ANALYTIC SELF-MAPS OF TORI

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ABSTRACT

This thesis is a study of the natural algebraic structure on the set of analytic maps between tori with conformal structure. In particular, the set of analytic self-maps of a torus is shown to be a monoidal semi-direct product. The active monoid of the product is actually a ring and is represented by a discrete lattice in the complex plane. For conformally equivalent tori, these lattices are identical. More generally, conformal invariants are assigned to these lattices and are used to determine relationships between the lattices assigned to two tori when there exists a non-constant analytic map from one torus into the other. These invariants are of particular importance as a criteria for lifting an analytic Γ-structure from one torus to another.

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INTRODUCTION

This thesis is a study of the natural algebraic structure on the set of analytic maps between tori with conformal structure. In particular, investigation of the set of analytic self-maps of a torus shows it to be a monoidal semi-direct product. The active monoid in this product is in fact a ring and is represented as a discrete lattice in the complex plane. The relationship between the lattices of the different tori is of interest when there exist non-constant analytic maps between the tori. If the tori are conformally equivalent, these distortion lattices, as they are called, are identical.

The study of these lattices is facilitated by the introduction of imaginary quadratic number fields. The necessary preliminaries to such an introduction are included in Chapter 2. Chapter 2 also contains a development of the monoidal theory required for this investigation.

Chapter 1 reviews the works of H.G. Helfenstein on which this study is based. The main result is the representation theorem for analytic maps between tori with conformal structure.

Chapter 3 establishes the structure on the set of analytic self-maps and determines the analytic automorphism group of an arbitrary torus with conformal structure. In each case, the torsion subgroup of the automorphism group is explicitly described.

In Chapter 4, the family of lattices of complex distortions associated with the set of analytic maps between two tori is defined and some of its properties determined. The symmetry described in Chapter 1, that there exists a non-constant analytic map from a torus τ_1 into the torus τ_2 iff there exists one from τ_2 into τ_1, is further displayed by the fact that the lattice of complex distortions $\mathcal{L}(\tau_1, \tau_2)$ of analytic maps from τ_1 into τ_2 is the reflection in the imaginary axis of $\mathcal{L}(\tau_2, \tau_1)$.

Chapter 5 investigates the distortion ring of a torus. A necessary and sufficient condition that the distortion ring of one torus can be imbedded in the distortion ring of some other torus is given, subject to the condition that there exist a non-constant analytic map between the two tori. This condition is phrased in terms of conformal invariants of the tori, and appears in Chapter 7 as well.

The notion of analytic homotopy is introduced in Chapter 6 in order to define analytic Γ-structure in Chapter 7. The notion of analytic Γ-structure is a variant of the concept of Γ-manifold, defined by H. Hopf in 1941 [6].

Chapter 7 gives a necessary and sufficient condition that an analytic Γ-structure on one torus can be lifted to another torus, if there exists a non-constant analytic map between the tori.

In the following chapters, the symbol [a,b] shall denote the greatest common divisor of the integers a and b, and is assumed to be positive. The greatest common divisor of three integers is also required, and shall be denoted by $[a,b,c] = [a,[b,c]] = [[a,b],c]$ for arbitrary integers a, b and c.

Finally, the abbreviation "iff" shall be used in place of the phrase "if and only if", and the symbol ∇ shall denote the end of a proof.

CHAPTER 1

CONFORMAL TORI AND THEIR ANALYTIC MAPS

§1.1 Conformal Structure

In order to formulate the notion of a conformal torus, we require a definition of conformal structure. Thus we have the following:

<u>(1.1.1) Definition</u>. A topological space X has *conformal structure* $\mathfrak{S}$ if there exists a set $\mathfrak{S} = \{(f,u) \mid U \text{ open in } X, f:U \to ¢\}$, where ¢ denotes the set of complex numbers, and $\mathfrak{S}$ satisfies the following conditions:

i) for all $x \in X$, there exists an element $(f,U) \in \mathfrak{S}$ such that $x \in U$ and f maps U homeomorphically onto an open subset of ¢. Such an (f,U) is called a *local parameter*.

ii) for each $(f,U) \in \mathfrak{S}$, f is analytic at each $x \in U$ in the sense that, if (g,V) is a local parameter such that $x \in V \subset U$, then $(f|V) \circ g^{-1}$ is analytic at g(x).

iii) if $\mathfrak{H} = \{(f,U) \mid U \text{ open in } X, f:U \to ¢\}$ is a set satisfying conditions (i) and (ii) and $\mathfrak{S} \subset \mathfrak{H}$, then $\mathfrak{S} = \mathfrak{H}$.

<u>(1.1.2) Definition</u>. Let X and Y be spaces with conformal structure $\mathfrak{S}$ and $\mathfrak{H}$ respectively. A continuous function $f:X \to Y$ is said to be *analytic* at $x \in X$ iff for each local parameter $(g,V) \in \mathfrak{H}$ such that $f(x) \in V$, then $(g \circ (f|f^{-1}(V)), f^{-1}(V)) \in \mathfrak{S}$. If f is analytic at every element of X, then f is said to be an *analytic function*

from X to Y. If, in addition, f is a homeomorphism and f^{-1} is analytic, then f is called a *conformal isomorphism* between X and Y, and X is said to be *conformally equivalent* to Y. When Y = X, a conformal isomorphism will be called a conformal automorphism.

The complex plane $\mathbb{C}$ has conformal structure consisting of the set of all restrictions of analytic functions to open subsets of $\mathbb{C}$. Let us consider two elements ω_1, ω_2 of $\mathbb{C}$ for which the imaginary part of ω_1/ω_2, denoted by $I(\omega_1/\omega_2)$, is not zero.

<u>(1.1.3) Definition</u>. The group of transformations of $\mathbb{C}$ generated by T_1 and T_2, where $T_1(x) = z + \omega_1$, $T_2(z) = z + \omega_2$ for all $z \in \mathbb{C}$, will be denoted by $\Gamma(\omega_1, \omega_2)$, and is defined for $\omega_1, \omega_2 \in \mathbb{C}$ for which $I(\omega_1/\omega_2)$ is non-zero. Associated with this group is the lattice $\mathcal{L}(\omega_1, \omega_2)$, given by

$$\mathcal{L}(\omega_1, \omega_2) = \{k\omega_1 + \ell\omega_2 | k, \ell \in Z\}$$

where Z is the set of integers.

For such a group $\Gamma(\omega_1, \omega_2)$, the quotient space has one and only one conformal structure such that the projection mapping $\pi : \mathbb{C} \to \mathbb{C}/\Gamma(\omega_1, \omega_2)$ becomes an analytic function. The space $\mathbb{C}/\Gamma(\omega_1, \omega_2)$ with this conformal structure is called a *torus with conformal structure*. Since the relation of conformal equivalence is an equivalence relation on the set of all spaces with conformal structure, the set of all tori with conformal structure is partitioned by this relation.

<u>(1.1.4) Definition</u>. An equivalence class of tori with conformal structure will be called a *conformal torus.*

It is well-known that for $\omega_1, \omega_2, \lambda_1, \lambda_2 \in \mathbb{C}$ with $I(\omega_1/\omega_2) \neq 0$, $I(\lambda_1/\lambda_2) \neq 0$, $\mathbb{C}/\Gamma(\omega_1, \omega_2)$ is conformally equivalent to $\mathbb{C}/\Gamma(\lambda_1, \lambda_2)$ iff there exist integers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = \pm 1$, and a $\mu \in \mathbb{C} \sim \{0\}$ such that $\mathcal{L}(\lambda_1, \lambda_2) = \mu \mathcal{L}(\omega_1, \omega_2)$. This condition is clearly sufficient. Conversely, if a function $\psi : \mathbb{C}/\Gamma(\omega_1, \omega_2) \to \mathbb{C}/\Gamma(\lambda_1, \lambda_2)$ is a conformal equivalence, then since $\mathbb{C}$ is simply connected, ψ can be lifted to an entire function $\tilde{\psi}$ from $\mathbb{C}$ onto $\mathbb{C}$ such that

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\tilde{\psi}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\
\mathbb{C}/\Gamma(\omega_1, \omega_2) \arrow[r, "\psi"] & \mathbb{C}/\Gamma(\lambda_1, \lambda_2)
\end{tikzcd}$$

is a commutative diagram. Thus there exist two functions m and n from $Z \times Z$ into Z such that for all $(k, \ell) \in Z \times Z$, $\tilde{\psi}(z + k\omega_1 + \ell\omega_2) = \tilde{\psi}(z) + m(k, \ell)\lambda_1 + n(k, \ell)\lambda_2$ for all $z \in \mathbb{C}$. This in turn implies that $\tilde{\psi}$ is a linear function, i.e. there exist $C, D \in \mathbb{C}$ such that $\tilde{\psi}(z) = Cz + D$ for all $z \in \mathbb{C}$. It is easily seen that D may be assumed to be zero without loss of generality. So we have $C\omega_1 = m(1,0)\lambda_1 + n(1,0)\lambda_2$ and $C\omega_2 = m(0,1)\lambda_1 + n(0,1)\lambda_2$. But now if $\Gamma(C\omega_1, C\omega_2) \neq \Gamma(\lambda_1, \lambda_2)$, then ψ can not be injective, a contradiction. Thus $(C\omega_1, C\omega_2)$ is a basis for $\mathcal{L}(\lambda_1, \lambda_2)$, from which we conclude that $m(1,0)n(0,1) - m(0,1)n(1,0) = \pm 1$.

In particular, for $\omega_1, \omega_2 \in \mathbb{C}$ with $I(\omega_1/\omega_2) \neq 0$, it is evident that $\mathbb{C}/\Gamma(\omega_1, \omega_2)$ is conformally equivalent to both $\mathbb{C}/\Gamma(1, \frac{\omega_1}{\omega_2})$ and $\mathbb{C}/\Gamma(1, -\frac{\omega_1}{\omega_2})$. Thus the set $\{\mathbb{C}/\Gamma(\omega_1, \omega_2) \mid I(\omega_1/\omega_2) > 0\}$ contains representatives of each conformal torus. If we let H denote the upper half-plane, $H = \{z \in \mathbb{C} \mid I(z) > 0\}$, then from the above remarks, we see that the set $\{\mathbb{C}/\Gamma(1,h) \mid h \in H\}$ contains representatives of each conformal torus. Furthermore, $\mathbb{C}/\Gamma(1,h)$ is conformally equivalent to $\mathbb{C}/\Gamma(1,h')$ iff there exist integers $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma = 1$ and $h = \dfrac{\alpha h' + \beta}{\gamma h' + \delta}$.

Let Q denote the set of rational numbers and Z^+ the positive integers. Let the multiplicative group of all square matrices of dimension two with entries in Q and with positive determinant be called $GL^+(2,Q)$. Analogously, let $GL^+(2,Z)$ be the multiplicative semigroup of square matrices of dimension two with entries in Z and with positive determinant. Finally, let $SL^+(2,Z)$ be the subset of $GL^+(2,Z)$ consisting of matrices with determinant equal to one. Since $SL^+(2,Z)$ can be considered as a subgroup of $GL^+(2,Q)$, it follows that both $GL^+(2,Q)$ and $SL^+(2,Z)$ act on H by defining the transformation $G:H \to H$ for each $G = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \varepsilon\, GL^+(2,Q)$ in the following manner: for all $z \varepsilon H$, let $G(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$. Thus we obtain a useful representation of the set of conformal tori as described in the following:

<u>(1.1.5) Theorem.</u> The set of all conformal tori is parametrized by the set $H/SL^+(2,Z)$ with the correspondence given by mapping the conformal torus containing $¢/\Gamma(\omega_1,\omega_2)$ to the orbit $SL^+(2,Z)(\omega_1/\omega_2)$ if $I(\omega_1/\omega_2) > 0$, otherwise to the orbit $SL^+(2,Z)(-\omega_1/\omega_2)$.

<u>(1.1.6) Definition.</u> Let $\Gamma(h)$ denote the transformation group $\Gamma(1,h)$.

With this notation, an element $\tau \varepsilon H/SL^+(2,Z)$ represents the conformal torus containing the set $\{¢/\Gamma(h) \mid h \varepsilon \tau\}$. Now for two conformal tori τ_1 and τ_2, either there exists a non-constant analytic map from $¢/\Gamma(h_1)$ onto $¢/\Gamma(h_2)$ for every $h_1 \varepsilon \tau_1$, $h_2 \varepsilon \tau_2$, or else the only analytic maps from $¢/\Gamma(h_1)$ into $¢/\Gamma(h_2)$ are the constant maps.

§1.2 Analytic Maps of Conformal Tori

<u>(1.2.1) Definition.</u> Let τ_1 and τ_2 be conformal tori. If there exist $h_1 \varepsilon \tau_1$, $h_2 \varepsilon \tau_2$ such that there is a non-constant

analytic map from $\mathbb{C}/\Gamma(h_1)$ to $\mathbb{C}/\Gamma(h_2)$, then we say that there is a non-constant analytic map from τ_1 to τ_2. Let $An(\tau_1,\tau_2)$ be the set of all analytic maps from τ_1 to τ_2, that is, for any $h_1 \varepsilon \tau_1$, $h_2 \varepsilon \tau_2$, the set of all analytic maps from $\mathbb{C}/\Gamma(h_1)$ to $\mathbb{C}/\Gamma(h_2)$ represents $An(\tau_1,\tau_2)$.

A necessary and sufficient condition for the existence of non-constant analytic maps between two conformal tori was given by H.G. Helfenstein [4]. In order to state this condition, we require some additional notation. We shall say that for $h_1, h_2 \varepsilon H$, h_1 and h_2 are *immersion-equivalent in* H iff there exists $T \varepsilon GL^+(2,Q)$ such that $h_1 = T(h_2)$. This is an equivalence relation and the equivalence classes of H under this relation are called *analytic immersion classes of* H.

(1.2.2) Definition. Conformal tori τ_1 and τ_2 are said to be *immersion-equivalent,* written $\tau_1 \sim \tau_2$, iff there exist $h_1 \varepsilon \tau_1$, $h_2 \varepsilon \tau_2$ such that h_1 and h_2 are immersion-equivalent in H. The equivalence classes into which $H/SL^+(2,Z)$ is partitioned are called *analytic immersion classes of conformal tori.*

(1.2.3) Theorem. Let τ_1 and τ_2 be conformal tori. Then there is a non-constant analytic map from τ_1 to τ_2 iff $\tau_1 \sim \tau_2$. Since immersion-equivalence is an equivalence relation, it follows that there exists a non-constant analytic map from τ_1 to τ_2 iff there exists a non-constant analytic map from τ_2 to τ_1.

In proving this theorem, it was shown that every analytic map between two tori could be lifted, via the projections, to an analytic map of $\mathbb{C}$ onto itself. Furthermore, this map must be linear.

(1.2.4) Corollary. Every analytic map between two conformal tori is either constant or a covering map.

In the same paper, a representation of $An(\tau_1,\tau_2)$ was given. It was shown that the properties of the sets of analytic maps depend on number theoretic attributes of the representatives of the conformal tori.

An element $z \varepsilon \mathbb{C}$ is said to be *ample* iff both the real part, $\mathcal{R}(z)$, of z and $|z|^2$ are elements of Q. Otherwise z is said to be *non-ample*. Since every element of the orbit $GL^+(2,Q)(z)$ is ample iff z is ample, we have the following

(1.2.5) Definition. A conformal torus is said to be *ample* iff it has an ample representative.

We shall see that the ample tori have rather distinct properties from those of the non-ample tori.

(1.2.6) Lemma. If τ_1 and τ_2 are conformal immersion-equivalent tori, then there exist representatives $h_1 \varepsilon \tau_1$, $h_2 \varepsilon \tau_2$ such that $h_1 = ah_2$ for some $a \varepsilon Z^+$.

The proof of this lemma also appears in [4]. It is based on the fact that if $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \varepsilon GL^+(2,Z)$ is such that $\det(T) = a$, then there exist F, $G \varepsilon SL^+(2,Z)$ such that

$FTG = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$, where $d = [\alpha, \beta, \gamma, \delta]$, and $c = a/d \in Z^+$. With this process in mind, it is easily seen that in general, a is not the only integer satisfying such a relationship between τ_1 and τ_2.

<u>(1.2.7) Definition</u>. Let τ_1 and τ_2 be conformal tori. Then let $\mathrm{Rep}(\tau_1,\tau_2)$ be the set of all pairs (a,h) where $a \in Z^+$ and $h \in \tau_2$ are such that $ah \in \tau_1$, if $\tau_1 \sim \tau_2$, else let $\mathrm{Rep}(\tau_1,\tau_2) = \{(0,h) | h \in \tau_2\}$.

<u>(1.2.8) Definition</u>. Let the *index* of two conformal tori τ_1 and τ_2, denoted $\mathrm{ind}(\tau_1,\tau_2)$, be zero if τ_1 and τ_2 are not immersion equivalent, otherwise let

$$\mathrm{ind}(\tau_1,\tau_2) = \min \{\det(T) | T \in GL^+(2,Z), \exists\, h \in \tau_2 \ni T(h) \in \tau_1\}.$$

From (1.2.6), it is apparent that if $\tau_1 \sim \tau_2$, $\mathrm{ind}(\tau_1,\tau_2)$ could equivalently be defined to be $\min \{a | (a,h) \in \mathrm{Rep}(\tau_1,\tau_2)\}$. In [3], H.G. Helfenstein has shown that if $\mathrm{ind}(\tau_1,\tau_2) \neq 0$, then for each $h_1 \in \tau_1$, $h_2 \in \tau_2$ there exists an analytic map $f: ¢/\Gamma(h_1) \to ¢/\Gamma(h_2)$ such that the number of sheets of f (for by (1.2.4), f is a covering map) is equal to $\mathrm{ind}(\tau_1,\tau_2)$, and for any other analytic map $g: ¢/\Gamma(h_1) \to ¢/\Gamma(h_2)$, the number of sheets of g is not less then $\mathrm{ind}(\tau_1,\tau_2)$.

<u>(1.2.9) Lemma</u>. For all conformal tori τ_1 and τ_2, $\mathrm{ind}(\tau_1, \tau_2) = \mathrm{ind}(\tau_2, \tau_1)$.

Proof: If τ_1 is not immersion-equivalent to τ_2, then $\mathrm{ind}(\tau_1,\tau_2)$ and $\mathrm{ind}(\tau_2,\tau_1)$ are both zero by definition. If $\tau_1 \sim \tau_2$, then for each $(a,h) \varepsilon \mathrm{Rep}(\tau_1, \tau_2)$, $J(h) = aJ(ah)$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \varepsilon SL^+(2,Z)$. ∇

It is convenient to introduce at this point some functions which will be required throughout the paper. For ample $h \varepsilon H$, there exist integers a, b, c, d such that $2\mathcal{R}(h) = a/b$ and $|h|^2 = c/d$, with $[a,b] = [c,d] = 1$. Without any loss in generality, we shall assume b, c and d to be positive.

(1.2.10) Definition. Let the functions $p, q, r, s, [q,s], q', s', ps', qs', q'r: H \to Z$ be defined as follows: for each $h \varepsilon H$, $p(h) = q(h) = r(h) = s(h) = 0$ if h is non-ample, else let $p(h) = a$, $q(h) = b$, $r(h) = c$, and $s(h) = d$. Now let $[q,s](h) = 1$ if h is non-ample, and if h is ample, let $[q,s](h) = [q(h), s(h)]$. Then put $q'(h) = q(h)/[q,s](h)$, $s'(h) = s(h)/[q,s](h)$ for all $h \varepsilon H$. Finally, let ps', qs' and $q'r$ denote the product functions given by $ps'(h) = p(h)s'(h)$, $qs'(h) = q(h)s'(h)$ and $q'r(h) = q'(h)r(h)$ for all $h \varepsilon H$.

We shall also introduce a matrix-valued function on H. For each $h \varepsilon H$, let

$$M(h) = \begin{bmatrix} ps'(h) & -q'r(h) \\ qs'(h) & 0 \end{bmatrix} .$$

Note that if h is non-ample, $M(h)$ is the zero matrix.

It is an interesting fact that there is associated with each analytic immersion class of conformal tori an abstract group which provides all the information necessary to describe the set of analytic maps between any two conformal tori in the immersion class. This group is the stabilizer subgroup I_h of $GL^+(2,Q)$ with respect to any representative $h \in \tau$ for a conformal torus τ in the immersion class. Since the action of $GL^+(2,Q)$ is transitive on each analytic immersion class, it follows that the structure of the stabilizer subgroup is invariant as h varies in the immersion class of H. The following representation of I_h is due to H.G. Helfenstein [4].

<u>(1.2.11) Theorem</u>. Let $h \in H$. Then the stabilizer subgroup I_h of $GL^+(2,Q)$ at h is given by:

$$I_h = \{\rho I + \sigma M(h) \mid (\rho,\sigma) \in Q \times Q,\ \rho^2 + (\sigma q(h))^2 \neq 0\} .$$

The following development also appears in [4], and it will show that the set of analytic maps between conformal tori is closely dependent on certain elements of $GL^+(2,Q)/I_h$, the set of left cosets of some stabilizer subgroup I_h .

<u>(1.2.12) Definition</u>. For each $a \in Z^+$, define the function $\tilde{a}:H \to Q$ as follows: for $h \in H$, let $\tilde{a}(h) = 1/[a,qs'(h)]$. It will be convenient to define $\tilde{a}$ for $a = 0$ as well. If $a = 0$, let $\tilde{a} \equiv 1$.

(1.2.13) Lemma. Let $a \epsilon Z^+$, and let $T = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. For each $h \epsilon H$, the elements of the left coset TI_h which also belong to $GL^+(2,Z)$ are exactly the matrices $k_1T + k_2\tilde{a}(h)TM(h)$, for arbitrary k_1, $k_2 \epsilon Z$ for which $k_1^2 + k_2^2q(h) \neq 0$.

