[13], e[33], e[34], e[44]$$

for $y_1, x_1, y_2, y_3, y_4, x_2, y_5$ and considering the coefficient of e_{14} .

6. Substituting

$$e[11], e[12], e[22], e[23], e[33], e[34], e[44]$$

for $y_1, y_2, y_3, x_1, y_4, x_2, y_5$ and considering the coefficients of e_{14} .

Finally, since $f_1(x_2, x_1, y_1, \dots, y_4, x_1) + f'_2 = 0$ and

$$\begin{aligned} f_2 + f'_2 + f_4(x_1, y_1, \dots, y_4, x_2)x_1 + f_5(x_1, y_1, \dots, y_4)x_2 + \\ f_5(x_1, y_2, y_3, y_4, x_2)y_1 + f_5(x_1, y_3, y_4, x_2, y_1)y_2 + \\ f_5(x_1, y_4, x_2, y_1, y_2)y_3 + f_5(x_1, x_2, y_1, y_2, y_3)y_4 = 0, \end{aligned}$$

$$\begin{aligned} f_1(x_2, x_1, y_1, \dots, y_4, x_1) - f_2 - f_4(x_1, y_1, \dots, y_4, x_2)x_1 - f_5(x_1, y_1, \dots, y_4)x_2 - \\ f_5(x_1, y_2, y_3, y_4, x_2)y_1 - f_5(x_1, y_3, y_4, x_2, y_1)y_2 + \\ f_5(x_1, y_4, x_2, y_1, y_2)y_3 - f_5(x_1, x_2, y_1, y_2, y_3)y_4 = 0. \end{aligned}$$

Thus considering the following monomials yields a system (V)

$$w_i(x_1, x_2, y_1, \dots, y_4, x_1), \quad i = 1, \dots, 20.$$

When $f^* = -f$, solving (I), ..., (V) yield $f_1 = \beta_2 g_1 + \alpha_{36} g_2$. Since g_i is of type $[1, 1, 1^5]$ and skew symmetric in the last 5 variables

$$g_j = \sum_{\sigma \in S_5} (-1)^\sigma q_j^\sigma,$$

where

$$q_i^\sigma := V_j[\sigma(w_1), \dots, \sigma(w_{42})]^t,$$

and σ acts on w_i to permute y_i 's.

On the other hand, every identity of type $[1, 1, 1^5]$ on H_3 which is skew in the last 5 variables is a linear combination of 40 h_i 's by the results in Chapter 3 that every identity of degree 7 on H_3 comes from T_6 , Q , K and $T_6(x_1, \dots, x_5; [x_6, x_7])$. We claim that

(*) Both g_1, g_2 are not identities.

If, say g_1 is an identity then there exist scalars $\beta_i \in F$ such that

$$g_1 = B_1[h_1, \dots, h_{40}]^t, \tag{5.43}$$

where $B_1 = (\beta_1, \dots, \beta_{40})$. Since

$$h_j = \sum_{\sigma \in S_5} (-1)^\sigma k_j^\sigma,$$

where

$$k_j^\sigma := A_j[\sigma(w_1), \dots, \sigma(w_{42})]^t,$$

for some 1×42 matrix $A_j = (a_{js})$, and σ acts on w_i to permute y_i 's. Thus (5.43) implies that

$$q_1 = B_1[k_1, \dots, k_{40}]^t, \quad (5.44)$$

where $r_l := r_l^{\text{id}}$ for $r = q, k$ and id denotes the identity permutation. Since

$$[k_1, \dots, k_{40}]^t = A[w_1, \dots, w_{42}],$$

where $A = [A_1, \dots, A_{40}]^t = (a_{js})$ which is a 40×42 matrix, by (5.44) and using the definition of q_j we obtain

$$V_1 = B_1 A.$$

Hence

$$A^t B_1^t = V_1^t.$$

That is, B_1^t is a solution of the linear equation $A^t X = V_1^t$. But as can easily be checked on a computer this equation has no solution. Thus the claim and the fact that $f_1 = \beta_2 g_1 + \alpha_{36} g_2$ is an identity imply that there exists a scalar $\alpha \in F$ such that $\beta_2 = \alpha \alpha_{36}$ and $f_1 = \alpha_{36}(g_1 + \alpha g_2)$. But arguing as above, $g_1 + \alpha g_2$ is not an identity since for any $\alpha \in F$ the equation $A^t X = V_1^t + \alpha V_2^t$ has no solution as was easily checked by computer. Thus $\alpha_{36} = 0$ and hence $\beta_2 = 0$. So $f_1 = f_4 = 0$, and therefore $f = 0$.

When $f^* = f$, solving $(I), \dots, (V)$ with $s = -1$ yield $f_1 = \beta_2 g_1 + \alpha_8 g_2 + \alpha_{42} g_3$, where

$$g_i = \sum_{\sigma \in S_5} (-1)^\sigma U_i[\sigma(w_1), \dots, \sigma(w_{42})].$$

As in the proof of skew symmetric case, it can easily be checked by computer that for any scalars $\lambda_i, \lambda \in F$ the following equations have no solution.

$$\begin{aligned} A^t X &= U_i^t, \quad i = 1, 2, 3, \\ A^t X &= U_1^t + \lambda_1 U_3^t, \\ A^t X &= U_2^t + \lambda_2 U_3^t, \\ A^t X &= U_1^t + \lambda_1 U_3^t + \lambda(U_2^t + \lambda_2 U_3^t). \end{aligned}$$

Hence $\alpha_8 = \alpha_{42} = 0$ and $\beta_2 = 0$. So $f_1 = f_4 = 0$, and therefore $f = 0$. ■

Let

$$\begin{aligned}
t_1 &:= T_8(y_1, \dots, y_7; x^2), \quad t_2 := \tilde{T}_8(y_1, \dots, y_7; x^2), \\
t_3 &:= \sum_{(1, \dots, 7)} T_8(x \circ y_1, \dots, y_7; x), \quad t_4 := \sum_{(1, \dots, 7)} \tilde{T}_8(x \circ y_1, \dots, y_7; x), \\
t_5 &:= \sum_{(1, \dots, 7)} T_8(x, y_1, \dots, y_6; x \circ y_7), \quad t_6 := \sum_{(1, \dots, 7)} \tilde{T}_8(x, y_1, \dots, y_6; x \circ y_7), \\
t_7 &:= xT_8(y_1, \dots, y_7; x), \quad t_8 := \tilde{T}_8(y_1, \dots, y_7; x)x, \\
t_9 &:= x\tilde{T}_8(y_1, \dots, y_7; x), \quad t_{10} := T_8(y_1, \dots, y_7; x)x, \\
t_{11} &:= \sum_{(1, \dots, 7)} y_1 T_8(x, y_2, \dots, y_7; x), \\
t_{12} &:= \sum_{(1, \dots, 7)} \tilde{T}_8(x, y_1, \dots, y_6; x)y_7, \\
t_{13} &:= \sum_{(1, \dots, 7)} y_1 \tilde{T}_8(x, y_2, \dots, y_7; x), \quad t_{14} := \sum_{(1, \dots, 7)} T_8(x, y_1, \dots, y_6; x)y_7, \\
t_{15} &:= \sum_{(1, \dots, 7)} T_8(x, y_1, \dots, y_6; [x, y_7]), \quad t_{16} := \sum_{(1, \dots, 7)} \tilde{T}_8(x, y_1, \dots, y_6; [x, y_7]).
\end{aligned}$$

Then

Proposition 5.3.5 *The set $\{t_i | i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\}$ forms a basis of the vector space $VIN_4[2, 1^7]$.*

Proof. Let $V := \text{Span}\{t_i | i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\}$. We show that $V = VIN_4[2, 1^7]$. First, we show there exist 12 polynomials such that every $f \in V$ is a linear combination of these polynomials.

Let $f \in VIN_n[2, 1^7]$. Then $\Delta(y_7, x)f \in VIN_4[3, 1^6]$. Hence there exist scalars $\alpha_1, \dots, \alpha_6 \in F$ such that $\Delta(y_7, x)f = \sum_{i=1}^6 \alpha_i q_i$, where $\{q_i | i = 1, \dots, 6\}$ is the basis of $VIN_4[3, 1^6]$ given in Proposition 5.3.3. Next we shall correct f so that $\Delta(y_7, x)f = 0$.

Let $\sum_{(1, \dots, k)}$ denote the sum over all cyclic permutations of $1, 2, \dots, k$ and

$$g_1 := \sum_{(1, \dots, 7)} T_8(x^2, y_1, \dots, y_6; y_7),$$

$$\begin{aligned}
g_2 &:= \sum_{(1,\dots,7)} T_8(x \circ y_1, \dots, y_7; x), \\
g_3 &:= \sum_{(1,\dots,7)} \sum_{(1,\dots,6)} (-1)^{(1,\dots,6)} T_8(x \circ y_1, x, y_2, \dots, y_6; y_7), \\
g_4 &:= T_8(y_1, \dots, y_7; x^2), \quad g_5 := T_8(y_1, \dots, y_7; x)x, \\
g_6 &:= xT_8(y_1, \dots, y_7; x).
\end{aligned}$$

Then

$$\begin{aligned}
\Delta(y_7, x)g_i &= q_i \text{ for } i = 4, 5, 6, \\
\Delta(y_7, x)g_1 &= q_1 + q_3, \quad \Delta(y_7, x)g_2 = q_2 + 2q_1, \quad \Delta(y_7, x)g_3 = q_2 + 2q_3. \quad (5.45)
\end{aligned}$$

Let

$$\begin{aligned}
\beta_i &:= \alpha_i, \quad \beta_3 := \frac{1}{4}(-\alpha_1 + 2\alpha_2 + \alpha_3) \\
\beta_2 &:= \alpha_2 - \beta_3, \quad \beta_1 := \alpha_3 - 2\beta_3
\end{aligned}$$

and

$$g := \sum_{i=1}^6 \beta_i g_i.$$

Then from (5.45), $\Delta(y_7, x)g = \Delta(y_7, x)f$. Hence $\Delta(y_7, x)(f - g) = 0$. So we may assume that $\Delta(y_7, x)f = 0$.

Since the operators Δ and $*$ are linear and they commute on any monomial, $(\Delta(y_7, x)f)^* = \Delta(y_7, x)(f^*)$. Thus $\Delta(y_7, x)f = 0$ implies $\Delta(y_7, x)f^* = 0$. Hence f can be decomposed into the sum of a $*$ -symmetric identity and a $*$ -skew symmetric identity $f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*)$ and $\Delta(y_7, x)(\frac{1}{2}(f + f^*)) = 0$, $\Delta(y_7, x)(\frac{1}{2}(f - f^*)) = 0$. So we may assume that $f^* = \pm f$ and $\Delta(y_7, x)f = 0$.

Hence

$$\begin{aligned}
f &= \sum_{i=1}^7 (f_1(x, y_1, \dots, \hat{y}_i, \dots, y_7)xy_i + f'_1(x, y_1, \dots, \hat{y}_i, \dots, y_7)y_ix) + \\
&\quad \sum_{i < j} f_2(x, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_7)[y_i, y_j] + f_3(y_1, \dots, y_7)x^2,
\end{aligned}$$

by the skew symmetry of f in the variables of y_i 's. $f_3 = \beta S_7$ for some $\beta \in F$ and

$$f_1 = \sum_{i=1}^7 \alpha_i y_1 \cdots \overset{(i)}{x} \cdots y_7,$$

$$f'_1 = \sum_{i=1}^7 \alpha'_i y_1 \cdots \overset{(i)}{x} \cdots y_7.$$

From $\Delta(y_7, x)f = 0$, we have

$$f_1 + f'_1 + \beta S_7(y_1, \dots, y_6, x) = 0,$$

$$\alpha_i + \alpha'_i + (-1)^{i-1} \beta = 0, i = 1, \dots, 7. \quad (5.46)$$

By the formulas in Lemma 2.1.1 we have

$$\alpha_1 + \alpha_2 + \alpha'_1 + \alpha'_2 = 0, \alpha_3 + \alpha_4 + \alpha'_3 + \alpha'_4 = 0, \alpha_5 + \alpha_6 + \alpha'_5 + \alpha'_6 = 0. \quad (5.47)$$

Since $\Delta(y_7, x)f = 0$ and $\Delta(y_6, x)f = 0$,

$$f_1(x, y_1, \dots, y_5, x) + f_2 = 0, f'_1(x, y_1, \dots, y_5, x) - f_2 = 0. \quad (5.48)$$

Let $w_i := y_1 \cdots \overset{(i)}{x} \cdots y_7$ and $s := \mp 1$. Then (5.48) and $f^* = \pm f$ imply that

$$\alpha(w_i) + s\alpha(w_i^*) = 0,$$

where $\alpha(w)$ denotes the coefficient of w in f . Thus by considering each w_i and w_i^* we have

$$\begin{aligned} \alpha_1 + s(-\alpha'_2) &= 0, \alpha'_1 + s(-\alpha_1) = 0, \alpha_2 + s(-\alpha_2) = 0, \\ \alpha_3 - s(-\alpha_2 - \alpha_7) &= 0, \alpha'_3 - s(-\alpha_1 + \alpha_7) = 0, \alpha_4 - s(\alpha_2 - \alpha_6) = 0, \\ \alpha'_4 - s(\alpha_1 + \alpha_6) &= 0, \alpha_5 - s(-\alpha_2 - \alpha_4) = 0, \alpha'_5 - s(-\alpha_1 + \alpha_5) = 0, \\ \alpha_6 - s(\alpha_2 - \alpha_4) &= 0, \alpha'_6 - s(\alpha_1 + \alpha_4) = 0, \alpha_7 - s(-\alpha_2 - \alpha_3) = 0, \\ \alpha'_7 - s(-\alpha_1 + \alpha_3) &= 0. \end{aligned} \quad (5.49)$$

Solving the equations (5.46), (5.47) and (5.49) in the $s = \mp 1$ cases respectively yields that in each case there are only three parameters. That is in each case f is a linear combination of three polynomials which are independent from f . This means $f - g$ is a linear combination

of six polynomials which are independent from $f - g$. Since g is a linear combination of six fixed polynomials, f is a linear combination of twelve polynomials which are also independent from f . Thus $\dim(VIN_4[2, 1^7]) \leq 12$.

On the other hand we shall show that there exist 12 such linear independent identities, so $V = VIN_4[2, 1^7]$.

