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A COMPREHENSIVE FREE VIBRATION ANALYSIS
OF SINGLE AND MULTI-SPAN BEAMS AND
SHAFTS WITH CLASSICAL AND NON-CLASSICAL
BOUNDARY CONDITIONS

by

RAM KUMAR SHARMA

A thesis submitted to the School of Graduate Studies
in partial fulfillment of the requirements for the
degree of Ph.D. in Civil Engineering

UNIVERSITY OF OTTAWA

OTTAWA, CANADA, 1974

ABSTRACT

In this thesis, a comprehensive free vibration analysis of single and multi-span beams and shafts with classical and non-classical boundary conditions is presented. Rigorous theoretical expressions for frequency equations and modal shapes have been derived for all the cases considered here.

The transverse vibrations of single and double span uniform beams and torsional vibrations of circular shafts have been investigated for various non-classical boundary conditions. The non-classical boundary conditions considered in case of beams comprise of a linear coil spring, a torsion spring, a concentrated mass, a rotational mass, or a possible combination of these. In case of torsional vibrations of shaft, these non-classical boundary conditions are taken in the form of a torsional spring, a rotational mass, a spring connected mass, a torsional spring coupling or a combination of these. The computed data to establish frequencies of such beam and shaft systems have been presented in tabular form for various parameters associated with these non-classical boundary conditions. In presenting these results, use has been made of a "Q factor" introduced as a measure of howfar the system frequency lies between the two classical limits associated with the system. It is further shown that any value of system parameter associated with non-classical boundary conditions can be

accounted for by varying Q from zero to unity. The number of discrete values of various influence parameters and Q factor, and the format of the tables have been so selected as to yield results for frequencies accurate within one percent when linear interpolation is used for values lying partway between the tabulated values. In the range where linear interpolation is not possible, the use of an alternate equation has been suggested which will be consistent with the accuracy requirement of one percent.

The vibration frequencies and model shape expressions are provided for three-span and four-span beams with all possible classical boundary conditions at the outer supports. For triple span beams, the vibration frequency data are provided for 39 different locations of interior supports whereas for four span beams, they are provided for 19 different locations of interior supports. The number of stations chosen along the overall beam length was dictated by the accuracy requirement if linear interpolation is used for values lying partway between the tabulated values.

For multi-span beams with uniform span lengths, the vibration frequency data are presented for first six modes of vibrations and upto 15 spans for all six possible combinations of outer end boundary conditions. The expressions for frequency and modal shapes have also been derived for multi-span beams with arbitrary spacings between

the supports - for all possible boundary conditions. It is further shown how the frequency equations and modal shape expressions can be quickly constructed utilizing various recursion formulas provided. Several cases of multi-span beams with uniform spans are discussed to show how the frequencies of beams with large number of spans can be inferred from those with small number of spans wherever possible.

The effect of axial loads on vibration frequencies of single span beams is also investigated and results are presented in graphical form in Chapter Eight for various boundary conditions. It is further shown in that chapter how the frequencies of beams on elastic foundation can be obtained by using the results provided for beams without elastic foundations.

The problem of free vibrations of tapered beams is discussed in Chapter Nine. Results for vibration frequencies of tapered beams are presented in graphical form for all possible taper ratios and various boundary conditions. Both conical and wedge-shaped tapered beams have been studied. The vibration frequency data are also provided for beams consisting of two sections, one of uniform cross-section and the other tapered with either conical or wedge-shaped form, with various boundary conditions.

Finally, the effects of rotary inertia and shear deformation on the vibration of single span uniform beams with classical and non-classical boundary conditions are investigated in Chapter Ten.

✓

Results are presented in graphical form for beams with classical boundary conditions. Frequency equations and modal shape expressions are provided for beams with non-classical boundary conditions.

Several illustrative examples have been solved at the end of each chapter to demonstrate the use of tables and the solution technique. All the computations were carried out on IBM 360/65 high speed digital computer at the University of Ottawa using FORTRAN IV with double precision.

ACKNOWLEDGEMENT

The author is deeply grateful to Dr. D.J. Gorman, his thesis supervisor, for his able guidance and many constructive suggestions during the entire course of this project.

The financial support received from the Government of Ontario in the form of a "Ontario Graduate Fellowship", from the National Research Council of Canada in the form of a research assistantship, and from the Department of Mechanical Engineering, University of Ottawa, in the form of a teaching assistantship is gratefully acknowledged.

The author wishes to thank his friends, Mr. K.G. Gupta for inking the charts appearing in Chapter Nine, and Mr. D.S. Nguyen for programming some of the equations appearing in Chapter Seven of this thesis. The author is also thankful to the staff of the Computing Centre of the University of Ottawa for their assistance and cooperation.

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LIST OF SYMBOLS

A	Area of beam cross-section
A_0	Area of tapered beam at larger end
A,B,C,D	Arbitrary constants
E	Young's modulus of elasticity
f	Frequency
G	Shear modulus of rigidity
I	Moment of inertia of beam cross-section
I_0	Moment of inertia of rotational mass
I^*	Non-dimensional system parameter associated with rotational mass
I_p	Polar moment of inertia
$I_n(z)$	n^{th} order modified Bessel function of first kind
$J_n(z)$	n^{th} order Bessel function of first kind
$K_n(z)$	n^{th} order modified Bessel function of second kind
K	Spring stiffness of linear coil spring
K^*	Non-dimensional system parameter associated with linear coil spring
k'	Shape factor of beam cross-section
L	Length of the beam
L_1	Full cone length of tapered beam

M	Concentrated mass
P	Axial load
$p(t)$	A function of time t only
Q	"Q factor"
$R(x)$	A function of x only
r	Non-dimensional system parameter associated with rotary inertia
s	Non-dimensional system parameter associated with shear deformation
S_1	Another parameter associated with shear deformation
t	Time
T	Spring stiffness of torsional spring
T^*	Non-dimensional system parameter associated with torsional spring
V	Shear force
x	Spatial coordinate
$Y(x,t)$	Elastic line of the beam
$Y_n(z)$	n^{th} order Bessel function of second kind
Z	Argument of Bessel functions

GREEK SYMBOLS

α	Ratio of two system parameters
α_{ij}	Elements of a matrix
β	Eigenvalue
β_1, β_2	Frequency parameters defined by equations (1.14) and (1.15)
γ	Non-dimensional length of last span
ξ	Non-dimensional spatial coordinate parameter
ψ	Slope due to bending
ω	Circular frequency
ρ	Density of beam material
ν_n	Non-dimensional length of n^{th} span
$\phi(x, \tau)$	Elastic line of vibrating shaft
θ	Slope of shaft
η	Non-dimensional system parameter associated with concentrated mass

SUBSCRIPTS

1,2	Refer to left and right hand end of beam respectively
n	n th span
cc	Clamped-clamped
cs	Clamped-simply supported
cf	Clamped-free
ss	Simple-simple
sf	Simple-free
ff	Free-free

CHAPTER 1

INTRODUCTION AND GENERAL THEORY

1.1 Introduction

The object of a "free vibration analysis" is to predict the vibration frequencies and modal shapes of systems when no time varying external forces are acting on them. The problem of vibration of a system under the action of time varying external forces is known as a forced vibration problem. These time varying external forces could either be periodic or stochastic. The examples of the former are the forces acting on gear trains and motor pump shafts due to unbalance or the forces transmitted to the foundation of a reciprocating engine. The examples of the later are the forces caused by wall pressure distributions across nuclear reactor heat exchanger tubes in a turbulent flow. If the exciting forces are periodic, the forced vibrations take place at the frequency of excitation and if the exciting forces are random in nature, the forced vibrations take place over a wide frequency range associated with the spectrum of frequencies possessed by the exciting forces.

If any of the free vibration frequencies are contained by these external time varying forces, a condition of resonance is encountered and dangerously high amplitudes are produced. These, in turn, give rise to severe stresses and strains causing fretting and fatigue of the system material and eventual failure. Consequently, the establishment of free vibration frequencies and modal shapes are of great practical interest for all types of vibration problems. A knowledge of free vibration frequencies helps the vibration analyst to modify his system parameters so that the condition of resonance is avoided. In fact in many cases this is the only option available to the designer as there is often very little that can be done to change the frequency of exciting forces. Sometimes however, if the exciting force is acting at a point, it may be possible to remove the point of application of this force to a nodal point so that it may not be able to impart an appreciable amount of energy to excite large amplitudes. Here again a knowledge of free vibration frequencies and modal shapes is required.

A knowledge of modal shapes also plays an important role in research and development work. For example, in the course of development work, very often the dynamic response of a system is measured by using such devices as accelerometers, strain gauges and pressure transducers, etc. It is only through a knowledge of modal shapes that one can ascertain the optimum location of placing these devices so that the



output from these may be fully utilized. For example, if an accelerometer is placed at a nodal point, it will obviously not record anything as it is activated by the displacement.

It is therefore obvious from the preceding remarks that free vibration analysis plays a very important role in seeking solutions to various vibration problems encountered by designers and engineers in the industry.

The "beam and shaft vibration problem" is an important class of problems in mechanics. The vibration frequencies and modal shapes of free lateral vibration of beams with classical boundary conditions have been known for many years and are well documented in the existing literature. Bishop and Johnson (1)* have prepared tables of the eigenvalues, eigenfunctions, and their derivatives for various classical end supports of the uniform beams. The classical end in case of beams implies either a free, simply-supported (pinned or hinged) or a fixed (clamped or built-in) end. The free vibration frequencies and modal shapes of torsional vibration of shafts with fixed or free ends are also available in the existing literature (2).

But engineers working in the industry frequently encounter situations where the beams and shafts do not meet the requirements of these classical ends. Very often these ends are supported against lateral deflection or elastically restrained against rotation. Furthermore, these

* Numbers in the brackets refer to the references provided at the end.

ends could be carrying a concentrated mass which in turn could possess rotary inertia. In fact there could be any possible and permissible combination of these non-classical boundary conditions. Although in recent years, various researchers have shown keen interest in these non-classical boundary conditions, there exists very little in the available literature in the form which can be readily utilized by the practicing engineer who may or may not be a vibration specialist. It is the purpose of this thesis to present the vibration frequencies and modal shapes of virtually any beam or shaft system whatever, having any permissible combination of these non-classical boundary conditions or resting on multiple supports.

It appears that in the existing literature which deals with only a very limited range of these non-classical boundary conditions, the emphasis has been on the various methods of solution rather than presenting the solution in the form so that it can be readily referred to by the practicing engineer. Furthermore, various combinations of these boundary conditions have been left untouched.

Newmark and Veletsos (3) have published an approximate formula for the natural frequency of beams with both ends elastically restrained against rotation. The accuracy claimed for first mode is within 4% and no modal shape expressions are given. Lee and Saibel (4) have utilized the energy technique to determine the free vibration frequencies of beams with intermediate elastic supports in the form of

coil springs and concentrated masses. They have not discussed any case in which these elastic restraints are located at the ends. Again no modal shape expressions are provided. Burgreen (5) has discussed the effects of equal-end-fixity on the vibrations of rods. The end fixity in his paper is provided by the torsional springs at the ends. He also discussed the concept of negative spring constants. The same problem with unequal end restraints was solved by Sharp and Cobble (6) using finite transform method. They have provided the modal shape expressions but no numerical results for frequencies are presented. Cobble (7) has utilized the same method in solving the damped clamped-clamped beam having distributed load and elastic support.

Baker (8) presented the frequency equations for single span uniform beams with a rigid mass at the center. His beam was classically supported at its outer ends. He also gave some eigenvalues for the first two modes of vibrations. No modal shape equations were presented. Hess (9) extended the work of Baker by taking one end of the beam as elastically restrained against rotation. The concentrated mass, as in the case of Baker, was located at the center of the beam. Chen (10) gave a new formulation to the problem of vibration of beams carrying a concentrated mass anywhere along the span. He utilized Dirac- δ -function in the differential equation to describe the effect of a concentrated mass. He did not give any numerical results. Srinath and Das (11) further extended the work on beams with an inspan concentrated mass taking the rotary

inertia of the mass into consideration. Dowell (12) gave a new method of solution for the free vibrations of a linear structure valid for any type of support conditions. He utilized the normal modes of the unsupported or unconstrained structure in a Rayleigh-Ritz analysis with the support or constraint conditions enforced by means of Lagrange multipliers. No numerical results were provided. Ditaranto (13) utilized the extension of Myklestad method for determining the natural frequencies of non-uniform beams with elastic supports. The special cases of various end fixities of a two-support beam are given as illustrations. No numerical data for frequencies are given nor are, modal shape expressions.

In all these papers cited above, there is very little of the form which the designer can use to his advantage. Furthermore, there is a wide range of these non-classical boundary conditions, acting singly or in combination, which has not been studied in the existing literature, for example, a mass supported by a coil spring, a coil spring acting in conjunction with a torsional spring, etc. The first Chapter of this thesis deals with single span uniform beams with non-classical boundary conditions. Some of these cases, not included here, were solved by the author in his Master's degree thesis at the University of Ottawa (14).

In case of two span beams, even with classical boundary conditions, the vibration frequencies and modal shapes have only recently been made available by Gorman (15). The Appendix B of this thesis has been taken from this Ref. (15). This reference has discussed all six possible

boundary conditions at the outer ends of the beam and frequencies are provided for any location of the intermediate support. He has also discussed the cases of intermediate concentrated masses and coil springs. The cases of torsional springs and rotational masses have not been reported. The vibration frequencies and modal shapes of two span beams, with non-uniform span lengths and one non-classical boundary condition in the form of a rotational mass or a torsional spring are discussed in the third Chapter of this thesis.

The vibration frequencies and modal shapes of three span and four span beams with non-uniform span lengths are also not available in the existing literature. They have been discussed in Chapters 4 and 5 of this thesis for all possible classical boundary conditions at the outer ends.

The vibration of continuous beams with uniform spans has drawn the attention of many researchers. Vet (16) has developed a recurrence formula for normal vibration modes of continuous beams with distinct flexural rigidities for each span. No numerical results are presented in his paper. Sengupta (17) has utilized the concept of propagation constants and knowledge of natural flexural waves to determine the frequencies of periodically supported beams. Bolton (18) has solved this problem recognizing the close connections between natural frequencies and elastic critical loads. He conducted the elastic critical load analysis on the system and inferred natural frequencies from there. The most comprehensive work reported in the

field of vibrations of continuous beams with uniform span lengths has been one by Jacobsen and Ayre (19). It can be easily recognised that there are six possible combinations of classical boundary conditions at the outer ends of a continuous beam. Jacobsen and Ayre have computed the values of a frequency parameter for three of these cases utilizing a rather complicated semi-graphical network technique. They have also inferred some of the more easily anticipated modal shapes but no expressions for them are given in their paper. Macduff and Felgar (20) utilized this work in their paper which in turn is referred to by Harris and Crede (21) in their well known handbook. Timoshenko (22) has also referred to

the work of Jacobsen and Ayre in his textbook on mechanical vibrations.

Chapter 6 of this thesis provides vibration frequencies and modal shapes for all possible outer end conditions of these beams upto 15 spans, expressions for modal shapes and frequencies are also given for non-uniform span lengths as well, though the numerical results are presented only for beams with uniform span lengths.

As in the case of free lateral vibration of beams, shafts may execute torsional vibrations with non-classical boundary conditions. The non-classical boundary conditions may constitute an attached rotational mass, a torsional spring, a spring connected mass or a spring coupling located at the end or anywhere along the shaft span. Torsional vibrations of shafts with all these boundary conditions are discussed in Chapter 7.

There is another class of beam vibration problem arising, when

ends of the beams are subjected to uniform compression or tension.

Ambarao (23) has presented some vibration frequencies for the beam under compression only. He presented plots for fundamental mode only. No tensile forces were considered by him.

Galit (24) published an approximate formula for the vibrations of compressed beams. His formula is only valid for first mode. Pelkington and Carr (25) extended this work by considering the axially distributed load also. Again only the first mode was considered. In Chapter 8 of this thesis this class of problem has been solved for six modes and the range of end load is taken from Euler load in compression to $10 \times$ Euler load in tension.

There are situations encountered by practicing engineers where the beams do not have a uniform cross-section. Their transverse dimensions may vary along the length of the beam. Cranch and Adler (26) have given solutions for fully-tapered cantilever beams. No other boundary conditions are considered and no solution is provided for the case when beam is truncated at the outer end. Convey, Becker and Dubil (27) have considered the solution for a truncated cantilever beam. They have provided numerical results for some specific taper ratios. All possible taper ratios have not been considered. Convey and Dubil (28) extended their work to include all possible boundary conditions for conical and wedge shaped beams. They have computed the values of a frequency parameter for some specific values of taper ratios. The full range of taper

ratios has not been covered. Georgian (29) has presented some experimentally obtained results for a cone cantilever beam which are in close-agreement with those of Convey and Dubil. Mable and Rogers (30) have also computed some values of a vibration parameter of a tapered beam with fixed-simply supported end conditions. They (31) later extended their work to include double-tapered cantilever beams. Gaines and Volterra (32) developed some approximate formulas for upper and lower bounds of frequencies for three modes of truncated cones and wedges, taking the effect of rotary inertia and shear deformation into consideration. Harnopp and Fung (33) obtained approximate lower and upper bounds of vibration frequencies for a variable cross-section cantilever by lumping the mass properties at discrete points and using an iteration technique.

None of these papers cover the full range of taper ratios for conical and wedge shaped beams. Furthermore, if the beam is composed of two sections, one uniform and the other tapered, the vibration frequencies are not available. Chapter 9 of this thesis presents the vibration frequencies for conical and wedge shaped beams for a full range of taper ratios and also when the beam is composed of a straight section and a tapered section.

There may arise situations when the assumptions inherent in Euler-Bernoulli beam theory no longer remain valid and consequently the effects of shear deformation and rotary inertia should be taken into consideration. These effects can significantly alter the

natural frequencies of free vibrations for higher modes and even for lower modes if the cross sectional dimensions of the beam are not small compared to its length. Trail-Nash and Collar (34) and also Huang (35) have treated the case of single span beams with classical boundary conditions taking into account the effects of shear deformation and rotary inertia. But they have not presented the numerical results which can be easily utilized by the practicing engineer. Furthermore, when the single span beam has non-classical boundary conditions, no expressions for frequency equations and modal shapes are available in the existing literature. In Chapter 10 of this thesis, the frequency equations and modal shape expressions are provided for single span beams with classical and non-classical boundary conditions. The correction factor when rotary inertia and shear deformation are accounted for, is also presented for various values of parameters associated with shear deformation and rotary inertia for all the six possible classical boundary conditions of a single span uniform beam. The relevant theory and boundary conditions for these short beams are discussed in Chapter 10.

Finally, it may be emphasized that overall work presented in this thesis, makes the complete and exhaustive solution of the various beam and shaft vibration problems available to designers and students for the first time. The complete set of vibration frequency data provided for entire range of various parameters of interest and modal shape expressions should prove very useful to designers. With the availability of

this work, the necessity of any further computational work in this area is obviated. With the help of the data provided in this thesis, one can obtain solutions to various beam and shaft vibration problems even if no computational facility is available. This work therefore will be a welcome contribution in the places where computers are not commonly available.

The rigorous theoretical treatment of the problems and their solution technique presented here should prove to be immensely useful to the students of mechanical vibrations in their learning process and better understanding of the problem. The solution technique utilizing a "Q factor" which is described later in this thesis, has proved to be a very simple but powerful technique. It may be appreciated that it is because of this technique that the frequency data could be so exhaustively presented in a concise form. The author feels that this technique also has a further potential of being advantageously used for vibrations of plates and shells with elastically supported boundary conditions.

Furthermore, with the availability of this work, one would be spared the time consuming process of digging into various research papers cited earlier in this Chapter, which deal with only a limited number of aspects of the problem in bits and pieces. With the availability of this work, the previous work of various researchers (3, 5, 16, 18, 19, 23, 24, 26, 27, 28, 30, 34, 35) has been superceded.

1.2 Development of Theory

1.2.1 Uniform beams undergoing free lateral vibrations

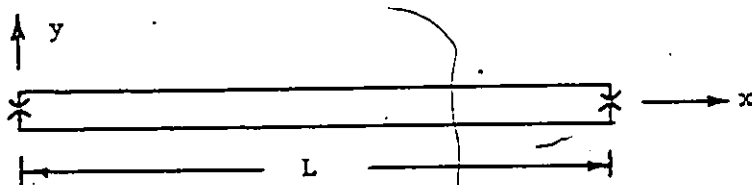


Figure 1.1

The governing partial differential equation of a uniform beam undergoing free lateral vibrations is given by

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} = - \rho A \frac{\partial^2 y(x,t)}{\partial t^2} \quad (1.1)$$

Where $y(x,t)$ is the elastic line of the beam, E is Young's modulus of elasticity of beam material, I is the moment of inertia of the beam cross section, ρ is the density of beam material and A its cross-sectional area.

Equation (1.1) is based on elementary beam theory which implies certain well known assumptions. These assumptions are:

- (1) The beam material is homogeneous, isotropic and obeys Hooke's law;
- (2) The beam is straight and of uniform cross-section;
- (3) Deflections are small;
- (4) Each plane section that is initially normal to the beam axis remains plane and normal to axis after deflection;
- (5) Cross-sectional dimensions of the beam are small compared to its length between nodal points, consequently the effects of shear deformation are neglected;
- (6) The effects of rotary inertia are neglected.

The partial differential equation (1.1) can be readily solved by the method of separation of variables. Assuming that the

solution of equation (1.1) is separable in time and space and is of the following form:

$$y(x,t) = R(x) \cdot p(t) \quad (1.2)$$

Where $R(x)$ is a function of x only and $p(t)$ is a function of t only. Substituting expression (1.2) into equation (1.1) and dividing through by $\rho A \cdot R(x) \cdot p(t)$, we obtain the following coupled ordinary differential equation

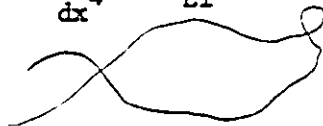
$$\frac{EI}{\rho A} \frac{1}{R(x)} \frac{d^4 R(x)}{dx^4} = - \frac{1}{p(t)} \frac{d^2 p(t)}{dt^2} \quad (1.3)$$

The left side of equation (1.3) is a function of x only and the right side is a function of t only. Since both x and t are independent variables, both sides of equation (1.3) should therefore be equal to a constant. Let this constant be a positive constant ω^2 . This then leads to the following two ordinary differential equations:

$$\frac{d^2 p(t)}{dt^2} + \omega^2 p(t) = 0 \quad (1.4)$$

and

$$\frac{d^4 R(x)}{dx^4} - \frac{\rho A \omega^2}{EI} R(x) = 0 \quad (1.5)$$



The choice of the sign of the constant is dictated by physical considerations. For a conservative system, the displacements must remain finite as t increases. The negative constant will yield the solution of equation (1.4) in an exponential form which will not be consistent with the above observations.

Equation (1.5) defines the shape of the vibrating beam whereas equation (1.4) ensures a simple harmonic motion under vibrations. It is advantageous to work with equation (1.5) in a non-dimensional form. Non-dimensionalizing the lateral displacement $R(x)$ and the spatial coordinate x with respect to beam length L , equation (1.5) can be re-written as:

$$\frac{L}{L^4} \frac{d^4 R(\frac{x}{L})}{d(\frac{x}{L})^4} - \frac{\rho A \omega^2}{EI} L R(\frac{x}{L}) = 0$$

or

$$\frac{d^4 R(\xi)}{d\xi^4} - \beta^4 R(\xi) = 0 \quad (1.6)$$

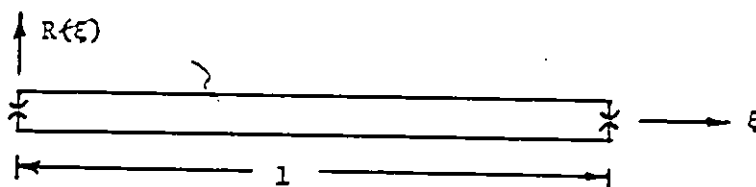


Figure 1.2

$$\text{Where } \beta^4 = \frac{\rho A \omega^2 L^4}{EI} \quad (1.7)$$

and $\xi = \frac{x}{L}$ is a non-dimensionalized spatial coordinate parameter.

The general solution of the differential equation (1.6) is given by:

$$R(\xi) = A \sin \beta \xi + B \cos \beta \xi + C \sinh \beta \xi + D \cosh \beta \xi \quad (1.8)$$

Equation (1.8) represents the shape of a uniform beam in free lateral vibration in terms of four arbitrary constants. The eigenvalue problem consists of finding those values of β which yield a non-trivial solution for equation (1.8). This is done by enforcing four boundary conditions, two at each end of the beam. This will then yield four algebraic homogeneous equations relating the four constants A, B, C, and D. The condition for the non-trivial solution for these constants to exist yields an equation known as a characteristic equation or a frequency equation of the beam.

This frequency equation being transcendental in nature will yield an infinite sequence of discrete eigenvalues β_n related to natural circular frequencies ω_n ($n = 1, 2, \dots$ etc) by equation (1.7). The task of a vibration analyst is then to enforce appropriate boundary conditions of his system to solve for these eigenvalues by a proper iteration technique.

The successive derivative of equation (1.8) which will be frequently utilized in this thesis are given below:

$$R'(\xi) = \beta[A \cos \beta\xi - \beta \sin \beta\xi + C \cosh \beta\xi + D \sinh \beta\xi]$$

$$R''(\xi) = \beta^2[-A \sin \beta\xi - \beta \cos \beta\xi + C \sinh \beta\xi + D \cosh \beta\xi]$$

$$R'''(\xi) = \beta^3[-A \cos \beta\xi + \beta \sin \beta\xi + C \cosh \beta\xi + D \sinh \beta\xi]$$

where $R'(\xi) \equiv \frac{dR}{d\xi}$, $R''(\xi) \equiv \frac{d^2R}{d\xi^2}$, and $R'''(\xi) \equiv \frac{d^3R}{d\xi^3}$

1.2.2 Uniform beams with axial forces at the ends

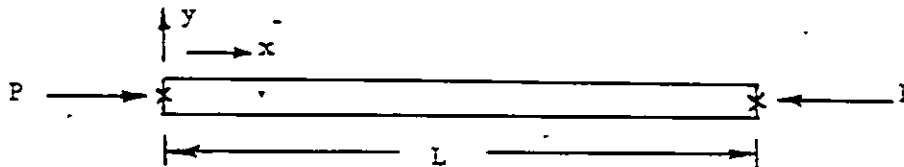


Figure 1.3

The partial differential equation governing the free lateral vibrations of beams with a compressive force P acting at its ends is given by:

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + P \frac{\partial^2 y(x,t)}{\partial x^2} = - \rho A \frac{\partial^2 y(x,t)}{\partial t^2} \quad (1.9)$$

As explained earlier, the method of separation of variable yields the following ordinary differential equation defining the shape of the vibrating beam

$$\frac{d^4 R(x)}{dx^4} + \frac{P}{EI} \frac{d^2 R(x)}{dx^2} - \frac{\rho A \omega^2}{EI} R(x) = 0 \quad (1.10)$$

When lateral deflection $R(x)$ and spatial coordinate parameter x are non-dimensionalized with respect to beam length L , equation (1.10) assumes the following form:

$$\frac{d^4 R(\xi)}{d\xi^4} + \frac{PL^2}{EI} \frac{d^2 R(\xi)}{d\xi^2} - \frac{\rho A \omega^2 L^4}{EI} R(\xi) = 0 \quad (1.11)$$

Utilizing equation (1.7) and introducing $\alpha = \frac{PL^2}{EI\pi^2}$ which is the ratio of compressive load P to the Euler load $EI\pi^2/L^2$, equation (1.11) is reduced to:

$$\frac{d^4 R}{d\xi^4} + \pi^2 \alpha \frac{d^2 R}{d\xi^2} - \beta^4 R = 0 \quad (1.12)$$

Where α will be negative if P is tensile. The general solution of equation (1.12) is given by

$$R(\xi) = A \sin \beta_1 \xi + B \cos \beta_1 \xi + C \sinh \beta_2 \xi + D \cosh \beta_2 \xi \quad (1.13)$$

where

$$\beta_1 = \sqrt{\frac{\pi^2 \alpha}{2} + \sqrt{\frac{\pi^4 \alpha^2}{4} + \beta^4}} \quad (1.14)$$

and

$$\beta_2 = \sqrt{-\frac{\pi^2 \alpha}{2} + \sqrt{\frac{\pi^4 \alpha^2}{4} + \beta^4}} \quad (1.15)$$

Here again, the task of a vibration analyst is to find those values of frequency parameter β which gives a non-trivial solution for equation (1.13) consistent with the appropriate boundary conditions.

1.2.3 Tapered beams

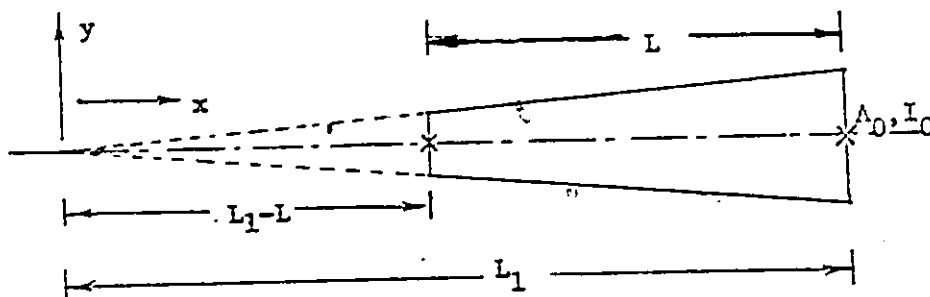


Figure 1.4

Since in tapered beams the flexural rigidity EI and cross-section area A are functions of the spatial coordinate x , the partial differential equation governing the free vibrations of such beams is given by ,

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] - \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (1.16)$$

The ordinary differential equation defining the shape of the deflected beam is obtained by solving equation (1.16) by the method of separation of variables and is presented below.

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 R(x)}{dx^2} \right] - \rho A(x) \omega^2 R(x) = 0 \quad (1.17)$$

It is assumed that the beam has a linear taper in its transverse dimensions. Let L be the length of the beam and L_1 be that length in which the beam tapers to a point. Let this be called a "full cone length". The coordinates are chosen as shown in Figure 1.4.

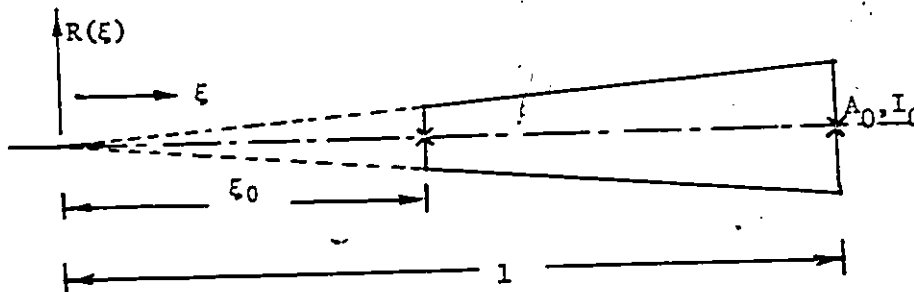


Figure 1.5

Non-dimensionalizing the beam deflection $R(x)$ and spatial coordinate parameter x with respect to full cone length L_1 of the beam, we obtain

$$\frac{d^2}{d\xi^2} \left(EI(\xi) \frac{d^2 R(\xi)}{d\xi^2} \right) - \rho A(\xi) \omega^2 L_1^4 R(\xi) = 0. \quad (1.18)$$

For a beam with linear taper in transverse dimensions, the cross-section area $A(\xi)$, and moment of inertia of the cross-section $I(\xi)$, are related to their respective values A_0 and I_0 at the larger end as follows

$$A(\xi) = A_0 \xi^2 \quad (1.19)$$

$$I(\xi) = I_0 \xi^4 \quad (1.20)$$

Where the coordinates are taken as shown in Figure 1.5.

Substitution of equations (1.19) and (1.20) into equation (1.18) yields:

$$\frac{d^2}{d\xi^2} \left[EI_0 \xi^4 \frac{d^2 R(\xi)}{d\xi^2} \right] - \rho A_0 \xi^2 \omega^2 L_1^4 R(\xi) = 0 \quad (1.21)$$

or

$$\frac{1}{\xi^2} \frac{d^2}{d\xi^2} \left[\xi^4 \frac{d^2 R(\xi)}{d\xi^2} \right] - \beta^4 R(\xi) = 0 \quad (1.22)$$

$$\text{where } \beta^4 = \frac{\rho A_0 \omega^2 L_1^4}{EI_0} \quad (1.23)$$

Equation (1.22) admits a general solution of the form (36),

$$R(\xi) = \xi^{-1} [A J_2(z) + B Y_2(z) + C I_2(z) + D K_2(z)] \quad (1.24)$$

$$\text{where } z = 2\beta \xi^{1/2} \quad (1.25)$$

and J_2 and Y_2 are second order Bessel functions of first and second kind respectively and I_2 and K_2 are second order modified Bessels functions of first and second kind respectively.

In order to enforce boundary conditions, their successive derivatives will be required which are given below.

$$R'(\xi) = -\beta \xi^{-3/2} [A J_3(z) + B Y_3(z) + C I_3(z) + D K_3(z)] \quad (1.26)$$

$$R''(\xi) = \beta^2 \xi^{-2} [A J_4(z) + B Y_4(z) + C I_4(z) + D K_4(z)] \quad (1.27)$$

The above two expressions will be required for conditions of zero slope and zero bending moment respectively. For a free boundary where the shear force is zero, we require,

$$\frac{d}{d\xi} \left(EI(\xi) \frac{d^2 R}{d\xi^2} \right) = 0 \quad (1.28)$$

Substituting from equations (1.19) and (1.20) into equation (1.28), we obtain:

$$E I_0 \frac{d}{d\xi} \left[\xi^4 \frac{d^2 R}{d\xi^2} \right] = 0 \quad (1.29)$$

where

$$\frac{d}{d\xi} \left(\xi^4 \frac{d^2 R}{d\xi^2} \right) = \beta^3 \xi^{3/2} [A J_3(z) + B Y_3(z) + C I_3(z) - D K_3(z)] \quad (1.30)$$

In case of wedge-shaped beams where the width of the beam is constant and depth varies linearly, the cross sectional area $A(\xi)$ and moment of inertia $I(\xi)$ are related to their respective values A_0 and I_0 at the larger end by the following equations:

$$A(\xi) = A_0 \xi \quad (1.31)$$

$$I(\xi) = I_0 \xi^3 \quad (1.32)$$

Substituting these equations (1.31) and (1.32) into equation (1.28), we obtain:

$$\frac{d^2}{d\xi^2} \left[\xi^3 \frac{d^2 R(\xi)}{d\xi^2} \right] - \beta^4 \xi R(\xi) = 0 \quad (1.33)$$

which has a general solution of the following form

$$R(\xi) = \xi^{-1/2} [A J_1(z) + B Y_1(z) + C I_1(z) + D K_1(z)] \quad (1.34)$$

Where $z = 2\beta \xi^{1/2}$

and J_1 and Y_1 are first order Bessel functions of first and second kind respectively and I_1 and K_1 are first order modified Bessel functions of first and second kind respectively.

The successive derivative required for various boundary conditions are

$$R'(\xi) = -\beta \xi^{-1} [A J_2(z) + B Y_2(z) - C I_2(z) + D K_2(z)] \quad (1.35)$$

$$R''(\xi) = \beta^2 \xi^{-3/2} [A J_3(z) + B Y_3(z) + C I_3(z) + D K_3(z)] \quad (1.36)$$

and

$$\frac{d}{d\xi} \left[\xi^3 \frac{d^2 R}{d\xi^2} \right] = \beta^3 \xi [A J_2(z) + B Y_2(z) + C I_2(z) - D K_2(z)] \quad (1.37)$$

The equation (1.37) will be required for a free boundary where the shear force is zero.

In determining these successive derivatives, use has been made of the following recurrence formulas (36):

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$\frac{d}{dx} [x^n Y_n(x)] = x^n Y_{n-1}(x)$$

$$\frac{d}{dx} [x^{-n} Y_n(x)] = -x^{-n} Y_{n+1}(x)$$

(1.38)

$$\frac{d}{dx} [x^n I_n(x)] = x^n I_{n-1}(x)$$

$$\frac{d}{dx} [x^{-n} I_n(x)] = -x^{-n} I_{n+1}(x)$$

$$\frac{d}{dx} [x^n K_n(x)] = -x^n K_{n-1}(x)$$

$$\frac{d}{dx} [x^{-n} K_n(x)] = -x^{-n} K_{n+1}(x)$$

1.2.4 Shafts undergoing free torsional vibrations

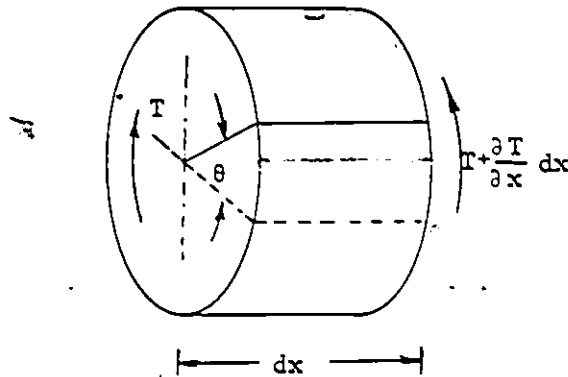


Figure 1.6

Figure 1.6 shows the coordinate system and a positive torque T acting on the shaft.

The partial differential equation governing the free torsional vibrations of a shaft is given by:

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{G}{\rho} \frac{\partial^2 \phi(x,t)}{\partial x^2} \quad (1.39)$$

Where $\phi(x,t)$ is the elastic line of the shaft,

G is the shear modulus of rigidity,

and ρ is the density of shaft material.

Assuming $\phi(x,t) = \theta(x) \cdot p(t)$, we obtain the following ordinary differential equation defining the shape of the vibrating shaft:

$$\frac{d^2 \theta(x)}{d_x^2} + \frac{\rho \omega^2}{G} \theta(x) = 0 \quad (1.40)$$

Non-dimensionalizing the spatial coordinate x with respect to shaft length L , we obtain:

$$\frac{d^2 \theta(\xi)}{d_\xi^2} + \frac{\rho \omega^2 L^2}{G} \theta(\xi) = 0$$

or

$$\frac{d^2 \theta(\xi)}{d_\xi^2} + \beta^2 \theta(\xi) = 0 \quad (1.41)$$

$$\text{where } \beta = \omega L \sqrt{\frac{\rho}{G}} \quad (1.42)$$

The general solution of equation (1.41) is given by,

$$\theta(\xi) = A \sin \beta \xi + B \cos \beta \xi \quad (1.43)$$

The object as before is to find those values of β which yield a non trivial solution for $\theta(\xi)$ in equation (1.43) consistent with the appropriate boundary conditions of the system.

1.3 Boundary Conditions

The general solution for transverse vibration of beams contains four arbitrary constants. This should therefore be supplemented by four boundary conditions. These boundary conditions could be either classical, non-classical or a combination of both. In case of torsional vibrations, since the general solution (1.43) contains only two arbitrary constants, this therefore needs to be supplemented by only two boundary conditions.

Both classical and non-classical boundary conditions have been utilized in this thesis. A description of these boundary conditions is presented below in that order.

1.3.1 Classical boundary conditions

In case of transverse vibrations of beam, the following three types of classical boundary conditions have been utilized:

(a) Fixed (clamped or built-in) end:

The slope and deflection at such ends are zero. Expressed mathematically

$$R(\xi) = 0 \quad (1.44)$$

$$R'(\xi) = 0 \quad (1.45)$$

(b) Pinned (hinged or simply supported) end:

There is no deflection or bending moment at such ends,
thus

$$R(\xi) = 0 \quad (1.46)$$

$$R''(\xi) = 0 \quad (1.47)$$

(c) Free end:

Since this end cannot support any moment or shear force,
we have:

$$R''(\xi) = 0 \quad (1.48)$$

$$R'''(\xi) = 0 \quad (1.49)$$

The shear force expression for tapered beams is expressed
as:

$$\frac{d}{d\xi} \left[E I(\xi) \frac{d^2 R}{d\xi^2} \right] = 0 \quad (1.50)$$

In case of torsional vibrations, there are two types of
classical boundary conditions. A fixed end will have no angular

rotation i.e. $\theta(\xi) = 0$ and a free end cannot sustain any torque so that $\theta'(\xi) = 0$.

1.3.2 Non classical boundary conditions

Part 1 - Transverse vibrations of beams

In case of transverse vibrations of beams, the following four types of non-classical boundary conditions have been utilized. There are several cases of course when they act in conjunction with one another.

(a) Ends elastically supported by a linear coil spring:

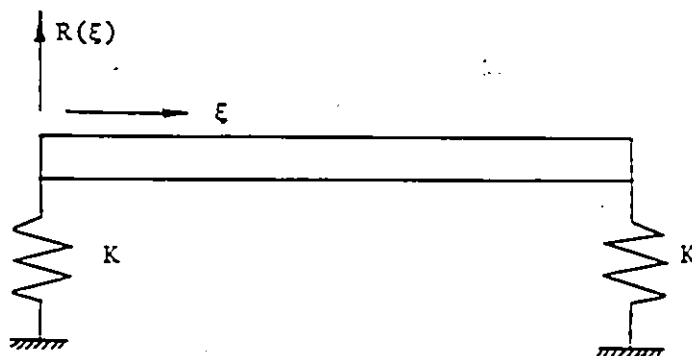


Figure 1.7

The shear force balance at the end $x = L$, requires

$$EI \left. \frac{d^3 y}{dx^3} \right|_{x=L} = -K y \Big|_{x=L} \quad (1.51)$$

Where K is the spring stiffness.

Equation (1.51) reduces to following dimensionless form

$$\frac{d^3 R(\xi)}{d\xi^3} - \frac{K L^3}{EI} R(\xi) \quad (1.52)$$

or

$$R'''(\xi) \Big|_{\xi=1} - K^* R(\xi) \Big|_{\xi=1} \quad (1.53)$$

where $K^* = \frac{K L^3}{EI}$ is non dimensional system parameter.

Since the entire analysis in this thesis is carried out in non dimensional form, the boundary conditions too will be utilized in their non dimensional forms as given in equation (1.53).

If the coil spring is at the left end of the beam, the shear force equilibrium is expressed by

$$R'''(\xi) \Big|_{\xi=0} = -K^* R(\xi) \Big|_{\xi=0} \quad (1.54)$$

(b) Ends elastically restrained against rotation by a torsion spring:

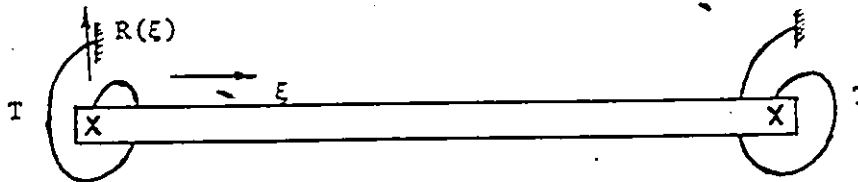


Figure 1.8

The bending moment balance at the end $x = L$ requires

$$EI \frac{d^2 y}{dx^2} \bigg|_{x=L} = -T \frac{dy}{dx} \bigg|_{x=L} \quad (1.55)$$

which in the non dimensional form reduces to

$$\frac{d^2 R}{d\xi^2} = -\frac{TL}{EI} \frac{dR}{d\xi} \quad (1.56)$$

or

$$R'' \bigg|_{\xi=1} = -T^* R' \bigg|_{\xi=1} \quad (1.57)$$

where T is the spring constant of the torsional spring

and $T^* = \frac{TL}{EI}$ is a non dimensional system parameter.

For a torsional spring attached to the left end of the beam

$$R'' \bigg|_{\xi=0} = T^* R' \bigg|_{\xi=0} \quad (1.58)$$

(c) Ends carrying concentrated masses:

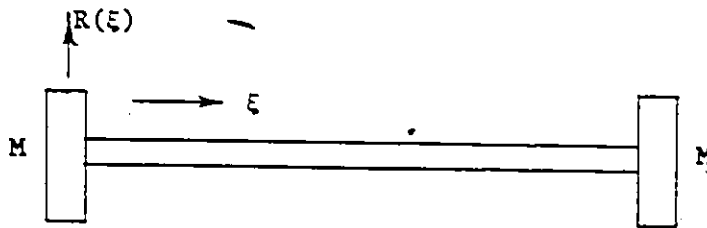


Figure 1.9

The shear force balance at the right hand end requires

$$EI \frac{\partial^3 y(x,t)}{\partial x^3} = -M \frac{\partial^2 y(x,t)}{\partial t^2} \quad (1.59)$$

Assuming the solution $y(x,t) = R(x) \sin(\omega t + \alpha)$, Equation (1.59) is reduced to

$$EI \frac{d^3 R}{dx^3} = -M \omega^2 R(x) \quad (1.60)$$

Reducing equation (1.60) in a non-dimensional form,

$$\frac{d^3 R(\xi)}{d\xi^3} = -\frac{M \omega^2 L^3}{EI} R(\xi) \quad (1.61)$$

Substituting for ω^2 from equation (1.7), we obtain

$$R''''(\xi) = - \frac{M \beta^4}{\rho AL} R(\xi) \\ = - \frac{\beta^4}{\eta} R(\xi) \Big|_{\xi=1} \quad (1.62)$$

where

$$\eta = \frac{\rho AL}{M}$$

is a non-dimensional system parameter.

For the left hand end, the boundary condition assumes the form,

$$R''''(\xi) \Big|_{\xi=0} = \frac{\beta^4}{\eta} R(\xi) \Big|_{\xi=0} \quad (1.64)$$

(d) Ends with rotational masses:

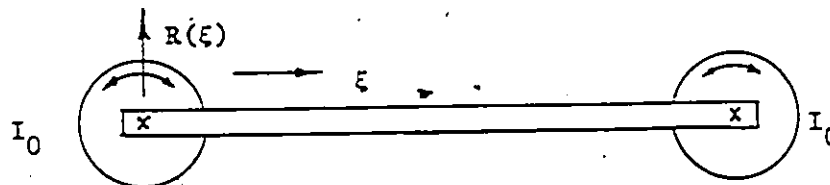


Figure 1.10

The bending moment balance at the right hand end requires

$$EI \frac{\partial^2 y(x,t)}{\partial x^2} = - I_o \frac{\partial^2}{\partial t^2} \left(\frac{\partial y(x,t)}{\partial x} \right) \quad (1.65)$$

Assuming $y(x,t) = R(x) \sin(\omega t + c)$, equation (1.65) reduces to:

$$EI \frac{d^2 R}{dx^2} = - I_o \omega^2 \frac{dR}{dx} \quad (1.66)$$

Non dimensionalizing (1.66) we obtain:

$$\frac{d^2 R}{d\xi^2} = \frac{I_o \omega^2 L}{EI} \frac{dR}{d\xi} \quad (1.67)$$

Substituting for ω^2 from equation (1.7), we obtain

$$\frac{d^2 R}{d\xi^2} = \frac{I_o \beta^4}{\rho AL^3} \frac{dR}{d\xi} \quad (1.68)$$

or

$$R' \Big|_{\xi=1} = \frac{\beta^4}{I^*} R' \Big|_{\xi=1} \quad (1.69)$$

$$\text{where } I^* = \frac{\rho AL^3}{I_o}$$

If the rotational mass is at the left hand end, we have

$$R'' \Big|_{\xi=0} = - \frac{\beta^4}{I^*} R' \Big|_{\xi=0} \quad (1.70)$$

The expressions for non classical boundary conditions, when they occur in conjunction with one another, will be provided as and when they are encountered later in this thesis.

Part 2 - Torsional vibrations of shafts

The following two types of non-classical boundary conditions commonly encountered in torsional vibrations of shafts, are discussed below,

(a) Shaft ends with torsion springs of stiffness K:

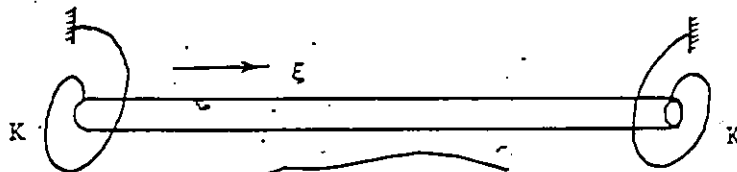


Figure 1.11

For the torsional spring at the right hand end, the torque equilibrium requires

$$G I_p \frac{d\theta}{dx} = -K \theta \quad (1.71)$$

which in the non dimensional form reduces to

$$\frac{d\theta}{d\xi} = -\frac{K L}{G I_p} \theta$$

$$\theta' \Big|_{\xi=1} = -K^* \theta \Big|_{\xi=1} \quad (1.72)$$

For torsional spring at the left hand end, we have

$$\theta' \Big|_{\xi=0} = K^* \theta \Big|_{\xi=0} \quad (1.73)$$

where, $K^* = \frac{K L}{G I_p}$ is a non dimensional system parameter

and I_p = polar moment of inertia of the shaft.

The symbol K^* used in case of torsional vibrations should not be confused with the same symbol used in case of transverse vibrations. The chances of this confusion are avoided by discussing the torsional vibrations of shaft in a separate chapter.

(b) Shaft ends with rotational masses:

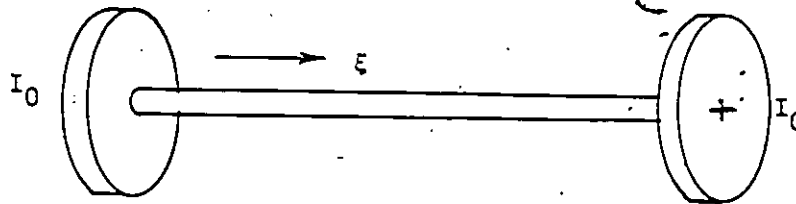


Figure 1.12

For this mass located at the right hand end, the torque equilibrium requires:

$$G I_p \frac{\partial \phi}{\partial x} = -I_0 \frac{\partial^2 \phi}{\partial t^2} \quad (1.74)$$

Assuming $\phi(x,t)$ of the form, $\phi(x,t) = \theta(x) \sin(\omega t + \alpha)$, reduces the equation (1.74) to the following form

$$G I_p \frac{d\theta}{dx} = -I_0 \omega^2 \theta \quad (1.75)$$

Non-dimensionalizing equation (1.75)

$$G I_p \frac{d\theta}{d\xi} = -I_0 \omega^2 L \theta$$

or

$$\frac{d\theta}{d\xi} = \frac{I_o \omega^2 L}{G I_p} \theta \quad (1.76)$$

Substituting for ω^2 from equation (1.42), we obtain

$$\frac{d\theta}{d\xi} = \frac{I_o s^2}{I_p \rho L} \theta$$

or

$$\left. \frac{d\theta}{d\xi} \right|_{\xi=1} = \frac{\beta^2 \theta}{I_o^*} \Big|_{\xi=1} \quad (1.77)$$

where $I_o^* = \frac{I_p \rho L}{I_o}$ is a non dimensional system parameter.

If the rotational mass is at the left hand end, we have:

$$\left. \frac{d\theta}{d\xi} \right|_{\xi=0} = - \frac{\beta^2 \theta}{I_o^*} \Big|_{\xi=0} \quad (1.78)$$

The cases when the non-classical boundary conditions occur in conjunction with one another, will be discussed as and when they are encountered.

CHAPTER 2

SINGLE SPAN UNIFORM BEAMS WITH NON-CLASSICAL

BOUNDARY CONDITIONS

Part 1 - Beams with one non-classical boundary condition

2.1 Beams with one End Clamped and Other Pinned with a Rotational

Mass of Moment of Inertia I_0

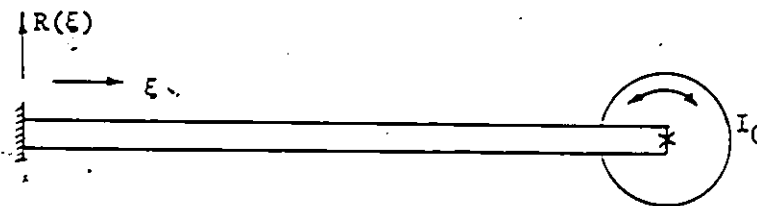


Figure 2.1

The pertinent boundary conditions for this class of beams are

$$R(0) = 0 \quad (2.1)$$

$$R'(0) = 0 \quad (2.2)$$

Substitution of the equations (2.1) and (2.2) in the general solution (1.8) yields $D = -B$ and $C = -A$, so that the shape function for the vibrating beam of this class becomes,

$$R(\xi) = A(\sin \beta \xi - \sinh \beta \xi) + B(\cos \beta \xi - \cosh \beta \xi) \quad (2.3)$$

The remaining two boundary conditions are:

$$R(1) = 0 \quad (2.4)$$

and

$$R''(1) = \frac{\beta^4}{I^*} R'(1) \quad (2.5)$$

Utilizing equations (2.4), (2.5) and (2.3) we obtain the following two homogeneous algebraic equations in terms of constants A and B

$$A(\sin \beta - \sinh \beta) + B(\cos \beta - \cosh \beta) = 0 \quad (2.6)$$

$$A \left[-\sinh \beta - \sin \beta + \frac{\beta^3}{I^*} (\cosh \beta - \cos \beta) \right]$$

$$B \left[(-\cosh \beta - \cos \beta) + \frac{\beta^3}{I^*} (\sin \beta + \sinh \beta) \right] = 0 \quad (2.7)$$

For a non trivial solution of constants A and B in equations (2.6) and (2.7), the determinant of their coefficient matrix must vanish. Therefore,

$$\begin{vmatrix} (\sin \beta - \sinh \beta) & (\cos \beta - \cosh \beta) \\ [-\sinh \beta - \sin \beta + \frac{\beta^3}{I^*} (\cosh \beta - \cos \beta)] & [(-\cosh \beta - \cos \beta) + \frac{\beta^3}{I^*} (\sin \beta + \sinh \beta)] \end{vmatrix} = 0 \quad (2.8)$$

Expanding the determinantal equation (2.8), we obtain:

$$I^* = \frac{\beta^3 (1 - \cos \beta \cosh \beta)}{(\sin \beta \cosh \beta - \sinh \beta \cos \beta)} \quad (2.9)$$

which represents the frequency equation of the system. The equation (2.9) being transcendental in nature will be satisfied by an infinity of discrete sequences of β for a given system parameter I^* . The system parameter I^* which is the ratio of the value ρAL^3 associated with the beam to the mass moment of inertia I_0 of the rotational mass can assume any value between zero and infinity.

In order to present some useful data for the frequency parameter associated with equation (2.9) in the form which can be readily utilized by the practicing engineer, the following approach has been followed.

The two limiting values of the system parameter I^* are zero and infinity. The condition of the system parameter I^* equal to zero is reached when the mass moment of inertia I_0 , of the rotation mass is infinitely large compared to ρAL^3 of the beam. For the first mode, the system will approach a simple spring mass system and its frequency will approach zero for infinite mass moment of inertia. For higher modes, the infinite mass moment of inertia of the rotational mass will make that end behave as a clamped end. On the other extremity when I^* approaches infinity, the beam will approach a clamped-simple boundary condition with mass moment of inertia I_0 of the rotational mass being infinitely smaller than ρAL^3 of the beam.

For any other value of I^* between zero and infinity, the beam will vibrate with the frequency in between those corresponding to these two limiting cases. The two limiting cases for the first mode are:

- (1) Simple spring mass system with $I^* = 0$ (or $I_0 = \infty$)
- (2) Clamped-simple beam with $I^* = \infty$ (or $I_0 = 0$)

For higher modes the two limiting cases are:

- (1) Clamped-clamped beam with $I^* = 0$ (or $I_0 = \infty$)
- (2) Clamped-simple beam with $I^* = \infty$ (or $I_0 = 0$)

The frequency for any value of I^* can therefore be expressed as a function of these two frequencies associated with limiting conditions.

$$f_{(n)} = f_{\text{low}(n)} + Q(f_{\text{upper}(n)} - f_{\text{lower}(n)}) \quad (2.10)$$

where parameter Q , called "Q factor", has been introduced as a measure of how far the beam frequency lies along the two limiting conditions. It can be recognised that Q varies from 0.0 to 1.0 between these limiting conditions. This, in a way is analogous to the quality parameter q used in steam tables.

For the problem in question, the equation (2.10) can be written down as

$$f_{(n)} = Q f_{\text{cs}(n)} \quad \text{for } n = 1 \quad (2.11)$$

and

$$f_{(n)} = f_{\text{cc}(n-1)} + Q[f_{\text{cs}(n)} - f_{\text{cc}(n-1)}] \quad \text{for } n > 1 \quad (2.12)$$

where subscript n denotes mode number and cc denotes clamped-clamped beam etc.

The relationship between Q and β can be established by utilizing equation (1.7)

$$\beta_{(n)} = \beta_{cs(n)} \sqrt{Q}, \quad \text{for } n = 1 \quad (2.13)$$

and

$$\beta_{(n)} = \beta_{cc(n-1)} \sqrt{1 + Q \left[\left(\frac{\beta_{cs(n)}}{\beta_{cc(n-1)}} \right)^2 - 1 \right]} \quad \text{for } n > 1 \quad (2.14)$$

By utilizing equations (2.13), (2.14), and (2.9), relationships between Q and I^* can be established. The procedure followed is essentially that of synthesizing the system characterized by its non-dimensional parameter I^* at regularly spaced intervals of Q between the two limiting cases.

The limiting conditions also become obvious by looking at frequency equation (2.9). For $I^* = 0$, the frequency equation approaches $\beta^3 (1 - \cos \beta \cosh \beta) = 0$ which is associated with a spring mass system for $\beta = 0$ and with a clamped-clamped beam for, $1 - \cos \beta \cosh \beta = 0$. Similarly if $I^* = \infty$, the equation (2.9) approaches $\sin \beta \cosh \beta - \sinh \beta \cos \beta = 0$, associated with clamped-simply supported beam.

By prescribing various values of Q between 0 and 1.0, the values of I^* can be obtained for various modes. The table 2.1 provides the values of I^* for various values of Q between 0.05 and 0.9 for first ten modes of vibration. The number of intervals for Q and the format of tables has been so selected that linear interpretation for values of Q

and I^* not covered by these discrete points will yield results accurate within 1% for the frequencies of the beam. It was found desirable to give a smaller spacing for Q near the origin to be consistent with the accuracy requirement. The figure of 1% for accuracy is considered to be quite good since generally the beam properties are not known to a greater accuracy.

The graphical presentation of these data was abandoned in favour of tabular presentation for it was found that the former was not consistent with the accuracy requirement and furthermore it would have taken a prohibitively larger space. As the plot of Q v/s I^* in Figure 2.1(a) shows that for higher modes, the variation in I^* becomes too high to be represented graphically with a reasonable degree of accuracy, because as Q approaches a value of one, the plots of Q v/s I^* extend farther and farther out in positive I^* direction for higher modes.

For $Q > 0.9$, linear interpolation is obviously not possible. Recognising the fact that as I^* approaches infinity, Q approaches to unity in an asymptotic fashion, the following equation, for values of $Q > 0.8$, is utilized and the analysis of tabulated data indicates that the application of this equation is also consistent with the accuracy requirement of 1%.

$$Q = 1 - \frac{0.2 I^* (0.8)}{I^*} \quad (2.15)$$

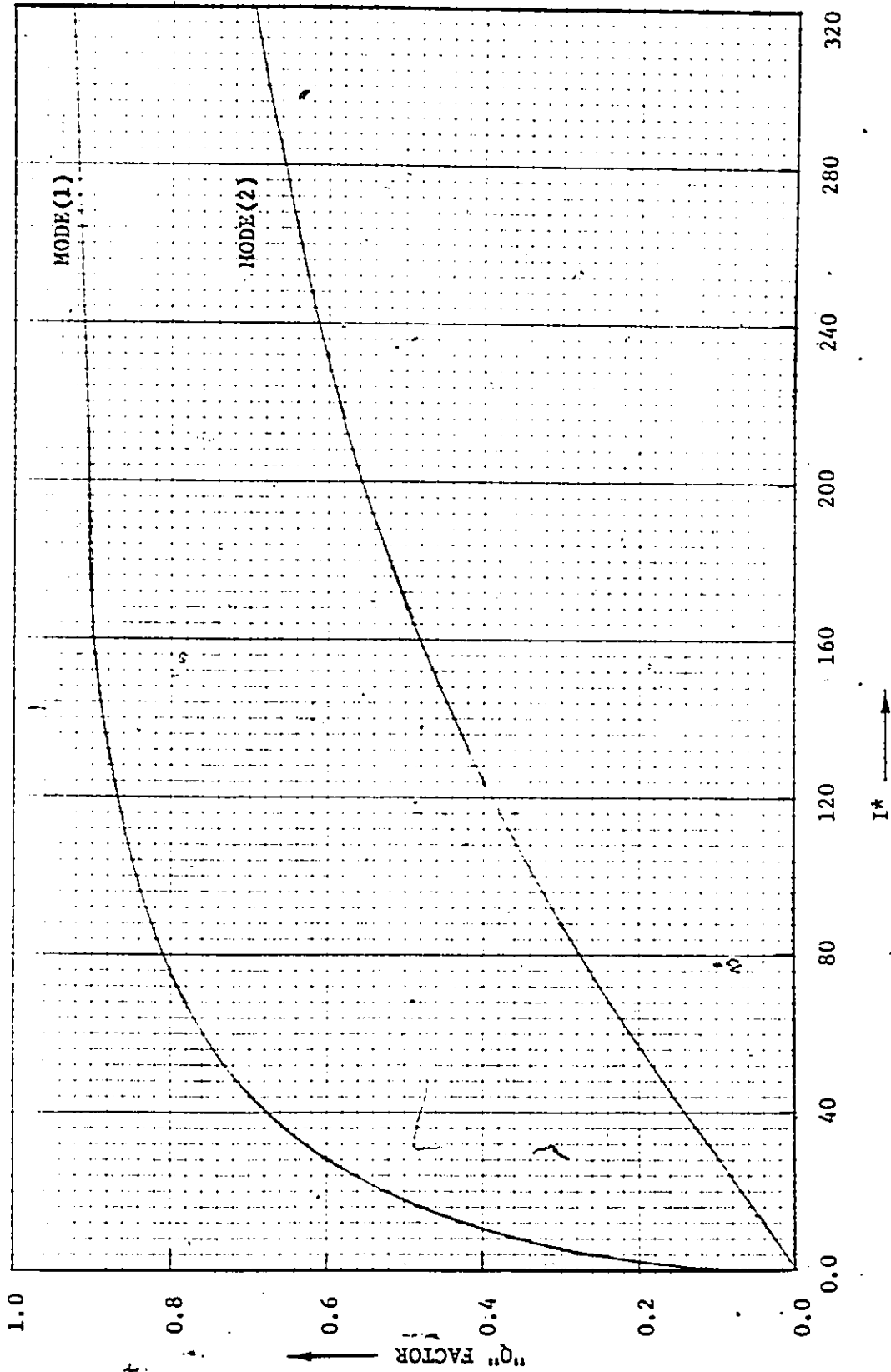


FIGURE : 2.1(a) : A plot of Q versus I^*

where $I^*_{(0.8)}$ is the value of system parameter I^* at $Q = 0.8$ and is found directly from tables, and I^* is the system parameter of interest for which Q is greater than 0.8.

That the utilization of linear interpolation or equation (2.15) is consistent with the accuracy requirement of 1%, was verified by plotting several plots of Q v/s I^* and several accuracy checks were carried out at several specific points.

We next focus our attention to modal shapes. Equation (2.3) gives the modal shape expression in terms of two arbitrary constants A and B . We are at liberty to set any of these constants equal to 1 since the exact magnitude of amplitude will be dictated by initial conditions. Setting the constant $A = 1$, and determining the value of $\frac{B}{A}$ from either of the equations (2.6) and (2.7), we can write the modal shape expression as

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma(\cos \beta \xi - \cosh \beta \xi) \quad (2.16)$$

where

$$\gamma = \frac{B}{A} \frac{\sinh \beta - \sin \beta}{\cos \beta - \cosh \beta} \quad (2.17)$$

which has been obtained from equation (2.6).

The use of Table 2.1 permits the designer to find the value of Q associates with his system parameter I^* or vice-versa. Then the equations (2.11) or (2.12) and (2.16) permit him to establish the natural frequencies and modal shapes of the system since the values of frequency parameter associated with limiting cases are available in existing literature. For the sake of completeness, these values of frequency parameter β for the limiting cases which correspond to classical boundary conditions are included in Table A-1 of the Appendix A. The use of those tables and the method of approach is demonstrated by working out some illustrative examples at the end of each chapter.

2.2 Beams Simply Supported at One End and Carrying a Rotational Mass at the Other

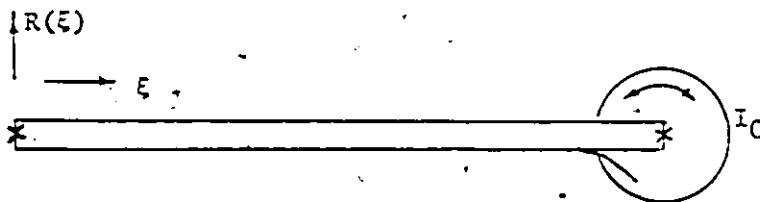


Figure 2.2

The two classical boundary conditions associated with the

two limiting values of system parameter I^* are:

- (1) A simple spring mass system for the first mode or a clamped-simply supported beam for higher modes with $I^* = 0$, and
- (2) A simple-simple beam with $I^* = \infty$.

Following the procedure similar to case 2.1, the frequency of this class of beams can be expressed as:

$$f_{(n)} = Q f_{ss(n)} \quad \text{for } n = 1 \quad (2.18)$$

and

$$f_{(n)} = f_{cs(n-1)} + Q[f_{ss(n)} - f_{cs(n-1)}] \quad \text{for } n > 1 \quad (2.19)$$

Use of equation (1.7) transforms equations (2.18) and (2.19) to:

$$\beta_{(n)} = \beta_{ss(n)} \sqrt{Q} \quad \text{for } n = 1 \quad (2.20)$$

$$\beta_{(n)} = \beta_{cs(n-1)} \sqrt{1 + Q \left[\left(\frac{\beta_{ss(n)}}{\beta_{cs(n-1)}} \right)^2 - 1 \right]} \quad \text{for } n > 1 \quad (2.21)$$

The two boundary conditions at the end $\xi = 0$, namely $R(0) = R'(0) = 0$, yield $\beta = D = 0$, so that the shape function of the beam can be written down as

$$R(\xi) = A \sin \beta \xi + C \sinh \beta \xi \quad (2.22)$$

The remaining two boundary conditions are supplied by the end $\xi = 1$; viz:

$$R(1) = 0 \quad (2.23)$$

and

$$R''(1) = \frac{\beta^4}{I^*} R'(1) \quad (2.24)$$

Substitution of equations (2.22) into equations (2.23) and (2.24) yields the following two homogeneous algebraic equations for the constants A and C.

$$A \sin \beta + C \sinh \beta = 0 \quad (2.25)$$

$$A \left[\frac{\beta^3}{I^*} \cos \beta + \sin \beta \right] + C \left[\frac{\beta^3}{I^*} \cosh \beta - \sinh \beta \right] = 0 \quad (2.26)$$

Equating the determinant of the coefficient matrix of constants A and C in equations (2.25) and (2.26) to zero, we obtain the following frequency equation for the beam in hand.

$$I^* = \frac{\beta^3 [\sin \beta \cosh \beta - \cos \beta \sinh \beta]}{2 \sin \beta \sinh \beta} \quad (2.27)$$

The relationship between Q and I^* is established by using equations (2.20) or (2.21) and equation (2.27). Table 2.2 provides the values of I^* for various values of Q for first ten modes of vibrations.

The modal shape expression is determined by utilizing equations (2.22) and (2.25) and is presented below.

$$R(\xi) = \sin \beta \xi + \gamma \sinh \beta \xi \quad (2.28)$$

where

$$\gamma = - \frac{\sin \beta}{\sinh \beta} \quad (2.29)$$

Table 2.2 together with equations (2.20), (2.21), (2.28) and (2.29) gives complete information for the vibration frequencies and modal shapes of this class of beams.

2.3 Beams Free at One End and With a Rotational Mass at the Other

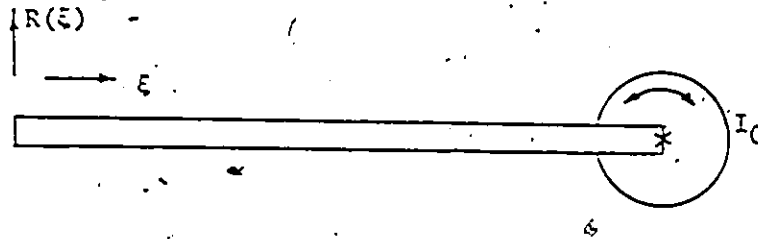


Figure 2.3

The enforcement of boundary conditions at the free end, namely $R''(0) = R'''(0) = 0$, in the general solution (1.8) results in $A = C$ and $B = D$ and the shape function assumes the following form.

$$R(\xi) = A (\sin \beta \xi + \sinh \beta \xi) + B (\cos \beta \xi + \cosh \beta \xi) \quad (2.30)$$

The remaining two boundary conditions, viz, $R(1) = 0$ and $R''(1) = \frac{\beta^4}{I^*} R'(1)$, yield the following two equations in constants A and B

$$A (\sin \beta + \sinh \beta) + B (\cos \beta + \cosh \beta) = 0 \quad (2.31)$$

$$A \left[\sinh \beta - \sin \beta - \frac{\beta^3}{I^*} (\cos \beta + \cosh \beta) \right]$$

$$+ B \left[\cosh \beta - \cos \beta + \frac{\beta^3}{I^*} (\sin \beta - \sinh \beta) \right] = 0 \quad (2.32)$$

The condition for a non-trivial solution for constants A and B in equations (2.31) and (2.32) yields the following frequency equation:

$$I^* = \frac{\beta^3 (1 + \cos \beta \cosh \beta)}{\sinh \beta \cos \beta - \sin \beta \cosh \beta} \quad (2.33)$$

Recognising the limiting cases, one can write,

$$\beta_{(n)} = \beta_{sf(n)} \sqrt{Q} \quad \text{for } n = 1 \quad (2.34)$$

and

$$\beta_{(n)} = \beta_{cf(n-1)} \sqrt{1 + Q \left[\left(\frac{\beta_{sf(n)}}{\beta_{cf(n-1)}} \right)^2 - 1 \right]} \quad \text{for } n > 1 \quad (2.35)$$

Utilizing equations (2.33) and (2.34) or (2.35), the values of I^* v/s Q are presented in Table 2.3 for first ten modes of vibrations.

The modal shape expression is arrived at by using equations (2.30) and (2.31)

$$R(\xi) = (\sin \beta \xi + \sinh \beta \xi) + \gamma (\cos \beta \xi + \cosh \beta \xi) \quad (2.36)$$

where

$$\gamma = - \frac{\sin \beta + \sinh \beta}{\cos \beta + \cosh \beta} \quad (2.37)$$

Table 2.3 together with equations (2.34), (2.35), (2.36), and (2.37) furnishes complete information for the vibration frequencies and modal shapes of this class of beams.

2.4 Beams Simply Supported at One End and With a Concentrated Mass at the Other

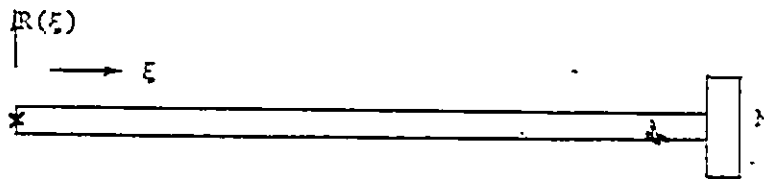


Figure 2.4

The requirement of zero displacement and bending moment at the beam end $\xi = 0$, results in the following displacement function of the beam.

$$R(\xi) = A \sin \beta \xi + C \sinh \beta \xi \quad (2.38)$$

At the other end of the beam where $\xi = 1$, we have:

$$R''(1) = 0 \quad (2.39)$$

$$R'''(1) = -\frac{\beta^4}{\eta} R(1) \quad (2.40)$$

Substitution of equation (2.38) into equations (2.39) and (2.40) yields the following two equations in terms of constants A and C

$$A(-\sin \beta) + C(\sinh \beta) = 0 \quad (2.41)$$

$$A\left(\frac{\beta}{\eta} \sin \beta - \cos \beta\right) + C\left(\frac{\beta}{\eta} \sinh \beta + \cosh \beta\right) = 0 \quad (2.42)$$

Equating the determinant of the coefficient matrix of constants A and C in equations (2.41) and (2.42) to zero, we obtain:

$$\eta = \frac{2\beta \sin \beta \sinh \beta}{\cos \beta \sinh \beta - \sin \beta \cosh \beta} \quad (2.43)$$

which represents the frequency equation of the system. Recognizing the two limiting cases of this class of beams as system parameter η varies from zero to infinity, the system frequency f can be expressed as

$$f_{(n)} = f_{ss(n)} + Q(f_{sf(n+1)} - f_{ss(n)}) \quad (2.44)$$

Utilizing the equation (1.7), (2.44) and (2.43), the relationship between Q and η can be established. Table 2.4 provides the necessary information for the first ten modes of vibrations of this class of beams.

The modal shape expression arrived at by using equations (2.38) and (2.41) is given below:

$$R(\xi) = \sin \beta \xi + \gamma \sinh \beta \xi \quad (2.45)$$

where

$$\gamma = \sin \beta / \sinh \beta \quad (2.46)$$

2.5 Beams with One End Free and the Other Carrying a Concentrated Mass

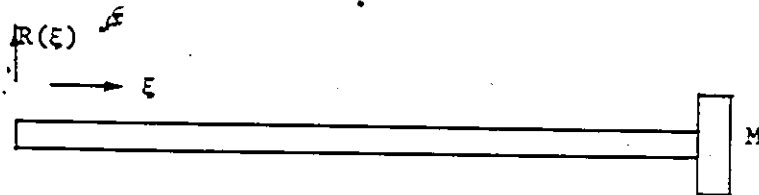


Figure 2.5

Since the free end $\xi = 0$, cannot sustain any bending moment or shear force, the shape function of the beam can be written down as:

$$R(\xi) = A(\sin \beta \xi + \sinh \beta \xi) + B(\cos \beta \xi + \cosh \beta \xi) \quad (2.47)$$

The other two boundary conditions at the end $\xi = 1$ are

$$R''(1) = 0 \quad (2.48)$$

$$R'''(1) = -\frac{\beta^4}{\eta} R(1) \quad (2.49)$$

Substitution of equation (2.47) into equations (2.48) and (2.49) yields:

$$A(-\sin \beta + \sinh \beta) + B(-\cos \beta + \cosh \beta) = 0 \quad (2.50)$$

$$A \left[-\cos \beta + \cosh \beta + \frac{\beta}{\eta} (\sin \beta + \sinh \beta) \right] + B$$

$$\left[\sin \beta + \sinh \beta + \frac{\beta}{\eta} (\cos \beta + \cosh \beta) \right] = 0 \quad (2.51)$$

The requirement of a non-trivial solution for the constants in equations (2.50) and (2.51) yields the following frequency equation of the system

$$\eta = \frac{\beta (\sinh \beta \cos \beta - \cosh \beta \sin \beta)}{1 - \cos \beta \cosh \beta} \quad (2.52)$$

In the limiting cases, this beam approaches to a simply

supported-free beam with $\eta \rightarrow 0$ and a free-free beam with $\eta \rightarrow \infty$. The first mode frequency for this class of beams is obviously zero. The frequency of such beams can therefore be expressed as

$$f_1 = 0 \quad (2.53)$$

and

$$f_{(n)} = f_{sf(n)} + Q[f_{ff(n)} - f_{sf(n)}] \quad \text{where } n > 1 \quad (2.54)$$

Recognising the fact that frequencies f are proportional to the square of eigenvalues β , the equation (2.54) establishes the relationship between β and Q . The use of equation (2.52) permits evaluation of the system parameter η for various values of Q . The computed values of η for various values of Q are presented in Table 2.5

Equations (2.47) and (2.50) permits the following expression for the modal shape.

$$R(\xi) = \sin \beta \xi + \sinh \beta \xi + \gamma(\cos \beta \xi + \cosh \beta \xi) \quad (2.55)$$

where

$$\gamma = \frac{\sin \beta - \sinh \beta}{\cosh \beta - \cos \beta} \quad (2.56)$$

Part 2 - Single span uniform beams with one compound non-classical boundary condition

In practice, this class of problems arise where there are two types of elastic restraints at one end of the beam. The examples are ends carrying concentrated masses supported on coil springs, torsional springs connected rotational masses or combinations of torsional and linear coil springs. The cases 2.6 through 2.13 deal with such kinds of non-classical boundary conditions. It may be recalled that in all the cases considered in Part 1 of this Chapter, the beam frequency was completely characterized by a single system parameter. This was quite a fortunate situation since all the pertinent information about frequencies could be presented in a single table. But this no longer remains possible now as the system frequency, with the existence of a compound non-classical boundary condition, is found to be a function of two system parameters. Therefore a set of tables is required to present all the data associated with any particular case of this type.

2.6 Beams Fixed at One End and Carrying a Concentrated Mass M Supported by a Coil Spring of Stiffness K



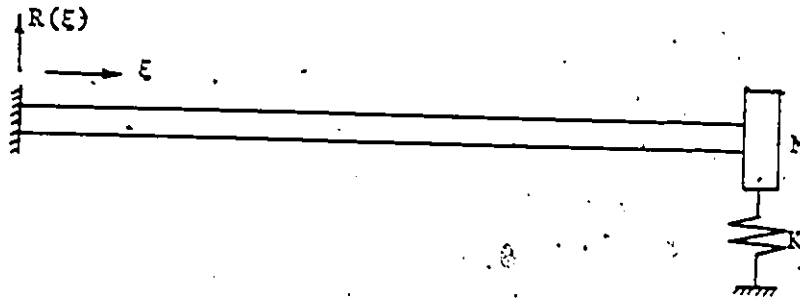


Figure 2.6

No displacement and slope condition at the fixed end $\xi = 0$ yields the following equation for the displacement curve of the vibrating beam.

$$R(\xi) = A(\sin \beta \xi - \sinh \beta \xi) + B(\cos \beta \xi - \cosh \beta \xi) \quad (2.57)$$

Two boundary conditions at the end $\xi = 1$, are:

$$R''(1) = 0 \quad (2.58)$$

$$R'''(1) = \left(-\frac{\beta^4}{\eta} + K^*\right) R(1) \quad (2.59)$$

Substituting equation (2.57) into equations (2.58) and (2.59), we obtain:

$$A(\sin \beta + \sinh \beta) + B(\cos \beta + \cosh \beta) = 0 \quad (2.60)$$

and

$$A \left[-\cos \beta - \cosh \beta - \left(\frac{K^*}{\beta^3} - \frac{\beta}{\eta} \right) (\sin \beta - \sinh \beta) \right] \\ + B \left[\sin \beta - \sinh \beta - \left(\frac{K^*}{\beta^3} - \frac{\beta}{\eta} \right) (\cos \beta - \cosh \beta) \right] = 0 \quad (2.61)$$

For a non-trivial solution for constants A and B in equations (2.60) and (2.61), the determinant of their coefficient matrix must vanish. Equating this determinant to zero yields the following equation relating frequency parameter β to system parameters K^* and η

$$\left(\frac{K^*}{\beta^3} - \frac{\beta}{\eta} \right) (\sinh \beta \cos \beta - \sin \beta \cosh \beta) - \cos \beta \cosh \beta - 1 = 0 \quad (2.62)$$

Introducing another non-dimensional parameter α such that

$$\alpha = \frac{\eta}{K^*} \quad (2.63)$$

The equation (2.62) is transformed into the following quadratic equation in K^*

$$K^{*2} + b K^* + c \quad (2.64)$$

where

$$b = \frac{-\beta^3 (1 + \cos \beta \cosh \beta)}{\sinh \beta \cos \beta - \sin \beta \cosh \beta} \quad (2.65)$$

$$c = -\frac{\beta^4}{\alpha} \quad (2.66)$$

It should be noticed that equation (2.64) is a function of the frequency parameter β and the dimensionless constant α . In order to synthesize the system characterized by K^* it is necessary to prescribe α and β . The values of α are prescribed from 0.1 to 1.0 in the intervals of 0.1 and then values of $\frac{1}{\alpha}$ are varied from 0.1 to 0.9 in the intervals of 0.1. So the range of the variation of α are, $0.1 \leq \alpha \leq 1.0$ and $0.1 \leq \frac{1}{\alpha} \leq 0.9$. Although in theory α can assume any value between zero and infinity, it is expected that most of the practical applications would be covered by the prescribed range of α . The values of β are prescribed at regularly spaced intervals between the two limiting conditions. The frequency of this class of beams can be expressed in terms of their limiting frequencies as follows:

$$f_{(n)} = Q f_{cs(n)} \quad \text{for } n = 1 \quad (2.67)$$

$$f_{(n)} = f_{cs(n-1)} + Q [f_{cs(n)} - f_{cs(n-1)}] \quad \text{for } n > 1 \quad (2.68)$$

It may be remembered that equations (2.67) and (2.68) also express the relationship between Q and β implicitly. For prescribed values of Q and α , the quadratic equation (2.64) yields two roots for K^* , one of them being negative is discarded because of its no physical significance. Tables 2.6 provide values of K^* for various values of Q ($0.05 \leq Q \leq 0.9$) and α [$0.1 \leq \alpha \leq 1.0$ and $0.1 \leq 1/\alpha \leq 0.9$].

The modal shape expression is derived by utilizing equations (2.57) and (2.60) as follows:

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma (\cos \beta \xi - \cosh \beta \xi) \quad (2.69)$$

where

$$\gamma = \frac{-(\sin \beta + \sinh \beta)}{\cos \beta + \cosh \beta} \quad (2.70)$$

2.7 Beams Pinned at One End, Carrying a Spring Supported Concentrated Mass at the Other

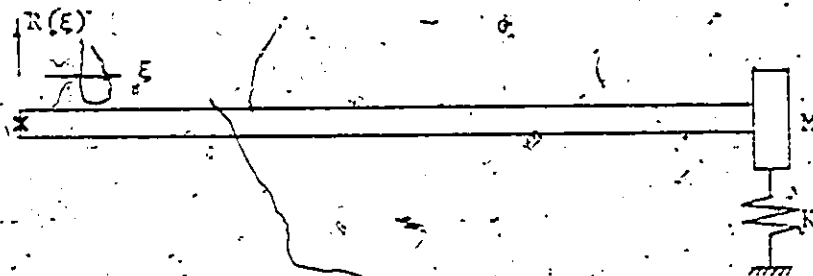


Figure 2.7

The enforcement of boundary conditions $R(0) = R'(0) = 0$ in the general shape function (1.8) yields,

$$R(\xi) = A \sin \beta \xi + C \sinh \beta \xi \quad (2.71)$$

The other two boundary conditions are supplied by the end $\xi = 1$, and they are

$$R''(1) = 0 \quad (2.72)$$

$$R'''(1) = \left(-\frac{\beta^4}{\eta} + K^*\right) R(1) \quad (2.73)$$

Utilizing equations (2.71), (2.72), and (2.73), we obtain:

$$A(-\sin \beta) + C(\sinh \beta) = 0 \quad (2.74)$$

and

$$A\left[-\cos \beta - \left(\frac{K^*}{\beta^3} - \frac{\beta}{\eta}\right) \sin \beta\right] + C\left[\cosh \beta - \left(\frac{K^*}{\beta^3} - \frac{\beta}{\eta}\right) \sinh \beta\right] = 0 \quad (2.75)$$

Equating the determinant of the coefficient matrix of constants A and C in equations (2.74) and (2.75) to zero, yields the following quadratic equation in K^* .

$$K^{*2} + b K^* + c = 0 \quad (2.76)$$

where

$$b = \frac{\beta^3 (\cos \beta \sinh \beta - \sin \beta \cosh \beta)}{2 \sin \beta \sinh \beta} \quad (2.77)$$

$$c = -\frac{\beta^4}{\alpha}$$

and $\alpha = \frac{\eta}{K^*}$ has been introduced as in the previous case.

The frequency in terms of Q factor is expressed as

$$f_{(n)} = Q f_{ss(n)} \quad \text{for } n = 1 \quad (2.78)$$

and

$$f_{(n)} = f_{ss(n-1)} + Q[f_{ss(n)} - f_{ss(n-1)}] \quad \text{for } n > 1 \quad (2.79)$$

Tables 2.7 furnish values of K^* for various values of Q

and α .

The modal shape expression is obtained by utilizing equations (2.71) and (2.74).

$$R(\xi) = \sin \beta \xi + \gamma \sinh \beta \xi \quad (2.80)$$

where

$$\gamma = \frac{\sin \beta}{\sinh \beta} \quad (2.81)$$

2.8 Beam Pinned With a Rotational Mass and a Torsional Spring at One End and Clamped at the Other

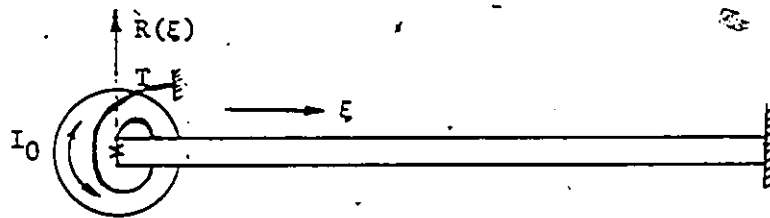


Figure 2.8

The requirement of zero displacement at the end $\xi = 0$, yields $B = -D$ in the general shape function (1.8) and we obtain:

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.82)$$

From dynamic equilibrium at the end $\xi = 0$, we obtain:

$$R''(0) = \left(-\frac{\beta^4}{I^*} + T^*\right) R'(0) \quad (2.83)$$

Substituting equation (2.82) into (2.83), and simplifying we obtain:

$$C = \frac{-2\beta B}{T^* - \frac{\beta^4}{I^*}} - A \quad (2.84)$$

Enforcement of remaining two boundary conditions viz $R(1) = R'(1) = 0$ and utilizing equation (2.84), we obtain:

$$A(\sin \beta - \sinh \beta) + B\left[\cos \beta - \cosh \beta - \frac{2\beta \sinh \beta}{\left(T^* - \frac{\beta^4}{I^*}\right)}\right] = 0 \quad (2.85)$$

and

$$A(\cos \beta - \cosh \beta) - B\left[\sin \beta + \sinh \beta + \frac{2\beta \cosh \beta}{\left(T^* - \frac{\beta^4}{I^*}\right)}\right] = 0 \quad (2.86)$$

For a non-trivial solution for constants A and B in equations (2.85) and (2.86), the determinant of their coefficient matrix

must vanish. Equating this determinant to zero and simplifying yields the following quadratic equation in T^*

$$T^{*2} + b T^* + c = 0 \quad (2.87)$$

where

$$b = \frac{\beta(\sin \beta \cosh \beta - \cos \beta \sinh \beta)}{(1 - \cos \beta \cosh \beta)}$$

$$c = -\frac{\beta^4}{\alpha}$$

$$\alpha = \frac{I^*}{T^*}$$

We next look for the two limiting cases and it can be easily recognised that the frequency equation in terms of Q factor can be written down as follows:

$$f_{(n)} = Q f_{cc(n)} \quad \text{for } n = 1 \quad (2.88)$$

$$f_{(n)} = f_{cc(n-1)} + Q[f_{cc(n)} - f_{cc(n-1)}] \quad \text{for } n > 1 \quad (2.89)$$

The equation (2.87) being quadratic in T^* yields two values of T^* for every given value of Q and α , one of which being negative is discarded. The information necessary to establish frequencies of this class of beams is provided in Tables 2.8.

The modal shape expression can be written down from equation (2.82) by setting the one of the arbitrary constant, say A , equal to unity

$$R(\xi) = \sin \beta \xi + \frac{B}{A} (\cos \beta \xi - \cosh \beta \xi) + \frac{C}{A} \sinh \beta \xi \quad (2.90)$$

The ratios of constants $\frac{B}{A}$ and $\frac{C}{A}$ is determined from equations (2.84) and (2.85) as follows:

$$\frac{B}{A} = \frac{(\sinh \beta - \sin \beta)}{[\cos \beta - \cosh \beta - \frac{2\beta \sinh \beta}{(T^* - \frac{\beta^4}{I^*})}] \quad (2.91)$$

and

$$\frac{C}{A} = \frac{-2\beta}{T^* - \frac{\beta^4}{I^*}} \frac{B}{A} - 1$$

Equation (2.90) is further simplified to

$$R(\xi) = (\sin \beta \xi - \sinh \beta \xi) + \frac{B}{A} \left[\cos \beta \xi - \cosh \beta \xi - \frac{2\beta \sinh \beta \xi}{(T^* - \frac{\beta^4}{I^*})} \right] \quad (2.92)$$

2.9 Beams With a Rotational Mass and Torsional Spring at One End and Simply Supported at the Other

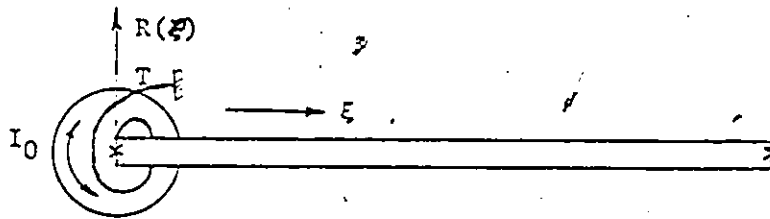


Figure 2.9

The boundary condition $R(0) = 0$ results in the following shape function of the beam

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.93)$$

From dynamic equilibrium at the end $\xi = 0$,

$$R''(0) = \left(-\frac{\beta^4}{I^*} + T^*\right) R'(0) \quad (2.94)$$

Utilizing equations (2.93) and (2.94), we obtain:

$$C \frac{-2\beta B}{\left(T^* - \frac{\beta^4}{I^*}\right)} = A \quad (2.95)$$

Enforcing the remaining two boundary conditions, viz $R(1) = R'(1) = 0$ at the end $\xi = 1$, and utilizing equation (2.95), the following two homogeneous algebraic equations in terms of constants A and B are obtained.

$$A(\sin \beta - \sinh \beta) + B\left[\cos \beta - \cosh \beta - \frac{2\beta \sinh \beta}{\left(T^* - \frac{\beta^4}{I^*}\right)}\right] = 0 \quad (2.96)$$

and

$$A(-\sin \beta - \sinh \beta) - B\left[\cos \beta + \cosh \beta + \frac{2\beta \sinh \beta}{\left(T^* - \frac{\beta^4}{I^*}\right)}\right] = 0 \quad (2.97)$$

Simplifying the determinantal equation and introducing

$\alpha = \frac{I^*}{T^*}$, we obtain the following quadratic equation in T^*

$$T^{*2} + b T^* + c = 0 \quad (2.98)$$

where

$$b = \frac{2\beta \sin \beta \sinh \beta}{\sin \beta \cosh \beta - \sinh \beta \cos \beta} \quad (2.99)$$

and

$$c = -\frac{\beta^4}{\alpha} \quad (2.100)$$

Recognizing the limits, one can write

$$f_{(n)} = Q f_{cs(n)} \quad \text{for } n = 1 \quad (2.101)$$

and

$$f_{(n)} = f_{cs(n-1)} + Q[f_{cs(n)} - f_{cs(n-1)}] \quad \text{for } n > 1 \quad (2.102)$$

As before, one root of equation (2.98) being negative is discarded. Tables 2.9 furnish value of T^* for various values of Q and α .

By utilizing equations (2.93), (2.95) and (2.96), the modal shape expression is simplified to

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma \left[\cos \beta \xi - \cosh \beta \xi - \frac{2\beta}{\left(T^* - \frac{\beta^4}{I^*}\right)} \sinh \beta \xi \right] \quad (2.103)$$

where

$$\gamma = \frac{\sinh \beta - \sin \beta}{\cos \beta - \cosh \beta - \frac{2\beta \sinh \beta}{\left(T^* - \frac{\beta^4}{I^*}\right)}} \quad (2.104)$$

2.10 Beams with a Rotational Mass and Torsional Spring at One End and Free at the Other

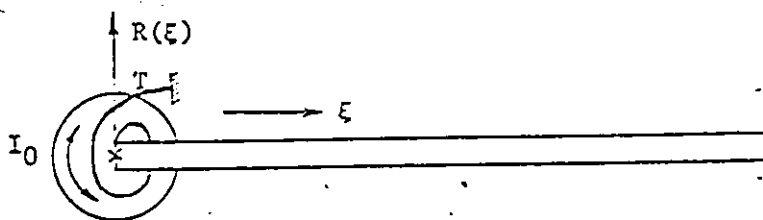


Figure 2.10

The shape function of the beam assumes the following form after utilizing boundary condition $R(0) = 0$.

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.105)$$

As in the previous case, the condition of dynamic equilibrium at $\xi = 0$ yields

$$C = \frac{-2\beta B}{(T^* - \frac{\beta^4}{I^*})} - A \quad (2.106)$$

Next, we utilize the two boundary conditions at the beam end $\xi = 1$, viz, $R''(1) = R'''(1) = 0$ to obtain the following two homogeneous algebraic equations in terms of constants A and B.

$$A(\sin \beta + \sinh \beta) + B(\cos \beta + \cosh \beta + \frac{2\beta}{(T^* - \frac{\beta^4}{I^*})} \sinh \beta) = 0 \quad (2.107)$$

and

$$A(-\cos \beta - \cosh \beta) + B[\sin \beta - \sinh \beta - \frac{2\beta \cosh \beta}{(T^* - \frac{\beta^4}{I^*})}] = 0 \quad (2.108)$$

Equating the determinant of coefficient matrix of constants A and B in equations (2.107) and (2.108) to zero and introducing $\alpha = \frac{I^*}{T^*}$, we obtain the following quadratic equation in T^*

$$T^{*2} + b T^* + c = 0 \quad (2.109)$$

where

$$b = \frac{\beta(\cos \beta \sinh \beta - \sin \beta \cosh \beta)}{(1 + \cos \beta \cosh \beta)} \quad (2.110)$$

$$c = -\frac{\beta^4}{\alpha} \quad (2.111)$$

Recognizing the limiting conditions, we may write the frequency of this system as a function of Q factor as

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1 \quad (2.112)$$

and

$$f_{(n)} = f_{cf(n-1)} + Q[f_{cf(n)} - f_{cf(n-1)}] \quad \text{for } n > 1 \quad (2.113)$$

Table 2.10 presents the values of system parameter T^* for various values of Q and α . Utilizing expressions (2.105), (2.106), and (2.107), the modal shape function can be expressed in the following simplified form

$$R(\xi) = (\sin \beta \xi - \sinh \beta \xi) + \gamma \left[\cos \beta \xi - \cosh \beta \xi - \frac{2\beta}{\left(T^* - \frac{\beta^4}{I^*}\right)} \sinh \beta \xi \right] \quad (2.114)$$

where

$$\gamma = \frac{-(\sin \beta + \sinh \beta)}{\left[\cos \beta + \cosh \beta + \frac{2\beta}{\left(T^* - \frac{\beta^4}{I^*}\right)} \sinh \beta \right]} \quad (2.115)$$

2.11 Beams Clamped at One End and With a Torsional and a Linear Coil Spring Support at the Other

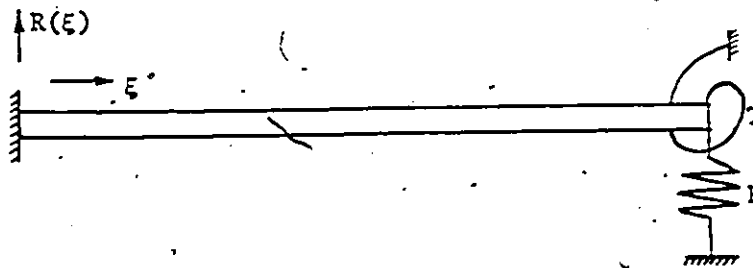


Figure 2.11

By using the two classical boundary conditions at the clamped end, the shape function of the vibrating beam is obtained in terms of remaining two constants as:

$$R(\xi) = A(\sin \beta \xi - \sinh \beta \xi) + B(\cos \beta \xi - \cosh \beta \xi) \quad (2.116)$$

The other two boundary conditions are supplied by the end $\xi = 1$ in the form of bending moment and shear force equilibrium

$$R''(1) = -T^* R'(1) \quad (2.117)$$

and

$$R'''(1) = K^* R(1) \quad (2.118)$$

Utilizing equations (2.116), (2.117), and (2.118), we obtain the following two algebraic homogeneous equations in terms of constants A and B.

$$\begin{aligned} & A\left[\sin \beta + \sinh \beta - \frac{T^*}{\beta} (\cos \beta - \cosh \beta)\right] \\ & + B\left[\cosh \beta + \cos \beta + \frac{T^*}{\beta} (\sin \beta + \sinh \beta)\right] = 0 \end{aligned} \quad (2.119)$$

and

$$A[-\cos \beta - \cosh \beta + \frac{K^*}{\beta^3} (\sinh \beta - \sin \beta)] + B[\sin \beta - \sinh \beta + \frac{K^*}{\beta^3} (\cosh \beta - \cos \beta)] = 0 \quad (2.120)$$

For a non trivial solution for constants A and B, the determinant of their coefficient matrix in the above two equations is set equal to zero. This yields a quadratic equation in K^*

$$K^{*2} + b K^* + c = 0 \quad (2.121)$$

where

$$b = \frac{[(-\frac{\beta}{c})(\sinh \beta \cos \beta - \sin \beta \cosh \beta) + \beta^3(\cos \beta \sinh \beta + \cosh \beta \sin \beta)]}{1 - \cos \beta \cosh \beta} \quad (2.122)$$

and

$$c = -\frac{\beta^4}{c} \left[\frac{1 + \cos \beta \cosh \beta}{\cosh \beta \cos \beta - 1} \right] \quad (2.123)$$

where $c = \frac{T^*}{K}$ has been introduced as another non-dimensional parameter.

In terms of Q factor, the frequency of this class of beams can be expressed as

$$f_{(n)} = f_{cf(n)} + Q[f_{cc(n)} - f_{cf(n)}] \quad (2.124)$$

The quadratic equation (2.121) can be solved for various values of Q and α and yields two values of K^* , one of them being negative is discarded. Tables 2.11 furnish all the necessary information for establishing the frequencies of this class of beams.

The modal shape expression is obtained by using equations (2.116) and (2.120) as,

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi \quad \gamma (\cos \beta \xi - \cosh \beta \xi) \quad (2.125)$$

where

$$\gamma = \frac{[\cos \beta + \cosh \beta + \frac{K^*}{\beta^3} (\sin \beta - \sinh \beta)]}{[\sin \beta - \sinh \beta + \frac{K^*}{\beta^3} (\cosh \beta - \cos \beta)]} \quad (2.126)$$

2.12 Beams with One End Pinned and Other Having a Torsional and a Linear Coil Spring Support

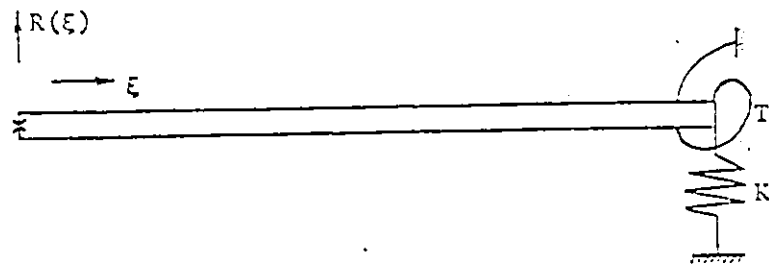


Figure 2.12

Using the requirement of zero displacement and zero moment at the end $\xi = 0$, we obtain:

$$R(\xi) = A \sin \beta \xi + C \sinh \beta \xi \quad (2.127)$$

The other two boundary conditions are

$$R''(1) = -T^* R'(1)$$

and

$$R'''(1) = K^* R(1)$$

Utilizing above two boundary conditions and equation (2.127)

we obtain:

$$A[-\sin \beta + \frac{T^*}{\beta} \cos \beta] + C[\sinh \beta + \frac{T^*}{\beta} \cosh \beta] = 0 \quad (2.128)$$

and

$$A[-\cos \beta - \frac{K^*}{\beta^3} \sin \beta] + C[\cosh \beta - \frac{K^*}{\beta^3} \sinh \beta] = 0 \quad (2.129)$$

Introducing $\alpha = \frac{T^*}{K^*}$ and equating the determinant of the above two equations to zero, we obtain:

$$K^{*2} + b K^* + c = 0 \quad (2.130)$$

where

$$b = \frac{[-\frac{2\beta}{\alpha} (\sin \beta \sinh \beta) - 2\beta^3 \cos \beta \cosh \beta]}{\cos \beta \sinh \beta - \sin \beta \cosh \beta} \quad (2.131)$$

$$c = \frac{-\beta^4}{\alpha} \quad (2.132)$$

The frequency of this class of beams in terms of Q factors is given by

$$f_{(n)} = f_{sf(n)} + Q[f_{cs(n)} - f_{sf(n)}] \quad (2.133)$$

Tables 2.12 present the computed values of K^* for various values of α and Q . It may be remembered that the negative root of the quadratic equation (2.130) is discarded.

The modal shape expression is arrived at by utilizing equations (2.127) and (2.129) as

$$R(\xi) = \sin \beta \xi + \gamma \sinh \beta \xi \quad (2.134)$$

where

$$\gamma = \frac{\cos \beta + \frac{K^*}{\beta^3} \sin \beta}{\cosh \beta - \frac{K^*}{\beta^3} \sinh \beta} \quad (2.135)$$

2.13 Beams with One End Free and Other Having a Torsional and a Linear Coil Spring Support

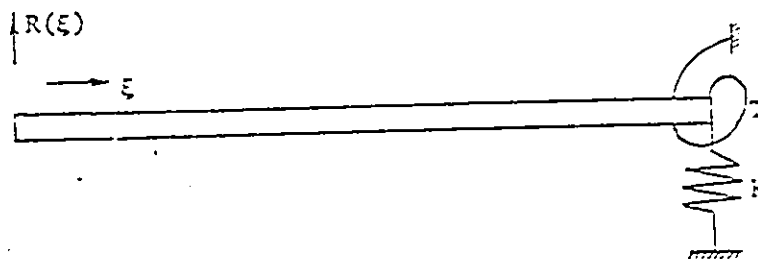


Figure 2.13

Enforcement of two classical boundary conditions at the free end yields:

$$R(\xi) = A(\sin \beta \xi + \sinh \beta \xi) + B(\cos \beta \xi + \cosh \beta \xi) \quad (2.136)$$

The other two boundary conditions at the end $\xi = 1$, are:

$$R''(1) = -T^* R'(1)$$

and

$$R'''(1) = K^* R(1)$$

Application of these two boundary conditions yield:

$$A\left[\sinh \beta - \sin \beta + \frac{T^*}{\beta} (\cos \beta + \cosh \beta)\right] + B\left[\cosh \beta - \cos \beta + \frac{T^*}{\beta}\right]$$

$$(\sinh \beta - \sin \beta) = 0 \quad (2.137)$$

and

$$A\left[\cosh \beta - \cos \beta - \frac{K^*}{\beta^3} (\sin \beta + \sinh \beta)\right] + B\left[\sinh \beta + \sin \beta - \frac{K^*}{\beta^3}\right]$$

$$(\cosh \beta + \cos \beta) = 0 \quad (2.138)$$

Letting $\alpha = \frac{T^*}{K^*}$ and setting the determinant of the above two equations to zero, we obtain the following quadratic equation in K^* .

$$K^{*2} + b K^* + c = 0 \quad (2.139)$$

where

$$b = \frac{\left(\frac{\beta}{\alpha}\right) (\sinh \beta \cos \beta - \cosh \beta \sin \beta) - \beta^3 (\sin \beta \cosh \beta + \cos \beta \sinh \beta)}{1 + \cos \beta \cosh \beta}$$

$$c = \frac{\beta^4 (1 - \cos \beta \cosh \beta)}{\alpha (1 + \cos \beta \cosh \beta)}$$

Recognising the fact that the frequency equation for a free-free beam has a double root for first mode frequency, the frequency of this class of beam is expressed in terms of a factor as indicated below:

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1 \text{ and } 2 \quad (2.140)$$

and

$$f_{(n)} = f_{ff(n-1)} + Q [f_{cf(n)} - f_{ff(n-1)}] \quad \text{for } n > 2 \quad (2.141)$$

Tables 2.13 present the values of K^* for various values of Q and α . The modal shape expression is obtained by using equations (2.136) and (2.138)

$$R(\xi) = \sin \beta \xi + \sinh \beta \xi + \gamma [\cos \beta \xi + \cosh \beta \xi] \quad (2.142)$$

where

$$\gamma = \frac{\cos \beta - \cosh \beta + \frac{K^*}{\beta^3} (\sin \beta + \sinh \beta)}{\sin \beta + \sinh \beta - \frac{K^*}{\beta^3} (\cos \beta + \cosh \beta)}$$

Part 3 - Beams with a non-classical boundary condition at each end

2.14 Beams with a Rotational Mass at One End and a Linear Coil Spring at the Other

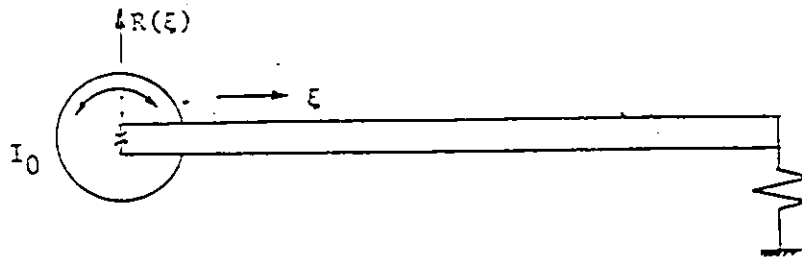


Figure 2.14

The boundary condition $R(0) = 0$ leads to

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.143)$$

From dynamic equilibrium at $\xi = 0$, we have:

$$R''(0) = -\frac{\beta^4}{I^*} R'(0) \quad (2.144)$$

which when substituted in equation (2.143) yields:

$$C = \frac{2 B I^*}{\beta^3} - A \quad (2.145)$$

The remaining two boundary conditions are supplied by the end $\xi = 1$ and these are:

$$R''(1) = 0$$

and

$$R'''(1) = K^* R(1)$$

Utilizing the relation (2.145) and these two boundary conditions, we obtain:

$$A[-\sin \beta - \sinh \beta] + B\left[\frac{2 I^* \sinh \beta}{\beta^3} - \cos \beta - \cosh \beta\right] = 0 \quad (2.146)$$

and

$$A[-\cos \beta - \cosh \beta - \frac{K^*}{\beta^3} (\sin \beta - \sinh \beta)] + B[\sin \beta - \sinh \beta +$$

$$+ \frac{2 I^* \cosh \beta}{\beta^3} - \frac{K^*}{\beta^3} (\cos \beta - \cosh \beta) - \frac{2 K^* I^*}{\beta^6} \sinh \beta] = 0 \quad (2.147)$$

Introducing $\alpha = \frac{I^*}{K^*}$ and equating the determinant of the coefficient matrix of constants A and B in equations (2.146) and (2.147) to zero, we obtain:

$$K^{*2} + b K^* + c = 0 \quad (2.148)$$

where

$$b = \frac{\beta^3 (1+c)}{\alpha} \frac{[\cos \beta \sinh \beta - \sin \beta \cosh \beta]}{2 \sin \beta \sinh \beta}$$

and

$$c = \frac{-\beta^6}{\alpha} \frac{(1 + \cos \beta \cosh \beta)}{2 \sin \beta \sinh \beta}$$

The frequency in terms of Q factor is given by

$$f_{(n)} = Q f_{ss(n)} \quad \text{for } n = 1 \quad (2.149)$$

and

$$f_{(n)} = f_{cf(n-1)} + Q[f_{ss(n)} - f_{cf(n-1)}]; n > 1 \quad (2.150)$$

Using equations (2.149) or (2.150) and (2.148), the computed values of K^* for various values of Q and α are presented in Table 2.14.

