THE ANALYTIC REPRESENTATIONS OF THE SOLUTIONS OF SOME POTENTIAL-THEORETIC PROBLEMS

A thesis submitted

by

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to

the Faculty of Pure and Applied Science

of the University of Ottawa

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Mathematics

May 1967

ACKNOWLEDGEMENTS

I gratefully acknowledge the guidance and encouragement of Professor J.L. Howland in the preparation of this thesis.

I sincerely thank Professor V. Linis, Chairman, Department of Mathematics, University of Ottawa, for providing the opportunity for research.

Thanks are also due the Government of the Province of Ontario and the International Business Machines Corporation for the award of Research Fellowships which enabled me to complete this work.

I am grateful to D. Anand Kumar, Chairman, Managing Committee, Thapar Institute of Engineering and Technology, Patiala, India, for granting me study leave.

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INTRODUCTION

The Dirichlet and Neumann problems commonly known as the First and Second Boundary Value Problems of classical potential theory with respect to a closed, bounded surface are suggested by physical problems [4] as well as the identity of Green [10]. These problems have been extensively discussed mainly around the turn of the century.

This thesis is devoted to the study of generalizations of these problems, referred to herein as the Neumann-Poincaré and Robin-Poincaré [31; (i) Ch. VII] problems respectively. The purpose is to investigate the various analytical representations of the solutions of the pair of integral equations associated with these problems.

The whole project is being studied in the background of computational schemes of linear algebra, some of which have originated in the last two decades.

The motivation for the above investigation is to develop an existence and uniqueness theory best suited to numerical computation and which may be extended to surfaces with edges and corners or to infinite smooth surfaces.

This work is divided into four chapters. In the first chapter, the problem is formulated and Neumann's [27] method of the arithmetic mean, one of the earliest attempts in solving the Dirichlet problem with respect to a closed convex surface, is briefly discussed. Mention is made of Robin's

[34] attack on the Neumann problem, the method is on somewhat identical lines.

The assumption of convexity of the surface proved to be very restrictive. Poincaré, motivated by the efforts of Schwarz [38] and Picard [43] in tackling the problem of certain minimal surfaces and his own experience with the vibrating membrane problem [31; (iii), p. 119], was led to the introduction of a parameter λ in the boundary conditions, since the Laplace's equation did not contain a convenient parameter as was the case with the vibration problem. This was a significant step in the solution of the potential problem. Poincaré reformulated the First and Second Boundary Value Problems now called the Neumann-Poincaré and Robin-Poincaré problems respectively.

Formal solutions of these problems, in terms of spherical harmonics, in the case of the unit sphere, led Poincaré to conjecture that the solutions of these problems were similarly meromorphic functions of the parameter λ. Birkhoff [1] has noted important physical interpretations of the generalized single layer problem called the induced potential problem, which corresponds to $|\lambda| < 1$ in the Robin-Poincaré problem.

The methods of Neumann [27] and Robin [34] were reformulated by Poincaré [31; (ii)] and the required poten-

tials were sought in the form of infinite series in powers of λ, the coefficients being potentials of double and single layers respectively. The convergence of these series was established by the method of the arithmetic mean, in the case of convex surfaces.

One of the direct methods of solving the pair of adjoint integral equations associated with these problems, is the method of successive substitution. The solution is obtained in terms of Neumann series in powers of λ and gives the corresponding moment of the double layer and the density of the single layer respectively.

Here an expression for truncation error is given, when a finite number of terms of this series is taken for the solution. The error estimate is given in terms of the sequence which arises when a special algorithm, the so-called quotient-difference algorithm, is used. The notable feature of this estimate is that it avoids the conventional form of error estimates involving derivatives of integrands. Furthermore a repeated application of the above generates an interesting algorithm. The error estimate, or for that matter, the solution of the integral equation, may be represented as a continued fraction, the elements of which are terms of certain sequences that are minimizing sequences in the sense above. The difficulties that remain to be overcome in this approach will be touched upon later.

In the second chapter, a résumé of important results on integral equations due to Fredholm [8; (i), (ii)] is given. The solution, in two dimensions at least, is given as quotient of entire functions. Plemelj [30] has thoroughly analyzed, the integral equations of potential theory, by the method of the Fredholm resolvent. The principal results, valid in two and three dimensions, are also noted. It is also shown by a simple method that the Fredholm resolvent is an analytic continuation of the Neumann and Robin series. The chapter concludes with the theorems of Marty [24] and their abstract interpretation.

In the third chapter the analytic representation of the solution from the eigen-value aspect is given in terms of Mittag-Leffler expansions. The connection between the residue at the pole and the solution of the corresponding homogeneous equation can easily be established. Plemelj [30, (ii)] has given a canonical decomposition near a singular point. Here a non-trivial extension is made to the case of a finite number of simple poles, and some further properties of the residues at these poles are noted. This effectively yields an approximation to the solution of the integral equation. The reason why we call it an approximation is that we retain only a finite number of terms in the expansion, the entire spectrum being denumerably infinite.

The Robin-Poincaré and the Neumann-Poincaré problems have been solved by means of the so-called fundamental functions of Poincaré. These functions are, in general, potentials of single layers with densities proportional to the eigen-functions of the Fredholm-Poincaré integral equations. In case of a co-ordinate surface in a system of co-ordinates in which the Laplace's equation separates, the method of separation of variables furnishes examples of fundamental functions. It may be that in the case of infinite co-ordinate surfaces or surfaces with corners, partial separability of Laplace's equation obtains. The thing to look for is the continuous spectrum as is the case in two-dimensions discussed by Carleman [5, (iii)].

The original discussion of the eigen-value aspect of the integral equations of potential theory is due to Blumenfeld and Mayer [2] who establish the existence of Poincaré fundamental functions and prove the convergence of the Mittag-Leffler expansions in the interior and exterior regions bounded by a closed, smooth surface, but not on the surface itself. The completion at this point has recently been achieved by Howland and Vaillancourt [18] who establish that the series indeed converges everywhere, uniformly and absolutely. They use a functional analytic approach and employ the fact that the kernel of the integral equation, corresponding to the Robin-Poincaré problem, is symmetrizable

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on the left, by a positive definite and symmetric transformation. The transformation in question gives rise to potentials due to single layers. Furthermore the second iterated kernel is completely continuous on the prehilbert space of continuous functions on the bounding surface in the ordinary sense, with L^2 norm. Then arguments due to Reid [32] and Mercer [25] are utilized to achieve the desired result.

Here we construct linear subspaces of increasing dimension, spanned by functions generated by the minimized iteration scheme [22] and endowed with the Dirichlet norm. An invariant subspace of infinite dimensions is defined, which is the closure, with respect to the norm, of the union of the linear finite dimensional subspaces and which is complete with respect to this norm.

The procedure of constructing the orthonormal system by the above scheme gives, for the kernel of the integral equation, a symmetric, tridiagonal matrix representation of infinite order. The finite 'sections' of this matrix are employed in the approximate solution of the homogeneous and non-homogeneous integral equations.

Here the existence of eigen-values is established with the aid of the maximum principle, which is concerned with the maximum of the Rayleigh quotient [15; p. 175] and some modification of Reid's [32] methods. The argu-

ments are made to depend upon the Fredholm alternative and the discreteness of the spectrum. The mini-max property [15; p. 31-33] of the eigen-values is also stated which implies the monotonicity of the approximate eigen-values when successive finite sections of the tridiagonal matrix, representing the kernel, are taken. The extremizing sequences yield eigen-functions.

Some expansion theorems are also stated which help to prove an important identity. Arguments due to Riesz and Nagy [33; p. 234-235] together with the maximum principle combine to establish the complete continuity of the linear transformation, associated with the potential theory, with respect to the Dirichlet norm.

However, Mikhlin [42; p. 78] establishes complete continuity of a non-symmetric, weakly singular transformation, with respect to the L^2 norm. This provides an alternate approach, since Reid's [32] results are then directly applicable, to establish the existence of eigen-functions. But for the present, from the computational aspect, the former view point is being pursued.

It is noticed that the assumptions on the linear transformation, made by Karush [20] are met by the linear transformation, associated with potential theory. In view of the equivalence, established by him, of the projection technique and the minimized iteration scheme, analogous

results on approximate solution, eigen-values and eigen-functions with truncation-error estimate have been stated. The above arguments depend on the equivalence of two definitions of complete continuity.

The results suffer from the fact that only convergence in norm obtains. It is conjectured that the convergence is in fact, pointwise and uniform on the surface.

In the fourth chapter a rational approximation to the solution of the integral equation has been carried out using Lanczos' [22] method of minimized iteration outlined above. The convergence is noted to be geometric with small ratio. Rational approximation to the solution is equivalent to the approximation of the kernel by degenerate kernels.

The four different analytic representations of the solution of integral equation, noted above, suggest that they may be linked with one parent representation, that of a continued fraction. Such a representation would have some advantages. The convergence properties may be formidable, in view of the dependence, of the elements of the continued fraction on the point of the surface and the parameter λ.

CHAPTER I

PROBLEMS OF CLASSICAL POTENTIAL THEORY

1. Statement of the Problem

This thesis is concerned with the First and Second Boundary Value Problems of classical potential theory in three dimensions.

Let R denote a finite region of three-dimensional space, bounded by a surface S of class B [23] and R' its exterior.

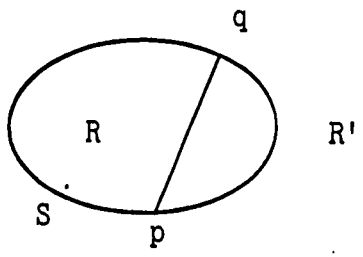

Fig. 1

The assumptions on the surface S assure the validity of the formulae of Green [21; pp. 212-218] and the Divergence Theorem [21; p. 113]. We will denote points on S by small letters p,q, etc. and those not on S, by capital letters P,Q, etc. P may be a point of R or R' according as the interior or exterior region is implied.

The Dirichlet Problem or the First Boundary Value Problem [21; p. 236].

We wish to find a function W(P) which is harmonic in

the regions R or R', subject to the conditions:

i) W(P) is regular at infinity, if $P \in R'$,

ii) on S, it takes pre-assigned continuous value, say, g(p).

In the two-dimensional case,

$$W(P) = 0 \text{ at infinity, for } P \in R'.$$

The Neumann Problem or the Second Boundary Value Problem [21; p. 246]

We wish to find a function V(P) which is harmonic in the region R or R' subject to the conditions:

i) V(P) is regular at infinity, if $P \in R'$,

ii) the normal derivative on S takes pre-assigned value, say, f(p).

In the case of two dimensions we assume that f(p) satisfies the condition

$$\int_S f(p)\, dS = 0,$$

where dS denotes the element of the arc, and add to the condition i) the requirement V(P) = 0 at infinity.

I shall consider these problems only in three dimensions.

2. The Potential of a Single Layer

The potential V(P) of a single layer of density $\mu(q)$ on S is defined by [21; p. 55]

$$V(P) = \frac{1}{2\pi} \int_S \mu(q)\, \frac{1}{r_{Pq}}\, dS_q , \tag{1}$$

where r_{Pq} is the distance between the points $P \in R$ or R' and $q \in S$ and dS_q is the element of surface area at q, the integration is taken over the whole surface S.

V(P) has important properties when $\mu(q)$ satisfies different conditions:

(i) If $\mu(q)$ is a continuous function of $q \in S$, then V(P) is also defined at points $P = p$ of S and is a continuous function of P as P crosses S [9; pt I, p. 247]. In particular the boundary values V(p) of V(P) on S are continuous. Furthermore, V(P) is harmonic throughout R and R' and is regular at infinity. Then, the limiting values

$$\frac{\partial V(p_o)}{\partial n_-}, \frac{\partial V(p_o)}{\partial n_+}$$

exist at every point of S, where n denotes the outward normal to S at p_o and the limits are as $P \to p_o$ from within or without S, and are taken uniformly with respect to p_o. These limiting values are continuous functions of position on S and are related [30; (ii) pp. 349-353] by the following:

$$(2) \qquad \frac{1}{2}\left(\frac{\partial V(p)}{\partial n_-} - \frac{\partial V(p)}{\partial n_+}\right) = \mu(p);$$

$$(3) \qquad \frac{1}{2}\left(\frac{\partial V(p)}{\partial n_-} + \frac{\partial V(p)}{\partial n_+}\right) = \frac{1}{2\pi}\int_S \mu(q)\,\frac{\partial}{\partial n_p}\left(\frac{1}{r_{pq}}\right) dS_q,$$

where n_p denotes the outward normal at p.

(ii) Suppose that $\mu(q)$ satisfies a Hölder condition [17; pp. 36-42] uniformly on S.

Then, the function V(P) is continuously differentiable throughout R and R'. The tangential derivatives of V(P) are continuous [17; pp. 36-42] as P crosses S.

(iii) The existence of derivatives of V(P) of higher orders has been discussed by E. Schmidt [37, (i); (ii)]. This requires that S and $\mu(q)$ possess derivatives of higher orders.

Potentials of this type are also sometimes referred to as due to simple distribution. They are normally employed in solving the Second Boundary Value Problem in which we seek a harmonic function in R or R' whose normal derivative is given on S.

3. The Potential of a Double Layer

The potential W(P) of a double layer of moment $\nu(q)$ on S is defined [21; p. 66] by

$$W(P) = \frac{-1}{2\pi} \int_S \nu(q) \frac{\partial}{\partial n_q} \left(\frac{1}{r_{Pq}}\right) dS_q , \tag{4}$$

where n_q is the direction of the outward normal at q, and r_{Pq} is the distance between P and q.