Suppose there exist scalars $\alpha_i \in F$ such that $\sum_{i \neq 6, 13, 14} \alpha_i t_i = 0$. Then considering the coefficients of the following monomials

$$\begin{aligned} x^2 y_1 \cdots y_7, & y_1 y_2 x^2 y_3 \cdots y_7, x y_1 \cdots y_4 x y_5 y_6 y_7, x y_1 y_2 x y_3 \cdots y_7, \\ x y_1 x y_2 \cdots y_7, & y_1 y_2 x y_3 x y_4 \cdots y_7, y_1 y_2 x y_3 \cdots y_6 x y_7, \end{aligned} \quad (5.50)$$

yields

$$\begin{aligned} \alpha_1 + \alpha_3 - \alpha_5 + \alpha_7 - \alpha_{15} &= 0, \\ \alpha_2 + 2\alpha_4 - \alpha_5 + \alpha_{11} + \alpha_{15} - \alpha_{16} &= 0, \\ 3\alpha_3 - 3\alpha_5 + \alpha_7 - \alpha_{15} &= 0, \\ 2\alpha_3 + \alpha_4 - \alpha_5 + \alpha_9 - \alpha_{12} - \alpha_{15} &= 0, \\ \alpha_3 - \alpha_7 + \alpha_{12} + 2\alpha_{15} - \alpha_{16} &= 0, \\ \alpha_{11} + \alpha_{12} + 2\alpha_{15} - 2\alpha_{16} &= 0, \\ 4\alpha_4 - \alpha_5 + \alpha_{11} + \alpha_{15} - \alpha_{16} &= 0. \end{aligned} \quad (5.51)$$

Since $t_i^* = t_{i+1}$, for $i = 1, 3, 7, 9, 11, 15$ and $t_5^* = -t_3 - t_4 - t_5$ from $t_5 + t_5^* = -t_3 - t_4$, $\sum_{i \neq 6, 13, 14} \alpha_i t_i^* = 0$ implies

$$\begin{aligned} \alpha_1 t_2 + \alpha_2 t_1 + (\alpha_4 - \alpha_5) t_3 + (\alpha_3 - \alpha_5) t_4 - \\ \alpha_5 t_5 + \alpha_8 t_7 + \alpha_7 t_8 + \alpha_{10} t_9 + \alpha_9 t_{10} + \\ \alpha_{12} t_{11} + \alpha_{11} t_{12} + \alpha_{16} t_{15} + \alpha_{15} t_{16} = 0. \end{aligned}$$

Thus, considering the coefficients of the monomials in (5.50) in the polynomial $\sum_{i \neq 6, 13, 14} \alpha_i t_i^*$ we have

$$\alpha_2 + \alpha_4 + \alpha_8 - \alpha_{16} = 0,$$

$$\begin{aligned}
\alpha_1 + 2(\alpha_3 - \alpha_5) + \alpha_5 + \alpha_{12} + \alpha_{16} - \alpha_{15} &= 0, \\
3(\alpha_4 - \alpha_5) + 3\alpha_5 + \alpha_8 - \alpha_{16} &= 0, \\
2(\alpha_4 - \alpha_5) + \alpha_3 + \alpha_{10} - \alpha_{11} - \alpha_{16} &= 0, \\
\alpha_4 - \alpha_5 - \alpha_8 + \alpha_{11} + 2\alpha_{16} - \alpha_{15} &= 0, \\
\alpha_{12} + \alpha_{11} + 2\alpha_{16} - 2\alpha_{15} &= 0, \\
4(\alpha_3 - \alpha_5) + \alpha_5 + \alpha_{12} + \alpha_{16} - \alpha_{15} &= 0.
\end{aligned} \tag{5.52}$$

Let $\{g_i | i = 1, \dots, 6\}$ be the basis of $VIN_4[3, 1^6]$ given in Proposition 5.3.3. Then

$$\begin{aligned}
\Delta(y_7, x)t_2 &= \Delta(y_7, x)t_1^* = g_4^* = \frac{1}{4}(-g_1 - g_3 + g_4) - g_4, \\
\Delta(y_7, x)t_4 &= (-g_2 + 2g_1)^* = g_2 + \frac{1}{2}(g_1 + g_3 - g_4) - g_1, \\
\Delta(y_7, x)t_1 &= g_4, \Delta(y_7, x)t_3 = -g_2 + 2g_1, \\
\Delta(y_7, x)t_5 &= 2g_4, \Delta(y_7, x)t_7 = g_6, \Delta(y_7, x)t_8 = g_6^* = -g_5, \\
\Delta(y_7, x)t_{10} &= g_5, \Delta(y_7, x)t_9 = g_5^* = -g_6, \Delta(y_7, x)t_{11} = g_6, \\
\Delta(y_7, x)t_{12} &= g_6^* = -g_5, \Delta(y_7, x)t_{15} = \Delta(y_7, x)t_{16} = 0.
\end{aligned}$$

Thus $(\sum_i \alpha_i t_i)^* = 0$ implies

$$\begin{aligned}
&\alpha_1 g_4 + \alpha_2 \left(\frac{1}{4}(-g_1 - g_3 + g_4) - g_4 \right) + \\
&\alpha_3 (2g_1 - g_2) + \alpha_4 \left(g_2 + \frac{1}{2}(g_1 + g_3 - g_4) - g_1 \right) + \\
&2\alpha_5 g_4 + \alpha_7 g_6 - \alpha_8 g_5 - \alpha_9 g_6 + \alpha_{10} g_5 + \alpha_{11} g_6 - \alpha_{12} g_5 = 0.
\end{aligned}$$

So

$$\begin{aligned}
\frac{-1}{4}\alpha_2 + 2\alpha_3 - \frac{1}{2}\alpha_4 &= 0, \quad -\alpha_3 + \alpha_4 = 0, \\
\frac{-1}{4}\alpha_2 + \frac{1}{2}\alpha_4 &= 0, \quad \alpha_1 - \frac{3}{4}\alpha_2 - \frac{1}{2}\alpha_4 + 2\alpha_5 = 0, \\
-\alpha_8 + \alpha_{10} - \alpha_{12} &= 0, \quad \alpha_7 - \alpha_9 + \alpha_{11} = 0.
\end{aligned} \tag{5.53}$$

Solving (5.51), (5.52) and (5.53) yields

$$\alpha_i = 0, i \neq 7, 8, 9, 10, 15, 16$$

and $\alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{15} = \alpha_{16}$. Thus among these 13 identities, 12 of them are linearly independent. Since we have shown that $\dim(VIN_4[2, 1^7]) \leq 12$ at the beginning of the proof, $t_7 + t_8 + t_9 + t_{10} + t_{15} + t_{16} = 0$. ■

Remark: Proposition 5.3.5 holds for H_5 . The proof is exactly the same.

Chapter 6

The Identities of Degree $2n + 1$ on H_n

6.1 Introduction

In Chapter 4 and Chapter 5 we have found out all identities of degree 7 and 9 on H_3 and H_4 respectively. A basis of $VIN_4[t]$, for each type t was given. In the present chapter the above results for the identities of degree 9 on H_4 will be extended to H_n for arbitrary $n \geq 4$. This is the content of Theorem 6.1 and Theorem 6.2.

Let $VIn[n_1, \dots, n_r]$ denote the set of all identities of degree $2n + 1$ on H_n which have type $[n_1, \dots, n_r]$, and $VIN_n[n_1, \dots, n_r]$ denote the set of all identities of degree $2n + 1$ on H_n which have type $[n_1, \dots, n_r]$ and are symmetric or skew symmetric in all their variables of degree m according as m is even or odd for any $m \in \{n_1, \dots, n_r\}$.

We also define

$$e(n) := \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 6.1 *If the characteristic of the field F does not divide $e(n)!$, $|F| > 2n$ and $n \geq 4$ then:*

1. *For any type $[n_1, \dots, n_r]$ with $\sum_{i=1}^r n_i = 2n + 1$ if*

$$[n_1, \dots, n_r] \neq [4, 1^{2n-3}], [3, 2, 1^{2n-4}], [3, 1^{2n-2}], [2^3, 1^{2n-5}], \\ [2, 2, 1^{2n-3}], [2, 1^{2n-1}] \text{ or } [1^{2n+1}]$$

then $VI_n[n_1, \dots, n_r] = \{0\}$.

2. The vector space $VIN_n[2^3, 1^{2n-5}] = \{0\}$.

Theorem 6.2 *If the characteristic of the field F does not divide $e(n)!$ and $|F| > 2n$ then*

1. $\{T_{2n}(x^2, x, y_1, \dots, y_{2n-3}; x)\}$ is a basis of the vector space $VI_n[4, 1^{2n-3}]$.
2. $\{T_{2n}(x^2, x, y, z_1, \dots, z_{2n-4}; y), T_{2n}(x \circ y, x, y, z_1, \dots, z_{2n-4}; x)\}$ is a basis of the space $VI_n[3, 2, 1^{2n-4}]$.
3. The following identities form a basis of the space $VIN_n[3, 1^{2n-2}]$:

$$\begin{aligned} &T_{2n}(x^2, y_1, \dots, y_{2n-2}; x), \quad T_{2n}(x, y_1, \dots, y_{2n-2}; x^2), \\ &\sum_{i=1}^{2n-2} (-1)^{i-1} T_{2n}(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_{2n-2}, x; x), \\ &\sum_{i=1}^{2n-2} (-1)^i T_{2n}(x^2, x, y_1, \dots, \hat{y}_i, \dots, y_{2n-2}; y_i), \\ &T_{2n}(x, y_1, \dots, y_{2n-2}; x)x, \quad xT_{2n}(x, y_1, \dots, y_{2n-2}; x). \end{aligned}$$

4. The identities

$$\begin{aligned} &T_{2n}(x^2, y, z_1, \dots, z_{2n-3}; y) + T_{2n}(y^2, x, z_1, \dots, z_{2n-3}; x), \\ &T_{2n}(x \circ y, x, z_1, \dots, z_{2n-3}; y) + T_{2n}(x \circ y, y, z_1, \dots, z_{2n-3}; x), \\ &\sum_{i=1}^{2n-3} (-1)^{i-1} (T_{2n}(x \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_{2n-3}, x, y; y) + \\ &\quad T_{2n}(y \circ z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(2n-3)}, y, x; x)), \\ &xT_{2n}(x, y, z_1, \dots, z_{2n-3}; y) + yT_{2n}(y, x, z_1, \dots, z_{2n-3}; x), \\ &T_{2n}(x, y, z_1, \dots, z_{2n-3}; y)x + T_{2n}(y, x, z_1, \dots, z_{2n-3}; x)y, \\ &T_{2n}(x, y, z_1, \dots, z_{2n-3}; [x, y]), \quad \tilde{T}_{2n}(x, y, z_1, \dots, z_{2n-3}; [x, y]) \end{aligned}$$

form a basis of the space $VIN_n[2, 2, 1^{2n-3}]$.

5. The identities

$$\begin{aligned}
& T_{2n}(y_1, \dots, y_{2n-1}; x^2), \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_{2n-1}; x), \\
& \tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x^2), \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} \tilde{T}_{2n}(x \circ y_i, y_1, \dots, \hat{y}_i, \dots, y_{2n-1}; x), \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-1}; y_i \circ x), \\
& T_{2n}(y_1, \dots, y_{2n-1}; x)x, \quad xT_{2n}(y_1, \dots, y_{2n-1}; x), \\
& \tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x)x, \quad x\tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x), \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} y_i T_{2n}(y_1, \dots, \hat{y}_i, \dots, y_{2n-1}, x; x), \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} \tilde{T}_{2n}(y_1, \dots, \hat{y}_i, \dots, y_{2n-1}, x; x)y_i, \\
& \sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-1}; [x, y_i])
\end{aligned}$$

form a basis of the space $VIN_n[2, 1^{2n-1}]$.

6. $S_{2n+1}(x_1, \dots, x_{2n+1})$ forms a basis of the space $VIN_n[1^{2n+1}]$.

To prove these theorems we use induction on n . For $n = 4$ it was shown in Chapter 5 that the Theorems hold. Assume that the theorems are also true for integers which are less than n then for n the proof will be completed in following sections.

6.2 The Proof of the Theorem 6.1

Lemma 6.2.1 *Let f be an identity of H_n of degree $2n + 1$ with $n > 4$. Assume that $f^* = \pm f$ and that f has no terms which start with $y_2 y_1$. If $\text{char } F \nmid e(n)!$ and*

$$f = xyf_1 + \sim, \text{ or } f = x^2 y f_2 + \sim,$$

where $\sim$ denote the sum of the terms of f which do not start with xy and x^2y respectively, then the following statements hold:

1. If $f_1 = f_1(z, y_1, y_2, \dots, y_{2n-5})$ is of type $[4, 1^{2n-5}]$ or $f_1 = f_1(z, w, y_1, y_2, \dots, y_{2n-6})$ is of type $[3, 2, 1^{2n-6}]$ then $f_1 = 0$.
2. If $f_1 = f_1(z, y_1, \dots, y_{2n-4})$ is of type $[3, 1^{2n-4}]$ and skew symmetric in all its variables of degree 1 then f_1 is a scalar multiple of $T_{2n-2}(z, y_1, \dots, y_{2n-4}; z)z$.
3. If $f_1 = f_1(z, w, y_1, \dots, y_{2n-5})$ is of type $[2, 2, 1^{2n-5}]$, symmetric in its variables of degree 2 and skew symmetric in its variables of degree 1 then f_1 is a scalar multiple of

$$T_{2n-2}(z, w, y_1, \dots, y_{2n-5}; w)z + T_{2n-2}(w, z, y_1, \dots, y_{2n-5}; z)w.$$

4. If $f_2 = f_2(z, y_1, y_2, \dots, y_{2n-4})$ is of type $[2, 1^{2n-4}]$ then $f_2 = 0$.

Proof. If f is an identity of H_n of degree $2n + 1$ and

$$f = xyf_1 + \sim, \text{ or } f = x^2yf_2 + \sim,$$

then f_1, f_2 are identities of degree $2n - 1$ and degree $2n - 2$ on H_{n-1} respectively by Lemma 2.1.2.

1. If f_1 is of type $[4, 1^{2n-5}]$ then by the induction hypothesis there exists a scalar $\alpha \in F$ such that

$$f_1 = \alpha T_{2n-2}(z^2, z, y_1, \dots, y_{2n-5}; z).$$

So

$$f_1 = \alpha T_{2n-4}(z^2, z, y_3, \dots, y_{2n-5}; z)y_1y_2 + \sim$$

and hence

$$f = \alpha xy T_{2n-4}(z^2, z, y_3, \dots, y_{2n-5}; z)y_1y_2 + \sim. \quad (6.1)$$

Since f has no terms which start with y_2y_1 and $f^* = \pm f$ by assumption, f has no terms which end in y_1y_2 . So (6.1) implies that $\alpha = 0$ and therefore $f_1 = 0$.

2. If f_1 is of type $[3, 2, 1^{2n-6}]$ then by the induction hypothesis there exists scalars $\alpha, \beta \in F$ such that

$$\begin{aligned} f_1 &= \alpha T_{2n-2}(z^2, z, w, y_1, \dots, y_{2n-6}; w) + \\ &\quad \beta T_{2n-2}(z \circ w, z, w, y_1, \dots, y_{2n-6}; z). \end{aligned}$$

So

$$\begin{aligned} f_1 &= \{\alpha T_{2n-4}(z^2, z, w, y_3, \dots, y_{2n-6}; w) + \\ &\quad \beta T_{2n-4}(z \circ w, z, w, y_3, \dots, y_{2n-6}; z)\} y_1 y_2 + \sim. \end{aligned}$$

Therefore we have

$$\begin{aligned} f &= xy\{\alpha T_{2n-4}(z^2, z, w, y_3, \dots, y_{2n-6}; w) + \\ &\quad \beta T_{2n-4}(z \circ w, z, w, y_3, \dots, y_{2n-6}; z)\} y_1 y_2 + \sim. \end{aligned} \quad (6.2)$$

Using the same argument as in the proof of part 1, (6.2) implies that

$$\alpha T_{2n-4}(z^2, z, w, y_3, \dots, y_{2n-6}; w) + \beta T_{2n-4}(z \circ w, z, w, y_3, \dots, y_{2n-6}; z) = 0,$$

hence $\alpha = \beta = 0$, since for $n \geq 5$, it can be shown by induction on n that

$$T_{2n-4}(z^2, z, w, y_3, \dots, y_{2n-6}; w), T_{2n-4}(z \circ w, z, w, y_3, \dots, y_{2n-6}; z)$$

are linear independent.

Let $t_1^{n-1}, \dots, t_6^{n-1}$ be the basis of the space $VIN_{n-1}[3, 1^{2n-4}]$ given in Theorem 6.2. Then

$$t_i^{n-1} = t_i^{n-2}[y_1, y_2] + \sim, \quad i = 1, 4, 6,$$

$$t_2^{n-1} = \{t_2^{n-2} + t_5^{n-2}\}[y_1, y_2] + \sim,$$

$$t_3^{n-1} = \begin{cases} t_3^{n-2}[y_1, y_2] + \sim & \text{if } n-1 \text{ is even} \\ (t_3^{n-2} - 2S_{2n-4}(z^2, z, y_3, \dots, y_{2n-4}))[y_1, y_2] + \sim & \text{if } n-1 \text{ is odd.} \end{cases}$$

By [MR, Lemma 1.4]

$$\sum_{i=1}^{2n} T_{2n}^i(x_1, \dots, x_{2n}) = e(n)S_{2n}(x_1, \dots, x_{2n}),$$

where $e(n) = n$ if n is even, $e(n) = n + 1$ if n is odd. So

$$S_{2n-4}(z^2, z, y_3, \dots, y_{2n-4}) = \frac{1}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-2}),$$

when $n - 1$ is odd. Thus if $n - 1$ is odd then

$$t_3^{n-1} = \{t_3^{n-2} - \frac{2}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-2})\}[y_1, y_2] + \sim.$$

Now, let $f_1 \in VIN_{n-1}[3, 1^{2n-4}]$. Then $f_1 = \sum_{i=1}^6 \alpha_i t_i^{n-1}$. To show that $f_1 = 0$ different arguments are given according as n is even or odd.

