

TRAVELING WAVE SOLUTIONS OF
INTEGRO-DIFFERENTIAL EQUATIONS OF
ONE-DIMENSIONAL NEURONAL NETWORKS

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Abstract

In this thesis, we study traveling wave solutions of the integro-differential equations

$$u_t(t, x) = f(u, w) + \alpha \int_{\mathbb{R}} K(x - y) H(u(t, y) - \theta) dy$$

and

$$u_t(t, x) = f(u, w) - A(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x - y) H(u(t, y) - \theta) dy$$

which are used for modeling propagation of activity in one-dimensional neuronal networks. Under some moderate continuity assumptions, necessary and sufficient conditions for the existence and uniqueness of monotone increasing (decreasing) traveling wave solutions of these integro-differential equations are established by applying geometric methods of ordinary differential equations. Symmetry of monotone traveling wave solutions in opposite directions in a one-dimensional network is obtained. We seek solutions of the form $u = U(x + v(w, \theta)t, w)$, where v is the wave speed. The dependence of the wave speed, v , on the threshold θ and on the parameter w respectively are discovered. These results enrich the understanding of the dynamical behavior of neuronal networks.

The existence and uniqueness of traveling wave solutions for the first equation has been studied by other authors, but the conditions required to ensure the existence of the desired solutions are not soundly derived. Some errors in previous research are found and corrected.

The second equation is a singular perturbation problem of the system of equations modeling propagation of activity in one-dimensional neuronal networks. It is a more realistic model than the first one, and has not been studied by other authors.

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Chapter 1

Introduction

1.1 Problems

One-dimensional neuronal networks, coupled with direct synapses, are modeled by the system of equations [5]

$$\begin{aligned}\frac{\partial u}{\partial t} &= f(u, w) + g_{\text{syn}}(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x, y) s(y, t) dy, \\ \frac{\partial w}{\partial t} &= \epsilon g(u, w), \\ \frac{\partial s}{\partial t} &= \alpha(1 - s)H(u - \theta) - \beta s.\end{aligned}\tag{1.1}$$

Here, $u(t, x)$, $w(t, x)$, and $s(t, x)$ are state variables of the neuronal network, x stands for the network spatial coordinate and t is the time variable. $K(x, y)$ is the weight function. Finally, α , g_{syn} , u_{syn} , θ , and β are constants. (See Chapter 2 for details.)

The study of traveling wave solutions of this kind of systems is of crucial interest and has been pursued in recent decades by numerical simulations or theoretical approaches [34, 32, 31, 30, 24]. A simplified form of this model,

studied by Zhang [38], is

$$u_t(t, x) = f(u, w) + \alpha \int_{\mathbb{R}} K(x - y)H(u(t, y) - \theta) dy. \quad (1.2)$$

However, the main theorem on the existence of traveling wave solutions proposed in [38] is not correct. We investigate this problem again in this thesis in order to offer a correct answer.

The second problem is the equation

$$u_t(t, x) = f(u, w) - A(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x - y)H(u(t, y) - \theta) dy, \quad (1.3)$$

which is a simplified form of (1.1) under the following assumption:

1. ϵ is small so w varies slowly and can be treated as a parameter.
2. α is about ten times bigger than β , so $s(t, y)$ rises fast to its stable value $\frac{\alpha}{\alpha + \beta}$, and can be approximately treated as a constant once $u(t, y) > \theta$.

It is a better model than (1.2) because it contains the factor of $(u - u_{\text{syn}})$ in the last term.

In this thesis, the traveling wave solutions we look for have the form

$$u(t, x) = U(x + v(w, \theta)t, w) = U(z, w), \quad z = x + v(w, \theta)t,$$

with initial condition

$$U(0, w) = \theta,$$

and boundary conditions

$$\lim_{z \rightarrow +\infty} U(z, w) = l_+(w), \quad \lim_{z \rightarrow -\infty} U(z, w) = l_-(w).$$

Monotone traveling wave solutions model the two important behaviors of neuronal networks. Monotone increasing traveling wave solutions model the

propagation of neurons firing in one-dimensional neuronal networks. Monotone decreasing traveling wave solutions describe the propagation of neurons returning to the rest phases in the one-dimensional neuronal networks.

1.2 Thesis contributions

The first contribution is that we establish necessary and sufficient conditions for the existence and uniqueness of monotone increasing (decreasing) traveling wave solutions of the integro-differential equation (1.2). The same problem has been studied by Zhang [38] using analytic methods. In [38] the following conditions:

- (1) $\alpha + f(u, 0) = 0$ has a unique positive solution $\beta > 1$ and $f'(\beta) < 0$.
- (2) There exists a unique number w_0 such that $2f(\theta, w_0) + \alpha = 0$, $2f(\theta, w) + \alpha < 0$ for all $w > w_0$, and $2f(\theta, w) + \alpha > 0$ for all $w < w_0$,

are used as sufficient conditions for the existence and uniqueness of monotone traveling wave solutions. However, these two conditions cannot ensure the existence of the desired solutions. A counterexample is presented. Our results give the correct conditions

- (1) $f(u, w) + \alpha = 0$ has only one solution $\phi_2(w)$,
- (2) $f(u, w) + \frac{1}{2}\alpha > 0$ for $u < \theta$,

which are necessary and sufficient for the existence and uniqueness of monotone increasing traveling wave solutions, and

- (1) $f(u, w) = 0$ has only one solution $\phi_1(w)$,

$$(2) \quad f(u, w) + \frac{1}{2}\alpha < 0 \text{ for } u > \theta,$$

which are necessary and sufficient for the existence and uniqueness of monotone decreasing traveling wave solutions.

The second contribution is that we prove the symmetry of the monotone traveling wave solutions of (1.2) in opposite directions along the one-dimensional network, and discover the dependence of the wave speed, $v = v(w, \theta)$, on the threshold θ and on the parameter w , respectively. These results are helpful in understanding wave propagation in neuronal networks.

The third contribution is that we establish necessary and sufficient conditions for the existence and uniqueness of monotone increasing (decreasing) traveling wave solutions of the integro-differential equation (1.3). Equation (1.3) is an original model which is a simplification of model (1.1). Compared to equation (1.2) it considers the effect of reverse potential, and describes neuronal networks more realistically than the former one.

The fourth contribution is that we prove the symmetry of the monotone traveling wave solutions of (1.3) in opposite directions along a one-dimensional network and discover the dependence of the wave speed $v = v(w, \theta)$ on the threshold θ and on the parameter w , respectively.

The fifth contribution is the bifurcation analysis of the general Morris–Lecar model, which includes a complicated case where the system has five fixed points. In this context, the existence of a limit cycle and the limit position of the limit cycle are proved by using an elementary method.

The sixth contribution is the methodological feature of this thesis, that is, the use of geometrical methods of ordinary differential equations. Since the kernel function $K(x, y) = K(x - y)$ is not smooth, not even continuous,

and the integral $\int_{-\infty}^z K(\xi)d\xi$ appears in the ordinary differential equations, only the first derivative of the solutions can be used in the proofs. Moreover, since the solutions we are interested in are heteroclinic orbits of ordinary differential equations, which are uniquely defined on the infinite interval, then the continuous dependence of solutions on the initial values and on the parameters cannot be used to prove the properties of the desired traveling wave solutions, which depend on the threshold θ and the parameter w . By using geometrical methods, these technical difficulties are overcome.

1.3 Thesis contents

Chapter 2 gives a general view of cellular neurophysiology and the chronological development of mathematical models for neuron membrane potential. Fundamental models, such as Hodgkin–Huxley equations, FitzHugh–Nagumo model, Morris–Lecar model, and whole cell models are introduced. The dynamical behaviors of the solutions to the Morris–Lecar model are studied in detail. The models for neuronal networks are reviewed, and we propose the mathematical problems, which are investigated in this thesis.

Chapter 3 presents the necessary and sufficient conditions for the existence and uniqueness of monotone traveling wave solutions of the integro-differential equation (1.2). The faults in [38] are discussed, and a counterexample, which satisfies the requirements of [38], is presented. The symmetry of monotone traveling wave solutions of (1.2) in opposite directions along a one-dimensional network are proved. The dependence of the wave speed v on the threshold θ and on the parameter w are discussed. The main content

of chapter 3 will appear in [17].

Chapter 4 presents the necessary and sufficient conditions for the existence and uniqueness of monotone traveling wave solutions of the integro-differential equation (1.3). The symmetry of the monotone traveling wave solutions of (1.3) in opposite directions along a one-dimensional network are proved. The dependence of the wave speed $v = v(w, \theta)$ on the threshold θ and on the parameter w are discovered, respectively. Thus we have proved that the results for equation (1.2) also hold for equation (1.3). Both equations (1.2) and (1.3) are models for excitatory coupled neuronal network.

Chapter 5 discusses the connection between our results and the dynamical features of excitatory coupled neuronal network. Future work is also considered in this chapter.

Chapter 2

Review of Cellular Neurophysiology and its Mathematical Problems

The purpose of this chapter is to introduce basic facts from neurophysiology for the benefit of the nonexpert. The contents of this chapter claims no originality. It is based on [5, 6, 11, 19, 20, 33]. The expert may skip this chapter and go directly to Chapters 3 and 4.

2.1 Basic facts from neurophysiology

The nervous system mainly consists of a special type of cells called neurons. Although neurons vary in shape and function, a neuron generally has two parts, *cell body* (or *soma*) and neurite. There are two kinds of neurite. They are *dendrite* and *axon*. A soma is the metabolic and nutrient center of a

neuron. A dendrite is a tree shaped neurite growing from a soma. It can receive signals and transfer them to the soma. A neuron has one or more dendrites but only one axon. An axon can transfer the signals to another neuron or other organs such as muscles or glands.

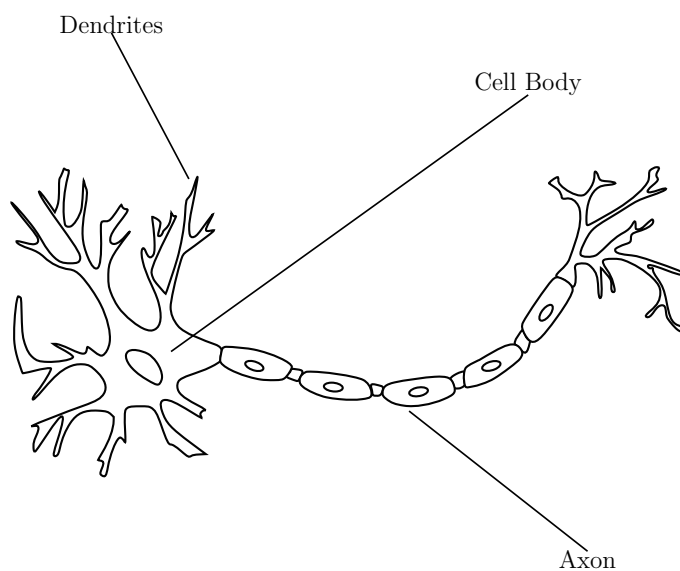


Figure 2.1: A Neuron. Adapted from: [carrie on Clker.com](http://carrieon.com).

Like other types of animal cells, a neuron is surrounded by a membrane, which mainly consists of lipid bilayer. The membrane separates the inside of a neuron from the outside. There are ions inside and outside the neuron. The most important of them are K^+ , Na^+ , Ca^{2+} , and Cl^- . The concentrations of ions is very different inside and outside the neuron. This difference in the concentration results in a potential difference between the inside and outside of the neuron, called *membrane potential*.

The neuron can maintain membrane potential for two reasons. The first is that the plasma membrane has low permeability to ions. The other reason is

the ion pumps in the neuron. These pumps are special types of proton which can pump ions from the low concentration side to the high concentration side. The membrane potential can vary rapidly under certain conditions. This is due to the ion channels embedded in the membrane. Ion channels are integral membrane proteins through which ions can move rapidly from high concentration side to low concentration side. This fast movement results in a quick change in membrane potential.

2.2 Mathematical description of membrane potential

Using circuit models to describe cell membrane potentials and ionic currents began in 1968 [2]. Because of its bilayer structure the membrane of the neuron can be considered as a capacitor. The ionic permeabilities of the membrane act as resistors in an electronic circuit. The electrochemical driving forces act as batteries driving the ionic currents. The typical equivalent electrical circuit for one neuron is as in Figure 2.2.

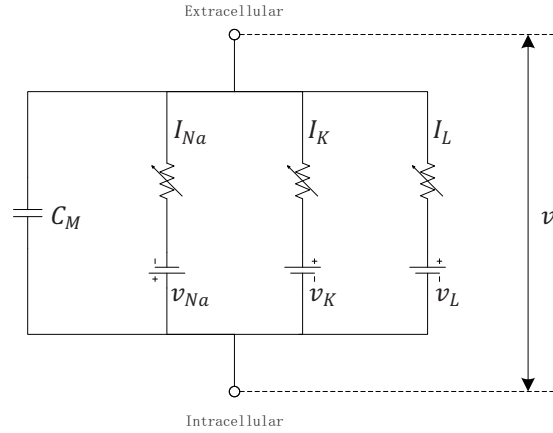


Figure 2.2: Equivalent electrical circuit.

In fact, a neuron is a highly complicated cell. Only in the most idealized sense, the electrical behavior of neurons is modeled. The circuit model is just a point model for a patch of membranes, or the whole cell is idealized as a point. The electrical current flowing out of the cell is defined positive. The membrane potential is defined as

$$V = V_m = V_{\text{in}} - V_{\text{out}}.$$

The conductance is defined as

$$g_i = \frac{1}{R_i},$$

where R_i is the membrane resistance to the ionic electrical current for the ion species i .

2.2.1 Resting membrane potential of neuron.

Nernst equation and reversal potential of ions.

For a given neuron and an ion species, the cross-membrane movement of ions is driven by concentration gradient and electrical field. Using physical laws, we can describe the ion movement as

$$J = -D \frac{d[C]}{dx} - \mu z [C] \frac{dV}{dx}, \quad (2.1)$$

where D is the diffusion coefficient, $[C]$ is the ion concentration, x is the site ordinate in the membrane from inside to outside, z is the ion valence, μ is the ion drift mobility, and J is the ion flux from inside to outside. According to the Einstein relation

$$D = \frac{\mu k T}{q},$$

equation (2.1) can be transformed into the Nernst–Planck equation

$$I = - \left\{ u z^2 F [C] \frac{dV}{dx} + u z R T \frac{d[C]}{dx} \right\} \quad (2.2)$$

with

$$k = \frac{R}{N_A}, \quad F = q N_A, \quad I = J z F, \quad u = \frac{\mu}{N_A},$$

where N_A is Avogadro's constant, q is the charge of electron, R is the ideal gas constant, and T is temperature in degree Kelvin. At equilibrium, $I = 0$, by integrating (2.2) the membrane potential can be written as

$$E_i = V_{\text{in}} - V_{\text{out}} = \frac{RT}{zF} \ln \frac{[C]_{\text{out}}}{[C]_{\text{in}}}. \quad (2.3)$$

Equation (2.3) describes the potential for the specified ion at equilibrium state, at which there is no net ion current flow through the membrane. This potential is called the *Nernst (equilibrium or reversal) potential* of this kind of ion at the specified temperature in the neuron.

Goldman–Hodgkin–Katz equation and resting potential

Consider the movement of the K^+ , Na^+ , and Cl^- ions in neuron membranes and assume that:

1. Each species of ion obeys equation (2.2),
2. Ions move independently,
3. The electric field in the membrane is constant, i.e. $\frac{dV}{dx}$ is constant.

Denoting $V = V_{\text{in}} - V_{\text{out}}$ and l the membrane thickness, then $\frac{dV}{dx} = \frac{V}{l}$, and equation (2.2) has the form

$$I_i = uz^2F[C_i] \frac{V}{l} - uzRT \frac{d[C_i]}{dx} \quad (2.4)$$

for the ion species i .

Under the above assumptions, the electric current across the membrane should be constant and satisfy the boundary conditions

$$[C_i](0) = \beta_i[C_i]_{\text{in}} \quad \text{and} \quad [C_i](l) = \beta_i[C_i]_{\text{out}},$$

where $\beta_i < 1$ is the relative solubility of ions in the membrane compared to that in an aqueous solution. I_i in equation (2.4) is an unknown constant. Solving the differential equation (2.4) gives

$$[C_i](x) = \frac{I_i l}{uz^2 F V} + A e^{uz^2 F V x / luzRT},$$

where A and I_i are unknown constants. By the boundary conditions we can determine the constant I_i as

$$I_i = PzF\sigma \frac{[C_i]_{\text{in}} - [C_i]_{\text{out}} e^{-\sigma}}{1 - e^{-\sigma}} \quad (2.5)$$

with

$$\sigma = \frac{zVF}{RT}, \quad P = \frac{\beta_i u RT}{lF}.$$

Here, P represents the membrane permeability for the specified ions. Since the membrane is permeable to K^+ , Na^+ , and Cl^- , the total electric current is the sum $I_K + I_{Na} + I_{Cl} = I = 0$. For K and Na , $z = 1$, and for Cl , $z = -1$. Applying (2.5) to these three kinds of ions, we get

$$V = V_{GHK} = \frac{RT}{F} \ln \frac{P_{K^+}[K^+]_{out} + P_{Na^+}[Na^+]_{out} + P_{Cl^-}[Cl^-]_{in}}{P_{K^+}[K^+]_{in} + P_{Na^+}[Na^+]_{in} + P_{Cl^-}[Cl^-]_{out}}. \quad (2.6)$$

Here $[K^+]$, $[Na^+]$, $[Cl^-]$ stand for the concentrations and P_i are the permeability for the ion i , $i = K^+$, Na^+ , and Cl^- . This equation is called the *Goldman–Hodgkin–Katz* (GHK) voltage equation [20], and gives the *resting (reverse) potential* of the neuron in terms of the ion concentrations and the permeabilities of membrane for the ions.