It is now possible to give an effective method of computing the index of two conformal tori.

(1.2.14) Theorem. Let τ_1 and τ_2 be conformal tori and let $(a,h) \epsilon \mathrm{Rep}(\tau_1,\tau_2)$. Then, defining $T = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, $\mathrm{ind}(\tau_1,\tau_2)$ is equal to $\min\{\det(k_1T + k_2\tilde{a}(h)TM(h))|$ $k_1, k_2 \epsilon Z, k_1^2 + k_2^2q(n) \neq 0\}$.

Proof: If τ_1 is not conformally equivalent to τ_2, then $\mathrm{ind}(\tau_1,\tau_2)$ is zero by definition. Suppose that $\tau_1 \sim \tau_2$. For any $S \epsilon GL^+(2,Z)$ such that there exist $h_1' \epsilon \tau_1$, $h_2' \epsilon \tau_2$ with $h_1' = S(h_2')$, there exist A, B $\epsilon SL^+(2,Z)$ such that $h = T^{-1}A^{-1}SB(h)$. Then $T^{-1}A^{-1}SB \epsilon I_h$, whence $A^{-1}SB \epsilon TI_h$. Thus there exist integers k_1 and k_2[4] with $k_1^2 + k_2^2q(h) \neq 0$, such that $A^{-1}SB = k_1T + k_2\tilde{a}(h)TM(h)$. Since A and B belong to $SL^+(2,Z)$, we have $\det(S) = \det(k_1T + k_2\tilde{a}(h)TM(h))$. ∇

Thus if $\tau_1 \sim \tau_2$, $\mathrm{ind}(\tau_1, \tau_2)$ is the minimum integer represented by the positive definite quadratic form

$S(x,y) = \det(xT + y\tilde{a}(h)TM(h))$ for any $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$. It follows that $\mathrm{ind}(\tau_1,\tau_2)$ can be computed in a finite number of steps.

<u>(1.2.15) Corollary</u>. Let τ_1 and τ_2 be non-ample conformal tori and let $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$. Then $\mathrm{ind}(\tau_1, \tau_2) = a$.

<u>(1.2.16) Lemma</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then for each element T of $GL^+(2,Z)$ such that there exists a representative $h \in \tau_2$ with $T(h) \in \tau_1$, the set of all analytic maps from τ_1 to τ_2 is parametrized by the set $TI_h \cap GL^+(2,Z)$.

In fact, the following representation theorem was established by H.G. Helfenstein [4].

<u>(1.2.17) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori, and choose any $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$. Then $\mathrm{An}(\tau_1, \tau_2)$, as represented by the lifts of the analytic maps from $\mathbb{C}/\Gamma(ah)$ to $\mathbb{C}/\Gamma(h)$ to analytic self-maps of $\mathbb{C}$, is the set of all linear functions of the form $F_{(k,\ell),D}(z) = C_a^h(k,\ell)z + D$, where $C_a^h(k,\ell) = k + \ell\tilde{a}(h)qs'(h)h$ and $k, \ell \in Z$, $D \in \mathbb{C}$.

If $a = 1$, the function C_1^h will be denoted simply by C^h. The values of C_a^h are called distortion coefficients.

As a consequence of the linearity of the lifts of analytic maps, we have the following

<u>(1.2.18) Corollary</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then an analytic map $f: \tau_1 \to \tau_2$ is uniquely determined by its image at a single point and its differential f_* at that point.

In fact, if one considers f as an analytic map of $\mathbb{C}/\Gamma(ah)$ onto $\mathbb{C}/\Gamma(h)$ for $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$, then if the vector form of f_* is given by $f_*(X) = AX$ for all vectors X, A must have the form $k_1 I + k_2 \tilde{a}(h) q s'(h) L(h)$ for some integers k_1 and k_2 with $k_1^2 + q(h) k_2^2 \neq 0$, and where

$$L(h) = \begin{bmatrix} R(h) & -I(h) \\ I(h) & R(h) \end{bmatrix} .$$

CHAPTER 2

ALGEBRAIC CONCEPTS AND TERMINOLOGY

In this chapter, we shall establish some theorems on monoids, specify what we mean by ring, and give some structure theorems for certain algebraic number fields. These are the algebraic objects which will be most prominent in the ensuing developments.

§2.1 Monoids

A *monoid* is just a semigroup with an identity element. When the need arises, we shall denote the monoid $\mathfrak{M}$ by $\langle M , * , e\rangle$, where M is the set on which the operation $*$ is defined, and e is the identity element. Otherwise, we shall simply talk about the monoid M, and denote the identity element by e_M. The symbol for the operation will be suppressed from the notation whenever it is possible to do so.

By monoid homomorphism we shall mean a semigroup homomorphism which preserves the identity. For a monoid M, let Hom(M) denote the set of all monoid homomorphisms from M into M. It is obvious that Hom(M) is a monoid under composition of maps.

<u>(2.1.1) Definition</u>. Let M and N be monoids and let $\psi : M \to \mathrm{Hom}(N)$ be a monoid homomorphism. Define an operation $\circledast$ on the Cartesian product $M \times N$ in the following way: for

elements (m_1,n_1), (m_2,n_2) of $M \times N$, let $(m_1,n_1) \circledast (m_2,n_2) = (m_1m_2, n_1\psi_{m_1}(n_2))$. Then $M \times N$ with this operation becomes a monoid, with identity element (e_M,e_N). It is called the *monoidal semi-direct product* of M and N by ψ and we denote it by $M \times_\psi N$.

This is, of course, analogous to the semi-direct product for groups. If M and N are groups, then $M \times_\psi N$ is exactly the semi-direct product.

Let M be a monoid and let A be a subset of M. Then for any $m \in M$, the set mA is defined to be $\{ma \mid a \in A\}$.

(2.1.2) Definition. Let M be a monoid and let N be a submonoid of M. Then N is said to be *normal* in M iff $mN = Nm$ for all $m \in M$.

(2.1.3) Theorem. Let N be a normal submonoid of the monoid M. If N is a group, then M/N is a monoid.

Proof: Since N is a group, the set of left cosets of M by N is a partition of M. If an operation $*$ is defined on M/N by setting $aN * bN = abN$ for all aN, $bN \in M/N$, then M/N with this operation is a monoid. That the operation is well-defined is a consequence of the normality of N in M. ∇

Now if $\psi: M \to N$ is a homomorphism of the monoid M into the monoid N, the set $\{m \in M \mid \psi(m) = e_N\}$ is called

$\mathrm{Ker}(\psi)$. It might be expected that $\mathrm{Ker}(\psi)$ would be normal in M, but in general this is not the case. For example, let $\det: GL^+(2,Z) \to Z^+$ be the determinant function. It is clearly a monoid homomorphism, and Ker(det) is the group $SL^+(2,Z)$. But $SL^+(2,Z)$ is not normal in $GL^+(2,Z)$, for let $T = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $T \in GL^+(2,Z)$ and $S \in SL^+(2,Z)$. Suppose that $T(SL^+(2,Z)) = (SL^+(2,Z))T$. Then $TST^{-1} \in SL^+(2,Z)$, but by computation, $TST^{-1} \notin SL^+(2,Z)$.

There are, however, situations where the kernel is known to be normal. In particular, let M and N be monoids and let $\psi: M \to \mathrm{Hom}(N)$ be a monoid homomorphism. Then we have the following short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "j"] & M \times_\psi N \arrow[r, "\pi"] & M \arrow[r] \arrow[l, bend left, "\beta"] & 0 \end{tikzcd}$$

with $j: N \to M \times_\psi N$ defined by $j(n) = (e_M, n)$ for all $n \in N$, and π is the projection map. Furthermore, it splits at M, since $\pi \circ \beta = 1_M$, the identity map on M, where β is given by $\beta(m) = (m, e_N)$ for all $m \in M$.

<u>(2.1.4) Lemma</u>. The submonoid $j(N) = \mathrm{Ker}(\pi)$ is normal in $M \times_\psi N$ if N is a group, or if N is an abelian monoid.

[illegible] ust be shown that for each $(m,n) \in M \times_\psi N$, [illegible], there exists $\ell \in N$ such that $(m,n) \circledast (e_M, k) = (e_M, \ell) \circledast (m,n)$. But $(m,n) \circledast (e_M,k) = (m, n\,\psi_m(k))$ and $(e_M, \ell) \circledast (m,n) = (m, \ell n)$. If N is abelian, let $\ell = \psi_m(k)$. If N is a group, let $\ell = n\psi_m(k)n^{-1}$. ∇

If we identify N with $j(N)$, then we obtain the following

(2.1.5) Lemma. If N is a group, then M can be imbedded in $(M \times_\psi N)/N$. Furthermore, $M \simeq (M \times_\psi N)/N$ iff ψ_m is surjective for every $m \in M$.

Proof: By (2.1.3), $(M \times_\psi N)/N$ is a monoid. Define the function $\phi: M \to (M \times_\psi N)/N$ by setting $\phi(m) = (m,e_N)N$. If $\phi(m) = \phi(k)$ for some m and $k \in M$, then $(m,e_N)N = (k,e_N)N$. Thus there exists $n \in N$ such that $(m,e_N) = (k,e_N) \circledast (e_M,n) = (k, \psi_k(n))$, whence $m = k$. It is clear that ϕ is a homomorphism and so ϕ is an imbedding of M in $(M \times_\psi N)/N$. Now ϕ is surjective iff for all $k, \ell \in N$, $(m,k)N = (m,\ell)N$ for all $m \in M$. For any $n \in N$, $(m,k) \circledast (e_M,n) = (m,k\psi_m(n)) = (m,\ell\ell^{-1}k\psi_m(n))$. Since ψ_m is surjective, there exists $\ell_1 \in N$ such that $\ell^{-1}k = \psi_m(\ell_1)$, whence $(m,\ell\ell^{-1}k\psi_m(n)) = (m,\ell\psi_m(\ell_1 n)) = (m,\ell) \circledast (e_M,\ell_1 n)$. Thus $(m,k)N \subset (m,\ell)N$ and the converse follows in a similar way. Therefore $(m,k)N = (m,\ell)N$, and thus ϕ is surjective. On the other hand, if ϕ is surjective, then for every $\ell \in N$, $(m,\ell)N = (m,e_N)N$ for every $m \in M$. Thus there exists $n \in N$ with $(m,\ell) = (m,e_N) \circledast (m,n)$ whence $\ell = \psi_m(n)$. ∇

The following lemma will conclude the section on monoids.

(2.1.6) Lemma. Let M_1, N_1, M_2, N_2 be monoids and let $\psi:M_1 \to \mathrm{Hom}(N_1)$, $\Psi:M_2 \to \mathrm{Hom}(N_2)$ be monoid homomorphisms. Then every monoid homomorphism $\theta:M_1 \times_\psi N_1 \to M_2 \times_\Psi N_2$ induces a monoid homomorphism $\tilde{\theta}:M_1 \to M_2$.

Proof: The map $\beta:M_1 \to M_1 \times_\psi N_1$ given by $\beta(m) = (m,e_{N_1})$ is a monoid homomorphism. If we now define the projection mapping $\pi:M_2 \times_\Psi N_2 \to M_2$ by $\pi(m,n) = m$ for all $m \in M_2$, $n \in N_2$, then it is easily seen that π is a monoid homomorphism. Finally, let $\tilde{\theta} = \pi \circ \theta \circ \beta$. Then $\tilde{\theta}$ is a monoid homomorphism from M_1 into M_2. ∇

§2.2 Rings and Ring Homomorphisms

(2.2.1) Definition. A ring $\mathcal{M}$ is a sequence $\langle M,*,+,e,0\rangle$ such that $\langle M,*,e\rangle$ is a monoid, $\langle M,+,0\rangle$ is an abelian group, and the left and right distributive laws of $*$ over $+$ are satisfied, i.e. for all ℓ, m, $n \in M$:

i) $\ell*(m + n) = (\ell * m) + (\ell * n)$ and

ii) $(\ell + m) * n = (\ell * n) + (m * n)$.

(2.2.2) Definition. Let $\mathcal{M}_1 = \langle M_1,*_1,+_1,e_1,0_1\rangle$ and $\mathcal{M}_2 = \langle M_2,*_2,+_2,e_2,0_2\rangle$ be rings. Then a ring homomorphism $\psi: \mathcal{M}_1 \to \mathcal{M}_2$ is a function $\psi: M_1 \to M_2$ such that

$\psi: \langle M_1,*_1,e_1\rangle \to \langle M_2,*_2,e_2\rangle$ is a monoid homomorphism, and

$\psi: \langle M_1,+_1,0_1\rangle \to \langle M_2,+_2,0_2\rangle$ is a group homomorphism.

Thus a ring in our sense is the usual ring with unity. We require a homomorphism to preserve the ring unit. Otherwise, our ring theoretic conventions are the usual.

§2.3 Algebraic Number Fields

In this section we shall describe a class of algebraic number fields and show that there is a correspondence between this class and the set of ample elements of ¢.

Let F be a principal ideal domain and denote its quotient field by $\tilde{F}$. Let K be any extension of $\tilde{F}$. Then an *F-module in* K is an additive subgroup L of K such that for each $n \in F$, $\ell \in L$, the product $n\ell \in L$. If there exists a finite number of elements $\alpha_1, \ldots\ldots, \alpha_t$ of L such that every $\ell \in L$ can be represented as a sum

$$\ell = \sum_{j=1}^{t} a_j \alpha_j$$

for $a_j \in F$, then L is said to be *finite F-module in* K.

For finite extensions K of $\tilde{F}$, we introduce the concept of integral dependence. That is, if K is a finite extension of $\tilde{F}$, then an element $\alpha \in K$ is said to be *integrally dependent* on F iff there exists a finite F-module in K containing the set $\{\alpha^j \mid j \in Z^+\}$.

(2.3.1) Definition. Let K be a finite extension of $\tilde{F}$. Then the set of all elements of K which are integrally dependent on F forms an integral domain D, called the *principal order* of K with respect to F.

It is clear that D is an F-module in K. If by the product MN of two F-modules M and N in K we mean the set consisting of all finite sums $\sum \alpha_j \beta_j$ with arbitrary $\alpha_j \in M$, $\beta_j \in N$, then a D-*ideal* in K is defined as a finite non-trivial F-module M in K for which $DM = M$. If, in addition, $M \subset D$, then M is called an *integral* D-*ideal* in K.

Certain integral D-ideals have a special importance, and these are called the prime ideals. An integral D-ideal M in K is said to be *prime* iff for each integral D-ideal N in K such that $M \subset N$, then either $M = N$ or $N = D$. One can show [2] that the D-ideals of K form a group with the operation of product of F-modules in K. Furthermore, every D-ideal can be expressed as a product of prime D-ideals.

(2.3.2) Definition. Let $\alpha \in D$. If there exists $\beta \in D$ such that the product $\alpha\beta$ is the multiplicative identity, then α is said to be a *unit* of D. The set of all units of D forms a multiplicative subgroup of D.

In the chapters to follow, we shall be interested in a specific principal ideal domain, namely Z, and its quotient field Q. The extensions which occur will be subfields of $\mathbb{C}$ obtained by adjoining a single element of $\mathbb{C}$ to Q. Such imaginary number fields will be denoted by Q[h] where $h \in \mathbb{C}$ is the element adjoined to Q. In fact, we shall only be interested in Q[h] for ample $h \in H$, and then Q[h] is an imaginary quadratic number field.

(2.3.3) Lemma. Let $h \in H$ be ample. Then

$$h^2 = \frac{p(h)}{q(h)} h - \frac{r(h)}{s(h)} .$$

Proof: It is sufficient to show that $M(h) \in I_h$. But this follows from (1.2.11). ∇

Thus for ample $h \in H$, if we put $\alpha = \frac{p(h)}{q(h)}$, $\beta = \frac{r(h)}{s(h)}$, then $\alpha, \beta \in Q$. Furthermore, $h^2 - \alpha h + \beta = 0$, whence the extension field $Q[h]$ is an imaginary quadratic number field.

(2.3.4) Definition. Let $m \in Z^+$. If for every prime $p \in Z^+$, $[p^2, m]$ divides p, then m is said to be *square-free*.

It can be shown [7] that every imaginary quadratic number field is of the form $Q[\sqrt{m}\, i]$ for some square-free $m \in Z^+$. If $m \neq 3 (\bmod 4)$, the principal order of $Q[\sqrt{m}\, i]$ with respect to Z is the set of elements of the form $k + \ell \sqrt{m}\, i$ for arbitrary $k, \ell \in Z$. If $m = 3 (\bmod 4)$, the principal order consists of all elements of the form $k + \frac{\ell}{2}(1 + \sqrt{m}\, i)$ for arbitrary $k, \ell \in Z$.

(2.3.5) Definition. Let m be a positive square-free integer. Then the principal order of $Q[\sqrt{m}\, i]$ with respect to Z will be denoted by $Z(m)$, and is usually referred to as the *ring of integers* of $Q[\sqrt{m}\, i]$.

CHAPTER 3

ANALYTIC SELF-MAPS OF A CONFORMAL TORUS

In the study of analytic maps between conformal tori, we shall pay special attention to the situation where both conformal tori are the same. One might expect a monoidal structure to exist on the set of all such morphisms and indeed, if we choose the proper representatives (see 1.2.17), that is precisely what we obtain. The structure of this monoid is that of a monoidal semi-direct product.

It is possible to determine the analytic automorphisms of a conformal torus and subsequently all analytic periodic self-maps can be determined.

§3.1 The Set of Analytic Self-Maps of a Conformal Torus

Let τ be a conformal torus. For any $h \varepsilon \tau$, the set of all analytic self-maps of $\mathbb{C}/\Gamma(h)$ is a monoid under composition of maps, with identity element $1_{\mathbb{C}/\Gamma(h)}$, the identity map of $\mathbb{C}/\Gamma(h)$. Since for any other $h' \varepsilon \tau$, $\mathbb{C}/\Gamma(h')$ is conformally equivalent to $\mathbb{C}/\Gamma(h)$, the monoid of analytic self-maps of $\mathbb{C}/\Gamma(h')$ is isomorphic to the monoid of analytic self-maps of $\mathbb{C}/\Gamma(h)$. Thus we have the next

<u>(3.1.1) Definition</u>. By the monoid of analytic self-maps $An(\tau)$ of a conformal torus τ we shall mean the monoid of analytic self-maps of $\mathbb{C}/\Gamma(h)$ for any $h \varepsilon \tau$.

For each $h \in H$, let $\pi_h : \mathbb{C} \to \mathbb{C}/\Gamma(h)$ denote the projection mapping, and let $(\overline{z})_h = \pi_h(z)$ for each $z \in \mathbb{C}$. When no confusion can result, the subscript h will be suppressed from the notation.

Now define addition in $\mathbb{C}/\Gamma(h)$ in the usual way: for each $\overline{z}_1$, $\overline{z}_2 \in \mathbb{C}/\Gamma(h)$, let $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$. With this operation, $\mathbb{C}/\Gamma(h)$ becomes an abelian group.

Let $An(\mathbb{C})$ denote the set of analytic self-maps of $\mathbb{C}$. Then by (1.2.15), the subset $\mathcal{F}(h)$ of $An(\mathbb{C})$ which consists of lifts of analytic self-maps of $\mathbb{C}/\Gamma(h)$ is given by

$$\mathcal{F}(h) = \{F_{(k,\ell),D} \mid (k,\ell) \in Z \times Z,\ D \in \mathbb{C}\} .$$

Now define a relation $F_{(k,\ell),\overline{D}}$ on $\mathbb{C}/\Gamma(h)$ for each $(k,\ell) \in Z \times Z$, $\overline{D} \in \mathbb{C}/\Gamma(h)$ in the following way: for each $\overline{z} \in \mathbb{C}/\Gamma(h)$, let $F_{(k,\ell),\overline{D}}(\overline{z}) = \overline{F_{(k,\ell),\, D}(z)}$.

<u>(3.1.2) Lemma</u>. For each $(k,\ell) \in Z \times Z$, $\overline{D} \in \mathbb{C}/\Gamma(h)$, $F_{(k,\ell),\overline{D}}$ is a function.

Proof: It must be shown that for $\omega \in \overline{z}$, $\Xi \in \overline{D}$, then $\overline{F_{(k,\ell),D}(z)} = \overline{F_{(k,\ell),\Xi}(\omega)}$. Since $\omega \in \overline{z}$, there exist integers m_1, n_1 such that $\omega = z + m_1 + n_1 h$. Similarly, since $\Xi \in \overline{D}$, there exist integers m_2, n_2 such that $\Xi = D + m_2 + n_2 h$. Thus $F_{(k,\ell),\Xi}(\omega) =$
$(k + \ell q s'(h)h)\omega + \Xi = (k + \ell q s'(h)h)z + D$
$+ (k + \ell q s'(h)h)(m_1 + n_1 h) + m_2 + n_2 h.$

Now if h is ample then by (2.3.3), it follows that the expression $(k + \ell qs'(h)h)(m_1 + n_1h)$ can be written as $m_3 + n_3h$, where m_3 and n_3 are the integers $km_1 - \ell n_1q'r(h)$ and $kn_1 + \ell n_1ps'(h) + \ell m_1qs'(h)$ respectively. Thus $F_{(k,\ell),E}(\omega) = F_{(k,\ell),D}(z) + (m_2 + m_3) + (n_2 + n_3)h$, and so $\overline{F_{(k,\ell),E}(\omega)} = \overline{F_{(k,\ell),D}(z)}$.