The modal shape expression is obtained following the procedure outlined in earlier cases and is provided below

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma [\cos \beta \xi - \cosh \beta \xi + \frac{2 I^*}{\beta^3} \sinh \beta \xi] \quad (2.151)$$

where

$$\gamma = \frac{\sin \beta + \sinh \beta}{\frac{2 I^*}{\beta^3} \sinh \beta - \cos \beta - \cosh \beta}$$

2.15 Beams with a Rotational Mass at One End and a Torsional Spring Support at the Other

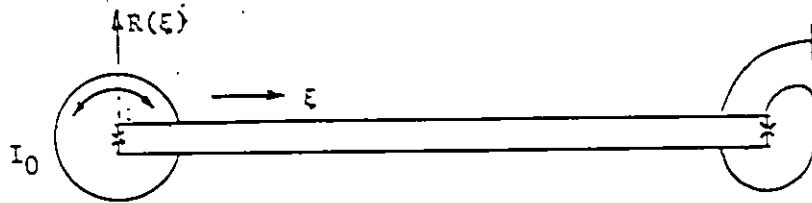


Figure 2.15

The zero deflection requirement at the end $\xi = 0$ gives the following shape function of the beam under consideration.

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.152)$$

The dynamic equilibrium condition $R''(0) = -\frac{\beta^4}{I^*} R'(0)$ leads to:

$$C = \frac{2 B I^*}{\beta^3} - A \quad (2.153)$$

The two boundary conditions at the end $\xi = 1$ are:

$$R(1) = 0$$

and

$$R'''(1) = -T^* R'(1)$$

Utilizing these two boundary conditions together with equations (2.152) and (2.153), we obtain:

$$A[\sin \beta - \sinh \beta] + B[\cos \beta - \cosh \beta + \frac{2 T^*}{\beta^3} \sinh \beta] = 0 \quad (2.154)$$

and

$$\begin{aligned} & A[-\sin \beta + \frac{T^*}{\beta} \cos \beta - \sinh \beta - \frac{T^*}{\beta} \cosh \beta] \\ & + B[-\cos \beta - \cosh \beta - \frac{T^*}{\beta} (\sin \beta + \sinh \beta) + \frac{2 T^*}{\beta^3} (\sinh \beta + \frac{T^*}{\beta} \cosh \beta)] = 0 \end{aligned} \quad (2.155)$$

Introducing $\alpha = \frac{T^*}{T}$ and utilizing the condition for a non-trivial solution for constants A and B in the above two equations, we obtain:

$$T^{*2} + b T^* + c = 0 \quad (2.156)$$

where

$$b = \frac{\beta^3 \left[-1 + \frac{2\alpha}{\beta^2} \sin \beta \sinh \beta + \cos \beta \cosh \beta \right]}{\alpha [\sin \beta \cosh \beta - \sinh \beta \cos \beta]}$$

and

$$c = \frac{-\beta^4}{\alpha}$$

The relationship between Q and frequency is given by .

$$f_{(n)} = Q f_{cs(n)} \quad \text{for } n = 1$$

and

$$f_{(n)} = f_{cs(n-1)} + Q[f_{cs(n)} - f_{cs(n-1)}] \quad \text{for } n > 1$$

The Tables 2.15 present the computed results for T^* for various values of α and Q.

Using equations (2.152), (2.153), and (2.154) the modal shape expression is simplified to

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma [\cos \beta \xi - \cosh \beta \xi + \frac{2 I^*}{\beta^3} \sinh \beta \xi]$$

(2.157)

where

$$\gamma = \frac{\sinh \beta - \sin \beta}{\cos \beta - \cosh \beta + \frac{2 I^*}{\beta^3} \sinh \beta}$$

2.16 Beams with a Torsional Spring Support at One End and with a Concentrated Mass at the Other



Figure 2.16

The boundary condition $R(0) = 0$ leads to

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.158)$$

and the boundary condition $R''(0) = T^* R'(0)$ yields

$$C = \frac{-2\beta B}{T^*} - A \quad (2.159)$$

The remaining two boundary conditions are supplied by the end carrying a concentrated mass and these are:

$$R''(1) = 0$$

and

$$R'''(1) = -\frac{\beta^4}{\eta} R(1)$$

Substituting these two boundary conditions in equation (2.158) and utilizing the relationship (2.159) we obtain:

$$A[\sin \beta + \sinh \beta] + B\left[\cos \beta + \cosh \beta + \frac{2\beta \sinh \beta}{T^*}\right] = 0 \quad (2.160)$$

and

$$\begin{aligned} A[-\cos \beta - \cosh \beta - \frac{\beta}{\eta} (\sinh \beta - \sin \beta)] + B[\sin \beta - \sinh \beta - \frac{2\beta \cosh \beta}{T^*} \\ + \frac{\beta}{\eta} (\cos \beta - \cosh \beta - \frac{2\beta \sinh \beta}{T^*})] = 0 \end{aligned} \quad (2.161)$$

Introducing $\alpha = \frac{n}{T^*}$ and equating the determinant of the coefficient matrix of constants A and B in equations (2.160) and (2.161) to zero, we obtain:

$$T^{*2} + b T^* + c = 0 \quad (2.162)$$

where

$$b = \frac{[\beta(1+\alpha)(\cos \beta \sinh \beta - \sin \beta \cosh \beta)]}{\alpha(1 + \cos \beta \cosh \beta)}$$

and

$$c = - \frac{2\beta^2 \sin \beta \sinh \beta}{\alpha(1 + \cos \beta \cosh \beta)}$$

The relationship between frequencies and Q factor is given

by:

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1$$

and

$$f_{(n)} = f_{ss(n-1)} + Q[f_{cf(n)} - f_{ss(n-1)}] \quad \text{for } n > 1$$

Tables 2.16 furnish the values of system parameter T^* for various values of Q and α .

The simplified form for the modal shape expression is expressed by,

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma \left[\cos \beta \xi - \cosh \beta \xi - \frac{2\beta}{T^*} \sinh \beta \xi \right] \quad (2.163)$$

where

$$\gamma = \frac{-(\sin \beta + \sinh \beta)}{\cos \beta + \cosh \beta + \frac{2\beta \sinh \beta}{T^*}}$$

2.17 Beams with a Rotational Mass at Each End

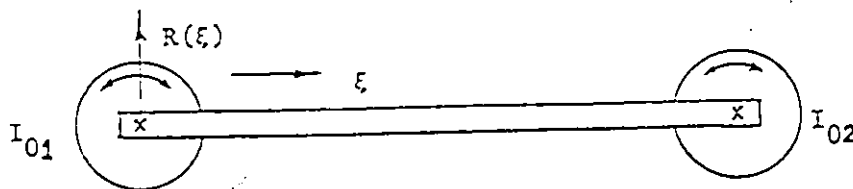


Figure 2.17

The condition $R(0) = 0$ leads to

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sin \beta \xi \quad (2.164)$$

and the condition $R(1) = 0$ leads to

$$C = \frac{-A \sin \beta + B(\cosh \beta - \cos \beta)}{\sinh \beta} \quad (2.165)$$

The other two boundary conditions are

$$R''(0) = -\frac{\beta^4}{I_1^*} R'(0)$$

and

$$R''(1) = \frac{\beta^4}{I_2^*} R'(1)$$

where I_1^* and I_2^* are associated with moment of inertia of the rotational mass at the left hand end and right hand end respectively.

Substituting these two conditions into equation (2.164) and using equation (2.165) we obtain:

$$A \left[\frac{\beta^3}{I_1^*} \left(1 - \frac{\sin \beta}{\sinh \beta} \right) \right] + B \left[-2 + \frac{\beta^3}{I_1^*} \left(\frac{\cosh \beta - \cos \beta}{\sinh \beta} \right) \right] = 0 \quad (2.166)$$

and

$$A \left[-2 \sin \beta - \frac{\beta^3}{I_2^*} \left(\cos \beta - \frac{\sin \beta \cosh \beta}{\sinh \beta} \right) \right] + B \left[-2 \cos \beta - \frac{\beta^3}{I_2^*} (-\sin \beta - \sinh \beta + \right. \\ \left. + \frac{\cosh^2 \beta - \cos \beta \cosh \beta}{\sinh \beta} \right] = 0 \quad (2.167)$$

Introducing $a = \frac{I_2^*}{I_1}$, and applying the condition of a non-trivial solution, we obtain:

$$I_1^{*2} + b I_1^* + c = 0 \quad (2.168)$$

where

$$b = \frac{\beta^3 \left(1 + \frac{1}{a} \right) (\sin \beta \cosh \beta - \cos \beta \sinh \beta)}{-2 \sin \beta \sinh \beta}$$

and

$$c = \frac{\beta^6 (1 - \cos \beta \cosh \beta)}{a (2 \sin \beta \sinh \beta)}$$

The Q factor is related to the frequencies of this class of beams by the following equations:

$$f_{(n)} = Q f_{ss(n)} \quad \text{for } n = 1 \text{ and } 2 \quad (2.169)$$

and

$$f_{(n)} = f_{cc(n-2)} + Q[f_{ss(n)} - f_{cc(n-2)}], \text{ for } n > 2 \quad (2.170)$$

The equation (2.169) is applicable for first two modes of vibrations. As Q varies from 0 to 1.0, the first mode frequency varies from zero to the first mode frequency of a simply supported beam and second mode frequency varies from zero to second mode frequency of the simply supported beam. For higher modes with $n = 0$, the two ends behave as clamped. The Tables 2.17 provide values of I_1^* for various values of Q and α . It may be noted that because of a symmetry in boundary conditions, the values of α are given only for the range $0.1 \leq \alpha \leq 1.0$. Prescribing values for $\frac{1}{\alpha}$ as was done in earlier cases would be redundant here.

It is interesting to note here that for this case the upper and lower limits for adjacent modes are found to overlap. The upper and lower limits are obviously simple - simple and clamped - clamped respectively. The determinant equation (2.168) is quadratic as expected. The solution for I_1^* in terms of eigenvalues β indicates that there is not always one positive root. This might seem perplexing at first but a further study indicates that this is to be expected and the reasons are explained with the aid of Figure 2.17(a).

It is known that for this case the lowest eigenvalue is zero and represents a limiting case for first mode vibration of the beam under study (I_1^* approaching zero). It should be recognized that there are two modes of vibration associated with this zero value, the symmetric mode corresponding to the counter rotation of the masses while the anti-symmetric mode corresponds to the situation in which the masses have angular rotation in the same direction. As I_1^* increases, the symmetric and anti-symmetric modes of clamped - clamped beam degenerate into the next higher symmetric and anti-symmetric modes of simple - simple beams.

A plot of β versus I_1^* is presented in Figure 2.17(a) for both symmetric and anti-symmetric modes. It is seen that eigenvalues for every mode begin at those pertaining to clamped -

clamped beams including zero eigenvalue. As I_1^* is increased for symmetric modes, the next higher simple - simple symmetric mode limit is reached. An analogous rule applies for anti-symmetric modes. The lowest eigenvalue zero is unique in the sense that it corresponds to a starting point for both symmetric and anti-symmetric modes. Even with springs that are not equal in stiffness, the limits remain unchanged. Designers should be aware of the implications of Figure 2.17(a). It is evident, for example, that if a resonant condition exists due to a driving force of a particular frequency, altering I_1^* to move one mode out of resonance may move another in, if the alteration is not judiciously chosen.

The modal shape expression is arrived at by utilizing equations (2.164), (2.165), and (2.166) and is given below:

$$R(\xi) = \sin \beta \xi - \frac{\sin \beta}{\sinh \beta} \sinh \beta \xi + \gamma [\cos \beta \xi - \cosh \beta \xi + \frac{(\cosh \beta - \cos \beta)}{\sinh \beta} \sinh \beta \xi] \quad (2.171)$$

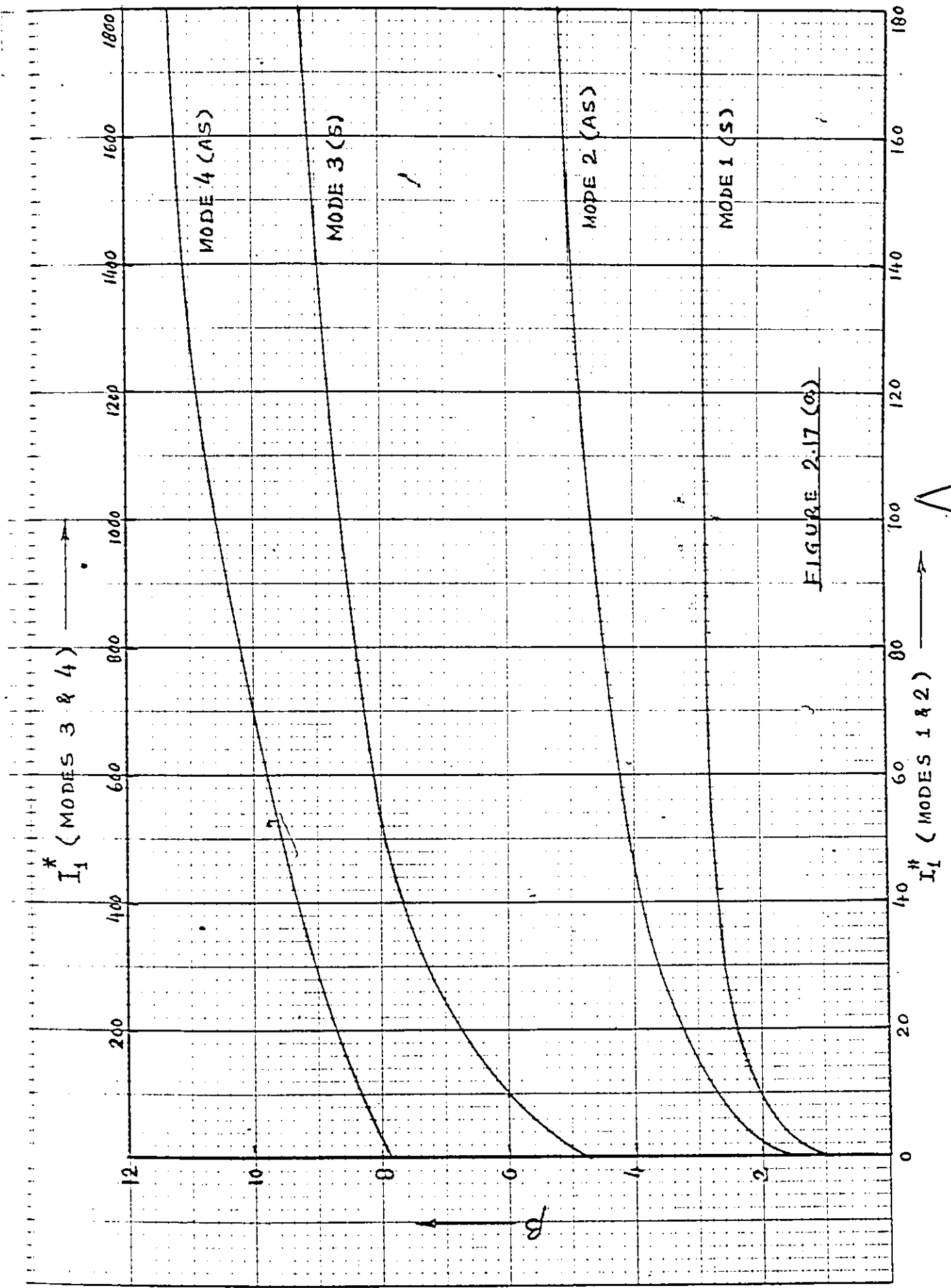


FIGURE 2.17 (cs)

where

$$\gamma = \frac{\left(\frac{\sin \beta}{\sinh \beta} - 1\right)}{\left(\frac{\cosh \beta - \cos \beta}{\sinh \beta}\right) - 2 \frac{I_1^*}{\beta^3}}$$

2.18 Beams with a Rotational Mass at One End and a Concentrated Mass at the Other

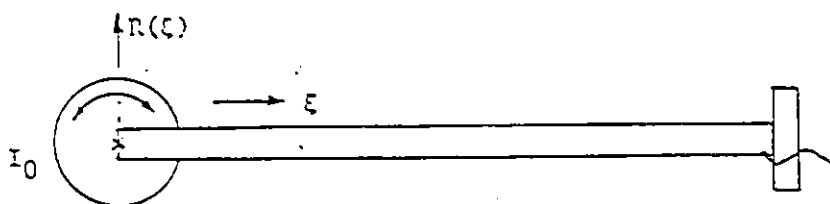


Figure 2.18

The boundary condition $R(0) = 0$ leads to

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.172)$$

and the boundary condition $R''(0) = -\frac{\beta^4}{I^*} R'(0)$ yields:

$$C = \frac{2 B I^*}{\beta^3} - A \quad (2.173)$$

The two boundary conditions at the end $\xi = 1$, are:

$$R''(1) = 0$$

and

$$R'''(1) = -\frac{\beta^4}{\eta} R(1)$$

Using these two conditions together with the relationship (2.173) we obtain:

$$A(\sin \beta + \sinh \beta) + B[\cos \beta + \cosh \beta - \frac{2 I^* \sinh \beta}{\beta^3}] = 0$$

and

$$A[\cos \beta + \cosh \beta - \frac{\beta}{\eta} (\sin \beta - \sinh \beta)] + B[-\frac{\beta}{\eta} (\cos \beta - \cosh \beta) - \frac{2 I^* \sinh \beta}{\eta \beta^2} - \sin \beta + \sinh \beta - \frac{2 I^* \cosh \beta}{\beta^3}] = 0 \quad (2.175)$$

Introducing $\alpha = \frac{I^*}{\eta}$ and enforcing the requirement of a non-zero solution for A and B, we obtain:

$$\eta^2 + b \eta + C = 0 \quad (2.176)$$

where

$$b = \frac{1 + \cos \beta \cosh \beta + \frac{2\alpha}{\beta^2} \sin \beta \sinh \beta}{\frac{\alpha}{\beta^3} [\cosh \beta \sin \beta - \sinh \beta \cos \beta]}$$

and

$$C = \frac{-\beta^4}{\alpha}$$

The frequency in terms of Q factor is given by

$$f_{(n)} = Q f_{sf(n+1)} \quad \text{for } n = 1$$

and

$$f_{(n)} = f_{cs(n-1)} + Q[f_{sf(n+1)} - f_{cs(n-1)}] \quad \text{for } n > 1$$

Tables 2.18 furnish all the information necessary to establish frequencies of this class of beams. The modal shape function is given by:

$$R(\xi) = \sin \beta \xi - \sinh \beta \xi + \gamma [\cos \beta \xi - \cosh \beta \xi + \frac{2I^*}{\beta^3} \sinh \beta \xi]$$

(2.177)

where

$$\gamma = \frac{\sin \beta + \sinh \beta}{\frac{2I^*}{\beta^3} \sinh \beta - \cos \beta - \cosh \beta}$$

2.19 Beams with a Concentrated Mass at Each End

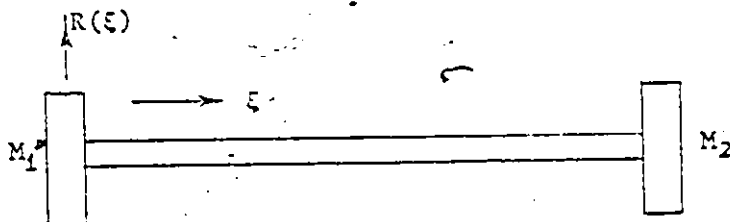


Figure 2.19

The condition $R''(0) = 0$ leads to

$$R(\xi) = A \sin \beta \xi + B(\cos \beta \xi - \cosh \beta \xi) + C \sinh \beta \xi \quad (2.178)$$

and the condition of dynamic equilibrium at the end $\xi = 0$, namely

$$R'''(0) = \frac{\beta^4}{n_1} R(0), \text{ yields:}$$

$$C = \frac{2\beta}{\eta_1 (*)} B + A \quad (2.179)$$

The two boundary conditions supplied by the end $\xi = 1$, are

$$R''(1) = 0$$

and

(2.180)

$$R'''(1) = - \frac{\beta^4}{\eta_2 (*)} R(1)$$

Utilizing equations (2.180) and the relation (2.179) we obtain:

$$A(\sinh \beta - \sin \beta) + B[\cosh \beta - \cos \beta + \frac{2\beta}{\eta_1} \sinh \beta] = 0 \quad (2.181)$$

and

$$A[\cosh \beta - \cos \beta + \frac{\beta}{\eta_2} (\sin \beta + \sinh \beta)]$$

$$+ B[\sin \beta + \sinh \beta + \frac{2\beta}{\eta_1} (\cosh \beta + \sinh \beta) + \frac{\beta}{\eta_2} (\cos \beta + \cosh \beta)] = 0$$

(2.182)

(*) The subscripts 1 and 2 refer to left hand end and right hand end respectively.

Introducing $\alpha = \frac{\eta_2}{\eta_1}$ and simplifying the determinantal equation yields the following quadratic equation in η_1 .

$$\eta_1^2 + b \eta_1 + c = 0 \quad (2.183)$$

where

$$b = \frac{\beta (1+\alpha) (\sin \beta \cosh \beta - \cos \beta \sinh \beta)}{\alpha (1 - \cos \beta \cosh \beta)}$$

and

$$c = \frac{2 \beta^2 \sin \beta \sinh \beta}{\alpha (1 - \cos \beta \cosh \beta)}$$

Recognising the limiting cases of this class of beams, we can write

$$f_{(n)} = f_{ss(n)} + Q[f_{ff(n+1)} - f_{ss(n)}]$$

By prescribing various values for Q and α , equation (2.183) can be solved for η_1 and the results are prescribed in Tables 2.19. Here again, like case 2.17, we need prescribe the values of α only in the range

$0.1 \leq \alpha \leq 1.0$, taking advantage of symmetry in boundary conditions.

The modal shape expression is simplified to,

$$R(\xi) = \sin \beta \xi + \sinh \beta \xi + \gamma \left[\cos \beta \xi + \cosh \beta \xi + \frac{2\beta}{\eta_1} \sinh \beta \xi \right] \quad (2.184)$$

where

$$\gamma = \frac{\sin \beta - \sinh \beta}{\cosh \beta - \cos \beta + \frac{2\beta}{\eta_1} \sinh \beta}$$

Illustrative example 2.1

The aluminum beam shown in Figure 2.6 has a length of 29.5 in., width of 1" and a depth of 0.25 in. The Youngs modulus of elasticity and density of beam material are 10.5×10^6 psi and 0.0965 lb/in^3 respectively. The beam is clamped at one end and carries a mass of 0.25 lb supported by a coil spring of stiffness 8.2 lb/in.

(a) Determine the fundamental and second frequency of the beam system.

The moment of inertia of the beam cross section $I = \frac{1}{12} \times 1 \times (0.25)^3$
 $= 0.0013 \text{ in}^4$

$$f_{cs(1)} = \frac{(\beta_{cs(1)})^2}{2\pi L^2} \sqrt{\frac{EI}{\rho A}}$$

$$= \frac{(3.926)^2}{2\pi \times (29.5)^2} \sqrt{\frac{10.5 \times 10^6 \times 0.0013 \times 386}{0.0965 \times 0.25}}$$

$$= 41.66 \text{ Hz}$$

$$f_{cs(2)} = 41.66 \times \left(\frac{7.068}{3.926}\right)^2 = 135 \text{ Hz}$$

where the values of $\beta_{cs(n)}$ are read off the Table A-1 provided in Appendix A.

$$\eta = \frac{\rho AL}{M} = \frac{0.0965 \times 0.25 \times 29.5}{0.25} = 2.85$$

$$K^* = \frac{KL^3}{EI} = \frac{8.2 \times (29.5)^3}{10.5 \times 10^6 \times 0.0013} = 15.4$$

$$\alpha = \frac{\eta}{K^*} = 0.185$$

Using Tables 2.6, obtain the value of Q for first and second modes for the given values of $\alpha = 0.185$ and $K^* = 15.4$ using linear interpolation if necessary.

For first mode $Q = 0.351$

and for second mode $Q = 0.0597$

$$f_1 = Q f_{cs(1)} = 0.351 \times 41.66 = \underline{14.6} \text{ Hz}$$

and

$$f_2 = f_{cs(1)} + Q[f_{cs(2)} - f_{cs(1)}]$$

$$= 41.66 + 0.0597 \times 135 - 41.66$$

$$= 41.66 + 8.06 - 41.66 = \underline{8.06} \text{ Hz}$$

(b) Determine the modal shape expression for the first mode.

The eigenvalue parameter β for the first mode is determined from first mode frequency calculated above.

$$\frac{\beta_1}{\beta_{cs(1)}} = \sqrt{\frac{f_1}{f_{cs(1)}}}$$

$$\beta_1 = 3.926 \sqrt{\frac{14.6}{41.66}}$$

$$= 2.32$$

Using equation (2.69), we obtain:

$$R(\xi) = \sin 2.32 \xi - \sinh 2.32 \xi + \gamma (\cos 2.32 \xi - \cosh 2.32 \xi)$$

where

$$\gamma = - \frac{(\sin 2.32 + \sinh 2.32)}{\cos 2.32 + \cosh 2.32}$$

- (c) Determine the percentage change in the fundamental frequency of the system if the coil spring is replaced by a very soft coil spring of stiffness 1 lb/in.

$$\text{Now } K^* = \frac{KL^3}{EI} = \frac{15.4}{8.2} = 1.88$$

$$c = \frac{\eta}{K^*} = \frac{2.85}{1.88}$$

or

$$\frac{1}{c} = \frac{1.88}{2.85} = 0.66$$

For $K^* = 1.88$ and $\frac{1}{c} = 0.66$, obtain value of Q from Tables 2.6

$$Q = 0.1825$$

$$f_1 = 0.1825 \times 41.66 = 7.6 \text{ Hz}$$

So the introduction of this soft spring has reduced the frequency by 48%.

Illustrative example 2.2

The steel beam shown in Figure 2.16 has the following geometric properties: length = 40", width = 1", depth = $\frac{1}{2}$ ".

On the left hand end of the beam is attached a torsion spring of stiffness 1500 lb in/rad. while the right hand end is carrying a concentrated mass of 5 lbs.

- (a) Determine fundamental frequency and modal shape of the beam system.

For steel take $E = 30 \times 10^6 \text{ lb/in}^2$. $\rho = 0.28 \text{ lb/in}^3$

Moment of inertia of the cross-section = $\frac{1}{12} \times 1 \times (\frac{1}{2})^3 = \frac{1}{96} = 0.0104 \text{ in}^4$

Mass of the beam = $\rho AL = 0.28 \times \frac{1}{2} \times 40 = 5.6 \text{ lb}$

$$\therefore n = \frac{\rho AL}{M} = \frac{5.6}{5} = 1.12$$

and

$$T^* = \frac{TL}{EI} = \frac{1500 \times 40}{30 \times 10^6 \times 0.0104} = \frac{4}{31.2} = 0.129$$

$$\text{So } \alpha = \frac{\eta}{T^*} = \frac{1.12}{.128}, \quad \text{or} \quad \frac{1}{\alpha} = \frac{.192}{1.12} = 0.17$$

The upper limit frequency

$$\begin{aligned} f_{cf} &= \frac{1}{2\pi} \frac{s_{cf}^2}{L^2} \sqrt{\frac{EI}{\rho A}} \\ &= \frac{1.875^2}{2\pi \times 40^2} \sqrt{\frac{30 \times 10^6 \times 0.0104 \times 386}{0.28 \times \frac{\pi}{4}}} \\ &= 10.25 \text{ Hz} \end{aligned}$$

Obtain value of Q from Tables 2.16. Using linear interpolation for $\frac{1}{\alpha} = 0.17$ and $T^* = 0.192$, thus, we obtain $Q = 0.11$, and

$$f_{(1)} = Q f_{cf(1)} = 1.13 \text{ Hz}$$

and

$$s = s_{cf} \sqrt{Q} = 0.62$$

The modal shape for first mode is given by equation (2.163), thus

$$R_1(\xi) = \sin 0.62 \xi - \sinh 0.62 \xi + \gamma [\cos 0.62 \xi - \cosh 0.62 \xi -$$

$$\frac{1.24}{.192} \sinh 0.26 \xi]$$

where

$$\gamma = \frac{-(\sin 0.62 + \sinh 0.62)}{\cos 0.62 + \cosh 0.62 + \frac{1.24}{.192} \sinh 0.62}$$

TABLE 2.1 : 1"

C	0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
NOISE										
(1)	1488	1977	2423	2847	3247	3621	3968	4288	4581	4847
(2)	14.88	19.77	24.23	28.47	32.47	36.21	39.68	42.88	45.81	48.47
(3)	80.69	114.8	153.5	197.7	242.3	284.7	324.7	362.1	396.8	428.8
(4)	158.8	297.4	446.3	595.2	744.1	893.0	1041.9	1190.8	1339.7	1488.6
(5)	328.8	612.1	895.4	1178.7	1462.0	1745.3	2028.6	2311.9	2595.2	2878.5
(6)	598.0	1093	1573	2048	2518	2983	3443	3898	4348	4793
(7)	962.4	1782	2594	3398	4193	4978	5753	6518	7273	8018
(8)	1488	2709	3927	5142	6354	7562	8766	9966	11162	12354
(9)	2219	3912	5604	7295	8985	10674	12362	14050	15738	17425
(10)	2963	5257	7550	9843	12136	14429	16722	19015	21308	23601

TABLE 2.2 : 1"

C	0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
NOISE										
(1)	1488	1977	2423	2847	3247	3621	3968	4288	4581	4847
(2)	14.88	19.77	24.23	28.47	32.47	36.21	39.68	42.88	45.81	48.47
(3)	80.69	114.8	153.5	197.7	242.3	284.7	324.7	362.1	396.8	428.8
(4)	158.8	297.4	446.3	595.2	744.1	893.0	1041.9	1190.8	1339.7	1488.6
(5)	328.8	612.1	895.4	1178.7	1462.0	1745.3	2028.6	2311.9	2595.2	2878.5
(6)	598.0	1093	1573	2048	2518	2983	3443	3898	4348	4793
(7)	962.4	1782	2594	3398	4193	4978	5753	6518	7273	8018
(8)	1488	2709	3927	5142	6354	7562	8766	9966	11162	12354
(9)	2219	3912	5604	7295	8985	10674	12362	14050	15738	17425
(10)	2963	5257	7550	9843	12136	14429	16722	19015	21308	23601

TABLE 2.3 : 1"

C	0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
NOISE										
(1)	1488	1977	2423	2847	3247	3621	3968	4288	4581	4847
(2)	14.88	19.77	24.23	28.47	32.47	36.21	39.68	42.88	45.81	48.47
(3)	80.69	114.8	153.5	197.7	242.3	284.7	324.7	362.1	396.8	428.8
(4)	158.8	297.4	446.3	595.2	744.1	893.0	1041.9	1190.8	1339.7	1488.6
(5)	328.8	612.1	895.4	1178.7	1462.0	1745.3	2028.6	2311.9	2595.2	2878.5
(6)	598.0	1093	1573	2048	2518	2983	3443	3898	4348	4793
(7)	962.4	1782	2594	3398	4193	4978	5753	6518	7273	8018
(8)	1488	2709	3927	5142	6354	7562	8766	9966	11162	12354
(9)	2219	3912	5604	7295	8985	10674	12362	14050	15738	17425
(10)	2963	5257	7550	9843	12136	14429	16722	19015	21308	23601

	$12\pi\epsilon_1(2,1) : \epsilon^2 (u=0.3)$	$12\pi\epsilon_1(2,1) : \epsilon^2 (u=0.4)$	$12\pi\epsilon_1(2,1) : \epsilon^2 (u=0.5)$
ζ_1	0.43	0.43	0.43
ζ_2	0.39	0.39	0.39
ζ_3	0.35	0.35	0.35
ζ_4	0.31	0.31	0.31
ζ_5	0.27	0.27	0.27
ζ_6	0.23	0.23	0.23
ζ_7	0.19	0.19	0.19
ζ_8	0.15	0.15	0.15
ζ_9	0.11	0.11	0.11
ζ_{10}	0.07	0.07	0.07
ζ_{11}	0.03	0.03	0.03
ζ_{12}	0.01	0.01	0.01
ζ_{13}	0.01	0.01	0.01
ζ_{14}	0.01	0.01	0.01
ζ_{15}	0.01	0.01	0.01
ζ_{16}	0.01	0.01	0.01
ζ_{17}	0.01	0.01	0.01
ζ_{18}	0.01	0.01	0.01
ζ_{19}	0.01	0.01	0.01
ζ_{20}	0.01	0.01	0.01
ζ_{21}	0.01	0.01	0.01
ζ_{22}	0.01	0.01	0.01
ζ_{23}	0.01	0.01	0.01
ζ_{24}	0.01	0.01	0.01
ζ_{25}	0.01	0.01	0.01
ζ_{26}	0.01	0.01	0.01
ζ_{27}	0.01	0.01	0.01
ζ_{28}	0.01	0.01	0.01
ζ_{29}	0.01	0.01	0.01
ζ_{30}	0.01	0.01	0.01
ζ_{31}	0.01	0.01	0.01
ζ_{32}	0.01	0.01	0.01
ζ_{33}	0.01	0.01	0.01
ζ_{34}	0.01	0.01	0.01
ζ_{35}	0.01	0.01	0.01
ζ_{36}	0.01	0.01	0.01
ζ_{37}	0.01	0.01	0.01
ζ_{38}	0.01	0.01	0.01
ζ_{39}	0.01	0.01	0.01
ζ_{40}	0.01	0.01	0.01
ζ_{41}	0.01	0.01	0.01
ζ_{42}	0.01	0.01	0.01
ζ_{43}	0.01	0.01	0.01
ζ_{44}	0.01	0.01	0.01
ζ_{45}	0.01	0.01	0.01
ζ_{46}	0.01	0.01	0.01
ζ_{47}	0.01	0.01	0.01
ζ_{48}	0.01	0.01	0.01
ζ_{49}	0.01	0.01	0.01
ζ_{50}	0.01	0.01	0.01
ζ_{51}	0.01	0.01	0.01
ζ_{52}	0.01	0.01	0.01
ζ_{53}	0.01	0.01	0.01
ζ_{54}	0.01	0.01	0.01
ζ_{55}	0.01	0.01	0.01
ζ_{56}	0.01	0.01	0.01
ζ_{57}	0.01	0.01	0.01
ζ_{58}	0.01	0.01	0.01
ζ_{59}	0.01	0.01	0.01
ζ_{60}	0.01	0.01	0.01
ζ_{61}	0.01	0.01	0.01
ζ_{62}	0.01	0.01	0.01
ζ_{63}	0.01	0.01	0.01
ζ_{64}	0.01	0.01	0.01
ζ_{65}	0.01	0.01	0.01
ζ_{66}	0.01	0.01	0.01
ζ_{67}	0.01	0.01	0.01
ζ_{68}	0.01	0.01	0.01
ζ_{69}	0.01	0.01	0.01
ζ_{70}	0.01	0.01	0.01
ζ_{71}	0.01	0.01	0.01
ζ_{72}	0.01	0.01	0.01
ζ_{73}	0.01	0.01	0.01
ζ_{74}	0.01	0.01	0.01
ζ_{75}	0.01	0.01	0.01
ζ_{76}	0.01	0.01	0.01
ζ_{77}	0.01	0.01	0.01
ζ_{78}	0.01	0.01	0.01
ζ_{79}	0.01	0.01	0.01
ζ_{80}	0.01	0.01	0.01
ζ_{81}	0.01	0.01	0.01
ζ_{82}	0.01	0.01	0.01
ζ_{83}	0.01	0.01	0.01
ζ_{84}	0.01	0.01	0.01
ζ_{85}	0.01	0.01	0.01
ζ_{86}	0.01	0.01	0.01
ζ_{87}	0.01	0.01	0.01
ζ_{88}	0.01	0.01	0.01
ζ_{89}	0.01	0.01	0.01
ζ_{90}	0.01	0.01	0.01
ζ_{91}	0.01	0.01	0.01
ζ_{92}	0.01	0.01	0.01
ζ_{93}	0.01	0.01	0.01
ζ_{94}	0.01	0.01	0.01
ζ_{95}	0.01	0.01	0.01
ζ_{96}	0.01	0.01	0.01
ζ_{97}	0.01	0.01	0.01
ζ_{98}	0.01	0.01	0.01
ζ_{99}	0.01	0.01	0.01
ζ_{100}	0.01	0.01	0.01

[illegible][illegible]

J	TIME 2.12E: 1° (n = 0.1)										TIME 2.12E: 1° (n = 0.1)									
	G ₁	G ₂	G ₃	G ₄	G ₅	G ₆	G ₇	G ₈	G ₉	G ₁₀	G ₁₁	G ₁₂	G ₁₃	G ₁₄	G ₁₅	G ₁₆	G ₁₇	G ₁₈	G ₁₉	G ₂₀
0	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
1	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
2	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
3	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
4	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
5	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
6	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
7	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
8	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
9	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
10	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
11	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
12	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
13	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
14	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
15	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
16	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
17	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
18	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
19	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
20	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
21	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
22	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
23	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
24	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
25	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
26	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
27	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
28	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
29	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
30	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
31	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
32	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
33	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
34	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
35	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
36	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
37	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
38	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
39	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
40	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
41	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
42	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
43	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
44	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
45	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
46	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
47	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
48	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
49	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
50	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
51	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
52	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
53	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
54	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
55	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
56	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
57	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
58	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
59	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
60	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
61	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93
62	0.53	0.10	0.22	0.30	0.43	0.53	0.63	0.72	0.83	0.93	0.53	0.10	0.22	0.30	0.43	0.53				

TABLE 2.19: π_1 ($\alpha = 0.1$)									
α	0.15	0.10	0.05	0.00	0.05	0.10	0.15	0.20	0.25
π_1	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_2	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_3	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_4	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_5	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_6	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_7	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_8	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_9	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{10}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{11}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{12}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{13}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{14}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{15}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{16}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{17}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{18}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{19}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{20}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{21}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{22}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{23}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{24}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{25}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{26}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{27}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{28}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{29}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{30}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{31}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{32}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{33}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{34}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{35}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{36}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{37}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{38}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{39}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{40}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{41}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{42}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{43}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{44}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{45}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{46}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{47}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{48}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{49}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{50}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{51}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{52}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{53}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{54}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{55}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{56}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{57}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{58}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{59}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{60}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{61}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{62}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{63}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{64}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{65}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{66}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{67}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{68}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{69}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{70}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{71}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{72}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{73}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{74}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{75}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{76}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{77}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{78}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{79}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{80}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{81}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{82}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{83}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{84}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{85}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{86}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{87}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{88}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{89}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{90}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{91}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{92}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{93}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{94}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{95}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{96}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{97}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{98}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{99}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15
π_{100}	0.55	0.45	0.35	0.25	0.15	0.05	0.00	0.05	0.15

TABLE 2.19 : η_1 ($\alpha = 1.0$)

Q:	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE:										
(1)	.3327	.7098	1.628	2.839	4.485	6.820	10.36	16.28	28.18	63.94
(2)	.5848	1.236	2.786	4.783	7.447	11.17	16.76	26.06	44.62	100.2
(3)	.8416	1.772	3.963	6.755	10.45	15.58	23.22	35.90	61.14	136.6
(4)	1.098	2.308	5.142	8.733	13.46	20.00	29.72	45.79	77.75	173.2
(5)	1.355	2.844	6.323	10.71	16.47	24.43	36.22	55.70	94.38	209.8
(6)	1.612	3.380	7.504	12.70	19.49	28.86	42.74	65.63	111.0	246.6
(7)	1.868	3.917	8.685	14.68	22.51	33.30	49.25	75.55	127.7	283.3
(8)	2.126	4.452	9.867	16.66	25.53	37.74	55.77	85.48	144.4	319.9
(9)	2.382	4.989	11.05	18.64	28.55	42.18	62.29	95.41	161.1	356.7
(10)	2.639	5.526	12.23	20.63	31.58	46.61	68.81	105.3	177.7	393.4

CHAPTER 3

TWO SPAN BEAMS WITH NON-CLASSICAL BOUNDARY CONDITIONS

The non-classical boundary conditions considered in this Chapter are provided by a torsional spring or a rotational mass. The other two supports are considered for all possible combinations of classical boundary conditions. Since the limiting frequencies of this class of beams are given by those of two span beams with classical boundary conditions, the problem here is formulated as a two-span beam problem.

Part 1 - Two span beams with one non-classical boundary condition provided by a torsional spring support

3.1 Beams Clamped at One End and Free at the Other and With an In-Span Torsional Spring Support

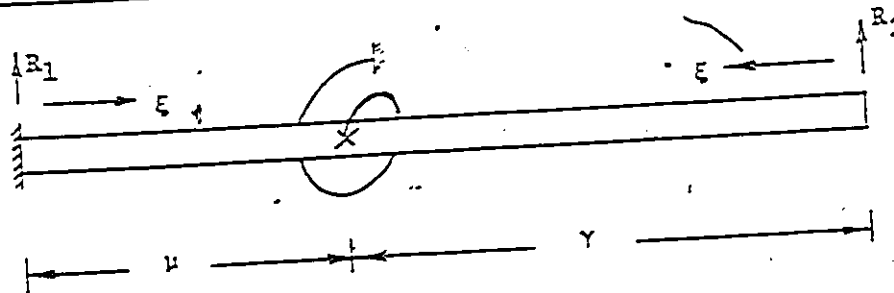


Figure 3.1

The modal shape functions for the two spans are expressed as given by equation (1.8)

$$R_1(\xi) = A_1 \sin \beta \xi + B_1 \cos \beta \xi + C_1 \sinh \beta \xi + D_1 \cosh \beta \xi$$

$$R_2(\xi) = A_2 \sin \beta \xi + B_2 \cos \beta \xi + C_2 \sinh \beta \xi + D_2 \cosh \beta \xi$$

where the coordinates for the two spans are taken as shown in Figure 3.1 and the torsional spring has been provided at a non-dimensional length μ , expressed as a fraction of overall beam length, from the fixed end. Therefore the distance of this torsional spring from the free end is given by $\gamma = 1 - \mu$, expressed in non-dimensional form. Utilizing the boundary conditions $R_1(0) = R_1'(0) = 0$ and $R_2''(0) = R_2'''(0) = 0$, the modal shape functions reduce to following forms.

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.1)$$

and

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi) \quad (3.2)$$

For the remaining four arbitrary constants A_1 , B_1 , A_2 , and B_2 the following four boundary conditions are supplied by the intermediate support. These are:

$$R_1(\mu) = 0 \quad (3.3)$$

$$R_2(\gamma) = 0 \quad (3.4)$$

$$R'_1(\mu) = -R'_2(\gamma) \quad (3.5)$$

and

$$R''_1(\mu) + T^* R'_1(\mu) = R''_2(\gamma) \quad (3.6)$$

where the last two of the boundary conditions are arrived at by considering slope and bending moment continuity at the intermediate support. The enforcement of the above four boundary conditions yields the following set of four algebraic homogeneous equations in terms of these four constants

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.7)$$

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.8)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu) + A_2(\cos \beta \gamma + \cosh \beta \gamma) + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \quad (3.9)$$

$$A_1 [-\sin \beta \mu - \sinh \beta \mu + \frac{T^*}{\beta} (\cos \beta \mu - \cosh \beta \mu)] + B_1 [-\cos \beta \mu - \cosh \beta \mu + \frac{T^*}{\beta} (-\sin \beta \mu - \sinh \beta \mu)] + A_2 (\sin \beta \gamma - \sinh \beta \gamma) +$$

$$B_2 (\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.10)$$

For a non-trivial solution for the constants A_1 , B_1 , A_2 , and B_2 appearing in equations (3.7) through (3.10), the determinant of their coefficient matrix must vanish. Equating this determinant to zero, we obtain:

$$B \left[(\sin \beta \mu + \sinh \beta \mu) D_1 - (\cos \beta \mu + \cosh \beta \mu) D_2 + (\sinh \beta \gamma - \sin \beta \gamma) D_3 + (\cos \beta \gamma - \cosh \beta \gamma) D_4 \right] = 0 \quad (3.11)$$

where D_1 , D_2 , D_3 , and D_4 are the minors of the coefficients of A_1 , B_1 , A_2 , and B_2 respectively appearing in equation (3.10), viz:

$$D_1 = \begin{vmatrix} \cos \beta \mu - \cosh \beta \mu & 0 & 0 \\ 0 & \sin \beta \gamma + \sinh \beta \gamma & \cos \beta \gamma + \cosh \beta \gamma \\ -\sin \beta \mu - \sinh \beta \mu & \cos \beta \gamma + \cosh \beta \gamma & -\sin \beta \gamma + \sinh \beta \gamma \end{vmatrix}$$

$$D_2 = \begin{vmatrix} \sin \beta \mu - \sinh \beta \mu & 0 & 0 \\ 0 & \sin \beta \gamma + \sinh \beta \gamma & \cos \beta \gamma + \cosh \beta \gamma \\ \cos \beta \mu - \cosh \beta \mu & \cos \beta \gamma + \cosh \beta \gamma & -\sin \beta \gamma + \sinh \beta \gamma \end{vmatrix}$$

$$D_3 = \begin{vmatrix} \sin \beta \mu - \sin \beta \mu & \cos \beta \mu - \cosh \beta \mu & 0 \\ 0 & 0 & \cos \beta \gamma + \cosh \beta \gamma \\ \cos \beta \mu - \cosh \beta \mu & -\sin \beta \mu - \sinh \beta \mu & -\sin \beta \gamma + \sinh \beta \gamma \end{vmatrix}$$

and

$$D_4 = \begin{vmatrix} \sin \beta \mu - \sinh \beta \mu & \cos \beta \mu - \cosh \beta \mu & 0 \\ 0 & 0 & \sin \beta \gamma + \sinh \beta \gamma \\ \cos \beta \mu - \cosh \beta \mu & -\sin \beta \mu - \sinh \beta \mu & \cos \beta \gamma + \cosh \beta \gamma \end{vmatrix}$$

The frequency parameter β of this class of beams is completely characterized by the non-dimensional system parameter T^* . The same procedure of synthesizing the system characterized by the parameter T^* at regularly placed intervals of frequency between the two limiting conditions as explained in Chapter 2 will be utilized here.

At this stage it is necessary to examine closely the appropriate upper and lower limiting frequencies for this class of beams. This examination is of utmost importance since it has not been carried out heretofore in the existing literature and in fact, a need for an answer to this problem has not previously existed.

With T^* approaching zero, the frequency of this class of beam approaches that of a clamped-simply supported free two span beam. The charts for values of the frequency parameter β for two span beams with

various classical boundary conditions (15) are included in Appendix B for the sake of completeness. So the lower limits of frequencies for this case can be easily read off these charts.

The upper limits of frequencies as T^* approaches infinity are not so obvious. In fact with $T^* = \infty$, the intermediate support behaves as a clamped end and the entire system of a two span beam is decoupled into two independent single span beam systems of non-dimensional lengths, μ and γ . With $T^* = \infty$, the left hand span of the beam has clamped-clamped boundary conditions and a length of μ whereas the right hand span of the beam has a length of $\gamma(\gamma-1-\mu)$, with clamped free boundary conditions.

"Whether the upper frequency limit will correspond to the left hand span, or to the right hand span or neither" is the dilemma which the analyst is confronted with in this problem.

The answer to the question, "which one of these two spans will dictate the upper frequency limit of the beam system in hand?" necessitated an extensive study of several β v/s T^* plots. Three such plots are shown in Figures 3.1(a), 3.1(b), and 3.1(c) for $\mu = 0.4, 0.6$, and 0.9 respectively. The range of T^* shown is, $-30 \leq T^* \leq 30$. For $T^* = 0$, the values of β correspond to those of a two span beam with clamped-simple-free boundary conditions. For the first mode as T^* becomes negative, a condition of instability is encountered for particular values of T^* depending on the values of μ . For $\mu = 0.4, 0.6$, and 0.9 , the first mode frequency of the beam system becomes zero for $T^* = -10, -6.67$, and -4.5 respectively. The condition of zero frequency is generally associated

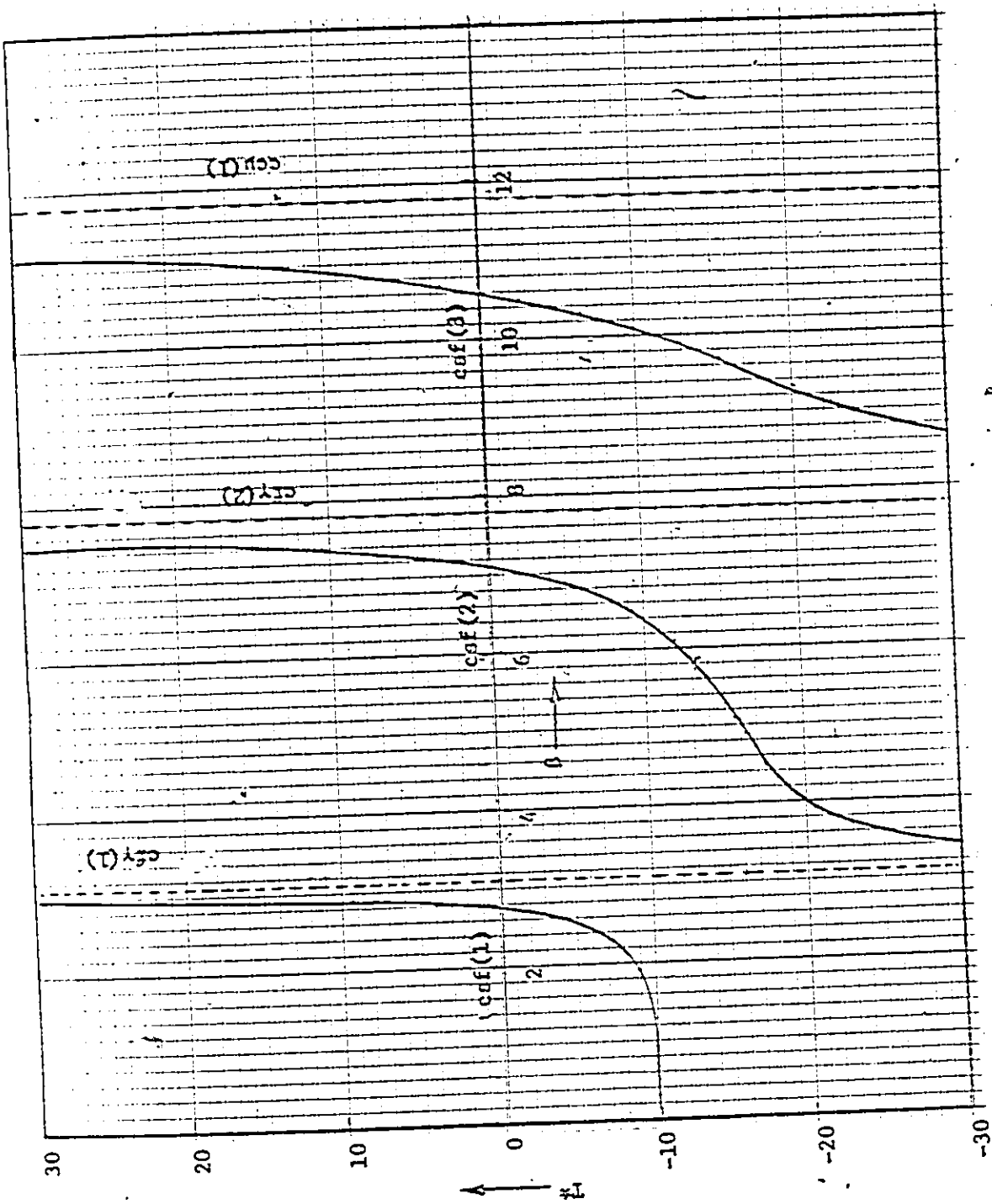


FIGURE 3.1(a) : A plot of $T^* v/s \beta$ for $\mu = 0.4$

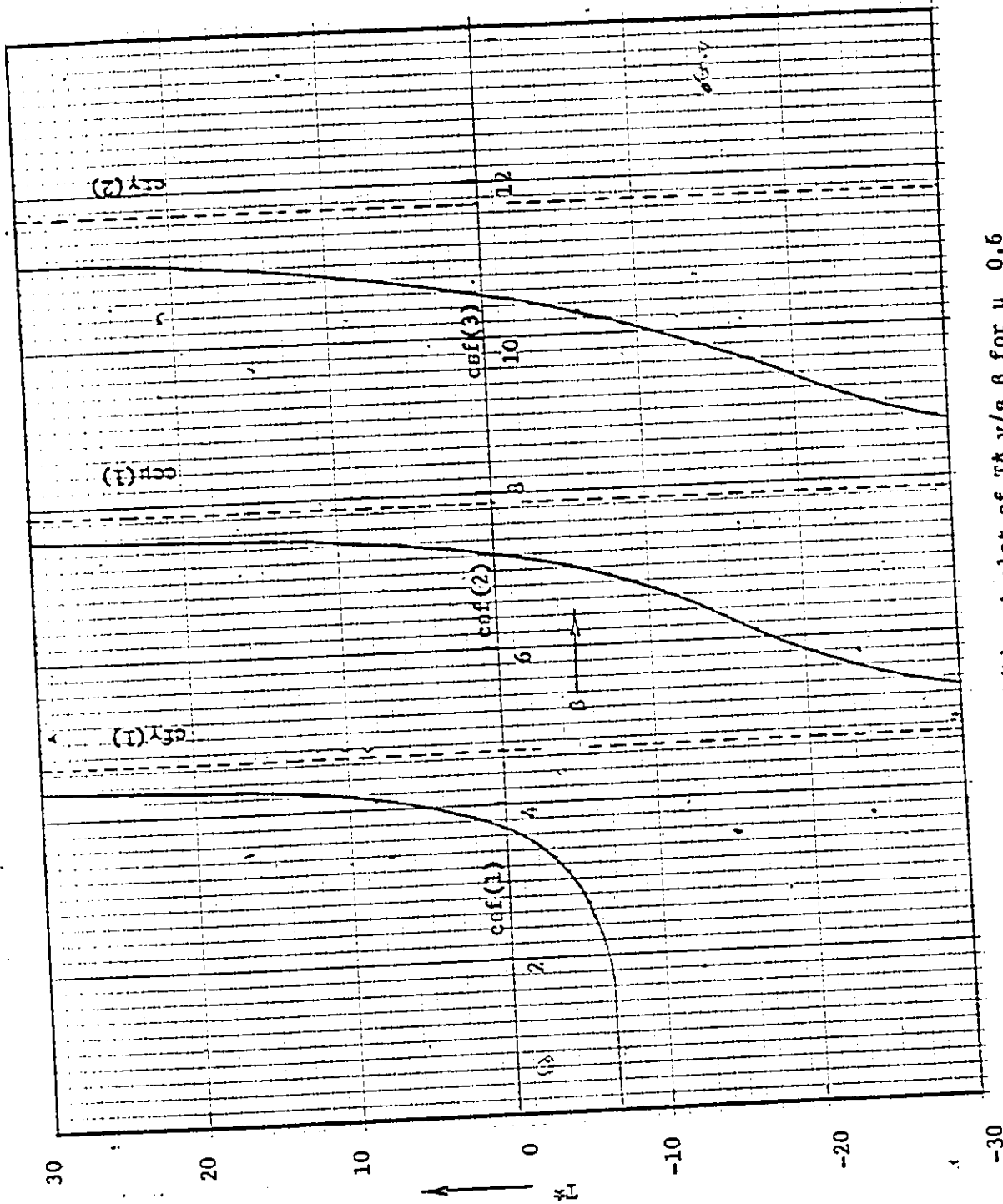


FIGURE 3.1(b) : A plot of $T^* v/s \beta$ for $\mu = 0.6$

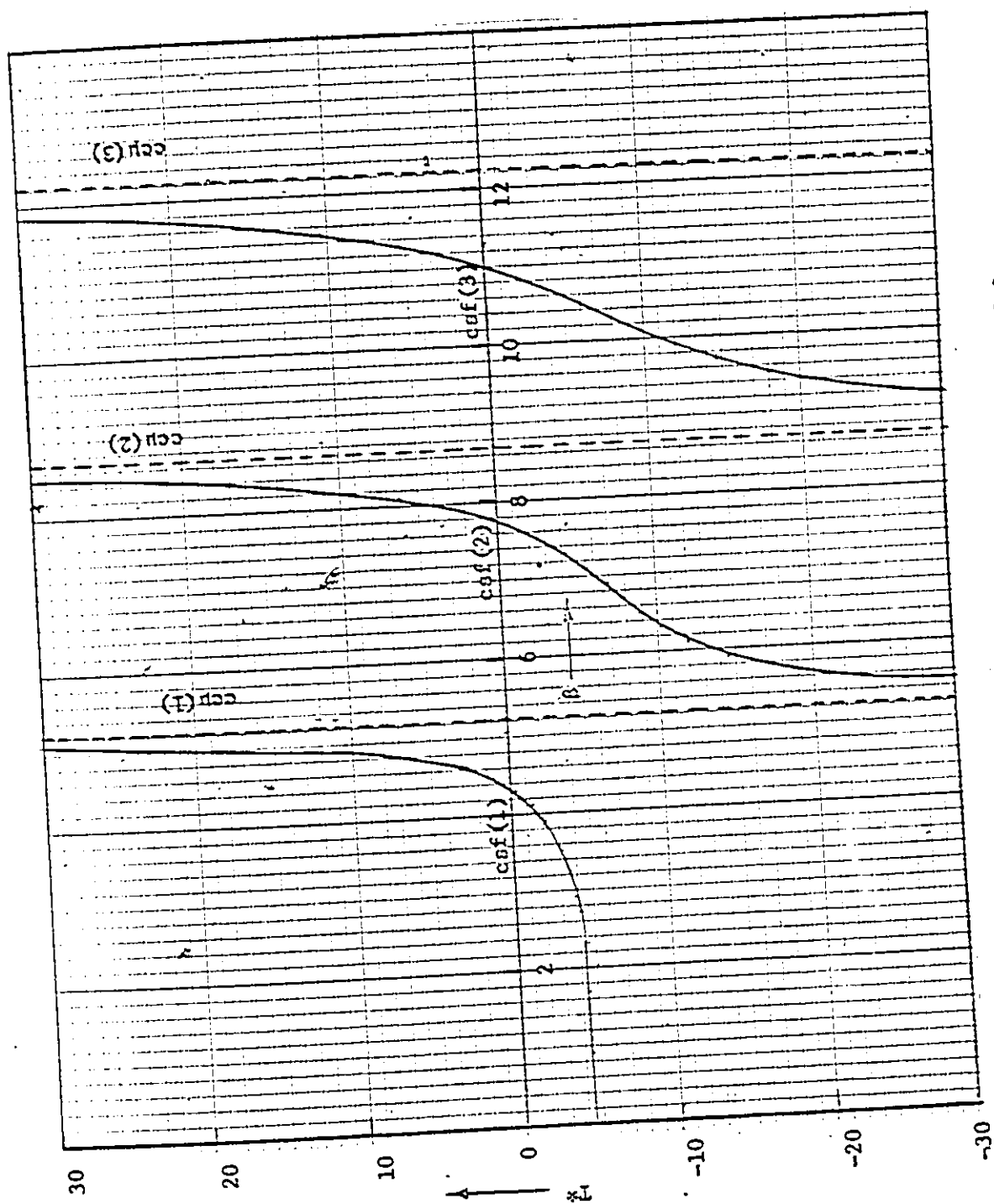


FIGURE 3.1(c) : A plot of $T^* v/\beta$ for $\mu = 0.9$

with compressive loads but it is interesting to see that a similar phenomenon exists for negative spring constants at least for the first mode for the beam system in question. A similar phenomenon was also observed by Burgreen (5) for single span beams with equal end fixity conditions. This negative value of T^* therefore gives the limiting value at which the vibration frequencies of the beam system becomes zero. The higher modes, as the curve of β versus T^* indicate, do not have any limiting values of negative spring constant which exclude the possibility of vibration. The negative values of parameter T^* at higher modes have, however, limited physical significance, since it is not generally possible to get past the lowest value of T^* that causes instability in the first mode.

For the first mode as T^* varies from its negative value which causes instability, the frequency of the system varies from zero to that of clamped-simple-free beam with $T^* = 0$ and as T^* is further increased, the frequency of the system approaches those associated with either segment of the beam depending upon the values of μ . For $\mu = 0.4$, as the Figure 3.1(a) indicates, the upper limits of frequencies for first three modes are given by $f_{cf\gamma(1)}$, $f_{cf\gamma(2)}$, and $f_{cc\mu(1)}$, whereas for $\mu = 0.6$ they are given by $f_{cf\gamma(1)}$, $f_{cf\mu(1)}$, and $f_{cf\gamma(2)}$ as shown in Figure 3.1(b) and for $\mu = 0.9$, they are given by $f_{cc\mu(1)}$, $f_{cc\mu(2)}$, and $f_{cc\mu(3)}$ as shown in Figure 3.1(c) where $f_{cf\gamma(n)}$ represents the n^{th} mode

frequency of a clamped free beam of length γL and $f_{ccu(n)}$ represents the n^{th} mode frequency of a clamped-clamped beam of length μL .

Excluding the possibility of an interest in negative values of T^* , the general rule that emerged out of the extensive study of such β versus T^* plots is that as T^* varies from zero to infinity, the frequencies of clamped-simple-free beams degenerate into next higher frequencies possessed by either the clamped-clamped segment of the beam of length μ or the clamped-free segment of length γ . Since for two span beams, present study is limited to the first four modes of vibration, it is only necessary to read first four frequencies for clamped-simple-free beams from charts in Appendix B which define the lower limits. For upper limits, the first four frequencies of left hand span and first four frequencies of the right hand span are calculated and these eight frequencies are then arranged in an ascending order. The first four of these eight frequencies govern the upper limits for respective modes of vibration of this class of beams.

To illustrate this point a little further, let us consider this case for $\mu = 0.4$. The values of frequency parameter β associated with lower limits are read off the charts provided in Appendix B for clamped-pinned-free two span beams and for the first four modes, these are 2.72, 7.1, 10.6, and 12.7. The values of this frequency parameter for the left hand segment and right hand segment are calculated using the table provided in Appendix A.

The first four values of this parameter for the left hand segment with clamped-clamped boundary conditions and a dimensionless length of $\mu = 0.4$, are 11.8, 19.6, 27.5, and 35.3 and the corresponding values of β for the right hand segment are 3.12, 7.7, 13.1, and 18.3. It may be noted that these values are obtained by dividing the values given in Appendix A by the respective non-dimensional lengths of the two segments, for example, for the left hand segment, the first value is $\frac{4.73}{0.4} = 11.8$, etc. These eight values of frequency parameter are arranged in ascending order as 3.12, 7.7, 11.8, 13.1, 18.3, 19.6, 27.5, and 35.3. The first four of these eight values, viz; 3.12, 7.7, 11.8, and 13.1 govern the upper limits of frequencies for corresponding lower limits of 2.72, 7.1, 10.6, and 12.7,

So utilizing the Q factor as before

$$f_{(n)} = f_{csf(n)} + Q[f_{(n)}^* - f_{csf(n)}] \quad (3.12)$$

where $f_{(n)}^*$ are the upper limit frequencies arranged in ascending order as explained. Utilizing equations (3.11) and (3.12), the values of T^* have been computed for various values of Q and μ and results are presented in Tables 3.1. It is interesting to note that for $\mu = 0.5$, the lower and upper limits of frequencies for third mode are too close to each other. This can be verified by reading off the first four values of β for

clamped-pinned-free double span beams from the charts provided in Appendix B. For $\mu = 0.5$, these are 3.2, 7.85, 9.4, and 14.1. The first four values of β of right hand segment of length $\gamma = 0.5$, are 3.75, 9.39, 15.7, and 21.99 and of the left hand segment with length $\mu = 0.5$ are 9.46, 15.7, ..., etc. The first four values of β out of these eight values arranged in ascending order are 3.75, 9.39, 9.46, and 15.7 and corresponding lower limits are 3.2, 7.85, 9.4, and 14.1. It can be easily seen that for the third mode the upper and lower limits are 9.46 and 9.4 which are very close together, so changing T^* does not affect the third mode frequency significantly. That is why for the third mode, Table 3.1 ($\mu = 0.5$), reads "upper and lower limits agree within 1%".

The modal shape expressions are arrived at by setting the arbitrary constant $A_1 = 1$ and the ratios $\frac{B_1}{A_1}$, $\frac{A_2}{A_1}$, and $\frac{B_2}{A_1}$ are determined from equations (3.7), (3.8), and (3.9). Thus:

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{\sinh \beta \mu - \sin \beta \mu}{\cos \beta \mu - \cosh \beta \mu} (\cos \beta \xi - \cosh \beta \xi)$$

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi + \sinh \beta \xi - \frac{\sin \beta \gamma + \sinh \beta \gamma}{\cos \beta \gamma + \cosh \beta \gamma} (\cos \beta \xi + \cosh \beta \xi) \right]$$

where

$$\gamma_1 = \frac{[\cosh \beta \mu - \cos \beta \mu + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\sin \beta \mu + \sinh \beta \mu)]}{[\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\sinh \beta \gamma - \sin \beta \gamma)]}$$

The use of the tables is demonstrated by working out, an illustrative example at the end of the Chapter.

3.2 Beams Clamped at One End and Simply Supported at the Other and With an In-Span Torsional Spring Support

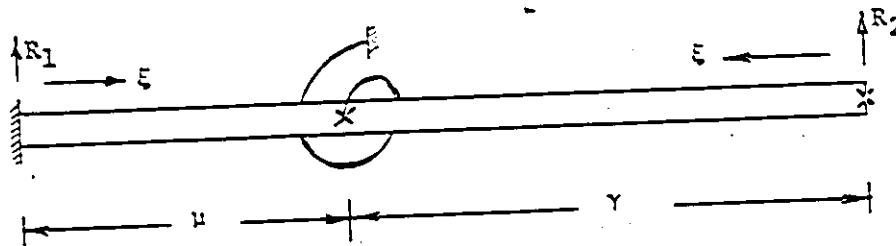


Figure 3.2

Utilizing the two classical boundary conditions at each of the two outer ends, the general solution for the deflection of the two spans can be written down as:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.13)$$

and

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.14)$$

The four boundary conditions at the intermediate support

are:

$$R_1(u) = 0, \quad R_2(\gamma) = 0$$

$$R'(u) = -R'_2(\gamma) \quad \text{and} \quad R''_1(u) + T^* R'_1(u) = R''(\gamma)$$

Enforcing these four boundary conditions in the general solutions (3.13) and (3.14), we obtain the following four equations in terms of constants A_1 , B_1 , A_2 , and C_2 :

$$A_1(\sin \beta u - \sinh \beta u) + B_1(\cos \beta u - \cosh \beta u) = 0 \quad (3.15)$$

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.16)$$

$$A_1(\cos \beta u - \cosh \beta u) + B_1(-\sin \beta u - \sinh \beta u) + A_2(\cos \beta \gamma) + C_2(\cosh \beta \gamma) = 0 \quad (3.17)$$

and

$$A_1[-\sin \beta \mu - \sinh \beta \mu + \frac{T^*}{\beta} (\cos \beta \mu - \cosh \beta \mu)] + B_1[-\cos \beta \mu - \cosh \beta \mu +$$

$$\frac{T^*}{\beta} (-\sin \beta \mu - \sinh \beta \mu) + A_2(\sin \beta \gamma) + C_2(-\sinh \beta \gamma) = 0 \quad (3.18)$$

Utilizing the condition for a non-trivial solution for the four constants in the above four equations, the following frequency equation for the beam system is obtained

$$T^* = \frac{\beta[(\sin \beta \mu + \sinh \beta \mu)D_1 - (\cos \beta \mu + \cosh \beta \mu)D_2 - \sin \beta \gamma D_3 - \sinh \beta \gamma D_4]}{[(\sin \beta \mu + \sinh \beta \mu)D_2 + (\cos \beta \mu - \cosh \beta \mu)D_1]} \quad (3.19)$$

where D_1 , D_2 , D_3 , and D_4 are respectively the minors of the coefficients of A_1 , B_1 , A_2 , and C_2 in equation (3.18).

Recognizing the lower and upper limits of frequency for this class of beams, one can write

$$f_{(n)} = f_{css(n)} + Q[f_{(n)}^* - f_{css(n)}] \quad (3.20)$$

where $f_{(n)}^*$ are the arranged upper limits of frequency corresponding to the two independent segments of the decoupled two span beam with $T^* = \infty$. The left hand span has a dimensionless length of μ .

with clamped-clamped boundary conditions whereas the right hand segment has a length of γ with clamped-simple boundary conditions.

Tables 3.2 present computed values of T^* for, 0.1 μ 0.9, and 0.05 Q 0.9. As explained in Chapter 2, linear interpolation may be used for intermediate values of μ and Q ($Q < 0.8$) not covered by tables. For values of $Q > 0.8$, the asymptotic equation (2.15) may be used. A number of accuracy checks indicated that both these approaches are consistent with an accuracy requirement of 1% for frequencies.

The modal shape expressions are obtained by utilizing equations (3.13), (3.14), (3.15), (3.16), and (3.17). These expressions in their simplified forms are presented below

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\cos \beta \xi - \cosh \beta \xi)$$

and

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right]$$

where

$$\gamma_1 = \frac{[\cosh \beta \mu - \cos \beta \mu + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\sin \beta \mu + \sinh \beta \mu)]}{\cos \beta \gamma - \cosh \beta \gamma \left(\frac{\sin \beta \gamma}{\sinh \beta \gamma} \right)}$$

3.3 Beams Clamped at Outer Ends and With an In-Span Torsional Spring

Support

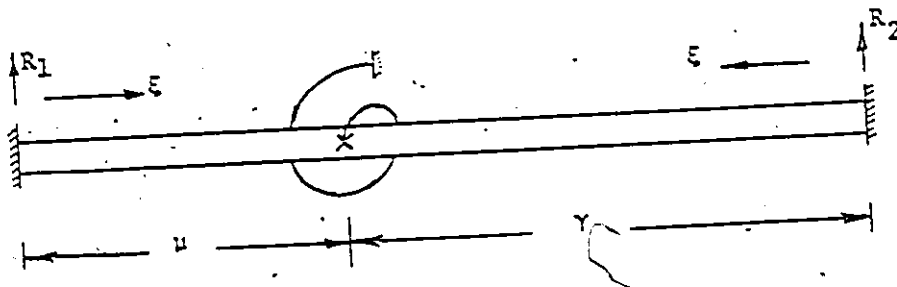


Figure 3.3

By using four boundary conditions, two at each outer end, the four of the eight constants appearing in the shape functions of the two spans are eliminated and we obtain:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.21)$$

$$R_2(\xi) = A_2(\sin \beta \xi - \sinh \beta \xi) + B_2(\cos \beta \xi - \cosh \beta \xi) \quad (3.22)$$

Further enforcing the remaining four boundary conditions,

viz;

$$R_1(\mu) = 0, \quad R_2(\gamma) = 0, \quad R'_1(\mu) = -R'_2(\gamma) \quad \text{and}$$

$$R''_1(\mu) + T^* R'_1(\mu) = R''_2(\gamma),$$

we obtain the following four equations in terms of constants $A_1, B_1, A_2,$
and B_2 .

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.23)$$

$$A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.24)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu) + A_2(\cos \beta \gamma - \cosh \beta \gamma)$$

$$+ B_2(-\sin \beta \gamma - \sinh \beta \gamma) = 0 \quad (3.25)$$

and

$$A_1[-\sin \beta \mu - \sinh \beta \mu + \frac{T^*}{\beta} (\cos \beta \mu - \cosh \beta \mu)] + B_1[-\cos \beta \mu - \cosh \beta \mu +$$

$$\frac{T^*}{\beta} (-\sin \beta \gamma - \sinh \beta \gamma)] + A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0$$

$$(3.26)$$

The condition for the non-trivial solution yields the following frequency equation:

$$T^* = \frac{\beta[(\sin \beta \mu + \sinh \beta \mu) D_1 - (\cos \beta \mu + \cosh \beta \mu) D_2 - (\sin \beta \gamma + \sinh \beta \gamma) D_3 + (\cos \beta \gamma + \cosh \beta \gamma) D_4]}{[(\sin \beta \mu + \sinh \beta \mu) D_2 + (\cos \beta \mu - \cosh \beta \mu) D_1]} \quad (3.27)$$

The frequencies of this class of beams can be expressed in terms of Q factor as follows:

$$f_{(n)} = f_{\text{csc}(n)} + Q[f_{(n)}^* - f_{\text{csc}(n)}] \quad (3.28)$$

where $f_{(n)}^*$ are the upper limit frequencies arranged in order as explained earlier.

Tables 3.3 furnish all the necessary information for establishing frequencies of such beam systems. Since the beam is symmetric with respect to its outer end boundary conditions, the values of μ are given only upto 0.5 since prescribing $\mu > 0.5$ will yield redundant information.

The upper and lower limits of frequency for second and fourth

mode for $\mu = 0.5$, are same as indicated in the tables. To explain this let us take a look at the second mode frequency parameter and the associated modal shape for clamped-pinned-clamped two span beam. The value of β for this mode is 9.46 and the modal shape is as shown in Figure

3.1(a)

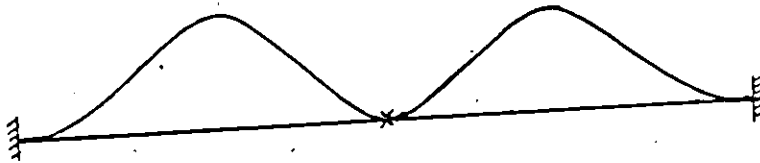


Figure 3.3(a)

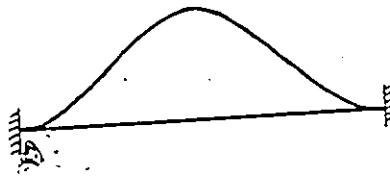


Figure 3.3(b)

Since the modal shape associated with the second mode frequency of clamped-simple-clamped double span beam is symmetric, the slope at the intermediate support is essentially zero, hence the presence of the torsional spring does not affect the frequency as the torsional spring

cannot provide any restoring moment with zero slope. In fact the modal shape remains the same even when the intermediate support becomes clamped with $T^* = \infty$ as shown in Figure 3.3(b). These remarks are in fact valid for second and fourth modes of all the symmetric cases discussed in this Chapter.

The modal shape expressions are obtained in their usual manner and are presented below in simplified form.

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\cos \beta \xi - \cosh \beta \xi)$$

and

$$R_2(\xi) = \gamma_1 [\sin \beta \xi - \sinh \beta \xi + \frac{(\sinh \beta \gamma - \sin \beta \gamma)}{(\cos \beta \gamma - \cosh \beta \gamma)} (\cos \beta \xi - \cosh \beta \xi)]$$

where

$$\gamma_1 = \frac{[\cosh \beta \mu - \cos \beta \mu + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\sin \beta \mu + \sinh \beta \mu)]}{[\cos \beta \gamma - \cosh \beta \gamma + \frac{(\sinh \beta \gamma - \sin \beta \gamma)}{(\cosh \beta \gamma - \cos \beta \gamma)} (\sin \beta \gamma + \sinh \beta \gamma)]}$$

3.4 Beams Simply Supported at One End and Free at the Other and With an In-Span Torsional Spring Support

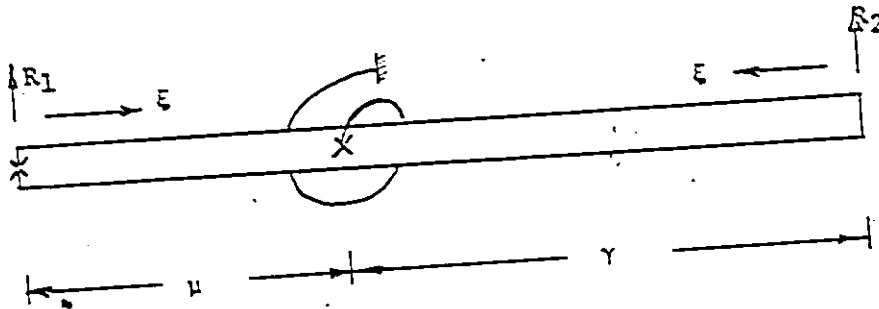


Figure 3.4

Utilizing the boundary conditions at the outer ends, it can be readily shown that

$$R_1(\xi) = A_1 \sin \beta \xi + C_1 \sinh \beta \xi \quad (3.29)$$

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi) \quad (3.30)$$

Further utilizing the remaining four boundary conditions at the intermediate support as provided in the previous case 3.3, we obtain the following four equations in terms of constants A_1 , C_1 , A_2 , and B_2 .

$$A_1 \sin \beta \mu + C_1 \sinh \beta \mu = 0 \quad (3.31)$$

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.32)$$

$$A_1(\cos \beta \mu) + C_1(\cosh \beta \mu) + A_2(\cos \beta \gamma + \cosh \beta \gamma) + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \quad (3.33)$$

and

$$A_1[-\sin \beta \mu + \frac{T^*}{\beta} (\cos \beta \mu)] + C_1[\sinh \beta \mu + \frac{T^*}{\beta} \cosh \beta \mu] + A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.34)$$

For a non-trivial solution for these constants, the following frequency equation must be satisfied:

$$T^* = \frac{-\beta[\sin \beta \mu.D_1 + \sinh \beta \mu.D_2 - \sin \beta \gamma.D_3 + \sinh \beta \gamma.D_3 + (\cos \beta \gamma - \cosh \beta \gamma)D_4]}{\cos \beta \mu.D_1 - \cosh \beta \mu.D_2}$$

where D_1 , D_2 , D_3 , and D_4 are respectively the minors of the coefficients of A_1 , C_1 , A_2 , and B_2 appearing in equation (3.34).

The relationship between β and Q is established from the

following equation

$$f_{(n)} = f_{ssf(n)} + Q[f_{(n)}^* - f_{ssf(n)}] \quad (3.36)$$

where $f_{ssf(n)}$ are read off the charts provided in Appendix B and $f_{(n)}^*$ are the arranged upper limits which can be easily calculated from the single span frequency parameters provided in Appendix A.

Tables 3.4 provide the values of system parameter T^* for various values of μ and Q .

The modal shape expressions are simplified to

$$R_1(\xi) = \sin \beta \xi - \frac{\sin \beta \mu}{\sinh \beta \mu} \sinh \beta \xi$$

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right]$$

where

$$\gamma_1 = \frac{\cosh \beta \mu \left(\frac{\sin \beta \mu}{\sinh \beta \mu} \right) - \cos \beta \mu}{\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\sinh \beta \gamma - \sin \beta \gamma)}$$

3.5 Beams Simply Supported at Outer Ends and With an In-Span Torsional

Spring Support

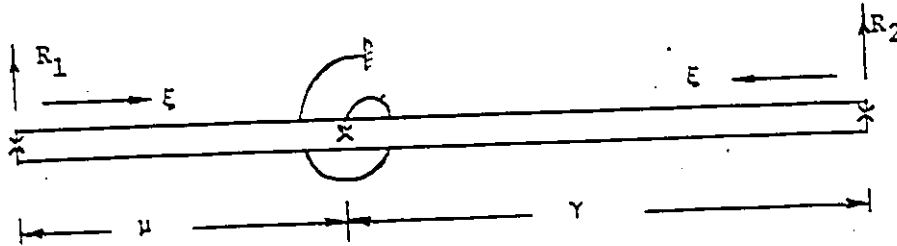


Figure 3.5

Utilizing the boundary conditions at the outer ends, it can be easily shown that

$$R_1(\xi) = A_1 \sin \beta \xi + C_1 \sinh \beta \xi \quad (3.37)$$

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.38)$$

Further, at the intermediate support, the deflection should be zero and slope and bending moment should be continuous. Utilizing these conditions at the intermediate support, we obtain:

$$A_1 \sin \beta \mu + C_1 \sinh \beta \mu = 0 \quad (3.39)$$

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.40)$$

$$A_1 \cos \beta \mu + C_1 \cosh \beta \mu + A_2 \cos \beta \gamma + C_2 \cosh \beta \gamma = 0 \quad (3.41)$$

and

$$A_1(-\sin \beta \mu + \frac{T^*}{\beta} \cos \beta \mu) + C_1(\sinh \beta \mu + \frac{T^*}{\beta} \cosh \beta \mu) + A_2 \sin \beta \gamma + C_2(-\sinh \beta \gamma) = 0 \quad (3.42)$$

Expanding the determinant of the coefficient matrix of constants A_1 , C_1 , A_2 , and C_2 in the above four equations and setting it equal to zero, we obtain:

$$T^* = \frac{\beta[-\sin \beta \mu.D_1 - \sinh \beta \mu.D_2 + \sin \beta \gamma.D_3 + \sinh \beta \gamma.D_4]}{\cosh \beta \mu.D_2 - \cos \beta \mu.D_1} \quad (3.43)$$

The frequencies of this system are related to Q factor

by,

$$f_{(n)} = f_{ss(n)} + Q[f_{(n)}^* - f_{ss(n)}] \quad (3.44)$$

where $f_{ss(n)}$ and $f_{(n)}^*$ have their usual meaning. Tables 3.5 furnish all the necessary information for establishing frequencies of this class of beams. Being a symmetric case, values of μ are only given upto 0.5.

The modal shape functions are obtained by setting $A_1 = 1$

and evaluating the ratios $\frac{C_1}{A_1}$, $\frac{A_2}{A_1}$, and $\frac{C_2}{A_1}$ from equations (3.39), (3.40) and (3.41). The modal shape expressions in their simplified form reduce to:

$$R_1(\xi) = \sin \beta \xi - \frac{\sin \beta \mu}{\sinh \beta \mu} \sinh \beta \xi$$

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right]$$

where

$$\gamma_1 = \frac{\cosh \beta \mu \left(\frac{\sin \beta \mu}{\sinh \beta \mu} \right) - \cos \beta \mu}{\cos \beta \gamma - \cosh \beta \gamma \left(\frac{\sin \beta \gamma}{\sinh \beta \gamma} \right)}$$

3.6 Beams Clamped at One End and Elastically Restrained Against Rotation at the Other With a Pinned Intermediate Support

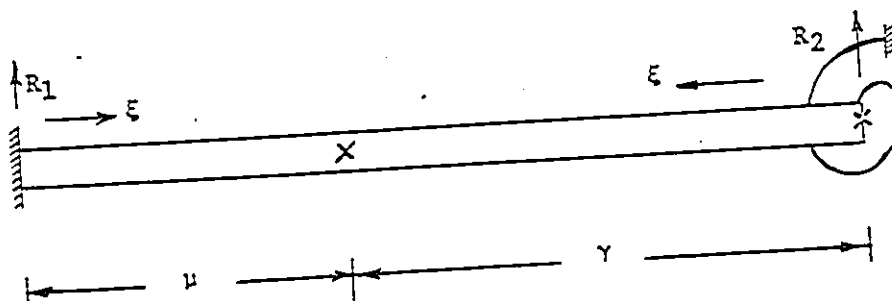


Figure 3.6

Utilizing the requirements of no deflection and slope at the end $\xi = 0$ for the left hand span we obtain:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.45)$$

For the right hand span, the boundary condition $R_2(0) = 0$ leads to

$$R_2(\xi) = A_2 \sin \beta \xi + B_2(\cos \beta \xi - \cosh \beta \xi) + C_2 \sinh \beta \xi \quad (3.46)$$

Utilizing the boundary condition $R_2(\gamma) = 0$, the constant B_2 can be expressed in terms of A_2 and C_2 as follows:

$$B_2 = A_2 \theta + C_2 \phi \quad (3.47)$$

where

$$\theta = \frac{\sin \beta \gamma}{\cosh \beta \gamma - \cos \beta \gamma}$$

$$\phi = \frac{\sinh \beta \gamma}{\cosh \beta \gamma - \cos \beta \gamma}$$

so that equation (3.46) is modified to

$$R_2(\xi) = A_2[\sin \beta \xi + \theta(\cos \beta \xi - \cosh \beta \xi)] + C_2[\sinh \beta \xi + \phi(\cos \beta \xi - \cosh \beta \xi)]$$

(3.48)

The four boundary conditions that still remain to be enforced are:

$$R_1(\mu) = 0, \quad R'_1(\mu) = -R'_2(\gamma), \quad R''_1(\mu) = R''_2(\gamma)$$

and

$$R'''_2(0) = \frac{1}{P} R''_2(0)$$

Utilizing these four boundary conditions, we obtain:

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.49)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu)$$

$$+ A_2[\cos \beta \gamma + \theta(-\sin \beta \gamma - \sinh \beta \gamma)]$$

$$+ C_2[\cosh \beta \gamma + \phi(-\sin \beta \gamma - \sinh \beta \gamma)] = 0 \quad (3.50)$$

$$\begin{aligned}
 & A_1[-\sin \beta \mu - \sinh \beta \mu] + B_1[-\cos \beta \mu - \cosh \beta \mu] \\
 & + A_2[\sin \beta \gamma + \theta(\cos \beta \gamma + \cosh \beta \gamma)] \\
 & + C_2[-\sinh \beta \gamma + \phi(\cos \beta \gamma + \cosh \beta \gamma)] = 0
 \end{aligned} \tag{3.51}$$

and

$$A_2 \left[\frac{T^*}{\beta} + 2\theta \right] + C_2 \left[\frac{T^*}{\beta} + 2\phi \right] = 0 \tag{3.52}$$

For a non-trivial solution for constants A_1 , B_1 , A_2 , and C_2 in equations (3.49) through (3.52), the determinant of their coefficient matrix must vanish. Utilizing this condition, we obtain:

$$T^* = \frac{\beta (2\phi \cdot D_4 - 2\theta \cdot D_3)}{D_3 - D_4} \tag{3.53}$$

Where D_3 and D_4 are the minors of the coefficient of A_2 and C_2 respectively in the last equation.

The upper and lower limits of frequency for this class of beams are obviously those of clamped-simple-simple and clamped-simple-clamped two span beams.

$$f_{(n)} = f_{css(n)} + Q[f_{csc(n)} - f_{css(n)}] \quad (3.54)$$

Utilizing the relationship between f and Q , and therefore between β and Q , from equation (3.54), the values of T^* for various values of Q and μ are computed and are presented in Tables 3.6.

The modal shape expressions are obtained in the usual manner and are given below.

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\cos \beta \xi - \cosh \beta \xi)$$

and

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)} (\sinh \beta \xi + \phi \cos \beta \xi - \phi \cosh \beta \xi) \right]$$

where

$$\gamma_1 = \frac{[\cosh \beta \mu - \cos \beta \mu + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\sin \beta \mu + \sinh \beta \mu)]}{[\cos \beta \gamma - \theta (\sin \beta \gamma + \sinh \beta \gamma) - \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)} (\cosh \beta \gamma - \phi \sin \beta \gamma - \phi \sinh \beta \gamma)]}$$

3.7 Beams with a Torsional Spring at One Outer End and Simply-Supported
at the Remaining Two Supports

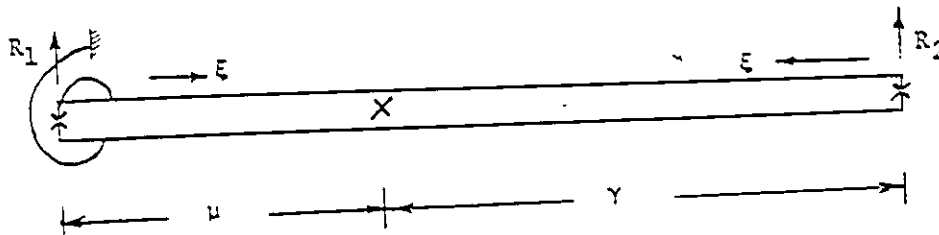


Figure 3.7

Utilizing the conditions $R_1(0) = 0$ and $R_1(\mu) = 0$, we obtain

$$R_1(\xi) = A_1[\sin \beta \xi + \theta(\cos \beta \xi - \cosh \beta \xi)] + C_1[\sinh \beta \xi + \phi(\cos \beta \xi - \cosh \beta \xi)] \quad (3.55)$$

where

$$\theta = \frac{\sin \beta \mu}{\cosh \beta \mu - \cos \beta \mu} \quad \text{and} \quad \phi = \frac{\sinh \beta \mu}{\cosh \beta \mu - \cos \beta \mu}$$

For the right hand span, the requirement of no deflection and no moment at the end $\xi = 0$, yields

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.56)$$

The remaining four boundary conditions are:

$$R_2(\gamma) = 0, \quad R'_1(\mu) = -R'_2(\gamma), \quad R''_1(\mu) = R''_2(\gamma)$$

and

$$R''_1(0) = T^* R'_1(0)$$

Substituting these into the modal shape functions (3.55) and (3.56) we obtain the following four equations

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.57)$$

$$A_1 [\cos \beta \mu + \theta (-\sin \beta \mu - \sinh \beta \mu)] + C_1 [\cosh \beta \mu + \phi (-\sin \beta \mu - \sinh \beta \mu)]$$

$$+ A_2 \cos \beta \gamma + C_2 \cosh \beta \gamma = 0 \quad (3.58)$$

$$A_1 [-\sin \beta \mu + \theta (-\cos \beta \mu - \cosh \beta \mu)] + C_1 [\sinh \beta \mu + \phi (-\cos \beta \mu - \cosh \beta \mu)]$$

$$+ A_2 \sin \beta \gamma + C_2 (-\sinh \beta \gamma) = 0 \quad (3.59)$$

$$A_1 \left[\frac{T^*}{\beta} + 2\theta \right] + C_1 \left[\frac{T^*}{\beta} + 2\phi \right] = 0 \quad (3.60)$$

The condition for the non-trivial solution for the constants in the above four equations yields

$$T^* = \frac{\beta (2\phi \cdot D_2 - 2\theta \cdot D_1)}{D_1 - D_2} \quad (3.61)$$

where D_1 and D_2 are respectively the minors of the coefficients of A_1 and C_1 from equation (3.60).

The frequencies of this system are related to its limiting frequencies by means of Q factor as follows

$$f_{(n)} = f_{ss(n)} + Q[f_{css(n)} - f_{ss(n)}] \quad (3.62)$$

Tables 3.7 present values of system parameter T^* for various values of Q and μ .

The modal shape expressions are reduced to the following simplified forms.

$$R_1(\xi) = \sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)}$$

$$(\sinh \beta \xi + \phi \cos \beta \xi - \phi \cosh \beta \xi)$$

and

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right]$$

where

$$\gamma_1 = \frac{[\theta(\sin \beta \mu + \sinh \beta \mu) - \cos \beta \mu + \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)} (\cosh \beta \mu - \phi \sin \beta \mu - \phi \sinh \beta \mu)]}{\cos \beta \gamma - \left(\frac{\sin \beta \gamma}{\sinh \beta \gamma} \right) \cosh \beta \gamma}$$

3.8 Beams With One Outer End Elastically Restrained Against Rotation,

the Other Free and With an Intermediate Pinned Support

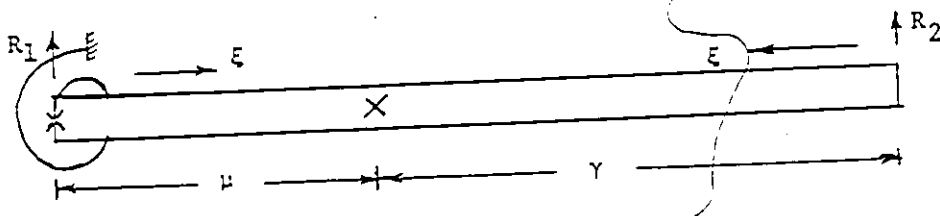


Figure 3.8

The boundary conditions $R_1(0) = 0$ and $R_1(\mu) = 0$ lead to

$$R_1(\xi) = A_1 [\sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi)] + C_1 [\sinh \beta \xi + \phi (\cos \beta \xi - \cosh \beta \xi)] \quad (3.63)$$

where

$$\theta = \frac{\sin \beta \mu}{\cosh \beta \mu - \cos \beta \mu} \quad \text{and} \quad \phi = \frac{\sinh \beta \mu}{\cosh \beta \mu - \cos \beta \mu}$$

and the boundary conditions $R''_2(0) = R'''_2(0) = 0$ lead to

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi) \quad (3.64)$$

The remaining four boundary conditions, namely, $R_2(\gamma) = 0$, $R'_1(\mu) = -R'_2(\gamma)$, $R''_1(\mu) = R''_2(\gamma)$ and $R'''_1(0) = T^* R'''_1(0)$ result in the following four equations

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.65)$$

$$A_1[\cos \beta \mu + \theta(-\sin \beta \mu - \sinh \beta \mu)] + C_1[\cosh \beta \mu + \phi(-\sin \beta \mu - \sinh \beta \mu)]$$

$$+ A_2(\cos \beta \gamma + \cosh \beta \gamma) + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \quad (3.66)$$

$$A_1[-\sin \beta \mu + \theta(-\cos \beta \mu - \cosh \beta \mu)] + C_1[\sinh \beta \mu + \phi(-\cos \beta \mu - \cosh \beta \mu)]$$

$$+ A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.67)$$

and

$$A_1 \left(\frac{T^*}{\beta} + 2 \right) + C_1 \left(\frac{T^*}{\beta} + 2\phi \right) = 0 \quad (3.68)$$

Utilizing the condition for the non-trivial solution of these four constants appearing in the above four equations, we obtain:

$$T^* = \frac{\beta (2\phi \cdot D_2 - 2\theta \cdot D_1)}{D_1 - D_2} \quad (3.69)$$

where D_1 and D_2 are the minors of the coefficients of A_1 and C_1 in the equation (3.68).

Relationship between frequency parameter β and Q are given by

$$f_{(n)} = f_{ssf(n)} + Q[f_{csf(n)} - f_{ssf(n)}] \quad (3.70)$$

Tables 3.8 furnish the designer all the information necessary to establish frequencies of this class of beams.

The modal shape expressions are given below in their simplified forms:

$$R_1(\xi) = \sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)}$$

$$(\sinh \beta \xi + \phi \cos \beta \xi - \phi \cosh \beta \xi)$$

$$R_2(\xi) = \gamma_1 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right]$$

where

$$\gamma_1 = \frac{[\theta(\sin \beta \mu + \sinh \beta \mu) - \cos \beta \mu + \frac{(2\theta \beta + T^*)}{(2\phi \beta + T^*)} (\cosh \beta \mu - \phi \sin \beta \mu - \phi \sinh \beta \mu)]}{[\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\sinh \beta \gamma - \sin \beta \gamma)]}$$

Part 2 - Beams with one non-classical boundary condition provided by a rotational mass

3.9 Beams With One Outer End Clamped and the Other Free and With an In-Span Rotational Mass

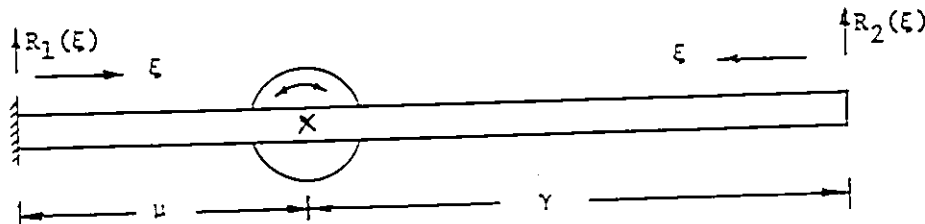


Figure 3.9

Enforcing two classical boundary conditions at each of the outer ends, it can be shown that:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi)$$

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi)$$

The remaining four boundary conditions are: $R_1(\mu) = 0$, $R_2(\gamma) = 0$, $R'_1(\mu) = -R'_2(\gamma)$, and $R''_1(\mu) - \frac{\beta^4}{I^*} R'_1(\mu) = R''_2(\gamma)$ where the last of these boundary condition is obtained from the condition of moment continuity at the intermediate support.

Substituting these four boundary conditions in the above two expressions for $R_1(\xi)$ and $R_2(\xi)$, we obtain:

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.71)$$

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.72)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu) + A_2(\cos \beta \gamma + \cosh \beta \gamma) + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \quad (3.73)$$

and

$$A_1[-\sin \beta \mu - \sinh \beta \mu - \frac{\beta^3}{I^*}(\cos \beta \mu - \cosh \beta \mu)] + B_1[-\cos \beta \mu - \cosh \beta \mu - \frac{\beta^3}{I^*}(-\sin \beta \mu - \sinh \beta \mu)] + A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.74)$$

For a non-trivial solution for the constants A_1 , B_1 , A_2 , and B_2 in the above four equations, the determinant of their coefficient matrix is set equal to zero which yields:

$$I^* = \frac{s^3 (\cos s \mu - \cosh s \mu) D_1 + (\sin s \mu + \sinh s \mu) D_2}{-(\sin s \mu + \sinh s \mu) D_1 + (\cos s \mu + \cosh s \mu) D_2 + (\sin s \gamma - \sinh s \gamma) D_3 + (\cosh s \gamma - \cos s \gamma) D_4} \quad (3.75)$$

where D_1 , D_2 , D_3 , and D_4 are respectively the minors of the coefficients of A_1 , B_1 , A_2 , and B_2 in the last row of the determinant.

In the limiting case when I^* approaches zero, the fundamental frequency of this system will approach zero as the moment of inertia I_0 of the rotation mass is infinitely larger than $\rho A L^3$ of the beam ($I^* = \frac{\rho A L^3}{I_0}$). For $I^* = \infty$, the beam frequency will approach that of a clamped-simple-free double span beam. Therefore, for the first mode, we can write:

$$f_{(n)} = Q f_{csf(n)} \quad \text{for } n = 1$$

For higher modes, the upper limits of frequency are now well defined as for $I^* = \infty$, the beam behaves as a clamped-simple-free double span beam. The lower limits of frequency are associated with the condition $I^* = 0$ which makes the intermediate support behave as a clamped support and therefore decouples the beam into two separate segments, capable of vibrating independently. For the beam system under consideration, the left segment of the beam has clamped-clamped boundary conditions and a non-dimensional length of μ whereas the right segment has a length of γ and clamped-free boundary conditions. The first three frequencies of each of these two segments are calculated and these six frequencies are arranged in ascending order. The first three of these six frequencies define the lower limits of frequencies for 2nd, 3rd, and 4th modes, the lower limit for fundamental mode being zero. In terms of Q factor, they are given as

$$f_{(n)} = f_{(n)}^* + Q[f_{csf(n)} - f_{(n)}^*] \quad \text{for } n > 1 \quad (3.76)$$

To illustrate this, let us take the case of such a beam where rotational mass is attached at a distance $\mu = 0.6$ from the left hand end. The first four values of β , as read off the charts provided in Appendix B, for clamped-simple-free beams with $\mu = 0.6$ are, 3.65, 7.25,

10.5, and 12.7. The first three values of δ for a clamped-clamped beam of length 0.6 are 7.9, 13.1, and 18.3, and for a clamped-free beam of length γ ($\gamma=0.4$) are 4.69, 11.73, and 19.63.

Therefore, for the first four modes of frequencies, the lower and upper limits of frequencies are 0, 4.69, 7.9, 11.73 and 3.65, 7.25, 10.5, and 12.7 respectively. It may be noted that lower limits are arranged in the ascending order from a total of six values, three of each segment with first lower limit always equal to zero for this class of beams.

Tables 3.9 present the computed values of I^* for various values of μ and Q . For values not covered by the tables, utilization of linear interpolation or the asymptotic equation (2.15) yield results for frequencies accurate within 1%.

The modal shape expression for this class of beams are same as given in case 3.1.

3.10 Beams Clamped at One End Simply-Supported at the Other and With an

In-Span Rotational Mass

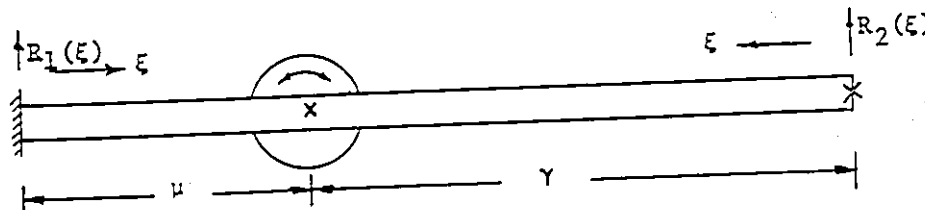


Figure 3.10

Utilizing the boundary conditions prevailing at outer ends,
we obtain:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.77)$$

and

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.78)$$

Further utilizing the four boundary conditions prevailing at the intermediate support as in case 3.9; we obtain the following four equations in terms of constants A_1 , B_1 , A_2 , and C_2 .

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.79)$$

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.80)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu) + A_2 \cos \beta \gamma + B_2 \cosh \beta \gamma = 0 \quad (3.81)$$

and

$$\begin{aligned} & A_1 \left[-\sin \beta \mu - \sinh \beta \mu - \frac{\beta^3}{I^*} (\cos \beta \mu - \cosh \beta \mu) \right] \\ & + B_1 \left[-\cos \beta \mu - \cosh \beta \mu - \frac{\beta^3}{I^*} (-\sin \beta \mu - \sinh \beta \mu) \right] \\ & + A_2 \sin \beta \gamma + C_2 (-\sinh \beta \gamma) = 0 \end{aligned} \quad (3.82)$$

Utilizing the condition for a non-trivial solution for these four constants yields:

$$I^* = \frac{s^3[(\cos s \mu - \cosh s \nu)D_1 + (\sin s \mu + \sinh s \nu)D_2]}{[-(\sin s \mu + \sinh s \nu)D_1 + (\cos s \mu + \cosh s \nu)D_2 + \sin s \gamma D_3 + \sinh s \gamma D_4]} \quad (3.83)$$

where D_1 , D_2 , D_3 , and D_4 have the same meaning assigned to them previously.

Utilizing Q factor, the frequencies of this system can be expressed as

$$f_{(n)} = Q f_{css(n)} \quad \text{for } n = 1 \quad (3.84)$$

and

$$f_{(n)} = f_{(n)}^* + Q[f_{css(n)} - f_{(n)}^*] \quad \text{for } n > 1$$

where $f_{(n)}^*$ are the lowest n frequencies arranged in ascending order associated with either the left hand segment or the right hand segment of the beam.

Tables 3.10 furnish all the information necessary to establish frequencies of this class of beams.

The modal shape expressions for this case are same as those provided in case 3.2.

3.11 Beams Clamped at Outer Ends and With a Rotational Mass Located
Anywhere in the Span

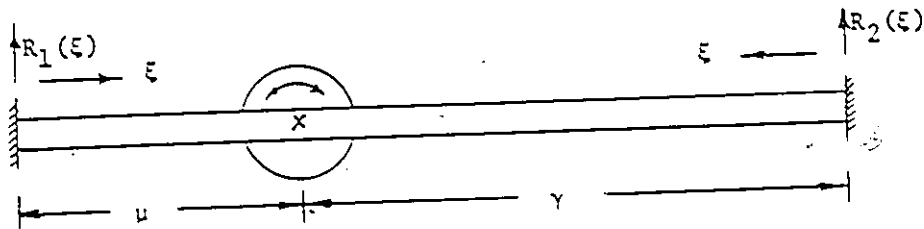


Figure 3.11

The requirement of no slope and no deflection at the outer ends yields:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.85)$$

$$R_2(\xi) = A_2(\sin \beta \xi - \sinh \beta \xi) + B_2(\cos \beta \xi - \cosh \beta \xi) \quad (3.86)$$

Utilizing the remaining four boundary conditions at the intermediate support as given in case 3.9, we obtain:

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.87)$$

$$A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.88)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu) + A_2(\cos \beta \gamma - \cosh \beta \gamma) + B_2(-\sin \beta \gamma - \sinh \beta \gamma) = 0 \quad (3.89)$$

and

$$A_1[-\sin \beta \mu - \sinh \beta \mu - \frac{\beta^3}{I^*}(\cos \beta \mu - \cosh \beta \mu)] + B_1[-\cos \beta \mu - \cosh \beta \mu - \frac{\beta^3}{I^*}(-\sin \beta \mu - \sinh \beta \mu)] + A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.90)$$

For a non-trivial solution for the constants A_1 , B_1 , A_2 , and B_2 in the above four equations, the following equation must be satisfied

$$I^* = \frac{\beta^3[(\cos \beta \mu - \cosh \beta \mu) D_1 + (\sin \beta \mu + \sinh \beta \mu) D_2]}{[-(\sin \beta \mu + \sinh \beta \mu) D_1 + (\cos \beta \mu + \cosh \beta \mu) D_2] + (\sin \beta \gamma + \sinh \beta \gamma) D_3 - (\cos \beta \gamma + \cosh \beta \gamma) D_4} \quad (3.91)$$

The frequencies of this class of beams are related to its limiting frequencies as follows

$$f_{(n)} = Q f_{csc(n)} \quad n = 1$$

and

$$f_{(n)} = f_{(n)}^* + Q[f_{csc(n)} - f_{(n)}^*] \quad \text{for } n > 1 \quad (3.92)$$

Tables 3.11 present results for this case. It may be noted that since this is a symmetric problem, the values of μ are given only upto 0.5. For $\mu = 0.5$, as explained earlier, the upper and lower limits of frequencies are the same for second and fourth modes. These modes being symmetric, the presence of a rotational mass is not felt as the slope at $\mu = 0.5$ remains zero for these two symmetric modes.

Modal shape expressions for this case are same as those of case 3.3.

3.12 Beams Simply Supported at One Outer End and Free at the Other, and With an In-Span Rotational Mass

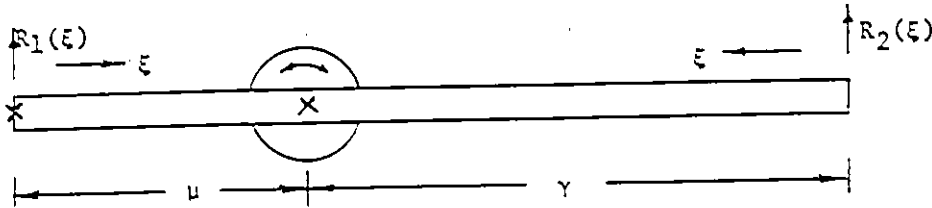


Figure 3.12

Utilizing the boundary conditions at the simply supported and free ends, we obtain:

$$R_1(\xi) = A_1 \sin \beta \xi + C_1 \sinh \beta \xi \quad (3.93)$$

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi) \quad (3.94)$$

Enforcing the boundary conditions at the intermediate support, we obtain:

$$A_1 \sin \beta \mu + C_1 \sinh \beta \mu = 0 \quad (3.95)$$

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.96)$$

$$\begin{aligned} & A_1 \cos \beta \mu + C_1 \cosh \beta \mu + A_2(\cos \beta \gamma + \cosh \beta \gamma) \\ & + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \end{aligned} \quad (3.97)$$

and

$$A_1(-\sin \beta \mu - \frac{\beta^3}{I^*} \cos \beta \mu) + C_1(\sinh \beta \mu - \frac{\beta^3}{I^*} \cosh \beta \mu) + A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.98)$$

The condition for a non-trivial solution yields:

$$I^* = \frac{\beta^3(\cos \beta \mu D_1 - \cosh \beta \mu D_2)}{[-\sin \beta \mu D_1 - \sinh \beta \mu D_2 + (\sin \beta \gamma - \sinh \beta \gamma) D_3 + (\cosh \beta \gamma - \cos \beta \gamma) D_4]} \quad (3.99)$$

The relationship between frequency and Q is established by

$$f_{(n)} = Q f_{ssf(n)} \quad \text{for } n = 1$$

and

$$f_{(n)} = f_{(n)}^* + Q[f_{ssf(n)} - f_{(n)}^*] \quad (3.100)$$

Tables 3.12 provide all the information for this case. The modal shape expressions are same as those of case 3.4.

