A new class of inequality measures based on weighted conditional incomplete moments

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Abstract

In the previous few decades, inequality measurements have particularly been interested in statistics and economics research. Among the foremost widely used measures of inequality are the measures based on complete, incomplete, and conditional moments. In this investigation, we introduce a new class of inequality measures based on weighted conditional incomplete moments. Also, we compare the new class of measures with the class of inequality measures based on weighted incomplete moments introduced by Abouelmagd and Ahmed (2014). The new class measures are shown to include many previously discussed ones such as Butler and McDonald (1987), Ahmad (1998), Lorenz measures as well as others. A statistical analysis of the new measures is presented. The new measures also characterize the income distributions well. We study these new measures under Pareto distribution. A real data application is given to illustrate the benefits of the proposed inequality measures based on weighted conditional over the previous measures.

Keywords: Income inequality measures, complete and incomplete moments, probability weighted moments, Lorenz curve, characterization on income inequality.

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1. Introduction

Equality of income distribution is found when each income unit receives the same proportional share from the whole population income. The income units could also be further defined as individual, household, and a consumption unit; see for example, Alesina et al (2016).

Inequality is additionally outlined as any deviation from equality. Therefore, if any unit received less than his proportionate share of the aggregate income, the distribution would be uneven, see for example Hauser and Norton (2017).

On the other hand, Stiglitz (2015) argues that economic growth will stop flowing until we deal with the problem of inequality.

Measuring inequality is thus not easy because it first requires answering a series of questions. What population is the focus? What variables might interest us? And what properties of that variable's distribution matter for our purposes, which we do summarize in a statistic?

Income and wealth distributions are typically non-normal and may take various shapes. In view of this, distribution-free approaches are especially well-suited to the task of comparing inequality measures. However, despite a large literature, nonparametric methods for inference on such measures perform poorly in finite samples. As the sample size increases, the focus shifts from finite-sample distortions to asymptotic problems caused by the failure of the assumptions necessary to confirm size control. These problems are typically related to heavy tails, a common situation in applied work; see for example Zenga (2007) and Dufour et al (2019).

It is a well-known fact that income distributions are commonly unimodal and skew with a heavy right tail. Therefore, different skew models such as the lognormal and the Pareto have been proposed as suitable descriptions of the income distribution, but they are usually applied in specific empirical situations. For general studies, more wide-ranging tools have been considered. The foremost commonly used theory relies on the Lorenz curve. The Lorenz curve can be defined as the relationship between the cumulative proportion of income units and the cumulative proportion of income received when units are arranged in ascending order of their income; see for example Habba et al (2018), Sen (1997), and Shahateet (2006). However, several economic questions depend on the shape as well as the mean. Butler and McDonald

(1987) used incomplete moments to characterize income inequality and provide the basis for interdistributional Lorenz curves. The incomplete moments represent the proportion of each respective moment that is accounted for by income levels less than pre-chosen values. In defense of their approach, Butler and McDonald argued that these incomplete moments do explain the shape of distributions and also can be used readily to build measures of income inequality. They reintroduced Gini coefficient, Pietra measure, and Lorenz curve in terms of incomplete moments. Gastwirth et al (1989) gave a statistical analysis of a special case of the Butler-McDonald measures. They established its asymptotic normality under minimal conditions. Ahmad (1998) studied a conditional moments version of the incomplete moments and presented a good account of its inferential properties. Abouelmagd and Ahmed (2014) introduced the new class of inequality measures based on weighted incomplete moments using probability weighted moments. They showed that this class is a generalization of Butler-McDonald (1987). They illustrated that the new class measures characterize the income distributions well. They did not present applications for the new measures under real data or income distribution.

In this article, we review inequality measures based on incomplete moments and conditional incomplete moments. Also, we redefine Probability Weighted Moments (PWM) in terms of expected value of order statistics and show how they can be used to estimate the inequality measures. In addition, we propose a general class of inequality measures using PWM which gives fast, straightforward and unbiased estimators. In section 2, we discuss the impact of income inequality on economic growth and vice versa. We review the desirable axioms of inequality measures to define which measure satisfies these axioms and determine the best measure. In addition, we discuss several famous inequality measures such as Lorenz curve and Gini coefficient. In addition, the PWM, a generalization of the usual moments of probability distribution, is introduced and how to use it to obtain unbiased estimators. We review the incomplete moments measure for income inequality presented by Butler-McDonald (1987) and conditional incomplete moments measure for income inequality proposed by Ahamd (1998). Finally, we discuss inequality measures based on weighted incomplete moments introduced by Abouelmagd and Ahmed (2014) and show its different special cases. In Section 3, we propose a generalized class of inequality using weighted conditional incomplete moments and give different special cases. In section 4, we propose the estimation of the proposed inequality measures based on weighted conditional incomplete moments measures. In section 5, we study the proposed measures of inequality under Pareto distribution. Also, we use real data from the Egyptian National Survey about individuals' income in 2009, 2011, 2013, and 2015 to measure income inequality using weighted incomplete moments measures and weighted conditional moments measures. Section 6 is devoted to the conclusion.

2. Literature review

To begin with, we discuss the impact of income inequality on economic growth and the impact of growth on income inequality. We review the desirable properties of inequality measure to know which measure satisfies these properties and determine the best measure. In addition, we discuss the most famous inequality measures, Lorenz curve and Gini coefficient. The PWM, a generalization of the usual moments of probability distribution, is reviewed and how to use it to obtain unbiased estimators. Finally, we review the Butler-McDonald measure (1989), Ahmad measure (1998), and the class of inequality measures based on weighted incomplete moments introduced by Abouelmagd and Ahmed (2014) of income inequality.

2.1. Inequality and economic growth

There are many reasons why nations want to reduce inequality even if there is no impact on economic development. These reasons may include inequality's impact on social cohesion. The studies on the impact of income inequality on economic development and the effect of economic growth on income inequality have created controversies and unconvincing results that cause doubts about the true effect of income inequality on economic development and vice versa. Specially, the variety of research findings, and the resulting lack of consensus among economists about the real effects. The inconsistency among studies may result from the range of income inequality measures that are being utilized in quantitative studies. Here, we discuss the impact of income inequality on economic growth and the effect of economic growth on income inequality.

2.1.1. Impact of economic growth volatility on income inequality

Chambers (2007, 2010) uses real data to prove that there is sturdy proof that nations' income distributions respond similarly to economic growth over the short and medium run, regardless of economic growth. However, in the long-run, prior economic growth affects developed and developing countries in consistently different ways. Specifically, prior long-run growth promotes inequality in developed countries, while having the opposite effect on developing countries.

Checchi and Garcia-Penalosa (2004) show that economic shocks hit the poor more profoundly than the rich, as a result of the poorer sections of the population not having adequate safeguards to deal with uncertainty and economic shocks. They concluded that neither economic growth nor inflationary trends have worsened the distribution of income, but the volatility of economic growth significantly affects the distribution of income.

Atkinson and Morelli (2011) explore the causality between income inequality and volatility during economic crises. Their research shows that income inequality increases if the probability of fluctuations in the economy increases. In keeping with their findings, the proportion of top income earners rises during periods of economic upswings; however, the decline during downswings is less in comparison to the income shares of the lower groups of income earners.

Ling-Zheng and Xia-Hai (2012) show that, for China, financial deepening worsens income inequality and that there is an inverted 'U' shaped relationship between income inequality and financial development. The impact of financial development on income inequality is favorable if the share of the growth of average income for the poorer population is higher than that enjoyed by the top 20 per cent.

Sotomayor (2019) accounted for the forces that drove inequality and poverty to decline by unprecedented extents and tried to determine the evidence for the relationship between economic growth and income inequality.

Ghosh (2020) examines how income inequality is affected by financial development, inflation, human capital formation, trade openness, and fiscal policies, apart from the volatility nexus. His empirical findings strongly suggest a long-term cointegrating relationship between income inequality and growth volatility, with a positive and statistically significant impact. His study reconfirms the theoretical underpinning in the literature that larger volatility in economic growth leads to a rise in inequality of the distribution of income. The findings continue to hold for the major ASEAN countries using other control variables, namely, financial development, human capital formation and trade oneness.