Another form for the resting potential of a membrane can be obtained in the following way. We consider the ion currents flow through the membrane in an equivalent circuit. The electrical current equation of the circuit is

$$I_m = -g_K(V - E_K) - g_{Na}(V - E_{Na}) - g_{Cl}(V - E_{Cl}) = C \frac{dV}{dx},$$

where g_i is the conductance of the membrane for ion i , and E_i is the Nernst potential of ion i . Since the resting potential V is constant, the total cross-membrane current must be zero. We have the resting potential of membrane expressed in conductances of membrane and Nernst potentials for ions

$$V = V_m = \frac{g_K E_K + g_{Na} E_{Na} + g_{Cl} E_{Cl}}{g_K + g_{Na} + g_{Cl}}.$$

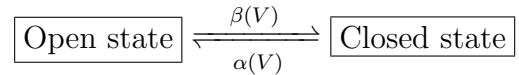
The primary character of neuron is that, under some conditions, the membrane potential of neuron can increase rapidly and transmit signals to other

neurons. This increased potential is known as *action potential* and one says that the neuron *fires*.

2.2.2 Voltage-dependent gating and gating variables

An important character of neuron membranes is their nonlinear property in the current-voltage relation. The conductance, g_i , of a membrane is dependent on the membrane potential. The voltage-gated model developed by Hodgkin and Huxley is used to elucidate the mechanism of nonlinear conductance of membranes [12].

The mathematical description of voltage-dependent gates is based on the assumption that the ionic channels in membranes are gated with particles and the states of gates, decided by the particles, change in response to variation of the membrane potential. The change in gate states can be expressed by the schematic diagram



Open states change to closed states at the rate $\beta(V)$, whereas closed states change to open states at the rate $\alpha(V)$. Let the fraction of open channels in all channels be f_0 and the fraction of closed channels be $1 - f_0$. The rate of channel transitions, $\alpha(V)$ and $\beta(V)$, are functions in the form

$$\alpha(V) = \alpha_0 e^{-aV}, \quad \beta(V) = \beta_0 e^{-bV}, \quad (2.7)$$

where α_0 , β_0 , a , and b are constants. The function f_0 satisfies the equation

$$\frac{df_0}{dt} = \alpha(1 - f_0) - \beta f_0 = \frac{-(f_0 - f_\infty)}{\tau}, \quad (2.8)$$

where

$$f_\infty(V) = \frac{\alpha(V)}{\alpha(V) + \beta(V)}, \quad \tau(V) = \frac{1}{\alpha(V) + \beta(V)}. \quad (2.9)$$

By putting (2.7) into (2.9) and simplifying the expressions, we obtain the following expressions which frequently appear in modeling equations,

$$f_\infty(V) = 0.5 \left[1 + \tanh \frac{V - V_0}{2s_0} \right], \quad \tau(V) = \frac{e^{V(a+b)/2}}{2\sqrt{\alpha_0\beta_0} \cosh \frac{V-V_0}{2s_0}},$$

$$V_0 = \frac{\ln \frac{\beta_0}{\alpha_0}}{b - a}, \quad s_0 = \frac{1}{b - a}.$$

If the function f_∞ is an increasing function of V , then the channel gate is called *activation* (more channels open for higher potential). If f_∞ is a decreasing function of V , then the channel gate is called *inactivation* (fewer channels open for higher potential). The function f_∞ has range $(0, 1)$ and τ is the time constant which reflects the speed of the transients of the gate state, from closed to open, at the fixed membrane potentials. The solution f_0 of equation (2.8) depends both on time t and potential V , which is the probability of open gate for a specified ion species, and is called *gating variable*. Furthermore, the gating variables determine the conductance of the membranes.

2.2.3 Hodgkin–Huxley Model

In live neurons, the conductance g_i of a membrane for the ion species i depends on the states of the specified voltage-gated channels, which are functions of time t and potential V . The model of the squid giant axon has been developed by Hodgkin and Huxley [12, 13, 14, 15, 16] in 1951. The HH model describe the behavior of ion channels and the propagation of action

potentials in neurons. If the current along the axon is included, then the electrical current equation of the equivalent circuit is (the cable equation)

$$C \frac{\partial V}{\partial t} = \frac{a}{2R_0} \frac{\partial^2 V}{\partial x^2} - \sum_i g_i (V - E_i) + I_{\text{app}},$$

where C is the membrane capacity, V is the membrane potential, E_i is the Nernst potential of ion i , g_i is the membrane conductance for ion i which depends on the potential V and time t , x is the ordinate along the axon, a is the radius of the cylindrical axon, R_0 is the resistance along the axon per unit length, the second partial derivative expresses the current along the axon, and I_{app} is the applied electrical current to the neuron. In the Hodgkin–Huxley model, the K^+ and Na^+ ionic currents and the leak current are considered. The conductance for the leak current, g_L , is constant. The key problem is to decide the conductances g_K and g_{Na} . Based on experimental data on the conductances $g_{Na}(V, t)$ and $g_K(V, t)$ with the properties of the gating variables, Hodgkin and Huxley set

$$g_K(V, t) = n^4(V, t)\bar{g}_K, \quad g_{Na}(V, t) = m^3(V, t)h(V, t)\bar{g}_{Na}.$$

The constants \bar{g}_K and \bar{g}_{Na} are the maximal conductances for K^+ and Na^+ , respectively, which can be measured by experiments. The gating variable functions, $m(V, t)$, $n(V, t)$, and $h(V, t)$, satisfy the differential equation (2.8) with specified $\alpha_m(V)$, $\alpha_n(V)$, $\alpha_h(V)$ and $\beta_m(V)$, $\beta_n(V)$, $\beta_h(V)$, of the form as (2.7), respectively. The system of modeling equations of the membrane potential, without current along the axon nor applied current, is the following Hodgkin–Huxley (HH) equation [5]:

$$c_M \frac{dV}{dt} = -\bar{g}_K n^4 (V - E_K) - \bar{g}_{Na} m^3 h (V - E_{Na}) - g_L (V - E_L),$$

$$\begin{aligned} \frac{dn}{dt} &= \alpha_n(1-n) - \beta_n n, & \alpha_n(V) &= \frac{0.01(V+55)}{1 - e^{-(V+55)/10}}, & \beta_n(V) &= 0.125 e^{-(V+65)/80}, \\ \frac{dm}{dt} &= \alpha_m(1-m) - \beta_m m, & \alpha_m(V) &= \frac{0.1(V+40)}{1 - e^{-(V+40)/10}}, & \beta_m(V) &= 4 e^{-(V+65)/18}, \\ \frac{dh}{dt} &= \alpha_h(1-h) - \beta_h h, & \alpha_h(V) &= 0.07 e^{-(V+65)/20}, & \beta_h(V) &= \frac{1}{(1 + e^{-(V+35)})10}. \end{aligned}$$

The parameters are taken as:

$$\begin{aligned} \bar{g}_K &= 36mS/cm^3, & \bar{g}_{Na} &= 120mS/cm^3, & g_L &= 0.3mS/cm^3, \\ E_K &= 50mV, & E_{Na} &= -77mV, & E_L &= -54.4mV, & c_M &= 1\mu F/cm^2. \end{aligned}$$

Here, mV is the abbreviation for millivolt, which is a unit of potential, mS is the abbreviation for millisiemens which is a unit of electrical conductance, and μF represents microfarads, which is a unit of capacitance. The values of $\alpha(V)$ and $\beta(V)$ are empirical experimental data. It can be found from equation (2.8) that

$$h_\infty(V) = \frac{\alpha_h(V)}{\alpha_h(V) + \beta_h(V)}$$

is a decreasing function and h is an inactive channel gating variable. The full form of the HH equation, including the term $\frac{\partial^2 V}{\partial x^2}$ of the current along the axon, is more complicated.

For the Hodgkin–Huxley model equation, only numerical solutions can be obtained. It is widely used to simulate neuron dynamics. A detailed description of the Hodgkin–Huxley model is contained in the book [11].

2.2.4 FitzHugh–Nagumo Model

While the Hodgkin–Huxley model is realistic and biologically sound, the HH equations are too complicated for mathematical analysis. It is a 4-dimensional system of partial differential equations with nonlinear right-hand

sides. The only way to obtain its solutions is by numerical methods. Moreover, its orbits (phase trajectories) are curves in 4-dimensional space; hence only their projection to lower dimensional space could be viewed. For this reason, simpler, reduced models for a single neuron were developed. FitzHugh, in the 1950s, tried to reduce the HH model to a 2-dimensional one [8, 9]. The facts that the gating variables $n(V, t)$ and $h(V, t)$ have slow kinetics compared to the gating variable $m(V, t)$, and $n(V, t) + h(V, t) = 0.8$ (approximately) were observed. Thus, the HH model was reduced to a 2-dimensional model of the form

$$\begin{aligned}
C_m \frac{dV}{dt} &= -\bar{g}_K n^4 (V - E_K) - \bar{g}_{Na} m_\infty^3 (V) (0.8 - n) (V - E_{Na}) \\
&\quad - g_L (V - E_L) + I_{app}, \\
\frac{dn}{dt} &= \frac{n_\infty(V) - n}{\tau_n(V)}.
\end{aligned} \tag{2.10}$$

In equation (2.10) the gating variable m was approximated by its steady value m_∞ and $h(V, t)$ was substituted by $0.8 - n$. By now, many other simplified models have been developed. FitzHugh's other important discovery is that the V -nullcline has a shape similar to the one of a cubic polynomial in which the cubic term has a negative coefficient and the n -nullcline can be approximated by a straight line in the varying range of V . This observation suggests to reduce the model equation to the simpler dimensionless form:

$$\begin{aligned}
\frac{dV}{dt} &= BV(V - a)(b - V) - cw + I, \\
\frac{dw}{dt} &= \epsilon(V - kw).
\end{aligned} \tag{2.11}$$

The parameters B , a , b , c , and k are constants. The small positive number, $\epsilon > 0$, emphasizes that the ‘‘gating variable’’ w varies slowly compared with

the potential V . Finally, I is the injected current. System (2.11) was independently proposed by FitzHugh in 1961 and Nagumo in 1962 and is known as the FitzHugh–Nagumo model [9, 29]. Equations (2.11) are amenable to qualitative or quantitative methods of ordinary differential equations. One can adjust the parameters to produce an oscillating solution.

2.2.5 Morris–Lecar Model

The study of the Ca^{2+} channel started in the 1950s. The results of a number of research groups suggested that there must be some mechanism different from that reflected in the experiments conducted by Hodgkin and Huxley for the squid giant axon. Morris and Lecar [28] in 1981 proposed a simple model to explain the observed electrical behaviors of the barnacle muscle fiber. The model was tested against a number of experiments in designed conditions and the simulations of the model provided a good explanation of the experimental data. Following [6], the model equations are

$$\begin{aligned}
 C \frac{dV}{dt} &= -g_{Ca}m_\infty(V - E_{Ca}) - g_Kw_\infty(V - E_K) \\
 &\quad - g_L(V - E_L) + I_{\text{app}}, \\
 \frac{dw}{dt} &= \frac{\phi(w_\infty - w)}{\tau}, \\
 m_\infty(V) &= 0.5 \frac{1 + \tanh(V - V_1)}{V_2}, \\
 w_\infty(V) &= 0.5 \frac{1 + \tanh(V - V_3)}{V_4}, \\
 \tau(V) &= \frac{1}{\cosh \frac{V - V_3}{2V_4}}.
 \end{aligned} \tag{2.12}$$

Typical parameters for oscillatory solutions are

$$\begin{aligned}
 V_K &= -84mV, & g_K &= 8mS/cm^2, & E_{Ca} &= 120mV, & g_{Ca} &= 4.4mS/cm^2, \\
 E_L &= -60mV, & g_L &= 2mS/cm^2, & V_1 &= -1.2mV, & V_2 &= 18mV, \\
 V_3 &= 2nV, & V_4 &= 30mV, & \phi &= 0.04/ms, & C &= 20\mu F/cm^2.
 \end{aligned}$$

The functions $m_\infty(V)$ and $w_\infty(V)$ are, at the equilibrium state for fixed V , the fractions of open channels for Ca^{2+} and K^+ , respectively, which are only dependent on voltage. Here, w is the gating variable for K^+ and $\tau(V)$ is the activation time constant. These constants are determined by experiments. Under these settings, the Morris–Lecar equations have a V -nullcline like a cubic curve, and the w -nullcline is an increasing curve in the (V, w) plane, see Figure 2.3.

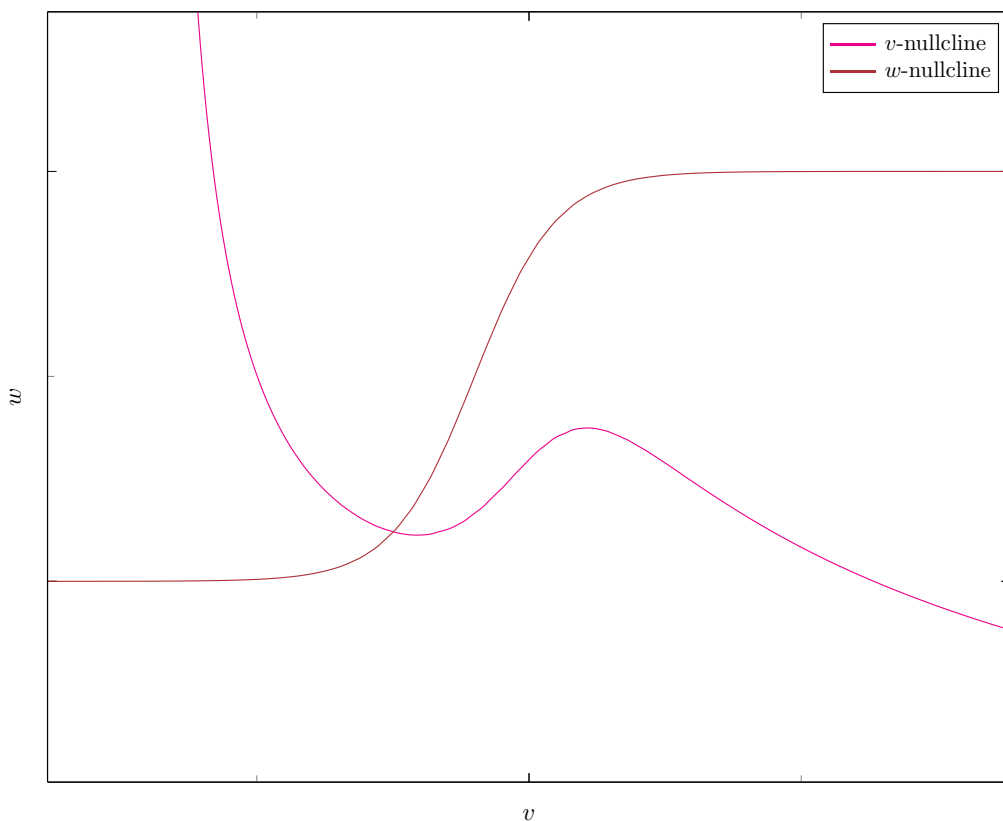


Figure 2.3: Nullclines of the Morris–Lecar equations.

The main character of the Morris–Lecar model is that it has a stable limit cycle for an extensive range of parameters. This limit cycle shows that the membrane potentials of the neuron oscillate periodically. The dynamical behavior of the solutions of system (2.12), which shows that the membrane potential variable V increases and decreases during the period of oscillation, describes the fact that the neuron fires repeatedly.

Many dynamical features of neuron membrane potentials can be displayed through numerical simulation by using the Morris–Lecar model with appropriate parameters. Moreover, this model can be analyzed by using many

methods of ordinary differential equations, especially, phase plane analysis and bifurcation theory, so that an overview of the changes of membrane potentials can be obtained.

Two-dimensional systems of ordinary differential equations, like the Fitz-Hugh–Nagumo system and the Morris–Lecar system, where one variable has a cubic-like nullcline and the other variable has an increasing (or decreasing) nullcline, have become a general module for building more comprehensive mathematical models of neurons [5, 33].

2.2.6 A whole cell model

Most neurons have several different ion channels. In one neuron there are different compartments that take up or release ions in different ways. Whole cell models have more variables and consist of more equations. In this kind of models, the Morris–Lecar model is used as the modular subsystem to describe the membrane potential and the channel variable for K^+ , coupled with equations for other channel variables or ionic currents, especially, the equations for the concentration of Ca^{2+} and the additional K^+ current activated by intracellular Ca^{2+} . A simple whole cell model can be constructed by adding the Morris–Lecar model with the K^+ current activated by the intracellular Ca^{2+} in the current equation and an equation for the concentration of Ca^{2+} .

This model can be written as in [6]

$$\begin{aligned}
C_m \frac{dV}{dt} &= -g_L(V - E_L) - g_K n(V - E_K) - g_{Ca} m_\infty(V)(V - E_{Ca}) \\
&\quad - g_{KCa} \frac{[Ca]}{[Ca] + k_{KCa}}(V - E_K), \\
\frac{dn}{dt} &= \frac{\Phi(n_\infty(V) - n)}{\tau_n(V)}, \\
\frac{d[Ca]}{dt} &= \epsilon(-\mu I_{Ca} - k_{Ca}[Ca]),
\end{aligned}$$

where g_{KCa} , ϵ , μ , k_{KCa} , and k_{Ca} are constants empirically determined by experiment. Whole cell models share a common dynamical feature that the active phase, during which the neuron rapidly spikes repeatedly, and the silent phase, in which the neuron stays in a resting behavior of near-steady-state, appear alternatively. This kind of dynamical behaviors is called bursting oscillations. Typical examples of whole cell models are contained in the book [6]. We are not pursuing this topic further because it is not discussed in the later part of this thesis.

2.2.7 Remarks

There are many models that describe the dynamics of membrane potentials of neurons. The models describing the propagations of membrane potentials along dendrite fibers or axons are systems of partial differential equations. What was introduced in the above section is only to show typical forms of neuron models. In numerical simulation, the dimensions and measure units of the variables and parameters are treated carefully. Fortunately, the theory of ordinary differential equations, including the qualitative theory, asymptotic analysis and numerical methods, offer a sound foundation for

the mathematical analysis of the Morris–Lecar model. Understanding the dynamical behaviors of the solutions of the Morris–Lecar equation is very helpful for building more realistic models.

