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ON P.P. AND RELATED RINGS

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ABSTRACT

A ring R in which every principal left ideal is projective is called a left P.P. ring. In this thesis, the behaviour of left P.P. rings and related rings is examined under various ring constructions. The ring constructions studied are polynomial rings, power series rings, skew-polynomial rings, direct limits of rings, rings of continuous functions, direct products of rings and rings of fractions. The concept of a weakly P.P. ring is introduced and it is studied similarly.

TABLE OF CONTENTS

INTRODUCTION

A ring R in which every principal left ideal is projective is called a left P.P. ring. Equivalently a ring R is left P.P. ring if, and only if, the left annihilator of any element in R is generated by an idempotent element. This type of ring has been explicitely studied by Kaplansky, Small, Bergman, Jøndrup, Armendariz and others. Jøndrup [9] proved that if a ring R is commutative, then R is a P.P. ring if, and only if its polynomial ring R[x] is P.P. Armendariz [2] generalized this property to non-commutative rings with no non-zero nilpotent elements, which he called reduced rings. It was this paper which prompted the work of this thesis. The purpose of this thesis is to study the stabilization of P.P. property in the case of power series rings, skew-polynomial rings and other types of ring constructions. These types of stabilizations are also studied for the stronger property of Baer rings and the weaker property of i-dense rings.

The first chapter presents most of the preliminary notions of ring theory that are used in the sequel, that is, annihilators, regular rings and reduced rings. The concepts of P.P. and Baer rings are also presented. Most of the details can be found in [10].

The second chapter begins with the introduction of power series rings and skew-polynomial rings. Proposition 2.4 shows that if R is a reduced ring, then R is Baer if, and only if R[[x]] is Baer. However a counterexample is given to show that P.P. property is not preserved by power series rings. A new property of rings called (*) property is introduced which is stronger than the P.P. property, but weaker than the Baer property. It is shown in Proposition 2.6 that a reduced ring R satisfies (*) property if, and only if R[[x]] is a P.P. ring. The concept of i-dense rings is studied relative to P.P. rings and its stabilization for polynomial rings is studied in Proposition 2.9. R*[x] by a ring whose elements are polynomials $f = \sum_{i=1}^{n} x^i a_i$, a_i from a ring R. R*[x] is called a skew-polynomial ring in x over R with respect to homomorphism $\alpha:R \to R$ if we define addition as usual and multiplication by the commutation rule $ax = xa^{\alpha}$, $a \in R$. Proposition 2.17 states that if R is a reduced ring and α a monomorphism which preserves idempotent elements, then R is a P.P. ring if, and only if R*[x] is a P.P. ring. These skew-polynomial rings give a nice generalization of the results of Armendariz [2] for polynomial rings.

At the start of the third chapter, semiprime and weakly P.P. rings are introduced. It is shown that weakly P.P. ring property is stable in the formation of polynomial rings. After the introduction of a concept of direct limit, it is seen, in Example 3.9, that the direct limit of P.P. rings is not necessarily P.P. However Proposition 3.10 gives a family of P.P. rings whose direct limit is a P.P. ring. In a similar fashion, it is shown in Proposition 3.11 that a direct product of P.P. rings is P.P. The Baer and i-dense properties are also studied for direct products of rings. If X is a completely regular topological space, the ring of all continuous functions from X to $\mathbb{R}$ is denoted by C(X). We have characterized those spaces X for which C(X) is P.P., by showing that C(X) is

P.P. if, and only if X is basically disconnected. Let $C(X,R)$ be the set of all continuous functions from a topological space X into a ring R which is given a discrete topology. We have proved in Proposition 3.19 that if R is left P.P., then $C(X,R)$ is left P.P.

In the fourth chapter, it is shown that if a ring R is P.P., then its ring of fractions, $S^{-1}R$, is P.P., S being any multiplicatively closed set. Similar results are also proved for Baer and i-dense rings, by putting certain conditions on S. At the end of this section, we have tried to expose the concept of P.P. rings in relation with other concepts such as hereditary, semi-hereditary and completely reducible rings.

The following propositions, while in most cases modelled on known results are believed to be new: Propositions 2.4, 2.6, 2.8, 2.17, 3.6, 3.8, 3.10, 3.11, 3.18, 3.19 and 4.2. Also Examples 2.5, 2.7 and 3.9 are new. These results were worked out from outlines suggested by W. Burgess.

CHAPTER I

PRELIMINARY NOTATIONS

§1. Description of annhilators

This section presents the basic definitions and properties of annihilators. Most of these are standard, however, they are presented here for easy reference (for further details see [10]). Throughout this thesis, we will be considering rings with identity 1.

Let R be a ring and S be a subset of R.

Definition. The set of all elements r in R satisfying $rS = o$, that is $rs = o$ for all $s \in S$, is called the left annihilator of S. It is denoted by $lann_R(S)$. The suffix R here specifies the particular ring R. In a similar fashion, the right annihilator of a set S is defined and is denoted by $rann_R(S)$.

The following proposition gives some of the basic properties of annihilators and the proofs are very easy.

Proposition 1.1. Let R be a ring, and S, T, S_i be subsets of R, $i \in A$. Then

(a) $S \subset lann_R(rann_R(S))$,

(b) $S \subset T$ implies $lann_R(S) \supset lann_R(T)$,

(c) S and $lann_R(rann_R(S))$ have the same right annihilators,

(d) $lann_R(\cup_A S_i) = \cap_A lann_R(S_i)$

At this juncture, the concepts of idempotent elements and nilpotent elements are also worth recalling.

Definition. An element e of a ring R is called an idempotent element if $e^2 = e$. An element r in a ring R is called a nilpotent element if $r^n = o$ for some positive integer n.

Proposition 1.2: Let e be an idempotent element in a ring R. Then $rann_R(Re) = (1-e)R$.

Proof. $Re(1-e)R = o$ shows that $(1-e)R \subset rann_R(Re)$. If $Rex = o$ for x in R, then $ex = o$ and $x = (1-e)x \in (1-e)R$.

Q.E.D.

The following proposition gives the incentive to define a very useful class of rings, Baer rings.

Proposition 1.3. In a ring R, the following statements are equivalent:

(a) Every right annihilator is of the form eR, e being an idempotent element in R.

(b) Every left annihilator is of the form Re, e being an idempotent element in R.

Proof. (b) implies (a). Given any subset S of R, we should show that $rann_R(S) = eR$ with e an idempotent element in a ring R. By (c) in Proposition 1.1, we can assume that S is a left annihilator, say $S = Rf$ with f an idempotent element in a ring R. Then by Proposition 1.2, $rann_R(S) = eR$ with $e = 1-f$.

(a) implies (b). It is easy to prove along similar lines as in the proof of (a) implies (b).

Based on the above proposition, a Baer ring is defined as follows.

Definition. A Baer ring is a ring satisfying the two equivalent conditions of Proposition 1.3. Hence a ring R is called a Baer ring if the left annihilator of any subset of R is generated by an idempotent element.

Examples of Baer rings.

(1) Integral domains and products of integral domains.

(2) The ring of all linear transformations of a vector space over a division ring.

(3) The ring of all bounded operators on a Hilbert space (see [10], p. 11).

We will mention some remarks about Baer rings.

Remarks (see [10]).

(1) If R is a Baer ring and e an idempotent element in R, then eRe is a Baer ring.

(2) The centre of a Baer ring is a Baer ring.

Since we will be dealing with the central idempotents and non-zero-divisors in this thesis, we give the respective definitions.

Definition. An idempotent element e in a ring R is called a central idempotent if it commutes with all the elements in R, that is, if it lies in the centre of the ring R. An element a in R is termed a non-zero-divisor if $ar \neq o$ and $ra \neq o$ for all $o \neq r \in R$; and is called a zero-divisor if $sa = o$ or $as = o$ for some $o \neq s \in R$.

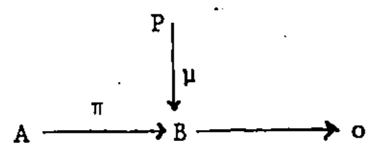
Closely related to Baer rings are P.P. rings. In this thesis, most of the work is related to this class of rings. To understand the definition of P.P. rings, it is important to quote two more definitions.

<u>Definition</u>. Suppose $\{M_i\}$ is a non-empty collection of R-modules with a corresponding collection of mappings $f_i : M_i \longrightarrow M_{i+1}$ such that kernel of f_i is equal to image of f_{i-1}. Then the sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

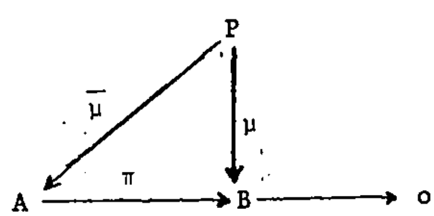
is called <u>an exact sequence</u> of R-modules.

An R-module P is called <u>projective R-module</u> if the diagram



exact

can be embedded in the diagram



in such a way that the latter diagram is commutative, that is, $\pi\bar{\mu} = \mu$.

If a is an element in a ring R, the right ideal aR will be projective if, and only if the kernel of the left multiplication map $R \to aR$ is a direct summand of R. An equivalent statement for this is given by the next proposition.

<u>Proposition 1.4</u>. A principal right ideal aR in a ring R is projective if, and only if rann $\{a\}$ is generated by an idempotent element of R.

<u>Proof</u>. Let a be an element in a ring R. Assume that aR is projective. Suppose μ_a is the left multiplication map from R into aR, K the kernel of this map. Consider the diagram

$$o \longrightarrow K \longrightarrow R \xrightarrow{\mu_a} aR \longrightarrow o \quad .$$

Since aR is projective, there exists a map $\nu: aR \longrightarrow R$ such that $\mu_a \nu$ = identity map on aR. Then there exists $B \subset R$ such that $K \oplus B = R$. Indeed, let $B = \nu(aR)$. If $r \in R$, $r = (r-\nu(ar)) + \nu(ar)$, where $r-\nu(ar) \in K$. Let $r \in K \cap B$. Then $r = \nu(as)$ for some $s \in R$ and $\mu_a(r) = ar = o$. Hence $as = o$ and $r = o$.

Conversely, if $K = eR$ for some idempotent element e in a ring R, then $R = eR \oplus (1-e)R$. Hence $aR \cong (1-e)R$.

Q.E.D.

The above proposition is equivalent to saying that aR is projective if, and only if the right annihilator of a equals the right annihilator of some idempotent element e in a ring R.

<u>Definition</u>. A ring in which every principal right ideal is projective is called a <u>right P.P. ring</u>.

Thus a ring R is called a right P.P. ring if, and only if the right annihilator of any element $a \in R$ is generated by some idempotent element e of R, that is $\text{rann}_R(a) = eR$,

$e^2 = e \in R$.

Analogously we can define the concept of a left P.P. ring. A ring which is left and right P.P. is called a P.P. ring. It is not always true that a right P.P. ring is a left P.P. ring. (However, if we assume that all idempotent elements in a ring R are central, then R is a left P.P. ring if, and only if R is a right P.P. ring. The following example gives a ring which is left P.P., but not right P.P.

Example 1.5. (see [6]). Let F be a field. Consider the rings $S' = \Pi_A F$, $I = \oplus F$ and $S = S'/I$. Let R be the ring of all 2×2-upper triangular matrices, $\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}$, with $\bar{a} \in S$, $\bar{b} \in S$ and $c \in S'$. We will prove that this ring R is left P.P. but not right P.P.

Consider $\text{rann}_R\left(\begin{bmatrix} o & \bar{1} \\ o & o \end{bmatrix}\right)$. It is not generated by an idempotent element in R. for, if it were generated by some idempotent element, say $\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}$; then $\begin{bmatrix} o & \bar{1} \\ o & o \end{bmatrix}\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix} = \begin{bmatrix} o & o \\ o & o \end{bmatrix}$, gives $\bar{c} = o$ and hence $c \in I$. Also from $\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}$, we get three equations $\bar{a}\bar{a} = \bar{a}$, $\bar{a}\bar{b} + \bar{b}\bar{c} = \bar{b}$ and $cc = c$. Since $c \in S'$, its components are either 1 or 0. If $\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}$ were the required idempotent element, then c would have generated I; but I is not generated by an idempotent. Hence R is not right P.P.

It is obvious from [6] that R is a left P.P. ring; since it is left semi-hereditary (see p. 66). However, we will give a direct proof. Let $\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}$ be an element in R. We want to find an idempotent element $\begin{bmatrix} \bar{e} & \bar{y} \\ o & c \end{bmatrix}$ which generates $\text{lann}_R\left(\begin{bmatrix} \bar{a} & \bar{b} \\ o & c \end{bmatrix}\right)$. Hence in particular we need equations

$$\overline{e}\overline{a} = o,\ fc = o,\ \overline{e}\overline{b} + \overline{y}\overline{c} = o \quad \text{and} \quad \overline{e}^2 = \overline{e},\ f^2 = f,\ \overline{e}\overline{y} + \overline{y}\overline{f} = \overline{y} \qquad \text{——— (1)}$$

Let f generate $\operatorname{ann}_{S'}(c)$. Let us try to find suitable $\overline{e}$ and $\overline{y}$. For this, pick representatives a, b in S' for $\overline{a}, \overline{b}$. Then we need to solve for each $\alpha \in A$, the equations

$$e_\alpha a_\alpha = o,\ e_\alpha b_\alpha + y_\alpha c_\alpha = o,\ e_\alpha^2 = e_\alpha,\ e_\alpha y_\alpha + y_\alpha f_\alpha = y_\alpha \qquad \text{——— (2)}$$

almost everywhere. Here the suffix α denotes the α^{th} component. Define $c'_\alpha \in S'$ by $c'_\alpha = \begin{cases} 1, & c_\alpha = o \\ o, & c_\alpha \neq o \end{cases}$. Let $e = (e_\alpha) \in S'$ be the idempotent element which generates $\operatorname{lann}_{S'}(\{a, bc'\})$. Then

$$e_\alpha = \begin{cases} o, & a_\alpha \neq o \text{ or } b_\alpha \text{ and } c'_\alpha \neq 0 \\ 1, & a_\alpha = o \text{ and } b_\alpha \text{ or } c'_\alpha = o \end{cases}.$$

When $a_\alpha \neq o$ and $e_\alpha = o$, the equations in (2) reduce to $y_\alpha c_\alpha = o$ and $y_\alpha f_\alpha = y_\alpha$. However, these equations can always be solved given the definition of f.

Next if $a_\alpha = o$ and $e_\alpha = 1$, we get $b_\alpha + y_\alpha c_\alpha = o$ and $y_\alpha + y_\alpha f_\alpha = y_\alpha$. Given $\overline{b}$ and $\overline{c}$, these equations can be solved if $f_\alpha = o$. Finally when $a_\alpha = o$, $f_\alpha = 1$ and $e_\alpha = 1$, we have no problem if $b_\alpha = o$. However if $b_\alpha \neq o$, we can see that the equations $b_\alpha + y_\alpha c_\alpha = o$ and $y_\alpha + y_\alpha f_\alpha = y_\alpha$ can be solved given f and b, by definition of c'. Thus we have seen that in all cases, all the equations in (1) have solutions. Hence the idempotent $\begin{bmatrix} \overline{e} & \overline{y} \\ o & f \end{bmatrix}$ satisfies our needs, and so R is a left P.P. ring.

