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ABSTRACT

In this thesis we investigate the distortion caused by an analytic map $f : B \to W$, between the Bergman metric ds_B of a bounded domain B in the complex z-plane and the Euclidean metric of the complex w-plane W.

We deduce sharp inequalities in the cases of the unit disc, the punctured hyperbolic plane and the annuli, and the minimizing maps are determined.

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Chapter I

INTRODUCTION

Conformal mappings with least distortion of a simply connected domain into the complex w-plane have been investigated in [1]. A distortion theorem for analytic covering maps of annuli has been proved recently and its result has been applied so far to the natural locally hyperbolic metric of annuli [2] and also to the pseudosphere [3].

In this thesis, this theorem is applied to the Bergman metrics of the annuli, the punctured hyperbolic plane and the unit disc of the complex z-plane.

Chapter II

DEFINITIONS AND NOTATIONS

1. The Bergman Metric.

Let B be a bounded domain of the complex z-plane, then an analytic function $w = f(z)$, defined at every point $z = x + iy$ of B and such that $f'(z) \neq 0$ everywhere, maps the domain B locally conformally onto the range of the values of $f(z)$ in the w-plane.

A function $f(z)$, holomorphic in B, is said to be square-integrable over B if

$$(1) \qquad (f, f) = \iint_B |f(z)|^2 \, d\omega$$

is finite, where $d\omega = dx\,dy$ and the integration is understood in the Lebesgue sense.

Let $f(z)$ and $g(z)$ be any two square-integrable functions over B then the expression

$$(2) \qquad (f, g) = \iint_B f(z)\,\overline{g(z)}\, d\omega$$

is called the inner product of f and g.

The existence of this inner product follows from the Cauchy-Schwarz inequality.

A sequence $\{\phi_n(z)\}$ of complex, single-valued regular functions of a complex variable $z = x + iy$ is said to be an orthonormal system with respect to a domain B of the complex plane if

$$(3) \qquad (\phi_n, \phi_m) = \iint_B \phi_n(z)\,\overline{\phi_m(z)}\, d\omega = \delta_{nm} ,$$

where the Kronecker δ equals 1 if $m = n$ and 0 otherwise.

An orthonormal system $\{\phi_n(z)\}$ is called complete or closed if the relation

$$(4) \qquad \sum_{n=1}^{\infty} |(f, \phi_n)|^2 = (f, f)$$

holds for every square-integrable holomorphic function f defined in B. By such a system $f(z)$ can be approximated in the mean to any degree of accuracy.

The function

$$(5) \qquad K_B(z, \overline{\zeta}) = \sum_{n=1}^{\infty} \phi_n(z) \overline{\phi_n(\zeta)}$$

of the two complex variables z and ζ is called the kernel of the complete orthonormal system $\{\phi_n\}$. The series on the right-hand side of (5) converges absolutely, uniformly with respect to z for fixed ζ, and uniformly with respect to ζ for fixed z in any closed subdomain B' of B. Its sum is, by Weierstrass' theorem, an analytic function of z and $\overline{\zeta}$. [(4), p. 9]

$K_B(z, \overline{\zeta})$ is independent of the special choice of the orthonormal system and is entirely characterized by the domain B[(4), p. 21]. $K_B(z, \overline{\zeta})$ is called the Bergman kernel function of the domain B.

The Bergman metric which is invariant with respect to conformal mappings [(4), p. 32] is given by the first fundamental form

$$(6) \qquad ds_B^2 = K_B(z, \overline{z}) |dz|^2 = K_B(z, \overline{z})(dx^2 + dy^2).$$

Since $K_B(z, \overline{z}) > 0$ this form is positive definite and hence ds_B defines a Riemann metric on B.

The curvature $C(z)$ of the Bergman metric at any point z in B is given by

$$C(z) = -\frac{2}{K} \frac{\partial^2}{\partial z \partial \overline{z}} (\log K), \tag{7}$$

where $K = K_B(z, \overline{z})$, the Bergman kernel function of B. This curvature is always negative. [(4), p. 36].

2. Distortion Maps.

Let $f : B \rightarrow W$ be any smooth covering map of B into the finite complex plane W. The function $w = f(z)$ is holomorphic in B and satisfies $f'(z) \neq 0$ at every point z of B.

We may define an integral distortion for any such map f by the relation

$$D_z[f] = \int \int_B [\log \delta(z)]^2 \, d\omega, \tag{8}$$

with integration in the Lebesgue sense and where

$$\begin{aligned} \delta(z) &= \text{local distortion at } z \\ &= |dw| / ds_B \\ &= \left|\frac{dw}{dz}\right| \frac{|dz|}{ds_B}, \quad \frac{|dz|}{ds_B} > 0 \text{ in } B. \end{aligned} \tag{9}$$

Obviously $D_z[f] = 0$ implies $\delta(z) \equiv 1$ almost everywhere in B, i.e. absence of distortion.

Since $w = f(z)$ we have

$$(10) \qquad \left|\frac{dw}{dz}\right| = |f'(z)|$$

and from (6) we obtain

$$(11) \qquad \frac{|dz|}{ds_B} = \{K_B\ (z, \bar{z})\}^{-1/2}$$

Let us set

$$(12) \qquad \phi(z) = \log\left[\frac{|dz|}{ds_B}\right] = -\frac{1}{2} \log \{K_B\ (z, \bar{z})\}$$

and

$$(13) \qquad \phi(z) = \ln \left|\frac{dw}{dz}\right| = \ln |f'(z)|$$

$$= \operatorname{Re} \log f'(z), \ (\operatorname{Re} = \text{real part of})$$

then

$$(14) \qquad \log \delta\ (z) = \phi\ (z) + \phi\ (z).$$

Any <u>simply connected plane region</u> with at least two boundary points is conformally equivalent to

$$(15) \qquad H = \{\ z\ \big|\ |z| < 1\}$$

$$= \underline{\text{the hyperbolic plane}}. \qquad [(5), \text{p. } 225]$$

Any <u>doubly connected Riemann surface</u> is conformally equivalent to one and only one of the following canonical domains: [(5), p. 247]

$$\text{(16)}\quad \begin{aligned} &\text{(a)} && \dot{E} = \{ z \mid 0 < |z| < \infty\} \\ &&& = \underline{\text{the punctured Euclidean plane}}. \\ &\text{(b)} && \dot{H} = \{ z \mid 0 < |z| < 1\} \\ &&& = \underline{\text{the punctured hyperbolic plane}}. \\ &\text{(c)} && A_q = \{ z \mid \sqrt{q} < |z| < \frac{1}{\sqrt{q}},\ 0 < q < 1\} \\ &&& = \underline{\text{an infinite set of conformally different annuli}}. \end{aligned}$$

These are the only surfaces whose covering group is a free cyclic group with one generator.