W(P) has important properties when $\nu(q)$ satisfies certain conditions:

(i) Suppose that $\nu(q)$ is a continuous function of the co-ordinates of $q \in S$. Then, W(P) is harmonic [9; pt I, p. 250] throughout R and R', and is regular at infinity [21; p. 216]. Furthermore, W(P) is continuous in the regions

[21; p. 168] R + S and R' + S, but experiences a jump when P crosses S. This discontinuity is expressed [30; (ii) pp. 355-357] by the relations:

$$(5) \qquad \frac{1}{2}\,(W_+(p) - W_-(p)) = \nu(p);$$

$$(6) \qquad \frac{1}{2}\,(W_+(p) + W_-(p)) = \frac{1}{2\pi}\int_S \nu(q)\,\frac{\partial}{\partial n_q}\,(\frac{1}{r_{pq}})\,dS_q$$

In addition, the following limit [21; p. 170] holds uniformly with respect to p.

$$\lim_{\varepsilon\to 0}\,(\frac{\partial W(p_+)}{\partial n} - \frac{\partial W(p_-)}{\partial n}) = 0,$$

where p_+ and p_- are points of R' and R lying on the normal to S at p at equal distance ε from p.

(ii) Suppose that $\nu(q)$ has continuous partial derivatives of second order on S, then, W(P) possesses continuous normal derivatives [21; p. 172] in the closed regions S_+ and S_- constructed from S by moving along the normal a distance $+\varepsilon$ and $-\varepsilon$ at each point of S, i.e. $\frac{\partial W}{\partial n}$ is continuous on S and Green's formula may be applied to W(P).

(iii) Suppose that $\nu(q)$ is bounded and integrable, then W(p) satisfies a Hölder condition [21; p. 300] uniformly on S.

(iv) Suppose that $\nu(q)$ satisfies a Hölder condition uniformly with respect to S, then W(p) also satisfies a Hölder condition with a different exponent [36; (i)].

Potentials of the type W(P) arise in magnetic distributions. They are normally employed in the solution of the First Boundary Value Problem i.e. the Dirichlet Problem.

4. The Method of Arithmetic Mean

C. Neumann was the first mathematician to seek a solution of the Dirichlet Problem with respect to the region R in the form of the potential of a double layer [27; (i), (ii)] on S. Neumann's method, called the Method of Arithmetic Mean (see for summary [21; pp. 281-283]) is applicable to closed smooth, convex surfaces S. Briefly the argument is as follows:

Let $W^o(p)$ denote the value of the following surface integral

$$W^o = \frac{1}{2\pi} \int_S \nu(q) \frac{\partial}{\partial n_q} \left(\frac{1}{r_{pq}}\right) dS_q; \tag{7}$$

then by virtue of the relation [21; p. 68]

$$\frac{\partial}{\partial n_q} \left(\frac{1}{r_{pq}}\right) dS_q = -d\Omega;$$

where $d\Omega$ is the element of the solid angle subtended at p by the element of the surface S at q, we have

$$-W^o = \frac{1}{2\pi} \int \nu(q)\, d\Omega. \tag{8}$$

Let W^- and W^+ denote the limiting values of W, then

$$W^- = -\nu + W^o; \quad W^+ = \nu + W^o \tag{9}$$

The integral in (8) may be interpreted geometrically as the arithmetic mean of the values of ν, transferred along the radii, to the hemisphere of the unit sphere about p. Since S is convex it follows that the extreme values of $-W^o$ lie strictly between those of the moment of the double distribution on the surface, and thus vary less in this sense, than do those of ν.

The method involves the construction of a sequence of double layer potentials by iteration, thus

$$W_1 = -\frac{1}{2\pi}\int_S \nu(q)\,\frac{\partial}{\partial n_q}\left(\frac{1}{r_{pq}}\right) dS_q;$$

$$W_2 = -\frac{1}{2\pi}\int_S W_1^o\,\frac{\partial}{\partial n_q}\left(\frac{1}{r_{pq}}\right) dS_q;$$

. . . . . .

$$W_n = -\frac{1}{2\pi}\int_S W_{n-1}^o\,\frac{\partial}{\partial n_q}\left(\frac{1}{r_{pq}}\right) dS_q; \quad \text{etc.}$$

which yield the limiting relations:

$$W_1^- = \nu + W_1^o \;;\; W_1^+ = -\nu + W_1^o;$$

$$W_2^- = W_1^o + W_2^o \;;\; W_2^+ = -W_1^o + W_2^o;$$

. . . . . .

$$W_n^- = W_{n-1}^o + W_n^o \;;\; W_n^+ = -W_{n-1}^o + W_n^o;$$

etc.

Neumann showed from the arithmetic mean property that

$$W_n^o \to c, \text{ a constant}$$

as $n \to \infty$, whence the uniform convergence of

$$\sum_{1}^{\infty} (W^{-}_{2n-1} - W^{-}_{2n}) \to \nu - c \ .$$

Consequently the series of potentials

$$\sum_{1}^{\infty} (W_{2n-1} - W_{2n}) + c = W_i$$

solves the Dirichlet Problem for the region R.

For the region R', the solution [21; p. 283] is given as

$$\sum_{1}^{\infty} W_n + \frac{c}{C} \left(\frac{1}{r} + C - V\right) = W_e + c$$

where V is regular at infinity.

A similar method for the solution of the Second Boundary Value Problem or the Neumann Problem with respect to a convex surface S, in terms of the potentials of single layer, was developed by Robin [34; (ii), (iii)].

5. Integral Equations of Potential Theory

Poincaré [31; (ii)] generalized the method of Robin and Neumann in a significant way and reformulated the problems with the introduction of a parameter λ into the boundary conditions.

The Robin-Poincaré Problem [31; (i) p. 134]

Let S be the surface defined in Section 1 and λ a given parameter. Let f(p) be a given continuous function of $p \in S$. We wish to find the potential V(P) of a single

layer on S which satisfies the condition

$$(10)\quad \frac{\partial V(p)}{\partial n_-} - \frac{\partial V(p)}{\partial n_+} = \lambda\left(\frac{\partial V(p)}{\partial n_-} + \frac{\partial V(p)}{\partial n_+}\right) + 2\, f(p) \text{ on } S.$$

We may note that no restrictions are made upon the behaviour of V(P) at infinity. The Second Boundary Value Problem or the Neumann Problem (see [21; p. 287]) is a particular case of the Robin-Poincaré Problem. For the region R, we have $\lambda = -1$ and f(p) as the value of the normal derivative of V on the boundary and for the region R', $\lambda = 1$, and f(p) is taken as the negative of the given value.

The Neumann-Poincaré Problem [31; (i), p. 291]

Let g(p) be a given continuous function defined for points $p \in S$. We wish to find the potential W(P) of a double layer on S which satisfies the condition

$$(11)\quad W_+(p) - W_-(p) = \lambda(W_+(p) + W_-(p)) + 2\, g(p) \text{ on } S.$$

Solutions of the Neumann-Poincaré Problem are regular at infinity (zero at infinity in two dimensions).

As before, the First Boundary Value Problem or Dirichlet Problem is a particular case of the Neumann-Poincaré Problem (see [21; p. 286]). For the region R, $\lambda = 1$ and g(p) = -(boundary value); and for R', $\lambda = -1$, and g(p) = (boundary value).

The Fredholm-Poincaré Integral Equations

Suppose that

$$(1)\qquad V(P) = \frac{1}{2\pi} \int \mu(q)\, \frac{1}{r_{Pq}}\, dS_q$$

and

$$(4) \qquad W(P) = \frac{1}{2\pi} \int \nu(q) \frac{\partial}{\partial n_q} \left(\frac{1}{r_{Pq}}\right) dS_q$$

are solutions of Robin-Poincaré and Neumann-Poincaré Problems respectively. As potentials of single and double layers V(P), W(P) satisfy conditions (2); (3) and (5); (6) respectively. Using (10) and (11) we see that $\mu(q)$ and $\nu(q)$ satisfy the pair of adjoint integral equations,

$$(12) \qquad \mu(p) = f(p) + \frac{\lambda}{2\pi} \int \mu(q) \frac{\partial}{\partial n_p} \left(\frac{1}{r_{pq}}\right) dS_q ;$$

$$(13) \qquad \nu(p) = g(p) + \frac{\lambda}{2\pi} \int \nu(q) \frac{\partial}{\partial n_q} \left(\frac{1}{r_{pq}}\right) dS_q ,$$

respectively.

6. Properties of the Kernel of Potential Theory

The Fredholm-Poincaré Integral equations (12) and (13) may be written as

$$(14) \qquad \mu(p) = f(p) + \lambda \int K(p, q)\, \mu(q)\, dS_q$$

and

$$(15) \qquad \nu(p) = g(p) + \lambda \int \nu(q)\, K(q, p)\, dS_q ,$$

respectively, where

$$(16) \qquad K(p, q) = \frac{1}{2\pi} \frac{\partial}{\partial n_p} \left(\frac{1}{r_{pq}}\right).$$

and

$$(17) \qquad K(q, p) = \frac{1}{2\pi} \frac{\partial}{\partial n_q} \left(\frac{1}{r_{pq}}\right)$$

K(p, q) and K(q, p) are called the kernels of the

integral equations (14) and (15) respectively. In case f(p) and g(p) are zero, we have a pair of adjoint homogeneous integral equations

$$\mu(p) = \lambda \int K(p, q)\, \mu(q)\, dS_q; \tag{18}$$

$$\nu(p) = \lambda \int \nu(q)\, K(q, p)\, dS_q. \tag{19}$$

An elegant and powerful method for proving the existence and studying the properties of the solutions of (14) and (15) is due to Fredholm [8, (i), (ii)]. This method, in particular, has been applied in detail, by Plemelj [30] to the integral equations (14) and (15) of the Classical Potential Theory. A brief summary of these results is given in Chapter II. Here we only note some important properties [21; pp. 299-300] common to the kernels K(p, q) and K(q, p), given by (16) and (17) respectively.

(i) K(p, q) is continuous for $p \neq q$

(ii) $|K(p, q)| \leq \frac{M}{r_{pq}}$; $r_{pq} \neq 0$, holds for all S, M being a constant

(iii) K(p, q) has continuous derivatives of the first order with respect to the co-ordinates of projection of p and q on any fixed tangent plane to S at a point and is subject to the inequality

$$\left|\frac{\partial}{\partial s} K(p, q)\right| \leq \frac{M}{r_{pq}^2}, \quad r_{pq} \neq 0,$$

where s is any arc on the surface S.

(iv) Let f(p) be a continuous function.

Then, $$\phi(p) = \int K(p, q)\, f(q)\, dS_q,$$

$$\psi(p) = \int f(q)\, K(q, p)\, dS_q,$$

satisfy uniform Hölder condition on S. If F is a bound of $|f(p)|$, there is a constant A independent of these functions such that [21; p. 300]

$$|\phi(p)| \leq AF \; ; \quad |\psi(p)| \leq AF.$$

(v) The second iterated kernel

$$K^{(2)}(p, q) = \int K(p, q_1)\, K(q_1, q)\, dS_{q_1}$$

is continuous for $p \neq q$, and

$$|K^{(2)}(p, q)| \leq B \log \frac{M}{r_{pq}}$$

holds uniformly on S. B and M are constants [21; p. 303]

(vi) The third iterated kernel

$$K^{(3)}(p, q) = \int K(p, q_1)\, K^{(2)}(q_1, q)\, dS_{q_1}$$

is everywhere continuous on S [21; p. 301]

(vii) The order of integration in iterated integrals over integrands containing K(p, q) as a factor may be inverted [21; p. 304]

(viii) Property (iv) holds if in the integrals any iterated kernel $K^{(i)}(p, q)$ be substituted for K(p, q) [21; p. 307]

(ix) The geometric interpretation of the kernels is simple, since

$$K(p, q) = \frac{1}{2\pi} \frac{\partial}{\partial n_p} \left(\frac{1}{r_{pq}}\right) = \frac{1}{2\pi} \frac{\cos(n_p, r_{pq})}{r_{pq}^2},$$

$$K(q, p) = \frac{1}{2\pi} \frac{\partial}{\partial n_q} \left(\frac{1}{r_{pq}}\right) = \frac{1}{2\pi} \frac{\cos(n_q, r_{pq})}{r_{pq}^2},$$

where $\cos(n_p, r_{pq})$ denotes the cosine of the angle between the inward normal at p and the chord $\vec{pq}$. Similarly $\cos(n_q, r_{pq})$ denotes the cosine of the angle between the inward normal at q and the chord $\vec{qp}$.

7. Neumann's Method (Method of Successive Approximations)

One of the direct methods of solution of the equations of potential theory is the Neumann Method [27; (i), (ii)] and is commonly known as the method of successive approximations. We may write equations (14) and (15) in a symbolic form as

$$\mu(p) = f(p) + \lambda K\mu(q), \tag{20}$$

and

$$\nu(p) = g(p) + \lambda K^* \nu(q), \tag{21}$$

corresponding to the Robin-Poincaré and Neumann-Poincaré Problems. For purposes of illustration we treat equation (20) and seek the solution in the form

$$\mu(p) = f_0(p) + \lambda f_1(p) + \lambda^2 f_2(p) + \ldots . \tag{22}$$

Substituting this in (20)

$$f_0(p) + \lambda f_1(p) + \lambda^2 f_2(p) \ldots =$$

$$= f(p) + \lambda \int_S K(p, q)[f_0(q) + \lambda f_1(q) \ldots] dS_q$$

and comparing the coefficients of like powers of λ, we have

$$f_0(p) = f(p)$$

$$f_1(p) = \int K(p, q) f(q) dS_q;$$

$$f_2(p) = \int K(p, q) f_1(q) dS_q;$$

. . . . . .