Q.E.D.

Example.1.6. Jøndrup [9] quotes the next example which is due to P.M. Cohn. We will explain that example in detail here. Let $Z_2[x] = (Z[x])_2$ be the ring of 2×2-matrices with entries from $Z[x]$, the polynomial ring over the set of integers, Z. We will show that $Z_2[x]$ is not a left P.P. ring.

Consider the element $\begin{bmatrix} 2 & o \\ x & o \end{bmatrix}$ of $Z_2[x]$. We are going to show that $\text{lann}_{Z_2[x]} \begin{bmatrix} 2 & o \\ x & o \end{bmatrix}$ is not generated by idempotent element. Suppose $\text{lann}_{Z_2[x]} \begin{bmatrix} 2 & o \\ x & o \end{bmatrix} = Z_2[x]\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an idempotent element in $Z_2[x]$. Then from $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & o \\ x & o \end{bmatrix} = 0$, we get

$$2a + bx = o \qquad \text{—— (1)}$$

$$2c + dx = 0 \qquad \text{—— (2)}$$

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have four equations:

$$a^2 + bc = a \qquad \text{—— (3)}$$

$$ab + bd = b \qquad \text{—— (4)}$$

$$ac + cd = c \qquad \text{—— (5)}$$

$$d^2 + bc = d \qquad \text{—— (6)}$$

From equations (1) and (2), we see that the coefficients of b and d are even. Again equation (4) shows that either $a+d = 1$ or $b = o$. If $a+d = 1$, then the constant term in a is odd, since the coefficient of d are even. However from equation (1), the constant term of a is zero, and hence we get a contradiction.

If we assume that $b \neq o$; then equation (1) shows that

$a = o$. Then from equation (5), it is clear that either $c = o$ or $d = 1$. If $d = 1$, we get a contradiction to the fact that the coefficients of d are even. Hence the idempotent element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} o & o \\ o & o \end{bmatrix}$. However, this contradicts the fact that $\text{lann}_{Z_2[x]}\left(\begin{bmatrix} 2 & o \\ x & o \end{bmatrix}\right)$ contains non-zero elements, for example $\begin{bmatrix} x & -2 \\ o & o \end{bmatrix}$. Thus $\text{lann}_{Z_2[x]}\left(\begin{bmatrix} 2 & o \\ x & o \end{bmatrix}\right)$ is not generated by an idempotent element.

Q.E.D.

Remark: (1) Since $Z_2[x] = (Z[x])_2$, this also gives an example of a P.P. ring whose polynomial ring is not P.P.

(2) Consider the ring, $Q_2[x] = (Q[x])_2$, of 2×2-matrices with entries from the polynomial ring $Q[x]$ over the set of rational numbers, Q. Then we can see that $\text{lann}_{Q_2[x]}\left(\begin{bmatrix} 2 & o \\ x & o \end{bmatrix}\right)$ is generated by the idempotent element

$$\begin{bmatrix} \frac{1}{2}x - \frac{1}{2}x^2 & x - 1 \\ \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}x^3 & 1 - \frac{1}{2}x + \frac{1}{2}x^2 \end{bmatrix}$$

This example is a special case of a result discussed later in Chapter 4.

§2. Regular rings

Definition. A ring R is called (von Newmann) regular if for every element a in R, there exists an element b in R such that $a = aba$.

We will call a ring R as strongly regular if for every element a in R, there exists an element b in R satisfying

$a = a^2b$. We can see that a commutative regular ring is strongly regular. The next proposition shows the relationship between strongly regular and regular ring.

Proposition 1.7. A strongly regular ring is regular.

Proof. Every strongly regular ring is without nilpotent elements, because if for some $x \in R$, $x^2 = o$, then there exists b in R such that $x^2b = x$ and that gives $x = o$. Also in such a ring R, if a, b in R are such that $ab = o$, then $ba = o$ (see p. 11). So if x is an element in R, there exists b in R such that $x^2b = x$ which gives $x(xb-1) = o$. This shows that $(xb-1)x = o$. Hence $xbx = x$; and R is a regular ring.

Q.E.D.

From the definition of a regular ring, we can see that ab and ba are idempotent elements having, respectively, the same left and right annihilator as a. Hence a regular ring R is a right and left P.P. ring. We give a few examples of regular rings which will assist us in the study of P.P. rings.

Examples of regular rings (See [11])

(1) Any division ring is regular.

(2) A direct product of regular rings is regular.

(3) The $n \times n$-matrix ring over a regular ring is regular. This is not true in the case of P.P. rings as we have seen in example 1.6.

(4) More generally, the ring of all linear transformations on a vector space (not necessarily finite dimensional) over a division ring is regular.

(5) A homomorphic image of a regular ring is regular. This, too, is false in the case of P.P. rings. Consider the ring Z of integers and its homomorphic image $Z/(4)$, the ring of integers modulo 4. We know that the ring Z is P.P. (but not regular); however $Z/(4)$ is not a P.P. ring.

§3. Reduced rings

We will call a ring <u>reduced</u> if it has no non-zero nilpotent elements. In a reduced ring R, it can be easily seen that $ab = o$ if, and only if $ba = o$ for any two elements a, b in R. Indeed, left multiplying by b and right multiplying by a; $ab = o$ gives $baba = o$, that is, $(ba)^2 = o$. Since R is a reduced ring, we have $ba = o$.

Because of this remark, the left and right annihilators coincide for any subset of a reduced ring R. Hence if S is a subset of a reduced ring R, we will write,

$$\text{rann}_R(S) = \text{lann}_R(S) = \text{ann}_R(S) .$$

We are going to write $R[x]$, for the ring of all polynomials in an indeterminate x with entries from a ring R. Both Baer and P.P. rings have been extensively studied, and it is known that both of these properties are not stable relative to the formation of polynomial rings, as seen from P.M. Cohn's example 1.6.

Jøndrup [9] studied the stabilization of P.P. rings with respect to polynomial rings in the case of commutative rings. He has shown that a commutative ring R is P.P. if, and only if $R[x]$

is P.P. He proved it by using the technique of localization of rings.

However Jøndrup's theorem does not touch upon the study of non-commutative rings. Armendariz [2] proved that if a ring R is reduced, then R is a P.P. (Baer) ring if, and only if $R[x]$ is a P.P. (Baer) ring. Since the condition of commutativity on P.P. rings implies the condition of reducedness, Armendariz's theorem generalizes that of Jøndrup. For our reference, we will give Armendariz's results, without proof here (see [2]). The first three are special cases of Propositions 2.1, 2.2 and 2.3 below, for power series rings.

Lemma 1.8. Let R be a reduced ring and $f,g \in R[x]$ with $f = \sum_{i=o}^{n} a_i x^i$, $g = \sum_{i=o}^{m} b_i x^i$. Then $fg = o$ if, and only if $a_i b_j = o$ for all $o \leq i \leq n$, $o \leq j \leq m$.

Corollary 1.9. If R is a reduced ring and $f \in R[x]$ such that $f^2 = f$, then $f \in R$.

Corollary 1.10. Let $f \in R[x]$ of degree n and $f = \sum_{i=o}^{n} a_i x^i$. Let $S_f = \{a_o, a_1, \ldots, a_n\}$. Let R be a reduced ring and $U \subseteq R[x]$. If $T = \cup_{f \in U} S_f$, then $ann_{R[x]}(U) = (ann_R(T))[x]$.

Proposition 1.11. Let R be a reduced ring. Then $R[x]$ is a P.P. ring if, and only if R is a P.P. ring

Proposition 1.12. Let R be a reduced ring. Then $R[x]$ is a Baer ring if, and only if R is a Baer ring.

In example 1.6, the ring Z_2 of 2×2-matrices over the ring Z of integers is not reduced (because it has non-zero nilpotent elements such as $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$); and $Z_2[x] = (Z[x])_2$ is not a P.P. ring. However there exist some rings which are not reduced; but the polynomial rings over those are P.P. rings. Here is an example of such a ring. Let F be a field. Consider the ring F_2 of all 2×2-matrices over F. F_2 is clearly not a reduced ring; but F_2 being a regular ring is P.P. However $F_2[x]$ is also a P.P. ring. Naturally this example again shows the need to weaken the condition of reducedness on the given rings.

It seems difficult to generalize Armendariz's result by finding some conditions weaker than reduced. Two rather trivial results follow. However a characterization of those rings R such that R left P.P. implies R[x] left P.P. eludes us; because the new conditions on a left P.P. ring force it to be reduced. Our first step of the generalizations is as follows:

Lemma 1.13. If all the idempotent elements in a ring R are central, then R is a left P.P. ring if, and only if R[x] is a left P.P. ring.

Proof. It is sufficient to show that if all the idempotent elements in a left P.P. ring R are central, then R is reduced. If a is an element in R with $a^2 = o$, then $a = o$. Indeed, $a^2 = o$ gives $a \in lann_R(a)$. Since R is left P.P., $lann_R(a) = Re$ for some idempotent element e in R. Hence $a = ae$ and $ea = o$. Since the idempotent elements are central, $a = ae = ea = o$.

Q.E.D.

The following two corollary follows immediately from this lemma.

Corollary 1.14. If for every element a in a ring R, $lann_R(a) = rann_R(a)$, then R is a left P.P. ring if, and only if R[x] is left P.P.

Proof. From Corollary 1.13, it is sufficient to show that all the idempotent elements in a ring R are central; that is, to show that for any idempotent element e in a ring R, ae = ea for all a in R. Since $e(1-e)a = o$, by hypothesis we have $(1-e)ae = o$, and so ae = eae. Since $(1-e)ea = o$, by hypothesis again we get $ea(1-e) = o$, and hence ea = eae.

Q.E.D.

If we try to distinguish the properties of the rings Z_2 and F_2, we can see that Z_2 is not regular but F_2 is regular. Hence it is quite natural to conjecture that "If a ring R is regular, then R[x] is P.P." The proof of this conjecture is not attempted in this thesis.

We will now prove the following proposition which simplifies most of the remaining work in this thesis.

Proposition 1.15. If R[x] is a left (right) P.P. ring, then R is a left (right) P.P. ring.

Proof. Let a be an element in R. Then a is an element of R[x]. Since R[x] is a left P.P. ring, there exists an idempotent element e in R[x] such that $lann_{R[x]}(a) = R[x]e$ ——(1)

Suppose $e = \sum_{i=o}^{n} a_i x^i$. We claim that $lann_R(a) = Ra_o$. Indeed if t is an element in Ra_o, for some element y in R we can write that $t = ya_o$. However since e is an element in $lann_{R[x]}(a)$, we have $ea = o$. Hence $a_o a = o$ giving $ya_o a = o$ and $ta = o$. Therefore $t \in lann_R(a)$ and hence $Ra_o \subset lann_R(a)$. ———(2)

Conversely, let b be an element in $lann_R(a)$. We have to show that $b \in Ra_o$. Then we should show that $b = ca_o$ for some element c in R. But it is sufficient to show that $b = ba_o$ and this follows since $ba_o = ca_o a_o = ca_o = b$. Since $lann_{R[x]}(a) = Re$, $b = be$. By comparing the coefficients on both sides, since b is an element in R, we have $b = ba_o$.

Q.E.D.

On the similar lines of the proof of the above proposition, we have:

Proposition 1.16. If $R[x]$ is a Baer ring, then R is Baer.

Proof. Let S be a subset in R. Then S is a subset of $R[x]$. Since $R[x]$ is Baer, there exists an idempotent element e in $R[x]$ such that $ann_{R[x]}(S) = R[x]e$. Suppose $e = \sum_{i=o}^{n} a_i x^i$. We claim that $ann_R(S) = Ra_o$. Indeed, if $t \in Ra_o$, then $t = ya_o$ for some y in R. Since $e \in ann_{R[x]}(S)$, we have $ea = o$ for all a in R. Hence $a_o a = o$ giving $ya_o a = o$ and $ta = o$. Therefore $Ra_o \subset ann_R(S)$.

Conversely, let $b \in ann_R(S)$. To show $b \in Ra_o$, that is to show $b = ca_o$ for some element c in R; it is sufficient to show that $b = ba_o$, and this follows since $ba_o = ca_o a_o = ca_o^2 = ca_o = b$.

Since $\mathrm{ann}_{R[x]}(S) = Re$, $b = be$. By comparing the coefficients on both sides, since $b \in R$, we have $b = ba_o$.

Q.E.D.

CHAPTER 2

POWER SERIES RINGS AND SKEW-POLYNOMIAL RINGS

§1. Results for power series rings

In this section, we will consider analogous questions to Armendariz's results [2] in the case of power series rings. Let R be a given ring. Then the ring of all formal power series $f = \sum_{n=o}^{\infty} a_n x^n$ with coefficients in R is called the ring of (formal) power series and is denoted by $R[[x]]$. Let us first prove the following key lemma.

Lemma 2.1. (cf. Lemma 1.8) Let R be a reduced ring and f, g be in $R[[x]]$ with $f = \sum_{i=o}^{\infty} a_i x^i$, $g = \sum_{j=o}^{\infty} b_j x^j$. Then $fg = o$ if, and only if $a_i b_j = o$ for all $i \geq o$, $j \geq o$.

Proof. Suppose that $fg = o$. Then $\left(\sum_{i=o}^{\infty} a_i x^i\right)\left(\sum_{j=o}^{\infty} b_j x^j\right) = o$. From this, we get a family of equations as follows:

$$\left.\begin{array}{l} a_o b_o = o \\ a_1 b_o + a_o b_1 = o \\ a_2 b_o + a_1 b_1 + a_o b_2 = o \\ a_3 b_o + a_2 b_1 + a_1 b_2 + a_o b_3 = o \\ \quad\vdots \end{array}\right\} \text{—— (A)}$$

Since R is given to be a reduced ring, $a_i b_j = o$ if, and only if $b_j a_i = o$. Consider the equation $a_1 b_o + a_o b_1 = o$. Left multiplying by b_o, we get $b_o a_1 b_o + b_o a_o b_1 = o$. Since $b_o a_o = o$, we have $b_o a_1 b_o = o$ and, hence $(a_1 b_o)^2 = o$. Again as R is a reduced ring, $a_1 b_o = o$. The same equation now gives that $a_o b_1 = o$.