3.13 Beams Simply Supported at Outer Ends and With an In-Span Rotational Mass

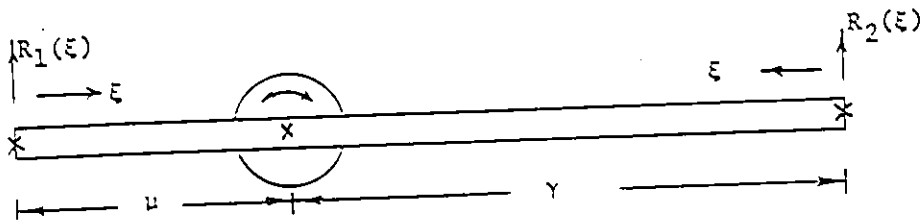


Figure 3.13

The requirements of simply supported outer ends yields:

$$R_1(\xi) = A_1 \sin \beta \xi + C_1 \sinh \beta \xi \quad (3.101)$$

and

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.102)$$

Boundary conditions at the intermediate support, namely,

$$R_1(\mu) = 0, \quad R_2(\gamma) = 0, \quad R'_1(\mu) = -R'_2(\gamma), \quad \text{and}$$

$$R''_1(\mu) - \frac{\beta^4}{I^*} R'_1(\mu) = R''_2(\gamma) \quad \text{result in the following four equations}$$

$$A_1 \sin \beta \mu + C_1 \sinh \beta \mu = 0 \quad (3.103)$$

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.104)$$

$$A_1 \cos \beta \mu + C_1 \cosh \beta \mu + A_2 \cos \beta \gamma + C_2 \cosh \beta \gamma = 0 \quad (3.105)$$

and

$$\begin{aligned} & A_1 \left(-\sin \beta \mu - \frac{\beta^3}{I^*} \cos \beta \mu \right) + C_1 \left(\sinh \beta \mu - \frac{\beta^3}{I^*} \cosh \beta \mu \right) \\ & + A_2 \sin \beta \gamma + C_2 (-\sinh \beta \gamma) = 0 \end{aligned} \quad (3.106)$$

Utilizing the condition for a non-trivial solution, we obtain:

$$I^* = \frac{\beta^3 (-\cosh \beta \mu \cdot D_2 + \cos \beta \mu \cdot D_1)}{[-\sin \beta \mu \cdot D_1 - \sinh \beta \mu \cdot D_2 + \sin \beta \gamma \cdot D_3 + \sinh \beta \gamma \cdot D_4]} \quad (3.107)$$

Recognising the limiting conditions, one can write

$$\begin{aligned} f_{(n)} &= Q_{ss(n)} \quad \text{for } n = 1 \\ f_{(n)} &= f_{(n)}^* + Q[f_{ss(n)} - f_{(n)}^*] \quad \text{for } n > 1 \end{aligned} \quad (3.108)$$

Tables 3.13 provide the necessary information for this case.

Being a symmetric system, the values of μ need be given only upto 0.5.

The modal shape expressions are same as those of case 3.5.

3.14 Beams Clamped at one Outer End and Carrying a Rotational Mass at the Other, the Intermediate Support Being Simply-Supported

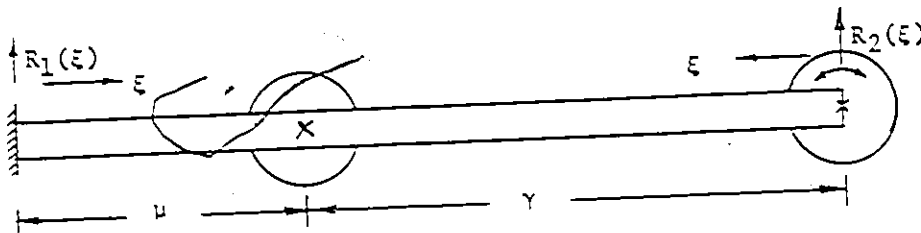


Figure 3.14

Utilizing the boundary conditions $R_1(0) = R'_1(0) = 0$,

$R_2(0) = 0$, and $R_2(\gamma) = 0$, we obtain:

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (3.109)$$

and

$$R_2(\xi) = A_2[\sin \beta \xi + \theta(\cos \beta \xi - \cosh \beta \xi)] \\ + C_2[\sinh \beta \xi + \phi(\cos \beta \xi - \cosh \beta \xi)] \quad (3.110)$$

where

$$\theta = \frac{\sin \beta \gamma}{\cosh \beta \gamma - \cos \beta \gamma} \quad \text{and} \quad \phi = \frac{\sinh \beta \gamma}{\cosh \beta \gamma - \cos \beta \gamma}$$

The other four boundary conditions to be imposed are:

$$R_1(\mu) = 0, \quad R'_1(\mu) = -R'_2(\gamma), \quad R''_1(\mu) = R''_2(\gamma)$$

and

$$R'''_2(0) = -\frac{\beta^4}{I^*} R'_2(0)$$

Substituting these into equations (3.109) and (3.110), we obtain:

$$A_1(\sin \beta \mu - \sinh \beta \mu) + B_1(\cos \beta \mu - \cosh \beta \mu) = 0 \quad (3.111)$$

$$A_1(\cos \beta \mu - \cosh \beta \mu) + B_1(-\sin \beta \mu - \sinh \beta \mu)$$

$$+ A_2[\cos \beta \gamma + \theta(-\sin \beta \gamma - \sinh \beta \gamma)]$$

$$+ C_2[\cosh \beta \gamma + \phi(-\sin \beta \gamma - \sinh \beta \gamma)] = 0 \quad (3.112)$$

$$A_1(-\sin \beta u - \sinh \beta u) + B_1(-\cos \beta u - \cosh \beta u)$$

$$+ A_2 \sin \beta \gamma + \theta(\cos \beta \gamma + \cosh \beta \gamma)$$

$$+ C_2[-\sinh \beta \gamma + \phi(\cos \beta \gamma + \cosh \beta \gamma)] = 0 \quad (3.113)$$

and

$$A_2(-\frac{\beta^3}{I^*} + 2\theta) + C_2(-\frac{\beta^3}{I^*} + 2\phi) = 0 \quad (3.114)$$

Imposing the condition of a non-trivial solution for these constants, we obtain:

$$I^* = \frac{\beta^3(D_3 - D_4)}{2\theta \cdot D_3 - 2\phi \cdot D_4} \quad (3.115)$$

where D_3 and D_4 have similar meanings as previously assigned to them.

The relationship between frequency and Q factor is given by

$$f_{(n)} = Q f_{css(n)} \quad \text{for } n = 1 \quad (3.116)$$

and

$$f_{(n)} = f_{csc(n-1)} + Q[f_{css(n)} - f_{csc(n-1)}] \quad \text{for } n > 1$$

The computed results for this case are provided in Tables

3.14. Modal shape expressions are as provided below:

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\cos \beta \xi - \cosh \beta \xi)$$

$$R_2(\xi) = \gamma_1 \left\{ \sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta I^* - \beta^3)}{(2\phi I^* - \beta^3)} \right.$$

$$\left. (\sinh \beta \xi + \phi \cosh \beta \xi - \phi \cosh \beta \xi) \right\}$$

where

$$\gamma_1 = \frac{(\cosh \beta \mu - \cos \beta \mu + \frac{(\sinh \beta \mu - \sin \beta \mu)}{(\cos \beta \mu - \cosh \beta \mu)} (\sin \beta \mu + \sinh \beta \mu))}{(\cos \beta \gamma - \theta (\sin \beta \gamma + \sinh \beta \gamma) - \frac{(2\theta I^* - \beta^3)}{(2\phi I^* - \beta^3)})}$$

$$(\cosh \beta \gamma - \phi \sin \beta \gamma - \phi \sinh \beta \gamma)$$

3.15 Beams With a Rotational Mass at One Outer End and Simply Supported at the Remaining Two Supports

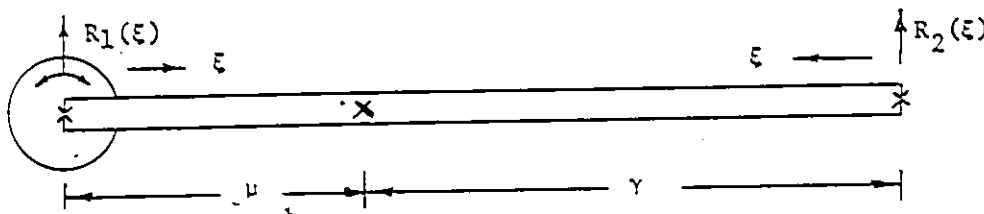


Figure 3.15

Utilizing the boundary conditions $R_1(0) = 0$, $R_1(\mu) = 0$, $R_2(0) = R_2'(0) = 0$, we obtain

$$R_1(\xi) = A_1[\sin \beta \xi + \theta(\cos \beta \xi - \cosh \beta \xi)] + C_1[\sinh \beta \xi + \phi(\cos \beta \xi - \cosh \beta \xi)] \quad (3.117)$$

where

$$\theta = \frac{\sin \beta \mu}{\cosh \beta \mu - \cos \beta \mu} \quad \text{and} \quad \phi = \frac{\sinh \beta \mu}{\cosh \beta \mu - \cos \beta \mu}$$

and

$$R_2(\xi) = A_2 \sin \beta \xi + C_2 \sinh \beta \xi \quad (3.118)$$

The remaining four boundary conditions are

$$R_2(\gamma) = 0, \quad R'_1(\mu) = -R'_2(\gamma), \quad R''_1(\mu) = R''_2(\gamma) \quad \text{and}$$

$$R''_1(0) = -\frac{\beta^4}{I^*} R'_1(0)$$

Substituting these in equations (3.117) and (3.118), we

obtain:

$$A_2 \sin \beta \gamma + C_2 \sinh \beta \gamma = 0 \quad (3.119)$$

$$\begin{aligned} A_1 [\cos \beta \mu + \theta(-\sin \beta \mu - \sinh \beta \mu)] + C_1 [\cosh \beta \mu + \phi(-\sin \beta \mu - \sinh \beta \mu)] \\ + A_2 \cos \beta \gamma + C_2 \cosh \beta \gamma = 0 \end{aligned} \quad (3.120)$$

$$\begin{aligned} A_1 [-\sin \beta \mu + \theta(-\cos \beta \mu - \cosh \beta \mu)] + C_1 [\sinh \beta \mu + \phi(-\cos \beta \mu - \cosh \beta \mu)] \\ + A_2 \sin \beta \gamma + C_2 (-\sinh \beta \gamma) = 0 \end{aligned} \quad (3.121)$$

and

$$A_1 \left(-\frac{\beta^3}{I^*} + 2\theta \right) + C_1 \left(-\frac{\beta^3}{I^*} + 2\phi \right) = 0 \quad (3.122)$$

The condition for a non-trivial solution for shape functions yields:

$$I^* = \frac{\beta^3 (D_1 - D_2)}{2\theta \cdot D_1 - 2\phi \cdot D_2} \quad (3.123)$$

The relationship between $f_{(n)}$ and Q and therefore between $\beta_{(n)}$ and Q is established by

$$f_{(n)} = Q f_{ss(n)} \quad \text{for } n = 1$$

$$f_{(n)} = f_{css(n-1)} + Q[f_{ss(n)} - f_{css(n-1)}] \quad \text{for } n > 1 \quad (3.124)$$

The values of I^* for various values of μ and Q as explained earlier are presented in Tables 3.15. The modal shape expressions are provided below.

$$R_1(\xi) = \sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta I^* - \beta^3)}{(2\phi I^* - \beta^3)} (\sinh \beta \xi + \phi \cos \beta \xi - \phi \cosh \beta \xi)$$

$$R_2(\xi) = \gamma_1 \left\{ \sin \delta \xi - \frac{\sin \delta \gamma}{\sinh \delta \gamma} \sinh \delta \xi \right\}$$

where

$$\gamma_1 = \frac{\{ \theta (\sin \delta \mu + \sinh \delta \mu) - \cos \delta \mu + \frac{(2\theta I^* - \delta^3)}{(2\phi I^* - \delta^3)} (\cosh \delta \mu - \phi \sin \delta \mu - \phi \sinh \delta \mu) \}}{\cos \delta \gamma - \left(\frac{\sin \delta \gamma}{\sinh \delta \gamma} \right) \cosh \delta \gamma}$$

3.16 Beams With Rotational Mass at One Outer End and Free at the Other
and With a Pinned Intermediate Support

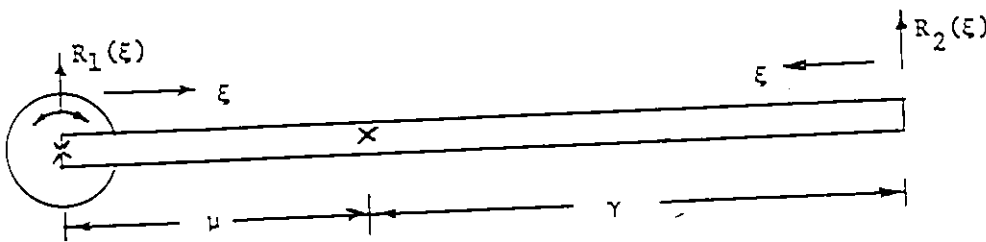


Figure 3.16

Utilizing the boundary conditions $R_1(0) = 0$, $R_1(\mu) = 0$, $R_2''(0) = 0$, and $R_2'''(0) = 0$, it can be readily shown that

$$R_1(\xi) = A_1[\sin \beta \xi + \theta(\cos \beta \xi - \cosh \beta \xi)] \\ + C_1[\sinh \beta \xi + \phi(\cos \beta \xi - \cosh \beta \xi)] \quad (3.125)$$

where

$$\theta = \frac{\sin \beta \mu}{\cosh \beta \mu - \cos \beta \mu} \quad \text{and} \quad \phi = \frac{\sinh \beta \mu}{\cosh \beta \mu - \cos \beta \mu}$$

and

$$R_2(\xi) = A_2(\sin \beta \xi + \sinh \beta \xi) + B_2(\cos \beta \xi + \cosh \beta \xi) \quad (3.126)$$

Further utilization of the remaining four boundary conditions, namely: $R_2(\gamma) = 0$, $R'_1(\mu) = -R'_2(\gamma)$, $R''_1(\mu) = R''_2(\gamma)$ and

$$R''_1(0) = -\frac{\beta^4}{I^*} R'_1(0); \text{ yields}$$

$$A_2(\sin \beta \gamma + \sinh \beta \gamma) + B_2(\cos \beta \gamma + \cosh \beta \gamma) = 0 \quad (3.127)$$

$$A_1[\cos \beta \mu + \theta(-\sin \beta \mu - \sinh \beta \mu)] + C_1[\cosh \beta \mu + \phi(-\sin \beta \mu - \sinh \beta \mu)]$$

$$+ A_2(\cos \beta \gamma + \cosh \beta \gamma) + B_2(-\sin \beta \gamma + \sinh \beta \gamma) = 0 \quad (3.128)$$

$$A_1[-\sin \beta \mu + \theta (-\cos \beta \mu - \cosh \beta \mu)] + C_1[\sinh \beta \mu + \phi (-\cos \beta \mu - \cosh \beta \mu)] \\ + A_2(\sin \beta \gamma - \sinh \beta \gamma) + B_2(\cos \beta \gamma - \cosh \beta \gamma) = 0 \quad (3.129)$$

and

$$A_1(-\frac{\beta^3}{I^*} + 2\theta) + C_1(-\frac{\beta^3}{I^*} + 2\phi) = 0 \quad (3.130)$$

For a non-trivial solution for these constants in the above four equations, the determinant of their coefficient matrix is set equal to zero which yields:

$$I^* = \frac{\beta^3(D_1 - D_2)}{2\theta D_1 - 2\phi D_2} \quad (3.131)$$

Recognising the limiting frequencies of this class of beams, one can write,

$$f_{(n)} = Q f_{ssf(n)} \quad \text{for } n = 1 \\ f_{(n)} = f_{csf(n-1)} + Q[f_{ssf(n)} - f_{csf(n-1)}] \quad \text{for } n > 1 \quad (3.132)$$

Tables 3.16 provide all the information necessary for establishing frequencies of this class of beams. The modal shape expressions are provided below

$$R_1(\xi) = \sin \beta \xi + \theta (\cos \beta \xi - \cosh \beta \xi) - \frac{(2\theta I^* - \beta^3)}{(2\phi I^* - \beta^3)}$$

$$(\sinh \beta \xi + \phi \cos \beta \xi - \phi \cosh \beta \xi)$$

$$R_2(\xi) = \gamma_1 \left\{ \sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right\}$$

where

$$\gamma_1 = \frac{\{\theta(\sin \beta \mu + \sinh \beta \mu) - \cos \beta \mu + \frac{(2\theta I^* - \beta^3)}{(2\phi I^* - \beta^3)} (\cosh \beta \mu - \phi \sin \beta \mu - \phi \sinh \beta \mu)\}}{\{\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\sinh \beta \gamma - \sin \beta \gamma)\}}$$

Illustrative Example 3.1

The beam shown in Figure 3.1 has a rectangular cross-section of 1" width and $\frac{1}{2}$ " depth and a length of 100". The intermediate support is pinned with a torsion spring of stiffness 1500 lb in/degree and is situated at a distance of 60" from left hand support which is built-in. The right hand end is free.

- (a) Determine the frequencies for the first two modes of vibration of the beam.

Moment of inertia I of cross-section $= \frac{1}{12} bd^3$

$= 0.0013 \text{ in}^4$

Cross-sectional area of the beam $A = \frac{1}{4} \text{ in}^2$

$$T^* = \frac{TL}{EI} = \frac{1500 \times 100}{30 \times 10^6 \times 0.0013} = 3.846$$

Obtain the values of frequency parameter β for clamped-simple-free two span beams from charts provided in Appendix B. For $\nu = 0.6$, these are $\beta_{csf(1)} = 3.7$ and $\beta_{csf(2)} = 7.25$

$$\therefore f_{csf(1)} = \frac{\beta_{csf(1)}^2}{2\pi L^2} \sqrt{\frac{EI}{\rho A}}$$

$$= \frac{(3.7)^2}{2\pi \times 100^2} \sqrt{\frac{30 \times 10^6 \times 0.0013 \times 386}{0.283 \times 0.25}}$$

$$= 3.178 \text{ Hz.}$$

and

$$f_{csf(2)} = \left(\frac{7.25}{3.7}\right)^2 \times 3.178 = 12.2 \text{ Hz.}$$

The lower limits of frequency i.e. with $T^* = 0$ are 3.178 and 12.2 H_z for first and second mode respectively. For upper limits, calculate the values of frequency parameter S for left hand and right hand segments of the beam separately. For the left hand segment these are $S_{ccu(1)} = \frac{4.73}{0.6} = 7.88$, $S_{ccu(2)} = 13.09$ and for the right hand segment, these are: $S_{cfv(1)} = \frac{1.875}{0.4} = 4.687$, and $S_{cfv(2)} = 11.73$. Arrange these four values in ascending order as 4.687, 7.88, 11.73, and 13.09. So, for first two modes, the upper limit frequencies $f_{(n)}^*$ are associated with parameter $S = 4.687$ and 7.88 respectively, hence

$$f_{(1)}^* = 5.1 H_z \quad \text{and} \quad f_{(2)}^* = 14.41 \text{ Hz}$$

From Tables 3.1 ($\mu = 0.6$) obtain the value of Q for $T^* = 3.846$ as

$$Q = 0.225 \quad (\text{for first mode})$$

$$\text{and} \quad Q = 0.143 \quad (\text{for second mode})$$

Now utilizing equation (3.12), we obtain:

$$\begin{aligned} f_{(1)} &= f_{csf(1)} + Q[f_{(1)}^* - f_{csf(1)}] \\ &= 3.178 + 0.225 [5.1 - 3.178] \\ &= \underline{3.61 H_z} \end{aligned}$$

and

$$f_{(2)} = f_{\text{csf}(2)} + Q[f_{(2)}^* - f_{\text{csf}(2)}]$$

$$= 12.2 + 0.143 [14.41 - 12.2]$$

$$= \underline{12.5 \text{ H}_2}$$

CHAPTER 4

TRIPLE SPAN BEAMS WITH CLASSICAL BOUNDARY CONDITIONS

Unlike single and double span beams, the eigenvalues of triple span beams with classical boundary conditions are not available in the existing literature. In this Chapter, an attempt is made to establish the free vibration frequencies and modal shapes of triple span beams. The intermediate supports of these beams are taken as simply supported whereas at the extreme outer ends, any combination of classical boundary conditions may prevail. It may be recognised that there are only six distinct combinations possible at outer ends. The two interior supports could be located anywhere between the two outer ends. A detailed analysis of this class of beams for all six possible classical boundary conditions at the outer ends regardless of the location of two interior supports is presented below.

4.1 Beams with Outer Ends Simply Supported

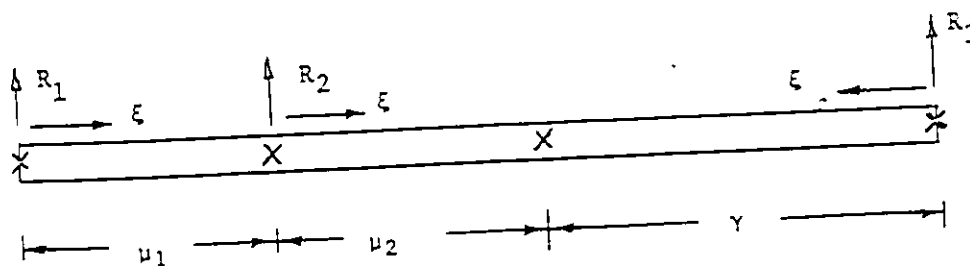


Figure 4.1

Figure 4.1 shows a triple span beam with outer ends simply supported. The dimensionless lengths of the three spans are u_1 , u_2 , and γ respectively where the spans are numbered from left to right. The span lengths have been non-dimensionalized with respect to overall beam length. Obviously the length γ of the third span is equal to $1-(u_1+u_2)$. The origins for the first two spans are located at their left ends whereas for the third span, it is taken at its right end as shown in Figure 4.1. This arrangement of co-ordinate systems is chosen in order to make the task of enforcing the boundary conditions a little easier.

The deflection expressions for the three spans may be written down as:

$$R_1(\xi) = A_1 \sin \beta \xi + B_1 \cos \beta \xi + C_1 \sinh \beta \xi + D_1 \cosh \beta \xi$$

$$R_2(\xi) = A_2 \sin \beta \xi + B_2 \cos \beta \xi + C_2 \sinh \beta \xi + D_2 \cosh \beta \xi$$

(4.1)

and

$$R_3(\xi) = A_3 \sin \beta \xi + B_3 \cos \beta \xi + C_3 \sinh \beta \xi + D_3 \cosh \beta \xi$$

In the above expressions, there are 12 arbitrary constants involved which necessitate 12 boundary conditions to be imposed. Four of these boundary conditions are supplied by two exterior supports and the remaining eight are provided by two interior supports.

Utilizing the boundary conditions $R_1(0) = 0$, $R_1''(0) = 0$, and $R_1(u_1) = 0$, the shape function for the first span reduces to

$$R_1(\xi) = A_1 \left(\sin \beta \xi - \frac{\sin \beta u_1}{\sinh \beta u_1} \sinh \beta \xi \right) \quad (4.2)$$

Similarly the imposition of boundary conditions $R_3(0) = 0$, $R_3''(0) = 0$, and $R_3(\gamma) = 0$ requires

$$R_3(\xi) = A_3 \left(\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right) \quad (4.3)$$

The requirements of $R_2(0) = 0$ and $R_2(u_2) = 0$ lead to

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.4)$$

where

$$\theta = \frac{\sin \beta u_2}{\sinh \beta u_2} \quad \text{and} \quad \phi = \frac{\cosh \beta u_2 - \cos \beta u_2}{\sinh \beta u_2}$$

Now the deflection expressions (4.2) through (4.4) contain only four arbitrary constants, the other eight constants having been

eliminated by utilizing eight boundary conditions. The four boundary conditions, still remaining to be imposed, are provided by the requirements of slope and bending moment continuity at the two interior supports.

These boundary conditions are $R'_1(u_1) = R'_2(0)$, $R''_1(u_1) = R''_2(0)$, $R'_2(u_2) = -R'_3(\gamma)$ and $R''_2(u_2) = R''_3(\gamma)$. Substituting these into equations (4.2) through (4.4), we obtain the following set of four homogeneous algebraic equations in terms of constants A_1 , A_2 , B_2 , and A_3 .

$$A_1 \left(\cos \beta u_1 - \frac{\sin \beta u_1}{\sinh \beta u_1} \cosh \beta u_1 \right) + A_2(\theta - 1) + B_2(-\phi) = 0 \quad (4.5)$$

$$A_1(-2 \sin \beta u_1) + 2B_2 = 0 \quad (4.6)$$

$$A_2(-\cos \beta u_2 + \theta \cosh \beta u_2) + B_2(\sin \beta u_2 + \sinh \beta u_2 - \phi \cosh \beta u_2) + A_3 \left(\frac{\sin \beta \gamma}{\sinh \beta \gamma} \cosh \beta \gamma - \cos \beta \gamma \right) = 0 \quad (4.7)$$

and

$$A_2(\sin \beta u_2 + \theta \sinh \beta u_2) + B_2(\cos \beta u_2 + \cosh \beta u_2 - \phi \sinh \beta u_2) + A_3(-\sin \beta \gamma - \theta \sinh \beta \gamma) = 0 \quad (4.8)$$

For a non-trivial solution for the constants A_1 , A_2 , B_2 , and A_3 in the equations (4.5) through (4.8), the determinant of their coefficient matrix must vanish. It can be recognised that this determinant is a function of β , μ_1 , and μ_2 . The eigenvalue problem consists of finding those values of β for certain specific values of μ_1 and μ_2 which permit this determinant to vanish. These eigenvalues β were determined utilizing the iteration technique based on the method commonly known as "method of false position". The technique essentially consists of increasing the values of β till the associated determinant changes sign. The point at which the determinant crosses the β -axis can then be arbitrarily approached to any desired accuracy by assuming a linear relationship between β and the associated determinant.

The eigenvalues β for various values of μ_1 and μ_2 for first four modes of vibration are provided in Tables 4.1. It may be noted that the beam has been divided into 40 equal segments providing 39 possible locations for the two interior supports. The eigenvalues presented in Tables 4.1 are provided for all these 39 possible locations of intermediate supports. The choice of dividing the beam into 40 equal increments was dictated by the accuracy requirement when linear interpolation is used to obtain eigenvalues for beams whose supports might lie between any of these stations. The effect of dividing the overall beam length into a number of equal segments on the eigenvalues β was extensively

studied and it is found that with this choice of 40 equal increments, the errors encountered in using linear interpolation is $\leq 2\%$ for the first mode and $\leq 1\%$ for higher modes.

It can be easily recognised that since $\mu_1 + \mu_2 + \gamma = 1$, the quantity $\mu_1 + \mu_2$ is always less than 1. Though the parameters μ_1 and μ_2 are both independent parameters, the upper bound on their sum is dictated by the above condition. It is for this reason that table of data for each mode lends itself to be presented in the form of a triangular array. In Tables 4.1, two such modes are combined to give one square array. It may be noted that in the present case, the outer ends have identical supports and therefore the Tables 4.1 provide data for $\mu_1 \leq 0.5$ since for a symmetric case one can always choose axes such that $\mu_1 \leq 0.5$.

The modal shape expressions are arrived at by setting the constant A_1 associated with first span equal to unity. Then the equations (4.5) and (4.6) are solved for constants A_2 and B_2 . The constant A_3 then can be determined from either of the equations (4.7) and (4.8). Both the expressions for A_3 are given, since for certain values of δ , either one of them can assume an indeterminate form. By giving the alternate expression for A_3 , the necessity of finding the limit associated with the indeterminate form is obviated and it is further found that both these expressions for A_3 will never take on indeterminate form simultaneously. The modal shape expressions for the three spans are as follows:

$$R_1(\xi) = \sin \beta \xi - \frac{\sin \beta u_1}{\sinh \beta u_1} \sinh \beta \xi$$

$$R_2(\xi) = A_2(\sin \beta \xi - \theta \sinh \beta \xi) + B_2(\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi)$$

and

$$R_3(\xi) = A_3(\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi)$$

where

$$A_2 = \frac{\sin \beta u_1 \cosh \beta u_1 - \cos \beta u_1 \sinh \beta u_1 + \phi \sin \beta u_1 \sinh \beta u_1}{(\theta - 1) \sinh \beta u_1}$$

$$B_2 = \sin \beta u_1$$

$$A_3 = \frac{A_2(\cos \beta u_2 - \theta \cosh \beta u_2) + B_2(-\sin \beta u_2 - \sinh \beta u_2 + \phi \cosh \beta u_2)}{(\frac{\sin \beta \gamma \cosh \beta \gamma}{\sinh \beta \gamma} - \cos \beta \gamma)}$$

or

$$A_3 = A_2 \frac{\sin \beta u_2}{\sin \beta \gamma} + B_2 \frac{\cos \beta u_2}{\sin \beta \gamma}$$

$$\theta = \frac{\sin \beta u_2}{\sinh \beta u_2} \quad \text{and} \quad \phi = \frac{\cosh \beta u_2 - \cos \beta u_2}{\sinh \beta u_2}$$

The use of tables and modal shape expressions are demonstrated by working out an illustrative example at the end of this Chapter.

4.2 Beams With Outer Ends Clamped

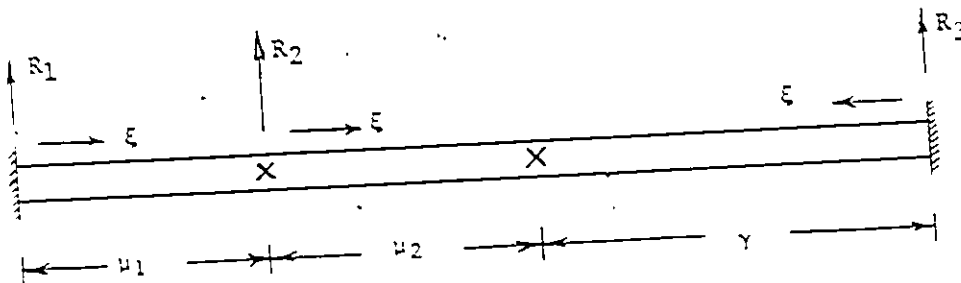


Figure 4.2

Following the procedure established in case 4.1 and utilizing the eight boundary conditions, viz; $R_1(0) = 0$, $R'_1(0) = 0$, $R_1(u_1) = 0$, $R_2(0) = 0$, $R_2(u_2) = 0$, $R_3(0) = 0$, $R'_3(0) = 0$, and $R_3(y) = 0$, the modal shape functions for the three spans are reduced to the following form

$$R_1(\xi) = A_1 \left[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (\cos \beta \xi - \cosh \beta \xi) \right] \quad (4.9)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.10)$$

and

$$R_3(\xi) = A_3 \left[\sin \beta \xi - \sinh \beta \xi + \frac{\sin \beta \gamma - \sinh \beta \gamma}{\cosh \beta \gamma - \cos \beta \gamma} (\cos \beta \xi - \cosh \beta \xi) \right] \quad (4.11)$$

A set of four homogeneous algebraic equations in terms of these four constants is obtained by utilizing the conditions of slope and bending moment continuity across two interior supports.

$$A_1 \left[(\cos \beta \mu_1 - \cosh \beta \mu_1) + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (-\sin \beta \mu_1 - \sinh \beta \mu_1) \right] + A_2 (\theta - 1) + B_2 (-\phi) = 0 \quad (4.12)$$

$$A_1 \left[(-\sin \beta \mu_1 - \sinh \beta \mu_1) + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (-\cos \beta \mu_1 - \cosh \beta \mu_1) \right] + 2B_2 = 0 \quad (4.13)$$

$$\begin{aligned}
 & A_2(\cos \beta u_2 - \theta \cosh \beta u_2) + B_2(-\sin \beta u_2 - \sinh \beta u_2 + \phi \cosh \beta u_2) \\
 & + A_3[(\cos \beta y - \cosh \beta y) + \frac{(\sin \beta y - \sinh \beta y)}{(\cosh \beta y - \cos \beta y)}(-\sin \beta y - \sinh \beta y)] = 0
 \end{aligned}
 \tag{4.14}$$

and

$$\begin{aligned}
 & A_2(-\sin \beta u_2 - \theta \sinh \beta u_2) + B_2(-\cos \beta u_2 - \cosh \beta u_2 + \phi \sinh \beta u_2) \\
 & + A_3[(\sin \beta y + \sinh \beta y) + \frac{(\sin \beta y - \sinh \beta y)}{(\cosh \beta y - \cos \beta y)}(\cos \beta y + \cosh \beta y)] = 0
 \end{aligned}
 \tag{4.15}$$

The values of β for various values of u_1 and u_2 which permit the determinant of the coefficient matrix of constants A_1 , A_2 , B_2 , and A_3 in the above four equations to vanish, are computed and are presented in Tables 4.2. It may be noted that, this being a symmetric case, the values of u_1 are obviously given for $u_1 \leq 0.5$.

The modal shape expressions are derived in their usual manner and are presented below in their simplified forms.

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta u_1 - \sinh \beta u_1)}{(\cosh \beta u_1 - \cos \beta u_1)} (\cos \beta \xi - \cosh \beta \xi)$$

$$R_2(\xi) = A_2(\sin \beta \xi - \theta \sinh \beta \xi) + B_2(\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi)$$

and

$$R_3(\xi) = A_3[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \gamma - \sinh \beta \gamma)}{(\cosh \beta \gamma - \cos \beta \gamma)} (\cos \beta \xi - \cosh \beta \xi)]$$

where

$$A_2 = \frac{2(1 - \cosh \beta \mu_1 \cos \beta \mu_1) - \phi(\sinh \beta \mu_1 \cos \beta \mu_1 - \sin \beta \mu_1 \cosh \beta \mu_1)}{(\theta - 1)(\cosh \beta \mu_1 - \cos \beta \mu_1)}$$

$$B_2 = \frac{(\sin \beta \mu_1 \cosh \beta \mu_1 - \sinh \beta \mu_1 \cos \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)}$$

$$A_3 = \frac{A_2(\sinh \beta \mu_2 \cos \beta \mu_2 - \sin \beta \mu_2 \cosh \beta \mu_2) + B_2(1 - \cos \beta \mu_2 \cosh \beta \mu_2 - \sin \beta \mu_2 \sinh \beta \mu_2)}{2 \sinh \beta \mu_2 (1 - \cosh \beta \gamma \cos \beta \gamma) / (\cosh \beta \gamma - \cos \beta \gamma)}$$

or

$$A_3 = \frac{A_2 \sin \beta \mu_2 + B_2 \cos \beta \mu_2}{[(\sin \beta \gamma \cosh \beta \gamma - \sinh \beta \gamma \cos \beta \gamma) / (\cosh \beta \gamma - \cos \beta \gamma)]}$$

and expressions for θ and ϕ are same as given in case 4.1.

4.3 Beams With Outer Ends Free

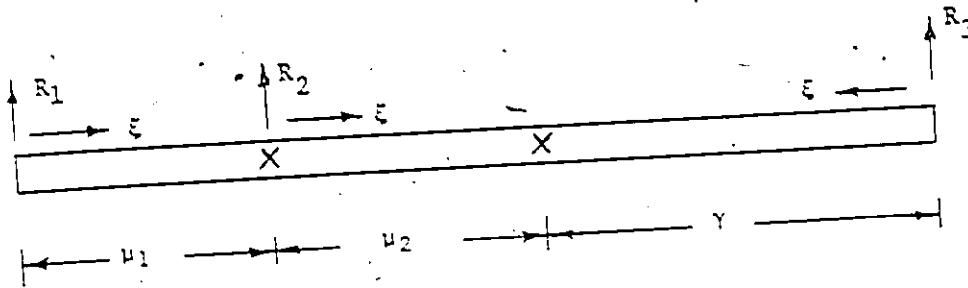


Figure 4.3

Utilizing the boundary conditions at outer ends and the requirement of zero deflection at the two intermediate supports, it can be readily shown that,

$$R_1(\xi) = A_1 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \mu_1 + \sinh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1)} (\cos \beta \xi + \cosh \beta \xi) \right] \quad (4.16)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.17)$$

and

$$R_3(\xi) = A_3 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right] \quad (4.18)$$

where the expressions for θ and ϕ are same as in case 4.1.

Further utilizing the conditions of slope and bending moment continuity at the two intermediate supports, viz, $R'_1(\mu_1) = R'_2(0)$, $R''_1(\mu_1) = R''_2(0)$, $R'_2(\mu_2) = -R'_3(\gamma)$ and $R''_2(\mu_2) = R''_3(\gamma)$; we obtain:

$$A_1 \left[\cos \beta \mu_1 + \cosh \beta \mu_1 - \frac{(\sin \beta \mu_1 + \sinh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1)} (-\sin \beta \mu_1 + \sinh \beta \mu_1) \right] + A_2(\theta-1) + B_2(-\phi) = 0 \quad (4.19)$$

$$A_1 \left[-\sin \beta \mu_1 + \sinh \beta \mu_1 - \frac{(\sin \beta \mu_1 + \sinh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1)} (-\cos \beta \mu_1 + \cosh \beta \mu_1) \right] + 2B_2 = 0 \quad (4.20)$$

$$A_2(\cos \beta \mu_2 - \theta \cosh \beta \mu_2) + B_2(-\sin \beta \mu_2 - \sinh \beta \mu_2 + \phi \cosh \beta \mu_2)$$

$$+ A_3 \left[\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (-\sin \beta \gamma + \sinh \beta \gamma) \right] = 0 \quad (4.21)$$

and

$$A_2(-\sin \beta \mu_2 - \theta \sinh \beta \mu_2) + B_2(-\cos \beta \mu_2 - \cosh \beta \mu_2 + \phi \sinh \beta \mu_2) \\ + A_3[\sin \beta \gamma - \sinh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \gamma - \cosh \beta \gamma)] = 0. \quad (4.22)$$

The eigenvalues β which permit the determinant of the coefficient matrix of the constants A_1 , A_2 , B_2 , and A_3 in the above four equations to vanish are completely characterized by μ_1 and μ_2 , the dimensionless lengths of first and second spans. Tables 4.3 furnish the eigenvalues of this case for first four modes of vibration for various possible values of μ_1 and μ_2 .

The following modal shape expressions have been obtained for this case. The expressions for $R_1(\xi)$, $R_2(\xi)$, and $R_3(\xi)$ appear in equations (4.16) through (4.18), where the constant A_1 is set equal to unity and the expressions for other constants are as given below

$$A_2 = \frac{2(1 + \cos \beta \mu_1 \cosh \beta \mu_1) + \phi(\sinh \beta \mu_1 \cos \beta \mu_1 - \sin \beta \mu_1 \cosh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1) (1-\theta)}$$

$$B_2 = \frac{\sin \beta u_1 \cosh \beta u_1 - \sinh \beta u_1 \cos \beta u_1}{\cos \beta u_1 + \cosh \beta u_1}$$

$$A_3 = \frac{A_2(\sin \beta u_2 \cosh \beta u_2 - \sinh \beta u_2 \cos \beta u_2) + B_2(\sin \beta u_2 \sinh \beta u_2 + \cos \beta u_2 \cosh \beta u_2 - 1)}{2 \sinh \beta u_2 [(1 + \cos \beta y \cosh \beta y) / (\cos \beta y + \cosh \beta y)]}$$

or

$$A_3 = \frac{A_2(\sin \beta u_2 \sinh \beta u_2) + B_2(\cos \beta u_2 \sinh \beta u_2)}{\sinh \beta u_2 [(\sin \beta y \cosh \beta y - \sinh \beta y \cos \beta y) / (\cos \beta y + \cosh \beta y)]}$$

4.4 Beams With One Outer End Clamped and the Other Simply Supported

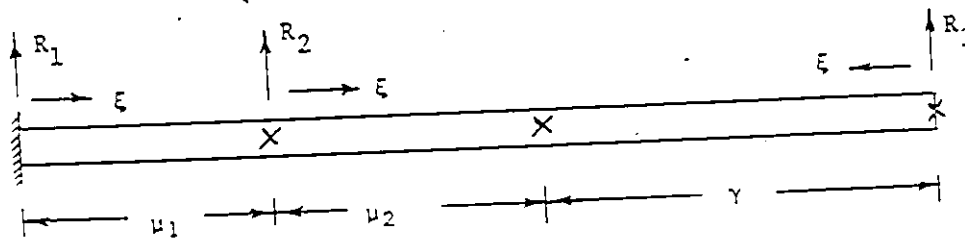


Figure 4.4

Following the procedure established in earlier cases, one can write

$$R_1(\xi) = A_1 \left[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta u_1 - \sinh \beta u_1)}{(\cosh \beta u_1 - \cos \beta u_1)} (\cos \beta \xi - \cosh \beta \xi) \right] \quad (4.23)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.24)$$

and

$$R_3(\xi) = A_3 \left(\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right) \quad (4.25)$$

Utilizing the conditions of slope and bending moment continuity across the two interior supports, viz, $R'_1(u_1) = R'_2(0)$, $R''_1(u_1) = R''_2(0)$, $R'_2(u_2) = -R'_3(\gamma)$ and $R''_2(u_2) = R''_3(\gamma)$, we obtain the following set of four homogeneous algebraic equations in terms of constants A_1 , A_2 , B_2 , and A_3 .

$$A_1 \left[\cos \beta u_1 - \cosh \beta u_1 + \frac{(\sin \beta u_1 - \sinh \beta u_1)}{(\cosh \beta u_1 - \cos \beta u_1)} (-\sin \beta u_1 - \sinh \beta u_1) \right] + A_2 (\theta - 1) + B_2 (-\phi) = 0 \quad (4.26)$$

$$A_1[-\sin \beta u_1 - \sinh \beta u_1 + \frac{(\sin \beta u_1 - \sinh \beta u_1)}{(\cosh \beta u_1 - \cos \beta u_1)} (-\cos \beta u_1 - \cosh \beta u_1)] + 2B_2 = 0 \quad (4.27)$$

$$A_2(\cos \beta u_2 - \theta \cosh \beta u_2) + B_2(-\sin \beta u_2 - \sinh \beta u_2 + \phi \cosh \beta u_2) + A_3(\cos \beta y - \frac{\sin \beta y}{\sinh \beta y} \cosh \beta y) = 0 \quad (4.28)$$

and

$$A_2(-\sin \beta u_2 - \theta \sinh \beta u_2) + B_2(-\cos \beta u_2 - \cosh \beta u_2 + \phi \sinh \beta u_2) + A_3(2 \sin \beta y) = 0 \quad (4.29)$$

The values of β which permit a non-trivial solution for the constants A_1 , A_2 , B_2 , and A_3 in the above four equations were computed for all possible values of u_1 and u_2 and are presented in Tables 4.4 for four modes of vibration. Unlike the previous three cases it is necessary to provide values of u_1 upto 0.95.

The modal shape expressions appear in equations (4.23) through (4.25) where the constant A_1 is set equal to unity as usual and

the expressions for constants A_2 and B_2 are same as in case 4.2 and the expressions for A_3 are same as in case 4.1.

4.5 Beams With One Outer End Clamped and Other Free

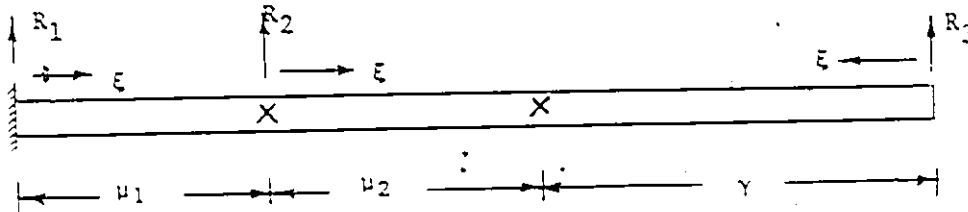


Figure 4.5

The eight out of a total of 12 constants appearing in the shape functions for the three spans are eliminated following the procedure outlined earlier, and we are left with,

$$R_1(\xi) = A_1 \left[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (\cos \beta \xi - \cosh \beta \xi) \right] \quad (4.30)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.31)$$

and

$$R_3(\xi) = A_3 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right] \quad (4.32)$$

Utilizing the boundary conditions involving slope and moment continuity at the two interior supports, we obtain:

$$A_1 \left[\cos \beta \mu_1 - \cosh \beta \mu_1 + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (-\sin \beta \mu_1 - \sinh \beta \mu_1) \right] + A_2(\theta - 1) + B_2(-\phi) = 0 \quad (4.33)$$

$$A_1 \left[-\sin \beta \mu_1 - \sinh \beta \mu_1 + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (-\cos \beta \mu_1 - \cosh \beta \mu_1) \right] + 2B_2 = 0 \quad (4.34)$$

$$A_2(\cos \beta \mu_2 - \theta \cosh \beta \mu_2) + B_2(-\sin \beta \mu_2 - \sinh \beta \mu_2 + \phi \cosh \beta \mu_2)$$

$$+ A_3 \left[\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (-\sin \beta \gamma + \sinh \beta \gamma) \right] = 0 \quad (4.35)$$

and

$$\begin{aligned}
 & A_2(-\sin S\mu_2 - \theta \sinh S\mu_2) + B_2(-\cos S\mu_2 - \cosh S\mu_2 + \phi \sinh S\mu_2) \\
 & + A_3\left[\sin SY - \sinh SY - \frac{(\sin SY + \sinh SY)}{(\cos SY + \cosh SY)} (\cos SY - \cosh SY)\right] = 0
 \end{aligned}
 \tag{4.36}$$

The values of S which satisfy the condition for a non-trivial solution for the constants A_1 , A_2 , B_2 , and A_3 in the above four equations for all possible combinations of μ_1 and μ_2 are presented in Tables 4.5.

Modal shape expressions $R_1(\xi)$, $R_2(\xi)$ and $R_3(\xi)$ appear in equations (4.30) through (4.32) where constant A_1 is set equal to unity, constants A_2 and B_2 have same expressions as provided for them in case 4.2 and the expression for A_3 is the same as in case 4.3.

4.6 Beams with One Outer End Simply Supported and the Other Free

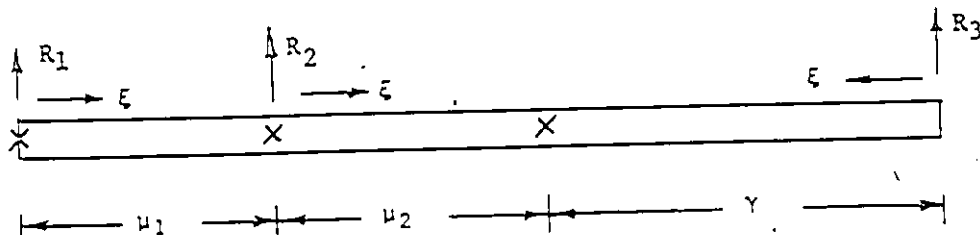


Figure 4.6

Utilizing the boundary conditions $R_1(0) = 0$, $R'_1(0) = 0$, $R_1(\mu_1) = 0$, $R_2(0) = 0$, $R_2(\mu_2) = 0$, $R_3(\gamma) = 0$, $R_3(0) = 0$ and $R'_3(0) = 0$, it can be readily shown that

$$R_1(\xi) = A_1 \left(\sin \beta \xi - \frac{\sin \beta \mu_1}{\sinh \beta \mu_1} \sinh \beta \xi \right) \quad (4.37)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (4.38)$$

and

$$R_3(\xi) = A_3 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right] \quad (4.39)$$

By enforcing the remaining four boundary conditions, viz;

$R'_1(\mu_1) = R'_2(0)$, $R''_1(\mu_1) = R''_2(0)$, $R'_2(\mu_2) = -R'_3(\gamma)$ and $R''_2(\mu_2) = R''_3(\gamma)$; we obtain the following four equations for the four unknowns A_1 , A_2 , B_2 , and A_3 .

$$A_1 \left(\cos \beta \mu_1 - \frac{\sin \beta \mu_1}{\sinh \beta \mu_1} \cosh \beta \mu_1 \right) + A_2 (\theta - 1) + B_2 (-\phi) = 0 \quad (4.40)$$

$$A_1(-2 \sin \beta \mu_1) + 2B_2 = 0 \quad (4.41)$$

$$\begin{aligned} & A_2(\cos \beta \mu_2 - \theta \cosh \beta \mu_2) + B_2(-\sin \beta \mu_2 - \sinh \beta \mu_2 + \phi \cosh \beta \mu_2) \\ & + A_3[\cos \beta \gamma + \cosh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (-\sin \beta \gamma + \sinh \beta \gamma)] = 0 \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} & A_2(-\sin \beta \mu_2 - \theta \sinh \beta \mu_2) + B_2(-\cos \beta \mu_2 - \cosh \beta \mu_2 + \phi \sinh \beta \mu_2) \\ & + A_3[\sin \beta \gamma - \sinh \beta \gamma - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \gamma - \cosh \beta \gamma)] = 0 \end{aligned} \quad (4.43)$$

Tables 4.6 furnish the values β for all possible values of μ_1 and μ_2 which are associated with the non-trivial solutions for shape functions. The expressions (4.37) through (4.39) determine the modal shapes for the three spans where constants A_2 and B_2 are determined from expressions given in case 4.1 and A_3 has the same expression as provided for it in case 4.3.

Illustrative Example 4.1

A steel tube of O.D. = $\frac{3}{4}$ " and I.D. = $\frac{1}{2}$ " is clamped at left hand end and is simply supported at the extreme right hand end as shown in Figure 4.4. It is simply supported at two intermediate supports. The lengths of three spans are 40", 35", and 25" respectively where spans are counted from the left hand end.

- (a) Determine its fundamental frequency and modal shape

For steel $E = 30 \times 10^6$ psi, and density $\rho = 0.283$ lb/in³

$$\text{Cross sectional area of the tube } A = \frac{\pi}{4} (D_o^2 - D_i^2)$$

$$= \frac{\pi}{4} (0.75^2 - 0.50^2) = .245 \text{ in}^2$$

$$\mu_1 = 0.4, \quad \mu_2 = 0.35$$

Obtain the eigenvalue β from Table 4.4.

$$\beta_1 = 9.824$$

$$\text{Moment of inertia of the cross-section of beam} = \frac{\pi}{64} (D_o^4 - D_i^4)$$

$$= \frac{\pi}{64} (.75^4 - 0.5^4)$$

$$= 0.0124 \text{ in}^4$$

Utilize expression (1.7)

$$f_1 = \frac{s^2}{2\pi L^2} \sqrt{\frac{EI}{\rho A}} = \frac{9.824^2}{2\pi \times 100^2} \sqrt{\frac{30 \times 10^6 \times 0.0124 \times 386}{0.283 \times 0.245}}$$

$$= 69.90 \text{ Hz}$$

Computing the values of θ , ϕ , A_2 , B_2 , and A_3 , utilizing the expressions provided for them in case 4.4, the modal shape expressions are given by:

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + 0.945 (\cos \beta \xi - \cosh \beta \xi)$$

$$R_2(\xi) = 1.02 (\sin \beta \xi + 0.018 \sinh \beta \xi) + 0.0031 (\cosh \beta \xi - \cos \beta \xi - 1.06 \sinh \beta \xi)$$

and

$$R_3(\xi) = 0.465 \left(\frac{\sin 2.456}{\sinh 2.456} \sinh \beta \xi - \sin \beta \xi \right)$$

(b) Determine the second mode frequency of the beam

From Table 4.4, obtain $\beta_2 = 12.46$

hence $f_2 = \left(\frac{12.46}{9.824}\right)^2 \times 69.90 = 112.4 \text{ Hz}$

- (c) Determine the fundamental frequency of the beam if the first intermediate support is moved to the left by 8". Now we have

$$u_1 = \frac{40-8}{100} = 0.32, \quad u_2 = \frac{35.8}{100} = 0.43$$

and $\gamma = u_1 - u_2 = 0.25$

Obtain $s_1 = 8.936$ from Tables 4.4 using linear interpolation.

Hence $f_1 = \left(\frac{8.936}{9.824}\right)^2 \times 69.90 = 57.83 \text{ Hz}$

TABLE 41 (USCAR 3 AND 4)

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[illegible]

TABLE 42 DUXES (JUD 2)

MODE 1	μ_2	MODE 2
0.000	0.000	0.000
0.001	0.001	0.001
0.002	0.002	0.002
0.003	0.003	0.003
0.004	0.004	0.004
0.005	0.005	0.005
0.006	0.006	0.006
0.007	0.007	0.007
0.008	0.008	0.008
0.009	0.009	0.009
0.010	0.010	0.010
0.011	0.011	0.011
0.012	0.012	0.012
0.013	0.013	0.013
0.014	0.014	0.014
0.015	0.015	0.015
0.016	0.016	0.016
0.017	0.017	0.017
0.018	0.018	0.018
0.019	0.019	0.019
0.020	0.020	0.020
0.021	0.021	0.021
0.022	0.022	0.022
0.023	0.023	0.023
0.024	0.024	0.024
0.025	0.025	0.025
0.026	0.026	0.026
0.027	0.027	0.027
0.028	0.028	0.028
0.029	0.029	0.029
0.030	0.030	0.030
0.031	0.031	0.031
0.032	0.032	0.032
0.033	0.033	0.033
0.034	0.034	0.034
0.035	0.035	0.035
0.036	0.036	0.036
0.037	0.037	0.037
0.038	0.038	0.038
0.039	0.039	0.039
0.040	0.040	0.040
0.041	0.041	0.041
0.042	0.042	0.042
0.043	0.043	0.043
0.044	0.044	0.044
0.045	0.045	0.045
0.046	0.046	0.046
0.047	0.047	0.047
0.048	0.048	0.048
0.049	0.049	0.049
0.050	0.050	0.050
0.051	0.051	0.051
0.052	0.052	0.052
0.053	0.053	0.053
0.054	0.054	0.054
0.055	0.055	0.055
0.056	0.056	0.056
0.057	0.057	0.057
0.058	0.058	0.058
0.059	0.059	0.059
0.060	0.060	0.060
0.061	0.061	0.061
0.062	0.062	0.062
0.063	0.063	0.063
0.064	0.064	0.064
0.065	0.065	0.065
0.066	0.066	0.066
0.067	0.067	0.067
0.068	0.068	0.068
0.069	0.069	0.069
0.070	0.070	0.070
0.071	0.071	0.071
0.072	0.072	0.072
0.073	0.073	0.073
0.074	0.074	0.074
0.075	0.075	0.075
0.076	0.076	0.076
0.077	0.077	0.077
0.078	0.078	0.078
0.079	0.079	0.079
0.080	0.080	0.080
0.081	0.081	0.081
0.082	0.082	0.082
0.083	0.083	0.083
0.084	0.084	0.084
0.085	0.085	0.085
0.086	0.086	0.086
0.087	0.087	0.087
0.088	0.088	0.088
0.089	0.089	0.089
0.090	0.090	0.090
0.091	0.091	0.091
0.092	0.092	0.092
0.093	0.093	0.093
0.094	0.094	0.094
0.095	0.095	0.095
0.096	0.096	0.096
0.097	0.097	0.097
0.098	0.098	0.098
0.099	0.099	0.099
0.100	0.100	0.100

TABLE 43. *INSECTA* 1 AND 20 21

MOON 1300M

MOORE 2

TABLE 4.4. DUGGES 3 AND 4

MODE 3	μ_2	MODE 4
0.113	0.113	0.113
0.114	0.114	0.114
0.115	0.115	0.115
0.116	0.116	0.116
0.117	0.117	0.117
0.118	0.118	0.118
0.119	0.119	0.119
0.120	0.120	0.120
0.121	0.121	0.121
0.122	0.122	0.122
0.123	0.123	0.123
0.124	0.124	0.124
0.125	0.125	0.125
0.126	0.126	0.126
0.127	0.127	0.127
0.128	0.128	0.128
0.129	0.129	0.129
0.130	0.130	0.130
0.131	0.131	0.131
0.132	0.132	0.132
0.133	0.133	0.133
0.134	0.134	0.134
0.135	0.135	0.135
0.136	0.136	0.136
0.137	0.137	0.137
0.138	0.138	0.138
0.139	0.139	0.139
0.140	0.140	0.140
0.141	0.141	0.141
0.142	0.142	0.142
0.143	0.143	0.143
0.144	0.144	0.144
0.145	0.145	0.145
0.146	0.146	0.146
0.147	0.147	0.147
0.148	0.148	0.148
0.149	0.149	0.149
0.150	0.150	0.150
0.151	0.151	0.151
0.152	0.152	0.152
0.153	0.153	0.153
0.154	0.154	0.154
0.155	0.155	0.155
0.156	0.156	0.156
0.157	0.157	0.157
0.158	0.158	0.158
0.159	0.159	0.159
0.160	0.160	0.160
0.161	0.161	0.161
0.162	0.162	0.162
0.163	0.163	0.163
0.164	0.164	0.164
0.165	0.165	0.165
0.166	0.166	0.166
0.167	0.167	0.167
0.168	0.168	0.168
0.169	0.169	0.169
0.170	0.170	0.170
0.171	0.171	0.171
0.172	0.172	0.172
0.173	0.173	0.173
0.174	0.174	0.174
0.175	0.175	0.175
0.176	0.176	0.176
0.177	0.177	0.177
0.178	0.178	0.178
0.179	0.179	0.179
0.180	0.180	0.180
0.181	0.181	0.181
0.182	0.182	0.182
0.183	0.183	0.183
0.184	0.184	0.184
0.185	0.185	0.185
0.186	0.186	0.186
0.187	0.187	0.187
0.188	0.188	0.188
0.189	0.189	0.189
0.190	0.190	0.190
0.191	0.191	0.191
0.192	0.192	0.192
0.193	0.193	0.193
0.194	0.194	0.194
0.195	0.195	0.195
0.196	0.196	0.196
0.197	0.197	0.197
0.198	0.198	0.198
0.199	0.199	0.199
0.200	0.200	0.200

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TABLE 45 (CONTINUED)

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623	1.319	1.310	1.496	1.453	1.413	1.376	1.342	1.310	1.280	1.252	1.227	1.204	1.182	1.161	1.141	1.121	1.102	1.083	1.064	1.046	1.028	1.011	994	977	961	945	929	914	898	883	868	853	838	824	810	796	782	768	754	740	727	714	701	688	675	662	649	636	624	611	599	587	575	563	551	539	527	515	503	491	479	467	455	443	431	419	407	395	383	371	359	347	335	323	311	299	287	275	263	251	239	227	215	203	191	179	167	155	143	131	119	107	95	83	71	59	47	35	23	11	-1	-13	-25	-37	-49	-61	-73	-85	-97	-109	-121	-133	-145	-157	-169	-181	-193	-205	-217	-229	-241	-253	-265	-277	-289	-301	-313	-325	-337	-349	-361	-373	-385	-397	-409	-421	-433	-445	-457	-469	-481	-493	-505	-517	-529	-541	-553	-565	-577	-589	-601	-613	-625	-637	-649	-661	-673	-685	-697	-709	-721	-733	-745	-757	-769	-781	-793	-805	-817	-829	-841	-853	-865	-877	-889	-901	-913	-925	-937	-949	-961	-973	-985	-997	-1009	-1021	-1033	-1045	-1057	-1069	-1081	-1093	-1105	-1117	-1129	-1141	-1153	-1165	-1177	-1189	-1201	-1213	-1225	-1237	-1249	-1261	-1273	-1285	-1297	-1309	-1321	-1333	-1345	-1357	-1369	-1381	-1393	-1405	-1417	-1429	-1441	-1453	-1465	-1477	-1489	-1501	-1513	-1525	-1537	-1549	-1561	-1573	-1585	-1597	-1609	-1621	-1633	-1645	-1657	-1669	-1681	-1693	-1705	-1717	-1729	-1741	-1753	-1765	-1777	-1789	-1801	-1813	-1825	-1837	-1849	-1861	-1873	-1885	-1897	-1909	-1921	-1933	-1945	-1957	-1969	-1981	-1993	-2005	-2017	-2029	-2041	-2053	-2065	-2077	-2089	-2101	-2113	-2125	-2137	-2149	-2161	-2173	-2185	-2197	-2209	-2221	-2233	-2245	-2257	-2269	-2281	-2293	-2305	-2317	-2329	-2341	-2353	-2365	-2377	-2389	-2401	-2413	-2425	-2437	-2449	-2461	-2473	-2485	-2497	-2509	-2521	-2533	-2545	-2557	-2569	-2581	-2593	-2605	-2617	-2629	-2641	-2653	-2665	-2677	-2689	-2701	-2713	-2725	-2737	-2749	-2761	-2773	-2785	-2797	-2809	-2821	-2833	-2845	-2857	-2869	-2881	-2893	-2905	-2917	-2929	-2941	-2953	-2965	-2977	-2989	-3001	-3013	-3025	-3037	-3049	-3061	-3073	-3085	-3097	-3109	-3121	-3133	-3145	-3157	-3169	-3181	-3193	-3205	-3217	-3229	-3241	-3253	-3265	-3277	-3289	-3301	-3313	-3325	-3337	-3349	-3361	-3373	-3385	-3397	-3409	-3421	-3433	-3445	-3457	-3469	-3481	-3493	-3505	-3517	-3529	-3541	-3553	-3565	-3577	-3589	-3601	-3613	-3625	-3637	-3649	-3661	-3673	-3685	-3697	-3709	-3721	-3733	-3745	-3757	-3769	-3781	-3793	-3805	-3817	-3829	-3841	-3853	-3865	-3877	-3889	-3901	-3913	-3925	-3937	-3949	-3961	-3973	-3985	-3997	-4009	-4021	-4033	-4045	-4057	-4069	-4081	-4093	-4105	-4117	-4129	-4141	-4153	-4165	-4177	-4189	-4201	-4213	-4225	-4237	-4249	-4261	-4273	-4285	-4297	-4309	-4321	-4333	-4345	-4357	-4369	-4381	-4393	-4405	-4417	-4429	-4441	-4453	-4465	-4477	-4489	-4501	-4513	-4525	-4537	-4549	-4561	-4573	-4585	-4597	-4609	-4621	-4633	-4645	-4657	-4669	-4681	-4693	-4705	-4717	-4729	-4741	-4753	-4765	-4777	-4789	-4801	-4813	-4825	-4837	-4849	-4861	-4873	-4885	-4897	-4909	-4921	-4933	-4945	-4957	-4969	-4981	-4993	-5005	-5017	-5029	-5041	-5053	-5065	-5077	-5089	-5101	-5113	-5125	-5137	-5149	-5161	-5173	-5185	-5197	-5209	-5221	-5233	-5245	-5257	-5269	-5281	-5293	-5305	-5317	-5329	-5341	-5353	-5365	-5377	-5389	-5401	-5413	-5425	-5437	-5449	-5461	-5473	-5485	-5497	-5509	-5521	-5533	-5545	-5557	-5569	-5581	-5593	-5605	-5617	-5629	-5641	-5653	-5665	-5677	-5689	-5701	-5713	-5725	-5737	-5749	-5761	-5773	-5785	-5797	-5809	-5821	-5833	-5845	-5857	-5869	-5881	-5893	-5905	-5917	-5929	-5941	-5953	-5965	-5977	-5989	-6001	-6013	-6025	-6037	-6049	-6061	-6073	-6085	-6097	-6109	-6121	-6133	-6145	-6157	-6169	-6181	-6193	-6205	-6217	-6229	-6241	-6253	-6265	-6277	-6289	-6301	-6313	-6325	-6337	-6349	-6361	-6373	-6385	-6397	-6409	-6421	-6433	-6445	-6457	-6469	-6481	-6493	-6505	-6517	-6529	-6541	-6553	-6565	-6577	-6589	-6601	-6613	-6625	-6637	-6649	-6661	-6673	-6685	-6697	-6709	-6721	-6733	-6745	-6757	-6769	-6781	-6793	-6805	-6817	-6829	-6841	-6853	-6865	-6877	-6889	-6901	-6913	-6925	-6937	-6949	-6961	-6973	-6985	-6997	-7009	-7021	-7033	-7045	-7057	-7069	-7081	-7093	-7105	-7117	-7129	-7141	-7153	-7165	-7177	-7189	-7201	-7213	-7225	-7237	-7249	-7261	-7273	-7285	-7297	-7309	-7321	-7333	-7345	-7357	-7369	-7381	-7393	-7405	-7417	-7429	-7441	-7453	-7465	-7477	-7489	-7501	-7513	-7525	-7537	-7549	-7561	-7573	-7585	-7597	-7609	-7621	-7633	-7645	-7657	-7669	-7681	-7693	-7705	-7717	-7729	-7741	-7753	-7765	-7777	-7789	-7801	-7813	-7825	-7837	-7849	-7861	-7873	-7885	-7897	-7909	-7921	-7933	-7945	-7957	-7969	-7981	-7993	-8005	-8017	-8029	-8041	-8053	-8065	-8077	-8089	-8101	-8113	-8125	-8137	-8149	-8161	-8173	-8185	-8197	-8209	-8221	-8233	-8245	-8257	-8269	-8281	-8293	-8305	-8317	-8329	-8341	-8353	-8365	-8377	-8389	-8401	-8413	-8425	-8437	-8449	-8461	-8473	-8485	-8497	-8509	-8521	-8533	-8545	-8557	-8569	-8581	-8593	-8605	-8617	-8629	-8641	-8653	-8665	-8677	-8689	-8701	-8713	-8725	-8737	-8749	-8761	-8773	-8785	-8797	-8809	-8821	-8833	-8845	-8857	-8869	-8881	-8893	-8905	-8917	-8929	-8941	-8953	-8965	-8977	-8989	-9001	-9013	-9025	-9037	-9049	-9061	-9073	-9085	-9097	-9109	-9121	-9133	-9145	-9157	-9169	-9181	-9193	-9205	-9217	-9229	-9241	-9253	-9265	-9277	-9289	-9301	-9313	-9325	-9337	-9349	-9361	-9373	-9385	-9397	-9409	-9421	-9433	-9445	-9457	-9469	-9481	-9493	-9505	-9517	-9529	-9541	-9553	-9565	-9577	-9589	-9601	-9613	-9625	-9637	-9649	-9661	-9673	-9685	-9697	-9709	-9721	-9733	-9745	-9757	-9769	-9781	-9793	-9805	-9817	-9829	-9841	-9853	-9865	-9877	-9889	-9901	-9913	-9925	-9937	-9949	-9961	-9973	-9985	-9997	-10009	-10021	-10033	-10045	-10057	-10069	-10081	-10093	-10105	-10117	-10129	-10141	-10153	-10165	-10177	-10189	-10201	-10213	-10225	-10237	-10249	-10261	-10273	-10285	-10297	-10309	-10321	-10333	-10345	-10357	-10369	-10381	-10393	-10405	-10417	-10429	-10441	-10453	-10465	-10477	-10489	-10501	-10513	-10525	-10537	-10549	-10561	-10573	-10585	-10597	-10609	-10621	-10633	-10645	-10657	-10669	-10681	-10693	-10705	-10717	-10729	-10741	-10753	-10765	-10777	-10789	-10801	-10813	-10825	-10837	-10849	-10861	-10873	-10885	-10897	-10909	-10921	-10933	-10945	-10957	-10969	-10981	-10993	-11005	-11017	-11029	-11041	-11053	-11065	-11077	-11089	-11101	-11113	-11125	-11137	-11149	-11161	-11173	-11185	-11197	-11209	-11221	-11233	-11245	-11257	-11269	-11281	-11293	-11305	-11317	-11329	-11341	-11353	-11365	-11377	-11389	-11401	-11413	-11425	-11437	-11449	-11461	-11473	-11485	-11497	-11509	-11521	-11533	-11545	-11557	-11569	-11581	-11593	-11605	-11617	-11629	-11641	-11653	-11665	-11677	-11689	-11701	-11713	-11725	-11737	-11749	-11761	-11773	-11785	-11797	-11809	-11821	-11833	-11845	-11857	-11869	-11881	-11893	-11905	-11917	-11929	-11941	-11953	-11965	-11977	-11989	-11999	-12009	-12019	-12029	-12039	-12049	-12059	-12069	-12079	-12089	-12099	-12109	-12119	-12129	-12139	-12149	-12159	-12169	-12179	-12189	-12199	-12209	-12219	-12229	-12239	-12249	-12259	-12269	-12279	-12289	-12299	-12309	-12319	-12329	-12339	-12349	-12359	-12369	-12379	-12389	-12399	-12409	-12419	-12429	-12439	-12449	-12459	-12469	-12479	-12489	-12499	-12509	-12519	-12529	-12539	-12549	-12559	-12569	-12579	-12589	-12599	-12609	-12619	-12629	-12639	-12649	-12659	-12669	-12679	-12689	-12699	-12709	-12719	-12729	-12739	-12749	-12759	-12769	-12779	-12789	-12799	-12809	-12819	-12829	-12839	-12849	-12859	-12869	-12879	-12889	-12899	-12909	-12919	-12929	-12939	-12949	-12959	-12969	-12979	-12989	-12999	-13009	-13019	-13029	-13039	-13049	-13059	-13069	-13079	-13089	-13099	-13109	-13119	-13129	-13139	-13149	-13159	-13169	-13179	-13189	-13199	-13209	-13219	-13229	-13239	-13249	-13259	-13269	-13279	-13289	-13299	-13309	-13319	-13329	-13339	-13349	-13359	-13369	-13379	-13389	-13399	-13409	-13419	-13429	-13439	-13449	-13459	-13469	-13479	-13489	-13499	-13509	-13519	-13529	-13539	-13549	-13559	-13569	-13579	-13589	-13599	-13609	-13619	-13629	-13639	-13649	-13659	-13669	-13679	-13689	-13699	-13709	-13719	-13729	-13739	-13749	-13759	-13769	-13779	-13789	-13799	-13809	-13819	-13829	-13839	-13849	-13859	-13869	-13879	-13889	-13899	-13909	-13919	-13929	-13939	-13949	-13959	-13969	-13979	-13989	-13999	-14009	-14019	-14029	-14039	-14049	-14059	-14069	-14079	-14089	-14099	-14109	-14119	-14129	-14139	-14149	-14159	-14169	-14179	-14189	-14199	-14209	-14219	-14229	-14239	-14249	-14259	-14269	-14279	-14289	-14299	-14309	-14319	-14329	-14339	-14349	-14359	-14369	-14379	-14389	-14399	-14409	-14419	-14429	-14439	-14449	-14

TABLE 45. MOORE 3 AND 4

MOORE 3		MOORE 4	
0.123	0.123	0.123	0.123
0.124	0.124	0.124	0.124
0.125	0.125	0.125	0.125
0.126	0.126	0.126	0.126
0.127	0.127	0.127	0.127
0.128	0.128	0.128	0.128
0.129	0.129	0.129	0.129
0.130	0.130	0.130	0.130
0.131	0.131	0.131	0.131
0.132	0.132	0.132	0.132
0.133	0.133	0.133	0.133
0.134	0.134	0.134	0.134
0.135	0.135	0.135	0.135
0.136	0.136	0.136	0.136
0.137	0.137	0.137	0.137
0.138	0.138	0.138	0.138
0.139	0.139	0.139	0.139
0.140	0.140	0.140	0.140
0.141	0.141	0.141	0.141
0.142	0.142	0.142	0.142
0.143	0.143	0.143	0.143
0.144	0.144	0.144	0.144
0.145	0.145	0.145	0.145
0.146	0.146	0.146	0.146
0.147	0.147	0.147	0.147
0.148	0.148	0.148	0.148
0.149	0.149	0.149	0.149
0.150	0.150	0.150	0.150
0.151	0.151	0.151	0.151
0.152	0.152	0.152	0.152
0.153	0.153	0.153	0.153
0.154	0.154	0.154	0.154
0.155	0.155	0.155	0.155
0.156	0.156	0.156	0.156
0.157	0.157	0.157	0.157
0.158	0.158	0.158	0.158
0.159	0.159	0.159	0.159
0.160	0.160	0.160	0.160
0.161	0.161	0.161	0.161
0.162	0.162	0.162	0.162
0.163	0.163	0.163	0.163
0.164	0.164	0.164	0.164
0.165	0.165	0.165	0.165
0.166	0.166	0.166	0.166
0.167	0.167	0.167	0.167
0.168	0.168	0.168	0.168
0.169	0.169	0.169	0.169
0.170	0.170	0.170	0.170
0.171	0.171	0.171	0.171
0.172	0.172	0.172	0.172
0.173	0.173	0.173	0.173
0.174	0.174	0.174	0.174
0.175	0.175	0.175	0.175
0.176	0.176	0.176	0.176
0.177	0.177	0.177	0.177
0.178	0.178	0.178	0.178
0.179	0.179	0.179	0.179
0.180	0.180	0.180	0.180
0.181	0.181	0.181	0.181
0.182	0.182	0.182	0.182
0.183	0.183	0.183	0.183
0.184	0.184	0.184	0.184
0.185	0.185	0.185	0.185
0.186	0.186	0.186	0.186
0.187	0.187	0.187	0.187
0.188	0.188	0.188	0.188
0.189	0.189	0.189	0.189
0.190	0.190	0.190	0.190
0.191	0.191	0.191	0.191
0.192	0.192	0.192	0.192
0.193	0.193	0.193	0.193
0.194	0.194	0.194	0.194
0.195	0.195	0.195	0.195
0.196	0.196	0.196	0.196
0.197	0.197	0.197	0.197
0.198	0.198	0.198	0.198
0.199	0.199	0.199	0.199
0.200	0.200	0.200	0.200

TABLE 45. MOORE 3 AND 4



TABLE 46 (CONTS.) (A40 2)

— 300M

[illegible]

TABLE 46 (MODES 3 AND 4)

MODE 3 μ_3		MODE 4	
0.123	0.124	0.125	0.126
0.127	0.128	0.129	0.130
0.131	0.132	0.133	0.134
0.135	0.136	0.137	0.138
0.141	0.142	0.143	0.144
0.145	0.146	0.147	0.148
0.151	0.152	0.153	0.154
0.155	0.156	0.157	0.158
0.161	0.162	0.163	0.164
0.165	0.166	0.167	0.168
0.171	0.172	0.173	0.174
0.175	0.176	0.177	0.178
0.181	0.182	0.183	0.184
0.185	0.186	0.187	0.188
0.191	0.192	0.193	0.194
0.195	0.196	0.197	0.198
0.201	0.202	0.203	0.204
0.205	0.206	0.207	0.208
0.211	0.212	0.213	0.214
0.215	0.216	0.217	0.218
0.221	0.222	0.223	0.224
0.225	0.226	0.227	0.228
0.231	0.232	0.233	0.234
0.235	0.236	0.237	0.238
0.241	0.242	0.243	0.244
0.245	0.246	0.247	0.248
0.251	0.252	0.253	0.254
0.255	0.256	0.257	0.258
0.261	0.262	0.263	0.264
0.265	0.266	0.267	0.268
0.271	0.272	0.273	0.274
0.275	0.276	0.277	0.278
0.281	0.282	0.283	0.284
0.285	0.286	0.287	0.288
0.291	0.292	0.293	0.294
0.295	0.296	0.297	0.298
0.301	0.302	0.303	0.304
0.305	0.306	0.307	0.308
0.311	0.312	0.313	0.314
0.315	0.316	0.317	0.318
0.321	0.322	0.323	0.324
0.325	0.326	0.327	0.328
0.331	0.332	0.333	0.334
0.335	0.336	0.337	0.338
0.341	0.342	0.343	0.344
0.345	0.346	0.347	0.348
0.351	0.352	0.353	0.354
0.355	0.356	0.357	0.358
0.361	0.362	0.363	0.364
0.365	0.366	0.367	0.368
0.371	0.372	0.373	0.374
0.375	0.376	0.377	0.378
0.381	0.382	0.383	0.384
0.385	0.386	0.387	0.388
0.391	0.392	0.393	0.394
0.395	0.396	0.397	0.398
0.401	0.402	0.403	0.404
0.405	0.406	0.407	0.408
0.411	0.412	0.413	0.414
0.415	0.416	0.417	0.418
0.421	0.422	0.423	0.424
0.425	0.426	0.427	0.428
0.431	0.432	0.433	0.434
0.435	0.436	0.437	0.438
0.441	0.442	0.443	0.444
0.445	0.446	0.447	0.448
0.451	0.452	0.453	0.454
0.455	0.456	0.457	0.458
0.461	0.462	0.463	0.464
0.465	0.466	0.467	0.468
0.471	0.472	0.473	0.474
0.475	0.476	0.477	0.478
0.481	0.482	0.483	0.484
0.485	0.486	0.487	0.488
0.491	0.492	0.493	0.494
0.495	0.496	0.497	0.498
0.501	0.502	0.503	0.504
0.505	0.506	0.507	0.508
0.511	0.512	0.513	0.514
0.515	0.516	0.517	0.518
0.521	0.522	0.523	0.524
0.525	0.526	0.527	0.528
0.531	0.532	0.533	0.534
0.535	0.536	0.537	0.538
0.541	0.542	0.543	0.544
0.545	0.546	0.547	0.548
0.551	0.552	0.553	0.554
0.555	0.556	0.557	0.558
0.561	0.562	0.563	0.564
0.565	0.566	0.567	0.568
0.571	0.572	0.573	0.574
0.575	0.576	0.577	0.578
0.581	0.582	0.583	0.584
0.585	0.586	0.587	0.588
0.591	0.592	0.593	0.594
0.595	0.596	0.597	0.598
0.601	0.602	0.603	0.604
0.605	0.606	0.607	0.608
0.611	0.612	0.613	0.614
0.615	0.616	0.617	0.618
0.621	0.622	0.623	0.624
0.625	0.626	0.627	0.628
0.631	0.632	0.633	0.634
0.635	0.636	0.637	0.638
0.641	0.642	0.643	0.644
0.645	0.646	0.647	0.648
0.651	0.652	0.653	0.654
0.655	0.656	0.657	0.658
0.661	0.662	0.663	0.664
0.665	0.666	0.667	0.668
0.671	0.672	0.673	0.674
0.675	0.676	0.677	0.678
0.681	0.682	0.683	0.684
0.685	0.686	0.687	0.688
0.691	0.692	0.693	0.694
0.695	0.696	0.697	0.698
0.701	0.702	0.703	0.704
0.705	0.706	0.707	0.708
0.711	0.712	0.713	0.714
0.715	0.716	0.717	0.718
0.721	0.722	0.723	0.724
0.725	0.726	0.727	0.728
0.731	0.732	0.733	0.734
0.735	0.736	0.737	0.738
0.741	0.742	0.743	0.744
0.745	0.746	0.747	0.748
0.751	0.752	0.753	0.754
0.755	0.756	0.757	0.758
0.761	0.762	0.763	0.764
0.765	0.766	0.767	0.768
0.771	0.772	0.773	0.774
0.775	0.776	0.777	0.778
0.781	0.782	0.783	0.784
0.785	0.786	0.787	0.788
0.791	0.792	0.793	0.794
0.795	0.796	0.797	0.798
0.801	0.802	0.803	0.804
0.805	0.806	0.807	0.808
0.811	0.812	0.813	0.814
0.815	0.816	0.817	0.818
0.821	0.822	0.823	0.824
0.825	0.826	0.827	0.828
0.831	0.832	0.833	0.834
0.835	0.836	0.837	0.838
0.841	0.842	0.843	0.844
0.845	0.846	0.847	0.848
0.851	0.852	0.853	0.854
0.855	0.856	0.857	0.858
0.861	0.862	0.863	0.864
0.865	0.866	0.867	0.868
0.871	0.872	0.873	0.874
0.875	0.876	0.877	0.878
0.881	0.882	0.883	0.884
0.885	0.886	0.887	0.888
0.891	0.892	0.893	0.894
0.895	0.896	0.897	0.898
0.901	0.902	0.903	0.904
0.905	0.906	0.907	0.908
0.911	0.912	0.913	0.914
0.915	0.916	0.917	0.918
0.921	0.922	0.923	0.924
0.925	0.926	0.927	0.928
0.931	0.932	0.933	0.934
0.935	0.936	0.937	0.938
0.941	0.942	0.943	0.944
0.945	0.946	0.947	0.948
0.951	0.952	0.953	0.954
0.955	0.956	0.957	0.958
0.961	0.962	0.963	0.964
0.965	0.966	0.967	0.968
0.971	0.972	0.973	0.974
0.975	0.976	0.977	0.978
0.981	0.982	0.983	0.984
0.985	0.986	0.987	0.988
0.991	0.992	0.993	0.994
0.995	0.996	0.997	0.998
1.001	1.002	1.003	1.004

Handwritten note: μ_3 MODE 4

CHAPTER 5

FOUR SPAN BEAMS WITH CLASSICAL BOUNDARY CONDITIONS

The problem of establishing the frequencies and modal shapes for four span beams is almost identical to that encountered with triple span beams in Chapter 4. It may be easily recognised that the eigenvalues δ for four span beams are a function of the three dimensionless lengths μ_1 , μ_2 , and μ_3 (e.g.; see case 5.1).

5.1 Beams With Outer Ends Simply Supported

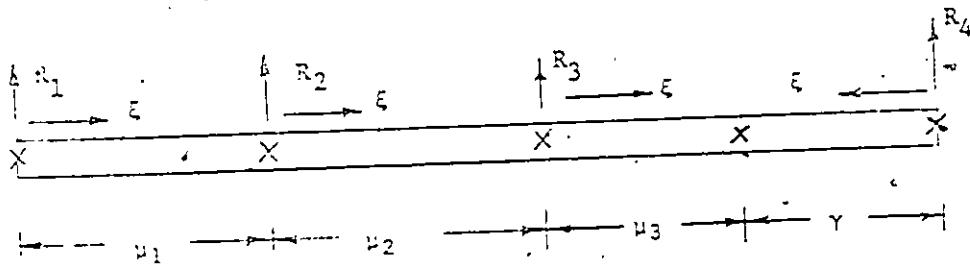


Figure 5.1

As in case of triple span beams, the origin for first three spans (counting from left) are chosen at their left ends while

for the last span it is taken at its right hand end. The parameters μ_1 , μ_2 , and μ_3 represent the dimensionless lengths of first three spans expressed as a fraction of overall beam length. The non-dimensional length of the fourth span is then obviously equal to γ where $\gamma = 1 - (\mu_1 + \mu_2 + \mu_3)$.

Following the procedure identical to that described in Chapter 4, it can be readily shown that

$$R_1(\xi) = A_1 \left(\sin \beta \xi - \frac{\sin \beta \mu_1}{\sinh \beta \mu_1} \sinh \beta \xi \right) \quad (5.1)$$

$$R_2(\xi) = A_2 (\sin \beta \xi - \theta_2 \sinh \beta \xi) + B_2 (\cos \beta \xi - \cosh \beta \xi + \phi_2 \sinh \beta \xi) \quad (5.2)$$

where

$$\theta_2 = \frac{\sin \beta \mu_2}{\sinh \beta \mu_2}, \quad \phi_2 = \frac{\cosh \beta \mu_2 - \cos \beta \mu_2}{\sinh \beta \mu_2}$$

$$R_3(\xi) = A_3 (\sin \beta \xi - \theta_3 \sinh \beta \xi) + B_3 (\cos \beta \xi - \cosh \beta \xi + \phi_3 \sinh \beta \xi) \quad (5.3)$$

where

$$\theta_3 = \frac{\sin \beta \mu_3}{\sinh \beta \mu_3}, \quad \phi_3 = \frac{\cosh \beta \mu_3 - \cos \beta \mu_3}{\sinh \beta \mu_3}$$

and

$$R_4(\xi) = A_4 \left(\sin \beta \xi - \frac{\sin \beta \gamma}{\sinh \beta \gamma} \sinh \beta \xi \right) \quad (5.4)$$

We are now left with six unknowns A_1 , A_2 , B_2 , A_3 , B_3 , and A_4 and we may write six equations in terms of these six constants utilizing the condition of continuity of slope and bending moment across three interior supports. The following set of six homogeneous algebraic equations is then obtained.

$$A_1 \left(\cos \beta \mu_1 - \frac{\sin \beta \mu_1}{\sinh \beta \mu_1} \cosh \beta \mu_1 \right) + A_2 (\theta_2 - 1) + B_2 (-\phi_2) = 0 \quad (5.5)$$

$$A_1 (-2 \sin \beta \mu_1) + 2B_2 = 0 \quad (5.6)$$

$$A_2 [\cos \beta \mu_2 - \theta_2 \cosh \beta \mu_2] + B_2 [-\sin \beta \mu_2 - \sinh \beta \mu_2 + \phi_2 \cosh \beta \mu_2] + A_3 (\theta_3 - 1) + B_3 (-\phi_3) = 0 \quad (5.7)$$

$$A_2 (-\sin \beta \mu_2 - \theta_2 \sinh \beta \mu_2) + B_2 (-\cos \beta \mu_2 - \cosh \beta \mu_2 + \phi_2 \sinh \beta \mu_2) + 2B_3 = 0 \quad (5.8)$$

$$A_3(\cos \beta u_3 - \theta_3 \cosh \beta u_3) + B_3(-\sin \beta u_3 - \sinh \beta u_3 + \phi_3 \cosh \beta u_3)$$

$$+ A_4(\cos \beta y - \frac{\sin \beta y}{\sinh \beta y} \cosh \beta y) = 0 \quad (5.9)$$

and

$$A_3(-\sin \beta u_3 - \theta_3 \sinh \beta u_3) + B_3(-\cos \beta u_3 - \cosh \beta u_3 + \phi_3 \sinh \beta u_3)$$

$$+ A_4(2 \sin \beta y) = 0 \quad (5.10)$$

The coefficient matrix of the six constants A_1, A_2, B_2, A_3, B_3 , and A_4 appearing in equations (5.5) through (5.10) can be written down as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad (5.11)$$

where the elements appearing in first column are the coefficients of A_1 , second column elements are coefficients of A_2 , ... etc. and last column elements are the coefficients of A_4 in equations (3.5) through (3.10).

The values of eigenvalues δ are found by permitting the determinant of the matrix (5.11) to vanish. It may be appreciated that these values of δ are a function of three dimensionless lengths μ_1 , μ_2 , and μ_3 . The next matter to be decided is the number of stations to be selected along the dimensionless beam length for the purpose of data storage. The more the stations that are utilized, the smaller will be the station to station increment in the parameters μ_1 , μ_2 , and μ_3 , and the greater will be the accuracy if linear interpolation is used to establish frequencies of beams whose internal supports do not lie exactly on these stations.

A detailed study indicated that the number of stations required to ensure one percent accuracy on interpolation, even for the first mode, is very large. The associated storage space is considered to be prohibitively excessive. For this reason the beam has been divided into twenty equal segments with nineteen internal stations. Data for this class of beams are provided in Tables 5.1 and can be used so long as the internal supports of the beam lie on any of these stations. The accurate eigenvalues of beams which do not fall into this

category can be obtained by utilizing the determinantal equation from matrix (5.11) which has been provided for all possible outer end conditions. It is however expected that most practical problems can be solved using the data provided here.

Data for frequency analysis of the first four modes for this case are provided in Tables 5.1.

Each table has a value of μ_1 associated with it. Associated with the table and its value of μ_1 is a triangular array of data so that eigenvalues may be obtained for any possible combinations of the values μ_2 and μ_3 for the first four modes. By combining first and second mode data and third and fourth mode data, four triangular arrays are combined to form two square arrays. Storage of the square arrays has been arranged so as to minimize storage space requirement. Advantage has been taken of symmetry, where possible, and redundant data has not been provided. Use of these tables will be demonstrated with an illustrative example at the end of this Chapter.

The modal shape expressions are obtained by setting the constant A_1 equal to unity and solving equations (5.5) through (5.10) for the other five constants A_2 , B_2 , A_3 , B_3 , and A_4 . The constant A_4 can be evaluated from either of the two expressions provided. Both the expressions are given so as to obviate the necessity of finding the limit if any one of these takes on an indeterminate form.

The values of constants A_2 , B_2 , A_3 , B_3 , and A_4 appearing in equations (5.2) through (5.4) can be obtained from following expressions for any eigenvalue β .

$$A_2 = \frac{\sin \beta \mu_1 \cosh \beta \mu_1 - \cos \beta \mu_1 \sinh \beta \mu_1 + \phi_2 \sin \beta \mu_1 \sinh \beta \mu_1}{(\theta_2 - 1) \sinh \beta \mu_1}$$

$$B_2 = \sin \beta \mu_1$$

$$B_3 = \frac{A_2}{2} (\sin \beta \mu_2 + \theta_2 \sinh \beta \mu_2) + \frac{B_2}{2} (\cos \beta \mu_2 + \cosh \beta \mu_2 - \phi_2 \sinh \beta \mu_2)$$

$$A_3 = \frac{A_2}{(1 - \theta_3)} (\cos \beta \mu_2 - \theta_2 \cosh \beta \mu_2) + \frac{B_2}{(1 - \theta_3)} (-\sin \beta \mu_2 - \sinh \beta \mu_2 + \phi_2 \cosh \beta \mu_2) - \frac{B_3 \phi_3}{(1 - \theta_3)}$$

$$A_4 = \frac{A_3 (\cos \beta \mu_3 - \theta_3 \cosh \beta \mu_3) + B_3 (-\sin \beta \mu_3 - \sinh \beta \mu_3 + \phi_3 \cosh \beta \mu_3)}{((\sin \beta \gamma \cosh \beta \gamma) / (\sinh \beta \gamma)) - \cos \beta \gamma}$$

or

$$A_4 = A_3 \frac{\sin \beta \mu_3}{\sin \beta \gamma} + B_3 \frac{\cos \beta \mu_3}{\sin \beta \gamma}$$

It is interesting to note that in the coefficient matrix (5.11), only the elements α_{11} , α_{21} , α_{56} , and α_{66} depend explicitly on outer end boundary conditions. It is for this reason and to avoid repetitiveness, only these four elements will be provided for the following cases 5.2 through 5.6. Furthermore, since only the first and last span modal shapes are explicitly dependent on outer end boundary conditions, the expressions for α_{11} , α_{21} , α_{56} , α_{66} , $R_1(\xi)$, $R_4(\xi)$, A_2 , B_2 , and A_4 will be provided for rest of the cases considered in this Chapter. In fact these are the only expressions needed to complete the vibration analysis of such beams. The values for α_{11} , α_{21} , α_{56} , and α_{66} are required to write the frequency equation and the rest are needed for establishing modal shapes. The remaining elements in the coefficient matrix (5.11) and the expressions for $R_2(\xi)$, $R_3(\xi)$, A_3 , and B_3 remain unchanged throughout this Chapter.

5.2 Beams With Outer Ends Clamped

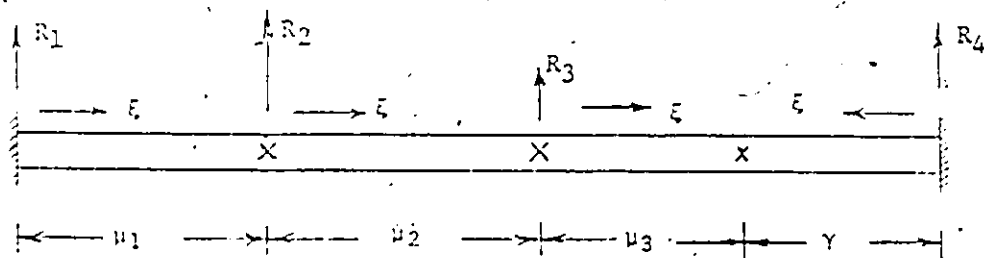


Figure 5.2 .

$$\alpha_{11} = \frac{2(1 - \cos \beta \mu_1 \cosh \beta \mu_1)}{\cos \beta \mu_1 - \cosh \beta \mu_1}$$

$$\alpha_{21} = \frac{2(\sinh \beta \mu_1 \cos \beta \mu_1 - \sin \beta \mu_1 \cosh \beta \mu_1)}{\cosh \beta \mu_1 - \cos \beta \mu_1}$$

$$\alpha_{56} = \frac{2(1 - \cos \beta \gamma \cosh \beta \gamma)}{\cos \beta \gamma - \cosh \beta \gamma}$$

$$\alpha_{66} = \frac{2(\sin \beta \gamma \cosh \beta \gamma - \sinh \beta \gamma \cos \beta \gamma)}{\cosh \beta \gamma - \cos \beta \gamma}$$

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (\cos \beta \xi - \cosh \beta \xi)$$

$$R_4(\xi) = A_4 \left[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \gamma - \sinh \beta \gamma)}{(\cosh \beta \gamma - \cos \beta \gamma)} (\cos \beta \xi - \cosh \beta \xi) \right]$$

$$A_2 = \frac{2(1 - \cosh \beta \mu_1 \cos \beta \mu_1) - \phi_2(\sinh \beta \mu_1 \cos \beta \mu_1 - \sin \beta \mu_1 \cosh \beta \mu_1)}{(\theta_2 - 1)(\cosh \beta \mu_1 - \cos \beta \mu_1)}$$

$$B_2 = \frac{(\sin \beta \mu_1 \cosh \beta \mu_1 - \sinh \beta \mu_1 \cos \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)}$$

$$[A_3(\sinh \beta \mu_3 \cos \beta \mu_3 - \sin \beta \mu_3 \cosh \beta \mu_3) + B_3(1 - \cos \beta \mu_3 \cosh \beta \mu_3 -$$

$$A_4 = \frac{\sin \beta \mu_3 \sinh \beta \mu_3]}{2 \sinh \beta \mu_3 [(1 - \cosh \beta \gamma \cos \beta \gamma)/(\cosh \beta \gamma - \cos \beta \gamma)]}$$

or.

$$A_4 = \frac{A_3(\sin \beta \mu_3) + B_3(\cos \beta \mu_3)}{[(\sin \beta \gamma \cosh \beta \gamma - \sinh \beta \gamma \cos \beta \gamma)/(\cosh \beta \gamma - \cos \beta \gamma)]}$$

It may be noted that the remaining elements of the coefficient matrix for this case are same as in case 5.1. The values of β which permit the determinant of this coefficient matrix to vanish are presented in Tables 5.2. The modal shape expressions for frequencies associated with any eigenvalue β can be determined from the above expressions. This may however, be recalled that expressions for $R_2(\xi)$, $R_3(\xi)$, A_3 , and B_3 are same as provided in case 5.1 and in fact remain same for all

the cases considered in this Chapter. Being a symmetric case, the values of μ_1 are provided only upto 0.5.

5.3 Beams With Outer Ends Free

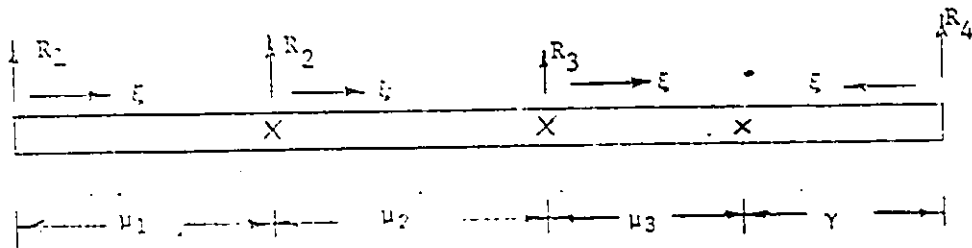


Figure 5.3

$$a_{11} = \frac{2(1 + \cos \beta\mu_1 \cosh \beta\mu_1)}{\cos \beta\mu_1 + \cosh \beta\mu_1}$$

$$a_{21} = \frac{2(\sinh \beta\mu_1 \cos \beta\mu_1 - \sin \beta\mu_1 \cosh \beta\mu_1)}{\cos \beta\mu_1 + \cosh \beta\mu_1}$$

$$a_{56} = \frac{2(1 + \cos \beta\gamma \cosh \beta\gamma)}{\cos \beta\gamma + \cosh \beta\gamma}$$

$$a_{66} = \frac{2(\sin \beta \gamma \cosh \beta \gamma - \sinh \beta \gamma \cos \beta \gamma)}{\cos \beta \gamma + \cosh \beta \gamma}$$

The remaining elements of coefficient matrix are same as given in equation (5.11) and the values of β associated with the non-trivial solution for modal shape functions are presented in Table 5.3. For modal shape expressions we need the following information for this class of beams.

$$R_1(\xi) = \sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \mu_1 + \sinh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1)} (\cos \beta \xi + \cosh \beta \xi)$$

$$R_4(\xi) = A_4 \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \gamma + \sinh \beta \gamma)}{(\cos \beta \gamma + \cosh \beta \gamma)} (\cos \beta \xi + \cosh \beta \xi) \right]$$

$$A_2 = \frac{2(1 + \cos \beta \mu_1 \cosh \beta \mu_1) + \phi_2(\sinh \beta \mu_1 \cos \beta \mu_1 - \sin \beta \mu_1 \cosh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1) (1 - \theta_2)}$$

$$B_2 = \frac{\sin \beta \mu_1 \cosh \beta \mu_1 - \sinh \beta \mu_1 \cos \beta \mu_1}{\cos \beta \mu_1 + \cosh \beta \mu_1}$$

$$A_4 = \frac{A_3(\sin \beta \mu_3 \cosh \beta \mu_3 - \sinh \beta \mu_3 \cos \beta \mu_3) + B_3(\sin \beta \mu_3 \sinh \beta \mu_3 \cos \beta \mu_3 \cosh \beta \mu_3 - 1)}{2 \sinh \beta \mu_3 [(1 + \cos \beta \gamma \cosh \beta \gamma) / (\cos \beta \gamma + \cosh \beta \gamma)]}$$

or

$$A_4 = \frac{A_3(\sin \beta \mu_3 \sinh \beta \mu_3) + B_3(\cos \beta \mu_3 \sinh \beta \mu_3)}{\sinh \beta \mu_3 [(\sin \beta \gamma \cosh \beta \gamma - \sinh \beta \gamma \cos \beta \gamma) / (\cos \beta \gamma + \cosh \beta \gamma)]}$$

The expressions for $R_2(\xi)$, $R_3(\xi)$, A_2 , and B_2 are same as provided in case 5.1.

5.4 Beams With One Outer End Clamped and Other Simply Supported

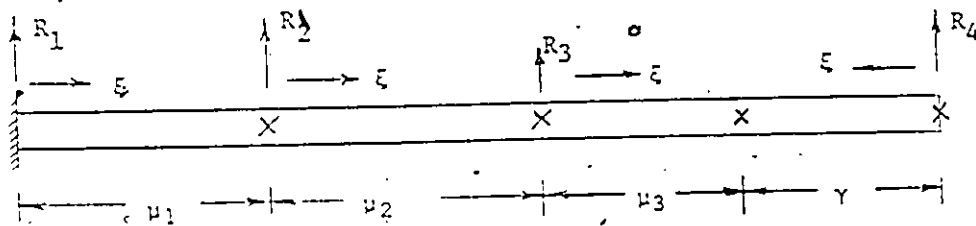


Figure 5.4

The expressions for α_{11} , α_{21} , $R_1(\xi)$, $R_2(\xi)$, $R_3(\xi)$, A_2 , B_2 , A_3 , and B_3 ; are same as for case 5.2; and for α_{56} , α_{66} , $R_4(\xi)$, and A_4 ; are same as for case 5.1.

Tables 5.4 furnish the frequency data for this case.

5.5 Beams With One Outer End Clamped and the Other Free

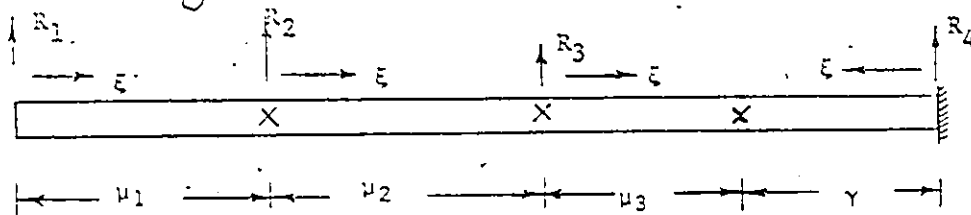


Figure 5.5

The expressions for α_{11} , α_{21} , $R_1(\xi)$, $R_2(\xi)$, $R_3(\xi)$, A_2 , B_2 , A_3 , and B_3 ; are same as provided in case 5.3; and for α_{56} , α_{66} , $R_4(\xi)$ and R_4 ; are same as for case 5.2.

The eigenvalues β for this case are provided in Tables 5.5.

5.6 Beams With One End Simply Supported and the Other Free

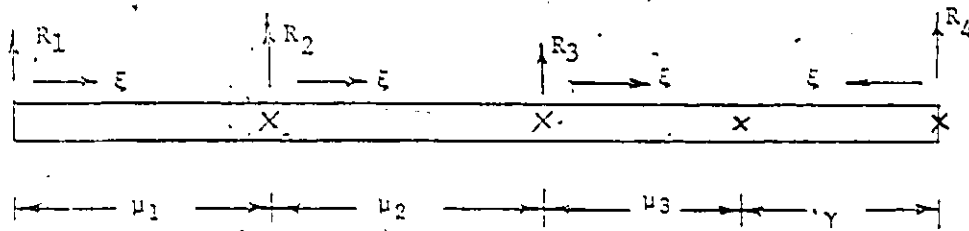


Figure 5.6

$\alpha_{11}, \alpha_{21}, R_1(\xi), R_2(\xi), R_3(\xi), A_2, B_2, A_3,$ and B_3 ; as for case 5.3
 $\alpha_{56}, \alpha_{66}, R_4(\xi),$ and A_4 ; as for case 5.1.

All the data necessary to establish frequencies for this case are provided in Tables 5.6.

Illustrative Example 5.1

The lengths of the four spans of a four span beam are 10", 20", 30" and 40" respectively, where spans are counted from left..

- (a) If extreme ends are free, determine the first and second mode frequencies of the beam.

$$u_1 = 0.1, \quad u_2 = 0.2, \quad u_3 = 0.3.$$

Using Tables 5.3, obtain $B_1 = 4.009$ and $B_2 = 10.33$, so that

$$f_{(1)} = \frac{(4.009)^2}{2 \times 100^2} \sqrt{\frac{EI}{cA}} = 11.64 \, E_z$$

$$f_{(2)} = \frac{(10.33)^2}{2 \cdot L^2} \sqrt{\frac{EI}{cA}} = 77.28 \, E_z$$

- (b) If the left hand end of the beam in part (a) is clamped, determine the first mode frequency of the new beam system.