2.1.2. Impact of inequality on economic growth

Researchers investigate if inequality-growth relationship shows constant signs when calculated based on alternative measures of income inequality. Therefore, two research hypotheses have been put forward: income inequality has a negative effect on the rate of income development, and the strength of the estimated effects of income inequalities on the proportion of income growth varies for different measures of income inequality. On the other hand, some other researchers try to determine the effects of poverty on economic growth. Theoretically, some authors have attempted to find a link between poverty and economic growth, with the assumption that low income puts people in a poverty trap as in Sachs (2005).

Dabla-Norris et al (2015) explore the effect of inequality on economic growth and notice that, when the income of the wealthiest 20 per cent increases, then economic growth gradually decreases, whereas, when the income of the poorest 20 per cent increases, this has a positive effect on economic growth in the long-run.

Bartak and Jabłoński (2020) verify that income inequality has a negative impact on income growth rates in developed OECD countries, and that the strength of the effect of income inequalities on income growth rates varies according to different measures of income inequality.

Breunig and Majeed (2020) suggest that the proposition that inequality influences economic development on its own may be even stronger. However, inequality tends to interact with high levels of poverty to impact economic growth negatively and significantly. They argue that their findings do not suggest that inequality has a positive role to play in economic development.

2.2. Axioms for inequality measures

If $x_1 \leq x_2 \leq \cdots \leq x_n$ is an ordered income distribution among n individuals denoted by a nonnegative vector:

$$X = (x_1, x_2, \dots x_n)$$

the inequality measure $\theta(X)$ is defined as a unique function of $x_1, x_2, \dots x_n$ satisfying certain desirable properties. We will discuss these properties; however, not all axioms can be met by a single measure. Therefore, researchers choose the one satisfying most of these axioms.

Property 1. Income scale independence: This axiom requires the inequality measure to be invariant to proportional income changes. In other words, this axiom implies that the inequality measure should remain unaffected if each income is increased (decreased) by the same proportion, therefore, the inequality measure should be independent of the scale of measurement.

$$\text{If} \quad Y = \alpha X (\alpha > 0), \quad then \;\; \theta(X) = \theta(Y).$$

Most standard measures pass this test except the variance since $Y = \sqrt{\alpha}\ X$ the $v(Y) = \alpha\, V(X)$ where α is a scalar and Y is a vector of incomes; see Ravallion (2004).

Property 2. Principle of population: The population principle requires inequality measures to be invariant to replications of the population. This means that the inequality measure should not be affected if a proportional number of people are added at all income levels. In other words, the principle of population (sometimes called population independence) states that multiplying a society leaves income inequality unchanged; see Jancewicz (2016).

Property 3. Transfer Principle: When an income is transferred from the poorest to the richest, this inequality must be increased (or at least not reduced) through measures of inequality. On the other hand, when an income is transferred from a richer person to a poorer person it should register a fall (or at least not increase). In other words, adding a constant to all incomes leaves income inequality unchanged; see Cowell (2011).

Property 4. Symmetry: This axiom implies that if two individuals interchange their income positions, the inequality measure remains unchanged. Moreover, inequality depends only on the frequency distribution of incomes and not on the order in which individuals are ranked within the distribution:

$$\theta(X) = \theta\big(\pi(X)\big),$$

where π is any permutation of X.

In addition, this axiom requires that the inequality measure be independent of any characteristic of individuals other than their income; Gaertner and Namezie (2003).

Property 5. Decomposability: This axiom requires that If we expect overall inequality to increase (decrease) then this population needs inequality to increase (decrease) in each subgroup of the population. Total inequality should be the sum of within and between-group inequality, however, some measures when we sum the within and between-group inequality, they don't sum to the total inequality.

Property 6. Range: The inequality measure lies in the range of zero to one. The inequality measure is nil when all individuals have the same income, but it takes the value unity when one individual gets all the income.

2.3. Examples of inequality measures

2.3.1. Lorenz curve

The Lorenz curve may be described as a tool used to represent income distributions: it tells us which proportion of total income is in the hands of a given percentage of the population. Also, The Lorenz curve can be defined as the relationship between the cumulative proportion of income units and the cumulative proportion of income received when units are adjusted in ascending order of their income. Alternatively, the Lorenz curve is a graphical representation of inequality of some quantity such as wealth or income. Individuals are arranged in ascending order by how much of this quantity they have, and the cumulative share of the total is then plotted against the cumulative share of the population. For complete equality of income, for example, it would be a straight line with a slope of one. The extent to which the curve falls below this perfect equality line is a measure of inequality. See for example Camargo (2019), Sen (1997), and Shahateet (2006).

1. **The first definition of the Lorenz curve**

   Let a vector of income X (a positive random variable) from a continuous distribution with cumulative distribution function (cdf) F(x)=F, density function$f(x)$, x(F)=$F^{-1}(x)$ quantile function and let $X_{i:n}$ denote the corresponding order statistics for a general distribution function F(x).

$$\mathrm{F}(x) = \int_0^x f(x)\, dx \qquad (1)$$

where $\mathrm{F}(x)$ can be interpreted as the proportion of units having an income less than or equal to x; $\mathrm{F}(x)$ varies from 0 to 1. Furthermore, if the mean μ of the distribution exists, the first-moment distribution function of X is defined as

$$F_1(x) = \frac{1}{\mu} \int_0^x x\, f(x)\, dx \qquad (2)$$

Also, $F_1(x)$ varies from 0 to 1 and $F_1(x)$ can be interpreted as the proportional share of total income of the units having income less than or equal to x.

If $f(x)$ is continuous, the derivative of $F_1(x)$ exists and is given by

$$\frac{dF_1(x)}{d\,x} = \frac{x\ f(x)}{\mu},$$

which implies that $F_1(x)$ is a monotonically nondecreasing function of x.

The Lorenz curve $\big(L(p)\big)$ is the relationship between the variables $\mathrm{F}(x)$ and $F_1(x)$ and is obtained by inverting the functions $\mathrm{F}(x)$ and $F_1(x)$, and eliminating x if the functions are invertible. Alternatively, the curve can be graphed by generating the values of $\mathrm{F}(x)$ and $F_1(x)$ by considering the arbitrary values of x.

In addition, the slope of the Lorenz curve is obtained as

$$\frac{dF_1}{dF} = \frac{x}{\mu}$$

which is always positive for positive income. Similarly, the second derivative of the curve is

$$\frac{d^2F_1}{dF^2} = \frac{1}{\mu f(x)} > 0,$$

These two derivatives mean that the slope of the Lorenz curve is positive and increases monotonically. From this it follows that $F_1 \leq F$. The straight line $F_1 = F$ is called the egalitarian line (equality line).