Chapter III

LEAST DISTORTION MAPS FOR H AND $\dot{H}$

1. Let B denote any of the surfaces H and $\dot{H}$, then a complete orthonormal system in B is given by

$$\phi_n(z) = (n/\pi)^{\frac{1}{2}}\, z^{n-1}, \quad n = 1, 2, \ldots . \tag{17}$$

The Bergman kernel function of B is

$$K_B(z, \bar{z}) = \sum_{n=1}^{\infty} \phi_n(z)\, \overline{\phi_n(z)} = \sum_{n=1}^{\infty} \frac{n}{\pi} (z\bar{z})^{n-1}$$

$$= \frac{1}{\pi}(1 - z\bar{z})^{-2} = \frac{1}{\pi(1 - |z|^2)^2},$$

and the Bergman metric is

$$ds_B = \frac{|dz|}{\sqrt{\pi}(1 - |z|^2)} \tag{18}$$

The curvature $C(z)$ of this metric is given by

$$\begin{aligned} C(z) &= -\frac{2}{K_B(z, \bar{z})} \frac{\partial^2}{\partial z \partial \bar{z}} [\log K_B(z, \bar{z})] \\ &= -2\pi(1 - z\bar{z})^2 . \frac{-2}{(1 - z\bar{z})^2} = -4\pi. \end{aligned} \tag{19}$$

If B is simply connected then the curvature of the Bergman metric of B is necessarily constant. But if the curvature is constant then B need not be simply connected as is shown by $B = \dot{H}$.

2. Theorem 1.

Any smooth covering map $f : H \rightarrow W$ satisfies

$$D_z[f] \geq \pi, \tag{20}$$

with equality holding then if and only if

$$f(z) = \frac{1}{\sqrt{\pi}} \{\exp (1 + i\lambda)\}z + \mu, \tag{21}$$

where λ and μ are arbitrary constants, real and complex respectively. Hence the minimal covering map is 'schlicht'.

Proof:

Let $z = r e^{i\theta} = x + iy$, so that

$x = r \cos \theta, \quad y = r \sin \theta,$

then

$$\begin{aligned} \phi(z) &= \ln \left(\frac{|dz|}{ds_H}\right) = \ln \{\pi^{1/2} (1 - |z|^2)\} \\ &= \frac{1}{2} \ln \pi + \ln (1 - r^2). \end{aligned} \tag{22}$$

Since $f'(z) \neq 0$, and $\phi(z) = \operatorname{Re} \log f'(z)$, so $\phi(z)$ is a harmonic function in H. Let its Fourier representation be

$$c_o(r) + \sum_{n=1}^{\infty} (c_n(r) \cos n\theta + d_n(r) \sin n\theta).$$

Since $\nabla^2 \phi = 0$, we get

$$c_n(r) = a_n r^n, \quad d_n(r) = b_n r^n,$$

where a_n and b_n are real constants, so that

$$(23) \qquad \Phi(z) = a_o + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n .$$

Thus

$$\phi(z) + \Phi(z) = \frac{1}{2} \ln\pi + \ln(1 - r^2) + a_o + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n .$$

The completeness relation yields now

$$D_z[f] = \int\int_H \{\phi(z) + \phi(z)\}^2 \, dx \, dy$$

$$= \int_0^1 r \, dr \int_0^{2\pi} [\frac{1}{2} \ln\pi + \ln(1 - r^2) + a_o$$

$$+ \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n]^2 \, d\theta$$

$$(24) \qquad = 2\pi \int_0^1 (a_o + \frac{1}{2} \ln\pi)^2 \, r \, dr$$

$$+ 4\pi (a_o + \frac{1}{2} \ln\pi) \int_0^1 \ln(1 - r^2) . r \, dr$$

$$+ 2\pi \int_0^1 \ln^2(1 - r^2) . \, r \, dr$$

$$+ \sum_{n=1}^{\infty} [a_n^2 \int_0^1 r^{2n+1} \, dr \int_0^{2\pi} \cos^2 n\theta \, d\theta$$

$$+ b_n^2 \int_0^1 r^{2n+1} \, dr \int_0^{2\pi} \sin^2 n\theta \, d\theta]$$

$$= 2\pi[\frac{1}{2}(a_o + \frac{1}{2} \ln\pi)^2 - (a_o + \frac{1}{2} \ln\pi) + 1] + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{n + 1}$$

or $$D_z[f] = \pi[(a_0 + \frac{1}{2}\ln\pi - 1)^2 + 1] + \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{n+1}$$

$$\geq \pi$$

The harmonic conjugate of $\Phi(z)$ is

(25) $$\Psi(z) = \sum_{n=1}^{\infty}(a_n \sin n\theta - b_n \cos n\theta)r^n + \lambda,$$

where λ is an arbitrary real constant.

From (23) and (25) we find that $f'(z)$ is of the form

(26) $$\begin{aligned} f'(z) &= \exp(\Phi(z) + i\,\Psi(z)) \\ &= \exp[a_0 + i\lambda + \sum_{n=1}^{\infty}(a_n - ib_n)z^n] \end{aligned}$$

Now $D_z[f] = \pi$ implies that

(27) $$a_0 = 1 - \frac{1}{2}\ln\pi, \quad a_n = b_n = 0, \quad n \geq 1$$

and

(28) $$f'(z) = \frac{1}{\sqrt{\pi}}\exp(1 + i\lambda)$$

Thus the analytic functions with least distortion are given by

$$f(z) = \frac{z}{\sqrt{\pi}}\exp(1 + i\lambda) + \mu,$$

where λ and μ are arbitrary constants, real and complex respectively.

3. Let $B = \dot{H}$ then we have the following theorem:

Theorem 2.