From Tables 5.5, obtain $B_1 = 10.36$, so that

$$f_{(1)} = 77.7 \, E_z$$

- (c) If all the supports are pinned in the above beam, then obtain

$B_1 = 8.69$, from Tables 5.6 and

$$f_{(1)} = 54.7 \, E_z$$

[illegible]

TABLE 5.1 ($v_1 = .40$)

MEZ 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	7.453	8.313	9.116	9.849	10.516	11.126	11.680	12.188	12.650	13.066
.10	8.471	9.237	9.938	10.578	11.156	11.680	12.152	12.572	12.940	13.256
.15	9.166	9.838	10.437	10.972	11.450	11.872	12.240	12.552	12.808	13.016
.20	9.615	10.185	10.672	11.100	11.472	11.792	12.052	12.260	12.416	12.528
.25	9.872	10.340	10.712	11.000	11.220	11.380	11.580	11.720	11.820	11.880
.30	9.972	10.340	10.612	10.800	10.920	10.980	11.080	11.140	11.180	11.200
.35	10.012	10.280	10.452	10.540	10.580	10.600	10.640	10.660	10.680	10.680
.40	10.032	10.200	10.280	10.320	10.340	10.360	10.380	10.380	10.380	10.380
.45	10.042	10.160	10.180	10.180	10.180	10.180	10.180	10.180	10.180	10.180
.50	10.042	10.160	10.180	10.180	10.180	10.180	10.180	10.180	10.180	10.180

TABLE 5.1 ($v_1 = .35$)

MEZ 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	4.922	5.313	5.616	5.849	6.016	6.126	6.180	6.188	6.150	6.066
.10	5.471	5.837	6.138	6.378	6.556	6.680	6.752	6.772	6.740	6.656
.15	5.872	6.185	6.437	6.600	6.680	6.712	6.712	6.680	6.616	6.528
.20	6.166	6.385	6.537	6.612	6.612	6.580	6.512	6.416	6.280	6.116
.25	6.372	6.500	6.572	6.572	6.500	6.400	6.260	6.080	5.820	5.520
.30	6.472	6.500	6.452	6.340	6.180	6.000	5.740	5.440	5.080	4.680
.35	6.512	6.452	6.300	6.040	5.740	5.400	4.940	4.440	3.880	3.280
.40	6.512	6.352	6.000	5.600	5.160	4.600	3.940	3.280	2.520	1.680
.45	6.472	6.212	5.760	5.260	4.720	4.040	3.280	2.440	1.520	.680
.50	6.402	6.142	5.640	5.100	4.520	3.840	3.080	2.240	1.320	.480
.55	6.302	6.042	5.500	4.920	4.300	3.640	2.880	2.040	1.120	.280

TABLE 5.1 ($v_1 = .40$)

MEZ 2

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	11.81	13.04	14.23	15.37	16.46	17.50	18.48	19.32	20.00	20.52
.10	13.29	14.76	16.14	17.42	18.60	19.68	20.64	21.48	22.16	22.68
.15	14.63	16.20	17.58	18.86	19.94	20.90	21.76	22.50	23.04	23.48
.20	15.72	17.29	18.67	19.85	20.93	21.80	22.54	23.18	23.62	23.96
.25	16.59	18.16	19.54	20.72	21.80	22.67	23.31	23.85	24.29	24.63
.30	17.26	18.83	20.21	21.39	22.47	23.34	24.08	24.62	25.06	25.40
.35	17.72	19.29	20.67	21.85	22.93	23.80	24.54	25.08	25.52	25.86
.40	18.00	19.57	20.95	22.13	23.21	24.08	24.82	25.36	25.80	26.14
.45	18.12	19.69	21.07	22.25	23.33	24.20	24.94	25.48	25.92	26.26
.50	18.12	19.69	21.07	22.25	23.33	24.20	24.94	25.48	25.92	26.26

TABLE 5.1 ($v_1 = .35$)

MEZ 3

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	12.37	13.60	14.78	15.92	17.00	18.04	19.02	19.94	20.80	21.60
.10	13.79	15.02	16.20	17.34	18.42	19.40	20.36	21.22	22.00	22.72
.15	15.28	16.51	17.69	18.83	19.91	20.88	21.74	22.50	23.16	23.72
.20	16.72	17.95	19.13	20.27	21.35	22.32	23.18	23.94	24.60	25.16
.25	17.99	19.22	20.40	21.54	22.62	23.59	24.45	25.21	25.87	26.43
.30	19.04	20.27	21.45	22.59	23.67	24.64	25.50	26.26	26.92	27.48
.35	19.88	21.11	22.29	23.43	24.51	25.48	26.34	27.10	27.76	28.32
.40	20.52	21.75	22.93	24.07	25.15	26.12	26.98	27.74	28.40	28.96
.45	20.96	22.19	23.37	24.51	25.59	26.56	27.42	28.18	28.84	29.40
.50	21.20	22.43	23.61	24.75	25.83	26.80	27.66	28.42	29.08	29.64

TABLE 5.1 ($v_1 = .35$)

MEZ 4

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	12.37	13.60	14.78	15.92	17.00	18.04	19.02	19.94	20.80	21.60
.10	13.79	15.02	16.20	17.34	18.42	19.40	20.36	21.22	22.00	22.72
.15	15.28	16.51	17.69	18.83	19.91	20.88	21.74	22.50	23.16	23.72
.20	16.72	17.95	19.13	20.27	21.35	22.32	23.18	23.94	24.60	25.16
.25	17.99	19.22	20.40	21.54	22.62	23.59	24.45	25.21	25.87	26.43
.30	19.04	20.27	21.45	22.59	23.67	24.64	25.50	26.26	26.92	27.48
.35	19.88	21.11	22.29	23.43	24.51	25.48	26.34	27.10	27.76	28.32
.40	20.52	21.75	22.93	24.07	25.15	26.12	26.98	27.74	28.40	28.96
.45	20.96	22.19	23.37	24.51	25.59	26.56	27.42	28.18	28.84	29.40
.50	21.20	22.43	23.61	24.75	25.83	26.80	27.66	28.42	29.08	29.64

TABLE 5.1 ($\nu_1 = .50$)

ν_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	7.653	7.645	7.641	7.639
.10	7.511	7.499	7.492	7.487
.15	7.398	7.384	7.375	7.369
.20	7.302	7.288	7.279	7.272
.25	7.217	7.204	7.194	7.187
.30	7.138	7.124	7.115	7.107
.35	7.060	7.045	7.036	7.028
.40	6.976	6.961	6.952	6.944
.45	6.892	6.877	6.868	6.860
.50	6.808	6.793	6.784	6.776
.55	6.724	6.709	6.700	6.692
.60	6.640	6.625	6.616	6.608
.65	6.556	6.541	6.532	6.524
.70	6.472	6.457	6.448	6.440
.75	6.388	6.373	6.364	6.356
.80	6.304	6.289	6.280	6.272
.85	6.220	6.205	6.196	6.188
.90	6.136	6.121	6.112	6.104
.95	6.052	6.037	6.028	6.020
1.00	5.968	5.953	5.944	5.936

ν_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	8.453	8.471	8.467	8.463
.10	8.313	8.298	8.290	8.284
.15	8.180	8.164	8.153	8.146
.20	8.068	8.051	8.040	8.031
.25	7.968	7.950	7.938	7.929
.30	7.871	7.852	7.839	7.829
.35	7.788	7.768	7.753	7.743
.40	7.709	7.688	7.673	7.663
.45	7.635	7.613	7.598	7.588
.50	7.565	7.542	7.527	7.517
.55	7.495	7.472	7.457	7.447
.60	7.425	7.402	7.387	7.377
.65	7.355	7.332	7.317	7.307
.70	7.285	7.262	7.247	7.237
.75	7.215	7.192	7.177	7.167
.80	7.145	7.122	7.107	7.097
.85	7.075	7.052	7.037	7.027
.90	7.005	6.982	6.967	6.957
.95	6.935	6.912	6.897	6.887
1.00	6.865	6.842	6.827	6.817

TABLE 5.1 ($\nu_1 = .50$)

ν_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	13.81	13.79	13.79	13.94
.10	13.60	13.58	13.58	13.78
.15	13.44	13.42	13.42	13.68
.20	13.28	13.26	13.26	13.58
.25	13.12	13.10	13.10	13.48
.30	12.96	12.94	12.94	13.40
.35	12.80	12.78	12.78	13.32
.40	12.64	12.62	12.62	13.24
.45	12.48	12.46	12.46	13.16
.50	12.32	12.30	12.30	13.08
.55	12.16	12.14	12.14	13.00
.60	12.00	11.98	11.98	12.92
.65	11.84	11.82	11.82	12.84
.70	11.68	11.66	11.66	12.76
.75	11.52	11.50	11.50	12.68
.80	11.36	11.34	11.34	12.60
.85	11.20	11.18	11.18	12.52
.90	11.04	11.02	11.02	12.44
.95	10.88	10.86	10.86	12.36
1.00	10.72	10.70	10.70	12.28

ν_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	15.26	15.29	15.28	15.28
.10	15.06	15.04	15.03	15.03
.15	14.87	14.85	14.84	14.84
.20	14.65	14.69	14.68	14.68
.25	14.43	14.47	14.46	14.46
.30	14.21	14.25	14.24	14.24
.35	13.99	14.03	14.02	14.02
.40	13.77	13.81	13.80	13.80
.45	13.55	13.59	13.58	13.58
.50	13.33	13.37	13.36	13.36
.55	13.11	13.15	13.14	13.14
.60	12.89	12.93	12.92	12.92
.65	12.67	12.71	12.70	12.70
.70	12.45	12.49	12.48	12.48
.75	12.23	12.27	12.26	12.26
.80	12.01	12.05	12.04	12.04
.85	11.79	11.83	11.82	11.82
.90	11.57	11.61	11.60	11.60
.95	11.35	11.39	11.38	11.38
1.00	11.13	11.17	11.16	11.16

TABLE 5.2 ($v_1 = .30$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.204	8.164	9.316	10.713	12.413	14.371	16.643	19.271	22.301	25.781	29.751	34.251
.10	8.301	9.507	10.857	12.413	14.133	16.073	18.293	20.853	23.803	27.193	31.073	35.493
.15	9.319	10.691	12.191	13.871	15.691	17.701	19.951	22.491	25.361	28.611	32.291	36.451
.20	10.251	11.751	13.351	15.091	16.991	19.101	21.471	24.151	27.111	30.411	34.091	38.211
.25	11.141	12.741	14.441	16.241	18.191	20.341	22.741	25.441	28.491	31.941	35.741	39.941
.30	11.991	13.691	15.441	17.291	19.291	21.501	23.991	26.741	29.811	33.261	37.061	41.261
.35	12.811	14.561	16.361	18.241	20.291	22.541	25.091	27.941	31.141	34.691	38.541	42.741
.40	13.611	15.411	17.261	19.241	21.341	23.641	26.241	29.141	32.591	36.241	40.141	44.241
.45	14.391	16.241	18.141	20.291	22.391	24.691	27.341	30.291	33.741	37.341	41.141	45.141
.50	15.141	16.991	18.941	21.291	23.291	25.591	28.291	31.241	34.741	38.341	42.141	46.041
.55	15.861	17.691	19.691	22.191	24.141	26.441	29.141	32.141	35.641	39.241	43.141	46.941
.60	16.561	18.391	20.391	22.991	24.891	27.141	29.841	32.841	36.341	40.041	43.841	47.641

TABLE 5.2 ($v_1 = .35$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.319	8.319	9.519	10.919	12.579	14.549	16.889	19.649	22.789	26.359	30.419	35.019
.10	8.419	9.619	10.919	12.419	14.179	16.149	18.389	20.949	23.899	27.299	31.199	35.619
.15	9.419	10.819	12.319	13.979	15.779	17.779	19.979	22.449	25.349	28.649	32.349	36.519
.20	10.319	11.819	13.419	15.179	17.079	19.149	21.449	24.149	27.149	30.449	34.049	38.219
.25	11.149	12.749	14.449	16.249	18.199	20.349	22.749	25.449	28.499	31.949	35.749	39.949
.30	11.999	13.699	15.449	17.299	19.299	21.509	23.999	26.749	29.819	33.269	37.069	41.269
.35	12.819	14.569	16.369	18.249	20.299	22.549	25.099	27.949	31.149	34.699	38.549	42.749
.40	13.619	15.419	17.269	19.249	21.349	23.649	26.249	29.149	32.599	36.249	40.149	44.249
.45	14.399	16.249	18.149	20.299	22.399	24.699	27.349	30.299	33.749	37.349	41.149	45.149
.50	15.149	16.999	18.949	21.299	23.299	25.599	28.299	31.249	34.749	38.349	42.149	46.049
.55	15.869	17.699	19.699	22.199	24.149	26.449	29.149	32.149	35.649	39.249	43.149	46.949
.60	16.569	18.399	20.399	22.999	24.899	27.149	29.849	32.849	36.349	40.049	43.849	47.649

TABLE 5.2 ($v_1 = .40$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.434	8.434	9.634	11.034	12.734	14.704	16.994	19.654	22.734	26.344	30.434	35.034
.10	8.534	9.734	11.034	12.534	14.284	16.254	18.494	21.054	23.994	27.394	31.294	35.714
.15	9.534	10.934	12.434	14.184	16.034	18.074	20.374	22.974	25.914	29.214	33.014	37.434
.20	10.434	11.934	13.534	15.284	17.184	19.294	21.644	24.294	27.294	30.644	34.244	38.434
.25	11.284	12.884	14.584	16.434	18.384	20.534	22.934	25.634	28.684	32.134	35.834	39.934
.30	12.134	13.734	15.434	17.384	19.434	21.684	24.134	26.884	29.934	33.384	37.134	41.334
.35	12.984	14.584	16.284	18.384	20.434	22.684	25.134	27.884	31.034	34.584	38.334	42.534
.40	13.784	15.384	17.084	19.384	21.434	23.684	26.134	28.884	32.034	35.584	39.334	43.734
.45	14.584	16.184	17.884	20.384	22.434	24.684	27.134	29.884	33.034	36.584	40.334	44.534
.50	15.384	16.984	18.684	21.384	23.434	25.684	28.134	30.884	34.034	37.584	41.334	45.334
.55	16.184	17.784	19.484	22.384	24.434	26.684	29.134	31.884	35.034	38.584	42.334	46.134
.60	16.984	18.584	20.284	23.384	25.434	27.684	30.134	32.884	36.034	39.584	43.334	46.934

TABLE 5.2 ($v_1 = .45$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.549	8.549	9.749	11.149	12.849	14.819	17.099	19.719	22.799	26.409	30.499	35.099
.10	8.649	9.849	11.149	12.649	14.399	16.369	18.609	21.169	24.109	27.509	31.409	35.829
.15	9.649	11.049	12.549	14.299	16.149	18.249	20.599	23.249	26.189	29.589	33.489	37.909
.20	10.549	12.049	13.649	15.499	17.399	19.549	21.999	24.699	27.699	31.099	34.699	38.699
.25	11.399	12.999	14.699	16.549	18.499	20.649	23.099	25.799	28.849	32.299	35.999	39.999
.30	12.249	13.849	15.549	17.499	19.599	21.849	24.299	27.049	29.999	33.449	37.199	41.399
.35	13.099	14.699	16.399	18.499	20.599	22.849	25.299	28.049	31.199	34.699	38.449	42.599
.40	13.899	15.499	17.199	19.499	21.599	23.849	26.299	29.049	32.199	35.699	39.449	43.799
.45	14.699	16.299	17.999	20.499	22.599	24.849	27.299	29.999	33.199	36.699	40.449	44.599
.50	15.499	17.099	18.799	21.499	23.599	25.849	28.299	31.049	34.199	37.699	41.449	45.399
.55	16.299	17.899	19.599	22.499	24.599	26.849	29.299	32.049	35.199	38.699	42.449	46.199
.60	17.099	18.699	20.399	23.499	25.599	27.849	30.299	33.049	36.199	39.699	43.449	46.999

TABLE 5.2 ($v_1 = .50$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.664	8.664	9.864	11.264	12.964	14.934	17.214	19.834	22.914	26.524	30.614	35.214
.10	8.764	9.964	11.264	12.764	14.514	16.484	18.724	21.284	24.224	27.624	31.524	35.944
.15	9.764	11.164	12.664	14.414	16.264	18.364	20.714	23.374	26.314	29.714	33.614	38.034
.20	10.664	12.164	13.764	15.614	17.514	19.664	22.114	24.814	27.754	31.154	34.754	38.754
.25	11.514	13.014	14.614	16.564	18.514	20.664	23.114	25.814	28.754	32.154	35.754	39.554
.30	12.364	13.864	15.464	17.414	19.514	21.764	24.214	26.964	29.914	33.314	36.914	40.554
.35	13.214	14.714	16.314	18.414	20.514	22.764	25.214	28.014	30.964	34.364	37.964	41.554
.40	14.014	15.514	17.114	19.414	21.514	23.764	26.214	29.014	31.964	35.364	38.964	42.554
.45	14.814	16.314	17.914	20.414	22.514	24.764	27.214	29.964	32.914	36.314	39.914	43.554
.50	15.614	17.114	18.714	21.414	23.514	25.764	28.214	30.964	33.914	37.314	40.914	44.554
.55	16.414	17.914	19.514	22.414	24.514	26.764	29.214	31.964	34.914	38.314	41.914	45.554
.60	17.214	18.714	20.314	23.414	25.514	27.764	30.214	32.964	35.914	39.314	42.914	46.554

TABLE 5.2 ($v_1 = .55$)

Model 1

v_1	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600
.05	7.779	8.779	9.979	11.379	13.079	15.049	17.329	19.949	23.029	26.639	30.729	35.329
.10	8.879	10.079	11.379	12.879	14.629	16.599	18.839	21.399	24.339	27.739	31.639	36.049
.15	9.879	11.279	12.779	14.529	16.379	18.479	20.829	23.489	26.429	29.829	33.729	38.149
.20	10.779	12.279	13.879	15.729	17.629	19.779	22.229	24.929	27.869	31.269	34.869	38.769
.25	11.629	13.129	14.729	16.679	18.629	20.779	23.229	25.929	28.869	32.269	35.869	39.569
.30	12.479	13.979	15.579	17.529	19.629	21.879	24.329	27.029	29.969	33.369	36.969	40.569
.35	13.329	14.829	16.429	18.529	20.629	22.879	25.329	28.029	30.969	34.369	37.969	41.569
.40	14.129	15.629	17.229	19.529	21.629	23.879	26.329	29.029	31.969	35.369	38.969	42.569
.45	14.929	16.429	18.029	20.529	22.629	24.779	27.229	29.969	32.919	36.319	39.919	43.569
.50	15.729	17.229	18.829	21.529	23.629	25.779	28.229	30.969	33.919	37.319	40.919	44.569
.55	16.529	18.029	19.629	22.529	24.629	26.779	29.229	31.969	34.919	38.319	41.919	45.569
.60	17.329	18.829	20.429	23.529	25.629	27.779	30.229	32.969	35.919	39.319	42.919	46.569

TABLE 5.2 ($v_1 = .35$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550
.05	9.402	9.641	9.881	10.120	10.359	10.598	10.837	11.076	11.315	11.554
.10	9.215	9.454	9.693	9.932	10.171	10.410	10.649	10.888	11.127	11.366
.15	9.028	9.267	9.506	9.745	9.984	10.223	10.462	10.701	10.940	11.179
.20	8.841	9.080	9.319	9.558	9.797	10.036	10.275	10.514	10.753	10.992
.25	8.654	8.893	9.132	9.371	9.610	9.849	10.088	10.327	10.566	10.805
.30	8.467	8.706	8.945	9.184	9.423	9.662	9.901	10.140	10.379	10.618
.35	8.280	8.519	8.758	8.997	9.236	9.475	9.714	9.953	10.192	10.431
.40	8.093	8.332	8.571	8.810	9.049	9.288	9.527	9.766	10.005	10.244
.45	7.906	8.145	8.384	8.623	8.862	9.101	9.340	9.579	9.818	10.057
.50	7.719	7.958	8.197	8.436	8.675	8.914	9.153	9.392	9.631	9.870
.55	7.532	7.771	8.010	8.249	8.488	8.727	8.966	9.205	9.444	9.683

TABLE 5.2 ($v_1 = .40$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	9.214	10.03	10.84	11.65	12.46	13.27	14.08	14.89	15.70
.10	9.027	9.84	10.65	11.46	12.27	13.08	13.89	14.70	15.51
.15	8.840	9.65	10.46	11.27	12.08	12.89	13.70	14.51	15.32
.20	8.653	9.46	10.27	11.08	11.89	12.70	13.51	14.32	15.13
.25	8.466	9.27	10.08	10.89	11.70	12.51	13.32	14.13	14.94
.30	8.279	9.08	9.89	10.70	11.51	12.32	13.13	13.94	14.75
.35	8.092	8.90	9.71	10.52	11.33	12.14	12.95	13.76	14.57
.40	7.905	8.71	9.52	10.33	11.14	11.95	12.76	13.57	14.38
.45	7.718	8.52	9.33	10.14	10.95	11.76	12.57	13.38	14.19
.50	7.531	8.34	9.15	9.96	10.77	11.58	12.39	13.20	14.01
.55	7.344	8.15	8.96	9.77	10.58	11.39	12.20	13.01	13.82

TABLE 5.2 ($v_1 = .40$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	15.35	16.76	18.17	19.58	20.99	22.40	23.81	25.22	26.63
.10	15.16	16.57	17.98	19.39	20.80	22.21	23.62	25.03	26.44
.15	14.97	16.38	17.79	19.20	20.61	22.02	23.43	24.84	26.25
.20	14.78	16.19	17.60	19.01	20.42	21.83	23.24	24.65	26.06
.25	14.59	16.00	17.41	18.82	20.23	21.64	23.05	24.46	25.87
.30	14.40	15.81	17.22	18.63	20.04	21.45	22.86	24.27	25.68
.35	14.21	15.62	17.03	18.44	19.85	21.26	22.67	24.08	25.49
.40	14.02	15.43	16.84	18.25	19.66	21.07	22.48	23.89	25.30
.45	13.83	15.24	16.65	18.06	19.47	20.88	22.29	23.70	25.11
.50	13.64	15.05	16.46	17.87	19.28	20.69	22.10	23.51	24.92
.55	13.45	14.86	16.27	17.68	19.09	20.50	21.91	23.32	24.73

TABLE 5.2 ($v_1 = .35$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	13.98	15.13	16.28	17.43	18.58	19.73	20.88	22.03	23.18
.10	13.79	14.94	16.09	17.24	18.39	19.54	20.69	21.84	22.99
.15	13.60	14.75	15.90	17.05	18.20	19.35	20.50	21.65	22.80
.20	13.41	14.56	15.71	16.86	18.01	19.16	20.31	21.46	22.61
.25	13.22	14.37	15.52	16.67	17.82	18.97	20.12	21.27	22.42
.30	13.03	14.18	15.33	16.48	17.63	18.78	19.93	21.08	22.23
.35	12.84	13.99	15.14	16.29	17.44	18.59	19.74	20.89	22.04
.40	12.65	13.80	14.95	16.10	17.25	18.40	19.55	20.70	21.85
.45	12.46	13.61	14.76	15.91	17.06	18.21	19.36	20.51	21.66
.50	12.27	13.42	14.57	15.72	16.87	18.02	19.17	20.32	21.47
.55	12.08	13.23	14.38	15.53	16.68	17.83	18.98	20.13	21.28

TABLE 5.2 ($v_1 = .40$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	15.35	16.76	18.17	19.58	20.99	22.40	23.81	25.22	26.63
.10	15.16	16.57	17.98	19.39	20.80	22.21	23.62	25.03	26.44
.15	14.97	16.38	17.79	19.20	20.61	22.02	23.43	24.84	26.25
.20	14.78	16.19	17.60	19.01	20.42	21.83	23.24	24.65	26.06
.25	14.59	16.00	17.41	18.82	20.23	21.64	23.05	24.46	25.87
.30	14.40	15.81	17.22	18.63	20.04	21.45	22.86	24.27	25.68
.35	14.21	15.62	17.03	18.44	19.85	21.26	22.67	24.08	25.49
.40	14.02	15.43	16.84	18.25	19.66	21.07	22.48	23.89	25.30
.45	13.83	15.24	16.65	18.06	19.47	20.88	22.29	23.70	25.11
.50	13.64	15.05	16.46	17.87	19.28	20.69	22.10	23.51	24.92
.55	13.45	14.86	16.27	17.68	19.09	20.50	21.91	23.32	24.73

TABLE 5.2 ($v_1 = .35$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	13.98	15.13	16.28	17.43	18.58	19.73	20.88	22.03	23.18
.10	13.79	14.94	16.09	17.24	18.39	19.54	20.69	21.84	22.99
.15	13.60	14.75	15.90	17.05	18.20	19.35	20.50	21.65	22.80
.20	13.41	14.56	15.71	16.86	18.01	19.16	20.31	21.46	22.61
.25	13.22	14.37	15.52	16.67	17.82	18.97	20.12	21.27	22.42
.30	13.03	14.18	15.33	16.48	17.63	18.78	19.93	21.08	22.23
.35	12.84	13.99	15.14	16.29	17.44	18.59	19.74	20.89	22.04
.40	12.65	13.80	14.95	16.10	17.25	18.40	19.55	20.70	21.85
.45	12.46	13.61	14.76	15.91	17.06	18.21	19.36	20.51	21.66
.50	12.27	13.42	14.57	15.72	16.87	18.02	19.17	20.32	21.47
.55	12.08	13.23	14.38	15.53	16.68	17.83	18.98	20.13	21.28

TABLE 5.2 ($v_1 = .40$)

v_1	.100	.150	.200	.250	.300	.350	.400	.450	.500
.05	15.35	16.76	18.17	19.58	20.99	22.40	23.81	25.22	26.63
.10	15.16	16.57	17.98	19.39	20.80	22.21	23.62	25.03	26.44
.15	14.97	16.38	17.79	19.20	20.61	22.02	23.43	24.84	26.25
.20	14.78	16.19	17.60	19.01	20.42	21.83	23.24	24.65	26.06
.25	14.59	16.00	17.41	18.82	20.23	21.64	23.05	24.46	25.87
.30	14.40	15.81	17.22	18.63	20.04	21.45	22.86	24.27	25.68
.35	14.21	15.62	17.03	18.44	19.85	21.26	22.67	24.08	25.49
.40	14.02	15.43	16.84	18.25	19.66	21.07	22.48	23.89	25.30
.45	13.83	15.24	16.65	18.06	19.47	20.88	22.29	23.70	25.11
.50	13.64	15.05	16.46	17.87	19.28	20.69	22.10	23.51	24.92
.55	13.45	14.86	16.27	17.68	19.09	20.50	21.91	23.32	24.73

TABLE 5.2 ($\mu_1 = .50$) MODE 1

μ_2	.050	.100	.150	.200	.250	.300	.350	.400
.05	9.224	9.215	9.210	9.207	9.205	9.203	9.200	9.197
.10	9.061	9.047	9.039	9.033	9.029	9.024	9.019	
.15	8.934	8.918	8.908	8.901	8.895	8.889	10.76	.40
.20	8.825	8.809	8.798	8.790	8.783	11.72	11.92	.35
.25	8.725	8.709	8.697	8.688	12.94	13.10	13.32	.30
.30	8.624	8.605	8.591	14.05	14.16	14.26	14.36	.25
.35	8.506	8.484	14.56	14.64	14.68	14.71	14.75	.20
.40	8.351	13.90	14.81	14.88	14.90	14.92	14.93	.15
	12.36	13.89	15.02	15.07	15.09	14.76	13.04	.10
	11.15	12.38	14.00	15.27	15.31	14.52	12.78	11.47
	.400	.350	.300	.250	.200	.150	.100	.050

MODE 2 $\leftarrow \mu_2$

TABLE 5.2 ($\mu_1 = .45$) MODE 1

μ_2	.050	.100	.150	.200	.250	.300	.350	.400	.450
.05	10.19	10.21	10.21	10.20	10.20	10.20	10.19	10.18	9.905
.10	10.03	10.01	10.00	9.997	9.991	9.984	9.974	9.949	
.15	9.881	9.863	9.851	9.842	9.834	9.825	9.812	10.38	.45
.20	9.753	9.734	9.720	9.709	9.700	9.689	10.92	11.06	.40
.25	9.630	9.608	9.592	9.579	9.568	11.81	11.95	12.16	.35
.30	9.494	9.467	9.446	9.429	13.05	13.23	13.43	13.70	.30
.35	9.315	9.275	9.244	14.30	14.72	14.96	15.16	15.38	.25
.40	9.016	8.973	13.66	15.27	15.96	16.09	16.15	15.11	.20
.45	8.598	12.16	13.65	15.59	16.41	16.45	14.75	13.03	.15
	10.95	12.11	13.67	15.63	16.38	14.48	12.75	11.46	.10
	10.26	10.95	12.18	13.76	15.27	14.22	12.55	11.24	10.26
	.450	.400	.350	.300	.250	.200	.150	.100	.050

MODE 2 $\leftarrow \mu_2$

TABLE 5.2 ($\mu_1 = .50$) MODE 3

μ_2	.050	.100	.150	.200	.250	.300	.350	.400
.05	15.35	15.33	15.33	16.62	16.16	15.35	15.33	15.32
.10	15.13	15.12	17.15	16.59	16.16	15.15	15.11	
.15	15.15	17.34	20.34	18.90	16.17	15.01	18.13	.40
.20	16.00	19.18	18.81	17.91	16.12	20.81	20.89	20.95
.25	16.93	16.63	16.40	16.14	20.56	21.21	21.27	21.31
.30	15.52	15.42	15.35	19.67	20.73	21.64	21.52	19.69
.35	15.08	15.03	20.79	20.89	21.04	21.08	20.87	20.93
.40	14.85	20.63	21.19	21.23	21.23	21.19	21.22	21.24
	18.61	20.73	21.48	21.47	20.98	21.48	21.35	19.10
	.400	.350	.300	.250	.200	.150	.100	.050

MODE 4 $\leftarrow \mu_2$

TABLE 5.2 ($\mu_1 = .45$) MODE 3

μ_2	.050	.100	.150	.200	.250	.300	.350	.400	.450
.05	16.97	17.00	16.99	16.98	17.00	17.00	16.99	16.97	16.55
.10	16.76	16.74	16.72	16.80	16.80	16.74	16.72	16.69	
.15	16.54	16.51	17.12	16.27	16.66	16.52	16.49	17.26	.45
.20	16.25	17.31	19.13	18.07	16.57	16.24	19.95	20.23	20.53
.25	17.59	17.47	17.33	17.12	16.54	20.14	22.08	22.46	22.24
.30	16.86	16.80	16.76	16.70	18.16	20.35	23.16	21.67	18.16
.35	16.54	16.50	16.47	19.67	20.16	21.19	19.96	20.37	.20
.40	16.20	16.14	20.30	22.22	22.79	22.25	22.47	22.82	21.70
.45	15.49	18.34	20.41	22.83	21.72	20.58	23.20	21.33	19.08
	17.06	18.34	20.43	21.00	18.54	20.31	20.93	18.79	17.06
	.450	.400	.350	.300	.250	.200	.150	.100	.050

MODE 4 $\leftarrow \mu_2$

TABLE 5.3 ($\mu_1 = .50$)

μ_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	3.648	3.643	3.641	3.639
.10	3.566	3.557	3.553	3.547
.15	3.494	3.484	3.477	3.469
.20	3.429	3.418	3.410	3.405
.25	3.370	3.359	3.350	3.344
.30	3.315	3.304	3.295	3.288
.35	3.264	3.253	3.244	3.237
.40	3.217	3.206	3.197	3.190
.45	3.174	3.163	3.154	3.147
.50	3.134	3.123	3.114	3.107
.55	3.092	3.081	3.072	3.065
.60	3.050	3.039	3.030	3.023
.65	3.008	2.997	2.988	2.981
.70	2.966	2.955	2.946	2.939
.75	2.924	2.913	2.904	2.897
.80	2.882	2.871	2.862	2.855
.85	2.840	2.829	2.820	2.813
.90	2.798	2.787	2.778	2.771
.95	2.756	2.745	2.736	2.729
1.00	2.714	2.703	2.694	2.687

TABLE 5.3 ($\mu_1 = .45$)

μ_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	4.027	4.035	4.031	4.029
.10	3.941	3.932	3.922	3.916
.15	3.855	3.843	3.830	3.823
.20	3.778	3.765	3.750	3.745
.25	3.708	3.695	3.678	3.672
.30	3.644	3.631	3.613	3.607
.35	3.584	3.571	3.551	3.545
.40	3.528	3.515	3.495	3.489
.45	3.475	3.462	3.442	3.436
.50	3.422	3.409	3.389	3.383
.55	3.369	3.356	3.336	3.330
.60	3.316	3.303	3.283	3.277
.65	3.263	3.250	3.230	3.224
.70	3.210	3.197	3.177	3.171
.75	3.157	3.144	3.124	3.118
.80	3.104	3.091	3.071	3.065
.85	3.051	3.038	3.018	3.012
.90	2.998	2.985	2.965	2.959
.95	2.945	2.932	2.912	2.906
1.00	2.892	2.879	2.859	2.853

TABLE 5.3 ($\mu_1 = .50$)

μ_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	9.154	9.146	9.141	9.138
.10	8.995	8.982	8.974	8.972
.15	8.871	8.857	8.853	8.850
.20	8.766	8.753	8.743	8.741
.25	8.773	8.760	8.750	8.748
.30	8.773	8.760	8.750	8.748
.35	8.773	8.760	8.750	8.748
.40	8.773	8.760	8.750	8.748
.45	8.773	8.760	8.750	8.748
.50	8.773	8.760	8.750	8.748
.55	8.773	8.760	8.750	8.748
.60	8.773	8.760	8.750	8.748
.65	8.773	8.760	8.750	8.748
.70	8.773	8.760	8.750	8.748
.75	8.773	8.760	8.750	8.748
.80	8.773	8.760	8.750	8.748
.85	8.773	8.760	8.750	8.748
.90	8.773	8.760	8.750	8.748
.95	8.773	8.760	8.750	8.748
1.00	8.773	8.760	8.750	8.748

TABLE 5.3 ($\mu_1 = .45$)

μ_1	MODE 1	MODE 2	MODE 3	MODE 4
.050	.100	.150	.200	.250
.05	10.11	10.14	10.13	10.12
.10	9.958	9.942	9.933	9.922
.15	9.812	9.795	9.786	9.778
.20	9.688	9.670	9.661	9.658
.25	9.571	9.555	9.542	9.535
.30	9.447	9.431	9.418	9.411
.35	9.323	9.307	9.294	9.287
.40	9.200	9.184	9.171	9.164
.45	9.077	9.061	9.048	9.041
.50	8.954	8.938	8.925	8.918
.55	8.831	8.815	8.802	8.795
.60	8.708	8.692	8.679	8.672
.65	8.585	8.569	8.556	8.549
.70	8.462	8.446	8.433	8.426
.75	8.339	8.323	8.310	8.303
.80	8.216	8.200	8.187	8.180
.85	8.093	8.077	8.064	8.057
.90	7.970	7.954	7.941	7.934
.95	7.847	7.831	7.818	7.811
1.00	7.724	7.708	7.695	7.688

TABLE 5.6 ($\gamma_2 = -13$)		TABLE 5.6 ($\gamma_2 = -13$)	
γ_1	γ_2	γ_1	γ_2
-550	-150	-550	-150
-540	-140	-540	-140
-530	-130	-530	-130
-520	-120	-520	-120
-510	-110	-510	-110
-500	-100	-500	-100
-490	-90	-490	-90
-480	-80	-480	-80
-470	-70	-470	-70
-460	-60	-460	-60
-450	-50	-450	-50
-440	-40	-440	-40
-430	-30	-430	-30
-420	-20	-420	-20
-410	-10	-410	-10
-400	0	-400	0
-390	10	-390	10
-380	20	-380	20
-370	30	-370	30
-360	40	-360	40
-350	50	-350	50
-340	60	-340	60
-330	70	-330	70
-320	80	-320	80
-310	90	-310	90
-300	100	-300	100
-290	110	-290	110
-280	120	-280	120
-270	130	-270	130
-260	140	-260	140
-250	150	-250	150
-240	160	-240	160
-230	170	-230	170
-220	180	-220	180
-210	190	-210	190
-200	200	-200	200
-190	210	-190	210
-180	220	-180	220
-170	230	-170	230
-160	240	-160	240
-150	250	-150	250
-140	260	-140	260
-130	270	-130	270
-120	280	-120	280
-110	290	-110	290
-100	300	-100	300
-90	310	-90	310
-80	320	-80	320
-70	330	-70	330
-60	340	-60	340
-50	350	-50	350
-40	360	-40	360
-30	370	-30	370
-20	380	-20	380
-10	390	-10	390
0	400	0	400
10	410	10	410
20	420	20	420
30	430	30	430
40	440	40	440
50	450	50	450
60	460	60	460
70	470	70	470
80	480	80	480
90	490	90	490
100	500	100	500
110	510	110	510
120	520	120	520
130	530	130	530
140	540	140	540
150	550	150	550
160	560	160	560
170	570	170	570
180	580	180	580
190	590	190	590
200	600	200	600
210	610	210	610
220	620	220	620
230	630	230	630
240	640	240	640
250	650	250	650
260	660	260	660
270	670	270	670
280	680	280	680
290	690	290	690
300	700	300	700
310	710	310	710
320	720	320	720
330	730	330	730
340	740	340	740
350	750	350	750
360	760	360	760
370	770	370	770
380	780	380	780

[illegible][illegible][illegible][illegible]

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

A		B		C		D		E		F		G		H		I		J		K		L		M		N		O		P		Q		R		S		T		U		V		W		X		Y		Z	
50	50.0	50.1	50.2	50.3	50.4	50.5	50.6	50.7	50.8	50.9	51.0	51.1	51.2	51.3	51.4	51.5	51.6	51.7	51.8	51.9	52.0	52.1	52.2	52.3	52.4	52.5	52.6	52.7	52.8	52.9	53.0	53.1	53.2	53.3	53.4	53.5	53.6	53.7	53.8	53.9	54.0	54.1	54.2	54.3	54.4	54.5	54.6	54.7	54.8	54.9	55.0
51	55.1	55.2	55.3	55.4	55.5	55.6	55.7	55.8	55.9	56.0	56.1	56.2	56.3	56.4	56.5	56.6	56.7	56.8	56.9	57.0	57.1	57.2	57.3	57.4	57.5	57.6	57.7	57.8	57.9	58.0	58.1	58.2	58.3	58.4	58.5	58.6	58.7	58.8	58.9	59.0	59.1	59.2	59.3	59.4	59.5	59.6	59.7	59.8	59.9	60.0	
52	60.1	60.2	60.3	60.4	60.5	60.6	60.7	60.8	60.9	61.0	61.1	61.2	61.3	61.4	61.5	61.6	61.7	61.8	61.9	62.0	62.1	62.2	62.3	62.4	62.5	62.6	62.7	62.8	62.9	63.0	63.1	63.2	63.3	63.4	63.5	63.6	63.7	63.8	63.9	64.0	64.1	64.2	64.3	64.4	64.5	64.6	64.7	64.8	64.9	65.0	
53	65.1	65.2	65.3	65.4	65.5	65.6	65.7	65.8	65.9	66.0	66.1	66.2	66.3	66.4	66.5	66.6	66.7	66.8	66.9	67.0	67.1	67.2	67.3	67.4	67.5	67.6	67.7	67.8	67.9	68.0	68.1	68.2	68.3	68.4	68.5	68.6	68.7	68.8	68.9	69.0	69.1	69.2	69.3	69.4	69.5	69.6	69.7	69.8	69.9	70.0	
54	70.1	70.2	70.3	70.4	70.5	70.6	70.7	70.8	70.9	71.0	71.1	71.2	71.3	71.4	71.5	71.6	71.7	71.8	71.9	72.0	72.1	72.2	72.3	72.4	72.5	72.6	72.7	72.8	72.9	73.0	73.1	73.2	73.3	73.4	73.5	73.6	73.7	73.8	73.9	74.0	74.1	74.2	74.3	74.4	74.5	74.6	74.7	74.8	74.9	75.0	
55	75.1	75.2	75.3	75.4	75.5	75.6	75.7	75.8	75.9	76.0	76.1	76.2	76.3	76.4	76.5	76.6	76.7	76.8	76.9	77.0	77.1	77.2	77.3	77.4	77.5	77.6	77.7	77.8	77.9	78.0	78.1	78.2	78.3	78.4	78.5	78.6	78.7	78.8	78.9	79.0	79.1	79.2	79.3	79.4	79.5	79.6	79.7	79.8	79.9	80.0	
56	80.1	80.2	80.3	80.4	80.5	80.6	80.7	80.8	80.9	81.0	81.1	81.2	81.3	81.4	81.5	81.6	81.7	81.8	81.9	82.0	82.1	82.2	82.3	82.4	82.5	82.6	82.7	82.8	82.9	83.0	83.1	83.2	83.3	83.4	83.5	83.6	83.7	83.8	83.9	84.0	84.1	84.2	84.3	84.4	84.5	84.6	84.7	84.8	84.9	85.0	
57	85.1	85.2	85.3	85.4	85.5	85.6	85.7	85.8	85.9	86.0	86.1	86.2	86.3	86.4	86.5	86.6	86.7	86.8	86.9	87.0	87.1	87.2	87.3	87.4	87.5	87.6	87.7	87.8	87.9	88.0	88.1	88.2	88.3	88.4	88.5	88.6	88.7	88.8	88.9	89.0	89.1	89.2	89.3	89.4	89.5	89.6	89.7	89.8	89.9	90.0	
58	90.1	90.2	90.3	90.4	90.5	90.6	90.7	90.8	90.9	91.0	91.1	91.2	91.3	91.4	91.5	91.6	91.7	91.8	91.9	92.0	92.1	92.2	92.3	92.4	92.5	92.6	92.7	92.8	92.9	93.0	93.1	93.2	93.3	93.4	93.5	93.6	93.7	93.8	93.9	94.0	94.1	94.2	94.3	94.4	94.5	94.6	94.7	94.8	94.9	95.0	
59	95.1	95.2	95.3	95.4	95.5	95.6	95.7	95.8	95.9	96.0	96.1	96.2	96.3	96.4	96.5	96.6	96.7	96.8	96.9	97.0	97.1	97.2	97.3	97.4	97.5	97.6	97.7	97.8	97.9	98.0	98.1	98.2	98.3	98.4	98.5	98.6	98.7	98.8	98.9	99.0	99.1	99.2	99.3	99.4	99.5	99.6	99.7	99.8	99.9	100.0	

V_1	.500	.510	.520	.530	.540	.550	.560	.570	.580	.590	.600	.610	.620	.630	.640	.650	.660	.670	.680	.690	.700	.710	.720	.730	.740	.750	.760	.770	.780	.790	.800	.810	.820	.830	.840	.850	.860	.870	.880	.890	.900	.910	.920	.930	.940	.950	.960	.970	.980	.990	1.000
V_1	5.073	5.116	5.158	5.199	5.239	5.278	5.316	5.353	5.389	5.424	5.458	5.491	5.522	5.552	5.580	5.607	5.633	5.658	5.682	5.705	5.727	5.748	5.768	5.787	5.805	5.822	5.838	5.853	5.868	5.882	5.895	5.908	5.920	5.931	5.941	5.950	5.958	5.965	5.971	5.977	5.982	5.986	5.990	5.993	5.995	5.997	5.998	5.999	6.000		

MEZ 3	
V_1	TABLE 5.8 ($V_1 = .11$)
.55	11.64 12.43 13.43 14.43 15.38 16.41 17.41 18.41 19.44 20.45 21.46 22.46 23.44 24.43
.56	12.32 13.12 14.12 15.12 16.09 17.10 18.10 19.14 20.15 21.16 22.16 23.13 24.11
.57	13.01 13.82 14.82 15.82 16.81 17.82 18.82 19.84 20.85 21.86 22.86 23.83 24.81
.58	13.69 14.51 15.51 16.51 17.50 18.51 19.51 20.54 21.55 22.56 23.56 24.53 25.51
.59	14.38 15.20 16.20 17.20 18.19 19.20 20.20 21.24 22.25 23.26 24.26 25.23 26.21
.60	15.07 15.89 16.89 17.89 18.88 19.89 20.89 21.92 22.93 23.94 24.94 25.91 26.89
.61	15.76 16.58 17.58 18.58 19.57 20.58 21.58 22.61 23.62 24.63 25.63 26.60 27.58
.62	16.45 17.27 18.27 19.27 20.26 21.27 22.27 23.30 24.31 25.32 26.32 27.29 28.27
.63	17.14 17.96 18.96 19.96 20.95 21.96 22.96 24.00 25.01 26.02 27.02 28.00 28.98
.64	17.83 18.65 19.65 20.65 21.64 22.65 23.65 24.69 25.70 26.71 27.72 28.70 29.68
.65	18.52 19.34 20.34 21.34 22.33 23.34 24.34 25.38 26.39 27.40 28.41 29.39 30.37
.66	19.21 20.03 21.03 22.03 23.02 24.03 25.03 26.07 27.08 28.09 29.10 30.08 31.06
.67	19.90 20.72 21.72 22.72 23.71 24.72 25.72 26.76 27.77 28.78 29.79 30.77 31.75
.68	20.59 21.41 22.41 23.41 24.40 25.41 26.41 27.45 28.46 29.47 30.48 31.46 32.44
.69	21.28 22.10 23.10 24.10 25.09 26.10 27.10 28.14 29.15 30.16 31.17 32.15 33.13
.70	21.97 22.79 23.79 24.79 25.78 26.79 27.79 28.83 29.84 30.85 31.86 32.84 33.82
.71	22.66 23.48 24.48 25.48 26.47 27.48 28.48 29.52 30.53 31.54 32.55 33.53 34.51
.72	23.35 24.17 25.17 26.17 27.16 28.17 29.17 30.21 31.22 32.23 33.24 34.22 35.20
.73	24.04 24.86 25.86 26.86 27.85 28.86 29.86 30.90 31.91 32.92 33.93 34.91 35.89
.74	24.73 25.55 26.55 27.55 28.54 29.55 30.55 31.59 32.60 33.61 34.62 35.60 36.58
.75	25.42 26.24 27.24 28.24 29.23 30.24 31.24 32.28 33.29 34.30 35.31 36.29 37.27
.76	26.11 26.93 27.93 28.93 29.92 30.93 31.93 32.97 33.98 34.99 35.99 36.97 37.95
.77	26.80 27.62 28.62 29.62 30.61 31.62 32.62 33.66 34.67 35.68 36.69 37.67 38.65
.78	27.49 28.31 29.31 30.31 31.30 32.31 33.31 34.35 35.36 36.37 37.38 38.36 39.34
.79	28.18 28.99 29.99 30.99 31.98 32.99 33.99 35.03 36.04 37.05 38.06 39.04 40.02
.80	28.87 29.69 30.69 31.69 32.68 33.69 34.69 35.73 36.74 37.75 38.76 39.74 40.72
.81	29.56 30.38 31.38 32.38 33.37 34.38 35.38 36.42 37.43 38.44 39.45 40.43 41.41
.82	30.25 31.07 32.07 33.07 34.06 35.07 36.07 37.11 38.12 39.13 40.14 41.12 42.10
.83	30.94 31.76 32.76 33.76 34.75 35.76 36.76 37.80 38.81 39.82 40.83 41.81 42.79
.84	31.63 32.45 33.45 34.45 35.44 36.45 37.45 38.49 39.50 40.51 41.52 42.50 43.48
.85	32.32 33.14 34.14 35.14 36.13 37.14 38.14 39.18 40.19 41.20 42.21 43.19 44.17
.86	33.01 33.83 34.83 35.83 36.82 37.83 38.83 39.87 40.88 41.89 42.90 43.88 44.86
.87	33.70 34.52 35.52 36.52 37.51 38.52 39.52 40.56 41.57 42.58 43.59 44.57 45.55
.88	34.39 35.21 36.21 37.21 38.20 39.21 40.21 41.25 42.26 43.27 44.28 45.26 46.24
.89	35.08 35.90 36.90 37.90 38.89 39.90 40.90 41.94 42.95 43.96 44.97 45.95 46.93
.90	35.77 36.59 37.59 38.59 39.58 40.59 41.59 42.63 43.64 44.65 45.66 46.64 47.62
.91	36.46 37.28 38.28 39.28 40.27 41.28 42.28 43.32 44.33 45.34 46.35 47.33 48.31
.92	37.15 37.97 38.97 39.97 40.96 41.97 42.97 44.01 45.02 46.03 47.04 48.02 49.00
.93	37.84 38.66 39.66 40.66 41.65 42.66 43.66 44.70 45.71 46.72 47.73 48.71 49.69
.94	38.53 39.35 40.35 41.35 42.34 43.35 44.35 45.39 46.40 47.41 48.42 49.40 50.38
.95	39.22 40.04 41.04 42.04 43.03 44.04 45.04 46.08 47.09 48.10 49.11 50.09 51.07
.96	39.91 40.73 41.73 42.73 43.72 44.73 45.73 46.77 47.78 48.79 49.80 50.78 51.76
.97	40.60 41.42 42.42 43.42 44.41 45.42 46.42 47.46 48.47 49.48 50.49 51.47 52.45
.98	41.29 42.11 43.11 44.11 45.10 46.11 47.11 48.15 49.16 50.17 51.18 52.16 53.14
.99	41.98 42.80 43.80 44.80 45.79 46.80 47.80 48.84 49.85 50.86 51.87 52.85 53.83
1.00	42.67 43.49 44.49 45.49 46.48 47.49 48.49 49.53 50.54 51.55 52.56 53.54 54.52

[illegible]

TABLE 5.6 ($v_1 = .45$)
 v_1 MCE 1

.05	2.813	2.811	2.810	2.819	2.818
.10	2.773	2.768	2.765	2.766	
.15	2.718	2.712	2.719		2.719
.20	2.647	2.641		6.412	6.411
.25	2.648		6.416	6.412	6.412
.30		6.416	6.411	6.417	6.415
.35	7.048	7.049	7.010	7.073	7.019
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .50$)
 v_1 MCE 3

.05	12.57	12.42	12.44	12.41	12.43
.10	12.46	12.45	12.44	12.43	12.43
.15	12.54	12.53	12.51	12.50	14.79
.20	12.51	12.49	12.51	12.50	15.79
.25	12.51	12.49	12.51	12.50	15.79
.30	12.51	12.49	12.51	12.50	15.79
.35	12.51	12.49	12.51	12.50	15.79
.40	12.51	12.49	12.51	12.50	15.79
.45	12.51	12.49	12.51	12.50	15.79

MCE 4 = v_1

TABLE 5.6 ($v_1 = .55$)
 v_1 MCE 1

.05	12.57	12.42	12.44	12.41	12.43
.10	12.46	12.45	12.44	12.43	12.43
.15	12.54	12.53	12.51	12.50	14.79
.20	12.51	12.49	12.51	12.50	15.79
.25	12.51	12.49	12.51	12.50	15.79
.30	12.51	12.49	12.51	12.50	15.79
.35	12.51	12.49	12.51	12.50	15.79
.40	12.51	12.49	12.51	12.50	15.79
.45	12.51	12.49	12.51	12.50	15.79

MCE 2 = v_1

TABLE 5.6 ($v_1 = .60$)
 v_1 MCE 1

.05	3.034	3.019	3.049	3.047	3.047
.10	2.995	2.979	2.995	2.994	2.993
.15	2.935	2.918	2.932	2.932	2.931
.20	2.875	2.858	2.871	2.871	2.871
.25	2.815	2.800	2.813	2.813	2.813
.30	2.811	2.811	2.811	2.811	2.811
.35	2.811	2.811	2.811	2.811	2.811
.40	2.811	2.811	2.811	2.811	2.811
.45	2.811	2.811	2.811	2.811	2.811

MCE 2 = v_1

TABLE 5.6 ($v_1 = .65$)
 v_1 MCE 1

.05	11.48	11.45	11.45	11.44	11.44
.10	11.71	11.70	11.69	11.69	
.15	11.60	11.59	11.58	11.57	15.47
.20	11.50	11.48	11.48	11.47	15.47
.25	11.39	11.38	11.37	11.37	15.47
.30	11.39	11.38	11.37	11.37	15.47
.35	11.39	11.38	11.37	11.37	15.47
.40	11.39	11.38	11.37	11.37	15.47
.45	11.39	11.38	11.37	11.37	15.47

MCE 2 = v_1

TABLE 5.6 ($v_1 = .70$)
 v_1 MCE 1

.05	2.455	2.452	2.451		
.10	2.415	2.412		6.003	6.003
.15	2.381		6.043	6.049	6.049
.20		6.144	6.146	6.150	6.150
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .75$)
 v_1 MCE 1

.05	11.02	11.01	11.01	11.01	
.10	10.49	10.48	10.48		
.15	10.79	10.78		15.01	15.01
.20	10.70		15.15	15.16	15.16
.25		15.21	15.18	15.18	15.18
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .80$)
 v_1 MCE 1

.05	2.416	2.413	2.412	2.412	
.10	2.512	2.518	2.516		
.15	2.542	2.533	6.333		
.20	2.568	6.403	6.416		
.25	6.409	6.403	6.401		
.30	6.313	6.374	6.377		
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .85$)
 v_1 MCE 1

.05	2.416	2.413	2.412	2.412	
.10	2.512	2.518	2.516		
.15	2.542	2.533	6.333		
.20	2.568	6.403	6.416		
.25	6.409	6.403	6.401		
.30	6.313	6.374	6.377		
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .90$)
 v_1 MCE 1

.05	10.30	10.19	10.19		
.10	10.19	10.17		14.18	14.18
.15	10.09		14.18	14.19	14.19
.20		14.18	14.19		
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .95$)
 v_1 MCE 1

.05	12.75	12.75	12.75	12.75	12.75
.10					
.15					
.20					
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .95$)
 v_1 MCE 1

.05	12.75	12.75	12.75	12.75	12.75
.10					
.15					
.20					
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .95$)
 v_1 MCE 1

.05	12.75	12.75	12.75	12.75	12.75
.10					
.15					
.20					
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .95$)
 v_1 MCE 1

.05	12.75	12.75	12.75	12.75	12.75
.10					
.15					
.20					
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

TABLE 5.6 ($v_1 = .95$)
 v_1 MCE 1

.05	12.75	12.75	12.75	12.75	12.75
.10					
.15					
.20					
.25					
.30					
.35					
.40					
.45					

MCE 2 = v_1

CHAPTER 6

MULTISPAN BEAMS WITH CLASSICAL BOUNDARY CONDITIONS

Part 1 - Beams with arbitrary span lengths

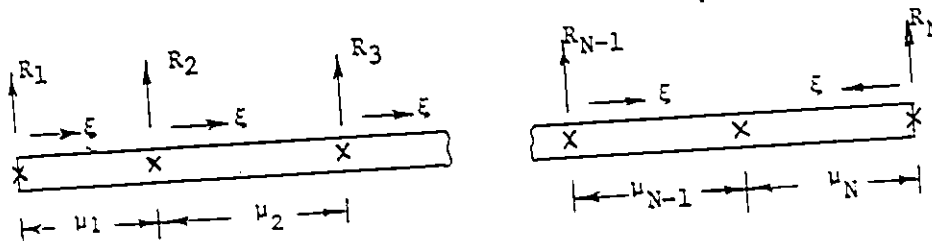


Figure 6.1

This is the most general family of multispan beam problems. While all the interior supports must necessarily be simple, any combination of classical boundary conditions may prevail at the two outer ends. It may be recalled that only six such combinations are possible.

Following the procedure established in Chapters 4 and 5, the modal shapes are expressed as a function of ξ for each span, where ξ represents the distance along the span expressed as a fraction of the beam overall length.

$$R_n(\xi) = A_n \sin \xi \xi + B_n \cos \xi \xi + C_n \sinh \xi \xi + D_n \cosh \xi \xi \quad (6.0)$$

The subscript n of the displacement function (6.0) indicates the span number where the spans are numbered from the left end as shown in Figure 6.1. It may be noted that all displacement functions except the one for the extreme right outer span, have their origins at the left end of the span. The beam is considered to have N spans. Each span has its length expressed by the dimensionless subscripted quantity ξ_n , where ξ_n equals the non-dimensional length of n^{th} span.

For illustrative purposes we will develop the equations for a beam with outer ends simply supported as shown in Figure 6.1 which will then be extended to include all possible outer end conditions.

Utilizing the boundary conditions $R_1(0) = 0$, $R'_1(0) = 0$, and $R_N(\xi_N) = 0$, we can write

$$R_1(\xi) = A_1 \left(\sin \xi \xi - \frac{\sin \xi \xi_1}{\sinh \xi \xi_1} \sinh \xi \xi \right) \quad (6.1)$$

The function $R_N(\xi)$ is evidently subjected to identical boundary conditions, we may therefore write

$$R_N(\xi) = A_N \left(\sin \xi \xi - \frac{\sin \xi \xi_N}{\sinh \xi \xi_N} \sinh \xi \xi \right) \quad (6.2)$$

where, of course, $\mu_N = 1 - (\mu_1 + \mu_2 + \dots + \mu_{N-1})$

For interior spans, $1 < n < N$, we have the boundary condition $R_n(0) = 0$ which results in $B_n = -D_n$ and we obtain

$$R_n(\xi) = A_n \sin \beta \xi + B_n (\cos \beta \xi - \cosh \beta \xi) + C_n \sinh \beta \xi \quad (6.3)$$

The constant C_n may be eliminated by utilizing $R_n(\mu_n) = 0$, after which we obtain

$$R_n(\xi) = A_n (\sin \beta \xi - \theta_n \sinh \beta \xi) + B_n (\cos \beta \xi - \cosh \beta \xi + \phi_n \sinh \beta \xi) \quad (6.4)$$

where

$$\theta_n = \frac{\sin \beta \mu_n}{\sinh \beta \mu_n} \quad \text{and} \quad \phi_n = \frac{\cosh \beta \mu_n - \cos \beta \mu_n}{\sinh \beta \mu_n}$$

From equations (6.1), (6.2), and (6.4) it is obvious that the modal shape expressions now contain, $2(N-1)$ unknown constants $A_1, A_2, B_2, A_3, B_3, \dots$, and A_N . There also remain $2(N-1)$ equations arising from the requirements to satisfy continuity of slope and bending moment across the, $N-1$, interior supports. Solution of the vibration problem involves finding those values of β for which the determinant of

the coefficient matrix vanishes. The first two equations of the set are obtained by enforcing the conditions $R'_1(u_1) = R_2(0)$ and $R''_1(u_1) = R''_2(0)$. The resulting equations are respectively:

$$A_1(\cos \beta u_1 - \frac{\sin \beta u_1}{\sinh \beta u_1} \cosh \beta u_1) + A_2(\epsilon_2 - 1) + B_2(-\epsilon_2) = 0 \quad (6.5)$$

$$A_1(-2 \sin \beta u_1) + A_2(0) + B_2(2) = 0 \quad (6.6)$$

Continuity of slope and bending moment across internal supports require respectively, for $2 \leq n \leq (N-2)$

$$A_n(\cos \beta u_n - \epsilon_n \cosh \beta u_n) + B_n(-\sin \beta u_n - \sinh \beta u_n + \epsilon_n \cosh \beta u_n) + A_{n+1}(\epsilon_{n+1} - 1) + B_{n+1}(-\epsilon_{n+1}) = 0 \quad (6.7)$$

and

$$A_n(-\sin \beta u_n - \epsilon_n \sinh \beta u_n) + B_n(-\cos \beta u_n - \cosh \beta u_n + \epsilon_n \sinh \beta u_n) + A_{n+1}(0) + B_{n+1}(2) = 0 \quad (6.8)$$

The final two equations relate to satisfying the requirement of continuity of slope and the bending moment, respectively, across the internal support adjacent to the extreme right hand span, these are:

$$A_{N-1}(\cos \beta u_{N-1} - \theta_{N-1} \cosh \beta u_{N-1}) + B_{N-1}(-\sin \beta u_{N-1} - \sinh \beta u_{N-1} + \phi_{N-1} \cosh \beta u_{N-1}) + A_N(\cos \beta u_N - \frac{\sin \beta u_N}{\sinh \beta u_N} \cosh \beta u_N) = 0 \quad (6.9)$$

and

$$A_{N-1}(-\sin \beta u_{N-1} - \theta_{N-1} \sinh \beta u_{N-1}) + B_{N-1}(-\cos \beta u_{N-1} - \cosh \beta u_{N-1} + \phi_{N-1} \sinh \beta u_{N-1}) + A_N(2 \sin \beta u_N) = 0 \quad (6.10)$$

The coefficient matrix of constants $A_1, A_2, B_2, \dots, A_N$ appearing in the above set of equations (6.5) through (6.10) can therefore be easily written down taking advantage of recursion in equations (6.4) and (6.8) for any arbitrary number of spans. For a beam of N spans, the above set contain $2(N-1)$ equations. It is extremely important to note that only the coefficients of A_1 in the first two equations and coefficients of A_N in the last two equations of the set depend explicitly on outer end boundary conditions. This is true regardless of the

number of spans in the beam. The elements of the $(2N-2) \times (2N-2)$ coefficient matrix for a beam of N spans can therefore be easily written down for any combination of outer end conditions. Representing the elements of this matrix by a_{ij} , it is easily recognised that only four elements, namely a_{11} , a_{21} , $a_{(2N-3), (2N-2)}$ and $a_{(2N-2), (2N-2)}$, depend explicitly on outer end conditions. They are presented below for the three possible end conditions at each end which may prevail.

Left end simply supported

$$a_{11} = \cos \beta u_1 - \frac{\sin \beta u_1}{\sinh \beta u_1} \cosh \beta u_1$$

$$a_{21} = -2 \sin \beta u_1$$

Left end clamped

$$a_{11} = \frac{-2(1 - \cos \beta u_1 \cosh \beta u_1)}{\cosh \beta u_1 - \cos \beta u_1}$$

$$a_{21} = \frac{2(\sinh \beta u_1 \cos \beta u_1 - \sin \beta u_1 \cosh \beta u_1)}{\cosh \beta u_1 - \cos \beta u_1}$$

Left end free

$$a_{11} = \frac{2(1 + \cos su_1 \cosh su_1)}{\cos su_1 + \cosh su_1}$$

$$a_{21} = \frac{2(\sinh su_1 \cos su_1 - \sin su_1 \cosh su_1)}{\cos su_1 + \cosh su_1}$$

Right end simply supported

$$a_{(2N-3), (2N-2)} = \cos su_N - \frac{\sin su_N}{\sinh su_N} \cosh su_N$$

$$a_{(2N-2), (2N-2)} = 2 \sin su_N$$

Right end clamped

$$a_{(2N-3), (2N-2)} = \frac{-2(1 - \cos su_N \cosh su_N)}{\cosh su_N - \cos su_N}$$

$$a_{(2N-3), (2N-2)} = \frac{-2(\sinh su_N \cos su_N - \sin su_N \cosh su_N)}{\cosh su_N - \cos su_N}$$

Right end free

$$a_{(2N-3), (2N-2)} = \frac{2(1 + \cos \beta u_N \cosh \beta u_N)}{\cos \beta u_N + \cosh \beta u_N}$$

$$a_{(2N-2), (2N-2)} = \frac{-2(\sinh \beta u_N \cos \beta u_N - \sin \beta u_N \cosh \beta u_N)}{\cos \beta u_N + \cosh \beta u_N}$$

To construct the matrix, then, the elements of first and last columns may be taken from the appropriate expressions provided above. The remaining elements in the first and last two rows are respectively taken from equations (6.5), (6.6), (6.9), and (6.10). They are:

$$a_{12} = \theta_2 - 1, \quad a_{13} = -\phi_2$$

$$a_{22} = 0, \quad a_{23} = 2$$

$$a_{(2N-3), (2N-4)} = \cos \beta u_{N-1} - \theta_{N-1} \cosh \beta u_{N-1}$$

$$a_{(2N-3), (2N-3)} = -\sin \beta u_{N-1} - \sinh \beta u_{N-1} + \phi_{N-1} \cosh \beta u_{N-1}$$

$$a_{(2N-2), (2N-4)} = -\sin \beta u_{N-1} - \theta_{N-1} \sinh \beta u_{N-1}$$

$$\alpha_{(2N-2), (2N-3)} = -\cos \beta_{N-1} - \cosh \beta_{N-1} + \phi_{N-1} \sinh \beta_{N-1}$$

The rows of coefficient matrix, other than the first two and last two rows may be generated as follows. For $n = 3, 5, 7, \dots$ and introducing $m = \frac{(n+1)}{2}$, the n^{th} row has following four elements.

$$\alpha_{n, n-1} = \cos \beta_m - \theta_m \cosh \beta_m$$

$$\alpha_{n, n} = -\sin \beta_m - \sinh \beta_m + \phi_m \cosh \beta_m$$

$$\alpha_{n, n+1} = \theta_{m+1} - 1$$

and

$$\alpha_{n, n+2} = -\phi_{m+1}$$

and $(n+1)^{\text{th}}$ row has the following four elements

$$\alpha_{n+1, n-1} = -\sin \beta_m - \theta_m \sinh \beta_m$$

$$\alpha_{n+1, n} = -\cos \beta_m - \cosh \beta_m + \phi_m \sinh \beta_m$$

$$\alpha_{n+1, n+1} = 0$$

and

$$a_{n+1,n+2} = 2$$

All other elements of the coefficient matrix not covered by above expressions are equal to zero. The information presented above is sufficient for writing down the coefficient matrix for any multispan beam regardless of outer end conditions and interior support spacings. With no restrictions placed on the spacing of the intermediate supports, the problem is quite general and it is not practical to present direct information, as was done with three and four span beams, necessary for establishing frequencies of any beam in this class. It appears that the most significant contribution that can be made here is to provide all the information for establishing the elements of the coefficient matrix as has been done above. The problem of seeking out the eigenvalues β can then be handled by the analyst for the specific values of parameters $u_1, u_2, \dots$, etc of his interest.

Now turning our attention to modal shape functions, it will be seen that they are readily obtained for any frequency once the associated eigenvalues is determined. The elements of the coefficient matrix a_{ij} are then determined based on eigenvalue β . These elements then represent the coefficients for a set of homogeneous simultaneous algebraic

equations. In order to affect a solution for the modal shape, we begin by arbitrarily assigning to A_1 the value of unity. The first two equations then automatically become non-homogeneous and may be written as

$$\alpha_{12} A_2 + \alpha_{13} B_2 = -\alpha_{11} \quad (6.11)$$

$$\alpha_{22} A_2 + \alpha_{23} B_2 = -\alpha_{21}$$

Utilizing Cramer's rule, it is readily shown that

$$A_2 = \frac{\alpha_{21} \alpha_{13} - \alpha_{11} \alpha_{23}}{\alpha_{12} \alpha_{23} - \alpha_{13} \alpha_{22}} \quad (6.12)$$

$$B_2 = \frac{\alpha_{11} \alpha_{22} - \alpha_{21} \alpha_{12}}{\alpha_{12} \alpha_{23} - \alpha_{13} \alpha_{22}}$$

Having solved for A_2 and B_2 , the second pair of equations is made non-homogeneous and solved for A_3 and B_3 . The so evaluated values of A_3 and B_3 are utilized in the next pair of equations to solve for A_4 and B_4 and so on. In the general case we can write solutions for the intermediate pair of equations (not including the first and last pairs of equations of the set) as follows

For, $3 \leq n \leq (N-1)$,

$$A_n = \frac{\alpha_{n,n+2} (\alpha_{n+1,n-1} A_{n-1} + \alpha_{n+1,n} B_{n-1}) - \alpha_{n+1,n+2} (\alpha_{n,n} B_{n-1} + \alpha_{n,n-1} A_{n-1})}{\alpha_{n,n+1} \alpha_{n+1,n+2} - \alpha_{n+1,n+1} \alpha_{n,n+2}}$$

$$B_n = \frac{\alpha_{n+1,n+1} (\alpha_{n,n} B_{n-1} + \alpha_{n,n-1} A_{n-1}) - \alpha_{n,n+1} (\alpha_{n+1,n-1} A_{n-1} + \alpha_{n+1,n} B_{n-1})}{\alpha_{n,n+1} \alpha_{n+1,n+2} - \alpha_{n+1,n+1} \alpha_{n,n+2}}$$

(6.13)

There are two final equations available for establishing the values of the constant A_N . They may be used as follows:

$$A_N = \frac{-(A_{N-1} \alpha_{2N-3,2N-4} + B_{N-1} \alpha_{2N-3,2N-3})}{\alpha_{2N-3,2(N-1)}}$$

or

$$A_N = \frac{-(A_{N-1} \alpha_{2N-2,2N-4} + B_{N-1} \alpha_{2N-2,2N-3})}{\alpha_{2N-2,2N-2}}$$

(6.14)

The constant A_N may be evaluated from either of the equations (6.14). Any of the above two equations may be used if other takes on an indeterminate form.

Having established values for the constants $A_2, B_2, \dots$, and A_N based on a value of unity for A_1 , it is then possible to write expressions

for the modal shapes. For internal spans i.e. for, $2 \leq n \leq (N-1)$, the displacements are expressed according to equation (6.4).

For the outer two spans the displacement expressions depend on the adjacent beam outer end conditions. For convenience they are summarized as follows.

Left end simply supported

$$R_1(\xi) = \sin \beta \xi - \frac{\sin \beta \mu_1}{\sinh \beta \mu_1} \sinh \beta \xi$$

Left end clamped

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \mu_1 - \sinh \beta \mu_1)}{(\cosh \beta \mu_1 - \cos \beta \mu_1)} (\cos \beta \xi - \cosh \beta \xi)$$

Left end free

$$R_1(\xi) = \sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \mu_1 + \sinh \beta \mu_1)}{(\cos \beta \mu_1 + \cosh \beta \mu_1)} (\cos \beta \xi + \cosh \beta \xi)$$

Right end simply supported

$$R_N(\xi) = A_N \left(\sin \beta \xi - \frac{\sin \beta \mu_N}{\sinh \beta \mu_N} \sinh \beta \xi \right)$$

Right end clamped

$$R_N(\xi) = A_N \left[\sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta \mu_N - \sinh \beta \mu_N)}{(\cosh \beta \mu_N - \cos \beta \mu_N)} (\cos \beta \xi - \cosh \beta \xi) \right]$$

Right end free

$$R_N(\xi) = A_N \left[\sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta \mu_N + \sinh \beta \mu_N)}{(\cos \beta \mu_N + \cosh \beta \mu_N)} (\cos \beta \xi + \cosh \beta \xi) \right]$$

This completes the analysis provided here for multispan beams with arbitrary interior support spacings.

Part 2 - Beams with uniform span lengths

As pointed out earlier in this Chapter, it was not found practical to present eigenvalues of multispan beams with arbitrary support spacings when total number of spans are more than four. However, when span spacings are uniform, the non-dimensional length of every span becomes $\mu_n = \frac{1}{N}$ where N is the total number of spans. The set of homogeneous equations (6.5) through (6.10) will then be a function of β and N only for uniform span length beams. The quantities $\beta \mu_n$ is then replaced by β/N in all the equations of the above mentioned set. By setting the

determinant of the coefficient matrix equal to zero, the values of β can be computed for any number of uniform spans N . This has been done for $3 \leq N \leq 15$ and the results are presented in Tables 6.1 through 6.6. In description of end conditions on the tables, the first boundary condition refers to the left end of the beam and the second refers to the right end. With use of equation (1.7) and the above tables, the first six frequencies of lateral vibration can be determined for any beam with the number of spans lying between 3 and 15 for uniformly spaced supports.

We next turn to the modal shapes. It is evident from the previously developed equations that, regardless of the outer end boundary conditions, we may write

$$R_n(\xi) = A_n(\sin \beta \xi - \theta \sinh \beta \xi) + B_n(\cos \beta \xi - \cosh \beta \xi + \phi \sinh \beta \xi) \quad (6.15)$$

for $1 < n < N$ where

$$\theta = \frac{\sin \beta^*}{\sinh \beta^*} \quad \text{and} \quad \phi = \frac{\cosh \beta^* - \cos \beta^*}{\sinh \beta^*}$$

and

$$\beta^* = \beta_n = \frac{\beta}{N}$$

Further examination of pair of equations (6.7) and (6.8) and recalling that for uniformly spaced supports $\delta u_n = \delta^*$, it is easily revealed that

$$A_n = A_{n-1} \frac{(\theta \cosh \delta^* - \cos \delta^* + \phi \sin \delta^*)}{(\theta - 1)} + B_{n-1} \frac{[\sin \delta^* + \sinh \delta^* + \phi(\cos \delta^* - \cosh \delta^*)]}{(\theta - 1)}$$

(6.16)

and

$$B_n = A_{n-1} \sin \delta^* + B_{n-1} \cos \delta^* \quad \text{for } 2 < n < N$$

With δ obtained from the tables and hence δ^* , values of A_3 and B_3 are obtained from A_2 and B_2 , then A_4 and B_4 from A_3 and B_3 etc, upto A_{n-1} , B_{n-1} . To complete the modal shape analysis we look next at those quantities which are explicitly dependent on the outer boundary conditions. We begin by looking at the left end of the beam. Three pieces of information are required here. They are the first span modal shape expression and values of A_2 and B_2 . They are presented for each of the three boundary conditions which may prevail.

Left end pinned

$$R_1(\xi) = \sin \beta \xi - \frac{\sin \beta^*}{\sinh \beta^*} \sinh \beta \xi$$

$$A_2 = \frac{2 \sin \beta^* \cosh \beta^* - (\sin \beta^* + \sinh \beta^*) \cos \beta^*}{(\sin \beta^* - \sinh \beta^*)}$$

$$B_2 = \sin \beta^*$$

Left end clamped

$$R_1(\xi) = \sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta^* - \sinh \beta^*)}{(\cosh \beta^* - \cos \beta^*)} (\cos \beta \xi - \cosh \beta \xi)$$

$$B_2 = \frac{\sin \beta^* \cosh \beta^* - \sinh \beta^* \cos \beta^*}{\cosh \beta^* - \cos \beta^*}$$

$$A_2 = \frac{B_2(\cosh \beta^* - \cos \beta^*)}{(\sin \beta^* - \sinh \beta^*)} + \frac{2 \sinh \beta^* (1 - \cos \beta^* \cosh \beta^*)}{(\sin \beta^* - \sinh \beta^*) (\cosh \beta^* - \cos \beta^*)}$$

Left end free

$$R_1(\xi) = \sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta^* + \sinh \beta^*)}{(\cos \beta^* + \cosh \beta^*)} (\cos \beta \xi + \cosh \beta \xi)$$

$$B_2 = \frac{(\sin \beta^* \cosh \beta^* - \sinh \beta^* \cos \beta^*)}{(\cos \beta^* + \cosh \beta^*)}$$

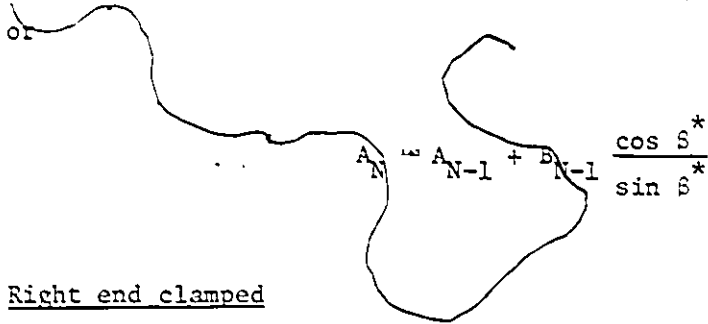
$$A_2 = \frac{B_2(\cosh^2 \beta^* - \cos^2 \beta^*) - 2 \sinh \beta^* (1 + \cos \beta^* \cosh \beta^*)}{(\sin \beta^* - \sinh \beta^*) (\cos \beta^* + \cosh \beta^*)}$$

At the right end of the beam two pieces of information are required. They are the N^{th} span modal shape expression and the value of A_N . As mentioned earlier, there are two valid expressions for A_N available. As one may give rise to an indeterminate form, both expressions are supplied.

Right end pinned

$$R_N(\xi) = A_N \left(\sin \beta \xi - \frac{\sin \beta^*}{\sinh \beta^*} \sinh \beta \xi \right)$$

$$A_N = A_{N-1}(-1) + B_{N-1} \frac{\{-1 + \sin \beta^* \sinh \beta^* + \cos \beta^* \cosh \beta^*\}}{(\cos \beta^* \sinh \beta^* - \sin \beta^* \cosh \beta^*)}$$



$$R_N(\xi) = A_N \left\{ \sin \beta \xi - \sinh \beta \xi + \frac{(\sin \beta^* - \sinh \beta^*)}{(\cosh \beta^* - \cos \beta^*)} (\cos \beta \xi - \cosh \beta \xi) \right\}$$

$$A_N = \frac{(\cosh \beta^* - \cos \beta^*)}{2 \sinh \beta^* (\cos \beta^* \cosh \beta^* - 1)} \{ A_{N-1} (\sin \beta^* \cosh \beta^* - \cos \beta^* \sinh \beta^*) \\ + B_{N-1} (\sin \beta^* \sinh \beta^* + \cos \beta^* \cosh \beta^* - 1) \}$$

or

$$A_N = (\cos \beta^* - \cosh \beta^*) \frac{\{ A_{N-1} \sin \beta^* + B_{N-1} \cos \beta^* \}}{(\sinh \beta^* \cos \beta^* - \sin \beta^* \cosh \beta^*)}$$

Right end free

$$R_N(\xi) = A_N \left\{ \sin \beta \xi + \sinh \beta \xi - \frac{(\sin \beta^* + \sinh \beta^*)}{(\cos \beta^* + \cosh \beta^*)} (\cos \beta \xi + \cosh \beta \xi) \right\}$$

$$A_N = \frac{(\cos \delta^* + \cosh \delta^*)}{2 \sinh \delta^* (1 + \cos \delta^* \cosh \delta^*)} \{A_{N-1} (\sin \delta^* \cosh \delta^* - \cos \delta^* \sinh \delta^*)$$

$$+ B_{N-1} (\sin \delta^* \sinh \delta^* + \cos \delta^* \cosh \delta^* - 1)\}$$

or

$$A_N = (\cos \delta^* + \cosh \delta^*) \frac{\{A_{N-1} \sin \delta^* + B_{N-1} \cos \delta^*\}}{(\sin \delta^* \cosh \delta^* - \sinh \delta^* \cos \delta^*)}$$

The procedure then for determining frequencies and modal shapes of any beam of the family under discussion is as follows:

- (1) For the mode of interest determine the eigenvalue δ from the appropriate table;
- (2) Write the modal shape expression for the first (left hand) span;
- (3) Using appropriate expressions provided evaluate A_2 and B_2 ;
- (4) Utilize the recursion formulae (equations 6.16) to evaluate A_n and B_n for $2 < n < N$;
- (5) Write the modal shape expressions for all of the internal spans, i.e. all spans except the extreme outer spans;
- (6) Evaluate A_N from appropriate expression;

- (7) Write the modal shape expression for the N^{th} span using the appropriate expression.

It may be noted that the iteration technique generally works fairly well for well-behaved functions, but it should be recognised that it is possible to miss adjacent eigenvalues if their spacing is small compared to the increments used in searching for a change in sign of the determinant. It will be observed that in Table 6.6 the first and second mode eigenvalues are very close together for beams with more than six spans. For this reason, first and second mode eigenvalues for beams of more than six spans are given with six digits instead of four. It has obviously been necessary to utilize unusually small increments when searching for these eigenvalues. It has also been found advisable to visually study plots of the determinant in order that no eigenvalues would be missed.

In addition to the computer analysis, it is wise, where possible, to observe how eigenvalues for beams with large numbers of spans can be obtained from eigenvalues for beams with smaller numbers of spans.

For illustration purposes we consider the first mode vibration of a six span clamped-clamped beam as shown in Figure 6.2. The first modal shape will evidently be anti-symmetric about the middle of the beam. But we recognise that the motion of the right half of this

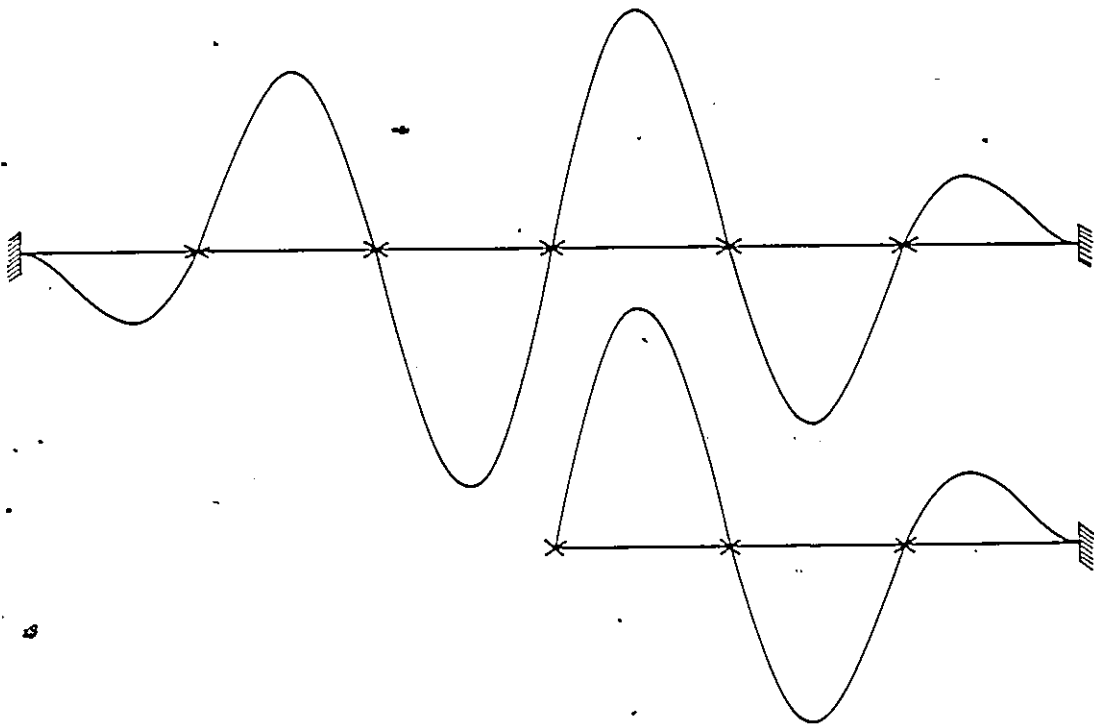


FIGURE 6.2 : Modal shapes for first mode vibration of six-span clamped-clamped and three-span pinned-clamped beams.

beam should be no different than that of a pinned-clamped beam of three spans as shown also in Figure 6.2. This is because no displacement and bending moment is possible at the left end of the three-span beam or the center support of the six-span beam. The three-span beam has half the length of the six-span beam and since their frequencies are identical, the eigenvalue of the shorter beam must be half that of the longer one (equation 1.7). This is in keeping with the observation in the tables that the eigenvalues are respectively 9.782, and 19.56.

For beams with even numbers of spans, and identical boundary conditions at each end, such as the beam under discussion, the even numbered modal shape will be symmetric about the centre of the beam. The second modal shape is shown in Figure 6.3. Here it is seen, following similar arguments to those presented for the first mode, that the second mode eigenvalue for the clamped-clamped beam should be equal to twice the first mode eigenvalue for a three-span clamped-clamped beam. That this is in fact the case is readily established by consulting the tables.

There are numerous other examples in the data presented where eigenvalues of one beam can be inferred from another. These cases cannot be discussed here, but they can be utilized, and have been utilized to check on the computer search for eigenvalues.

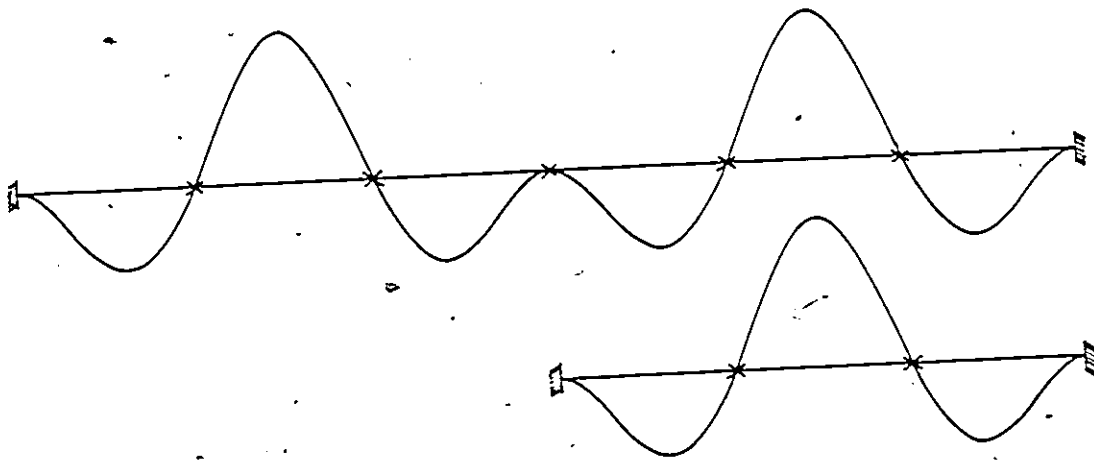


FIGURE 6.3 : Modal shapes for second mode vibration of six-span clamped-clamped beam and first mode vibration of three-span clamped-clamped beam.

It will be noted that the first mode eigenvalues for beams with one free end, and the first two eigenvalues for beams with two free ends are substantially reduced from those of other beams. This is particularly obvious if we study the last row of numbers in each table, i.e. the eigenvalues for fifteen span beams.

For beams with one free end, the first mode motion will primarily involve the span with the free end. Focusing our attention on this span for the moment, and recalling that the first mode eigenvalue for a clamped-free beam is 1.875, we recognise that the first mode eigenvalue for a beam of N spans cannot exceed $1.875 N$. This is because, for first mode motion, the bending moment at the inner end of the span in question will be somewhat less than that provided by a clamped condition.

For higher mode vibration of beams with one free end, the limit discussed above is no longer applicable and the eigenvalues are not so greatly influenced by the existence of the free end, particularly as the number of spans in the beam increases. This is easily verified by comparing the final row of numbers of Tables 6.4 and 6.5 with those of Tables 6.1, 6.2, and 6.3. Let us suppose we have a multi-span beam with clamped-pinned or pinned-pinned end conditions, and we decide to add one more span to the pinned end, the additional span having a free outer end.

Then, while the additional span will have the effect of bringing about some change in existing frequencies its major effect will

be the introduction of a new and much lower fundamental frequency. A similar change could be effected by removing the simple support from one end of a multispans clamped-pinned or pinned-pinned beam.

As noted earlier, the first and second mode eigenvalues for free-free multispans beams are very close together (Table 6.6). This is in keeping with the observations made above. For free-free beams of even number of spans the first modal shape will be anti-symmetric about the centre of the beam. The second modal shape will be symmetric about the centre. For beams of odd number of spans the reverse situation will prevail, i.e. the first modal shape will be symmetric about the centre of the beam and the second mode anti-symmetric, etc.

We observe that, regardless of the number of spans, the first and second mode frequencies of free-free beams are determined primarily by the end spans. The major difference in the two modes is that one switches from a symmetric to anti-symmetric mode in going from first to second mode or from anti-symmetric to symmetric, depending on the number of spans. In either case, there is not much difference in the associated bending energy for a multispans beam and hence the eigenvalues of the two modes are not very different.

There will be no difficulty in obtaining the first and second mode frequencies with error less than one percent using Table 6.6. The problem of establishing exact modal shapes for first and second

mode vibrations of free-free multispan beams will be more difficult because of the high dependence of the modal shape on the eigenvalue. For beams with even numbers of spans the problem can be reduced to analyzing a pinned-free or clamped-free beam of half the length as discussed earlier. For beams with odd numbers of spans this latter procedure cannot be utilized to advantage since it would involve analyzing beams with one half-span. It is recognized, however, that since the modes will be either symmetric or anti-symmetric about the centre of the beam, depending on whether the mode is odd numbered or even numbered, one only need determine the shape of one half of the beam. It is evidently advantageous to work with the left half of the beam (Figure 6.1) since the recursion formula need be used less often and less arithmetical error will be introduced.

The expressions derived and presented herein provide a powerful and exact means for establishing vibration frequencies and modal shapes for uniform beams with equally spaced intermediate supports. While the analysis is applicable to all beams of this type, it will be particularly useful to designers of heat exchangers and boilers. The novel aspects of the work are that it provides frequency data for all types of outer boundary conditions and that it provides exact expressions for modal shapes. Because of the comprehensive nature of the work it is expected that it will find useful application in both the industry and research laboratories. A paper based on the second part of this Chapter was presented by the author before the VII SECTAM Conference and published in its proceedings (37).

Illustrative example 6.1

Determine the fundamental frequency of a 10 span beam of length 20' with uniform span spacings. The cross-section of the beam is circular with O.D. = $\frac{3}{4}$ " and I.D. = $\frac{1}{2}$ ". The intermediate supports can be considered as pinned.

- (a) When beam is clamped at both the extreme ends.

From Table 6.2, obtain $\beta = 31.86$

$$\begin{aligned} f_{(1)} &= \frac{2}{2\pi L^2} \sqrt{\frac{EI}{\rho A}} \\ &= \frac{31.86^2}{2\pi \times (240)^2} \sqrt{\frac{30 \times 10^6 \times 0.0124 \times 386}{0.283 \times 0.245}} \\ &= 127.6 \text{ Hz} \end{aligned}$$

- (b) When one extreme end is clamped and other pinned.

From Table 6.3, obtain $\beta = 31.53$, so that

$$f_{(1)} = 124.9 \text{ Hz}$$

(c) When one extreme end is clamped and other free.

From Table 6.4, obtain $S = 15.39$, so that

$$f_{(1)} = 29.77 \text{ Hz}$$

It may be noted that the removal of the pinned support from the beam of case (b) has resulted in a tremendous reduction in its fundamental frequency.

TABLE 6.1 Values of β for "Pinned-Pinned" beams

No. of spans n .	Mode:	1	2	3	4	5	6
	3	9.425	10.67	12.89	18.85	20.12	22.29
	4	12.57	13.57	15.71	17.85	25.13	26.18
	5	15.71	16.55	18.50	20.76	22.75	31.42
	6	18.85	19.56	21.34	23.56	25.79	27.61
	7	21.99	22.61	24.22	26.35	28.62	30.76
	8	25.13	25.68	27.15	29.16	31.41	33.66
	9	28.27	28.76	30.10	32.01	34.20	36.48
	10	31.42	31.86	33.09	34.88	37.00	39.27
	11	34.56	34.96	36.10	37.79	39.83	42.05
	12	37.70	38.07	39.13	40.72	42.68	44.85
	13	40.84	41.18	42.17	43.67	45.55	47.66
	14	43.98	44.30	45.22	46.65	48.44	50.50
	15	47.12	47.42	48.29	49.64	51.36	53.35

TABLE 6.2 Values of β for "Clamped-Clamped" beams

No. of spans N.	Mode:	1	2	3	4	5	6
	3	10.67	12.89	14.19	20.12	22.29	23.56
	4	13.57	15.71	17.85	18.92	26.18	28.27
	5	16.55	18.50	20.76	22.75	23.65	32.30
	6	19.56	21.34	23.56	25.79	27.61	28.38
	7	22.61	24.22	26.35	28.62	30.76	32.44
	8	25.68	27.15	29.16	31.41	33.66	35.71
	9	28.76	30.10	32.01	34.20	36.48	38.68
	10	31.86	33.09	34.88	37.00	39.27	41.53
	11	34.96	36.10	37.79	39.83	42.05	44.33
	12	38.07	39.13	40.72	42.68	44.85	47.12
	13	41.18	42.17	43.67	45.55	47.66	49.91
	14	44.30	45.22	46.65	48.44	50.50	52.70
	15	47.42	48.29	49.64	51.36	53.35	55.51

TABLE 6.3 Values of δ for "Clamped-Pinned" beams

No. of spans N.	Mode:	1	2	3	4	5	6
	3	9.782	11.78	13.80	19.23	21.21	23.18
	4	12.84	14.58	16.83	18.62	25.43	27.18
	5	15.93	17.44	19.63	21.83	23.41	31.66
	6	19.04	20.36	22.43	24.69	26.78	28.18
	7	22.15	23.32	25.25	27.49	29.73	31.69
	8	25.27	26.32	28.11	30.27	32.55	34.73
	9	28.40	29.34	31.00	33.08	35.34	37.60
	10	31.53	32.39	33.93	35.92	38.13	40.41
	11	34.66	35.45	36.89	38.78	40.93	43.19
	12	37.79	38.52	39.87	41.66	43.74	45.98
	13	40.93	41.60	42.86	44.57	46.58	48.77
	14	44.06	44.69	45.88	47.51	49.44	51.58
	15	47.20	47.79	48.91	50.46	52.33	54.41

TABLE 6.4 Values of β for "Clamped-Free" beams

No. of spans N.	Mode:	1	2	3	4	5	6
	3	4.624	10.71	12.85	14.16	20.12	22.29
	4	6.157	13.61	15.71	17.80	18.89	26.18
	5	7.695	16.58	18.53	20.74	22.69	23.62
	6	9.234	19.59	21.38	23.56	25.75	27.55
	7	10.77	22.63	24.26	26.37	28.60	30.72
	8	12.31	25.70	27.19	29.19	31.41	33.63
	9	13.85	28.78	30.14	32.04	34.22	36.46
	10	15.39	31.87	33.12	34.92	37.03	39.27
	11	16.93	34.97	36.13	37.83	39.86	42.06
	12	18.47	38.08	39.15	40.76	42.71	44.87
	13	20.01	41.19	42.19	43.71	45.59	47.69
	14	21.55	44.31	45.25	46.68	48.48	50.53
	15	23.08	47.43	48.31	49.67	51.40	53.38

TABLE 6.5 Values of β for "Pinned-Free" beams

No. of spans n .	Mode:					
	1	2	3	4	5	6
3	4.609	9.810	11.78	13.74	19.23	21.21
4	6.155	12.86	14.61	16.80	18.56	25.43
5	7.695	15.94	17.48	19.63	21.79	23.34
6	9.234	19.04	20.40	22.45	24.67	26.73
7	10.77	22.16	23.36	25.28	27.49	29.69
8	12.31	25.28	26.35	28.15	30.29	32.53
9	13.85	28.40	29.37	31.04	33.11	35.34
10	15.39	31.53	32.41	33.97	35.95	38.14
11	16.93	34.66	35.47	36.93	38.81	40.95
12	18.47	37.80	38.54	39.90	41.70	43.77
13	20.01	40.93	41.62	42.90	44.61	46.62
14	21.55	44.06	44.71	45.91	47.54	49.48
15	23.08	47.20	47.80	48.94	50.50	52.36

TABLE 6.6 Values of β , for "Free-Free" beams

Mode:	No. of spans N.					
	1	2	3	4	5	6
3	4.235	4.943	10.74	12.82	14.12	20.12
4	6.024	6.283	13.65	15.71	17.75	18.85
5	7.650	7.740	16.62	18.55	20.72	22.64
6	9.219	9.249	19.62	21.41	23.56	25.71
7	10.77	10.77	22.66	24.30	26.39	28.59
8	12.31	12.31	25.72	27.23	29.22	31.41
9	13.85	13.85	28.80	30.18	32.08	34.23
10	15.39	15.39	31.89	33.16	34.96	37.05
11	16.93	16.93	34.98	36.16	37.87 ^s	39.89
12	18.47	18.47	38.09	39.18	40.80	42.75
13	20.01	20.01	41.20	42.22	43.75	45.63
14	21.55	21.55	44.32	45.27	46.72	48.52
15	23.08	23.08	47.44	48.33	49.70	51.44

CHAPTER 7

TORSIONAL VIBRATIONS OF CIRCULAR SHAFTS WITH

NON-CLASSICAL BOUNDARY CONDITIONS

7.1 Shafts with Classical Boundary Conditions

As mentioned in first Chapter, the eigenvalues of torsional vibrations of circular shafts with classical boundary conditions are well established and are available in the existing literature. Since they will be frequently used in determining the frequencies for non-classical boundary conditions, they are included here in Table 7.1, for the sake of completeness

Table 7.1

Mode	β_{cc}	β_{CF}	β_{FF}
1	π	$\pi/2$	0
2	2π	$3\pi/2$	π
3	3π	$5\pi/2$	2π
4	4π	$7\pi/2$	3π

It should however be remembered that eigenvalues β are related to frequencies of torsional vibrations by $\beta = \omega L \sqrt{\frac{\rho}{G}}$ and should not be confused with the expression (1.7) which relates β to frequencies of lateral vibrations.

Part 1 - Shafts with one non-classical boundary condition

7.2 Shafts Clamped at One End and With a Torsional Spring Attached at the Other

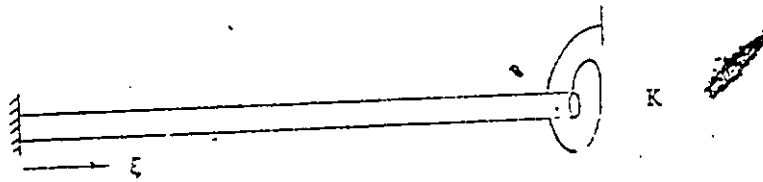


Figure 7.2

The general expression for the angular displacement is given by

$$\theta(\xi) = A \sin \beta \xi + B \cos \beta \xi \quad (7.1)$$

The appropriate boundary conditions to be enforced for this case are

$$\theta(0) = 0 \quad \text{and} \quad \theta'(1) = -K^* \theta(1)$$

where $K^* = \frac{KL}{GIp}$ as explained in Chapter 1.

The first boundary condition results in $B = 0$ in equation (7.1) and the second boundary condition yields

$$\beta \cos \beta = -K^* \sin \beta$$

or

$$K^* = -\beta / \tan \beta \quad (7.2)$$

It may be noted that as K^* approaches zero, the frequency equation (7.2) approaches that governing the vibration of a clamped-free shaft and as K^* approaches infinity the equation approaches that of a clamped-clamped shaft. Frequencies for both these limiting cases are readily obtained from Table 7.1. The eigenvalues β and therefore the frequencies of this system are completely characterized by the dimensionless system parameter K^* . Introducing the 'Q factor' as was done in Chapter 2, the frequencies of this case can be related to its classical limits as follows

$$f_{(n)} = f_{CF(n)} + Q[f_{cc(n)} - f_{cf(n)}] \quad (7.3)$$

where subscript n denotes mode number.

It can be easily recognised that Q factor is a measure of how far the shaft frequencies lie between the two limiting values associated with classical boundary conditions. Recalling the fact that eigenvalues are proportional to frequencies of torsional vibrations, equation (7.3) can be rewritten as

$$\beta_{(n)} = \beta_{cf(n)} \{1 + Q[\frac{\beta_{cc(n)}}{\beta_{cf(n)}} - 1]\} \quad (7.4)$$

Utilizing equations (7.2) and (7.4), the values of K^* can be computed* for various values of Q. The data associated with this case are presented in Table 7.2 for first six modes of vibration. The format for the computed data and spacings for Q between zero and one have been so selected as to permit one percent accuracy when linear interpolation is used for parameters lying partway between the tabulated values. The linear interpolation can be used for $Q < 0.8$, for $Q > 0.8$, the intermediate values may be obtained from the following formula

$$Q = 1 - \frac{0.2 K^*}{K^* - 0.8} \quad (7.5)$$

* The computer programs for cases 1 through 19 in this Chapter were prepared by Mr. D.S. Nguyen.

The equation (7.5) is identical to one utilized in Chapter 2 for lateral vibration analysis. The term $K_{0.8}^*$ indicates the value of K^* for $Q = 0.8$ to be readily obtained from tables. For beams with rotational mass, K^* in equation (7.5) is of course replaced by I_0^* . The use of the tables is demonstrated by an illustrative example provided at the end of this Chapter.

The modal shape expression is given by $\theta(\xi) = \sin \delta \xi$, where constant A has been set equal to unity. This is quite permissible since the exact amount of amplitude will be dictated by initial conditions.

7.3 Shafts Free at One End and With Torsional Spring Attached at the Other

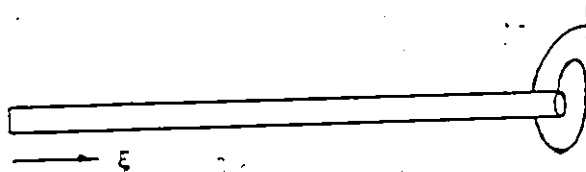


Figure 7.3

Utilizing the boundary conditions $\theta'(0) = 0$, we obtain

$$\theta(\xi) = B \cos \beta \xi$$

and enforcing the condition of torque equilibrium at the end $\xi = 1$,
(i.e. $\theta'(1) = -K^* \theta(1)$), we obtain the following frequency equation for
this case

$$K^* = \beta \tan \beta \quad (7.6)$$

The frequencies of this case are related to their limiting
values associated with classical boundary conditions by,

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1$$

(7.7)

and

$$f_{(n)} = f_{FF(n)} + Q[f_{cf(n)} - f_{FF(n)}] \quad \text{for } n > 1$$

The computed values of K^* for various values of Q are pre-
sented in Table 7.3 for first six modes. The modal shape function is
obviously given by $\theta(\xi) = \cos \beta \xi$ where the arbitrary constant B is
assigned a value of unity.

7.4 Shafts Clamped at One End and With Rotational Mass Attached at the Other

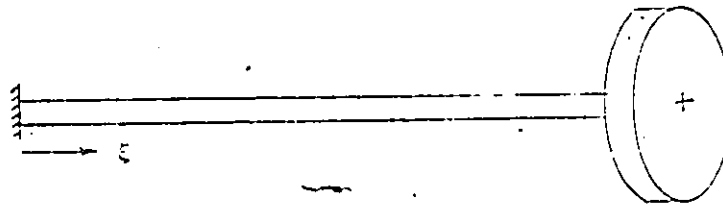


Figure 7.4

The requirement of no angular displacement at the clamped end results in

$$\theta(\xi) = A \sin \beta \xi \quad (7.8)$$

Enforcing the boundary condition involving torque equilibrium at the right hand end, i.e. $\theta'(L) = \frac{\beta^2}{I_o^*} \theta(L)$, we obtain:

$$I_o^* = \beta \tan \beta \quad (7.9)$$

where $I_o^* = \frac{I_p \rho L}{I_o}$ is a non-dimensional system parameter.

Recognising the limiting conditions, we can write

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1$$

and

(7.10)

$$f_{(n)} = f_{cc(n-1)} + Q[f_{cf(n)} - f_{cc(n-1)}] \quad \text{for } n > 1$$

It should be noted that for the first mode, the lower limit of frequency is zero corresponding to $I_o^* = 0$. The zero value of I_o^* is associated with the situation when mass moment of inertia I_o of the rotational mass is infinitely larger than the quantity $I_p \rho L$ associated with the shaft. For higher modes the end with the rotational mass behaves as a nodal point and hence the lower limit of frequency for higher modes corresponds to clamped-clamped boundary conditions. The upper limit frequencies are always associated with clamped free boundary conditions.

Table 7.4 provides the values of system parameter I_o^* for various values of Q factor for first six modes.

The modal shape is given by $\theta(\xi) = \sin \beta \xi$.

7.5 Shafts Free at One End and With a Rotational Mass Attached at the Other

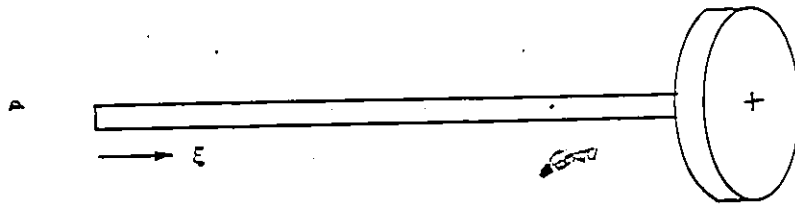


Figure 7.5

Utilizing the boundary conditions $\theta'(0) = 0$ and $\theta'(l) = \frac{\beta^2}{I_o^*} \theta(l)$, it can be easily shown that

$$I_o^* = -\frac{\beta}{\tan \beta} \quad (7.11)$$

Recognising the two limits of frequencies associated with $I_o^* = 0$ and $I_o^* = \infty$, one can write

$$f_{(n)} = f_{cf(n)} + Q[f_{ff(n+1)} - f_{cf(n)}] \quad (7.12)$$

Values of I_o^* are presented in Table 7.5 for various values of Q for first six modes. The modal shape expression is given by

$$\theta(\xi) = \cos \beta \xi$$

Part 2 - Shafts with two non-classical boundary conditions

A number of cases described in this part are fairly commonly encountered in practical engineering works.

7.6 Shafts With Rotational Mass Attached at Each End

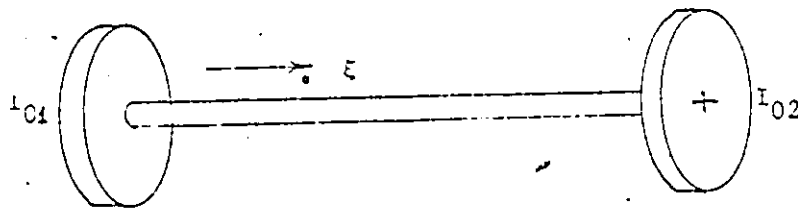


Figure 7.6

We consider the shaft shown in the above figure which carries a rotational mass of moment of inertia I_{o1} at its left end and a mass of moment of inertia I_{o2} at its right end.

The boundary conditions to be enforced at each end are:

$$\theta'(0) = -\frac{\beta^2}{I_{o1}} \theta(0) \quad \text{and} \quad \theta'(l) = -\frac{\beta^2}{I_{o2}} \theta(l)$$

Substituting these boundary conditions in the general solution for angular displacement of the shaft, i.e.

$$\theta(\xi) = A \sin \beta \xi + B \cos \beta \xi$$

we obtain:

$$A + B \left(\frac{\beta}{I_{o1}^*} \right) = 0 \quad (7.13)$$

$$A \left(\beta \cos \beta - \frac{\beta^2}{I_{o2}^*} \sin \beta \right) + B \left(\beta \sin \beta + \frac{\beta^2}{I_{o2}^*} \cos \beta \right) = 0 \quad (7.14)$$

For a non-trivial solution to exist for the constants A and B appearing in equations (7.13) and (7.14), the determinant of their coefficient matrix must vanish, i.e.

$$I_{o1}^{*2} + \frac{\beta(1-\alpha)}{\alpha \tan \beta} I_{o1}^* - \frac{\beta^2}{\alpha} = 0 \quad (7.15)$$

where the constant α has been introduced as a ratio of two system parameters I_{o1}^* and I_{o2}^* such that $\alpha = \frac{I_{o2}^*}{I_{o1}^*}$

The equation (7.15) is a quadratic equation in I_{o1}^* and once the values of β and α are selected, two associated roots of I_{o1}^* can be obtained. The negative root is discarded for obvious reasons.

For any fixed value of α , the lower limit frequency for first mode of vibration is zero and is approached as the moment of inertia of the rotational masses approach infinity. This corresponds to a mode of vibration where the masses rotate in opposite directions. For all higher modes, the masses with infinite moment of inertia make the shaft ends behave as nodal points and therefore the lower limits for higher modes are those of clamped-clamped conditions. For all modes the upper limits are represented by free-free end conditions. In terms of Q factor, therefore, we can write

$$f_{(n)} = Q f_{FF(n+1)} \quad \text{for } n = 1 \quad (7.16)$$

$$f_{(n)} = f_{cc(n-1)} + Q[f_{FF(n+1)} - f_{cc(n-1)}] \quad \text{for } n > 1$$

The data for the torsional vibration analysis of such shafts are presented in Tables 7.6 for various values of Q and α . The values of α are given from 0.1 to 1.0 in the intervals of 0.1. Since this is a symmetric case, it is not necessary to give value of α greater than 1.0.

The modal shape expression is arrived at by setting the

constant A equal to unity and then determining the constant B from equation (7.13). We obtain the following expression for modal shape.

$$\theta(\xi) = \sin \beta \xi - \frac{I_{o1}^*}{\beta} \cos \beta \xi$$

7.7 Shafts With Torsional Springs Attached at Each End

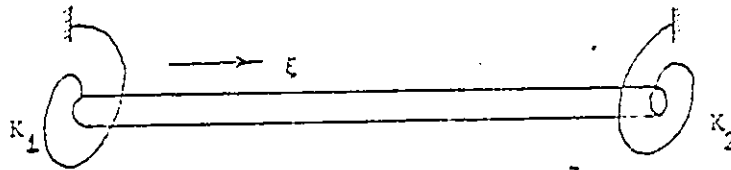


Figure 7.7

The torsional stiffness of the two springs attached at left and right ends of the shaft are K₁ and K₂ respectively. The pertinent boundary conditions for this class of problems are

$$\theta'(0) = K_1^* \theta(0) \quad \text{and} \quad \theta'(1) = -K_2^* \theta(1)$$

Substitution of these boundary conditions in the general

expression for the angular displacement $\theta(\xi)$ results in the following two homogeneous algebraic equations in terms of constants A and B.

$$A (\dot{\beta}) - B (K_1^*) = 0 \quad (7.17)$$

and

$$A (\beta \cosh \beta + K_2^* \sin \beta) + B (K_2^* \cos \beta - \beta \sin \beta) = 0 \quad (7.18)$$

The condition for a non-trivial solution for the constants A and B in above two equations yields the following frequency equation of the system

$$K_1^* + \frac{\beta(1+\alpha)}{\alpha \tan \beta} K_1^* - \frac{\beta^2}{\alpha} = 0 \quad (7.19)$$

$$\text{where } \alpha = \frac{K_2^*}{K_1^*}$$

For a fixed value of α , the lower and upper limits of frequency are associated with free-free and clamped-clamped boundary conditions respectively. Therefore we can write

$$f_{(n)} = f_{FF(n)} + Q[f_{cc(n)} - f_{FF(n)}] \text{ for } n > 1 \quad (7.20)$$

where this should however be remembered that

$$f_{FF}(1) = 0$$

The equation (7.20) implicitly establishes the relationship between β and Q . Therefore equation (7.19) can be solved for various values of Q and α . As before, the negative root for K_1^* is disregarded. The values of K_1^* for various values of Q and α are presented in Tables 7.7. The modal shape expression is derived in the usual manner and is given below

$$\theta(\xi) = \sin \beta \xi + \left(\frac{\beta}{K_1^*} \right) \cos \beta \xi$$

7.8 Shafts With Torsional Spring at One End and Rotational Mass at the Other

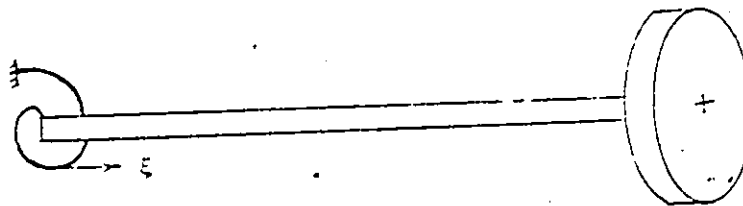


Figure 7.8

The boundary conditions to be enforced here are:

$$\theta'(0) = K^* \theta(0) \quad \text{and} \quad \theta'(1) = \frac{\beta^2}{I_o^*} \theta(1)$$

Substituting these boundary conditions in the general expression for angular displacement, we obtain the following two equations

$$A(\beta) - B(K^*) = 0$$

$$A\left[\frac{\beta^2}{I_o^*} \sin \beta - \beta \cos \beta\right] + B\left[\frac{\beta^2}{I_o^*} \cos \beta + \beta \sin \beta\right] = 0$$

For a non-trivial solution for the constants A and B, the following frequency equation must be satisfied.

$$I_o^{*2} - I_o^* \frac{\beta \tan \beta (1+\alpha)}{\alpha} - \frac{\beta^2}{\alpha} = 0 \quad (7.21)$$

$$\text{where } \alpha = \frac{K^*}{I_o^*}$$

The relationship between Q and the frequency of this kind of system is given by

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1 \quad (7.22)$$

$$f_{(n)} = f_{cf(n-1)} + Q[f_{cf(n)} - f_{cf(n-1)}] \quad \text{for } n > 1$$

The data necessary for torsional vibration analysis of this type of problems are presented in Tables 7.8. This being an unsymmetric case, the ratio α of the two parameters is varied between 0.1 and 1 in intervals of 0.1 and then $\frac{1}{\alpha}$ is varied between 0.1 and 0.9 intervals of 0.1. While in practice, the ratio α can assume any value between zero and infinity, it is expected that all practical problems can be solved by the given range of α .