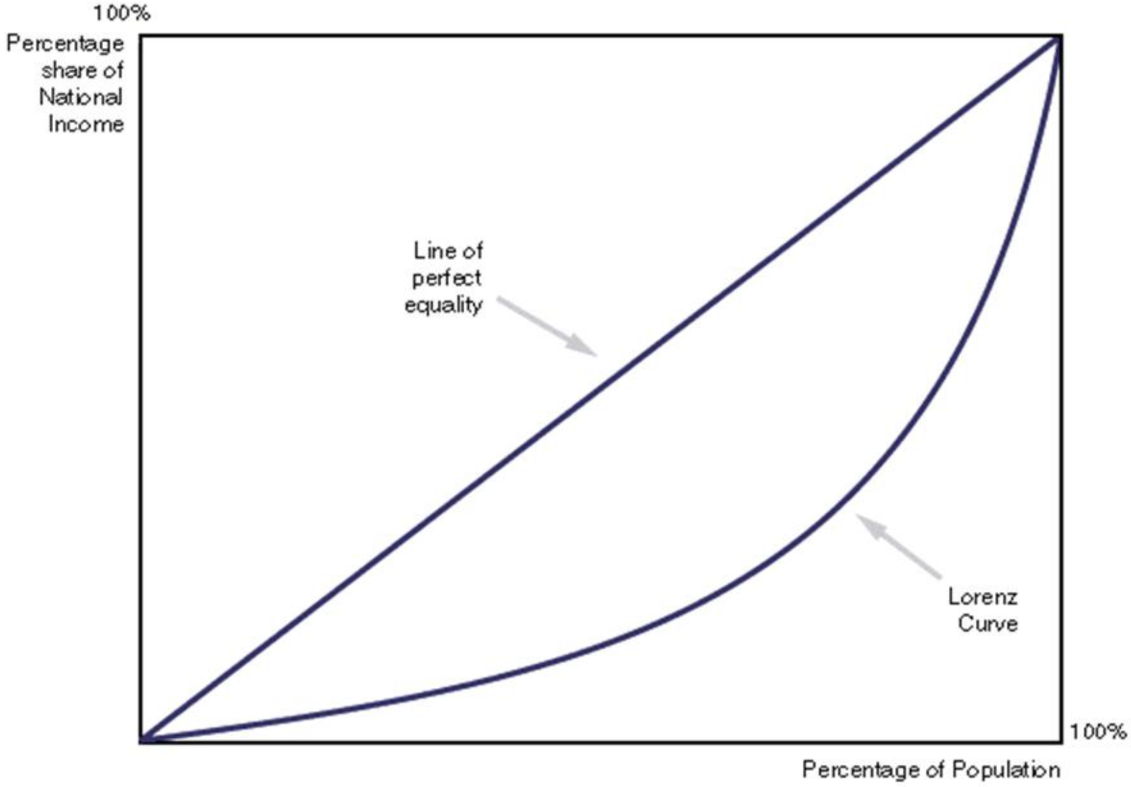

Figure 1. Lorenz curve

If the curve coincides with the equality line, this means that each unit receives the same income; this is the case of perfect equality of incomes. In the case of perfect inequality of incomes, all income is received by only one unit in the population; see for example Idrees and Ahmad (2017) and Fellman (2018).

Kakwani and Podder (1973) represented the relation as

$$p = \mathrm{F}(x) \qquad and \qquad L(p) = F_1(x)$$

and $0 \leq p \leq 1$. Also, $L(p)$ is interpreted as the fraction of total income received by the lowest p^{th} fraction of the families. In addition, it satisfies the following conditions

a) If $p = 0, \quad L(p) = 0;$

b) If $p = 1, \quad L(p) = 1;$

c) $L^{'}(p) = \frac{x}{\mu} \geq 0 \; and \; L^{''}(p) = \frac{x}{\mu\, f(x)} > 0;$

d) $L(p) < p.$

2. **The second definition of the Lorenz curve**

Gastwirth (1971) provided an alternative definition of the Lorenz curve, which is expressed in terms of the inverse of the distribution function

$$F^{-1}(t) = Min[x{:}\, F(x) \geq t] \tag{3}$$

This definition guarantees the existence of x for all values of $F(x)$, even though in the case of a discrete distribution corresponding to some values of $F(x)$, x does not exist. Also, if the density function is continuous, this definition of the inverse is identical to the usual definition of an inverse function, because x = $F^{-1}(t)$ and f$(x)dx = dt$ where t varies from zero to unity when x varies from zero to infinity.

The Lorenz curve can be rewritten as

$$L(p) = \frac{\int_0^p F^{-1}(t)\, dt}{\int_0^1 F^{-1}(t)\, dt} = \frac{\int_0^p F^{-1}(t)\, dt}{\mu}, \tag{4}$$

where $0 \leq p \leq 1$.

The advantages of the Lorenz curve are (Zegna, 2007):

i. Lorenz curve is suitable for any population and variable.
ii. Lorenz curve is easy to analyze.
iii. Lorenz curve always starts at (0,0) and ends at (1,1).
iv. Lorenz curve is scale invariant. It is based on how the variable values are distributed.
v. Lorenz curve can be used to look at a distribution of a single year or the changes in the Lorenz curve over time.

The disadvantages of the Lorenz curve are:

i. The Lorenz curve can't rise above the line of perfect equality and can't sink below the X axis.
ii. The Lorenz curve is not defined if the mean of probability distribution is zero or infinite.

2.3.2. Gini index

The Gini coefficient is the foremost utilized single measure of inequality. It is a measure of inequality of any quantity such as income or wealth, lies between zero (if there is no inequality) and one (if a single individual receives all of it). It depends on the Lorenz curve where it is a ratio of the areas on a Lorenz curve. Using the Lorenz curve, this coefficient is the ratio of the area between the diagonal and the Lorenz curve and the whole area under the diagonal. The Gini coefficient is defined as A/(A+B), where A is the area between the line of perfect equality and the Lorenz curve and the area under the Lorenz curve is B. Since A+B = $\frac{1}{2}$, the Gini coefficient, G= A/$\frac{1}{2}$= 2A= 1-2B; see Deutsch and Silber(1997), Fontanari et al (2018), and Gastwirth (1972).

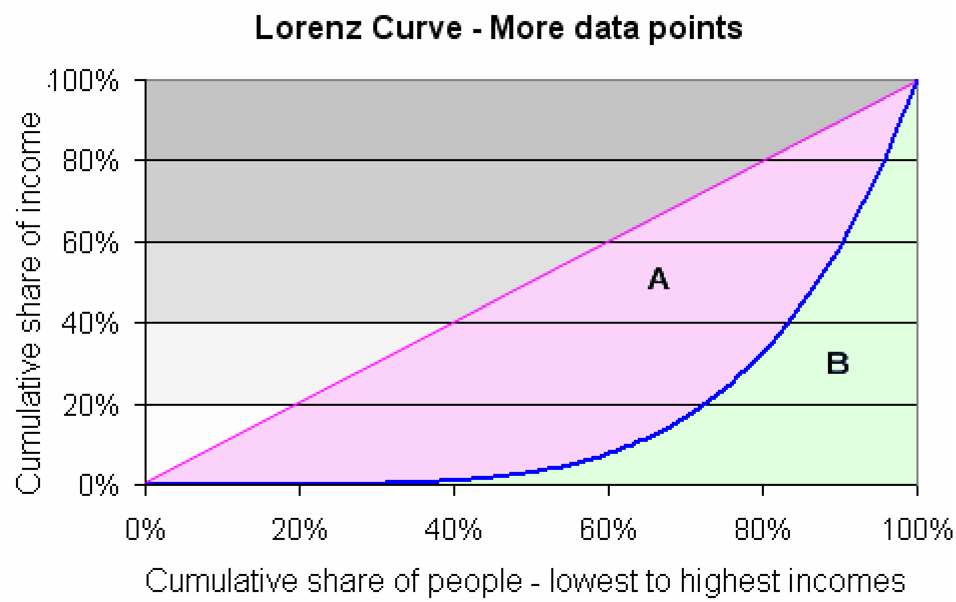

Figure 2 Gini index

Gini proposed two forms to measure income inequality. In 1912, he presented the first one which is defined as

$$G = \frac{\Delta}{2\mu} \quad (5)$$

In the continuous case, Δ is

$$\Delta = \int_0^\infty \int_0^\infty |x - y| f(x) f(y) dx\, dy.$$

and the estimated Δ is

$$\Delta = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|,$$

x_i being the income of the i^{th} individual, and n the total number of individuals.

Note that Δ is the arithmetic average of the $n(n-1)$ differences taken as absolute value. The 2μ is the maximum value of Δ, which is obtained when one individual gets all the income, so the Gini index will be one. On the other hand, the minimum value of Δ is zero when all individuals have the same income, so the Gini index will be zero.

In 1914, Gini presented the second form of the Gini measure that is equal to one minus twice the area under the Lorenz curve.

If the Lorenz curve is presented by the function y=L(x), the value of B can be found with integration and

$$G\ = 1 - 2\int L(x)dx \qquad (6)$$

If A = 0 the Gini coefficient becomes zero which means perfect equality, whereas if B = 0 the Gini coefficient becomes one which means complete inequality.

We can prove that both of these forms are equal, i.e.,

$$G = \frac{1}{2\mu}\int_0^\infty \int_0^\infty |x-y| f(x)f(y)dx\, dy = \frac{1}{\mu}\int_0^\infty [xF(x) - \mu F_1(x)]\ f(x)\, dx$$

Note that $F(x)$ is the probability distribution function, and that $F_1(x)$ is the first-moment distribution function. Integrating the first term by parts yields

$$\frac{1}{\mu}\int_0^\infty xF(x)\, f(x)\, dx = 1 - \int_0^\infty F_1(x)\ f(x)\, dx$$

which, on substituting into G, yields

$$G = 1 - 2\int_0^\infty F_1(x)\ f(x)\, dx \qquad (7)$$

which equals to one minus twice the area under the Lorenz curve.