Case 1. If $n - 1$ is even then

$$f_1 = \{ \sum_{i \neq 2,5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) \}[y_1, y_2] + \sim$$

and

$$f = xy \{ \sum_{i \neq 2,5} t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) \}[y_1, y_2] + \sim.$$

Therefore f has no terms which start with $y_1 y_2$, and $f^* = \pm f$ implies that

$$\sum_{i \neq 2,5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) = 0.$$

So $\alpha_i = 0$ for $i \neq 5$, since $t_1^{n-2}, \dots, t_6^{n-2}$ are linearly independent by the induction hypothesis [Note that when $n-2 = 3$, $t_1^{n-2}, \dots, t_6^{n-2}$ are also linear independent. See Proposition 4.2.4]. Hence $f_1 = \alpha_5 T_{2n-2}(z, y_1, \dots, y_{2n-4}; z)z$.

Case 2. If $n - 1$ is odd then we have

$$\sum_{i \neq 2,3,5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) + \alpha_3 \{ t_3^{n-2} - \frac{2}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-2}) \} = 0.$$

Thus $\alpha_i = 0$ for $i \neq 5$ by the linear independence of $t_1^{n-2}, \dots, t_6^{n-2}$. So part 2 holds in any case.

3. By the induction hypothesis, if $f_1 \in VIN_{n-1}[2, 2, 1^{2n-5}]$ then

$$\begin{aligned} f_1 &= \sum_{i=1}^7 \alpha_i t_i^{n-1} \\ &= (\sum_{i \neq 3,5} \alpha_i t_i^{n-2} + \alpha_3(t_3^{n-2} + t_5^{n-2})) [y_1, y_2] + \sim, \end{aligned}$$

where $t_1^k, \dots, t_7^k$ is the given basis of $VIN_k[2, 2, 1^{2k-3}]$ for $k = n-1, n-2$ respectively by the induction hypothesis. So arguing as above we have $\alpha_i = 0$ for $i \neq 5$. Therefore

$$f_1 = \alpha_5(T_{2n-2}(z, w, y_1, \dots, y_{2n-4}; w)z + T_{2n-2}(w, z, y_1, \dots, y_{2n-4}; z)w).$$

The proof of 4 is easy, completing the proof of the Lemma. ■

Proposition 6.2.1 *If $\text{char } F \nmid e(n)!$ then the vector space $VIn[5, 1^{2n-4}] = \{0\}$. Therefore $VIn[5, n_1, \dots, n_r] = \{0\}$ for any positive integers $n_i \leq 5$, $i = 1, \dots, r$ such that $\sum_{i=1}^r n_i = 2n-4$.*

Proof. Let $f(y_0, y_1, \dots, y_{2n-4})$ be an identity of type $[5, 1^{2n-4}]$. By Lemma 2.1.1, if

$$\begin{aligned} f &= y_i y_j f_1(y_0, y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_{2n-4}) + \\ &\quad y_0^2 y_k f_2(y_0, y_1, \dots, \widehat{y}_k, \dots, y_{2n-4}) + \sim, \end{aligned}$$

where at least one of i, j is nonzero and $\sim$ denotes the sum of the terms of f which do not start with either $y_i y_j$ or $y_0^2 y_k$, then both f_1 and f_2 are identities of H_{n-1} . Thus by the induction hypothesis and the fact that H_{n-1} has no identity of degree $2n-2$ which is lower than $[2, 1^{2n-4}]$,

$$f = \sum_{i=1}^r (y_0 y_i f_i(y_0, \dots, \widehat{y}_i, \dots, y_r) + y_i y_0 f'_i(y_0, \dots, \widehat{y}_i, \dots, y_r))$$

where $f_i, f'_i \in VIN_{n-1}[4, 1^{2n-5}]$. Thus if $f^* = \pm f$ then $f_i = f'_i = 0$ by Lemma 6.2.1. So $f = 0$ when $f^* = \pm f$. Since f can be decomposed into the sum of a $*$ -symmetric identity and a $*$ -skew symmetric identity, $f = 0$.

The second part of the proposition follows from the first part. ■

Using Proposition 6.2.1 and Proposition 1.2.1 one can show easily that

Corollary 6.2.1 *If $\text{char } F$ does not divide $e(n)!$ and $|F| > 2n$ then for each $[n_1, \dots, n_r] \leq [5, 1^{2n-4}]$ the vector space $VIn[n_1, \dots, n_r] = \{0\}$. ■*

From now on we assume that F does not divide $e(n)!$ and $|F| > 2n$.

Proposition 6.2.2 *The vector space $VI_n[4, 3, 1^{2n-6}] = \{0\}$.*

Proof. Let $f(x, y, z_1, \dots, z_{2n-6}) \in VI_n[4, 3, 1^{2n-6}]$. Then using the same argument as in the proof of Proposition 6.2.1

$$f = xyf_1(x, y, z_1, \dots, z_{2n-6}) + yxf_2(x, y, z_1, \dots, z_{2n-6}),$$

where $f_1, f_2 \in VI_{n-1}[3, 2, 1^{2n-6}]$. We may assume that $f^* = \pm f$. So, the fact that f has no terms which start with $z_i z_j$ implies that f has no terms which end in $z_i z_j$ either. Thus by Lemma 6.2.1, $f_1 = f_2 = 0$. Hence $f = 0$. ■

Proposition 6.2.3 *The vector space $VI_n[4, 2, 1^{2n-5}] = \{0\}$.*

Proof. Let $f(x, y, z_1, \dots, z_{2n-5})$ be an identity of type $[4, 2, 1^{2n-5}]$. To show $f = 0$ we may assume that $f^* = \pm f$. By the induction hypothesis we have

$$\begin{aligned} f = & xyf_1(x, y, z_1, \dots, z_{2n-5}) + yxf'_1(x, y, z_1, \dots, z_{2n-5}) + \\ & \sum_{i=1}^{2n-5} xz_i f_{2i}(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-5}) + \\ & \sum_{i=1}^{2n-5} z_i x f'_{2i}(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-5}) + \\ & \sum_{i=1}^{2n-5} yz_i f_{3i}(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-5}) + \\ & \sum_{i=1}^{2n-5} z_i y f'_{3i}(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-5}) + \\ & x^2 y f_4(x, y, z_1, \dots, z_{2n-5}), \end{aligned}$$

where $f_{2i}, f'_{2i} \in VI_{n-1}[3, 2, 1^{2n-6}]$, $f_{3i}, f'_{3i} \in VI_{n-1}[4, 1, 1^{2n-6}]$ and $f_4 \in VI_{n-1}[2, 1^{2n-4}]$. So, f has no terms which start with $z_j z_k$ for $i \neq j, k$ by Lemma 6.2.1, this implies that $f_4 = f_{ri} = f'_{ri} = 0$ for $r = 2, 3$. Hence

$$f = xyf_1(x, y, z_1, \dots, z_{2n-5}) + yxf'_1(x, y, z_1, \dots, z_{2n-5}).$$

Since $\Delta(y, x)f \in VI_n[5, 1^{2n-4}] = \{0\}$, $f_1 = -f'_1$. Thus

$$f = [x, y]f_1(x, y, z_1, \dots, z_{2n-5}).$$

Because $\Delta(z_1, y)f \in VI_n[4, 3, 1^{2n-6}] = \{0\}$ we have $\Delta(z_1, y)f_1 = 0$. So f_1 is skew symmetric in variables $y, z_1, \dots, z_{2n-5}$. That is, $f_1 \in VIN_{n-1}[3, 1^{n-4}]$. Therefore $f_1 = \alpha T_{2n-2}(x, y, z_1, \dots, z_{2n-5}; x)x$ and

$$f = \alpha[x, y]T_{2n-2}(x, y, z_1, \dots, z_{2n-5}; x)x.$$

So f has no terms which do not end in x . Hence, f has no terms which do not start with x because $f^* = \pm f$. Therefore $\alpha = 0$ and f is zero identically. ■

Using Corollary 6.2.1 and Proposition 6.2.3 one can easily show that

Corollary 6.2.2 For each type $[n_1, \dots, n_r] \leq [4, 2, 1^{2n-5}]$ the vector space $VI_n[n_1, \dots, n_r] = \{0\}$.

Proposition 6.2.4 The vector space $VI_n[3, 3, 2, 1^{2n-7}] = \{0\}$.

Proof. Let $f(x_1, x_2, y, z_1, \dots, z_{2n-7})$ be an identity of type $[3, 3, 2, 1^{2n-7}]$. Then by the induction hypothesis and since II_{n-1} has no identity of degree $2n-4$ of type $[2, 2, 1^{2n-6}]$ or lower,

$$f = x_1x_2f_1 + x_2x_1f_2 + x_1yf_3 + yx_1f_4 + x_2yf_5 + yx_2f_6,$$

where $f_3, f_4, f_5, f_6 \in VI_{n-1}[3, 2, 1^{2n-7}]$. So they are zero by Lemma 6.2.1. Since $\Delta(y, x_i)f \in VI_n[4, 3, 1^{2n-6}] = \{0\}$, for $i = 1, 2$, $\Delta(y, x_i)f_1 = 0$. Hence f_1 is symmetric in the variables of degree 2 by Theorem 1.1. Moreover f_1 is also skew symmetric in the variables of degree 1 since $VI_{n-1}[2^4, 1^{2n-9}] = \{0\}$. Thus $f_1 \in VIN_{n-1}[2^3, 1^{2n-7}] = \{0\}$. Symmetrically we can show that $f_2 = 0$. Thus $f = 0$. ■

Proposition 6.2.5 The vector space $VI_n[3, 3, 1^{2n-5}] = \{0\}$.

Proof. Let $f(x, y, z_1, \dots, z_{2n-5})$ be an identity of type $[3, 3, 1^{2n-5}]$. We may assume that $f^* = \pm f$. Since $\Delta(z_i, z_j)f \in VI_n[3^2, 2, 1^{2n-7}] = \{0\}$ and $\Delta(x, y)f \in VI_n[4, 2, 1^{2n-5}] = \{0\}$, $VI_n[3, 3, 1^{2n-5}] = VIN_n[3, 3, 1^{2n-5}]$. So by the induction hypothesis we have

$$f = xyf_1(x, y, z_1, \dots, z_{2n-5}) - yxf_1(y, x, z_1, \dots, z_{2n-5}) +$$

$$\begin{aligned}
& \sum_{i=1}^{2n-5} x z_i f_{2i}(x, y, z_1, \dots, \widehat{z_i}, \dots, z_{2n-5}) + \\
& \sum_{i=1}^{2n-5} z_i x f'_{2i}(x, y, z_1, \dots, \widehat{z_i}, \dots, z_{2n-5}) - \\
& \sum_{i=1}^{2n-5} y z_i f_{2i}(y, x, z_1, \dots, \widehat{z_i}, \dots, z_{2n-5}) - \\
& \sum_{i=1}^{2n-5} z_i y f'_{2i}(y, x, z_1, \dots, \widehat{z_i}, \dots, z_{2n-5}) + \\
& x^2 y f_3(x, y, z_1, \dots, z_{2n-5}) - y^2 x f_3(y, x, z_1, \dots, z_{2n-5}),
\end{aligned}$$

where $f_{2i}, f'_{2i} \in VIN_{n-1}[3, 2, 1^{2n-6}]$, $f_3 \in VIN_{n-1}[2, 1^{2n-4}]$. So, by Lemma 6.2.1, $f_3 = f_{2i} = f'_{2i} = 0$, for $i = 1, \dots, 2n-5$. Hence

$$f = xyf_1(x, y, z_1, \dots, z_{2n-5}) - yxf_1(y, x, z_1, \dots, z_{2n-5}).$$

This implies that f has no terms which do not start with xy or yx , therefore it has no terms which do not end in xy or yx by $f^* = \pm f$.

On the other hand, since $\Delta(x, y)f = 0$ implies that $\Delta(x, y)f_1 = 0$, $f_1 \in VIN_{n-1}[2, 2, 1^{2n-5}]$. So by Lemma 6.2.1

$$f_1 = \alpha(T_{2n-2}(x, y, z_1, \dots, z_{2n-5}; y)x + T_{2n-2}(y, x, z_1, \dots, z_{2n-5}; x)y).$$

So there exists a monomial in f_1 therefore a monomial in f which does not end in either xy or yx with coefficient α . Thus $\alpha = 0$. So $f = 0$. ■

Proposition 6.2.6 *The vector space $VIN_n[3, 2^3, 1^{2n-8}] = \{0\}$.*

Proof. Let $f(x, y_1, y_2, y_3, z_1, \dots, z_{2n-8})$ be an identity of type $[3, 2^3, 1^{2n-8}]$. Then

$$\begin{aligned}
f &= xy_3 f_1(x, y_1, y_2, y_3, z_1, \dots, z_{2n-8}) + \dots \\
&+ y_3 x f'_1(x, y_1, y_2, y_3, z_1, \dots, z_{2n-8})x + \dots \\
&+ y_2 y_3 f_2(x, y_1, y_2, y_3, z_1, \dots, z_{2n-8}) + \dots
\end{aligned}$$

We may assume that $f^* = \pm f$. Since $f_2 \in VI_{n-1}[3, 2, 1^{2n-6}]$ and f has no terms which start with $z_i z_j$, $f_2 = 0$ by Lemma 6.2.1. So

$$\begin{aligned} f &= xy_3 f_1(x, y_1, y_2, z_1, \dots, z_{2n-8}) + \dots \\ &\quad + y_3 x f'_1(x, y_1, y_2, z_1, \dots, z_{2n-8})x + \dots. \end{aligned}$$

Since $\Delta(y_1, x)f \in VI_n[4, 2, 2, 1^{2n-7}] = \{0\}$, $\Delta(y_1, x)f_1 \in VI_n[4, 2, 2, 1^{2n-7}] = \{0\}$. Hence f_1 is symmetric in the variables x and y_1 by Theorem 1.1. While $\Delta(y_2, x)f = 0$ and $\Delta(z_i, y_3)f = 0$ imply that f_1 is symmetric in the variables x, y_2 , and is skew symmetric in the variables z_i, y_3 . Thus

$$f_1(x, y_1, y_2, y_3, z_1, \dots, z_{2n-8}) \in VIN_{n-1}[2^3, 1^{2n-7}] = \{0\}.$$

Therefore $f_1 = 0$. Similarly we can show that $f'_1 = 0$. So $f = 0$. ■

Proposition 6.2.7 *The vector space $VI_n[3, 2^2, 1^{2n-6}] = \{0\}$. Therefore $VI_n[3, 2^2, n_1, \dots, n_r] = \{0\}$, for any type $[3, 2^2, n_1, \dots, n_r]$ with*

$$[4, 1^{2n-5}] < [3, 2^2, n_1, \dots, n_r] \leq [3, 2^2, 1^{2n-6}].$$

Proof. Let $f(x, y_1, y_2, z_1, \dots, z_{2n-6})$ be an identity of type $[3, 2^2, 1^{2n-6}]$ then

$$\begin{aligned} f &= xy_2 f_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\ &\quad + y_2 x f'_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\ &\quad + x z_1 f_2(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\ &\quad + z_1 x f'_2(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\ &\quad + y_1 y_2 f_3(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\ &\quad + y_2 z_1 f_4(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\ &\quad + z_1 y_2 f'_4(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\ &\quad + x^2 y_2 f_5(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots. \end{aligned}$$

We may assume that $f^* = \pm f$. Thus by Lemma 6.2.1, f has no terms which start with $z_i z_j$, which implies that $f_4 = f'_4 = f_5 = 0$ since $f_4, f'_4 \in VI_{n-1}[3, 2, 1^{2n-6}]$ and $f_5 \in$

$V I_{n-1}[2, 1^{2n-4}]$. Hence

$$\begin{aligned}
f &= x y_2 f_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\
&\quad + y_2 x f'_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\
&\quad + x z_1 f_2(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\
&\quad + z_1 x f'_2(x, y_1, y_2, z_2, \dots, z_{2n-6}) + \dots \\
&\quad + y_1 y_2 f_3(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots.
\end{aligned} \tag{6.3}$$

Since $\Delta(z_1, y_1)f \in V I_n[3, 3, 2, 1^{2n-7}] = \{0\}$, $\Delta(z_1, y_1)f_3 = 0$ by (6.3). Similarly, we have $\Delta(z_1, y_2)f_3 = 0$ by (6.3). So f_3 is skew symmetric in y_1, y_2, z_1 . Since $V I_n[3, 2^3, 1^{2n-8}] = \{0\}$, $\Delta(z_1, z_i)f = 0$ implies that $\Delta(z_j, z_1)f = 0$ and hence f is skew symmetric in all z_i 's. So $f_3 \in V I N_{n-1}[3, 1^{2n-4}]$. Thus by Lemma 6.2.1

$$f_3 = \alpha T_{2n-2}(x, y_1, y_2, z_1, \dots, z_{2n-6}; x)x.$$

The monomial $x y_1 y_2 z_1 \dots z_{2n-6} x x$ has coefficient $(-1)^n \alpha$ in f_3 . So the monomial $y_1 y_2 x y_1 y_2 z_1 \dots z_{2n-6} x x$ has coefficient $(-1)^n \alpha$ in f . Thus $\alpha = 0$ by (6.3) and $f^* = \pm f$. Thus $f_3 = 0$.