Introducing the iterated kernels

$$K^{(1)}(p, q) = K(p, q)$$

$$K^{(2)}(p, q) = \int K(p, q_1) K(q_1, q) dS_{q_1};$$

$$K^{(3)}(p, q) = \int K(p, q_1) K^{(2)}(q_1, q) dS_{q_1};$$

. . . . . .

we may write

$$f_0(p) = f(p)$$

$$f_1(p) = \int K(p, q) f(q) dS_q.$$

More generally:

$$(23) \quad f_n(p) = \int K^{(n)}(p, q) f(q) dS_q, \quad n = 1, 2, \ldots$$

and the series solution of (20) becomes

$$\mu(p) = f(p) + \lambda \int K^{(1)}(p,q) f(q) dS_q + \lambda^2 \int K^{(2)}(p,q) f(q) dS_q + \ldots$$

which may be written in a compact form indicating the dependence of the solution on the parameter λ in the following form:

$$(24)\quad \mu(p,\lambda) = f(p) + \lambda K^{(1)}f + \lambda^2 K^{(2)}f + \lambda^3 K^{(3)}f + \dots .$$

Let us define $R(p, q, \lambda)$ by the following relation:

$$(25)\quad R(p,q,\lambda) = K^{(1)}(p,q) + \lambda K^{(2)}(p,q) + \lambda^2 K^{(3)}(p,q) + \dots$$

and call it the Neumann Resolvent for the kernel $K(p,q)$. Since the series of continuous functions on the right hand side of (24) converges uniformly, term by term integration is permissible, and we may write the solution as:

$$(26)\quad \mu(p,\lambda) = f(p) + \lambda \int R(p, q, \lambda)\, f(q)\, dS_q .$$

A similar treatment may be given to (21) and the solution written as an infinite series in powers of λ,

$$(27)\quad \nu(p, \lambda) = g(p) + \lambda\, K^{*(1)}g + \lambda^2\, K^{*(2)}g + \dots$$

The analytic properties of $\mu(p,\lambda)$ and $\nu(p, \lambda)$ will be discussed, in the following pages, from various aspects. The Taylor series representation of the solution (24) or (27) is valid for small values of the parameter λ and thus heavily restricts its usefulness.

8. Poincaré Method (Method of Undetermined Coefficients

A more informative method of solution of the Robin-Poincaré and Neumann-Poincaré Problem is the method of Poincaré [31; (ii), (iii)] or the method of undetermined coefficients in which the required potentials $V(P)$ and $W(P)$ are sought in the form of the series

$$(28) \qquad V(P) = V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \dots ,$$

$$(29) \qquad W(P) = W_0 + \lambda W_1 + \lambda^2 W_2 + \lambda^3 W_3 + \dots ,$$

where V(P) and W(P) satisfy the boundary conditions (10) and (11) respectively;

$$(10) \qquad \frac{1}{2}\left(\frac{\partial V}{\partial n_-} - \frac{\partial V}{\partial n_+}\right) - \frac{\lambda}{2}\left(\frac{\partial V}{\partial n_-} + \frac{\partial V}{\partial n_+}\right) = f,$$

$$(11) \qquad \frac{1}{2}(W_+ - W_-) - \frac{\lambda}{2}(W_+ + W_-) = g.$$

Upon substitution of (28) and (29) in (10) and (11) and comparing coefficients, we have

$$(30) \quad \begin{cases} \dfrac{1}{2}\left(\dfrac{\partial V_0}{\partial n_-} - \dfrac{\partial V_0}{\partial n_+}\right) = f; \\ \dfrac{1}{2}\left(\dfrac{\partial V_1}{\partial n_-} - \dfrac{\partial V_1}{\partial n_+}\right) - \dfrac{1}{2}\left(\dfrac{\partial V_0}{\partial n_-} + \dfrac{\partial V_0}{\partial n_+}\right) = 0 ; \\ \dfrac{1}{2}\left(\dfrac{\partial V_2}{\partial n_-} - \dfrac{\partial V_2}{\partial n_+}\right) - \dfrac{1}{2}\left(\dfrac{\partial V_1}{\partial n_-} + \dfrac{\partial V_1}{\partial n_+}\right) = 0 ; \\ \dots\dots\dots\dots\dots\dots \\ \text{etc.} \end{cases}$$

and

$$(31) \quad \begin{cases} \dfrac{1}{2}(W_{0_+} - W_{0_-}) = g; \\ \dfrac{1}{2}(W_{1_+} - W_{1_-}) - \dfrac{1}{2}(W_{0_+} + W_{0_-}) = 0; \\ \dfrac{1}{2}(W_{2_+} - W_{2_-}) - \dfrac{1}{2}(W_{1_+} + W_{1_-}) = 0; \\ \dots\dots\dots\dots\dots\dots \\ \text{etc.} \end{cases}$$

Thus the V_i and W_i may be constructed recursively, as single and double layer potentials respectively. The method of arithmetic mean establishes the convergence of the series

$$W - \frac{c}{1-\lambda}; \quad |\lambda| < 1 \tag{32}$$

in the case of a convex surface S, where c is a constant.

In this procedure we successively approximate the potentials V(P) and W(P) as opposed to the Neumann Method in which the $\mu(p)$ and the $\nu(q)$ are successively approximated and then the potentials V(P) and W(P) evaluated through (1) and (4).

9. Method of Schwarz

Schwarz [38] was the first to demonstrate the existence of the fundamental harmonic of a membrane while discussing the vibration problem. His method may be related to the present situation to show that the Neumann series (24) has at least one pole [9; pt II, p. 101].

Let

$$\mu(p,\lambda) = f_0(p) + \lambda f_1(p) + \lambda^2 f_2(p) + \lambda^3 f_3(p) + \dots , \tag{24}$$

where

$$f_n(p) = \int K^{(n)}(p,q)\, f_0(q)\, dq; \quad f_0(p) = f(p) \tag{23}$$

be the Neumann series solution of the Robin-Poincaré Problem. Multiplying (24) by a bounded function h(p) and integrating term-wise, we have

(33) $$(\mu(p, \lambda), h) = a_0 + \lambda a_1 + \lambda^2 a_2 + \ldots$$

where

$$a_n = (f_n, h) = \int f_n(p)\, h(p)dp.$$

Suppose that (24) has no pole, then (33) is convergent everywhere. This is not true, since we may choose f and h such that

(i) The a's in (33) are positive and satisfy Schwarz' inequality

$$a_n^2 \leq a_{n-1}\, a_{n+1}.$$

(ii) The ratio $\frac{a_n}{a_{n-1}}$ increases with n, but being bounded above, tends to a finite limit ℓ,

(34) $$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \ell \neq 0.$$

Thus (33) has a finite radius of convergence $\lambda_0 = \frac{1}{\ell}$ and hence a singularity. Therefore (24) has a pole and consequently the Neumann resolvent (25) too, has a pole.

Furthermore, if

(35) $$\lim_{n\to\infty} \lambda_0^n f_n = \mu_0$$

then μ_0 is a solution of the homogeneous equation (18) corresponding to λ_0. μ_0 is called the eigen-function corresponding to the eigen-value λ_0.

On somewhat similar lines Schmidt [37; (i)] proved that a real symmetric kernel possesses an eigen-value and hence a countable discrete spectrum of eigen-values. This model helped Marty [24; (i), (ii)] to establish the existence of a spectrum for kernels of Fredholm type that are symmetrizable by a symmetric definite kernel.

Blumenfeld and Mayer [2] have applied Marty's arguments to the equations of potential theory using a symmetrizing kernel noted by Plemelj [30; (i)]. They obtained a proof of the existence of Poincaré's fundamental functions and established expansion theorems for the Robin-Poincaré and Neumann-Poincaré Problems. These results and recent improvements [40] are discussed in Chapters II and III.

10. Error Estimation in Truncation of Neumann Series Solution

Now we seek bounds for the error in the solution of

$$\mu(p) = f + \lambda K\mu \tag{20}$$

when the first n terms of the Neumann series (24) are taken for its solution. The bound is given in terms of an iteration sequence which arises when a special algorithm, the so-called quotient-difference algorithm [35, 14] is used. This algorithm is a generalization of the Bernoulli-Aitken method of finding the root of the smallest modulus of a given polynomial.

Theorem: If μ_n denotes the nth. approximation of

$$\mu = f_0 + \lambda f_1 + \lambda^2 f_2 + \ldots \quad , \tag{24}$$

the Neumann series solution of (20), then

$$(36) \qquad |\mu - \mu_n| < \frac{|\lambda|^n |f_n|}{\sum_{i=0} x_i \lambda^i}$$

where $\{x_i\}$ is the iteration sequence generated by the quotient-difference scheme for the smallest root of

$$(37) \qquad f_n + \lambda f_{n+1} + \lambda^2 f_{n+2} + \ldots = 0.$$

Proof: Let

$$\mu_n = f + \lambda f_1 + \ldots + \lambda^{n-1} f_{n-1}$$

and

$$\mu_{n+1} = f + \lambda f_1 + \ldots + \lambda^{n-1} f_{n-1} + \lambda^n f_n;$$

obviously,

$$(38) \qquad \frac{\mu - \mu_{n+1}}{\mu - \mu_n} = \frac{\lambda f_{n+1} + \lambda^2 f_{n+2} + \ldots}{f_n + \lambda f_{n+1} + \ldots}.$$

Let the series expansion of the right hand side of (38) be

$$\sum_{m=0} a_m \lambda^{m+1}$$

where

$$(39) \quad a_m = \begin{vmatrix} f_{n+1} & f_n & 0 & 0 \\ f_{n+2} & f_{n+1} & ' & 0 \\ ' & ' & ' & ' \\ ' & ' & & f_n \\ f_{n+m+1} & f_{n+m} & \cdots & f_{n+1} \end{vmatrix} / (-1)^m (f_n)^{m+1}$$

$m = 0,1,2, \ldots$

If we try to solve

$$f_n + \lambda f_{n+1} + \lambda^2 f_{n+2} + \ldots = 0.$$

for the smallest root with the help of the quotient-difference algorithm we are led to a sequence $\{x_i\}$ where

$$\lim_{i\to\infty} \frac{x_{i+1}}{x_i} = \frac{1}{\lambda_0} , \tag{40}$$

and x_i are given by

$$x_i = \begin{vmatrix} f_{n+1} & f_n & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & f_n \\ f_{n+i} & f_{n+i-1} & \cdots & f_{n+1} \end{vmatrix} \Big/ (-1)^i (f_n)^i, \tag{41}$$

for $i = 1, 2, \ldots$.

Comparing (39) and (41), we have

$$\begin{cases} x_1 = -a_0 = -\dfrac{f_{n+1}}{f_n} \\ x_2 = -a_1 = \begin{vmatrix} f_{n+1} & f_n \\ f_{n+2} & f_{n+1} \end{vmatrix} \Big/ f_n^2 \\ \cdots\cdots\cdots\cdots\cdots \\ x_i = -a_{i-1} \end{cases} \tag{42}$$

therefore we can write (38) as

$$\frac{\mu - \mu_{n+1}}{\mu - \mu_n} = -\sum_{i=1} x_i \lambda^i .$$

Hence $$\mu\left(1 + \sum_{i=1} x_i \lambda^i\right) = \mu_{n+1} + \mu_n \sum_{i=1} x_i \lambda^i,$$

$$\mu = \frac{\mu_{n+1} + \mu_n \sum_{i=1} x_i \lambda^i}{1 + \sum_{i=1} x_i \lambda^i},$$

$$\mu = \frac{\mu_n(1 + \sum_{i=1} x_i \lambda^i) + (\mu_{n+1} - \mu_n)}{1 + \sum_{i=1} x_i \lambda^i},$$

$$\mu = \mu_n + \frac{\lambda^n f_n}{\sum_{i=0} x_i \lambda^i}. \tag{43}$$

Here we have substituted the value

$$\mu_{n+1} - \mu_n = \lambda^n f_n ,$$

and taken $x_0 = 1$, in accordance with the algorithm.

Thus, we get the bound for the error as

$$|\mu - \mu_n| < \frac{|\lambda|^n |f_n|}{\sum_{i=0} x_i \lambda^i} \qquad \text{q.e.d.}$$

It is of interest to note that the above expression for the truncation error avoids the conventional form employing the derivatives of the integrand, which in the present case is not of much help, since the integrand involves the unknown quantity μ. Furthermore, a repeated application of (36) generates an interesting algorithm and the error may be evaluated in terms of certain sequences analogous to $\{x_i\}$ that are minimizing sequences in the sense above. The result may formally be written in the form of a continued fraction, whose elements are given by the above sequences.

CHAPTER II

THE METHOD OF I. FREDHOLM AND H. MARTY

11. The Fredholm Theory

We briefly outline the method of solving integral equations originally due to Fredholm [8; (i), (ii)]. Main results are stated from the point of view of their application to Potential Theory [21; Ch. XI]; [33; Ch. IV].

Let us consider the pair of adjoint integral equations:

$$\phi(p) = \lambda \int_S K(p, q)\, \phi(q)\, dS_q + f(p) \tag{44}$$

$$\psi(p) = \lambda \int_S K(q, p)\, \psi(q)\, dS_q + f(p) \tag{45}$$

where p, q represent points in a finite dimensional space, and the integrals are taken over a closed bounded region S of this space.