Let $f : \dot{H} \to W$ be any smooth covering map then

$$D_z[f] \geq \pi\left(\frac{9}{4} - \frac{\pi^2}{6}\right), \tag{29}$$

with equality holding only for

$$f(z) = \frac{z^2}{2\sqrt{\pi}} \exp\left(\frac{3}{2} + i\lambda\right) + \mu, \tag{30}$$

where λ is an arbitrary real constant and μ is an arbitrary complex constant.

Proof: In this case also

$$\phi(z) = \frac{1}{2} \ln \pi + \ln(1 - r^2)$$

with $z = re^{i\theta}$.

Since $f'(z) \neq 0$ and $f(z)$ is holomorphic in $\dot{H}$, $\phi(z) = \operatorname{Re} \log f'(z)$ is harmonic in $\dot{H}$, let

$$a_o(r) + \sum_{n=1}^{\infty} (a_n(r) \cos n\theta + b_n(r) \sin n\theta)$$

be the Fourier expansion of $\phi(z)$.

Integrating $\nabla^2\phi = 0$, we find that

$a_o(r) = E_o \ln r + F_o$, $a_n(r) = E_n r^n$, $b_n(r) = F_n r^n$, $n \geq 1$,

where E_n, F_n are real constants depending upon $f(z)$.

We can now write

$$\phi(z) + \phi(z) = F_o + \frac{1}{2} \ln \pi + E_o \ln r + \ln(1 - r^2)$$
$$+ \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \tag{31}$$

By the Lebesgue-Fubini theorem and the completeness relation we get

$$D_z[f] = \int_0^1 r\,dr \int_0^{2\pi} (\phi + \phi)^2\, d\theta$$

$$= 2\pi[(F_o + \frac{1}{2}\ln\pi)^2 \int_0^1 r\,dr + 2\,E_o(F_o + \frac{1}{2}\ln\pi) \int_0^1 \ln r.r dr$$

$$+ 2(F_o + \frac{1}{2}\ln\pi) \int_0^1 \ln(1 - r^2).r\,dr$$

$$(32) \qquad + E_o^2 \int_0^1 \ln^2 r.r\,dr + 2\,E_o \int_0^1 \ln r.\ln(1 - r^2).r\,dr$$

$$+ \int_0^1 \ln^2(1 - r^2).r\,dr] + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{E_n^2 + F_n^2}{n + 1}$$

$$= 2\pi[\frac{1}{2}(F_o + \frac{1}{2}\ln\pi - \frac{1}{2}E_o - 1)^2 + \frac{1}{8}(E_o - \frac{\pi^2}{3} + 2)^2$$

$$+ \frac{\pi^2}{6} - \frac{\pi^4}{72}] + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{E_n^2 + F_n^2}{n + 1}$$

The harmonic conjugate of $\phi(z)$ is

$$(33) \qquad \psi(z) = E_o\theta + \lambda + \sum_{n=1}^{\infty} (E_n \sin n\theta - F_n \cos n\theta)r^n ,$$

where λ is an arbitrary real constant.

Now we have

$$f'(z) = \exp(\phi(z) + i\psi(z))$$

$$(34) \qquad = z^{E_o} \exp\{F_o + i\lambda + \sum_{n=1}^{\infty} (E_n - iF_n)z^n\}$$

Since $f(z)$ is single valued analytic function, E_o must be an integer.

For the minimizing map E_o must be a nearest integer to

$\frac{\pi^2}{3} - 2 = 1.2898$, i.e. $E_o = 1$.

Thus, by taking

$$E_o = 1, \quad F_o = \frac{3}{2} - \frac{1}{2} \ln\pi, \tag{35}$$

we have

$$D_z[f] \geq 2\pi[\frac{1}{8} (\frac{\pi^2}{3} - 3)^2 + \frac{\pi^2}{6} - \frac{\pi^4}{72}] = \pi(\frac{9}{4} - \frac{\pi^2}{6})$$

Equality holds only if $E_n = F_n = 0$ $(n = 1, 2, \ldots)$, i.e., only if

$$\begin{aligned} f'(z) &= z \exp(\frac{3}{2} - \frac{1}{2} \ln\pi + i\lambda) \\ &= \frac{z}{\sqrt{\pi}} \exp(\frac{3}{2} + i\lambda) \end{aligned} \tag{36}$$

Integrating this we obtain

$$f(z) = \frac{z^2}{2\sqrt{\pi}} \exp(\frac{3}{2} + i\lambda) + \mu,$$

where μ is an arbitrary complex constant.

4. The following points are worth noting:

Although the Bergman metrics for H and $\dot{H}$ are the same

yet (a) $\text{Min} \iint_H \ln^2 \delta(z)\, dx\, dy \neq \text{Min} \iint_{\dot{H}} \ln^2 \delta(z)\, dx\, dy$,

(b) the minimal covering maps in the first case are one-to-one while in the second case they are not one-to-one.

Chapter IV

MINIMAL MAPS OF ANNULI

1. Let $B = A_q$,

$$A_q = \{ z \mid \sqrt{q} < |z| < \frac{1}{\sqrt{q}}, \quad 0 < q < 1\}. \tag{37}$$

The Bergman kernel function of A_q is

$$K_{A_q}(z, \overline{\zeta}) = \frac{1}{\pi z\overline{\zeta}} \{\frac{1}{2\ln\frac{1}{q}} + \sum_{-\infty}^{\infty}{}' \frac{nq^n}{1 - q^{2n}} (z\,\overline{\zeta})^n\} , \tag{38}$$

so that

$$\begin{aligned} K_{A_q}(z, \overline{z}) &= \frac{1}{\pi|z|^2} \{\frac{1}{2\ln\frac{1}{q}} + \sum_{-\infty}^{\infty}{}' \frac{nq^n}{1 - q^{2n}} |z|^{2n}\} \\ &= \frac{1}{\pi|z|^2} [\frac{\eta_1}{\ln\frac{1}{q}} + p\{\ln (q|z|^2)\}], \end{aligned} \tag{39}$$

where the Weierstrassian p-function $p(u)$ has periods $2 \ln \frac{1}{q}$ and $2\pi i$, and $2\eta_1$ is the increment of the Weierstrassian ζ-function related to the period $2 \ln (\frac{1}{q})$, i.e. $\zeta(\ln \frac{1}{q}) = \eta_1$.