Now consider the equation $a_2 b_o + a_1 b_1 + a_o b_2 = o$. Left multiplying by b_o, we have $b_o a_2 b_o + b_o a_1 b_1 + b_o a_o b_2 = o$. Since $b_o a_1 = o$ and $b_o a_o = o$, we get $b_o a_2 b_o = o$ giving $(a_2 b_o)^2 = o$. Again using that R is reduced, it follows that $a_2 b_o = o$. Using this argument successively in all other remaining equations in (A), we see that $a_i b_o = o$ for all $i \geq o$. Thus the original system of equations in (A) will be reduced to the new system of equations as follows:

$$\left.\begin{array}{l} a_1 b_1 + a_o b_2 = o \\ a_2 b_1 + a_1 b_2 + a_o b_3 = o \\ \vdots \end{array}\right\} \quad \text{——— (B)}$$

Right multiplying the equation $a_1 b_1 + a_o b_2 = o$ by a_o, we get $a_1 b_1 a_o + a_o b_2 a_o = o$. Since $b_1 a_o = o$, this reduces to $a_o b_2 a_o = o$ and, therefore $(a_o b_2)^2 = o$. Since R is a reduced ring, $a_o b_2 = o$. The same equation now gives us $a_1 b_1 = o$.

If we right multiply the equation $a_2 b_1 + a_1 b_2 + a_o b_3 = o$ by a_o, we have $a_2 b_1 a_o + a_1 b_2 a_o + a_o b_3 a_o = o$. Since $b_1 a_o = o$ and $b_2 a_o = o$, this gives $a_o b_3 a_o = o$, and hence $(a_o b_3)^2 = o$. Again as R is a reduced ring, $a_o b_3 = o$. Right multiplication of the equation $a_2 b_1 + a_1 b_2 = o$ by b_1 gives use $b_1 a_2 b_1 + b_1 a_1 b_2 = o$, and hence $b_1 a_2 b_1 = o$, which follows since $b_1 a_1 = o$. Hence $(a_2 b_1)^2 = o$ and, therefore $a_2 b_1 = o$, since R is a reduced ring.

Using this fact successively in other remaining equations in (B), we have $a_i b_1 = o$ for all $i \geq o$. Similar repetition yields $a_i b_2 = o$, $a_i b_3 = o, \ldots, a_i b_j = o$, for all $i \geq o$, $j \geq o$.

This converse is quite obvious, since $a_i b_j = o$ implies $fg = o$.

Q.E.D.

We will now prove below two important corollaries of the above lemma.

Corollary 2.2. (cf. Cor. 1.9) If R is a reduced ring and f is an element in R[[x]] such that $f^2 = f$, then f is an element in R.

Proof. Let $f = \sum_{i=o}^{\infty} a_i x^i$. Then $1-f = (1-a_o) + \sum_{j=1}^{\infty} a_i x^i$. Given $f^2 = f$, we have $f-f^2 = o$, and hence $f(1-f) = o$. Therefore by Lemma 2.1, we get $a_o(1-a_o) = o$, $a_i^2 = o$ for all $i \geq 1$. Thus $a_o = a_o^2$ and $a_i = o$ for all $i \geq 1$; since R is a reduced ring. Consequently $f = a_o$ is an element in the ring R.

Q.E.D.

Let f be an element in R[[x]] such that $f = \sum_{i=o}^{\infty} a_i x^i$. Let $S_f = \{a_o, a_1, \ldots\}$, the set of all coefficients of f.

Corollary 2.3. (cf. Cor. 1.10) Let R be a reduced ring and U be a subset of R[[x]]. If $T = \cup_{f \in U} S_f$; then $\mathrm{ann}_{R[[x]]}(U) = (\mathrm{ann}_R(T))[[x]]$.

Proof. Let g be an element in R[[x]] such that $g = \sum_{i=o}^{\infty} b_i x^i$. If $g \in \mathrm{ann}_{R[[x]]}(U)$, then $gU = o$, that is, $gf = o$ for all f in U. Application of Lemma 2.1 gives $b_i a_j = o$ for all $f \in U$, where a_j are coefficients of f in U. Therefore the b_i are elements in $ann_R(T)$ and thus g is an element in $(ann_R(T))[[x]]$.

To prove converse, suppose g is an element in $(ann_R(T))[[x]]$. Let $g = \sum_{i=o}^{\infty} b_i x^i$ where the b_i are elements in $ann_R(T)$. Then $b_i a_j = o$ for all elements a_j in T. By applying Lemma 2.1 we see that $gf = o$, for any element g in $R[[x]]$ and f in U. Thus g is an element in $ann_{R[[x]]}(U)$.

Q.E.D.

Remark (A). Similar to Proposition 1.15, we can see that if $R[[x]]$ is a Baer ring, then R is Baer.

Let S be a subset in R. Then S is a subset of $R[[x]]$. Since $R[[x]]$ is Baer, there exists an idempotent element $e = \sum_{i=o}^{\infty} a_i x_i$ in $R[[x]]$ such that $ann_{R[[x]]}(S) = R[[x]]e$. We claim that $ann_R(S) = Ra_o$. Indeed, if $t \in Ra_o$, then $t = ya_o$ for some y in R. Since $e \in ann_{R[[x]]}(S)$, we have $ea = o$ for all a in R. Hence $a_o a = o$ giving $ya_o a = o$ and $ta = o$. Thus $Ra_o \subset ann_R(S)$.

Conversely, let $b \in ann_R(S)$. To show $b \subset Ra_o$, that is to show $b = ca_o$ for some $c \in R$. It is sufficient to show that $b = ba_o$, and this follows since $ba_o = ca_o a_o = ca_o^2 = ca_o = b$. Since $ann_{R[[x]]}(S) = Re$, $b = be$. By comparing the coefficients on both sides, since $b \in R$, we have $b = ba_o$.

Q.E.D.

The following proposition demonstrates that the Baer property is stable with respect to the formation of power series ring.

<u>Proposition 2.4</u>. (Cf. Proposition 1.12) Let R be a reduced ring. Then R is a Baer ring if, and only if R[[x]] is a Baer ring.

<u>Proof</u>. Suppose R is a Baer ring. Take U a subset of R[[x]]. By Corollary 2.3, if $T = \cup_{f \in U} S_f$, then $\mathrm{ann}_{R[[x]]}(U) = (\mathrm{ann}_R(T))[[x]]$. Since R is a Baer ring, $\mathrm{ann}_R(T) = Re$ for some idempotent element e in a ring R. Hence $\mathrm{ann}_{R[[x]]}(U)$ is generated by e. Thus R[[x]] is a Baer ring.

The converse is a special case of Remark (A) above.

Q.E.D.

From the above Proposition 2.4, we are tempted to conjecture that if R is a P.P. ring, then R[[x]] is a P.P. ring when R is a reduced ring. However here is a counterexample.

<u>Example 2.5</u>. Let F be a field. Let
$R = \{(a_0, a_1, a_2, \ldots) : a_i \in F$ such that the sequence is eventually constant$\}$.

It is easy to prove that R is a commutative ring with unity $i = (1,1,1,\ldots)$ under the operations of ordinary addition and multiplication of sequences. R is also a regular ring. Indeed, for any element $a = (a_0, a_1, a_2, \ldots)$ in R, we can choose $x = (x_0, x_1, \ldots)$ satisfying $axa = a$, by defining

$$x_i = \begin{cases} a_\alpha^{-1} & \text{if } a_\alpha \neq o \\ 0 & \text{if } a_\alpha = o \end{cases} .$$

Evidently R is a P.P. ring. However R is not a Baer ring. Because consider the countable subset

$$S = \{x_1 = (1,o,o,\ldots),\ x_2 = (o,o,1,o,o,\ldots),\ x_3 = (o,o,o,o,1,o,o,\ldots),\ldots\}.$$

Let $g = (a_1,a_2,\ldots)$ be an element in $ann_R(S)$. If g is an element of $ann_R(x_1)$, then $gx_1 = o$ gives that $(a_1,a_2,\ldots)(1,o,o,\ldots) = (a_1,o,o,\ldots) = (o,o,\ldots)$, and hence $a_1 = o$. If g is an element in $ann_R(x_2)$, the similar argument leads us to conclude $a_3 = o$. Proceeding in the similar fashion, we see that $a_i = o$ if i is odd. However, if i is even, a_i can take any value in R. Thus an element g in $ann_R(S)$ has the form $(o,a_2,o,a_4,o,\ldots)$.

If $ann_R(S)$ were generated by an idempotent element, the form of an element g in $ann_R(S)$ indicates that the idempotent element should have been $(o,1,o,1,o,1,o,\ldots)$. Certainly $(o,1,o,1,\ldots)$ is not an element in R. This shows that $ann_R(S)$ is not generated by an idempotent element in R, and hence R is not Baer.

For this ring R, we can easily show that $R[[x]]$ is not a P.P. ring. Consider the elements a_i in R as follows: $a_o = (1,o,o,\ldots)$, $a_1 = (o,o,1,o,\ldots)$, $a_2 = (o,o,o,o,1,o,\ldots)$, ... and let $f(x) = a_o + a_1x + a_2x^2 + \ldots$ be the power series. Consider now $ann_{R[[x]]}(f(x))$. If $S_f = \{a_o,a_1,\ldots\}$, Corollary 2.3 gives that $ann_{R[[x]]}(f(x)) = (ann_R(S_f))[[x]]$. From the foregoing discussion, we know that $ann_R(S_f)$ is not generated by an idempotent element in R. Consequently, $ann_{R[[x]]}(f(x))$ is not generated by an idempotent element in R. Thus $R[[x]]$ is not a P.P. ring.

Q.E.D.

Let us denote the following property by (*): "The left annihilator of every countable subset in a ring R is generated by an idempotent element in R."

Proposition 2.6. Let R be a reduced ring. Then (*) holds in R if, and only if R[[x]] is a P.P. ring.

Proof. Let $f(x)$ be an element in $R[[x]]$ given by $f(x) = a_o + a_1x + a_2x^2 + \ldots$. Let $S_f = \{a_o, a_1, a_2, \ldots\}$. Then S_f is a countable subset of R. From Corollary 2.3 we know that $\mathrm{ann}_{R[[x]]}(f) = (\mathrm{ann}_R(S_f))[[x]]$.

If R satisfies (*) property, $\mathrm{ann}_R(S_f)$ is generated by an idempotent element, say e, in R. The $\mathrm{ann}_{R[[x]]}(f)$ is generated by an idempotent element e in $R[[x]]$; by Corollary 2.2. Hence $R[[x]]$ is a P.P. ring.

Conversely if S_f is a countable subset of R; then we can find corresponding f in $R[[x]]$ such that $\mathrm{ann}_{R[[x]]}(f) = (\mathrm{ann}_R(S_f))[[x]]$. Since $R[[x]]$ is a P.P. ring, $\mathrm{ann}_{R[[x]]}(f)$ is generated by an idempotent element e in $R[[x]]$; which is in R, by Corollary 2.2. Hence $\mathrm{ann}_R(S_f)$ is generated by an idempotent element e in R. Thus R satisfies (*) property.

Q.E.D.

We shall now look at the next example from which we can infer that the (*) property is stronger than the P.P. property, but weaker than the Baer property.

Example 2.7. Let F be a field. $\mathcal{R}$ be the set of all real numbers.

Let $R = \{x = \{x_\alpha\}_{\alpha \in \mathcal{R}} : x_\alpha \in F$ are constant on the complement of some countable set in $\mathcal{R}.\}$

It can be easily verified that R is a commutative ring with unity under the operations of the usual addition and multiplication of sequences defined as follows: If $x = \{x_\alpha\}_{\alpha \in \mathcal{R}}$ and $y = \{y_\alpha\}_{\alpha \in \mathcal{R}}$, then $x+y = \{x_\alpha + y_\alpha\}_{\alpha \in \mathcal{R}}$ and $xy = \{x_\alpha y_\alpha\}_{\alpha \in \mathcal{R}}$. Clearly if x, y are elements in R, then x + y and xy are elements in R, because they are constant on the complement of some countable set.

R is a regular ring. To show this, for any element $a = \{a_\alpha\}_{\alpha \in \mathcal{R}}$ in R, we can choose an element $x = \{x_\alpha\}_{\alpha \in \mathcal{R}}$ in R satisfying $a^2x = a$ by defining: $x_\alpha = \begin{cases} a_\alpha^{-1}, & \text{if } a_\alpha \neq o \\ o, & \text{if } a_\alpha = o \end{cases}$. Thus R is a P.P. ring.

Now we shall show that the annihilator of any countable subset of R is generated by an idempotent element in R. Let $\mathcal{N}$ be the set of natural numbers. Let $S = \{x^{(n)} : n \in \mathcal{N}\}$ be a countable subset of R and let $y = \{y_\alpha\}_{\alpha \in \mathcal{R}}$ be an element in $ann_R(S)$. Clearly $y \in ann_R(S)$ if, and only if, and only if $yx^{(n)} = o$ for all n. Hence $y \in ann_R(S)$ if, and only if $y_\alpha = o$ whenever $x_\alpha^{(n)} \neq o$ for some n. Define $e = \{e_\alpha\}_{\alpha \in \mathcal{R}}$ by the following:

$$e_\alpha = \begin{cases} o, & \text{if } x_\alpha^{(n)} \neq o \text{ for some } n \\ 1, & \text{otherwise .} \end{cases}$$

Then e will be an idempotent element in R, and $ann_R(S)$ will be generated by an idempotent element $e = \{e_\alpha\}_{\alpha \in \mathcal{R}}$

in R if we show that $y = ye$ for all $y \in ann_R(S)$ and either $\{\alpha : e_\alpha = 1\}$ or $\{\alpha : e_\alpha = o\}$ are countable. Obviously by definition of e, if e is an element in R then e is an element in $ann_R(S)$ and $y = ye$ all $y \in ann_R(S)$.

Suppose $\{\alpha : e_\alpha = o\}$ is uncountable. By definition of e, $\{\alpha : x_\alpha^{(n)} \neq o$ for some $n\}$ is uncountable. Since we have $\{\alpha : x_\alpha^{(n)} \neq o$ for some $n\} = \cup_n \{\alpha : x_\alpha^{(n)} \neq o\}$; for some m, $\{\alpha : x_\alpha^{(m)} \neq o\}$ is uncountable. By the definition of R, for this m, $\{\alpha : x_\alpha^{(m)} \neq o\}^c$ is countable. Hence $\{\alpha : e_\alpha = 1\} = \{\alpha : e_\alpha = o\}^c = (\cup_n \{\alpha : x_\alpha^{(n)} \neq o\})^c = \cap_n \{\alpha : x_\alpha^{(n)} \neq o\}^c$ shows that $\{\alpha : e_\alpha = 1\}$ is countable, because one set in the intersection is countable.

A similar explanation can be given if we assume that $\{\alpha : e_\alpha = 1\}$ is uncountable, and then show that $\{\alpha : e_\alpha = o\}$ is countable. Thus the ring R is certainly a P.P. ring since it satisfies the (*) property.

However R is not a Baer ring. Let Λ be the set of all positive real numbers. Consider the uncountable subset S of R consisting of the elements $\{x_\alpha^{(\lambda)}\}_{\alpha \in R, \lambda \in \Lambda}$ such that

$$x_\alpha^{(\lambda)} = \begin{cases} 1, & \text{if } \alpha = \lambda \\ o, & \text{otherwise} \end{cases} .$$

We can see that if $y = \{y_\alpha\}_{\alpha \in R}$ is an element in $ann_R(S)$, then $y_\alpha = \begin{cases} o , & \text{if } \alpha = \lambda \\ \text{non-zero}, & \text{otherwise} \end{cases}$.