The modal shape expression is given by

$$\theta(\xi) = \sin \beta \xi + \frac{\beta}{K} \cos \beta \xi$$

7.9 Shafts Clamped at One End and With Torsional Spring and a Rotational Mass at the Other

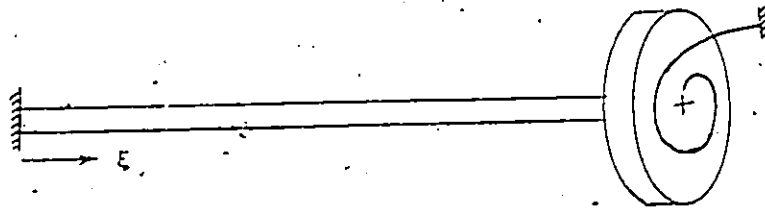


Figure 7.9

The boundary conditions at the two ends, $\xi = 0$ and $\xi = 1$ are respectively $\theta(0) = 0$ and $\theta'(1) = \left(\frac{\beta^2}{I_0} - K^*\right) \theta(1)$.

Since the first of these boundary condition results in $B = 0$ in the general displacement function, the second yields the following equation

$$A\left(\beta \cos \beta - \frac{\beta^2}{I_0} \sin \beta + K^* \sin \beta\right) = 0 \quad (7.23)$$

Since $A = 0$ gives the trivial solution, the quantity in the brackets must be zero, or

$$\beta \cos \beta - \frac{\beta^2}{I_0} \sin \beta + K^* \sin \beta = 0$$

Introducing $\alpha = \frac{K^*}{I_0}$, we obtain:

$$I_0 \alpha^2 + \frac{\beta I_0}{\alpha \tan \beta} - \frac{\beta^2}{\alpha} = 0 \quad (7.24)$$

Recognising the limits, one can write

$$f_{cc}(n) = Q f_{cc}(n) \quad \text{for } n = 1 \quad (7.25)$$

$$f_{cc}(n) = f_{cc}(n-1) + Q[f_{cc}(n) - f_{cc}(n-1)] \quad \text{for } n > 1$$

The computed results for this case are provided in Tables 7.9.

The modal shape function is obviously $\theta(\xi) = A \sin \beta \xi$ where the arbitrary constant A has been set equal to unity.

7.10 Shafts Free at One End and With a Torsional Spring and a Rotational Mass at the Other

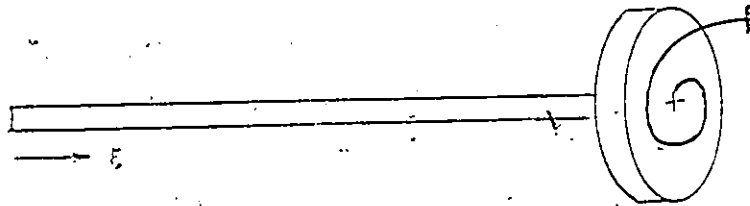


Figure 7.10

Utilizing the boundary condition at the free end results in $A = 0$ in the general displacement function, and then enforcement of the boundary conditions at the end $\xi = 1$, i.e. $\theta'(1) = \left(\frac{\beta^2}{I_o^*} - K^* \right) \theta(1)$, yields the following frequency equation of the system.

$$I_o^{*2} - I_o^* \frac{\beta \tan \beta}{\alpha} - \frac{\beta^2}{\alpha} = 0 \quad (7.26)$$

$$\text{where } \alpha = \frac{K^*}{I_o^*}$$

Recognising the frequencies associated with the two limiting values of the system parameter I_0^* for fixed values of α , we can write

$$\left\{ \begin{array}{l} f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1 \\ \end{array} \right. \quad (7.27)$$

$$f_{(n)} = f_{cf(n-1)} + Q[f_{cf(n)} - f_{cf(n-1)}] \quad \text{for } n > 1$$

Computed values of I_0^* for values of Q , $0.05 \leq Q \leq 0.9$, and for values of α , $0.1 \leq \alpha \leq 1.0$ and $0.1 \leq \frac{1}{\alpha} \leq 0.9$ are presented in Tables 7.10.

The modal shape is given by $\theta(\xi) = \cos \beta \xi$.

7.11 Shafts Clamped at One End and With Torsion Spring Connected Rotational Mass at the Other

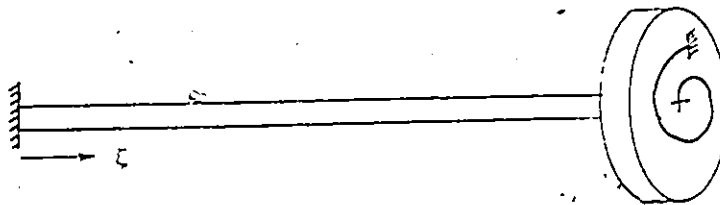


Figure 7.11

This kind of boundary condition prevails when the rotational mass mounted on the shaft is permitted to undergo constrained rotation relative to the shaft. The rotary motion of the shaft is opposed by a spring system which apposes relative motion with a restoring torque proportional to the angular displacement from its neutral configuration. This type of arrangement is often utilized in conjunction with an energy dissipating device to damp out torsional vibrations of the shaft. The figure 7.11 shows a schematic diagram of such an arrangement without its energy dissipating device. The damping effects of this device are neglected here as, for light damping, they are known to be of small orders. It can be readily shown that by choosing the time scale properly, the displacement function can be written as

$$\phi(\xi, t) = (A \sin \beta \xi + \beta \cos \beta \xi) \cos \omega t \quad (7.28)$$

The absolute angular motion of the rotational mass will be sinusoidal in nature and of the same frequency as the end of the shaft and may be expressed as

$$\eta(t) = C \cos (\omega t + \alpha) \quad (7.29)$$

where C is an arbitrary constant.

Equating the torque at the shaft outer end, to the restoring moment provided by the spring we obtain

$$\left. \frac{\partial \phi(\xi, \tau)}{\partial \xi} \right|_{\xi=1} = K^* (\eta(\tau) - \phi(\xi, \tau)) \Big|_{\xi=1} \quad (7.30)$$

The condition at the clamped end requires that $B = 0$, and substituting for ϕ and η from equations (7.28) and (7.29) into equation (7.30) we obtain:

$$A(\beta \cos \beta + K^* \sin \beta) \cos \omega t = C K^* \cos (\omega t + \alpha) \quad (7.31)$$

To satisfy the equality indicated in the equation (7.31), the phase angle α should be equal to zero. The motion of the rotating mass and the end of the shaft are therefore in phase. Equation (7.31) may therefore be re-written as,

$$A \left(\frac{\beta \cos \beta}{K^*} + \sin \beta \right) - C = 0 \quad (7.32)$$

The requirement of dynamic equilibrium between the torques at the shaft outer end permits us to write

$$G I_p \left. \frac{\partial \phi(\xi, t)}{\partial \xi} \right|_{\xi=1} = -I_o \left. \frac{\partial^2 \eta(t)}{\partial t^2} \right|_{\xi=1} \quad (7.33)$$

Substituting for ϕ and η and rearranging we obtain

$$A(\cos \beta) - C \left(\frac{\beta}{I_o^*} \right) = 0 \quad (7.34)$$

$$\text{where } I_o^* = \frac{I_p \rho L}{I_o} \text{ as before.}$$

For a non-trivial solution for the constants A and C appearing in equations (7.32) and (7.34), the determinant of their coefficient matrix must vanish. Utilizing this condition we obtain:

$$I_o^{*2} - I_o^* (\beta \tan \beta) - \frac{\beta^2}{\alpha} = 0 \quad (7.35)$$

$$\text{where } \alpha = \frac{K^*}{I_o}$$

The limiting conditions of this case are associated with clamped free boundary conditions and we can write

$$f_{(n)} = Q f_{cf(n)} \quad \text{for } n = 1$$

and

(7.36)

$$f_{(n)} = f_{cf(n-1)} + Q[f_{cf(n)} - f_{cf(n-1)}] \quad \text{for } n > 1$$

The tables 7.11 provide all the information necessary to establish torsional vibration frequencies of this class of shafts. The modal shape is obviously given by $\theta(\xi) = \sin \beta \xi$ and the value of rotational mass vibration amplitude $C = \frac{I_o^* \cos \beta}{\beta}$ which has been determined from equation (7.34).

7.12 Shafts Free at One End and With a Torsional Spring Connected Rotational Mass at the Other

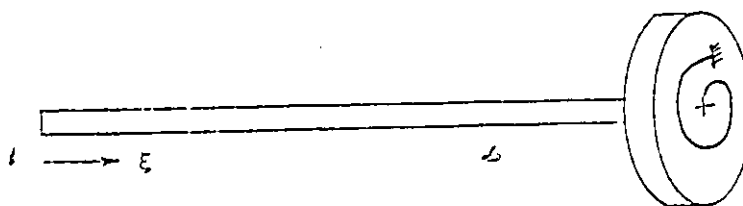


Figure 7.12

The requirement of no torque at the free end results in $A = 0$ in the displacement function. Therefore we can write the following for the angular displacement of the shaft

$$\phi(\xi, t) = (B \cos \beta \xi) \cos \omega t \quad (7.37)$$

Following the procedure outlined in the previous case, we can write the following equation for the absolute angular motion of the rotational mass

$$\eta(t) = C \cos \omega t \quad (7.38)$$

Where C is a constant representing the amplitude of vibration of the rotational mass. Further enforcement of boundary conditions represented by equations (7.30) and (7.33) in the previous case, we obtain the following two equations in terms of constants B and C .

$$B(\cos \beta - \frac{\beta \sin \beta}{K^*}) - C = 0 \quad (7.39)$$

and

$$B(-I_o^* \sin \beta) + C(\beta) = 0 \quad (7.40)$$

The condition for a non-trivial solution for the constants B and C in the above two equations permits us to write the following frequency equation of the system

$$I_o^{*2} + I_o^* \left(\frac{\beta}{\tan \beta} \right) - \frac{\beta^2}{\alpha} = 0 \quad (7.41)$$

$$\text{where } \alpha = \frac{K^*}{I_o^*}$$

We can write the following equations expressing the frequencies in terms of Q factors and recognising the associated limits.

$$f_{(n)} = Q f_{FF(n+1)} \quad \text{for } n = 1 \quad (7.42)$$

and

$$f_{(n)} = f_{FF(n)} + Q[f_{FF(n+1)} - f_{FF(n)}] \quad \text{for } n > 1$$

Tables 7.12 provide values of I_o^* for various values of Q and α . The modal shape is given by $\theta(\xi) = \cos \beta \xi$ and the rotational mass vibration amplitude C is determined from equation (7.40) and is given by $C = -I_o^* \frac{\sin \beta}{\beta}$.

Part 3 - Shafts with intermediately located torsion springs or rotational masses.

7.13 Shafts Clamped at One End and Free at the Other With an Intermediately Located Rotational Mass

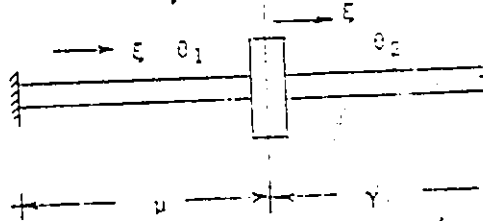


Figure 7.13

Let us consider the shaft shown in the Figure 7.13 where the rotational mass is located at a dimensionless distance μ from the left end. The value of μ is obviously equal to the actual distance divided by the overall shaft length.

The angular displacements of the two sections of the shaft may be represented as

$$\theta_1(\xi) = A_1 \sin \beta \xi + B_1 \cos \beta \xi \quad (7.42)$$

and

$$\theta_2(\xi) = A_2 \sin \beta \xi + B_2 \cos \beta \xi \quad (7.43)$$

where the subscripts 1 and 2 refer to the left and right hand segments of the shaft respectively.

Denoting the dimensionless length of the second segment as γ , where, of course, $\gamma = 1 - \mu$, and enforcing the shaft outer end boundary conditions $\theta_1(0) = 0$ and $\theta'_2(\gamma) = 0$, we obtain $B_1 = 0$ and $A_2 = B_2 \tan \beta \gamma$, so that the displacement functions can now be written as

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.44)$$

and

$$\theta_2(\xi) = B_2 (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi) \quad (7.45)$$

The two other boundary conditions arise as a result of continuity of angular displacement where the segments join and dynamic equilibrium of the rotating mass. They may be written respectively, as

$$\theta_1(\mu) = \theta_2(0) \quad \text{and} \quad \theta'_1(\mu) - \theta'_2(0) = \frac{\beta^2}{I_0^*} \theta_1(\mu)$$

Enforcing the above two boundary conditions, we obtain

$$A_1 (\sin \beta \mu) - B_2 = 0 \quad (7.46)$$

and

$$A_1 \left(\cos \beta \mu - \frac{\beta}{I_o^*} \sin \beta \mu \right) - B_2 \tan \beta \gamma = 0 \quad (7.47)$$

Equating the determinant of the coefficient matrix of constants A_1 and B_2 in equations (7.46) and (7.47) to zero gives the following eigenvalue equation

$$I_o^* = \frac{\beta \sin \beta \mu}{\cos \beta \mu - \tan \beta \gamma \sin \beta \mu} \quad (7.48)$$

As in analogous cases encountered in Chapter 3 for lateral vibration of beams, there appear initially to be some uncertainty in the selection of frequency limits for this kind of shafts. The upper limit frequencies for this case are evidently those for a clamped free shaft as I_o^* approaches infinity (or mass moment of inertia I_o of the rotational mass approaches zero). The first mode lower limit is obviously zero since as I_o^* approaches zero (or I_o approaches infinity) the rotational mass will completely dominate the character of the oscillations. On the other hand, for all higher modes, the intermediate point on the shaft will

approach a nodal point as I_0^* approaches zero (or I_0 approaches infinity). This makes the intermediate point behave as a clamped end and consequently the two segments of the shafts are decoupled and are capable of vibrating independently. There are two distinct frequencies associated with each segment. The left segment will have a frequency associated with clamped-clamped boundary conditions and a dimensionless length of μ whereas the right segment will have a frequency associated with clamped-free boundary conditions and a dimensionless length of γ .

An intensive study of the variation of shaft eigenvalues with I_0^* reveals the correct procedure for analyzing such shafts. The steps to be taken are as follows.

- (1) Establish numerical values for the first five frequencies of each segment with clamped conditions at the intermediate position.
- (2) Arrange these frequencies in order of ascending magnitude.
- (3) Use the first five frequencies in this new ascending order for the lower limits of modes two through six as the first mode lower limit is zero.

These ordered frequencies will be designated by an asterisk thus f_1^* , f_2^* , etc.

Therefore, utilizing Q factor we can write

$$f_{(n)} = Q f_{cF(n)} \quad \text{for } n = 1 \quad (7.49)$$

$$f_{(n)} = f_{(n-1)}^* + Q[f_{cF(n)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

where f_n^* are the lower limit frequencies arranged in the order explained above based on f_{cc} for shaft segment of length μ and f_{cF} for shaft segment of length γ .

It is evident from equation (7.48) that the eigenvalues of this case are completely characterized by the dimensionless parameters I_o^* and μ . Eigenvalues are also related to Q factor as implied in equation (7.49).

Values of parameter I_o^* are presented in Tables 7.15 for various values of μ and Q . The spacings of μ and Q are so selected as to keep the error associated with linear interpolation approximately below 1%.

To obtain expressions for the modal shape, we set the constants A_1 equal to one and using equation (7.46) to solve for B_2 , we obtain:

$$\theta_1(\xi) = \sin \beta \xi$$

and

$$\theta_2(\xi) = \sin \beta \mu (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi)$$

7.14 Shafts Free at Both Outer Ends and With an Intermediate Rotational Mass

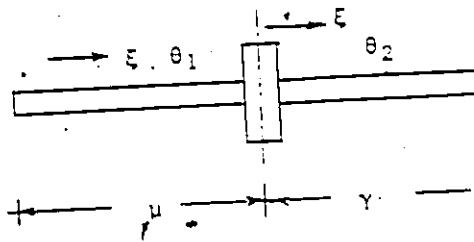


Figure 7.14

Since the free ends of the shaft cannot sustain any torque, the displacement function for the two sections of the shaft can be expressed as:

$$\theta_1(\xi) = B_1 \cos \beta \xi \quad (7.50)$$

$$\theta_2(\xi) = B_2 (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi) \quad (7.51)$$

Condition of continuity of displacement at the intermediate mass results in

$$B_1 (\cos \beta \mu) - B_2 = 0 \quad (7.52)$$

and the condition $\theta'_1(\mu) - \theta'_2(0) = \frac{\delta^2}{I_0^*} \theta_1(\mu)$ involving the dynamic equilibrium of the mass yields,

$$B_1 \left(\frac{\delta}{I_0^*} \cos \delta \mu + \sin \delta \mu \right) + B_2 (\tan \delta \gamma) = 0. \quad (7.53)$$

For a non-trivial solution for the constants in equations (7.52) and (7.53) we obtain:

$$I_0^* = \frac{-\delta \cos \delta \mu}{\sin \delta \mu + \cos \delta \mu \tan \delta \gamma} \quad (7.54)$$

Recognising the associated limits for this case we can write

$$f_{(n)} = f_n^* + Q[f_{FF(n+1)} - f_n^*] \quad (7.55)$$

where the lower limits f_n^* are the limits based on left segment of the shaft of length μ and right segment of length γ with clamped-free boundary conditions and are arranged in the ascending order as explained in previous case.

Values of I_0^* for various values of Q and μ are presented in Tables 7.14. It may be noted that because of the symmetry of the problem the values of μ are given only upto 0.5.

The modal shape expressions are given by

$$\theta_1(\xi) = \cos \beta \xi$$

and

$$\theta_2(\xi) = \cos \beta \mu (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi)$$

7.15 Shafts Clamped at Both Ends and With an Intermediate Rotational

Mass

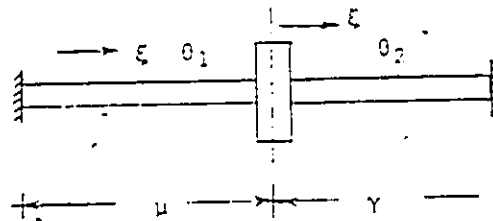


Figure 7.15

Utilizing the boundary conditions at shaft outer ends we

obtain:

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.56)$$

and

$$\theta_2(\xi) = A_2 [\sin \beta \xi - \tan \beta \gamma \cos \beta \xi] \quad (7.57)$$

The two boundary conditions involving continuity of angular displacement and dynamic equilibrium of the mass at the intermediate point result in,

$$A_1 \sin \beta \mu + A_2 \tan \beta \gamma = 0 \quad (7.58)$$

and

$$A_1 \left(\cos \beta \mu - \frac{\beta}{I_o^*} \sin \beta \mu \right) - A_2 = 0 \quad (7.59)$$

Setting the determinant of the coefficient matrix of the constants in the above two equations to zero we obtain

$$I_o^* = \frac{\beta \sin \beta \mu \tan \beta \gamma}{\sin \beta \mu + \tan \beta \gamma \cos \beta \mu} \quad (7.60)$$

Introducing the Q factor as before,

$$f_{(n)} = Q f_{cc(n)} \quad \text{for } n = 1 \quad (7.61)$$

$$f_{(n)} = f_{(n-1)}^* + Q[f_{cc(n)} - f_{(n-1)}^*] \quad \text{for } n > 1 \quad (7.62)$$

where lower limits f_n^* are based on f_{cc} for shaft segment of length μ and f_{cc} for shaft segment of length $\gamma = (1-\mu)$.

Tables 7.15 present the computed values of I_o^* for various values of Q and μ , where, $0.1 \leq \mu \leq 0.5$, because of symmetry of the problem.

The modal shape expressions are derived in their usual manner and are given below for the two sections of the shaft

$$\theta_1(\xi) = \sin \beta \xi$$

and

$$\theta_2(\xi) = - \frac{\sin \beta \mu}{\tan \beta \gamma} (\sin \beta \xi - \tan \beta \gamma \cos \beta \xi)$$

7.16 Shafts Clamped at One Outer End, Free at the Other, and With Intermediate Torsion Spring

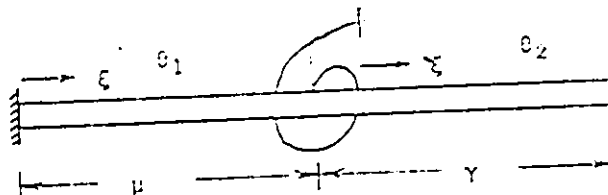


Figure 7.16

Utilizing the boundary conditions $\theta_1(0) = 0$ and $\theta'_2(\gamma) = 0$ prevailing at shaft outer ends, we obtain

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.63)$$

and

$$\theta_2(\xi) = B_2 (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi) \quad (7.64)$$

The two boundary conditions at the intermediate point are given by $\theta_1(\mu) = \theta_2(0)$ and $\theta'_1(\mu) - \theta'_2(0) = -K^* \theta_1(\mu)$, representing the continuity of displacement and equilibrium of shaft torque and the restoring torque of the spring, respectively. Substitution of $\theta_1(\xi)$ and $\theta_2(\xi)$ from equations (7.63) and (7.64) in the above two boundary conditions yields,

$$A_1 (\sin \beta \mu) - B_2 = 0 \quad (7.65)$$

and

$$A_1 \left[\cos \beta \mu + \frac{K^*}{\beta} \sin \beta \mu \right] - B_2 \tan \beta \gamma = 0 \quad (7.66)$$

For a non-trivial solution for the constants in the above two equations we obtain:

$$K^* = \frac{\beta (\sin \beta \mu \tan \beta \gamma - \cos \beta \mu)}{\sin \beta \mu} \quad (7.67)$$

It may be easily recognised that as K^* approaches zero, the lower limits of frequency of this shaft are well defined and correspond to clamped-free boundary conditions. As K^* approaches infinity, the upper limits are not that readily apparent as the intermediate point approaches a clamped boundary condition. Here again, as discussed in case 7.13, it is necessary to establish the first six frequencies of each individual segment with a clamped intermediate condition. These frequencies are then arranged in an ascending order and denoting them by f_1^* , f_2^* , ..., etc, they are utilized for the upper limits for first, second, and third modes etc.

Therefore, in terms of Q factor we can write

$$f_{(n)} = f_{cF(n)} + Q[f_{(n)}^* - f_{cF(n)}] \quad (7.68)$$

where upper limit frequencies f_n^* are based on f_{cc} for shaft segment of length μ and f_{cF} for shaft segment of length γ as explained earlier.

The modal shape expressions are same as provided for case 7.13. The computed values of K^* for various values of μ and Q are presented in Tables 7.16.

7.17 Shafts Free at Each End and With Intermediate Torsion Spring

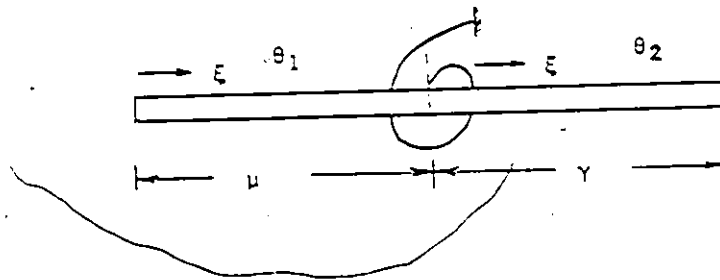


Figure 7.17

Two of the four constants appearing in displacement functions of the two segments of the shaft are eliminated by utilizing the boundary conditions at the shaft outer ends. The resulting expressions for the displacement functions are:

$$\theta_1(\xi) = B_1 \cos \beta \xi \quad (7.69)$$

and

$$\theta_2(\xi) = B_2 (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi) \quad (7.70)$$

Further utilizing the two boundary conditions at the intermediate point as given in case 7.16, we obtain:

$$B_1 \cos S \mu - B_2 = 0 \quad (7.71)$$

and

$$B_1 \left(\frac{K^*}{S} \cos S \mu - \sin S \mu \right) - B_2 (\tan S \gamma) = 0 \quad (7.72)$$

The condition for a non-trivial solution for the constants in equations (7.71) and (7.72) yields:

$$K^* = \frac{S (\sin S \mu + \cos S \mu \tan S \gamma)}{\cos S \mu} \quad (7.73)$$

Recognising the limits, we can write

$$f_{(n)} = Q f_n^* \quad \text{for } n = 1 \quad (7.74)$$

$$f_{(n)} = f_{FF(n)} + Q[f_{(n)}^* - f_{FF(n)}] \quad \text{for } n > 1$$

where upper limits f_n^* are based on the frequencies associated with each of the two segments of shaft with clamped-free boundary conditions and are arranged in ascending order as explained before. Table 7.17 present all the pertinent data for this problem. Values of μ are given for, $0.1 \leq \mu \leq 0.5$, taking advantage of the symmetry. The modal shape expressions are same as provided in case 7.14.

7.18 Shafts Clamped at Each End and With an Intermediate Torsion

Spring

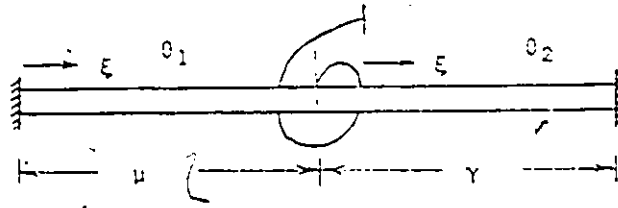


Figure 7.18

The boundary conditions at the shaft outer ends result in

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.75)$$

$$\theta_2(\xi) = A_2 (\sin \beta \xi - \tan \beta \gamma \cos \beta \xi) \quad (7.76)$$

and the remaining two boundary conditions at the intermediate point yield,

$$A_1 \sin \beta \mu + A_2 \tan \beta \gamma = 0 \quad (7.77)$$

$$A_1 \left(\cos \beta \mu + \frac{K^*}{\beta} \sin \beta \mu \right) - A_2 = 0 \quad (7.78)$$

For a non-trivial solution of the two constants in the above two equations, the following frequency equation must be satisfied.

$$K^* = \frac{-\delta (\sin \delta u + \tan \delta \gamma \cos \delta u)}{\sin \delta u \tan \delta \gamma} \quad (7.79)$$

The eigenvalues δ are completely characterized by the system parameters K^* and u . From a synthesis point of view, we can say that the system is completely defined by its eigenvalue δ and parameter u . The relationship between δ and Q is implicitly given by

$$f_{(n)} = f_{cc(n)} + Q[f_{(n)}^* - f_{cc(n)}] \quad (7.80)$$

where f_n^* are based on f_{cc} for shaft segment of length u and f_{cc} for shaft segment of length $\gamma = (1-u)$.

Tables 7.18 present the values of K^* for various values of Q and u for first six modes of vibration. The modal shape expressions for this case are identical to those provided in case 7.15.

7.19 Double Segment Shafts Clamped at One End, Free at the Other, With Segments Joined by a Torsion Spring

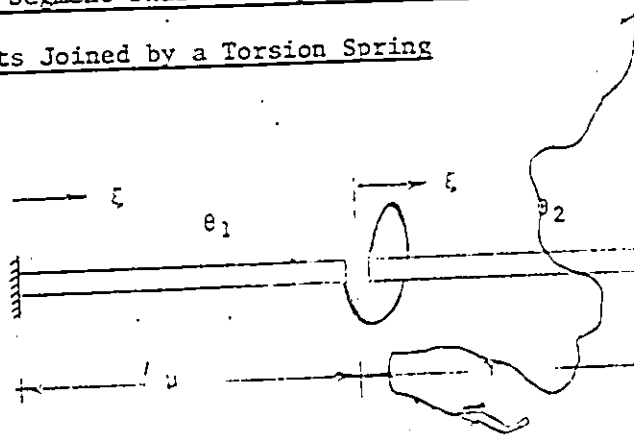


Figure 7.19

The Figure 7.19 shows a shaft of dimensionless length, 1 and having two segments of length μ and $\gamma = (1-\mu)$ where the shaft segments are coupled by a torsion spring. The spring represents a torsion spring coupling where the mass of the associated flange has been neglected.

The boundary conditions at the shaft outer ends yield the following expressions for angular displacements of the two segments

$$\theta_1(\xi) = A_1 \sin \delta \xi \quad (7.81)$$

and

$$\theta_2(\xi) = B_2 (\tan \delta \gamma \sin \delta \xi + \cos \delta \xi) \quad (7.82)$$

The remaining two boundary conditions are supplied by the junction of the two segments. They are

$$\theta'_1(\mu) = -K^* [\theta_1(\mu) - \theta_2(0)]$$

and

$$\theta'_1(\mu) = \theta'_2(0)$$

Substitution of $\theta_1(\xi)$ and $\theta_2(\xi)$ in the above two boundary conditions yields

$$A_1 (S \cos S \mu + K^* \sin S \mu) - B_2 (K^*) = 0 \quad (7.83)$$

and

$$A_1 \cos S \mu + B_2 (-\tan S \gamma) = 0 \quad (7.84)$$

Utilizing the condition for a non-trivial solution for the constants A_1 and B_2 in equations (7.83) and (7.84) we obtain:

$$K^* = \frac{-S \cos S \mu \tan S \gamma}{\sin S \mu \tan S \gamma - \cos S \mu} \quad (7.85)$$

In the limit as K^* approaches infinity, the torsional spring coupling behaves as a rigid coupling and the upper limit frequencies for the various modes of this shaft correspond to those of a clamped-free shaft. The lower limit frequencies are not quite so obvious as K^* approaches zero, the two segments are disconnected and are therefore capable of independent vibrations. For this condition we have a set of easily established clamped-free frequencies to associate with the left segment of dimensionless length μ . For the right hand segment of length γ we associate a set of free free frequencies. Having established both sets of frequencies, these are then arranged in their increasing order of

magnitude. These are denoted by $f_{(n)}^*$ as before. The first six of these twelve frequencies then form a set of lower limits for the first six modes.

For the problem in hand, we therefore have,

$$f_{(n)} = f_n^* + Q [f_{cF(n)} - f_{(n)}^*] \quad (7.86)$$

Results for this problem for various values of μ and Q are presented in Tables 7.19. The modal shape expressions are determined by setting the constant A_1 equal to unity and then solving for B_2 from equation (7.84).

$$\theta_1(\xi) = \sin \beta \xi$$

$$\theta_2(\xi) = \frac{\cos \beta \mu}{\tan \beta \gamma} (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi)$$

7.20 Double Segment Shafts Free at Both Ends with Segments Formed by a Torsion Spring

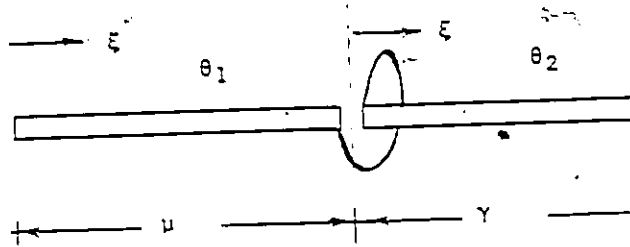


Figure 7.20

The condition of the requirement of zero torque at the free ends yields

$$\theta_1(\xi) = B_1 \cos \beta \xi \quad (7.87)$$

and

$$\theta_2(\xi) = B_2 (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi) \quad (7.88)$$

Utilizing the two boundary conditions at the junction of the two segments as given in case 7.19, we obtain

$$B_1 (-\beta \sin \beta \mu + K^* \cos \beta \mu) - B_2 (K^*) = 0 \quad (7.89)$$

and

$$B_1 (-\sin \beta \mu) - B_2 (\tan \beta \gamma) = 0 \quad (7.90)$$

For a non-trivial solution for the constants B_1 and B_2 in the above two equations, the following eigenvalue equation must be satisfied

$$K^* = \frac{\beta \sin \beta \mu \tan \beta \gamma}{\tan \beta \gamma \cos \beta \mu + \sin \beta \mu} \quad (7.91)$$

Recognising the associated upper and lower limits of frequency, we can write

$$f_{(n)} = Q f_{FF(n+1)} \quad \text{for } n = 1 \quad (7.92)$$

and

$$f_{(n)} = f_{(n-1)}^* + Q[f_{FF(n+1)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

where the lower limit frequencies $f_{(n)}^*$ are based on f_{FF} for shaft segment of length μ and f_{FF} for shaft segment of length $\gamma = (1-\mu)$.

Computed values of K^* for various values of Q and μ are presented in Tables 7.20. Taking advantage of symmetry, the values of μ are given in the range, $0.1 \leq \mu \leq 0.5$, only.

The modal shape expressions are given by

$$\theta_1(\xi) = \cos \beta \xi$$

$$\theta_2(\xi) = -\frac{\sin \beta \mu}{\tan \beta \gamma} (\tan \beta \gamma \sin \beta \xi + \cos \beta \xi)$$

7.21 Double Segment Shafts Clamped at Both Ends With Segments Joined by a Torsion Spring

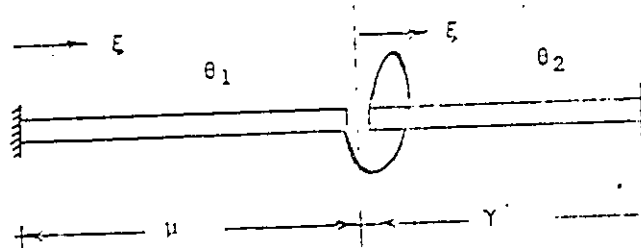


Figure 7.21

The conditions of no angular displacement at the fixed ends results in

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.93)$$

and

$$\theta_2(\xi) = A_2 (\sin \beta \xi - \tan \beta \gamma \cos \beta \xi) \quad (7.94)$$

Utilizing the two boundary conditions at the junction of the two segments as given in case 7.19, we obtain:

$$A_1 (\beta \cos \beta \mu + K^* \sin \beta \mu) + A_2 (K^* \tan \beta \gamma) = 0 \quad (7.95)$$

and

$$A_1 (\cos \beta \mu) - A_2 = 0 \quad (7.96)$$

Equating the determinant of the coefficient matrix of constants A_1 and A_2 in the above two equations to zero we obtain

$$K^* = \frac{-\beta \cos \beta \mu}{(\sin \beta \mu + \cos \beta \mu \tan \beta \gamma)} \quad (7.97)$$

Recognising the limits we can write

$$f_{(n)} = f_{(n)}^* + Q[f_{cc(n)} - f_{(n)}^*] \quad (7.98)$$

where the lower limits $f_{(n)}^*$ are associated with left and right segments of the shaft with lengths μ and γ respectively, and with clamped-free boundary conditions.

Tables 7.21 present all the information necessary to establish the frequencies of torsional vibrations of this shaft. The modal shapes are given by the following expressions

$$\theta_1(\xi) = \sin \beta \xi$$

$$\theta_2(\xi) = \cos \beta \mu (\sin \beta \xi - \tan \beta \gamma \cos \beta \xi)$$

Part 4 - Shafts with intermediately located rotational masses or torsion spring couplings and rotational masses at one end

7.22 Shafts Clamped at One End With Rotational Mass at the Other and With Intermediate Torsion Spring Coupling

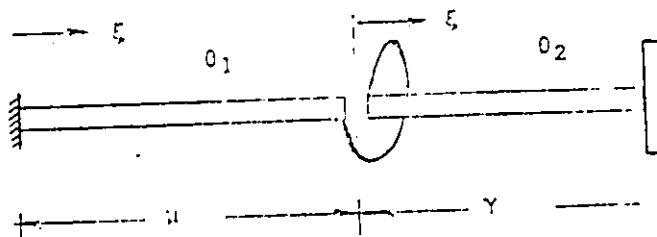


Figure 7.22

Enforcing the boundary condition $\theta_1(0) = 0$ at the clamped end, we obtain:

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.99)$$

The boundary condition to be satisfied at the right hand end is given by $\theta'_2(\gamma) = \frac{\beta^2}{I_o^*} \theta_2(\gamma)$ arising out of dynamic equilibrium of the mass. Substituting this boundary condition in the general expression $\theta_2(\xi)$, the constant B_2 is eliminated and we obtain the following expression for the angular displacement of the second segment of the shaft.

$$\theta_2(\xi) = A_2 \left[\sin \beta \xi + \frac{(I_o^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_o^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right] \quad (7.100)$$

The two other boundary conditions are supplied by the junction of the two segments and are same as given in case 7.19. Utilizing these boundary conditions we obtain:

$$A_1 \cos \beta \mu - A_2 = 0 \quad (7.101)$$

and

$$A_1 (\beta \cos \beta \mu + K^* \sin \beta \mu) - A_2 \left[\frac{K^* (I_o^* \cos \beta \gamma - \beta \sin \beta \gamma)}{I_o^* \sin \beta \gamma + \beta \cos \beta \gamma} \right] = 0 \quad (7.102)$$

Introducing $\alpha = \frac{K^*}{I_o^*}$ and setting the determinant of the coefficient matrix of the above two equations to zero, we obtain the following eigenvalue equation

$$I_o^{*2} + b I_o^* + c = 0 \quad (7.103)$$

where

$$b = \frac{\beta [(1+\alpha) \cos \beta \mu \sin \beta \gamma + \alpha \cos \beta \gamma \sin \beta \mu]}{\alpha (\sin \beta \mu \sin \beta \gamma - \cos \beta \mu \cos \beta \gamma)}$$

and

$$c = \frac{\beta^2 \cos \beta \mu \cos \beta \gamma}{\alpha (\sin \beta \mu \sin \beta \gamma - \cos \beta \mu \cos \beta \gamma)}$$

There are three parameters involved in the quadratic equation (7.103), namely I_o^* , α , and μ . For a fixed value of α , the upper limit of frequencies for any mode of shaft vibrations are evidently those corresponding to clamped-free boundary condition. This is because of the fact that as K^* and I_o^* approach infinity, the torsional coupling behaves as a rigid coupling and mass moment of inertia I_o of the rotational mass approaches zero. As I_o^* and K^* approach zero, the first mode lower limit will be zero governed by the right hand segment. For higher modes the left hand segment of length μ behaves as a clamped-free shaft and right

hand segment of length γ also behaves as a clamped-free shaft and both the segments are now decoupled. Accordingly, the lower limits frequencies, designated by $f_{(n)}^*$ are those frequencies associated with each of the two segments and are arranged in the ascending order as discussed in Part 3 of this Chapter. Therefore, we can write,

$$f_{(n)} = Q f_{cF(n)} \quad \text{for } n = 1 \quad (7.104)$$

and

$$f_{(n)} = f_{(n-1)}^* + Q [f_{cF(n)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

where $f_{(n)}^*$ are the lower limit frequencies as explained above.

Equation (7.103) can be solved for fixed values of α , μ , and β . Value of β are prescribed between the two limiting values utilizing the equations (7.104) in terms of Q factor. The negative root of the quadratic equation (7.103) is disregarded for obvious reasons. The values of I_0^* for various values of Q , α , and μ are presented in Tables 7.22, for first three modes of vibrations. The modal shape expressions are arrived at by setting the constant A_1 equal to unity and then solving equation (7.101) for A_2 . They are

$$\theta_1(\xi) = \sin \beta \xi$$

$$\theta_2(\xi) = \cos \beta \mu \left[\sin \beta \xi + \frac{(I_o^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_o^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right]$$

7.23 Shafts Free at One End, With a Rotational Mass at the Other and
With Intermediate Torsion Spring Coupling

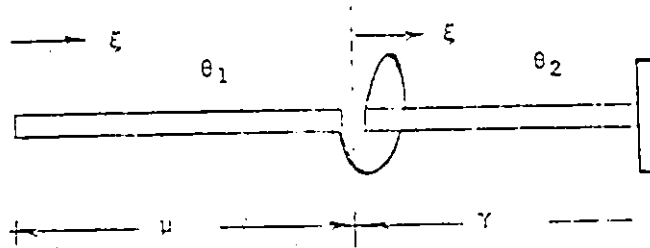


Figure 7.23

Utilizing the boundary condition at the free end, we can write

$$\theta_1(\xi) = B_1 \cos \beta \xi \quad (7.105)$$

The boundary condition at the right hand end is the same as given in case 7.22 and therefore, the displacement function of the second segment is given by:

$$\vartheta_2(\xi) = A_2 \left[\sin \beta \xi + \frac{(I_0^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_0^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right] \quad (7.106)$$

Utilizing the remaining two boundary conditions at the junction of two segments, we obtain:

$$B_1 (-\sin \beta u) - A_2 = 0 \quad (7.107)$$

and

$$B_1 (-\beta \sin \beta u + K^* \cos \beta u) - A_2 \left[\frac{K^* (I_0^* \cos \beta \gamma - \beta \sin \beta \gamma)}{I_0^* \sin \beta \gamma + \beta \cos \beta \gamma} \right] = 0 \quad (7.108)$$

Introducing $\alpha = \frac{K^*}{I_0^*}$ and utilizing the condition for a non-trivial solution for the constants B_1 and A_2 in equations (7.107) and (7.108) we obtain the following frequency equation of the system.

$$I_0^{*2} + b I_0^* + c = 0 \quad (7.109)$$

where

$$b = \frac{\alpha \beta \cos \beta u \cos \beta \gamma - \beta (1+\alpha) \sin \beta u \sin \beta \gamma}{\alpha (\cos \beta u \sin \beta \gamma + \sin \beta u \cos \beta \gamma)}$$

and

$$c = - \frac{\beta^2 \sin \beta \mu \cos \beta \gamma}{\alpha (\cos \beta \mu \sin \beta \gamma + \sin \beta \mu \cos \beta \gamma)}$$

Recognising the two associated limits, we can write

$$f_{(n)} = Q f_{FF(n+1)} \quad \text{for } n = 1 \quad (7.110)$$

$$f_{(n)} = f_{(n-1)}^* + Q[f_{FF(n+1)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

where lower limits $f_{(n)}^*$ are associated with left segment of length μ and free-free boundary conditions and with right segment of length γ with clamped-free boundary conditions.

Utilizing equations (7.109) and (7.110), the values of system parameter I_o^* can be computed for various values of μ , α , and Q and they are presented in Tables 7.23.

The modal shape expressions are provided below for this case

$$\theta_1(\xi) = \cos \beta \xi$$

$$\theta_2(\xi) = - \sin \beta \mu \left[\sin \beta \xi + \frac{(I_o^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_o^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right]$$

7.24 Shafts Clamped at One End, With Rotational Mass at the Other
and With an Intermediate Rotational Mass

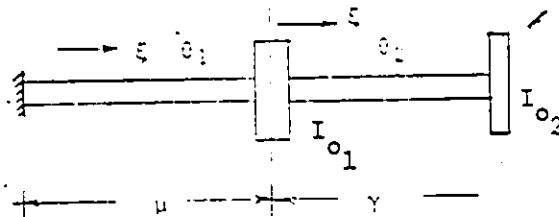


Figure 7.24

By utilizing the boundary conditions at the shaft outer ends, we can write

$$\theta_1(\xi) = A_1 \sin \beta \xi \quad (7.111)$$

and

$$\theta_2(\xi) = A_2 \left[\sin \beta \xi + \frac{I_{o2}^* \cos \beta y - \beta \sin \beta y}{I_{o2}^* \sin \beta y + \beta \cos \beta y} \cos \beta \xi \right] \quad (7.112)$$

The remaining two boundary conditions at the location of the intermediate mass are $\theta_1(u) = \theta_2(0)$ and $\theta'_1(u) = \theta'_2(0) - \frac{\beta^2}{I_{o1}} \theta_1(u)$.

Substituting for $\theta_1(\xi)$ and $\theta_2(\xi)$ from equations (7.111) and (7.112) into the above two boundary conditions we obtain:

$$A_1 (\sin \beta u) - A_2 \left(\frac{I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma}{I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma} \right) = 0 \quad (7.113)$$

and

$$A_1 \left(\cos \beta u - \frac{\beta}{I_{o1}^*} \sin \beta u \right) - A_2 = 0 \quad (7.114)$$

Introducing $\alpha = \frac{I_{o1}^*}{I_{o2}^*}$ and setting the determinant of coefficient matrix

of the two constants in the above two equations to zero, we obtain:

$$I_{o2}^{*2} + b I_{o2}^* + c = 0 \quad (7.115)$$

where

$$b = \frac{\beta (1+\alpha) \sin \beta u \cos \beta \gamma + \alpha \beta \cos \beta u \sin \beta \gamma}{\alpha (\sin \beta u \sin \beta \gamma - \cos \beta u \cos \beta \gamma)}$$

and

$$c = - \frac{\beta^2 \sin \beta u \sin \beta \gamma}{\alpha (\sin \beta u \sin \beta \gamma - \cos \beta u \cos \beta \gamma)}$$

For fixed value of α , as the system parameter $I_{o_2}^*$ approaches infinity (or I_{o_2} approaches zero), the upper limits of frequency are well defined and correspond to clamped free boundary conditions. But as $I_{o_2}^*$ approaches zero, the points where masses are located become nodal points and therefore the lower limits have to be arranged in an ascending order as described earlier from two sets of frequencies associated with each of the two segments with clamped-clamped boundary conditions. We can therefore write

$$f_{(n)} = Q f_{cF(n)} \quad \text{for } n = 1 \quad (7.116)$$

$$f_{(n)} = f_{(n-1)}^* + Q[f_{cF(n)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

For first mode the lower limit is obviously zero. The computed values of $I_{o_2}^*$ for various values of Q , $0.05 \leq Q \leq 0.9$, and μ , $0.1 \leq \mu \leq 0.9$, and α , $0.1 \leq \alpha \leq 1.0$ and $0.1 \leq \frac{1}{\alpha} \leq 0.9$, are presented in Tables 7.24.

The modal shape expressions are given by:

$$\theta_1(\xi) = \sin \beta \xi$$

$$\theta_2(\xi) = \left(\cos \beta \mu - \frac{\beta}{I_{o_1}^*} \sin \beta \mu \right) \left[\sin \beta \xi + \frac{(I_{o_2}^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_{o_2}^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right]$$

7.25 Shafts Free at One End, With a Rotational Mass at the Other, and
With an Intermediately Located Rotational Mass

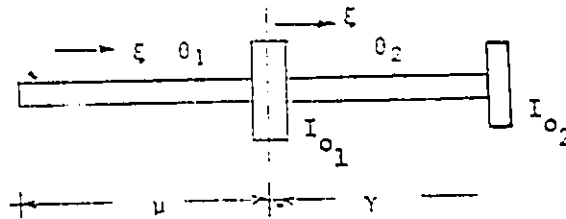


Figure 7.25

Utilizing the boundary conditions at the shaft outer end,
the deflection functions for the two sections of the shaft can be written
as

$$\theta_1(\xi) = B_1 \cos \beta \xi \quad (7.117)$$

and

$$\theta_2(\xi) = A_2 \left[\sin \beta \xi + \frac{(I_{o_2}^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_{o_2}^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right] \quad (7.118)$$

Enforcing the boundary conditions involving continuity of angular displacement and dynamic equilibrium of the mass at the intermediate point, we obtain

$$B_1 \cos \beta \mu - A_2 \left(\frac{I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma}{I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma} \right) = 0 \quad (7.119)$$

and

$$B_1 \left(\frac{\beta}{I_{o1}^*} \cos \beta \mu + \sin \beta \mu \right) + A_2 = 0 \quad (7.120)$$

Introducing $\alpha = \frac{I_{o1}^*}{I_{o2}^*}$ and setting the determinant of the

coefficient matrix of the constants B_1 and A_2 in the equations (7.119) and (7.120) to zero, we obtain the following frequency equation of this case.

$$I_{o2}^{*2} + b I_{o2}^* + c = 0 \quad (7.121)$$

where

$$b = \frac{\beta [(1+\alpha) \cos \beta \mu \cos \beta \gamma - \alpha \sin \beta \mu \sin \beta \gamma]}{\alpha (\cos \beta \mu \sin \beta \gamma + \sin \beta \mu \cos \beta \gamma)}$$

and

$$c = - \frac{\beta^2 \cos \beta \mu \sin \beta \gamma}{\alpha (\cos \beta \mu \sin \beta \gamma + \sin \beta \mu \cos \beta \gamma)}$$

Recognising the lower and upper limits of frequency associated with this case, we can write

$$f_{(n)} = Q f_{FF(n+1)} \quad \text{for } n = 1 \quad (7.122)$$

and

$$f_{(n)} = f_{(n-1)}^* + Q [f_{FF(n+1)} - f_{(n-1)}^*] \quad \text{for } n > 1$$

where the lower limit frequencies $f_{(n)}^*$ are arranged in an ascending order from a total of four frequencies associated with the two segments of the shaft with appropriate boundary conditions and respective lengths.

Tables 7.25 furnish all the information necessary to establish frequencies of this case. The modal shape expressions are determined in the usual manner and are given below

$$\theta_1(\xi) = \cos \beta \xi$$

$$\theta_2(\xi) = - \left(\frac{\beta}{I_{o1}} \cos \beta \mu + \sin \beta \mu \right) \left[\sin \beta \xi + \frac{(I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right]$$

7.26 Shafts With Rotational Mass at Each Outer End and With an Intermediately Located Torsion Spring Coupling

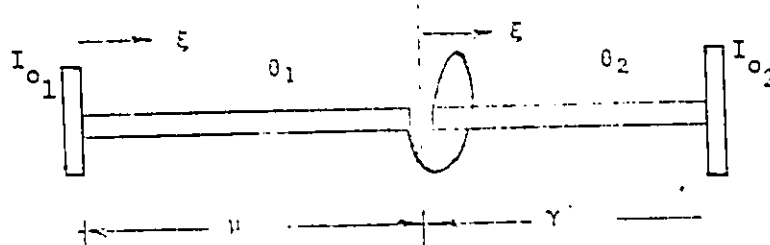


Figure 7.26

This kind of problem is very likely to be encountered in practical engineering field. An electric motor driving a centrifuge pump is a typical example of this case. The torsional frequencies of the motor-pump assembly can be altered by introducing a torsion coupling at an intermediate position. Such a system is illustrated in Figure 7.26. The effects of the mass of coupling are considered negligible compared to those of the shaft and the rotary masses at the ends of the shaft.

$$\text{Enforcing the boundary conditions } \theta'_1(0) = -\frac{\beta^2}{I_{o1}} \theta(0)$$

$$\text{and } \theta'_2(\gamma) = \frac{\beta^2}{I_{o2}} \theta_2(\gamma) \text{ prevailing at the left and right outer ends}$$

of the shaft, we obtain:

$$\theta_1(\xi) = A_1 \left(\sin \beta \xi + \frac{I_{o1}^*}{\beta} \cos \beta \xi \right) \quad (7.123)$$

and

$$\theta_2(\xi) = A_2 \left[\sin \beta \xi + \frac{(I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma)}{(I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma)} \cos \beta \xi \right] \quad (7.124)$$

The two boundary conditions to be satisfied at the coupling are; as before, $\theta'_1(\mu) = \theta'_2(0)$ and $\theta'_1(\mu) = -K^* [\theta_1(\mu) - \theta_2(0)]$, arising out of torque continuity across the spring coupling and equilibrium between shaft torque and spring restoring torque respectively. Utilizing these two boundary conditions and equations (7.123) and (7.124) we obtain

$$A_1 \left(\cos \beta \mu + \frac{I_{o1}^*}{\beta} \sin \beta \mu \right) + A_2 (-1) = 0 \quad (7.125)$$

and

$$A_1 \left[\beta - \left(\frac{K^* I_{o1}^*}{\beta} \right) \cos \beta \mu + (I_{o1}^* + K^*) \sin \beta \mu \right] - A_2 K^* \left[\frac{I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma}{I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma} \right] = 0 \quad (7.126)$$

Introducing $\alpha_1 = \frac{I_{o2}^*}{I_{o1}^*}$ and $\alpha_2 = \frac{K^*}{I_{o1}^*}$ and utilizing

the condition for a non-trivial solution for the constants A_1 and A_2 in equations (7.125) and (7.126) we obtain the following cubic equation in terms of I_{o1}^* which represents the characteristic equation of the system.

$$I_{o1}^{3*} + b I_{o1}^{*2} + c I_{o1}^* + d = 0 \quad (7.127)$$

where

$$b = \frac{\beta [\alpha_2(1+\alpha_1) \cos \beta \gamma \cos \beta \mu - \{\alpha_1 \alpha_2 + \alpha_2 + (\alpha_1)\} \sin \beta \gamma \sin \beta \mu]}{\alpha_1 \alpha_2 (\cos \beta \mu \sin \beta \gamma + \cos \beta \gamma \sin \beta \mu)}$$

$$c = - \frac{\beta^2 [(\alpha_1 + \alpha_2) \sin \beta \gamma \cos \beta \mu + (\alpha_2 + 1) \sin \beta \mu \cos \beta \gamma]}{\alpha_1 \alpha_2 (\cos \beta \mu \sin \beta \gamma + \sin \beta \mu \cos \beta \gamma)}$$

and

$$d = - \frac{\beta^3 \cos \beta \gamma \cos \beta \mu}{\alpha_1 \alpha_2 (\cos \beta \mu \sin \beta \gamma + \sin \beta \mu \cos \beta \gamma)}$$

Equation (7.127) being a cubic equation in I_{o1}^* yields three roots for fixed values of α_1 , α_2 , and μ but only one of these roots

as would rather be expected, was found to be positive. It will be observed that there are four variables involved in this problem, namely $I_{o_1}^*$; $I_{o_2}^*$, K^* , and μ . The table storage space required would become excessive if a format similar to that employed for previous cases is followed. It is for this reason that data for first mode of vibration only are provided and values of μ are given in intervals of 0.2 between 0.1 and 0.9.

It is recognised that in general, the first mode of vibration is the one of greatest interest to designers. In special cases where an interest in higher modes exist, equation (7.127) may be used to solve for the appropriate eigenvalues. Since we restrict ourselves here to first mode vibration only, there can be only one nodal point along the shaft and it is reasonable to expect that greater spacing in the values of μ would be permissible. A study of the results did indicate this to be so. To further minimize space requirements only ten values of the Q factor are given in each table. Linear interpolation may be used for values of $Q < 0.8$ and for values of $Q > 0.8$, the asymptotic approximation (7.5) may be used as discussed earlier in this Chapter.

A study of the results has indicated that error associated with linear interpolation will in general be not greater than 1%, except for interpolation between adjacent values of $Q > 0.2$. In this latter

situation error upto about 3% may be encountered. This small error could be reduced considerably if desired, by plotting adjacent points of the Q curve and "fairing in" the curve which would be an improvement over the linear interpolation.

We next look at the lower and upper limits of frequency for this shaft system as I_{o1}^* ranges from zero to infinity for fixed values of α_1 and α_2 . As I_{o1}^* approaches zero, I_{o1} and I_{o2} approach infinity and K, the torsion spring constant approaches zero. The lower limits frequency is then evidently zero with the rotational masses rotating in opposite directions.. As I_{o1}^* approaches infinity, I_{o1} and I_{o2} approach zero and K approaches infinity. The upper limit therefore obviously corresponds to that associated with free free boundary conditions. For any such shaft system, the first mode frequency can therefore be written as:

$$f_1 = Q f_{FF(2)}$$

and the eigenvalue is given by

$$\delta = Q \pi \quad (7.128)$$

Utilizing equations (7.128) and (7.127), the computed values of I_{o1}^* for fixed values of α_1 , α_2 , and Q are presented in Tables

7.26. It may be observed that because of symmetry, tables for values of $\alpha_1 > 1.0$ are not required.

The modal shape expressions are arrived at by setting the constant A_1 equal to unity in equation (7.123) and then solving equation (7.125) for A_2 .

$$\theta_1(\xi) = \sin \beta \xi - \frac{I_{o1}^*}{\beta} \cos \beta \xi$$

$$\theta_2(\xi) = \left(\cos \beta \eta + \frac{I_{o1}^*}{\beta} \sin \beta \eta \right) \left[\sin \beta \xi + \frac{I_{o2}^* \cos \beta \gamma - \beta \sin \beta \gamma}{I_{o2}^* \sin \beta \gamma + \beta \cos \beta \gamma} \cos \beta \xi \right]$$

Illustrative Example 7.1

A generator is driven off the main engine through a $\frac{1}{2}$ " steel shaft 20" long. The moment of inertia of the generator rotor is 0.1 lb.in.sec.²

- (a) Determine the fundamental natural frequency of torsional vibration if the moment of inertia of the engine is so large compared to that of generator rotor that the engine end of the shaft may be assumed to be fixed (free of torsional vibrations).

This corresponds to case 7.4

$$\begin{aligned} \text{Polar moment of inertia of the shaft } I_p &= \frac{\pi d^4}{32} = \frac{\pi \times (\frac{1}{2})^4}{32 \times} \\ &= 0.00613 \text{ in}^4 \end{aligned}$$

$$\therefore I_o^* = \frac{I_p \rho L}{I_o} = \frac{0.00613 \times 0.28 \times 20}{0.1} = 0.344$$

Utilizing Table 7.1, we obtain:

$$\begin{aligned} f_{cF(1)} &= \frac{\beta_{cF(1)}}{2\pi L} \sqrt{\frac{G}{\rho}} = \frac{1}{4L} \sqrt{\frac{12 \times 10^6 \times 386}{0.28}} \\ &= 1607.7 \text{ Hz} \end{aligned}$$

From Table 7.4, obtain $Q \approx 0.353$

Hence,

$$\begin{aligned} f_{(1)} &= Q f_{cF(1)} = 0.353 \times 1607.7 \\ &= 567.5 \text{ Hz} \end{aligned}$$

- (b) Determine the second mode frequency

Utilizing Table 7.1, obtain,

$$f_{cc(1)} = 3215.4 \text{ Hz}$$

$$f_{cF(2)} = 4823.2 \text{ Hz}$$

Obtain Q from Table 7.4 as $Q = 0.067$

$$\begin{aligned} f_2 &= f_{cc(1)} + Q [f_{cF(2)} - f_{cc(1)}] \\ &= 3215.4 + 0.067 (4823.2 - 3215.4) \\ &= \underline{3323} \text{ Hz} \end{aligned}$$

- (c) The modal shape for first mode is

$$\theta(\xi) = \sin \beta \xi = \sin \frac{0.353 \pi}{2} \xi = \sin 0.55 \xi$$

- (d) Determine the fundamental frequency if the moment of inertia of the engine is 5 times that of generator.

This corresponds to case 7.6.

Here $I_{o_1}^* = 0.344 \times 5 = 1.720$

$I_{o_2}^* = 0.344$

$\alpha = \frac{I_{o_2}^*}{I_{o_1}^*} = \frac{1}{5} = 0.2$

$f_{FF(2)} = \frac{s_{FF(2)}}{2\pi L} \sqrt{\frac{G}{\rho}} = \frac{\pi}{2\pi L} \sqrt{\frac{G}{\rho}} = 3215.4 \text{ Hz}$

Obtain Q from Table 7.6 for $\alpha = 0.2$

$Q = 0.389$

$f_{(1)} = 0.389 \times 3215.4 = 1250.8 \text{ Hz}$

- (e) If the engine and generator are connected through a shaft coupling of stiffness 1000 in.lb/degree, located midway through the shaft, determine the new fundamental frequency of torsional vibrations.

This corresponds to case 7.26

now,

$\alpha_1 = 0.2$ [same as in (d)]

$$K^* = \frac{KL}{G I_p} = \frac{1000 \times 20}{12 \times 10^6 \times 0.00613} = .272$$

$$c_2 = \frac{K^*}{I_{o_1}} = \frac{0.272}{1.72} = 0.158$$

Obtain the value of Q for $c_1 = 0.2$, $c_2 = 0.158$, and $\mu = 0.5$ from Table 7.26 as $Q = 0.151$

$$f = Q f_f (2)$$

$$= 0.151 \times 3215.4 = \underline{485.5 \text{ Hz}}$$

It may be noted that introduction of shaft coupling has reduced the frequency by 61%.

	1920	1921	1922	1923	1924	1925	1926	1927	1928	1929	1930	1931	1932	1933	1934	1935	1936	1937	1938	1939	1940	1941	1942	1943	1944	1945	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053	2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067	2068	2069	2070	2071	2072	2073	2074	2075	2076	2077	2078	2079	2080	2081	2082	2083	2084	2085	2086	2087	2088	2089	2090	2091	2092	2093	2094	2095	2096	2097	2098	2099
1920	1921	1922	1923	1924	1925	1926	1927	1928	1929	1930	1931	1932	1933	1934	1935	1936	1937	1938	1939	1940	1941	1942	1943	1944	1945	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053	2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067	2068	2069	2070	2071	2072	2073	2074	2075	2076	2077	2078	2079	2080	2081	2082	2083	2084	2085	2086	2087	2088	2089	2090	2091	2092	2093	2094	2095	2096	2097	2098	2099	

[illegible]

TABLE 7.11 : χ^2 (a = .05, b = .05)		TABLE 7.11 : χ^2 (a = .05, b = .05)	
df	0.5	0.5	0.5
1	0.000	0.000	0.000
2	0.000	0.000	0.000
3	0.000	0.000	0.000
4	0.000	0.000	0.000
5	0.000	0.000	0.000
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000
11	0.000	0.000	0.000
12	0.000	0.000	0.000
13	0.000	0.000	0.000
14	0.000	0.000	0.000
15	0.000	0.000	0.000
16	0.000	0.000	0.000
17	0.000	0.000	0.000
18	0.000	0.000	0.000
19	0.000	0.000	0.000
20	0.000	0.000	0.000
21	0.000	0.000	0.000
22	0.000	0.000	0.000
23	0.000	0.000	0.000
24	0.000	0.000	0.000
25	0.000	0.000	0.000
26	0.000	0.000	0.000
27	0.000	0.000	0.000
28	0.000	0.000	0.000
29	0.000	0.000	0.000
30	0.000	0.000	0.000
31	0.000	0.000	0.000
32	0.000	0.000	0.000
33	0.000	0.000	0.000
34	0.000	0.000	0.000
35	0.000	0.000	0.000
36	0.000	0.000	0.000
37	0.000	0.000	0.000
38	0.000	0.000	0.000
39	0.000	0.000	0.000
40	0.000	0.000	0.000
41	0.000	0.000	0.000
42	0.000	0.000	0.000
43	0.000	0.000	0.000
44	0.000	0.000	0.000
45	0.000	0.000	0.000
46	0.000	0.000	0.000
47	0.000	0.000	0.000
48	0.000	0.000	0.000
49	0.000	0.000	0.000
50	0.000	0.000	0.000
51	0.000	0.000	0.000
52	0.000	0.000	0.000
53	0.000	0.000	0.000
54	0.000	0.000	0.000
55	0.000	0.000	0.000
56	0.000	0.000	0.000
57	0.000	0.000	0.000
58	0.000	0.000	0.000
59	0.000	0.000	0.000
60	0.000	0.000	0.000
61	0.000	0.000	0.000
62	0.000	0.000	0.000
63	0.000	0.000	0.000
64	0.000	0.000	0.000
65	0.000	0.000	0.000
66	0.000	0.000	0.000
67	0.000	0.000	0.000
68	0.000	0.000	0.000
69	0.000	0.000	0.000
70	0.000	0.000	0.000
71	0.000	0.000	0.000
72	0.000	0.000	0.000
73	0.000	0.000	0.000
74	0.000	0.000	0.000
75	0.000	0.000	0.000
76	0.000	0.000	0.000
77	0.000	0.000	0.000
78	0.000	0.000	0.000
79	0.000	0.000	0.000
80	0.000	0.000	0.000
81	0.000	0.000	0.000
82	0.000	0.000	0.000
83	0.000	0.000	0.000
84	0.000	0.000	0.000
85	0.000	0.000	0.000
86	0.000	0.000	0.000
87	0.000	0.000	0.000
88	0.000	0.000	0.000
89	0.000	0.000	0.000
90	0.000	0.000	0.000
91	0.000	0.000	0.000
92	0.000	0.000	0.000
93	0.000	0.000	0.000
94	0.000	0.000	0.000
95	0.000	0.000	0.000
96	0.000	0.000	0.000
97	0.000	0.000	0.000
98	0.000	0.000	0.000
99	0.000	0.000	0.000
100	0.000	0.000	0.000

[illegible][illegible]

[illegible]

TABLE 2.11 : χ^2 (1/2) - (1/2) - (1/2)

χ^2	0.15	0.10	0.05	0.025	0.01	0.005	0.001
0.15	0.15	0.10	0.05	0.025	0.01	0.005	0.001
0.20	0.20	0.15	0.075	0.0375	0.015	0.0075	0.0015
0.25	0.25	0.20	0.10	0.05	0.02	0.01	0.002
0.30	0.30	0.25	0.125	0.0625	0.025	0.0125	0.0025
0.35	0.35	0.30	0.15	0.075	0.03	0.015	0.003
0.40	0.40	0.35	0.175	0.0875	0.035	0.0175	0.0035
0.45	0.45	0.40	0.20	0.1	0.04	0.02	0.004
0.50	0.50	0.45	0.225	0.1125	0.045	0.0225	0.0045
0.55	0.55	0.50	0.25	0.125	0.05	0.025	0.005
0.60	0.60	0.55	0.275	0.1375	0.055	0.0275	0.0055
0.65	0.65	0.60	0.30	0.15	0.06	0.03	0.006
0.70	0.70	0.65	0.325	0.1625	0.065	0.0325	0.0065
0.75	0.75	0.70	0.35	0.175	0.07	0.035	0.007
0.80	0.80	0.75	0.375	0.1875	0.075	0.0375	0.0075
0.85	0.85	0.80	0.40	0.2	0.08	0.04	0.008
0.90	0.90	0.85	0.425	0.2125	0.085	0.0425	0.0085
0.95	0.95	0.90	0.45	0.225	0.09	0.045	0.009
1.00	1.00	0.95	0.475	0.2375	0.095	0.0475	0.0095
1.05	1.05	1.00	0.50	0.25	0.10	0.05	0.01
1.10	1.10	1.05	0.525	0.2625	0.105	0.0525	0.0105
1.15	1.15	1.10	0.55	0.275	0.11	0.055	0.011
1.20	1.20	1.15	0.575	0.2875	0.115	0.0575	0.0115
1.25	1.25	1.20	0.60	0.3	0.12	0.06	0.012
1.30	1.30	1.25	0.625	0.3125	0.125	0.0625	0.0125
1.35	1.35	1.30	0.65	0.325	0.13	0.065	0.013
1.40	1.40	1.35	0.675	0.3375	0.135	0.0675	0.0135
1.45	1.45	1.40	0.70	0.35	0.14	0.07	0.014
1.50	1.50	1.45	0.725	0.3625	0.145	0.0725	0.0145
1.55	1.55	1.50	0.75	0.375	0.15	0.075	0.015
1.60	1.60	1.55	0.775	0.3875	0.155	0.0775	0.0155
1.65	1.65	1.60	0.80	0.4	0.16	0.08	0.016
1.70	1.70	1.65	0.825	0.4125	0.165	0.0825	0.0165
1.75	1.75	1.70	0.85	0.425	0.17	0.085	0.017
1.80	1.80	1.75	0.875	0.4375	0.175	0.0875	0.0175
1.85	1.85	1.80	0.90	0.45	0.18	0.09	0.018
1.90	1.90	1.85	0.925	0.4625	0.185	0.0925	0.0185
1.95	1.95	1.90	0.95	0.475	0.19	0.095	0.019
2.00	2.00	1.95	0.975	0.4875	0.195	0.0975	0.0195
2.05	2.05	2.00	1.00	0.5	0.2	0.1	0.02
2.10	2.10	2.05	1.025	0.5125	0.205	0.1025	0.0205
2.15	2.15	2.10	1.05	0.525	0.21	0.105	0.021
2.20	2.20	2.15	1.075	0.5375	0.215	0.1075	0.0215
2.25	2.25	2.20	1.10	0.55	0.22	0.11	0.022
2.30	2.30	2.25	1.125	0.5625	0.225	0.1125	0.0225
2.35	2.35	2.30	1.15	0.575	0.23	0.115	0.023
2.40	2.40	2.35	1.175	0.5875	0.235	0.1175	0.0235
2.45	2.45	2.40	1.20	0.6	0.24	0.12	0.024
2.50	2.50	2.45	1.225	0.6125	0.245	0.1225	0.0245
2.55	2.55	2.50	1.25	0.625	0.25	0.125	0.025
2.60	2.60	2.55	1.275	0.6375	0.255	0.1275	0.0255
2.65	2.65	2.60	1.30	0.65	0.26	0.13	0.026
2.70	2.70	2.65	1.325	0.6625	0.265	0.1325	0.0265
2.75	2.75	2.70	1.35	0.675	0.27	0.135	0.027
2.80	2.80	2.75	1.375	0.6875	0.275	0.1375	0.0275
2.85	2.85	2.80	1.40	0.7	0.28	0.14	0.028
2.90	2.90	2.85	1.425	0.7125	0.285	0.1425	0.0285
2.95	2.95	2.90	1.45	0.725	0.29	0.145	0.029
3.00	3.00	2.95	1.475	0.7375	0.295	0.1475	0.0295
3.05	3.05	3.00	1.50	0.75	0.3	0.15	0.03
3.10	3.10	3.05	1.525	0.7625	0.305	0.1525	0.0305
3.15	3.15	3.10	1.55	0.775	0.31	0.155	0.031
3.20	3.20	3.15	1.575	0.7875	0.315	0.1575	0.0315
3.25	3.25	3.20	1.60	0.8	0.32	0.16	0.032
3.30	3.30	3.25	1.625	0.8125	0.325	0.1625	0.0325
3.35	3.35	3.30	1.65	0.825	0.33	0.165	0.033
3.40	3.40	3.35	1.675	0.8375	0.335	0.1675	0.0335
3.45	3.45	3.40	1.70	0.85	0.34	0.17	0.034
3.50	3.50	3.45	1.725	0.8625	0.345	0.1725	0.0345
3.55	3.55	3.50	1.75	0.875	0.35	0.175	0.035
3.60	3.60	3.55	1.775	0.8875	0.355	0.1775	0.0355
3.65	3.65	3.60	1.80	0.9	0.36	0.18	0.036
3.70	3.70	3.65	1.825	0.9125	0.365	0.1825	0.0365
3.75	3.75	3.70	1.85	0.925	0.37	0.185	0.037
3.80	3.80	3.75	1.875	0.9375	0.375	0.1875	0.0375
3.85	3.85	3.80	1.90	0.95	0.38	0.19	0.038
3.90	3.90	3.85	1.925	0.9625	0.385	0.1925	0.0385
3.95	3.95	3.90	1.95	0.975	0.39	0.195	0.039
4.00	4.00	3.95	1.975	0.9875	0.395	0.1975	0.0395
4.05	4.05	4.00	2.00	1.0	0.4	0.2	0.04
4.10	4.10	4.05	2.025	1.0125	0.405	0.2025	0.0405
4.15	4.15	4.10	2.05	1.025	0.41	0.205	0.041
4.20	4.20	4.15	2.075	1.0375	0.415	0.2075	0.0415
4.25	4.25	4.20	2.10	1.05	0.42	0.21	0.042
4.30	4.30	4.25	2.125	1.0625	0.425	0.2125	0.0425
4.35	4.35	4.30	2.15	1.075	0.43	0.215	0.043
4.40	4.40	4.35	2.175	1.0875	0.435	0.2175	0.0435
4.45	4.45	4.40	2.20	1.1	0.44	0.22	0.044
4.50	4.50	4.45	2.225	1.1125	0.445	0.2225	0.0445
4.55	4.55	4.50	2.25	1.125	0.45	0.225	0.045
4.60	4.60	4.55	2.275	1.1375	0.455	0.2275	0.0455
4.65	4.65	4.60	2.30	1.15	0.46	0.23	0.046
4.70	4.70	4.65	2.325	1.1625	0.465	0.2325	0.0465
4.75	4.75	4.70	2.35	1.175	0.47	0.235	0.047
4.80	4.80	4.75	2.375	1.1875	0.475	0.2375	0.0475
4.85	4.85	4.80	2.40	1.2	0.48	0.24	0.048
4.90	4.90	4.85	2.425	1.2125	0.485	0.2425	0.0485
4.95	4.95	4.90	2.45	1.225	0.49	0.245	0.049
5.00	5.00	4.95	2.475	1.2375	0.495	0.2475	0.0495
5.05	5.05	5.00	2.50	1.25	0.5	0.25	0.05
5.10	5.10	5.05	2.525	1.2625	0.505	0.2525	0.0505
5.15	5.15	5.10	2.55	1.275	0.51	0.255	0.051
5.20	5.20	5.15	2.575	1.2875	0.515	0.2575	0.0515
5.25	5.25	5.20	2.60	1.3	0.52	0.26	0.052
5.30	5.30	5.25	2.625	1.3125	0.525	0.2625	0.0525
5.35	5.35	5.30	2.65	1.325	0.53	0.265	0.053
5.40	5.40	5.35	2.675	1.3375	0.535	0.2675	0.0535
5.45	5.45	5.40	2.70	1.35	0.54	0.27	0.054
5.50	5.50	5.45	2.725	1.3625	0.545	0.2725	0.0545
5.55	5.55	5.50	2.75	1.375	0.55	0.275	0.055
5.60	5.60	5.55	2.775	1.3875	0.555	0.2775	0.0555
5.65	5.65	5.60	2.80	1.4	0.56	0.28	0.056
5.70	5.70	5.65	2.825	1.4125	0.565	0.2825	0.0565
5.75	5.75	5.70	2.85	1.425	0.57	0.285	0.057
5.80	5.80	5.75	2.875	1.4375	0.575	0.2875	0.0575
5.85	5.85	5.80	2.90	1.45	0.58	0.29	0.058
5.90	5.90	5.85	2.925	1.4625	0.585	0.2925	0.0585
5.95	5.95	5.90	2.95	1.475	0.59	0.295	0.059
6.00	6.00	5.95	2.975	1.4875	0.595	0.2975	0.0595
6.05	6.05	6.00	3.00	1.5	0.6	0.3	0.06
6.10	6.10	6.05	3.025	1.5125	0.605	0.3025	0.0605
6.15	6.15	6.10	3.05	1.525	0.61	0.305	0.061
6.20	6.20	6.15	3.075	1.5375	0.615	0.3075	0.0615
6.25	6.25	6.20	3.10	1.55	0.62	0.31	0.062
6.30	6.30	6.25	3.125	1.5625	0.625	0.3125	0.0625
6.35	6.35	6.30	3.15	1.575	0.63	0.315	0.063
6.40	6.40	6.35	3.175	1.5875	0.635	0.3175	0.0635
6.45	6.45	6.40	3.20	1.6	0.64	0.32	0.064
6.50	6.50	6.45	3.225	1.6125	0.645	0.3225	0.0645
6.55	6.55	6.50	3.25	1.625	0.65	0.325	0.065
6.60	6.60	6.55	3.275	1.6375	0.655	0.3275	0.0655
6.65	6.65	6.60	3.30	1.65	0.66	0.33	0.066
6.70	6.70	6.65	3.325	1.6625	0.665	0.3325	0.0665
6.75	6.75	6.70	3.35	1.675	0.67	0.335	0.067
6.80	6.80	6.75	3.375	1.6875	0.675	0.3375	0.0675
6.85	6.85	6.80	3.40	1.7	0.68	0.34	0.068
6.90	6.90	6.85	3.425	1.7125	0.685	0.3425	0.0685
6.95	6.95	6.90	3.45	1.725	0.69	0.345	0.069
7.00	7.00	6.95	3.475	1.7375	0.695	0.3475	0.0695
7.05	7.05	7.00	3.50	1.75	0.7	0.35	0.07
7.10	7.10	7.05	3.525	1.7625	0.705	0.3525	0.0705
7.15	7.15	7.10	3.55	1.775	0.71	0.355	0.071
7.20	7.20	7.15	3.575	1.7875	0.715	0.3575	0.0715
7.25	7.25	7.20	3.60	1.8	0.72	0.36	0.072
7.30	7.30	7.25	3.625	1.8125	0.725	0.3625	0.0725
7.35	7.35	7.30	3.65	1.825	0.73	0.365	0.073
7.40	7.40	7.35	3.675	1.8375	0.735	0.3675	0.0735
7.45	7.45	7.40	3.70	1.85	0.74	0.37	0.074
7.50	7.50	7.45	3.725	1.8625	0.745	0.3725	0.0745
7.55	7.55	7.50	3.75	1.875	0.75	0.375	0.075
7.60	7.60	7.55	3.775	1.8875	0.755	0.3775	0.0755
7.65	7.65	7.60	3.80	1.9	0.76		

[illegible][illegible]

[illegible]

$$\log_e \Gamma(10) - \frac{1}{2} \log_e \left(\frac{0}{0_2} - 3, \tau = 3 \right)$$

2.5	2.55	2.6	2.65	2.7	2.75	2.8
1.0	1.05	1.1	1.15	1.2	1.25	1.3
1.35	1.4	1.45	1.5	1.55	1.6	1.65
1.7	1.75	1.8	1.85	1.9	1.95	2.0
2.05	2.1	2.15	2.2	2.25	2.3	2.35
2.4	2.45	2.5	2.55	2.6	2.65	2.7
2.75	2.8	2.85	2.9	2.95	3.0	3.05
3.1	3.15	3.2	3.25	3.3	3.35	3.4
3.45	3.5	3.55	3.6	3.65	3.7	3.75
3.8	3.85	3.9	3.95	4.0	4.05	4.1
4.15	4.2	4.25	4.3	4.35	4.4	4.45
4.5	4.55	4.6	4.65	4.7	4.75	4.8
4.85	4.9	4.95	5.0	5.05	5.1	5.15
5.2	5.25	5.3	5.35	5.4	5.45	5.5
5.55	5.6	5.65	5.7	5.75	5.8	5.85
5.9	5.95	6.0	6.05	6.1	6.15	6.2
6.25	6.3	6.35	6.4	6.45	6.5	6.55
6.6	6.65	6.7	6.75	6.8	6.85	6.9
6.95	7.0	7.05	7.1	7.15	7.2	7.25
7.3	7.35	7.4	7.45	7.5	7.55	7.6
7.65	7.7	7.75	7.8	7.85	7.9	7.95
8.0	8.05	8.1	8.15	8.2	8.25	8.3
8.35	8.4	8.45	8.5	8.55	8.6	8.65
8.7	8.75	8.8	8.85	8.9	8.95	9.0
9.05	9.1	9.15	9.2	9.25	9.3	9.35
9.4	9.45	9.5	9.55	9.6	9.65	9.7
9.75	9.8	9.85	9.9	10.0	10.05	10.1
10.15	10.2	10.25	10.3	10.35	10.4	10.45
10.5	10.55	10.6	10.65	10.7	10.75	10.8
10.85	10.9	10.95	11.0	11.05	11.1	11.15
11.2	11.25	11.3	11.35	11.4	11.45	11.5
11.55	11.6	11.65	11.7	11.75	11.8	11.85
11.9	11.95	12.0	12.05	12.1	12.15	12.2
12.25	12.3	12.35	12.4	12.45	12.5	12.55
12.6	12.65	12.7	12.75	12.8	12.85	12.9
13.0	13.05	13.1	13.15	13.2	13.25	13.3
13.35	13.4	13.45	13.5	13.55	13.6	13.65
13.7	13.75	13.8	13.85	13.9	14.0	14.05
14.1	14.15	14.2	14.25	14.3	14.35	14.4
14.45	14.5	14.55	14.6	14.65	14.7	14.75
14.8	14.85	14.9	15.0	15.05	15.1	15.15
15.2	15.25	15.3	15.35	15.4	15.45	15.5
15.55	15.6	15.65	15.7	15.75	15.8	15.85
15.9	15.95	16.0	16.05	16.1	16.15	16.2
16.25	16.3	16.35	16.4	16.45	16.5	16.55
16.6	16.65	16.7	16.75	16.8	16.85	16.9
17.0	17.05	17.1	17.15	17.2	17.25	17.3
17.35	17.4	17.45	17.5	17.55	17.6	17.65
17.7	17.75	17.8	17.85	17.9	18.0	18.05
18.1	18.15	18.2	18.25	18.3	18.35	18.4
18.45	18.5	18.55	18.6	18.65	18.7	18.75
18.8	18.85	18.9	19.0	19.05	19.1	19.15
19.2	19.25	19.3	19.35	19.4	19.45	19.5
19.55	19.6	19.65	19.7	19.75	19.8	19.85
19.9	19.95	20.0	20.05	20.1	20.15	20.2
20.25	20.3	20.35	20.4	20.45	20.5	20.55
20.6	20.65	20.7	20.75	20.8	20.85	20.9
21.0	21.05	21.1	21.15	21.2	21.25	21.3
21.35	21.4	21.45	21.5	21.55	21.6	21.65
21.7	21.75	21.8	21.85	21.9	22.0	22.05
22.1	22.15	22.2	22.25	22.3	22.35	22.4
22.45	22.5	22.55	22.6	22.65	22.7	22.75
22.8	22.85	22.9	23.0	23.05	23.1	23.15
23.2	23.25	23.3	23.35	23.4	23.45	23.5
23.55	23.6	23.65	23.7	23.75	23.8	23.85
23.9	23.95	24.0	24.05	24.1	24.15	24.2
24.25	24.3	24.35	24.4	24.45	24.5	24.55
24.6	24.65	24.7	24.75	24.8	24.85	24.9
25.0	25.05	25.1	25.15	25.2	25.25	25.3
25.35	25.4	25.45	25.5	25.55	25.6	25.65
25.7	25.75	25.8	25.85	25.9	26.0	26.05
26.1	26.15	26.2	26.25	26.3	26.35	26.4
26.45	26.5	26.55	26.6	26.65	26.7	26.75
26.8	26.85	26.9	27.0	27.05	27.1	27.15
27.2	27.25	27.3	27.35	27.4	27.45	27.5
27.55	27.6	27.65	27.7	27.75	27.8	27.85
27.9	27.95	28.0	28.05	28.1	28.15	28.2
28.25	28.3	28.35	28.4	28.45	28.5	28.55
28.6	28.65	28.7	28.75	28.8	28.85	28.9
29.0	29.05	29.1	29.15	29.2	29.25	29.3
29.35	29.4	29.45	29.5	29.55	29.6	29.65
29.7	29.75	29.8	29.85	29.9	30.0	30.05
30.1	30.15	30.2	30.25	30.3	30.35	30.4
30.45	30.5	30.55	30.6	30.65	30.7	30.75
30.8	30.85	30.9	31.0	31.05	31.1	31.15
31.2	31.25	31.3	31.35	31.4	31.45	31.5
31.55	31.6	31.65	31.7	31.75	31.8	31.85
31.9	31.95	32.0	32.05	32.1	32.15	32.2
32.25	32.3	32.35	32.4	32.45	32.5	32.55
32.6	32.65	32.7	32.75	32.8	32.85	32.9
33.0	33.05	33.1	33.15	33.2	33.25	33.3
33.35	33.4	33.45	33.5	33.55	33.6	33.65
33.7	33.75	33.8	33.85	33.9	34.0	34.05
34.1	34.15	34.2	34.25	34.3	34.35	34.4
34.45	34.5	34.55	34.6	34.65	34.7	34.75
34.8	34.85	34.9	35.0	35.05	35.1	35.15
35.2	35.25	35.3	35.35	35.4	35.45	35.5
35.55	35.6	35.65	35.7	35.75	35.8	35.85
35.9	35.95	36.0	36.05	36.1	36.15	36.2
36.25	36.3	36.35	36.4	36.45	36.5	36.55
36.6	36.65	36.7	36.75	36.8	36.85	36.9
37.0	37.05	37.1	37.15	37.2	37.25	37.3
37.35	37.4	37.45	37.5	37.55	37.6	37.65
37.7	37.75	37.8	37.85	37.9	38.0	38.05
38.1	38.15	38.2	38.25	38.3	38.35	38.4
38.45	38.5	38.55	38.6	38.65	38.7	38.75
38.8	38.85	38.9	39.0	39.05	39.1	39.15
39.2	39.25	39.3	39.35	39.4	39.45	39.5
39.55	39.6	39.65	39.7	39.75	39.8	39.85
39.9	39.95	40.0	40.05	40.1	40.15	40.2
40.25	40.3	40.35	40.4	40.45	40.5	40.55
40.6	40.65	40.7	40.75	40.8	40.85	40.9
41.0	41.05	41.1	41.15	41.2	41.25	41.3
41.35	41.4	41.45	41.5	41.55	41.6	41.65
41.7	41.75	41.8	41.85	41.9	42.0	42.05
42.1	42.15	42.2	42.25	42.3	42.35	42.4
42.45	42.5	42.55	42.6	42.65	42.7	42.75
42.8	42.85	42.9	43.0	43.05	43.1	43.15
43.2	43.25	43.3	43.35	43.4	43.45	43.5
43.55	43.6	43.65	43.7	43.75	43.8	43.85
43.9	43.95	44.0	44.05	44.1	44.15	44.2
44.25	44.3	44.35	44.4	44.45	44.5	44.55
44.6	44.65	44.7	44.75	44.8	44.85	44.9
45.0	45.05	45.1	45.15	45.2	45.25	45.3
45.35	45.4	45.45	45.5	45.55	45.6	45.65
45.7	45.75	45.8	45.85	45.9	46.0	46.05
46.1	46.15	46.2	46.25	46.3	46.35	46.4
46.45	46.5	46.55	46.6	46.65	46.7	46.75
46.8	46.85	46.9	47.0	47.05	47.1	47.15
47.2	47.25	47.3	47.35	47.4	47.45	47.5
47.55	47.6	47.65	47.7	47.75	47.8	47.85
47.9	47.95	48.0	48.05	48.1	48.15	48.2
48.25	48.3	48.35	48.4	48.45	48.5	48.55
48.6	48.65	48.7	48.75	48.8	48.85	48.9
49.0	49.05	49.1	49.15	49.2	49.25	49.3
49.35	49.4	49.45	49.5	49.55	49.6	49.65
49.7	49.75	49.8	49.85	49.9	50.0	50.05
50.1	50.15	50.2	50.25	50.3	50.35	50.4
50.45	50.5	50.55	50.6	50.65	50.7	50.75
50.8	50.85	50.9	51.0	51.05	51.1	51.15
51.2	51.25	51.3	51.35	51.4	51.45	51.5
51.55	51.6	51.65	51.7	51.75	51.8	51.85
51.9	51.95	52.0	52.05	52.1	52.15	52.2
52.25	52.3	52.35	52.4	52.45	52.5	52.55
52.6	52.65	52.7	52.75	52.8	52.85	52.9
53.0	53.05	53.1	53.15	53.2	53.25	53.3
53.35	53.4	53.45	53.5	53.55	53.6	53.65
53.7	53.75	53.8	53.85	53.9	54.0	54.05
54.1	54.15	54.2	54.25	54.3	54.35	54.4
54.45	54.5	54.55	54.6	54.65	54.7	54.75
54.8	54.85	54.9	55.0	55.05	55.1	55.15
55.2	55.25	55.3	55.35	55.4	55.45	55.5
55.55	55.6	55.65	55.7	55.75	55.8	55.85
55.9	55.95	56.0	56.05	56.1	56.15	56.2
56.25	56.3	56.35	56.4	56.45	56.5	56.55
56.6	56.65	56.7	56.75	56.8	56.85	56.9
57.0	57.05	57.1	57.15	57.2	57.25	57.3
57.35	57.4	57.45	57.5	57.55	57.6	57.65
57.7	57.75	57.8	57.85	57.9	58.0	58.05
58.1	58.15	58.2	58.25	58.3	58.35	58.4
58.45	58.5	58.55	58.6	58.65	58.7	58.75
58.8	58.85	58.9	59.0	59.05	59.1	59.15
59.2	59.25	59.3	59.35	59.4	59.45	59.5
59.55	59.6	59.65	59.7	59.75	59.8	59.85
59.9	59.95	60.0	60.05	60.1	60.15	60.2
60.25	60.3	60.35	60.4	60.45	60.5	60.55
60.6	60.65	60.7	60.75	60.8	60.85	60.9
61.0	61.05	61.1	61.15	61.2	61.25	61.3
61.35	61.4	61.45	61.5	61.55	61.6	61.65
61.7	61.75	61.8	61.85	61.9	62.0	62.05
62.1	62.15	62.2	62.25	62.3	62.35	62.4
62.45	62.5	62.55	62.6	62.65	62.7	62.75
62.8	62.85	62.9	63.0	63.05	63.1	63.15
63.2	63.25	63.3	63.35	63.4	63.45	63.5
63.55	63.6	63.				

CHAPTER 8

BEAMS WITH AXIAL LOADING AND BEAMS ON ELASTIC FOUNDATIONS

Part 1 - Free lateral vibrations of beams subjected to axial compressive or tensile loads

8.1 Beams With Both Ends Simply Supported

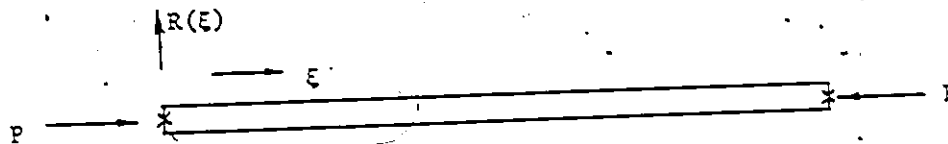


Figure 8.1

The general displacement function of such a beam, as derived in Chapter 1, is given by equation (1.13) and is re-written below for ready reference

$$R(\xi) = A \sin \beta_1 \xi + B \cos \beta_1 \xi + C \sinh \beta_2 \xi + D \cosh \beta_2 \xi \quad (8.1)$$

where

$$\beta_1 = \sqrt{\frac{\pi^2 \alpha}{2} + \sqrt{\frac{\pi^4 \alpha^2}{4} + \beta^4}} \quad (8.2)$$

$$\beta_2 = \sqrt{-\frac{\pi^2 \alpha}{2} + \sqrt{\frac{\pi^4 \alpha^2}{4} + \beta^4}} \quad (8.3)$$

and $\alpha = \frac{P}{(\pi^2 EI/L^2)}$ is the ratio of the compressive load to Euler's load

and will be negative for tensile loads at the ends.

The requirement of no deflection at the end $\xi = 0$ leads to $B = -D$ and the conditions of no moment at the same end leads to $D(\beta_1^2 + \beta_2^2) = 0$ where B has been eliminated by using the first boundary condition. Since $\beta_1^2 + \beta_2^2$ can never be zero, it therefore follows that $B = D = 0$ and the deflection function is therefore reduced to

$$R(\xi) = A \sin \beta_1 \xi + C \sinh \beta_2 \xi \quad (8.4)$$

Now utilizing the two boundary conditions at the end $\xi = 1$, namely $R(1) = 0$ and $R''(1) = 0$, we obtain

$$A \sin \beta_1 + C \sinh \beta_2 = 0 \quad (8.5)$$

and

$$-A \beta_1^2 \sin \beta_1 + C \beta_2^2 \sinh \beta_2 = 0 \quad (8.6)$$

For a non-trivial solution for the constants A and C in the above two equations, the determinant of their coefficient matrix is set equal to zero and we obtain

$$\sin \beta_1 \sinh \beta_2 (\beta_1^2 + \beta_2^2) = 0 \quad (8.7)$$

Since for all values of β , $\sinh \beta$ and $(\beta_1^2 + \beta_2^2)$ cannot be equal to zero, it follows therefore that

$$\sin \beta_1 = 0 \quad (8.8)$$

Equation (8.6) then represents the eigenvalue equation of this class of beams and is readily satisfied by $\beta_1 = n\pi$, where $n = 1, 2, \dots$

Substituting from β_1 from equation (8.2) and simplifying, we obtain

$$\beta_n^4 = n^2 \pi^4 [n^2 - \alpha] \quad (8.9)$$

where n denotes mode number. It may be noted that for no axial forces at the ends, i.e. $\alpha = 0$, equation (8.9) gives the frequency equation for simply supported beams. The eigenvalues β and therefore the frequencies of this class of beams can be readily obtained from equation (8.9) for any value of α and for any mode n .

The modal shape of the beam is determined by setting the constant A equal to unity and then solving equation (8.5) for the remaining constant C . Thus, we obtain:

$$R(\xi) = \sin \beta_1 \xi - \frac{\sin \beta_1}{\sinh \beta_2} \sinh \beta_2 \xi$$

$$= \sin \beta_1 \xi, \quad \text{since} \quad \sin \beta_1 = 0$$

8.2 Beams with Both Ends Clamped

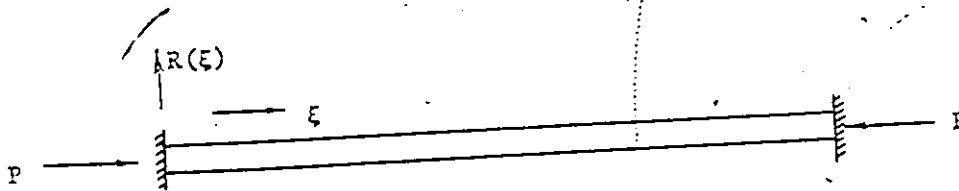


Figure 8.2

Enforcing the boundary conditions $R(0) = 0$ and $R'(0) = 0$ at the left end, the shape function (8.1) of the beam reduces to

$$R(\xi) = A \left(\sin \beta_1 \xi - \frac{\beta_1}{\beta_2} \sinh \beta_2 \xi \right) + B \left(\cos \beta_1 \xi - \cosh \beta_2 \xi \right) \quad (8.10)$$

Further utilization of the boundary conditions $R(1) = 0$ and $R'(1) = 0$, we obtain:

$$A \left(\sin \beta_1 - \frac{\beta_1}{\beta_2} \sinh \beta_2 \right) + B \left(\cos \beta_1 - \cosh \beta_2 \right) = 0 \quad (8.11)$$

and

$$A (\beta_1 \cos \beta_1 - \beta_1 \cosh \beta_2) + B (-\beta_1 \sin \beta_1 - \beta_2 \sinh \beta_2) = 0 \quad (8.12)$$

The condition for a non-trivial solution for the constants in the above two equations yields the following frequency equation of the system

$$\cos \beta_1 \cosh \beta_2 + \frac{(\beta_1^2 - \beta_2^2)}{2\beta_1 \beta_2} \sin \beta_1 \sinh \beta_2 - 1 = 0 \quad (8.13)$$

It may be noted that, unlike case 8.1, the eigenvalues δ for this case cannot be expressed in the concise nature of a closed form solution. Equation (8.13) was therefore solved by iteration for various values of α .

The next thing to be decided is the range for the parameter α associated with compressive and tensile axial loads. On the compressive side, the limiting value of α is quite obvious. This corresponds to the value of buckling load for the first mode at which the frequency of vibrations essentially reduces to zero. The same limiting value of α has been chosen for higher modes also, as in practice, the compressive loads greater than the first buckling loads are normally avoided.

However, there is no such limiting value of α on tensile side which may preclude the possibility of vibrations. In fact, the effect of tensile forces is to increase the vibration frequency. This is to be expected since they have the effect of increasing the elastic restoring forces when the beam is disturbed from its neutral configuration.

The tensile forces can be indefinitely increased till the material of the beam fractures. This value would therefore depend on the geometric and material properties of the beam. However, by choosing a value of $10 \times$ Euler load, as the limiting value of tensile loads, it is expected that most practical applications would be covered by the prescribed range. Consequently the range of α prescribed is, $-10 \leq \alpha \leq 4$ for this case.

Figure 8.2(a) presents the plots of β/β_{cc} for the above mentioned range of α for first six modes of vibrations, where β_{cc} corresponds to the eigenvalue with no axial loads. Since the eigenvalues of this class of beams, as presented in Figure 8.2(a), are normalized with respect to those with $\alpha = 0$, all the curves of the Figure 8.2(a) pass through the point $\beta/\beta_{cc} = 1$ and $\alpha = 0$. For better accuracy in reading off the plots, a part of the first mode curve, is shown enlarged in the same figure.

The modal shape expression is given by

$$R(\xi) = \sin \beta_1 \xi - \frac{\beta_1}{\beta_2} \sinh \beta_2 \xi + \beta_1 \left(\frac{\cos \beta_1 - \cosh \beta_2}{\beta_1 \sin \beta_1 + \beta_2 \sinh \beta_2} \right) (\cos \beta_1 \xi - \cosh \beta_2 \xi)$$

where β_1 and β_2 can be calculated from equations (8.2) and (8.3) once the value of β has been read off the charts provided.

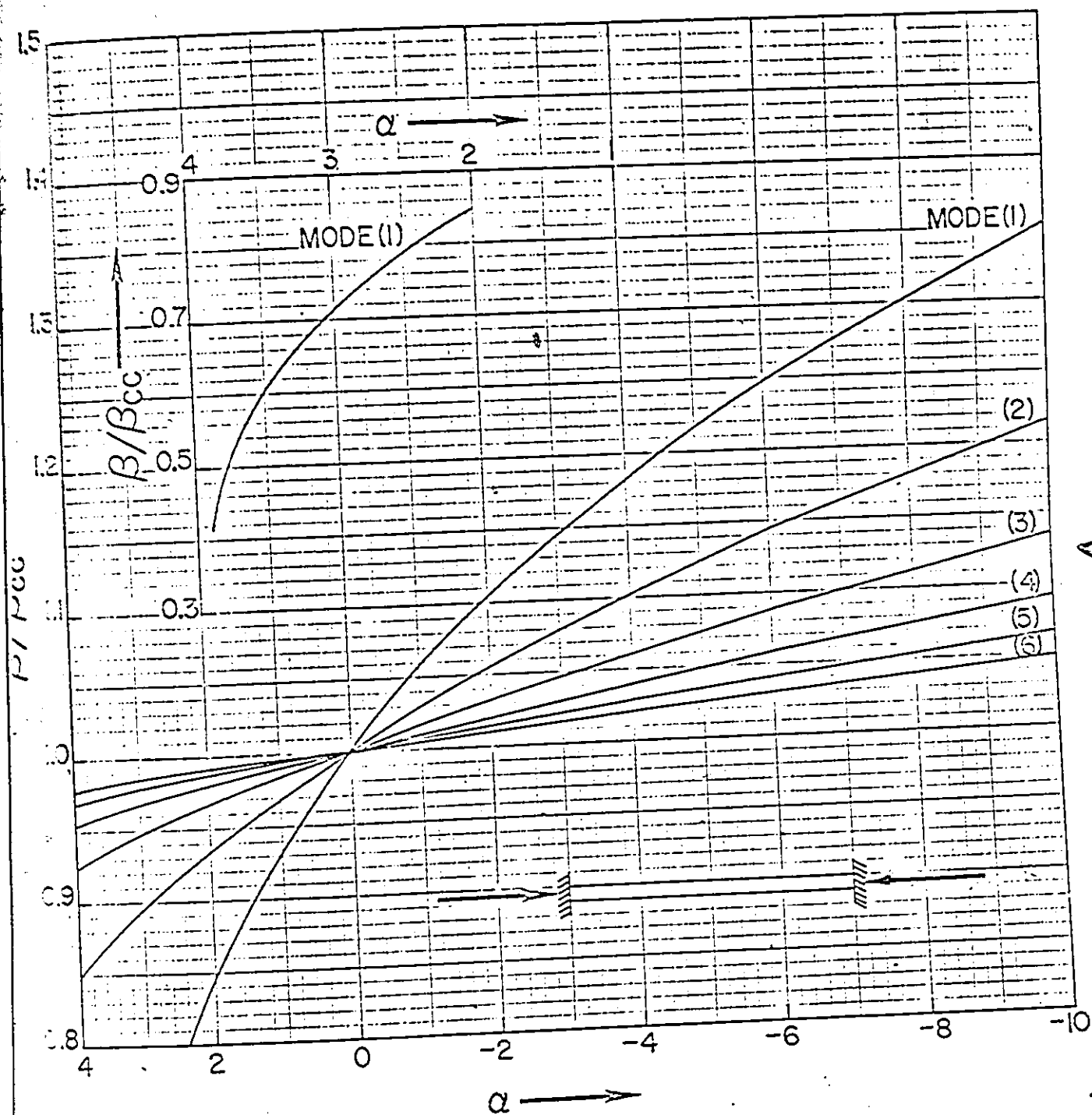


FIGURE 8.2(a)

8.3 Beams With One End Fixed and Other Simply Supported

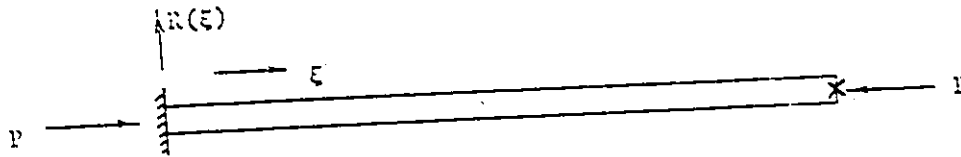


Figure 8.3

Utilizing the boundary conditions at the clamped end, it can be shown that the displacement function of this beam is expressed by equation (8.10). Then, enforcing the boundary conditions $R(1) = 0$ and $R''(1) = 0$ at the simply supported end, we obtain:

$$A(\sin \beta_1 - \frac{\beta_1}{\beta_2} \sinh \beta_2) + B(\cos \beta_1 - \cosh \beta_2) = 0 \quad (8.14)$$

and

$$A(-\beta_1^2 \sin \beta_1 - \beta_1 \beta_2 \sinh \beta_1) + B(-\beta_1^2 \cos \beta_1 - \beta_2^2 \cosh \beta_2) = 0 \quad (8.15)$$

Setting the determinant of the coefficient matrix of the constants A and B in the above two equations, we obtain the following frequency equation of the beam system

$$\beta_1 \cos \beta_1 \sinh \beta_2 - \beta_2 \sin \beta_1 \cosh \beta_2 = 0 \quad (8.16)$$

The equation (8.16) is solved for β after prescribing various values of α . The range of α , as discussed in case 8.2 corresponds to, $-10 \times \text{Euler load} \leq \alpha P \leq \text{buckling load in first mode i.e. } -10 \leq \alpha \leq 2$. The plots of β/β_{cs} versus α are presented in Figure 8.3(a).

The modal shape expression is given by

$$R(\xi) = \sin \beta_1 \xi - \frac{\beta_1}{\beta_2} \sinh \beta_2 \xi + \frac{(\sin \beta_1 - \frac{\beta_1}{\beta_2} \sinh \beta_2)}{(\cosh \beta_2 - \cos \beta_1)} (\cos \beta_1 \xi - \cosh \beta_2 \xi)$$

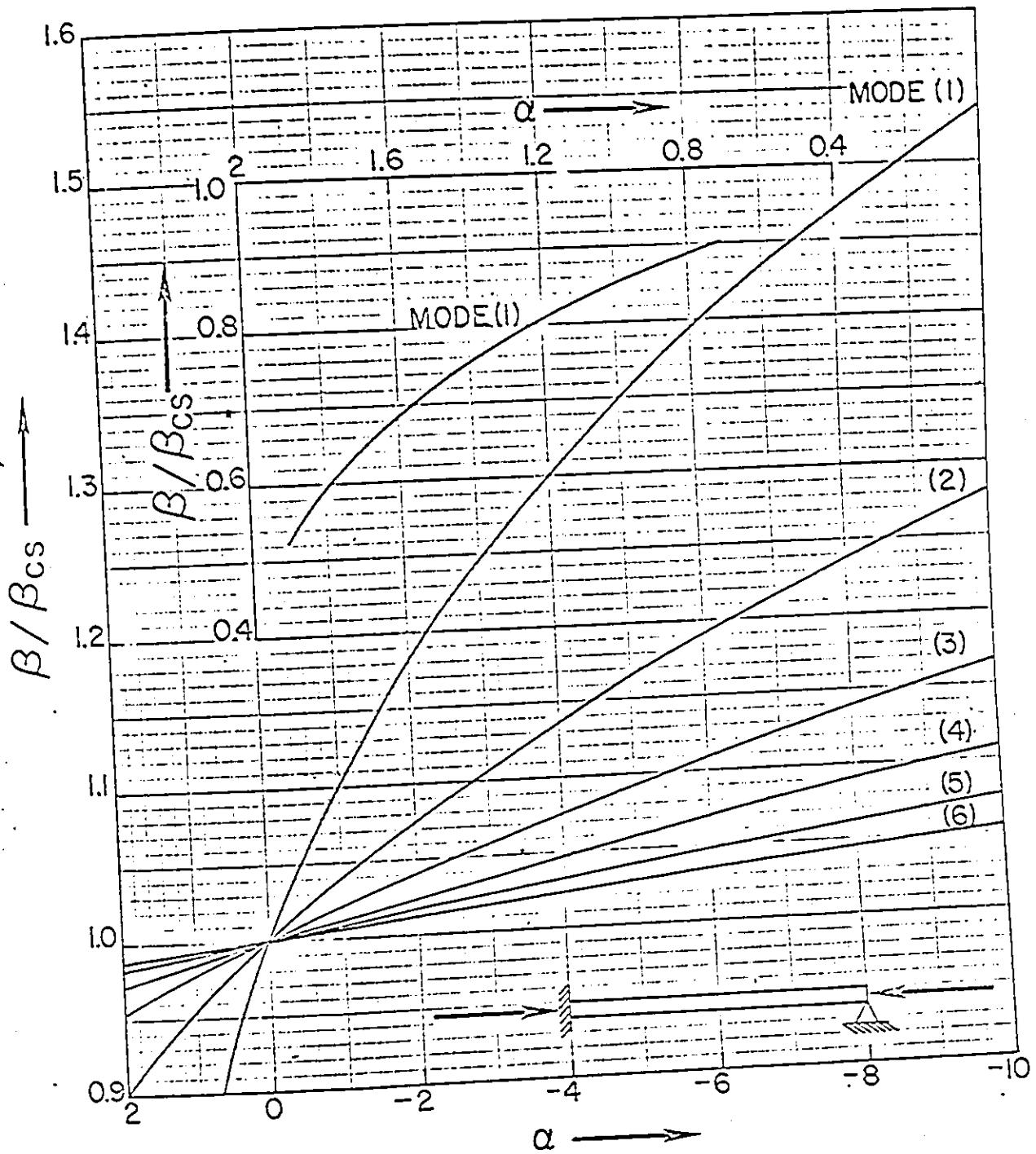


FIGURE 8.3(a)

Part 2 - Uniform beams on elastic foundations

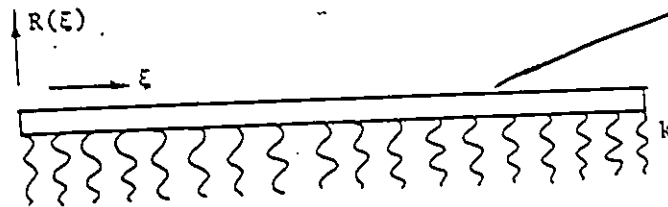


Figure 8.4

A beam is said to be resting on an elastic foundation if it experiences a restoring force throughout its entire length proportional to its displacement from the neutral configuration. Such a beam is shown in the Figure 8.4.

The modulus k of the elastic foundation is assumed to be uniform along the beam and has the units of force per unit deflection, per unit length of the beam. The inertia effects of the foundation are neglected and it is further assumed that the contact between the beam and foundation is always there.

The partial differential equation governing the vibration of such a beam is given by

$$\frac{EI}{\rho A} \frac{\partial^4 y(x,t)}{\partial x^4} + \frac{k}{\rho A} y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} \quad (8.17)$$

Utilizing the method of 'separation of variables' and non-dimensionalizing the displacement and spatial coordinate parameter x with respect to beam length L , the following ordinary differential equation defining the shape of the vibrating beam is obtained.

$$\frac{d^4 R(\xi)}{d \xi^4} - s^{*4} R(\xi) = 0 \quad (8.18)$$

$$\text{where } s^{*4} = \frac{\rho A \omega^2 L^4}{EI} - k^* \quad (8.19)$$

$$\text{where } k^* = \frac{k L^4}{EI}$$

It is interesting to note that equation (8.18) is identical to that obtained for uniform beams with no elastic foundation. Therefore, the eigenvalues of such beams are independent of elastic foundation and all the results obtained in this thesis for uniform beams are also valid if the beam is resting on elastic foundation. The modal shape expressions are also independent of whether or not elastic foundation is provided.