2.3 Dynamical behavior of the Morris–Lecar model

The general mathematical form of the Morris–Lecar model is

$$\begin{aligned}\frac{dv}{dt} &= f(v, w) + I_{\text{app}}, \\ \frac{dw}{dt} &= \epsilon g(v, w), \quad \epsilon > 0.\end{aligned}\tag{2.13}$$

The following conditions are satisfied:

1. f and g are continuously differentiable functions of (v, w) .
2. The v -nullcline $f(v, w) + I_{\text{app}} = 0$ defines a curve C_1 like the cubic curve $v(v - a)(a + b - v) - w = 0$, $a, b > 0$. It has one w -minimum point $L = (v^-, w^-)$ and one w -maximum point $R = (v^+, w^+)$. Moreover, $f_w < 0$, $f_v \neq 0$, except for $f_v(v^-, w^-) = f_v(v^+, w^+) = 0$. The w -nullcline $g(v, w) = 0$ defines a monotone increasing curve C_2 and $g_w < 0$.
3. I_{app} is a constant or a function of v and t .

Many results and methods from the theory of ordinary differential equations can be used to study the dynamical behavior of equation (2.13). We mention the following points, which are interesting for computational simulations in neuron science.

The nullcline C_1 divides the phase plane into an upper and a lower part. In the upper part $f(v, w) < 0$, whereas in the lower part $f(v, w) > 0$. The nullcline C_2 divides the phase plane into left and right parts. In the left part, $g(v, w) < 0$, whereas $g(v, w) > 0$ in the right part.

The characteristic equation of (2.13) is

$$\begin{vmatrix} \frac{\partial f}{\partial v} - \lambda & \frac{\partial f}{\partial w} \\ \epsilon \frac{\partial g}{\partial v} & \epsilon \frac{\partial g}{\partial w} - \lambda \end{vmatrix} = \lambda^2 - \lambda \left(\frac{\partial f}{\partial v} + \epsilon \frac{\partial g}{\partial w} \right) + \epsilon \left(\frac{\partial f}{\partial v} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial v} \right).$$

Let (v_0, w_0) be a fixed point of system (2.13). Denote

$$I = \frac{\partial f}{\partial v}(v_0, w_0) + \epsilon \frac{\partial g}{\partial w}(v_0, w_0),$$

and

$$D = \frac{\partial f}{\partial v} \frac{\partial g}{\partial w}(v_0, w_0) - \frac{\partial f}{\partial w} \frac{\partial g}{\partial v}(v_0, w_0).$$

At the fixed point (v_0, w_0) , the characteristic equation is transformed into

$$\lambda^2 - I\lambda + \epsilon D = 0.$$

The following three observations are helpful for the consideration of the dynamical behavior of system (2.13).

Firstly, if (v_0, w_0) is on the left branch of C_1 then

$$\frac{\partial f}{\partial v}(v_0, w_0) < 0, \tag{2.14}$$

because $f(v, w_0) > 0$ for $v < v_0$ and $f(v, w_0) < 0$ for $v > v_0$. Furthermore,

$$I = \frac{\partial f}{\partial v}(v_0, w_0) + \epsilon \frac{\partial g}{\partial w}(v_0, w_0) < 0, \tag{2.15}$$

because $g_w < 0$. If (v_0, w_0) is on the middle segment of C_1 then

$$\frac{\partial f}{\partial v}(v_0, w_0) > 0,$$

because $f(v, w_0) < 0$ for $v < v_0$, and $f(v, w_0) > 0$ for $v > v_0$. If (v_0, w_0) is on the right branch of C_1 then inequalities (2.14) and (2.15) also hold for the same reasons as in the case of the fixed point being on the left branch of C_1 .

Secondly, the sign of D is decided by the slopes of the curves C_1 and C_2 at their point of intersection. The inequality $D > 0$ is equivalent to the condition that the slope of C_2 is bigger than the slope of C_1 because the following inequalities are equivalent:

$$\frac{\partial f}{\partial v} \frac{\partial g}{\partial w}(v_0, w_0) > \frac{\partial f}{\partial w} \frac{\partial g}{\partial v}(v_0, w_0) \iff -\frac{\frac{\partial f}{\partial v}(v_0, w_0)}{\frac{\partial f}{\partial w}(v_0, w_0)} < -\frac{\frac{\partial g}{\partial v}(v_0, w_0)}{\frac{\partial g}{\partial w}(v_0, w_0)}.$$

Thirdly, we have

$$\frac{\partial g}{\partial v}(v_0, w_0) > 0$$

because C_2 is an increasing curve, $g(v, w_0) < 0$ for $v < v_0$, and $g(v, w_0) > 0$ for $v > v_0$. If

$$\frac{\partial f}{\partial v}(v_0, w_0) - \epsilon \frac{\partial g}{\partial w}(v_0, w_0) = 0,$$

then

$$\Delta = I^2 - 4\epsilon D = \left[\left(\frac{\partial f}{\partial v} - \epsilon \frac{\partial g}{\partial w} \right) (v_0, w_0) \right]^2 + 4\epsilon \frac{\partial f}{\partial w} \frac{\partial g}{\partial v}(v_0, w_0) < 0. \quad (2.16)$$

These observations from the geometrical view point are useful in studying the stability of the fixed point. We list the configuration patterns of the nullclines in the phase plane and the stability of the fixed points in the following propositions.

Proposition 2.3.1. *If C_2 intersects C_1 on the left (or right) branch, then the fixed point is a stable node or spiral.*

Proof. Since $I < 0$ and $D > 0$ in this case, the characteristic roots have negative real parts for any $\epsilon > 0$. It is possible from (2.16) that for some $\epsilon > 0$ such that $I < 0$ and $\Delta < 0$ the fixed point is a stable spiral. \square

Proposition 2.3.2. *If C_2 intersects C_1 in the middle segment with smaller slope, then the fixed point is a saddle.*

Proof. Since $D < 0$, the characteristic roots are real numbers with different signs. \square

Proposition 2.3.3. *If C_2 intersects C_1 in the middle segment with bigger slope, then the fixed point changes from unstable node to unstable spiral, stable spiral surrounded by unstable limit cycle, and stable node gradually while ϵ increases.*

Proof. In this case, $\frac{\partial f}{\partial v}(v_0, w_0) > 0$ and $D > 0$ for any $\epsilon > 0$. Since $D > 0$ and for small ϵ , both $\Delta > 0$ and $I > 0$ and the characteristic roots are positive numbers, the fixed point is an unstable node. There is a unique $\epsilon_1 > 0$ such that $I = 0$ and, furthermore, $\Delta < 0$. This fact shows that there is an interval such that $\Delta < 0$ and $I > 0$ for $\epsilon_0 < \epsilon < \epsilon_1$. The fixed point is an unstable spiral. For $\epsilon_1 < \epsilon$ and close to ϵ_1 , $I < 0$ and $\Delta < 0$, the fixed point is a stable spiral. By the Hopf bifurcation theorem, there is an unstable limit cycle for $\epsilon > \epsilon_1$ in the vicinity of ϵ_1 wrapping around the fixed point with small amplitude. If ϵ increases sufficiently, then $I < 0$ and $\Delta > 0$. Hence the fixed point is a stable node. \square

2.3.1 Qualitative behavior patterns of solutions of general Morris–Lecar models

(1) **The resting state.** If the two nullclines intersect at the left branch of C_1 only then the fixed point of (2.13) is a stable equilibrium. It is a node when $\epsilon > 0$ is small enough. As ϵ increases, the characteristic roots become complex numbers and the orbits approach the equilibrium with fluctuations. Proposition 2.3.1 corresponds to this pattern. A typical phase portrait is shown in Figure 2.4.

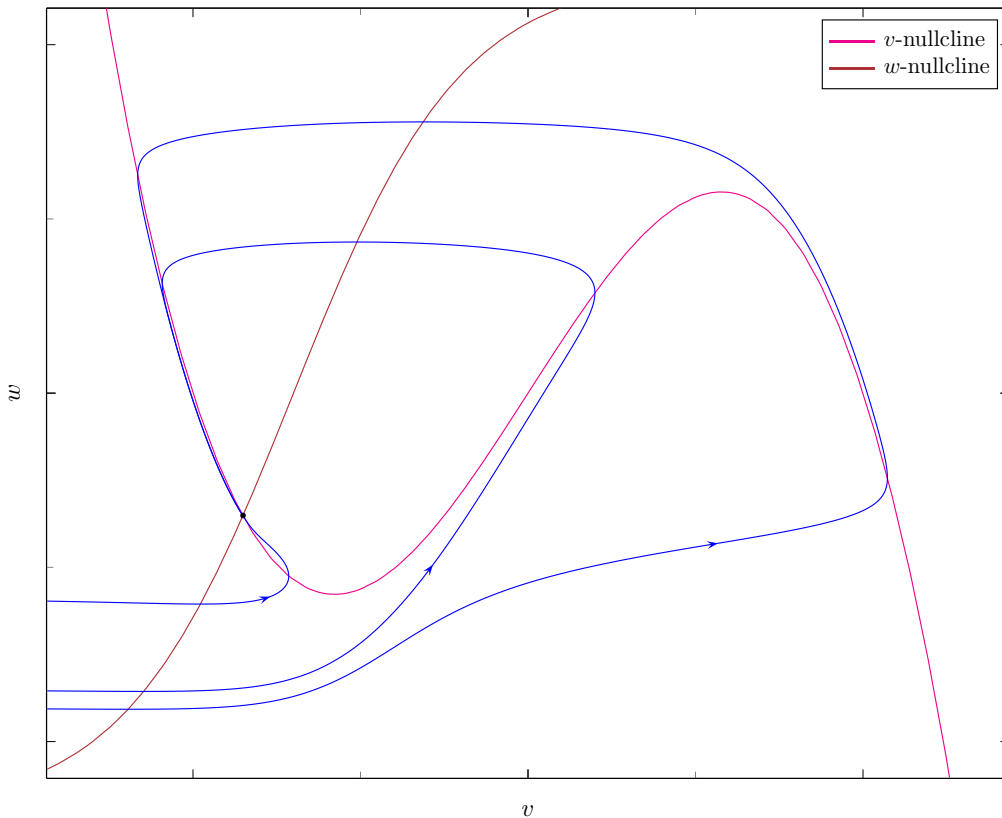


Figure 2.4: Typical phase portrait of pattern (1).

(2) Oscillation. If the two nullclines uniquely intersect at the middle segment of C_1 then the fixed point of (2.13) is an unstable node and there exists a stable limit cycle when $\epsilon > 0$ is small enough. Otherwise, if ϵ is not small, it is possible that there is no limit cycle or there are two limit cycles by the bifurcation theory. Proposition 2.3.3 describes the properties of the fixed point in this case. The existence and the properties of limit cycles will be discussed later. A typical phase portrait is shown in Figure 2.5.

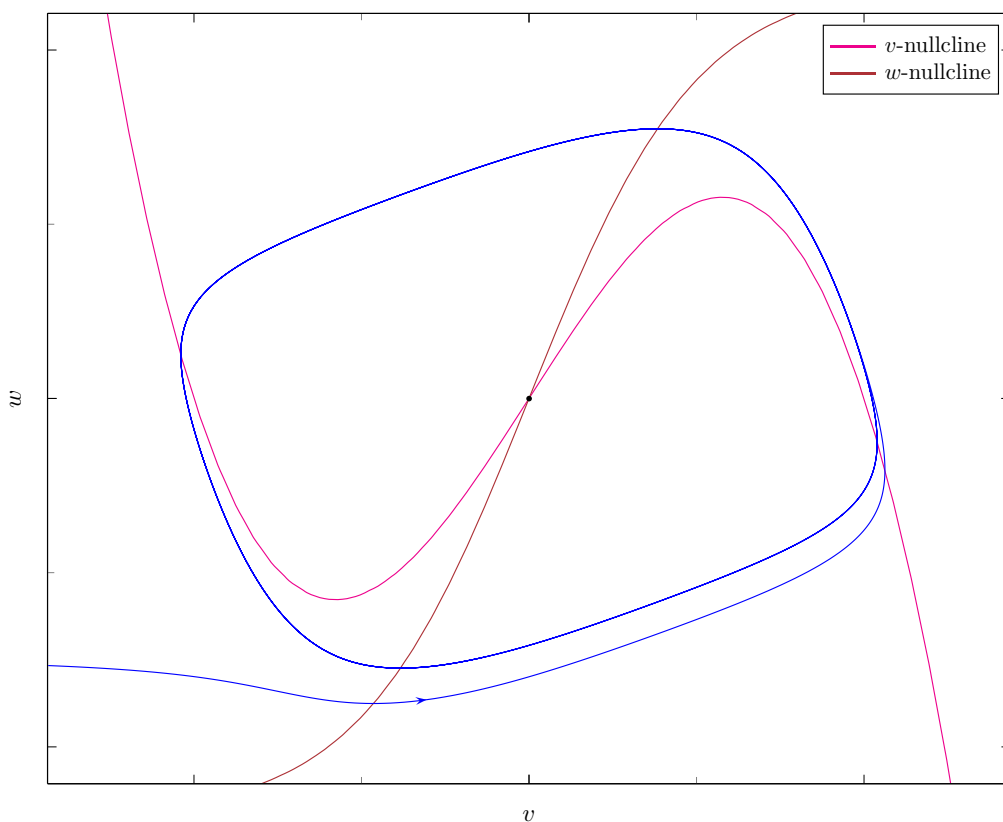


Figure 2.5: Typical phase portrait of pattern (2).

These two cases are the basic patterns of dynamical behaviors of the Morris–Lecar model used in computational neuron science. The following

patterns may appear for certain parameters.

(3) Fixed point on the right branch. If the two nullclines intersect on the right branch uniquely, then the fixed point is a stable equilibrium. It is a node when $\epsilon > 0$ is small enough. As ϵ increases, the characteristic roots become complex numbers and the orbits approach equilibrium with fluctuation. This pattern is shown in Figure 2.6.

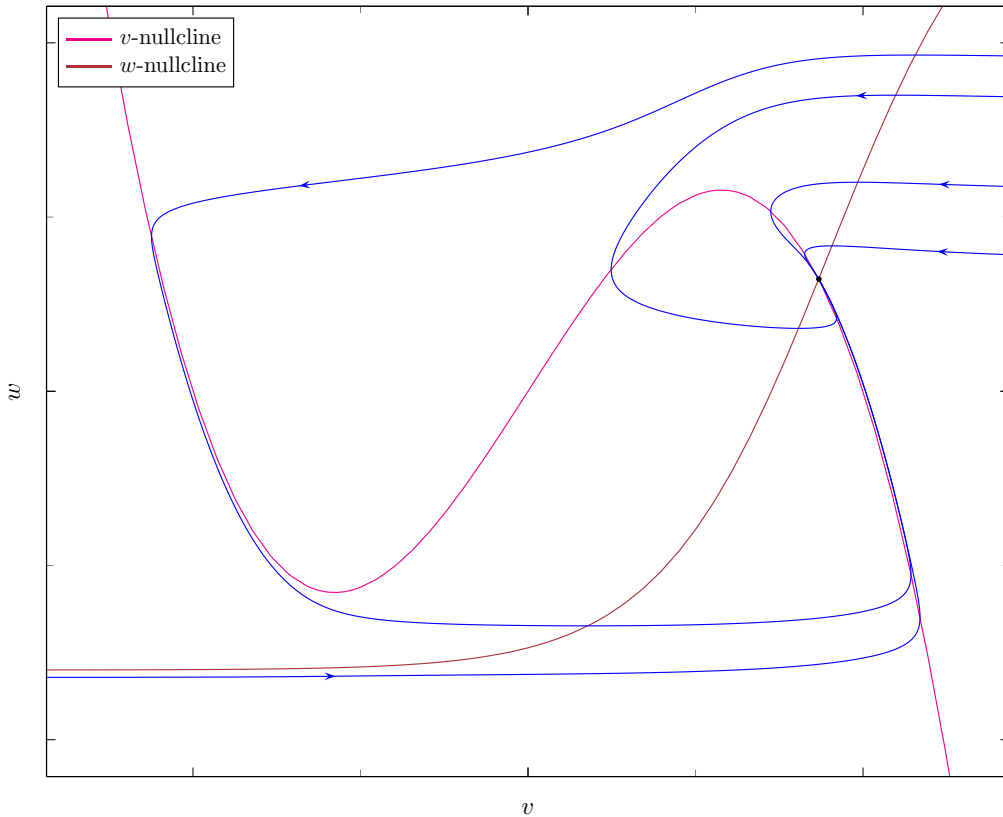


Figure 2.6: Typical phase portrait of pattern (3).

(4) Two stable equilibria. If the two nullclines intersect at three points on the left, middle, and right branches, respectively, then the system has two

stable equilibria on the left and right branches, respectively. The stable manifolds of the saddle point separates the phase plane into two parts. The orbits in the right part approach the node on the right branch of C_1 . The orbits on the left part approach the node on the left branch of C_1 . See Figure 2.7. At the fixed point in the middle segment of C_1 the slope of C_1 is greater than that of C_2 , and the fixed point is a saddle point by Proposition 2.3.2.