If $ann_R(S)$ were generated by an idempotent element $e = \{e_\alpha\}_{\alpha \in R}$, then e would be such that

$$e_\alpha = \begin{cases} o, & \text{if } \alpha = \lambda \\ 1, & \text{otherwise} \end{cases} .$$

However the set $\Lambda = \{\alpha : e_\alpha = o\}$ and $\Lambda^c = \{\alpha : e_\alpha = 1\}$ are both uncountable. Hence e is certainly not an element in the ring R. Thus R is not a Baer ring.

Q.E.D.

§2. i-dense rings

In this section, the following generalization of P.P. rings is studied.

Definition. The ring R is left (right) i-dense if every non-zero left (right) annihilator contains a non-zero idempotent element. If a ring is left and right i-dense, it is called i-dense.

Proposition 2.8. If R is a right P.P. ring, then R is left i-dense.

Proof. If not, suppose for some subset S of R, its annihilator $lann_R(S) \neq o$, does not contain any non-zero idempotent element. Let $o \neq a$ be an element in $lann_R(S)$. Since $\{a\} \subset lann_R(S)$, $rann_R(a) \supset rann_R(lann_R(S))$. As R is given to be a right P.P. ring, suppose $rann_R(a) = eR$ for some idempotent element $o \neq e$ in R. Hence $eR \supset rann_R(lann_R(S))$. This will again mean that $lann_R(eR) \subset lann_R(rann_R(lann_R(S))) = lann_R(S)$. Thus $R(1-e) \subset lann_R(S)$. This gives that $(1-e)$ is an element of $lann_R(S)$. Clearly if $1-e = o$, then $e = 1$. However this is impossible, since $e = 1$ will mean $eR = R$ and then $rann_R(a) = R$, and hence $a = o$. This contradicts our assumption that $a \neq o$. Thus R is a left i-dense ring.

Q.E.D.

In analogy with P.P. rings, we have the following proposition.

Proposition 2.9. Let R be a reduced ring. Then R is an i-dense ring if, and only if R[x] is an i-dense ring.

Proof. Let f be an element in R[x] such that $f = \sum_{i=o}^{n} a_i x^i$. Let $S_f = \{a_o, a_1, \ldots, a_n\}$, and let $U \subseteq R[x]$. If $T = \cup_{f \in U} S_f$, then $ann_{R[x]}(U) = ann_R(T)\,[x]$. (Refer to Proposition 1.10)

Suppose R is an i-dense ring. Let $o \neq ann_{R[x]}(U)$. Then $ann_R(T)$ is non-zero. Since R is an i-dense ring, $ann_R(T)$ will contain a non-zero idempotent element, say $o \neq e$ in R.

Obviously the same idempotent element will be an element in $ann_{R[x]}(U)$. Hence R[x] is an i-dense ring.

The converse is a special case of the following Proposition 2.10.

Proposition 2.10. (Cf. Proposition 1.15) If R[x] is a left i-dense ring, then R is a left i-dense ring.

Proof. Let T be a subset of a ring R with non-zero $lann_R(T)$. Then T is a subset of the left i-dense ring R[x]. Suppose $o \neq lann_{R[x]}(T)$ contains some idempotent element $e = \sum_{i=o}^{n} a_i x^i$ in R[x]. Since $e = e^2$, $e(1-e) = o$, and hence $\left[a_o + \sum_{i=1}^{n} a_i x^i\right]\left[(1-a_o) + \sum_{i=1}^{n} a_i x^i\right] = o$. Then we get the system of equations as follows:

$$a_o(1-a_o) = o, \quad a_1(1-a_o) + a_o a_1 = o, \quad a_2(1-a_o) + a_1^2 + a_o a_2 = o, \ \ldots \qquad \text{——— (1).}$$

If $a_o = o$, then from (1), all the remaining a_i are zero and thus $e = o$ in R. If $a_o = 1$, then again from (1), all other a_i are zero, and hence $e = a_o = 1$ in R. Suppose that $e \neq 1$ and $a_o \neq o$. Then $\mathrm{lann}_R(T)$ contains the idempotent $o \neq a_o = a_o^2$ in R. Thus R is a left i-dense ring.

Q.E.D.

§3. Skew-polynomial rings

This is a very interesting type of polynomial ring. Starting from any given ring R, let us construct a ring $R^*[x]$ whose elements can all be expressed as polynomials

$$f = a_o + xa_1 + x^2a_2 + \dots + x^na_n \qquad \text{——— (1)}$$

To multiply two elements given by (1), say $f = \sum x^i a_i$ and $g = \sum x^j b_j$, we have by distributivity $fg = \sum x^i(a_ix^j)b_j$. Hence it will only be necessary for us to prescribe a_ix^j. Let ax for any element a in R be such that

$$ax = xa^\alpha \qquad \text{——— (2)}$$

where $a \to a^\alpha$ is a homomorphism of the ring R into itself.

We observe that this is enough to fix the multiplication in $R^*[x]$, since we can work out ax^r by induction on r as follows:

$$ax^r = (ax)x^{r-1} = (xa^\alpha)x^{r-1} = x(a^\alpha x)x^{r-2} =$$
$$= x^2a^{\alpha^2}x^{r-2} = \dots = x^ra^{\alpha^r} \qquad \text{——— (3)}$$

Let us now investigate the consequences of (2). We have

$$(a+b)x = x(a+b)^{\alpha} \quad \text{and} \quad ax + bx = xa^{\alpha} + xb^{\alpha},$$

$$\text{and hence} \quad (a+b)^{\alpha} = a^{\alpha} + b^{\alpha} \qquad \text{—— (4)}$$

Also $(ab)x = x(ab)^{\alpha}$ and $a(bx) = a(xb^{\alpha}) = (ax)b^{\alpha} = xa^{\alpha}b^{\alpha}$, and so

$$(ab)^{\alpha} = a^{\alpha}b^{\alpha} . \qquad \text{—— (5)}$$

Further $1x = x1 = x1^{\alpha}$ and therefore $1^{\alpha} = 1$. —— (6)

The set of all expressions (1) can be made into a ring by defining addition as usual and multiplication by the commutation rule (2). Then the resulting ring is called the <u>skew-polynomial ring in</u> x <u>over</u> R <u>with respect to</u> α and we will denote it by $R^*[x]$.

The skew-polynomial rings are much more complicated than the usual polynomial rings. Our immediate objective is to prove a supply of skew-polynomial rings which are P.P. rings. To simplify the matter still further and to get a specific class of rings, we will prove the next proposition. The following proposition describes a particular class of rings by putting some conditions on the endomorphism α.

<u>Proposition 2.11</u>. Let $\alpha : R \to R$ be a ring homomorphism and R be a reduced P.P. ring. Then $ann_R(a) = ann_R(a^{\alpha})$ for all $a \in R$ if, and only if the homomorphism α is one-one and in addition it preserves the idempotent elements in R.

<u>Proof</u>. Let us assume that $ann_R(a) = ann_R(a^{\alpha})$ for all $a \in R$. We first observe that α is one-one. For, if not, let a be a non-zero

element belonging to $\ker \alpha$. Then $a^{\alpha} = o$. However this indicates that $ann_R(a^{\alpha}) = R$, by assumption this leads us to the fact that $ann_R(a) = R$. This means the element a is zero. Hence $\ker \alpha = o$. Now if e is an idempotent element in R, we need to prove $e = e^{\alpha}$. Now e^{α} is an idempotent element in R and, as R is a reduced ring, all idempotent elements are central. Now $ann_R(1-e) = Re = ann_R(1-e^{\alpha}) = Re^{\alpha}$. Then by Exercise (2) in ([1], p. 102), we have $e = e^{\alpha}$.

To prove the converse, we assume that α is one-one and that α preserves the idempotent elements in R. To show $ann_R(a) = ann_R(a^{\alpha})$, it is sufficient to show that for any element r in R, $ra = o$ if, and only if $ra^{\alpha} = o$. We will establish this fact as follows:

Suppose $ra^{\alpha} = o$. This means that r is an element in $ann_R(a^{\alpha})$. Since R is a P.P. ring, we can assume that $ann_R(a^{\alpha}) = eR$, for some idempotent element e in R. Hence

$$ea^{\alpha} = o \quad \text{and} \quad re = r \qquad \text{—— (7)}$$

We now see that $(ra)^{\alpha} = (rea)^{\alpha} = r^{\alpha}e^{\alpha}a^{\alpha}$, since α is a homomorphism. As α preserves the idempotent elements we have by (7), $(ra)^{\alpha} = r^{\alpha}ea^{\alpha} = r^{\alpha}o = o$. Since α is a monomorphism, $(ra)^{\alpha} = o$ implies that $ra = o$. In order to prove the remark in an other way, let us assume that $ra = o$. Then r is an element in $ann_R(a)$. Since R is a P.P. ring, we can again assume that $ann_R(a) = fR$ for some idempotent element f in R. Hence we have

$$r = rf \quad \text{and} \quad fa = o. \qquad \text{—— (8)}$$

Now $fa = o$ implies that $(fa)^{\alpha} = o$, and hence, $f^{\alpha}a^{\alpha} = o$. Since α preserves the idempotent elements, we have $fa^{\alpha} = o$. Using this and (8) we can see that $ra^{\alpha} = rfa^{\alpha}$. Hence $ra^{\alpha} = rfa^{\alpha} = o$.

Q.E.D.

By studying the previous proposition, the question naturally arises whether this type of endomorphism exists or not. Next we will give a very simple example in a ring, which has non-trivial idempotents, exhibiting the properties of the previous proposition. Of course, we can find many examples proceeding on similar lines.

<u>Example 2.12</u>. Let C be the field of all complex numbers, with the usual operations. Clearly C is reduced and P.P. We have also proved in this thesis (see Proposition 3.11) that a product of reduced and P.P. rings is reduced and P.P. Let $R = C \times C$. If $z = x+iy$ is any complex number, we will denote as usual the conjugate complex number of z by $\bar{z} = x-iy$. Let us define a homomorphism $f : C \to C$ such that $f(z) = \bar{z}$. Obviously f is a ring homomorphism. Now we can define homomorphism $\alpha = f \times f$ from $R = C \times C$ into itself by $\alpha(z_1,z_2) = (\bar{z}_1,\bar{z}_2)$. It can be easily verified that the homomorphism α preserves the idempotent elements in the ring, viz. (o,o), (o,1), (1,o), (1,1). α is clearly one-one. The proposition proved just now indicates that $ann_R(a) = ann_R(a^{\alpha})$ for any element a in R. This can also be easily verified by direct computation.

Q.E.D.

The next proposition shows that if $R^*[x]$ is a left P.P. ring then R is a left P.P. ring. We note that there are no conditions on the ring homomorphism α.

Proposition 2.13. If $R^*[x]$ is a left P.P. ring, then R is a left P.P. ring.

Proof. Let a be an element in R. Then a is an element in $R^*[x]$. Since $R^*[x]$ is a left P.P. ring, $lann_{R^*[x]}(a) = R^*[x]e$, for some idempotent element $e = \sum_{i=o}^{n} x^i a_i$ in $R^*[x]$. Then proceeding as in the proof of Proposition 1.15, we can easily prove that $lann_R(a) = Ra_o$, $a_o^2 = a_o$. Hence R is a left P.P. ring.

Q.E.D.

Let R be a ring with unity and $\alpha : R \to R$ be a ring homomorphism. Let us denote the condition (*) as:

"For any two elements a and b in a ring R, $ab = o$ if, and only if $a^\alpha b = o$ if, and only if $ab^\alpha = o$."

This condition holds for α satisfying the conditions of Proposition 2.11.

Suppose that a ring R is reduced and $\alpha : R \to R$ satisfies the (*) property and is one-one. As explained at the start of this section, let us form a skew-polynomial ring $R^*[x]$ from the given ring R with conditions above and with the commutation rule

$$ax = xa^\alpha \qquad (9)$$

We will prove the following key lemma which characterizes the zero-divisors in the above skew-polynomial ring.

Lemma 2.14. Let f and g be two elements in a ring $R^*[x]$ as defined above, such that $f = \sum_{i=o}^{n} x^i a_i$, $g = \sum_{j=o}^{m} x^j b_j$. Then $fg = o$ if, and only if $a_i b_j = o$ for all $0 \le i \le n$, $o \le j \le m$.

<u>Proof</u>. Let us suppose that $fg = o$ and $n = m$. We have

$$fg = (a_o b_o) + (a_o x b_1 + x a_1 b_o) + (a_o x^2 b_2 + x^2 a_2 b_o + x a_1 x b_1)$$
$$+ (a_o x^3 b_3 + x a_1 x^2 b_2 + x^2 a_2 x b_1 + x^3 a_3 b_o) + \dots .$$

Using the relation $ax = xa^{\alpha}$, we have $ax^r = x^r a^{\alpha^r}$, and hence the above equation reduces to:

$$fg = (a_o b_o) + x(a_o^{\alpha} b_1 + a_1 b_o) + x^2(a_o^{\alpha^2} b_2 + a_2 b_o + a_1^{\alpha} b_1)$$
$$+ x^3(a_o^{\alpha^3} b_3 + a_1^{\alpha^2} b_2 + a_2^{\alpha} b_1 + a_3 b_o) + \dots$$
$$+ x^i(a_o^{\alpha^i} b_i + a_1^{\alpha^{i-1}} b_{i-1} + \dots + a_{i-1}^{\alpha} b_1 + a_i b_o) + \dots .$$

Since $fg = o$, this gives rise to a family of equations:

$$\left.\begin{array}{l} a_o b_o = o \\ a_o^{\alpha} b_1 + a_1 b_o = o \\ a_o^{\alpha^2} b_2 + a_1^{\alpha} b_1 + a_2 b_o = o \\ a_o^{\alpha^3} b_3 + a_1^{\alpha^2} b_2 + a_2^{\alpha} b_1 + a_3 b_o = o \\ \vdots \end{array}\right\} \text{—— (a)}$$

Consider the equation $a_o^{\alpha} b_1 + a_1 b_o = o$. Multiplying by b_o, we get $b_o a_o^{\alpha} b_1 + b_o a_1 b_o = o$. Due to the (*) property, since $a_o b_o = o$, this yields $b_o a_1 b_o = o$. Multiplying by a_1, we have $(a_1 b_o)^2 = o$, and hence, since R is a reduced ring, $a_1 b_o = o$. Therefore the equation $a_o^{\alpha} b_1 + a_1 b_o = o$ shows that $a_o^{\alpha} b_1 = o$. By the (*) property, this establishes

$$b_1 a_o = o \quad \text{and so} \quad a_o b_1 = o \qquad \text{—— (2)}$$

Left multiplying the equation $a_o^{\alpha^2} b_2 + a_1^{\alpha} b_1 + a_2 b_o = o$ by b_o we get $b_o a_o^{\alpha^2} b_2 + b_o a_1^{\alpha} b_1 + b_o a_2 b_o = o$, and hence $b_o a_2 b_o = o$ by (*) property and $a_o b_o = o$, $a_1 b_1 = o$ which follows since R is a reduced ring. Hence we have $(a_2 b_o)^2 = o$ and $a_2 b_o = o$.