The only difference in this case is that the eigenvalues s are related to frequencies through equation (8.19) which has an additional term k^* different from the beams with no elastic foundation.

Denoting the frequency of the beam without elastic foundation as f_c , it can be readily shown that

$$f = f_c \sqrt{1 + \frac{k^*}{\delta^4}} \quad (8.20)$$

where δ is the eigenvalue of the beam without elastic foundation. It may be noted that as k^* approaches zero, the frequency f of the beam with elastic foundation approaches the frequency f_c of the beam without elastic foundation. As k^* approaches infinity, the frequency f also approaches infinity.

Illustrative Example 8.1

If the beam of example 2.1 is clamped at one end and simply supported at the other and is subjected to a compressive load of 400 lbs at each end.

- (a) Determine the first mode frequency of the beam.

$$\text{Euler load} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 30 \times 10^6 \times 0.0013}{(29.5)^2}$$

$$= 442.3 \text{ lbs}$$

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$$\alpha = \frac{400}{442.3} = .90$$

From Figure 8.3(a) obtain $\frac{\beta}{\beta_{cs}} = 0.87$

$$\text{hence } \beta = 0.87 \times 3.926 = 3.4156$$

and

$$f_{(1)} = \frac{\beta^2}{2\pi L^2} \sqrt{\frac{EI}{\rho A}} = \underline{31.55 \text{ Hz}}$$

- (b) Determine the fundamental frequency if a tensile force of 1000 lbs is acting at each end of the beam in part (a), now

$$\alpha = \frac{-1000}{445.3} = -2.245$$

and $\frac{\beta}{\beta_{cs}}$ from Figure 8.3(a) is 1.2

$$\text{so that } \beta = 1.2 \times 3.926 = 4.71$$

and

$$f_{(1)} = \underline{60.05 \text{ Hz}}$$

- (c) If the beam in part (a) is clamped at each end, determine the second mode frequency.

From Figure 8.2(a), obtain

$$\frac{\beta_{cc}(2)}{\beta_{cc}(2)} = 0.945 \quad \text{for} \quad \alpha = 0.9$$

so that

$$\beta_2 = 0.946 \times 7.853 = 7.43$$

and

$$f_2 = \underline{149.3 \text{ Hz}}$$

- (d) If the beam in part (a) is supported on an elastic foundation of modulus = 2 lb/in², determine its natural frequency for first and second modes.

Utilizing equation (8.20)

$$\tilde{f} = f_c \sqrt{1 + \frac{k^*}{\beta^4}} \quad \text{where} \quad k^* = \frac{k l^4}{EI} = \frac{2 \times 24.5^4}{10.5 \times 10^6 \times 0.0013}$$

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$$f_1 = 41.66 \sqrt{1 + \frac{111}{(3.926)^4}} = 1.21 \times 41.66 = \underline{50.5 \text{ Hz}}$$

$$f_2 = 135 \sqrt{1 + \frac{111}{(7.068)^4}} = \underline{138 \text{ Hz}}$$

CHAPTER 9

FREE TRANSVERSE VIBRATIONS OF TAPERED BEAMS

Part 1 - Conical beams

The term "conical beams" is generally referred to the beams whose transverse dimensions vary linearly along the length of the beam. The vibration analysis of these beams for six combinations of boundary conditions of practical interest is presented in this part of the Chapter.

9.1 Beams Fixed at Larger End and Simply Supported at the Smaller End

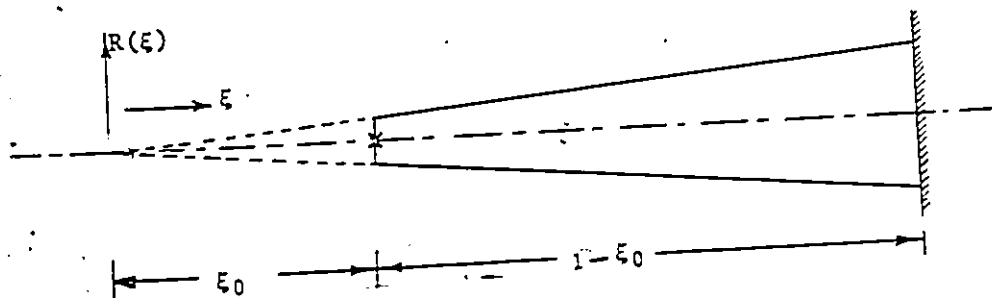


Figure 9.1

The general solution for the deflection curve is given by equation (1.24) and is presented below for ready reference

$$R(\xi) = \xi^{-1} [A J_2(z) + B Y_2(z) + C I_2(z) + D K_2(z)] \quad (9.1)$$

where

$$z = 2 \beta \xi^{1/2}$$

and β is related to frequencies by

$$\beta^4 = \frac{\rho A_0 \omega^2 L_1^4}{EI_0} \quad (9.2)$$

Defining the ratio of transverse dimension $\frac{b}{b_0}$ or $\frac{t}{t_0} = \xi_0$, it is easily recognised that the truncated end of the beam is situated at a distance ξ_0 from the origin. With the choice of coordinate system as shown Figure 9.1, the location of the truncated end from the origin is always equal to the ratio of the transverse dimensions of smaller and larger ends.

The four boundary conditions to be considered for this class of beams are $R(\xi_0) = 0$, $R''(\xi_0) = 0$, $R(1) = 0$ and $R'(1) = 0$.

Enforcement of these four boundary conditions yield the following four homogeneous equations in terms of four constants. They are respectively:

$$A J_2(z_0) + B Y_2(z_0) + C I_2(z_0) + D K_2(z_0) = 0 \quad (9.3)$$

$$A J_4(z_0) + B Y_4(z_0) + C I_4(z_0) + D K_4(z_0) = 0 \quad (9.4)$$

$$A J_2(2\beta) + B Y_2(2\beta) + C I_2(2\beta) + D K_2(2\beta) = 0 \quad (9.5)$$

and

$$A J_3(2\beta) + B Y_3(2\beta) - C I_3(2\beta) + D K_3(2\beta) = 0 \quad (9.6)$$

where

$$z = 2 \beta \xi_0^{1/2}$$

In arriving at the above four equations, the use has been made of various recursion formulas given in Chapter 1.

The condition of existence of a non-trivial solution for the constants A, B, C, and D in the above four equations yields the following determinantal equation

$$\begin{vmatrix} J_2(z_0) & Y_2(z_0) & I_2(z_0) & K_2(z_0) \\ J_4(z_0) & Y_4(z_0) & I_4(z_0) & K_4(z_0) \\ J_2(2\beta) & Y_2(2\beta) & I_2(2\beta) & K_2(2\beta) \\ J_3(2\beta) & Y_3(2\beta) & -I_3(2\beta) & K_3(2\beta) \end{vmatrix} = 0 \quad (9.7)$$

This frequency equation can be solved by prescribing values of the parameter ξ_0 . It can be easily recognised that as ξ_0 approaches unity, the beam frequencies approach those of uniform beams. As ξ_0 approaches zero, the beam becomes a full cone beam instead of truncated. Since the Bessel functions Y and K blow up for ξ_0 equal to zero, the value of ξ_0 were prescribed from 0.02 to 1 and the resulting eigenvalues β were computed. These values of β are based on a full cone length of the beam as expressed by equation (9.2). If eigenvalues β^* are based on actual beam length so that frequencies of the beam are expressible by

$$\beta^{*4} = \frac{\omega^2 L^4 \rho A_0}{EI_0}$$

the computed value of β should be multiplied by a factor $(1 - \xi_0)$, in order to obtain β^* . In Figure 9.1(a), the plots, of β^* versus ξ_0 are presented for the first six modes of vibrations.

It may be observed that the values of δ^* approach those of uniform beams as ξ_0 approaches unity. Obtaining the values of δ^* for any value of interest of ξ_0 , the frequencies can be calculated using,

$$\delta^{*4} = \frac{\rho A_0 \omega^2 L^4}{EI_0} \quad (9.7a)$$

We next focus our attention to modal shape expressions. They can be determined by setting any one of the constants appearing in equation (9.1), say A, to unity and then solving any three of the four equations (9.3) through (9.6) for the remaining three constants. It should be remembered that δ appearing in those equation is related to δ^* read off the chart 9.1(a) by $\delta(1 - \xi_0) = \delta^*$.

9.2 Beams Simply Supported at Both Ends

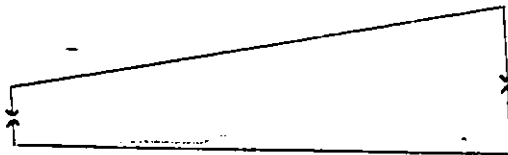


Figure 9.2

Enforcing the four boundary conditions, $R(\xi_0) = 0$, $R''(\xi_0) = 0$, $R(1) = 0$ and $R''(1) = 0$, we obtain the following four equations for the constants A, B, C, and D

$$A J_2(z) + B Y_2(z) + C I_2(z) + D K_2(z) = 0 \quad (9.8)$$

$$A J_4(z) + B Y_4(z) + C I_4(z) + D K_4(z) = 0 \quad (9.9)$$

$$A J_2(2\beta) + B Y_2(2\beta) + C I_2(2\beta) + D K_2(2\beta) = 0 \quad (9.10)$$

and

$$A J_4(2\beta) + B Y_4(2\beta) + C I_4(2\beta) + D K_4(2\beta) = 0 \quad (9.11)$$

and the resulting determinantal equation is

$$\begin{vmatrix} J_2(z_0) & Y_2(z_0) & I_2(z_0) & K_2(z_0) \\ J_4(z_0) & Y_4(z_0) & I_4(z_0) & K_4(z_0) \\ J_2(2\beta) & Y_2(2\beta) & I_2(2\beta) & K_2(2\beta) \\ J_4(2\beta) & Y_4(2\beta) & I_4(2\beta) & K_4(2\beta) \end{vmatrix} = 0 \quad (9.12)$$

The plots of β^* versus ξ_0 are presented in Figure 9.2(a) for the first six modes of vibrations.

9.3 Beams Clamped at Smaller End and Simply Supported at the Larger End

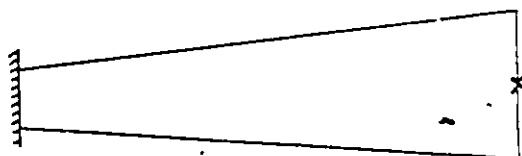


Figure 9.3

Following the procedure established in cases 9.1 and 9.2, following determinantal equation is obtained for the eigenvalues of this beam.

$$\begin{vmatrix}
 J_2(z_0) & Y_2(z_0) & I_2(z_0) & K_2(z_0) \\
 J_3(z_0) & Y_3(z_0) & -I_3(z_0) & K_3(z_0) \\
 J_2(2\beta) & Y_2(2\beta) & I_2(2\beta) & K_2(2\beta) \\
 J_4(2\beta) & Y_4(2\beta) & I_4(2\beta) & K_4(2\beta)
 \end{vmatrix} = 0 \quad (9.13)$$

Results for this case are presented in Figure 9.3(a).

9.4 Beams with Both Ends Clamped

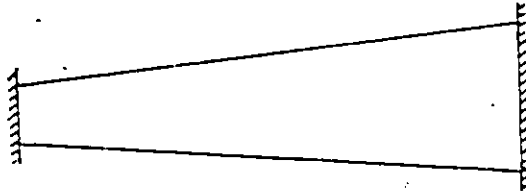


Figure 9.4

The following frequency equation is obtained for this case

$$\begin{vmatrix} J_2(z_0) & Y_2(z_0) & I_2(z_0) & K_2(z_0) \\ J_3(z_0) & Y_3(z_0) & -I_3(z_0) & K_3(z_0) \\ J_2(2\beta) & Y_2(2\beta) & J_2(2\beta) & K_2(2\beta) \\ J_3(2\beta) & Y_3(2\beta) & -I_3(2\beta) & K_3(2\beta) \end{vmatrix} = 0 \quad (9.14)$$

Results are presented in Figure 9.4(a).

9.5 Cantilever Beam Fixed at Larger End

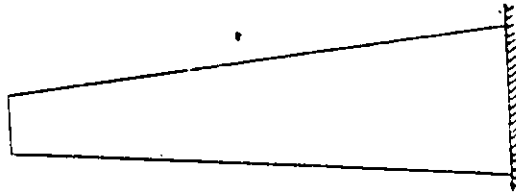


Figure 9.5

Utilizing the appropriate boundary conditions at the two ends and following the procedure established earlier, we obtain the following frequency equation for the beam in hand.

$$\begin{vmatrix}
 J_4(z_0) & Y_4(z_0) & I_4(z_0) & K_4(z_0) \\
 J_3(z_0) & Y_3(z_0) & I_3(z_0) & -K_3(z_0) \\
 J_2(2\beta) & Y_2(2\beta) & I_2(2\beta) & K_2(2\beta) \\
 J_3(2\beta) & Y_3(2\beta) & -I_3(2\beta) & K_3(2\beta)
 \end{vmatrix} = 0$$

(9.15)

The plots of eigenvalue β^* v/s ξ_0 are presented in Figure 9.5(a) for first six modes of vibration.

9.6 Cantilever Beams with Larger End Free



Figure 9.6

Following the steps indicated in case 9.1, it can be readily shown that the frequency equation is given by

$$\begin{vmatrix} J_2(z_0) & Y_2(z_0) & I_2(z_0) & K_2(z_0) \\ J_3(z_0) & Y_3(z_0) & -I_3(z_0) & K_3(z_0) \\ J_4(2\beta) & Y_4(2\beta) & I_4(2\beta) & K_4(2\beta) \\ J_3(2\beta) & Y_3(2\beta) & I_3(2\beta) & -K_3(2\beta) \end{vmatrix} = 0 \quad (9.16)$$

Results are presented in graphical form in Figure

9.6(a).

FIGURE 9.1(a)
Frequency chart for beam with linear variation in
cross section (Simple-clamped support).

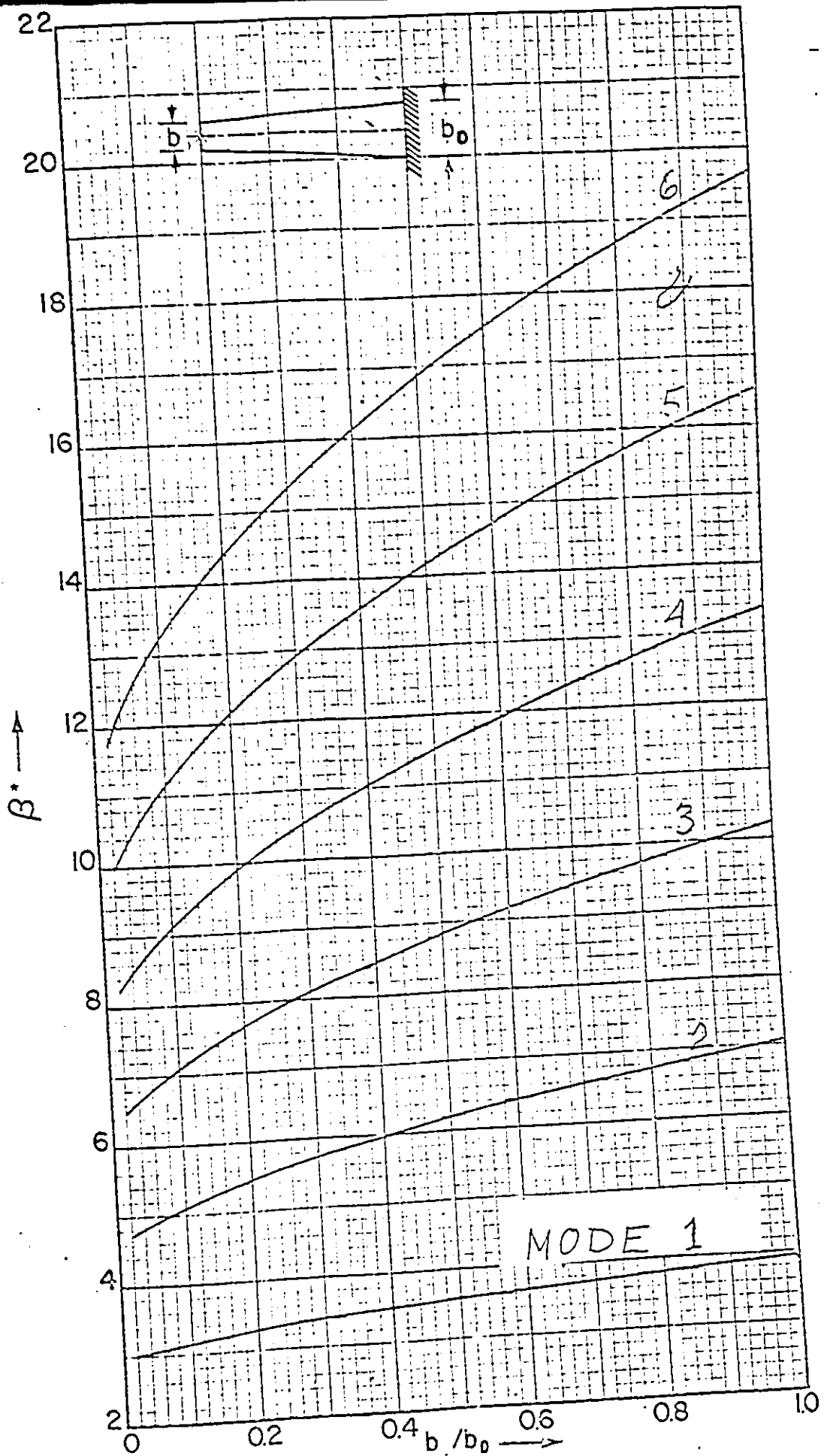


FIGURE 9.2 (a) Frequency chart for beam with linear variation in cross section (Simple-simile support).

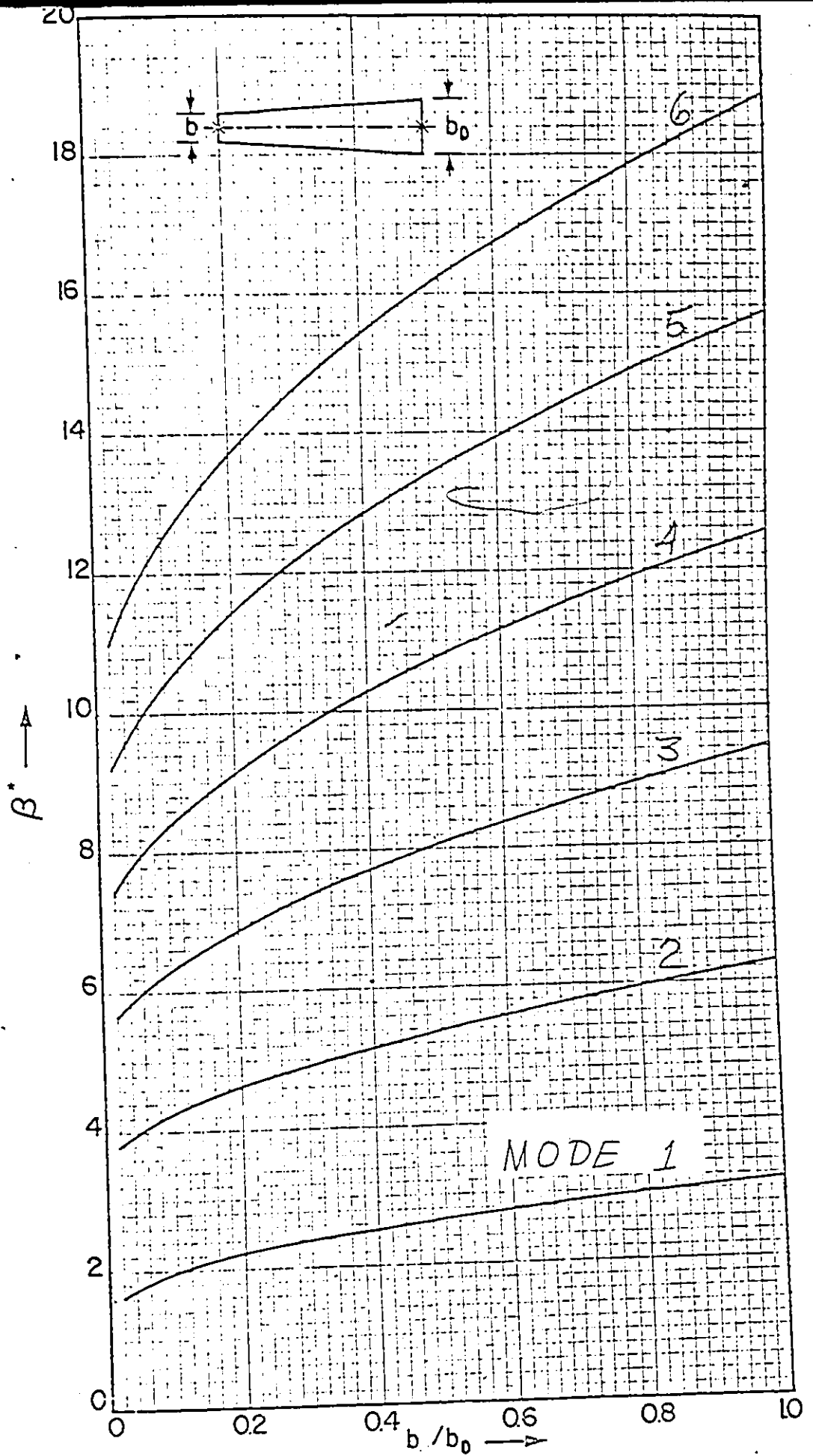


FIGURE 9.3(a) Frequency chart for beam with linear variation in cross section (Clamped-simple support).

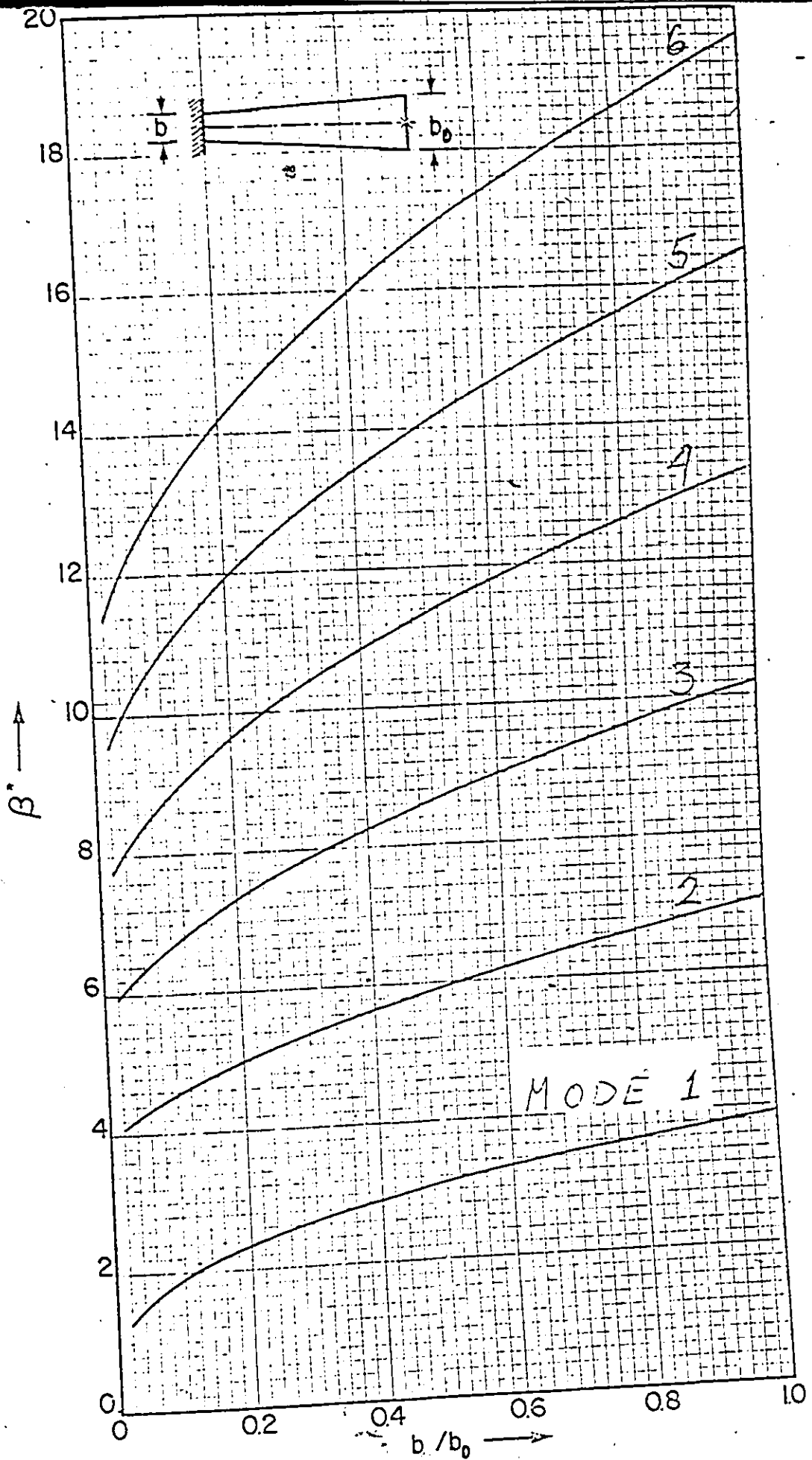


FIGURE 9.4(a) Frequency chart for beam with linear variation in cross section (Clamped-clamped support).

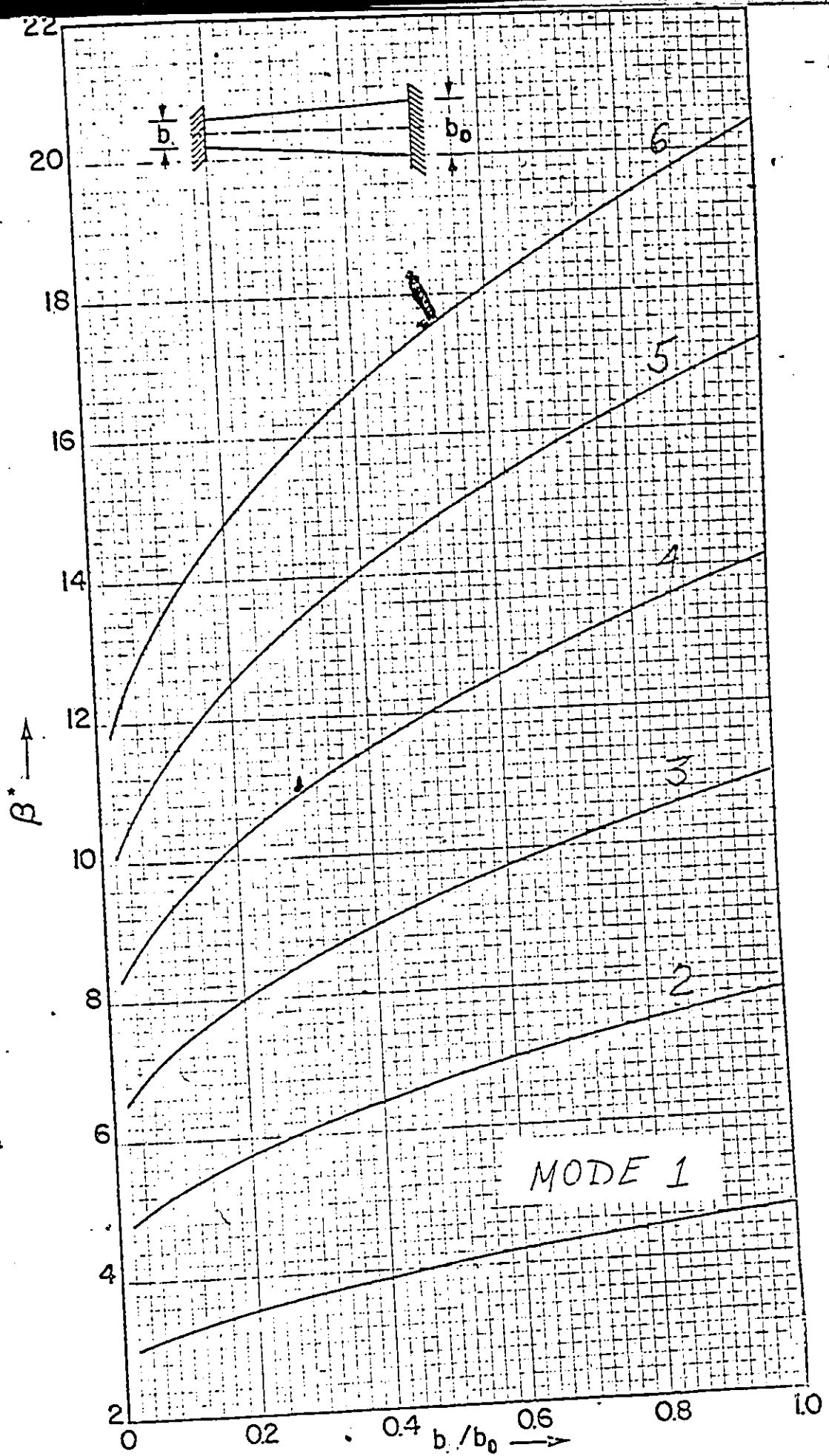


FIGURE 9.5 (a) Frequency chart for beam with linear variation in cross section (Free-clamped support).

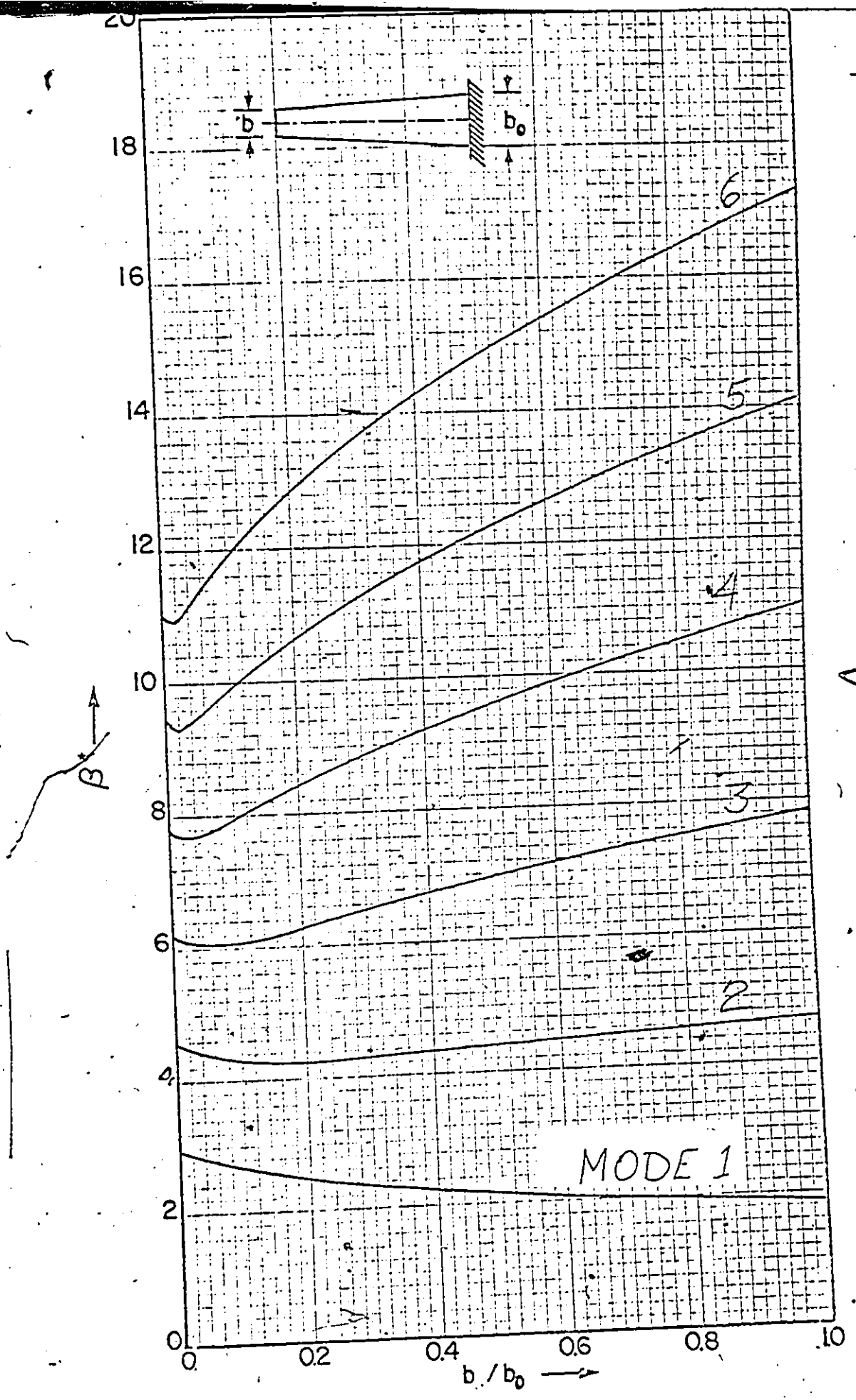
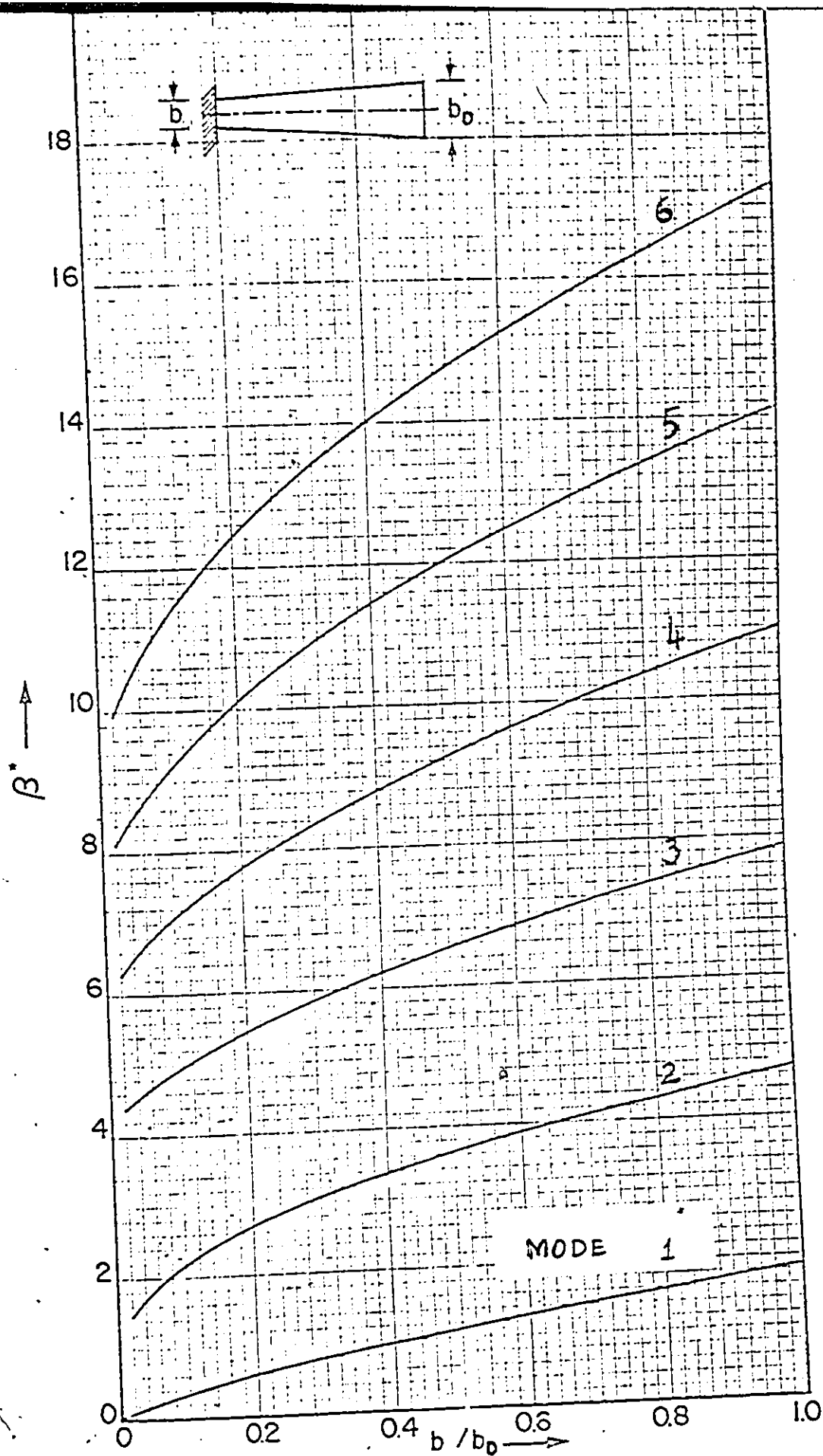


FIGURE 9.3(a) Frequency chart for beam with linear variation in cross section (Clamped-free support).



Part 2 - Beams composed of two sections, one of uniform cross section
and the other with linear taper in transverse dimensions

9.7 Beams Fixed at Larger End and Free at the Smaller

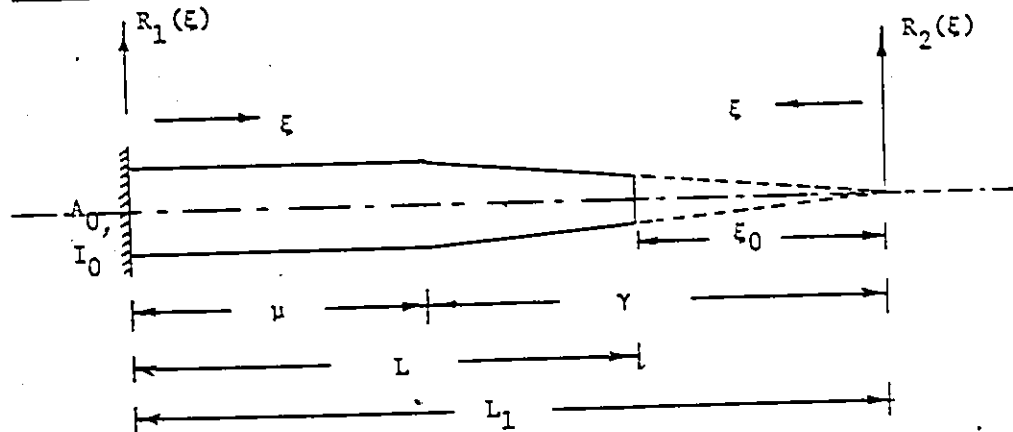


Figure 9.7

The problem is formulated here as a two span beam and the co-ordinates for the two spans are chosen as shown in Figure 9.7. The differential equation for the tapered span is given by the following equation in its non-dimensional form.

$$\frac{d^2}{d\xi^2} \left(EI(\xi) \frac{d^2 R_2(\xi)}{d\xi^2} \right) - \rho A(\xi) \omega^2 L_1^4 R_2(\xi) = 0 \quad (9.17)$$

It should be noted that spatial co-ordinate x and deflections R₁(x) and R₂(x) are now non-dimensionalized with respect to length L₁, where L₁ is

equal to the sum of the lengths of uniform section and the full cone length of the tapered section. The non-dimensionalized lengths of these two sections are denoted by μ and γ as shown in Figure 9.7. Consequently, the area A and second moment of area I at any cross section of the tapered section are given by,

$$A(\xi) = \frac{A_0 \xi^2}{\gamma} \quad \text{and} \quad I(\xi) = \frac{I_0 \xi^4}{\gamma}$$

where A_0 and I_0 correspond to the uniform cross section. Substituting these in equation (9.17) we obtain,

$$\frac{d^2}{d\xi^2} \left(EI_0 \frac{\xi^4}{\gamma} \frac{d^2 R_2(\xi)}{d\xi^2} \right) - \frac{\rho A_0 \xi^2}{\gamma} \omega^2 L_1^4 R_2(\xi) = 0$$

$$\frac{1}{\xi^2} \left(\frac{d^2}{d\xi^2} \xi^4 \frac{d^2 R_2(\xi)}{d\xi^2} \right) - \beta^4 \gamma^2 R_2(\xi) = 0 \quad (9.18)$$

where

$$\beta^4 = \frac{\rho A_0 \omega^2 L_1^4}{EI_0}$$

Equation (9.18) admits a general solution of the form,

$$R_2(\xi) = \xi^{-1} [A_2 J_2(z) + B_2 Y_2(z) + C_2 I_2(z) + D_2 K_2(z)] \quad (9.19)$$

where

$$z = 2\beta \gamma^{1/2} \xi^{1/2}$$

It may be noted that as γ approaches unity, the solution $R_2(\xi)$ approaches that of a beam tapered throughout its entire length. The deflection function for the uniform span is given by

$$R_1(\xi) = A_1 \sin \beta \xi + B_1 \cos \beta \xi + C_1 \sinh \beta \xi + D_1 \cosh \beta \xi \quad (9.20)$$

Utilizing the boundary conditions at the fixed end, equation (9.20) is reduced to

$$R_1(\xi) = A_1 (\sin \beta \xi - \sinh \beta \xi) + B_1 (\cos \beta \xi - \cosh \beta \xi) \quad (9.21)$$

Equations (9.19) and (9.21) now contain six arbitrary constants, they should therefore be supplemented by six more boundary conditions. Two of these are provided by the truncated end $\xi = \xi_0$ of the

tapered section and the other four are supplied by the junction of tapered and uniform cross sections involving continuity of deflection, slope, moment and shear force.

The two boundary conditions at the truncated free end are:

$$R''_2(\xi_0) = 0 \quad \text{and} \quad \frac{d}{d\xi} \left(EI(\xi) \frac{d^2 R_2}{d\xi^2} \right) = 0 \quad \Big|_{\xi=\xi_0}$$

Utilizing these, we obtain

$$A_2 J_4(z_0) + B_2 Y_4(z_0) + C_2 I_4(z_0) + D_2 K_4(z_0) = 0 \quad (9.22)$$

and

$$A_2 J_3(z_0) + B_2 Y_3(z_0) + C_2 I_3(z_0) - D_2 K_3(z_0) = 0 \quad (9.23)$$

where

$$z_0 = 2\beta \gamma^{1/2} \xi_0^{1/2}$$

The four boundary conditions at the junction are

$$R_1(\mu) = R_2(\gamma)$$

$$R'_1(\mu) = -R'_2(\gamma)$$

$$R''_1(\mu) = R''_2(\gamma)$$

and

$$EI_0 R'''_1(\mu) = - \frac{d}{d\xi} \left[EI(\xi) \frac{d^2 R_2}{d\xi^2} \right] \bigg|_{\xi=\gamma}$$

Enforcing these four boundary conditions, we obtain

$$A_1(\alpha_{31}) + B_1(\alpha_{32}) - \frac{1}{\gamma} [A_2 J_2(2\beta\gamma) + B_2 Y_2(2\beta\gamma) + C_2 I_2(2\beta\gamma) + D_2 K_2(2\beta\gamma)] = 0 \quad (9.24)$$

$$A_1(\alpha_{41}) + B_1(\alpha_{42}) - \frac{1}{\gamma} [A_2 J_3(2\beta\gamma) + B_2 Y_3(2\beta\gamma) + C_2 I_3(2\beta\gamma) + D_2 K_3(2\beta\gamma)] = 0 \quad (9.25)$$

$$A_1(\alpha_{51}) + B_1(\alpha_{52}) - \frac{1}{\gamma} [A_2 J_4(2\beta\gamma) + B_2 Y_4(2\beta\gamma) + C_2 I_4(2\beta\gamma) + D_2 K_4(2\beta\gamma)] = 0 \quad (9.26)$$

and

$$A_1(\alpha_{61}) + B_1(\alpha_{62}) + \frac{1}{\gamma} [A_2 J_3(2\beta\gamma) + B_2 Y_3(2\beta\gamma) + C_2 I_3(2\beta\gamma) - D_2 K_3(2\beta\gamma)] = 0 \quad (9.27)$$

where

$$\alpha_{31} = \sin \beta u - \sinh \beta u$$

$$\alpha_{32} = \cos \beta u - \cosh \beta u$$

$$\alpha_{41} = \cos \beta u - \cosh \beta u$$

$$\alpha_{42} = -\sin \beta u - \sinh \beta u$$

$$\alpha_{51} = -\sin \beta u - \sinh \beta u$$

$$\alpha_{52} = -\cos \beta u - \cosh \beta u$$

$$\alpha_{61} = -\cos \beta u - \cosh \beta u$$

$$\alpha_{62} = \sin \beta u - \sinh \beta u$$

For a non-trivial solution for the six constants appearing in equations (9.22) through (9.27), the determinant of their coefficient matrix is set equal to zero, thus yielding the frequency equation of the beam system.

It may be noted that this equation is a function of β , μ , and ξ_0 . Furthermore, the value of ξ_0 which represents the non-dimensional distance of the truncated end from the origin of co-ordinates for tapered section can never be greater than $\gamma - 1 - \mu$.

Now denoting the ratio of transverse dimensions of narrow end to corresponding dimensions of the larger end by ξ_1 , it is readily recognised that

$$\xi_0 = (1 - \mu) \xi_1$$

where the parameter ξ_1 can vary from 0.0 to 1.0.

The frequency equation can be solved for β by prescribing values of μ and ξ_1 where both μ and ξ_1 can vary from 0.0 to 1.0. If the eigenvalues β^* are related to frequencies based on actual length L of the beam instead of the fictitious length L_1 , it is further recognised that

$$\beta^* = \beta(1 - \xi_0)$$

The computed values of β^* for various values of μ and ξ_1 are presented in Tables 9.7 for first four modes of vibrations. It may be noted that the values of β^* approach those of uniform beams as ξ_1 approaches a value of unity. The format of tables and spacings for μ and ξ_1 are so chosen as to give frequencies of the beam with error not exceeding 1% if linear interpolation is used for values not covered by the tables.

9.8 Beams Clamped at Larger End and Simply Supported at the Smaller

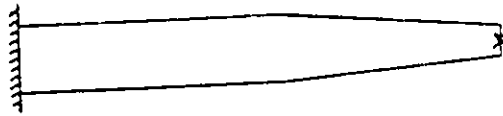


Figure 9.8

It may be appreciated that only equations (9.22), (9.23), and coefficients of A_1 and B_1 in equations (9.24) through (9.27) are explicitly dependent on outer end boundary conditions. Only these therefore would be presented for the rest of the boundary conditions considered for this class of beams. Since the boundary conditions at the left hand end are same as in case 9.7, only the first two equations of the set of six homogeneous equations will be affected. Resulting equations for this case are same as for case 9.7 except equation (9.23) which should be replaced by:

$$A_2 J_2(2\beta\gamma) + B_2 Y_2(2\beta\gamma) + C_2 I_2(2\beta\gamma) + D_2 K_2(2\beta\gamma) = 0 \quad (9.28)$$

Following the procedure explained in case 9.7, the computed

values of β^* are presented in Tables 9.8 for various values of the taper ratios ξ_1 and parameter μ .

9.9 Beams Clamped at Both Ends

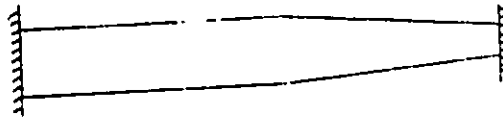


Figure 9.9

In this case the first two equations of the set of six homogeneous equations are:

$$A_2 J_2(2\beta\gamma) + B_2 Y_2(2\beta\gamma) + C_2 I_2(2\beta\gamma) + D_2 K_2(2\beta\gamma) = 0 \quad (9.29)$$

and

$$A_2 J_3(2\beta\gamma) + B_2 Y_3(2\beta\gamma) - C_2 I_3(2\beta\gamma) + D_2 K_3(2\beta\gamma) = 0 \quad (9.30)$$

Rest of the equations are same as provided in case 9.7,
and the results are presented in Tables 9.9.

9.10 Beams Simply Supported at Both Ends



Figure 9.9

Only the coefficients of A_1 and B_1 in equations (7.24) through (7.27) explicitly depend on left hand boundary conditions which can be calculated from the shape function for the first span which is given by

$$R_1(\xi) = A_1 \sin \beta \xi + C_1 \sinh \beta \xi$$

The rest of the equations are same as provided in case 9.8,
and the results are presented in Tables 9.9.

9.11 Beams Simply Supported at Larger End and Clamped at the Smaller

End

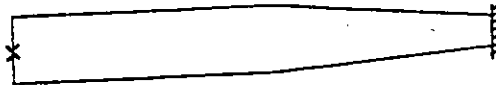


Figure 9.11

The set of six homogeneous equations is similar to that provided in case 9.9 except for the coefficients of first two constants in equations (9.24) through (9.27) which may be obtained from the deflection shape function provided in case 9.10. The results are presented in Tables 9.11.

9.12 Beams Free at the Larger End and Clamped at the Smaller



Figure 9.12

Except for the coefficients α_{31} , α_{32} , α_{41} , α_{62} , α_{51} , α_{52} , α_{61} , and α_{82} , the equations are same provided in case 9.9. These coefficients may be determined from the deflection shape of the first span which is given by

$$R_1(\xi) = A_1(\sin \beta \xi + \sinh \beta \xi) + B_1(\cos \beta \xi + \cosh \beta \xi)$$

The results are provided in Tables 9.12.

Part 3 - Tapered beams with constant width and linearly varying depth.

These types of beams are commonly referred to as wedge shaped beams in the literature. The frequency equations and computed eigenvalues of these beams for six combinations of boundary conditions of practical interest are presented in this part of the Chapter.

9.13 Beams Simply Supported on Both Ends (Figure 9.2)

The deflection function of such beams is provided by equation (1.34) of the first Chapter which is re-written below

$$R(\xi) = \xi^{-1/2} [A J_1(z) + B Y_1(z) + C I_1(z) + D K_1(z)] \quad (9.31)$$

where

$$z = 2\beta \xi^{1/2}$$

Following the procedure outlined in case 9.1, the following determinantal equation for frequencies is obtained for this case.

$$\begin{vmatrix} J_1(z_0) & Y_1(z_0) & I_1(z_0) & K_1(z_0) \\ J_3(z_0) & Y_3(z_0) & I_3(z_0) & K_3(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_3(2\beta) & Y_3(2\beta) & I_3(2\beta) & K_3(2\beta) \end{vmatrix} = 0 \quad (9.32)$$

where

$$z_0 = 2\beta \xi_0^{1/2}$$

The plots of β^* versus ξ_0 are presented in Figure 9.13 for first six modes of vibrations.

9.14 Beams Clamped at the Larger End and Simply Supported at the Smaller (Figure 9.3)

The frequency equation for this case is given by

$$\begin{vmatrix} J_1(z_0) & Y_1(z_0) & I_1(z_0) & K_1(z_0) \\ J_3(z_0) & Y_3(z_0) & I_3(z_0) & K_3(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_5(2\beta) & Y_2(2\beta) & -I_2(2\beta) & K_2(2\beta) \end{vmatrix} = 0 \quad (9.33)$$

The results are presented in Figure 9.14.

9.15 Beams Clamped at Both Ends (Figure 9.4)

The frequency equation for this case is given by,

$$\begin{vmatrix} J_1(z_0) & Y_1(z_0) & I_1(z_0) & K_1(z_0) \\ J_2(z_0) & Y_2(z_0) & -I_2(z_0) & K_2(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_2(2\beta) & Y_2(2\beta) & -I_2(2\beta) & K_2(2\beta) \end{vmatrix} = 0 \quad (9.34)$$

and the results are presented in Figure 9.15.

9.16 Beams Clamped at Smaller End and Simply Supported at the Larger End (Figure 9.3)

The frequency equation is expressed by

$$\begin{vmatrix} J_1(z_0) & Y_1(z_0) & I_1(z_0) & K_1(z_0) \\ J_2(z_0) & Y_2(z_0) & -I_2(z_0) & K_2(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_3(2\beta) & Y_3(2\beta) & I_3(2\beta) & K_3(2\beta) \end{vmatrix} = 0 \quad (9.35)$$

and the results are presented in Figure 9.16

9.17 Beams Clamped at Larger End and Free at the Smaller End (Figure 9.5)

The following frequency equation is obtained for this case.

$$\begin{vmatrix} J_3(z_0) & Y_3(z_0) & I_3(z_0) & K_3(z_0) \\ J_2(z_0) & Y_2(z_0) & I_2(z_0) & -K_2(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_2(2\beta) & Y_2(2\beta) & -I_2(2\beta) & K_2(2\beta) \end{vmatrix} = 0 \quad (9.36)$$

and the results are provided in Figure 9.17.

9.18 Beams Clamped at Smaller End and Free at the Larger (Figure 9.6)

The frequency equation is given by,

$$\begin{vmatrix} J_1(z_0) & Y_1(z_0) & I_1(z_0) & K_1(z_0) \\ J_2(z_0) & Y_2(z_0) & -I_2(z_0) & -K_2(z_0) \\ J_3(2\beta) & Y_3(2\beta) & I_3(2\beta) & K_3(2\beta) \\ J_2(2\beta) & Y_2(2\beta) & I_2(2\beta) & -K_2(2\beta) \end{vmatrix} = 0 \quad (9.37)$$

Results are presented in Figure 9.18.

Part 4 - Wedge shaped beams composed of two sections, one of uniform cross-section and the other with a linear variation in depth.

9.19 Beams Clamped at the Larger End and Free at the Smaller (Figure 9.7)

In this case, the area A and second moment of area I of any cross section of the tapered section are related to their corresponding values A_0 and I_0 of the uniform cross section by

$$A(\xi) = \frac{A_0 \xi}{\gamma} \quad \text{and} \quad I(\xi) = \frac{I_0 \xi^3}{\gamma^3}$$

Substituting these in the governing differential equation (9.17) we obtain

$$\frac{d^2}{d\xi^2} \left[\xi^3 \frac{d^2 R_2(\xi)}{d\xi^2} \right] - \beta^4 \gamma^2 \xi R_2(\xi) = 0$$

which has a general solution of the form

$$R_2(\xi) = \xi^{-\frac{1}{2}} [A_2 J_1(z) + B_2 Y_1(z) + C_2 I_1(z) + D_2 K_1(z)] \quad (9.38)$$

where

$$z = 2\beta \gamma^{\frac{1}{2}} \xi^{\frac{1}{2}}$$

Utilizing the boundary conditions at the clamped end, we can write

$$R_1(\xi) = A_1(\sin \beta \xi - \sinh \beta \xi) + B_1(\cos \beta \xi - \cosh \beta \xi) \quad (9.39)$$

At this stage there are six arbitrary constants in equations (9.38) and (9.39). We therefore need six boundary conditions to evaluate them. Two of these boundary conditions are supplied by the truncated end and the remaining four are supplied by the junction of tapered and uniform cross sections.

Utilizing the two boundary conditions at the truncated end,

i.e.

$$R''_2(\xi_0) = 0 \quad \text{and} \quad \frac{d}{d\xi} \left[EI(\xi) \frac{d^2 R_2}{d\xi^2} \right] = 0 \quad \Big|_{\xi=\xi_0}$$

we obtain

$$A_2 J_3(z_0) + B_2 Y_3(z_0) + C_2 I_3(z_0) + D_2 K_3(z_0) = 0 \quad (9.40)$$

and

$$A_2 J_2(z_0) + B_2 Y_2(z_0) + C_2 I_2(z_0) - D_2 K_2(z_0) = 0 \quad (9.41)$$

where

$$z_0 = 2\beta \gamma^{\frac{1}{2}} \xi_0^{\frac{1}{2}}$$

The four boundary conditions, viz: $R_1(\mu) = R_2(\gamma)$,

$R'_1(\mu) = -R'_2(\gamma)$, $R''_1(\mu) = R''_2(\gamma)$ and

$$EI_0 R'''_1(\mu) = - \frac{d}{d\xi} \left[EI(\xi) \frac{d^2 R_2}{d\xi^2} \right] \Big|_{\xi=\gamma}$$

are supplied by the junction involving continuity of deflection, slope,

bending moment and shear force respectively. Utilizing these four boundary conditions we obtain the following four homogeneous equations.

$$A_1(\alpha_{31}) + B_1(\alpha_{32}) - \gamma^{\frac{1}{2}}[A_2 J_1(2\beta\gamma) + B_2 Y_1(2\beta\gamma) + C_2 I_1(2\beta\gamma) + D_2 K_1(2\beta\gamma)] = 0 \quad (9.42)$$

$$A_1(\alpha_{41}) + B_1(\alpha_{42}) - \gamma^{\frac{1}{2}}[A_2 J_2(2\beta\gamma) + B_2 Y_2(2\beta\gamma) - C_2 I_2(2\beta\gamma) + D_2 K_2(2\beta\gamma)] = 0 \quad (9.43)$$

$$A_1(\alpha_{51}) + B_1(\alpha_{52}) - \gamma^{\frac{1}{2}}[A_2 J_3(2\beta\gamma) + B_2 Y_3(2\beta\gamma) + C_2 I_3(2\beta\gamma) + D_2 K_3(2\beta\gamma)] = 0 \quad (9.44)$$

and

$$A_1(\alpha_{61}) + B_1(\alpha_{62}) - \gamma^{\frac{1}{2}}[A_2 J_2(2\beta\gamma) + B_2 Y_2(2\beta\gamma) + C_2 I_2(2\beta\gamma) - D_2 K_2(2\beta\gamma)] = 0 \quad (9.45)$$

where

$\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}, \alpha_{51}, \alpha_{52}, \alpha_{61}, \alpha_{62}$ are same as provided

in case 9.7.

For a non-trivial solution for the constants A_1, B_1, A_2, B_2, C_2 , and D_2 in equations (9.40) through (9.45), the determinant of their coefficient matrix is set equal to zero. It can be easily observed that the resulting frequency equation is a function of μ, β , and ξ_0 . The quality ξ_0 as used above can obviously not be greater than γ . If ξ_1 is

defined as the ratio of the depth at the end $\xi = \xi_0$ to that at the junction, it is seen that $\xi_0 = \gamma \xi_1$ so that as the taper ratio ξ_1 varies from 0.0 to 1.0, ξ_0 varies from 0.0 to γ . The value of β for prescribed values of μ , 0.1 μ 0.9 and taper ratio ξ_1 , $0.05 \leq \xi_1 \leq 0.9$ can then be determined following the iteration procedure as explained earlier. The values of β so obtained will be based on a length which is sum of the actual length of the beam and the distance ξ_0 of the truncated end from the origin for the tapered span. It is however advantageous to use those values of β which are based on the actual length of the beam. Denoting them by β^* , it is easily recognised that $\beta^* = \beta (1 - \xi_0)$ as the eigenvalues are proportional to the length of the beam.

These values of β^* are presented in Tables 9.19 as a function of μ and ξ_1 . Linear interpolation may be used to obtain values of β^* for those values of μ and ξ_1 which lie partway between the tabulated values.

It may be appreciated that only the equations (9.40) and (9.41) are explicitly dependent on the boundary conditions at the truncated end. Similarly the coefficients a_{31} , a_{32} , a_{41} , a_{42} , a_{51} , a_{52} , a_{61} and a_{62} are explicitly dependent on the left hand end boundary conditions. These coefficients are same as provided in case 9.7 through 9.12 for corresponding boundary conditions and will not be repeated here. For other boundary conditions that may prevail at the truncated end, the equations corresponding to (9.40) and (9.41) are presented below

Right Hand End Simply Supported

Since $R_2(\xi_0) - R'_2(\xi_0) = 0$ for this end we have

$$A_2 J_1(z_0) + B_2 Y_1(z_0) + C_2 I_1(z_0) + D_2 K_1(z_0) = 0$$

and

$$A_2 J_3(z_0) + B_2 Y_3(z_0) + C_2 I_3(z_0) + D_2 K_3(z_0) = 0$$

Right Hand End Clamped

Corresponding to $R_2(\xi_0) = 0$ and $R'_2(\xi_0) = 0$ we have

$$A_2 J_1(z_0) + B_2 Y_1(z_0) + C_2 I_1(z_0) + D_2 K_1(z_0) = 0$$

and

$$A_2 J_2(z_0) + B_2 Y_2(z_0) - C_2 I_2(z_0) + D_2 K_2(z_0) = 0$$

so the relevant frequency equation can be easily determined for any boundary conditions of interest. For the following boundary conditions, the corresponding tables present the relevant data necessary to establish frequencies of this class of beams.

9.20 Beams Fixed at Larger End and Simply Supported at the Other
(Figure 9.8)

Results are given in Tables 9.20

9.21 Beams Clamped at Both Ends (Figure 9.9)

Results are given in Tables 9.21.

9.22 Beams Simply Supported at Both Ends (Figure 9.10)

Results are given in Tables 9.22..

9.23 Beams Simply Supported at Larger End and Clamped at the Smaller
(Figure 9.11)

Results are given in Tables 9.23.

9.24 Beams Free at the Larger End and Clamped at the Smaller (Figure 9.12)

Results are given in Tables 9.24.

Illustrative Example 9.1

The beam of example 2.1 has a cross section of 1" x 0.25" at the fixed end and linearly tapers to a cross section of $\frac{1}{2}$ " x 0.125" at the other end which is simply supported.

- (a) Determine its fundamental natural frequency

$$\xi_0 = \frac{b_0}{b_1} = \frac{1}{2} = 0.5$$

From Figure 9.1(a), $\beta^* = 3.6$

hence,

$$f_{(1)} = \frac{\beta^{*2}}{2\pi L^2} \sqrt{\frac{EI}{\rho A}} = \underline{35.0 \text{ Hz}}$$

- (b) Determine the second mode frequency of the beam in part (a) if both ends are clamped.

From Figure 9.4(a), $\beta^* = 6.7$

and,

$$f_{(2)} = \underline{121.4 \text{ Hz}}$$

- (c) Determine the fundamental frequency of the beam in part (a) if the smaller end is fixed and the larger end is simply supported.

From Figure 9.3(a), $\beta^* = 3.1$

$$\therefore f_{(1)} = \underline{26 \text{ Hz}}$$

- (d) Determine the first natural frequency of the beam in part (a) if the width remains constant at 1" and depth varies linearly from $\frac{1}{4}$ " at the fixed end to $\frac{1}{8}$ " at the free end.

This is the case of a wedge shaped beam from Figure 9.17, obtain

$$\beta^* = 2 \text{ for } \xi_0 = 0.5, \text{ and}$$

$$\therefore f_{(1)} = \underline{10.8 \text{ Hz}}$$

Illustrative Example 9.2

If the beam shown in Figure 9.7 has a uniform cross section of $1" \times \frac{1}{2}"$ for a length of 20" and then its cross section tapers to $\frac{1}{2}" \times \frac{1}{4}"$ in a length of 40". The beam material is steel.

- (a) Determine its fundamental natural frequency.

$$\text{Here } L_1 = 20 + 40 \times 2 = 100"$$

$$\therefore \mu = \frac{20}{100} = 0.2$$

and

$$\xi_1 = \frac{b}{b_0} = \frac{1}{2} = 0.5$$

Obtain $\beta_1^* = 2.272$ from tables 9.7 for $\mu = 0.2$ and $\xi_1 = 0.5$

$$f_1 = \frac{\beta_1^{*2}}{2\pi L^2} \sqrt{\frac{EI_0}{\rho A_0}}$$

where

$$I_0 = \frac{1}{12} \times \left(\frac{1}{2}\right)^3 = 0.0104 \text{ in}^4$$

$$A_0 = 0.5 \text{ in}^2$$

$$f_1 = \frac{(2.272)^2}{2\pi (60)^2} \sqrt{\frac{30 \times 10^6 \times 0.0104 \times 386}{0.283 \times 0.5}} = \underline{21 \text{ Hz}}$$

- (b) If the cross section at the smaller end of the beam in part (a) is $1" \times \frac{1}{4}"$, determine its fundamental frequency.

This corresponds to case 9.19 of the wedge-shaped beams.

From Tables 7.19, obtain β_1^* for $\mu = 0.2$ and $\xi_1 = 0.5$

$$\beta_1^* = 2.060$$

so that

$$f_1 = \left(\frac{2.060}{2.272}\right)^2 \times 21 = \underline{17.4 \text{ Hz}}$$

- (c) Determine the second mode frequency of the beam of part (b).

Again from Tables 9.19, obtain

$$\beta_2^* = 4.5$$

$$\therefore f_2 = \left(\frac{4.5}{2.272}\right)^2 \times 21 = \underline{83 \text{ Hz}}$$

FIGURE 9.13 Frequency chart for wedge shaped beam (Simple-Simple Support).

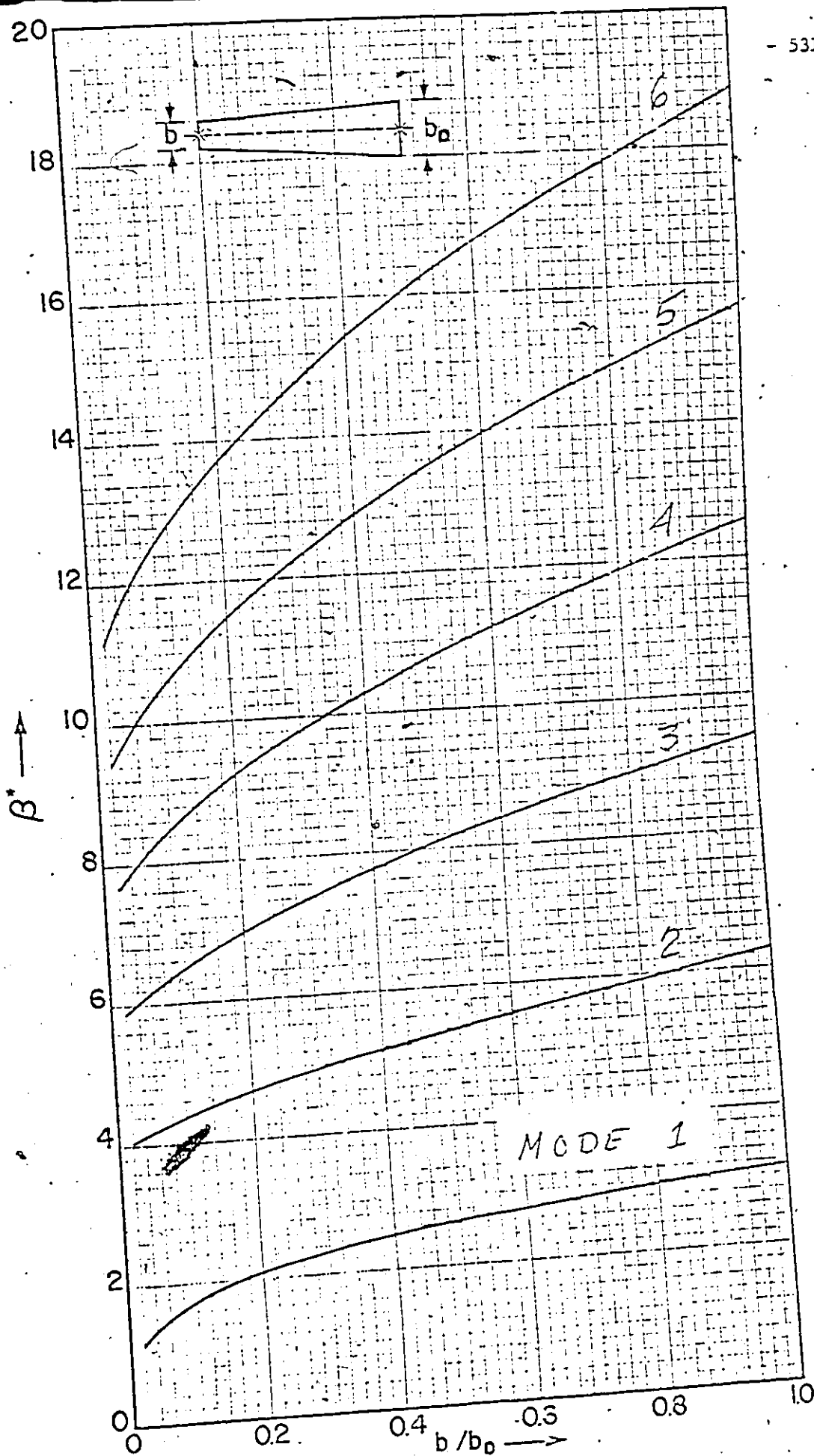
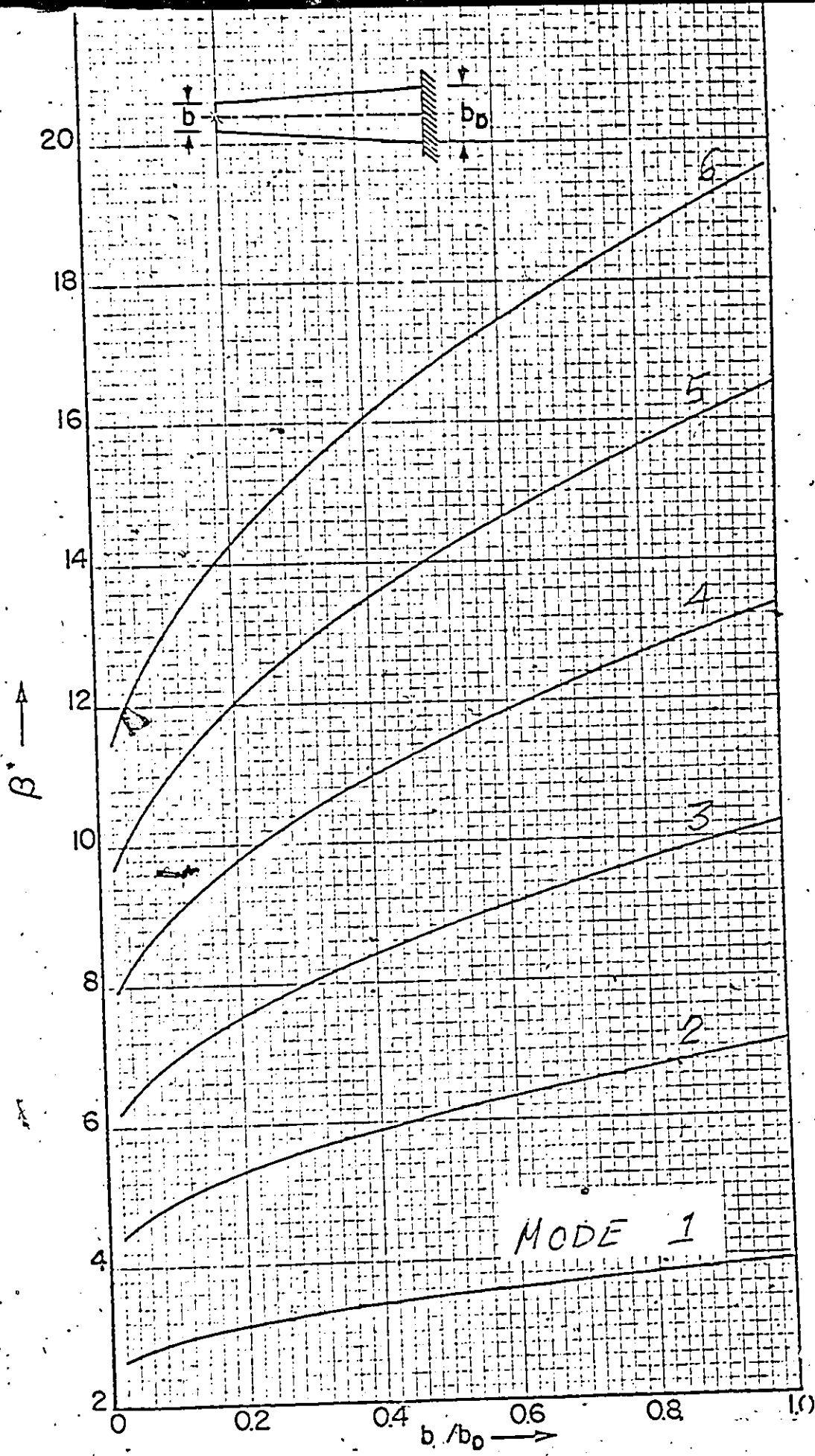


Figure 9.14 Frequency chart for wedge shaped beam (Simply-clamped support).



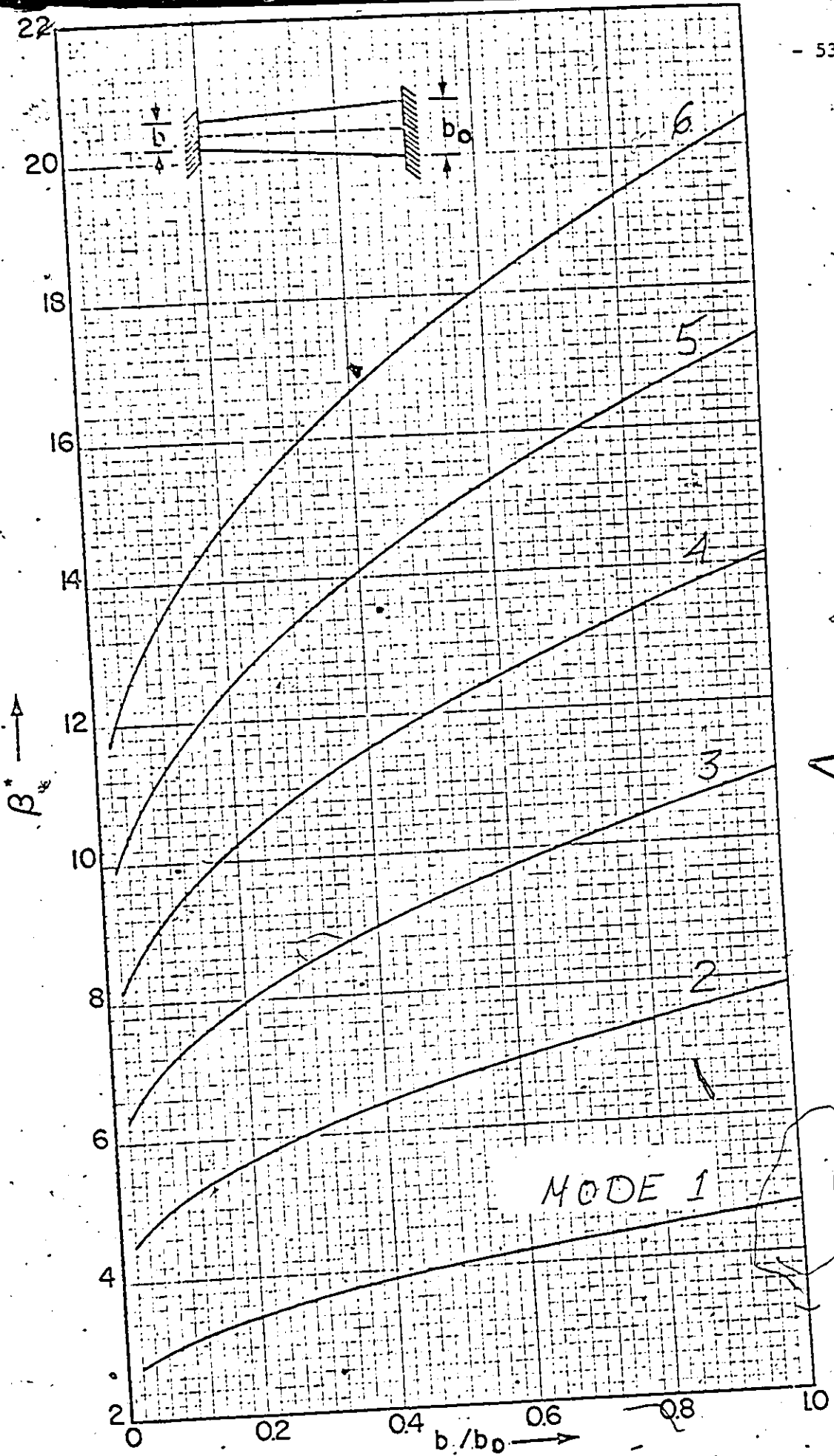


Figure 9.15. Frequency chart for wedge shaped beam (Clamped-clamped)

FIGURE 9.16 Frequency chart for wedge shaped beam (Clamped-Simple Support).

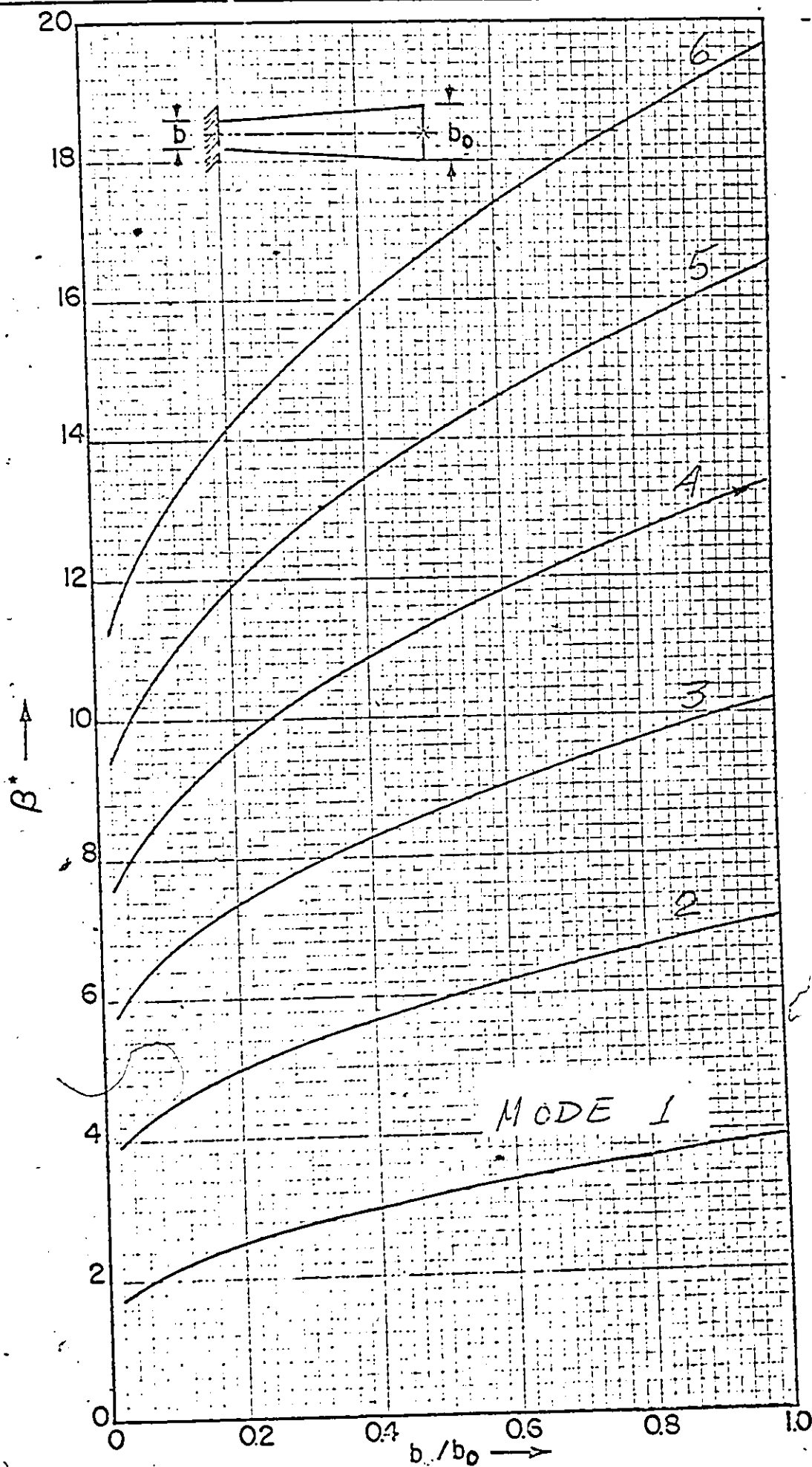
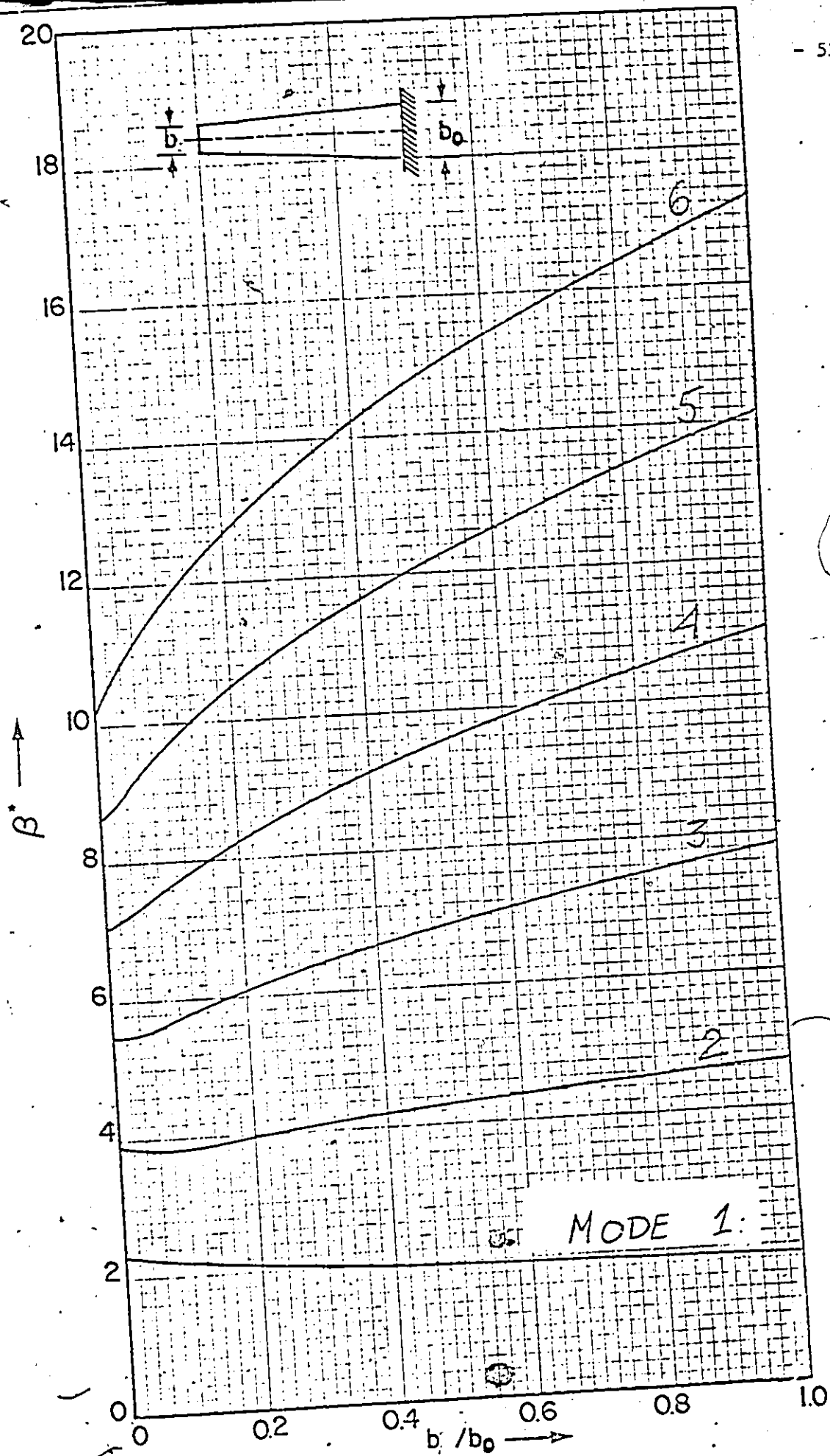
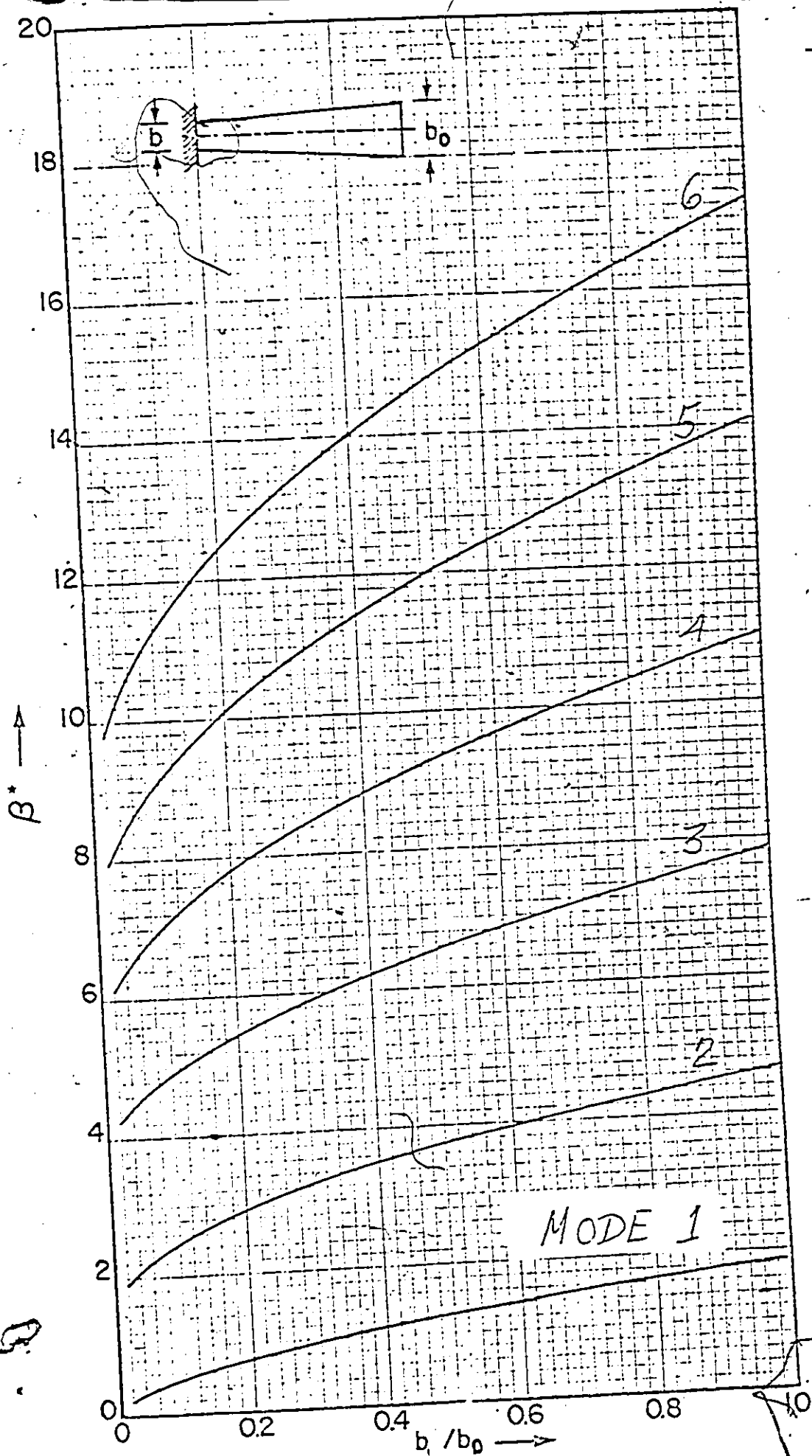


Figure 9.17 Frequency chart for wedge shaped beam (Free-Clamped Support).





MODE 1

TABLE 9.7 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	2.939	2.818	2.622	2.472	2.352	2.252	2.165	2.085	2.008	1.933
0.2	2.989	2.875	2.680	2.521	2.388	2.272	2.169	2.076	1.990	1.917
0.3	2.931	2.831	2.649	2.493	2.358	2.241	2.137	2.045	1.967	1.905
0.4	2.794	2.710	2.554	2.415	2.293	2.186	2.093	2.012	1.946	1.897
0.5	2.622	2.557	2.432	2.318	2.216	2.126	2.048	1.982	1.928	1.891
0.6	2.448	2.399	2.304	2.217	2.138	2.067	2.006	1.954	1.914	1.886
0.7	2.284	2.249	2.183	2.120	2.063	2.012	1.968	1.931	1.902	1.882
0.8	2.133	2.112	2.070	2.031	1.995	1.962	1.933	1.910	1.891	1.879
0.9	1.997	1.987	1.968	1.949	1.932	1.916	1.903	1.891	1.883	1.877

MODE : 2

0.1	4.613	4.514	4.455	4.464	4.496	4.536	4.580	4.627	4.677	4.719
0.2	4.777	4.669	4.584	4.577	4.602	4.642	4.689	4.734	4.761	4.744
0.3	4.996	4.884	4.781	4.761	4.776	4.802	4.824	4.826	4.798	4.743
0.4	5.311	5.195	5.064	5.007	4.977	4.952	4.916	4.865	4.802	4.735
0.5	5.617	5.498	5.324	5.205	5.111	5.026	4.945	4.865	4.790	4.726
0.6	5.703	5.595	5.409	5.258	5.131	5.022	4.926	4.842	4.772	4.718
0.7	5.548	5.466	5.315	5.182	5.066	4.967	4.881	4.809	4.752	4.711
0.8	5.284	5.231	5.131	5.039	4.957	4.885	4.823	4.772	4.732	4.704
0.9	4.989	4.964	4.916	4.870	4.829	4.792	4.759	4.733	4.712	4.699

MODE : 3

0.1	6.310	6.337	6.523	6.733	6.938	7.134	7.321	7.500	7.663	7.802
0.2	6.585	6.588	6.747	6.945	7.142	7.326	7.489	7.632	7.767	7.877
0.3	6.955	6.932	7.045	7.201	7.351	7.490	7.624	7.760	7.877	7.904
0.4	7.351	7.289	7.322	7.418	7.536	7.667	7.801	7.907	7.944	7.906
0.5	7.726	7.640	7.625	7.704	7.815	7.920	7.990	8.003	7.959	7.899
0.6	8.313	8.211	8.136	8.137	8.147	8.135	8.095	8.032	7.958	7.890
0.7	8.848	8.732	8.550	8.414	8.299	8.196	8.101	8.014	7.939	7.881
0.8	8.738	8.653	8.496	8.357	8.238	8.134	8.046	7.972	7.913	7.871
0.9	8.331	8.289	8.210	8.136	8.069	8.010	7.959	7.916	7.884	7.863

MODE : 4

0.1	8.075	8.272	8.713	9.116	9.482	9.818	10.13	10.41	10.67	10.90
0.2	8.452	8.620	9.033	9.415	9.755	10.06	10.33	10.60	10.81	10.97
0.3	8.907	9.028	9.370	9.700	10.01	10.30	10.56	10.75	10.91	11.03
0.4	9.374	9.445	9.736	10.05	10.33	10.55	10.72	10.87	11.01	11.04
0.5	10.00	10.02	10.22	10.41	10.56	10.73	10.89	11.03	11.09	11.04
0.6	10.64	10.57	10.61	10.72	10.87	11.01	11.12	11.14	11.10	11.04
0.7	11.37	11.28	11.24	11.27	11.29	11.29	11.24	11.17	11.09	11.03
0.8	12.02	11.90	11.72	11.57	11.44	11.33	11.27	11.14	11.07	11.02
0.9	11.63	11.58	11.47	11.37	11.28	11.20	11.13	11.08	11.03	11.01

TABLE 9.8 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	3.173	3.252	3.377	3.479	3.565	3.641	3.709	3.772	3.834	3.898
0.2	3.193	3.260	3.370	3.465	3.551	3.633	3.713	3.793	3.867	3.917
0.3	3.115	3.185	3.313	3.431	3.543	3.651	3.751	3.836	3.896	3.917
0.4	2.988	3.089	3.272	3.435	3.581	3.706	3.806	3.875	3.913	3.925
0.5	2.874	3.032	3.295	3.502	3.662	3.780	3.857	3.901	3.921	3.926
0.6	2.829	3.069	3.405	3.623	3.763	3.847	3.894	3.916	3.925	3.926
0.7	2.927	3.249	3.595	3.764	3.850	3.894	3.914	3.923	3.926	3.927
0.8	3.272	3.579	3.797	3.872	3.904	3.917	3.923	3.926	3.926	3.927
0.9	3.786	3.869	3.909	3.920	3.924	3.926	3.926	3.927	3.927	3.927

MODE : 2

0.1	5.075	5.277	5.596	5.858	6.087	6.295	6.489	6.672	6.841	6.981
0.2	5.211	5.405	5.720	5.984	6.217	6.426	6.610	6.765	6.897	7.023
0.3	5.411	5.603	5.914	6.164	6.371	6.540	6.682	6.814	6.949	7.050
0.4	5.687	5.855	6.110	6.301	6.455	6.593	6.733	6.877	7.001	7.061
0.5	5.906	6.018	6.192	6.341	6.491	6.652	6.814	6.952	7.037	7.066
0.6	5.913	6.000	6.178	6.372	6.574	6.764	6.915	7.011	7.056	7.068
0.7	5.772	5.921	6.230	6.517	6.748	6.907	7.001	7.047	7.064	7.068
0.8	5.713	6.049	6.531	6.802	6.945	7.016	7.049	7.063	7.068	7.069
0.9	6.377	6.749	6.964	7.027	7.058	7.062	7.067	7.068	7.068	7.069

MODE : 3

0.1	6.989	7.335	7.869	8.302	8.678	9.017	9.327	9.610	9.862	10.09
0.2	7.248	7.587	8.116	8.542	8.904	9.217	9.494	9.752	9.982	10.13
0.3	7.589	7.907	8.391	8.769	9.094	9.390	9.664	9.885	10.04	10.12
0.4	7.917	8.192	8.629	8.998	9.325	9.598	9.799	9.944	10.08	10.19
0.5	8.251	8.524	8.965	9.303	9.547	9.718	9.857	10.00	10.14	10.21
0.6	8.778	8.998	9.294	9.482	9.626	9.773	9.934	10.08	10.18	10.21
0.7	9.103	9.192	9.345	9.509	9.699	9.895	10.05	10.15	10.20	10.21
0.8	8.945	9.080	9.388	9.692	9.929	10.08	10.16	10.19	10.21	10.21
0.9	9.013	9.488	9.934	10.09	10.16	10.19	10.21	10.21	10.21	10.21

MODE : 4

0.1	8.924	9.420	10.17	10.78	11.29	11.76	12.17	12.55	12.90	13.19
0.2	9.296	9.783	10.52	11.10	11.58	12.02	12.41	12.76	13.03	13.25
0.3	9.719	10.17	10.85	11.41	11.88	12.29	12.61	12.90	13.14	13.29
0.4	10.16	10.59	11.26	11.77	12.16	12.48	12.78	13.03	13.18	13.32
0.5	10.73	11.11	11.63	12.04	12.40	12.71	12.94	13.09	13.23	13.28
0.6	11.21	11.51	12.01	12.40	12.68	12.86	13.00	13.16	13.28	13.28
0.7	11.89	12.14	12.45	12.63	12.77	12.93	13.10	13.25	13.30	13.37
0.8	12.22	12.31	12.47	12.68	12.92	13.12	13.25	13.31	13.33	13.35
0.9	12.00	12.33	12.86	13.13	13.27	13.33	13.33	13.36	13.37	13.34

TABLE 9.9 β^*

MODE : 1

ϵ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	3.243	3.376	3.597	3.782	3.945	4.093	4.231	4.362	4.492	4.619
0.2	3.255	3.373	3.574	3.751	3.915	4.072	4.224	4.371	4.506	4.622
0.3	3.173	3.294	3.513	3.716	3.907	4.086	4.249	4.392	4.512	4.630
0.4	3.050	3.205	3.483	3.729	3.944	4.127	4.279	4.403	4.519	4.644
0.5	2.951	3.169	3.523	3.796	4.007	4.169	4.297	4.411	4.533	4.660
0.6	2.933	3.232	3.636	3.894	4.067	4.196	4.309	4.427	4.556	4.676
0.7	3.065	3.424	3.794	3.981	4.105	4.213	4.328	4.458	4.589	4.691
0.8	3.417	3.711	3.924	4.031	4.129	4.242	4.374	4.512	4.629	4.705
0.9	3.836	3.911	3.980	4.065	4.187	4.335	4.481	4.598	4.677	4.718

MODE : 2

0.1	5.196	5.474	5.905	6.254	6.557	6.830	7.083	7.319	7.532	7.714
0.2	5.327	5.597	6.025	6.378	6.684	6.955	7.189	7.390	7.572	7.740
0.3	5.523	5.790	6.211	6.542	6.813	7.037	7.235	7.427	7.610	7.746
0.4	5.790	6.025	6.376	6.641	6.862	7.071	7.279	7.477	7.632	7.751
0.5	5.987	6.155	6.422	6.658	6.894	7.131	7.347	7.516	7.640	7.762
0.6	5.977	6.121	6.408	6.704	6.986	7.225	7.403	7.533	7.650	7.777
0.7	5.845	6.067	6.499	6.861	7.125	7.303	7.431	7.546	7.672	7.795
0.8	5.846	6.265	6.797	7.070	7.227	7.341	7.453	7.580	7.712	7.815
0.9	6.559	6.884	7.080	7.176	7.273	7.392	7.530	7.665	7.772	7.834

MODE : 3

0.1	7.156	7.588	8.238	8.754	9.198	9.595	9.955	10.28	10.57	10.84
0.2	7.412	7.838	8.484	8.993	9.419	9.787	10.17	10.42	10.68	10.87
0.3	7.746	8.148	8.743	9.205	9.601	9.961	10.28	10.53	10.71	10.89
0.4	8.059	8.417	8.924	9.435	9.828	10.14	10.37	10.56	10.75	10.89
0.5	8.389	8.748	9.302	9.705	9.990	10.21	10.41	10.61	10.77	10.89
0.6	8.898	9.184	9.559	9.809	10.03	10.25	10.48	10.66	10.78	10.91
0.7	9.173	9.314	9.566	9.835	10.12	10.36	10.55	10.68	10.80	10.92
0.8	9.013	9.222	9.666	10.04	10.30	10.46	10.58	10.70	10.83	10.94
0.9	9.207	9.718	10.13	10.29	10.40	10.50	10.63	10.77	10.89	10.97

MODE : 4

0.1	9.129	9.715	10.58	10.27	11.84	12.36	12.82	13.23	13.62	13.94
0.2	9.500	10.08	10.92	11.58	12.13	12.62	13.06	13.44	13.75	13.99
0.3	9.914	10.45	11.26	11.89	12.43	12.88	13.24	13.57	13.83	14.01
0.4	10.34	10.87	11.66	12.23	12.68	13.06	13.41	13.67	13.86	14.00
0.5	10.91	11.36	11.99	12.49	12.92	13.27	13.51	13.71	13.89	13.97
0.6	11.36	11.75	12.37	12.82	13.13	13.34	13.56	13.76	13.87	14.11
0.7	12.02	12.33	12.71	12.95	13.18	13.42	13.64	13.82	13.92	14.12
0.8	12.29	12.42	12.71	13.03	13.34	13.55	13.71	13.85	13.95	14.23
0.9	12.13	12.57	13.11	13.41	13.54	13.65	13.75	13.87	14.01	14.10

TABLE 9.10 β^*

MODE : 1

ϵ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	1.503	1.793	2.141	2.376	2.557	2.707	2.835	2.947	3.044	3.118
0.2	1.549	1.848	2.207	2.449	2.636	2.788	2.915	3.019	3.095	3.134
0.3	1.611	1.921	2.293	2.542	2.730	2.877	2.991	3.073	3.121	3.140
0.4	1.698	2.023	2.408	2.657	2.836	2.965	3.053	3.107	3.133	3.141
0.5	1.824	2.166	2.555	2.789	2.941	3.038	3.096	3.126	3.138	3.141
0.6	2.011	2.365	2.733	2.925	3.032	3.091	3.121	3.135	3.140	3.142
0.7	2.293	2.630	2.920	3.039	3.095	3.122	3.134	3.139	3.141	3.142
0.8	2.694	2.924	3.065	3.110	3.128	3.136	3.140	3.141	3.142	3.142
0.9	3.063	3.109	3.132	3.138	3.140	3.141	3.141	3.142	3.142	3.142

MODE : 2

0.1	4.381	4.568	4.876	5.135	5.365	5.573	5.763	5.934	6.082	6.206
0.2	4.620	4.800	5.095	5.338	5.546	5.725	5.877	6.007	6.131	6.248
0.3	4.883	5.044	5.299	5.498	5.660	5.798	5.927	6.057	6.185	6.270
0.4	5.103	5.225	5.413	5.564	5.701	5.840	5.984	6.125	6.231	6.278
0.5	5.173	5.260	5.418	5.577	5.744	5.915	6.072	6.192	6.260	6.281
0.6	5.082	5.187	5.409	5.637	5.853	6.034	6.165	6.241	6.274	6.283
0.7	4.951	5.151	5.522	5.818	6.028	6.160	6.233	6.268	6.280	6.283
0.8	4.998	5.388	5.854	6.081	6.192	6.245	6.270	6.279	6.283	6.283
0.9	5.755	6.050	6.208	6.254	6.271	6.279	6.282	6.283	6.283	6.283

MODE : 3

0.1	6.402	6.726	7.233	7.647	8.006	8.325	8.612	8.870	9.104	9.316
0.2	6.730	7.039	7.517	7.898	8.220	8.505	8.766	9.012	9.215	9.355
0.3	7.043	7.318	7.744	8.093	8.403	8.688	8.935	9.122	9.261	9.389
0.4	7.312	7.569	7.990	8.343	8.639	8.869	9.036	9.172	9.311	9.410
0.5	7.687	7.941	8.331	8.608	8.802	8.949	9.088	9.236	9.363	9.419
0.6	8.165	8.330	8.553	8.708	8.852	9.009	9.175	9.316	9.398	9.423
0.7	8.306	8.387	8.550	8.739	9.948	9.145	9.293	9.379	9.415	9.424
0.8	8.119	8.288	8.649	8.964	9.185	9.316	9.383	9.413	9.423	9.425
0.9	8.319	8.803	9.199	9.333	9.387	9.410	9.420	9.423	9.425	9.425

MODE : 4

0.1	8.383	8.853	9.570	10.14	10.63	11.07	11.46	11.81	12.14	12.41
0.2	8.785	9.231	9.907	10.44	10.90	11.32	11.69	12.00	12.27	12.48
0.3	9.161	9.581	10.23	10.76	11.20	11.57	11.87	12.14	12.36	12.51
0.4	9.614	10.02	10.63	11.08	11.43	11.75	12.04	12.26	12.41	12.54
0.5	10.14	10.46	10.94	11.34	11.69	11.98	12.17	12.31	12.46	12.55
0.6	10.56	10.86	11.34	11.70	11.93	12.09	12.23	12.39	12.51	12.56
0.7	11.25	11.45	11.69	11.85	12.00	12.17	12.34	12.48	12.55	12.56
0.8	11.41	11.50	11.69	11.92	12.17	12.36	12.48	12.54	12.56	12.57
0.9	11.23	11.60	12.12	12.37	12.48	12.53	12.55	12.56	12.57	12.57

TABLE 9.11 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	1.634	1.969	2.395	2.701	2.950	3.166	3.357	3.530	3.686	3.817
0.2	1.683	2.026	2.460	2.770	3.021	3.234	3.419	3.577	3.708	3.823
0.3	1.748	2.102	2.545	2.857	3.102	3.303	3.466	3.600	3.715	3.834
0.4	1.839	2.206	2.656	2.959	3.185	3.358	3.494	3.610	3.725	3.849
0.5	1.972	2.351	2.793	3.066	3.254	3.393	3.508	3.621	3.744	3.865
0.6	2.165	2.546	2.944	3.159	3.300	3.412	3.522	3.642	3.770	3.880
0.7	2.445	2.787	3.079	3.221	3.325	3.429	3.547	3.677	3.802	3.893
0.8	2.810	3.019	3.164	3.252	3.348	3.466	3.601	3.733	3.840	3.906
0.9	3.094	3.140	3.199	3.288	3.418	3.568	3.707	3.813	3.882	3.917

MODE : 2

0.1	4.493	4.757	5.179	5.528	5.832	6.105	6.351	6.573	6.765	6.936
0.2	4.728	4.983	5.389	5.718	5.996	6.233	6.439	6.623	6.801	6.958
0.3	4.983	5.213	5.570	5.847	6.076	6.279	6.472	6.664	6.836	6.962
0.4	5.188	5.369	5.650	5.882	6.099	6.314	6.524	6.711	6.851	6.969
0.5	5.241	5.383	5.641	5.894	6.147	6.386	6.587	6.739	6.858	6.981
0.6	5.146	5.313	5.651	5.974	6.255	6.473	6.630	6.752	6.870	6.997
0.7	5.039	5.318	5.798	6.146	6.378	6.532	6.651	6.767	6.895	7.015
0.8	5.155	5.608	6.092	6.319	6.453	6.561	6.676	6.806	6.936	7.033
0.9	5.906	6.157	6.309	6.396	6.495	6.619	6.760	6.893	6.994	7.051

MODE : 3

0.1	6.561	6.972	7.596	8.095	8.521	8.898	9.234	9.536	9.814	10.06
0.2	6.885	7.278	7.871	8.334	8.724	9.068	9.385	9.675	9.905	10.08
0.3	7.187	7.543	8.084	8.522	8.907	9.249	9.535	9.752	9.937	10.10
0.4	7.447	7.787	8.329	8.768	9.120	9.387	9.594	9.788	9.973	10.10
0.5	7.818	8.153	8.642	8.979	9.222	9.429	9.638	9.839	9.991	10.11
0.6	8.266	8.490	8.793	9.025	9.253	9.491	9.711	9.877	9.999	10.12
0.7	8.370	8.505	8.777	9.074	9.365	9.602	9.770	9.893	10.01	10.14
0.8	8.196	8.447	8.935	9.301	9.533	9.680	9.796	9.915	10.05	10.16
0.9	8.525	9.016	9.372	9.513	9.615	9.724	9.853	9.991	10.11	10.19

MODE : 4

0.1	8.582	9.142	9.973	10.63	11.18	11.66	12.10	12.50	12.86	13.17
0.2	8.978	9.514	10.30	10.92	11.45	11.92	12.33	12.68	12.99	13.21
0.3	9.347	9.858	10.63	11.24	11.74	12.15	12.50	12.82	13.05	13.23
0.4	9.796	10.29	11.01	11.53	11.95	12.33	12.66	12.89	13.08	13.24
0.5	10.30	10.70	11.30	11.79	12.20	12.51	12.73	12.93	13.12	13.25
0.6	10.71	11.10	11.69	12.09	12.36	12.57	12.78	12.99	13.13	13.25
0.7	11.36	11.62	11.93	12.17	12.40	12.65	12.87	13.02	13.14	13.27
0.8	11.47	11.61	11.93	12.28	12.58	12.78	12.93	13.04	13.17	13.29
0.9	11.37	11.84	12.38	12.61	12.74	12.85	12.97	13.10	13.23	13.32

TABLE 9.12 β^*

MODE : 1

ϵ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	0.204	0.343	0.572	0.776	0.960	1.132	1.295	1.452	1.606	1.758
0.2	0.205	0.344	0.577	0.781	0.969	1.146	1.315	1.479	1.638	1.787
0.3	0.208	0.349	0.587	0.796	0.989	1.172	1.134	1.515	1.674	1.809
0.4	0.213	0.358	0.603	0.820	1.020	1.208	1.388	1.557	1.709	1.826
0.5	0.221	0.372	0.627	0.852	1.060	1.254	1.437	1.602	1.742	1.839
0.6	0.232	0.391	0.659	0.895	1.112	1.312	1.494	1.651	1.773	1.849
0.7	0.248	0.418	0.705	0.956	1.183	1.386	1.561	1.702	1.801	1.858
0.8	0.274	0.462	0.777	1.048	1.285	1.485	1.643	1.756	1.828	1.865
0.9	0.326	0.548	0.916	1.217	1.456	1.629	1.744	1.814	1.852	1.870