The Bergman metric of A_q is

$$\begin{aligned} ds_{A_q} &= \frac{1}{\pi^{1/2} |z|} [\frac{\eta_1}{\ln\frac{1}{q}} + p\{\ln(q|z|^2)\}]^{1/2} |dz| \\ &= \frac{1}{\pi^{1/2} r} [\frac{\eta_1}{\ln\frac{1}{q}} + p\{\ln(qr^2)\}]^{1/2} |dz|, \quad \text{with} \quad z = r e^{i\theta}. \end{aligned} \tag{40}$$

The Gaussian curvature of this metric at any point z is given by the formula

$$(41) \qquad c(z) = -\frac{2}{K}\,\frac{\partial^2 (\log K)}{\partial z \partial \bar{z}} ,$$

where

$$K = K_{A_q}(z, \bar{z}) = \frac{1}{\pi z \bar{z}} \{p(\ell n q|z|^2) + \frac{\eta_1}{\ell n(\frac{1}{q})}\}.$$

Since $\log K = -\log \pi - \log z - \log \bar{z} + \log \{p(\log(qz\bar{z})) + \frac{\eta_1}{\ell n(\frac{1}{q})}\}$,

we obtain

$$c(z) = -2\pi \frac{\{p(\ell n q r^2) + \frac{\eta_1}{\ell n(\frac{1}{q})}\}\, p''(\ell n q r^2) - p'^2(\ell n q r^2)}{\{p(\ell n q r^2) + \frac{\eta_1}{\ell n(\frac{1}{q})}\}^3}$$

with $z = re^{i\theta}$.

Setting $\quad \ell n \frac{1}{q} = s, \quad \ell n q r^2 = u, \quad \frac{\eta_1}{\ell n(\frac{1}{q})} = \lambda$

and since

$$p'^2(u) = 4\, p^3(u) - g_2\, p(u) - g_3$$

$$p''(u) = 6\, p^2(u) - \frac{1}{2} g_2 ,$$

we have $\quad c(z) = -2\pi \dfrac{2p^3(u) + \frac{1}{2} g_2\, p(u) + g_3 + \frac{1}{2}\lambda(12\, p^2(u) - g_2)}{[p(u) + \lambda]^3}$

$$= -2\pi \frac{2p^2(u)\{p(u) + 3\lambda\} + \frac{1}{2} g_2 \{p(u) - \lambda\} + g_3}{\{p(u) + \lambda\}^3},$$

where $\qquad 0 < u < 2s.$

Since

(i) $p(u)$ varies from $+\infty$ to $p(s)$ as u varies from 0 to s, and $p(u)$ varies from $p(s)$ to $+\infty$ as u varies from s to $2s$, that is $p(u) \geq p(s) > 0$ for all values of u in $(0, 2s)$.

(ii) $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$

$g_3 = 4\, e_1\, e_2\, e_3$, e_1, e_2, e_3 are real and

$e_1 > 0 \geq e_2 > e_3$ so that $g_3 \geq 0$, $g_2 > 0$ [(6), p. 633]

(iii) $p(s) + \lambda = p(s) + \frac{\zeta(s)}{s}$

$$= \frac{2}{s^2} + \sum_{k=2}^{\infty} \frac{2k-2}{2k-1}\, c_k\, s^{2k-2},$$

where $c_2 = \frac{1}{20} g_2$, $c_3 = \frac{1}{28} g_3$, $c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m\, c_{k-m}$, $k \geq 4$

Also $$p(s) - \lambda = \sum_{k=2}^{\infty} \frac{2k}{2k-1}\, c_k\, s^{2k-2} > 0$$

Thus we have

$$p(u) + \lambda \geq p(s) + \lambda > 0,$$

and $$2\, p^2(u)\, \{p(u) + 3\lambda\} + \frac{1}{2} g_2\, \{p(u) - \lambda\} + g_3 > 0$$

which shows that $c(z)$ is always negative.

2. We apply now the general distortion theorem of [2] to A_q provided with the Bergman metric and supply strong numerical evidence for the validity of the following:

<u>"Theorem" 3.</u>

For an annulus with the Bergman metric (40), any analytic map $f : A_q \to W$ satisfies the following inequality which is best possible:

$$D_z[f] \geq 2\pi\{\frac{\delta_o(s)}{\alpha_o(s)} |A_o(s) - \chi(s)|^2 + c_o(s)\} \tag{42}$$

with the equality holding then only if

$$f(z) = \exp(B_o + i\chi_o) \frac{z^{A_o+1}}{A_o + 1} + \chi_1 , \tag{43}$$

where χ_o and χ_1 are arbitrary constants, real and complex respectively.

The symbols occurring in the theorem have the following meaning:

$$s = \ln (\frac{1}{q}), \quad \alpha_o = \sinh s,$$

$$\beta = \frac{1}{2} (s \cosh s - \sinh s),$$

$$\gamma = \frac{1}{4} s^2 \sinh s - \frac{1}{2} s \cosh s + \frac{1}{2} \sinh s$$

$$\delta_o(s) = \frac{1}{4} (\sinh^2 s - s^2)$$

(44)

$$e_1 = \int_0^s \ln \{p(u) + \frac{\zeta(s)}{s}\} \cosh (s - u)\, du$$

$$e_2 = \int_0^s \ln \{p(u) + \frac{\zeta(s)}{s}\} (s - u) \sinh (s - u)\, du$$

$$e_3 = \int_0^s \ln^2 \{p(u) + \frac{\zeta(s)}{s}\} \cosh (s - u)\, du$$

(45)

$$B_o(s) = \frac{1}{2 \sinh s} [e_1 - (A_o + 1)(s \cosh s - \sinh s) - \sinh s \ln \pi]$$

$$c_o(s) = \frac{1}{4} e_3 - \frac{1}{4\delta_o} [\gamma e_1^2 + \frac{1}{4} \alpha_o e_2^2 - \beta e_1 e_2]$$

$$(46)\quad \chi(s) = -1 + \frac{1}{4\delta_o}(\alpha_o e_2 - 2\beta e_1) = -1 + \int_0^s \ln\left[p(u) + \frac{\zeta(s)}{s}\right] \times$$

$$\left[\frac{-u\cosh(2s-u) - (2s-u)\cosh u + \sinh(2s-u) + \sinh u}{2(\sinh^2 s - s^2)}\right] du$$

$A_o(s)$ = nearest integer to $\chi(s)$.