The advantages of Gini coefficient; see Zagier (1983) and Plata-Pérez et al (2015).

i. If all incomes were doubled, the measure wouldn't change.
ii. If the population were to vary, the measure of inequality should not change.
iii. Symmetry. The measure concentrates on the measured variable and two units swap incomes, there should be no change in the measure of inequality.
iv. Gini coefficient satisfies the principle of transfer sensitivity. Under this criterion, the transfer of income from rich to poor reduces measured inequality.

v. Gini coefficient can be used to indicate how a distribution changes over time and if this change shows that equality is increasing or decreasing.

The disadvantages of the Gini coefficient:

i. Gini is not affected by the shape of the Lorenz curve.

ii. Gini doesn't indicate how inequality is distributed, only the total amount of inequality.

iii. Gini index isn't easily decomposable across groups. Gini is only decomposable if the partitions are non-overlapping. That is, the total Gini of society is not equal to the sum of the Gini coefficients of its subgroups.

iv. When using Gini, we can't test for the significance of changes in the index over time.

2.4. Probability weighted moments

Let $X_1, X_2, \dots, X_n$ be a random sample from a continuous distribution with density function $f(x)$, quantile function $x(F) = F^{-1}(x) = Q(F)$, $0 < F < 1$, cumulative distribution function $F(x) = F$, mean $\mu = E(X)$, and σ is the standard deviation of the distribution. The probability weighted moments are a generalization of the usual moments of a probability distribution. The probability weighted moments of a random variable X with distribution function F(x) = P(x≤ x) are the quantities

$$M_{p,r,s} = E[X^p\{F(X)\}^r\{1 - F(X)\}^s] \tag{8}$$

where p, r and s are real numbers. PWM are likely the most useful when the quantile function $x(F)$ can be written in closed form, so we can rewrite

$$M_{p,r,s} = \int_0^1 [x(F)]^p\, F^r\, (1 - F)^s dF \tag{9}$$

It is easily noticeable that $M_{p,r,s}$ is the *p-th* moment of the (r +1)*th* order statistic in a sample of size (r +s +1). Note that using $M_{p,r,s}$ involves information about the value and rank of a random variable in a group of (r +s +1) sample units; see Bartolucci et al (1999), Greenwood et al (1979), and Hosking (1990).

The quantities $M_{p,0,0}$ are the usual non-central moments. When r and s are integers, $F^r(1 - F)^s$ may be expressed as a linear combination of either powers of F or powers of $(1 - F)$, so it is natural to summarize a distribution either by the moments $M_{1,r,0}$ or $M_{1,0,s}$, where

$$\beta_r = M_{1,r,0} = E[X\{F(X)\}^r], \quad r = 0,1,2,\dots \tag{10}$$

and

$$\beta_s = M_{1,0,s} = E[X\{1-F(X)\}^s], \quad s = 0,1,2,\dots \tag{11}$$

Note that the expected value of order statistics is

$$E(X_{r:n}) = \frac{n!}{(r-1)!\,(n-r)!}\int_0^1 x(F)\,F^{r-1}\,(1-F)^{n-r}dF$$

Therefore, the probability-weighted moments can be written in terms of expected value of order statistics as

$$\beta_r = \int_0^1 x(F)F^r dF = \frac{E(X_{r+1:r+1})}{r+1}, \quad r = 0,1,\dots \tag{12}$$

and

$$\beta_s = \int_0^1 x(F)(1-F)^s dF = \frac{E(X_{1:s+1})}{s+1}, \quad s = 0,1,\dots \tag{13}$$

Note that β_r and β_s are used in many applications such as linear moments, estimation of the generalized extreme value distribution, analysis of hydrological extremes, and estimating the three-parameter Weibull distribution; see for example, Hosking et.al (1985), Moisello (2007), and Muhammad and Riaz (2006).

Hosking et al (1985) showed two unbiased estimators of β_r:

$$b_r = \frac{1}{n}\sum_{j=1}^{n}\frac{(j-1)(j-2)(j-3)\dots(j-r)}{(n-1)(n-2)(n-3)\dots(n-r)}x_{(j)} \tag{14}$$

is an unbiased estimator of β_r (Greenwood et al 1979)

and

$$\widehat{\beta_r}(p_{j,n}) = n^{-1}\sum_{j=1}^{n} p_{j,n}^r\, x_j\,, \tag{15}$$

where $p_{j,n}$ is a plotting position and can be either

$$p_{j,n} = \frac{j-a}{n}\ ,0 < a < 1\ \ or \quad p_{j,n} = \frac{n-a}{n+1-2a}\ ,\frac{-1}{2} < a < \frac{1}{2}$$

These two estimates are equivalent.

2.5. Butler-McDonald measure

The answers of several economic questions require knowing the mean and variance of a distribution, as well as the shape of the distribution. Butler-McDonald (1989) argued that the shape of many of those distributions may be usefully described by what they called the normalized incomplete moments of the underlying distribution. Furthermore, the effect of different shapes of distribution in the population is easily understood using moments rather than relying on indicators such as the Lorenz curve and the Gini index.

Let X denote the (random) income in an economic system (population) having cumulative distribution function (cdf) $F(x)$, probability density function (pdf) $f(x)$, y_r, y_s pre-chosen values and rth non-central moment

$$\mu_X^{(r)} = E(X^r) = \int_0^\infty x^r dF(x) \tag{16}$$

The Butler-McDonald measure of income inequality is given by

$$M_{BM}^r = \frac{\int_0^{y_r} x^r dF(x)}{E(X^r)} = \frac{I(y;r)}{\mu_X^{(r)}} = \phi(y_r;r) \tag{17}$$

The first two incomplete moments provide particularly useful information for many applications about the shape of the distribution, so the most important case is when r = 0 or 1. Gastwirth et al (1989) indicated that $\widehat{M}_{BM}^1$ can be used to test the hypothesis that there is no income inequality versus there is income inequality. In terms of the income distribution setting $r = 0$ yields the fraction of the population with income less than x, whereas r = 1 yields the proportion of total income accounted for by those with income less than x, see; Butler and McDonald (1989). Note that $\phi(y_r;r)$ is a distribution function and for some distributions it exhibits the same properties and form as the original df. Gastwirth et al (1989) established the asymptotic normality of an estimate of $M_{BM}^{(0)}$ and a variation of it when the means are replaced by the medians. They also compared the asymptotic relative efficiency of their estimate of $M_{BM}^{(0)}$ to the Wilcoxon and t statistics for location and scale alternatives.

Butler-McDonald (1989) represented income quantiles, Lorenz curve, Gini coefficient, and Pietra measure in terms of incomplete moments. They expressed the income quintiles of a distribution as the proportion of total income held by a given fifth of the population. The first quintile can be expressed in terms of the normalized incomplete moments functions as the regular first quintile measure $= \emptyset\,(\emptyset^{-1}(0.2; 0); 1.)$ An alternative quintile is to rank what proportion of the people holds the lowest fifth of income: The alternative first quantile measure $= \emptyset\,(\emptyset^{-1}(0.2; 1); 0.)$ They reintroduced Lorenz curve as a plot of $\big(\emptyset(x; 0),$ $\emptyset(x; 1)\big)$. In addition, they represented the Pietra measure (P) of income inequality which is equal to twice the area of the largest Lorenz curve and the 45° line of perfect inequality in terms of incomplete moments as

$$P = E(|Y - \mu|)/2\mu \tag{18}$$

$$P = \frac{1}{2\mu}\left[\int_{\mu}^{\infty}(y-\mu)f(y)dy - \int_{0}^{\mu}(y-\mu)f(y)dy\right]$$

$$P = \frac{1}{2\mu}\left[\int_{0}^{\infty} yf(y)dy - 2\int_{0}^{\mu} yf(y)dy - \mu\int_{0}^{\infty} f(y)dy + 2\mu\int_{0}^{\mu} f(y)dy\right]$$

$$P = \frac{1}{\mu}\left[\mu\int_{0}^{\mu} f(y)dy - \int_{0}^{\mu} yf(y)dy\right]$$

$$P = \emptyset(\mu; 0) - \emptyset(\mu; 1), \tag{19}$$

which is the difference of the fraction of the population having income less the mean and the fraction of total income held by that group.