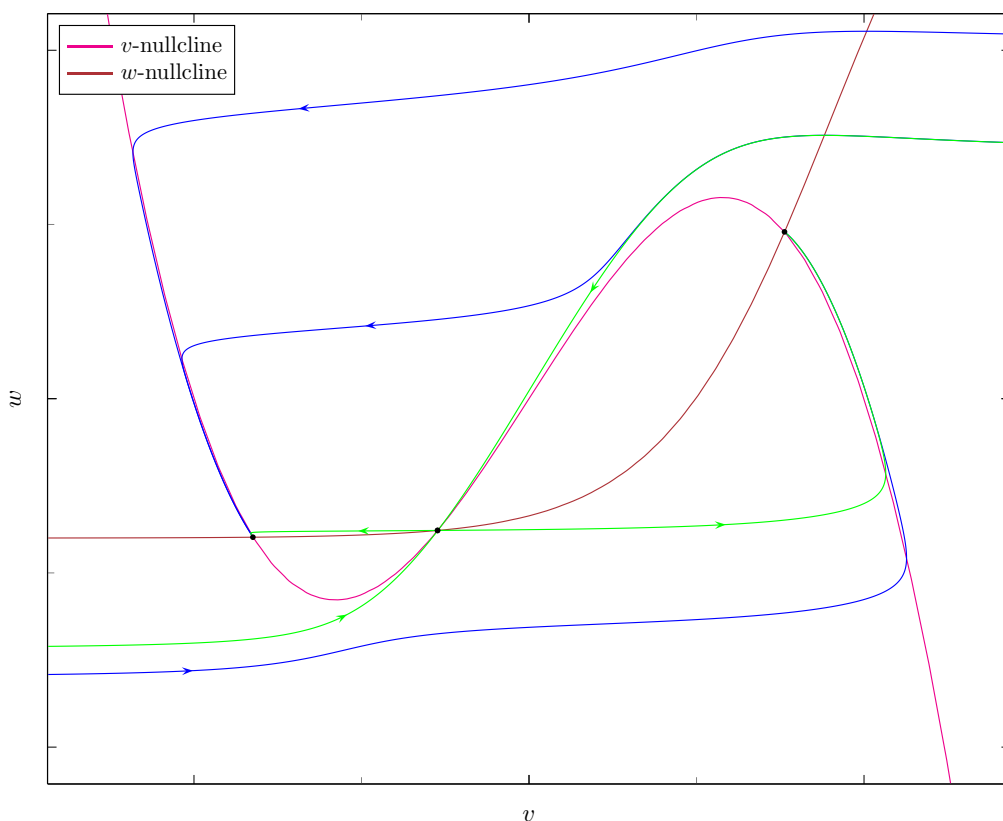


Figure 2.7: Typical phase portrait of pattern (4).

(5) Saddle-node invariant cycle. If the two nullclines intersect at three fixed points (one fixed point is in the left branch of C_1 , and the other two are in the middle segment of C_1) as in Figure 2.8, then the fixed point on

the left branch of C_1 is a stable equilibrium, the fixed point on the bottom-left part of the middle segment of C_1 is a saddle point, and the fixed point on the top-right part of the middle segment is an unstable node for small $\epsilon > 0$. In this case, the orbit starting from the saddle point, as its unstable manifold, leaving the saddle in the direction of both $v-w$ increasing then wrapping around the unstable node in counterclockwise way, approaches the stable node. Meanwhile, the orbit starting at the saddle point, as the other unstable manifold of the saddle, leaving the saddle in the direction of both $v-w$ decreasing, approaches to the stable node in a clockwise way. See Figure 2.8. These two orbits, the saddle and the stable node, form a *saddle-node invariant cycle (SNC)* enclosing the unstable node. The stable manifold of the saddle outside the cycle, as a separator, separates the orbits outside the cycle into two categories: the orbits in the right side of the separator, go around the unstable node and approach the stable node, whereas the orbits in the left side of the separator go to the stable node without wrapping around the unstable node.

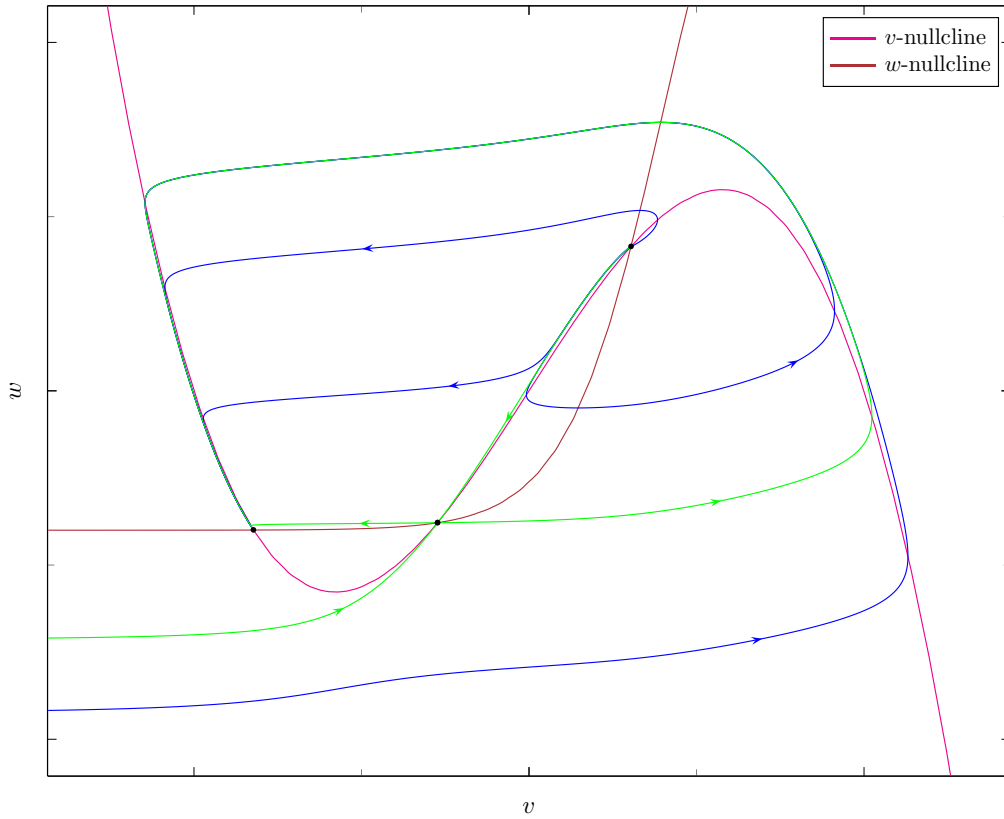


Figure 2.8: Typical phase portrait of pattern (5).

(6) Node-saddle-node-saddle-node. If C_2 intersects C_1 in five points, the first point A being on the left branch, the next three points of intersection, B , C , and D , being in the middle segment, and the fifth one, E , being on the right branch of C_1 , as shown in Figure 2.9, then the dynamical behavior of the solutions of (2.13) is quite complicated.

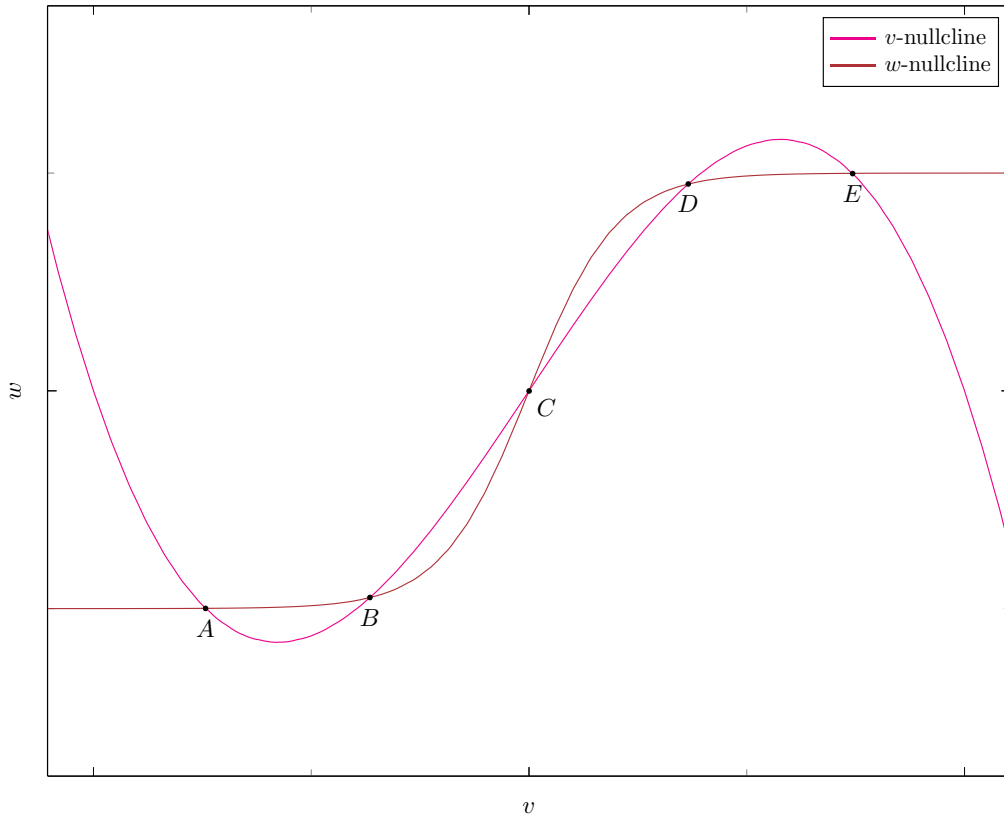


Figure 2.9: Fixed points in pattern (6).

For small ϵ , the fixed points A and E are stable nodes, B and D are saddle points, and C is an unstable node. The unstable manifolds of the saddle points approach to nodes A and E , respectively. The four orbits and the four points A , B , D , and E form an invariant cycle. See Figure 2.10. These four stable manifolds of the saddle points, B and D , divide the phase plane into two parts. All the orbits in the right part approach to node E . The orbits in the left part approach to node A . See Figure 2.10.

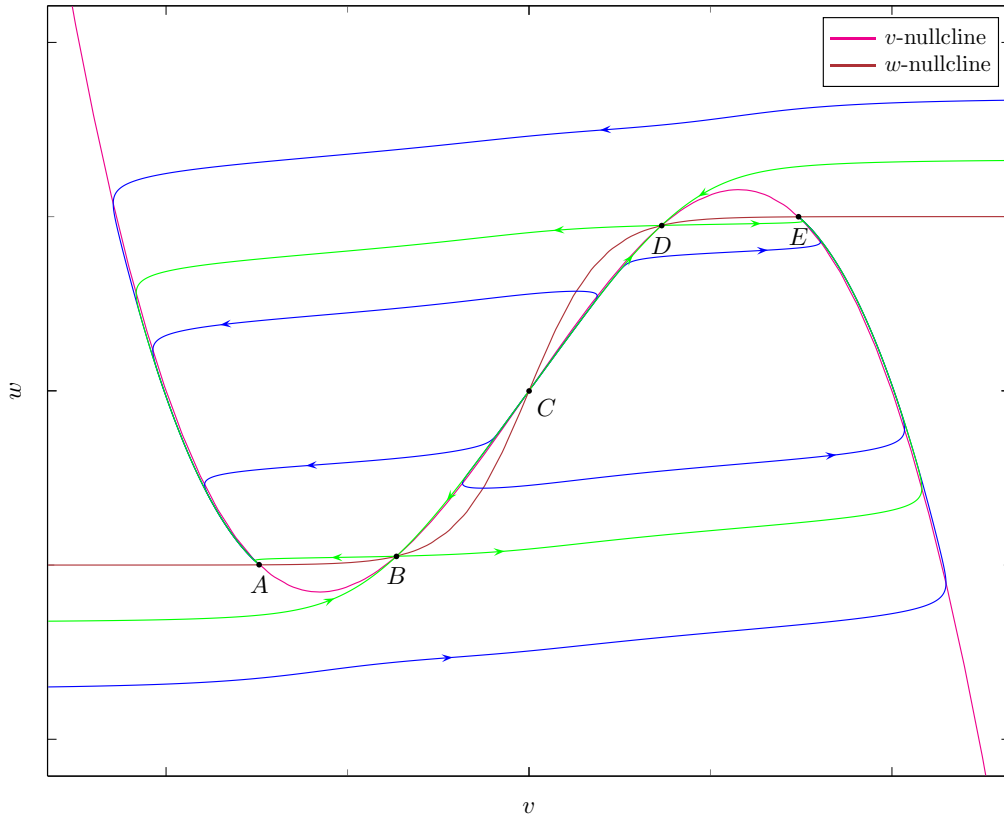


Figure 2.10: Invariant cycle connecting A , B , D , and E .

As ϵ varies, the property of the fixed point C will change from an unstable node to unstable spiral, stable spiral, and stable node and the dynamical behavior of the orbits will change drastically. For example, it is possible that there is a critical case where the heteroclinic orbit BD , from saddle point B to saddle point D , and the heteroclinic orbit DB , from saddle point D to saddle point B , form an invariant cycle, named saddle separatrix cycle and denoted by SSC. The other stable manifold of the saddle point B and the other stable manifold of the saddle point D divide the phase plane into two parts: the orbits in the right part approach to the stable fixed point E and

the orbits in the left part approach to the stable fixed point A . Inside the cycle SSC, the orbits spiral and approach to the cycle SSC. This case is very rare and subtle, and a little perturbation of the parameters can destroy the invariant cycle SSC.

More complicated cases than that of an SSC are also possible. We will exhibit them in the next section on bifurcation analysis. All cases share a common feature that two stable states coexist. In some special cases, three stable states coexist: two stable equilibrium states and one stable oscillation. Analyzing this pattern in detail may be an interesting theoretical issue because there are few reports on this pattern both in theoretical study and computational simulations.

2.3.2 Bifurcation analysis

By varying the parameter I_{app} in (2.13) we may change the position of the curve C_1 in the phase plane and shift the fixed point to different positions. This also changes the characteristic equation and the characteristic roots at the fixed point. The parameter ϵ may also affect the characteristic roots of system (2.13) at the fixed point. When the complex characteristic roots change across the imaginary axis at a nonzero pure imaginary point, Hopf bifurcation theory [25] can be employed to explore the changes of the dynamical behavior of systems according to the changes of parameters. The bifurcation arising by varying I_{app} and ϵ in (2.13) has been studied thoroughly in a computational way [5]. The following cases are worthy of mention.

(1) Hopf bifurcation. In the oscillation pattern, Proposition 2.3.3 assumes that while the parameter ϵ increases beyond ϵ_1 , the characteristic roots of the Jacobian of system (2.13) at the unique fixed point first change from two positive real numbers (the unstable node) to complex roots with positive real parts (unstable spiral), then to a pair of imaginary roots (only for $\epsilon = \epsilon_1$), then to complex roots with negative real parts (stable spiral), and lastly to two negative real roots (stable node). The Hopf bifurcation theorem gives conditions that for $0 < \epsilon - \epsilon_1$ small enough, there is an unstable limit cycle wrapping around the fixed point with small oscillation amplitude of the order $\sqrt{\epsilon - \epsilon_1}$ [25].

It is possible, Hopf bifurcation theory is local, that there is also a stable limit cycle with bigger oscillation amplitude enclosing the extreme points $L = (v^-, w^-)$ and $R = (v^+, w^+)$.

(2) Bifurcation analysis for node-saddle-node pattern. In the pattern of a saddle-node invariant cycle, both I_{app} and ϵ can play the roles of bifurcation parameters. If I_{app} is fixed, then the fixed points are invariant with respect to ϵ . Denote the fixed point on the left branch of C_1 by A , the one at the bottom-left part of the middle segment of C_1 by B , and the one at the top-right part by C . By gradually increasing ϵ , the property of the fixed point C can gradually change from unstable node to unstable spiral, stable spiral, and stable node. The dynamical behavior of system (2.13) can be explored by numerical simulations. The unstable manifold RU of the fixed point B , leaving B in the direction of both v and w increasing, will change its route correspondingly. RU approaches the stable node A for small ϵ . This

insures that a SNC exists. As ϵ increases, RU approaches B as a homoclinic orbit for a value of ϵ , and, as ϵ becomes bigger, RU wraps around the fixed point C and converges to a stable limit cycle. The SNC disappears for such ϵ . The orbit between RU and the stable manifold of the saddle point B will converge to the stable limit cycle. Meanwhile, as the real parts of the characteristic roots at the fixed point C change signs, an unstable limit cycle, with small oscillation amplitude around the point C , appears by the Hopf bifurcation theorem.

We use the following system as an example,

$$\begin{aligned}\frac{dv}{dt} &= \frac{2}{3}v(1-v)(1+v) - w, \\ \frac{dw}{dt} &= \epsilon\{1.2[\tanh(3.5(v-0.4)) - w + 0.8]\}.\end{aligned}\tag{2.17}$$

When ϵ is small, the fixed point C is an unstable node. Figure 2.11 shows the phase portrait of (2.17) when $\epsilon = 0.1$.

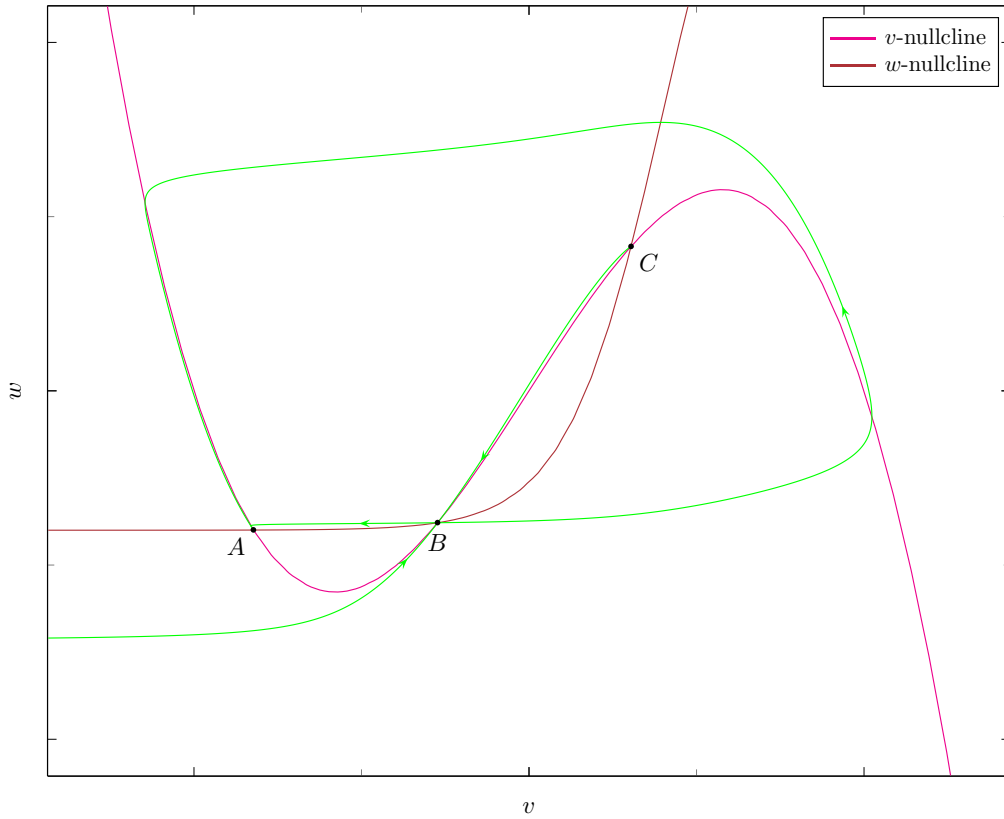


Figure 2.11: Phase portrait of (2.17), when $\epsilon = 0.1$.