If h is non-ample, then $qs'(h) = 0$ and so $F_{(k,\ell),E}(\omega) = k\omega + E = k(z + m_1 + n_1h) + D + m_2 + n_2h = F_{(k,\ell),D}(z) + (km_1 + m_2) + (kn_1 + n_2)h$, whence $\overline{F_{(k,\ell),E}(\omega)} = \overline{F_{(k,\ell),D}(z)}$. ∇

<u>(3.1.3) Lemma</u>. For each $F_{(k,\ell),D} \in \mathfrak{F}(h)$, $F_{(k,\ell),\overline{D}}$ is the unique analytic self-map of $\mathbb{C}/\Gamma(h)$ for which

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F_{(k,\ell)D}"] \arrow[d, "\pi_h"'] & \mathbb{C} \arrow[d, "\pi_h"] \\
\mathbb{C}/\Gamma(h) \arrow[r] & \mathbb{C}/\Gamma(h)
\end{tikzcd}$$

commutes. Furthermore, the subset of $\mathfrak{F}(h)$ consisting of all lifts of $F_{(k,\ell),\overline{D}}$ is given by $\{F_{(k,\ell),E} \mid E \in \overline{D}\}$. Thus $An(\tau) = \{F_{(k,\ell),\overline{D}} \mid (k,\ell) \in Z \times Z,\ \overline{D} \in \mathbb{C}/\Gamma(h)\}$.

Proof: Since π_h is a covering map, if there exists a map making the diagram commute, it is unique. But $F_{(k,\ell),\overline{D}}$ is the function defined to make the diagram commutative.

Now suppose that $F_{(k,\ell),D}$ and $F_{(m,n),E}$ project onto the same analytic self-map of $\mathbb{C}/\Gamma(h)$.. If h is ample, then we must have $k + \ell q s'(h)h = m + nqs'(h)h$ and $\overline{D} = \overline{E}$. Thus $k + \ell q s'(h)R(h) = m + nqs'(h)R(h)$ and $\ell q s'(h)I(h) = nqs'(h)I(h)$. Since $qs'(h) \neq 0$, $\ell = n$, whence $k = m$. Thus $(k,\ell) = (m,n)$ and $\overline{D} = \overline{E}$. If h is non-ample, then since $qs'(h) = 0$, $k = m$ and $\overline{D} = \overline{E}$. ∇

§3.2 The Natural Monoidal Structure on the Set of Analytic Self-Maps

<u>(3.2.1) Lemma</u>. For each $\alpha, \beta \in Q$, define an operation $*_{(\alpha,\beta)}$ on $Q \times Q$ as follows: for each $(r_1,r_2), (s_1,s_2) \in Q \times Q$, let

$$(r_1,r_2)*_{(\alpha,\beta)}(s_1,s_2) = (r_1s_1 - r_2s_2\alpha\ ,\ r_1s_2 + r_2s_1 + r_2s_2\beta).$$

Then $\langle Q \times Q, *_{(\alpha,\beta)}, (1,0)\rangle$ is an abelian monoid. The non-zero elements of $Q \times Q$ form a group under $*_{(\alpha,\beta)}$ iff the quadratic form $\gamma x^2 + xy\gamma\beta + y^2\gamma\alpha$ is definite, where $\gamma \in Z^+$ is such that both $\gamma\alpha$, $\gamma\beta$ belong to Z.

Proof: It is readily verified that $*_{(\alpha,\beta)}$ is an associative abelian operation, and that $(1,0)$ is an identity for this operation. Thus $\langle Q \times Q, *_{(\alpha,\beta)}, (1,0)\rangle$ is an abelian monoid. To prove that every non-zero element of $Q \times Q$ is invertible iff the given quadratic form is definite, we compute what the inverse of an arbitrary element must be if it exists. Let $(r_1,r_2), (s_1,s_2) \in Q \times Q \sim \{(0,0)\}$

be such that $(r_1,r_2) *_{(\alpha,\beta)} (s_1,s_2) = (1,0)$. Since $*_{(\alpha,\beta)}$ is abelian, this will imply that $(s_1,s_2) = (r_1,r_2)^{-1}$. By definition, $(r_1,r_2) *_{(\alpha,\beta)} (s_1,s_2) = (r_1s_1 - r_2s_2\alpha , r_1s_2 + r_2s_1 + r_2s_2\beta) = (1,0)$. Thus $r_1s_1 - r_2s_2\alpha = 1$ and $r_1s_2 + r_2s_1 + r_2s_2\beta = 0$. We consider the three possible cases:

i) $r_1 = 0$. Then $r_2 \neq 0$ and α cannot be zero since $r_2s_2\alpha = -1$. So $s_2 = -1/\alpha r_2$ and $s_1 = \beta/\alpha r_2$.

ii) $r_2 = 0$. Then $r_1 \neq 0$ and $s_1 = 1/r_1$, $s_2 = 0$.

iii) $r_1, r_2 \neq 0$. Then $s_1 = \dfrac{1 + r_2s_2\alpha}{r_1}$ and so $(r_1^2 + r_1r_2\beta + r_2^2\alpha)\, s_2 = -r_2$, whence $r_1^2 + r_1r_2\beta + r_2^2\alpha \neq 0$. Finally, $(r_1^2 + r_1r_2\beta + r_2^2\alpha)s_1 = r_1 + r_2\beta$. Thus we conclude that $(r_1,r_2)^{-1}$ exists iff $r_1^2 + r_1r_2\beta + r_2^2\alpha \neq 0$. ∇

<u>(3.2.2) Corollary</u>. If $\alpha, \beta \in Z$, then $\langle Z \times Z, *_{(\alpha,\beta)}, (1,0)\rangle$ is a submonoid of $\langle Q \times Q, *_{(\alpha,\beta)}, (1,0)\rangle$.

Let τ be a conformal torus and let $h \in \tau$. If $\alpha = \alpha(h) = qs'(h)q'r(h)$ and $\beta = \beta(h) = ps'(h)$, then $\alpha, \beta \in Z$.

<u>(3.2.3) Definition</u>. For $\alpha = \alpha(h)$, $\beta = \beta(h)$, the operation $*_{(\alpha,\beta)}$ shall be denoted by $*_h$.

Thus $\langle Z \times Z, *_h, (1,0)\rangle$ is an abelian monoid, for each $h \in H$.

(3.2.4) Lemma. For $F_{(k,\ell),\bar{D}}, F_{(m,n),\bar{E}} \in An(\tau)$,

$$F_{(k,\ell),\bar{D}} \circ F_{(m,n),\bar{E}} = F_{(k,\ell) *_h (m,n),\, \overline{D + C^h(k,\ell)E}}$$

for every $h \in \tau$, and $(k,\ell) \in Z \times Z$.

Proof: For each $\bar{z} \in \text{¢}/\Gamma(h)$, we have

$F_{(k,\ell),\bar{D}} \circ F_{(m,n),\bar{E}}(\bar{z}) = F_{(k,\ell),\bar{D}}(\overline{F_{(m,n),E}(z)})$

$= \overline{F_{(k,\ell),D} \circ F_{(m,n),E}(z)}$. But $F_{(k,\ell),D} \circ F_{(m,n),E}(z) =$ $C^h(k,\ell)C^h(m,n)z + C^h(k,\ell)E + D$ and by (2.3.3) this is equal to

$\{(km - \ell nqs'(h)q'r(h)) + (kn + \ell m + \ell nps'(h))qs'(h)h\}z$

$+ C^h(k,\ell)E + D$

$= C^h((k,\ell) *_h (m,n))z + D + C^h(k,\ell)E = F_{(k,\ell)*_h(m,n),D+C^h(k,\ell)E}(z)$.

Now recall that $\mathrm{Hom}(\langle \text{¢}/\Gamma(h), \oplus, \bar{0}\rangle)$ is the set of all monoid homomorphisms of $\text{¢}/\Gamma(h)$ into itself, as described in §2.1.

(3.2.5) Definition. Let $\psi: Z \times Z \to \mathrm{Hom}(\text{¢}/\Gamma(h))$ be the function given by $\psi_{(k,\ell)} = F_{(k,\ell)\,\bar{0}}$ for each $(k,\ell) \in Z \times Z$.

(3.2.6) Lemma. ψ is a monoid homomorphism of $\langle Z \times Z, *_h, (1,0)\rangle$ into $\langle \mathrm{Hom}(\text{¢}/\Gamma(h)), \circ, 1_{\text{¢}/\Gamma(h)}\rangle$.

Proof: Firstly, we have $\psi_{(1,0)} = F_{(1,0),\bar{0}} = 1_{\text{¢}/\Gamma(h)}$.

Now for any $(k,\ell), (m,n) \in Z \times Z$:

$\psi_{(k,\ell)*_h(m,n)}(\bar{z}) = F_{(k,\ell)*_h(m,n),\bar{0}}(\bar{z}) = F_{(k,\ell),\bar{0}} \circ F_{(m,n),\bar{0}}(\bar{z})$

for all $\bar{z} \in ¢/\Gamma(h)$, by (3.2.4) and thus

$$\psi_{(k,\ell)*_h(m,n)} = \psi_{(k,\ell)} \circ \psi_{(m,n)} \quad .$$

∇

Thus we obtain the monoidal semi-direct product of $\langle Z \times Z, *_h, (1,0)\rangle$ and $\langle ¢/\Gamma(h), \oplus , \bar{0}\rangle$ by ψ , which we denote by $Z \times Z \times_\psi ¢/\Gamma(h)$. Let the function $\Psi: Z \times Z \times_\psi ¢/\Gamma(h) \rightarrow An(\tau)$ be defined as follows: for each $(k,\ell) \in Z \times Z$, $\bar{D} \in ¢/\Gamma(h)$, let $\Psi(k,\ell,\bar{D}) = F_{(k,\ell),\bar{D}}$.

<u>(3.2.7) Theorem</u>. If τ is an ample conformal torus, then Ψ is a monoid isomorphism of $Z \times Z \times_\psi ¢/\Gamma(h)$ onto $An(\tau)$.

Proof: The identity of $Z \times Z \times_\psi ¢/\Gamma(h)$ is $(1,0,\bar{0})$ and we have $\Psi(1,0,\bar{0}) = F_{(1,0),\bar{0}} = 1_{¢/\Gamma(h)}$, the identity of $An(\tau)$. For any (k,ℓ), $(m,n) \in Z \times Z$, and $\bar{D}$, $\bar{E} \in ¢/\Gamma(h)$, we have

$$\begin{aligned}\Psi((k,\ell,\bar{D}) \circledast (m,n,\bar{E})) &= F_{(k,\ell)*_h(m,n),\overline{D + C^h(k,\ell)E}} \\ &= F_{(k,\ell),\bar{D}} \circ F_{(m,n),\bar{E}} \\ &= \Psi(k,\ell,\bar{D}) \circ \Psi(m,n,\bar{E}).\end{aligned}$$

Thus Ψ is a monoid homomorphism. By (3.1.3), Ψ is bijective, hence it is a monoid isomorphism.

∇

Denote $Z \times \{0\} \times_\psi ¢/\Gamma(h)$ by $Z \times_\psi ¢/\Gamma(h)$ and let the restriction of Ψ to $Z \times \{0\} \times ¢/\Gamma(h)$ be denoted by Ψ as well.

<u>(3.2.8) Theorem</u>. If τ is a non-ample conformal torus, then Ψ is a monoid isomorphism of $Z \times_\psi \mathbb{C}/\Gamma(h)$ onto $An(\tau)$.

Proof: This proceeds exactly as did the proof of (3.2.7). ∇

Let S^1 be the set $\{z \in \mathbb{C} \mid |z| = 1\}$ with addition on S^1 defined to be addition of arguments, modulo 2π. Then S^1 is an abelian group and we may form the group $S^1 \times S^1$ with addition defined componentwise. It is readily seen that the group $\langle \mathbb{C}/\Gamma(h), \oplus, \bar{0} \rangle$ is isomorphic to $S^1 \times S^1$, whence the monoids $Hom(\mathbb{C}/\Gamma(h))$ and $Hom(S^1 \times S^1)$ are isomorphic. It follows that we can consider the monoid homomorphism $\psi: Z \times Z \to Hom(\mathbb{C}/\Gamma(h))$ as a monoid homomorphism $\psi: Z \times Z \to Hom(S^1 \times S^1)$. Thus we have

<u>(3.2.9) Theorem</u>. If τ is an ample conformal torus, then $An(\tau)$ is a monoidal semi-direct product $Z \times Z \times_\psi S^1 \times S^1$. If τ is a non-ample conformal torus, then $An(\tau)$ is a monoidal semi-direct product $Z \times_\psi S^1 \times S^1$.

Furthermore, if τ is ample, then with the notation of (2.1.4), the sequence

$$0 \to S^1 \times S^1 \xrightarrow{j} Z^* \times Z^* \times_\psi S^1 \times S^1 \underset{\beta}{\overset{\pi}{\rightleftarrows}} \langle Z \times Z, *_h \rangle \to 0$$

is a short split exact sequence of monoids, where $Z^* = Z \sim \{0\}$. Since $S^1 \times S^1$ is a group, (2.1.5) applies, and we obtain $(Z^* \times Z^* \times_\psi S^1 \times S^1)/(S^1 \times S^1) \cong \langle Z^* \times Z^*, *_h \rangle$.

Similarly, if τ is non-ample, then we obtain the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & S^1 \times S^1 \arrow[r, "j"] & Z^* \times_\psi S^1 \times S^1 \arrow[r, "\pi"] & Z^* \arrow[r] \arrow[l, bend left, "\beta"] & 0 \end{tikzcd}$$

and analagously, $(Z^* \times_\psi S^1 \times S^1)/(S^1 \times S^1) \simeq Z^*$.

<u>(3.2.10) Theorem</u>. Let τ be a conformal torus. Then for any $h, h' \in \tau$, $\langle Z \times Z, *_h \rangle = \langle Z \times Z, *_{h'} \rangle$.

Proof: As mentioned in §3.1, the monoids $Z \times Z \times_\psi \text{¢}/\Gamma(h)$ and $Z \times Z \times_\psi \text{¢}/\Gamma(h')$ are isomorphic. Then by (2.1.6) the theorem follows. ∇

§3.3 Analytic Automorphisms of a Conformal Torus

<u>(3.3.1) Definition</u>. Let τ be a conformal torus. Then if $f \in An(\tau)$, f is said to be an *analytic automorphism* of τ iff $f^{-1} \in An(\tau)$. The set of all analytic automorphisms of τ is denoted by $Aut(\tau)$.

It is clear that $Aut(\tau)$ is a group under composition of maps. To determine this group, we must determine the invertible elements of the monoid $An(\tau)$. In the non-ample case, this is readily done. For consider $(k,\overline{D}), (\ell,\overline{E}) \in Z \times \text{¢}/\Gamma(h)$ for any $h \in \tau$. Suppose that $(k,\overline{D}) \circledast (\ell,\overline{E}) = (1,\overline{0})$. Then $(k\ell, \overline{D} \oplus \overline{kE}) = (1,\overline{0})$ whence $k\ell = 1$ and $\overline{D} = -\overline{kE}$. This implies that $k = \ell = \pm 1$. For $k = 1$, $\overline{D} = -\overline{E}$ and for $k = -1$, then $\overline{D} = \overline{E}$. Thus we have

<u>(3.3.2) Lemma</u>. If τ is a non-ample conformal torus, then $Aut(\tau) = \{(\pm 1,\overline{D}) \mid \overline{D} \in \text{¢}/\Gamma(h)\}$ for $h \in \tau$.

When τ is ample, $\mathrm{Aut}(\tau)$ is not so easily determined. It is necessary to determine the invertible elements of $\langle Z \times Z, *_h \rangle$ first. Recall that this monoid is a submonoid of $\langle Q \times Q, *_h \rangle$.

<u>(3.3.3) Lemma</u>. Let τ be an ample conformal torus. Then for any $h \in \tau$, the quadratic form $x^2 + xy\beta(h) + y^2\alpha(h)$ is positive definite.

Proof. It is necessary and sufficient to show that $\beta^2 - 4\alpha < 0$. But $\beta^2 - 4\alpha = (ps')^2 - 4qs'q'r =$ $(2qs')^2((\frac{p}{2q})^2 - \frac{r}{s}) = (2qs')^2(R(h)^2 - |h|^2) < 0$ since $qs'(h)I(h) > 0$.
∇

<u>(3.3.4) Corollary</u>. Let τ be an ample conformal torus. Then for any $h \in \tau$, $\langle Q \times Q \sim \{(0,0)\}, *_h, (1,0) \rangle$ is an abelian group.

Proof: By (3.2.1), it is necessary and sufficient to show that $x^2 + xy\beta + y^2\alpha$ is a definite quadratic form. But by (3.3.3) it is positive definite.
∇

Thus every non-zero element of $Q \times Q$ is invertible. Our task is to determine the non-zero elements (k,ℓ) of $Z \times Z$ for which $(k,\ell)^{-1}$ belongs to $Z \times Z$. By (3.2.1),

$$(k,\ell)^{-1} = \left(\frac{k + \ell\beta}{k^2 + k\ell\beta + \ell^2\alpha}, \frac{-\ell}{k^2 + k\ell\beta + \ell^2\alpha}\right).$$

If $(k,\ell)^{-1} \in Z \times Z$, then $k^2 + k\ell\beta + \ell^2\alpha$ divides both k and ℓ, whence $[k,\ell] = 1$. Since $x^2 + xy\beta + y^2\alpha$ is positive definite, we must have $k^2 + k\ell\beta + \ell^2\alpha = 1$. Conversely, if $k^2 + k\ell\beta + \ell^2\alpha = 1$, then certainly $(k,\ell)^{-1} \in Z \times Z$. Thus the group consisting of all invertible elements of $\langle Z \times Z, *_h \rangle$ is $\{(k,\ell) \in Z \times Z \mid k^2 + k\ell\beta + \ell^2\alpha = 1\}$. We remark here that the map from $Z \times Z$ into $Z^+ \cup \{0\}$ which takes (k,ℓ) to $k^2 + k\ell\beta + \ell^2\alpha$ is a monoid homomorphism of $\langle Z \times Z, *_h \rangle$ into $\langle Z^+ \cup \{0\}, \circ \rangle$.

At this stage, it is obvious that for every conformal torus τ, $\mathrm{Aut}(\tau)$ will contain the elements $(\pm 1, 0, \bar{D})$ for every $\bar{D} \in ¢/\Gamma(h)$. What is perhaps not so obvious is that in general, these are the only analytic automorphisms of τ. In order to establish this, we shall choose special representatives for the conformal tori. It is well-known that every conformal torus has exactly one representative h such that $-\frac{1}{2} < R(h) \leq \frac{1}{2}$, $|h| \geq 1$ and $|h| = 1$ only if $R(h) \geq 0$.

<u>(3.3.5) Definition</u>. Let τ be a conformal torus. Then the representative $h \in \tau$ for which $-\frac{1}{2} < R(h) \leq \frac{1}{2}$, $|h| \geq 1$ and $|h| = 1$ only if $R(h) \geq 0$ is called the *fundamental representative* of τ.

Thus for the fundamental representative h of τ we shall calculate the invertible elements of $\langle Z \times Z, *_h \rangle$. Since we are considering ample conformal tori τ, the fundamental representative is ample. Thus $2R(h) = \frac{p(h)}{q(h)}$, and $|h|^2 = \frac{r(h)}{s(h)}$. For this section we shall assume that all functions defined on H are evaluated at h and hence no specific indication of the variable h shall appear.

Thus we have $-\frac{1}{2} < \frac{p}{2q} \leq \frac{1}{2}$ and $\frac{r}{s} \geq 1$, whence

$$|p| \leq q \text{ and } r \geq s . \tag{1}$$

For $(k,\ell) \in Z \times Z \sim \{(0,0)\}$ we have $R(k + \ell qs'h) = k + \ell qs'R(h) = k + \ell\frac{ps'}{2} = \frac{1}{2}(2k + \ell ps')$. Let us define

$$\gamma = \gamma_h(k,\ell) = 2k + \ell ps'. \tag{2}$$

Then $R(k + \ell qs'h) = \frac{\gamma}{2}$. Now suppose that $(k,\ell)^{-1} \in Z \times Z$, whence $k^2 + k\ell\beta + \ell^2\alpha = 1$. Observe that $|k + \ell qs'h|^2 = k^2 + k\ell\beta + \ell^2\alpha$. Thus $|R(k + \ell qs'h)| \leq 1$ and so $|\gamma| \leq 2$. Since γ is an integer-valued function, it follows that if $(k,\ell)^{-1}$ belongs to $Z \times Z$, then $\gamma(k,\ell) = 0, \pm 1$ or ± 2. Let us introduce the expression

$$\Delta = \Delta_h = 4qq'r - p^2s' . \tag{3}$$

From (1), $4qq's \leq 4qq'r$ and $p^2s' \leq q^2s' = qq's$, since both q and s are positive. Thus $4qq's \leq 4qq'r = \Delta + p^2s' \leq \Delta + qq's$, whence

$$\Delta \geq 3qq's. \tag{4}$$

We now determine the conformal tori and pairs (k,ℓ) for which γ can attain the allowable values.

Case 1: Suppose $\gamma(k,\ell) = \pm 2$. Then $R(k + \ell qs'h) = \pm 1$. Since $|k + \ell qs'h| = 1$, this implies that $\ell qs'h = 0$. But $qs'h \neq 0$, and so $\ell = 0$. Thus $k = \pm 1$ and so $(\pm 1,0)$ are the only pairs (k,ℓ) for which $\gamma(k,\ell) = \pm 2$. There are no restrictions on the conformal tori in this case.