Furthermore, they reintroduced the Gini coefficient in terms of incomplete moments as follows:

$$G = E(|Y - X|)/2\mu = \frac{1}{2\mu}\iint |y-x| f(x)f(y)dx\,dy \tag{20}$$

$$G = \frac{1}{2\mu}\left[\int_{0}^{\infty} f(x)\left[\int_{x}^{\infty}(y-x)f(y)dy - \int_{0}^{x}(y-x)f(y)dy\right]\right]dx$$

$$= \frac{1}{2\mu}\left[2\int_{0}^{\infty} x\,f(x)\int_{0}^{x} f(y)dy\,dx - 2\int_{0}^{\infty} f(x)\int_{0}^{x} yf(y)dy\,dx\right]$$

$$= \int_0^{\infty} f(x) \left[\frac{x}{\mu} \ \emptyset(x;0) \ - \ \emptyset(x;1)\right] dx \qquad (21)$$

2.6. Ahmad measure

Ahmad (1998) introduced a measure of income inequality based on conditional moments as

$$M_A^{(r)} = E(\, X^r \mid X^r \le y\,) = \frac{\int_0^{y_r} x^r \, dF(x)}{F(y)} = \psi(y_r; r) \qquad (22)$$

Note here that for $r = 0$, $M_A^{(0)}$=0. He illustrated that while $M_{BM}^{(0)}$ and $M_A^{(1)}$ perform well in terms of the concept of asymptotic relative efficiency (a concept that measures how good a statistical procedure is) $M_{BM}^{(1)}$ seems to perform rather poorly. He also proved mathematically that $M_A^{(1)}$ when used to estimate income inequality based on sampled data, does perform well relative to $M_{BM}^{(0)}$ and $M_{BM}^{(1)}$. It seems also to be natural since it is a straight conditional incomplete moment and not a relative incomplete moment as $M_{BM}^{(1)}$. $M_A^{(1)}$ may be explained as the fraction of poor people in the population who have income less than the y_r income of the population. However, Ahmad measure has no upper limit and it takes unlimited values.

2.7. Inequality measures based on weighted incomplete moments

Abouelmagd and Ahmad (2014) introduced a general class of inequality measures based on PWM

A. Measure based on $\boldsymbol{\beta_r}$

They proposed first a linear version of a measure of inequality based on PWM:

$$M_{PWM1}^{r} = \frac{\int_0^{y_r} xF^r dF(x)}{\int_0^{1} xF^r dF(x)} = \frac{\int_0^{y_r} xF^r dF(x)}{E(XF^r)} = \frac{I_L(y_r;r)}{\beta_r} = \frac{(r+1)I_L(y_r;r)}{E(X_{r+1:r+1})} = \phi_L(y_r;r) \qquad (24)$$

This version gives more weight for income in the largest part of the distribution.

B. Measure based on $\boldsymbol{\beta_s}$

They proposed second a linear version of a measure of inequality based on PWM:

$$M_{PWM2}^{s}=\frac{\int_0^{y_s}x(1-F)^s dF(x)}{\int_0^1 x(1-F)^s dF(x)}=\frac{\int_0^{y_s}x(1-F)^r dF(x)}{E(X(1-F)^r)}=\frac{I_L(y_s;s)}{\beta_s}=\frac{(s+1)I_L(y_s;s)}{E(X_{1:s+1})}=\phi_L(y_s;s) \qquad (25)$$

This version gives more weight for income in the lowest part of the distribution.

3. A unified general class of inequality measures based on weighted conditional incomplete moments

3.1. A modification of Ahmad measure

We propose an adjustment of Ahmad measure to make it scaled. The modified measure can be written as

$$M_{AA}^{(r)}=E(X^r \mid X^r\le y)=\frac{\left(\frac{\int_0^{y_r}x^r dF(x)}{F(y_r)}\right)}{E(X^r)}=\psi(y_r;r). \qquad (26)$$

In terms of the income distribution $M_{AA}^{(r)}$ can be explained as the fraction of poor people in the population who have income less than y_r of the population.

3.2. Generalization of the modified measure

A. Measures based on $\boldsymbol{\beta_r}$

The Adjusted Ahmed measure based on PWM can be defined as

$$A_{PWM1}^{r}=\frac{\left(\frac{\int_0^{y_r}xF^r dF(x)}{F(y_r)}\right)}{\int_0^1 xF^r dF(x)}=\frac{\left(\frac{\int_0^{y_r}xF^r dF(x)}{F(y_r)}\right)}{E(XF^r)}=\psi_L(y_r;r). \qquad (27)$$

This can be explained as the fraction of total weighted income accounted for poor people with weighted income less than y_r. More generally A_{PWM1}^{r} can be explained as the proportion of total weighted income accounted for by those with weighted income less than y_r. This version will give more weight for income in the largest part of the distribution.

B. Measure based on $\boldsymbol{\beta_s}$

The Adjusted Ahmad measure based on β_s can be defined as

$$A_{PWM2}^{s}=\frac{\left(\frac{\int_0^{y_s}x(1-F)^s dF(x)}{F(y_s)}\right)}{\int_0^1 x(1-F)^s dF(x)}=\frac{\left(\frac{\int_0^{y_s}x(1-F)^s dF(x)}{F(y_s)}\right)}{E(X(1-F)^r)}=\psi_L(y_s;s). \qquad (28)$$

This can be explained as the fraction of total weighted income accounted for poor people with weighted income less than y_r. More generally A^s_{PWM2} can be explained as the proportion of total weighted income accounted for by those with weighted income less than y_s. This version will give more weight for income in the lowest part of the distribution.

4. Estimation

Using the sample income data $X_1, X_2, \ldots, X_n$ and the empirical distribution function the above measures can be estimated as follows.

$$\widehat{M}^r_{PWM1} = \frac{\sum_{i=1}^n I(w_{ri}X_i < y_r) w_{ri} X_i}{\sum_{i=1}^n w_{ri} X_i} \tag{29}$$

and

$$\widehat{M}^s_{PWM2} = \frac{\sum_{i=1}^n I(w_{si}X_i < y_s) w_{si} X_i}{\sum_{i=1}^n w_{si} X_i} \tag{30}$$

where

$$w_{ri} = \left(\frac{i-0.5}{n}\right)^r \quad \text{and} \quad w_{si} = \left(1 - \frac{(i-0.5)}{n}\right)^s$$

Moreover,

$$\hat{A}^r_{PWM1} = \frac{\left(\frac{\sum_{i=1}^n I(w_{ri}X_i < y_r) w_{ri} X_i}{\hat{F}(y_r)}\right)}{\sum_{i=1}^n w_{ri} X_i} \tag{31}$$

and

$$\hat{A}^s_{PWM1} = \frac{\left(\frac{\sum_{i=1}^n I(w_{si}X_i < y_s) w_{si} X_i}{\hat{F}(y_r)}\right)}{\sum_{i=1}^n w_{si} X_i} \tag{32}$$

where $\hat{F}$ is the empirical distribution function.

5. Applications

5.1. Theoretical application

It is widely known that income distribution in general is unimodal and skew with a heavy right tail. Hence, the Pareto distribution has been proposed as a suitable description of the income distribution.

If $X_1, X_2, \dots\dots, X_n$ are random variables from a Pareto distribution with density function $pdf = ak^a\, x^{-(a+1)}$, cumulative distribution function $F = 1 - \left(\frac{k}{x}\right)^a$, quantile function X(F) = k $(1-F)^{\frac{-1}{a}}$, r^{th} noncertral moments $\mu_r^{`} = ak^r(a-r)^{-1}$ and

$\mu_X^{(h,r)} = \int_0^\infty x\, F^r dF$ and $\mu_X^{(h,s)} = \int_0^\infty x\,(1-F)^s dF$

then we can derive the general class of income inequality by using a Pareto distribution.