Proof:

Let $\quad \phi(z) = \ln(|dz|/ds_{A_q})$

$$(47)\qquad = \frac{1}{2}\ln\pi + \ln r - \frac{1}{2}\ln\left[p(\ln q r^2) + \frac{\zeta(s)}{s}\right]$$

be square integrable on A_q and let

$$(48)\quad \phi(z) = \ln|f'(z)| = \text{Re} \log f'(z),\quad \text{Re} = \text{real part of.}$$

Since $f(z)$ is holomorphic in A_q and $f'(z) \neq 0$, $\phi(z)$ is harmonic in A_q, let

$$(49)\qquad \sum_{-\infty}^{\infty} a_m(r)\, e^{im\theta}$$

be the Fourier representation of $\phi(z)$.

Substituting (49) in the relation

$$0 = \nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

we see that $a_m(r)$ satisfies the differential equations:

$$r^2 a_m''(r) + r\, a_m'(r) - m^2 a_m(r) = 0 \quad \text{for } m \neq 0$$

and

$$r\, a_o''(r) + a_o'(r) = 0,$$

where primes denote differentiation with respect to r. Integrating these differential equations, we obtain

$$a_m(r) = E_m r^m + F_m r^{-m}, \qquad m \neq 0$$

(50)

$$a_o(r) = E_o \ln r + F_o,$$

where E_m and F_m are complex constants depending upon $f(z)$. Since $\phi(z)$ is a real valued function, we have

$$a_{-m}(r) = \overline{a_m(r)}$$

$$E_{-m} = \overline{F}_m,\ \overline{E}_m = F_{-m},\ E_o = \overline{E}_o,\ F_o = \overline{F}_o\ .$$

Thus $\phi(z)$ can be written as

(51) $$\phi(z) = E_o \ln r + F_o + \sum_{-\infty}^{\infty}{}' (E_m r^m + F_m r^{-m}) e^{im\theta}, \quad \sum{}' \text{ omits } m = 0.$$

The harmonic conjugate of $\phi(z)$ is

(52) $$\psi(z) = E_o \theta + \chi_o - i \sum_{-\infty}^{\infty}{}' (E_m r^m - F_m r^{-m}) e^{im\theta},$$

where χ_o is a real constant.

Hence

$$f'(z) = \exp\,(\phi(z) + i\psi(z))$$

(53)

$$= \exp\,(F_o + i\chi_o)\ z^{E_o} \exp\,(2 \sum_{-\infty}^{\infty}{}' E_m z^m).$$

Since $f(z)$ is a single valued analytic function, we conclude that E_o must be an integer.

Now

$$\phi(z) + \phi(z) = (F_o + \frac{1}{2}\ln\pi) + (A_o + 1)\ln r - \frac{1}{2}\ln[p(\ln qr^2) + \frac{\zeta(s)}{s}]$$

(54)

$$+ \sum_{-\infty}^{\infty}{}' (E_m r^m + F_m r^{-m})e^{im\theta},$$

so by the Lebesgue-Fubini theorem and the completeness relation, we obtain

$$D_z[f] = \int_{\sqrt{q}}^{1/\sqrt{q}} r\, dr \int_0^{\pi/2} (\phi + \phi)^2 \, d\theta$$

$$= 2\pi[(F_o + \frac{1}{2}\ln\pi)^2\alpha_o + 2(F_o + \frac{1}{2}\ln\pi)(A_o + 1)\beta$$

(55)

$$+ (A_o + 1)^2\gamma - (F_o + \frac{1}{2}\ln\pi)e_1 - \frac{1}{2}(A_o + 1)e_2 + \frac{1}{4}e_3$$

$$+ \sum_{-\infty}^{\infty}{}' \{|E_m|^2\alpha_m + (E_m \bar{F}_m + \bar{E}_m F_m)\alpha_o + |F_m|^2 \alpha_{-m}\}],$$

where $\alpha_m = \frac{1}{m+1} \sinh (m+1)s, \quad m \neq -1.$

and $\alpha_{-1} = s.$

Let $\delta_m = \alpha_m \alpha_{-m} - \alpha_o^2, \quad m \neq 0.$

Clearly $\alpha_m > 0$ for all m and $\delta_m > 0$, by the Schwarz inequality.

Hence

$$D_z[f] = 2\pi[\frac{\delta_o}{\alpha_o}|A_o - \chi|^2 + \frac{1}{\alpha_o}|\alpha_o(F_o + \frac{1}{2}\ln\pi) + \beta(A_o + 1) - \frac{1}{2}e_1|^2$$

(56)

$$+ \frac{1}{4}e_3 - \frac{1}{4\delta_o}\{\gamma e_1^2 - \beta e_1 e_2 + \frac{1}{4}\alpha_o e_2^2\}$$

$$+ \sum_{-\infty}^{\infty}{}' \{\frac{\delta_m}{\alpha_m}|F_m|^2 + \frac{1}{\alpha_m}|\alpha_o F_m + \alpha_m E_m|^2\}]$$

so that we have

$$D_z[f] \geq 2\pi\{\frac{\delta_0}{\alpha_0} |A_0 - \chi|^2 + C_0\}.$$

Equality holds only if

$$\text{each } E_m = 0, \text{ for } m \neq 0 \text{ and } F_0 = B_0,$$

that is, only if

$$f'(z) = \exp(B_0 + i\chi_0) z^{A_0}. \qquad (57)$$

Integrating (57), we obtain

$$f(z) = \exp(B_0 + i\chi_0) \frac{z^{A_0+1}}{A_0 + 1} + \chi_1 , \text{ provided } A_0 \neq -1.$$

where χ_1 is an arbitrary complex constant.

Lemma 1.

When $q \to 0$, then $\chi = \frac{\pi^2}{3} - 2 = 1.289868$ so that $A_0 = 1$.