Using this argument successively in the remaining equations from the family (a), we have the relations $a_i b_o = o$ for all $o \leq i \leq n$. The equations in (a) now after simplification can be written as:

$$\left.\begin{aligned} & a_1 b_1 = o \\ & a_o^{\alpha^2} b_2 + a_1^{\alpha} b_1 = o \\ & a_o^{\alpha^3} b_3 + a_1^{\alpha^2} b_2 + a_2^{\alpha} b_1 = o \\ & \vdots \end{aligned}\right\} \text{—— (b)}$$

Right multiplication of the equation $a_1^{\alpha} b_1 + a_o^{\alpha^2} b_2 = o$ by a_o gives $a_1^{\alpha} b_1 a_o + a_o^{\alpha^2} b_2 a_o = o$, which using (2) gives $a_o^{\alpha^2} b_2 a_o = o$. Since R is a reduced ring, we can write this as $a_o^{\alpha^2} a_o b_2 = o$, and hence, $(a_o^{\alpha})^{\alpha}(a_o b_2) = o$. Applying the (*) property this simplifies to $a_o^{\alpha}(a_o b_2) = o$ and again to $a_o(a_o b_2) = o$. Thus $(a_o b_2) a_o = o$, and so $(a_o b_2)^2 = o$. Since R is a reduced ring, $a_o b_2 = o$ and $b_2 a_o = o$. From the same equation $a_1^{\alpha} b_1 + a_o^{\alpha^2} b_2 = o$, we get $a_1^{\alpha} b_1 = o$. Evidently by the (*) property, $a_1 b_1 = o$. Ultimately by the successive repetition of above procedure with the remaining equations from a family (b), we obtain new relations as: $a_i b_1 = o$ for all $o \leq i \leq n$.

The subsequent repetition of the above process again shows that $a_i b_j = o$ for all $o \le i \le n$, $o \le j \le n$. The converse in the above lemma is quite clear.

Q.E.D.

As the consequence of the above lemma, we have the following two corollaries:

Corollary 2.15. Let f be an element in the ring $R^*[x]$ considered in Lemma 2.14. Let $f^2 = f$. Then f is an element in R.

Proof. Suppose $f = a_o + xa_1 + x^2a_2 + \dots + x^na_n$. By hypothesis, $f^2 = f$ which means $f(1-f) = o$. The application of Lemma 2.14 shows that $a_o(1-a_o) = o$ and $a_i^2 = o$ for all $i \ge 1$. Since R is a reduced ring, this leads to the conclusion that a_o is an idempotent element in R and all other a_i are zero for $i \ge 1$. Hence we can conclude that $f = a_o = a_o^2$ is an element in R.

Q.E.D.

The above corollary characterizes idempotent elements of such a ring $R^*[x]$. Let an element f in such a ring $R^*[x]$ be $f = \sum_{i=o}^{n} x^i a_i$. Let $S_f = \{a_o, a_1, \dots, a_n\}$.

Corollary 2.16. Let U be a subset of the ring $R^*[x]$ in Lemma 2.14. It $T = \cup_{f \in U} S_f$, then we have $\text{ann}_{R^*[x]}(U) = (\text{ann}_R(T))^*[x]$.

Proof. Let g be an element in the above ring $R^*[x]$ such that $g = \sum_{i=o}^{m} x^i b_i$. If g belongs to $\text{ann}_{R^*[x]}(U)$, we get $gU = o$. In other words, we have $gf = o$ for all elements f in U. From the Lemma 2.14, we get $b_i a_j = o$ for all f in U and all a_j in R.

Hence all b_j are elements of $ann_R(T)$. Naturally this indicates that g belongs to $(ann_R(T))*[x]$.

To prove the converse, let us consider an element g in $(ann_R(T))*[x]$. Then this means all b_j are in $ann_R(T)$. Consequently this gives that $b_i a_j = o$ for all a_j. Again application of Lemma 2.14 shows $gf = o$, for all g in $R*[x]$ and all f in U. Hence we conclude that g is in $ann_{R*[x]}(U)$. This completes the proof of the corollary.

Q.E.D.

Corollary 2.16 gives the incentive to study the stabilization of the P.P. ring property for the original ring and its skew-polynomial ring. Though it is not true in general, it holds for the ring with conditions as in Lemma 2.14. This is established in the following proposition.

<u>Proposition 2.17</u>. Let R be a reduced ring satisfying the (*) property and $\alpha : R \to R$ be a ring monomorphism with the commutation rule $ax = xa^{\alpha}$. Let $R*[x]$ be its skew-polynomial ring. Then R is a P.P. ring if, and only if $R*[x]$ is a P.P. ring.

<u>Proof</u>. Let us first assume that R is a P.P. ring. If a and b are any two elements in the ring R such that $ann_R(a) = Re_1$ and $ann_R(b) = Re_2$ where e_1 and e_2 are idempotent elements, then we can easily see that $ann_R(\{a,b\}) = Re$ where $e = e_1 + e_2 - e_1e_2$ which is also an idempotent element. Inductively we can establish that for any finite subset T in a ring R, $ann_R(T) = Re$ for some idempotent element e in a ring R. If $f \in R*[x]$, the set S_f

is finite. By Corollary 2.16, we get

$$\mathrm{ann}_{R^*[x]}(f) = (\mathrm{ann}_R(S_f))^*[x] = (Re)^*[x] = R^*[x]e ,$$

for some idempotent element e in a ring R. Thus we reach to the conclusion that $R^*[x]$ is a P.P. ring.

The converse is a special case of Proposition 2.13.

Q.E.D.

On the similar lines of the proof of Proposition 1.16 and Proposition 2.13, we can prove the next proposition. We will state the proposition without proof.

Proposition 2.18. If $R^*[x]$ is a Baer ring, then R is Baer.

Q.E.D.

The next proposition can be proved very easily on the lines of the proof of Proposition 2.17.

Proposition 2.19. Let R be a reduced ring satisfying the (*) property and $\alpha : R \to R$ a ring monomorphism with the commutation rule $ax = xa^{\alpha}$. Let $R^*[x]$ be its skew-polynomial ring. Then R is a Baer ring if, and only if $R^*[x]$ is Baer.

Proof. Let us assume that R is a Baer ring. Let U be a subset of $R^*[x]$. By Corollary 2.16, if $T = \cup_{f \in U} S_f$, then $\mathrm{ann}_{R^*[x]}(U) = (\mathrm{ann}_R(T))^*[x]$. Since R is a Baer ring, $\mathrm{ann}_R(T) = Re$ for some idempotent element e in a ring R. Hence $\mathrm{ann}_{R^*[x]}(U)$ is generated by e in a ring $R^*[x]$. Thus $R^*[x]$ is a Baer ring.

The converse is a special case of Proposition 2.18.

Q.E.D.

We know that ordinary polynomial rings are the special cases of the skew-polynomial rings, if we take the map α as an identity map. From all the propositions proved here, it is quite obvious that all the results proved in Armendariz [2] can be given as the corollaries to the above results.

CHAPTER 3

WEAKLY P.P. AND OTHER RINGS

§1. Semiprime rings and weakly P.P. rings

Up to now, we have developed our work with the study of P.P. rings, and we have mostly used the condition of reducedness for the rings. We will now try to find some analogies to the condition of reducedness and P.P. rings, respectively semiprimeness and weakly P.P. rings. It will then be possible to develop something more general, very much like our previous work. Let us now recall the notion of a semiprime ring:

Definition. A ring R is said to be a semiprime ring if it has no non-zero nilpotent ideals. In other workds, if I is an ideal in a ring R and if $I^2 = o$, then $I = o$.

The above definition shows that if a ring R is commutative, then R is semiprime if, and only if, it is reduced. For, in the case of commutative rings, the property of a ring having no nilpotent ideals means that a ring has no nilpotent elements. We can also see that the ring $\mathbb{Z}_2$ of 2 × 2-matrices over a ring $\mathbb{Z}$ of integers is semiprime but not reduced.

The notion of a semiprime ring involves ideals and from this we get a clue to what we should define instead of P.P. rings. Before giving that definition, note that in a semiprime ring, left and right annihilators of ideals coincide.

Definition. The ring R is said to be a weakly P.P. ring if for every element a in R, $ann_R(RaR)$ is generated by a central idempotent element e. This means that $ann_R(RaR) = eR$ for some central idempotent element.

Also it is clear that if R is left weakly P.P., then it is semiprime. Indeed, for some ideal I of R, suppose $I^2 = o$. Then this means $II = o$, that is, $a \in lann_R(RbR)$ for all $a,b \in I$. Since R is weakly P.P., $lann_R(RbR) = Re$ for some central idempotent element in R. Then for all $a \in I$, $a = ae$ and $eb = o$. In particular, $b = be = eb = o$. Hence $I = o$.

With the introduction of these notions, let us evaluate the validity of analogous propositions to those proved earlier for P.P. and reduced rings. As before, let us construct the first key-lemma and prove it.

Lemma 3.1. Let R be a semiprime ring. Let f and g be two elements in the ring $R[[x]]$ such that $f = \sum_{i=o}^{\infty} a_i x^i$ and $g = \sum_{j=o}^{\infty} b_j x^j$. Then $fR[[x]]g = o$ if, and only if $a_i R b_j = o$ for all $i \geq o$, $j \geq o$.

Proof. Clearly $fR[[x]]g = o$ if, and only if $fRg = o$. Then $\left(\sum_{i=o}^{\infty} a_i x^i\right) R \left(\sum_{j=o}^{\infty} b_j x^j\right) = o$. This gives rise to a family of equations for all $r \in R$, as follows:

$$\left.\begin{array}{l} a_o R b_o = o \\ a_1 r b_o + a_o r b_1 = o \\ a_2 r b_o + a_1 r b_1 + a_o r b_2 = o \\ \vdots \\ a_n r b_o + a_{n-1} r b_1 + \ldots + a_1 r b_{n-1} + a_o r b_n = o \\ \vdots \end{array}\right\} \text{——— (A)}$$

Since R is a semiprime ring, $a_oRb_o = o$ if, and only if $b_oRa_o = o$. Consider the equation $a_1rb_o + a_orb_1 = o$. Right multiplying by sa_o, $s \in R$, we get $a_1rb_osa_o + a_orb_1sa_o = o$, which means $a_orb_1sa_o = o$. This shows that $(a_oRb_1)^2 = o$. Since R is a semiprime ring. $a_oRb_1 = o$, and hence $b_1Ra_o = o$. The same equation now gives $a_1Rb_o = o$ and $b_oRa_1 = o$.

Proceeding in the similar fashion as in the proof of Lemma 2.1 and reducing the equations by the substitutions obtained, we get $a_iRb_j = o$ for all $i \geq o$, $j \geq o$.

The converse is clear.

Q.E.D.

Corollary 3.2. Let R be a semiprime ring. Let f be a central idempotent element in the ring $R[[x]]$ with $f = \sum_{i=o}^{\infty} a_ix^i$. Then f is an element in the ring R.

Proof. Since $f^2 = f$ is a central idempotent element in $R[[x]]$, we can write $fgf = fg$ for all $g \in R[[x]]$. This will give us $fg(1-f) = o$, for all $g \in R[[x]]$. Consequently with the application of Lemma 3.1, we immediately conclude that $a_or(1-a_o) = o$ for all $r \in R$ and $a_iRa_i = o$ for all $i = 1,2,\ldots$. Since we are assuming that R has a unit element, we see that $a_o^2 = a_o$ and $a_iRa_i = o$ for all $i = 1,2,\ldots$. Therefore $a_o^2 = a_o$ and $a_i = o$ for all $i = 1,2,\ldots$. Hence $f = a_o = a_o^2$ is an element in R.

Q.E.D.

The above corollary characterizes the central idempotent elements in the semiprime ring $R[[x]]$. Let f be an element in $R[[x]]$ with $f = \sum_{i=o}^{\infty} a_ix^i$. Let $S_f = \{a_o,a_1,\ldots\}$.

<u>Corollary 3.3</u>. Let R be a semiprime ring. Let U be a subset of $R[[x]]$ and let $T = \cup_{f \in U} S_f$. Then we have

$$\mathrm{ann}_{R[[x]]}(R[[x]]U\ R[[x]]) = (\mathrm{ann}_R(RTR))[[x]] .$$

<u>Proof</u>. Let g be an element in $R[[x]]$ with $g = \sum_{j=o}^{\infty} b_j x^j$. Clearly g is an element in $\mathrm{ann}_{R[[x]]}(R[[x]]U\ R[[x]])$ if, and only if $gR[[x]]U = o$, which means $gR[[x]]f = o$ for all elements f in U. However Lemma 3.1 reduces this to the fact that $b_j Ra = o$ for all j and for all elements a in T. Therefore g is an element in $\mathrm{ann}_{R[[x]]}(R[[x]]U\ R[[x]])$ if, and only if $b_j RaR = o$, which means b_j are elements in $\mathrm{ann}_R(RaR)$ for all j and all elements a in T.

Q.E.D.

Since the polynomial rings are particular type of power series rings, the next corollary follows immediately.

<u>Corollary 3.4</u>. Let R be a semiprime ring. (1) Let f, g be two elements in a ring $R[x]$ such that $f = \sum_{i=o}^{n} a_i x^i$ and $g = \sum_{j=o}^{m} b_j x^j$. Then $fR[x]g = o$ if, and only if $a_i R b_j = o$ for all $o \leq i \leq n$ and $o \leq j \leq m$.

(2) Let f be a central idempotent element in a ring $R[x]$ with $f = \sum_{i=o}^{n} a_i x^i$. Then f is an element in the ring R.

(3) Let U be a subset of $R[x]$ and let $T = \cup_{f \in U} S_f$. Then $\mathrm{ann}_{R[x]}(R[x]U\ R[x]) = (\mathrm{ann}_R(RTR))[x]$.

Q.E.D.

We will now see that if $R[[x]]$ is a weakly P.P. ring the so is R.

Proposition 3.5. If $R[x]$ is a weakly P.P. ring, then R is weakly P.P. Also, if $R[[x]]$ is weakly P.P., so is R.

Proof. Let a be an element in the ring R. Then $a \in R[x]$. Since $R[x]$ is a weakly P.P. ring, $ann_{R[x]}(R[x]aR[x]) = R[x]e$ for some central idempotent element $e = \sum_{i=o}^{n} a_i x^i$ in $R[x]$. Since $a_o(1-a_o) = o$, and since e is central, a_o is a central idempotent element in R. We claim that $ann_R(RaR) = Ra_o$. Indeed, if t is an element in Ra_o, $t = ya_o$ for some $y \in R$. However since $e \in ann_{R[x]}(R[x]aR[x])$, we have $eR[x]aR[x] = o$. Hence $a_oR[x]aR[x] = o$ giving $ya_oRaR = o$, that is, $t \in ann_R(RaR)$, and $Ra_o \subset ann_R(RaR)$.