As ϵ increases, the fixed point C changes from unstable node to unstable spiral. Figure 2.12 shows the phase portrait of (2.17) when $\epsilon = 0.5$.

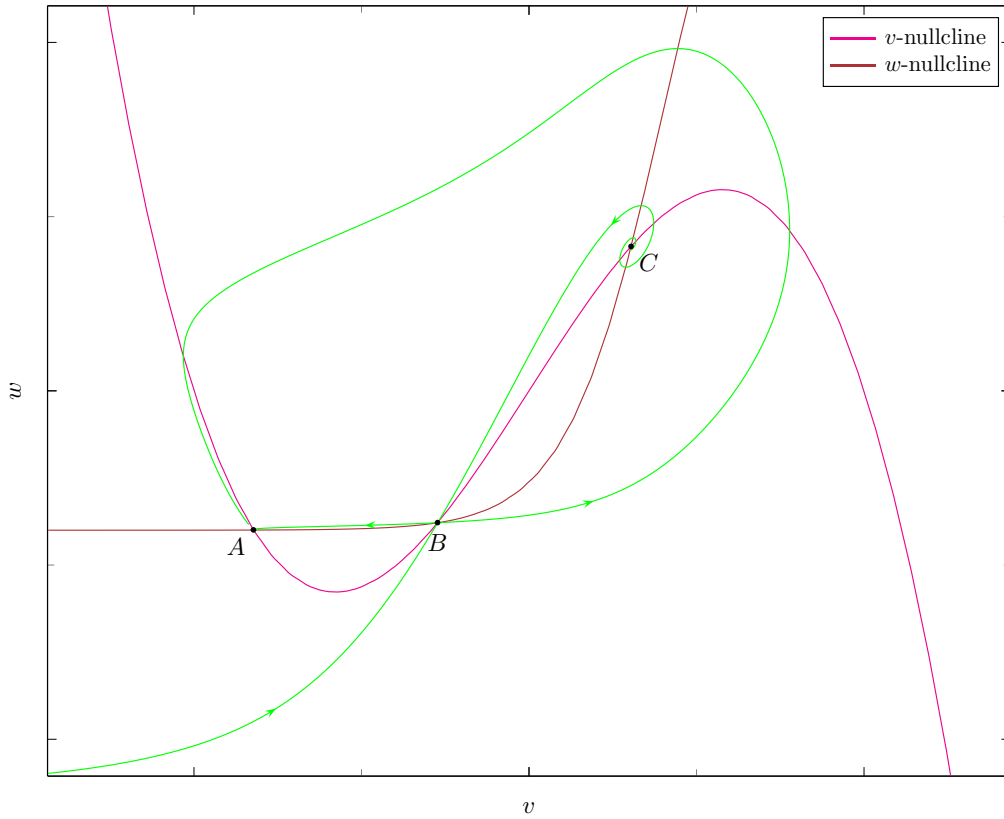


Figure 2.12: Phase portrait of (2.17), when $\epsilon = 0.5$.

If ϵ increases more, the fixed point C becomes a stable spiral. In this case, the system has a limit cycle wrapping around the fixed point C . Figure Figure 2.13 shows the phase portrait of (2.17) when $\epsilon = 1.1$.

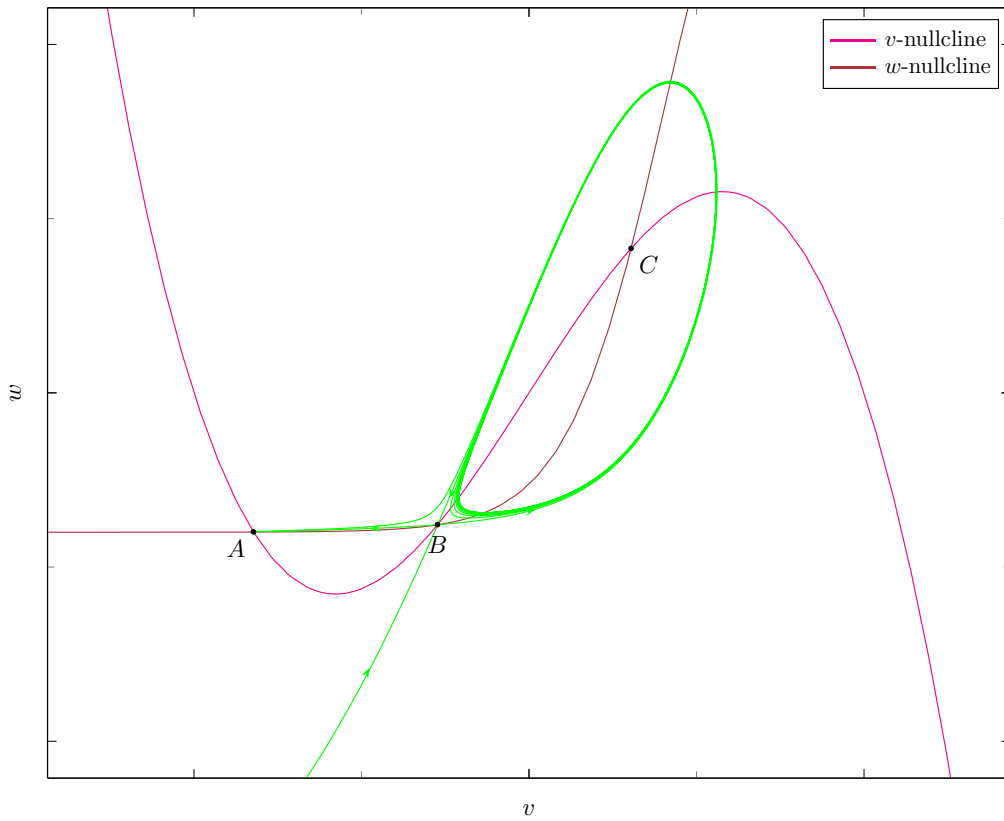


Figure 2.13: Phase portrait of (2.17) when $\epsilon = 1.1$.

If ϵ increases a bit more, the fixed point C can still be a stable spiral, but the limit cycle disappears. Figure 2.14 shows the phase portrait of (2.17) when $\epsilon = 1.6$.

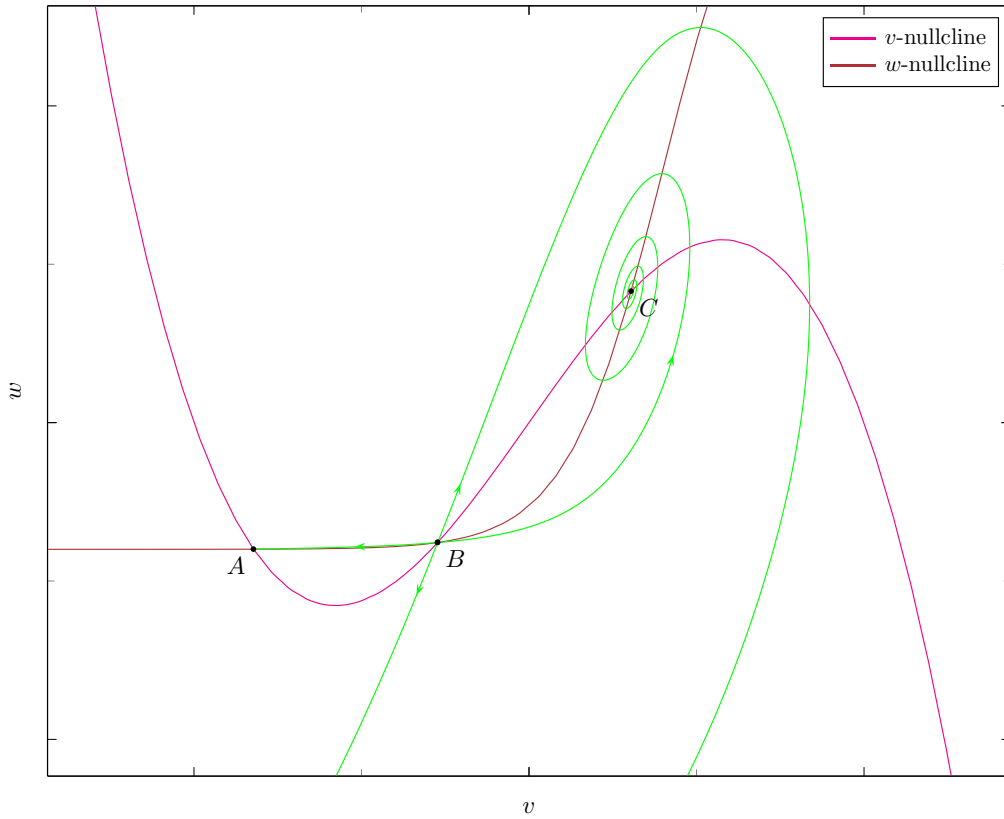


Figure 2.14: Phase portrait of (2.17), when $\epsilon = 1.6$.

If ϵ increases a lot more, the fixed point C becomes a stable node. Figure 2.15 shows the phase portrait of (2.17) when $\epsilon = 8$.

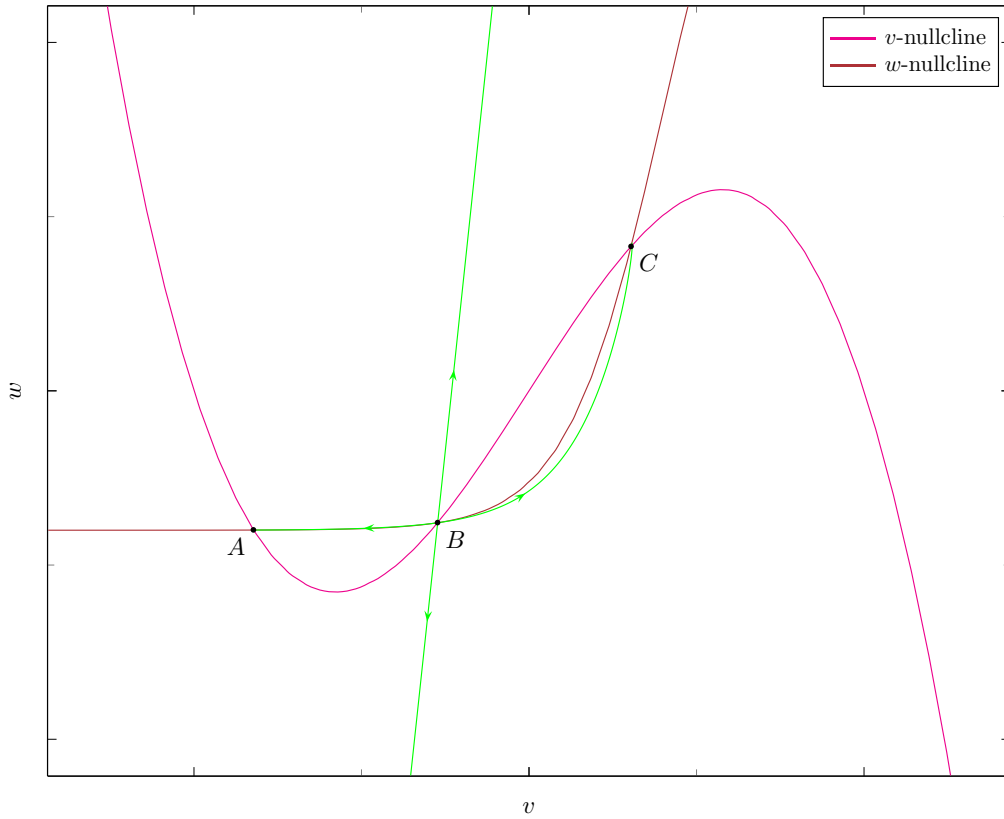


Figure 2.15: Phase portrait of (2.17) when $\epsilon = 8$.

Bifurcations of system (2.17) are summarized in the following table.

ϵ	C	Figure
0.1	Unstable node	2.11
0.5	Unstable spiral	2.12
1.1	Stable spiral surrounded by a limit cycle	2.13
1.6	Stable spiral	2.14
8	Stable node	2.15

(3) **The saddle-heteroclinic bifurcation analysis for the node-saddle-node-saddle node pattern.** We use the model

$$\begin{aligned} \frac{dv}{dt} &= \frac{2}{3}v(1-v)(1+v) - w, \\ \frac{dw}{dt} &= \epsilon \left\{ \frac{1}{2}[\tanh(5v) - w] \right\}, \end{aligned} \quad (2.18)$$

and change ϵ gradually to see the features of the dynamical behavior. System (2.18) has five fixed points as Figure 2.9 shows. The fixed points A and E are always stable nodes or spirals. When ϵ is small, the fixed point C is an unstable node. Figure 2.16 shows the phase portrait of (2.18) when $\epsilon = 0.1$ in which two stable states coexist.

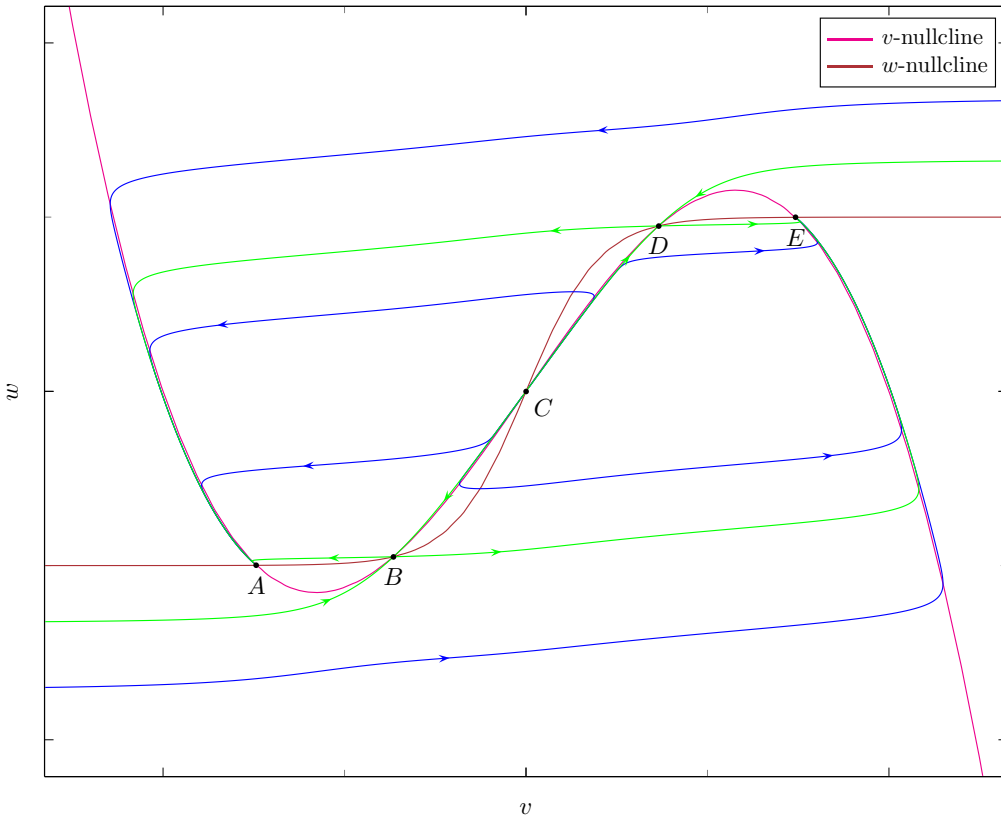


Figure 2.16: Phase portrait of (2.18), when $\epsilon = 0.1$.

As ϵ increases, the fixed point C changes from unstable node to unstable spiral. Figure 2.17 shows the phase portrait of (2.18) when $\epsilon = 0.8$, where a bistable state coexists.

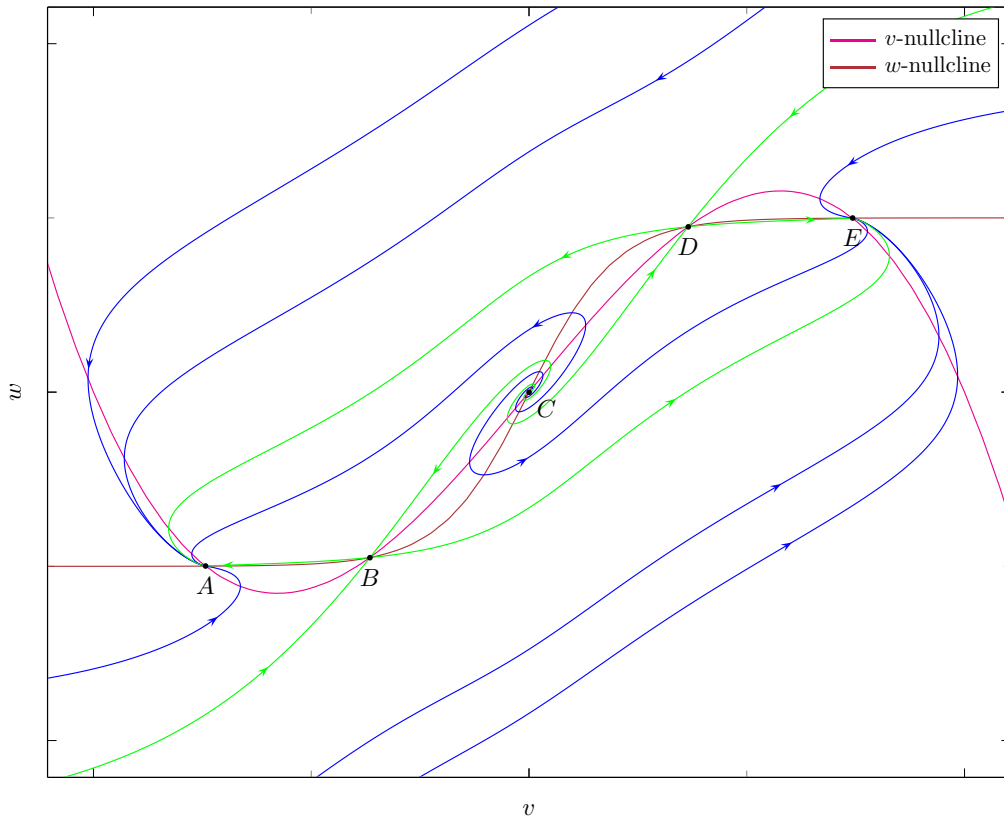


Figure 2.17: Phase portrait of (2.18), when $\epsilon = 0.8$.