Case 2: Suppose $\gamma(k,\ell) = \pm 1$. Then $R(k + \ell qs'h) = \pm \frac{1}{2}$, whence $(\ell qs'I(h))^2 = \frac{3}{4}$. Since $I(h)^2 = |h|^2 - R(h)^2$, this implies that $(\ell qs')^2(\frac{r}{s} - (\frac{p}{2q})^2) = \frac{3}{4}$, hence that $\ell^2 s'(4qq'r - p^2 s') = 3$. From (3), we have $\ell^2 \Delta s' = 3$ and so Δ, ℓ^2 and s' divide 3. Thus $\ell = \pm 1$ and $\Delta = 1$ or 3. By (4) we see that $\Delta = 3$. Thus $qq's \leq 1$ and $q = s = 1$. This implies that $I(h)^2 = \frac{3}{4}$, whence $|h|^2 \leq 1$. But h is a fundamental representative, hence $|h|^2 \geq 1$. Therefore $|h| = \frac{r}{s} = 1$. From (3), $\Delta = 4 - p^2$, and so $p = \pm 1$. For $p = -1$, we obtain $R(h) = -\frac{1}{2}$, which is not possible since h is fundamental and $|h| = 1$. For $p = 1$, $R(h) = \frac{1}{2}$ and $I(h) = \frac{\sqrt{3}}{2}$. Thus for the conformal torus represented by $h = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, γ_h may take the values ± 1 in addition to ± 2. It has been determined already that if $\gamma(k,\ell) = \pm 1$, then $\ell = \pm 1$. From (2) we have $2k + \ell = \pm 1$. Thus for $\ell = 1$, the possible values for k are 0 and -1 For $\ell = -1$, k may be 1 or 0. Thus the possible pairs (k,ℓ) for which $\gamma_h(k,\ell) = \pm 1$ are $(0,\pm 1)$, $(1,-1)$ and $(-1,1)$.

Case 3: Suppose that $\gamma(k,\ell) = 0$. Then $R(k + \ell qs'h) = k + \ell qs'R(h) = 0$, whence $\ell qs'I(h) = \pm 1$. Since $I(h) \geq \frac{\sqrt{3}}{2}$, we must have $I(h) = 1$ and so $\ell qs' = \pm 1$.

Thus $q = s = 1$ and $\ell = \pm 1$. If $\ell = 1$ then $k = -R(h)$. If $\ell = -1$, then $k = R(h)$. In either situation, since k is an integer, it can only be zero. Thus $k = R(h) = 0$. We have shown that only for the conformal torus $h = i$ can γ_h take the value 0 in addition to ± 2. The possible pairs (k,ℓ) for which $\gamma_i(k,\ell)$ can equal zero are $(0,\pm 1)$.

<u>(3.3.6) Theorem</u>. Let τ be a conformal torus. The automorphism group $\mathrm{Aut}(\tau)$ is a semi-direct product of a cyclic group of order four and the group $S^1 \times S^1$ if τ is represented by i, a semi-direct product of a cyclic group of order six and $S^1 \times S^1$ if τ is represented by $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, and for all other conformal tori, $\mathrm{Aut}(\tau)$ is a semi-direct product of a cyclic group of order two and $S^1 \times S^1$.

Proof: The semi-direct product structure is just the restriction of the monoidal semi-direct product structure to the group of analytic automorphisms. For a non-ample torus τ, then as was seen in (3.3.2), $\mathrm{Aut}(\tau) = \{(\pm 1,0)\} \times_{\psi} \mathbb{C}/\Gamma(h)$ for any $h \in \tau$. As a submonoid of $\langle Z \times Z, *_h, (1,0) \rangle$, $\{(\pm 1,0)\}$ is a cyclic group of order two.

Now if τ is ample, let $(k,\ell,D), (m,n,E) \in \mathrm{An}(\tau)$. Suppose that $(k,\ell,\overline{D}) \circledast (m,n,\overline{E}) = (m,n,\overline{E}) \circledast (k,\ell,\overline{D}) = (1,0,\overline{0})$. Then $(m,n,\overline{E}) = (k,\ell,\overline{D})^{-1}$. Now $(k,\ell,\overline{D}) \circledast (m,n,\overline{E}) = ((k,\ell) *_h (m,n), \overline{D} \oplus \psi_{(k,\ell)}(\overline{E})) = (1,0,\overline{0})$, whence $(m,n) = (k,\ell)^{-1}$

and $\psi_{(k,\ell)}(\bar{E}) = -\bar{D}$. As well, $(m,n,\bar{E}) \circledast (k,\ell,\bar{D}) = ((m,n)*_h(k,\ell), \bar{E} \oplus \psi_{(m,n)}(\bar{D})) = (1,0,\bar{0})$ whence $(m,n) = (k,\ell)^{-1}$ and $-\bar{E} = \psi_{(m,n)}(\bar{D})$.

Thus if $(k,\ell)^{-1} \in Z \times Z$, then for every $\bar{D} \in$ ¢$/\Gamma(h)$, $(k,\ell,D)^{-1}$ exists. It is clear that every ample torus τ admits the conformal automorphisms $(\pm 1,0,\bar{D})$ for all $\bar{D} \in$ ¢$/\Gamma(h)$, for any $h \in \tau$. Furthermore, as in the non-ample case, $\langle \{(\pm 1,0)\}, *_h, (1,0)\rangle$ is a cyclic group of order two.

We have seen that for all conformal tori, except possibly the two represented by i and $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, these are the only analytic automorphisms. We now consider the remaining two conformal tori.

Case 1: Let τ be the conformal torus represented by $h = i$. Then $p(h) = 0$, $q(h) = s(h) = r(h) = 1$, and so the operation $*_h$ is given by $(k,\ell)*_h(m,n) = (km - \ell n, kn + \ell m)$ for all (k,ℓ), (m,n) in $Z \times Z$. By the argument preceding the theorem, the candidates for invertible elements in $\langle Z \times Z, *_h\rangle$ are $(\pm 1,0)$ and $(0,\pm 1)$. It is readily verified that $\langle\{(\pm 1,0),(0,\pm 1)\}, *_h, (1,0)\rangle$ is a cyclic group of order four, generated by either $(0,\pm 1)$. In fact, $(0,-1)^2 = (-1,0)$, $(0,-1)^3 = (0,1)$, and $(0,-1)^4 = (1,0)$. Thus for the conformal torus τ represented by i, $\mathrm{Aut}(\tau)$ is a semi-direct product of a cyclic group of order four and $S^1 \times S^1$.

Case 2: Let τ be the conformal torus represented by $h = \frac{1}{2} + \frac{\sqrt{3}}{2} i$. Then $p(h) = q(h) = r(h) = s(h) = 1$. Thus the operation $*_h$ is given by $(k,\ell)*_h(m,n) = (km - \ell n, kn + \ell m + \ell n)$ for all $(k,\ell), (m,n) \in Z \times Z$. The candidates for invertible elements of $\langle Z \times Z, *_h, (1,0)\rangle$ are $(\pm 1,0)$, $(0,\pm 1)$, $(1,-1)$ and $(-1,1)$. Again it readily follows that $\langle\{(\pm 1,0),(0,\pm 1),(1,-1),(-1,1)\}, *_h, (1,0)\rangle$ is a cyclic group of order six, generated by $(1,-1)$ or $(0,1)$. In fact, $(1,-1)^2 = (0,-1)$, $(1,-1)^3 = (-1,0)$, $(1,-1)^4 = (-1,1)$, $(1,-1)^5 = (0,1)$ and $(1,-1)^6 = (1,0)$. Thus $\operatorname{Aut}(\tau)$ is a semi-direct product of a cyclic group of order six and $S^1 \times S^1$. ∇

§3.4 Periodic Analytic Self-Maps

Since an analytic map between two conformal tori was seen, in §1.2, to be a covering map, it follows that if a periodic analytic self-map of a conformal torus exists, it must be a one-sheeted map, hence an analytic automorphism. This fact also follows from a purely group theoretic argument.

With the classification of the analytic automorphisms given in the previous section, it is now possible to determine, for every conformal torus τ, the torsion subgroup of $\operatorname{Aut}(\tau)$, namely the set of all periodic analytic self-maps of τ.

Let τ be a conformal torus and let $h \in \tau$. Then for any $\overline{D} \in \mathbb{C}/\Gamma(h)$, and any $n \in Z^+$, $(1,0,\overline{D})^n = (1,0,\overline{nD})$.

It is evident that if $(1,0,\overline{D})$ is to be periodic, of period n, then $\overline{nD} = \overline{D}$. This is of course only a necessary condition. If $\overline{D}$ is such that n is the least positive integer such that $\overline{nD} = \overline{D}$, then the condition is also sufficient.

<u>(3.4.1) Lemma</u>. Let $h \in H$ and $n \in Z^+$, $n > 1$. Then the set of all z lying within the parallelogram with vertices 0, 1, h and h + 1 and such that $nz = z + m + kh$ for some integers m and k, and such that n is the least positive integer greater than 1 for which this is true is given by:

$$\left\{ \frac{m + kh}{n - 1} \;\middle|\; 0 \le m,\ k \le n - 1,\ [m,k,n - 1] = 1 \right\}.$$

Proof: It is clear that $S = \{\overline{\frac{m + kh}{n - 1}} \mid 0 \le m,k \le n-1\}$ is the set of all elements of $¢/\Gamma(h)$ whose order divides $n - 1$. Let $z \in S$ and suppose that there exists an integer ℓ with $2 < \ell < n$, such that $\ell - 1$ divides $n - 1$, say $n - 1 = g(\ell - 1)$, and for which there exist integers a,b such that $\ell z = z + a + bh$. Then

$$z = \frac{m + kh}{n - 1} = \frac{a + bh}{\ell - 1} \text{ and so } \frac{m + kR(h)}{n - 1} = \frac{a + bR(h)}{\ell - 1}, \text{ and}$$

$$\frac{kI(h)}{n - 1} = \frac{bI(h)}{\ell - 1}.$$ Since $I(h) > 0$, $\frac{k}{b} = \frac{n - 1}{\ell - 1}$, whence

$\frac{m}{a} = \frac{n - 1}{\ell - 1}$, if a and b are non-zero. If either is zero, the situation is very simple. So assume that they are both non-zero. Then g divides k, m and of course $n - 1$, hence g divides $[m,k,n - 1]$. Conversely, if g divides $[m,k,n-1]$ then for $a = \frac{m}{g}$, $b = \frac{k}{g}$ and $\ell = \frac{n - 1}{g} + 1$, we have $\ell z = z + a + bh$. Thus those and only those elements $\frac{m + kh}{n - 1}$ for which $[m,k,n - 1] = 1$ are of period n. Thus for any positive integer $n > 1$, every conformal torus τ has maps of period n and the set of all such maps of the form $(1,0,\bar{D})$ is the set $P_n(h) = \{(1,0,\bar{D}) \mid D = \frac{m + kh}{n - 1}$, $[m,k,n - 1] = 1\}$. ∇

Consider now the analytic automorphisms of τ of the form $(-1,0,\bar{D})$. Observe that, for any $h \in \tau$, $(-1,0,\bar{D})^2 = ((-1,0)*_h(-1,0),\bar{D} \oplus -\bar{D}) = (1,0,\bar{0})$ and so every such map is periodic of period three.

Thus all periodic maps of every conformal torus, except possible the conformal tori represented by i and $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, have been determined.

It is clear that the torsion subgroup of $\mathrm{Aut}(\tau)$ for both of the remaining conformal tori contains the above described maps.

For the conformal torus represented by i, we may have additional elements of the form $(0,\pm 1,\bar{D})$. In this monoid, we know that $(0,\pm 1)$ are both of order four. It is readily verified that for each $\bar{D} \in \mathbb{C}/\Gamma(h)$, $(0,\pm 1,\bar{D})^4 = (1,0,\bar{0})$.

Thus every analytic automorphism of the form $(0,\pm 1,\overline{D})$, $\overline{D} \in \mathbb{C}/\Gamma(h)$ is a periodic map, of period five.

Finally, for the conformal torus represented by $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, it is possible to have periodic maps of the form $(0,\pm 1,\overline{D})$, $(1,-1,\overline{D})$, and $(-1,1,\overline{D})$. We know that $(0,1)$ and $(1,-1)$ are of order six, and $(0,-1)$ and $(-1,1)$ are easily shown to be of order three. In fact, for every $\overline{D} \in \mathbb{C}/\Gamma(\frac{1}{2} + \frac{\sqrt{3}}{2} i)$, it is routine to show that $(0,1,\overline{D})^6 = (1,-1,\overline{D})^6 = (0,-1,\overline{D})^3 = (-1,1,\overline{D})^3 = (1,0,\overline{0})$. Thus every analytic automorphism of the form $(1,-1,\overline{D})$ or $(0,1,\overline{D})$ is periodic of period seven, while every analytic automorphism of the form $(-1,1,\overline{D})$ or $(0,-1,\overline{D})$ is periodic of period four.

Thus we obtain the following

<u>(3.4.2) Theorem</u>. Let τ be a conformal torus. If τ is not the conformal torus with fundamental representative either i or $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, then the torsion subgroup T of $\mathrm{Aut}(\tau)$ is given by

$$T = \{\bigcup_{n=2}^{\infty} P_n(h)\} \cup \{(-1,0,\overline{D}) \mid \overline{D} \in \mathbb{C}/\Gamma(h)\}$$

for any $h \in \tau$. The elements of P_n are of period n, while the elements of the form $(-1,0,\overline{D})$ are of period three.

If τ is represented by i, then the torsion subgroup of $\mathrm{Aut}(\tau)$ contains, in addition to the elements of T, the set $\{(0,\pm 1,\overline{D}) \mid \overline{D} \in ¢/\Gamma(i)\}$. Each such element is of period five.

If τ is represented by $\frac{1}{2} + \frac{\sqrt{3}}{2} i$, then the torsion subgroup of $\mathrm{Aut}(\tau)$ contains, in addition to T, the set $\{(0,1,\overline{D}), (1,-1,\overline{D}) \mid \overline{D} \in ¢/\Gamma(\frac{1}{2} + \frac{\sqrt{3}}{2} i)\}$, with elements of period seven, and the set

$$\{(0,-1,\overline{D}), (-1,1,\overline{D}) \mid \overline{D} \in ¢/\Gamma(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i)\}$$

with elements of period four.

CHAPTER 4

THE LATTICE OF COMPLEX DISTORTIONS

To each ordered pair of immersion-equivalent conformal tori there is associated a family of complex lattices, and each such lattice provides a representation of the analytic maps between the conformal tori. It is our purpose in this chapter to investigate the relationship between members of this family of lattices and to further display the symmetry of the immersion equivalence relation by showing that the lattices associated with the pair τ_1 and τ_2 are the reflections in the imaginary axis of the lattices associated with the pair τ_2 and τ_1.

§4.1 The Distortion Coefficients of Analytic Maps

Let τ_1 and τ_2 be immersion-equivalent conformal tori. In (1.2.17) it was shown that for each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, $\mathrm{An}(\tau_1,\tau_2)$ is given by $\{F_{(k,\ell),\bar{D}} \mid (k,\ell) \in Z \times Z,\ \bar{D} \in \mathbb{C}/\Gamma(h)\}$, where

$$F_{(k,\ell),\bar{D}}((\bar{z})_{ah}) = (\overline{C_a^h(k,\ell)z + D})_h,\quad C_a^h(k,\ell) = k + \ell q s'(h)\tilde{a}(h)h.$$

<u>(4.1.1) Definition</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. For each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, let $\mathcal{L}(a,h) = \{C_a^h(k,\ell) \mid (k,\ell) \in Z \times Z\}$. Then $\mathcal{L}(a,h)$ is said to be a *lattice of complex distortions* associated with $\mathrm{An}(\tau_1,\tau_2)$.

It is clear that if τ_1 and τ_2 are non-ample, then for each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, $\mathcal{L}(a,h) = Z \subset \mathbb{C}$. Thus for non-ample conformal tori, there is nothing more to be determined about their associated lattices. For the ample tori, the situation is more interesting. For example, every lattice of complex distortions associated with $\mathrm{An}(\tau_1,\tau_2)$ for ample immersion-equivalent conformal tori τ_1 and τ_2 contains, in addition to Z, a purely imaginary sublattice. In general, the full lattice properly contains the sublattice generated by Z and the largest purely imaginary sublattice.

In order to provide an adequate framework for such an investigation, we shall examine the analytic immersion class.

(4.1.2) Theorem. Let $\mathfrak{M}$ be an ample analytic immersion class in H. Then there exists a unique positive square-free integer m such that $\sqrt{m}\, i \in \mathfrak{M}$.

Proof: Let $h \in \mathfrak{M}$. Then there are integers a, b, c, d and m such that $h = \frac{a}{b} + \frac{c\sqrt{m}}{d} i$, with $cd > 0$, $m > 0$ and m is square-free. Define

$$T = \begin{bmatrix} \frac{d}{c} & -\frac{ad}{bc} \\ 0 & 1 \end{bmatrix} .$$

Then $T \in GL^+(2,Q)$ and $T(h) = \sqrt{m}\, i$. Thus $\sqrt{m}\, i \in \mathfrak{M}$.

Suppose that n is a positive square-free integer such that $\sqrt{n}\, i \in \mathfrak{M}$. Then there exist integers $\alpha, \beta, \gamma, \delta$ such that $\alpha\sqrt{m}\, i + \beta = \sqrt{n}\, i(\gamma\sqrt{m}\, i + \delta)$, whence $\sqrt{\frac{n}{m}} \in Q$. But this is only possible if $n = m$. ∇

(4.1.3) Definition. Let m be a positive, square-free integer. The ample analytic immersion class containing $\sqrt{m}\, i$ is denoted by $\mathfrak{M}_m$.

Thus for each $h \in \mathfrak{M}_m$, there exists a $T \in GL^+(2,Q)$ such that $h = T(\sqrt{m}\, i)$.

(4.1.4) Corollary. For each $h \in \mathfrak{M}_m$, $Q[h] = Q[\sqrt{m}\, i]$.

It is now possible to uniquely assign an imaginary quadratic number field to each ample analytic immersion class.

(4.1.5) Definition. To each analytic immersion class $\mathfrak{M}_m$ assign the imaginary quadratic number field $K(m) = Q[\sqrt{m}\, i]$. The ring of integers of $K(m)$ is denoted by $Z(m)$.

By (4.1.4), $K(m)$ is equal to $Q[h]$ for each $h \in \mathfrak{M}_m$.

(4.1.6) Lemma. Let $h \in H$ be ample. Then for each $T \in SL^+(2,Z)$, $qs'(h)I(h) = qs' \circ T(h)I(T(h))$.

Proof: Let $\mathfrak{M}_m$ be the ample analytic immersion class containing h and $T(h)$. Then there are integers $\alpha, \beta, \gamma, \delta$ such that $h = \frac{\alpha}{\beta} + \frac{\gamma}{\delta}\sqrt{m}\, i$, with $[\alpha,\beta] = [\gamma,\delta] = 1$. We now compute

$$q(h) = \frac{\beta}{[2\alpha,\beta]} = \frac{\beta}{[2,\beta]} = \frac{(\beta\delta)^2}{[2\beta\delta^2,(\beta\delta)^2]}$$

and $s(h) = \dfrac{(\beta\delta)^2}{[(\alpha\delta)^2 + m(\beta\gamma)^2,(\beta\delta)^2]}$. Thus we have

$$[q,s](h) = \frac{(\beta\delta)^2[2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2 m]}{[2\beta\delta^2,(\beta\delta)^2][(\alpha\delta)^2 + m(\beta\gamma)^2,(\beta\delta)^2]} \quad \text{whence}$$

$$qs'(h)I(h) = \frac{\gamma\beta^2\delta\sqrt{m}}{[(\alpha\delta)^2 + m(\beta\gamma)^2,(\beta\delta)^2,2\beta\delta^2]} \; .$$

Now since $T \in SL^+(2,Z)$, there exist integers a, b, c, d such that $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Thus $T(h) = \dfrac{ah + b}{ch + d}$, whence

$$T(h) = \frac{(a\alpha + b\beta)(c\alpha + d\beta)\delta^2 + ac(\beta\gamma)^2 m + \beta^2\gamma\delta\sqrt{m}\, i}{(c\alpha + d\beta)^2\delta^2 + (c\beta\gamma)^2 m} \; .$$

If we define

$$\lambda = 2(a\alpha + b\beta)(c\alpha + d\beta)\delta^2 + 2ac(\beta\gamma)^2 m$$

$$\mu = (c\alpha + d\beta)^2\delta^2 + (c\beta\gamma)^2 m$$

$$\nu = (a\alpha + b\beta)^2\delta^2 + (a\beta\gamma)^2 m$$

then we obtain $q \circ T(h) = \dfrac{\mu}{[\lambda,\mu]}$ and

$$s \circ T(h) = \frac{\mu^2}{[\{(a\alpha + b\beta)(c\alpha + d\beta)\delta^2 + ac(\beta\gamma)^2 m\}^2 + \gamma^2\beta^4\delta^2 m,\ \mu^2]} \; .$$

Fortunately, we have $((a\alpha+b\beta)(c\alpha+d\beta)\delta^2+ac(\beta\gamma)^2 m)^2+\gamma^2\beta^4\delta^2 m = \nu\mu$.

Thus $s \circ T(h) = \dfrac{\mu}{[\nu,\mu]}$. It can readily be shown that

$qs' \circ T(h) = \dfrac{\mu}{[\lambda,\mu,\nu]}$ whence

$$qs' \circ T(h)I(T(h)) = \frac{\gamma\beta^2\delta\sqrt{m}}{[\lambda,\mu,\nu]} \; .$$

Thus $qs'(h)I(h) = qs' \circ T(h)I(T(h))$ iff $[\lambda,\mu,\nu] = [(\alpha\delta)^2 + m(\beta\gamma)^2, (\beta\delta)^2, 2\beta\delta^2]$. Suppose that for $\ell \varepsilon Z$, ℓ divides $[(\alpha\delta)^2 + m(\beta\gamma)^2,(\beta\delta)^2,2\beta\delta^2]$. Then ℓ divides each of $(\alpha\delta)^2 + m(\beta\gamma)^2$, $(\beta\delta)^2$ and $2\beta\delta^2$, whence ℓ divides $2ac((\alpha\delta)^2 + m(\beta\gamma)^2) + 2\alpha\beta\gamma^2(ad - bc) + 2bd(\beta\delta)^2 = 2(a\alpha + b\beta)(c\alpha + d\beta)\delta^2 + 2ac(\beta\gamma)^2m = \lambda$; ℓ divides $a^2((\alpha\delta)^2 + m(\beta\gamma)^2) + b^2(\beta\delta)^2 + 2ab\alpha\beta\delta^2 = (a\alpha + b\beta)^2\delta^2 + (a\beta\gamma)^2 m = \nu$; and ℓ divides $c^2((\alpha\delta)^2 + m(\beta\gamma)^2) + d^2(\beta\delta)^2 + 2cd\alpha\beta\delta^2 = (c\alpha + d\beta)^2\delta^2 + (c\beta\gamma)^2m = \mu$. Thus ℓ divides $[\lambda,\mu,\nu]$. It follows that $[(\alpha\delta)^2 + m(\beta\gamma)^2,(\beta\delta)^2,2\beta\delta^2]$ divides $[\lambda,\mu,\nu]$.