Next we show that $f_2 = f'_2 = 0$ by showing that $f_2, f'_2 \in V I N_{n-1}[2^3, 1^{2n-7}]$. By the induction hypothesis $V I N_{n-1}[2^3, 1^{2n-7}] = \{0\}$, and it suffices to show that f_2, f'_2 is symmetric in x, y_1, y_2 . Since $\Delta(y_1, x)f = 0$ and $f_3 = 0$, $\Delta(y_1, x)f_2 = \Delta(y_1, x)f'_2 = 0$. So f_2, f'_2 are symmetric in x, y_1 by Theorem 1.1. Similarly we have f_2, f'_2 are symmetric in x, y_2 . Thus $f_2, f'_2 \in V I N_{n-1}[2^3, 1^{2n-7}]$. So

$$\begin{aligned}
f &= x y_2 f_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots \\
&\quad + y_2 x f'_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots.
\end{aligned}$$

This and $\Delta(y_2, x)f = 0$ imply that $f_1 = -f'_1$. Thus

$$f = [x, y_2]f_1(x, y_1, y_2, z_1, \dots, z_{2n-6}) + \dots. \tag{6.4}$$

We know $\Delta(y_1, x)f = 0$ and $\Delta(z_1, y_2)f = 0$ by (6.4). This implies that $\Delta(y_1, x)f_1 = 0$ and $\Delta(z_1, y_2)f_1 = 0$. Hence $f_1 \in V I N_{n-1}[2, 2, 1^{2n-5}]$. So

$$\begin{aligned}
f_1 &= \alpha(T_{2n-2}(x, y_1, y_2, z_1, \dots, z_{2n-6}; y_1)x + \\
&\quad T_{2n-2}(y_1, x, y_2, z_1, \dots, z_{2n-6}; x)y_1)
\end{aligned}$$

and

$$\begin{aligned} f = & \alpha[x, y_2](T_{2n-2}(x, y_1, y_2, z_1, \dots, z_{2n-6}; y_1)x + \\ & T_{2n-2}(y_1, x, y_2, z_1, \dots, z_{2n-6}; x)y_1) + \dots \end{aligned} \quad (6.5)$$

Since the monomial $xy_2y_1xy_1y_2z_1 \cdots z_{2n-6}x$ has coefficient α and the monomial $xz_{2n-6} \cdots z_1y_2y_1xy_1y_2x$ has coefficient 0 by (6.5), $\alpha = 0$ and $f = 0$. ■

Proposition 6.2.8 *The vector space $V I_n[2^4, 1^{2n-7}] = \{0\}$. Therefore $V I_n[2^4, n_1, \dots, n_r] = \{0\}$ for any type $[2^4, n_1, \dots, n_r]$ with*

$$[3, 1^{2n-5}] < [2^4, n_1, \dots, n_r] \leq [2^4, 1^{2n-6}].$$

Proof. Let $f(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7})$ be an identity of type $[2^4, 1^{2n-7}]$. Then $\Delta(x_4, x_1)f \in V I_n[3, 2, 2, 1^{2n-6}]$. By Proposition 6.2.7, $V I_n[3, 2, 2, 1^{2n-6}] = \{0\}$. So $\Delta(x_4, x_1)f = 0$ and hence f is symmetric in x_1, x_4 by Theorem 1.1. Similarly f is symmetric in other x_i 's. So f is symmetric in all its variables of degree two. Next we claim

$$\text{If } f = x_i z_j f_1 + z_j x_i f_2 + \sim, \text{ then } f_1, f_2 \in V I_{n-1}[2^3, 1^{2n-5}]. \quad (6.6)$$

Obviously $f_1, f_2 \in V I_{n-1}[2^3, 1^{2n-5}]$ by Lemma 2.1.1. f_1, f_2 are symmetric in the variables of degree 2 since f is. By the induction hypothesis $V I_{n-1}[2^4, 1^{2n-7}] = \{0\}$. Hence f_1, f_2 are skew symmetric in their variables of degree 1. So the claim holds. Since $V I_{n-1}[2^3, 1^{2n-5}] = \{0\}$, the claim implies that $f_1 = f_2 = 0$. Therefore

$$\begin{aligned} f = & x_3 x_4 f_1(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7}) + \\ & x_4 x_3 f_1(x_1, x_2, x_4, x_3, z_1, \dots, z_{2n-7}) + \dots \end{aligned}$$

This equation and $\Delta(x_3, x_4)f = 0$ imply that

$$f_1(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7}) + f_1(x_1, x_2, x_4, x_3, z_1, \dots, z_{2n-7}) = 0.$$

So

$$f = [x_3, x_4]f_1(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7}) + \dots$$

From this equation and $\Delta(x_1, x_2)f \in VIN_n[3, 2, 2, 1^{2n-6}] = \{0\}$ we know that f_1 is symmetric in the variables of degree 2 by Theorem 1.1. While $\Delta(z_i, x_3)f = 0$, for $i = 1, \dots, 2n-7$ implies that $\Delta(z_i, x_3)f_1 = 0$. So f_1 is skew symmetric in x_3, z_i . Since f_1 is skew symmetric in x_3, x_4 also, $f_1 \in VIN_{n-1}[2, 2, 1^{2n-5}]$. Thus by Lemma 6.2.1,

$$\begin{aligned} f_1 &= \alpha(T_{2n-2}(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7}; x_2)x_1 + \\ &\quad T_{2n-2}(x_2, x_1, x_3, x_4, z_1, \dots, z_{2n-7}; x_1)x_2). \end{aligned}$$

Hence

$$\begin{aligned} f &= \alpha[x_3, x_4](T_{2n-2}(x_1, x_2, x_3, x_4, z_1, \dots, z_{2n-7}; x_2)x_1 + \\ &\quad T_{2n-2}(x_2, x_1, x_3, x_4, z_1, \dots, z_{2n-7}; x_1)x_2) + \dots \end{aligned}$$

From this equation we know that the monomial $x_3x_4x_2x_1x_3x_2x_4z_1 \cdots z_{2n-7}x_1$ has coefficient $-\alpha$ while its $*$ has coefficient 0. So if $f^* = \pm f$ then $\alpha = 0$ and therefore $f = 0$ for any f in $VIN_n[2^4, 1^{2n-7}]$. ■

Proposition 6.2.9 *The vector space $VIN_n[2^3, 1^{2n-5}] = \{0\}$.*

Proof. Let $f(x_1, x_2, x_3, z_1, \dots, z_{2n-5}) \in VIN_n[2^3, 1^{2n-5}]$. It suffices to show $f = 0$ when $f^* = \pm f$. Since

$$VIN_{n-1}[2^3, 1^{2n-7}] = VIN_{n-1}[2, 2, 1^{2n-6}] = \{0\},$$

$$\begin{aligned} f &= x_2x_3f_1(x_1, x_2, x_3, z_1, \dots, z_{2n-5}) \\ &\quad + x_3x_2f_1(x_1, x_3, x_2, z_1, \dots, z_{2n-5}) + \dots \\ &\quad + z_1x_3f_2(x_1, x_2, x_3, z_2, \dots, z_{2n-5}) \\ &\quad + x_3z_1f'_2(x_1, x_2, x_3, z_2, \dots, z_{2n-5}) + \dots \\ &\quad + x_3^2x_2f_3(x_1, x_2, z_1, \dots, z_{2n-5}) + \dots \end{aligned}$$

From Lemma 6.2.1, $f_3 = 0$. So f has no terms which start with x_i^2 . Thus $\Delta(z_1, x_3)f = 0$ implies that $f_2 + f'_2 = 0$, and $\Delta(z_2, x_3)f = 0$ implies that f_2 is skew symmetric in the variables of degree 1. Since f is symmetric in the variables of degree 2 we have f_2 is also

symmetric in the variables of degree 2. Hence $f_2 \in VIN_{n-1}[2, 2, 1^{2n-5}]$. So by Lemma 6.2.1,

$$f_2 = \alpha(T_{2n-2}(x_1, x_2, x_3, z_2, \dots, z_{2n-5}; x_2)x_1 + T_{2n-2}(x_2, x_1, x_3, z_2, \dots, z_{2n-5}; x_1)x_2).$$

In f_2 the monomial $x_2x_3x_2z_2 \cdots z_{2n-5}x_1x_1$ has coefficient $-\alpha$. So the monomial $z_1x_3x_2x_3x_2z_2 \cdots z_{2n-5}x_1x_1$ has coefficient $-\alpha$ in f . But its reverse has coefficient 0 in f . So $\alpha = 0$ and hence $f_2 = 0$. Thus

$$f = x_2x_3f_1(x_1, x_2, x_3, z_1, \dots, z_{2n-5}) + x_3x_2f_1(x_1, x_3, x_2, z_1, \dots, z_{2n-5}) + \cdots$$

By this expression and $\Delta(z_1, x_2)f = \Delta(z_1, x_3)f = 0$ we have $f_1 \in VIN_{n-1}[2, 1^{2n-3}]$. Especially, f_1 is skew symmetric in x_2, x_3, z_i . Since $f^* = \pm f$ and f has no terms which start with z_iz_j, x_iz_j, z_jx_i we obtain that f_1 has no terms which end in z_iz_j, x_iz_j, z_jx_i . So by the symmetry of f_1 it also has no terms which end in x_iz_j . Thus $f_1 = 0$ and therefore $f = 0$. ■

From Corollary 6.2.2, Proposition 6.2.7, Proposition 6.2.8 and Proposition 6.2.9 we have Theorem 6.1.

6.3 The Proof of the Theorem 6.2

Lemma 6.3.1 *Let $f(x_1, \dots, x_s, y_1, \dots, y_t)$ be an identity of degree $2n+1$ on H_n which is skew symmetric in all variables y_i 's and has degree 1 in these y_i 's. If $t \geq 4$ and*

$$f = [y_1, y_2]f_1[y_{t-1}, y_t] + \sim,$$

where $\sim$ denotes the sum of the terms of f which do not start with one of y_1y_2, y_2y_1 or which do not end with one of $y_{t-1}y_t, y_t y_{t-1}$, then

1. f_1 is an identity on H_{n-2} .
2. If $f^* = \pm f$ then $f_1^* = \pm f_1$.

3. If $\Delta(y_i, x_j)f = 0$ then $\Delta(y_i, x_j)f_1 = 0$.

The proof is easy. We skip it. ■

Lemma 6.3.2 *The identities*

$$T_{2n}(y_1, \dots, y_{2n-2}, x; x), \tilde{T}_{2n}(y_1, \dots, y_{2n-2}, x; x)$$

*are *-skew symmetric.*

Proof. From the definition of T_{2n} and $\tilde{T}_{2n}$, $S_{2n} = T_{2n} + \tilde{T}_{2n}$. Thus

$$\begin{aligned} S_{2n}(x, y_1, \dots, y_{2n-2}, x) = \\ T_{2n}(y_1, \dots, y_{2n-2}, x; x) + \tilde{T}_{2n}(y_1, \dots, y_{2n-2}, x; x) \end{aligned}$$

So

$$T_{2n}(y_1, \dots, y_{2n-2}, x; x) + \tilde{T}_{2n}(y_1, \dots, y_{2n-2}, x; x) = 0. \quad (6.7)$$

On the other hand, by the definition of T_{2n} and $\tilde{T}_{2n}$, $T_{2n}^* = \tilde{T}_{2n}$ when n is even. Thus this equation and (6.7) imply that

$$T_{2n}(y_1, \dots, y_{2n-2}, x; x), \tilde{T}_{2n}(y_1, \dots, y_{2n-2}, x; x)$$

are *-skew symmetric. When n is odd, we always have that T_{2n} and $\tilde{T}_{2n}$ are both *-skew symmetric by their definitions. ■

Proposition 6.3.1 *The identity $T_{2n}(x^2, x, y_1, \dots, y_{2n-3}; x)$ forms a basis of the vector space $VI_n[4, 1^{2n-3}]$.*

Proof. Since $VI_n[4, 2, 1^{2n-5}] = \{0\}$ by Theorem 6.1, $VI_n[4, 1^{2n-3}] = VIN_n[4, 1^{2n-3}]$. Let $f(x, y_1, \dots, y_{2n-3}) \in VIN_n[4, 1^{2n-3}]$. Then

$$f = \sum_{i < j} (-1)^{i+j-1} [y_i, y_j] f_2(x, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_{2n-3}) + \sim$$

and $f_2 \in VIN_{n-1}[4, 1^{2n-5}]$. Therefore

$$f_2(x, y_3, \dots, y_{2n-3}) = \alpha T_{2n-2}(x^2, x, y_3, \dots, y_{2n-3}; x)$$

for some $\alpha \in F$. Subtracting a multiple of T_{2n} from f if necessary, we may assume that $f_2 = 0$. Since

$$T_{2n-2}(x^2, x, y_3, \dots, y_{2n-3}; x) = T_{2n-4}(x^2, x, y_3, \dots, y_{2n-5}; x)[y_{2n-4}, y_{2n-3}] + \sim,$$

$$f = \alpha[y_1, y_2]T_{2n-4}(x^2, x, y_3, \dots, y_{2n-5}; x)[y_{2n-4}, y_{2n-3}] + \sim.$$

Hence from Lemma 6.3.1 and Lemma 6.3.2, $\alpha = 0$ if $f^* = f$.