If ϵ increases more, the fixed point C can still be an unstable spiral, and the system has a limit cycle wrapping around the fixed point C . Figure 2.18 shows the phase portrait of (2.18) when $\epsilon = 1.42$. Two stable states and one stable oscillation coexist.

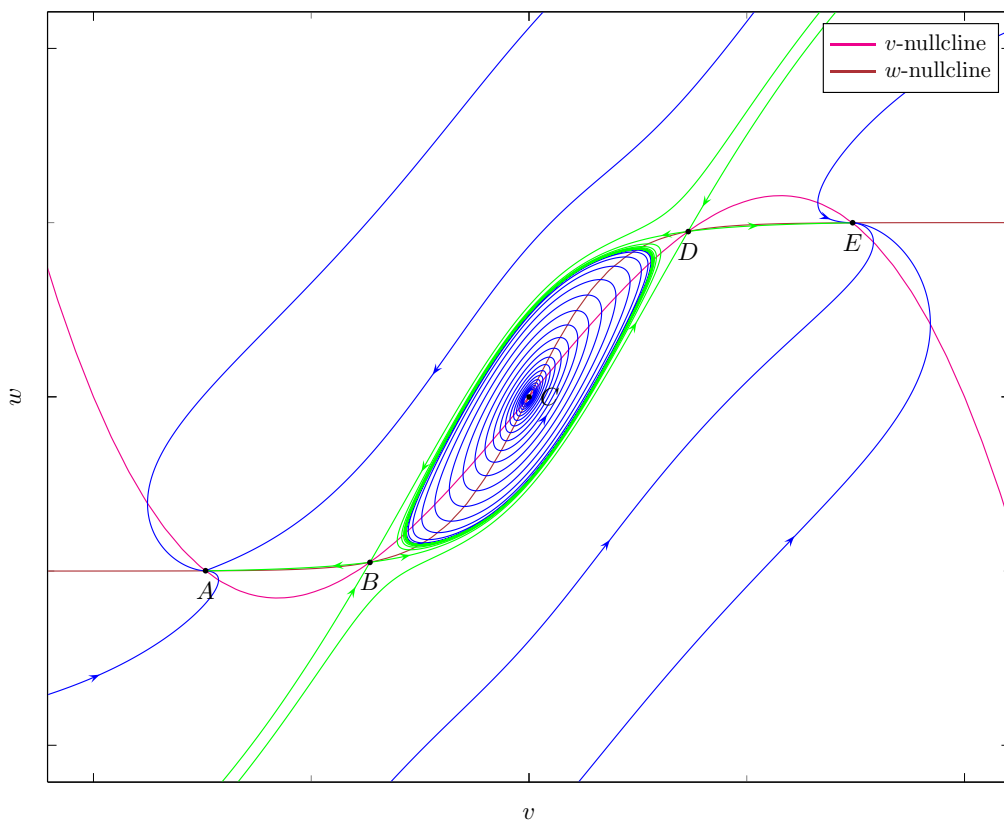


Figure 2.18: Phase portrait of (2.18), when $\epsilon = 1.42$.

When ϵ changes from 1.49 to 1.50, the sign of the real parts of the eigenvalues of the Jacobian of the fixed point C changes from plus to minus. By the Hopf bifurcation theorem, there exists a small unstable limit cycle wrapping the fixed point C for some ϵ in the interval (1.49, 1.50).

If ϵ increases a little more, the fixed point C becomes a stable spiral, and there is no limit cycle. Figure 2.19 shows the phase portrait of (2.18) when $\epsilon = 1.92$.

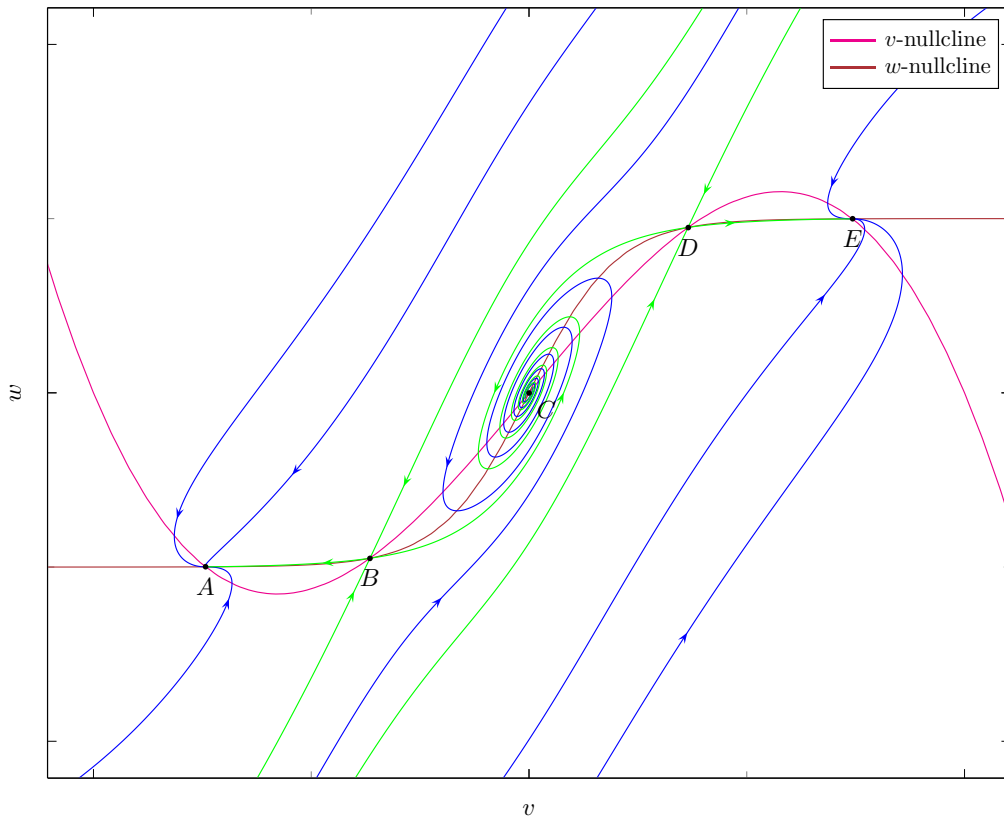


Figure 2.19: Phase portrait of (2.18), when $\epsilon = 1.92$.

If ϵ increases a lot more, the fixed point C becomes a stable node. Figure 2.20 shows the phase portrait of (2.18) when $\epsilon = 8$.

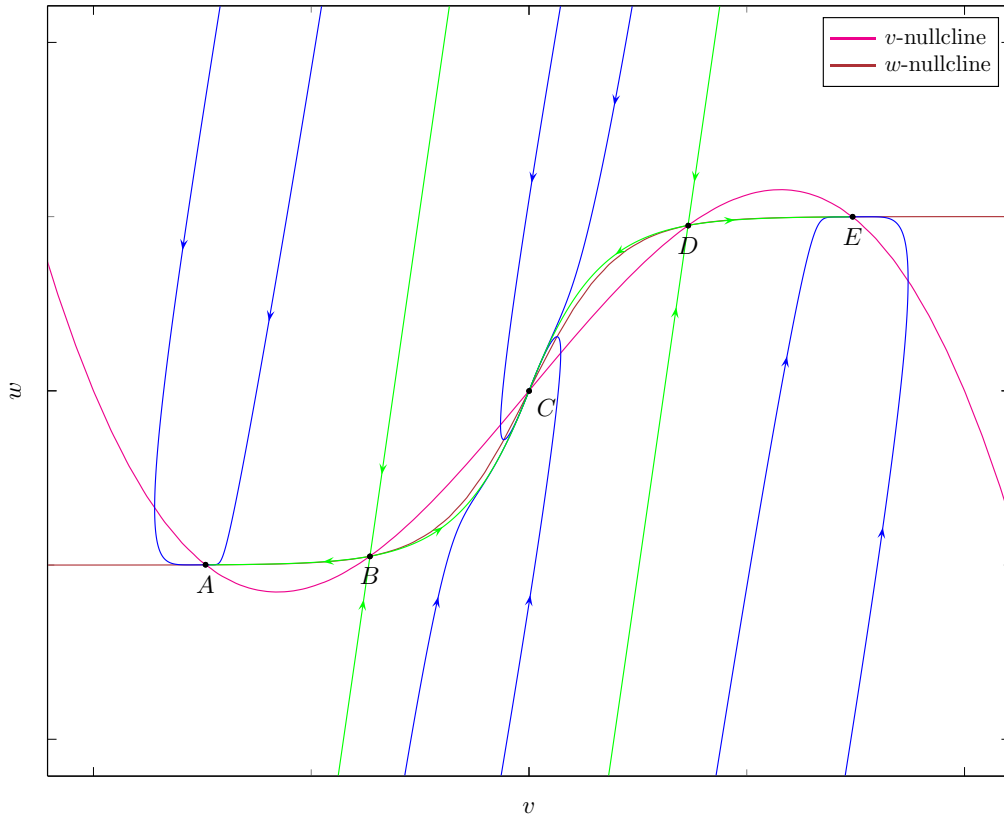


Figure 2.20: Phase portrait of (2.18), when $\epsilon = 8$.

Bifurcations of system (2.18) are summarized in the following table.

ϵ	C	Figure
0.1	Unstable node	2.16
0.8	Unstable spiral	2.17
1.42	Unstable spiral surrounded by a limit cycle	2.18
1.92	Stable spiral	2.19
8	Stable node	2.20

2.3.3 Dynamical feature of neurons

The dynamical behavior of system (2.13) reflects the dynamical features of neurons. The patterns listed in section 2.3.1 are the possible dynamical patterns of neurons. We now discuss the important ones.

The resting state pattern reflects that in this regime the neuron will return to its resting state even if the initial state is far away from equilibrium.

In the oscillation pattern, the unique stable limit cycle corresponds to the fact that the neuron fires repeatedly with almost fixed period and amplitude. Hodgken [15] called this oscillation pattern *type 2 (or class 2) spiking*.

The pattern of fixed point on the right branch shows that in this regime the neuron will arrive and stay at the depolarized state from any initial state. When the applied current I_{app} is large enough such that the two nullclines intersect at the right branch of C_1 the neuron will have this behavior.

The pattern of two stable equilibria stands for the case in which a depolarized state and a hyperpolarized state could be reached by starting at different initial states, respectively. The practical applications of this pattern are rarely reported.

The pattern of saddle-node invariant cycle and the pattern of saddle-homoclinic bifurcation say that under the initial states, outside the cycle and right to the separator, the neuron will fire one time and return to the resting state. This course could be arbitrary long but the amplitude is almost fixed. Hodgken [15] called this kind of dynamic *type 1 (or class 1) spiking*. If starting at the left side to the separator the neuron will return to the resting state without firing.

2.3.4 Limit cycle

Stable limit cycles are meaningful for the Morris–Lecar model because they represent neurons that fire repetitively. For the pattern of oscillation it has been shown in Proposition 2.3.3 that if C_2 intersects C_1 in the middle segment of C_1 with a bigger slope and ϵ is large, then the fixed point is stable. Consequently, there are no stable limit cycles around the fixed point. Limit cycles exist in the pattern of oscillation provided ϵ is small enough. We give a proof following the method in Lefschetz’s book [23].

Theorem 2.3.4. *System (2.13) has a stable limit cycle if the w -nullcline C_2 uniquely intersects the v -nullcline C_1 in the middle segment with a bigger slope and ϵ is small enough. Moreover the limit cycle approaches the limit position that is the cycle formed by horizontal lines starting from the w -maximum point and w -minimum point of C_1 , respectively, and the left and right branches of C_1 between the two lines. This cycle is called singular periodic orbit.*

Proof. We shift C_1 up by β to get a curve denoted by C_u . We shift C_1 down by β to get a curve denoted by C_d . We choose β small enough such that C_2 intersects C_u and C_d in their middle parts. Let $A = (v_a, w_a)$ be the left minimum of the curve C_1 . Let $A_1 = (v_1, w_a)$ be the point on the left branch of C_d with the same w coordinate as A . Let $A_2 = (v_2, w_a - \beta)$ be the point on C_2 with the same w coordinate as the left minimum of C_d . Let $B_1 = (v_3, w_a - \beta)$ be the point on the right branch of C_u with the same w coordinate as A_2 . Let $C = (v_c, w_c)$ be the right maximum of the curve C_1 . Let $A_4 = (v_4, w_c)$ be the point on the right branch of c_u with the same w

coordinate as C . Let $A_5 = (v_5, w_c + \beta)$ be the point on C_2 with the same w coordinate as the right maximum of C_u . Let A_6 be the point on the left branch of C_d with the same w coordinate as A_5 . Let A_1A_6 denote the part of the left branch of C_d between A_1 and A_6 . Let A_3A_4 denote the part of right branch of C_u between A_3 and A_4 . The segments A_1A_2 , A_2A_3 , A_4A_5 , A_5A_6 and the curves A_1A_6 , A_3A_4 consist of a closed curve denoted by C_0 (the red closed curve in Figure 2.21). We consider the vector fields on this close curve. On the segment A_5A_6 $w' < 0$, so the vector field points down. On the segment A_2A_3 , $w' > 0$, so the vector field points up. On the curve A_1A_6 , $v' > 0$, $w' < 0$, so the vector field points bottom-right. By choosing ϵ sufficiently small we can make the vector field points to the left side of A_1A_6 . On the segment A_1A_2 $v' > 0$, $w' < 0$, so the vector field points bottom-right. By choosing ϵ small enough, we can make the vector field points to the right side of A_1A_2 . On the segment A_4A_5 , $v' < 0$ and $w' > 0$, so the vector field on A_4A_5 points top-left. When ϵ is small enough the vector field points to the left side of A_4A_5 . On the curve A_3A_4 , $v' < 0$ and $w' > 0$, so the vector field on A_3A_4 points to top-left. When ϵ is small enough, the vector field points to the left side of A_3A_4 . In summary, we can find an $\epsilon_1 > 0$ such that the vector fields on C_0 point inside if $\epsilon < \epsilon_1$.

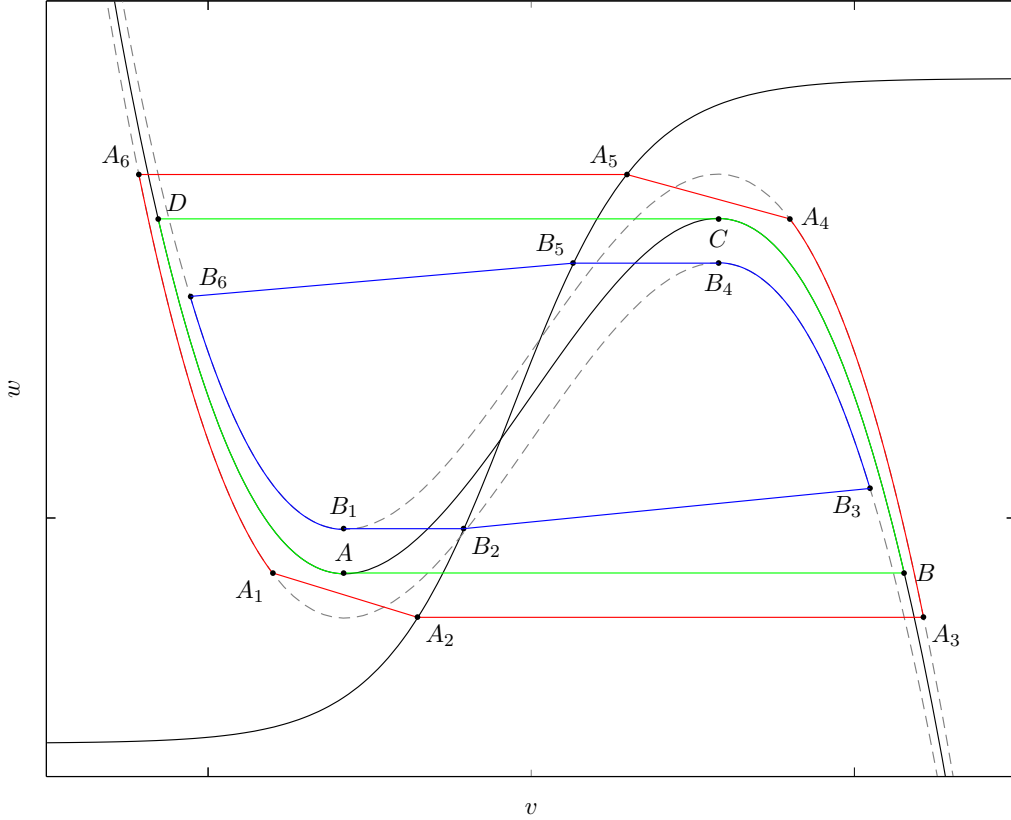


Figure 2.21: Positive invariant polygon.