Conversely, suppose that ℓ divides $[\lambda,\mu,\nu]$. Then we have the following situation: ℓ divides λ, whence

$$\ell \text{ divides } 2ac((\alpha\delta)^2 + (\beta\gamma)^2m) + 2\alpha\beta\delta^2(ad + bc) + 2bd(\beta\delta)^2 \tag{5}$$

and ℓ divides ν, whence

$$\ell \text{ divides } a^2((\alpha\delta)^2 + (\beta\gamma)^2m) + b^2(\beta\delta)^2 + 2ab\alpha\beta\delta^2 \tag{6}$$

and ℓ divides μ, whence

$$\ell \text{ divides } c^2((\alpha\delta)^2 + (\beta\gamma)^2m) + (d\beta\delta)^2 + 2cd\alpha\beta\delta^2 . \tag{7}$$

From (6) and (7), we obtain

$$\ell \text{ divides } (\beta\delta)^2(ad + bc) + 2ac\alpha\beta\delta^2 . \tag{8}$$

Then from (5) and (6)

$$\ell \text{ divides } 2b(\beta\delta)^2 + 2a\alpha\beta\delta^2 , \tag{9}$$

and from (5) and (7)

$$\ell \text{ divides } 2d(\beta\delta)^2 + 2c\alpha\beta\delta^2 . \tag{10}$$

Hence, from (10), we obtain that ℓ divides $2ad(\beta\gamma)^2 + 2ac\alpha\beta\delta^2$, and with (8),

$$\ell \text{ divides } (\beta\delta)^2 . \tag{11}$$

From (10) and (11), we have ℓ divides $2c\alpha\beta\delta^2$, and from (9) and (10) we have ℓ divides $2a\alpha\beta\delta^2$, whence ℓ divides $2\alpha\beta\delta^2$. But then ℓ divides $[2\alpha\beta\delta^2,(\beta\delta)^2] = [2\alpha,\beta]\beta\delta^2=[2,\beta]\beta\delta^2$. Thus

$$\ell \text{ divides } 2\beta\delta^2 . \tag{12}$$

Finally, with (11) and (12), we obtain from (6) that ℓ divides $a^2((\alpha\delta)^2 + m(\beta\gamma)^2)$, from (7) that ℓ divides $c^2((\alpha\delta)^2 + (\beta\gamma)^2m)$, whence

$$\ell \text{ divides } (\alpha\delta)^2 + m(\beta\gamma)^2 . \tag{13}$$

But now from (11), (12) and (13) we conclude that ℓ divides $[(\alpha\delta)^2 + (\beta\gamma)^2m,(\beta\delta)^2,2\beta\delta^2]$. Thus $[\lambda,\mu,\nu]$ divides $[(\alpha\delta)^2 + m(\beta\gamma)^2, (\beta\delta)^2, \nu\beta\delta^2]$. ∇

A rather more elegant proof of this lemma is given below. However, the methods introduced in the proof of (4.1.6) will be of use in the later developments.

(4.1.7) Lemma. Let $h \in H$ be ample and let $T \in SL^+(2,Z)$. Then the set of integers represented by the quadratic form $\det(xI + yM(h))$ is equal to the set of integers represented by $\det(xI + yM \circ T(h))$.

Proof: Let $k,\ell \in Z$ with $k^2 + \ell^2 \neq 0$. Then by (1.2.11), $kI + \ell M(h) \in I_h$, whence $T \circ (kI + \ell M(h)) \circ T^{-1} \in I_{T(h)}$.

Thus $\det(kI + \ell M(h))$ is an integer represented by $\det(xI + yM\circ T(h))$. Since $h = T^{-1}(T(h))$, the argument is symmetrical, i.e. h can be replaced by $T(h)$ and vice-versa. Thus the sets of integers represented by the two quadratic forms are the same. ∇

It is known [7] that if two quadratic forms represent the same set of integers, then the discriminants of the two forms are equal.

(4.1.8) Corollary. Let $h \in H$ be ample and let $T \in SL^+(2,Z)$. Then $(ps'(h))^2 - 4qs'(h)q'r(h) = (ps'\circ T(h))^2 - 4qs'\circ T(h)q'r\circ T(h)$.

Proof: Observe that $\det(xI + yM(h)) = x^2 + xyps'(h) + y^2qs'(h)q'r(h)$, whence the discriminant is equal to $(ps'(h))^2 - 4qs'(h)q'r(h)$. Similarly, the discriminant of $\det(xI + yM\circ T(h))$ is $(ps'\circ T(h))^2 - 4qs'\circ T(h)q'r\circ T(h)$. ∇

It is now easy to see that (4.1.6) follows. For each conformal torus τ is an element of $H/SL^+(2,Z)$. If τ is ample, then for any $h \in \tau$, $qs'(h)I(h) = qs'(h)\sqrt{|h|^2-(R(h))^2}$

$$= qs'(h)\sqrt{\frac{r(h)}{s(h)} - \left(\frac{p(h)}{2q(h)}\right)^2} = \frac{1}{2}\sqrt{4qs'(h)q'r(h) - (ps'(h))^2}\ .$$

By (4.1.8), the expression $4qs'(h)q'r(h) - (ps'(h))^2$ is invariant as h varies in τ. Thus the expression $qs'(h)I(h)$ also remains constant as h varies in τ.

(4.1.9) Definition. For each $a \in Z^+$, let $SL_a^+(2,Z)$ be the subgroup of $SL^+(2,Z)$ given by

$$SL_a^+(2,Z) = \left\{\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in SL^+(2,Z) \mid \gamma \equiv 0 \pmod{a}\right\}\ .$$

(4.1.10) Lemma. Let τ_1 and τ_2 be immersion-equivalent conformal tori. If $(a,h) \varepsilon \operatorname{Rep}(\tau_1,\tau_2)$, then for every $T \varepsilon SL_a^+(2,Z)$, $(a,T(h)) \varepsilon \operatorname{Rep}(\tau_1,\tau_2)$.

Proof: For each $T = \begin{bmatrix} \alpha & \beta \\ a\eta & \delta \end{bmatrix} \varepsilon SL^+(2,Z)$, $\eta \varepsilon Z$, we must show that $aT(h) \varepsilon \tau_1$. But

$$aT(h) = a\left(\frac{\alpha h + \beta}{a\eta\ h + \delta}\right) = \frac{\alpha(ah + \beta}{\eta(ah)+ \delta} = \begin{bmatrix} \alpha & a\beta \\ \eta & \delta \end{bmatrix}(ah) \quad \text{and}$$

$\alpha\delta - \eta a\beta = 1$. Thus $aT(h) \varepsilon \tau_1$. ∇

If h is non-ample, it is easily seen that these are all such elements of τ_2 with this property. If h is ample, however, the determination of all such elements requires the integral solutions (k,ℓ) of the quadratic form $ak^2 + k\ell a(\tilde{a}(h))ps'(h) + \ell^2 a(\tilde{a}(h))^2 qs'(h)q'r(h) = a$. The fact that (1,0) is a solution is all that Lemma (4.1.10) establishes.

(4.1.11) Lemma. Let τ_1 and τ_2 be immersion-equivalent conformal tori. If $(a,h) \varepsilon \operatorname{Rep}(\tau_1, \tau_2)$, then $\tilde{a}(h)=\tilde{a}\circ T(h)$ for each $T \varepsilon SL_a^+(2,Z)$.

Proof: It is true for non-ample tori by definition. Suppose that τ_1 and τ_2 are ample. Then choose any $(a,h) \varepsilon \operatorname{Rep}(\tau_1,\tau_2)$. Let $\mathfrak{M}_m$ be the analytic immersion class containing h. Then there exist integers, x, y, v, w such that $h = \frac{x}{y} + \frac{v}{w}\sqrt{m}\ i$, and $[x,y] = [v,w] = 1$.

Now calculate

$$q(h) = \frac{y}{[2x,y]} = \frac{y}{[2,y]} = \frac{(yw)^2}{[2yw^2,(yw)^2]} \quad \text{and}$$

$$s(h) = \frac{(yw)^2}{[(xw)^2 + (vy)^2m,(yw)^2]} \quad .$$

It follows now that

$$qs'(h) = \frac{(yw)^2}{[2yw^2,(yw)^2,(xw)^2 + (vy)^2m]} \quad .$$

Now let $T = \begin{bmatrix} \alpha & \beta \\ a\gamma & \delta \end{bmatrix} \in SL_a^+(2,Z)$. Then

$$T(h) = \frac{\alpha h + \beta}{a\gamma h + \delta} \quad \text{, whence}$$

$$T(h) = \frac{(\alpha x + \beta y)(a\gamma x + \delta y)w^2 + a\alpha\gamma(vy)^2m + y^2vw\sqrt{m}\, i}{(a\gamma x + \delta y)^2w^2 + (a\gamma vy)^2m}$$

and so we can compute $q \circ T(h)$ and $s \circ T(h)$. Firstly, we define

$$\lambda = 2(\alpha x + \beta y)(a\gamma x + \delta y)w^2 + 2a\alpha\gamma(vy)^2m$$

$$\mu = (a\gamma x + \delta y)^2w^2 + (a\gamma vy)^2m$$

$$\nu = (\alpha x + \beta y)^2w^2 + (\alpha vy)^2m \quad .$$

Then $q \circ T(h) = \frac{\mu}{[\lambda,\mu]}$ and $s \circ T(h) = \frac{\mu}{[\nu,\mu]}$, whence

$$qs' \circ T(h) = \frac{\mu}{[\lambda,\mu,\nu]} \quad .$$

For since h and $T(h)$ are elements of τ_2, by (4.1.6) $qs'(h)I(h) = qs' \circ T(h)I(T(h))$. So we compute

$$qs'(h)I(h) = \frac{y^2vw\sqrt{m}}{[2yw^2,(yw)^2,(xw)^2 + (vy)^2m]}$$

and

$qs'\circ T(h)I(T(h)) = \frac{y^2vw\sqrt{m}}{[\lambda,\mu,\nu]}$. Thus $[2yw^2,(yw)^2,(xw)^2 + (vy)^2m] = [\lambda,\mu,\nu]$. By definition, $\tilde{a}(h) = \tilde{a}\circ T(h)$ iff $[a,qs'(h)] = [a,qs'\circ T(h)]$. Since $\mu = (a\gamma)^2((xw)^2 + (vy)^2m) + \delta^2(yw)^2 + a\gamma\delta x(2yw^2)$ and since $[\lambda,\mu,\nu]$ divides each of the quantities in brackets, we have $[a,qs'\circ T(h)] = [a,\delta^2 \frac{(yw)^2}{[\lambda,\mu,\nu]}] = [a,\delta^2qs'(h)]$, whence $[a,qs'(h)]$ divides $[a,qs'\circ T(h)]$. Since $h = T^{-1}(T(h))$, the symmetry of the argument has $[a,qs'\circ T(h)]$ dividing $[a,qs'(h)]$ and thus $\tilde{a}(h) = a\circ T(h)$. ∇

Thus we obtain

<u>(4.1.12) Corollary</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then for each $(a,h) \varepsilon \mathrm{Rep}(\tau_1,\tau_2)$, $\tilde{a}(h)qs'(h)I(h) = \tilde{a}\circ T(h)qs'\circ T(h)I(T(h))$ for every $T \varepsilon SL_a^+(2,Z)$.

<u>(4.1.13) Lemma</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori, and let $(a,h) \varepsilon \mathrm{Rep}(\tau_1,\tau_2)$. Then for every $T \varepsilon SL_a^+(2,Z)$,

$$\frac{1}{2}(ps'\circ T(h) - ps'(h))\tilde{a}(h)$$

is an integer.

Proof: This is true by definition if τ_1 and τ_2 are non-ample. Suppose now that τ_1 and τ_2 are immersion-equivalent ample conformal tori. Let $\mathfrak{M}_m$ be the analytic immersion class containing τ_1 and τ_2. Then for any $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, there exist integers x, y, v, w such that $h = \frac{x}{y} + \frac{v}{w}\sqrt{m}\, i$, with $[x,y] = [v,w] = 1$.

Let $T = \begin{bmatrix} \alpha & \beta \\ a\gamma & \delta \end{bmatrix} \in SL_a^+(2,Z)$. Since our notation here coincides with that of (4.1.11), we shall make use of the calculations done there to obtain $ps'(h) = \frac{2xyw^2}{[\lambda,\mu,\nu]}$ and

$$ps' \circ T(h) = \frac{2(\alpha x + \beta y)(a\gamma x + \delta y)\omega^2 + 2a\gamma\alpha(vy)^2 m}{[\lambda,\mu,\nu]}\text{, and so}$$

$$ps' \circ T(h) - ps'(h) = \frac{2(\alpha x + \beta y)(a\gamma x + \delta y)\omega^2 + 2a\alpha\gamma(vy)^2 m - 2xyw^2}{[\lambda,\mu,\nu]}$$

$$= 2a\alpha\gamma\frac{(xw)^2+(vy)^2 m}{[\lambda,\mu,\nu]} + \frac{2(yw)^2\beta\delta}{[\lambda,\mu,\nu]} + 2a\beta\gamma\left(\frac{2xyw^2}{[\lambda,\mu,\nu]}\right).$$

Since $[\lambda,\mu,\nu]$ divides $(xw)^2 + (vy)^2 m$, $(yw)^2$ and $2xyw^2$, and since $qs'(h) = \frac{(yw)^2}{[\lambda,\mu,\nu]}$, we see that $2[a,qs'(h)]$ divides $ps' \circ T(h) - ps'(h)$. But by definition, $\frac{\tilde{a}(h)}{2} = \frac{1}{2[a,qs'(h)]}$, and so $\frac{1}{2}\tilde{a}(h)(ps' \circ T(h) - ps'(h))$ is an integer. ∇

<u>(4.1.14) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then for each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, $T \in SL_a^+(2,Z)$, $\mathcal{L}(a,h) = \mathcal{L}(a,T(h))$.

Proof: By the symmetry of the situation, it is sufficient to show that $\tilde{a} \circ T(h) qs' \circ T(h) T(h) \in \mathcal{L}(a,h)$.

By (4.1.12) and (4.1.11),

$$\tilde{a} \circ T(h) qs' \circ T(h) T(h) = \tilde{a} \circ T(h) qs' \circ T(h) \left(\frac{p \circ T(h)}{2q \circ T(h)} + I(T(h)) i\right)$$

$$= \frac{1}{2}\tilde{a} \circ T(h) ps' \circ T(h) + \tilde{a} \circ T(h) qs' \circ T(h) I(T(h)) i$$

$$= \frac{1}{2}\tilde{a}(h)(ps' \circ T(h) - ps'(h)) + \tilde{a}(h) qs'(h) h.$$

Thus we need only show that the first term is an integer. This however follows from (4.1.13). ∇

§4.2 The Symmetry Between $\mathrm{Rep}(\tau_1, \tau_2)$ and $\mathrm{Rep}(\tau_2, \tau_1)$

Let τ_1 and τ_2 be immersion-equivalent ample conformal tori.

(4.2.1) Lemma. For each $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$, there exists a positive integer $\kappa = \kappa(a,h)$ such that:

i) $ps'(ah) = \frac{\kappa}{a} ps'(h)$,

ii) $q'r(ah) = \kappa q'r(h)$, and

iii) $qs'(ah) = \frac{\kappa}{a^2} qs'(h)$.

Proof: Let $\kappa = \dfrac{a^2}{[a^2 q'(h), as'(h), qs'(h)]}$. Then $\kappa \in \mathbb{Q}$, and satisfies the three conditions. It only remains to show that $\kappa \in Z^+$. In fact, $\kappa > 0$ and so we simply show that it is an integer. If ℓ divides $a^2 q'(h)$, $as'(h)$ and $qs'(h)$, then ℓ divides $a^2[q'(h), s'(h)] = a^2$. Thus $\kappa \in Z^+$. ∇

<u>(4.2.2) Definition</u>. Let J denote the element $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ of $SL^+(2,Z)$.

It was shown in (1.2.9) that for each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, $(a,J(ah)) \in \mathrm{Rep}(\tau_2,\tau_1)$. It is a routine series of calculations to see that for ample $h \in H$, $p \circ J(h) = \frac{-\,ps'(h)}{[p(h),r(h)]}$, and $q \circ J(h) = \frac{q'r(h)}{[p(h),r(h)]}$. Furthermore, $|J(h)|^2 = \frac{s(h)}{r(h)}$, whence $r \circ J(h) = s(h)$, $s \circ J(h) = r(h)$. Since $[p(h),q(h)] = 1$, it follows that $[q,s] \circ J(h) = \frac{r(h)}{[p(h),r(h)]}$. Thus we obtain $qs' \circ J(h) = q'r(h)$, and $ps' \circ J(h) = -ps'(h)$.

<u>(4.2.3) Lemma</u>. For each $(a,h) \in \mathrm{Rep}(\tau_1,\ \tau_2)$, $\kappa(a,h) = \frac{a^2}{(a,J(ah))}$.

Proof: By (4.2.1), $ps'(ah) = \kappa(a,h)\ \frac{ps'(h)}{a}$, and $ps'(aJ(ah)) = \kappa(a,J(ah))\ \frac{ps' \circ J(ah)}{a}$. But $ps' \circ J(ah) = -ps'(ah)$ and $aJ(ah) = J(h)$, whence $ps' \circ J(h) = -\,\kappa(a,J(ah))\frac{ps'(ah)}{a} = -\,\kappa(a,J(ah)\ \kappa(a,h)\ \frac{ps'(h)}{a^2}$. Since $ps' \circ J(h) = -ps'(h)$, we obtain $\kappa(a,J(ah)\ \kappa(a,h) = a^2$. ∇

<u>(4.2.4) Lemma</u>. For each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$, $\kappa = \frac{a[a,\ q'r(h)]}{[a,qs'(h)]}$.

Proof: By (4.2.1), this is equivalent to proving that $\frac{q'r(ah)}{[a,q'r(ah)]} = \frac{aq'r(h)}{[a,qs'(h)]}$. But

$$q'r(ah) = \frac{a^2q(h)r(h)}{[[a^2,s(h)]q(h),[a,q(h)]s(h)]} \qquad \text{and so}$$

$$\frac{q'r(ah)}{[a,q'r(ah)]} = \frac{aq'r(h)[q,s](h)}{[[a^2,s(h)]q(h),[a,q(h)]s(h),aq(h)r(h)]}$$

$$= \frac{aq'r(h)[a,qs'(h)][q,s](h)}{[[a,qs'(h)][a^2,s(h)]q(h),[a,q(h)]s(h),aq(h)r(h)]}.$$

It only remains to show that

$$[a,qs'(h)][q,s](h) = [[a^2,s(h)]q(h),[a,q(h)]s(h),aq(h)r(h)].$$

Suppose that ℓ divides $[a,qs'(h)][q,s](h)$. Then ℓ divides $aq(h)$, $as(h)$ and $q(h)s(h)$. Thus ℓ divides $a^2q(h)$, $q(h)s(h)$, $as(h)$ and $aq(h)r(h)$, whence ℓ divides $[[a^2,s(h)]q(h),[a,q(h)]s(h),aq(h)r(h)]$. Conversely, if ℓ divides each of $a^2q(h)$, $q(h)s(h)$, $as(h)$ and $aq(h)r(h)$, then ℓ divides each of $as(h)$, $q(h)s(h)$ and $aq(h)$. To see that ℓ divides $aq(h)$, we observe that ℓ divides $[aq(h)s(h),aq(h)r(h)]$ which is equal to $aq(h)[r(h),s(h)] = aq(h)$. Thus ℓ divides $[a,qs'(h)][q,s](h)$. ∇

(4.2.5) Lemma. For each $(a,h) \in \operatorname{Rep}(\tau_1,\tau_2)$,

$$\frac{qs'\circ J(ah)}{qs'(h)} = \frac{ar(h)[a,qs'\circ J(ah)]}{s(h)[a,qs'(h)]}.$$

Proof: Since $qs'\circ J(ah) = q'r(ah)$, then by (4.2.1), $qs'\circ J(ah) = \kappa(a,h)q'r(h)$. Thus

$$\frac{qs'\circ J(ah)}{qs'(h)} = \frac{\kappa(a,h)q'r(h)}{qs'(h)} = \frac{\kappa(a,h)r(h)}{s(h)}.$$ By (4.2.4),

$$\frac{\kappa(a,h)r(h)}{s(h)} = \frac{a[a,\ \kappa(a,h)q'r(h)]r(h)}{[a,qs'(h)]s(h)} = \frac{a[a,q'r(ah)]r(h)}{[a,qs'(h)]s(h)}$$

with the last equality a result of (4.2.1). Thus we have

$$\frac{qs'\circ J(ah)}{qs'(h)} = \left(\frac{ar(h)}{s(h)}\right)\left(\frac{[a,qs'\circ J(ah)]}{[a,qs'(h)]}\right).$$ ∇

It is now possible to display the symmetry between the lattices associated with $\mathrm{Rep}(\tau_1,\tau_2)$ and those associated with $\mathrm{Rep}(\tau_2,\tau_1)$. We shall mean by $\overline{N}$ the set of complex conjugates of elements belonging to N, where N is any subset of $\mathbb{C}$.