Conversely. let b $ann_R(RaR)$. To show $b \in Ra_o$, we have to show. $b = ca_o$ for some $c \in R$. But it is sufficient to show that $b = ba_o$, since $ba_o = ca_oa_o = ca_o^2 = ca_o = b$. Since $ann_{R[x]}(R[x]aR[x]) = Re$, $b = be$. By comparing the coefficients on both sides, since b is an element in R, we have $b = ba_o$.

The remaining part of the proof is clear from the proof of Proposition 1.16.

Q.E.D.

The foregoing corollaries are very useful in proving the next proposition.

Proposition 3.6. Let R be a semiprime ring. Then R is a weakly P.P. ring if, and only if $R[x]$ is weakly P.P.

Proof. Suppose R is weakly P.P. If a, b are any two elements in R such that $ann_R(RaR) = Re_1$ and $ann_R(RbR) = Re_2$, where e_1

and e_2 are central idempotents in R, then $e = e_1e_2$ is also a central idempotent in R such that $\text{ann}_R\,(R\{a,b\}R) = Re$. Therefore we can prove, by induction, that for any finite subset T of a ring R, $\text{ann}_R(RTR) = Re$ for some central idempotent element e in R. Thus if f is an element in $R[x]$ with $f = \sum_{i=o}^{n} a_i x^i$ and $S_f = \{a_o, a_1, \ldots, a_n\}$, then $\text{ann}_R(RS_fR) = Re$ for some central idempotent element e in R. However from Corollary 3.4, since S_f is a finite subset of R, we have $\text{ann}_{R[x]}(R[x]fR[x]) = (\text{ann}_R(RS_fR))[x] = Re[x] = R[x]e$. Since f is an arbitrary element in $R[x]$, we have thus established that $R[x]$ is a weakly P.P. ring.

The converse is the special case of Proposition 3.5.

Q.E.D.

From all this discussion above, we can see that the notions of weakly Baer and weakly i-dense rings can also be defined and results like those proved earlier can also be obtained here.

§2. Direct limit of rings

A partially ordered set I is said to be a directed set if for each pair i,j in I, there exists k in I such that $i \leq k$ and $j \leq k$. (See [3], p. 32)

Let I be a directed set and let A be a ring. $\{M_i\}_{i \in I}$ be a family of A-modules indexed by I. For each pair i,j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism and suppose that the following axioms are satisfied:

(1) μ_{ii} is the identity mapping of M_i, for all i in I.

(2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a direct system $\bar{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the direct limit of the direct system $\bar{M}$. Let C be the direct sum of the M_i, and identify each module M_i with the canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and x_i in M_i. Let $M = C/D$. Let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more exactly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$ is called the direct limit of the direct system $\bar{M}$ and is written as $\varinjlim M_i$.

From the construction, it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$. Now let $\{R_i\}_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I, let $\mu_{ij} : R_i \to R_j$ be now a ring homomorphism satisfying the conditions (1) and (2) above. Regarding each R_i as Z-module, we can then form a direct limit $R = \varinjlim R_i$. Then R inherits a ring structure from R_i so that the mappings $R_i \to R$ are ring homomorphisms. R is then the direct limit of the system (R_i, μ_{ij}). We will quote the following result without proof:

Lemma 3.7. Every element of R can be written in the form $\mu_i(x_i)$ for some i in I and some x_i in R_i. Also if $\mu_i(x_i) = o$, then there exists $j \geq i$ such that $\mu_{ij}(x_i) = o$ in R_j.

Let $\{R_i\}_{i \in I}$ be a family of subrings of a ring R, such that for each pair of indices i, j in I, there exists k in I

such that $R_i + R_j \subset R_k$. Define $i \leq j$ to mean $R_i \subseteq R_j$ and let $\mu_{ij} : R_i \to R_j$ be the embedding of R_i in R_j. Then

$$\varinjlim R_i = \sum R_i = \cup R_i \ .$$

Owing to the above discussion of direct limit, we will check whether a direct limit of left P.P. rings is left P.P. or not. The answer for this is 'no', however we can see that the left annihilator of any element in a direct limit contains many idempotent elements, and, in fact, it is generated by idempotent elements. This is indicated in the next proposition.

Proposition 3.8. Let $\{R_i\}_{i \in I}$ be a direct system of rings. If each R_i is a left P.P. ring, the left annihilator of any element in the direct limit $R = \varinjlim R_i$ is generated by idempotent elements.

Proof. Let us start the proof by considering $\mathrm{lann}_R(x)$ of any element x in a direct limit R. Suppose b is an element in $\mathrm{lann}_R(x)$. Then $bx = o$. From the above discussion, there exists j in I and b_j, x_j in R_j such that $b = \mu_j(b_j)$ and $x = \mu_j(x_j)$. Due to the fact that $bx = o$, we have $\mu_j(b_j)\mu_j(x_j) = o$, and hence, $\mu_j(b_j x_j) = o$. Again there exists $k \geq j$ such that $\mu_{jk}(b_j x_j) = o$ in the ring R_k. We also get that

$$\mu_j = \mu_k \circ \mu_{jk} \ . \qquad \text{——— (1)}$$

Since $\mu_{jk}(b_j x_j) = o$, we have $\mu_{jk}(b_j)\mu_{jk}(x_j) = o$, and hence $\mu_{jk}(b_j) \in \mathrm{lann}_{R_k}(\mu_{jk}(x_j))$. Since R_k is a P.P. ring, suppose $\mathrm{lann}_{R_k}(\mu_{jk}(x_j)) = R_k e_k$ for some idempotent element in the ring R_k. Then

$$e_k(\mu_{jk}(x_j)) = o \quad \text{and} \quad \mu_{jk}(b_j) = \mu_{jk}(b_j)e_k \qquad \text{——— (2)}$$

Now consider the element $\mu_k(e_k)$. Evidently this is and idempotent element in R. Again from (1), we get $\mu_k(e_k)x = \mu_k(e_k)\mu_j(x_j) = \mu_k(e_k)\mu_k(\mu_{jk}(x_j)) = \mu_k(e_k\mu_{jk}(x_j))$. By applying (2) this reduces to $\mu_k(e_k)x = \mu_k(o) = o$. Thus $\mu_k(e_k)$ is an element in $lann_R(x)$. Also for the element $b \in lann_R(x)$ considered above, we get $b = \mu_j(b_j) = (\mu_k \circ \mu_{jk})(b_j) = \mu_k(\mu_{jk}(b_j))$, by (1). Using (2), we have $b = \mu_k(\mu_{jk}(b_j)e_k) = \mu_k(\mu_{jk}(b_j))\mu_k(e_k) = b\mu_k(e_k)$. However since the idempotent element $\mu_k(e_k)$ depends on the element b in $lann_R(x)$; and since b varies in $lann_R(x)$; we may get different idempotent elements $\mu_k(e_k)$ in $lann_R(x)$. Hence the proof.

Q.E.D.

From the Proposition 3.8, the question naturally arises whether a direct limit of left P.P. rings is a left P.P. ring. However we will indicate an example which shows that a direct limit of P.P. rings is not necessarily a P.P. ring.

Example 3.9. To start, let us consider a prime number p in the ring Z of integers. We will construct a new ring R such that $R = \{(a,\bar{b}_1,\bar{b}_2,\ldots) : a \in Z,\ \bar{b}_i \in Z/(p) \text{ and eventually } \bar{b}_i = \bar{a}\}$.

It can be easily verified that R is a ring under the operations of ordinary addition and multiplication of sequences. However this ring R is not a P.P. ring. To prove this, take the element $x = (p,\bar{o},\bar{o},\ldots)$ in R. It is evident that $y \in ann_R(x)$ if, and only if $y = (o,\bar{b}_1,\bar{b}_2,\ldots)$ where $\bar{b}_i$ are in $Z/(p)$ but eventually $\bar{o}$.

If $ann_R(x)$ were generated by an idempotent element e in R, the idempotent element e should have the form $e = (o,\bar{1},\bar{1},\ldots)$ and eventually $\bar{o}$. However it does not justify the choice of an idempotent element, because we do not know where $\bar{1}$ stops. If it stops at the n^{th} place, we can find $y \in ann_R(x)$ which has non-zero component at the $(n+1)^{th}$ place. Hence $ann_R(x)$ is not generated by an idempotent element and thus R is not a P.P. ring.

Now consider the ring R_i of the form

$$R_i = Z \times Z/(p) \times Z/(p) \times \ldots \times Z/(p)$$

the product of $Z/(p)$ taken i times.

Since Z and $Z/(p)$ are P.P. rings, and since the product of P.P. rings is always a P.P. ring (see Proposition 3.11), each R_i is a P.P. ring. Let the set N of all natural numbers be the directed set. If $j \geq i$, we can define a homomorphism $\mu_{ij} : R_i \to R_j$ such that $\mu_{ij}(a,\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_i) = (a,\bar{b}_1,\ldots,\bar{b}_i,\bar{a},\bar{a},\ldots,\bar{a})$, $\bar{a}$ repeated $(j-1)$ times. Clearly μ_{ij} is a ring homomorphism. Indeed, consider

$$\mu_{ij}(a,\bar{b}_1,\ldots,\bar{b}_i) + \mu_{ij}(b,\bar{c}_1,\ldots,\bar{c}_i)$$

$$= (a,\bar{b}_1,\ldots,\bar{b}_i,\bar{a},\bar{a},\ldots,\bar{a}) + (b,\bar{c}_1,\ldots,\bar{c}_i,\bar{b},\bar{b}_1,\ldots,\bar{b})$$

$$= (a+b;\bar{b}_1+\bar{c}_1,\ldots,\bar{b}_i+\bar{c}_i,\bar{a}+\bar{b},\ldots,\bar{a}+\bar{b})$$

$$= \mu_{ij}(a+b,\bar{b}_1+\bar{c}_1,\ldots,\bar{b}_i+\bar{c}_i) \quad .$$

It is readily seen that $\varinjlim R_i = R$. Thus in this example, we have demonstrated that each R_i is a P.P. ring, but their direct limit R is not a P.P. ring.

Q.E.D.

Though we have proved that a direct limit of left P.P. rings, in general, is not a left P.P. ring, we can give some examples where this is the case. In the next proposition, we give a family of P.P. rings, whose direct limit is a P.P. ring.

<u>Proposition 3.10</u>. Let R be a reduced ring and let $\underline{x} = \{x_\alpha\}_{\alpha \in \Lambda}$ by any infinite set of indeterminates, Λ being an index set. Then R is a P.P. ring if, and only if $R[\underline{x}]$ is a P.P. ring.

Proof. We can write $R[\underline{x}] = \cup_F R[\{x_\alpha\}_{\alpha \in F}]$, the union over all finite subsets F of Λ.

Suppose $R[\underline{x}]$ is a P.P. ring. Consider $\text{ann}_R(a)$ of an element $a \in R$. Then a is an element of $R[\underline{x}]$. Since $R[\underline{x}]$ is a P.P. ring, we have $\text{ann}_{R[\underline{x}]}(a) = eR[\underline{x}]$, for some idempotent element e in $R[\underline{x}]$. Since R is a reduced ring, we can prove that if e is an element of $R[\underline{x}]$, e will be an element of $R[\{x_\alpha\}_{\alpha \in F}]$ for some finite subset F of Λ. Hence by induction, from Corollary 1.7, e is an element in R. Hence $\text{ann}_R(a) = R \cap \text{ann}_{R[\underline{x}]}(a) = R \cap eR[\underline{x}] = eR$. Thus R is a P.P. ring

Conversely, suppose that R is a P.P. ring. Let f be an element in $R[\underline{x}]$. Then f will be an element of $R[\{x_\alpha\}_{\alpha \in F}]$ for some finite subset F of Λ. However by a generalization obtained by induction on Proposition 1.9, since R is a reduced ring, we can show that R is P.P. if, and only if $R[\{x_\alpha\}_{\alpha \in F}]$ is. Hence

$ann_{R[\{x_\alpha\}_{\alpha\in F}]}(f) = eR[\{x_\alpha\}_{\alpha\in F}]$; for some idempotent element e in $R[\{x_\alpha\}_{\alpha\in F}]$. But again this will mean that e is an element in R. Moreover, it is evident that e depends not on element g in $R[\underline{x}]$ which satisfies a relation $gf = o$, but only on the coefficients of polynomial f. Consequently, $ann_{R[\underline{x}]}(f) = R[\underline{x}]e$. Thus $R[\underline{x}]$ is a P.P. ring.

Q.E.D.

§3. Direct product of rings

The following three propositions are concerned about the stabilization of the left P.P. property, Baer ring property, and left i-dense ring property for the direct products.

Proposition 3.11. $R = \Pi_A R_\alpha$ is a left P.P. ring if, and only if R_α is a left P.P. ring for all α in index set A.

Proof. Consider an element $x = (x_\alpha)$ in $\Pi_A R_\alpha$. If each R_α is a left P.P. ring, $lann_{R_\alpha}(x_\alpha) = R_\alpha e_\alpha$ for some idempotent element e_α in R. Let $y = (y_\beta)$ be an element in $lann_R(x)$. Then $yx = o$ if, and only if $y_\beta x_\alpha = o$ for all α,β in A. However $y_\beta = y_\beta e_\alpha$ and $e_\alpha x_\alpha = o$. Take $e = (e_\alpha)$ in R. Then clearly e is an idempotent element in R. Also we have $ye = (y_\beta)(e_\alpha) = (y_\beta e_\alpha) = (y_\beta) = y$. We can see that $ex = (e_\alpha)(x_\alpha) = (e_\alpha x_\alpha) = o$. Thus $e = (e_\alpha)$ is an idempotent element which generates $lann_R(x)$; and hence $R = \Pi_A R_\alpha$ is a left P.P. ring.

Conversely, let $x_\alpha \in R_\alpha$; and consider $lann_{R_\alpha}(x_\alpha)$. Clearly the element having α^{th} component x_α and all other remaining components 1 is in $\Pi_A R_\alpha$. Since $R = \Pi_A R_\alpha$ is a left P.P. ring; $lann_R(x) = Re$ for some idempotent element e in R. Then e should have α^{th} component e_α and all other remaining components 0. Then it can be easily seen that $lann_{R_\alpha}(x_\alpha) = R_\alpha e_\alpha$; where e_α is an idempotent element in R_α. Hence R_α is left P.P.

Q.E.D.

Let us now prove a similar result to the above proposition, for Baer rings.

Proposition 3.12. $R = \Pi_A R_\alpha$ is a Baer ring if, and only if each R_α is a Baer ring for all α in index set A.

Proof. Consider $ann_R(U)$ of any subset U of a ring R. Let $u = (u_\alpha)$ be an element in U. If $ann_R(\{u_\alpha : u \in U\})$ is generated by an idempotent element e_α, then $ann_R(U)$ is generated by the idempotent element $e = (e_\alpha)$ as above.