If $f^* = -f$ then

$$f - \alpha \tilde{T}_{2n}(x^2, x, y_1, \dots, y_{2n-3})$$

has no terms which start with $y_1 y_2$ therefore has no terms which start with $y_i y_j$ and is $\ast$ -skew symmetric. Thus we may assume that $f^* = \pm f$ and has no terms which start with $y_i y_j$. So by Lemma 6.2.1, f has no terms which start with $x^2 y_i$ and hence

$$f = \sum_{i=1}^{2n-3} (-1)^{i-1} x y_i \{f_1(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-3}) + y_i x \{f'_1(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-3})\}.$$

Since $\Delta(y_1, x)f \in VIN_n[5, 1^{2n-4}] = \{0\}$, $f_1 + f'_1 = 0$. So

$$f = \sum_{i=1}^{2n-3} (-1)^{i-1} [x, y_i] f_1(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-3}).$$

By Lemma 6.2.1,

$$f_1 = \alpha T_{2n-2}(x, y_{\sigma(2)}, \dots, y_{\sigma(2n-3)}; x)x$$

and

$$f = \alpha \sum_{i=1}^{2n-3} (-1)^{i-1} [x, y_i] T_{2n-2}(x, y_1, \dots, \hat{y}_i, \dots, y_{2n-3}; x)x. \quad (6.8)$$

Therefore f has no terms which end in xy_i by (6.8). But the monomial $y_1 x x y_2 x y_3 \cdots y_{2n-3} x$ has coefficient $-\alpha$. Thus $\alpha = 0$ by $f^* = \pm f$ and hence $f = 0$. ■

Proposition 6.3.2 *The identities*

$$\begin{aligned} T_{2n}(x^2, x, y, z_1, \dots, z_{2n-4}; y), \\ T_{2n}(x \circ y, x, y, z_1, \dots, z_{2n-4}; x) \end{aligned}$$

form a basis of the vector space $VIN_n[3, 2, 1^{2n-4}]$.

Proof. Let

$$\begin{aligned} t_1^n &:= T_{2n}(x^2, x, y, z_1, \dots, z_{2n-4}; y), \\ t_2^n &:= T_{2n}(x \circ y, x, y, z_1, \dots, z_{2n-4}; x). \end{aligned}$$

Then

$$\begin{aligned} t_1^n &= T_{2n-2}(x^2, x, y, z_1, \dots, z_{2n-6}; y)[z_{2n-5}, z_{2n-4}] + \sim \\ &= t_1^{n-1}[z_{2n-5}, z_{2n-4}] + \sim, \\ t_2^n &= T_{2n}(x \circ y, x, y, z_1, \dots, z_{2n-6}; y)[z_{2n-5}, z_{2n-4}] + \sim \\ &= t_2^{n-1}[z_{2n-5}, z_{2n-4}] + \sim. \end{aligned} \tag{6.9}$$

Since t_1^4, t_2^4 are linearly independent it can be shown that t_1^n, t_2^n are linearly independent for $n \geq 4$ by induction on n and (6.9). Thus we need only to show that t_1^n, t_2^n span $VIN_n[3, 2, 1^{2n-4}]$.

Let $f \in VIN_n[3, 2, 1^{2n-4}]$. We may assume that $f^* = \pm f$. Since f is skew symmetric in its variables of degree 1,

$$f = \sum_{1 \leq i < j \leq 2n-4} (-1)^{i+j-1} f_4(x, y, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_{2n-4})[z_i, z_j] + \sim, \tag{6.10}$$

where $f_4 \in VIN_{n-1}[3, 2, 1^{2n-6}]$. Hence by the induction $f_4 = \alpha t_1^{n-1} + \beta t_2^{n-1}$ for some $\alpha, \beta \in F$.

Next, we correct f by a suitable linear combination g of t_1^n and t_2^n so that $f - g$ has no terms which end in $z_i z_j$, and $f - g, f$ have the same symmetry.

Let

$$\begin{aligned}\tilde{l}_1^n &:= \tilde{T}_{2n}(x^2, x, y, z_1, \dots, z_{2n-4}; y), \\ \tilde{l}_2^n &:= \tilde{T}_{2n}(x \circ y, x, y, z_1, \dots, z_{2n-4}; x).\end{aligned}$$

Then

$$f_4 = [z_1, z_2](\alpha \tilde{l}_1^{n-2} + \beta \tilde{l}_2^{n-2}) + \sim,$$

and from (6.9)

$$f = [z_1, z_2](\alpha \tilde{l}_1^{n-2} + \beta \tilde{l}_2^{n-2})[z_{2n-5}, z_{2n-4}] + \sim.$$

Thus from Lemma 6.3.1, $\alpha \tilde{l}_1^{n-2} + \beta \tilde{l}_2^{n-2}$ and f have the same symmetry. Since $\tilde{l}_i^n = -l_i^n$ and l_i^n is $\ast$ -skew symmetric, $\alpha \tilde{l}_1^{n-2} + \beta \tilde{l}_2^{n-2}$ is $\ast$ -skew symmetric when α or $\beta \neq 0$. So if $f^\ast = f$ then $f_4 = 0$. That is f has no terms which end in $z_i z_j$. If $f^\ast = -f$, let $g := \alpha l_1^n + \beta l_2^n$ then g is $\ast$ -skew symmetric and $g \in VIN_n[3, 2, 1^{2n-4}]$. From (6.9) and (6.10), $f - g$ has no terms which end in $z_{2n-5} z_{2n-4}$ therefore has no terms which end in $z_i z_j$ by the skew symmetry of f and g in their variables of degree 1.

Now we may assume that f has no terms which end in $z_i z_j$ and $f^\ast = \pm f$. Applying Lemma 6.2.1 to $f^\ast$ we have

$$f =$$

$$\begin{aligned}& f_1(x, y, z_1, \dots, z_{2n-4})xy + f'_1(x, y, z_1, \dots, z_{2n-4})yx + \\& \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})xz_i + f'_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})z_i x) + \\& \sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})yz_i + f'_3(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})z_i y) + \\& f_5(x, y, z_1, \dots, z_{2n-4})yx^2.\end{aligned}\tag{6.11}$$

Since $\Delta(z_1, y)f \in VIN_n[3, 3, 1^{2n-5}] = \{0\}$, by considering the sum of the terms which end in yx^2 we have $\Delta(z_1, y)f_5 = 0$ from (6.11). This implies that f_5 is skew in y, z_1 , therefore skew in y, z_i . So, since $f^\ast = \pm f$ and has no terms which end in $z_i z_j$,

$$f_5 = \alpha \sum_{\sigma \in S_{2n-3}} xy_{\sigma(1)} \cdots y_{\sigma(2n-3)} + \beta \sum_{\sigma \in S_{2n-3}} y_{\sigma(1)}x \cdots y_{\sigma(2n-3)},$$

where $(y_1, \dots, y_{2n-3}) = (y, z_1, \dots, z_{2n-4})$.

We know that f_5 is an identity of degree $2n-2$ on $\tilde{H}_{n-1}$. We apply Lemma 3.3 of [MR] and consider for the monomial $y_1 x y_2 \cdots y_{2n-3}$ and the position $(i, j) = (2, 3)$. This yields $\beta = 0$. We apply Lemma 3.1 of [MR] to the monomial $y_1 x y_2 \cdots y_{2n-3}$. This gives $\alpha = 0$. So

$$\begin{aligned} f = & f_1(x, y, z_1, \dots, z_{2n-4})xy + f'_1(x, y, z_1, \dots, z_{2n-4})yx + \\ & \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})xz_i + f'_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})z_i x) + \\ & \sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})yz_i + f'_3(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})z_i y) \quad (6.12) \end{aligned}$$

Since $\Delta(z_i, x)f = 0$ and $\Delta(z_i, y)f = 0$, from (6.12) we have $f_i = -f'_i$ for $i = 2, 3$. We also have that $f_1 = -f'_1$. Indeed, from the proof of Proposition 6.3.1 we know that every identity g in $VI_n[4, 1^{2n-3}]$ which has no terms ending in $z_i z_j$ is zero identically if $g^* = \pm g$. Since f has no terms which end in $z_i z_j$ and $f^* = \pm f$, $\Delta(y, x)f$ has no terms which end in $z_i z_j$ and $(\Delta(y, x)f)^* = \Delta(y, x)f$. Thus $\Delta(y, x)f \in VI_n[4, 1^{2n-3}]$ implies that $\Delta(y, x)f = 0$, hence $f_1 + f'_1 = 0$ from (6.12). Thus

$$\begin{aligned} f = & f_1(x, y, z_1, \dots, z_{2n-4})[x, y] + \\ & \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})[x, z_i] + \\ & \sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})[y, z_i]. \quad (6.13) \end{aligned}$$

We claim that $f_3 = 0$. Indeed, from (6.13) and $\Delta(z_1, y)f = 0$ we have $f_3 \in VIN_{n-1}[3, 1^{2n-2}]$. Hence applying Lemma 6.2.1 to f^* yields $(f_3)^* = \alpha T_{2n-2}(x, y, z_1, \dots, z_{2n-4}; x)x$. Thus $f_3 = -\alpha x T_{2n-2}(x, y, z_1, \dots, z_{2n-4}; x)$ by Lemma 6.3.2. Thus the expression of f_3 , $f^* = \pm f$ and the fact that f has no terms which end in z^2 imply that $\alpha = 0$. So $f_3 = 0$, and

$$\begin{aligned} f = & f_1(x, y, z_1, \dots, z_{2n-4})[x, y] + \\ & \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \dots, \hat{z}_i, \dots, z_{2n-4})[x, z_i]. \quad (6.14) \end{aligned}$$

Now using the same argument as in the proof of the fact that f_3 is 0, one can show $f_2 = 0$ and get

$$f = f_1(x, y, z_1, \dots, z_{2n-1})[x, y]. \quad (6.15)$$

From (6.15) we know that f_1 is skew symmetric in y , and the z_i 's. Thus $f^* = \pm f$ and f has no terms which end in $z_i z_j$, $x z_i$ or $z_i x$ imply that every monomial in f_1 has coefficient zero. So $f_1 = 0$ and hence $f = 0$. ■

Let

$$\begin{aligned} t_1^n &:= T_{2n}(x^2, y_1, \dots, y_{2n-2}; x), \\ t_2^n &:= \sum_{i=1}^{2n-2} (-1)^{i-1} T_{2n}(x \circ y_i, y_1, \dots, \widehat{y_i}, \dots, y_{2n-2}; x), \\ t_3^n &:= \sum_{i=1}^{2n-2} (-1)^i T_{2n}(x^2, x, y_1, \dots, \widehat{y_i}, \dots, y_{2n-2}; y_i), \\ t_4^n &:= T_{2n}(x, y_1, \dots, y_{2n-2}; x^2), \\ t_5^n &:= T_{2n}(x, y_1, \dots, y_{2n-2}; x)x, \\ t_6^n &:= x T_{2n}(x, y_1, \dots, y_{2n-2}; x). \end{aligned} \quad (6.16)$$

Then

Proposition 6.3.3 *The set $\{t_1^n, \dots, t_6^n\}$ is a basis of the vector space $V/N_n[3, 1^{2n-2}]$.*

Proof. From the definition of t_i^n we have for $i = 1, 4, 6$

$$\begin{aligned} t_i^n &= t_i^{n-1}[y_{2n-3}, y_{2n-2}] + \sim, \\ t_2^n &= (t_2^{n-1} + t_5^{n-1})[y_{2n-3}, y_{2n-2}] + \sim, \\ t_3^n &= \begin{cases} t_3^{n-1}[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is even} \\ (t_3^{n-1} - 2S_{2n-2}(z^2, z, y_1, \dots, y_{2n-4}))[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} t_3^{n-1}[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is even} \\ (t_3^{n-1} - \frac{2}{n-1}(t_4^{n-1} - t_1^{n-1} - t_3^{n-1}))[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

[See the proof of part 4 in Lemma 6.2.1]. Using these formulas one can easily show that t_i^n for $i = 1, \dots, 6$ are linearly independent by induction on $n \geq 3$.

Next we show these t_i^n span $V/N_n[3, 1^{2n-2}]$. Let $\tilde{t}_i^n$ be the identity which is obtained by changing each T in t_i^n to be $\tilde{T}$. Then we have the following formulas. For $i = 1, 4, 6$

$$\begin{aligned} t_i^n &= [y_1, y_2] \tilde{t}_i^{n-1} + \sim \\ t_2^n &= [y_1, y_2] (\tilde{t}_2^{n-1} + \tilde{t}_6^{n-1}) + \sim \\ t_3^n &= [y_1, y_2] \begin{cases} (\tilde{t}_3^{n-1} - \frac{2}{n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is even} \\ (\tilde{t}_3^{n-1} - \frac{2}{1-n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Let $k_{n-1} := S_{2n-2}(x^2, x, y_3, \dots, y_{2n-2})$ then

$$k_{n-1} = -\frac{1}{e(n-1)}(t_1^{n-1} + t_3^{n-1} - t_4^{n-1}) \quad (6.17)$$

by the definition of the t_i^n and Lemma 1.4 of [MR] ($e(n-1) \neq 0$ since we assume that $n > 4$).

We only need to check t_3^n . The others are obvious. First, we have

$$\begin{aligned} \tilde{t}_i^{n-1} + t_i^{n-1} &= (-1)^i k_{2n-2} \text{ for } i = 1, 4, \\ \tilde{t}_3^{n-1} + t_3^{n-1} &= -(2n-4)k_{2n-2}. \end{aligned} \quad (6.18)$$

Indeed, from the definition of T_{2n} , $\tilde{T}_{2n}$ and t_i^n ,

$$\begin{aligned} T_{2n-2}(x^2, x, y_{\sigma(3)}, \dots, y_{\sigma(2n-2)}) + \tilde{T}_{2n-2}(x^2, x, y_{\sigma(3)}, \dots, y_{\sigma(2n-2)}) \\ &= S_{2n-2}(y_{\sigma(2n-2)}, x^2, x, y_{\sigma(3)}, \dots, y_{\sigma(2n-3)}) \\ &= -S_{2n-2}(x^2, x, y_{\sigma(3)}, \dots, y_{\sigma(2n-2)}) \\ &= -(-1)^\sigma S_{2n-2}(x^2, x, y_3, \dots, y_{2n-2}). \end{aligned}$$

Thus

$$\tilde{t}_3^{n-1} + t_3^{n-1} = -(2n-4)k_{n-1}.$$

The others in (6.18) are obvious. By the definition of t_3^n we have

$$t_3^n = [y_1, y_2](\tilde{t}_3^{n-1} - 2k_{n-1}) + \sim.$$

On the other hand by (6.17), (6.17), we have

$$\sum_{i=1,3,4} (-1)^{i-1} (t_i^{n-1} + \tilde{t}_i^{n-1}) = (-2(n-1))k_{n-1}$$

This formula and (6.17) imply

$$\begin{aligned} k_{n-1} &= \frac{1}{e(n-1) - 2(n-1)} (\tilde{t}_1^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) \\ &= \begin{cases} \frac{1}{2-n} (\tilde{t}_1^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) & \text{if } n \text{ is even} \\ \frac{1}{1-n} (\tilde{t}_1^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \quad (6.19)$$

Thus (6.19) implies

$$t_3^n = [y_1, y_2] \begin{cases} (\tilde{t}_3^{n-1} - \frac{2}{n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is even} \\ (\tilde{t}_3^{n-1} - \frac{2}{1-n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is odd.} \end{cases}$$

We also claim that

(*) $\tilde{t}_i^n$ for $i = 1, \dots, 6$ is a basis of $VIN_m[3, 1^{2n-2}]$, for $4 \leq m < n$.

Indeed, this follows from (6.18), (6.19) and the induction hypothesis.

Now let $f(x, y_1, y_2, \dots, y_{2n-3}) \in VIN_n[3, 1^{2n-2}]$ and $f^* = \pm f$. Then

$$\begin{aligned} f &= \sum (f_1(x, y_1, \dots, y_{2n-3}) x y_{2n-2} + f'_1(x, y_1, \dots, y_{2n-3}) y_{2n-2} x) + \\ &\quad \sum f_2(x, y_1, \dots, y_{2n-4}) [y_{2n-3}, y_{2n-2}] + \\ &\quad \sum f_3(x, y_1, \dots, y_{2n-3}) y_{2n-2} x^2, \end{aligned}$$

where $f_2(x, y_1, \dots, y_{2n-4}) \in VIN_{n-1}[3, 1^{2n-4}]$ and $f_3(x, y_1, \dots, y_{2n-3}) \in VIN_{n-1}[1^{2n-2}]$.