MODE : 2

0.1	1.858	2.251	2.761	3.137	3.449	3.724	3.972	4.200	4.408	4.580
0.2	1.931	2.336	2.861	3.245	3.563	3.840	4.083	4.295	4.465	4.589
0.3	2.019	2.440	2.980	3.372	3.690	3.956	4.176	4.350	4.481	4.597
0.4	2.131	2.570	3.126	3.517	3.821	4.058	4.238	4.374	4.489	4.610
0.5	2.282	2.742	3.306	3.678	3.943	4.132	4.270	4.385	4.502	4.625
0.6	2.500	2.978	3.521	3.836	4.035	4.173	4.286	4.400	4.524	4.641
0.7	2.831	3.299	3.743	3.957	4.088	4.195	4.305	4.429	4.556	4.656
0.8	3.326	3.668	3.909	4.021	4.115	4.223	4.380	4.480	4.595	4.670
0.9	3.823	3.905	3.975	4.055	4.169	4.309	4.449	4.564	4.642	4.682

MODE : 3

0.1	4.836	5.147	5.645	6.057	6.418	6.743	7.041	7.309	7.538	7.717
0.2	5.120	5.427	5.917	6.318	6.661	6.955	7.200	7.401	7.577	7.742
0.3	5.450	5.742	6.198	6.551	6.831	7.056	7.248	7.434	7.613	7.747
0.4	5.793	6.039	6.403	6.668	6.884	7.085	7.287	7.481	7.634	7.753
0.5	6.025	6.194	6.455	6.682	6.908	7.138	7.350	7.517	7.642	7.763
0.6	6.021	6.159	6.433	6.718	6.993	7.228	7.405	7.534	7.651	7.779
0.7	5.882	6.093	6.512	6.867	7.128	7.305	7.432	7.547	7.674	7.797
0.8	5.868	6.277	6.801	7.072	7.227	7.342	7.455	7.581	7.713	7.816
0.9	6.564	6.885	7.080	7.177	7.273	7.393	7.531	7.667	7.773	7.835

MODE : 4

0.1	6.956	7.412	8.109	8.666	9.146	9.571	9.951	10.28	10.58	10.84
0.2	7.343	7.791	8.466	8.993	9.428	9.796	10.12	10.42	10.68	10.87
0.3	7.749	8.159	8.759	9.217	9.608	9.964	10.28	10.52	10.71	10.88
0.4	8.079	8.434	8.984	9.439	9.828	10.14	10.37	10.56	10.75	10.89
0.5	8.398	8.753	9.303	9.704	9.989	10.20	10.41	10.61	10.77	10.89
0.6	8.898	9.183	9.557	9.807	10.03	10.25	10.48	10.66	10.78	10.90
0.7	9.171	9.313	9.565	9.834	10.17	10.36	10.55	10.68	10.80	10.93
0.8	9.012	9.221	9.665	10.04	10.30	10.46	10.58	10.69	10.83	10.95
0.9	9.207	9.717	10.13	10.29	10.40	10.50	10.63	10.77	10.89	10.97

TABLE 9.19 β^*

MODE : 1

ϵ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	2.320	2.258	2.173	2.115	2.071	2.034	2.001	1.970	1.938	1.904
0.2	2.391	2.327	2.235	2.166	2.110	2.060	2.015	1.973	1.933	1.896
0.3	2.411	2.349	2.251	2.175	2.111	2.056	2.006	1.961	1.922	1.890
0.4	2.381	2.324	2.230	2.155	2.091	2.035	1.987	1.946	1.912	1.886
0.5	2.317	2.268	2.185	2.117	2.059	2.009	1.967	1.931	1.903	1.883
0.6	2.234	2.195	2.127	2.070	2.022	1.981	1.946	1.917	1.895	1.881
0.7	2.144	2.115	2.064	2.021	1.984	1.952	1.926	1.905	1.889	1.879
0.8	2.052	2.033	2.000	1.971	1.946	1.925	1.908	1.894	1.884	1.877
0.9	1.962	1.953	1.936	1.922	1.910	1.899	1.891	1.884	1.879	1.876

MODE : 2

0.1	4.009	4.027	4.115	4.211	4.305	4.393	4.476	4.555	4.632	4.695
0.2	4.185	4.170	4.239	4.325	4.413	4.500	4.582	4.655	4.707	4.717
0.3	4.351	4.346	4.402	4.481	4.562	4.636	4.696	4.733	4.739	4.718
0.4	4.606	4.590	4.621	4.674	4.723	4.759	4.774	4.769	4.746	4.714
0.5	4.903	4.868	4.849	4.848	4.844	4.830	4.807	4.776	4.742	4.710
0.6	5.124	5.068	4.994	4.941	4.894	4.850	4.808	4.769	4.734	4.706
0.7	5.165	5.108	5.016	4.945	4.886	4.835	4.791	4.754	4.724	4.703
0.8	5.065	5.022	4.950	4.890	4.840	4.799	4.764	4.735	4.714	4.699
0.9	4.896	4.874	4.835	4.802	4.774	4.750	4.731	4.715	4.704	4.697

MODE : 3

0.1	5.785	5.953	6.276	6.560	6.814	7.045	7.260	7.460	7.638	7.787
0.2	6.033	6.184	6.490	6.766	7.011	7.229	7.419	7.583	7.733	7.853
0.3	6.348	6.478	6.752	6.995	7.202	7.381	7.544	7.699	7.829	7.876
0.4	6.699	6.790	7.000	7.193	7.371	7.541	7.699	7.824	7.885	7.880
0.5	7.036	7.089	7.253	7.432	7.602	7.750	7.855	7.905	7.905	7.877
0.6	7.486	7.515	7.640	7.771	7.871	7.930	7.948	7.936	7.906	7.873
0.7	8.093	8.065	8.055	8.051	8.035	8.008	7.973	7.934	7.897	7.868
0.8	8.335	8.276	8.184	8.112	8.052	8.000	7.954	7.916	7.885	7.863
0.9	8.169	8.134	8.072	8.021	7.978	7.942	7.912	7.888	7.870	7.859

MODE : 4

0.1	7.632	7.966	8.524	8.988	9.392	9.755	10.08	10.38	10.65	10.89
0.2	7.976	8.292	8.831	9.276	9.656	9.987	10.28	10.56	10.79	10.95
0.3	8.381	8.660	9.144	9.545	9.897	10.21	10.49	10.70	10.87	11.00
0.4	8.794	9.032	9.476	9.860	10.18	10.44	10.64	10.81	10.97	11.02
0.5	9.316	9.517	9.894	10.19	10.42	10.61	10.79	10.95	11.03	11.02
0.6	9.916	10.02	10.25	10.46	10.66	10.85	10.98	11.05	11.05	11.02
0.7	10.49	10.55	10.72	10.89	11.01	11.08	11.09	11.08	11.05	11.01
0.8	11.34	11.30	11.26	11.23	11.19	11.16	11.11	11.07	11.03	11.00
0.9	11.40	11.35	11.27	11.21	11.15	11.11	11.07	11.04	11.01	11.00

TABLE 9.20 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	2.903	3.038	3.223	3.362	3.477	3.577	3.665	3.747	3.824	3.896
0.2	2.960	3.089	3.265	3.399	3.511	3.612	3.705	3.791	3.867	3.917
0.3	2.981	3.111	3.293	3.435	3.557	3.666	3.762	3.843	3.898	3.923
0.4	2.996	3.140	3.344	3.501	3.631	3.739	3.823	3.882	3.914	3.926
0.5	3.048	3.214	3.440	3.602	3.723	3.811	3.871	3.906	3.922	3.926
0.6	3.177	3.359	3.582	3.722	3.812	3.868	3.901	3.918	3.925	3.926
0.7	3.407	3.571	3.741	3.828	3.877	3.904	3.917	3.924	3.926	3.927
0.8	3.699	3.787	3.863	3.896	3.912	3.920	3.924	3.926	3.926	3.927
0.9	3.892	3.907	3.918	3.923	3.925	3.926	3.926	3.927	3.927	3.927

MODE : 2

0.1	4.825	5.088	5.471	5.772	6.029	6.258	6.468	6.662	6.838	6.981
0.2	4.983	5.242	5.620	5.919	6.175	6.399	6.594	6.758	6.897	7.025
0.3	5.185	5.442	5.813	6.097	6.327	6.516	6.675	6.818	6.954	7.051
0.4	5.439	5.677	6.005	6.241	6.430	6.593	6.746	6.891	7.007	7.062
0.5	5.670	5.866	6.132	6.335	6.516	6.686	6.842	6.966	7.040	7.066
0.6	5.798	5.970	6.231	6.453	6.650	6.817	6.943	7.021	7.057	7.068
0.7	5.895	6.106	6.422	6.659	6.832	6.948	7.016	7.051	7.065	7.068
0.8	6.217	6.470	6.754	6.905	6.988	7.032	7.055	7.064	7.068	7.069
0.9	6.870	6.953	7.019	7.046	7.058	7.064	7.067	7.068	7.068	7.069

MODE : 3

0.1	6.772	7.178	7.772	8.239	8.637	8.992	9.313	9.603	9.850	10.09
0.2	7.039	7.440	8.027	8.485	8.867	9.194	9.482	9.747	9.978	10.13
0.3	7.365	7.748	8.295	8.712	9.060	9.371	9.650	9.875	10.03	10.17
0.4	7.688	8.036	8.543	8.945	9.288	9.570	9.782	9.942	10.09	10.19
0.5	8.024	8.366	8.865	9.232	9.500	9.697	9.858	10.01	10.14	10.20
0.6	8.499	8.800	9.182	9.429	9.619	9.792	9.959	10.09	10.18	10.21
0.7	8.892	9.080	9.342	9.561	9.765	9.947	10.08	10.16	10.20	10.21
0.8	9.042	9.257	9.591	9.839	10.01	10.11	10.17	10.20	10.21	10.21
0.9	9.701	9.900	10.07	10.14	10.18	10.20	10.21	10.21	10.21	10.21

MODE : 4

0.1	8.733	9.287	10.09	10.72	11.26	11.74	12.16	12.54	12.89	13.19
0.2	9.105	9.651	10.44	11.05	11.55	12.00	12.40	12.75	13.03	13.25
0.3	9.515	10.03	10.77	11.36	11.85	12.26	12.60	12.89	13.14	13.29
0.4	9.943	10.44	11.17	11.70	12.12	12.46	12.77	13.02	13.18	13.29
0.5	10.48	10.93	11.53	11.98	12.36	12.69	12.92	13.09	13.24	13.28
0.6	10.97	11.35	11.91	12.33	12.63	12.84	13.01	13.17	13.25	13.33
0.7	11.59	11.93	12.33	12.58	12.78	12.96	13.13	13.26	13.30	13.37
0.8	12.09	12.27	12.54	12.79	13.01	13.17	13.27	13.31	13.33	13.40
0.9	12.51	12.78	13.10	13.20	13.27	13.33	13.33	13.35	13.46	13.40

TABLE 9.21 β^*

ϵ_1	MODE : 1									
	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	3.037	3.239	3.526	3.748	3.935	4.099	4.249	4.387	4.519	4.640
0.2	3.089	3.282	3.559	3.775	3.960	4.128	4.283	4.425	4.548	4.647
0.3	3.107	3.301	3.584	3.810	4.005	4.177	4.329	4.456	4.560	4.655
0.4	3.126	3.335	3.639	3.876	4.071	4.232	4.363	4.471	4.568	4.666
0.5	3.188	3.418	3.735	3.963	4.137	4.272	4.382	4.481	4.581	4.678
0.6	3.325	3.561	3.856	4.046	4.184	4.295	4.396	4.498	4.601	4.690
0.7	3.545	3.744	3.964	4.102	4.211	4.314	4.419	4.527	4.627	4.701
0.8	3.788	3.899	4.029	4.133	4.239	4.352	4.467	4.573	4.658	4.712
0.9	3.925	3.969	4.068	4.191	4.324	4.451	4.558	4.639	4.693	4.721
MODE : 2										
0.1	5.020	5.363	5.854	6.234	6.556	6.841	7.100	7.338	7.550	7.728
0.2	5.176	5.516	6.004	6.384	6.704	6.981	7.218	7.421	7.602	7.761
0.3	5.378	5.715	6.194	6.552	6.839	7.077	7.284	7.478	7.651	7.770
0.4	5.624	5.937	6.363	6.670	6.922	7.146	7.356	7.541	7.678	7.776
0.5	5.835	6.099	6.465	6.753	7.009	7.241	7.436	7.583	7.688	7.785
0.6	5.946	6.192	6.569	6.880	7.140	7.343	7.491	7.601	7.698	7.797
0.7	6.058	6.350	6.772	7.065	7.267	7.408	7.517	7.615	7.718	7.810
0.8	6.408	6.710	7.035	7.214	7.335	7.440	7.543	7.649	7.750	7.826
0.9	6.956	7.057	7.175	7.277	7.386	7.504	7.619	7.720	7.796	7.840
MODE : 3										
0.1	7.014	7.503	8.203	8.744	9.201	9.606	9.969	10.30	10.59	10.84
0.2	7.282	7.768	8.463	8.993	9.431	9.806	10.14	10.44	10.70	10.88
0.3	7.605	8.072	8.724	9.214	9.625	9.989	10.31	10.55	10.74	10.91
0.4	7.917	8.350	8.971	9.455	9.853	10.17	10.40	10.61	10.79	10.91
0.5	8.253	8.684	9.290	9.714	10.02	10.26	10.47	10.67	10.83	10.92
0.6	8.715	9.087	9.548	9.857	10.12	10.36	10.57	10.72	10.83	10.93
0.7	9.052	9.307	9.679	9.995	10.27	10.48	10.63	10.74	10.84	10.94
0.8	9.207	9.510	9.951	10.24	10.43	10.56	10.66	10.77	10.87	10.95
0.9	9.865	10.08	10.28	10.40	10.50	10.61	10.71	10.83	10.91	10.97
MODE : 4										
0.1	9.011	9.648	10.56	11.26	11.85	12.37	12.83	13.25	13.63	13.95
0.2	9.385	10.02	10.91	11.58	12.14	12.63	13.08	13.45	13.77	14.01
0.3	9.791	10.39	11.24	11.90	12.44	12.89	13.26	13.59	13.85	14.01
0.4	10.22	10.81	11.64	12.23	12.69	13.09	13.44	13.69	13.90	14.01
0.5	10.75	11.28	11.98	12.50	12.95	13.30	13.54	13.76	13.89	14.04
0.6	11.21	11.68	12.36	12.83	13.16	13.40	13.63	13.82	13.94	14.05
0.7	11.83	12.23	12.70	13.02	13.28	13.53	13.73	13.89	14.01	14.10
0.8	12.24	12.50	12.89	13.23	13.49	13.70	13.82	13.95	13.98	14.14
0.9	12.71	13.03	13.34	13.54	13.61	13.76	13.87	13.97	14.09	14.18

TABLE 9.22 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	1.856	2.045	2.294	2.475	2.622	2.749	2.861	2.961	3.050	3.119
0.2	1.940	2.132	2.383	2.565	2.713	2.837	2.944	3.033	3.099	3.136
0.3	2.043	2.238	2.490	2.670	2.812	2.927	3.017	3.083	3.124	3.140
0.4	2.172	2.368	2.616	2.786	2.912	3.006	3.071	3.113	3.134	3.141
0.5	2.337	2.527	2.758	2.903	3.001	3.066	3.107	3.129	3.139	3.141
0.6	2.544	2.715	2.902	3.007	3.069	3.106	3.126	3.137	3.141	3.142
0.7	2.785	2.908	3.026	3.082	3.112	3.128	3.136	3.140	3.141	3.142
0.8	3.006	3.060	3.105	3.124	3.133	3.138	3.140	3.141	3.142	3.142
0.9	3.123	3.131	3.137	3.140	3.141	3.141	3.141	3.142	3.142	3.142

MODE : 2

0.1	4.196	4.441	4.801	5.087	5.333	5.551	5.749	5.926	6.078	6.206
0.2	4.411	4.625	5.002	5.275	5.503	5.696	5.861	6.003	6.133	6.249
0.3	4.645	4.872	5.192	5.430	5.621	5.782	5.927	6.065	6.191	6.270
0.4	4.859	5.056	5.326	5.527	5.698	5.855	6.005	6.139	6.236	6.278
0.5	4.997	5.166	5.411	5.611	5.793	5.959	6.101	6.205	6.263	6.282
0.6	5.069	5.248	5.523	5.749	5.937	6.085	6.188	6.248	6.275	6.283
0.7	5.199	5.426	5.742	5.957	6.102	6.193	6.245	6.271	6.281	6.283
0.8	5.581	5.810	6.046	6.163	6.225	6.257	6.273	6.280	6.283	6.283
0.9	6.141	6.201	6.248	6.267	6.276	6.280	6.282	6.283	6.283	6.283

MODE : 3

0.1	6.213	6.594	7.154	7.596	7.972	8.303	8.599	8.863	9.102	9.315
0.2	6.517	6.887	7.424	7.838	8.182	8.483	8.755	9.006	9.211	9.356
0.3	6.815	7.158	7.653	8.039	8.371	8.666	8.919	9.114	9.262	9.391
0.4	7.093	7.419	7.903	8.285	8.596	8.839	9.023	9.174	9.317	9.411
0.5	7.446	7.768	8.220	8.534	8.761	8.938	9.097	9.249	9.368	9.419
0.6	7.887	8.140	8.460	8.679	8.864	9.040	9.203	9.330	9.401	9.423
0.7	8.153	8.329	8.592	8.820	9.027	9.198	9.318	9.387	9.417	9.424
0.8	8.391	8.535	8.875	9.108	9.258	9.346	9.393	9.415	9.423	9.425
0.9	9.003	9.172	9.314	9.373	9.401	9.415	9.421	9.424	9.425	9.425

MODE : 4

0.1	8.207	8.731	9.499	10.10	10.60	11.05	11.44	11.81	12.14	12.41
0.2	8.588	9.094	9.827	10.39	10.87	11.30	11.68	12.00	12.27	12.47
0.3	8.957	9.441	10.15	10.71	11.17	11.54	11.85	12.14	12.36	12.51
0.4	9.391	9.863	10.53	11.01	11.40	11.73	12.03	12.25	12.41	12.54
0.5	9.887	10.29	10.85	11.28	11.65	11.95	12.16	12.32	12.46	12.55
0.6	10.33	10.70	11.24	11.62	11.89	12.07	12.24	12.40	12.52	12.56
0.7	10.95	11.25	11.60	11.82	12.02	12.20	12.37	12.49	12.55	12.56
0.8	11.32	11.50	11.79	12.05	12.26	12.41	12.50	12.54	12.56	12.57
0.9	11.80	12.06	12.33	12.45	12.51	12.54	12.56	12.56	12.57	12.57

TABLE 9.23 β^*

MODE : 1

ϵ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	2.010	2.261	2.604	2.862	3.078	3.266	3.436	3.589	3.727	3.841
0.2	2.095	2.348	2.692	2.950	3.163	3.346	3.506	3.642	3.754	3.848
0.3	2.198	2.453	2.795	3.047	3.249	3.416	3.554	3.667	3.763	3.857
0.4	2.327	2.580	2.911	3.145	3.324	3.465	3.579	3.678	3.773	3.869
0.5	2.489	2.731	3.032	3.230	3.377	3.493	3.593	3.690	3.789	3.881
0.6	2.685	2.895	3.138	3.292	3.408	3.510	3.609	3.711	3.811	3.892
0.7	2.897	3.047	3.213	3.327	3.429	3.531	3.637	3.743	3.836	3.902
0.8	3.070	3.147	3.253	3.353	3.462	3.576	3.689	3.788	3.865	3.911
0.9	3.149	3.189	3.292	3.421	3.556	3.678	3.777	3.849	3.895	3.919

MODE : 2

0.1	4.384	4.709	5.180	5.546	5.857	6.132	6.377	6.595	6.785	6.952
0.2	4.597	4.918	5.377	5.728	6.017	6.262	6.473	6.660	6.835	6.980
0.3	4.825	5.128	5.550	5.862	6.114	6.332	6.532	6.720	6.879	6.987
0.4	5.024	5.292	5.661	5.941	6.183	6.406	6.608	6.776	6.897	6.993
0.5	5.146	5.385	5.738	6.028	6.283	6.502	6.677	6.806	6.905	7.003
0.6	5.218	5.474	5.863	6.169	6.408	6.587	6.717	6.820	6.918	7.016
0.7	5.372	5.674	6.076	6.335	6.509	6.634	6.737	6.837	6.940	7.029
0.8	5.760	6.024	6.295	6.447	6.559	6.661	6.766	6.874	6.972	7.043
0.9	6.209	6.290	6.397	6.499	6.612	6.733	6.848	6.945	7.016	7.056

MODE : 3

0.1	6.449	6.914	7.582	8.098	8.532	8.912	9.250	9.551	9.828	10.07
0.2	6.751	7.205	7.848	8.335	8.738	9.090	9.410	9.697	9.925	10.10
0.3	7.042	7.468	8.071	8.536	8.933	9.277	9.562	9.782	9.970	10.12
0.4	7.314	7.728	8.327	8.783	9.143	9.419	9.639	9.838	10.01	10.13
0.5	7.668	8.075	8.622	8.992	9.265	9.496	9.713	9.903	10.04	10.13
0.6	8.082	8.401	8.806	9.101	9.363	9.602	9.801	9.944	10.05	10.14
0.7	8.304	8.552	8.934	9.256	9.521	9.719	9.857	9.962	10.06	10.16
0.8	8.676	8.794	9.225	9.491	9.661	9.781	9.883	9.986	10.09	10.17
0.9	9.148	9.330	9.505	9.617	9.721	9.831	9.945	10.05	10.14	10.19

MODE : 4

0.1	8.479	9.088	9.957	10.62	11.18	11.67	12.11	12.51	12.87	13.17
0.2	8.859	9.450	10.28	10.92	11.46	11.93	12.35	12.70	13.00	13.22
0.3	9.225	9.798	10.62	11.24	11.75	12.16	12.52	12.84	13.07	13.25
0.4	9.661	10.22	10.98	11.53	11.97	12.36	12.69	12.92	13.12	13.27
0.5	10.14	10.62	11.29	11.81	12.23	12.54	12.78	12.98	13.16	13.27
0.6	10.57	11.03	11.67	12.10	12.40	12.64	12.86	13.05	13.18	13.28
0.7	11.17	11.52	11.95	12.25	12.52	12.77	12.96	13.09	13.19	13.29
0.8	11.47	11.73	12.15	12.49	12.73	12.89	13.01	13.11	13.21	13.31
0.9	12.00	12.30	12.58	12.73	12.85	12.95	13.06	13.17	13.26	13.33

TABLE 9.24 β^*

MODE : 1

ξ_1	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.1	0.383	0.546	0.785	0.974	1.137	1.284	1.420	1.548	1.670	1.787
0.2	0.387	0.553	0.795	0.986	1.152	1.303	1.442	1.573	1.697	1.810
0.3	0.396	0.565	0.811	1.007	1.177	1.330	1.472	1.605	1.726	1.827
0.4	0.407	0.581	0.834	1.035	1.209	1.366	1.509	1.640	1.753	1.839
0.5	0.423	0.604	0.865	1.072	1.251	1.410	1.552	1.676	1.779	1.856
0.6	0.445	0.634	0.907	1.122	1.304	1.463	1.600	1.715	1.802	1.862
0.7	0.477	0.678	0.966	1.189	1.374	1.592	1.656	1.754	1.823	1.867
0.8	0.526	0.746	1.056	1.288	1.470	1.612	1.718	1.794	1.842	1.867
0.9	0.623	0.879	1.223	1.455	1.614	1.721	1.791	1.834	1.859	1.872

MODE : 2

0.1	2.292	2.603	3.025	3.346	3.617	3.856	4.073	4.272	4.453	4.602
0.2	2.405	2.715	3.142	3.466	3.737	3.974	4.184	4.365	4.510	4.614
0.3	2.535	2.849	3.280	3.602	3.867	4.089	4.273	4.419	4.528	4.621
0.4	2.694	3.012	3.439	3.749	3.992	4.182	4.329	4.441	4.536	4.631
0.5	2.895	3.210	3.617	3.893	4.093	4.241	4.356	4.453	4.549	4.643
0.6	3.150	3.444	3.794	4.010	4.158	4.272	4.371	4.468	4.568	4.655
0.7	3.460	3.691	3.937	4.083	4.193	4.292	4.392	4.496	4.593	4.666
0.8	3.760	3.882	4.017	4.119	4.220	4.327	4.437	4.540	4.623	4.676
0.9	3.921	3.965	4.058	4.172	4.299	4.421	4.525	4.604	4.657	4.686

MODE : 3

0.1	4.774	5.145	5.683	6.104	6.465	6.784	7.073	7.332	7.554	7.731
0.2	5.028	5.397	5.932	6.345	6.689	6.981	7.226	7.428	7.606	7.763
0.3	5.320	5.679	6.185	6.559	6.852	7.090	7.293	7.482	7.653	7.772
0.4	5.623	5.947	6.381	6.689	6.936	7.156	7.361	7.543	7.679	7.777
0.5	5.861	6.125	6.487	6.768	7.019	7.246	7.439	7.584	7.689	7.786
0.6	5.976	6.216	6.584	6.889	7.145	7.346	7.493	7.602	7.700	7.798
0.7	6.079	6.364	6.779	7.069	7.269	7.410	7.518	7.617	7.719	7.812
0.8	6.416	6.714	7.037	7.215	7.337	7.441	7.544	7.650	7.751	7.826
0.9	6.957	7.057	7.175	7.278	7.387	7.505	7.621	7.722	7.796	7.840

MODE : 4

0.1	6.875	7.384	8.117	8.685	9.167	9.590	9.966	10.30	10.59	10.84
0.2	7.231	7.735	8.451	8.993	9.437	9.811	10.14	10.44	10.70	10.88
0.3	7.605	8.078	8.734	9.222	9.630	9.991	10.31	10.55	10.74	10.91
0.4	7.931	8.362	8.977	9.457	9.853	10.17	10.40	10.61	10.79	10.91
0.5	8.260	8.688	9.291	9.714	10.02	10.26	10.47	10.67	10.82	10.92
0.6	8.716	9.086	9.547	9.856	10.11	10.36	10.57	10.72	10.83	10.93
0.7	9.051	9.306	9.678	9.995	10.27	10.48	10.63	10.74	10.84	10.94
0.8	9.206	9.509	9.951	10.24	10.43	10.56	10.66	10.77	10.87	10.96
0.9	9.865	10.08	10.28	10.40	10.50	10.61	10.72	10.83	10.92	10.98

CHAPTER 10

THE EFFECTS OF ROTARY INERTIA AND SHEAR DEFORMATION

ON VIBRATIONS OF SINGLE SPAN BEAMS WITH

CLASSICAL AND NON-CLASSICAL BOUNDARY CONDITIONS

The vibration frequencies and modal shape expressions of beams presented so far in this thesis are valid for Euler-Bernoulli beams, which, in the opinion of the author, are of more common occurrence. However, there may arise situations when the beam systems do not conform to the limitations imposed by Euler-Bernoulli beam theory. This may be specially true in higher modes where the distance between nodal points goes on diminishing with the appearance of more nodes at higher frequencies. Consequently, the assumption that the cross sectional dimensions of the beam are small compared to its length, may be violated. This assumption could even be violated in the very first mode for short beams or for multispan beams when adjacent supports are too close to each other. The designer utilizing the tables presented in this thesis, should therefore use them judiciously and not without regard to the limitations imposed by Euler-Bernoulli beam theory.

If the beam system does not conform to the Euler-Bernoulli model, consideration to the effects of rotary inertia and shear deformation should be given. In this Chapter, a study of these effects on vibration frequencies of single span uniform beams with classical and non-classical boundary conditions is made.

The consideration of rotary inertia and shear deformation effects inject two parameters r and s_1 defined later in this Chapter, into the governing differential equation. The parameter r is associated with the rotary inertia and s_1 is associated with the shear deformation. It is seen from the charts provided at the end of this Chapter for single span uniform beams with classical boundary conditions that the effect of rotary inertia and shear deformation is to reduce the natural frequency of vibrations. This is to be rather expected as the effect of rotary inertia is to make the system more sluggish whereas the effect of shear deformation is to make it less rigid. Both these effects, therefore, reduce the natural frequency of vibrations.

It may be observed from the charts 10.3.2(a) through 10.3.6(c) that for $r \leq 0.01$, these effects are insignificant at least in the first mode. The parameter r is taken as a ratio of the radius of gyration, K_r , of the cross-section to the length L of the beam. For most engineering applications, it will be found that $r \leq 0.01$, however.

when r becomes greater than 0.01 as the distance between nodal points decreases with higher modes, consideration to the use of the charts provided should be given.

The values of parameters r and s_1 can be calculated for any beam system from its geometric and elastic properties. The correct frequency of vibration of single span beams with classical boundary conditions can then be established utilizing the charts provided, upto the first four modes of vibrations.

It may however be emphasized that there can be no forth-right answer to the question, "exactly when or after which mode should the effects of shear deformation and rotary inertia be taken into account?" This would obviously depend on the geometric and elastic properties of the system. However, a close look at the charts provided does suggest that whenever r is > 0.01 , the consideration to the shear deformation and rotary inertia effects is warranted.

For single span beams with non-classical boundary conditions, only the frequency equations and modal shape expressions are provided and no attempt has been made to present numerical results because of the presence of too many influence parameters. Since the same problems are discussed in Chapter Two without shear deformation and rotary inertia effects, the eigenvalues β obtained from these tables will yield an upper bound for the frequency. The iteration can therefore be started from this upper bound value of β so obtained, utilizing the

frequency equations provided in this Chapter. This will save considerable time and labour.

The author could find only one of the fifteen charts presented here, available in the existing literature published by Huang (35). This chart is for a clamped-free beam with $s_1 = 2$ and $0 \leq r \leq 0.1$.

The corresponding chart presented here compares favourably with it as can be seen from Figure 10.3.5(a), page 636. This should however be remembered that P/P_0 in the reference chart corresponds to $(\beta/\beta_0)^2$ of the chart of this thesis.

10.1 General Theory

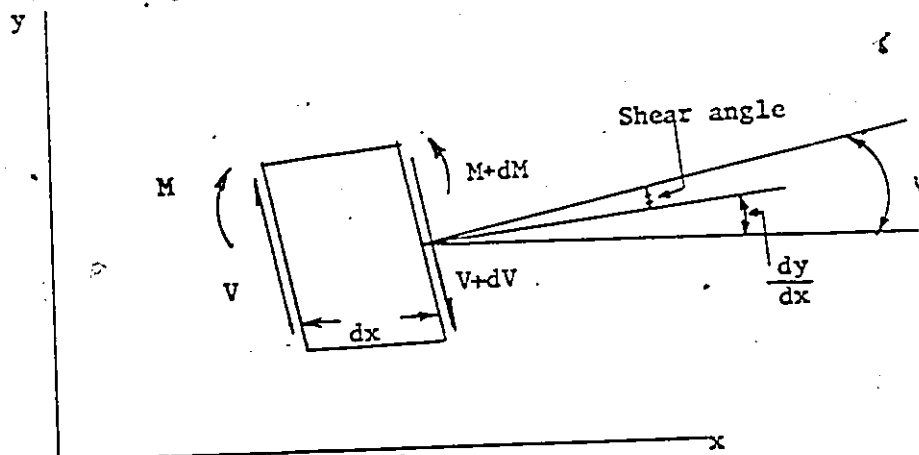


Figure 10.1 - Effect of shear deformation and rotary inertia on a small element of the beam

Figure 10.1 shows a free body diagram of a small element of the beam. The Euler-Bernoulli assumption that planes perpendicular to the center line of the beam remain plane and perpendicular to the center line after deflection is no longer valid if shear deformation is taken into consideration.

Due to shear, the rectangular elements tend to go into diamond shape without rotation of the face and the slope ψ due to bending is diminished by the shear angle γ .

The equilibrium considerations of the small element shown in Figure 10.1 yield the following two coupled partial differential equations (2)

$$EI \frac{\partial^2 \psi}{\partial x^2} + k' A G \left(\frac{\partial y}{\partial x} - \psi \right) I \rho \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (10.1)$$

and

$$\rho A \frac{\partial^2 y}{\partial t^2} - k' A G \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) = 0 \quad (10.2)$$

where use has been made of the following two relations (2)

$$V = k' A G \left(\psi - \frac{\partial y}{\partial x} \right)$$

and

$$M = EI \frac{\partial \psi}{\partial x}$$

The parameter k' is a numerical shape factor for the cross section of the beam and other quantities have the same meaning as assigned to them earlier in this thesis.

Eliminating y or ψ from equations (10.1) and (10.2), the following two equations in terms of y and ψ are obtained.

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} - \left(I \rho + \frac{EI \rho}{k' G} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{I \rho^2}{k' G} \frac{\partial^4 y}{\partial t^4} = 0 \quad (10.3)$$

and

$$EI \frac{\partial^4 \psi}{\partial x^4} + \rho A \frac{\partial^2 \psi}{\partial t^2} - \left(I \rho + \frac{EI \rho}{k' G} \right) \frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{I \rho^2}{k' G} \frac{\partial^4 \psi}{\partial t^4} = 0 \quad (10.4)$$

Assuming that y and ψ are executing simple harmonic motion under vibrations, we can write $y = Y \sin \omega t$ and $\psi = \Psi \sin \omega t$. Utilizing this assumption, the partial differential equations (10.1) through (10.4) are reduced to the following four ordinary differential equations

$$EI \frac{d^2 \psi}{dx^2} + k' A G \left(\frac{dy}{dx} - \psi \right) + I \rho \omega^2 \psi = 0 \quad (10.5)$$

$$\rho A \omega^2 Y + k' A G \left(\frac{d^2 Y}{dx^2} - \frac{d\psi}{dx} \right) = 0 \quad (10.6)$$

$$EI \frac{d^4 Y}{dx^4} - \rho A \omega^2 Y + \omega^2 \left(I \rho + \frac{EI \rho}{k' G} \right) \frac{d^2 Y}{dx^2} + \frac{I \rho^2 \omega^4}{k' G} Y = 0 \quad (10.7)$$

$$EI \frac{d^4 \psi}{dx^4} - \rho A \omega^2 \psi + \omega^2 \left(I \rho + \frac{EI \rho}{k' G} \right) \frac{d^2 \psi}{dx^2} + \frac{I \rho^2 \omega^4}{k' G} \psi = 0 \quad (10.8)$$

Non-dimensionalizing the deflection y and spatial co-ordinate parameter x with respect to beam length L , and denoting $\xi = \frac{x}{L}$, $R(\xi) = \frac{Y(x)}{L}$ and $\psi(\xi) = \psi(x/L)$, the differential equations (10.5) through (10.8) reduce to following four equations in their non-dimensional forms

$$s^2 \frac{d^2 \psi}{d\xi^2} - (1 - r^2 \beta^4 s^2) \psi + \frac{dR}{d\xi} = 0 \quad (10.9)$$

$$\frac{d^2 R}{d\xi^2} + \beta^4 s^2 R - \frac{d\psi}{d\xi} = 0 \quad (10.10)$$

$$\frac{d^4 R}{d\xi^4} + \beta^4 (r^2 + s^2) \frac{d^2 R}{d\xi^2} - \beta^4 (1 - \beta^4 r^2 s^2) R = 0 \quad (10.11)$$

and

$$\frac{d^4 \psi}{d\xi^4} + \beta^4 (r^2 + s^2) \frac{d^2 \psi}{d\xi^2} - \beta^4 (1 - \beta^4 r^2 s^2) \psi = 0 \quad (10.12)$$

where the non-dimensional parameter

$$r^2 = \frac{I}{A L^2} \quad \text{is a measure of rotary inertia,}$$

$$s^2 = \frac{EI}{k' G A L^2} \quad \text{is a measure of shear deformation, and}$$

$$\beta^4 = \frac{\rho A \omega^2 L^4}{EI} \quad \text{is the usual frequency parameter.}$$

The equation (10.11) permits two general solutions depending on whether $\beta^4 r^2 s^2 < 1$ or $\beta^4 r^2 s^2 > 1$.

If, $\beta^4 r^2 s^2 < 1$, the solution $R(\xi)$ of equation (10.11) is given by

$$R(\xi) = A \sin \beta_1 \xi + B \cos \beta_1 \xi + C \sinh \beta_2 \xi + D \cosh \beta_2 \xi \quad (10.13)$$

where

$$\beta_1 = \frac{\beta^2}{\sqrt{2}} \sqrt{(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{\beta^4}}}$$

and

$$\beta_2 = \frac{\beta^2}{\sqrt{2}} \sqrt{-(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{\beta^4}}}$$

and constants A, B, C, and D are arbitrary constants to be determined from boundary conditions. A similar solution for ψ can be written down for equation (10.12) but because of interdependence of equations (10.11) and (10.12), the arbitrary constants in the solution of equation (10.12) are related to the constants of equation (10.13) by equations (10.9) or (10.10). Utilizing this fact, the solution for ψ can be written down as follows

$$\psi(\xi) = A \alpha_1 \cos \beta_1 \xi + B(-\alpha_1) \sin \beta_1 \xi + C \alpha_2 \cosh \beta_2 \xi + D \alpha_2 \sinh \beta_2 \xi \quad (10.14)$$

where

$$\alpha_1 = \frac{\beta_1^2 - \beta^4 s^2}{\beta_1}$$

and

$$\alpha_2 = \frac{\beta_2^2 + \beta^4 s^2}{\beta_2}$$

For the case when $\beta^4 s^2 > 1$, the solutions $R(\xi)$ and $\psi(\xi)$ for equations (10.11) and (10.12) are obtained as follows

$$R(\xi) = A \sin \beta_1 \xi + B \cos \beta_1 \xi + C \sin \beta'_2 \xi + D \cos \beta'_2 \xi \quad (10.15)$$

where

$$\beta'_2 = \frac{\beta_2^2}{\sqrt{2}} \sqrt{(r^2 + s^2) - \sqrt{(r^2 - s^2)^2 + \frac{4}{\beta_2^4}}}$$

and

$$\psi(\xi) = A \alpha_1 \cos \beta_1 \xi + B(-\alpha_1) \sin \beta_1 \xi + C \alpha'_2 \cos \beta_2 \xi + D(-\alpha'_2) \sin \beta_2 \xi \quad (10.16)$$

where

$$\alpha'_2 = \frac{\beta_2^2 - \beta_2^4 s^2}{\beta_2}$$

The application of appropriate boundary conditions in the general solutions (10.13) or (10.15) will yield four algebraic equations in terms of constants A, B, C, and D. In order that a solution other than zero may exist, the determinant of the coefficient matrix of these four constants must vanish which in turn yields the required frequency equation.

It may be noted that there will be two frequency equations for each case depending on whether $\beta^4 r^2 s^2 < 1$ or > 1 . These frequency equations can then be solved for the frequency parameter β for various values of r and s . The range of values for r and s considered will be discussed later in this Chapter.

10.2 Boundary Conditions

The free body diagram of a beam element taken from the center of the beam, is shown in Figure 10.1. If the shear deformation is neglected, the centerline of the beam will coincide with the perpendicular to the face of the cross-section. Due to shear, the rectangular element tends to go into diamond shape and the slope of the center line is diminished by the shear angle $\gamma = \psi - \frac{dy}{dx}$. The following quantities can then be defined:

y - deflection of the centerline of the beam

$\frac{dy}{dx}$ - slope of the centerline of the beam

ψ - slope due to bending

$\gamma = \psi - \frac{dy}{dx}$ - loss of slope, equal to shear angle.

The bending moment and shear force expressions for the beam are given by (2)

$$M = EI \frac{d\psi}{dx}$$

and

$$V = k' A G \gamma = k' A G \left(\psi - \frac{dy}{dx} \right)$$

10.2.1 Classical End-Supports

(a) Hinged end: The bending moment and deflection of the beam should be zero at such supports. Expressed mathematically $y = 0$ and $\frac{d\psi}{dx} = 0$ which in the non-dimensional form reduce to $R = 0$ and $\psi' = 0$.

(b) Clamped end: The requirement of zero deflection implies $y = 0$ or $R = 0$. If the warping of the cross-section due to shear deformation is permitted at the built-in end, then only the slope due to bending will be zero, i.e. $\psi = 0$. In this thesis, it is assumed that shear strain is permitted at the clamped end and consequently in all the cases considered, the boundary conditions for clamped end are taken as $R = 0$ and $\psi = 0$.

However if the warping of cross-section is prevented at the built-in end, the expression for total slope should be equated to zero, i.e. $\frac{dy}{dx} = 0$ or $R' = 0$.

(c) Free end: Since a free end cannot sustain any moment or shear force, consequently $M = V = 0$ at such ends.

Looking at the expressions for M and V provided earlier, it becomes immediately apparent that $\frac{d\psi}{dx} = 0$ and $\psi - \frac{dy}{dx} = 0$ at free ends. The corresponding expressions in non-dimensional form are $\psi' = 0$ and $R' - \psi = 0$.

10.2.2 Non-Classical End Supports

(a) End supports provided by a coil spring (Figure 1.7)

The shear force balance at the right hand end requires

$$k' A G \left(\psi - \frac{dy}{dx} \right) = k y$$

which in the non-dimensional form reduces to

$$R' - \psi = - \frac{k L}{k' A G} R$$

$$= - k^* R$$

or

$$R' - \psi + k^* R = 0 \quad \Big|_{\xi=1} \quad (10.17)$$

where

$$k^* = \frac{k L}{k' A G}$$

is a non-dimensional system parameter introduced as a measure of spring stiffness. It should be noted that this expression is different from the one used earlier when distortion and rotary inertia effects were neglected.

For the left hand end support we have

$$R' - \psi - k^* R = 0 \quad \Big|_{\xi=0} \quad (10.18)$$

(b) End supports provided by a torsion spring (Figure 1.8)

The moment balance at the right hand end requires

$$EI \frac{d\psi}{dx} = -T \psi$$

which in non-dimensional form reduces to

$$\psi' = -\frac{TL}{EI} \psi$$

or

$$\psi' + T^* \psi = 0 \quad \Big|_{\xi=1} \quad (10.19)$$

where

$$T^* = \frac{TL}{EI}$$

For the left hand end support, we have

$$\psi' - T^* \psi = 0 \quad \Big|_{\xi=0} \quad (10.20)$$

(c) End supports with a rotational mass (Figure 1.10)

Left hand end:

$$\psi' + \frac{\beta^4}{I_o^*} \psi = 0 \quad \bigg|_{\xi=0}$$

where

$$I_o^* = \frac{\rho A L^3}{I_o} \quad (10.21)$$

Right hand end:

$$\psi' - \frac{\beta^4}{I_o^*} \psi = 0 \quad \bigg|_{\xi=1} \quad (10.22)$$

The procedure of arriving at these boundary conditions have been explained in Chapter 1.

(d) End supports with a concentrated mass (Figure 1.9)

Left hand end: Utilizing the shear force balance and recognizing that shear force is now given by, $V = k' A G (\psi - \frac{dy}{dx})$, the following equation for boundary condition is obtained

$$R' - \psi + \frac{\beta^4}{\eta} R = 0 \quad \bigg|_{\xi=0}$$

where

$$\eta^* = \frac{\rho AL}{M} \frac{k' AL^2 G}{EI} \quad (10.23)$$

Right hand end:

$$R' - \psi - \frac{\beta^4}{\eta^*} R = 0 \quad \Big|_{\xi=1} \quad (10.24)$$

It should be noted that, the parameter η^* is related to the parameter η used in Chapter 2 by,

$$\eta^* = \frac{\eta}{S^2}$$

The expressions for the coincident non-classical boundary conditions can easily be written down using the above information.

10.3 Frequencies and Modal Shapes of Beams with Classical Boundary

Conditions

As mentioned earlier, the governing differential equation (10.11) of free lateral vibrations of beams admits two solutions for $R(\xi)$ and $\psi(\xi)$ depending on whether $\beta^4 r^2 s^2 < 1$ or > 1 . Solutions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.13) and (10.14) respectively.

when $\beta^4 r^2 s^2 < 1$ and by equations (10.15) and (10.16) respectively when $\beta^4 r^2 s^2 > 1$. Consequently there will be two sets of frequency equations and modal shapes for the entire range of $\beta^4 r^2 s^2$.

10.3.1 Beams Simply Supported at Both Ends

Case (a) when $\beta^4 r^2 s^2 < 1$:

The substitution of boundary conditions at the left hand support, i.e. $R(0) = 0$ and $\psi'(0) = 0$, in equations (10.13) and (10.14) yields $B = D = 0$ and we obtain

$$R(\xi) = A \sin \beta_1 \xi + C \sinh \beta_2 \xi \quad (10.25)$$

and

$$\psi(\xi) = A \alpha_1 \cos \beta_1 \xi + C \alpha_2 \cosh \beta_2 \xi \quad (10.26)$$

The two boundary conditions at the right hand end are $R(1) = 0$ and $\psi'(1) = 0$. Substituting these in equations (10.25) and (10.26) we obtain the following two equations.

$$A \sin \beta_1 + C \sinh \beta_2 = 0 \quad (10.27)$$

and

$$A (-\alpha_1 \beta_1 \sin \beta_1) + C \alpha_2 \beta_2 \sinh \beta_2 = 0 \quad (10.28)$$

Utilizing the condition for a non-trivial solution for constants A and C in the above two equations, we obtain the following frequency for this case.

$$\sin \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2) = 0 \quad (10.29)$$

The equation (10.29) is obviously satisfied for all possible values of β if

$$\sin \beta_1 = 0$$

or

$$\beta_1 = n\pi \quad \text{where} \quad n = 1, 2, \dots \quad (10.30)$$

Substituting for β_1 , we obtain

$$n\pi = \frac{\beta^2}{\sqrt{2}} \sqrt{(r^2 + s^2)} + \sqrt{(r^2 - s^2)^2 + \frac{4}{\beta^4}} \quad (10.31)$$

Simplifying equation (10.31) yields

$$r^2 s^2 \beta^8 - \{(r^2 + s^2) n^2 \pi^2 + 1\} \beta^4 + n^4 \pi^4 = 0 \quad (10.32)$$

or

$$\beta^4 = \frac{1 + (r^2 + s^2) n^2 \pi^2}{2r^2 s^2} \pm \frac{1}{2r^2 s^2} \sqrt{\{1 + (r^2 + s^2) n^2 \pi^2\}^2 - 4r^2 s^2 n^4 \pi^4} \quad (10.33)$$

It may be noted that both the roots for β^4 are positive which seems perplexing at first sight. A further investigation showed that one of these roots corresponds to the other spectrum of frequencies for the case when $\beta^4 r^2 s^2 > 1$. For, the case in hand with $\beta^4 r^2 s^2 < 1$, the frequency parameter β can be obtained from

$$\beta^4 = \frac{1 + (r^2 + s^2) n^2 \pi^2}{2r^2 s^2} - \frac{1}{2r^2 s^2} \sqrt{\{1 + (r^2 + s^2) n^2 \pi^2\}^2 - 4r^2 s^2 n^4 \pi^4} \quad (10.34)$$

The equation (10.34) will give the frequency parameter β for any simply supported beam for any mode n , and any value of r and s when $\beta^4 r^2 s^2 < 1$. The modal shapes are obtained by setting one of the constants, say A , in equations (10.25) and (10.26) to unity and substituting for $\frac{C}{A}$ from equation (10.27). Thus we obtain:

$$R(\xi) = \sin \beta_1 \xi$$

and

$$\psi(\xi) = \alpha_1 \cos \beta_1 \xi$$

Case (b) when $\beta^4 r^2 s^2 > 1$

Utilizing the boundary conditions at the left hand end,

we obtain

$$R(\xi) = A \sin \beta_1 \xi + C \sin \beta'_2 \xi \quad (10.35)$$

$$\psi(\xi) = A \alpha_1 \cos \beta_1 \xi + C \alpha'_2 \cos \beta'_2 \xi \quad (10.36)$$

Substitution of boundary conditions $R(1) = 0$ and $\psi'(1) = 0$ in equations (10.35) and (10.36) we obtain:

$$A \sin \beta_1 + C \sin \beta'_2 = 0 \quad (10.37)$$

$$-A \alpha_1 \beta_1 \sin \beta_1 - C \alpha'_2 \beta'_2 \sin \beta'_2 = 0 \quad (10.38)$$

Equating the determinant of the coefficient matrix of constants A and C in the above two equations to zero, we obtain the following frequency equation for this case.

$$\sin \beta_1 \sin \beta'_2 (\alpha_1 \beta_1 - \alpha'_2 \beta'_2) = 0 \quad (10.39)$$

which suggests that,

$$\sin \beta_1 = 0 \quad \text{or} \quad \beta_1 = n\pi$$

$$\sin \beta'_2 = 0 \quad \text{or} \quad \beta'_2 = n\pi \quad (10.40)$$

We had already obtained the expression for β^4 for $\beta_1 = n\pi$, for $\beta'_2 = n\pi$, we have

$$n\pi = \frac{\beta^2}{\sqrt{2}} \sqrt{(r^2+s^2) - \sqrt{(r^2-s^2)^2 + \frac{4}{s^4}}} \quad (10.41)$$

which, on simplification, yields,

$$r^2 s^2 \beta^8 - \{(r^2+s^2) n^2 \pi^2 + 1\} \beta^4 + n^4 \pi^4 = 0 \quad (10.42)$$

It is interesting to note that equation (10.42) and (10.32) are identical and therefore have identical solutions.

$$\beta^4 = \frac{1 + (r^2+s^2) n^2 \pi^2}{2r^2 s^2} + \frac{1}{2r^2 s^2} \sqrt{\{1 + (r^2+s^2) n^2 \pi^2\}^2 - 4r^2 s^2 n^4 \pi^4}$$

In fact the first root satisfies equation (10.31) while the second roots satisfies equation (10.41); so for this case, we have

$$\beta^4 = \frac{1 + (r^2+s^2) n^2 \pi^2}{2r^2 s^2} + \frac{1}{2r^2 s^2} \sqrt{\{1 + (r^2+s^2) n^2 \pi^2\}^2 - 4r^2 s^2 n^4 \pi^4} \quad (10.43)$$

which is valid for any value of n , r , and s when $\beta^4 r^2 s^2 > 1$. It should be remembered that $n = 1$, here, does not necessarily correspond to the first mode of the beam. There may have been some modes of the beam in the range $\beta^4 r^2 s^2 < 1$, already.

Equations (10.34) and (10.43) are sufficient to find frequencies of free lateral vibration of any simply supported beam for any mode with any values of r and s . The modal shape is given by

$$R(\xi) = \sin \beta'_2 \xi \text{ and } \psi'(\xi) = \alpha'_2 \cos \beta'_2 \xi.$$

10.3.2 Simply Supported - Clamped Beams

Here the first boundary condition refers to the left hand end and the second, to the right hand end.

Case (a) when $\beta^4 r^2 s^2 < 1$

The boundary conditions at the left hand end are same as in previous case and consequently the expressions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.25) and (10.26).

The boundary conditions at the fixed end are $R(1) = 0$ and $\psi(1) = 0$. Substituting these in equations (10.25) and (10.26) we obtain:

$$A \sin \beta_1 + C \sinh \beta_2 = 0 \quad (10.44)$$

$$A \alpha_1 \cos \beta_1 + C \alpha_2 \cosh \beta_2 = 0 \quad (10.45)$$

The above two equations yield the following frequency equation for this case

$$\alpha_2 \sin \beta_1 \cosh \beta_2 - \alpha_1 \cos \beta_1 \sinh \beta_2 = 0 \quad (10.46)$$

Modal shapes are established in the usual manner and are given by

$$R(\xi) = \sin \beta_1 \xi - \frac{\sin \beta_1}{\sinh \beta_2} \sinh \beta_2 \xi$$

$$\psi(\xi) = \alpha_1 \cos \beta_1 \xi - \frac{\sin \beta_1}{\sinh \beta_2} \alpha_2 \cosh \beta_2 \xi$$

Case (b) when $\beta^4 r^2 s^2 > 1$

The boundary conditions at the left hand end yield the expressions for $R(\xi)$ and $\psi(\xi)$ as given by equations (10.35) and (10.36). The boundary conditions at the right hand end yield:

$$A \sin \beta_1 + C \sin \beta'_2 = 0 \quad (10.47)$$

$$A \alpha_1 \cos \beta_1 + C \alpha'_2 \cos \beta'_2 = 0 \quad (10.48)$$

which yields the following frequency equation

$$\alpha'_2 \sin \beta_1 \cos \beta'_2 - \alpha_1 \sin \beta'_2 \cos \beta_1 = 0 \quad (10.49)$$

and modal shape expressions are given by

$$R(\xi) = \sin \beta_1 \xi - \frac{\sin \beta_1}{\sin \beta'_2} \sin \beta'_2 \xi$$

$$\psi(\xi) = \alpha_1 \cos \beta_1 \xi - \frac{\sin \beta_1}{\sin \beta'_2} \alpha'_2 \cos \beta'_2 \xi$$

The frequency equation (10.46) and (10.49) yield the frequency parameter β for any value of r and s . In fact equation (10.49) becomes effective as soon as $\beta^4 r^2 s^2 > 1$ and equation (10.46) remains valid for $\beta^4 r^2 s^2 < 1$.

As mentioned earlier, the parameter r^2 and s^2 are given by

$$r^2 = \frac{I}{A L^2}$$

or

$$r = \frac{k_r}{L}$$

where k_r is the radius of gyration.

The values of r selected are, $0 \leq r \leq 0.2$, which covers the range valid for most practical applications. For $r = 0$, the beam

corresponds to an Euler-Bernoulli beam and for $r = 0.2$, the beam has a length of 5 x the radius of gyration of the cross section, which represents a pretty short beam.

The expression for s^2 is given by

$$s^2 = \frac{E}{k' G} r^2$$

or

$$s = s_1 r$$

where

$$s_1 = \sqrt{\frac{E}{k' G}}$$

$$= \sqrt{\frac{2(1+\nu)}{k'}} \quad \text{where } \nu \text{ is Poisson's ratio.}$$

The range for s_1 selected is from 1.5 to 4.0 in the intervals of 0.5. It is known that the value of shape factor k' is never greater than one and Poisson's ratio ν is not expected to be less than 0.2 for most engineering materials used for vibration applications. So the lower limit of s_1 will be $s_1 = \sqrt{2.4} = 1.55$. Therefore, it was decided to take $s_1 = 1.5$ as the starting point. The upper side of the range corresponds to a value of $\nu = 0.5$ and $k' = \frac{1}{5}$, so that $s_1 = \sqrt{3 \times 5}$ which is

less than 4. In fact for most material $\nu = \frac{1}{3}$, then the limiting value of 4 for s_1 , corresponds to a shape factor value of $k' = \frac{1}{6}$

Timoshenko (36) has suggested a value of $K' = \frac{2}{3}$ for rectangular sections and $\frac{3}{4}$ for circular sections. For I-sections he further suggests that it is generally less than 0.5. The chosen range of k' , i.e., $\frac{1}{6} \leq k' \leq 1$; therefore seems to include most of practical applications. Hence the parameter s_1 is varied from 1.5 to 4.0 in intervals of 0.5.

The ratio $\frac{\beta}{\beta_0}$, where β_0 is the frequency parameter of the beam when shear deformation and rotary inertia effects are neglected, is shown plotted against r ; $0.0 \leq r \leq 0.2$, for various values of s_1 (from 1.5 to 4.0 in the intervals of 0.5), in Figures 10.3.2(a) through 10.3.2(c) for first four modes of vibrations. It may be noted that this ratio decreases with increasing magnitudes of r and s_1 and also with higher modes. This indicates that the effect of shear deformation and rotary inertia is to decrease the natural frequencies and is quite predominant in higher modes.

10.3.3 Simply Supported-Free Beams

Case (a) when $\beta^4 r^2 s^2 < 1$.

The application of boundary conditions at the left hand end yields expressions (10.25) and (10.26) for $R(\xi)$ and $\psi(\xi)$ respectively. The boundary conditions at the free end are $\psi'(1) = 0$ and

$R'(1) - \psi(1) = 0$. Substitution of these boundary conditions in equations (10.25) and (10.26) yields

$$-A \alpha_1 \beta_1 \sin \beta_1 + C \alpha_2 \beta_2 \sinh \beta_2 = 0 \quad (10.50)$$

and

$$A (\beta_1 - \alpha_1) \cos \beta_1 + C (\beta_2 - \alpha_2) \cosh \beta_2 = 0 \quad (10.51)$$

which yield the following frequency equation,

$$\alpha_1 \beta_1 (\alpha_2 - \beta_2) \sin \beta_1 \cosh \beta_2 + \alpha_2 \beta_2 (\alpha_1 - \beta_1) \cos \beta_1 \sinh \beta_2 = 0 \quad (10.52)$$

The modal shape expressions are given by:

$$R(\xi) = \sin \beta_1 \xi + \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \frac{\sin \beta_1}{\sinh \beta_2} \sinh \beta_2 \xi$$

and

$$\psi(\xi) = \alpha_1 \cos \beta_1 \xi + \frac{\alpha_1 \beta_1}{\beta_2} \frac{\sin \beta_1}{\sinh \beta_2} \cosh \beta_2 \xi$$

Case (b) when $\beta^4 r^2 s^2 > 1$

Equations (10.35) and (10.36) represent the expressions for $R(\xi)$ and $\psi(\xi)$ when boundary conditions at the left hand end are applied. Enforcing the boundary conditions at the free end yields following equations:

$$-A \alpha_1 \beta_1 \sin \beta_1 - C \alpha'_2 \beta'_2 \sin \beta'_2 = 0 \quad (10.53)$$

and

$$A (\beta_1 - \alpha_1) \cos \beta_1 + C (\beta'_2 - \alpha'_2) \cos \beta'_2 = 0 \quad (10.54)$$

which yields the following frequency equation for this case

$$\alpha_1 \beta_1 (\alpha'_2 - \beta'_2) \sin \beta_1 \cos \beta'_2 + \alpha'_2 \beta'_2 (\beta_1 - \alpha_1) \cos \beta_1 \sin \beta'_2 = 0 \quad (10.55)$$

The modal shape expressions are

$$R(\xi) = \sin \beta_1 \xi - \frac{\alpha_1 \beta_1 \sin \beta_1}{\alpha'_2 \beta'_2 \sin \beta'_2} \sin \beta'_2 \xi$$

$$\psi(\xi) = \alpha_1 \cos \beta_1 \xi - \frac{\alpha_1 \beta_1 \sin \beta_1}{\beta'_2 \sin \beta'_2} \cos \beta'_2 \xi$$

The frequency equations (10.52) and (10.55) are solved by iteration for various values of r and s_1 and results are presented in Figures 10.3.3(a) through 10.3.3(c). For any value of r and s_1 , the values of $\frac{s}{s_0}$ can be read off these charts. This permits evaluation of the constants α_1 , β_1 , α_2 , β_2 , α'_2 , and β'_2 which when substituted into modal shape expressions provide the required modal shape.

10.3.4 Clamped-Clamped Beams

Case (a) when $\beta^4 r^2 s^2 < 1$

Utilizing the boundary conditions $R(0) = 0$ and $\psi(0) = 0$,

we obtain

$$R(\xi) = A(\sin \beta_1 \xi - \frac{\alpha_1}{\alpha_2} \sinh \beta_2 \xi) + B(\cos \beta_1 \xi - \cosh \beta_2 \xi) \quad (10.56)$$

and

$$\psi(\xi) = A \alpha_1 (\cos \beta_1 \xi - \cosh \beta_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi - \alpha_2 \sinh \beta_2 \xi) \quad (10.57)$$

Applying the boundary conditions at the right hand end,

i.e., $R(1) = 0$ and $\psi(1) = 0$, we obtain,

$$A(\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sinh \beta_2) + B(\cos \beta_1 - \cosh \beta_2) = 0 \quad (10.58)$$

and

$$A\{\alpha_1(\cos \beta_1 - \cosh \beta_2)\} + B(-\alpha_1 \sin \beta_1 - \alpha_2 \sinh \beta_2) = 0 \quad (10.59)$$

The condition for a non-trivial solution for A and B in the above two equations yields the following frequency equation of the system

$$2 - 2 \cos \beta_1 \cosh \beta_2 + \sin \beta_1 \sinh \beta_2 \left(\frac{\alpha_2^2 - \alpha_1^2}{\alpha_1 \alpha_2} \right) = 0 \quad (10.60)$$

The modal shapes are given by equations (10.56) and (10.57) when A is set equal to unity and the ratio B/A is given by equation (10.58) or (10.59), so that

$$\frac{B}{A} = \frac{\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sinh \beta_2}{\cosh \beta_2 - \cos \beta_1}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

The application of boundary conditions at the left hand end yield:

$$R(\xi) = A(\sin \beta_1 \xi - \frac{\alpha_1}{\alpha'_2} \sin \beta'_2 \xi) + B(\cos \beta_1 \xi - \cos \beta'_2 \xi) \quad (10.61)$$

$$\psi(\xi) = A \alpha_1 (\cos \beta_1 \xi - \cos \beta'_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi + \alpha'_2 \sin \beta'_2 \xi) \quad (10.62)$$

Now utilizing the boundary conditions at the right hand end we obtain:

$$A(\sin \beta_1 - \frac{\alpha_1}{\alpha'_2} \sin \beta'_2) + B(\cos \beta_1 - \cos \beta'_2) = 0 \quad (10.63)$$

and

$$A \alpha_1 (\cos \beta_1 - \cos \beta'_2) + B(-\alpha_1 \sin \beta_1 + \alpha'_2 \sin \beta'_2) = 0 \quad (10.64)$$

which yield the following frequency equation of the system.

$$2 - 2 \cos \beta_1 \cos \beta'_2 - \sin \beta_1 \sin \beta'_2 \left(\frac{\alpha'^2_2 + \alpha^2_1}{\alpha_1 \alpha'_2} \right) = 0 \quad (10.65)$$

The modal shape expressions are given by equations (10.61)

and (10.62) with

$$\frac{B}{A} = \frac{\sin \beta_1 - \frac{\alpha_1}{\alpha'_2} \sin \beta'_2}{\cos \beta'_2 - \cos \beta_1}$$

The frequency equations (10.60) and (10.65) are solved for various values of r and s_1 and results are shown plotted for β/β_0 versus r for various values of s_1 in Figures 10.3.4(a) through 10.3.4(c).

10.3.5 Clamped-Free Beams

Case (a) when $\beta^4 r^2 s^2 < 1$

After utilizing the boundary conditions at the left hand end, the expressions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.56) and (10.57). The boundary conditions at the free end are: $\psi'(1) = 0$ and $R' - \psi = 0$. Substitution of these in equations (10.56) and (10.57) yield:

$$A(-\alpha_1 \beta_1 \sin \beta_1 - \alpha_1 \beta_2 \sinh \beta_2) + B(-\alpha_1 \beta_1 \cos \beta_1 - \alpha_2 \beta_2 \cosh \beta_2) = 0 \quad (10.66)$$

and

$$A[(\beta_1 - \alpha_1) \cos \beta_1 + \left(\frac{\alpha_1 \alpha_2 - \alpha_1 \beta_2}{r \alpha_2} \right) \cosh \beta_2] + B[(\alpha_1 - \beta_1) \sin \beta_1 + (\alpha_2 - \beta_2) \sinh \beta_2] = 0 \quad (10.67)$$

The frequency equation is obtained by setting the determinant of the above two equations to zero and thus:

$$\begin{aligned} & \beta_1(\beta_1 - \alpha_1) + \beta_2(\alpha_2 - \beta_2) + \sin \beta_1 \sinh \beta_2 \{ \beta_1(\beta_2 - \alpha_2) + \beta_2(\beta_1 - \alpha_1) \} \\ & + \cos \beta_1 \cosh \beta_2 \left\{ \frac{\alpha_2 \beta_2 (\beta_1 - \alpha_1)}{\alpha_1} + \frac{\alpha_1 \beta_1 (\alpha_2 - \beta_2)}{\alpha_2} \right\} = 0 \quad (10.68) \end{aligned}$$

The modal shape expressions are given by equations (10.56) and (10.57) where $\frac{B}{A}$ is obtained from equation (10.66).

Case (b) when $\beta^4 r^2 s^2 > 1$

The application of the boundary conditions at the left hand end yields equations (10.61) and (10.62) for $R(\xi)$ and $\psi(\xi)$ respectively. The boundary conditions at the free end yield the following two equations in terms of constants A and B.

$$A \{ -\alpha_1 \beta_1 \sin \beta_1 + \alpha_1 \beta'_2 \sin \beta'_2 \} + B \{ -\alpha_1 \beta_1 \cos \beta_1 + \alpha'_2 \beta'_2 \cos \beta'_2 \} = 0 \quad (10.69)$$

$$\begin{aligned} A \{ (\beta_1 - \alpha_1) \cos \beta_1 + \left(\alpha_1 - \frac{\alpha_1 \beta'_2}{\alpha'_2} \right) \cos \beta'_2 \} + B \{ (\alpha_1 - \beta_1) \sin \beta_1 \\ + (\beta'_2 - \alpha'_2) \sin \beta'_2 \} = 0 \quad (10.70) \end{aligned}$$

which yield the following frequency equation for this case

$$\begin{aligned} & \beta_1(\beta_1 - \alpha_1) + \beta'_2(\beta'_2 - \alpha'_2) + \sin \beta_1 \sin \beta'_2 \{ \beta'_2(\alpha_1 - \beta_1) + \beta_1(\alpha'_2 - \beta'_2) \} \\ & + \cos \beta_1 \cos \beta'_2 \left\{ \frac{\alpha_1 \beta_1 (\alpha'_2 - \beta'_2)}{\alpha'_2} - \frac{\alpha'_2 \beta'_2 (\beta_1 - \alpha_1)}{\alpha_1} \right\} = 0 \end{aligned} \quad (10.71)$$

The modal shape expressions are provided by equations (10.61) and (10.62) where $\frac{B}{A}$ is determined from equation (10.69).

The frequency equations (10.68) and (10.71) are solved for various values of r and s_1 and results are shown plotted in Figures 10.3.5(a) through 10.3.5(c).

10.3.6 Free-Free Beams

Case (a) when $\beta^4 r^2 s^2 < 1$

Utilizing the boundary conditions at the left hand end, namely $\psi'(0) = 0$ and $R'(0) - \psi(0) = 0$, we obtain the following expressions for $R(\xi)$ and $\psi(\xi)$.

$$R(\xi) = A \left(\sin \beta_1 \xi + \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2} \sinh \beta_2 \xi \right) + B \left(\cos \beta_1 \xi + \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \cosh \beta_2 \xi \right) \quad (10.72)$$

and

$$\begin{aligned} \psi(\xi) = & A(\alpha_1 \cos \beta_1 \xi + \frac{\alpha_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \cosh \beta_2 \xi) \\ & + B(-\alpha_1 \sin \beta_1 \xi + \frac{\alpha_1 \beta_1}{\beta_2} \sinh \beta_2 \xi) \end{aligned} \quad (10.73)$$

The boundary conditions $\psi'(1) = 0$ and $R'(1) - \psi(1) = 0$ yield the following two equations for constants A and B.

$$A(-\alpha_1 \beta_1 \sin \beta_1 + \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \sinh \beta_2) + B\{\alpha_1 \beta_1 (\cosh \beta_2 - \cos \beta_1)\} = 0 \quad (10.74)$$

and

$$A[(\beta_1 - \alpha_1)(\cos \beta_1 - \cosh \beta_2)] + B[(\alpha_1 - \beta_1) \sin \beta_1 + \alpha_1 \beta_1 (\frac{1}{\alpha_2} - \frac{1}{\beta_2}) \sinh \beta_2] = 0 \quad (10.75)$$

which, for a non-trivial solution of A and B, yield the following frequency equation for free-free beams

$$2 - 2 \cos \beta_1 \cosh \beta_2 + \sin \beta_1 \sinh \beta_2 \left[\frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{\alpha_1 \beta_1 (\alpha_2 - \beta_2)} - \frac{\alpha_1 \beta_1 (\alpha_2 - \beta_2)}{\alpha_2 \beta_2 (\alpha_1 - \beta_1)} \right] = 0 \quad (10.76)$$

The modal shapes are given by equations (10.72) and (10.73) where $\frac{B}{A}$ is computed from either of the equations (10.74) and (10.75)

Case (b) when $s^4 r^2 s^2 > 1$

As before, the two boundary conditions at the left hand end result in

$$R(\xi) = A(\sin \beta_1 \xi + \frac{\alpha_1 - \beta_1}{\beta_1' - \alpha_1'} \sin \beta_2' \xi) + B(\cos \beta_1 \xi - \frac{\alpha_1 \beta_1}{\alpha_1' \beta_2'} \cos \beta_2' \xi) \quad (10.77)$$

and

$$\begin{aligned} \psi(\xi) = & A(\alpha_1 \cos \beta_1 \xi + \frac{\alpha_1' (\alpha_1 - \beta_1)}{(\beta_2' - \alpha_1')} \cos \beta_2' \xi) \\ & + B(-\alpha_1 \sin \beta_1 \xi + \frac{\alpha_1 \beta_1}{\beta_2'} \sin \beta_2' \xi) \end{aligned} \quad (10.78)$$

The other two boundary conditions at the right hand end yield the following two equations

$$A(-\alpha_1 \beta_1 \sin \beta_1 - \frac{\alpha_1' \beta_2' (\alpha_1 - \beta_1)}{(\beta_2' - \alpha_1')} \sin \beta_2') + B\{\alpha_1 \beta_1 (\cos \beta_2' - \cos \beta_1)\} = 0 \quad (10.79)$$

and

$$A\{(\beta_1 - \alpha_1)(\cos \beta_1 - \cos \beta_2')\} + B[(\alpha_1 - \beta_1) \sin \beta_1 + \alpha_1 \beta_1 (\frac{1}{\alpha_1'} - \frac{1}{\beta_2'}) \sin \beta_2'] = 0 \quad (10.80)$$

Setting the determinant of the two equations to zero, we obtain the following frequency equation for this case.

$$2 - 2 \cos \beta_1 \cos \beta'_2 - \sin \beta_1 \sin \beta'_2 \left[\frac{\alpha'_2 \beta'_2 (\beta_1 - \alpha_1)}{\alpha_1 \beta_1 (\beta'_2 - \alpha'_2)} + \frac{\alpha_1 \beta_1 (\beta'_2 - \alpha'_2)}{\alpha'_2 \beta'_2 (\beta_1 - \alpha_1)} \right] = 0 \quad (10.81)$$

As usual the modal shape expressions are given by equations (10.77) and (10.78) where $\frac{B}{A}$ is evaluated from either of the equations (10.79) and (10.80).

The frequency equations (10.76) and (10.81) are solved and the results are shown plotted for various values of r and s_1 in Figures 10.3.6(a) through 10.3.6(c).

This completes the free vibration analysis of single span uniform beams with all possible cases of classical end for supports, taking rotary inertia and shear deformation into account. The use of these charts provided is demonstrated by working out an illustrative example at the end of this Chapter.

10.4 Single Span Uniform Beams with Non-Classical Boundary Conditions

Besides the parameters r and s_1 , that we already have, the non-classical boundary conditions inject further parameters into

the frequency equations and therefore, unlike the previous cases, no attempt has been made to make the relevant frequency data available for all possible values of influence parameters. However, the expressions for frequency and modal shapes are developed which the analyst can use to his advantage for particular values of parameters of his interest.

Part 1 - Beams with a single non-classical boundary condition

10.4.1 Beams Clamped at One End and Supported by a Coil Spring at the Other

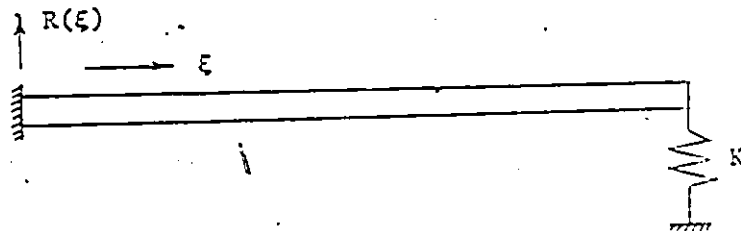


Figure 10.4.1

Case (a) when $\beta^4 r^2 s^2 < 1$

The application of two boundary conditions at the left hand end eliminates two of the four constants from the expressions of

$R(\xi)$ and $\psi(\xi)$ and the resulting expressions are given by equations (10.56) and (10.57) of case 10.3.4.

The boundary conditions at the right hand end are $\psi'(1) = 0$ and $R'(1) - \psi(1) + K^* R(1) = 0$, which, when substituted in equations (10.56) and (10.57) yield:

$$A(-\alpha_1 \beta_1 \sin \beta_1 - \alpha_1 \beta_2 \sinh \beta_2) + B(-\alpha_1 \beta_1 \cos \beta_1 - \alpha_2 \beta_2 \cosh \beta_2) = 0 \quad (10.82)$$

and

$$A[(\beta_1 - \alpha_1) \cos \beta_1 + (\alpha_1 - \frac{\alpha_1 \beta_2}{\alpha_2}) \cosh \beta_2 + K^* (\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sinh \beta_2)] + B[(\alpha_1 - \beta_1) \sin \beta_1 + (\alpha_2 - \beta_2) \sinh \beta_2 + K^* (\cos \beta_1 - \cosh \beta_2)] = 0 \quad (10.83)$$

Equations (10.82) and (10.83) result in the following frequency equation for this system.

$$K^* = \frac{\beta_1(\beta_1 - \alpha_1) + \beta_2(\alpha_2 - \beta_2) + \sin \beta_1 \sinh \beta_2 \{ \beta_1(\beta_2 - \alpha_2) + \beta_2(\beta_1 - \alpha_1) \} + \cos \beta_1 \cosh \beta_2 \{ \frac{\alpha_1^2 \beta_1(\alpha_2 - \beta_2) + \alpha_2^2 \beta_2(\beta_1 - \alpha_1)}{\alpha_1 \alpha_2} \}}{\frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_1 \alpha_2} [\alpha_1 \cos \beta_1 \sinh \beta_2 - \alpha_2 \sin \beta_1 \cosh \beta_2]} \quad (10.84)$$

It may be noted that as K^* approaches zero, the equation (10.84) approaches that of a clamped-free beam whereas K^* approaches infinity, the frequency equation approaches that of a clamped-simple beam. The frequencies of this class of beams is completely characterized by the system parameter k^* , r , and s_1 , and given these parameters, it can be solved for β .

The modal shapes are given by equations (10.56) and (10.57) where $\frac{B}{A}$ is evaluated from equation (10.82).

Case (b) when $\beta_1^4 r^2 s^2 > 1$.

Utilizing boundary conditions at the left hand support, the expressions for $R(\xi)$ and $\psi(\xi)$ are obtained as given by equations (10.61) and (10.62). The boundary conditions at the right hand end yield:

$$A[-\alpha_1 \beta_1 \sin \beta_1 + \alpha_1 \beta'_2 \sin \beta'_2] + B[-\alpha_1 \beta_1 \cos \beta_1 + \alpha'_2 \beta'_2 \cos \beta'_2] = 0 \quad (10.85)$$

and

$$A[(\beta_1 - \alpha_1) \cos \beta_1 + (\alpha_1 - \frac{\alpha_1 \beta'_2}{\alpha'_2}) \cos \beta'_2 + K^*(\sin \beta_1 - \frac{\alpha_1}{\alpha'_2} \sin \beta'_2)] + B[(\alpha_1 - \beta_1) \sin \beta_1 + (\beta'_2 - \alpha'_2) \sin \beta'_2 + K^*(\cos \beta_1 - \cos \beta'_2)] = 0 \quad (10.86)$$

Setting the determinant of the coefficient matrix of constants A and B in equations (10.85) and (10.86) to zero, we obtain the following frequency equation for this case.

$$\begin{aligned} & \delta_1(\delta_1 - \alpha_1) + \delta'_2(\delta'_2 - \alpha'_2) + \sin \delta_1 \sin \delta'_2 [\delta'_2(\alpha_1 - \delta_1) - \delta_1(\delta'_2 - \alpha'_2)] \\ & + \cos \delta_1 \cos \delta'_2 \left[\frac{\alpha_1 \delta_1 (\alpha'_2 - \delta'_2)}{\alpha'_2} - \frac{\alpha'_2 \delta'_2 (\delta_1 - \alpha_1)}{\alpha_1} \right] \\ & = \frac{\alpha_1 \delta_1 - \alpha'_2 \delta'_2}{\alpha_1 \alpha'_2} [\alpha_1 \cos \delta_1 \sin \delta'_2 - \alpha'_2 \sin \delta_1 \cos \delta'_2] \end{aligned} \quad (10.86a)$$

The modal shape expressions are given by equations (10.61) and (10.62) where $\frac{B}{A}$ is computed from equation (10.85).

It is quite clear from what has been described so far that the procedure of arriving at the frequency equations and modal shapes is quite mechanical and therefore in rest of the cases described in this Chapter, only the final results for frequency equations and modal shapes will be provided and intermediate steps will not be shown. Frequent use of the various expressions of boundary conditions, provided earlier in this Chapter, will obviously be made.

10.4.2 Beams Clamped at One End and With a Concentrated Mass at the Other

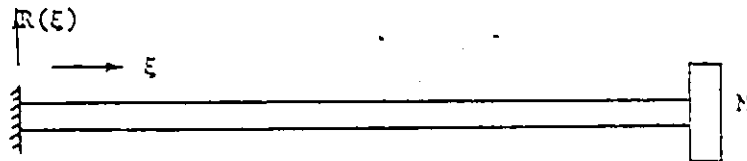


Figure 10.4.2

A close look at the boundary conditions at the right hand end suggests that the modal shape expressions are same as provided in case 10.4.1 and the frequency equation expression is determined by replacing K^* in the previous case with $(-\frac{\beta^4}{\eta})$.

10.4.3 Beams Clamped at One End and With a Concentrated Mass-Supported on a Coil Spring at the Other (Figure 2.6)

Modal shape expressions are same as in case 10.4.1 and frequency equation is obtained by replacing K^* in case 10.4.1 with $(K^* - \frac{\beta^4}{\eta})$.

10.4.4 Beams Clamped at One End and With a Rotational Mass at the Other (Figure 2.1)

Case (a) when $\beta^4 r^2 s^2 < 1$.

Frequency equation is given by,

$$I_o^* = \frac{\beta^4 [2 - 2 \cos \beta_1 \cosh \beta_2 + \sin \beta_1 \sinh \beta_2 (\frac{\alpha_2^2 - \alpha_1^2}{\alpha_1 \alpha_2})]}{\frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_1 \alpha_2} [\alpha_2 \sin \beta_1 \cosh \beta_2 - \alpha_1 \cos \beta_1 \sinh \beta_2]} \quad (10.87)$$

Modal shape expressions are given by equations (10.56) and (10.57), with

$$\frac{B}{A} = \frac{\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sinh \beta_2}{\cosh \beta_2 - \cos \beta_1}$$

Case (b), when $\beta^4 r^2 s^2 > 1$

Frequency equation is

$$I_o^* = \frac{\beta^4 [2 - 2 \cos \beta_1 \cos \beta'_2 - \sin \beta_1 \sin \beta'_2 (\frac{\alpha'^2_2 + \alpha^2_1}{\alpha_1 \alpha'_2})]}{(\beta_1 - \frac{\alpha'^2_2 \beta'_2}{\alpha_1}) \sin \beta_1 \cos \beta'_2 + \cos \beta_1 \sin \beta'_2 (\beta'_2 - \frac{\alpha_1 \beta_1}{\alpha'_2})} \quad (10.88)$$

The modal shape expressions are given by equations (10.61) and (10.62) with

$$\frac{B}{A} = \frac{\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sin \beta'_2}{\cos \beta'_2 - \cos \beta_1}$$

10.4.5 Beams Clamped at One End and With a Torsion Spring at the Other

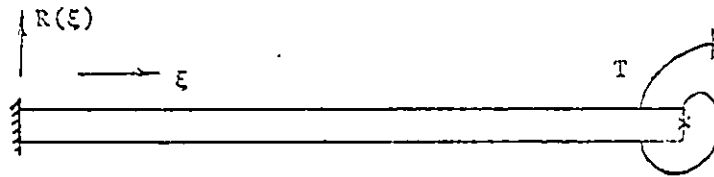


Figure 10.4.5

The modal shape expressions are same as in case 10.4.4 and the frequency equation is obtained by replacing I_o^* in case 10.4.4 with $(-\frac{\beta^4}{T})$.

10.4.6 Beams Clamped at One End and With a Rotational Mass and Torsion Spring at the Other

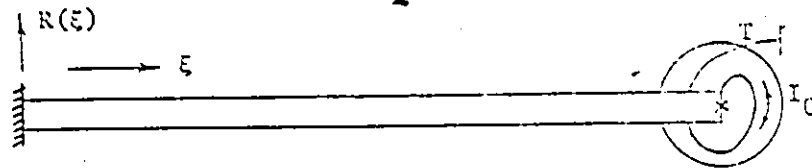


Figure 10.4.6

Modal shape expressions are same as in case 10.4.4 and frequency equation is obtained by replacing I_0^* in case 10.4.4 with $(I_0^* - \frac{\beta^4}{T^*})$.

10.4.7 Beams Simply Supported at One End and With a Linear Coil Spring at the Other

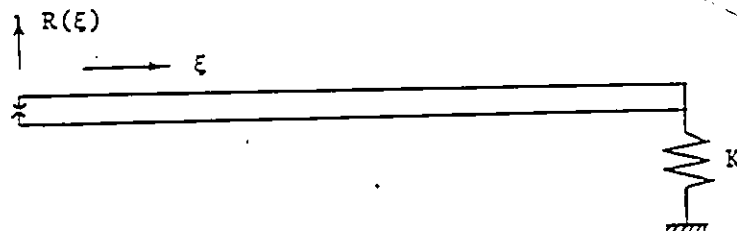


Figure 10.4.7

Case (a) when $\beta^4 r^2 s^2 < 1$

For a beam with left hand end simply supported the expressions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.25) and (10.26) of case 10.3.1. Applying the boundary conditions at the right hand end, we obtain:

$$A(-\alpha_1 \beta_1 \sin \beta_1) + C \alpha_2 \beta_2 \sinh \beta_2 = 0 \quad (10.89)$$

and

$$A(\beta_1 \cos \beta_1 - \alpha_1 \cos \beta_1 + K^* \sin \beta_1) + C(\beta_2 \cosh \beta_2 - \alpha_2 \cosh \beta_2 + K^* \sinh \beta_2) = 0 \quad (10.90)$$

From the above two equations, the following frequency equation is obtained

$$K^* = \frac{\alpha_1 \beta_1 (\alpha_2 - \beta_2) \sin \beta_1 \cosh \beta_2 + \alpha_2 \beta_2 (\alpha_1 - \beta_1) \cos \beta_1 \sinh \beta_2}{(\alpha_1 \beta_1 + \alpha_2 \beta_2) \sin \beta_1 \sinh \beta_2} \quad (10.91)$$

The modal shape expressions are given by equations (10.25) and (10.26) where $\frac{C}{A}$ is obtained from equation (10.89), so that

$$\frac{C}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1}{\alpha_2 \beta_2 \sinh \beta_2}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

For this case, with left hand end simply supported, the expressions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.35) and (10.36) of case 10.3.1. Enforcement of the boundary conditions at the right hand end yields:

$$A(-\alpha_1 \beta_1 \sin \beta_1) + C(-\alpha'_2 \beta'_2 \sin \beta'_2) = 0 \quad (10.92)$$

and

$$A(\beta_1 \cos \beta_1 - \alpha_1 \cos \beta_1 + K^* \sin \beta_1) + C(\beta'_2 \cos \beta'_2 - \alpha'_2 \cos \beta'_2 + K^* \sin \beta'_2) = 0 \quad (10.93)$$

From which, the following frequency equation is obtained

$$K^* = \frac{\alpha_1 \beta_1 (\alpha'_2 - \beta'_2) \sin \beta_1 \cos \beta'_2 + \alpha'_2 \beta'_2 (\beta_1 - \alpha_1) \cos \beta_1 \sin \beta'_2}{(\alpha_1 \beta_1 - \alpha'_2 \beta'_2) \sin \beta_1 \sin \beta'_2} \quad (10.94)$$

The modal shape expressions are provided by equations (10.35) and (10.36) where

$$\frac{C}{A} = - \frac{\alpha_1 \beta_1 \sin \beta_1}{\alpha'_2 \beta'_2 \sin \beta'_2}$$

10.4.8 Beams Simply-Supported at One End and With a Concentrated Mass at the Other (Figure 2.4)

The modal shape expressions are same as in case 10.4.7 and the frequency equation is obtained by replacing K^* in case 10.4.7 with $(-\frac{S^4}{\eta})$.

10.4.9 Beams Simply Supported at One End and With a Concentrated Mass and Linear Coil Spring at the Other (Figure 2.7)

The modal shape expressions are same as in case 10.4.7 and the frequency equation is obtained by replacing K^* in case 10.4.7 with $(K^* - \frac{S^4}{\eta})$.

10.4.10 Beams Simply Supported at One End and With a Rotational Mass at the Other (Figure 2.2)

Case (a) when $\beta^4 r^2 s^2 < 1$

As before the equations (10.25) and (10.26) represent $R(\xi)$ and $\psi(\xi)$ after the boundary conditions at the left hand end are utilized. Utilizing the boundary conditions at the right hand end, namely $R(1) = 0$ and $\psi'(1) - \frac{\beta^4}{I_o} \psi(1) = 0$, we obtain

$$A \sin \beta_1 + C \sinh \beta_2 = 0 \quad (10.95)$$

and

$$A(-\alpha_1 \beta_1 \sin \beta_1 - \frac{\beta_1^4}{I_o^*} \alpha_1 \cos \beta_1) + C(\alpha_2 \beta_2 \sinh \beta_2 - \frac{\beta_2^4}{I_o^*} \alpha_2 \cosh \beta_2) = 0 \quad (10.96)$$

which results in the following frequency equation of the system

$$I_o^* = \frac{\beta_1^4 (\alpha_2 \sin \beta_1 \cosh \beta_2 - \alpha_1 \cos \beta_1 \sinh \beta_2)}{\sin \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2)} \quad (10.97)$$

The modal shapes are given by equations (10.25) and (10.26)

with

$$\frac{C}{A} = - \frac{\sin \beta_1}{\sinh \beta_2}$$

Case (b) when $\beta_1^4 r^2 s^2 > 1$

Now $R(\xi)$ and $\psi(\xi)$ are expressed by equations (10.35) and (10.36) when the boundary conditions at the left hand end are utilized. Applying the boundary conditions at the right hand end, we obtain

$$A \sin \beta_1 + C \sin \beta_2 = 0 \quad (10.98)$$

and

$$A(-\alpha_1 \beta_1 \sin \beta_1 - \frac{\beta_1^4}{I_o^*} \alpha_1 \cos \beta_1) + C(-\alpha'_2 \beta'_2 \sin \beta'_2 - \frac{\beta_2^4}{I_o^*} \alpha'_2 \cos \beta'_2) = 0 \quad (10.99)$$

Utilizing the condition for a non trivial solution, we obtain the following frequency equation of the beam system

$$I_o^* = \frac{\beta_1^4 [\alpha_1 \cos \beta_1 \sin \beta'_2 - \alpha'_2 \cos \beta'_2 \sin \beta_1]}{\sin \beta_1 \sin \beta'_2 (\alpha'_2 \beta'_2 - \alpha_1 \beta_1)} \quad (10.100)$$

The modal shapes are given by equations (10.35) and (10.36)

with

$$\frac{C}{A} = - \frac{\sin \beta_1}{\sin \beta'_2}$$

10.4.11 Beams Simply Supported at One End and With a Torsion Spring at the Other

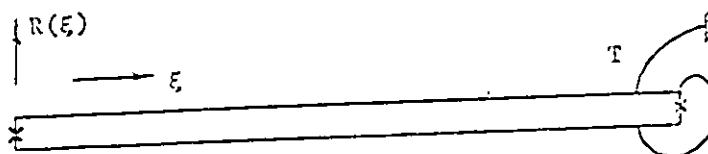


Figure 10.4.11

The modal shape expressions are same as in case 10.4.10 and the frequency equation is obtained by replacing I_o^* in case 10.4.10 with $(-\frac{\beta^4}{T^*})$.

10.4.12 Beams Simply Supported at One End and With a Torsion Spring and a Rotational Mass at the Other

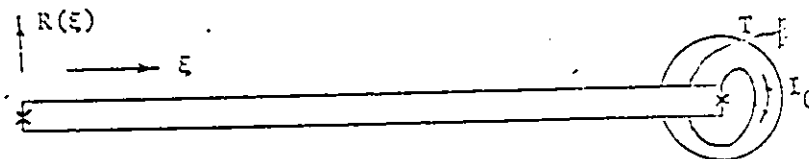


Figure 10.4.12

The modal shape expressions are same as given for case 10.4.10 and the frequency equation is obtained by replacing I_o^* in case 10.4.10 with $(I_o^* - \frac{\beta^4}{T^*})$.

10.4.13 Beams Free at One End and With a Rotational Mass at the Other (Figure 2.3)

Case (a) when $\beta^4 r^2 s^2 < 1$

Utilizing the boundary conditions at the free end, the expressions for $R(\xi)$ and $\psi(\xi)$ are given by equations (10.72) and (10.73) of case 10.3.6. Substitution of boundary conditions at the right hand end into equations (10.72) and (10.73) yields:

$$A(\sin \beta_1 + \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2} \sinh \beta_2) + B(\cos \beta_1 + \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \cosh \beta_2) = 0 \quad (10.101)$$

and

$$A \left[-\alpha_1 \beta_1 \sin \beta_1 + \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \sinh \beta_2 - \frac{\beta^4}{I_o^*} \left(\alpha_1 \cos \beta_1 + \frac{\alpha_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \cosh \beta_2 \right) \right] \\ + B \left[\alpha_1 \beta_1 (\cosh \beta_2 - \cos \beta_1) - \frac{\beta^4}{I_o^*} \left(-\alpha_1 \sin \beta_1 + \frac{\alpha_1 \beta_1}{\beta_2} \sinh \beta_2 \right) \right] = 0 \quad (10.102)$$

Which result in the following frequency equation

$$\beta^4 [\beta_2 (\beta_2 - \alpha_2) + \beta_1 (\alpha_1 - \beta_1) + \sin \beta_1 \sinh \beta_2 \{ \beta_2 (\alpha_1 - \beta_1) - \beta_1 (\beta_2 - \alpha_2) \} \\ + \cos \beta_1 \cosh \beta_2 \{ \frac{\alpha_1 \beta_1 (\beta_2 - \alpha_2)}{\alpha_2} + \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{\alpha_1} \}] \\ I_o^* = \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_1 \alpha_2} [\alpha_2 \beta_2 (\alpha_1 - \beta_1) \cos \beta_1 \sinh \beta_2 - \alpha_1 \beta_1 (\beta_2 - \alpha_2) \sin \beta_1 \cosh \beta_2] \quad (10.103)$$

The modal shape expressions are given by equations (10.72) and (10.73) with

$$\frac{B}{A} = - \frac{(\sin \beta_1 + \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2} \sinh \beta_2)}{(\cos \beta_1 + \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \cosh \beta_2)}$$

Case (b) where $\beta^4 r^2 s^2 > 1$

Equations (10.77) and (10.78) of case 10.3.6 are obtained for $R(\xi)$ and $\psi(\xi)$ when the boundary conditions at the left hand end are applied. The boundary conditions at the right hand end result in;

$$A \left[\sin \beta_1 + \frac{(\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \sin \beta'_2 \right] + B \left[\cos \beta_1 - \frac{\alpha_1 \beta_1}{\alpha'_2 \beta'_2} \cos \beta'_2 \right] = 0 \quad (10.104)$$

and

$$A \left[-\alpha_1 \beta_1 \sin \beta_1 - \frac{\alpha'_2 \beta'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \sin \beta'_2 - \frac{\beta^4}{I_o^*} \left(\alpha_1 \cos \beta_1 + \frac{\alpha'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \cos \beta'_2 \right) \right] \\ + B \left[-\alpha_1 \beta_1 \cos \beta_1 + \alpha_1 \beta_1 \cos \beta'_2 - \frac{\beta^4}{I_o^*} \left(-\alpha_1 \sin \beta_1 + \frac{\alpha_1 \beta_1}{\beta'_2} \sin \beta'_2 \right) \right] = 0 \quad (10.105)$$

and the frequency equation is obtained by equating the determinant of the above two equations to zero.

$$I_o^* = \frac{\begin{aligned} & \delta^4 [\delta'_2 (\alpha'_2 - \delta'_2) + \delta_1 (\alpha_1 - \delta_1) + \sin \delta_1 \sin \delta'_2 \{ \delta_1 (\delta'_2 - \alpha'_2) + \delta'_2 (\delta_1 - \alpha_1) \} \\ & + \cos \delta_1 \cos \delta'_2 \{ \frac{\alpha_1 \delta_1 (\delta'_2 - \alpha'_2)}{\delta_2} - \frac{\alpha'_2 \delta'_2 (\alpha_1 - \delta_1)}{\alpha_1} \} \end{aligned}}{\begin{aligned} & \frac{\alpha'_2 \delta'_2 - \alpha_1 \delta_1}{\alpha_1 \alpha'_2} [\alpha_1 \delta_1 (\alpha'_2 - \delta'_2) \sin \delta_1 \cos \delta'_2 + \alpha'_2 \delta'_2 (\alpha_1 - \delta_1) \cos \delta_1 \sin \delta'_2] \end{aligned}} \quad (10.106)$$

Modal shape expressions are provided by equations (10.77)

and (10.78) with

$$\frac{B}{A} = \frac{\sin \delta_1 + \frac{(\alpha_1 - \delta_1)}{(\delta'_2 - \alpha'_2)} \sin \delta'_2}{\frac{\alpha_1 \delta_1}{\alpha'_2 \delta'_2} \cos \delta'_2 - \cos \delta_1}$$

10.4.14 Beams Free at One End and With a Torsion Spring at the Other

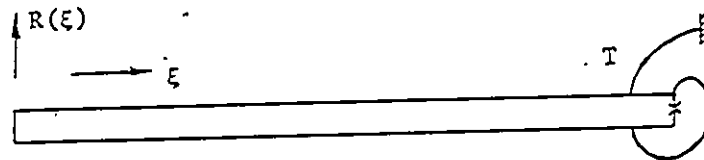


Figure 10.4.14

The modal shape expressions are same as provided in case 10.4.13 and the frequency equation is obtained by replacing I_o^* in case 10.4.13 with $(-\frac{\beta^4}{T^*})$.

10.4.15 Beams Free at One End and With a Rotational Mass and a Torsion Spring at the Other

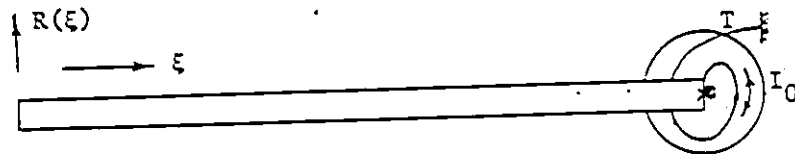


Figure 10.4.15

Modal shape expressions are same as in case 10.4.13 and frequency equation is obtained by replacing I_o^* in case 10.4.13 with $(I_o^* - \frac{\beta^4}{T^*})$.

10.4.16 Beams Clamped at One End and With a Torsion Spring and
Coil at the Other (Figure 2.11)

Case (a) when $\beta^4 r^2 s^2 < 1$.

Expressions for $R(\xi)$ and $\psi(\xi)$ reduce to equations (10.56)* and (10.57) of case 10.3.4 after utilizing the boundary conditions at the left hand end. Enforcing the boundary conditions at the right hand end, namely, $\psi'(1) + T^*(1) = 0$ and $R'(1) - \psi(1) + K^* R(1) = 0$, we obtain the following two equations

$$\begin{aligned} & A[-\alpha_1 \beta_1 \sin \beta_1 - \alpha_1 \beta_2 \sinh \beta_2 + T^*(\alpha_1 \cos \beta_1 - \alpha_1 \cosh \beta_2)] \\ & + B[-\alpha_1 \beta_1 \cos \beta_1 - \alpha_2 \beta_2 \cosh \beta_2 + T^*(-\alpha_1 \sin \beta_1 - \alpha_2 \sinh \beta_2)] = 0 \end{aligned} \quad (10.107)$$

and

$$\begin{aligned} & A[(\beta_1 - \alpha_1) \cos \beta_1 + (\alpha_1 - \frac{\alpha_1 \beta_2}{\alpha_2}) \cosh \beta_2 + K^*(\sin \beta_1 - \frac{\alpha_1}{\alpha_2} \sinh \beta_2)] \\ & + B[(\alpha_1 - \beta_1) \sin \beta_1 + (\alpha_2 - \beta_2) \sinh \beta_2 + K^*(\cos \beta_1 - \cosh \beta_2)] = 0 \end{aligned} \quad (10.108)$$

For a non-trivial solution for the constants A and B in the

above two equations, the determinant of their coefficient matrix must vanish which results in the following frequency equation of the system.

$$\begin{aligned}
 & T^* [\cos \beta_1 \sinh \beta_2 \{ \frac{\alpha_2 (\beta_1 - \alpha_1)}{\alpha_1} + (\alpha_2 - \beta_2) \} + \sin \beta_1 \cosh \beta_2 \{ \frac{\alpha_1 (\alpha_2 - \beta_2)}{\alpha_2} - (\alpha_1 - \beta_1) \}] \\
 & + K^* [\frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_1 \alpha_2} \{ \alpha_2 \sin \beta_1 \cosh \beta_2 - \alpha_1 \cos \beta_1 \sinh \beta_2 \}] \\
 & + T^* K^* [2 - 2 \cos \beta_1 \cosh \beta_2 + \frac{\alpha_2^2 - \alpha_1^2}{\alpha_1 \alpha_2} \sin \beta_1 \sinh \beta_2] \\
 & + \beta_1 (\beta_1 - \alpha_1) + \beta_2 (\alpha_2 - \beta_2) - \sin \beta_1 \sinh \beta_2 \{ \beta_2 (\alpha_1 - \beta_1) + \beta_1 (\alpha_2 - \beta_2) \} \\
 & + \cos \beta_1 \cosh \beta_2 \{ \frac{\alpha_2 \beta_2 (\beta_1 - \alpha_1)}{\alpha_1} + \frac{\alpha_1 \beta_1 (\alpha_2 - \beta_2)}{\alpha_2} \} = 0 \quad (10.109)
 \end{aligned}$$

The modal shape expressions are given by equations (10.56) and (10.57) with

$$\frac{B}{A} = \frac{-\alpha_1 \beta_1 \sin \beta_1 - \alpha_1 \beta_2 \sinh \beta_2 + T^* (\alpha_1 \cos \beta_1 - \alpha_1 \cosh \beta_2)}{\alpha_1 \beta_1 \cos \beta_1 + \alpha_2 \beta_2 \cosh \beta_2 + T^* (\alpha_1 \sin \beta_1 + \alpha_2 \sinh \beta_2)}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

Following the usual procedure, the following frequency equation is obtained

$$\begin{aligned}
 & T^* [\sin \beta_1 \cos \beta'_2 \{ \frac{\alpha_1 (\alpha'_2 - \beta'_2)}{\alpha'_2} - (\alpha_1 - \beta_1) \} - \cos \beta_1 \sin \beta'_2 \{ \frac{\alpha'_2 (\beta_1 - \alpha_1)}{\alpha_1} + (\alpha'_2 - \beta'_2) \}] \\
 & + K^* [\frac{\alpha_1 \beta_1 - \alpha'_2 \beta'_2}{\alpha_1 \alpha'_2} (\alpha'_2 \sin \beta_1 \cos \beta'_2 - \alpha_1 \cos \beta_1 \sin \beta'_2)] \\
 & + T^* K^* [2 - 2 \cos \beta_1 \cos \beta'_2 - \sin \beta_1 \sin \beta'_2 (\frac{\alpha_1^2 + \alpha'^2_2}{\alpha_1 \alpha'_2})] \\
 & + \beta_1 (\beta_1 - \alpha_1) - \beta'_2 (\alpha'_2 - \beta'_2) + \sin \beta_1 \sin \beta'_2 \{ \beta'_2 (\alpha_1 - \beta_1) + \beta_1 (\alpha'_2 - \beta'_2) \} \\
 & + \cos \beta_1 \cos \beta'_2 \{ \frac{\alpha_1 \beta_1 (\alpha'_2 - \beta'_2)}{\alpha'_2} - \frac{\alpha'_2 \beta'_2 (\beta_1 - \alpha_1)}{\alpha_1} \} = 0 \quad (10.110)
 \end{aligned}$$

and the modal shapes are given by equations (10.61) and (10.62) in case 10.3.4 where

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 - \alpha_1 \beta'_2 \sin \beta'_2 + T^* (\alpha_1 \cos \beta'_2 - \alpha_1 \cos \beta_1)}{\alpha'_2 \beta'_2 \cos \beta'_2 - \alpha_1 \beta_1 \cos \beta_1 + T^* (\alpha'_2 \sin \beta'_2 - \alpha_1 \sin \beta_1)}$$

10.4.17 Beams Simply Supported at One End and With a Torsion
Spring and a Coil Spring at the Other (Figure 2.12)

Case (a) when $\delta^4 r^2 s^2 < 1$

The frequency equation is given by,

$$\begin{aligned} & T^* [\cos \beta_1 \cosh \beta_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1)] - K^* [\sin \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2)] \\ & + T^* K^* [\alpha_1 \cos \beta_1 \sinh \beta_2 - \alpha_2 \sin \beta_1 \cosh \beta_2] \\ & + \sin \beta_1 \cosh \beta_2 \alpha_1 \beta_1 (\alpha_2 - \beta_2) + \cos \beta_1 \sinh \beta_2 \alpha_2 \beta_2 (\alpha_1 - \beta_1) = 0 \end{aligned} \quad (10.111)$$

The modal shape expressions are provided by equations
(10.25) and (10.26) in case 10.3.1 where

$$\frac{C}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 - T^* \alpha_1 \cos \beta_1}{\alpha_2 \beta_2 \sinh \beta_2 + T^* \alpha_2 \cosh \beta_2}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

The frequency equation is given by,

$$\begin{aligned} & T^* [\cos \beta_1 \cos \beta'_2 (\alpha_1 \beta'_2 - \alpha'_2 \beta_1)] + K^* [(\alpha'_2 \beta'_2 - \alpha_1 \beta_1) \sin \beta_1 \sin \beta'_2] \\ & + T^* K^* [\alpha_1 \cos \beta_1 \sin \beta'_2 - \alpha'_2 \sin \beta_1 \cos \beta'_2] \\ & + \sin \beta_1 \cos \beta'_2 \alpha_1 \beta_1 (\alpha'_2 - \beta'_2) + \cos \beta_1 \sin \beta'_2 \alpha'_2 \beta'_2 (\beta_1 - \alpha_1) = 0 \end{aligned} \quad (10.112)$$

The modal shape expressions are given by equations (10.35) and (10.36) provided in case 10.3.1 where

$$\frac{C}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 - T^* \alpha_1 \cos \beta_1}{T^* \alpha'_2 \cos \beta'_2 - \alpha'_2 \beta'_2 \sin \beta'_2}$$

10.4.18 Beams Free at One End and With a Torsion and Coil Spring at the Other (Figure 2.13)

Case (a) when $\beta^4 r^2 s^2 < 1$

The frequency equation is given by,

$$T^* [\sin \beta_1 \cosh \beta_2 \{ \frac{\alpha_2 (\alpha_1 - \beta_1)^2}{(\beta_2 - \alpha_2)} + \alpha_1 (\alpha_1 - \beta_1) \} + \cos \beta_1 \sinh \beta_2 \{ \frac{\alpha_1^2 \beta_1 (\beta_2 - \alpha_2)}{\alpha_2 \beta_2}$$

$$- \frac{\alpha_1 \beta_1 (\beta_1 - \alpha_1)}{\beta_2} \}]$$

$$- K^* [\sin \beta_1 \cosh \beta_2 \{ \alpha_1 \beta_1 - \frac{(\alpha_1 \beta_1)^2}{\alpha_2 \beta_2} \} + \cos \beta_1 \sinh \beta_2 \{ \frac{\alpha_1 \beta_1 (\beta_1 - \alpha_1)}{(\beta_2 - \alpha_2)}$$

$$- \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \}]$$

$$+ T^* K^* [\alpha_1 + \frac{\alpha_1 \beta_1 (\alpha_1 - \beta_1)}{\beta_2 (\beta_2 - \alpha_2)} + \cos \beta_1 \cosh \beta_2 \{ \frac{\alpha_1^2 \beta_1}{\alpha_2 \beta_2} + \frac{\alpha_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \}$$

$$- \sin \beta_1 \sinh \beta_2 \{ \frac{\alpha_1 \beta_1}{\beta_2} + \frac{\alpha_1 (\beta_1 - \alpha_1)}{(\beta_2 - \alpha_2)} \}]$$

$$+ 2\alpha_1 \beta_1 (\beta_1 - \alpha_1) - 2\alpha_1 \beta_1 (\beta_1 - \alpha_1) +$$

$$\sin \beta_1 \sinh \beta_2 \{ \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)^2}{(\beta_2 - \alpha_2)} - \frac{(\alpha_1 \beta_1)^2 (\beta_2 - \alpha_2)}{\alpha_2 \beta_2} \} = 0 \quad (10.113)$$

The modal shape expressions are given by equations (10.72) and (10.73) given in case 10.3.6 where

$$\frac{B}{A} = \frac{-\alpha_1 \beta_1 \sin \beta_1 + \frac{\alpha_2 \beta_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \sinh \beta_2 + T^* \{ \alpha_1 \cos \beta_1 + \frac{\alpha_2 (\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)} \cosh \beta_2 \}}{\alpha_1 \beta_1 \cos \beta_1 - \alpha_1 \beta_1 \cosh \beta_2 + T^* \{ \alpha_1 \sin \beta_1 - \frac{\alpha_1 \beta_1}{\beta_2} \sinh \beta_2 \}}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

The frequency equation is

$$T^* [\sin \beta_1 \cos \beta'_2 \{ \frac{\alpha'_2 (\alpha_1 - \beta_1)^2}{\beta'_2 - \alpha'_2} + \alpha_1 (\alpha_1 - \beta_1) \} + \cos \beta_1 \sin \beta'_2$$

$$\{ \frac{\alpha_1^2 \beta_1 (\beta'_2 - \alpha_2)}{\alpha'_2 \beta'_2} - \frac{\alpha_1 \beta_1 (\beta_1 - \alpha_1)}{\beta'_2} \}] - K^* [\sin \beta_1 \cos \beta'_2 \{ \alpha_1 \beta_1 + \frac{(\alpha_1 \beta_1)^2}{\alpha'_2 \beta'_2} \} +$$

$$\cos \beta_1 \sin \beta'_2 \{ \frac{\alpha_1 \beta_1 (\beta_1 - \alpha_1)}{(\beta'_2 - \alpha'_2)} + \frac{\alpha'_2 \beta'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \}]$$

$$+ T^* K^* [\alpha_1 + \frac{\alpha_1 \beta_1 (\beta_1 - \alpha_1)}{\beta'_2 (\beta'_2 - \alpha'_2)} + \cos \beta_1 \cos \beta'_2 \{ \frac{\alpha'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} - \frac{\alpha_1^2 \beta_1}{\alpha'_2 \beta'_2} \}]$$

$$- \sin \beta_1 \sin \beta'_2 \left\{ \frac{\alpha_1 \beta_1}{\beta'_2} + \frac{\alpha_1 (\beta_1 - \alpha_1)}{(\beta'_2 - \alpha'_2)} \right\}$$

$$+ 2\alpha_1 \beta_1 (\beta_1 - \alpha_1) - 2\alpha_1 \beta_1 (\beta_1 - \alpha_1) \cos \beta_1 \cos \beta'_2 -$$

$$\sin \beta_1 \sin \beta'_2 \left[\frac{\alpha'_2 \beta'_2 (\alpha_1 - \beta_1)^2}{\beta'_2 - \alpha'_2} + \frac{(\alpha_1 \beta_1)^2 (\beta'_2 - \alpha'_2)}{\alpha'_2 \beta'_2} \right] = 0$$

(10.114)

The modal shape expressions are given by equations (10.77) and (10.78) provided in case 10.3.6 where

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 + \frac{\alpha'_2 \beta'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \sin \beta'_2 - T^* (\alpha_1 \cos \beta_1 + \frac{\alpha'_2 (\alpha_1 - \beta_1)}{(\beta'_2 - \alpha'_2)} \cos \beta'_2)}{\alpha_1 \beta_1 (\cos \beta'_2 - \cos \beta_1) + T^* (-\alpha_1 \sin \beta_1 + \frac{\alpha_1 \beta_1}{\beta'_2} \sin \beta'_2)}$$

Part 2 - Beams with non-classical bounding conditions at each end

10.4.19 Beams With a Rotational Mass at One End and a Coil Spring at the Other (Figure 2.14)

Case (a) when $\beta^4 r^2 s^2 < 1$

The boundary conditions at the left hand end, namely

$R(0) = 0$ and $\psi'(0) + \frac{\beta^4}{I_o^*} \psi(0) = 0$, result in the following expressions

for $R(\xi)$ and $\psi(\xi)$

$$R(\xi) = A(\sin \beta_1 \xi - \theta \sinh \beta_2 \xi) + B(\cos \beta_1 \xi - \cosh \beta_2 \xi + \frac{I_o^*}{\beta^4} \phi \sinh \beta_2 \xi) \quad (10.115)$$

and

$$\begin{aligned} \psi(\xi) = & A(\alpha_1 \cos \beta_1 \xi - \beta \alpha_2 \cosh \beta_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi - \alpha_2 \sinh \beta_2 \xi \\ & + \frac{I_o^*}{\beta^4} \phi \alpha_2 \cosh \beta_2 \xi) \end{aligned} \quad (10.116)$$

where

$$\theta = \frac{\alpha_1}{\alpha_2} \quad \text{and} \quad \phi = \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_2}$$

Substituting the boundary conditions at the right hand end into above two equations and setting the determinant of the resulting equations to zero, we obtain the following frequency equations for this case

$$K^*[(\alpha_1 \beta_1 + \alpha_2 \beta_2)(\sin \beta_1 \cosh \beta_2 - \theta \cos \beta_1 \sinh \beta_2)]$$

$$+ \frac{I_o^* \phi}{\beta^4} [\alpha_2 \beta_2 (\alpha_1 - \beta_1) \cos \beta_1 \sinh \beta_2 + \alpha_1 \beta_1 (\alpha_2 - \beta_2) \sin \beta_1 \cosh \beta_2]$$

$$- \frac{I_o^* K^* \phi}{\beta^4} [(\alpha_1 \beta_1 + \alpha_2 \beta_2) \sin \beta_1 \sinh \beta_2] + \alpha_1 \beta_1 (\beta_1 - \alpha_1) + \alpha_2 \beta_2 \theta (\alpha_2 - \beta_2)$$

$$- \sin \beta_1 \sinh \beta_2 \{ \theta \alpha_2 \beta_2 (\alpha_1 - \beta_1) + \alpha_1 \beta_1 (\alpha_2 - \beta_2) \}$$

$$+ \cos \beta_1 \cosh \beta_2 \{ \theta \alpha_1 \beta_1 (\alpha_2 - \beta_2) + \alpha_2 \beta_2 (\beta_1 - \alpha_1) \} = 0 \quad (10.117)$$

The modal shape expressions are given by (10.115) and (10.116) where

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 + \theta \alpha_2 \beta_2 \sinh \beta_2}{-\alpha_1 \beta_1 \cos \beta_1 - \alpha_2 \beta_2 \cosh \beta_2 + \frac{I_o^* \phi}{\beta^4} \alpha_2 \beta_2 \sinh \beta_2}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

After utilizing the boundary conditions at the left hand end, following expressions for $R(\xi)$ and $\psi(\xi)$ are obtained.

$$R(\xi) = A(\sin \beta_1 \xi - \theta' \sin \beta'_2 \xi) + B(\cos \beta_1 \xi - \cos \beta'_2 \xi + \frac{\phi' I^*}{\beta^4} \sin \beta'_2 \xi) \quad (10.118)$$

and

$$\psi(\xi) = A(\alpha_1 \cos \beta_1 \xi - \theta' \alpha'_2 \cos \beta'_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi + \alpha'_2 \sin \beta'_2 \xi + \frac{\phi' I^*}{\beta^4} \alpha'_2 \cos \beta'_2 \xi) \quad (10.119)$$

where

$$\phi' = \frac{\alpha_1 \beta_1 - \alpha'_2 \beta'_2}{\alpha'_2} \quad \text{and} \quad \theta' = \frac{\alpha_1}{\alpha'_2}$$

Utilizing the boundary conditions at the right hand end and simplifying, the following frequency equation is obtained

$$K^*[(\alpha_1 \beta_1 - \alpha'_2 \beta'_2)(\sin \beta_1 \cos \beta'_2 - \theta' \cos \beta_1 \sin \beta'_2)]$$

$$+ \frac{I_o^* \phi'}{\beta^4} [(\beta_1 - \alpha_1) \alpha'_2 \beta'_2 \cos \beta_1 \sin \beta'_2 + \alpha_1 \beta_1 (\alpha'_2 - \beta'_2) \sin \beta_1 \cos \beta'_2]$$

$$- \frac{I_o^* K^* \phi'}{\beta^4} [\sin \beta_1 \sin \beta'_2 (\alpha_1 \beta_1 - \alpha'_2 \beta'_2)] + \alpha_1 \beta_1 (\beta_1 - \alpha_1) - \alpha'_2 \beta'_2 \theta' (\alpha'_2 - \beta'_2)$$

$$+ \sin \beta_1 \sin \beta'_2 \{\theta' \alpha'_2 \beta'_2 (\alpha_1 - \beta_1) + \alpha_1 \beta_1 (\alpha'_2 - \beta'_2)\}$$

$$+ \cos \beta_1 \cos \beta'_2 \{\theta' \alpha_1 \beta_1 (\alpha'_2 - \beta'_2) - \alpha'_2 \beta'_2 (\beta_1 - \alpha_1)\} = 0 \quad (10.120)$$

The modal shape expressions are given by equations (10.118) and (10.119) where

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 - \theta' \alpha'_2 \beta'_2 \sin \beta'_2}{-\alpha_1 \beta_1 \cos \beta_1 + \alpha'_2 \beta'_2 \cos \beta'_2 - \frac{I_o^* \phi'}{\beta^4} \alpha'_2 \beta'_2 \sin \beta'_2}$$

10.4.20 Beams With a Rotational Mass at One End and Concentrated Mass at the Other (Figure 2.18)

The modal shape expressions are same as provided in case 10.4.19 and the frequency equation for this case can be written down by replacing K^* in case 10.4.19 with $(-\frac{s^4}{\eta})$.