A. Measures based on $\boldsymbol{\beta_r}$

$$M_{PWM1}^r = \frac{1}{\int_0^1 [x\{F(x)\}^r]\, dF(x)} \int_0^{y_r} [x\{F(x)\}^r]\, dF(x).$$

$$= \frac{1}{\int_0^1 x\, F^r dF} \int_0^1 I(x \le y_r)\, x\, F^r dF$$

$$= \frac{1}{\int_0^1 x(F) F^r dF} \int_0^1 I(x \le y_r)\, x(F)\, F^r dF$$

$$= \frac{1}{\int_0^1 \mathrm{k}\,(1-F)^{\frac{-1}{a}} F^r dG} \int_0^1 I(\mathrm{k}\,(1-F)^{\frac{-1}{a}}) \le y_r)\mathrm{k}\,(1-F)^{\frac{-1}{a}}\ F^r dF$$

$$= \frac{1}{\int_0^1 \left((1-F)^{\frac{-1}{a}}\right) F^r dF} \int_0^1 I\left(F \le 1 - \left(\frac{k}{y_r}\right)^a\right)\left((1-F)^{\frac{-1}{a}}\right) F^r$$

Thus,

$$M_{PWM1}^r = \frac{1}{\int_0^1 \left((1-\mathrm{F})^{\frac{-1}{\mathrm{a}}}\right) \mathrm{F^r dF}} \int_0^{1-\left(\frac{\mathrm{k}}{\mathrm{y_r}}\right)^{\mathrm{a}}} (1-\mathrm{F})^{\frac{-1}{\mathrm{a}}}\ \mathrm{F^r dF}$$

And by using Karl Pearson's incomplete beta function, we find that

$$M_{PWM1}^{r} = \frac{\mathrm{I}\left(\frac{-1}{a}+1,\ \ r+1\ \ ;\left(1-\left(\frac{k}{\beta\left(\frac{-r}{a}+1,\ \ r+1\right)}\right)^{a}\right)\right)}{\beta\left(\frac{-1}{a}+1,\ \ r+1\right)}$$

and

$$A_{PWM1}^{r} = \frac{\mathrm{I}\left(\frac{-1}{a}+1,\ \ r+1\ \ ;\left(1-\left(\frac{k}{\beta\left(\frac{-r}{a}+1,\ \ r+1\right)}\right)^{a}\right)\right)}{F\left(y_r\right)} \Bigg/ \beta\left(\frac{-1}{a}+1,\ \ r+1\right)$$

Thus in the Pareto case,

$$A_{PWM1}^{r} = \frac{\mathrm{I}\left(\frac{-r}{a_1}+1, s+1;\left(1-\left(\frac{k_1}{\beta\left(\frac{-r}{a_1}+1,\ \ s+1\right)}\right)^{a_1}\right)\right)}{1-\left(\frac{k}{y_r}\right)^{a}} \Bigg/ \beta\left(\frac{-1}{a}+1,\ \ r+1\right)$$

B. Measures based on β_s

$$\mathrm{M}_{\mathrm{PWM2}}^{\mathrm{s}} = \frac{1}{\int_0^1 [\mathrm{x}\{1-\mathrm{F}(\mathrm{x})\}^{s}]\, \mathrm{dF}(\mathrm{x}).} \int_0^{\mathrm{y_r}} [\mathrm{x}\{1-\mathrm{F}(\mathrm{x})\}^{\mathrm{s}}]\, \mathrm{dF}(\mathrm{x}).$$

Therefore,

$$\mathrm{M}_{\mathrm{PWM2}}^{\mathrm{s}} = \frac{1}{\int_0^1 \mathrm{x}\,(1-\mathrm{F})^{\mathrm{s}}\mathrm{dF}} \int_0^1 \mathrm{I}(\mathrm{x} \le \mathrm{y_r})\ \mathrm{x}\,(1-\mathrm{F})^{\mathrm{s}}\mathrm{dF}$$

$$= \frac{1}{\int_0^1 \mathrm{x(F)}(1-\mathrm{F})^{\mathrm{s}}\mathrm{dF}} \int_0^1 \mathrm{I}(\mathrm{x} \le \mathrm{y_r})\ \mathrm{x(F)}(1-\mathrm{F})^{\mathrm{s}}\mathrm{dF}$$

$$= \frac{1}{\int_0^1 \mathrm{k}\,(1-F)^{\frac{-1}{a}}(1-F)^{s}dG} \int_0^1 I(\mathrm{k}\,(1-F)^{\frac{-1}{a}}) \le\ y_r)\mathrm{k}\,(1-F)^{\frac{-1}{a}}\ (1-F)^{s}dF$$

$$= \frac{1}{\int_0^1 \left((1-F)^{\frac{-1+s}{a}} \right) dF} \int_0^1 I\left(F \le 1 - \left(\frac{k}{y_r}\right)^a\right) \left((1-F)^{\frac{-1+s}{a}} \right) dF$$

$$= \frac{1}{\int_0^1 \left((1-F)^{\frac{-1+s}{a}} \right) F^0 dF} \int_0^{1-\left(\frac{k}{y_r}\right)^{a_1}} (1-F)^{\frac{-1+s}{a}} F^0 dF$$

And by using Karl Pearson’s incomplete beta function, we find that

$$\mathrm{M}_{\mathrm{PWM2}}^{s} = \frac{\mathrm{I}\left(\frac{-1+s}{a}+1,\ 1\ ;\left(1-\left(\frac{k}{\beta\left(\frac{-1+s}{a}+1,\ 1\right)}\right)^{a}\right)\right)}{\beta\left(\frac{-1+s}{a}+1,\ 1\right)}$$

Moreover,

$$A_{PWM2}^{s} = \frac{\mathrm{I}\left(\frac{-1+s}{a}+1,\ 1\ ;\left(1-\left(\frac{k}{\beta\left(\frac{-1+s}{a}+1,\ 1\right)}\right)^{a}\right)\right)}{F(y_r)} \Bigg/ \beta\left(\frac{-1+s}{a}+1,\ 1\right)$$

$$= \frac{\mathrm{I}\left(\frac{-1+s}{a}+1,\ 1\ ;\left(1-\left(\frac{k}{\beta\left(\frac{-1+s}{a}+1,\ 1\right)}\right)^{a}\right)\right)}{1-\left(\frac{k}{y_r}\right)^{a}} \Bigg/ \beta\left(\frac{-1+s}{a}+1,\ 1\right)$$

5.2. Real data application

We will use real data to compute and compare inequality measures discussed in sections three and four. The income data is from the survey of national income and individuals’ incomes in Egypt in 2009, 2011, 2013, and 2015, we consider the applicability to calculate and compare among income inequality indicators. The data set was obtained from the Central Agency for Public Mobilization and Statistics (CAPMS), which is responsible for systematically researching and analyzing incomes and consumptions of people every two years regularly. The CAPMS sample is a random sample from all Egyptian Governorates and the sample size is around 25000 units. The CAPMS gave us the data of incomes and consumptions of 12000 approximately individuals of the survey in the four years. Also, these data contain all details about incomes and consumptions of the individuals. For example, the individual income consists of salary and wages, agrarian income, and inherited income etc. Also, the individual consumption consists of the expenditure on cloths, food, and drinks etc.

The following tables will be calculated for the four years to Butler-McDonald measure(M_{BM}^r), generalized Butler-MacDonald Measure based on β_r (M_{PWM1}^r), and generalized Butler-MacDonald measure based on β_s (M_{PWM2}^r), using Ahmad measure (M_A^r), modified Ahmad measure (M_{MA}^r), generalized Ahmad Measure based on β_r (A_{PWM1}^r), and generalized Ahmad measure based on β_s (A_{PWM2}^s).

TABLE 1. Butler-MacDonald measure ($\mathbf{M_{BM}^r}$)

r	2009	2011	2013	2015
1	0.4294541	0.4334777	0.4319503	0.4284827
2	0.2905342	0.3166433	0.3176358	0.2050843
3	0.0611858	0.1417554	0.1397938	0.006139318
4	0.007200647	0.03876484	0.02947916	0.000346164
5	0.001558777	0.0122654	0.006965823	9.28E-05

Several points are noteworthy from the results of Table 1:

1. To begin with, the Sensitivity for (M_{BM}^r) measure decreases sharply when r^{th} increases. In 2009, income inequality is 0.4294541 when *r =1*, but it begins to decrease gradually when r *=2* until it reaches 0. 001558777 when r=5. The results of 2011, 2013, and 2015 show the same level of sensitivity. The first and second moments characterize the income distribution and the decline of M_{BM}^r was expected when r increases.
2. The values of income inequality for (M_{BM}^r) measure lie between zero and one from r = 1 to r = 5 for all years.
3. It is clear that income inequality slightly fluctuates for the same rank among years. When we compare the values of (M_{BM}^r) at r = 1 among years, we find that Butler-MacDonald measure (M_{BM}^r) is 0.4294541 in 2009, then slightly rose to 0.4334777 in 2011, and remains at the same level in 2013, after that slightly drops to 0.4284827 in 2015.

4. In 2009, income inequality is equal to 0.4294541 when r = 1, this value can be seen as a middle-income inequality in the population.