To prove the converse, consider $ann_{R_\alpha}(U_\alpha)$ of subset U_α of a ring R. As in the previous proposition, form an element x in R with α^{th} component $x_\alpha \in U_\alpha$ and all other remaining components 1. Let U be the set of all these x constructed as above. Since R is a Baer ring, $ann_R(U) = Re$, where e is an idempotent element in R. Clearly e has α^{th} component e_α and all other remaining components o. Then it is easy to see that $ann_{R_\alpha}(U_\alpha) = R_\alpha e_\alpha$, where e_α is an idempotent element in R_α. Thus R_α is a Baer ring.

Q.E.D.

In the next proposition, we are going to show that the direct product of i-dense rings is an i-dense ring, and conversely.

Proposition 3.13. $R = \Pi_A R_\alpha$ is a left i-dense ring if, and only if each R_α is a left i-dense ring for all α in index set A.

Proof. Let U be a subset of R. Let $o \neq y = (y_\beta)$ be an element in $lann_R(U)$. Then $yu = o$ for all $u \in U$. Hence $y_\alpha u_\alpha = o$ for all $u = (u_\alpha)$ in U, and for all α in A. Hence $y_\alpha \in lann(\{u_\alpha : u \in U\})$. Then $lann(\{u_\alpha : u \in U\}) \neq o$ for some β, since $lann_R(U) \neq o$. As each R_α is i-dense, $o \neq lann_{R_\beta}(\{u_\beta : u \in U\})$ contains a non-zero idempotent element, say e_β. Then $e = (f_\alpha)$, where $f_\alpha = o$, $\alpha \neq \beta$, $f_\beta = e_\beta$; is a non-zero idempotent element in $lann_R(u)$ for all $u \in U$. Hence $e \in lann_R(U)$. Thus R is i-dense.

Conversely, suppose $lann_{R_\alpha}(U_\alpha) \neq o$ for a subset U_α of R_α. As in the previous proposition, form an element x in R with α^{th} component x_α in U_α and all other remaining components 1. Let U be the set of all these x constructed as above. Since $lann_{R_\alpha}(U_\alpha) \neq o$, $lann_R(U) \neq o$. R is given to be left i-dense. Suppose $lann_R(U)$ contains a non-zero idempotent element $e = (e_\alpha)$. Clearly e has α^{th} component e_α and all other remaining components zero. Then $e_\alpha \neq o$ is contained in $lann_{R_\alpha}(U_\alpha)$. Thus R_α is left i-dense.

Q.E.D.

We will discuss now the relation between direct product and power series rings in the next proposition.

Proposition 3.14. Let R_α, α in index set A, be a family of rings.

Then $(\Pi_A R_\alpha)[[x]]$ and $\Pi_\alpha(R_A[[x]])$ are isomorphic.

Proof. Define the map $\xi : (\Pi_A R_\alpha)[[x]] \to \Pi_\alpha(R_A[[x]])$ by:

$$\xi[(a_{o\alpha})_{\alpha \in A} + (a_{1\alpha})_{\alpha \in A}x + \ldots] = (a_{o\alpha} + a_{1\alpha}x + \ldots)_{\alpha \in A} .$$

Clearly ξ is well defined, because

$$(a_{o\alpha})_{\alpha \in A} + (a_{1\alpha})_{\alpha \in A}x + \ldots = (b_{o\alpha})_{\alpha \in A} + (b_{1\alpha})_{\alpha \in A}x + \ldots$$

only if $a_{i\alpha} = b_{i\alpha}$ for all α in A and for all $i = o,1,2,\ldots$.

Also ξ is onto, since we can collect the corresponding coefficients and for every element q in $\Pi_\alpha(R_A[[x]])$, we can show the existence of an element p in $(\Pi_A R_\alpha)[[x]]$ such that $\xi(p) = q$. ξ is also monomorphism. Indeed if

$$\xi[(a_{o\alpha})_{\alpha \in A} + (a_{1\alpha})_{\alpha \in A}x + \ldots] = \xi[(b_{o\alpha})_{\alpha \in A} + (b_{1\alpha})_{\alpha \in A}x + \ldots],$$

then

$$(a_{o\alpha} + a_{1\alpha}x + \ldots)_{\alpha \in A} = (b_{o\alpha} + b_{1\alpha}x + \ldots)_{\alpha \in A} .$$

Hence $a_{o\alpha} + a_{1\alpha}x + \ldots = b_{o\alpha} + b_{1\alpha}x + \ldots$, for all α in A. This means $a_{i\alpha} = b_{i\alpha}$ for all $\alpha \in A$ and for all $i = o,1,2,\ldots$. Thus $(a_{i\alpha})_{\alpha \in A} = (b_{i\alpha})_{\alpha \in A}$ for all $i = o,1,\ldots$. Therefore we have $(a_{o\alpha})_{\alpha \in A} + (a_{1\alpha})_{\alpha \in A}x + \ldots = (b_{o\alpha})_{\alpha \in A} + (b_{1\alpha})_{\alpha \in A}x + \ldots$, and hence ξ is the isomorphism.

Q.E.D.

Remark:

(1) Consider $(\Pi_A R_\alpha)[x]$ and $\Pi_\alpha(R_A[x])$. The above defined isomorphism ξ restricted to the polynomial ring, does not work

here since the degrees of the polynomials are not preserved if the index set A is infinite.

(2) If each R_α is reduced, the Proposition 3.13 and Proposition 2.4 gives that $(\Pi_A R_\alpha)[[x]]$ is Baer if, and only if each R_α is Baer.

§4. Rings of continuous functions

Let X be a topological space and C(X) be the ring of continuous functions from X to the real field $\mathbb{R}$. Under the point-wise operations of addition and multiplication, C(X) becomes a ring. Zero and unity elements of C(X) are the constant functions o and 1 respectively.

Definition. The topological space X is called completely regular if X is a Hausdorff space, and for any neighbourhood U of a point x in X, there exists a function f in C(X) such that f vanishes outside U but not at x.

For any topological space X, there exists a completely regular space Y and a continuous mapping T of X onto Y such that the mapping $g \to g_o T$ is an isomorphism of C(Y) onto C(X). (See [8], p. 41.) Hence in studying C(X), we will assume without loss of generality that the space X is completely regular.

A subset S of X is dense in X if the closure of S, $c\ell(S)$, is X. If a subset S of X is dense in X, then the homomorphism $f = f|_S$ from C(X) into C(S) is a monomorphism, since two continuous functions on X, which coincide on a dense subset of X, coincide on X, (See [8], p. 48).

While studying comparatively topological properties of a space X and algebraic properties of C(X), it is natural to consider the subsets having the form $f^{-1}(a) = \{x \in X : f(x) = a\}$ where f is in C(X) and a in $\mathbb{R}$. If we substitute o for a, we get a subset $f^{-1}(o)$ of X which is often called a <u>zero set of</u> f, and is denoted by z(f). Any set which is a zero set of some function in C(X) is called a <u>zero set in</u> X. Obviously zero sets are closed. It is also true that every zero set is a G_δ (countable intersection of open sets), since {o} is a G_δ in $\mathbb{R}$. Moreover, $z(f) = z(|f|) = z(f^n)$, for all $n \in \mathbb{N}$. Also $z(o) = X$ and $z(1) = \phi$. Again $z(fg) = z(f) \cup Z(g)$, and a countable intersection of zero sets is a zero set (see [8], p. 16).

The complement of z(f) is called the <u>cozero set of</u> f, and is denoted by coz (f) or coz f. Clearly cozero sets are open. Any set of the form $\{x : f(x) \geq o\}$ is a zero set; since $\{x : f(x) \geq o\} = z(f-|f|)$. Similarly $\{x : f(x) \leq o\} = z(f+|f|)$. Thus the open sets pos $f = \{x : f(x) > o\}$ and neg $f = \{x : f(x) < o\}$ are cozero sets. But if the function is an idempotent e, its only values are o and 1, then the zero set and cozero set are both open and closed, which is clear from $z(e) = \{x : e(x) = o\} = e^{-1}(-1,1)$. We call a set <u>clopen</u> if it is both open and closed.

Conversely if a subset S of X is clopen, then let us define a function $e : X \to \mathbb{R}$ in such a way that

$$e(x) = \begin{cases} o, & \text{if } x \in S \\ 1, & \text{otherwise} \end{cases} .$$

Clearly e is an idempotent element and is continuous on X. Again

$z(e) = S$ and $\operatorname{coz} e = \sim S$ (we will denote complement of a set by the symbol $\sim$). Thus S is the zero set of an idempotent element.

The next proposition shows the relationship between dense ideals of $C(X)$ and dense sets of X.

<u>Proposition 3.15</u>. If D is an ideal in $C(X)$, then the following are equivalent:

(1) D is dense in $C(X)$.

(2) For all g in $C(X)$, if $\operatorname{coz}(D) = \cup\{\operatorname{coz} f : f \in D\}$, then $\operatorname{coz}(D) \subset z(g)$ implies $g = o$.

(3) $\operatorname{coz}(D)$ is dense.

<u>Proof</u>. To prove (1) implies (2), suppose x is an element in $\operatorname{coz} f$ for some f in D. Then $x \in \operatorname{coz}(D)$ and $x \in z(g)$. Thus $g(x)f(x) = o$ which means $gf = o$ for f in D. But D is dense in $C(X)$, hence $g = o$.

For the proof of (2) implies (3), consider an element x in $\sim c\ell(\operatorname{coz}(D))$. Then by complete regularity there exists an h in $C(X)$ such that $h(x) \neq o$ and $h(c\ell(\operatorname{coz}(D))) = o$. This gives $c\ell(\operatorname{coz}(D)) \subset z(h)$ which further implies $\operatorname{coz}(D) \subset z(h)$. Hence by (2), we have $h = o$, a contradiction. So $X = c\ell(\operatorname{coz}(D))$.

To prove (3) implies (2), suppose $\operatorname{coz}(D) \subset z(g)$. Then $c\ell(\operatorname{coz}(D)) = X \subset z(g)$, for all g in $C(X)$; which implies $g = o$.

Finally to prove (2) implies (1), suppose $gD = o$ for some $g \in C(X)$. So $gf = o$ for all f in D. If $x \in \operatorname{coz}(D)$, that is, $x \in \operatorname{coz}(f)$, for some f in D, then $f(x) \neq o$. But

$gf = o$ implies $g(x) = o$. Hence coz (D) $\subset z(g)$; which again gives that $g = o$.

Q.E.D.

With this much introduction, let us now turn to the study of P.P. rings. Let us first recall the following definitions.

Definition (See [8]) A space X is said to be extremally disconnected if every open set has an open closure. This is equivalent to saying that, for any two disjoint open sets U and V in X, $c\ell(U) \cap c\ell(V) = \phi$.

Definition. (See [8]) A space X is said to be basically disconnected if every cozero set has an open closure.

Clearly any extremally disconnected space is basically disconnected, but the converse fails. An example of such a space is non-discrete p-space. Now let us switch our attention over the characterization of those spaces X for which C(X) is i-dense, Baer and P.P.

Proposition 3.16. (See [5]) C(X) is i-dense if, and only if every non-empty open set of X contains a non-empty clopen set.

Proof. Suppose C(X) is i-dense. Let $\phi \neq U$ be open in X. Since X is completely regular, the zero set neighbourhoods form a basis for the neighbourhoods of each point, (see [8], p. 38). For $x \in U$, let $z(f)$ be a zero set neighbourhood of x in U. Now $g \in$ ann (f) means coz $g \subset z(f)$. By complete regularity, ann (f) $\neq o$. So for some idempotent element $e \neq o$, coz $e \subset z(f)$.

Conversely, if $I = \text{ann}(S)$ is a non-zero annihilator, $\cup_{f \in S} \text{coz } f$ is not dense. Hence there is a non-empty clopen set E in $\sim[c\ell(\cup_{f \in S} \text{coz } f)]$. Define e by

$$e(x) = \begin{cases} o, & \text{if } x \notin E \\ 1, & \text{if } x \in E \end{cases}.$$

Then e is a non-zero idempotent element in $\text{ann}(S)$.

Q.E.D.

The following proposition gives the characterization of X when $C(X)$ is a Baer ring.

Proposition 3.17. (See [5]) $C(X)$ is a Baer ring if, and only if X is extremally disconnected.

Proof. Let S be a subset of $C(X)$. If $\text{ann}(S)$ is non-zero, $\cup_{f \in S} \text{coz } f$ is not dense. Also $\sim[c\ell(\cup_{f \in S} \text{coz } f)]$ is open. If X is extremally disconnected, $c\ell\{\sim[c\ell(\cup_{f \in S} \text{coz } f)]\} = E$ is open. Define e by

$$e(x) = \begin{cases} o, & \text{if } x \notin E \\ 1, & \text{if } x \in E \end{cases}.$$

Then e is the idempotent element generating $\text{ann}(S)$, since if $g \in \text{ann}(S)$, then $gS = o$, and hence $gf = o$ for every $f \in S$, which gives $\text{coz } g \subset z(f)$ for every $f \in S$. Thus $\text{coz } e \subset z(f)$ which implies $e \in \text{ann}(S)$. It remains to show that $ge = g$ if $g \in \text{ann}(S)$. If $x \in E$, then $(ge)(x) = g(x)e(x) = g(x)$. If $x \notin E$, then $g(x) = o$ and $(ge)(x) = g(x)e(x) = o$; since $\cup_{f \in S} \text{coz } f$ is dense in $[c\ell(\cup_{f \in S} \text{coz } f)]$. Thus $C(X)$ is Baer.

Conversely suppose C(X) is Baer. Let $\phi \neq U$ be open in X. Consider $S = \{f : \text{coz } f \subset U\}$. Then $U = \cup_{f \in S} \text{coz } f$. As C(X) is Baer, ann (S) = eC(X), for $e^2 = e \in C(X)$. Now $e \in \text{ann}(S)$ if, and only if $\text{coz } e \subset z(f)$ for every $f \in S$. That is, $\text{coz } f \subset z(e)$ for every $f \in S$. Hence $U \subset z(e)$. But as z(e) is a clopen set, $c\ell(U) \subset z(e)$. If possible suppose $z(e) \neq c\ell(U)$. Then there exists $x \in z(e)$ such that $x \notin c\ell(U)$. By definition of complete regularity, there exists a continuous function h such that h(x) = 1 and h(y) = o for all y in $c\ell(U)$. Then $x \in \text{coz } h$ and $c\ell(U) \subset z(h)$. Clearly $h \in \text{ann}(S)$, because $\text{coz } h \subset z(e)$. But as $x \in z(e)$, e(x) = o and for the function h, h(x) = 1. Thus h = he, which means ann (S) is not generated by the idempotent element e. Hence $z(e) = c\ell(U)$. Therefore $c\ell(U)$ is open. Consequently X is extremally disconnected.

Q.E.D.

A similar type of result is proved in the following proposition for P.P. rings.