Hypothesis: We suppose that

1. The kernel K(p, q) is a given, bounded continuous, real function of the co-ordinates p and q; and is defined for $p, q \in S$.

2. The given non-homogeneous term f(p) is a continuous real function of the co-ordinates of p S.

Definitions: We define for n = 1, 2, 3, ...

$$K\begin{pmatrix} p_1 \cdots p_n \\ q_1 \cdots q_n \end{pmatrix} = \begin{vmatrix} K(p_1,q_1) & K(p_1,q_2) & \cdots & K(p_1,q_n) \\ \vdots & & & \vdots \\ K(p_n,q_1) & \cdots & & K(p_n,q_n) \end{vmatrix},$$

$$\delta_n = \frac{1}{n!} \iint \cdots \int K \begin{pmatrix} p_1 \cdots p_n \\ p_1 \cdots p_n \end{pmatrix} dS_{p_1} \cdots dS_{p_n},$$

$$\delta_0 = 1$$

$$N_0(p,q) = K(p, q)$$

$$N_n(p,q) = \frac{1}{n!} \iint \cdots \int K \begin{pmatrix} p, p_1 \, p_2 \cdots p_n \\ q, p_1 \, p_2 \cdots p_n \end{pmatrix} dS_{p_1} \cdots dS_{p_n}.$$

The iterated kernels are defined by

$$K^{(1)}(p, q) = K(p, q)$$

$$K^{(n)}(p, q) = \int_S K^{(n-1)}(p,q_1) \, K(q_1,q) \, dS_{q_1}.$$

Preliminary Results:

1. The functions $N_n(p,q)$ satisfy the recurrence formulae

$$(46) \quad N_n(p,q) = \delta_n \, K(p,q) - \int_S K(p,q_1) \, N_{n-1}(q_1, q) \, dS_{q_1},$$

$$N_n(p,q) = \delta_n \, K(p,q) - \int_S K(q_1,q) \, N_{n-1}(p,q_1) \, dS_{q_1},$$

from which we may write

$$N_n(p,q) = \delta_n \, K(p,q) - \delta_{n-1} \, K^{(2)}(p,q) \cdots (-1)^n \, K^{(n+1)}(p,q).$$

2. The series

$$(47) \qquad \delta(\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n \, \delta_n \, ;$$

$$(48) \qquad N(p,q,\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n \, N_n(p,q)$$

are uniformly and absolutely convergent for all λ. The convergence is established with the help of Hadamard's result on determinants. [12; (ii)]

Series (47) and (48) are entire functions. They define the determinant $\delta(\lambda)$ and the minor determinant of the first order $N(p,q,\lambda)$ of the kernel $K(p,q)$.

3. If $\lambda = c$ is a zero of $\delta(\lambda)$ of order j, then $\lambda = c$ is a zero of $N(p,q,\lambda)$ of order at most $j - 1$.

4. The Fredholm determinants $\delta(\lambda)$ and $N(p,q,\lambda)$ satisfy the following equations

$$(49)\quad N(p,q,\lambda) = K(p,q)\ \delta(\lambda) + \lambda \int_S K(p,q_1)\ N(q_1,q,\lambda)\,dS_{q_1},$$

$$(49')\quad N(p,q,\lambda) = K(p,q)\ \delta(\lambda) + \lambda \int_S N(p,q_1,\lambda)\ K(q_1,q)\,dS_{q_1}.$$

5. The series defining the Neumann resolvent [33; p. 175]

$$R(p,q,\lambda) = K(p,q) + \lambda\, K^{(2)}(p,q) + \ldots = \sum_0^\infty \lambda^n\, K^{(n+1)}(p,q),$$

has a finite radius of convergence λ_0 and for $|\lambda| < |\lambda_0|$,

$$R(p,q,\lambda) = \frac{N(p,q,\lambda)}{\delta(\lambda)}\ .$$

I shall give a simple proof of this which depends on the recurrence relation (46).

Theorem: The Neumann resolvent $R(p,q,\lambda)$ is identical with the Fredholm resolvent whenever the former is defined

i.e. $$R(p,q,\lambda) = \frac{N(p,q,\lambda)}{\delta(\lambda)}\ ,\ |\lambda| < |\lambda_0|.$$

Proof: Let

$$\frac{\sum(-\lambda)^n N_n}{\sum(-\lambda)^n \delta_n} = \sum c_n(-\lambda)^n.$$

The coefficients c_n are given by

$$(50) \qquad c_n = \begin{vmatrix} N_0 & \delta_0 & 0 & & 0 \\ N_1 & \delta_1 & \delta_0 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \delta_0 \\ \vdots & \vdots & \vdots & & \\ N_n & \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{vmatrix} \Big/ (-1)^n \delta_0^{n+1}$$

using the recurrence relations (46) in the form

$$(51) \qquad N_n = \delta_n K - K N_{n-1}; \qquad N_0 = \delta_0 K ,$$

where KN_{n-1} is to be interpreted in the light of (46); and may further be written as

$$KN_{n-1} = \delta_{n-1} K^{(2)} - K^{(2)} N_{n-2} ,$$

where $K^{(2)}$ is the second iterated kernel. Replacing the first column in (50) with the help of (51), we have

$$c_n = \begin{vmatrix} \delta_0 K - 0 & \delta_0 & 0 & & 0 \\ \delta_1 K - KN_0 & \delta_1 & \delta_0 & & \vdots \\ \delta_2 K - KN_1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \delta_0 \\ \vdots & \vdots & \vdots & & \\ \delta_n K - KN_{n-1} & \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{vmatrix} \Big/ (-1)^n \delta_0^{n+1} .$$

This decomposes into two determinants [3; p. 19] one of which vanishes because one column is a multiple of the

other, so

$$c_n = \begin{vmatrix} 0 & \delta_0 & 0 & & 0 \\ KN_0 & \delta_1 & \delta_0 & & \\ KN_1 & \delta_2 & & & \\ \vdots & \vdots & & & \\ KN_{n-1} & \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{vmatrix} \Big/ (-1)^{n-1}\, \delta_0^{n+1}.$$

Using the same argument on the first column of the above determinant, we have

$$c_n = \begin{vmatrix} 0 & \delta_0 & 0 & & \\ 0 & \delta_1 & \delta_0 & & \\ K^{(2)}N_0 & \delta_2 & & & \\ \vdots & \vdots & & & \\ K^{(2)}N_{n-2} & \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{vmatrix} \Big/ (-1)^{n-2}\, \delta_0^{n+1},$$

and finally

$$c_n = \begin{vmatrix} 0 & \delta_0 & 0 & & 0 \\ 0 & \delta_1 & \delta_0 & & \\ 0 & \vdots & & & \\ \vdots & \vdots & & & \\ 0 & & & & \\ K^{(n)}N_0 & \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{vmatrix} \Big/ \delta_0^{n+1}.$$

Since $\delta_0 = 1$, $N_0 = K$, we have

$$c_n = \frac{(-1)^n\, \delta_0^n\, K^{(n)}N_0}{\delta_0^{n+1}} = (-1)^n\, K^{(n+1)}.$$

Thus,

$$(52)\quad \frac{N(p,q,\lambda)}{\delta(\lambda)} = \frac{\sum (-\lambda)^n N_n}{\sum (-\lambda)^n \delta_n} = \sum c_n(-\lambda)^n = \sum(-\lambda)^n(-1)^n K^{(n+1)}$$

$$= \sum \lambda^n K^{(n+1)} = R(p,q,\lambda).$$

q.e.d.

6. The series $\sum \lambda^n K^{(n+1)}$ defining $R(p,q,\lambda)$ is dominated absolutely and uniformly by the series

$$M + \lambda M^2 S + \lambda^2 M^3 S^2 + \ldots$$

where

$$|K(p,q)| \leq M ; \qquad S = \int_S dS.$$

Thus the solutions of (44) and (45) are defined for $|\lambda| < 1/MS$, and in this region are analytic functions of λ.

12. Equivalence to Numerical Integration

The development of Fredholm Theory came as a consequence of the fact that the integral equation (44) was treated as a limiting case of a linear system of algebraic equations. As such it has a striking resemblance to the technique of numerical integration. In the two dimensional case, Hilbert, [15; Ch. I & II] using the trapezoidal rule of integration, arrived at Fredholm's results. Nyström [28] applied the technique of numerical interpolation to the potential theory problems in two dimensions. A separate bibliography has been prepared in which some of the numerical techniques for handling integral equations, in general,

are discussed at length. Here, we make some comments of an elementary nature.

Any linear formula for approximate integration has the form

$$(53) \qquad \int_a^b \chi(p)dp = \sum_{i=1}^{n} A_i \chi(p_i) + \rho ,$$

where A_i and p_i are numbers which are constant for the given interval and for a given formula, ρ is the error. Here

$$A_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} A_i = b - a.$$

Various quadrature formulae are in use, e.g. rectangular, trapezoidal, Simpson, Chebyshev or Gauss. The choice depends on the accuracy desired.

Using (53) in (44), the integral equation may be written as

$$(54) \qquad \phi(p) - \lambda \sum A_i K(p,p_i) \phi(p_i) = f(p) + \lambda\rho(p) .$$

In particular, putting $p = p_j$, $j = 1, 2, \ldots n$,

$$\phi(p_j) - \lambda \sum A_i K(p_j,p_i) \phi(p_i) = f(p_j) + \lambda\rho_j ,$$

where $$\rho_j = \rho(p_j) .$$

Neglecting ρ_j on the right, we obtain a system of n equations in the unknowns

$$(54') \qquad \overline{\phi}(p_1), \overline{\phi}(p_2), \ldots \overline{\phi}(p_n)$$

$$(55) \qquad \overline{\phi}(p_j) - \lambda \sum A_i K(p_j,p_i) \overline{\phi}(p_i) = f(p_j);$$

$$j = 1, 2, \ldots n.$$

Solving this system of equations we find approximations (54') to the values of the function at p_i. Consequently we may write

$$(56) \qquad \bar{\phi}(p) = f(p) + \lambda \sum_{i=1}^{n} A_i \, K(p,p_i) \, \bar{\phi}(p_i)$$

as an approximation to $\phi(p)$.

Kantorovich and Krylov [19; pp. 103-107] have noted expressions for the error ρ, when different quadrature formulae are used.

It is generally desirable to use the most exact of the formulae i.e. Gauss or Chebyshev. When $f(p)$ and $K(p,q)$ are periodic of period $(b - a)$, the rectangular formula is more convenient and equally accurate. Higher quadrature formulae may be used when the integrands satisfy regularity conditions.

13. Fredholm's Theorems

An important feature of the development outlined in § 11 is that the equation (52) shows that the quotient $\frac{N(p,q,\lambda)}{\delta(\lambda)}$ is an analytic continuation, over the complex λ-plane, of the function $R(p,q,\lambda)$. We may define $\Gamma(p,q,\lambda)$, by the following

$$(57) \qquad \Gamma(p,q,\lambda) = \frac{N(p,q,\lambda)}{\delta(\lambda)}$$

and call $\Gamma(p,q,\lambda)$ the Fredholm resolvent of the kernel $K(p,q)$. Clearly, $\Gamma(p,q,\lambda)$ is a meromorphic function of the

parameter λ whose poles are just the zeros of $\delta(\lambda)$. Thus the series $\sum \lambda^n K^{(n+1)}(p,q)$ is convergent for all $|\lambda| < |\lambda_0|$, where λ_0 is the zero of $\delta(\lambda)$ with least absolute value. Then $\Gamma(p,q,\lambda)$ furnishes the solution of (44) and (45) in the form

$$\phi(p) = f(p) + \lambda \int \Gamma(p,q,\lambda)\, f(q)dS_q; \tag{58}$$

$$\psi(p) = f(p) + \lambda \int \Gamma(q,p,\lambda)\, f(q)dS_q, \tag{59}$$

respectively, for all values of λ, $\delta(\lambda) \neq 0$.

The precise statement of Fredholm's fundamental results is the following:

Theorem 1: Suppose λ is not a zero of $\delta(\lambda)$. Then, (i) (44) and (45) possess unique continuous solutions given by (58) and (59). (ii) The corresponding homogeneous equations

$$\phi(p) = \lambda \int K(p,q)\, \phi(q)dS_q; \tag{60}$$

$$\psi(q) = \lambda \int K(q,p)\, \psi(q)dS_q, \tag{61}$$

possess only the trivial solutions.

Theorem 2: Suppose that $\lambda = c$ is a zero of the determinant $\delta(\lambda)$ of order m (necessarily finite). Then, the homogeneous equations (60) and (61) with $\lambda = c$ will each have at least one and at most m linearly independent non-trivial solutions $\phi_i(p)$ and $\psi_i(p)$, $i = 1, 2, \ldots m$ respectively.

Theorem 3: Suppose that $\lambda = c$ is a zero of $\delta(\lambda)$. Then,

for the existence of a solution of $\phi(p)$ of (44) with $\lambda = c$, it is necessary and sufficient that $f(p)$ be orthogonal to all the solutions $\psi_i(p)$ of the adjoint homogeneous equation (61), i.e.

$$(62) \qquad \int_S f(p)\, \psi_i(p) dS_p = 0, \qquad i = 1, \ldots m.$$

If these conditions are fulfilled, the general solution of (44) is formed by adding to the particular solution $\phi(p)$ any linear combination of the solutions $\phi_i(p)$ of the corresponding homogeneous equation (60).

The following are the important consequences of the above three theorems.