Proof:

As $q \to 0$, $s \to \infty$ so the real period $2 \ln \frac{1}{q}$ of the Weierstrassian p-function becomes infinite, and

$$p(u) + \frac{\zeta(s)}{s} \text{ tends to } \frac{1}{4} \operatorname{cosech}^2 \frac{u}{2}. \qquad [(6), \text{ p. } 651]$$

The corresponding value of the 'characteristic function' χ is given by

$$\chi = -1 + \int_0^\infty \ln [\frac{1}{4} \operatorname{cosech}^2 \frac{u}{2}](1 - u)e^{-u}\, du$$

$$= -1 - 2 \ln 2 \int_0^\infty (1 - u)e^{-u} du - 2 \int_0^\infty \ln(\sinh \frac{u}{2})(1 - u)e^{-u} du$$

$$= -1 + \int_0^\infty \coth \frac{u}{2}.ue^{-u}\, du$$

or $$\chi = -1 + \int_0^\infty [u\, e^{-u} + 2u \frac{e^{-u}}{1 - e^{-u}}]\, du$$

$$= -2 - 2 \int_0^1 \frac{\ell n(1 - t)}{t}\, dt$$

$$= -2 + 2 (\frac{\pi^2}{6}) = \frac{\pi^2}{3} - 2$$

$$= 1.289868$$

Hence $$A_o = 1.$$

Thus wide annuli (q-small) are mapped on double-sheeted rings, just as in the case of the locally hyperbolic metric.

Lemma 2.

When $q \to 1$ then $\chi = .49614$ and $A_o = 0$.

Proof:

From (46) we have

$$\chi(s) = -1 + \frac{1}{4\delta_o} (\alpha_o\, e_2 - 2\, \beta\, e_1)$$

$$= -1 + \frac{1}{2(\sinh^2 s - s^2)} \int_0^s \ell n\{p(u) + \frac{\zeta(s)}{s}\}[\sinh(2s - u)$$

$$- u \cosh(2s - u) + \sinh u - (2s - u)\cosh u]\, du.$$

Let us define

$$P^+ = \int_0^s \ell n[u^2\, p(u) + u^2 \frac{\zeta(s)}{s}]\, e^u\, du,$$

(58) $$P^- = \int_0^s \ell n[u^2\, p(u) + u^2 \frac{\zeta(s)}{s}] e^{-u}\, du,$$

$$Q^+ = \int_0^s \ell n[u^2\, p(u) + u^2 \frac{\zeta(s)}{s}]\, u\, e^u\, du,$$

$$Q^- = \int_0^s \ell n[u^2\, p(u) + u^2 \frac{\zeta(s)}{s}]\, u\, e^{-u}\, du,$$

then

$$(59)\quad \chi(s) = -1 + \frac{1}{4(\sinh^2 s - s^2)}\Big[(1 - e^{-2s})Q^+ + (1 - e^{2s})Q^-$$

$$+ (1 - 2s - e^{-2s})P^+ + (e^{2s} - 2s - 1)P^- + 8\sinh^2 s$$

$$- 8\sum_{n=1}^{\infty} \frac{s^{2n}}{(2n-1)(2n-1)!}\Big]$$

$$= -1 + \frac{1}{4(\sinh^2 s - s^2)}\Big[(1 - e^{-2s})Q^+ + (1 - e^{2s})Q^-$$

$$+ (1 - 2s - e^{-2s})P^+ + (e^{2s} - 2s - 1)P^-$$

$$- 8\sum_{n=1}^{\infty} \frac{s^{2n+2}}{(2n+1)(2n+1)!} + 8(\sinh^2 s - s^2)\Big]$$

$$= 1 + \frac{1}{4(\sinh^2 s - s^2)}\Big[(1 - e^{-2s})Q^+ + (1 - e^{2s})Q^-$$

$$+ (1 - 2s - e^{-2s})P^+ + (e^{2s} - 2s - 1)P^-$$

$$- 8\sum_{n=1}^{\infty} \frac{s^{2n+2}}{(2n+1)(2n+1)!}\Big],$$

When s is small the integrands in the above integrals can be represented by absolutely and uniformly convergent power series. Integrating these series term by term and making s tend to zero, we obtain

$$\lim_{s\to 0} \chi(s) = 1 - \frac{17}{30} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n + 7}{(n+2)(2n+3)(2n+5)}$$

$$= \frac{13}{30} + 2\sum_{m=1}^{\infty} \frac{4m + 5}{(4m+1)(4m+2)(4m+3)} - 2\sum_{m=1}^{\infty} \frac{4m + 7}{(4m+3)(4m+4)(4m+5)}$$

$$= .49614 .$$

Lemma 3.

When $0.165 \leq q \leq 0.999$ that is, when $1.800 \geq s \geq 0.001$ then we can strongly conjecture that

$$A_0(s) = \left| \begin{array}{ll} 1 & s > \sigma \\ 0 & \sigma > s \\ 0 \text{ or } 1 & s = \sigma \end{array} \right.$$

The number σ is the solution of $A_0(s) = \frac{1}{2}$ and satisfies

$0.450 < \sigma < 0.500.$

Proof:

From (59) we have

$$\chi(s) = -1 + \frac{1}{4(\sinh^2 s - s^2)} [(1 - e^{-2s})Q^+ + (1 - e^{2s})Q^-$$

$$+ (1 - 2s - e^{-2s})P^+ + (e^{2s} - 2s - 1)P^- + 8 \sinh^2 s$$

$$- 8 \sum_{n=1}^{\infty} \frac{s^{2n}}{(2n-1)(2n-1)!}].$$

Now if $$0 \leq u \leq 1.86$$

$$u^2 p(u) = \sum_0^6 a_n u^{4n} + \varepsilon(u),$$

where $|\varepsilon(u)| < 2 \times 10^{-7}$ and

$a_0 = 9.9999998 \ (-1)$

$a_1 = 4.9999962 \ (-2)$

$a_2 = 8.3335277 \ (-4)$

$a_3 = 6.4041266 \ (-6)$

$a_4 = 4.8143820 \ (-8)$

$a_5 = 2.2972921 \ (-10)$

$a_6 = 4.9451145 \ (-12)$ [(6), p. 662].

Also for $0 < s \leq 1.86$

$$s\,\zeta(s) = \sum_{0}^{6} b_n\, s^{4n} + \varepsilon(s),$$

where $$|\varepsilon(s)| < 3 \times 10^{-6}$$

and

$b_o = 9.9999999\ (-1)$

$b_1 = -1.6666674\ (-2)$

$b_2 = -1.1903670\ (-4)$

$b_3 = -5.8645163\ (-7)$

$b_4 = -2.5749262\ (-9)$

$b_5 = -5.6700800\ (-11)$

$b_6 = 9.7001580\ (-13)$ [(6), p. 662].