<u>(4.2.6) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent ample conformal tori. Then for each $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$,

$$\mathcal{L}(a,h) = \overline{-\mathcal{L}(a,J(ah))} .$$

Proof: Let $h' = J(ah)$. Since $\mathcal{L}(a,h)$ and $\mathcal{L}(a,h')$ have the lattice bases $\{1, \tilde{a}(h)qs'(h)h\}$ and $\{1,\tilde{a}(h')qs'(h')h'\}$ respectively, it is sufficient to show that

$$-\tilde{a}(h')qs'(h')\overline{h'} = \tilde{a}(h)qs'(h)h.$$

As $-\overline{h'} = \dfrac{h}{a|h|^2} = \dfrac{h}{a}\,\dfrac{s(h)}{r(h)}$, by (4.2.5) we have

$$-\tilde{a}(h')qs'(h')h' = \frac{qs'(h')}{[a,qs'(h')]}\,\frac{s(h)}{ar(h')}\,h$$

$$= \tilde{a}(h)qs'(h)h. \qquad \nabla$$

CHAPTER 5

THE DISTORTION RINGS OF AN ANALYTIC IMMERSION CLASS

Let τ be a conformal torus. In §3.2 it was shown that An(τ) is a monoidal semi-direct product $Z \times Z \times_\psi ¢/\Gamma(h)$ for any $h \in \tau$, if τ is ample, or $Z \times_\psi ¢/\Gamma(h)$ if τ is non-ample. In a manner analagous to §4.1, we shall define a family of distortion rings and corresponding distortion lattices associated with τ. In fact, the multiplicative monoids of the distortion rings have been investigated in Chapter 3. It was seen that there is essentially only one multiplicative monoid associated with τ. In this chapter, it will be seen that there is really one distortion ring associated with τ and in fact only one distortion lattice for τ.

It is also possible to define a pre-order on each ample analytic immersion class of conformal tori, based on the relationship between the distortion rings associated with the conformal tori of the class. This pre-ordering is the basis of the results of the next chapter.

§5.1 The Distortion Ring of a Conformal Torus

As in §3.1, we shall only consider the subset $\{(1,h) \mid h \in \tau\}$ of Rep(τ_1,τ_2). In §3.2 it was shown that for each $h \in \tau$, $\langle Z \times Z, *_h \rangle$ is an abelian monoid. Moreover, for $h, h' \in \tau$, then $\langle Z \times Z, *_h \rangle$ is isomorphic to $\langle Z \times Z, *_{h'} \rangle$.

It can be easily verified that if addition is defined on $Z \times Z$ in the usual way, then for each $h \in \tau$, $\langle Z \times Z, *_h, +\rangle$ is an abelian ring.

(5.1.1) Definition. Let τ be a conformal torus. For each $h \in \tau$, let $\mathcal{D}(h) = \langle Z \times Z, *_h, +\rangle$ if τ is ample, else let $\mathcal{D}(h) = Z$. $\mathcal{D}(h)$ is called a *distortion ring* associated with τ. Correspondingly, define $\mathcal{L}(h) = \mathcal{L}(1,h) = \{C^h(k,\ell) \mid (k,\ell) \in Z \times Z\}$. For each $h \in \tau$, $\mathcal{L}(h)$ is called a *lattice of complex distortions* associated with τ.

It is rather interesting to observe that the multiplicative monoid $An(\tau)$ with the natural addition defined on it is not a ring. All axioms but one are satisfied. It turns out that multiplication is not left distributive over addition.

Now since for every non-ample conformal torus τ, the associated distortion rings and lattices are just the ring Z and lattice $Z \subset ¢$ respectively, we shall be primarily concerned with ample conformal tori.

(5.1.2) Theorem. Let $h \in H$ be ample. Then for any $T \in SL^+(2,Z)$, $\mathcal{D}(h)$ is isomorphic to $\mathcal{D}(T(h))$ and $\mathcal{L}(h) = \mathcal{L}(T(h))$.

Proof: Let $\Psi: \mathcal{D}(h) \to \mathcal{D}(T(h))$ be defined by

$$\Psi(k,\ell) = (k - \ell\lambda, \ell)$$

where $\lambda = \frac{1}{2}(ps' \circ T(h) - ps'(h))$. By (4.1.11), $\lambda \in Z$. It is obvious that Ψ preserves the addition, and the choice of λ ensures that Ψ preserves the multiplication. By the definition, Ψ is bijective. Thus Ψ is a ring isomorphism.

That $\mathcal{L}(h) = \mathcal{L}(T(h))$ is a corollary to (4.1.12). For $\mathcal{L}(h) = \mathcal{L}(1,h)$, $\mathcal{L}(T(h)) = \mathcal{L}(1,T(h))$ and $SL_1^+(2,Z) = SL^+(2,Z)$. ∇

(5.1.3) Definition. Let τ be a conformal torus. Then define $\mathcal{L}(\tau)$ to be the unique lattice of complex distortions associated with τ, given by $\mathcal{L}(h)$ for any $h \in \tau$. Similarly, let $\mathcal{D}(\tau) = \mathcal{D}(h)$ for any $h \in \tau$.

We observe now that the distortion lattice associated with τ has a symmetry which is independent of τ.

(5.1.4) Theorem. Let τ be a conformal torus. Then $\mathcal{L}(\tau)$ is symmetrical with respect to the imaginary axis, that is, $\mathcal{L}(\tau) = -\overline{\mathcal{L}(\tau)}$.

Proof: By (4.2.6), $\mathcal{L}(\tau) = \mathcal{L}(1,h) = -\overline{\mathcal{L}(1,J(h))}$ for any $h \in \tau$. Since $J \in SL^+(2,Z)$, $\mathcal{L}(1,J(h)) = \mathcal{L}(1,h) = \mathcal{L}(\tau)$. ∇

In fact, $\mathcal{L}(\tau)$ is symmetric with respect to both the real and imaginary axis. This is obvious if τ is non-ample. If τ is ample, this follows from the fact that $qs'(h)\bar{h} + qs'(h)h = ps'(h)$, whence $qs'(h)\bar{h} \in \mathcal{L}(h)$.

Let us now return to the distortion ring in the context of the imaginary quadratic number field assigned to the analytic immersion class containing the ample conformal torus.

(5.1.5) Lemma. Let τ be an ample conformal torus, and let $\mathfrak{M}_m$ be the analytic immersion class containing τ. Then $\mathcal{D}(\tau)$ is a subring of $Z(m)$.

Proof: Let $h \in \tau$. Then the map $\phi: \mathcal{D}(h) \to Q[h]$ which takes $(k,\ell) \in \mathcal{D}(h)$ to $k + \ell qs'(h)h \in Q[h]$ is an injective ring homomorphism. Since $h \in \mathfrak{M}_m$, $Q[h] = K(m)$. Thus we need only show that every element of the form $k + \ell qs'(h)h \in Z(m)$. But $Z(m)$ is a Z-module and so if $qs'(h)h \in Z(m)$, it will follow that for all integers, k, ℓ, $k + \ell qs'(h)h \in Z(m)$. By §2.3, $qs'(h)h \in Z(m)$ iff the Z-module generated by all powers of $qs'(h)h$ in $K(m)$ is finite. Since $(qs'(h)h)^2 = ps'(h)qs'(h)h - qs'(h)q'r(h)$, the Z-module generated by all powers of $qs'(h)h$ has rank two. ∇

(5.1.6) Corollary. Let $h \in \mathfrak{M}_m$. Then if $m \not\equiv 3 \pmod 4$, $ps'(h)$ is an even integer.

Proof: There exists an integer ℓ such that

$$\ell^2 m = 4qs'(h)q'r(h) - (ps'(h))^2$$

and so $h = \frac{1}{2}\left(\frac{ps'(h) + \ell\sqrt{m}\, i}{qs'(h)}\right)$. Since $qs'(h)h \in Z(m)$, it follows that $\frac{1}{2}(ps'(h) + \ell\sqrt{m}\, i) \in Z(m)$. By §2.3, if $m \not\equiv 3 (\mathrm{mod}) 4)$, then 2 divides $ps'(h)$ and ℓ. ∇

(5.1.7) Corollary. For each $h \in \tau$, $\mathcal{L}(h) = \phi(\mathcal{D}(h))$.

Since the distortion ring of every ample conformal torus in the analytic immersion class $\mathfrak{M}_m$ is a subring of the ring of integers $Z(m)$, it is of interest to know which conformal tori in $\mathfrak{M}_m$ have $Z(m)$ for their ring of distortions.

<u>(5.1.8) Theorem</u>. Let τ be a conformal torus in the ample analytic immersion class $\mathfrak{M}_m$. Then $\mathcal{D}(\tau) = Z(m)$ iff for any $h \varepsilon \tau$, there exist integers a, b such that $h = \frac{a + \sqrt{m}\, i}{b}$ and

$$[2ab, b^2, a^2 + m] = \begin{cases} b & \text{if } m \neq 3(\text{mod } 4) \\ 2b & \text{if } m \neq 3(\text{mod } 4) \end{cases} .$$

Proof: Suppose that $\mathcal{D}(\tau) = Z(m)$. Then for any $h \varepsilon \tau$, $\mathcal{D}(h) = Z(m)$. Since $h \varepsilon \mathfrak{M}_m$, there exist integers $\alpha, \beta, \gamma, \delta$ such that $h = \frac{\alpha}{\beta} + \frac{\gamma}{\delta}\sqrt{m}\, i$ and $[\alpha,\beta] = [\gamma,\delta] = 1$. By (4.2.6), $qs'(h)h \varepsilon Z(m)$, hence there exist integers n and n' such that $qs'(h)h = n + n'(\frac{1 + \sqrt{m}\, i}{2})$ if $m=3(\text{mod } 4)$, or such that $qs'(h)h = n + n'\sqrt{m}\, i$ if $m \neq$ ~~3(mod~~ 4). By hypothesis, $Z(m) = \mathcal{D}(h)$, whence there exist integers ℓ, ℓ' such that $\sqrt{m}\, i = \ell + \ell' qs'(h)h = \ell + \ell' n + \ell' n' \sqrt{m}\, i$ if $m \neq 3(\text{mod } 4)$, or such that $\sqrt{m}\, i = 2(\ell + \ell' n) + \ell' n'(1 + \sqrt{m}\, i)$ if $m = 3(\text{mod } 4)$. In either case, $\ell' = n' = 1$. Thus there exists an integer, which we shall call n, such that $qs'(h)h = n + \sqrt{m}\, i$ if $m \neq 3(\text{mod } 4)$ or $qs'(h)h = \frac{n + \sqrt{m}\, i}{2}$ if $m = 3(\text{mod } 4)$. As done previously, we compute the values

$$qs'(h) = \frac{(\beta\delta)^2}{[2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2m]}$$

and

$$qs'(h)h = \frac{\alpha\beta\delta^2 + \beta^2\gamma\delta\sqrt{m}\, i}{[2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2m]} .$$

Suppose now that $m \not\equiv 3 \pmod 4$. Then $\beta^2\gamma\delta$ divides $(\beta\delta)^2$, whence γ divides δ. But $[\gamma,\delta] = 1$ and so $\gamma = 1$. Furthermore, $\beta^2\delta$ divides $\alpha\beta\delta^2$ and so β divides $\alpha\delta$. Since $[\alpha,\beta] = 1$, this implies that β divides δ. Let $\delta = \beta t$. Then $h = \frac{\alpha}{\beta} + \frac{\sqrt{m}\, i}{t\beta} = \frac{\alpha t + \sqrt{m}\, i}{\beta t}$. If we set $a = \alpha t$, $b = \beta t$, then $h = \frac{a + \sqrt{m}\, i}{b}$. As well $[2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2m] = \beta^2[2bt,b^2,a^2 + m] = \beta^2 b$, whence $[2bt,b^2,a^2 + m] = b$. But $t = [\alpha t, \beta t] = [a,b]$. Thus $b = [2ab,2b^2,b^2,a^2 + m] = [2ab,b^2,a^2 + m]$.

Now suppose that $m \equiv 3 \pmod 4$. Then $qs'(h)h = \frac{n + \sqrt{m}\, i}{2}$ from which we obtain, by equating real and imaginary parts, that $2\beta^2\gamma\delta = [2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2m]$ and $\beta^2\gamma\delta$ divides $\alpha\beta\delta^2$. Thus 2γ divides δ which again implies that $\gamma = 1$. Since β divides $\alpha\delta$, we conclude that β divides δ, again suppose that $\delta = \beta t$. Thus $h = \frac{\alpha t + \sqrt{m}\, i}{\beta t}$ as before, but now 2 divides βt. Then $h = \frac{a + \sqrt{m}\, i}{b}$ and $[2\beta\delta^2,(\beta\delta)^2,(\alpha\delta)^2 + (\beta\gamma)^2m] = \beta^2[2\beta t^2,(\beta t)^2,(\alpha t)^2 + m] = 2\beta^2\beta t$. Thus $[2bt,b^2,a^2 + m] = 2b$, and $t = [\alpha t, \beta t] = [a,b]$, whence $[2ab,b^2,a^2 + m] = 2b$.

Conversely, suppose that $h = \frac{a + \sqrt{m}\, i}{b}$ for integers a and b such that

$$[2ab,b^2,a^2 + m] = \begin{cases} b & m \neq 3 (\text{mod } 4) \\ 2b & m = 3 (\text{mod } 4) \end{cases} .$$

We must show that $\underline{q}s'(h) = b$ if $m \neq 3(\text{mod } 4)$ or else $2\underline{q}s'(h) = b$ if $m = 3(\text{mod } 4)$. But

$$\underline{q}s'(h) = \frac{b^2}{[a^2 + m,b^2,2ab]}$$

and thus it follows. ∇

<u>(5.1.9) Corollary</u>. Let τ be an ample conformal torus in the analytic immersion class $\mathfrak{M}_m$. A necessary condition that $\mathcal{D}(\tau) = Z(m)$ is that for any $h \in \tau$ there exist integers a and b such that $h = \frac{a + \sqrt{m}\, i}{b}$ and $[a,b]$ divides m.

Another necessary and sufficient condition that the distortion ring be the ring of integers of the field assigned to the immersion class is known [1]. If we consider $\Gamma(h)$ as an additive subgroup of $Q[h]$ by identifying T_1 with 1 and T_2 with h, then $\Gamma(h)$ is a Z-module, and we have

<u>(5.1.10) Theorem</u>. Let τ be an ample conformal torus contained in the analytic immersion class $\mathfrak{M}_m$. Then $\mathcal{D}(\tau) = Z(m)$ iff $\Gamma(h)$ is a $Z(m)$-ideal in $K(m)$, for any $h \in \tau$.

Proof: Let $h \in \tau$. It is clear that $\Gamma(h) \subset K(m)$. We must show that $\mathcal{D}(h) = Z(m)$ iff $Z(m)\Gamma(h) = \Gamma(h)$. If $\mathcal{D}(h) = Z(m)$, then $Z(m)\Gamma(h) = \mathcal{D}(h)\ \Gamma(h) = \Gamma(h)$. Conversely, suppose that $Z(m)\Gamma(h) = \Gamma(h)$. Since $\mathcal{D}(h) \subset Z(m)$, we need only show that every element of $Z(m)$ belongs to $\mathcal{D}(h)$. Let $\mu \in Z(m)$. Then $\mu, \mu h \in \Gamma(h)$ by assumption. Thus the function $F(z) = \mu z$ is the lift of an analytic self-map of $\mathbb{C}/\Gamma(h)$, whence $\mu \in \mathcal{L}(h)$. Since $\mathcal{D}(h)$ is identified with $\Phi(\mathcal{D}(h)) = \mathcal{L}(h)$ in $K(m)$, we have $\mu \in \mathcal{D}(h)$. Thus $Z(m) \subset \mathcal{D}(h)$. ∇

As a consequence of (5.1.8), we observe that if the positive square-free integer m is not equal to $3 (\bmod 4)$, then $\mathcal{D}(\sqrt{m}\, i) = Z(m)$, otherwise $\mathcal{D}(\frac{1 + \sqrt{m}\, i}{2}) = Z(m)$. In particular, we have $\mathcal{D}(i) = Z(1)$, and $\mathcal{D}(\frac{1 + \sqrt{3}\, i}{2}) = Z(3)$.

It also follows that if $t \in Z^+$ divides m, then $\mathcal{D}(\frac{\sqrt{m}\, i}{t}) = Z(m)$ if $m \not\equiv 3 (\bmod 4)$ and $\mathcal{D}(\frac{t + \sqrt{m}\, i}{2t}) = Z(m)$ otherwise. One might ask if every torus whose distortion ring is the ring of integers of $K(m)$ has a purely imaginary representative if $m \not\equiv 3 (\bmod 4)$ or the corresponding format if $m = 3 (\bmod 4)$. That this is not the case can be seen by considering the conformal torus τ whose fundamental representative is $\frac{1 + \sqrt{5}\, i}{2}$. For if τ is to be represented by $\frac{\sqrt{5}\, i}{t}$ for $t \in Z^+$ such that t divides 5, then $t = 1$ or 5. Since $\sqrt{5}\, i$ is a fundamental representative of a conformal torus, it is clear that $\sqrt{5}\, i$ does not represent τ. Furthermore $\frac{\sqrt{5}\, i}{5} = J(\sqrt{5}\, i)$, and so it represents the same torus as $\sqrt{5}\, i$.

§5.2 Comparison of Distortion Rings in an Analytic Immersion Class

In (4.1.6) it was shown that for each $h \in H$ and $T \in SL^+(2,Z)$, $qs'(h)I(h) = qs' \circ T(h)I(T(h))$. For a conformal torus τ, then, the expression $qs'(h)I(h)$ remains constant as h varies throughout τ.

<u>(5.2.1) Definition</u>. Let τ be a conformal torus. Assign a real number $C(\tau)$ to τ in the following way: If τ is non-ample, let $C(\tau) = 1$; if τ is ample, let $C(\tau) = qs'(h)I(h)$ for any $h \in \tau$.

Now suppose that τ_1 and τ_2 are immersion-equivalent ample conformal tori. Then let $(a,h) \in \text{Rep}(\tau_1,\tau_2)$ and observe that $qs'(ah)I(ah) = aqs'(ah)I(h)$.

<u>(5.2.2) Lemma</u>. For each $(a,h) \in \text{Rep}(\tau_1,\tau_2)$,

$$\frac{C(\tau_1)}{C(\tau_2)} = \frac{\kappa(a,h)}{a} .$$

Proof: Since $C(\tau_1) = qs'(ah)I(ah)$ and $C(\tau_2) = qs'(h)I(h)$, we have

$$\frac{C(\tau_1)}{C(\tau_2)} = \frac{aqs'(ah)}{qs'(h)} .$$

By (4.2.1), $qs'(ah) = \frac{\kappa(a,h)}{a^2} qs'(h)$, and so the lemma follows.

Now the ratio $C(\tau_1)/C(\tau_2)$ is independent of the representatives of τ_1 and τ_2, and so $C(\tau_1)/C(\tau_2)$ is an invariant of $\mathrm{Rep}(\tau_1,\tau_2)$. Thus the expression $\kappa(a,h)/a$ is constant as (a,h) varies in $\mathrm{Rep}(\tau_1,\tau_2)$. It follows that if (a,h), $(a,h') \in \mathrm{Rep}(\tau_1,\tau_2)$, then $\kappa(a,h) = \kappa(a,h')$. This implies that κ depends only on a, and of course the conformal tori τ_1 and τ_2. ∇

<u>(5.2.3) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent ample conformal tori. Then there exists a ring homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$ iff $C(\tau_1)$ divides $C(\tau_2)$. If such a homomorphism exists, then it is injective, and is an isomorphism iff $C(\tau_1) = C(\tau_2)$. Furthermore, there exists a homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$ iff there exist exactly two homomorphisms from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$.