1. The homogeneous equations (60) and (61) have solutions for $\lambda = c$, when c is a zero of $\delta(\lambda)$. Also every zero of $\delta(\lambda)$ is a pole at least of order one of $\Gamma(p,q,\lambda)$. We may, however, have $m > n$ linearly independent non-trivial solutions of the homogeneous equations associated with a pole of the resolvent of order n.

2. Whenever the homogeneous equations (60) and (61) have no non-trivial solution, the corresponding non-homogeneous equations have unique solutions.

3. The Fredholm alternative is valid i.e. either the homogeneous equations have a non-trivial solution or the non-homogeneous equations have unique solutions.

4. The two homogeneous equations (60) and (61)

have the same number of linearly independent solutions and these can be so chosen as to form a biorthogonal set.

If $\{\phi_i,\lambda_i\}$ and $\{\psi_i,\lambda_i\}$ are the solutions of (60) and (61), then

$$\int \phi_i(p)\,\psi_j(p)dS_p = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

5. If $\lambda = \lambda_0$ is a pole of $\Gamma(p,q,\lambda)$ of order m, then

$$\Gamma(p,q,\lambda) = \frac{A_m(p,q)}{(\lambda-\lambda_0)^m} + \frac{A_{m-1}(p,q)}{(\lambda-\lambda_0)^{m-1}} + \ldots + \frac{A_1(p,q)}{(\lambda-\lambda_0)} + B(p,q,\lambda)$$

where $A_m(p,q)$ are continuous and not identically zero and $B(p,q,\lambda)$ is holomorphic in a neighbourhood of λ_0.

6. $A_m(p,q)$ and $A_m(q,p)$ are the solutions of (60) (61) respectively, for any fixed q for which they do not vanish identically.

Definitions: The values $\lambda = \lambda_i$, i = 0, 1, 2, ... for which the homogeneous equations (60) and (61) have non-trivial solutions i.e. the poles of $\Gamma(p,q,\lambda)$, arranged in order of magnitude ($|\lambda_n| \leq |\lambda_{n+1}|$) are called the eigenvalues of the integral equations (44) and (45); each eigenvalue is repeated as many times as there are linearly independent non-trivial solutions of the corresponding homogeneous equation ϕ_i and ψ_i associated with it. These

solutions are called eigen-functions.

14. Further Development of the Fredholm Theory

Now we replace the hypothesis of Section 11 with that of the hypothesis on the kernel of the potential theory, Section 6, property (vi).

Hypothesis:

1. $K(p,q)$ is a given real function of the co-ordinates p,q, continuous for $p \neq q$, but unbounded as $p \to q$. However, the third iterated kernel

$$(63) \quad K^{(3)}(p,q) = \int_S \int_S K(p,q_1)\, K(q_1,q_2)\, K(q_2,q)\, dS_{q_1}\, dS_{q_2}$$

is a bounded and continuous function of the co-ordinates of $p, q \in S$.

2. $f(p)$ is a real, continuous function of $p \in S$.

Under this hypothesis the quantities δ_n and N_n lose their significance because of the undefined quantity $K(p_1,p_1)$, and consequently the same applies to $\Gamma(p,q,\lambda)$ and its quotient form (57). However, we may define the quantities with respect to $K^{(3)}(p,q)$ which is bounded and continuous.

Let

$$(64) \qquad S(p,q,\lambda) = K(p,q) + \lambda\, K^{(2)}(p,q)$$

and $\Gamma^{(3)}(p,q,\lambda)$ be the resolvent corresponding to $K^{(3)}(p,q)$, then, it can be shown [9; pt II, p. 60] or [21; p. 307] that

$$(65)\quad R(p,q,\lambda) = S(p,q,\lambda) + \lambda^2\, \Gamma^{(3)}(p,q,\lambda^3) + \lambda^3 \int_S S(p,q_1,\lambda)\, \Gamma^{(3)}(q_1,q,\lambda^3)\, dS_{q_1}$$

and consequently the resolvent and its analytic continuation may be defined under the general hypothesis (1.) and (65) may be written as

$$(66)\quad R(p,q,\lambda) = K(p,q) + \lambda\, K^{(2)}(p,q) + \frac{M(p,q,\lambda)}{\eta(\lambda)}\ ,$$

where M and η are entire functions of λ^3. The second member on the right of (65) is a meromorphic function of λ.

Theorems: With this definition of the resolvent, the Fredholm Theorems hold if we replace the words "zero of $\delta(\lambda)$" with the expression "pole of the resolvent $R(p,q,\lambda)$". The definitions of eigen-value and eigen-function given above may be extended to the present case [21; p. 307].

The Fredholm theory has been developed and applied to the integral equations of potential theory by J. Plemelj [30; (ii)]. His results valid in two or three dimensions are:

1. $R(p,q,\lambda)$ has only simple, real poles, $\lambda = -1$ is a pole, $\lambda = 1$ is not. All poles are greater than or equal to one in absolute value.

2. If λ_0 is a k-tuple zero of $\delta(\lambda)$, it is (k-1)-tuple zero of $N(p,q,\lambda)$ and the residue of $R(p,q,\lambda)$ has the form

$$\phi_1\psi_1 + \phi_2\psi_2 + \ldots + \phi_k\psi_k$$

where ϕ_i and ψ_i are linearly independent solutions of the homogeneous equations. They may be so chosen that ϕ's are the densities and ψ's the boundary values of the corresponding potentials of single layers, and that they are biorthogonal.

3. The three theorems of Fredholm, hold for the equations of potential theory.

15. The Theorems of H. Marty on Integral Equations With Symmetrizable Kernels.

Theorems of Marty [9; pt II, p. 145] are particularly important in the context of the integral equations of potential theory. We suppose that G(p,q) and K(p,q) satisfy the hypothesis of Section 14 of the extended Fredholm theory.

Definition: Let K(p,q) and G(p,q) be two arbitrary kernels satisfying condition 1 of Section 14. We obtain kernels $S_1(p,q)$ and $S_2(p,q)$, in general distinct, by composition of K(p,q) and G(p,q) on the left and on the right respectively as

$$S_1(p,q) = \int G(p,q_1)\, K(q_1,q)\, dS_{q_1},$$

$$S_2(p,q) = \int K(p,q_1)\, G(q_1,q)\, dS_{q_1}.$$

If G(p,q) is symmetric and furthermore, S_1 or S_2 is symmetric, then K is said to be symmetrizable either on the right or on the left, by G.

Theorem 1: Every kernel K(p,q) whose resolvent has only

real, simple poles is symmetrizable on both sides, [24; (iv)]; [9, pt II, p. 145].

Theorem 2: If $\phi(p)$ is a solution of the homogeneous equation

$$\phi(p) = \lambda \int K(p,q)\, f(q)\, dS_q$$

and $K(p,q)$ is symmetrizable on the right by $G(p,q)$, then

$$\psi(p) = \int G(p,q)\, \phi(q) dq$$

is a solution of the adjoint homogeneous equation [9; pt II, p. 146]

$$\psi(p) = \lambda \int K(q,p)\, \psi(q)\, dS_q .$$

Theorem 3: Suppose that $K(p,q)$ is symmetrizable by composition with a symmetric definite kernel.

Then all the poles of the resolvent of $K(p,q)$ are simple and real. [24; (iii)]; [9; pt II, p. 147].

Theorem 4: Every non-zero kernel $K(p,q)$ which is symmetrizable by composition with a symmetric definite kernel has at least one eigen-value. [24; (iii)]; [9; pt II, p. 147]

By repeating the arguments of Theorem 4 we can conclude that the kernel $K(p,q)$ has a countable number of real eigen-values and corresponding to these, a countable number of eigen-functions. In case there exists at most a finite number of eigen-values, we may write [2; p. 2019]

$$K(p,q) = \sum \frac{\phi_i(p)\, \psi_i(q)}{\lambda_i} \tag{67}$$

The above theorems have an elegant interpretation in abstract operator theory [18].

Definition: Let H denote a class of real functions $f(p)$, defined and continuous for $p \in S$. Let

$$Kf \to \int_S K(p,q)\, f(q)dS_q, \quad Gf \to \int_S G(p,q)\, f(q)dS_q,$$

and

$$K : H \to H, \quad G : H \to H.$$

$$(f,g) = \int_S f(p)\, g(p)dS_p$$

defines the scalar product in H.

Hypothesis:

1. Suppose that G is symmetric and positive definite i.e.

$$(Gf,g) = (f,Gg); \quad (Gf,f) > 0.$$

2. Suppose that $S_1 = GK$ is a symmetric operator.

Conclusion: If we define a second scalar product in H as

$$[f,g] = (Gf,g) = (f,Gg)$$

then K is symmetric with respect to this new scalar product, i.e. $[Kf,g] = [f,Kg]$, since,

$$[Kf,g] = (GKf,g) = (f,GKg) = [f,Kg].$$

The method outlined above is applicable to the kernel of the potential theory using the symmetrizing kernel G corresponding to the potential of a single layer. G is the principal solution of Laplace's equation in three-dimensions i.e. $G = \frac{1}{r_{pq}}$. This idea is worked out in detail by Vaillancourt [40], a summary of which appears in the next chapter (Section 16).

CHAPTER III

EIGEN-VALUE ASPECT (PARTIAL FRACTION REPRESENTATION)

16. Separation of Variables

The Robin-Poincaré and Neumann-Poincaré Problems stated in Section 5 have been solved by means of the so-called fundamental functions of Poincaré. These functions are, in general, potentials of single layers with densities proportional to the eigen-functions of the Fredholm-Poincaré integral equations. In case S is a co-ordinate surface in a system of co-ordinates in which the Laplace's equation separates, the method of separation of variables furnishes examples of fundamental functions.

Suppose that S is a co-ordinate surface $\rho = \rho_0 =$ constant, in the (ρ,θ,ϕ) system of curvilinear co-ordinates and that

$$\nabla^2 V = 0 \tag{68}$$

is separated in this system. We obtain a family of particular solutions [26; p. 1264]

$$V_i(\rho,\theta,\phi) = F_i^1(\rho)\, G_i^m(\theta,\phi) \text{ in } R \tag{69}$$

$$= F_i^2(\rho)\, G_i^m(\theta,\phi) \text{ in } R'$$

where $F_i^j(\rho)$ are selected in such a way that the functions $V_i(\rho,\theta,\phi)$ are harmonic throughout R and R' and regular at

infinity. Now

$$\frac{\partial V_i}{\partial n_-} = \frac{\partial F_i^1}{\partial \rho}\Big|_{\rho=\rho_0} G_i^m(\theta,\phi)\,,$$

$$\frac{\partial V_i}{\partial n_+} = \frac{\partial F_i^2}{\partial \rho}\Big|_{\rho=\rho_0} G_i^m(\theta,\phi)\,,$$

therefore

$$\frac{\partial V_i}{\partial n_-}\Big/\frac{\partial V_i}{\partial n_+} = \frac{\partial F_i^1}{\partial \rho}\Big|_{\rho=\rho_0} \Big/ \frac{\partial F_i^2}{\partial \rho}\Big|_{\rho=\rho_0} \tag{70}$$

Case of a Sphere

In the particular case of a sphere, the above quantities become [26; p. 1274]

$$F_i^1 = \rho^i, \quad F_i^2 = \rho^{-(i+1)}, \quad G_i^m(\theta,\phi) = Y_{mi}(\theta,\phi) \tag{71}$$

The eigen-values are

$$\lambda_i = (2i + 1)$$

and for each eigen-value λ_i, there correspond $(2i + 1)$ linearly independent eigen-functions, given in the normalized form by

$$\phi_i^{(m)}(p) = \frac{1}{2}\,(2i + 1)\sqrt{\frac{\varepsilon_m (i - m)!}{2\pi(i + m)!}}\; Y_{mi}(p)$$

$$\psi_i^{(m)}(p) = \sqrt{\frac{\varepsilon_m (i - m)!}{2\pi(i + m)!}}\; Y_{mi}(p) \tag{72}$$

where Y_{mi} are the spherical harmonics

$$Y_{mi} = P_i^m(\cos\theta) \begin{matrix} \cos m\phi & \text{(even)} \\ \sin m\phi & \text{(odd)} \end{matrix}$$

and ε_m is the Neumann factor,

$$\varepsilon_0 = 1, \quad \varepsilon_m = 2, \quad m = 1, 2, \ldots$$

$\phi_i^{(m)}$ and $\psi_i^{(m)}$ in (72) correspond to the single-layer and double-layer distributions respectively. The functions $V_i(\rho,\theta,\phi)$ in (69) are just the Poincaré fundamental functions with respect to the surface S and are, in general, single-layer potentials. The corresponding densities ϕ_i are solutions of the homogeneous integral equation (60).

The adjoint homogeneous equation (61) may also be discussed in an analogous manner and in place of $V_i(p)$ we may consider a family of double layer potentials $W_i(p)$, generated by the double layer distributions corresponding to the eigen-functions of the homogeneous equation (61). $W_i(p)$ are known to be simple multiples of $V_i(p)$ [30; (ii)].

The original discussion of the eigen-value aspect of the integral equations of potential theory is due to Blumenfeld and Mayer [2], who establish the existence of Poincaré fundamental functions and prove the convergence of the solutions of (14) and (15), when expressed as Mittag-Leffler type expansions, in the regions R and R' and not

on S itself. The completion at this point has recently been achieved by Howland and Vaillancourt [18] who establish that the series indeed converges everywhere, uniformly and absolutely, through a functional analytic approach. Here use is made of the fact that $K(p,q)$ is symmetrizable on the left by a positive definite and symmetric transformation G (Section 15); furthermore, $K^{(2)}(p,q)$, the second iterate of $K(p,q)$, is completely continuous on the prehilbert space of continuous functions on S, in the ordinary sense, with L^2 norm. Then arguments of Reid [32] and Mercer [25] are utilized to achieve the desired result. Here, in Section 18, we make some adaptations of Reid's [32] results with a view to applying them to the integral equations (14) and (18).