The above data were programmed for an IBM 1620 electronic computer and the following results were obtained:

$q = 0.999,\ \ s = .001,\ \ \chi = 0.4913464$

$q = 0.861,\ \ s = .150,\ \ \chi = 0.4921556$

$q = 0.705,\ \ s = .350,\ \ \chi = 0.4956587$

$q = 0.670,\ \ s = .400,\ \ \chi = 0.4969685$

$q = 0.638,\ \ s = .450,\ \ \chi = 0.4984512$

(60) $$A_o = 0.$$

$q = 0.607,\ \ s = .500\ \ \ \chi = 0.5000691$

$q = 0.577,\ \ s = .550\ \ \ \chi = 0.5018367$

$q = 0.549,\ \ s = .600\ \ \ \chi = 0.5037480$

$q = 0.522,\ \ s = .650\ \ \ \chi = 0.5057962$

$q = 0.497,\ \ s = .700\ \ \ \chi = 0.5079743$

$q = 0.472, \quad s = .750 \quad \chi = 0.5102745$

$q = 0.449, \quad s = .800 \quad \chi = 0.5126882$

$q = 0.427, \quad s = .850 \quad \chi = 0.5152072$

$q = 0.407, \quad s = .900 \quad \chi = 0.5178096$

$q = 0.368, \quad s = 1.000 \quad \chi = 0.5232992$

$q = 0.333, \quad s = 1.100 \quad \chi = 0.5290353$

$q = 0.301, \quad s = 1.200 \quad \chi = 0.5349377$

$q = 0.273, \quad s = 1.300 \quad \chi = 0.5409058$

$q = 0.247, \quad s = 1.400 \quad \chi = 0.5468310$

$q = 0.223, \quad s = 1.500 \quad \chi = 0.5525969$

$q = 0.202, \quad s = 1.600 \quad \chi = 0.5580802$

$q = 0.183, \quad s = 1.700 \quad \chi = 0.5631511$

$q = 0.165, \quad s = 1.800 \quad \chi = 0.5676709$

(61) $$A_o = 1.$$

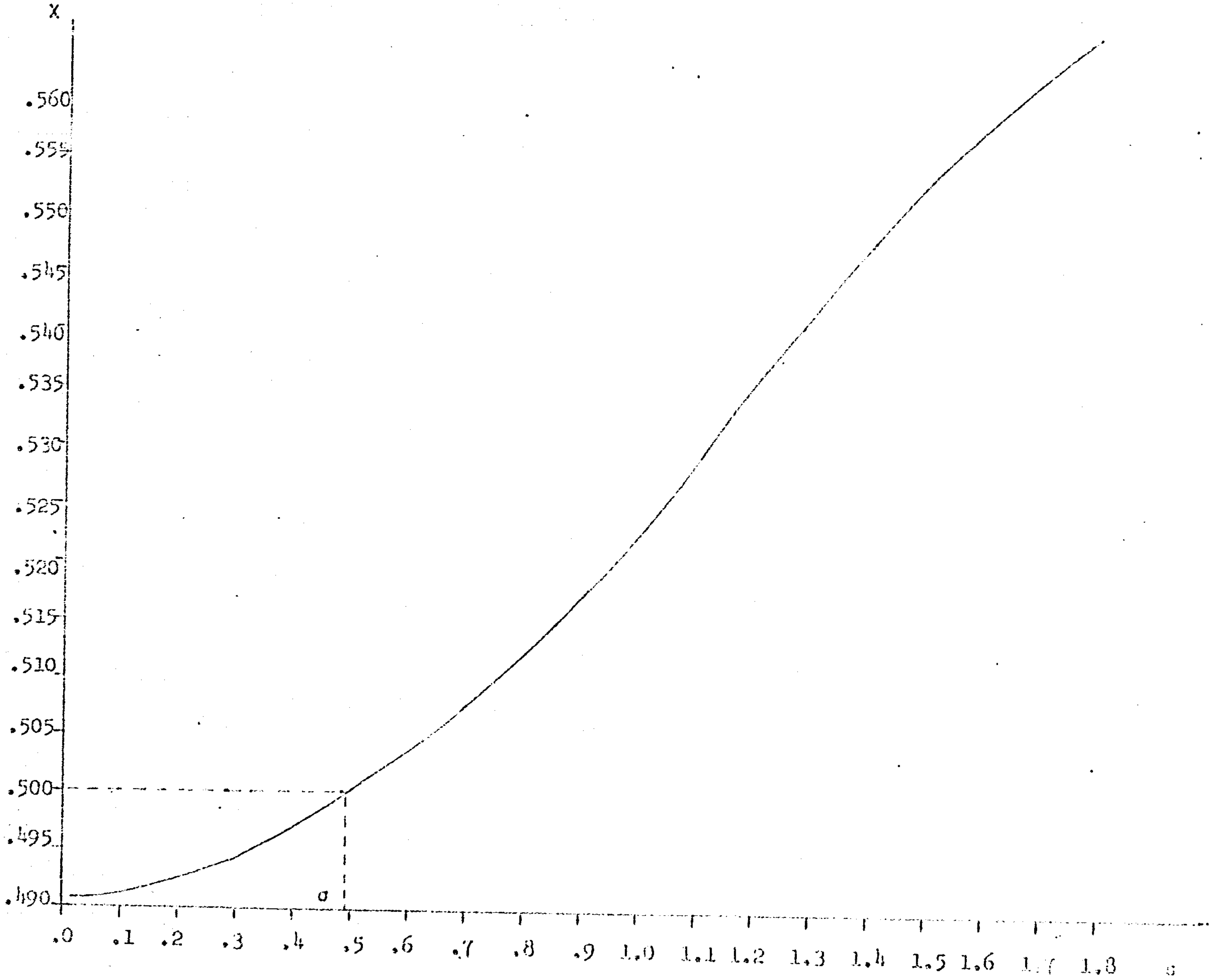
From these results we can strongly conjecture that

$$A_o = \left| \begin{array}{ll} 0 & s > \sigma \\ 1 & s < \sigma \\ 0 \text{ or } 1 & s = \sigma, \end{array} \right.$$

where σ lies between (.45, .50). See the graph on the next page:

The graph shows that $\chi(s)$ increases as s increases, that is, when q is large $\chi(s)$ is small and $A_o = 0$, while for small values of q, $\chi(s)$ is large and $A_o = 1$. Hence the minimizing maps for narrow annuli are one-to-one and for wide annuli two-to-one. We have, however, not succeeded in excluding the possibility of oscillation of the function $\chi(s)$ in between the computed values.