Let $B_1 = (v_a, w_a + \beta)$ be the left minimum of C_u . Let $B_2 = (v_7, w_a + \beta)$ be the point on C_2 with the same w coordinate as the left minimum of B_1 . Let $B_3 = (v_8, w_a + 2\beta)$ be the point on the right branch of C_d . Let $B_4 = (v_c, w_c - \beta)$ be the right maximum of the curve C_d . Let $B_5 = (v_9, w_c - \beta)$ be the point on C_2 with the same w coordinate as B_4 . Let $B_6 = (v_{10}, w_c - 2\beta)$ be the point on C_u . Let B_1B_6 denote the part of the left branch of C_d between B_1 and B_6 . Let B_3B_4 denote the part of right branch of C_u between B_3 and B_4 . The segments B_1B_2 , B_2B_3 , B_4B_5 , B_5B_6 and the curves B_1B_2 , B_3B_4 consist of a closed curve denoted by C_i (the blue closed curve in Figure 2.21).

We consider the vector fields on this close curve. On the segment B_1B_2 , $w' < 0$, so the vector field points down. On the segment B_4B_5 , $w' > 0$, so the vector field points up. On the curve B_1B_6 , $v' < 0$ and $w' < 0$, so the vector field points bottom-left. On the curve B_3B_4 , $v' > 0$ and $w' > 0$, so the vector field on B_3B_4 points top-right. On the segment B_2B_3 , $v' > 0$ and $w' > 0$, so the vector field points top-right. By choosing ϵ sufficiently small, we can make the vector field point to the right side of B_2B_3 . On the segment B_5B_6 $v' < 0$, $w' < 0$, so the vector field on B_5B_6 point bottom-left. When ϵ is small enough, the vector field points to the left side of B_5B_6 . In summary, we can find an $\epsilon_2 > 0$ such that the vector fields on C_i point outside if $\epsilon < \epsilon_2$.

The ringshaped region between C_i and C_0 is a positive invariant set if $\epsilon < \min\{\epsilon_1, \epsilon_2\}$. By the Poincaré–Bendixson theorem, there exists a limit cycle in this region. As β goes to zero, this region converges to the singular orbit (the green curve in Figure 2.21), and so does the limit cycle. \square

2.3.5 Singular perturbation

The Morris–Lecar system (2.13) contains a small parameter ϵ . It indicates that the variable w changes more slowly than the variable v . In the real world, this is true because the channel gating is a biological process, whereas, firing an action potential is an electrical process. When ϵ is small, the stable limit cycle of system (2.13) exists and is close to the singular periodic orbit. The solutions of system (2.13) will be wrapping around the singular periodic orbit. By geometric singular perturbation theory [7] and the theory of differential equations with small parameters [27], we can dissect the wrapping process into four pieces according to the different time scales following many authors

[5, 33, 34, 35].

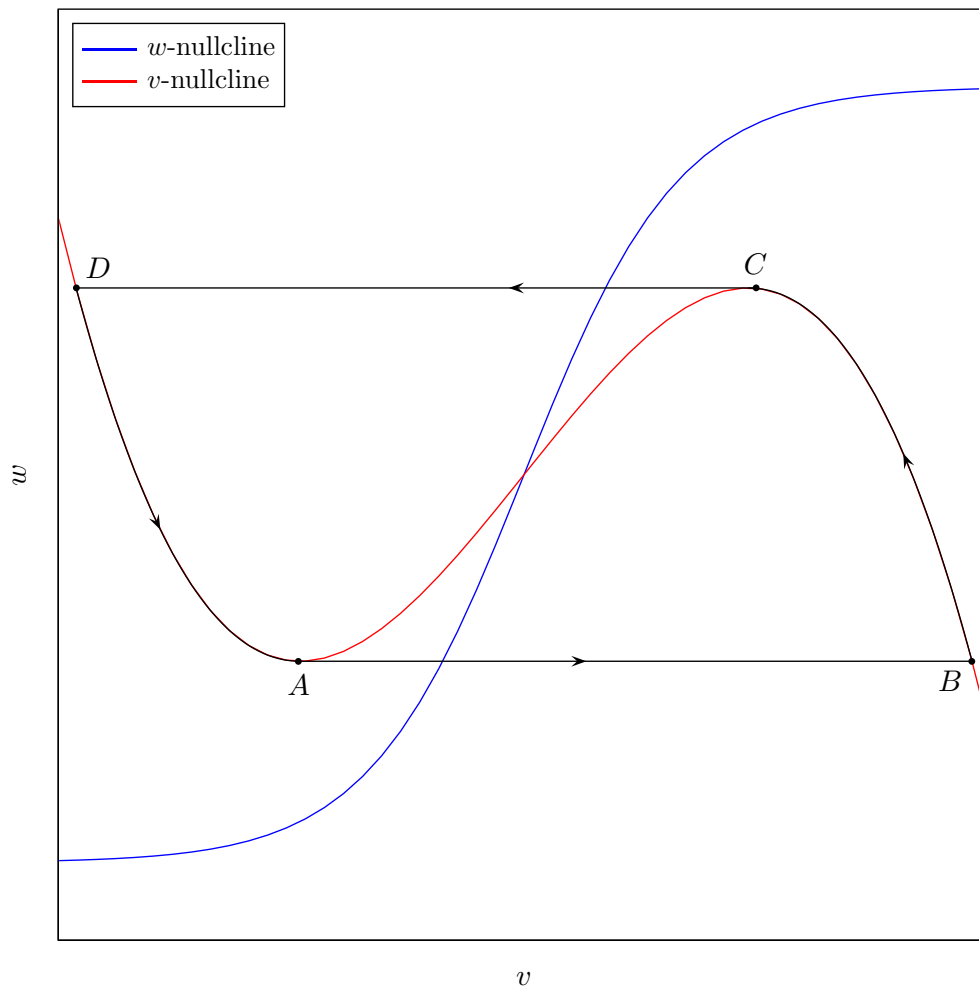


Figure 2.22: Singular orbit of system(2.13).

We introduce the time variable $\tau = \epsilon t$, which is called *slow time*, and the original time t is called *fast time*. Along the segment A to B , the variable v increases much faster than the variable w because $f(v, w) \gg \epsilon$ and $g(v, w) >$

0 during this course. As $\epsilon \rightarrow 0$, system (2.13) reduces to

$$\begin{aligned}\frac{dv}{dt} &= f(v, w), \\ \frac{dw}{dt} &= 0.\end{aligned}\tag{2.19}$$

This equation approximately governs the course from A to B . This is a fast process which describes the fact that the neuron fires (the membrane potential increases rapidly). When the orbit arrives at the right branch of C_1 , it intersects C_1 in the up direction, increasing along the right branch and goes up to the maximum point. In this case $f(v, w) < 0$ is small in absolute value. This means that both variables v and w change slowly. We use the slow time scale $\tau = \epsilon t$ to transform the system into

$$\begin{aligned}\epsilon \frac{dv}{d\tau} &= f(v, w), \\ \frac{dw}{d\tau} &= g(v, w).\end{aligned}$$

As $\epsilon \rightarrow 0$, we get the system

$$\begin{aligned}0 &= f(v, w), \\ \frac{dw}{d\tau} &= g(v, w).\end{aligned}\tag{2.20}$$

This equation approximately governs the course from B to C . This is a slow process that describes the evolution of the neuron in the active phase. Similarly, the course from C to D is governed by (2.19) to describe the fact that the neuron rapidly returns to the silent phase. The course from D to A is governed by (2.20) to describe the evolution of the neuron in the silent phase. This method can be applied to study neuron networks.

2.4 Neuron Networks

2.4.1 Synaptic connection: biological foundation

In the real world, neurons communicate through synapses. Biological signals are carried by action potentials. There are two types of synapses, the gap junction and the chemical synapses. The important and frequent type is the chemical synapses. The mechanisms and biological process in synapses are discussed in many excellent books [6, 11, 20].

The basic biological mechanisms of chemical synapses is as follows:

1. The action potential comes down along an axon to a synapse and elicits a sequence of events that make the presynaptic cell release chemicals, named *neurotransmitters*, into the synaptic cleft.
2. The neurotransmitters diffuse across the cleft and bind to the specialized receptors anchored on the postsynaptic membrane.
3. The binding of the neurotransmitters to the receptors causes a rapid opening of ion channels. The opening of ion channels causes a change in conductance of the membrane for specific ions, and a change of the membrane potential of the postsynaptic cells, depolarization or hyperpolarization, completing the transferring of the signals.

This process can repeat if the presynaptic cell repeatedly fires. This process actually happens in the synapse and the changes in synapse will cause further events in postsynaptic cell. One neuron may have many synaptic connections with other neurons, both presynaptic and postsynaptic.

The neurotransmitters mainly are two kinds of molecules, glutamate and γ -aminobutyric acid (GABA). There are two classes of receptors for glutamate neurotransmitters. They are AMPA receptors and NMDA receptors. The receptors of GABA neurotransmitters mainly belong to GABAA class or GABAB class. We do not care for their full names although they have definite meanings. There are many kinds of neurotransmitters and receptors. Those mentioned above are general and typical ones in synapses.

The glutamate transmitters are used in the synapses that increase the possibility of firing an action potential in the postsynaptic cell. This kind of synapse is called *excitatory synapse*. The GABA transmitters are used in the synapses which decrease the possibility of firing action potential in postsynaptic cell. This kind of synapse is called *inhibitory synapse*.

The AMPA receptors and the GABAA receptors quickly and directly produce the responses to open the ion channels in postsynaptic cells. The other two classes of receptors, NMDA and GABAB, slowly and indirectly produce responses to open the ion channels in postsynaptic cells.

2.4.2 Synaptic connection: mathematical modeling

In modeling neuronal networks, we need to describe the responses in postsynaptic cells, especially the conductance change and the added ion current into the postsynaptic cells.

Let an individual cell be modeled by the Morris–Lecar equation

$$\begin{aligned}\frac{dv}{dt} &= f(v, w), \\ \frac{dw}{dt} &= \epsilon g(v, w).\end{aligned}\tag{2.21}$$

We make the same assumption on the gating variable as Terman [33]. We assume that the gating variable of the fast synaptic receptors satisfies the equation

$$\frac{ds}{dt} = \alpha(1 - s)H_\infty(v_{\text{pre}} - v_T) - \beta s, \quad (2.22)$$

where the gating variable s represents the proportion of open channels in the postsynaptic cell, v_{pre} is the membrane potential of the presynaptic cell, v_T is a threshold in the presynaptic cell for the release of transmitters, function H_∞ is defined as $H_\infty(x) = 0$ for $x \leq 0$, and $H_\infty(x) = 1$ for $x > 0$ and the parameters $\alpha, \beta > 0$ are the rates of the channel gate opening or closing, respectively. Equation (2.22) describes the fact that once the propagation of the action potential in the presynaptic cell arrives at the synapse, the membrane potential v_{pre} exceeds the threshold v_T , and the channels in the postsynaptic cell will open according to equation (2.22) because $H_\infty(v_{\text{pre}} - v_T) = 1$. When the presynaptic cell is in the silence phase, $v_{\text{pre}} < v_T$, so $H_\infty(v_{\text{pre}} - v_T) = 0$, and the gating variable s decays according to a negative exponential law.

The change of conductance in the membrane of the postsynaptic cell causes an ion current to the postsynaptic cell in the form

$$I_{\text{syn}} = g_{\text{syn}}s(v_{\text{po}} - v_{\text{syn}}).$$

The constant g_{syn} is the maximum conductance, v_{syn} is the reversal potential of the postsynaptic receptor, v_{po} is the potential of postsynaptic cell. The reversal potential, different from the resting potential, is just for the synaptic receptors and measured by experiment for specific ions. Thus, two cells mutually coupled by a synaptic connection will be modeled by the system of

differential equations:

$$\frac{dv_i}{dt} = f(v_i, w_i) - g_{\text{syn}} s_i (v_i - v_{\text{syn}}), \quad (2.23a)$$

$$\frac{dw_i}{dt} = \epsilon g(v_i, w_i), \quad (2.23b)$$

$$\frac{ds_i}{dt} = \alpha(1 - s_i) H_\infty(v_j - v_T) - \beta s_i, \quad i \neq j, \quad (2.23c)$$

$i, j = 1, 2$. Here, the negative sign before the last term of (2.23a) represents the ion current (positive charge) flow into the neuron. Equation (2.23c) means that, if the presynaptic cell j fires, then $v_j > v_T$. The gating variable s_i of the postsynaptic cell will increase according to the governing equation (2.23c).

This system has six equations to describe the dynamics of the two neurons. When the propagation of the action potential arrives at the terminal of presynaptic cell 1 ($= j$), its membrane potential (at the terminal) exceeds the threshold v_T . This causes the variable s_2 of cell 2 ($= i$) to increase according to equation (2.23c). The increase of the gating variable s_2 causes the change of ion current injected into cell 2. This event corresponds to the last term of equation (2.23a).

For more synapses of one postsynaptic cell, equation (2.23c) changes into

$$\frac{ds_{ij}}{dt} = \alpha_j(1 - s_{ij}) H_\infty(v_j - v_{T_j}) - \beta_j s_{ij}. \quad (2.24)$$

Here s_{ij} stands for cell i 's fraction of open receptor channels in the synapse coupling with the presynaptic cell j , v_{T_j} is the threshold potential in the synapse coupling with cell j . The total synaptic current of cell i is

$$I_{\text{syn}} = - \sum_j g_{ij} s_{ij} (v_i - v_{j_{\text{syn}}}).$$

Here the reverse potential $v_{j_{\text{syn}}}$ is only for the synapse coupling cells i and j . In this case, equation (2.23a) changes into

$$\frac{dv_i}{dt} = f(v_i, w_i) - \sum_j g_{ij} s_{ij} (v_i - v_{j_{\text{syn}}}).$$

To describe synapses with NMDA and GABAB receptors we introduce a middle dynamical variable for the synapse. The simple form is an equation for the middle variable inserted before (2.23c) as in [5],

$$\frac{dx_i}{dt} = \alpha_x (1 - x_i) H_\infty(v_j - v_T) - \beta_x x_i, \quad (2.25a)$$

$$\frac{ds_i}{dt} = \alpha (1 - s_i) H_\infty(x_i - \theta_x) - \beta s_i. \quad (2.25b)$$

Equation (2.25a) describes the event that the neuron transmitter activates the middle process in which the middle variable x_i rises. When x_i crosses the threshold θ_x the action of the gating variable s_i is governed by equation (2.25b).

2.4.3 Neuron network models

The models for neuron networks are systems of differential equations. Even a model for two mutually coupled cells consists of, at least, six ordinary differential equations. The model for a network coupled with a finite number of neurons is

$$\frac{dv_i}{dt} = f_i(v_i, w_i) - \sum_j g_{ij} s_{ij} (v_i - v_{j_{\text{syn}}}), \quad (2.26a)$$

$$\frac{dw_i}{dt} = \epsilon_i g_i(v_i, w_i), \quad (2.26b)$$

$$\frac{ds_{ij}}{dt} = \alpha_j (1 - s_{ij}) H_\infty(v_j - v_{T_j}) - \beta_j s_{ij}, \quad (2.26c)$$

$i, j = 1, 2, \dots, N$. All the functions and parameters are specified for every individual neuron. This is quite close to the real situation but it is a complicated system for mathematical analysis. The only way to know the dynamics of its solutions is by numerical methods through heavy computational work.

In order to study neuron networks theoretically, we need to break the system into smaller systems, and some idealization on the network is also necessary. We suppose the neurons are homogeneous so that the functions and parameters are the same for each neuron. Thus, only one function, $f(v, w)$, and $\epsilon g(v, w)$, appears in (2.26a) and (2.26b), respectively. Next, we suppose only one class of synapses exists among the neurons so that the synapses are homogeneous and the parameters in equation (2.26c) are independent of the subscripts j . The equation becomes

$$\frac{ds_{ij}}{dt} = \alpha(1 - s_{ij})H_{\infty}(v_j - v_T) - \beta s_{ij}. \quad (2.27)$$

In this equation, the presynaptic membrane potential v_j reflects the fact that cell j is coupled with cell i . We can understand the variable s_{ij} as the influence of cell j on cell i . This influence is the same for another cell k if cell j is the presynaptic cell coupled with cells i and k , respectively. This influence is to change the ion currents of cell i or k by the terms

$$\begin{aligned} I_{kj} &= -g_{kj}s_{kj}(v_k - v_{j_{\text{syn}}}), \\ I_{ij} &= -g_{ij}s_{ij}(v_i - v_{j_{\text{syn}}}), \end{aligned} \quad (2.28)$$

respectively. By the assumption of homogeneous synapses, the reverse potential $v_{j_{\text{syn}}}$ can be replaced by v_{syn} and the maximum conductance g_{ij} can be replaced by g_{syn} in (2.28). Moreover the subscript i in (2.27) is only to indicate the fact that cell j is the presynaptic cell coupled with cell i . The

real influence emitted by cell j is governed by the equation

$$\frac{ds_j}{dt} = \alpha(1 - s_j)H_\infty(v_j - v_T) - \beta s_j.$$

If cell i is influenced by cell j then the term

$$I_{ij} = -g_{\text{syn}}s_j(v_i - v_{\text{syn}})$$

will appear in the equation governing the membrane potential v_i of cell i .

We can use a matrix

$$W = (w_{ij})$$

to indicate if cell i is influenced by cell j through the entry

$$w_{ij} = \begin{cases} 1, & \text{influenced; (coupled with } j) \\ 0, & \text{not influenced. (no coupling)} \end{cases}$$

Thus, for homogeneous neurons and homogeneous synapses, the system of the network equations is

$$\begin{aligned} \frac{dv_i}{dt} &= f_i(v_i, w_i) - g_{\text{syn}} \sum_j w_{ij} s_j (v_i - v_{j_{\text{syn}}}), \\ \frac{dw_i}{dt} &= \epsilon g(v_i, w_i), \\ \frac{ds_i}{dt} &= \alpha(1 - s_i)H_\infty(v_i - v_T) - \beta s_i. \end{aligned}$$

Here, we dropped the subscript i in s_{ij} , and describe the gating variable s_j by the third equation. We represent the coupling between cells i and j by w_{ij} , which can be viewed as a weight to represent the probability of cell i coupling with the presynaptic cell j , $i, j = 1, 2, \dots, N$.