Proof: Suppose firstly that $C(\tau_1)$ divides $C(\tau_2)$. We prove that there exist two injective ring homomorphisms from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$. By (5.2.2), $\kappa(a,h)$ divides a for any $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$. Define two maps from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$ as follows: for each $(k,\ell) \in \mathcal{D}(h)$, let $\theta(k,\ell) = (k,\ell\frac{a}{\kappa})$, and let $\Psi(k,\ell) = (k + \ell\frac{a}{\kappa}\, ps'(ah), -\ell\frac{a}{\kappa})$. By assumption, both $\theta(k,\ell)$ and $\Psi(k,\ell)$ are elements of $Z \times Z$. Now let (k,ℓ), $(m,n) \in \mathcal{D}(h)$. Then

$$(k,\ell) *_h (m,n) = (km - \ell nqs'(h)q'r(h),\ kn + \ell m + \ell nps'(h)).$$

By (4.2.1),

$$\theta((k,\ell) *_h (m,n)) = (km - \ell n \frac{a^2}{\kappa^2} qs'(ah)q'r(ah), \frac{a}{\kappa}(kn + \ell m + \ell n \frac{a}{\kappa} ps'(ah))$$

and

$$\Psi((k,\ell) *_h (m,n)) = (c_1,c_2)$$

where

$$c_1 = km - \ell n \frac{a^2}{\kappa^2} qs'(ah)q'r(ah) + \frac{a}{\kappa} ps'(ah)(kn+ \ell m+\ell n \frac{a}{\kappa} ps'(ah))$$

$$c_2 = -\frac{a}{\kappa} (kn + \ell m + \ell n \frac{a}{\kappa} ps'(ah)) .$$

But $\theta(k,\ell) *_{ah}\theta(m,n) = (k, \frac{a}{\kappa} \ell) *_{ah} (m, \frac{a}{\kappa} n)$

$$= (km - \frac{a^2}{\kappa^2} \ell nqs'(ah)q'r(ah), \frac{a}{\kappa} (kn + \ell m + \ell nps'(ah))$$

and

$$\Psi(k,\ell) *_{ah}\Psi(m,n) = (k + \ell\frac{a}{\kappa} ps'(ah), - \frac{a}{\kappa}\ell) *_{ah}(m+n \frac{a}{\kappa} ps'(ah), - \frac{a}{\kappa}\ell)$$

$$= (km + \frac{a}{\kappa} ps'(ah)(kn + \ell m + \ell n \frac{a}{\kappa} ps'(ah) - \frac{a^2}{\kappa^2} \ell nqs'(ah)q'r(ah), - \frac{a}{\kappa} (kn + \ell m + \ell n \frac{a}{\kappa} ps'(ah)) = (c_1, c_2)$$

Thus $\theta((k,\ell) *_h (m,n)) = \theta(k,\ell) *_{ah} \theta(m,n)$ and $\Psi((k,\ell) *_h (m,n)) = \Psi(k,\ell) *_{ah} \Psi(m,n)$ for all (k,ℓ), $(m,n) \in \mathcal{D}(h)$. Thus both θ and Ψ preserve multiplication. As well, $\theta(1,0) = \Psi(1,0) = (1,0)$. Furthermore, for each (k,ℓ), $(m,n) \in \mathcal{D}(h)$, we have

$$\theta((k,\ell) + (m,n)) = \theta((k + m, \ell + n)) = (k + m, \frac{a}{\kappa} (\ell + n))$$

$$= (k, \tfrac{a}{\kappa}\ell) + (m, \tfrac{a}{\kappa}n) = \theta(k,\ell) + \theta(m,n), \text{ and}$$

$$\Psi((k,\ell) + (m,n)) = \Psi(k + m, \ell + n) = (k + m + (\ell + n)\tfrac{a}{\kappa}\,ps'(ah), - \tfrac{a}{\kappa}(\ell + n))$$

$$= (k + \ell\tfrac{a}{\kappa}\,ps'(ah), - \tfrac{a}{\kappa}\ell) + (m + n\tfrac{a}{\kappa}\,ps'(ah), - \tfrac{a}{\kappa}n)$$

$$= \Psi(k,\ell) + \Psi(m,n) .$$

Obviously, $\theta(0,0) = \Psi(0,0) = (0,0)$. Thus both θ and Ψ are ring homomorphisms of $\mathcal{D}(h)$ into $\mathcal{D}(ah)$.

Suppose that for some $(k,\ell), (m,n) \varepsilon \mathcal{D}(h)$, $\theta(k,\ell) = \theta(m,n)$. Then $(k, \frac{a}{\kappa}\ell) = (m, \frac{a}{\kappa}n)$, whence $k = m$, $\ell = n$. Thus θ is injective. If now $\Psi(k,\ell) = \Psi(m,n)$, then $(k + \ell\frac{a}{\kappa}\,ps'(ah), - \frac{a}{\kappa}\ell) = (m + n\frac{a}{\kappa}\,ps'(ah), - \frac{a}{\kappa}n)$ whence $\ell = n$ and $k = m$. Thus Ψ is also injective.

It is clear that θ and Ψ are isomorphisms iff $\kappa = a$, i.e. $C(\tau_1) = C(\tau_2)$.

Conversely, we wish to prove that if there exists a ring homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$, then $C(\tau_1)$ divides $C(\tau_2)$ and furthermore, that if there exists one homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$, then there are exactly two. So suppose that there exists a ring homomorphism $\theta\colon \mathcal{D}(\tau_2) \to \mathcal{D}(\tau_1)$. Then for any $(a,h) \varepsilon \mathrm{Rep}(\tau_1,\tau_2)$, $\theta\colon \mathcal{D}(h) \to \mathcal{D}(ah)$. Now θ induces two functions $\theta_1, \theta_2\colon Z \times Z \to Z$ in the following way: for each $(k,\ell) \varepsilon Z \times Z$, let

$$\theta(k,\ell) = (\theta_1(k,\ell),\ \theta_2(k,\ell)) .$$

Then $\theta(k,\ell) = \theta(k,0) + \theta(0,\ell) = k\theta(1,0) + \ell\theta(0,1)$

$$= (k,0) + \ell(\theta_1(0,1),\ \theta_2(0,1)).$$

Thus to determine θ, we must determine $\theta_1(0,1)$ and $\theta_2(0,1)$.

For $(0,1) \in \mathcal{D}(h)$, we have

$$(0,1) *_h (0,1) = (-\ qs'(h)q'r(h), ps'(h))$$

$$= (-\ \frac{a^2}{\kappa^2}\ qs'(ah)q'r(ah),\ \frac{a}{\kappa}\ ps'(ah))$$

by (4.2.1). Thus

$$\theta((0,1) *_h (0,1)) = (-\ \frac{a^2}{\kappa^2}\ qs'(ah)q'r(ah) + \frac{a}{\kappa}\ ps'(ah)\theta_1(0,1),$$

$$\frac{a}{\kappa}\ ps'(ah)\ \theta_2(0,1)).$$

But $\theta((0,1) *_h (0,1)) = \theta(0,1)^2 = (\theta_1(0,1),\ \theta_2(0,1))^2$

$= (\theta_1(0,1)^2 - \theta_2(0,1)^2 qs'(ah)q'r(ah),\ 2\theta_1(0,1)\theta_2(0,1) + \theta_2(0,1)^2 ps'(ah))$.

Thus we obtain

$$-\ \frac{a^2}{\kappa^2}\ qs'(ah)q'r(ah) + \frac{a}{\kappa}\ ps'(ah)\ \theta_1(0,1) = \theta_1(0,1)^2 - \theta_2(0,1)^2 qs'(ah)q'r(ah) \tag{14}$$

and $\frac{a}{\kappa}\ ps'(ah)\ \theta_2(0,1) = 2\theta_1(0,1)\theta_2(0,1) + \theta_2(0,1)^2 ps'(ah).$ (15)

From (15) we have

$$\theta_2(0,1)\ (\frac{a}{\kappa}\ ps'(ah) - 2\theta_1(0,1) - \theta_2(0,1)ps'(ah)) = 0.$$

Thus either $\theta_2(0,1) = 0$ or else $\frac{a}{\kappa}\,ps'(ah) - 2\,\theta_1(0,1) - \theta_2(0,1)ps'(ah) = 0$. If $\theta_2(0,1) = 0$, then from (14),

$$\theta_1(0,1)^2 - \frac{a}{\kappa}\,ps'(ah)\,\theta_1(0,1) + \frac{a^2}{\kappa^2}\,qs'(ah)q'r(ah) = 0,$$

whence

$$\theta_1(0,1) = \frac{1}{2}\Big(\frac{a}{\kappa}\,ps'(ah) \pm \frac{a}{\kappa}\sqrt{(ps'(ah))^2 - 4qs'(ah)q'r(ah)}\Big) .$$

But from (3.3.3), this implies that $\theta_1(0,1)$ is not real, which is certainly a contradiction, since $\theta_1(0,1) \in Z$.

Thus $\theta_2(0,1) \neq 0$, and so

$$\frac{a}{\kappa}\,ps'(ah) - 2\,\theta_1(0,1) - \theta_2(0,1)ps'(ah) = 0.$$

Solving for $\theta_1(0,1)$, we obtain

$$\theta_1(0,1) = \frac{1}{2}\Big(\frac{a}{\kappa} - \theta_2(0,1)\Big)ps'(ah). \tag{16}$$

If we substitute this value for $\theta_1(0,1)$ into (14) we obtain, after simplification,

$$\Big(\theta_2(0,1)^2 - \frac{a^2}{\kappa^2}\Big)\Big(qs'(ah)q'r(ah) - \Big(\frac{ps'(ah)}{2}\Big)^2\Big) = 0 .$$

But this means that $\theta_2(0,1)^2 = \frac{a^2}{\kappa^2}$, whence $\theta_2(0,1) = \pm\frac{a}{\kappa}$. Since $\theta_2(0,1) \in Z$, this implies that κ divides a, whence $C(\tau_1)$ divides $C(\tau_2)$.

It is now possible to determine all possible ring homomorphisms from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$. From (16), and the fact that $\theta_2(0,1) = \pm\frac{a}{\kappa}$, we see that $\theta_1(0,1) = 0$

if $\theta_2(0,1) = \frac{a}{\kappa}$, and $\theta_1(0,1) = \frac{a}{\kappa}\,ps'(ah)$ if $\theta_2(0,1) = -\frac{a}{\kappa}$. Thus if θ is a ring homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$, then θ is one of two possible maps:

i) for all $(k,\ell) \in \mathcal{D}(h)$, $\theta(k,\ell) = (k, \frac{a}{\kappa}\ell)$, or

ii) for all $(k,\ell) \in \mathcal{D}(h)$, $\theta(k,\ell) = (k+\frac{a}{\kappa}\,ps'(ah), -\frac{a}{\kappa}\ell)$.

As we have seen, each of these is indeed a ring homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$ when $C(\tau_1)$ divides $C(\tau_2)$ and in fact each is an injective map. ∇

Thus each ample analytic immersion class of conformal tori does have an inherent pre-ordering, or quasi-ordering. That is, for τ_1, τ_2 in the same ample analytic immersion class, define $\tau_1 < \tau_2$ iff $C(\tau_2)$ divides $C(\tau_1)$. That the relation is not a partial ordering follows from the fact that $\tau_1 < \tau_2$ and $\tau_2 < \tau_1$ do not imply that $\tau_1 = \tau_2$. For consider the conformal tori τ_1 and τ_2 represented by $\sqrt{5}\,i$ and $\frac{1+\sqrt{5}\,i}{2}$ respectively. Then both τ_1 and τ_2 are contained in $\mathcal{M}_5$ and $\mathcal{D}(\tau_1) = \mathcal{D}(\tau_2) = Z(5)$, whence $C(\tau_1) = C(\tau_2)$. However, $\tau_1 \neq \tau_2$.

We make one last observation in this chapter, this time concerning the monoids $An(\tau_1)$ and $An(\tau_2)$. By (2.1.6), it follows that a monoid homomorphism of $An(\tau_1)$ into $An(\tau_2)$

induces a monoid homomorphism from (τ_1) into (τ_2). In certain circumstances, we have a converse to this result.

<u>(5.2.4) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. If there exists a ring homomorphism from $\mathcal{D}(\tau_1)$ into $\mathcal{D}(\tau_2)$, then there exists a monoid homomorphism from $An(\tau_1)$ into $An(\tau_2)$.

Proof: There is nothing to prove if τ_1 and τ_2 are non-ample. If τ_1 and τ_2 are ample, let $(a,h) \varepsilon Rep(\tau_1,\tau_2)$. Suppose now that there exists a ring homomorphism from $\mathcal{D}(\tau_1)$ into $\mathcal{D}(\tau_2)$. Then by (5.2.3) there exist exactly two ring homomorphisms from $\mathcal{D}(\tau_1)$ into $\mathcal{D}(\tau_2)$ and they are, respectively, the map which takes every $(k,\ell) \varepsilon \mathcal{D}(ah)$ to $(k + \ell\frac{K}{a} ps'(h), -\frac{K}{a}\ell) \varepsilon \mathcal{D}(h)$, and the map which takes $(k,\ell) \varepsilon \mathcal{D}(ah)$ to $(k, \frac{K}{a}\ell) \varepsilon \mathcal{D}(h)$. Let the latter homomorphism be denoted by $\theta\colon \mathcal{D}(ah) \to \mathcal{D}(h)$. Then define the map $\gamma\colon An(\tau_1) \to An(\tau_2)$ as follows: for each $(k,\ell,(\bar{D})_{ah}) \varepsilon An(\tau_1)$, let

$$\gamma(k,\ell),(\bar{D})_{ah}) = (\theta(k,\ell\ ,(\bar{D})_h).$$

It is necessary to show that for each $E \varepsilon (\bar{D})_{ah}$, $(\bar{E})_h = (\bar{D})_h$. But $E \varepsilon (\bar{D})_{ah}$ implies that there exist integers n and n' such that $E = D + n + n'ah$, whence $(\bar{E})_h = (\bar{D})_h$. Thus γ is a well-defined function from $An(\tau_1)$ into $An(\tau_2)$.

To show that $\tilde{\theta}$ preserves the multiplication, let $(k,\ell,(\bar{D})_{ah})$, $(m,n,(\bar{E})_{ah}) \in An(\tau_1)$. Then

$$\tilde{\theta}((k,\ell,(\bar{D})_{ah}) \circledast (m,n,(\bar{E})_{ah})) = \tilde{\theta}((k,\ell) *_{ah} (m,n), (\overline{C^{ah}(k,\ell)E+D})_{ah})$$

$$= (\theta(k,\ell) *_h \theta(m,n), (\overline{C^{h}(k,\ell\tfrac{\kappa}{a})E + D})_h)$$

$$= ((k,\ell\tfrac{\kappa}{a}) *_h (m,n\tfrac{\kappa}{a}), (\overline{C^{h}(k,\ell\tfrac{\kappa}{a})E + D})_h)$$

$$= (k,\ell\tfrac{\kappa}{a}, (\bar{D})_h) \circledast (m,n\tfrac{\kappa}{a}, (\bar{E})_h)$$

$$= \tilde{\theta}(k,\ell,(\bar{D})_{ah}) \circledast \tilde{\theta}(m,n,(\bar{E})_{ah}).$$

Thus $\tilde{\theta}$ is a monoid homomorphism of $An(\tau_1)$ into $An(\tau_2)$. ∇

CHAPTER 6

ANALYTIC HOMOTOPIES IN A CONFORMAL TORUS

When investigating the analytic maps between two conformal tori, it seems natural to introduce the notion of an analytic homotopy. As a result, the set of analytic maps is partitioned in a rather useful fashion.

It would seem natural as well to consider the concept of analytic homotopy type. However, the equivalence relation defined on the set of conformal tori by partitioning it into subsets by analytic homotopy type turns out to be trivial.

§6.1 Analytic Homotopy Classes

(6.1.1) Definition. Let X and Y be spaces with conformal structure. Then a continuous function $F: X \times I \to Y$ is called an *analytic homotopy* iff for each $t \in I$, the function $F_t: X \to Y$, defined by $F_t(x) = F(x,t)$ for all $x \in X$, is an analytic function. If both $f, g: X \to Y$ are analytic functions, we say that f is analytically homotopic to g, written $f \simeq g$, iff there exists an analytic homotopy $F: X \times I \to Y$ such that $F_0 = f$ and $F_1 = g$.

If τ_1 and τ_2 are conformal tori belonging to the same analytic immersion class, then $An(\tau_1,\tau_2)$ is the set of analytic maps between any representatives of τ_1 and τ_2.

Suppose that h_1, h_2' ε τ_1, and h_2, h_2' ε τ_2. Then if p_1 denotes a conformal equivalence from $\mathbb{C}/\Gamma(h_1)$ to $\mathbb{C}/\Gamma(h_1')$, and p_2 denotes a conformal equivalence from $\mathbb{C}/\Gamma(h_2)$ to $\mathbb{C}/\Gamma(h_2')$, it is easily seen that two analytically homotopic maps $f, g\colon \mathbb{C}/\Gamma(h_1') \to \mathbb{C}/\Gamma(h_2')$ are lifted to analytically homotopic maps $\tilde{f}, \tilde{g}\colon \mathbb{C}/\Gamma(h_1) \to \mathbb{C}/\Gamma(h_2)$ as in the commutative diagrams:

$$\begin{tikzcd}
\mathbb{C}/\Gamma(h_1) \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & \mathbb{C}/\Gamma(h_2) \arrow[d, "p_2"] \\
\mathbb{C}/\Gamma(h_1') \arrow[r, "f"'] & \mathbb{C}/\Gamma(h_2')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}/\Gamma(h_1) \arrow[r, "\tilde{g}"] \arrow[d, "p_1"] & \mathbb{C}/\Gamma(h_2) \arrow[d, "p_2"] \\
\mathbb{C}/\Gamma(h_1) \arrow[r, "g"'] & \mathbb{C}/\Gamma(h_2')
\end{tikzcd}$$

Thus there exists an analytic homotopy F between f and g. But since $\tilde{f} = p_2^{-1} \circ f \circ p_1$, $\tilde{g} = p_2^{-1} \circ g \circ p_1$, it follows that $\tilde{f}$ and $\tilde{g}$ are analytically homotopic via $\tilde{F} = p_2^{-1} \circ F \circ (p_1\,,\, 1_I)$. Thus we may speak of analytically homotopic maps between conformal tori, and so $An(\tau_1,\tau_2)$ is partitioned into analytic homotopy classes.

<u>(6.1.2) Theorem</u>. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then f, g ε $An(\tau_1,\tau_2)$ are analytically homotopic iff the distortion coefficients of f and g are equal.

Proof: Let (a,h) ε $Rep(\tau_1,\tau_2)$. Then suppose that f and g are analytic maps from $\mathbb{C}/\Gamma(ah)$ to $\mathbb{C}/\Gamma(h)$. If there exists an analytic homotopy F between f and g, then for each $t \in I$, $F_t: \mathbb{C}/\Gamma(ah) \to \mathbb{C}/\Gamma(h)$ must be an analytic map. Thus for all $(\overline{z},t) \in \mathbb{C}/\Gamma(ah) \times I$, $F(\overline{z},t) = \overline{D(t)z} + \overline{E(t)}$ for some continuous functions $D, E: I \to \mathbb{C}$. Since $D(I) \subset \mathcal{L}(a,h)$, D must be a constant function. But $f(\overline{z}) = F_0(\overline{z}) = \overline{Dz} \oplus \overline{E(0)}$ and $g(\overline{z}) = F_1(\overline{z}) = \overline{Dz} \oplus \overline{E(1)}$, and so the distortion coefficients of f and g are equal.

Conversely, if f and g are analytic maps differing only by a constant, say $f(\overline{z}) = \overline{Dz} \oplus \overline{E}_1$ and $g(\overline{z}) = \overline{Dz} \oplus \overline{E}_2$ for all $\overline{z} \in \mathbb{C}/\Gamma(ah)$, then $F: \mathbb{C}/\Gamma(h) \times I \to \mathbb{C}/\Gamma(h)$ given by

$$f(\overline{z},t) = \overline{Dz} \oplus (\overline{1 - t)E_0} \oplus \overline{tE_1}$$

is an analytic homotopy between f and g. ∇

(6.1.3) Corollary. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then the analytic homotopy classes of $An(\tau_1,\tau_2)$ are parametrized by any distortion lattice associated with $An(\tau_1,\tau_2)$.

§6.2 Analytic Homotopy Type

(6.2.1) Definition. Let X ~~and Y be~~ spaces with conformal structure. Then X and Y are said to have the same *analytic homotopy type* iff there exist analytic maps $f:X \to Y$ and $g:Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Let τ be a conformal torus. Then for any $h, h' \in \tau$, $¢/\Gamma(h)$ is conformally equivalent to $¢/\Gamma(h')$ and so they have the same analytic homotopy type. Thus we may extend the notion of analytic homotopy type to conformal tori.

(6.2.2) Definition. Let τ_1 and τ_2 be conformal tori. Then τ_1 and τ_2 are said to have the same *analytic homotopy type* iff $¢/\Gamma(h_1)$ and $¢/\Gamma(h_2)$ have the same analytic homotopy type for any $h_1 \in \tau_1$, $h_2 \in \tau_2$.

However, we have the following

(6.2.3) Theorem. Conformal tori τ_1 and τ_2 have the same analytic homotopy type iff $\tau_1 = \tau_2$.

Proof: If $\tau_1 = \tau_2$ there is nothing to prove. Suppose now that τ_1 and τ_2 have the same analytic homotopy type, whence $\tau_1 \sim \tau_2$. Let $a = \mathrm{ind}(\tau_1,\tau_2)$. Then there exists $h \in \tau_2$ such that $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$. If f and g are analytic functions $f:¢/\Gamma(ah) \to ¢/\Gamma(h)$, $g:¢/\Gamma(h) \to ¢/\Gamma(ah)$ such that $f \circ g = 1_{¢/\Gamma(h)}$, $g \circ f = 1_{¢/\Gamma(ah)}$, then f and g must be single-sheeted. Thus $a = 1$, whence $\tau_1 = \tau_2$. ∇

CHAPTER 7

ANALYTIC Γ-STRUCTURES ON A CONFORMAL TORUS

The analytic versions of topological H-space and the more general Γ-manifold [6] were considered by H.G. Helfenstein [5]. In this chapter, we determine all possible analytic Γ-structures on a conformal torus. Since every conformal torus τ is an analytic covering of every other conformal torus immersion-equivalent to τ, it is natural to examine the problem of lifting an analytic Γ-structure from one conformal torus to another. We give necessary and sufficient conditions for the lifting of an analytic Γ-structure.

§7.1 Analytic Γ-structures

In §6.1, the definition of an analytic homotopy was given. We now define analytic Γ-structure on a space with conformal structure and, with this definition, proceed to describe what is meant by an analytic Γ-structure on a conformal torus.

(7.1.1) Definition. Let X be a space with conformal structure. Then X is said to have an *analytic Γ-structure* μ iff there exists a continuous function $\mu: X \times X \to X$ satisfying the following conditions:

i) for each $w \in X$, the functions μ_w, $\mu^w : X \to X$, defined by $\mu_w(x) = \mu(w,x)$ and $\mu^w(x) = \mu(x,w)$ for all $x \in X$, are analytic, non-analytically null homotopic maps.

ii) for all $v, w \in X$, $\mu_v \simeq \mu_w$, $\mu^v \simeq \mu^w$.