Thus, there exist $\alpha_i \in F$ such that

$$\begin{aligned} f_2 &= \sum_{i=1}^6 \alpha_i \tilde{t}_i^{n-1} \\ &= [y_1, y_2] \{ \alpha_1 \tilde{t}_1^{n-2} + \alpha_2 (\tilde{t}_2^{n-2} + \tilde{t}_6^{n-2}) + \alpha_3 (\tilde{t}_3^{n-2} - \frac{2}{e(n-1) - 2(n-1)} (\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2})) + \\ &\quad \alpha_4 \tilde{t}_4^{n-2} + \alpha_5 \tilde{t}_5^{n-2} \} + \sim. \end{aligned} \quad \text{by (6.17)}$$

So by Lemma 6.3.1, $f^* = \pm f$ implies that $g_2^* = \pm g_2$, where

$$\begin{aligned} g_2 &:= \alpha_1 \tilde{t}_1^{n-2} + \alpha_2 (\tilde{t}_2^{n-2} + \tilde{t}_6^{n-2}) + \alpha_4 \tilde{t}_4^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \\ &\quad \alpha_3 (\tilde{t}_3^{n-2} - (\frac{2}{e(n-2) - 2(n-2)} (\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2}))) \end{aligned}$$

$$\begin{aligned}
&= (\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_1^{n-2} + \alpha_2 \tilde{t}_2^{n-2} + \\
&\quad (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_3^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \\
&\quad (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_4^{n-2} + \alpha_2 \tilde{t}_6^{n-2}.
\end{aligned} \tag{6.20}$$

When n is odd,

$$(t_i^n)^* = -t_i^n, i = 1, 2, 3, 4 \quad (t_5^n)^* = -t_5^n, (t_6^n)^* = -t_5^n,$$

and

$$(\tilde{t}_i^n)^* = -\tilde{t}_i^n, i = 1, 2, 3, 4 \quad (\tilde{t}_5^n)^* = -\tilde{t}_6^n, (\tilde{t}_6^n)^* = -\tilde{t}_5^n,$$

by the definition of $\tilde{T}_{2n}$. Thus

$$\begin{aligned}
g_2^* &= -((\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_1^{n-2} + \alpha_2 \tilde{t}_2^{n-2} + \alpha_5 \tilde{t}_6^{n-2} + \alpha_2 \tilde{t}_5^{n-2} + \\
&\quad (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_3^{n-2} + (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_4^{n-2}).
\end{aligned}$$

Hence by claim (*), $g^* = \pm g$ implies

$$\begin{aligned}
\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3 &= \mp (\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3), \\
\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3 &= \mp (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3), \\
\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3 &= \mp (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3), \\
\alpha_2 &= \mp \alpha_2 = \mp \alpha_5.
\end{aligned}$$

That is, if n is odd then:

1. When $f^* = f$, $\alpha_i = 0$, for $i = 1, \dots, 5$. So $f_2 = \alpha_6 t_6^{n-1}$.
2. When $f^* = -f$, $\alpha_2 = \alpha_5$. So $f_2 = \sum_{i \neq 2,5} \alpha_i t_i^{n-1} + \alpha_2 (t_2^{n-1} + t_5^{n-1})$.

Let $g := \alpha_6(t_6^n - t_5^n)$ if $f^* = f$. Then $g = f_2[y_{2n-3}, y_{2n-2}] + \sim$ by (6.17). g is $*$ -symmetric and $f - g$ is $*$ -symmetric and has no terms which end in $y_i y_j$. If $f^* = -f$, letting

$$g := (\alpha_1 - \frac{2\alpha_3}{n+1}) t_1^n + \alpha_2 t_2^n + \frac{(n-1)\alpha_3}{n+1} t_3^n + (\alpha_4 + \frac{2\alpha_3}{n+1}) t_4^n + \alpha_6(t_6^n + t_5^n),$$

then $g = f_2[y_{2n-3}, y_{2n-2}] + \sim$ by (6.17); g is $\ast$ -skew symmetric and $f - g$ is $\ast$ -symmetric and has no terms which end in $y_i y_j$. Thus we may assume that $f^\ast = \pm f$ and f has no terms which end in $y_i y_j$ when n is odd.

Next we discuss the case when n is even. If n is even then

$$(t_2^n)^\ast = -t_2^n, (t_5^n)^\ast = -t_6^n.$$

Since n is even, $T_{2n}^\ast = \tilde{T}_{2n}$. Thus by (6.18) we have

$$\begin{aligned} (t_1^n)^\ast &= -k_n - t_1^n, (t_4^n)^\ast = k_n - t_4^n, \\ (t_3^n)^\ast &= -(2(n-1))k_n - t_3^n, \end{aligned}$$

where $k_n = S_{2n}(x^2, x, y_1, \dots, y_{2n-2})$. So, in the vector space $\text{Span}\{t_i^n | i = 1, \dots, 6\}$ every $\ast$ -symmetric element is a linear combination of the elements $t_1^n + t_3^n - t_4^n$, $t_5^n - t_6^n$ and every $\ast$ -skew symmetric element is a linear combination of the elements

$$t_3^n - 2(n-1)t_1^n, t_1^n + t_4^n, t_5^n + t_6^n, t_2^n.$$

If n is even then $(\tilde{t}_2^n)^\ast = -\tilde{t}_2^n$, $(\tilde{t}_5^n)^\ast = -\tilde{t}_6^n$. Since n is even, $T_{2n}^\ast = \tilde{T}_{2n}$. Thus by (6.18) and (6.19) we have

$$\begin{aligned} (\tilde{t}_1^n)^\ast &= -k_n - \tilde{t}_1^n \\ &= \frac{-1}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_1^n, \\ (\tilde{t}_3^n)^\ast &= -(2(n-1))k_n - \tilde{t}_3^n \\ &= \frac{-2(n-1)}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_3^n, \\ (\tilde{t}_4^n)^\ast &= k_n - \tilde{t}_4^n \\ &= \frac{1}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_4^n. \end{aligned}$$

From (6.20),

$$\begin{aligned} g_2^\ast &= (\alpha_1 - \frac{2}{e(n-2) - 2(n-2)}\alpha_3)\{\frac{-1}{e(n-2) - 2(n-2)}(\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2}) - \tilde{t}_1^{n-2}\} + \\ &\quad (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)}\alpha_3)\{\frac{-2(n-3)}{e(n-2) - 2(n-2)}(\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2}) - \tilde{t}_3^{n-2}\} + \\ &\quad (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)}\alpha_3)\{\frac{1}{e(n-2) - 2(n-2)}(\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2}) - \tilde{t}_4^{n-2}\} - \\ &\quad \alpha_2 \tilde{t}_2^{n-2} - \alpha_5 \tilde{t}_6^{n-2} - \alpha_2 \tilde{t}_5^{n-2}. \end{aligned}$$

Since $n - 2$ is even, $e(n - 2) = n - 2$. Let

$$A_1 := \alpha_1 + \frac{2}{n-2}\alpha_3, \quad A_3 := \alpha_3 + \frac{2}{n-2}\alpha_3, \quad A_4 := \alpha_4 - \frac{2}{n-2}\alpha_3,$$

then by claim (*), $g_2^* = \pm g_2$ implies that

$$\begin{aligned} \left(\frac{1}{n-2} - 1\right)A_1 + \frac{2n-6}{n-2}A_3 - \frac{1}{n-2}A_4 &= \pm A_1, \\ \frac{1}{n-2}A_1 + \left(\frac{2n-6}{n-2} - 1\right)A_3 - \frac{1}{n-2}A_4 &= \pm A_3, \\ -\frac{1}{n-2}A_1 - \frac{2n-6}{n-2}A_3 + \left(\frac{1}{n-2} - 1\right)A_4 &= \pm A_4, \\ -\alpha_2 &= \pm \alpha_2 = \pm \alpha_5. \end{aligned} \tag{6.21}$$

Thus if $f^* = f$ then $g_2^* = g_2$, solving (6.21) yields

$$\alpha_2 = \alpha_5 = 0 = A_1 = A_3 = A_4.$$

Therefore $\alpha_i = 0$ for $i = 1, \dots, 5$, i.e. $f_2 = \alpha_6 t_6^{n-1}$ when $f^* = f$. Let $g := \alpha_6(t_6^n - t_5^n)$ then g is $\star$ -symmetric and $g = f_2[y_{2n-3}, y_{2n-2}] + \sim$. Thus $f - g$ has no terms which end in $y_i y_j$.

If $f^* = -f$ then $g_2^* = g_2$. Solving (6.21) yields

$$\alpha_2 = \alpha_5, \quad A_1 + (2n-6)A_3 - A_4 = 0.$$

That is

$$\alpha_2 = \alpha_5, \quad \alpha_4 = \alpha_1 + \frac{(2n-6)n+4}{n-2}\alpha_3$$

and

$$f_2 = \alpha_1 t_1^{n-1} + \alpha_2(t_2^{n-1} + t_5^{n-1})\alpha_3 t_3^{n-1}(\alpha_1 + \frac{(2n-6)n+4}{n-2}\alpha_3)t_4^{n-1} + \alpha_6 t_6^{n-1}.$$

Let

$$g = \alpha_1(t_1^n + t_4^n) + \alpha_2 t_2^n + \alpha_3((2-2n)t_1^n + t_3^n) + \alpha_6(t_5^n + t_6^n),$$

then g is $\star$ -skew symmetric and

$$\begin{aligned} g &= (\alpha_1(t_1^{n-1} + t_4^{n-1}) + \alpha_2(t_2^{n-1} + t_5^{n-1}) + \\ &\quad \alpha_3((2-2n)t_1^{n-1} + t_3^{n-1}) + \alpha_6 t_6^{n-1})[y_{2n-3}, y_{2n-2}] + \sim. \end{aligned}$$

Thus by (6.17)

$$\begin{aligned} f - g &= \gamma t_4^{n-1} [y_{2n-3}, y_{2n-2}] + \sim \\ &= [y_1, y_2] \gamma t_4^{n-2} [y_{2n-3}, y_{2n-2}] + \sim. \end{aligned}$$

Since $f - g$ is $\ast$ -skew symmetric, γt_4^{n-2} is $\ast$ -skew symmetric by Lemma 6.3.1. This implies that $\gamma = 0$ because t_4^{n-2} is not $\ast$ -skew symmetric when $n - 2$ is even.

Since $f - g$ is skew symmetric in the variables y_i 's, $f - g$ has no terms which end in $y_i y_j$. We may assume that f has this property. So we have reduced the proof to the case $f^\ast = \pm f$ and f has no terms which end in $y_i y_j$. Therefore from Lemma 6.2.1,

$$f = \sum_{i=1}^{2n-2} (-1)^i (f_1(x, y_1, \dots, \widehat{y}_i, \dots, y_{2n-2}) x y_i + f'_1(x, y_1, \dots, \widehat{y}_i, \dots, y_{2n-2}) y_i x). \quad (6.22)$$

Since f has no terms which end in $y_i y_j$, $\Delta(y_1, x)f$ has no such terms either. However $\Delta(y_1, x)f \in VIN_n[4, 1^{2n-3}]$. Thus $\Delta(y_1, x)f = 0$ by the proof of Proposition 6.3.1. Applying $\Delta(y_1, x)$ to formula (6.22) yields $f_1 = -f'_1$. Thus from (6.22) and $f^\ast = \pm f$,

$$f = \sum_{i < j}^{2n-2} (\pm 1) [x, y_i] h(x, y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_{2n-2}) [x, y_j]. \quad (6.23)$$

Since $\Delta(y_1, x)f = 0$ and f is skew in the variables y_i 's, h is alternating by (6.23), therefore $h = \beta S_{2n-3}(x, y_2, \dots, y_{2n-3})$ for some $\beta \in F$ and

$$f = \beta [x, y_1] S_{2n-3}(x, y_2, \dots, y_{2n-3}) [x, y_{2n-2}] + \sim.$$

But

$$S_{2n-3}(x, y_2, \dots, y_{2n-4}, 1) = S_{2n-4}(x, y_2, \dots, y_{2n-4})$$

from [DR]. So

$$\begin{aligned} f(x, y_1, \dots, y_{2n-4}, 1, y_{2n-2}) &= \\ \beta [x, y_1] S_{2n-3}(x, y_2, \dots, y_{2n-3}) [x, y_{2n-2}] &+ \sim. \end{aligned}$$

However,

$$f(x, y_1, \dots, y_{2n-4}, 1, y_{2n-2}) \in VIN_n[3, 1^{2n-3}] = \{0\},$$

so $\beta = 0$ and hence $f = 0$. Therefore t_i^n for $i = 1, \dots, 6$ is a basis of $VIN_n[3, 1^{2n-2}]$. ■

Let

$$\begin{aligned}
t_1^n &:= T_{2n}(x^2, y, z_1, \dots, z_{2n-3}; y) + T_{2n}(y^2, x, z_1, \dots, z_{2n-3}; x), \\
t_2^n &:= T_{2n}(x \circ y, x, z_1, \dots, z_{2n-3}; y) + T_{2n}(x \circ y, y, z_1, \dots, z_{2n-3}; x), \\
t_3^n &:= \sum_{i=1}^{2n-3} (-1)^{i-1} (T_{2n}(x \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_{2n-3}, x, y; y) + \\
&\quad T_{2n}(y \circ z_i, z_1, \dots, \hat{z}_i, \dots, z_{2n-3}, y, x; x)), \\
t_4^n &:= xT_{2n}(x, y, z_1, \dots, z_{2n-3}; y) + yT_{2n}(y, x, z_1, \dots, z_{2n-3}; x), \\
t_5^n &:= T_{2n}(x, y, z_1, \dots, z_{2n-3}; y)x + T_{2n}(y, x, z_1, \dots, z_{2n-3}; x)y, \\
t_6^n &:= T_{2n}(x, y, z_1, \dots, z_{2n-3}; [x, y]), \quad t_7^n := \tilde{T}_{2n}(x, y, z_1, \dots, z_{2n-3}; [x, y]).
\end{aligned}$$

Then we have

Proposition 6.3.4 $\{t_1^n, \dots, t_7^n\}$ is a basis of the vector space $VIN_n[2, 2, 1^{2n-3}]$.

Proof. From the definition of t_i^n we have the following formulas:

$$\begin{aligned}
t_i^n &= t_i^{n-1}[z_{2n-4}, z_{2n-3}] + \sim \text{ for } i = 1, 2, 4, 6, 7, \\
t_3^n &= (t_3^{n-1} + t_5^{n-1})[z_{2n-4}, z_{2n-3}] + \sim.
\end{aligned} \tag{6.24}$$

Since $t_1^4, \dots, t_7^4$ are linear independent, using (6.24) one can show that $t_1^n, \dots, t_7^n$ are linear independent by induction on n .