<u>Proposition 3.18</u>. C(X) is a P.P. ring if, and only if X is basically disconnected.

<u>Proof</u>. If f is an element in C(X) with ann (f) = o, then coz f is not dense. If we assume that X is basically disconnected, $c\ell(\text{coz } f)$ is open. Let $E = c\ell(\text{coz } f)$; and define e by

$$e(x) = \begin{cases} o, & x \in E \\ 1, & x \notin E \end{cases} .$$

Then e is an idempotent element generated ann (f). Indeed,

$g \in$ ann (f) if, and only if $gf = o$ if, and only if coz $g \subset z(f)$. However coz $e = \sim E$ and $\sim E \subset z(f)$. Hence coz $e \subset z(f)$ which shows that $e \in$ ann (f). The proof of this will be completed if we show that $ge = g$ for any $g \in$ ann (f). If $x \notin E$, $(ge)(x) = g(x)e(x) = g(x)$. If $x \in E$, $(ge)(x) = g(x)e(x) = o$ and $g(x) = o$, since coz $g \subset \sim E$. Thus $C(X)$ is a P.P. ring.

Conversely suppose that $C(X)$ is a P.P. ring. Let f be an element in $C(X)$, and consider coz f. Suppose that ann $(f) = eC(X)$ for some idempotent element e in $C(X)$. Then $e \in$ ann (f) if, and only if coz $e \subset z(f)$ if, and only if $z(e) \supset$ coz f. Since $z(e)$ is a clopen set, $c\ell(\text{coz } f) \subset z(e)$. If possible, suppose $z(e) = c\ell(\text{coz } f)$. Then there exists x such that $x \in Z(e)$ and $x \notin c\ell(\text{coz } f)$. By definition of complete regularity, there exists a continuous function h such that $h(x) = 1$ and $h(y) = o$ for all $y \in c\ell(\text{coz } f)$. So $x \in$ coz h and $c\ell(\text{coz } f) \subset z(h)$. Since coz $(h) \subset z(e)$, $h \in$ ann (f). Since $x \in z(e)$, $e(x) = o$. However $h(x) = 1$. Hence $h = he$, which shows that ann (f) is not generated by the idempotent element e. This gives the contradiction to our assumption. Thus $z(e) = c\ell(\text{coz } f)$ and hence $c\ell(\text{coz } f)$ is open. Therefore X is basically disconnected.

Q.E.D.

§5. Rings of continuous functions with values in a discrete ring

In the previous section, we dealt with the ring $C(X)$. Now we will look the problem through a different angle. Suppose X is a topological space and R is a ring. Let R be with the discrete topology. $C(X,R)$ be the set of all continuous functions from X into R. Then we know that $C(X,R)$ is a ring under the usual operations. We will investigate in the next proposition when $C(X,R)$ is a P.P. ring.

<u>Proposition 3.19</u>. If R is a left P.P. ring, then $C(X,R)$ is left P.P.

<u>Proof</u>. Let f be any element in $C(X,R)$. Consider an element g in $lann_{C(X,R)}(f)$, so that $gf = o$. This means $g(t)f(t) = o$ for all t in X. Hence $g(t) \in lann_R(f(t))$, for all t in X. Since R is a left P.P. ring, for every $t \in X$, $lann_R(f(t)) = Re_t$ for some idempotent element e_t in R. Then $g(t) = g(t)e_t$ and $e_t f(t) = o$. For some fixed $r \in R$, let $X_r = \{x : f(x) = r\}$. Since f is continuous each X_r is open, because R is discrete Let $lann_R(f(x)) = lann_R(r) = Re_r$ for some idempotent element e_r in R. Define e on X_r by $e(x) = e_r$ for any $x \in X_r$. Thus e is continuous on X_r. Since X can be written as a union of all X_r, for distinct $r \in R$, e is continuous. e is also an idempotent element in $C(X,R)$. Also for all g in $lann_{C(X,R)}(f)$, $g = ge$. Thus $lann_{C(X,R)}(f)$ is generated by the idempotent element e; and hence $C(X,R)$ is a left P.P. ring.

Q.E.D.

CHAPTER 4

RINGS OF FRACTIONS AND HEREDITARY RINGS

§1. Rings of fractions

In this section, all rings are assumed to be commutative. We will first state the definition of ring of fractions, (see [3]). Let R be a ring. A subset S of R is said to be multiplicatively closed if the identity 1 belongs to S and S is closed under multiplication. Let us define a relation $\sim$ on $R \times S$ by: $(a,s) \sim (b,t)$ if, and only if $(at-bs)u = o$ for some $u \in S$. This relation is clearly reflexive and symmetric. It is also transitive. Indeed, suppose $(a,s) \sim (b,t)$ and $(b,t) \sim (c,u)$. Then there exists v, w in S such that $(at-bs)v = o$ and $(bu-ct)w = o$. By eliminating b from these two equations, we obtain $(au-cs)tvw = o$. Since S is closed under multiplication, $tvw \in S$ and thus $(a,s) \sim (c,u)$. Hence $\sim$ is an equivalence relation. Denote the equivalence class of (a,s) by a/s and let $S^{-1}R$ denote the set of equivalence classes. Then $S^{-1}R$ is a ring if the addition and multiplication is defined as follows:

$$(a/s)+(b/t) = (at+bs)/st \quad \text{and} \quad (a/s)(b/t) = ab/st .$$

The ring $S^{-1}R$ is then called the ring of fractions of R with respect to S.

If N and P are submodules of the R-module M, we have $(N:P) = \{x \in R : xP \subset N\}$. The next result is very useful and we will state it without proof, (see [3], p. 43).

Proposition 4.1. If N, P are submodules of an R-module M and if P is finitely generated, then $S^{-1}(N:P) = (S^{-1}N:S^{-1}P)$.

In particular, if $N = o$ and $P = (a)$, the principal ideal generated by an element a in a ring R, then we have

$$S^{-1}(\text{ann}_R(a)) = \text{ann}_{S^{-1}R}(S^{-1}a) \qquad \text{——— (1)}$$

In the next proposition, we show that $S^{-1}R$ is a P.P. ring if R is a P.P. ring and this for any multiplicatively closed set S.

Proposition 4.2. If S is a multiplicatively closed subset of a P.P. ring R, then $S^{-1}R$ is a P.P. ring.

Proof. Let a be an element in a ring R. Since R is a P.P. ring, suppose $\text{ann}_R(a) = eR$, for some idempotent element e in R. Applying (1), we have $\text{ann}_{S^{-1}R}(S^{-1}a) = S^{-1}(eR) = \frac{e}{1} S^{-1}R$. However $\frac{e}{1}$ is an idempotent element in $S^{-1}R$. Hence $S^{-1}R$ is a P.P. ring.

Q.E.D.

Example 4.3. We can see from the next example that the converse of the above Proposition 4.2 is not true. Take R any ring. Let P be a minimal prime ideal of R. Then $S^{-1}R = R_P$ if $S = R-P$ is a P.P. ring, since R_P is a field.

This example also shows that if $S^{-1}R$ is an i-dense ring, then R is not necessarily an i-dense ring; since we know that if a ring R is right P.P., it is left i-dense.

Q.E.D.

If we suppose that S is a multiplicatively closed set of non-zero-divisors of R, we get the following two important propositions. Before that, we will give the following lemma.

Lemma 4.4. Let S be a multiplicatively closed set of non-zero-divisors of a ring R. If T is a subset of $S^{-1}R$, then $\mathrm{ann}_{S^{-1}R}(T) = S^{-1}(\mathrm{ann}_R(B))$; where $B = \{b \in R : \frac{b}{t} \in T\}$.

Proof. Clearly,

$$\mathrm{ann}_{S^{-1}R}(T) = \{\frac{a}{s} \in S^{-1}R : \frac{a}{s}T = o\}$$
$$= \{\frac{a}{s} : \frac{a}{s}\frac{b}{t} = o \text{ for all } \frac{b}{t} \in T\}$$
$$= \{\frac{a}{s} : ab = o \text{ for all } \frac{b}{t} \in T\}.$$

Indeed, if $\frac{a}{s}\frac{b}{t} = o$ then there exists $u \in S$ such that $abu = o$; and since u is a non-zero-divisor, $ab = o$. Hence we have

$$\mathrm{ann}_{S^{-1}R}(T) = \{\frac{a}{s} : a \in \mathrm{ann}_R(b) \text{ for all } \frac{b}{t} \in T\} = \{\frac{a}{s} : a \in \mathrm{ann}_R(B)\}$$
$$= S^{-1}(\mathrm{ann}_R(B)); \text{ where } B = \{b \in R : \frac{b}{t} \in T\}.$$

Q.E.D.

The following proposition shows that if we put some conditions on the multiplicatively closed set S, then R is a Baer ring implies that $S^{-1}R$ is a Baer ring.

Proposition 4.5. Let S be a set of non-zero-divisors. If a ring R is a Baer ring, then $S^{-1}R$ is a Baer ring.

Proof. Let T be a subset of $S^{-1}R$. Then by Lemma 4.4.,

$$\text{ann}_{S^{-1}R}(T) = S^{-1}(\text{ann}_R(B)); \quad \text{where } B = \{b \in R : \tfrac{b}{t} \in T\}.$$

Since R is a Baer ring, suppose that $\text{ann}_R(B) = S^{-1}(eR) = \frac{e}{1} S^{-1}R$. Clearly $\frac{e}{1}$ is an idempotent element in $S^{-1}R$. Thus $S^{-1}R$ is a Baer ring.

Q.E.D.

Similar result is obtained for i-dense ring as follows.

Proposition 4.6. If R is an i-dense ring, and if S is a set of non-zero-divisors of R, then $S^{-1}R$ is an i-dense ring.

Proof. Let T be a subset of $S^{-1}R$ with $o \neq \text{ann}_{S^{-1}R}(T)$. By applying Lemma 4.4, we have $\text{ann}_{S^{-1}R}(T) = S^{-1}(\text{ann}_R(B))$, where $B = \{b \in R : \frac{b}{t} \in T\}$. Clearly $\text{ann}_R(B) \neq o$; because if $\text{ann}_R(B) = o$, then $\text{ann}_{S^{-1}R}(T) = o$, contradicting our assumption. Since R is i-dense, suppose a non-zero idempotent element e is in $\text{ann}_R(B)$. Since S is a set of non-zero-divisors and $o \neq e$, $\frac{e}{1}$ will be a non-zero idempotent element in $S^{-1}R$, and $\frac{e}{1} \in \text{ann}_{S^{-1}R}(T)$. Thus $S^{-1}R$ is an i-dense ring.

Q.E.D.

§2. Hereditary and other rings

In this section, we are going to throw some light on the structure of semihereditary rings and their global properties. Let us state the definitions of a few more rings, (See [11]).

Definition. A ring R is left hereditary if every left ideal is projective. A ring R is called left semihereditary if every finitely generated left ideal is projective.

There are similar definitions for right hereditary and right semihereditary rings. From the above definitions, it is obvious that any left hereditary ring is left semihereditary, however, the converse is not true. The left semihereditary property would seem to be only a slight strengthening of the left P.P. property. But we shall see that it behaves quite differently.

For commutative rings one also speaks of hereditary and semihereditary rings. We call a commutative hereditary domain a Dedekind domain and a commutative semihereditary domain a Prufer domain. Any principal ideal domain R is hereditary and hence a Dedekind domain, since it is quite evident that every non-zero ideal is isomorphic to R. It is also clear that every regular ring is both left and right semihereditary.

We know that any module M can be represented as the image of a free module F, with kernel K as:

$$o \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow o \quad . \qquad\qquad (1)$$

We can see that another simple type of module to study is one for which K is free. However, in general, it is not independent of the particular type of resolution (1). But it is true if we take "projective" instead of "free". Hence consider a projective resolution as:

$$o \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow o \quad . \qquad\qquad (2)$$

We will term two modules M and N as equivalent if there exist projective modules P and Q such that $M \oplus P$ is isomorphic to $N \oplus Q$. We denote by P(M) the equivalence class of K in (2).

Let M be a module. We will call the class $P^n(M) = \{P_o, P_1, \ldots, P_n\}$, class of projective modules if there is a projective resolution

$$o \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_o \longrightarrow M \longrightarrow o .$$

We define the projective dimension, d(M), or $d_R(M)$, of M as the smallest positive integer n such that $P^n(M)$ is the class of projective modules. If there does not exist any such n, we will say that the projective dimension of M is ∞.

Definition. The global dimension of a ring R, written as D(R), is the supremum of d(M) taken over all R-modules M.

Let us state the Artin-Wedderburn theorem: "If every R-module is projective, then R is isomorphic to a finite product of $n_i \times n_i$ matrix rings over division rings D_i for some positive integers n_i." A ring of this type is called completely reducible.

It is easy to see that D(R) = o if, and only if R is a completely reducible ring (that is, a left Artinian ring with Jacobson radical zero). Also D(R) = 1 if, and only if R is a hereditary ring.

Let R[x] be a polynomial ring over R. The indeterminate x is in the centre of R and is a non-zero-divisor. Kaplansky ([11], p. 174) gives a very important result stating the relationship between the global dimensions of rings R and R[x]. The result reads as: " $D(R[x]) = 1 + D(R)$." This result shows that if R is a completely reducible ring, then global dimension of R[x] is one; hence R[x] is a semihereditary ring, which is, in particular, a P.P. ring.

We know that R is a left semihereditary if, and only if every finitely generated submodule of a projective R-module is projective (see [12], p. 80). Also if R is left hereditary, every submodule of a projective R-module is projective (see [12], p. 74). These and many other properties can be discribed in terms of module categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories (see [1], p. 251). Then a covariant function $F : \mathcal{C} \to \mathcal{D}$ is a <u>category equivalence</u> if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphism $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists a category equivalence from one to the other, and we denote it by $\mathcal{C} \approx \mathcal{D}$.

<u>Definition</u>. Two rings R and S are <u>(Morita) equivalent</u>, denoted by $R \approx S$ if the module categories ${}_RM \approx {}_SM$; that is, if there are additive equivalences between these categories of modules.

Hence if $R \approx S$, then the behviour of ${}_RM$ and ${}_SM$ is the same "to within isomorphism".

Let us denote by R_n, the n × n-matrix ring over a ring R. Properties of rings which can be expressed in module theoretic terms and which are preserved by equivalence of module categories, are preserved by Morita equivalence. Now $R \approx R_n$ for any $n \geq 1$. From this we get the following table:

R left P.P.	$\Leftarrow$	R left semihereditary	$\Leftarrow$	R left hereditary	$\Leftarrow$	R completely reducible
$\not\Downarrow$		$\Updownarrow$		$\Updownarrow$		
R_n left P.P.	$\Leftarrow$	R_n left semihereditary	$\Leftarrow$	R_n left hereditary		

The important point to note here will be that we do not have the above table completed for P.P. rings; that is, we do not have "R left P.P. if, and only if R_n left P.P.". This is proved by the famous Example 1.6, due to P.M. Cohn.

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