If there exists an element $u \in X$ such that:

iii) $\mu(u,u) = u$, and

iv) $\mu_u \simeq \mu^u \simeq 1_X$ (relative to u),

then μ is said to be an *analytic* H-*structure* on X. The element u is called an *analytic homotopy unit* for μ. If $u \in X$ satisfies (iii) and $\mu_u \simeq 1_X$ (relative to u) then u is called a left analytic homotopy unit for μ. A right analytic homotopy unit for μ is defined in an analogous fashion.

By a left *analytic homotopy inversion* ξ for the analytic Γ-structure μ we shall mean an analytic map $\xi : X \to X$ such that the function $\mu \circ (\xi, 1_X) \circ \Delta$ is null-homotopic. If $\mu \circ (1_X, \xi) \circ \Delta$ is null-homotopic, then ξ is called a right analytic homotopy inversion for μ. Finally, ξ is an analytic homotopy inversion for μ iff ξ is both a left and a right analytic homotopy inversion for μ.

If the diagram

$$\begin{tikzcd}
X \times X \times X \arrow[r, "{(\mu, 1_X)}"] \arrow[d, "{(1_X, \mu)}"'] & X \times X \arrow[d, "\mu"] \\
X \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

is analytically homotopy commutative, then μ is said to be ***analytically homotopy-associative.*** If the diagram commutes, then μ is said to be associative.

(7.1.2) Definition. Let μ and η be analytic Γ-structures on X and Y respectively. Then an analytic function $\psi : X \to Y$ is an ***analytic*** Γ-***homomorphism*** iff $\eta \circ (\psi, \psi) = \psi \circ \mu$. If there exists an analytic Γ-homomorphism $\Psi : Y \to X$ such that $\psi \circ \Psi = 1_Y$, $\Psi \circ \psi = 1_X$, then ψ is an ***analytic*** Γ-***isomorphism.***

In order to remove the distinction between analytic Γ-structures on a space which are essentially the same, we have the following

(7.1.3) Definition. If μ and η are analytic Γ-structures on X, then μ is ***equivalent*** to η iff 1_X is an analytic Γ-isomorphism.

Thus two analytic Γ-structures are equivalent iff they are analytically homotopic.

Let τ be a conformal torus. Then for any h, h' ε τ, there exists a conformal equivalence $p : \text{¢}/\Gamma(h') \to \text{¢}/\Gamma(h)$. If μ is an analytic Γ-structure on $\text{¢}/\Gamma(h')$, then there exists an analytic Γ-structure $\tilde{\mu}$ on $\text{¢}/\Gamma(h)$ such that p is an analytic Γ-isomorphism. For let $\tilde{\mu} = p \circ \mu \circ (p^{-1}, p^{-1})$. It is easily seen that $\tilde{\mu}$ is an analytic Γ-structure on $\text{¢}/\Gamma(h)$ and that p is an analytic Γ-isomorphism. Let $[\mu]$ denote the set $\{p_h \circ \mu \circ (p_h^{-1}, p_h^{-1}) \mid p_h : \text{¢}/\Gamma(h') \to \text{¢}/\Gamma(h)$ is a conformal equivalence, $h \varepsilon \tau\}$.

<u>(7.1.4) Definition</u>. The set $[\mu]$ is called an *analytic Γ-structure* on the conformal torus τ.

We proceed to determine all analytic Γ-structures on a given conformal torus τ. For suppose that μ is an analytic Γ-structure on $¢/\Gamma(h)$. Since $¢ \times ¢$ is simply-connected, and π_h is an analytic covering map, μ can be lifted to an analytic Γ-structure $\tilde{\mu}$ on ¢. For each $z \in ¢$, the maps $\tilde{\mu}_z, \tilde{\mu}^z : ¢ \to ¢$ must be lifts of analytic self-maps of $¢/\Gamma(h)$ and so they are non-constant linear maps with distortions belonging to $\mathcal{L}(h)$. This implies that there exist linear maps $C, D: ¢ \to ¢$, with distortions belonging to $\mathcal{L}(h)$, such that

$$\tilde{\mu}(z_1,z_2) = C(z_1)z_2 + D(z_1)$$

for all $(z_1,z_2) \in ¢ \times ¢$. Suppose $C(z) = C_1z + D_1$, $D(z) = C_2z + D_2$ for all $z \in ¢$. Then

$$\tilde{\mu}(z_1,z_2) = C_1z_1z_2 + D_1z_2 + C_2z_1 + D_2$$

for all $(z_1,z_2) \in ¢ \times ¢$. Since $\mathcal{L}(h)$ is a discrete subset of ¢, it follows that $C_1 = 0$, and then one must have $D_1 \in \mathcal{L}(h)$. Thus

$$\tilde{\mu}(z_1,z_2) = C_2z_1 + D_1z_2 + D_2$$

for all $(z_1,z_2) \in ¢ \times ¢$. From this, we see that μ has the form

$$\mu(\bar{z}_1,\bar{z}_2) = \overline{Cz_1} \oplus \overline{Dz_2} \oplus \bar{E}$$

for $C, D \in \mathcal{L}(h)$, $\overline{E} \in \mathbb{C}/\Gamma(h)$. It is clear that any function of this form is an analytic Γ-structure on $\mathbb{C}/\Gamma(h)$.

If we define

$$\Lambda = \{\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \mid (k^2 + \ell^2)(m^2 + n^2) \neq 0;\ k, \ell, m, n \in Z\}$$

and let $\lambda(\Lambda)$ be the subset of Λ obtained by requiring $\ell^2 + n^2 = 0$, then the function $\mu_{A,\overline{D}}$: $\mathbb{C}/\Gamma(h) \times \mathbb{C}/\Gamma(h) \rightarrow \mathbb{C}/\Gamma(h)$, for $A \in \Lambda$, $\overline{D} \in \mathbb{C}/\Gamma(h)$, given by

$$\mu_{A,\overline{D}}(\overline{z}_1, \overline{z}_2) = \overline{C^h(k,\ell)z_1} \oplus \overline{C^h(n,m)z_2} \oplus \overline{D}$$

is an analytic Γ-structure on $\mathbb{C}/\Gamma(h)$. Let λ: $\Lambda \rightarrow \lambda(\Lambda)$ be the projection mapping defined by

$$\lambda(\begin{bmatrix} k & \ell \\ m & n \end{bmatrix}) = \begin{bmatrix} k & 0 \\ 0 & n \end{bmatrix}$$

for all $\begin{bmatrix} k & \ell \\ m & n \end{bmatrix} \in \Lambda$.

Then we obtain the following

(7.1.5) Theorem. Let τ be a conformal torus. The set of all analytic Γ-structures on τ is equal to

$$\{\mu_{A,\overline{D}} \mid A \in \Lambda, \overline{D} \in \mathbb{C}/\Gamma(h)\} \text{ if } \tau \text{ is ample,}$$

or

$$\{\mu_{A,\overline{D}} \mid A \in \lambda(\Lambda), \overline{D} \in \mathbb{C}/\Gamma(h)\} \text{ if } \tau \text{ is non-ample,}$$

for $h \in \tau$.

(7.1.6) Theorem. Let τ be a conformal torus. Then two analytic Γ-structures, represented by $\mu_{A,\overline{D}}$ and $\mu_{B,\overline{E}}$ on $¢/\Gamma(h)$ for $h \in \tau$, are analytically homotopic iff $A = B$ for τ ample, or $\lambda(A) = \lambda(B)$ if λ is non-ample.

Proof: It is clear that for ample τ, $A = B$ implies that $\mu_{A,\overline{D}} \simeq \mu_{B,\overline{E}}$. For non-ample τ, $\lambda(A) = \lambda(B)$ is sufficient to conclude that $\mu_{A,\overline{D}} \simeq \mu_{B,\overline{E}}$. Conversely now, suppose that $\mu_{A,\overline{D}} \simeq \mu_{B,\overline{E}}$. Then for each $\bar{z} \in ¢/\Gamma(h)$, $\mu^{\bar{z}}_{A,\overline{D}} \simeq \mu^{\bar{z}}_{B,\overline{E}}$. If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then by (6.1.2) we must have $C^h(a_4,a_3) = C^h(b_4,b_3)$ and and $\overline{D} \oplus \overline{C^h(a_1,a_2)z} = \overline{E} \oplus \overline{C^h(b_1,b_2)z}$. Putting $\bar{z} = 0$ gives $\overline{D} = \overline{E}$, whence $C^h(a_1,a_2) = C^h(b_1,b_2)$. This implies by (3.1.3) that $a_1 = b_1$ and $a_4 = b_4$, whence $\lambda(A) = \lambda(B)$. If τ is ample, we may also conclude that $a_2 = b_2$ and $a_3 = b_3$, whence $A = B$. ∇

(7.1.7) Theorem. Up to equivalence, there is exactly one analytic H-structure $[\mu]$ on each conformal torus. Every analytic Γ-structure which is equivalent to $[\mu]$ is an analytic H-structure.

Proof: Suppose that τ is ample. Let $\mu_{A,D}$ be an H-structure on $¢/\Gamma(h)$ for any $h \in \tau$. If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, then from $\mu^{\bar{z}}_{A,\overline{D}} \simeq 1_{¢/\Gamma(h)}$ for all $\bar{z} \in ¢/\Gamma(h)$, we have $C^h(a_1,a_2) = 1$.

Similarly, $C^h(a_4,a_3) = 1$, whence $a_1 = a_4 = 1$, $a_3 = a_2 = 0$. Now let $\bar{u} \in ¢/\Gamma(h)$ be the analytic homotopy unit for $\mu_{A,\bar{D}}$. Then $\bar{u} \oplus \bar{u} \oplus \bar{D} = \bar{u}$. Since it is clear that for any $\bar{D} \in ¢/\Gamma(h)$, $\mu_{I,\bar{D}}$ is an analytic H-structure with unit $-\bar{D}$, it follows that all analytic H-structures are analytically homotopic. The case for non-ample tori is similar. ∇

Certainly, every analytic H-structure on a conformal torus admits an exact analytic homotopy inverse. It turns out that there are analytic Γ-structures which are not analytic H-structures but which admit analytic homotopy inversions.

<u>(7.1.8) Theorem</u>. Let τ be a conformal torus. The set of analytic Γ-structures on τ which admit left analytic homotopy inversions, and the inversions, is represented by

$$\{(\mu_{A,\bar{D}}, \xi_{\bar{E}}) \mid A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Lambda,\ \bar{D} \in ¢/\Gamma(h),\ \exists (k,\ell) \in Z \times Z \ni (a_1,a_2) = (k,\ell)*_h(a_4,a_3),\ \xi_{\bar{E}} = -F_{k,\ell,\bar{E}} \text{ for } \bar{E} \in ¢/\Gamma(h)\}$$

if τ is ample and $h \in \tau$. Otherwise, it is represented by

$$\{(\mu_{A,\bar{D}}, \xi_{\bar{E}}) \mid A = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix} \in \lambda(\Lambda),\ \bar{D} \in ¢/\Gamma(h),\ \exists\, k \in Z \ni a_1 = ka_4,\ \xi_{\bar{E}} = -F_{k,\bar{E}},\ \bar{E} \in ¢/\Gamma(h)\}.$$

Proof: We shall assume that τ is ample, since the non-ample case is similar. Suppose that there exists a left analytic homotopy inversion ξ for $\mu_{A,\bar{D}}$ on $\mathbb{C}/\Gamma(h)$, $h \in \tau$. Then ξ is an analytic self-map of $\mathbb{C}/\Gamma(h)$ and so there exists $(m,n) \in Z \times Z$, $\bar{E} \in \mathbb{C}/\Gamma(h)$ such that

$$\xi(\bar{z}) = \overline{C^h(m,n)z} \oplus \bar{E}$$

for all $\bar{z} \in \mathbb{C}/\Gamma(h)$. By assumption, $\mu_{A,\bar{D}}(\xi, 1_{\mathbb{C}/\Gamma(h)})$ is analytically null-homotopic. But for each $\bar{z} \in \mathbb{C}/\Gamma(h)$,

$$\mu_{A,\bar{D}}(\xi, 1_{\mathbb{C}/\Gamma(h)})(\bar{z}) = \overline{C^h(a_1,a_2)C^h(m,n)z} \oplus \overline{C^h(a_1,a_2)E}$$

$$\oplus \overline{C^h(a_4,a_3)z} \oplus \bar{D}$$

where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Lambda$. Thus $C^h(a_1,a_2)C^h(m,n) = -C^h(a_4,a_3)$, whence $(a_1,a_2) *_h (m,n) = -(a_4,a_3)$, or equivalently, $(a_1,a_2) *_h (-m,-n) = (a_4,a_3)$. The converse is obtained by retracing the argument. ∇

By the symmetry of the analytic Γ-structure, an analogous theorem for right analytic homotopy inversions is obtained.

<u>(7.1.9) Corollary</u>. Every analytic self-map of a conformal torus τ is a left (right) analytic homotopy inversion for some analytic Γ-structure on τ.

<u>(7.1.10) Corollary</u>. Up to equivalence, the only analytic Γ-structures on a conformal torus τ which admit both left and right analytic homotopy inversions are the functions

$$\mu(\bar{z}_1,\bar{z}_2) = \overline{A(z_1 + Uz_2)}$$

for $A, U \in \mathcal{D}(\tau)$ such that U is a unit.

<u>(7.1.11) Corollary</u>. Up to equivalence, the only analytic Γ-structures on a conformal torus τ which admit two-sided analytic homotopy inversions are the functions

$$\mu(\bar{z}_1,\bar{z}_2) = \bar{z}_1 \;\textcircled{$\pm$}\; \bar{z}_2 \quad .$$

Thus every abelian analytic Γ-structure admits two-sided analytic homotopy inversions.

<u>(7.1.12) Theorem</u>. An analytic Γ-structure on a conformal torus is analytically homotopy-associative iff it is an analytic H-structure, in which use it is associative.

Proof: Let $\mu_{A,\bar{D}}$, $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Lambda$, be an analytic Γ-structure on $\mathbb{C}/\Gamma(h)$ for $h \in \tau$. If $\mu_{A,\bar{D}}$ is analytically homotopy-associative, then $C^h(a_1,a_2) = C^h(a_1,a_2)^2$ and $C(a_2,a_3) = C(a_4,a_3)^2$, whence $C^h(a_1,a_2) = C^h(a_4,a_3) = 1$. Thus $a_1 = a_4 = 1$ and if τ is ample, $a_2 = a_3 = 0$. If τ is non-ample, then A can be assumed to belong to $\lambda(\Lambda)$, whence $a_2 = a_3 = 0$. Thus $A = I$ and $\mu_{I,\bar{D}}$ is an analytic H-structure. Conversely, it is easily seen that every analytic H-structure on a conformal torus is associative. ∇

§7.2 Criteria for Lifting an Analytic Γ-structure

It is well-known that a topological H-structure on a space X can be lifted to an H-structure on any nice covering space of X . Furthermore, within the category of spaces with conformal structure and analytic maps, it follows that an analytic H-structure on a space can be lifted to an analytic H-structure on any covering space of X in the category.

It is immediately clear, however, that a Γ-structure cannot in general be lifted. For the essence of the proof in the case of H-structures is the fact that the induced structure on the loop space of an H-space is the group structure of the fundamental group. This in turn relies on the fact that an H-structure has both a left and a right homotopy unit.

In the category of conformal tori and analytic maps we are able to determine exactly which analytic Γ-structures on a conformal torus may be lifted to a covering conformal torus. We remark once more that every non-constant map of this category is a covering map.

<u>(7.2.1) Theorem</u>. Let τ_1 and τ_2 be ample immersion-equivalent conformal tori. Choose $(a,h) \in \mathrm{Rep}(\tau_1,\tau_2)$. Then an analytic Γ-structure $[\mu]$ on τ_2, represented by $\mu_{A,\overline{E}}$ on $\mathbb{C}/\Gamma(h)$, can be lifted via an arbitrary non-constant analytic map from $\mathbb{C}/\Gamma(ah)$ to $\mathbb{C}/\Gamma(h)$, to an analytic Γ-structure on τ_1

iff $\kappa(a,h)$ divides $a[a_2,a_3]$, where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \varepsilon \Lambda$.

Proof: Consider the diagram:

$$\begin{tikzcd}
C \times C \arrow[rrr, "\tilde{\mu}^*"] \arrow[dr] \arrow[ddd, "\tilde{f} \times \tilde{f}"'] & & & \mathbb{C} \arrow[dl] \arrow[ddd, "\tilde{f}"] \\
& \mathbb{C}/\Gamma(ah) \times \mathbb{C}/\Gamma(ah) \arrow[r, "\mu^*"] \arrow[d, "f \times f"'] & \mathbb{C}/\Gamma(ah) \arrow[d, "f"] & \\
& \mathbb{C}/\Gamma(h) \times \mathbb{C}/\Gamma(h) \arrow[r, "\mu_{A,\bar{E}}"'] & \mathbb{C}/\Gamma(h) & \\
C \times C \arrow[ur] \arrow[rrr, "\tilde{\mu}"'] & & & \mathbb{C} \arrow[ul]
\end{tikzcd}$$

where the connecting maps are the respective projections.

The analytic Γ-structure $\mu_{A,\bar{E}}$ can be lifted to the analytic Γ-structure $\tilde{\mu}$ on $\mathbb{C}$ via the projection π_h . Every non-constant analytic map $f:\mathbb{C}/\Gamma(ah) \to \mathbb{C}/\Gamma(h)$ can be lifted to a linear map $\tilde{f}:\mathbb{C} \to \mathbb{C}$, say $\tilde{f}(z) = Fz + G$ for all $z \varepsilon \mathbb{C}$, where $F \varepsilon \mathcal{L}(a,h)$, $G \varepsilon \mathbb{C}$. Then $\tilde{\mu}$ induces an analytic Γ-structure $\tilde{\mu}^*$ on $\mathbb{C}$ such that $\tilde{f}$ is an analytic Γ-isomorphism. Thus if $\mu_{A,\bar{E}}$ is to be lifted to $\mathbb{C}/\Gamma(ah)$, $\tilde{\mu}^*$ must cover the lift of $\mu_{A,\bar{E}}$. If we denote $C = C^h(a_1,a_2)$ and $D = C^h(a_4,a_3)$, then for all $(z_1,z_2) \varepsilon \mathbb{C} \times \mathbb{C}$, $\tilde{\mu}(z_1,z_2) = Cz_1 + Dz_2 + E$. The fact that the outer square of the diagram commutes is expressed by

$$F\tilde{\mu}^*(z_1,z_2) + G = C(Fz_1 + G) + D(Fz_2 + G) + E$$

for all $(z_1,z_2) \varepsilon \mathbb{C} \times \mathbb{C}$. Thus $\tilde{\mu}^*(z_1,z_2) = Cz_1 + Dz_2 +$

$(CG + DG + E - G)\frac{1}{F}$ since $F \neq 0$, as a consequence of the fact that f is non-constant. Thus $\tilde{\mu}^*$ is the lift of an analytic Γ-structure on $\mathbb{C}/\Gamma(ah)$ iff both C and D belong to $\mathcal{L}(ah)$, which is true iff there exists

$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \varepsilon \Lambda$ such that $C = b_1 + b_2qs'(ah)ah$ and

$D = b_4 + b_3qs'(ah)ah$, that is, $C = C^{ah}(b_1,b_2)$ and

$D = C^{ah}(b_4,b_3)$. But by (4.2.1),

$$C = a_1 + \frac{a_2a}{\kappa(a,h)}qs'(ah)ah \quad \text{and} \quad D = a_4 + \frac{a_3a}{\kappa(a,h)}qs'(ah)ah.$$

Suppose now that $\mu_{A,\bar{E}}$ can be lifted. Then, equating the imaginary parts of the two equations giving C and D respectively, we conclude that $\kappa(a,h)$ divides both aa_2 and aa_3, whence $\kappa(a,h)$ divides $a[a_2,a_3]$. The converse follows by reversing the argument. ∇

Thus the lifts of an analytic Γ-structure obtained by using different covering maps are equivalent and so we may speak of lifting an analytic Γ-structure without making specific mention of the analytic covering.

(7.2.2) Theorem. Let τ_1 and τ_2 be non-ample immersion-equivalent conformal tori. Then every analytic Γ-structure on τ_2 can be lifted to τ_1.

Proof: This theorem follows from a proof similar to the preceeding theorem. ∇

(7.2.3) Corollary. Let τ_1 and τ_2 be conformal tori with analytic Γ-structures $[\mu]$ and $[\eta]$ defined on τ_1 and τ_2 respectively. Then there exists an analytic Γ-homomorphism from τ_1 to τ_2 iff every analytic map from τ_1 to τ_2 is an analytic Γ-homomorphism.

Proof: This was essentially proven in the proof of (7.2.1). ∇

(7.2.4) Corollary. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then every analytic Γ-structure on τ_2 can be lifted to τ_1 iff $C(\tau_1)$ divides $C(\tau_2)$.

Proof: If τ_1 and τ_2 are non-ample, then $C(\tau_1) = C(\tau_2) = 1$ by definition. Thus this case follows by (7.2.2). If τ_1 and τ_2 are ample, then by (5.2.2), $C(\tau_1)/C(\tau_2) = \kappa(a,h)/a$ for each $(a,h) \in \mathrm{Rep}(\tau_1, \tau_2)$. Since every analytic Γ-structure on τ_2 can be lifted to τ_1 if $\kappa(a,h)$ divides a, by (7.2.1~~), the the~~orem follows. ∇

(7.2.5) Corollary. Let τ_1 and τ_2 be immersion-equivalent conformal tori. Then every analytic Γ-structure on τ_2 can be lifted to τ_1 iff there exists a ring homomorphism from $\mathcal{D}(\tau_2)$ into $\mathcal{D}(\tau_1)$.

Proof: This follows from (5.2.3) and (7.2.4) in the ample case, and the non-ample case is obvious. ∇

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