17. Eigen-Value Theory

In this section we approach the eigen-value question from the computational and more basic point of view i.e. that of a pole and its residue. Plemelj [30] has thoroughly analysed the question by means of Fredholm's resolvents and has given explicit formulae for the canonical decomposition near a singular point. These and other results are summarized in Section 14. Here we make a non-trivial extension of these results to the case of a finite number of poles, and calculate their residues in terms of iterated kernels i.e. the coefficients of the Neumann resolvent $R(p,q,\lambda)$.

This, effectively, yields a method of approximating $R(p,q,\lambda)$ and consequently, the solution of the integral equations (14) and (15).

The Case of a Simple Pole

Suppose that $\lambda = \lambda_0$ is a simple pole of $R(p,q,\lambda)$, then we may write

$$(73) \quad R(p,q,\lambda) = \frac{A(p,q)}{\lambda-\lambda_0} + B(p,q,\lambda); \quad A(p,q) \not\equiv 0,$$

where $B(p,q,\lambda)$ remains finite for $\lambda = \lambda_0$. $A(p,q)$ is a continuous function and satisfies the pair of adjoint homogeneous equations for $\lambda = \lambda_0$. [21; p. 296]

$$(74) \quad A(p,q) = \lambda_0 \int A(p,q_1)\, K(q_1,q)\, dS_{q_1},$$

$$(75) \quad A(p,q) = \lambda_0 \int K(p,q_1)\, A(q_1,q)\, dS_{q_1}.$$

A repeated application of either (74) or (75) results in the following:

$$(76) \quad A(p,q) = \lambda_0^2 \int K^{(2)}(p,q_1)\, A(q_1,q)\, dS_{q_1},$$

where $K^{(2)}$ is the second iterate of K. We may write (76) as

$$(77) \quad \int K^{(2)}(p,q_1)\, A(q_1,q)\, dS_{q_1} = \frac{A(p,q)}{\lambda_0^2}.$$

Taking an analogy from (77), we hope to extend the result of a single pole to that of a finite number of poles.

The Case of a Finite Number of Simple Poles

Let $K(p,q)$ be the kernel of the potential theory satisfying the hypothesis of the Section 14 and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the simple poles of

$$R(p, q, \lambda) = K(p, q) + \lambda K^{(2)}(p, q) + \lambda^2 K^{(3)}(p, q) + \ldots$$

and let a "weighted" expansion of $R(p, q, \lambda)$ be the following:

$$(78) \quad R(p,q,\lambda) = K(p,q) + \lambda K^{(2)}(p,q) + \lambda^2 \sum_{m=1}^{n} \frac{A_m(p,q)}{\lambda_m^2(\lambda_m-\lambda)} .$$

Then, we wish to establish the following:

(i) $A_m(p,q)$ for $m = 1, 2, \ldots, n$ furnish continuous solutions of the pair of homogeneous integral equations (18) and (19) corresponding to $\lambda = \lambda_m$ for $m = 1, 2, \ldots, n$.

(ii) Also,

$$(79) \quad \int A_m(p,p)dS_p = 1, \quad \text{for } m = 1, 2, \ldots, n,$$

where the trace T_i of the ith. iterated kernel $K^{(i)}$ is defined as usual by

$$(80) \quad T_i = \int_S K^{(i)}(p,p)dS_p = \sum_{m=1}^{n} \frac{1}{\lambda_m^i} , \; m = 1,2, \ldots, n, \; i \geq 3$$

and conversely.

Proof: Expanding the right hand side of (78) in powers of λ and equating terms with those in $R(p,q,\lambda)$, we have the following:

$$
(81)\qquad
\begin{cases}
K^{(3)}(p,q) = \sum \dfrac{A_m(p,q)}{\lambda_m^3}, \\
K^{(4)}(p,q) = \sum \dfrac{A_m(p,q)}{\lambda_m^4}, \\
\cdots\cdots\cdots\cdots \\
K^{(n+2)}(p,q) = \sum \dfrac{A_m(p,q)}{\lambda_m^{n+2}}, \\
K^{(n+3)}(p,q) = \sum \dfrac{A_m(p,q)}{\lambda_m^{n+3}}.
\end{cases}
$$

The first n equations of (81) form a system of linear non-homogeneous equations for n unknowns $A_m(p,q)$, $m = 1, 2, \ldots n$. The determinant of the system is

$$
(82)\qquad \Delta = \begin{vmatrix}
\frac{1}{\lambda_1^3} & \frac{1}{\lambda_2^3} & \cdots & \frac{1}{\lambda_n^3} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\lambda_1^{n+2}} & \frac{1}{\lambda_2^{n+2}} & \cdots & \frac{1}{\lambda_n^{n+2}}
\end{vmatrix}
$$

which may be written as

$$
(83)\qquad \Delta = \frac{1}{(\lambda_1 \lambda_2 \cdots \lambda_n)^{n+2}} \begin{vmatrix}
\lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \\
\vdots & & \vdots \\
\lambda_1 & \cdots & \lambda_n \\
1 & \cdots & 1
\end{vmatrix}.
$$

Δ is of Vandermonde Type [3; p. 34] and has the value

$$(84)\quad \frac{1}{(\lambda_1 \cdots \lambda_n)^{n+2}} \prod_{j<i} (\lambda_i - \lambda_j);\quad i,j = 1,2, \ldots, n.$$

For distinct λ_i, $\Delta \neq 0$, hence the system (81) can be solved uniquely for $A_m(p,q)$, $m = 1,2, \ldots, n$ as a linear combination of the iterated kernels

$$(85)\quad K^{(3)}(p,q),\ K^{(4)}(p,q),\ \ldots,\ K^{(n+2)}(p,q).$$

Since each of the iterated kernels involved is continuous, $A_m(p,q)$ will also be continuous functions.

Application of Cramer's rule [3; p. 54] gives the explicit values of $A_m(p,q)$, $m = 1,2, \ldots, n$.

$$(86)\quad A_m(p,q) = \begin{vmatrix} \frac{1}{\lambda_1^3} & \frac{1}{\lambda_{m-1}^3} & K^{(3)}(p,q) & \frac{1}{\lambda_{m+1}^3} & \cdots & \frac{1}{\lambda_n^3} \\ \vdots & & & & & \\ \frac{1}{\lambda_1^{n+2}} & \cdots & \frac{1}{\lambda_{m-1}^{n+2}} & K^{(n+2)}(p,q) & \frac{1}{\lambda_{m+1}^{n+2}} & \cdots & \frac{1}{\lambda_n^{n+2}} \end{vmatrix} \div \Delta$$

where the mth. column in the determinant Δ of (82) has been replaced by the column given by (85).

Now we show that $A_m(p,q)$ identically satisfies the homogeneous integral equations for $\lambda = \lambda_m$, $m = 1,2, \ldots, n$,

$$(87)\quad A_m(p,q) = \lambda_m \int K(p,q_1)\, A_m(q_1,q)\, dS_{q_1},$$

$$(88)\quad A_m(p,q) = \lambda_m \int A_m(p,q_1)\, K(q_1,q)\, dS_{q_1}.$$

Substituting the value of $A_m(p,q)$ from (86) into (87)

and using additional results of the form

$$(89) \qquad K^{(4)}(p,q) = \int K(p,q_1)\, K^{(3)}(q_1,q)\, dS_{q_1}$$

etc. in the determinant on the right, we get the following determinantal equation

$$\begin{vmatrix} \frac{1}{\lambda_1^3} & \cdots & \frac{1}{\lambda_{m-1}^3} & K^{(3)}(p,q) & \frac{1}{\lambda_{m+1}^3} & \cdots & \frac{1}{\lambda_n^3} \\ & & & & & & \\ \frac{1}{\lambda_1^{n+2}} & \cdots & \frac{1}{\lambda_{m-1}^{n+2}} & K^{(n+2)}(p,q) & \frac{1}{\lambda_{m+1}^{n+2}} & \cdots & \frac{1}{\lambda_n^3} \end{vmatrix} =$$

$$= \lambda_m \begin{vmatrix} \frac{1}{\lambda_1^3} & \cdots & \frac{1}{\lambda_{m-1}^3} & K^{(4)}(p,q) & \frac{1}{\lambda_{m+1}^3} & \cdots & \frac{1}{\lambda_n^3} \\ & & & & & & \\ \frac{1}{\lambda_1^{n+2}} & \cdots & \frac{1}{\lambda_{m-1}^{n+2}} & K^{(n+3)}(p,q) & \frac{1}{\lambda_{m+1}^{n+2}} & \cdots & \frac{1}{\lambda_n^{m+2}} \end{vmatrix}$$

which can be simplified, by induction, to the following:

$$(90) \quad K^{(3)}(p,q) - (\sum \lambda_i)\, K^{(4)}(p,q) + (\sum \lambda_i\lambda_j)\, K^{(5)}(p,q) + \cdots$$

$$\cdots + (-1)^n \lambda_1\lambda_2 \cdots \lambda_n\, K^{(n+2)}(p,q) = 0,$$

and it can easily be seen that (90) holds in view of the assumption that $\lambda_1, \cdots \lambda_n$ are the poles of $\sum_i K^{(i)}(p,q)\, \lambda^i$. This proves the statement (i) above. In order to establish (ii), we put $p = q$ in (86) and integrate.

By using (80), the condition reduces to

$$(91)\quad \int A_m(p,p)\,dS_p = \begin{vmatrix} \frac{1}{\lambda_1^3} & \cdots & \frac{1}{\lambda_{m-1}^3} & \sum_{m=1}^{n} & \frac{1}{\lambda_m^3} & \frac{1}{\lambda_m^3} & \cdots & \frac{1}{\lambda_n^3} \\ \vdots & & & & & & & \\ \frac{1}{\lambda_1^{n+2}} & \cdots & \frac{1}{\lambda_{m-1}^{n+2}} & \sum_{m=1}^{n} & \frac{1}{\lambda_m^{n+2}} & \frac{1}{\lambda_m^{n+2}} & \cdots & \frac{1}{\lambda_n^{n+2}} \end{vmatrix} \div \Delta$$

The determinant in the numerator on the right has n terms in the mth. column, and thus may be decomposed into n determinants all of which vanish because two columns become identical, except the one in which the mth. column has for its elements,

$$\frac{1}{\lambda_m^3} \quad \frac{1}{\lambda_m^4} \quad \cdots \quad \frac{1}{\lambda_m^{n+2}} .$$

This determinant is exactly Δ and consequently the right hand side in (91) reduces to unity.

The converse statement may easily be seen in the light of set of equations (81).

We may call $A_m(p,q)$ of the above development, the residue at the pole $\lambda = \lambda_m$, $m = 1,2, \ldots, n$. We have shown that the residue $A_m(p,q)$ is a solution of the homogeneous equation (18). It may also be shown to be so for the adjoint homogeneous equation (19) corresponding to $\lambda = \lambda_m$. Some of the easily verifiable properties of the residues may be noted.

(i) $A_m(p,q)$ if integrated with respect to p or q yield respectively solutions of (18) and (19) corresponding to $\lambda = \lambda_m$ and for that matter, $A_m(p,q)$ when integrated after multiplication with any continuous function of p or q yield respectively solutions of (18) and (19) corresponding to $\lambda = \lambda_m$.

(ii) Property (i) above suggests the following decomposition

$$A_m(p,q) \approx \phi_m(p)\, \psi_m(q) \tag{92}$$

where ϕ_m and ψ_m are the solutions of the homogeneous equations (18) and (19). They may be chosen to form an orthonormal set.

(iii) The iterated kernels have the following decomposition:

$$K^{(i)}(p,q) = \sum_{m=1}^{n} \frac{\phi_m(p)\, \psi_m(q)}{\lambda_m^i}, \qquad i \geq 3. \tag{93}$$

With the help of the approximation to the resolvent $R(p,q,\lambda)$ as given in (78), we can formally write the solution of the non-homogeneous integral equations (14) or (15) by using (58) or (59) respectively.

If λ is not an eigen-value of the kernel, then (58) becomes:

$$\phi(p) = f_0 + \lambda f_1 + \lambda^2 f_2 + \lambda^3 \sum_{m=1}^{n} \frac{(\psi_m, f)}{\lambda_m^2(\lambda_m - \lambda)}, \tag{94}$$

where

$$(95) \qquad (\psi_m, f) = \int \psi_m(p)\, f(p)\, dS_p,$$

and

$$(96) \qquad f_i = \int K^{(i)}(p,q)\, f(q)\, dS_q.$$

We may note that the form of the solution (94) has an advantage over Fredholm type solution (quotient of two entire functions) because it displays the meromorphic character of the solution with respect to the parameter and indicates the residue at each pole. Furthermore, the Fredholm alternative may be argued for the integral equation (14). Similar arguments may be made in relation to (15).

In the next section we turn to the remark made at the end of Chapter II, and pursue the idea of symmetrizability of the kernel K(p,q) of the potential theory. This gives insight into the question of existence of eigen-values which we assumed in this section.

18. Certain Inequalities and the Space H

In Section 15 we noted that the kernel K of potential theory is symmetric with respect to the scalar product denoted by square brackets. In the following, we utilize this fact together with an elegant scheme for the computation of the solutions of (18) and (14) in functional spaces.