As in the proof of Proposition 6.3.3 we define $\tilde{t}_i^n$ to be the identity obtained by changing each T in t_i^n to be $\tilde{T}$. Since S_k is alternating,

$$\tilde{t}_i^n + t_i^n = 0, \text{ for } i = 1, 2, 3, 4, 5.$$

Thus

$$\tilde{t}_i^n = -t_i^n, \text{ for } i = 1, 2, 3, 4, 5, \quad \tilde{t}_6^n = t_7^n, \quad \tilde{t}_7^n = t_6^n. \tag{6.25}$$

From the definition of t_i^n and $\tilde{t}_i^n$ we also have the formulas

$$\begin{aligned}
t_i^n &= \tilde{t}_i^{n-1}[z_{2n-4}, z_{2n-3}] + \sim, \text{ for } i = 1, 2, 5, \\
t_3^n &= (\tilde{t}_3^{n-1} + \tilde{t}_4^{n-1})[z_{2n-4}, z_{2n-3}] + \sim, \\
t_6^n &= \tilde{t}_7^{n-1}[z_{2n-4}, z_{2n-3}] + \sim, \quad t_7^n = \tilde{t}_6^{n-1}[z_{2n-4}, z_{2n-3}] + \sim.
\end{aligned} \tag{6.26}$$

Let $f \in VIN_n[2, 2, 1^{2n-3}]$ then

$$\begin{aligned}
f = & f_1(x_1, x_2, z_1, \dots, z_{2n-3})x_1x_2 + f_1(x_2, x_1, z_1, \dots, z_{2n-3})x_2x_1 + \\
& f_2(x_1, x_2, z_2, \hat{z}_1, \dots, z_{2n-3})z_2x_2 + f_2'(x_1, x_2, z_2, \hat{z}_1, \dots, z_{2n-3})x_2z_2 + \dots \\
& + f_3(x_1, x_2, z_1, \dots, z_{2n-5})[z_{2n-4}, z_{2n-3}] + \dots \\
& + f_4(x_1, z_1, \dots, z_{2n-3})x_1x_2^2 + f_4(x_2, z_1, \dots, z_{2n-3})x_2x_1^2 + \\
& + f_5(x_1, z_1, \dots, z_{2n-4})z_{2n-3}x_2^2 + \dots.
\end{aligned} \tag{6.27}$$

So by the induction hypothesis $f_3 = \sum_{i=1}^7 \alpha_i t_i^{n-1}$. Hence, from (6.25) and (6.26)

$$\begin{aligned}
f_3 &= [z_1, z_2] \left\{ \sum_{i=1,2,5} \alpha_i \tilde{t}_i^{n-2} + \alpha_3(\tilde{t}_3^n + \tilde{t}_4^{n-2}) + \alpha_6 t_7^{n-2} + \alpha_7 t_6^{n-2} \right\} + \sim \\
&= [z_1, z_2] \left\{ \sum_{i=1,2,5} -\alpha_i t_i^{n-2} - \alpha_3(t_3^n + t_4^{n-2}) + \alpha_6 t_7^{n-2} + \alpha_7 t_6^{n-2} \right\} + \sim.
\end{aligned}$$

Let

$$g_3 := \sum_{i=1,2,5} -\alpha_i t_i^{n-2} - \alpha_3(t_3^n + t_4^{n-2}) + \alpha_6 t_7^{n-2} + \alpha_7 t_6^{n-2} + \sim.$$

We may assume that $f^* = \pm f$. To complete the proof we shall argue in two cases.

Case 1. n is odd. When n is odd $T_{2n}^* = -T_{2n}$ and $\tilde{T}_{2n}^* = -\tilde{T}_{2n}$ imply

$$(\tilde{t}_i^n)^* = -t_i^n, \text{ for } i = 1, 2, 3, 6, 7,$$

and

$$(t_4^n)^* = -t_5^n, (t_5^n)^* = -t_4^n.$$

Thus from Lemma 6.3.1, $f^* = \pm f$ implies that $g_3^* = \pm g_3$. Since

$$g_3 := \sum_{i=1,2,5} -\alpha_i t_i^{n-2} - \alpha_3(t_3^n + t_4^{n-2}) + \alpha_6 t_7^{n-2} + \alpha_7 t_6^{n-2} + \sim,$$

$$g_3^* = \alpha_1 t_1^{n-2} + \alpha_2 t_2^{n-2} + \alpha_3(t_3^n + t_5^{n-2}) + \alpha_5 t_4^{n-2} - \alpha_6 t_7^{n-2} - \alpha_7 t_6^{n-2} + \sim.$$

Thus $f^* = \pm f$ implies that

$$\alpha_i = \mp \alpha_i \text{ for } i = 1, 2, 3, 6, 7, \alpha_5 = \mp \alpha_3. \tag{6.28}$$

So if $f^* = f$ then from (6.28), $\alpha_i = 0$ for $i = 1, 2, 3, 5, 6, 7$ and $f_3 = \alpha_4 t_4^{n-1}$. Let $g := \alpha_4(t_4^n - t_5^n)$ then g is also $*$ -symmetric and $g = \alpha_4 t_4^{n-1}[z_{2n-4}, z_{2n-3}] + \sim$. Hence $f - g$ is $*$ -symmetric and has no terms which end in $z_i z_j$. If $f^* = -f$ then $\alpha_3 = \alpha_5$ by (6.28). So let $g := \sum_{i \neq 4} \alpha_i t_i^n + \alpha_4(t_4^n + t_5^n)$. Then g is $*$ -skew symmetric and

$$g = \left\{ \sum_{i \neq 3,5} \alpha_i t_i^{n-1} + \alpha_3(t_3^n + t_5^n) \right\} [z_{2n-4}, z_{2n-3}] + \sim.$$

Hence $f - g$ has no terms which end in $z_i z_j$. So we may assume that f has no terms which end in $z_i z_j$ and $f^* = \pm f$ when n is odd.

Case 2. n is even. Using the same argument as in case 1, we have

$$f_3 = \begin{cases} \alpha_4 t_4^{n-1} + \alpha_6(t_6^{n-1} + t_7^{n-1}) & \text{if } f^* = f \\ \alpha_3(t_3^{n-1} + t_5^{n-1}) + \alpha_6(t_6^{n-1} - t_7^{n-1}) & \text{if } f^* = -f. \end{cases}$$

Thus let

$$g_3 = \begin{cases} \alpha_4(t_4^{n-1} - t_5^{n-1}) + \alpha_6(t_6^{n-1} + t_7^{n-1}) & \text{if } f^* = f \\ \alpha_3(t_3^{n-1} + \alpha_6(t_6^{n-1} - t_7^{n-1})) & \text{if } f^* = -f. \end{cases}$$

Then $f - g$ has no terms which end in $z_i z_j$ in both $f^* = \pm f$ and it has the same $*$ -symmetry as f .

Now we assume that f has no terms which end in $z_i z_j$. By Lemma 6.2.1, $f_4 = f_5 = 0$. Thus (6.27) becomes

$$\begin{aligned} f = & f_1(x_1, x_2, z_1, \dots, z_{2n-3})x_1x_2 + f_1(x_2, x_1, z_1, \dots, z_{2n-3})x_2x_1 + \\ & f_2(x_1, x_2, z_2, \hat{z}_i, \dots, z_{2n-3})z_1x_2 + f_2'(x_1, x_2, z_2, \hat{z}_i, \dots, z_{2n-3})x_2z_1 + \dots \end{aligned} \quad (6.29)$$

Since $\Delta(z_1, x_2)f \in VIN_n[3, 2, 1^{2n-4}]$, it has no terms which end in $z_i z_j$ and is $*$ -symmetric or $*$ -skew symmetric because f has these properties, $\Delta(z_1, x_2)f = 0$ by the proof of Proposition 6.3.2. Thus $f_2 = -f_2'$ by (6.29). Since f has no terms which end in $z_i z_j$, $\Delta(z_2, x_2)f = 0$ implies that $\Delta(z_2, x_2)f_2 = 0$ by (6.29). So f_2 is skew symmetric in all its variables of degree 1, i.e $f_2(x_1, x_2, z_2, \dots, z_{2n-3})$ is skew symmetric in the variables x_2 and z_i 's. Thus $f^* = \pm f$ and the fact that f has no terms which end or start with $z_i z_j$ imply f_2 has no terms which start with $z_i z_j$. But f_2 is skew symmetric in x_2 and z_i 's, f_2 has no terms which start with

$x_2 z_i$ or $z_i x_2$. So

$$\begin{aligned} f_2(x_1, x_2, z_2, \dots, z_{2n-3}) &= [x_1, x_2] h_2(x_1, z_2, \dots, z_{2n-3}) + \\ &\quad \sum_{i=2}^{2n-3} (-1)^{i-1} [x_1, z_i] h_2(x_1, x_2, z_2, \dots, \hat{z}_i, \dots, z_{2n-3}). \end{aligned}$$

Thus from (6.29)

$$\begin{aligned} f &= f_1(x_1, x_2, z_1, \dots, z_{2n-3}) x_1 x_2 + f_1(x_2, x_1, z_1, \dots, z_{2n-3}) x_2 x_1 + \\ &\quad [x_1, x_2] h_2(x_1, z_2, \dots, z_{2n-3}) [z_1, x_2] - \\ &\quad [x_1, z_2] h_2(x_1, x_2, z_3, \dots, z_{2n-3}) [z_1, x_2] + \dots. \end{aligned} \quad (6.30)$$

Since f has no terms which start with $[z_2, x_2]$ and end in $[z_1, x_2]$ and $\Delta(z_2, x_1)f = 0$, by considering the sum of the terms in $\Delta(z_2, x_1)f$ which start with $[x_1, x_2]$ and end in $[z_1, x_2]$ yields $\Delta(z_2, x_1)h_2 = 0$. Thus h_2 is alternating, hence there exists a scalar $\gamma \in F$ such that $h_2 = \gamma S_{2n-3}(x_1, x_2, z_2, \dots, z_{2n-3})$. Next, we show that $\gamma = 0$. Since H_n has no identity of degree $2n$ with type lower than $[2, 1^{2n-2}]$ for $n \geq 4$, $f(x_1, x_2, z_1, 1, z_3, \dots, z_{2n-3}) \in VIN_n[2, 2, 1^{2n-4}] = \{0\}$. So $h_2(x_1, 1, z_3, \dots, z_{2n-3}) = 0$ by (6.30). This implies that $\gamma = 0$ because

$$S_{2n-3}(x_1, 1, z_3, \dots, z_{2n-3}) = S_{2n-4}(x_1, z_3, \dots, z_{2n-3}).$$

Thus

$$f = f_1(x_1, x_2, z_1, \dots, z_{2n-3}) x_1 x_2 + f_1(x_2, x_1, z_1, \dots, z_{2n-3}) x_2 x_1,$$

where f_1 is multilinear. However this equation and $\Delta(z_1, x_i)f = 0$ imply that $\Delta(z_1, x_i)f_1 = 0$. So f_1 is alternating. But f has no terms which start with $z_i z_j$ so does f_1 . This property and the fact that f_1 is alternating imply $f_1 = 0$. So $f = 0$. ■

Let

$$\begin{aligned} t_1^n &:= T_{2n}(y_1, \dots, y_{2n-1}; x^2), \quad t_2^n := \tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x^2), \\ t_3^n &:= \sum_{(1, \dots, 2n-1)} T_{2n}(x \circ y_1, \dots, y_{2n-1}; x), \\ t_4^n &:= \sum_{(1, \dots, 2n-1)} \tilde{T}_{2n}(x \circ y_1, \dots, y_{2n-1}; x), \\ t_5^n &:= \sum_{(1, \dots, 2n-1)} T_{2n}(x, y_1, \dots, y_{2n-2}; x \circ y_{2n-1}), \end{aligned}$$

$$\begin{aligned}
t_6^n &:= \sum_{(1, \dots, 2n-1)} \tilde{T}_{2n}(x, y_1, \dots, y_{2n-2}; x \circ y_{2n-1}), \\
t_7^n &:= x T_{2n}(y_1, \dots, y_{2n-1}; x), \quad t_8^n := \tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x)x, \\
t_9^n &:= x \tilde{T}_{2n}(y_1, \dots, y_{2n-1}; x), \quad t_{10}^n := T_{2n}(y_1, \dots, y_{2n-1}; x)x, \\
t_{11}^n &:= \sum_{(1, \dots, 2n-1)} y_1 T_{2n}(x, y_2, \dots, y_{2n-1}; x), \\
t_{12}^n &:= \sum_{(1, \dots, 2n-1)} \tilde{T}_{2n}(x, y_1, \dots, y_{2n-2}; x) y_{2n-1}, \\
t_{13}^n &:= \sum_{(1, \dots, 2n-1)} y_1 \tilde{T}_{2n}(x, y_2, \dots, y_{2n-1}; x), \\
t_{14}^n &:= \sum_{(1, \dots, 2n-1)} T_{2n}(x, y_1, \dots, y_{2n-2}; x) y_{2n-1}, \\
t_{15}^n &:= \sum_{(1, \dots, 2n-1)} T_{2n}(x, y_1, \dots, y_{2n-2}; [x, y_{2n-1}]), \\
t_{16}^n &:= \sum_{(1, \dots, 2n-1)} \tilde{T}_{2n}(x, y_1, \dots, y_{2n-2}; [x, y_{2n-1}]).
\end{aligned}$$

Then

$$\begin{aligned}
t_3^n &= (t_3^{n-1} + t_{10}^{n-1})[y_{2n-2}, y_{2n-1}] + \sim, \\
t_4^n &= (t_4^{n-1} + t_8^{n-1})[y_{2n-2}, y_{2n-1}] + \sim, \\
t_i^n &= t_i^{n-1}[y_{2n-2}, y_{2n-1}] + \sim,
\end{aligned} \tag{6.31}$$

where

$$i = \begin{cases} 1, 2, 5, 7, 9, 11, 15 & \text{if } n \text{ is even} \\ 1, 2, 6, 7, 9, 11, 16 & \text{if } n \text{ is odd,} \end{cases}$$

$$t_{12}^n = \begin{cases} (t_{12}^{n-1} + t_{10}^{n-1})[y_{2n-2}, y_{2n-1}] + \sim & \text{if } n \text{ is even} \\ (t_{12}^{n-1} - t_8^{n-1})[y_{2n-2}, y_{2n-1}] + \sim & \text{if } n \text{ is odd.} \end{cases}$$

We also have

$$\begin{aligned}
t_1^n &= [y_1, y_2] t_2^{n-1} + \sim, \quad t_2^n = [y_1, y_2] t_1^{n-1} + \sim, \\
t_3^n &= [y_1, y_2] (t_4^{n-1} + t_9^{n-1}) + \sim, \quad t_4^n = [y_1, y_2] (t_3^{n-1} + t_7^{n-1}) + \sim, \\
t_5^n &= [y_1, y_2] (t_6^{n-1} - t_7^{n-1} - t_9^{n-1}) + \sim, \quad t_8^n = [y_1, y_2] t_{10}^{n-1} + \sim, \\
t_{10}^n &= [y_1, y_2] t_8^{n-1} + \sim, \quad t_{11}^n = [y_1, y_2] (t_7^{n-1} - t_{11}^{n-1}) + \sim, \\
t_{12}^n &= -[y_1, y_2] t_{12}^{n-1} + \sim, \quad t_{16}^n = [y_1, y_2] t_{15}^{n-1} + \sim, \\
t_{15}^n &= [y_1, y_2] (t_{16}^{n-1} + t_7^{n-1} + t_3^{n-1}) + \sim.
\end{aligned}$$

Thus we can show that

$$t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n = 0. \quad (6.32)$$

Indeed, for $n = 4$ this holds [see the proof of Proposition 5.3.5]. Suppose it is also true for $n - 1$ then for n

$$\begin{aligned} t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n = \\ xy_{2n-1}f_1 + y_{2n-1}xf_1' + \dots \\ + [y_1, y_2](t_7^{n-1} + t_8^{n-1} + t_9^{n-1} + t_{10}^{n-1} + t_{15}^{n-1} + t_{16}^{n-1}) + \dots \\ + \alpha x^2 S_{2n-1}(y_1, \dots, y_{2n-1}). \end{aligned}$$

Let

$$g := t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n.$$

Then by the induction hypothesis

$$g = xy_{2n-1}f_1 + y_{2n-1}xf_1' + \dots + \alpha x^2 S_{2n-1}(y_1, \dots, y_{2n-1}).$$

From this equation we know that there are no terms which start with $y_i y_j$ while this implies that $\alpha = 0$ since g is $\ast$ -symmetric. By the definitions of t_i^n ,

$$\begin{aligned} t_7^n + t_8^n + t_9^n + t_{10}^n = \\ xS_{2n-1}(x, y_1, \dots, y_{2n-1}) + S_{2n-1}(x, y_1, \dots, y_{2n-1})x \end{aligned}$$

Thus $\Delta(y_{2n-1}, x)(t_7^n + t_8^n + t_9^n + t_{10}^n) = 0$. Obviously $\Delta(t_{15}^n + t_{16}^n) = 0$, thus $\Delta(y_{2n-2}, x)g = 0$. So, by (6.32), $f_1 + f_1' = 0$ and

$$g = [x, y_{2n-1}]f_1 + \dots$$

Again, $\Delta(y_{2n-1}, x)g = 0$ implies $\Delta(y_1, x)g = 0$. Thus $\Delta(y_1, x)f_1 = 0$. So $f_1 = \beta S_{2n-1}(x, y_1, \dots, y_{2n-2})$ for some $\beta \in F$. Therefore the fact that g has no terms which start in $y_i y_j$ and $g^\ast = g$ imply that $\beta = 0$, hence $g = 0$.