TABLE 2. Generalized Butler-MacDonald Measure based on $\boldsymbol{\beta_r}$ $(\boldsymbol{M^r_{PWM1}})$

r	2009	2011	2013	2015
1	0.1649598	0.1650852	0.1622601	0.1630331
2	5.08E-06	4.20E-06	3.37E-06	1.90E-06
3	3.19E-11	5.79E-11	3.89E-11	5.00E-13
4	4.02E-17	2.85E-16	1.13E-16	6.77E-20
5	3.91E-23	8.29E-22	1.82E-22	8.86E-27

Several points are noteworthy from the results of Table 2:

1. To begin with, the sensitivity for $(\boldsymbol{M^r_{PWM1}})$ measure drops sharply when r^{th} increases. In 2009, income inequality is 0.1649598 when *r = 1*, but it begins to fall dramatically when r = *2* and continues to drop until approximately reaching zero when r = *5*. The results of 2011, 2013, and 2015 showed the same level of sensitivity.
2. In addition, the values of income inequality for (M^r_{BM}) measure have the same range from zero to one from r = 1 to r = 5 for all years.
3. It is easy to see that income inequality remains the same for the same rank among years. When we compare the values of r = 1 among years, we find that $(\boldsymbol{M^r_{PWM1}})$ is 0.1649598 in 2009, then slightly rose to 0.1650852 in 2011, and fell to 0.1622601 in 2013, after that it slightly rose to 0.1630331 in 2015.
4. In 2009, income inequality is equal 0.1649598 when r = 1, this value can be seen as a low-income inequality in the population.

TABLE 3. Generalized Butler-MacDonald measure based on $\boldsymbol{\beta_s}$ ($\boldsymbol{M^r_{PWM2}}$)

r	2009	2011	2013	2015
1	0.2644943	0.2683925	0.2696901	0.2654495
2	4.70E-06	4.74E-06	4.11E-06	1.05E-06
3	1.27E-11	3.42E-11	2.17E-11	1.69E-13
4	1.17E-17	1.10E-16	3.99E-17	1.91E-20
5	9.54E-24	2.45E-22	5.09E-23	2.20E-27

Several points are noteworthy from the results of Table 3:

1. To begin with, the sensitivity for ($\boldsymbol{M^r_{PWM2}}$) measure declines sharply when r^{th} increases. In 2009, income inequality is 0.2644943 when *r = 1*, but it begins to fall dramatically when r = *2* and continues to drop until approximately reaching zero when *r = 5*. The results of 2011, 2013, and 2015 present the same level of sensitivity.
2. The values of income inequality for (M^r_{BM}) measure lies between zero and one from r = 1 to r = 5 for all years.
3. It is clear that income inequality approximately remains the same for the same rank among years. When we compare the values of r = 1 among years, we find that ($\boldsymbol{M^r_{PWM2}}$) is 0.2644943 in 2009, then slightly rose to 0.2683925 in 2011, and slightly increased to 0.2696901 in 2013, after that slightly declined to 0.2654495 in 2015.
4. In 2009, income inequality is equal 0. 2644943 when r = 1, this value can be seen as a low inequality in the population.

TABLE 4. Ahmad measure $(\boldsymbol{M_A^r})$

r	2009	2011	2013	2015
1	13517.7	16779.64	20220.23	28585.68
2	287017997	402192956	573528807	1733504424
3	1.33E+13	1.51E+13	2.80E+13	2.09E+14
4	1.24E+18	9.07E+17	2.27E+18	7.72E+19
5	2.57E+23	9.96E+22	3.26E+23	1.41E+26

Several points are noteworthy from the results of Table 4:

1. To begin with, the sensitivity for $(\boldsymbol{M_A^r})$ measure drops sharply when r^{th} increases. In 2009, income inequality is 13517.7 when $r = 1$, but it begins to increase steeply when r = 2 until reaching a huge number when r = 5. The results of 2011, 2013, and 2015 present the same level of sensitivity.
2. In addition, the values of income inequality for $(\boldsymbol{M_A^r})$ measure do not lie between zero and one where it is unlimited from r = 1 to r = 5 for all years. These values could not be explained precisely, because it does not have upper limits.

TABLE 5. Modified Ahmad measure $(\boldsymbol{M_{MA}^r})$

r	2009	2011	2013	2015
1	0.6546032	0.6645052	0.6716104	0.6583761
2	0.342869	0.3918702	0.401599	0.2147581
3	0.0625092	0.1530133	0.1493879	0.006149578
4	0.007230274	0.03985942	0.02995463	0.00034628
5	0.001560909	0.01241007	0.007001629	9.28E-05

Several points are noteworthy from the results of Table 5:

1. To begin with, the sensitivity for $(\boldsymbol{M_{MA}^r})$ measure drops gradually when r^{th} increases. In 2009, income inequality is 0.6546032 when $r = 1$, but it begins to fall steeply when

r = *3* and continues to drop until it reaches 0. 001560909 when r = 5. The results of 2011, 2013, and 2015 show the same level of sensitivity.

2. It is worth noting that the proposed measure satisfies the range property while Ahmad measure does not.
3. It is clear that income inequality slightly fluctuates for the same rank among years. When we compare the values of r = 1 among years, we find that $(\boldsymbol{M^r_{MA}})$ is 0.6546032 in 2009, then slightly rose to 0.6645052 in 2011, and slightly increased to 0.6716104 in 2013, after that it slightly declined to 0.6583761 in 2015.
4. In 2009, income inequality is equal to 0.6546032 when r = 1, this value can be seen as a high-income inequality in the population.

TABLE 6. Generalized Ahmad Measure based on $\boldsymbol{\beta_r}$ $(\boldsymbol{A^r_{PWM1}})$

r	2009	2011	2013	2015
1	2.15E-05	6.56E-05	1.68E-05	2.09E-05
2	5.12E-10	1.35E-09	2.83E-10	1.66E-10
3	2.78E-15	1.62E-14	2.76E-15	4.17E-17
4	3.45E-21	7.59E-20	7.64E-21	5.65E-24
5	3.34E-27	2.17E-25	1.22E-26	7.39E-31

Several points are noteworthy from the results of Table 6:

1. To begin with, the sensitivity for $(\boldsymbol{A^r_{PWM1}})$ measure drops sharply when r^{th} increases. In 2009, income inequality is 2.15E-05 when $r = 1$, but it begins to fall dramatically when r = *2* and continues to drop until it approximately reaches zero when $r = 5$. The results of 2011, 2013, and 2015 show the same level.
2. In addition, the values of income inequality for $(\boldsymbol{A^r_{PWM1}})$ measure lie between zero and one from r = 1 to r = 5 for all years.
3. In 2009, income inequality is equal 2.15E-05, when r = 1, this value can be seen as a very low-income inequality in the population.
4. Finally, when we compare income inequality among years, it is easily seen that income inequality has the same level.

TABLE 7. Generalized Ahmad measure based on $\boldsymbol{\beta_s}$ (A^s_{PWM2})

r	2009	2011	2013	2015
1	3.44E-05	0.00010659	2.78E-05	3.40E-05
2	4.74E-10	1.52E-09	3.45E-10	9.13E-11
3	1.11E-15	9.56E-15	1.54E-15	1.41E-17
4	1.00E-21	2.93E-20	2.69E-21	1.59E-24
5	8.16E-28	6.43E-26	3.40E-27	1.84E-31

Several points are noteworthy from the results of Table 5:

1. To begin with, the sensitivity for $(\boldsymbol{A^r_{PWM2}})$ measure drops sharply when r^{th} increases. In 2009, income inequality is 3.44E-05 when $r = 1$, but it begins to fall dramatically when r = *2* and continues to drop until it approximately reaches zero when $r = 5$. The results of 2011, 2013, and 2015 show the same level.
2. In addition, the values of income inequality for (M^r_{BM}) measure lie between zero and one from r = 1 to r = 5 for all years.
3. In 2009, income inequality is equal 3.44E-05 when r = 1, this value can be seen as a very low-income inequality in the population.
4. Finally, when we compare income inequality among years, it is easily seen that income inequality has the same level.

6. Conclusion

In recent years, the probability weighted method (PWM) played an important role in parameter estimation of the distributions. The power of PWM lies in its simplicity and efficiency of the estimates in comparison with the traditional method of moments and maximum likelihood method. On the other hand, Butler-McDonald (1989), Ahmed (1998) and Abouelmagd and Ahmed (2014) introduced measures of inequality which take into consideration the shape of the distribution based on traditional non-central moments and weighted incomplete moments.