GRAPH OF THE VARIATION OF THE 'CHARACTERISTIC FUNCTION' χ IN RELATION TO THE VARIATION IN s



3. Particular Case.

Let $A_{(\rho,1)}$ be the ring $\{ z \mid 0 < \rho < |z| < 1 \}$, and the Bergman kernel function of this ring is

$$K_{A_{(\rho,1)}}(z,\overline{z}) = \frac{1}{\pi|z|^2} \{ p(\ln|z|^2) + \frac{\eta_1}{\ln\frac{1}{\rho}} \}, \tag{62}$$

where $2\ln\frac{1}{\rho}$ and $2\pi i$ are the periods of $p(u)$ and $\eta_1 = \zeta(\ln\frac{1}{\rho})$.

Theorem 4.

If $\rho \to 0$ so that $A_{(\rho,1)} \to \dot{H}$ then

(a) $\lim\limits_{\rho\to 0} K_{A_{(\rho,1)}}(z,\overline{z}) = K_{\dot{H}}(z,\overline{z})$

(b) $\lim\limits_{\rho\to 0} \operatorname{Min} \iint_{A(\rho,1)} \ln^2\delta \, dx\, dy = \operatorname{Min} \iint_{\dot{H}} \ln^2\delta \, dx\, dy$

$$= \pi(\frac{9}{4} - \frac{\pi^2}{6})$$

Proof:

(a) If ρ tends to zero then $\ln\frac{1}{\rho}$ tends to infinity, that is, the real period of the Weierstrassian p-function becomes infinite and we have the following results [(6), p. 652]

$$\lim_{\rho\to 0} p(u) = \frac{1}{12} + \frac{1}{4} \operatorname{cosech}^2 \frac{u}{2}$$

$$\lim_{\rho\to 0} \frac{\eta_1}{\ln\frac{1}{\rho}} = -\frac{1}{12}, \text{ so that} \tag{63}$$

$$\lim_{\rho\to 0} K_{A_{(\rho,1)}}(z, \overline{z}) = \frac{1}{\pi|z|^2} \{\frac{1}{4} \operatorname{cosech}^2 (\frac{1}{2} \ln|z|^2)\}$$

$$= \frac{1}{\pi|z|^2} \frac{1}{(|z| - \frac{1}{|z|})^2} = \frac{1}{\pi(1 - |z|^2)^2} = K_{\dot{H}}(z,\overline{z})$$

(b) For $\lim\limits_{\rho\to 0} A_{(\rho,1)}$, we have the following values of the symbols:

$$\alpha_o = \lim_{\rho\to 0} \int_\rho^1 r\, dr = \lim_{\rho\to 0} \frac{1-\rho^2}{2} = \frac{1}{2}$$

$$\beta = \lim_{\rho\to 0} \int_\rho^1 \ln r . r\, dr = -\frac{1}{4}$$

$$\gamma = \lim_{\rho\to 0} \int_\rho^1 \ln^2 r . r\, dr = \frac{1}{4}$$

$$\delta_o = \alpha_o\gamma - \beta^2 = \frac{1}{16}$$

$$e_1 = \lim_{\rho\to 0} \int_\rho^1 \ln[p(\ln r^2) + \frac{\eta_1}{\ln\frac{1}{\rho}}]\, r\, dr$$

(64) $$= 2\int_0^1 \{\ln r - \ln(1-r^2)\} r\, dr = \frac{1}{2}$$

$$e_2 = \lim_{\rho\to 0} \int_\rho^1 \ln[p(\ln r^2) + \frac{\eta_1}{\ln\frac{1}{\rho}}] \ln r . r\, dr$$

$$= 2\int_0^1 \{\ln r - \ln(1-r^2)\} \ln r . r\, dr = -\frac{1}{2} + \frac{\pi^2}{12}$$

$$e_3 = \lim_{\rho\to 0} \int_\rho^1 \ln^2\{p(\ln r^2) + \frac{\eta_1}{\ln\frac{1}{\rho}}\} . r\, dr$$

$$= 4\int_0^1 \{\ln r - \ln(1-r^2)\}^2 . r\, dr = 1 + \frac{\pi^2}{3}$$

$$\text{Min } D_z[f] = \lim_{\rho\to 0} \text{Min} \iint_{A_{(\rho,1)}} \ln^2 [\delta(z)]\, dx\, dy$$

(65) $$= 2\pi[\text{Min}\{\frac{\delta_o}{\alpha_o}|A_o - X|^2 + \frac{1}{\alpha_o}|\alpha_o(B_o + \frac{1}{2}\ln\pi) + \beta(A_o + 1) - \frac{1}{2}e_1|^2$$

$$+ \frac{1}{4}e_3 - \frac{1}{4\delta_o}(\gamma e_1^2 + \alpha_o e_2^2 - 2\beta e_1 e_2)\}]$$

Since

$$x = -1 + \frac{1}{2\delta_o}(\alpha_o e_2 - \beta e_1) = -1 + 8\left(-\frac{1}{4} + \frac{\pi^2}{24} + \frac{1}{8}\right) = \frac{\pi^2}{3} - 2 = 1.2898$$

$$A_o = 1$$

and

$$(65) \qquad \lim_{\rho \to 0} \operatorname{Min} \iint_{A(\rho,1)} \ln^2\delta \, dx \, dy$$

$$= 2\pi\left[\frac{1}{8}\left(3 - \frac{\pi^2}{3}\right)^2 + \frac{1}{4}\left(1 + \frac{\pi^2}{3}\right) - 4\left(\frac{1}{16} + \frac{1}{8} - \frac{\pi^2}{24} + \frac{\pi^4}{288} - \frac{1}{8} + \frac{\pi^2}{48}\right)\right]$$

$$= \pi\left(\frac{9}{4} - \frac{\pi^2}{6}\right)$$

$$= \operatorname{Min} \iint_H \ln^2\delta \, dx \, dy.$$

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