Let H_i be the linear subspace spanned by the set of

functions

$$P_0 f,\ P_1 f,\ \dots,\ P_{i-1} f, \tag{97}$$

orthogonal with respect to the scalar product, where P_i is the polynomial of degree i in K, generated by the minimized iteration scheme [22]. We denote by H the invariant subspace which is the closure, with respect to the norm generated by the scalar product, of the linear subspace spanned by all H_i

$$H = \overline{\bigcup_{i=1}^{\infty} H_i},$$

and is complete with respect to the norm. A procedure for completing an incomplete functional space is due to Friedrichs (see for summary [33; p. 331-332]).

We now recall the essential argument by which the set (97) is generated.

Since K is symmetric with respect to the scalar product, we let

$$P_0 = 1,$$

and

$$KP_0 f = P_0' f,$$

where prime indicates composition with K on the left. Now we choose a new polynomial P_1 as a linear combination of P_0' and P_0

$$P_1 f = P_0' f - \alpha_0 P_0 f$$

such that $[P_1 f, P_1 f]$ is a minimum. This gives

$$\alpha_0 = \frac{[P_0'f,\ P_0f]}{[P_0f,\ P_0f]};$$

obviously, $[P_1f,\ P_0f] = 0$. Again, we choose P_2 as a linear combination of P_1', P_1 and P_0,

$$P_2f = P_1'f - \alpha_1P_1f - \beta_0P_0f,$$

such that $[P_2f,\ P_2f]$ is a minimum. This leads to

$$\alpha_1 = \frac{[P_1'f,\ P_1f]}{[P_1f,\ P_1f]}\ ; \quad \beta_0 = \frac{[P_1'f,\ P_0f]}{[P_0f,\ P_0f]},$$

$$[P_2f,\ P_1f] = 0, \quad [P_2f,\ P_0f] = 0.$$

Once again, we choose P_3 as follows

$$P_3f = P_2'f - \alpha_2P_2f - \beta_1P_1f - \gamma_0P_0f,$$

and minimize $[P_3f,\ P_3f]$, which yields

$$\alpha_2 = \frac{[P_2'f,\ P_2f]}{[P_2f,\ P_2f]}, \quad \beta_1 = \frac{[P_2'f,\ P_1f]}{[P_1f,\ P_1f]}, \quad \gamma_0 = 0,$$

$$[P_3f,\ P_2f] = 0,\ [P_3f,\ P_1f] = 0,\ [P_3f,\ P_0f] = 0.$$

Thus there are only two correction terms in P_3f. Clearly, P_3 is of degree 3 in K. This scheme leads to a general recurrence relation,

(98) $$P_if = (K - \alpha_{i-1})P_{i-1}f - \beta_{i-2}P_{i-2}f,$$

where P_i is a polynomial of degree i in K. By their very construction, P_i form an orthogonal set of polynomials with respect to the scalar product. They may further be normalized

$$[P_i f, P_j f] = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

with the help of a scale factor with $P_i f$. In this case the following recurrence relation holds

$$\beta_{i-1} P_i f = (K - \alpha_{i-1}) P_{i-1} f - \beta_{i-2} P_{i-2} f, \tag{98'}$$

where

$$\alpha_i = [P_i' f, P_i f]; \quad \beta_i = [P_{i+1}' f, P_i f].$$

Thus the scheme furnishes a particularly useful orthogonal system complete in H. From the above we get the following matrix representation for K, which is evidently symmetric and tri-diagonal

$$\begin{pmatrix} \alpha_0 & \beta_0 & 0 & 0 & \cdots \\ \beta_0 & \alpha_1 & \beta_1 & 0 & \cdots \\ 0 & \beta_1 & \alpha_2 & \beta_2 & \cdots \\ 0 & 0 & \beta_2 & \alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{99}$$

It is with the finite "sections" of this matrix that we will be concerned in the main, while seeking approximate solutions of (18) and (14).

Now K is a linear operator in H

$$Kf = \int_S K(p,q) f(q) dS_q; \quad f \in H$$

and is symmetric with respect to the scalar product which defines the Dirichlet norm in H. By a straight-forward

application of Green's formulae [11; p. 108] it may be shown that

$$[f,f] > 0; \quad f \not\equiv 0,$$

since

$$(100) \qquad [f,f] = \iint f(p) \frac{1}{r_{pq}} f(q) dS_p dS_q$$

$$= J + J',$$

where J and J' are the Dirichlet integrals for the regions R and R' (Fig. 1) respectively and are essentially positive quantities. Also it can be shown that

$$(101) \qquad [Kf,f] = J - J'.$$

Thus, we have

$$(102) \qquad [Kf,f] \leq [f,f].$$

The Schwarz's inequality also holds

$$(103) \qquad [f,g] \leq [f,f]^{1/2} [g,g]^{1/2}, \quad f,g \in H,$$

and from the arithmetic and geometric mean property

$$(104) \qquad [f,g] \leq \frac{1}{2}\{[f,f] + [g,g]\}, \quad f,g \in H.$$

The above inequalities play an important part in the discussion of the existence of eigen-values and eigen-functions which follows in the next section.

19. Existence of Eigen-values and Eigen-functions

We noted in Section 16 that the existence of eigen-values and eigen-functions of K was established by Blumenfeld and Mayer [2]. They used the method of Marty [24], the basic concept being that of Schwarz [38].

Howland [18] showed that these eigen-values and eigen-functions satisfy a maximum principle, but did not use this principle to establish their existence. This, however, may easily be done in the light of arguments due to Reid [32] which require the following properties of K (Chapter II)

(i) K satisfies the Fredholm alternative for all values of λ.

(ii) K has a discrete spectrum with no finite accumulation point i.e. the Fredholm resolvent (66) is a meromorphic function of λ in the whole λ-plane.

The maximum principle referred to above is concerned with the maximum of the Rayleigh quotient, defined by the following ratio

$$M(f) = \frac{[Kf,f]}{[f,f]}; \quad f \in H. \tag{105}$$

In view of the inequalities given in Section 18, we may suppose that there exists a $f \in H$ such that the quantities involved in (105) are positive and that $[Kf,f]$ has a finite positive least upper bound λ_1^{-1} on the set

$$\{f;\ [f,f] = 1\}, \quad f \in H$$

and hence there exists a sequence $\{f_n\}$ such that

$$[Kf_n,f_n] \to \lambda_1^{-1}, \quad [f_n,f_n] = 1. \tag{106}$$

Then λ_1^{-1} can be shown to be an eigen-value of K. The proof

is made by contradiction employing the Fredholm alternative [32; p. 44-45].

The above result may be stated in a concise way.

Theorem 1: The least upper bound of $M(f)$, $f \in H$ is a positive eigen-value of K, where the extremizing sequence $\{f_n\}$ is normalized with respect to the scalar product, i.e.

$$[f_n, f_n] = 1.$$

Let us denote by ϕ_1, the function for which the least upper bound λ_1^{-1} is attained. ϕ_1 then defines the eigen-function corresponding to the eigen-value λ_1^{-1}; and we have

$$\phi_1 = \lambda_1 K \phi_1. \tag{107}$$

It can easily be seen that if ϕ_1 and ϕ_2 are eigen-functions of K corresponding to λ_1^{-1} and λ_2^{-1}; $\lambda_1^{-1} \neq \lambda_2^{-1}$, then

$$[\phi_1, \phi_2] = 0. \tag{108}$$

We may also note that K is fully symmetrizable [32; p. 47] with respect to the scalar product i.e. the potential of a single layer of density ϕ_i, corresponding to each eigen-function of K for non-zero λ_i^{-1}, cannot be zero on S unless $\phi_i \equiv 0$. Thus the following more general result obtains [32].

Theorem 2: If the set

$$\{\phi;\ [K\phi,\ \phi] > 0\}$$

is non-void, then there exists a maximal set of linearly independent orthonormal set of eigen-functions ϕ_i, corresponding to positive eigen-values

$$\lambda_1^{-1} > \lambda_2^{-1} \ \ldots > \lambda_n^{-1} \ \ldots ,$$

such that

$$\max_{\phi \in E_i} [K\phi,\ \phi] = \lambda_i^{-1}, \tag{109}$$

where E_i is the set of all $\phi \in H$ such that

$$[\phi,\phi] = 1;\ \ [\phi,\phi_j] = 0,\ j = 1,2,\ \ldots\ ,\ i - 1.$$

The set $\{\phi_i,\ \lambda_i^{-1}\}$ is denumerably infinite and $\lambda_i^{-1} \to 0$.

In the above theorem the eigen-values are obtained by solving a series of maximum problems, each one depending on the solution of the preceding one. From this, however, a somewhat different characterization may be obtained which avoids any reference to the previous eigen-values [32].

If λ_n^{-1} exists and $f_1,\ f_2,\ \ldots\ ,\ f_{n-1}$ are arbitrary elements of H and

$$\lambda^{-1}(f_1,\ \ldots\ ,\ f_n) = \text{l.u.b.}_{f \in E_n} [Kf,f], \tag{110}$$

where E_n is the set of all f, such that

$$[f,f] = 1;\ \ [f,f_j] = 0,\ j = 1,2,\ \ldots\ ,\ n - 1,$$

then

$$\lambda_n^{-1} = \text{Min } \lambda^{-1}(f_1, \ldots, f_n). \tag{111}$$

This is known as the mini-max property of the eigen-values. This property [15; p. 31-33] implies the monotonicity of the approximate eigen-values when successive finite sections of (99) are taken for the matrix corresponding to K. We will have some occasion to make use of this fact in the next chapter.

20. Expansion Theorems

The set $\{\phi_i, \lambda_i^{-1}\}$ is infinite, where ϕ_i are the eigen-functions of K corresponding to the eigen-values λ_i^{-1}, and

$$[\phi_i, \phi_j] = \delta_{ij}.$$

Then we may easily establish the following analogous expansions [32].

Theorem 1: For arbitrary $f, g \in H$

$$[f,f] \geq \sum_i |[f,\phi_i]|^2 \tag{112}$$

$$[Kf,f] = \sum_i \lambda_i^{-1} |[f,\phi_i]|^2 \tag{113}$$

$$[Kg,f] = \sum_i \lambda_i^{-1} [f,\phi_i][g,\phi_i] = \sum_i [Kf,\phi_i][g,\phi_i]. \tag{114}$$

Proof: Let

$$f^{(n)} = f - \sum_{i=1}^{n} [f,\phi_i]\phi_i \tag{115}$$

then, obviously

$$(116)\quad 0 \le [f^{(n)}, f^{(n)}] = [f,f] - \sum_{i=1}^{n} |[f,\phi_i]|^2$$

which yields (112), known as the Bessel's inequality.

Also

$$(117)\quad [Kf^{(n)},f^{(n)}] = [Kf,f] - \sum_{i=1}^{n} \lambda_i^{-1}|[f,\phi_i]|^2$$

and since

$$[f^{(n)},\phi_i] = 0, \quad i = 1,2, \ldots , n ,$$

then, by the maximum principle

$$(118)\quad [Kf^{(n)},f^{(n)}] \le |\lambda_{n+1}^{-1}||[f^{(n)},f^{(n)}]|$$

$$\le |\lambda_{n+1}^{-1}|\ [f,f], \text{ by (116)}$$

$$\to 0 \text{ as } n \to \infty .$$

Thus (117) gives Hilbert quadratic expansion (113). Similarly the bilinear expansion (114) may be obtained.

q.e.d.

With the help of the above expansions we may establish the following identity.

Theorem 2: For $f \in H$,

$$(119)\quad [Kf^{(n)},f^{(n)}] = [(K - K_n)f,f],$$

where

$$(120)\quad f^{(n)} = f - \sum_{i=1}^{n} c_i\ \phi_i; \quad c_i = [f,\phi_i] ,$$

and

$$(121)\quad K_n = \sum_{i=1}^{n} \lambda_i^{-1}\, \phi_i\, \psi_i; \qquad \psi_i = \int \frac{1}{r_{pq}}\, \phi_i(q)\, dS_q,$$

ψ_i being the solution of the adjoint homogeneous equation (19) corresponding to λ_i^{-1}.

Proof: We may expand $f \in H$ in terms of the first n eigenfunctions ϕ_i of K

$$f = \sum_{i=1}^{n} c_i\, \phi_i + f^{(n)},$$

where $c_i = [f,\phi_i]$; $[f^{(n)},\phi_i] = 0$, $i = 1, 2, \ldots, n$,

thus

$$f^{(n)} = f - \sum_{i=1}^{n} c_i\, \phi_i$$

and from (117)

$$\begin{aligned}
[Kf^{(n)},f^{(n)}] &= [Kf,f] - \sum_{i=1}^{n} \lambda_i^{-1}\, c_i^2 \\
&= [Kf,f] - \sum_{i=1}^{n} [\lambda_i^{-1}\, c_i\, \phi_i,\, f] \\
&= [Kf - \sum_{i=1}^{n} \lambda_i^{-1}\, c_i\, \phi_i,\, f] \\
&= [(K - K_n)f, f],
\end{aligned}$$

where $$K_n f = \sum_{i=1}^{n} \lambda_i^{-1}\, c_i\, \phi_i;$$

since $$c_i = [f,\phi_i] = \iint f(p)\, \frac{1}{r_{pq}}\, \phi_i(q)\, dS_p\, dS_q$$

and $$\psi_i = \int \frac{1}{r_{pq}} \phi_i(q) dS_q.$$

$$c_i = \int \psi_i(p)\, f(p) dS_p$$

and

$$K_n = \sum_{i=1}^{n} \lambda_i^{-1}\, \phi_i\, \psi_i.$$

q.e.d.