Therefore, the new class of inequality measures take in consideration the shape of the distribution based on weighted conditional moments. Two versions of this measure are defined and studied. Ahmad (1998) proved the mathematical advantages of his measure over the Butler-McDonald measure (1987) and (1989). In this paper we introduce a modification of Ahmad measure and general weighted conditional measures which can be used to derive most of the previous measures like the Lorenz curve, Gini coefficient and Entropy measures. The new class measures lie between zero and one whereas Ahmad measure does not satisfy this axiom. The estimators of these new measures are introduced, and we study these new measures under Pareto distribution. Moreover, the proposed measures characterize the income distribution which is proved in Abouelmagd and Ahmad (2014). A real data application is given that illustrates the benefits of the proposed measures. Finally, we applied the measures to the last four Egypt national surveys about income, and we found that the inequality has gradually increased.

REFERENCES

Abouelmagd, T.H.M., and I.A. Ahmed. "Characterization of Income Distribution." *International Journal of Business and Statistical Analysis* 1, no. 01 (2014): 45-48.

Ahmad, I.A. (1998). "Measuring inequality based on incomplete moments," *Unpublished.*

Alesina, Alberto, Stelios Michalopoulos, and Elias Papaioannou. "Ethnic inequality." *Journal of Political Economy* 124, no. 2 (2016): 428-488.

Atkinson, Anthony B., and Salvatore Morelli. "Economic crises and inequality." *UNDP-HDRO Occasional Papers* 2011/6 (2011).

Bartak, Jakub, and Łukasz Jabłoński. "Inequality and growth: What come from the different inequality measures?" *Bulletin of Economic Research* 72, no. 2 (2020): 185-212.

Bartolucci, Alfred A., Karan P. Singh, Anne D. Bartolucci, and Sejong Bae. "Applying medical survival data to estimate the three-parameter Weibull distribution by the method of probability-weighted moments." *Mathematics and computers in simulation* 48, no. 4-6 (1999): 385-392.

Breunig, Robert, and Omer Majeed. "Inequality, poverty and economic growth." *International Economics* 161 (2020): 83-99.

Butler, Richard J., and James B. McDonald. "Interdistributional income inequality." *Journal of Business & Economic Statistics* 5, no. 1 (1987): 13-18.

Butler, Richard J., and James B. McDonald. "Using incomplete moments to measure inequality." *Journal of Econometrics* 42, no. 1 (1989): 109-119.

Camargo, Julio A. "The Lorenz curve: a suitable framework to define satisfactory indices of landscape composition." *Landscape Ecology* 34, no. 12 (2019): 2735-2742.

Chambers, Dustin. "Does a rising tide raise all ships? The impact of growth on inequality." *Applied Economics Letters* 17, no. 6 (2010): 581-586.

Chambers, Dustin. "Trading places: Does past growth impact inequality?" *Journal of Development Economics* 82, no. 1 (2007): 257-266.

Checchi, Daniele, and Cecilia García-Peñalosa. "Risk and the distribution of human capital." *Economics Letters* 82, no. 1 (2004): 53-61.

Cowell, Frank. *Measuring inequality*. Oxford University Press, 2011.

Dabla-Norris, Ms Era, Ms Kalpana Kochhar, Mrs Nujin Suphaphiphat, Mr Frantisek Ricka, and Evridiki Tsounta. *Causes and consequences of income inequality: A global perspective*. International Monetary Fund, 2015.

Deutsch, Joseph, and Jacques Silber. "Gini's "transvariazione" and the measurement of distance between distributions." *Empirical Economics* 22, (1997):547-554

Dufour, Jean-Marie, Emmanuel Flachaire, and Lynda Khalaf. "Permutation tests for comparing inequality measures." *Journal of Business & Economic Statistics* 37, no. 3 (2019): 457-470.

Fellman, Johan. "Income inequality measures." *Theoretical Economics Letters* 8, (2018): 557–574.

Fontanari, Andrea, Nassim Nicholas Taleb, and Pasquale Cirillo. "Gini estimation under infinite variance." *Physica A: Statistical Mechanics and its Applications* 502 (2018): 256-269.

Gaertner, Wulf, and Ceema Namazie. "Income inequality, risk, and the transfer principle: A questionnaire–experimental investigation." *Mathematical Social Sciences* 45, no. 2 (2003): 229-245.

Gastwirth, Joseph L. "A general definition of the Lorenz curve." *Econometrica* 39, no. 6 (1971): 1037-1039.

Gastwirth, Joseph L. "The estimation of the Lorenz curve and Gini index." *The Review of Economics and Statistics* (1972): 306-316.

Gastwirth, Joseph L., Tapan K. Nayak, and Jane-Ling Wang. "Statistical properties of measures of between-group income differentials." *Journal of Econometrics* 42, no. 1 (1989): 5-19.

Ghosh, Sudeshna. "Impact of economic growth volatility on income inequality: ASEAN experience." *Quality & Quantity* 54, (2020): 807–850.

Greenwood, J. Arthur, J. Maciunas Landwehr, Nicolas C. Matalas, and James R. Wallis. "Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form." *Water Resources Research* 15, no. 5 (1979): 1049-1054.

Habba, Maryam, Mustapha Ameur, and Younes Jabrane. "A novel Gini index based evaluation criterion for image segmentation." *Optik* 168 (2018): 446-457.

Hauser, Oliver P., and Michael I. Norton. "(Mis) perceptions of inequality." *Current Opinion in Psychology* 18 (2017): 21-25.

Hosking, Jonathan Richard Morley, James R. Wallis, and Eric F. Wood. "Estimation of the generalized extreme-value distribution by the method of probability-weighted moments." *Technometrics* 27, no. 3 (1985): 251-261.

Hosking, Jonathan RM. "L-moments: Analysis and estimation of distributions using linear combinations of order statistics." *Journal of the Royal Statistical Society: Series B (Methodological)* 52, no. 1 (1990): 105-124.

Idrees, Muhammad, and Eatzaz Ahmad. "Measurement of Income Inequality: A Survey." *Forman Journal of Economic Studies* 13 (2017): 1–32.

Jancewicz, Barbara. "Income inequalities: axioms of income inequality measures and people's perceptions." *Decyzje* 25 (2016): 21-42.

Kakwani, Nanak C., and Nripesh Podder. "On the estimation of Lorenz curves from grouped observations." *International Economic Review* (1973): 278-292.

Ling-Zheng, Y. U., and W. E. I. Xia-Hai. "Has financial development worsened income inequality in China? Evidence from threshold regression model [J]." *Journal of Finance and Economics* 3, no. 009 (2012).

Moisello, Ugo. "On the use of partial probability weighted moments in the analysis of hydrological extremes." *Hydrological Processes: An International Journal* 21, no. 10 (2007): 1265-1279.

Muhammad, Faqir, and Muhammad Riaz. "Probability weighted moments approach to quality control charts." *Stochastics and Quality Control* 21, no. 2 (2006): 251-260.

Plata-Pérez, L., Joss Sánchez-Pérez, and F. Sánchez-Sánchez. "An elementary characterization of the Gini index." *Mathematical Social Sciences* 74 (2015): 79-83.

Ravallion, Martin. *Competing concepts of inequality in the globalization debate*. The World Bank, 2004.

Sachs, Jeffrey. *The end of poverty: How we can make it happen in our lifetime*. Penguin UK, 2005.

Sen, Amartya, Master Amartya Sen, Sen Amartya, James Eric Foster, and James E. Foster. *On economic inequality*. Oxford University Press, 1997.

Shahateet, Mohammed. "How serious is regional economic inequality in Jordan? Evidence from two national household surveys." *American Journal of Applied sciences* 3, (2006): 1735-1744.

Sotomayor, Orlando. "Growth with reduction in poverty and inequality: did Brazil show the way?." *The Journal of Economic Inequality* 17, no. 4 (2019): 521-541.

Stiglitz, Joseph E "The Great Divide: Unequal Societies and What We Can Do About Them." *Journal of Social Research* 39, no. 1 (2016): 219-230.

Zagier, Don. "Inequalities for the Gini coefficient of composite populations." *Journal of Mathematical Economics* 12, no. 2 (1983): 103-118.

Zenga, Michele. "Inequality curve and inequality index based on the ratios between lower and upper arithmetic means." Stat. Appl 5, (2007): 3–27.