10.4.21 Beams With a Torsion Spring at One End and a Concentrated Mass at the Other (Figure 2.16)

The frequency equation and modal shapes of case 10.4.20 are modified by replacing I_o^* with $(-\frac{s^4}{T})$.

10.4.22 Beams With a Torsion Spring at One End and Coil Spring at the Other

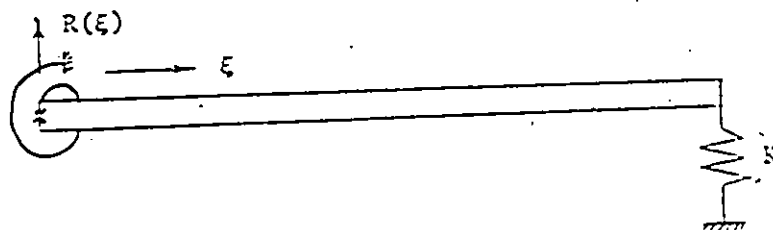


Figure 10.4.22

Frequency equations and modal shape expressions of case 10.4.19 are modified by replacing I_o^* with $(-\frac{\beta^4}{T^*})$.

10.4.23 Beams With Rotational Mass at One End and a Torsion Spring at the Other (Figure 2.15)

Case (a) when $\beta^4 r^2 s^2 < 1$

Application of boundary conditions at the left hand end results in expressions (10.115) and (10.116) of case 10.4.19 for $R(\xi)$ and $\psi(\xi)$ respectively. Substituting these in the boundary conditions of the right hand end and equating the determinant of the resulting equations to zero, we obtain the following frequency equation of this case

$$T^*[-\alpha_1 - \theta \alpha_2 + \sin \beta_1 \sinh \beta_2 (\alpha_1 \theta - \alpha_2) + \cos \beta_1 \cosh \beta_2 (\alpha_1 + \theta \alpha_2)]$$

$$+ \frac{I_o^*}{\beta^4} \phi [\sin \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2)] + \frac{T^* I_o^* \phi}{\beta^4}$$

$$[\alpha_2 \sin \beta_1 \cosh \beta_2 - \alpha_1 \cos \beta_1 \sinh \beta_2]$$

$$- \sin \beta_1 \cosh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2) + \cos \beta_1 \sinh \beta_2 \theta (\alpha_1 \beta_1 + \alpha_2 \beta_2) = 0$$

(10.121)

Modal shape expressions are given by equations (10.115) and (10.116) with

$$\frac{B}{A} = \frac{-\sin \beta_1 + \theta \sinh \beta_2}{\cos \beta_1 - \cosh \beta_2 + \frac{I_o^*}{\beta^4} \phi \sinh \beta_2}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

For this case equations (10.118) and (10.119) are obtained for $R(\xi)$ and $\psi(\xi)$ respectively after utilizing the boundary conditions at the left hand support. Further utilizing the boundary conditions at the right hand end and proceeding in the usual manner, the following frequency equation is obtained for this case.

$$T^* [-\alpha_1 - \theta' \alpha'_2 + \sin \beta_1 \sin \beta'_2 (\alpha_1 \theta' + \alpha'_2) + \cos \beta_1 \cos \beta'_2 (\alpha_1 + \theta' \alpha'_2)]$$

$$+ \frac{I_o^*}{\beta^4} \phi' [\sin \beta_1 \sin \beta'_2 (\alpha_1 \beta_1 - \alpha'_2 \beta'_2)] + \frac{T^* I_o^* \phi}{\beta^4} [\alpha'_2 \sin \beta_1 \cos \beta'_2 -$$

$$\alpha_1 \cos \beta_1 \sin \beta'_2] + \sin \beta_1 \cos \beta'_2 (\alpha'_2 \beta'_2 - \alpha_1 \beta_1) +$$

$$(\cos \beta_1 \sin \beta'_2 - \theta' (\alpha_1 \beta_1 - \alpha'_2 \beta'_2)) = 0$$

(10.122),

Modal shape expressions are given by equations (10.118) and (10.119) with

$$\frac{B}{A} = \frac{\sin \beta_1 - \theta' \sin \beta'_2}{\cos \beta_1 - \cos \beta'_2 - \frac{I_o^* \phi'}{\beta^4} \sin \beta'_2}$$

10.4.24 Beams With Rotational Masses at Each End (Figure 2.17)

The frequency equations and modal shape expressions for this case are obtained from the previous case 10.4.23 by replacing I_o^* and T^* with $I_{o_1}^*$ and $(-\frac{\beta^4}{I_{o_2}^*})$ respectively where subscripts 1 and 2 refer to left hand and right hand support respectively.

10.4.25 Beams with Torsion Springs at Each End

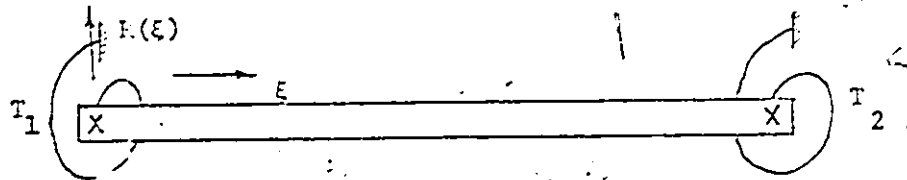


Figure 10.4.25

The frequency equations and modal shapes for this case are easily obtained from the pertinent expressions provided in case 10.4.23 by replacing I_o^* and T^* with $(-\frac{\beta_1^4}{T_1^*})$ and T_2^* respectively.

10.4.26 Beams With Linear Coil Springs at Each End

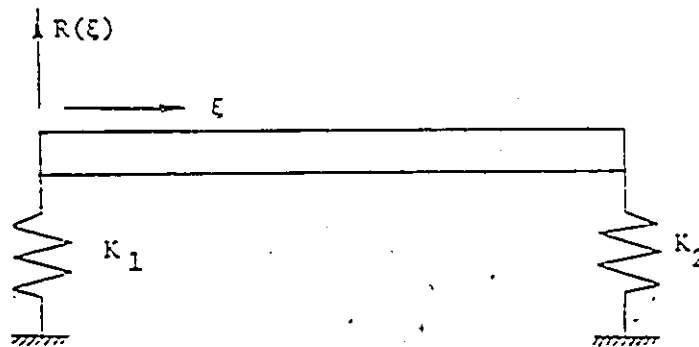


Figure 10.4.26

Case (a) when $\beta^4 r^2 s^2 < 1$

Utilizing the boundary conditions at the left hand end,

we obtain

$$R(\xi) = A(\sin \beta_1 \xi + \theta_1 \sinh \beta_2 \xi) + B(\cos \beta_1 \xi + \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \cosh \beta_2 \xi - K_1^* \phi_1 \sinh \beta_2 \xi) \quad (10.123)$$

and

$$\psi(\xi) = A(\alpha_1 \cos \beta_1 \xi + \theta_1 \alpha_2 \cosh \beta_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi + \frac{\alpha_1 \beta_1}{\beta_2} \sinh \beta_2 \xi - K_1^* \phi_1 \alpha_2 \cosh \beta_2 \xi) \quad (10.124)$$

where

$$\theta_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \quad \text{and} \quad \phi_1 = \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_2 \beta_2 (\alpha_2 - \beta_2)}$$

Utilizing the above two equations and the boundary conditions at the right hand end, the following frequency equation is obtained

$$\begin{aligned} & K_1^* \phi_1 [\alpha_1 \beta_1 (\beta_2 - \alpha_2) \sin \beta_1 \cosh \beta_2 + \alpha_2 \beta_2 (\beta_1 - \alpha_1) \cos \beta_1 \sinh \beta_2] \\ & + K_2^* [\beta_1 \cos \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2) - \sin \beta_1 \cosh \beta_2 \{ \frac{(\alpha_1 \beta_1)^2}{\alpha_2 \beta_2} + \alpha_1 \beta_1 \}] \\ & + K_1^* K_2^* \phi_1 [\sin \beta_1 \sinh \beta_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2)] + \alpha_1 \beta_1 (\beta_1 - \alpha_1) \\ & - \alpha_1 \beta_1 \theta_1 (\beta_2 - \alpha_2) + \sin \beta_1 \sinh \beta_2 [\theta_1 \alpha_2 \beta_2 (\alpha_1 - \beta_1) - \frac{(\alpha_1 \beta_1)^2 (\beta_2 - \alpha_2)}{\alpha_2 \beta_2}] \\ & + \cos \beta_1 \cosh \beta_2 \{ \theta_1 \alpha_1 \beta_1 (\beta_2 - \alpha_2) - \alpha_1 \beta_1 (\beta_1 - \alpha_1) \} = 0 \end{aligned} \quad (10.125)$$

The modal shape expressions are given by equations (10.123)

and (10.124) where

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 - \theta_1 \alpha_2 \beta_2 \sinh \beta_2}{-\alpha_1 \beta_1 \cos \beta_1 + \alpha_1 \beta_1 \cosh \beta_2 - K_1^* \phi_1 \alpha_2 \beta_2 \sinh \beta_2}$$

Case (b) when $\beta^4 r^2 s^2 > 1$

Utilizing the boundary conditions at the left hand end,

we obtain

$$R(\xi) = A(\sin \beta_1 \xi + \theta_2 \sin \beta'_2 \xi) + B(\cos \beta_1 \xi - \frac{\alpha_1 \beta_1}{\alpha'_2 \beta'_2} \cos \beta'_2 \xi - K_1^* \phi_2 \sin \beta'_2 \xi) \quad (10.126)$$

and

$$\psi(\xi) = A(\alpha_1 \cos \beta_1 \xi + \theta_2 \alpha'_2 \cos \beta'_2 \xi) + B(-\alpha_1 \sin \beta_1 \xi + \frac{\alpha_1 \beta_1}{\beta'_2} \sin \beta'_2 \xi - K_1^* \phi_2 \alpha'_2 \cos \beta'_2 \xi) \quad (10.127)$$

where

$$\phi_2 = \frac{\alpha'_2 \beta'_2 - \alpha_1 \beta_1}{\alpha'_2 \beta'_2 (\alpha'_2 - \beta'_2)} \quad \text{and} \quad \theta_2 = \frac{\beta_1 - \alpha_1}{\alpha'_2 - \beta'_2}$$

Utilizing the remaining two boundary conditions at the right hand-end and setting the determinant of the resulting two equations to zero, we obtain the following frequency equations for this case.

$$\begin{aligned}
 & K_1^* \phi_2 [\alpha'_2 \beta'_2 (\alpha_1 - \beta_1) \cos \beta_1 \sin \beta'_2 - \alpha_1 \beta_1 (\alpha'_2 - \beta'_2) \sin \beta_1 \cos \beta'_2] \\
 & + K_2^* [\theta_2 \cos \beta_1 \sin \beta'_2 (\alpha_1 \beta_1 - \alpha'_2 \beta'_2) - \sin \beta_1 \cos \beta'_2 \\
 & \quad \{ \alpha_1 \beta_1 - \frac{(\alpha_1 \beta_1)^2}{\alpha'_2 \beta'_2} \}] \\
 & + K_1^* K_2^* \phi_2 [\sin \beta_1 \sin \beta'_2 (\alpha_1 \beta_1 - \alpha'_2 \beta'_2)] + \alpha_1 \beta_1 (\beta_1 - \alpha_1) - \alpha_1 \beta_1 \theta_2 (\beta'_2 - \alpha'_2) \\
 & - \sin \beta_1 \sin \beta'_2 [\theta_2 \alpha'_2 \beta'_2 (\alpha_1 - \beta_1) + \frac{(\alpha_1 \beta_1)^2 (\beta'_2 - \alpha'_2)}{\alpha'_2 \beta'_2}] \\
 & + \cos \beta_1 \cos \beta'_2 [\theta_2 \alpha_1 \beta_1 (\beta'_2 - \alpha'_2) - \alpha_1 \beta_1 (\beta_1 - \alpha_1)] = 0 \quad (10.128)
 \end{aligned}$$

The modal shape expressions are given by equations (10.126) and (10.127) with

$$\frac{B}{A} = \frac{\alpha_1 \beta_1 \sin \beta_1 + \phi_2 \alpha'_2 \beta'_2 \sin \beta'_2}{-\alpha_1 \beta_1 \cos \beta_1 + \alpha_1 \beta_1 \cos \beta'_2 + K_1^* \phi_2 \alpha'_2 \beta'_2 \sin \beta'_2}$$

10.4.27 Beams With a Concentrated Mass at Each End (Figure 2.19)

The modal shape expressions and frequency equations for this case are easily obtained from the previous case 10.4.26 by replacing K_1^* and K_2^* with $(-\frac{\beta_1^4}{\eta_1^*})$ and $(-\frac{\beta_2^4}{\eta_2^*})$ respectively.

Illustrative Example 10.1

Determine natural frequency of a steel cantilever beam of uniform cross section of 1" x 1" and length 8" when shear deformation and rotary inertia effects are taken into consideration.

Here,

$$r = \sqrt{\frac{I}{A L^2}} = \sqrt{\frac{1}{12 \times 64}} = 0.036$$

$$s_1 = \sqrt{\frac{2(1+\nu)}{k'}} = \sqrt{\frac{8}{3} \times \frac{3}{2}} = 2.0$$

where the Poisson's ratio ν is taken equal to $\frac{1}{3}$ for steel, and shape factor $k' = \frac{2}{3}$ is taken for a rectangular cross section (36).

From Figure 10.3.5(a), obtain

$$\frac{\beta}{\beta_0} = 0.994 \text{ for first mode}$$

$$= 0.955 \text{ for second mode}$$

Hence,

$$f_1 = \frac{(0.994 \times 1.875)^2}{2\pi \times 64} \sqrt{\frac{30 \times 10^6 \times 1 \times 386}{12 \times 0.28}}$$

$$= 507 \text{ Hz}$$

$$f_2 = \frac{(0.955 \times 4.694)^2}{2\pi \times 64} \sqrt{\frac{EI}{\rho A}}$$

$$= 2933 \text{ Hz}$$

Where values of β_0 for cantilever beams are read off the Table A-1 provided in Appendix A. These are 1.875 and 4.694 for first and second mode respectively. If rotary inertia and shear deformation are neglected, then

$$f_1 = 513 \text{ Hz}$$

$$f_2 = 3217 \text{ Hz}$$

The reduction in frequency due to shear deformation and rotary inertia is 1.2% for first mode and 9.7% for the second mode for this beam.

CONCLUSIONS

The primary objective of the work presented in this thesis was to greatly simplify the problem of free vibrations of beams and shafts for vibration analysts and design engineers. This has been achieved by providing the frequency equations and modal shape expressions and the associated numerical results for a wide variety of problems. This should however be mentioned that only a limited class of problems have been solved in this thesis, though it is hoped that they will include most of the problems commonly encountered in practice.

The presentation of extensive numerical results was made convenient, perhaps possible, by using a methodical approach and the new "Q" factor. This "Q" factor was first used by the author in his master's thesis at the University of Ottawa. The subsequent introduction of the "Q" factor to the general public was carried out by Gorman in reference 39.

This thesis demonstrates the further potential of the technique of solution using the "Q" factor in solving vibration problems of beams and shafts with non-classical boundary conditions. The author feels that this technique can also be advantageously used for solving plates and shells vibration problems as well as longitudinal vibration problems of beams and shafts with non-classical boundary conditions.

The results presented in this thesis in chapters 2 through 9 are only valid for beam systems which conform to the Euler-Bernoulli model which assumes that cross-sectional dimensions are small compared to the length. If the beam system does not conform to the Euler-Bernoulli

model, consideration should be given to the effects of rotary inertia and shear deformation. The effect of rotary inertia and shear deformation is to reduce the natural frequency of vibrations. The results presented in chapter 10, indicate that these effects are insignificant at least in the first mode for $r \leq 0.01$, where r is the ratio of the radius of gyration K_r , of the cross-section to the length L of the beam. Whenever r is > 0.01 , the consideration to the shear deformation and rotary inertia effects is warranted.

It may however be emphasized that there can be no forth-right answer to the question, "exactly when or after which mode should the effects of shear deformation and rotary inertia be taken into account?" This would obviously depend on the geometric and elastic properties of the system. The assumption that the cross-sectional dimensions of the beam are small compared to its length may be violated at higher modes as the distance between the nodal points goes on diminishing with the appearance of more nodes at higher frequencies. This assumption could even be violated in the very first mode for short beams or for multi-span beams when adjacent supports are too close to each other.

In conclusion, it may be remarked that the work which was initiated in the author's master's thesis and concludes with this thesis, will be immensely useful to the practicing engineer in particular and all students of mechanical vibrations in general.

SUMMARY

1. The exact expressions for the frequency equations and mode shapes have been derived for the following classes of problems.

Lateral Vibrations

- (a) Single-span beams with non-classical boundary conditions.
- (b) Double-span beams of arbitrary span length with non-classical boundary conditions.
- (c) Triple and four span beams of arbitrary span length with classical boundary conditions.
- (d) Multi-span beams with classical boundary conditions for uniformly and arbitrarily spaced supports.
- (e) Tapered beams with linearly varying depth or width as well as beams consisting of both uniform and tapered cross-sections.
- (f) Beams with axial loading as well as beams on elastic foundation.

Torsional Vibrations

- (a) Circular shafts with non-classical boundary conditions.
2. The shear deformation and rotary inertia effects can significantly alter the natural frequencies of beam systems which do not conform to the Euler-Bernoulli model. The frequency equations and mode shape expressions have been provided for single span beams with classical and non-classical boundary conditions taking the shear deformation and rotary inertia effects into account.

3. The frequency equations for all the above cases have been solved for a wide variety of parameters covering a broad range of practical interest. The numerical results have been provided in the form of design charts and Tables.

The presence of too many influence parameters in the frequency equations of multi-span beams with arbitrarily spaced supports precluded the possibility of providing extensive numerical results like other cases. These limitations also apply to the single span beams with non-classical boundary conditions taking the shear deformation and rotary inertia effects into account. However, the expressions provided for the frequency equations for these cases can be solved by the analyst for particular values of various parameters of his interest.

4. In some cases, e.g., single span beams with rotational masses at each end (case 2.17), it is recognized that the upper and lower limits of frequencies of adjacent modes overlap. This recognition is important on the part of the designer so that in trying to move one mode out, another mode is not moved in.

5. A procedure has been devised to determine the upper and lower limiting lateral frequencies of double span beams and torsional frequencies of shafts, with intermediately located rotational masses or torsional springs or couplings respectively. An understanding of this procedure is very important since these limits unlike many simple cases can not be inferred at the onset. The procedure has been explained with the aid of illustrative examples.

The double span beams with classical boundary conditions were solved by Gorman (15). The double span beams with non-classical boundary conditions have been solved in this thesis using the same technique of reference 15, but with modified boundary conditions.

All the results presented in this thesis are original on the part of the author, except as mentioned otherwise, prepared in collaboration with his thesis supervisor Dr. D.J. Gorman. It was rather fortunate that these results proved to be of such important industrial significance as to warrant their being published eventually in the form of a book (38).

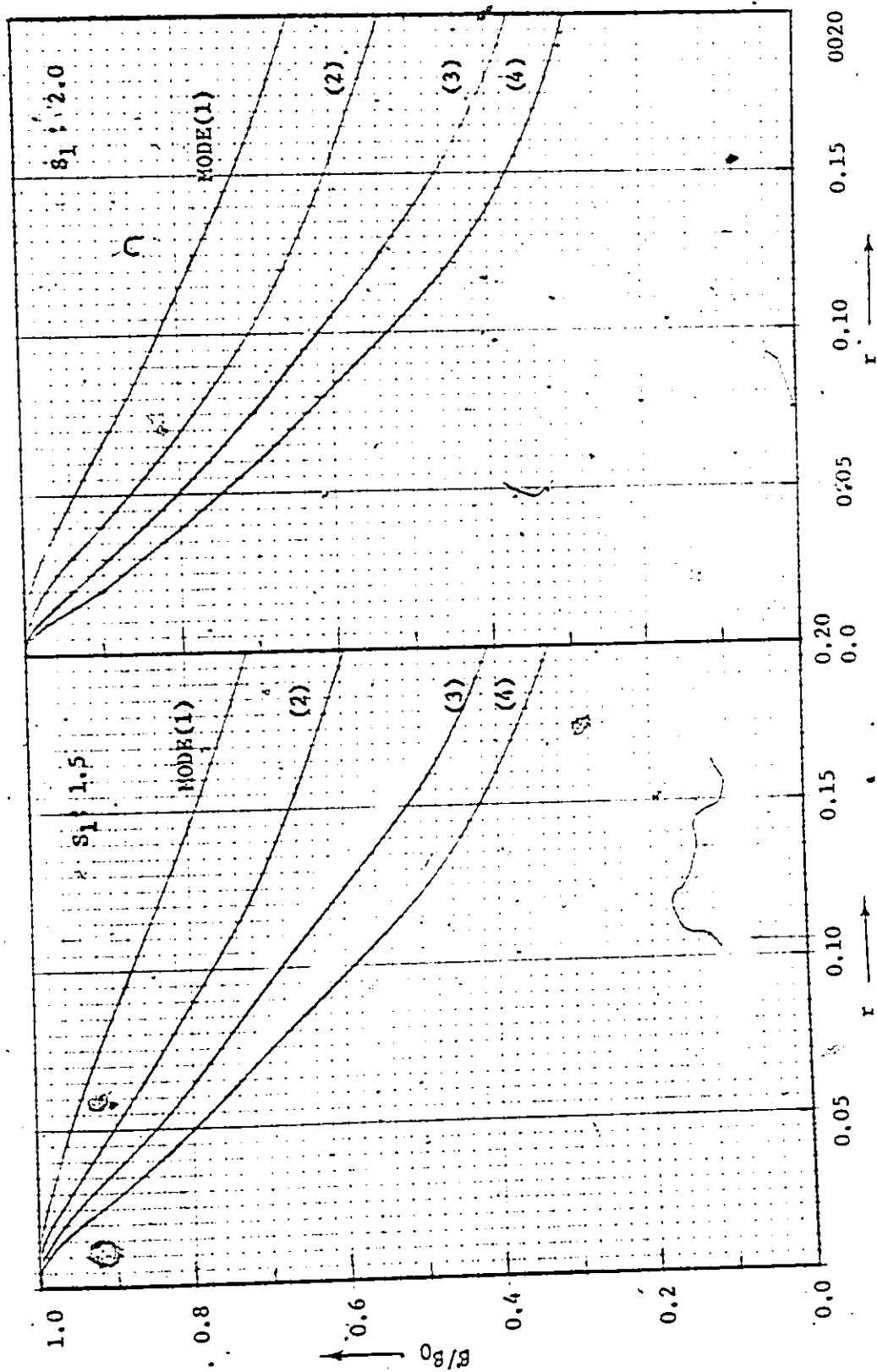


FIGURE 10.3.2(a) ; Effects of rotary inertia and shear deformation On free vibrations of uniform beams (clamped - simple supports)

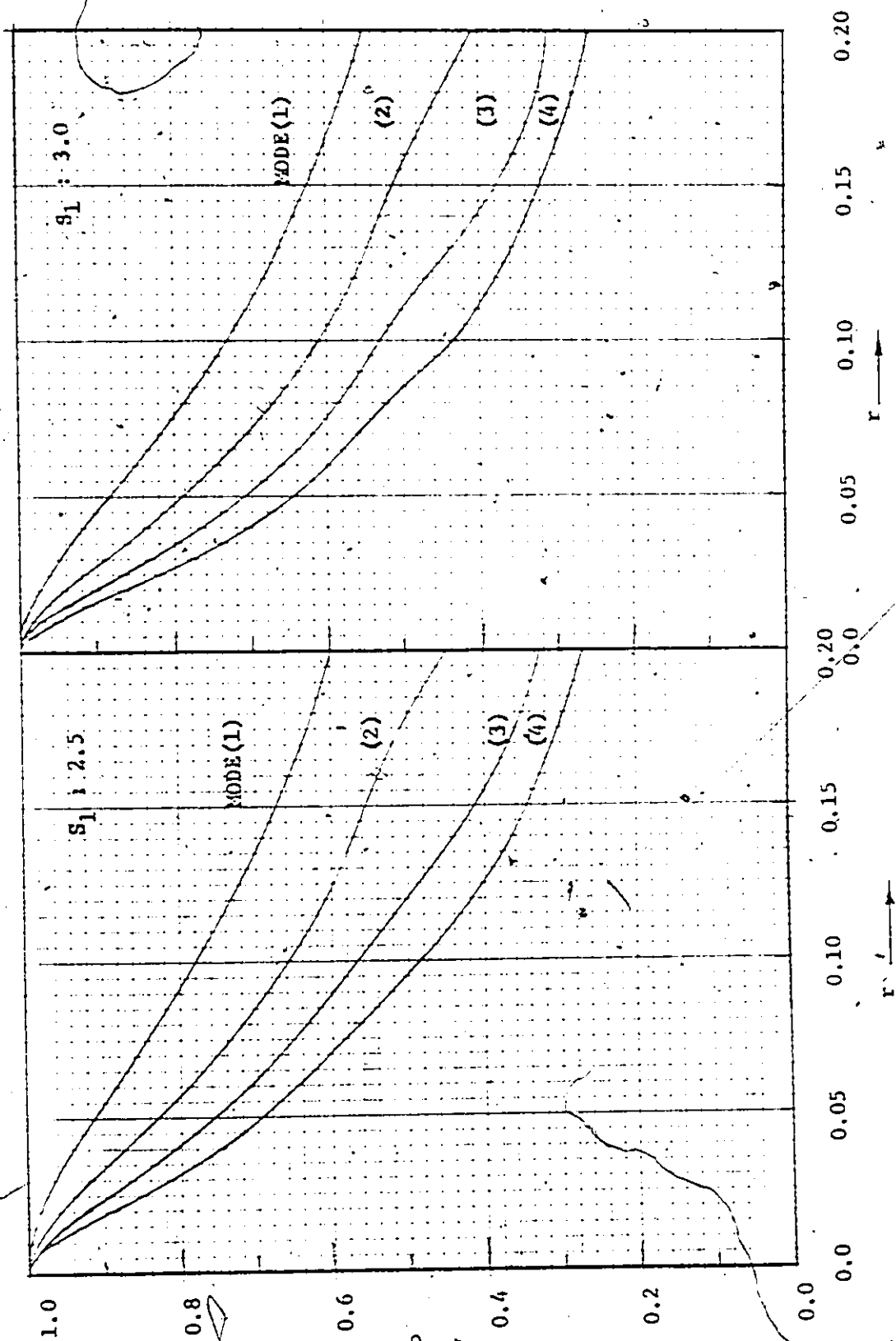


FIGURE 10.3.2(b) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-simple supports)

A

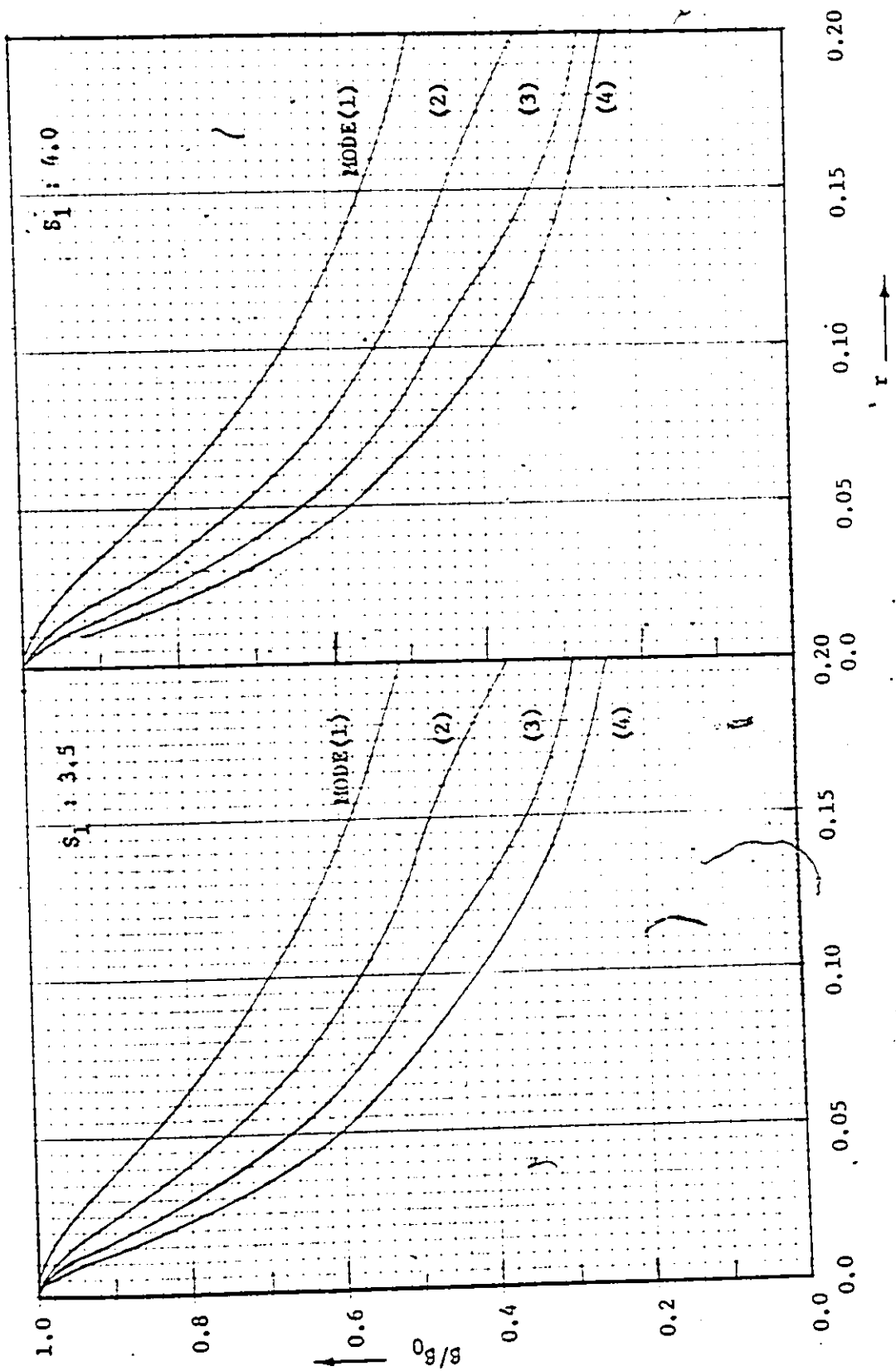


FIGURE 10.3.2(c) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-simple supports)

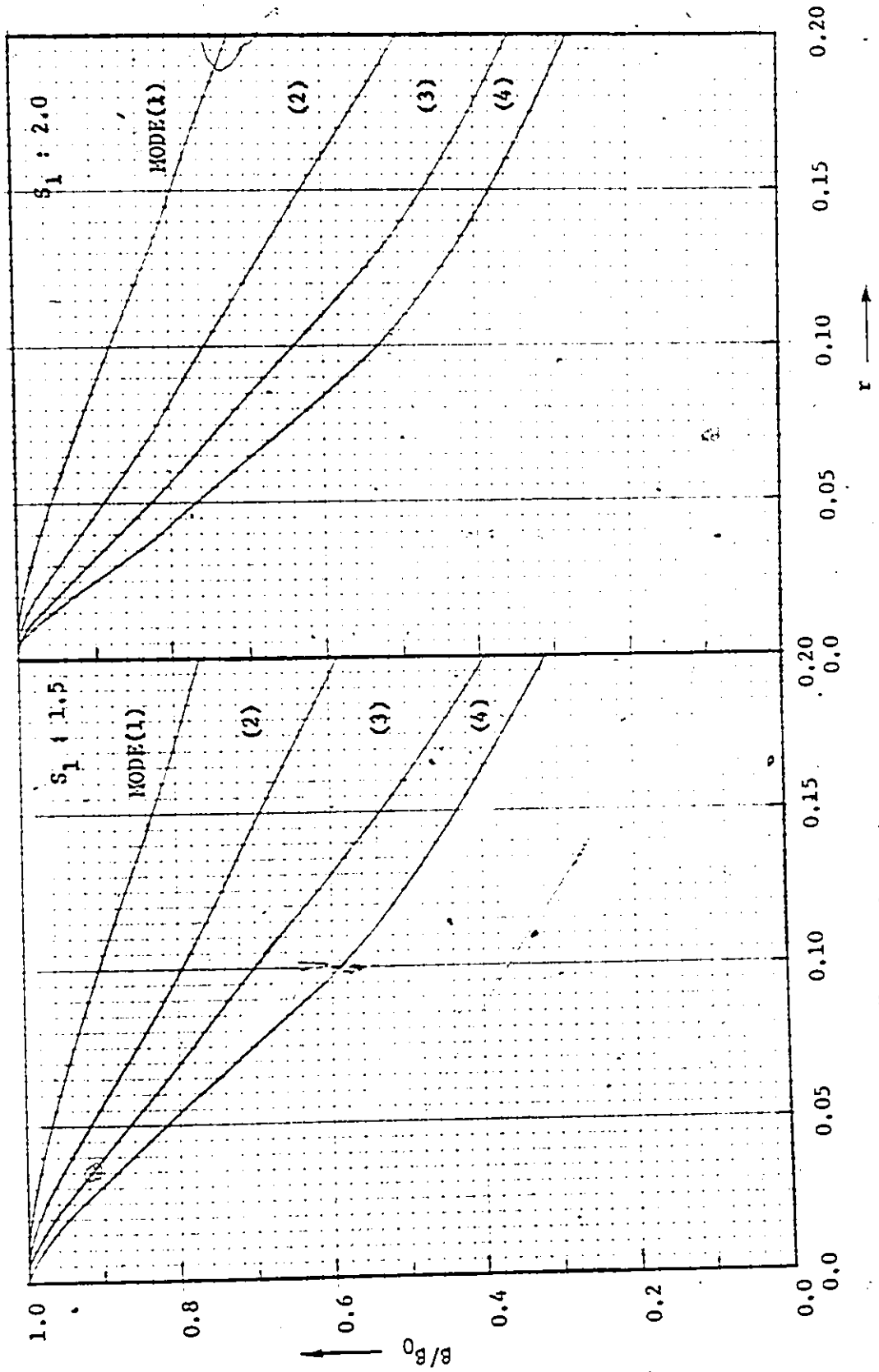


FIGURE 10.3.3(a) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (simple-free supports)

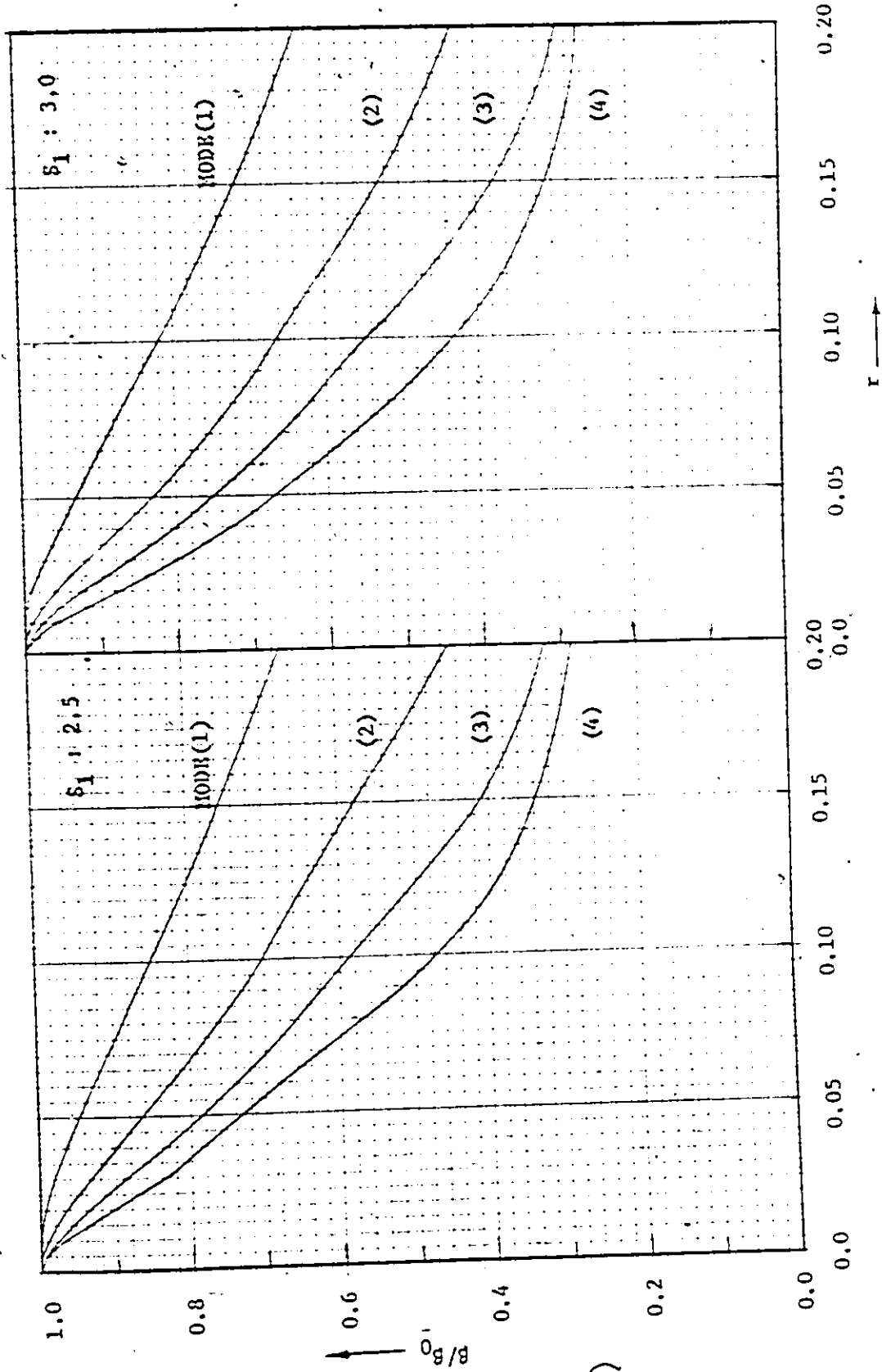


FIGURE 10.3.3(b) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (simple-free supports)

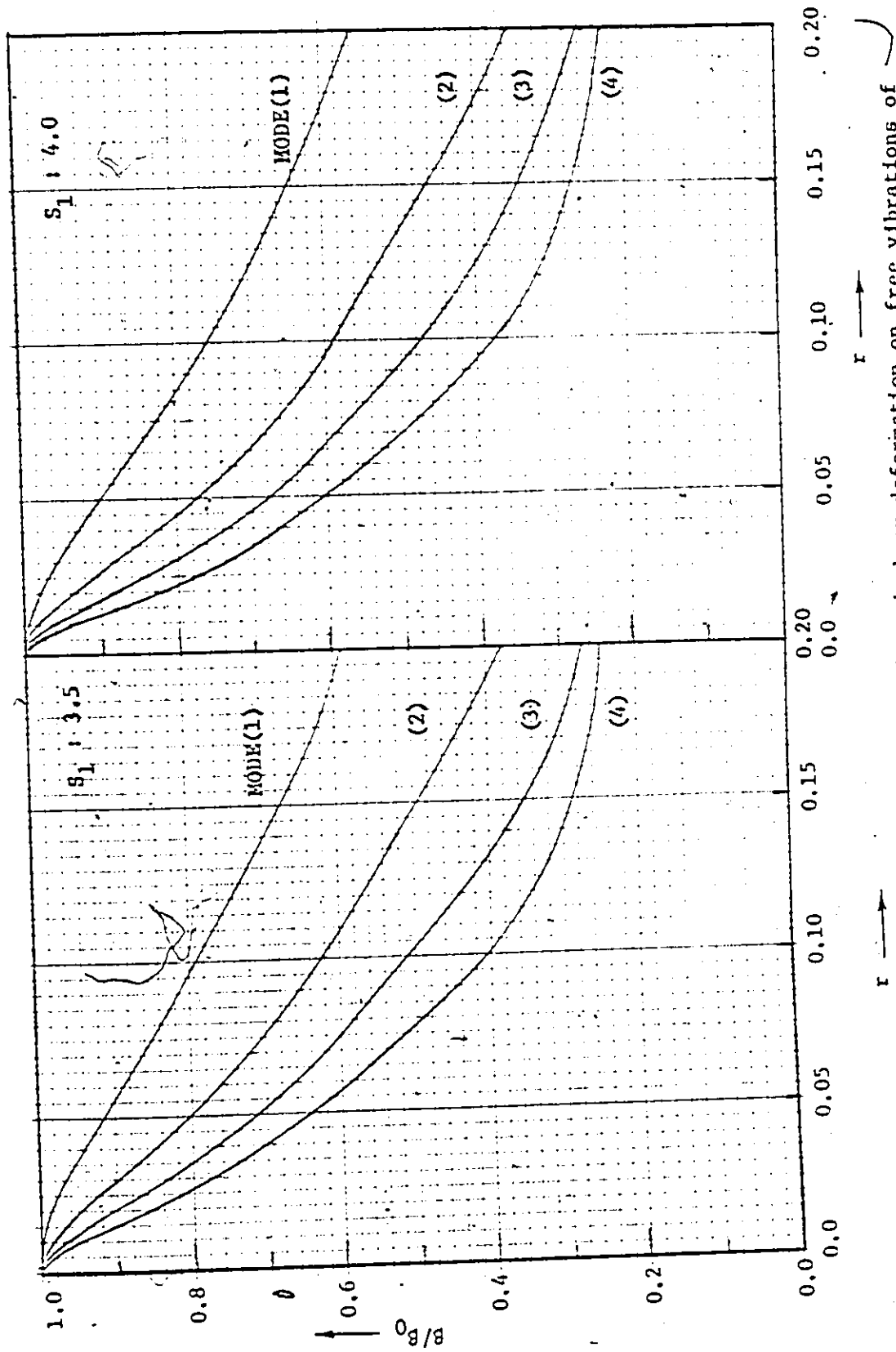


FIGURE 10.3.3(c) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (simple-free supports)

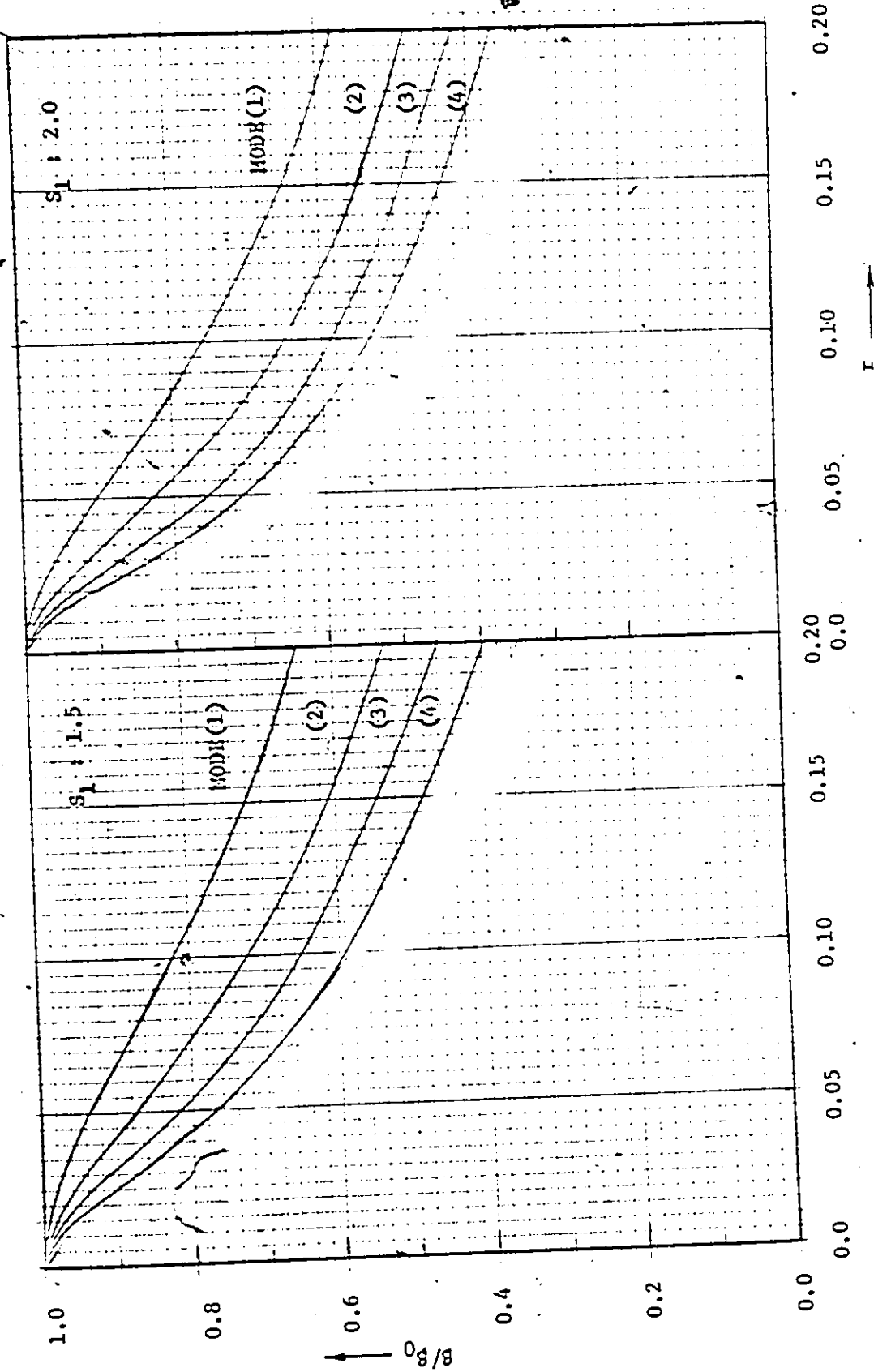


FIGURE 10.3.4(a) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-clamped supports)

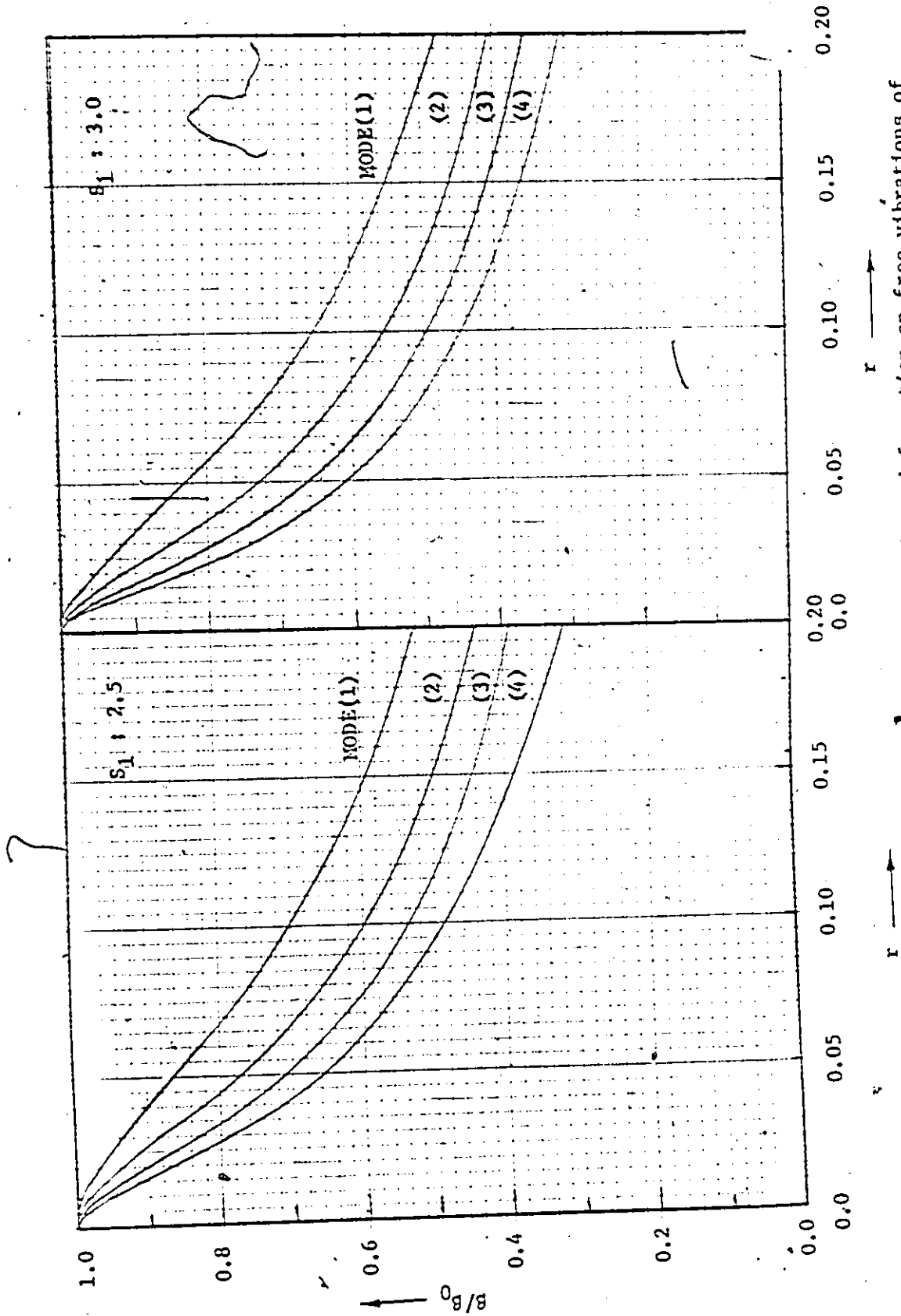


FIGURE 10.3.4(b) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-clamped supports)

1

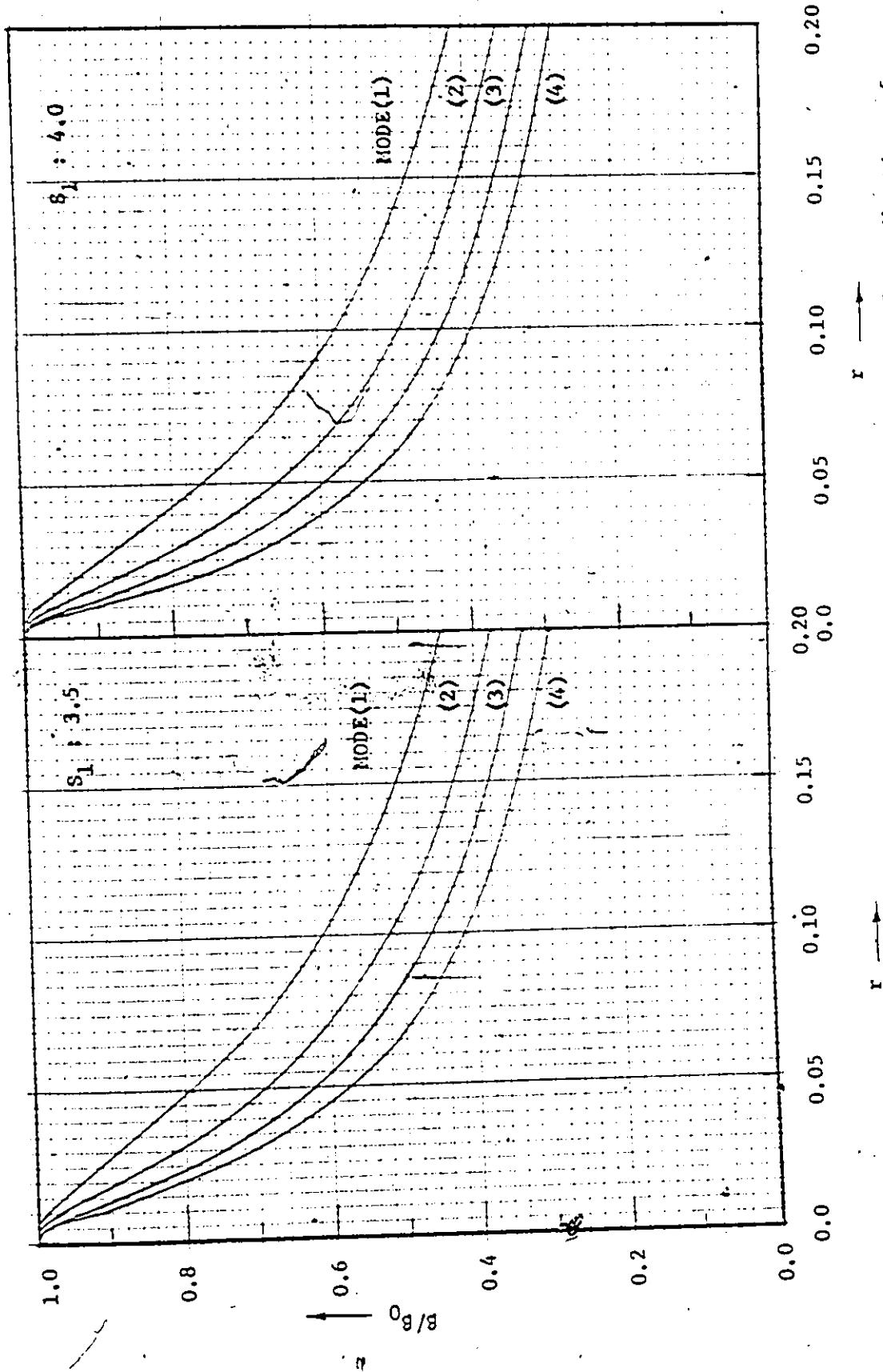


FIGURE 10.3.4(c) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-clamped supports)

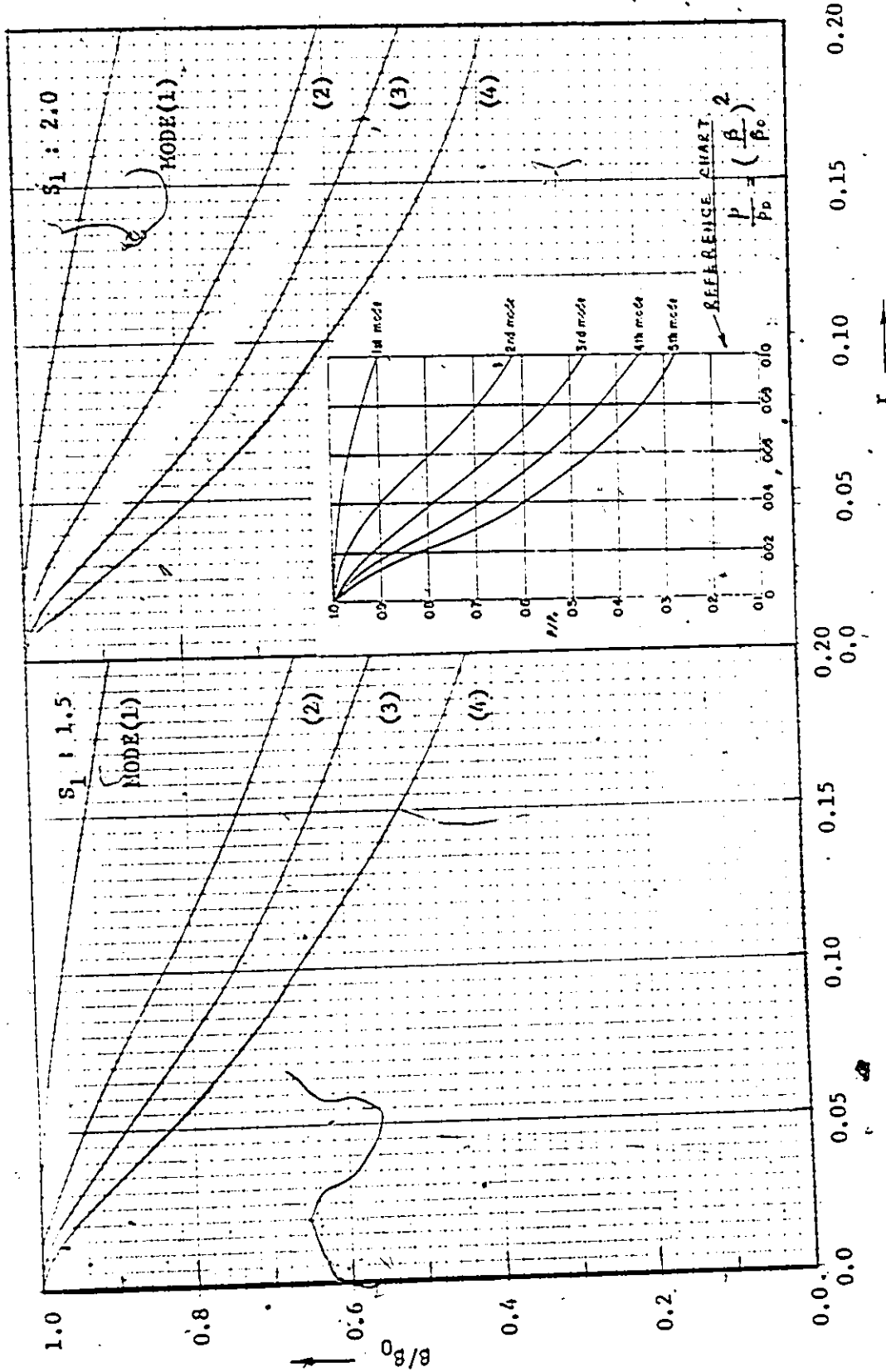


FIGURE 10.3.5(a) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-free supports)

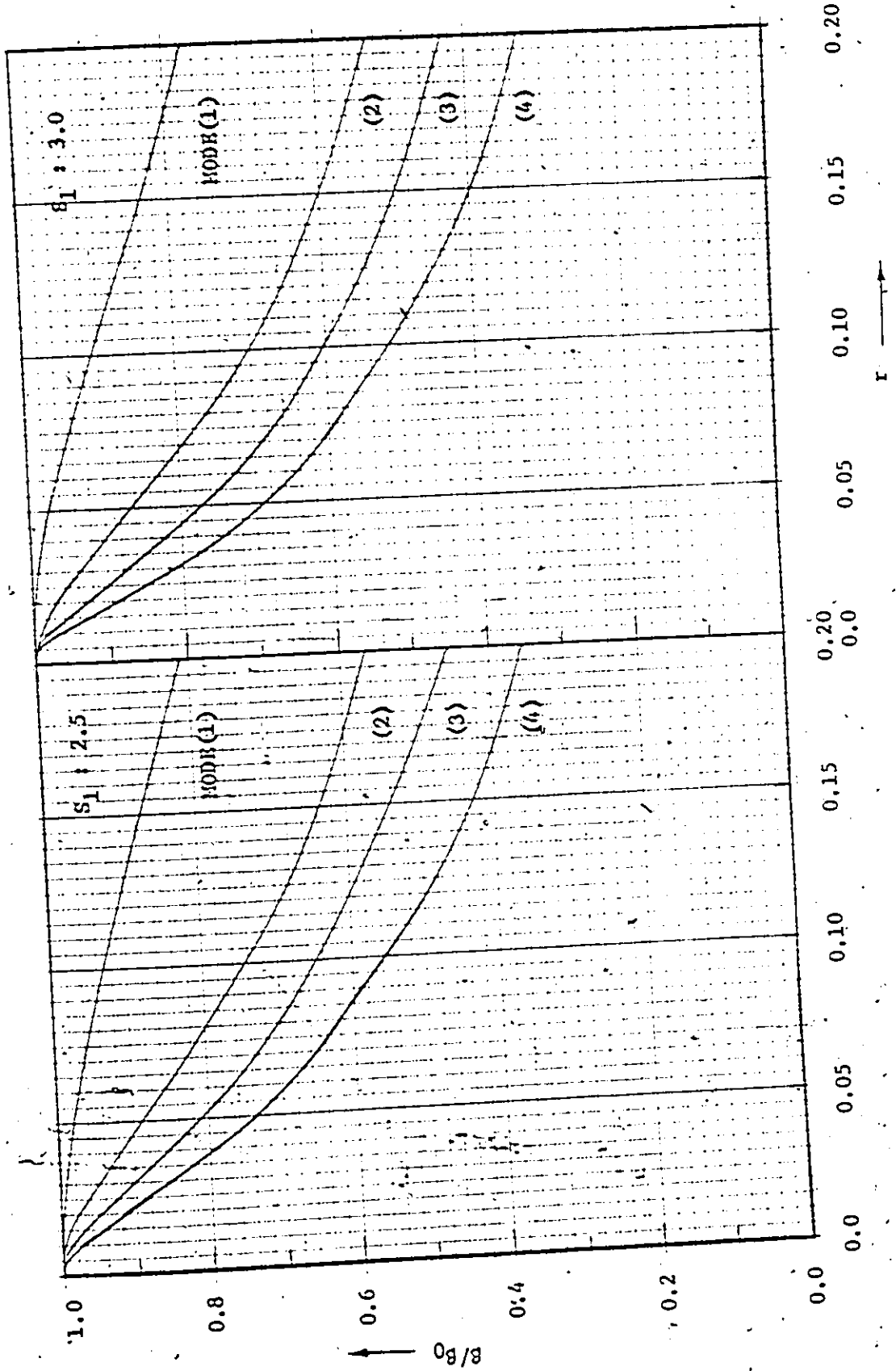


FIGURE 10.3.5(b) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-free supports).

A

d

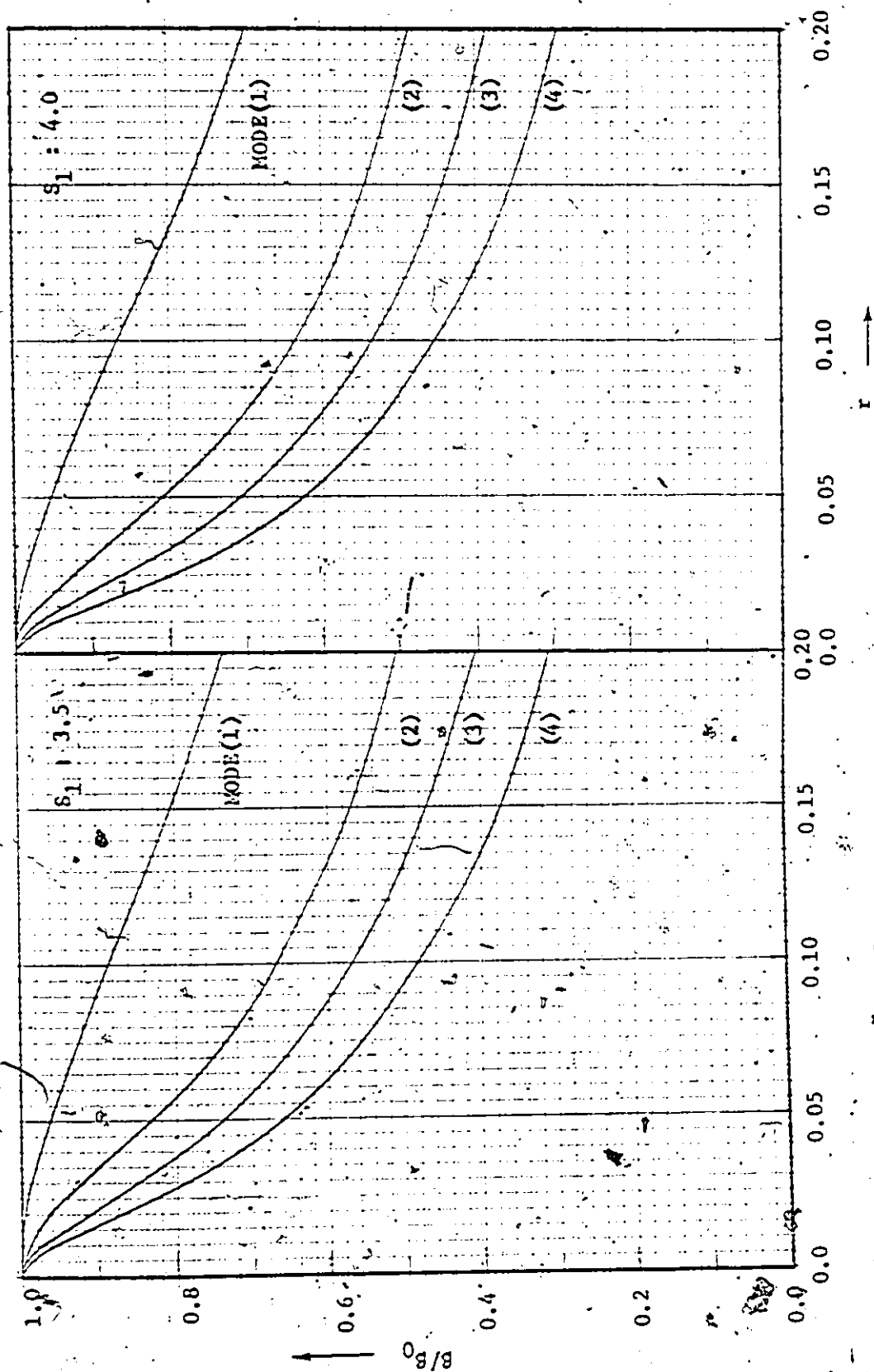


FIGURE 10.3.5(c): Effects of rotary inertia and shear deformation on free vibrations of uniform beams (clamped-free supports)

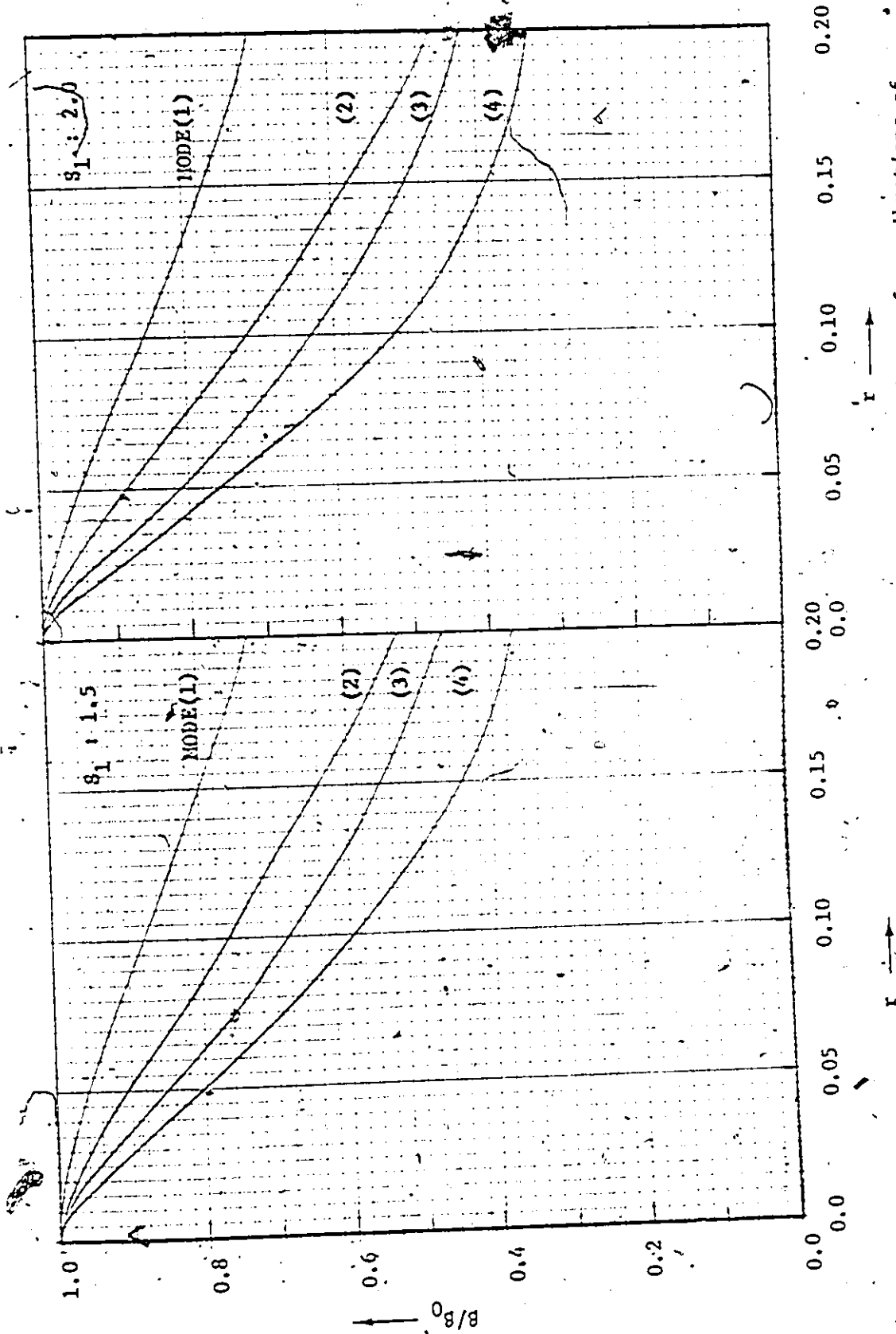


FIGURE 10.3.6(a) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (free-free supports)

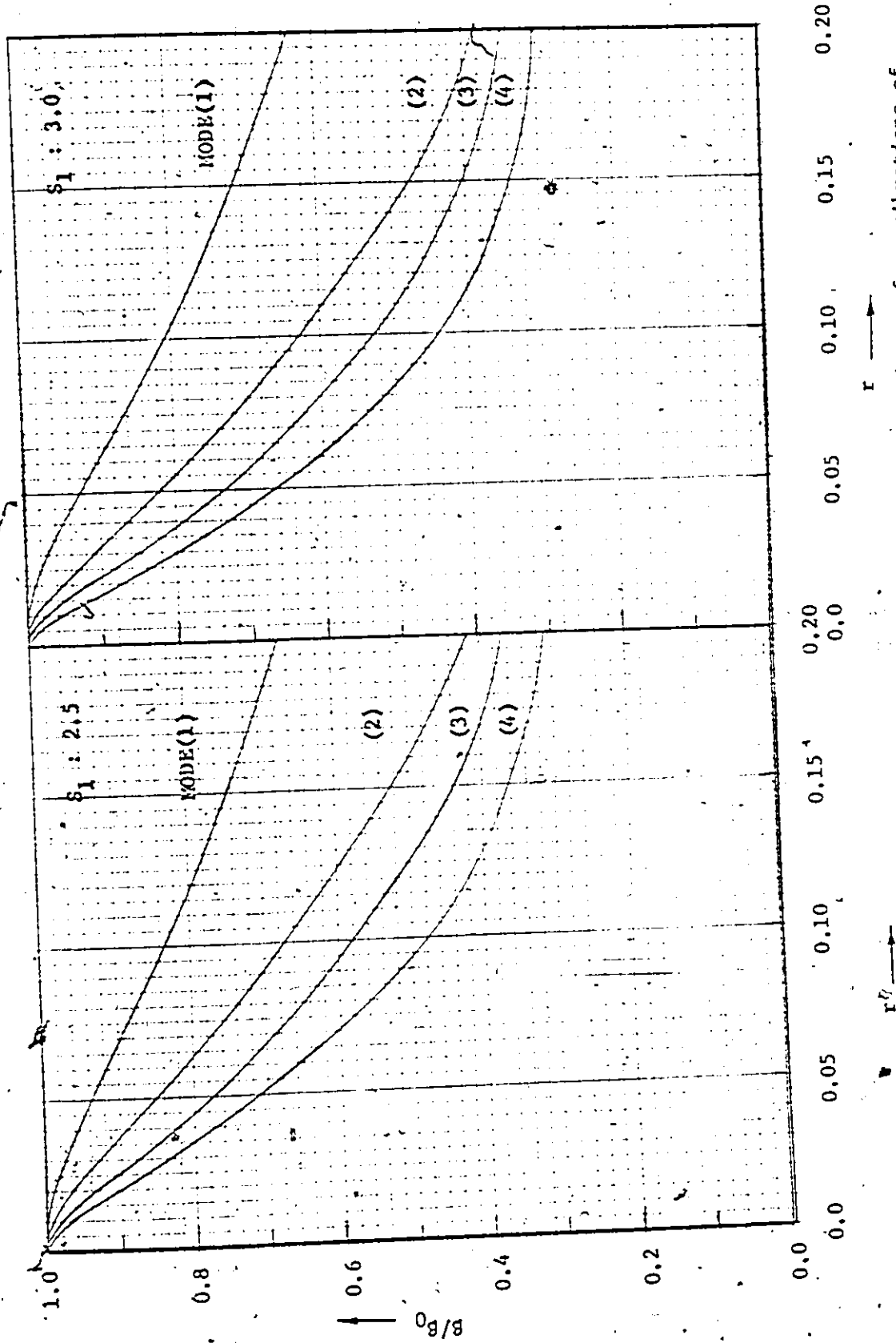


FIGURE 10.3.6(b) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (free-free supports)

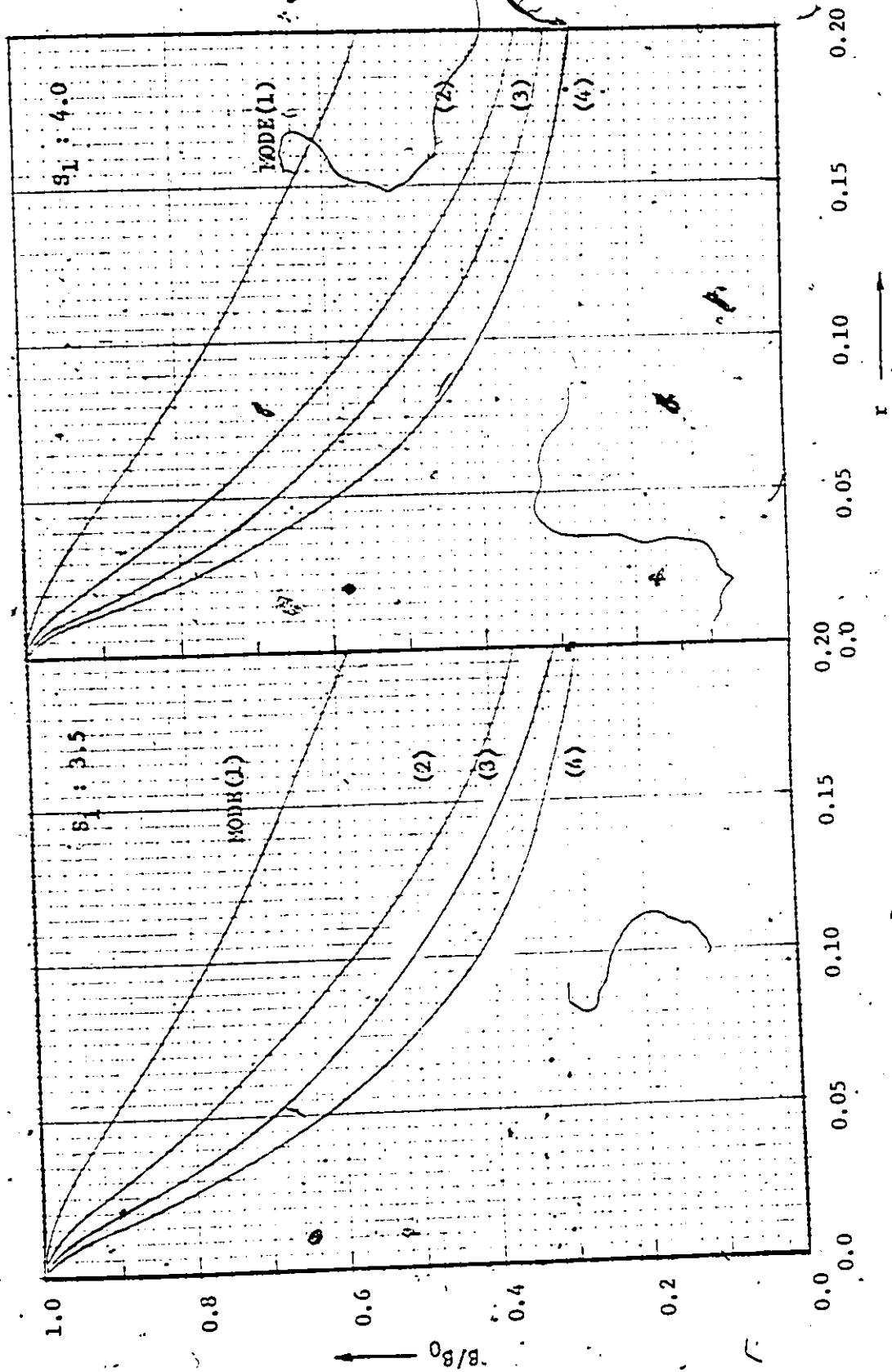


FIGURE 10.3.6(c) : Effects of rotary inertia and shear deformation on free vibrations of uniform beams (free-free supports)

APPENDIX A

Eigenvalues for single span uniform beams with classical end supports.

Mode	β_{CF}	β_{OB}	β_{OC}	β_{OP}	β_{OS}	β_{PF}
1	1.875	3.926	4.730	0	3.141	0
2	4.694	7.068	7.853	3.926	6.283	4.730
3	7.854	10.210	10.995	7.068	9.424	7.853
4	10.995	13.351	14.137	10.210	12.566	10.995
5	14.137	16.493	17.278	13.351	15.707	14.137
6	17.278	19.634	20.420	16.493	18.849	17.278
7	20.420	22.776	23.561	19.634	21.991	20.420
8	23.561	25.918	26.703	22.776	25.132	23.561
9	26.703	29.059	29.845	25.918	28.274	26.703
10	29.845	32.201	32.986	29.059	31.415	29.845

TABLE A-1

Eigenvalues β for the first ten modes of vibration of uniform beams with classical end conditions.

APPENDIX B

Eigenvalues for double span uniform beams with classical end supports.

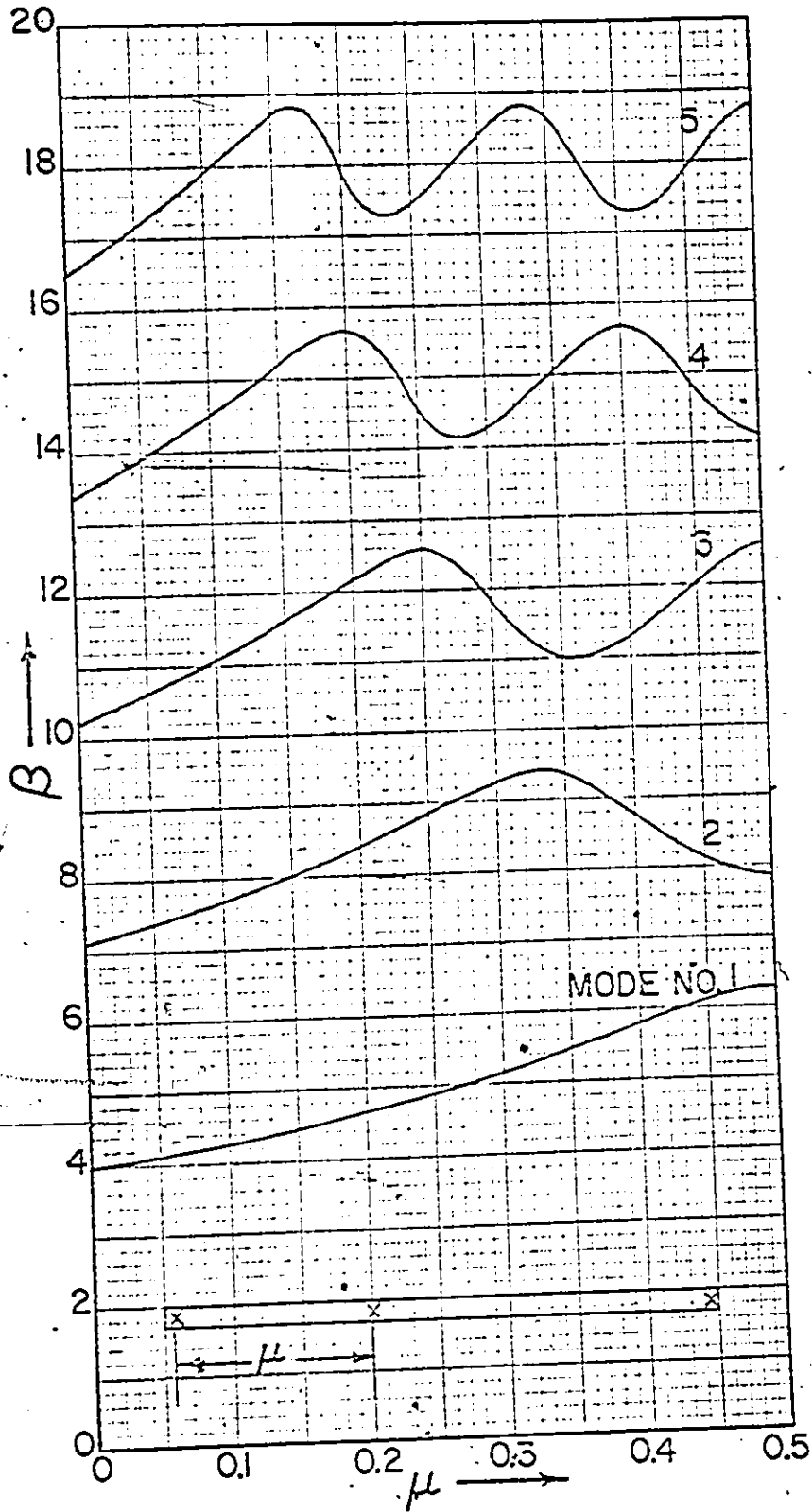


Fig. 1 Eigenvalues vs intermediate support spacing for beams with pinned-pinned-pinned support.

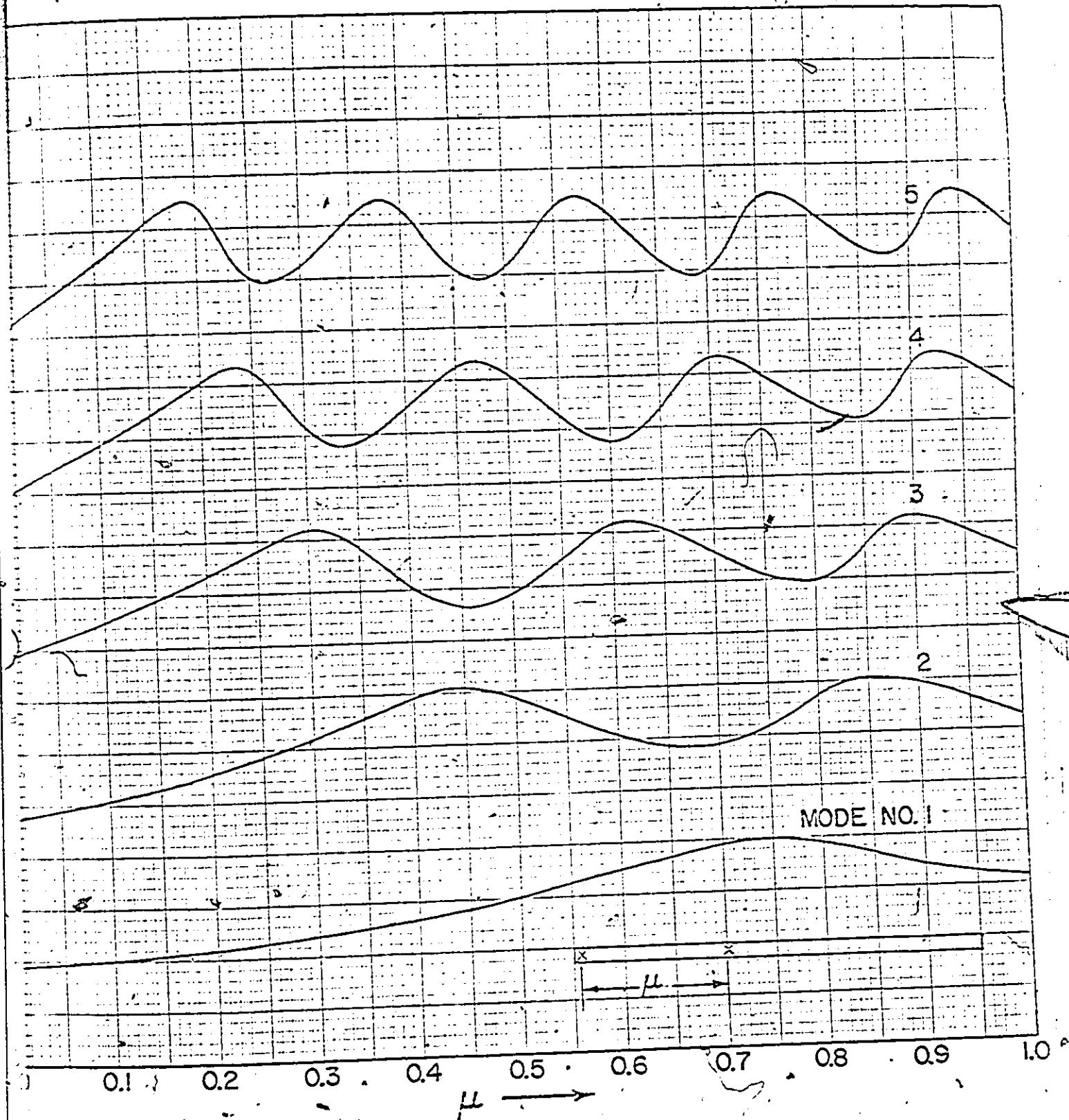


Fig. 2 Eigenvalues vs intermediate support spacing for beams with pinned-pinned-free support.

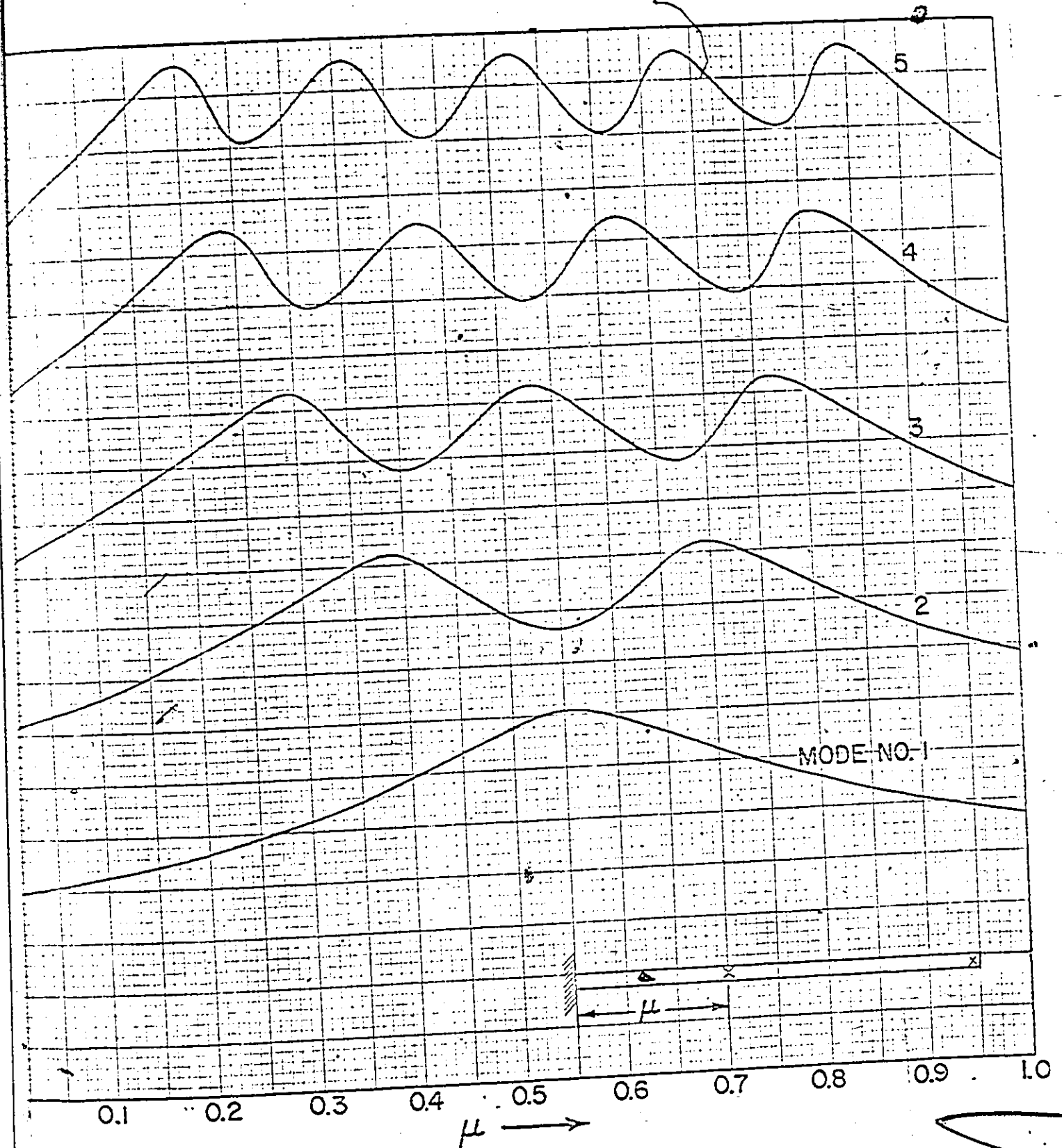


Fig. 3 Eigenvalues vs intermediate support spacing for beams with clamped-pinned-pinned support.

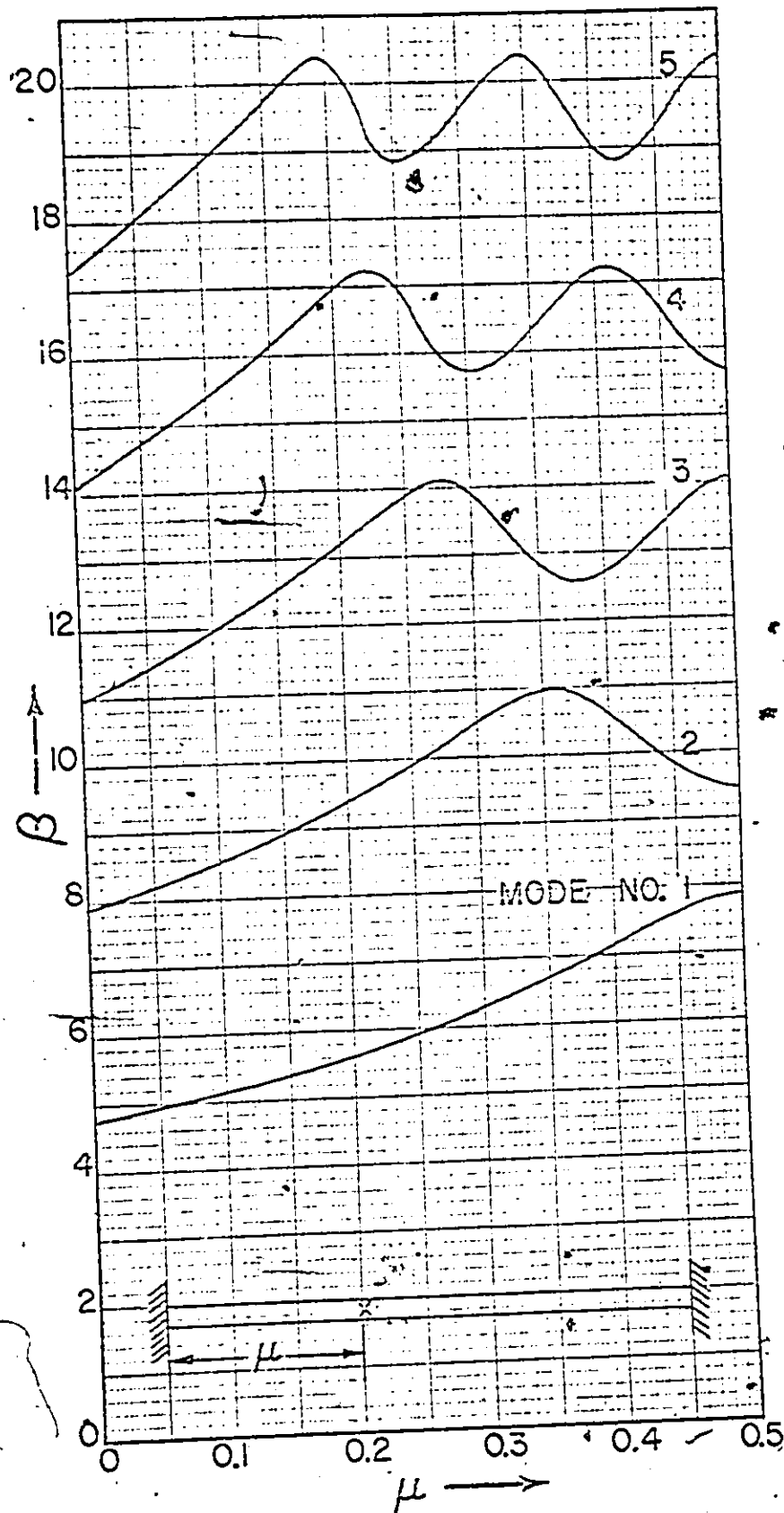


Fig. 4 Eigenvalues vs intermediate support spacing for beams with clamped-pinned-clamped support.

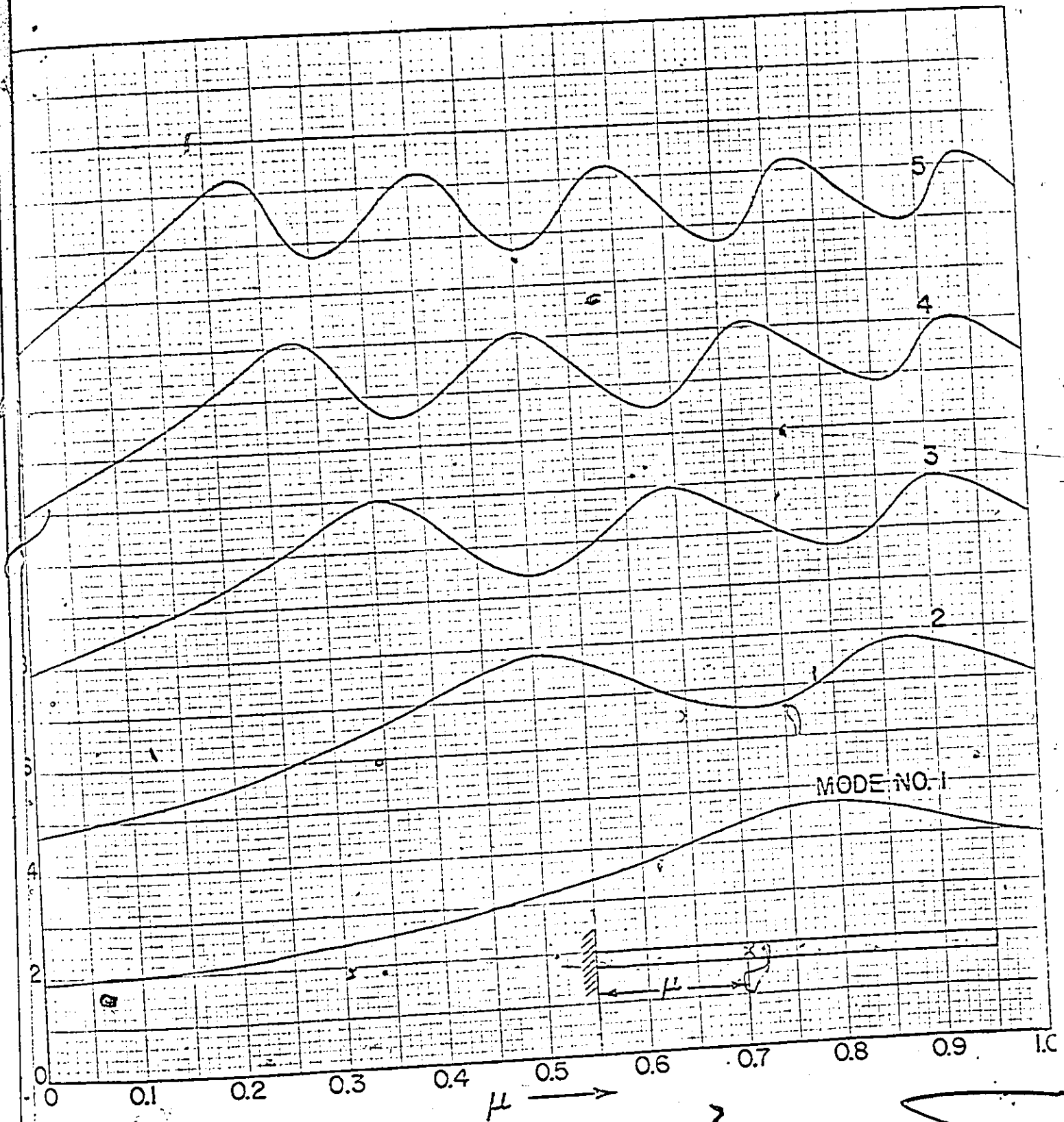


Fig. 5 Eigenvalues vs intermediate support spacing for beams with clamped-pinned-free support.

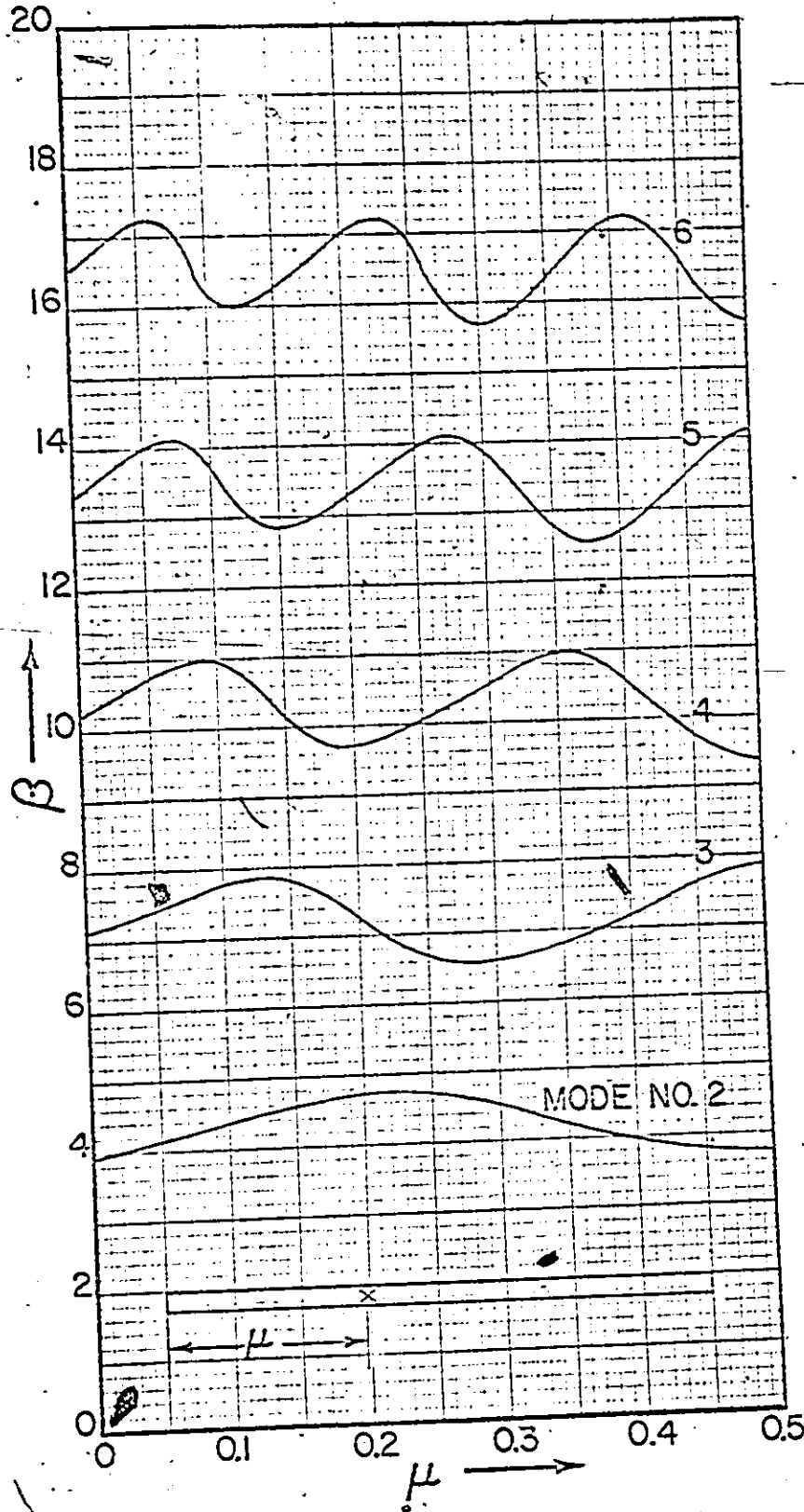


Fig. 6 Eigenvalues vs intermediate support spacing for beams with free-pinned-free support.

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