Definitions: We may also write

$$K_n f = \sum_{i=1}^{n} \lambda_i^{-1}\, c_i\, \phi_i$$

$$= \sum_{i,j}^{n} [K\phi_i, \phi_j][f, \phi_i]\phi_i; \quad [\phi_i, \phi_j] = \delta_{ij},$$

which defines the transformation of finite rank [33; p. 158].

In the following the concept of completely continuous transformation is essential. There are various definitions of this [33; p. 178,206]. For the present we recall that

(i) All linear transformations of finite rank as well as their uniform limits are completely continuous.

(ii) Every linear transformation which can be approximated in norm arbitrarily closely by a completely continuous linear transformation is itself completely continuous.

Theorem 3: K is completely continuous with respect to the Dirichlet norm.

Proof: From Theorem 2, we have

(119) $$[Kf^{(n)}, f^{(n)}] = [(K - K_n)f, f];$$

also from (116)

$$[f^{(n)},f^{(n)}] \leq [f,f],$$

therefore

$$\frac{[(K - K_n)f,f]}{[f,f]} \leq \frac{[Kf^{(n)},f^{(n)}]}{[f^{(n)},f^{(n)}]},$$

$$\sup_{f\in H} \frac{[(K - K_n)f,f]}{[f,f]} \leq \sup_{f\in H} \frac{[Kf^{(n)},f^{(n)}]}{[f^{(n)},f^{(n)}]}.$$

Since $[f^{(n)},\phi_j] = 0$, $j = 1,2, \ldots n$, then by the maximum principle, we have

$$\sup_{f\in H} \frac{[(K - K_n)f,f]}{[f,f]} \leq \lambda_{n+1}^{-1}.$$

We define the norm of K in the usual manner

$$||K|| = \sup_{f\in H} |[Kf,Kf]|^{1/2}$$

under the condition that the norm of f,

$$||f|| = |[f,f]|^{1/2} = 1.$$

By using the following result in case of symmetric transformations [33; p. 231]

$$\sup_{f\in H} |[(K - K_n)f,f]| = \sup_{f\in H} |[(K - K_n)f, (K - K_n)f]|^{1/2}$$

$$= ||K - K_n||,$$

(122) yields

$$||K - K_n|| \leq \lambda_{n+1}^{-1}. \tag{123}$$

Since $\lambda_{n+1}^{-1} \to 0$ as $n \to \infty$, we conclude that K_n tends to

K uniformly in norm, therefore K is completely continuous.

q.e.d.

Evidently the argument above depends on the orthonormal system constituted by the eigen-functions ϕ_i of K. What can we say about the convergence in the norm in case some other basis is referred to ? In order to answer this, we recall an equivalent definition of complete continuity [33; p. 178]. A transformation is said to be completely continuous if it transforms every infinite bounded set into a compact set. The following theorem [33; p. 204] shows that the convergence in norm obtains under more general conditions.

Theorem: If K is a completely continuous linear transformation, the 'reduced' transformations K_n defined by

$$K_n f = \sum_{i,j}^{n} [f,\phi_i][K\phi_i,\phi_j]\phi_j \tag{124}$$

tend uniformly in norm to K when $n \to \infty$, $\{\phi_i\}$ being a complete orthonormal sequence in H.

Thus in particular, we may conclude that the convergence in norm obtains when the orthonormal basis constituted by $P_i f$ is used. This fact together with the complete continuity of K will be needed in the next chapter.

CHAPTER IV

RATIONAL APPROXIMATION

21. Convergence of Eigen-values and Eigen-functions

Rational functions furnish important approximate representations of analytic functions. From a practical standpoint, the solution of integral equations (18) and (14) can, at best, be accomplished approximately. To this end, we propose to apply the elegant method of minimized iteration [22], outlined in the beginning of Section 18, to the integral equation associated with the Robin-Poincaré Problem which is of more general interest [18; 1]. This effectively gives the approximate solution as a rational function of λ^{-1}.

As before, we may write (18) and (14) in the following form:

(125) $$\mu = \lambda K\mu,$$

(126) $$\mu = \lambda K\mu + f,$$

where $f \neq 0$ is a given initial function prescribed for (126) and arbitrary for (125). The procedure for obtaining the solution may be described as follows:

For a given f we generate the subspaces H_i and H as in Section 18 and define π_i a projection operator onto H_i and reduce the problem to a finite-dimensional one by

replacing K by $\pi_i K$ on H_i viz. solve the following:

$$\mu = \lambda \pi_i K \mu \qquad \text{on } H_i; \tag{127}$$

$$\mu = \lambda \pi_i K \mu + f \qquad \text{on } H_i \tag{128}$$

and then allow $i \to \infty$.

We note that $\pi_i K$ is symmetric with respect to the scalar product, i.e. for $f, g \in H_i$

$$[\pi_i K f, g] = [Kf, g] = [f, Kg] = [f, \pi_i K g],$$

and that the Rayleigh quotient for the finite dimensional operator $\pi_i K$ has the form

$$M(\mu) = \frac{[K\mu, \mu]}{[\mu, \mu]}, \tag{129}$$

independent of i, provided μ is restricted to H_i.

Note that H reduces K and that we can consider only those eigen-functions of K which lie in H. Also it is possible that one arbitrary function f may not yield all the eigen-functions of (125) and another may be necessary.

Let the eigen-values of $\pi_i K$ in the finite dimensional subspace H_i be denoted by

$$\lambda_{1i}^{-1} > \lambda_{2i}^{-1} > \ldots > \lambda_{ii}^{-1} \tag{130}$$

and the corresponding eigen-functions by

$$\mu_{1i}, \mu_{2i}, \cdots, \mu_{ii}; \tag{131}$$

we may assume that

$$[\mu_{ji}, \mu_{ji}] = 1; \quad [f, \mu_{ji}] \geq 0, \; j = 1, 2, \ldots i. \tag{132}$$

Since K has a countable discrete real spectrum, let the eigen-values be

$$\lambda_1^{-1} > \lambda_2^{-1} > \cdots , \tag{133}$$

and the corresponding eigen-functions be

$$\mu_1, \mu_2, \cdots , \tag{134}$$

such that

$$[\mu_j, \mu_j] = 1; \quad [f, \mu_j] > 0; \quad j = 1, 2, \cdots . \tag{135}$$

Then the following result on convergence due to Karush [20; p. 842] obtains

Theorem 1: $\{\lambda_{ji}^{-1}\}$ is a monotonically increasing sequence and the following limits hold:

$$\lim_{i\to\infty} \lambda_{ji}^{-1} = \lambda_j^{-1}; \quad j = 1, 2, \cdots , \tag{136}$$

$$\lim_{i\to\infty} \mu_{ji} = \mu_j; \quad j = 1, 2, \cdots , \tag{137}$$

in the sense of norm.

Since K is completely continuous with respect to the scalar product the following additional results on the rate of convergence of the eigen-values and eigen-functions may be stated [20; p. 844].

Theorem 2: Let δ be an arbitrary number with $0 < \delta < 1$. Then for each eigen-value λ_k^{-1} of K there is a constant A independent of i such that

$$|\lambda_k^{-1} - \lambda_{ki}^{-1}| \leq A\delta^i, \tag{138}$$

and for each eigen-function μ_k of K,

$$|\mu_k - \mu_{ki}| \leq A\delta^i \tag{139}$$

in the sense of norm.

In the next section we turn our attention to the equation (126) and seek its approximate solution in H_i and the estimates analogous to those given above.

23. Approximate Solution of the Robin-Poincaré Problem

We now seek the solution of the integral equation

$$\mu = \lambda K\mu + f. \tag{126}$$

It is known that if λ^{-1} is not an eigen-value of K, then $(\lambda^{-1}I - K)$ has a bounded reciprocal and a unique solution of (126) is given by

$$\mu = \lambda^{-1}(\lambda^{-1}I - K)^{-1}f. \tag{140}$$

Let $P_i(K)$ be the polynomial of degree i in K, generated by the minimized iteration scheme, then with the help of the operator identity

$$\frac{P_i(\lambda^{-1}) - P_i(K)}{(\lambda^{-1} - K)} = \sum_{m=0}^{i-1} P_{i-m-1}(\lambda^{-1})K^m, \tag{141}$$

we have

$$\lambda^{-1}(\lambda^{-1} - K)^{-1}f - \lambda^{-1}(\lambda^{-1} - K)^{-1}\frac{P_i(K)f}{P_i(\lambda^{-1})} = \tag{142}$$

$$= \frac{\lambda^{-1}}{P_i(\lambda^{-1})} \sum_{m=0}^{i-1} P_{i-m-1}(\lambda^{-1})K^m f$$

Now $P_i(\lambda^{-1})$ can be shown to be the characteristic polynomial of $\pi_i K$ on H_i [20] and thus by taking $P_i(K)f = 0$, we get the approximate solution $\mu^{(i)} \in H_i$, given by

$$(143) \qquad \mu^{(i)} = \frac{\lambda^{-1}}{P_i(\lambda^{-1})} \sum_{m=0}^{i-1} P_{i-m-1}(\lambda^{-1}) K^m f$$

which is essentially a rational function of λ^{-1}. The advantage of (143) over the Neumann series solution (24) is that the weights

$$w_{im}(\lambda^{-1}) = \frac{P_{i-m-1}(\lambda^{-1})}{P_i(\lambda^{-1})}$$

of $K^m f$ in (143) help the series to converge for all values of λ^{-1}. Furthermore, $K^m f$ may be expressed as a linear combination of the polynomials $P_m f$ generated by the minimized iteration and the solution written as

$$(144) \qquad \mu^{(i)} = \frac{\lambda^{-1}}{P_i(\lambda^{-1})} \sum_{m=0}^{i-1} \overline{P}_{i-m-1}(\lambda^{-1}) P_m(K) f$$

where the coefficients, $\overline{P}_j$ are called the 'reversed polynomials' [22; p. 269]. The following result gives the rate vergence of $\mu^{(i)}$, in the light of the equivalence of the two methods, viz. that of projection operator $\pi_i K$ and the minimized iteration, established in [20; p. 849-850].

Theorem 1: If λ^{-1} is not an eigen-value of K, then the

ith approximate solution $\mu^{(i)}$ of

(126) $$\mu = \lambda K\mu + f$$

onstructed through the minimized iteration scheme satisfies the following

(145) $$|\mu^{(i)} - \mu| \leq A\delta^i; \quad 0 < \delta < 1,$$

in the sense of norm, where A is a suitable constant, independent of i.

The zeros of the polynomial $P_i(\lambda^{-1})$ coincide with that of the characteristic polynomial of $\pi_i K$ and hence give approximate eigen-values to λ_j^{-1}, $j = 1,2. \ldots , i.$

The following theorem indicates how the eigen-functions may be computed.

Theorem 2: If $\mu_j \in H$ is the jth eigen-function of K corresponding to λ_j^{-1}, then

(146) $$\mu_j = \frac{\sum_{i=0}^{\infty} P_i(\lambda_j^{-1})\, P_i(K)f}{[\sum_{i=0}^{\infty} P_i^2(\lambda_j^{-1})]^{1/2}}.$$

Proof: Let

(147) $$f = \sum_{j=1}^{\infty} c_j\, \mu_j,$$

where

(148) $$c_j = [f, \mu_j], \quad [\mu_j, \mu_j] = 1,$$

and

(149) $$\mu_j = \sum_{i=0}^{\infty} d_{ji}\, P_i(K)f.$$

Then

$$d_{ji} = [\mu_j,\ P_i(K)f]$$

$$= [P_i(K)\mu_j,\ f]$$

$$= c_j\, P_i(\lambda_j^{-1});$$

thus

(150) $$\mu_j = \sum_{i=0}^{\infty} c_j\, P_i(\lambda_j^{-1})\, P_i(K)f.$$

Also

$$1 = [\mu_j,\ \mu_j] = \sum_{i=0}^{\infty} c_j^2\, P_i^2(\lambda_j^{-1})$$

or

(151) $$c_j^2 = \frac{1}{\sum_{i=0}^{\infty} P_i^2(\lambda_j^{-1})}$$

substituting the value of c_j in (150) we get the desired expression for μ_j. The c_j are the Fourier coefficients of f relative to the orthonormal system $\{\mu_j\}$. q.e.d.

However, in practice, approximate values of μ_j will be obtained, since λ_j^{-1} in (146) is a zero of the polynomial $P_i(\lambda^{-1})$ and the summation stops at i - 1.

From the above development we may conclude that the eigen-values, eigen-functions, and the solution of the

integral equations of potential theory may be computed approximately by the minimized iteration scheme.

23. Concluding Remarks

In the foregoing, we studied four different representations of the solution of integral equations corresponding to the Robin-Poincaré and Neumann-Poincaré Problems. We conjecture that these representations may be linked with one parent representation, that of a continued fraction. There exists a very well developed analytic theory of continued fractions [29; 41]. Moreover, continued fractions are also efficient computationally.

We may note that there is a striking similarity between the principal results of classical potential theory, obtained by Fredholm theory and those related to the successive approximants of continued fractions. In view of the fundamental work of Stieltjes [39] on continued fractions and that of Hadamard [12; (i)] on polar singularities of an analytic function, represented by a Taylor series, it would seem probable to arrive at a theory, comparable to that of Fredholm.

Furthermore, a few significant results [39; p. 396] in the theory of continued fractions indicate a possible approach for the investigation of solutions in the case of infinite smooth surfaces or surfaces with edges and corners. In two dimensions, however, Carleman [5, (iii)] has obtained Fredholm's results for contours having a finite number of corners.

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