It can be shown by induction on n and the formulas above that both $\{t_i^n | i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\}$ and $\{t_i^n | i = 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 16\}$ are linearly independent if $n \geq 4$.

Proposition 6.3.5 *The set $\{t_i^n | i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\}$ is a basis of the space $VIN_n[2, 1^{2n-1}]$.*

Proof. Since for $n = 5$ the proposition holds [see the remark after Proposition 5.3.5] we may assume that $n > 5$. Let $f \in VIN_n[2, 1^{2n-1}]$. Then

$$f = \sum_{i=1}^{2n-1} (f_1 x y_{2n-1} + f'_1 y_{2n-1} x) + \sum_{i < j} \pm f_2 [y_i, y_j] + \beta S_{2n-1}(y_1, \dots, y_{2n-1}) x^2$$

for some $\beta \in F$. We may assume that $f^* = \pm f$.

Case 1. If n is odd then

$$\begin{aligned} (t_i^n)^* &= -t_i^n, \text{ for } i \neq 7, 8, 9, 10, 11, 12, \\ (t_7^n)^* &= -t_{10}^n, (t_8^n)^* = -t_9^n, (t_{11}^n)^* = -t_{12}^n. \end{aligned} \quad (6.33)$$

Since $f_2 \in VIN_{n-1}[2, 1^{2n-3}]$, by the induction hypothesis there exist scalars $\alpha_i \in F$ such that

$$f_2 = \alpha_1 t_1^{n-1} + \dots + \alpha_4 t_4^{n-1} + \alpha_6 t_6^{n-1} + \dots + \alpha_{12} t_{12}^{n-1} + \alpha_{16} t_{16}^{n-1}.$$

Thus by (6.32) $f_2 = [y_1, y_2]g_2 + \sim$, where

$$\begin{aligned} g_2 &= \alpha_1 t_2^{n-2} + \alpha_2 t_1^{n-2} + \alpha_3 (t_4^{n-2} + t_9^{n-2}) + \alpha_4 (t_3^{n-2} + t_7^{n-2}) + \\ &\quad \alpha_6 t_5^{n-2} + \alpha_8 t_{10}^{n-2} + \alpha_{10} t_8^{n-2} + \alpha_{11} (t_7^{n-2} - t_{11}^{n-2}) - \alpha_{12} t_{12}^{n-2} + \alpha_{16} t_{15}^{n-2}. \end{aligned}$$

Thus by (6.33)

$$\begin{aligned} (g_2)^* &= -\alpha_1 t_2^{n-2} - \alpha_2 t_1^{n-2} + \alpha_3 (-t_4^{n-2} - t_9^{n-2}) + \alpha_4 (-t_3^{n-2} - t_{10}^{n-2}) - \\ &\quad \alpha_6 t_5^{n-2} - \alpha_8 t_7^{n-2} - \alpha_{10} t_9^{n-2} + \alpha_{11} (-t_{10}^{n-2} + t_{12}^{n-2}) + \alpha_{12} t_{11}^{n-2} - \alpha_{16} t_{15}^{n-2}. \end{aligned}$$

Since $f^* = \pm f$, by Lemma 6.3.1, $g_2^* = \pm g_2$. Thus, when $g_2^* = g_2$, $\alpha_i = 0$, for $i \neq 7, 9, 11, 12$ and $\alpha_{11} = -\alpha_{12}$ since t_i^{n-2} for $i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15$ are linear independent when $n - 2 > 3$. So

$$f_2 = \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \alpha_{12} (-t_{11}^{n-1} + t_{12}^{n-1}).$$

Let

$$g := \alpha_7(t_7^n - t_{10}^n) - \alpha_9(t_8^n - t_9^n) + \alpha_{12}(-t_{11}^n + t_{12}^n).$$

Then g is $\ast$ -symmetric and

$$g = \{\alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \alpha_{12}(-t_{11}^{n-1} + t_{12}^{n-1}) - \alpha_9 t_8^{n-1}\}[y_{2n-2}, y_{2n-1}] + \sim.$$

So

$$\begin{aligned} f - g &= \sum_{i=1}^{2n-1} (f_1 x y_{2n-1} + f'_1 y_{2n-1} x) + \gamma S_{2n-1}(y_1, \dots, y_{2n-1}) x^2 + \\ &\quad \sum_{i < j} \pm (\beta_8 t_8^{n-1} + \beta_{10} t_{10}^{n-1}) [y_i, y_j]. \end{aligned} \quad (6.34)$$

Similarly, $g_2^\ast = -g_2$ implies that

$$\begin{aligned} f_2 &= \alpha_1 t_1^{n-1} + \alpha_2 t_2^{n-1} + \alpha_3(t_3^{n-1} + t_{10}^{n-1}) + \alpha_4(t_4^{n-1} + t_8^{n-1}) + \\ &\quad \alpha_6 t_6^{n-1} + \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \alpha_{16} t_{16}^{n-1} + \alpha_{11}(t_{11}^n + t_{12}^n). \end{aligned}$$

Let

$$\begin{aligned} g &= \alpha_1 t_1^n + \alpha_2 t_2^n + \alpha_3 t_3^n + \alpha_4 t_4^n + \alpha_6 t_6^n + \\ &\quad \alpha_7(t_7^n + t_{10}^n) + \alpha_8(t_8^n + t_9^n) + \alpha_{16} t_{16}^n + \alpha_{11}(t_{11}^n + t_{12}^n). \end{aligned}$$

Then $g^\ast = -g$ and we also have (6.34).

Case 2. If n is even then

$$(t_i^n)^\ast = t_{i+1}^n, \quad i = 1, 3, 5, 7, 9, 11, 15.$$

Since $f_2 \in VIN_{n-1}[2, 1^{2n-3}]$, by the induction hypothesis

$$\begin{aligned} f_2 &= \alpha_1 t_1^{n-1} + \dots + \alpha_5 t_5^{n-1} + \\ &\quad \alpha_7 t_7^{n-1} + \dots + \alpha_{12} t_{12}^{n-1} + \alpha_{15} t_{15}^{n-1}. \end{aligned}$$

Thus by (6.32) $f_2 = [y_1, y_2]g_2 + \sim$, where

$$\begin{aligned} g_2 &= \alpha_1 t_2^{n-2} + \alpha_2 t_1^{n-2} + \alpha_3(t_4^{n-2} + t_9^{n-2}) + \alpha_4(t_3^{n-2} + t_7^{n-2}) + \\ &\quad \alpha_5(t_6^{n-2} - t_7^{n-1} - t_9^{n-1}) + \alpha_8 t_{10}^{n-2} + \alpha_{10} t_8^{n-2} + \\ &\quad \alpha_{11}(t_7^{n-2} - t_{11}^{n-2}) + \alpha_{12}(-t_{12}^{n-2}) + \alpha_{15}(t_{16}^{n-2} + t_7^{n-1} + t_9^{n-1}). \end{aligned}$$

Since $f^* = \pm f$, $g_2^* = \pm g_2$. Thus, $g_2^* = g_2$ implies that

$$\begin{aligned}\alpha_5 &= \alpha_{15} = 0, \quad \alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_4, \quad \alpha_8 = \alpha_3, \\ \alpha_{10} &= \alpha_4 + \alpha_{11} = \alpha_3 + \alpha_{11}, \quad \alpha_{11} = \alpha_{12},\end{aligned}$$

and

$$\begin{aligned}f_2 &= \alpha_1(t_1^{n-1} + t_2^{n-1}) + \alpha_3(t_3^{n-1} + t_4^{n-1}) + \alpha_7 t_7^{n-1} + \alpha_3 t_8^{n-1} + \\ &\quad \alpha_9 t_9^{n-1} + (\alpha_3 + \alpha_{11}) t_{10}^{n-1} + \alpha_{11}(t_{11}^{n-1} + t_{12}^{n-1}).\end{aligned}$$

Let

$$\begin{aligned}g &= \alpha_1(t_1^n + t_2^n) + \alpha_3(t_3^n + t_4^n) + \alpha_7(t_7^n + t_8^n) + \\ &\quad \alpha_9(t_9^n + t_{10}^n) + \alpha_{11}(t_{11}^n + t_{12}^n).\end{aligned}$$

Then g is $*$ -symmetric and we also have (6.34).

When $f^* = -f$, $g_2^* = -g_2$ implies that

$$\begin{aligned}f_2 &= \alpha_1(t_1^{n-1} - t_2^{n-1}) + \alpha_3(t_3^{n-1} + t_5^{n-1} + t_{11}^{n-1} - t_{12}^{n-1}) + \\ &\quad \alpha_4(t_4^{n-1} + t_5^{n-1}) + \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \\ &\quad \alpha_{10}(t_{10}^{n-1} - t_{11}^{n-1} + t_{12}^{n-1}) + \alpha_{15} t_{15}^{n-1}.\end{aligned}$$

Let

$$\begin{aligned}g &:= \alpha_1(t_1^n - t_2^n) + \alpha_3(t_3^n + t_5^n + t_{11}^n - t_{12}^n) + \alpha_4(t_4^n + t_5^n) + \\ &\quad (\alpha_7 - \alpha_{15})(t_7^n - t_8^n) + (\alpha_9 - \alpha_{15})(t_9^n - t_{10}^n) + \\ &\quad \alpha_{10}(-t_{11}^n + t_{12}^n) + \alpha_{15}(t_7^n + t_9^n + t_{15}^n).\end{aligned}$$

Since

$$\begin{aligned}t_3^n + t_4^n &= S_{2n}(x, x \circ y_1, \dots, y_{2n-1}) \\ &= -t_5^n - t_6^n\end{aligned}$$

and (6.32), g is $*$ -skew symmetric and (6.34) holds.

Thus we may assume that $f^* = \pm f$ and

$$f = \sum_{i=1}^{2n-1} (f_1 x y_{2n-1} + f'_1 y_{2n-1} x) + \gamma S_{2n-1}(y_1, \dots, y_{2n-1}) x^2 + \sum_{i < j} \pm (\beta_8 t_8^{n-1} + \beta_{10} t_{10}^{n-1}) [y_i, y_j]. \quad (6.35)$$

From (6.35), the monomial $y_1 \cdots y_{2n-1} x^2$ has coefficient γ in f . But its reverse has coefficient 0 because

$$t_8^{n-1} = \tilde{T}_{2n-2}(y_1, \dots, y_{2n-3}; x)x, \quad t_{10}^{n-1} = T_{2n-2}(y_1, \dots, y_{2n-3}; x)x.$$

Using the same argument and considering the coefficients of monomials w and w^* for

$$w = y_1 \cdots y_6 x y_7 \cdots y_{2n-3} x y_{2n-2} y_{2n-1}, \quad y_1 \cdots y_4 x y_5 \cdots y_{2n-3} x y_{2n-2} y_{2n-1},$$

respectively yield $\beta_8 = \beta_{10} = 0$. Thus

$$f = \sum_{i=1}^{2n-1} (f_1 x y_{2n-1} + f'_1 y_{2n-1} x).$$

From this equation we know that f has no terms which end in $y_i y_j$. So $\Delta(y_{2n-1}, x)f$ has no such terms also. But $\Delta(y_{2n-1}, x)f \in VIN_n[3, 1^{2n-2}]$, and $\Delta(y_{2n-1}, x)f = 0$ from the proof of Proposition (6.3.3). Thus $f_1 + f'_1 = 0$. Since f is skew symmetric in the variables y_i 's, $\Delta(y_{2n-1}, x)f = 0$ implies $\Delta(y_1, x)f = 0$. So $\Delta(y_1, x)f_1 = 0$. Thus f_1 is skew symmetric in $x, y_1, \dots, y_{2n-2}$ by Theorem 1.1. Hence $f_1 = dS_{2n-1}(x, y_1, \dots, y_{2n-2})$. However, f has no terms which end in $y_i y_j$ and $f^* = \pm f$, $d = 0$. So $f = 0$. ■

Since every identity in $VIN_n[1^{2n+1}]$ is multilinear and alternating, it is a scalar multiple of the standard identity $S_{2n+1}(x_1, \dots, x_{2n+1})$. Thus, Theorem (6.2) was proved.

Appendix I

The Dimension of Space $VIN_n[n_1, \dots, n_r]$

The identities of H_2

Vector Space	Dimension	Reference	Page
$VI_2[3, 2]$	1	Prop. 3.1.1	27
$VI_2[3, 1, 1]$	4	Prop. 3.1.2	28
$VI_2[2, 2, 1]$	8	Prop. 3.1.3	29
$VIN_2[2, 1^3]$	7	Prop. 3.1.4	31
$VIN_2[1^5]$	1		32

The identities of H_3

Vector Space	Dimension	Reference	Page
$VI_3[4, 2, 1]$	1	Prop. 4.1.2	35
$VIN_3[4, 1^3]$	2	Prop. 4.1.3	36
$VIN_3[3, 2, 2]$	3	Prop. 4.2.2	40
$VIN_3[3, 2, 1^2]$	10	Prop. 4.2.3	43
$VIN_3[3, 1^4]$	7	Prop. 4.2.4	50
$VIN_3[2^3, 1]$	7	Prop. 4.3.1	53
$VIN_3[2^2, 1^3]$	16	Prop. 4.3.2	70
$VIN_3[2, 1^5]$	10	Prop. 4.3.3	70
$VIN_3[1^7]$	1		73

The identities of $H_n(n \geq 4)$

Vector Space	Dimension	Reference	Page
$VI_n[4, 1^{2n-3}]$	1	Prop. 5.3.1, 6.3.1	105, 139
$VI_n[3, 2, 1^{2n-4}]$	2	Prop. 5.3.2, 6.3.2	106, 141
$VIN_n[3, 1^{2n-2}]$	6	Prop. 5.3.3, 6.3.3	109, 141
$VIN_n[2^2, 1^{2n-3}]$	7	Prop. 5.3.4, 6.3.4	111, 151
$VIN_n[2, 1^{2n-1}]$	12	Prop. 5.3.5, 6.3.5	118, 157
$VIN_n[1^{2n+1}]$	1		160

Appendix II

Reliability of Computations

Like many if not most mathematical results, the contents of this thesis are not presented in the order they were obtained. We set out to prove that all identities of degree $2n + 1$ of H_n follow from those of degree $2n$ ($n > 3$). Corollary 1.3.2 allows us to restrict ourselves to homogeneous polynomials which are symmetric (respectively skew symmetric) in variables of equal even (respectively odd) degree. If need be we can further specialize to $+$ -symmetric and $+$ -skew symmetric polynomials with this property, where $+$ is the reversal involution.

Such a polynomial is a linear combination of simpler polynomials. For example, if $p(x_1, x_2, x_3, y)$ is of degree 1 in each x_i and k in y and is skew symmetric in x_1, x_2, x_3 then p is a linear combination of $\binom{k+3}{3}$ polynomials each of which corresponds to choosing a place for x_1, x_2 and x_3 and then taking the alternating sum. Subtracting a linear combination of known identities we may sometimes assume conditions on the coefficients. Then we must make substitutions to obtain further conditions on the coefficients and solve the resulting homogeneous system of linear equations. If it has only the trivial solution we are done and may pass to the next multidegree in our ordering. If not, at least one identity has been obtained or more substitutions must be made to rule it out.

The substitutions step and the solving of systems of linear equations step were done in Mathematica on a Next and Maple on the main frame. Some cases were done on both systems. All substitutions were checked by hand using the graph-theoretical version which allows one to visually decide which monomials correspond to unicursal paths. While we did not double check the second step for all cases, we feel confident in our results since errors are more likely to produce spurious candidate identities which are rejected when we try to show that they are indeed identities. Each linear equation was obtained by requesting the coefficients of a monomial. In the most cases, if we mistype a monomial we get the equation $0 = 0$ and nothing is lost. Moreover, the new class of identities $T_{2n}(x_1, \dots, x_{2n-1}; [x_{2n}, x_{2n+1}])$ was discovered by implementing the procedure described above.

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