If the number of cells in the network become large then the continuous form is convenient. Suppose the network is distributed on the space domain

$D \subset \mathbb{R}$, and $v(x, t)$ is the membrane potential at the site $x \in D$ and time moment t . The system of equations is

$$\begin{aligned} \frac{\partial v}{\partial t}(x, t) &= f(v(x, t), w(x, t)) \\ &\quad - g_{\text{syn}}(v(x, t) - v_{\text{syn}}) \int_D W(x, y) s(y, t) dy, \\ \frac{\partial w}{\partial t}(x, t) &= \epsilon g(v(x, t), w(x, t)), \\ \frac{\partial s}{\partial t}(x, t) &= \alpha(1 - s(x, t)) H_{\infty}(v(x, t) - v_T) - \beta s(x, t), \end{aligned} \tag{2.29}$$

where the weight function satisfies $W(x, y) \geq 0$ and $\int_D W(x, y) dy \leq 1$.

2.4.4 Excitatory / inhibitory coupling

We use the Morris–Lecar system to find the effect of the applied current on the dynamical behavior of a neuron. We suppose that the system

$$\begin{aligned} \frac{dv}{dt} &= f(v, w), \\ \frac{dw}{dt} &= \epsilon g(v, w), \quad \epsilon > 0, \end{aligned}$$

has the fixed point (v_0, w_0) at the left branch of C_1 near the minimum point. This means that the neuron is in the rest state (v_0, w_0) . If a positive current is applied to the neuron then the equation changes to

$$\begin{aligned} \frac{dv}{dt} &= f(v, w) + I, \\ \frac{dw}{dt} &= \epsilon g(v, w), \quad \epsilon > 0. \end{aligned} \tag{2.30}$$

The curve C_{11} , defined by the equation $f(v, w) + I = 0$, is a shift upward of the original C_1 . If I is appropriately large such that the fixed point of system (2.30) is in the middle segment of C_{11} and the original fixed point (v_0, w_0)

is below the minimum point of C_{11} then the orbit of system (2.30), starting at (v_0, w_0) , will rapidly go to the right branch and wind around the singular periodic orbit of system (2.30). This means that the neuron fires repeatedly. In this case, we say that the applied current excites the neuron fires.

Conversely, if a negative current, $I < 0$, is applied to the neuron, then the curve C_{11} will be a shift downward of C_1 and the orbit of system (2.30), starting at (v_0, w_0) , will go to the left and converge to the new fixed point (v_1, w_1) with $v_1 < v_0$. In this case, we say that the applied current inhibits the neuron firing.

In excitatory synapses, the reversal potential, v_{syn} , of the synaptic receptor for the specific ions is greater than the resting potential of postsynaptic cell. The ion current through the synaptic channel is

$$I_{\text{syn}} = -g_{\text{syn}}s(v_{\text{po}} - v_{\text{syn}}).$$

It is positive for v_{po} varying in some intervals. In the inhibitory synapses, the situation is just the converse, the ion current is negative for v_{po} varying in some intervals.

In the Morris–Lecar model, the current circuit equation is written as

$$C \frac{dV}{dt} = -g_{Ca}m_{\infty}(V - V_{Ca}) - g_K w(V - V_K) - g_L(V - V_L) + I_{\text{app}}.$$

The synaptic current is added to the equation as the term

$$I_{\text{app}} = I_{\text{syn}} = -g_{\text{syn}}s(v_{\text{po}} - v_{\text{syn}}).$$

It is positive in excitatory synapses, and negative in the inhibitory synapses. The effects of synaptic currents on the postsynaptic cell are consistent with the above mathematical analysis.

2.4.5 Two mathematical problems on neuronal network

The study of traveling wave solutions of equations modeling neuronal networks led to many results [3, 24, 30, 31, 32, 35]. The equation for infinite networks, like equation (2.29) has been studied in [34, 37, 38]. If we consider a network coupled by excitatory synapses, then it is reasonable to guess that there exists a traveling wave solution, which reflects the firing dynamics of neurons. A numerical simulation can show the neurons in an excitatory coupled network fire to form a traveling wave. But to prove this conjecture mathematically is difficult. Zhang [38] studied the simplified equation

$$\frac{\partial v}{\partial t}(x, t) = f(v(x, t), w) - \alpha \int_D W(x - y)H(v(y, t) - \theta) dy, \quad (2.31)$$

where the last term on right-hand side of (2.29) is simplified into a form without the variables w and s . But there are some serious faults in the results and the proof of his main theorem. This simplified model equation is investigated again in the next chapter, the wrong results on the existence and uniqueness of traveling wave solutions for this equation contained in [38] are corrected, and the necessary and sufficient conditions for traveling wave solutions are established. Furthermore, the properties of the traveling wave solutions are considered. Thus, we get a comprehensive understanding of this model equation. The details on this equation will be defined in the next chapter.

A more realistic model equation than (2.31) is

$$\frac{\partial v}{\partial t}(x, t) = f(v(x, t), w) - \alpha(v(x, t) - v_{\text{syn}}) \int_D W(x - y)H(v(y, t) - \theta) dy.$$

Here the term $(v - v_{\text{syn}})$ in (2.29) is reserved. This equation is studied in the Chapter 4.

Chapter 3

Existence and Uniqueness of Traveling Wave Solutions for a Simplified Equation

3.1 Introduction

We consider the one-dimensional integral equation

$$u_t(t, x) = f(u, w) + \alpha \int_{\mathbb{R}} K(x - y)H(u(t, y) - \theta) dy, \quad (3.1)$$

as a simplified model of a neuronal network. This is the same as (2.31) except that v is changed to u , because we want to use v for the wave speed. The space variables $x, y \in \mathbb{R}$ specify the sites of cells in the network (at each site there is a collection of neurons), t is time, $u(t, x)$ stands for the potential averaged over the population of cells at x , the constant w is the ion channel parameter, $H(s)$ is the Heaviside step function, the constant θ

is the threshold, and $\alpha > 0$ is the synaptic constant.

The integral term in (3.1) represents the global influence of the network on the cell at site x . The kernel, $K(x)$, is an even function which satisfies the conditions

$$0 \leq K(x) \leq C e^{-\gamma|x|}, \quad \gamma > 0, \quad \int_{\mathbb{R}} K(x) dx = 1, \quad \int_{\mathbb{R}} |K'(x)| dx < +\infty, \quad (3.2)$$

where C and γ are constants.

This equation has been investigated by Zhang [38], where a theorem on the existence and uniqueness of traveling wave solution for system (3.1) was proposed as follows:

“Suppose that $f \in C^1(\mathbb{R}^2)$ and $f_w < 0$. Let α be appropriately large such that $\alpha + f(u, 0) = 0$ has a unique positive solution $\beta > 1$ and that $f'(\beta) < 0$. Suppose that the threshold θ satisfy $\theta \in (\rho_-(a), \rho_+(a))$ and there exists a unique number w_0 such that $2f(\theta, w_0) + \alpha = 0$, $2f(\theta, w) + \alpha < 0$ for all $w > w_0$, and $2f(\theta, w) + \alpha > 0$ for all $w < w_0$. Then, for each fixed $w \in (w_-, w_+)$ with $w \neq w_0$, there exists a unique traveling wave solution $U = U(v, w, z)$ to the integro-differential equation

$$\begin{aligned} vU_z &= f(U(z, w), w) \\ &+ \alpha \int_{\mathbb{R}} K(z - (y + v(w, \theta)t)) H(U(y + v(w, \theta)t, w) - \theta) dy \end{aligned}$$

with the wave speed $-v(w)$, such that $U(v, w, 0) = \theta$ and $U_z(v, w, z) \neq 0$ on \mathbb{R} . Moreover the following exponentially fast limits hold:

$$\begin{aligned} \lim_{z \rightarrow -\infty} U(v, w, z) &= \phi_-(w), & \lim_{z \rightarrow +\infty} U(v, w, z) &= \phi_+(w), \\ \lim_{z \rightarrow \pm\infty} U_z(v, w, z) &= 0, \end{aligned}$$

for $w \in (w_-, w_0)$, and

$$\begin{aligned} \lim_{z \rightarrow -\infty} U(v, w, z) &= \phi_+(w), & \lim_{z \rightarrow +\infty} U(v, w, z) &= \phi_-(w), \\ \lim_{z \rightarrow \pm\infty} U_z(v, w, z) &= 0, \end{aligned}$$

for $w \in (w_0, w_+)$.”

This theorem is the main result of [38]. The proof consists of eight lemmas. However, the theorem is not true. The conditions required in the theorem cannot guarantee the existence of the wanted traveling wave solution. The requirement that there is a unique w_0 such that $2f(\theta, w) + \alpha < 0$ for all $w > w_0$, and $2f(\theta, w) + \alpha > 0$ for all $w < w_0$ is not used because $f_w < 0$ already ensures that. In this chapter, we study the same problem again by using a geometric methods for ODE [10, 19, 33]. The problem of existence and uniqueness of traveling wave solutions of equation (3.1) is transformed into a problem of existence and uniqueness of heteroclinic orbits of plane autonomous systems, which may have six fixed points in the phase plane. Counterexamples of non-existence of traveling wave solutions are provided in this chapter, and an example to the results of [38] is presented, which satisfies all the condition required by [38], but does not have the wanted traveling wave solution. Necessary and sufficient conditions for the existence and uniqueness of traveling wave solutions are established. The result corrects the faults in [38]. Moreover, the properties of traveling wave solutions are studied.

Assume the function $f(u, w)$ satisfies the following conditions, which are the same as in [38].

- (a) $f \in C^1(\mathbb{R}^2)$, $f(0, 0) = f(a, 0) = f(1, 0) = 0$, $0 < a < 1$, $f_w < 0$.

- (b) The cubic-like curve $C_1 = \{(u, w); f(u, w) = 0\}$ has a unique w -minimum (ρ_1^-, w_1^-) and a unique w -maximum (ρ_1^+, w_1^+) with $\rho_1^- < \rho_1^+$. On the left segment of C_1 , the function $u = \phi_1(w)$ is defined implicitly by the equation $f(\phi_1(w), w) = 0$ for $u < \rho_1^-$ and $w_1^- < w$. Moreover, $f_u(\phi_1(w), w) < 0$.
- (c) The cubic-like curve $C_2 = \{(u, w); f(u, w) + \alpha = 0\}$ has a unique w -minimum (ρ_2^-, w_2^-) and a unique w -maximum (ρ_2^+, w_2^+) with $\rho_2^- < \rho_2^+$. On the right segment of C_2 , the function $u = \phi_2(w)$ is defined implicitly by the equation $f(\phi_2(w), w) + \alpha = 0$ for $\rho_2^+ < u$ and $w < w_2^+$. Moreover, $\phi_2(0) = \beta > 1$ and $f_u(\phi_2(w), w) < 0$.
- (d) The threshold θ satisfies the inequality $\rho_1^- < \theta < \rho_1^+$.
- (e) The parameter w satisfies the inequality $w_1^- < w < w_2^+$.

Typical curves C_1 and C_2 are shown in Fig. 3.1. Here $\rho_2^- = \rho_1^-$ and $\rho_2^+ = \rho_1^+$. An example of such a curves is $f(u, w) = u(1 - u)(u - a) - w$.

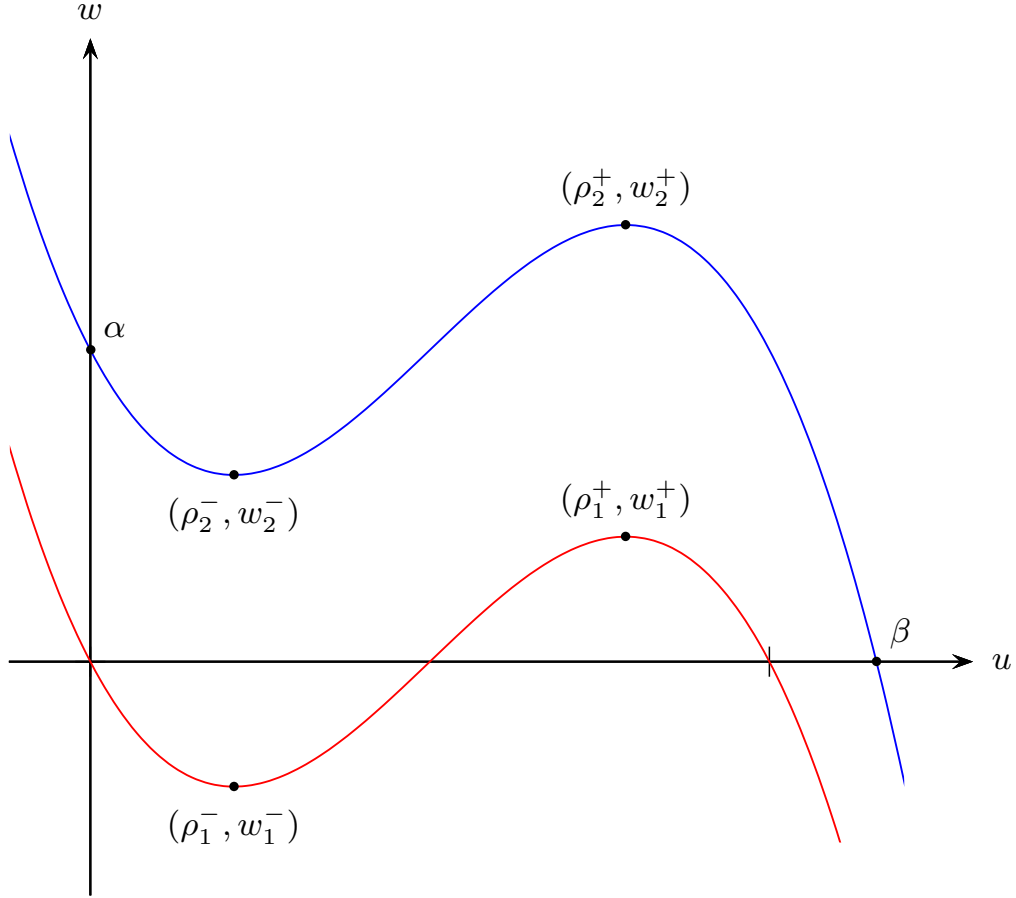


Figure 3.1: The pattern of the curves C_1 and C_2 .

Our notation corresponding to Zhang's notation in [38] is in the following table:

constants in [38]:	$\rho_-(a)$	$\rho_+(a)$	w^-	w^+	$\phi_-(w)$	$\phi_+(w)$
constants in this chapter:	ρ_1^-	ρ_1^+	w_1^-	w_2^+	$\phi_1(w)$	$\phi_2(w)$

In order to simplify the reasoning, we assume that the kernel function is continuous and $K(x) > 0$. The general case for piecewise continuous $K(x) \geq 0$ will be discussed in section 3.5.3.

Now we define traveling wave solutions as in [38].

Definition 3.1.1. A non-constant bounded smooth solution of equation (3.1) is called a *traveling wave solution* if the function

$$u(t, x, w) = U(x + v(w, \theta)t, w) = U(z, w), \quad z = x + v(w, \theta)t, \quad (3.3)$$

with some abuse of notation, satisfies equation (3.1) for some constant $v = v(w, \theta) > 0$ and boundary conditions

$$\lim_{z \rightarrow +\infty} U(z, w) = l_+(w), \quad \lim_{z \rightarrow -\infty} U(z, w) = l_-(w). \quad (3.4)$$

Finding the traveling wave solution of equation (3.1) is equivalent to finding the solution $U(z, w)$ of the integro-differential equation

$$vU_z = f(U(z, w), w) + \alpha \int_{\mathbb{R}} K(z - (y + v(w, \theta)t)) H(U(y + v(w, \theta)t, w) - \theta) dy \quad (3.5)$$

with boundary conditions (3.4). Our strategy is to prove the existence of traveling wave solutions for equation (3.1) by transforming the integro-differential equation (3.5) into an ODE and employing the geometric theory of ODEs.

Let $\xi = z - (y + v(w, \theta)t)$, then the integral part of equation (3.5) is transformed into

$$\int_{\mathbb{R}} K(\xi) H(U(z - \xi, w) - \theta) d\xi,$$

and equation (3.5) can be written as

$$vU_z = f(U(z, w), w) + \alpha \int_{\mathbb{R}} K(\xi) H(U(z - \xi, w) - \theta) d\xi. \quad (3.6)$$

We are interested in monotone solutions of equation (3.5). For monotone increasing solutions with $U_z > 0$ and $U(0, w) = \theta$, we have $U(z - \xi) > \theta$ for

$\xi < z$ and $U(z - \xi) < \theta$ for $\xi \geq z$; hence the equation

$$H(U(z - \xi, w) - \theta) = 1$$

holds only for $z > \xi$. Then we can get

$$\int_{\mathbb{R}} K(\xi) H(U(z - \xi, w) - \theta) d\xi = \int_{-\infty}^z K(\xi) d\xi,$$

and equation (3.6) becomes

$$vU_z = f(U(z, w), w) + \alpha \int_{-\infty}^z K(\xi) d\xi, \quad U_z > 0, \quad U(0, w) = \theta. \quad (3.7)$$

If we prove that equation (3.7) has solutions satisfying the boundary conditions (3.4), these solutions will be monotone increasing traveling wave solutions of equation (3.1).

3.2 Monotone increasing traveling wave solutions

As in [38], let

$$z(\tau) = \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau}, \quad 0 < k < \frac{\gamma}{2}, \quad \tau \in (-1, 1), \quad (3.8)$$

and

$$z(-1) = \lim_{\tau \rightarrow -1} \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau} = -\infty, \quad z(1) = \lim_{\tau \rightarrow 1} \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau} = \infty. \quad (3.9)$$

We denote

$$I(z) = \int_{-\infty}^z K(\xi) d\xi.$$

If we substitute expression (3.9) into the above integral, we obtain

$$I(\tau) = \int_{-1}^{\tau} \frac{1}{k(1-\eta^2)} K \left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta} \right) d\eta. \quad (3.10)$$

With equations (3.9) and (3.10), equation (3.7) is transformed into an equivalent autonomous system in the phase space (τ, U) . Then, a monotone increasing traveling wave solution of (3.5) must satisfy the system

$$\begin{aligned} \frac{d\tau}{dz} = \tau' &= k(1-\tau^2), & \tau(0) &= 0, \\ \frac{dU}{dz} = U' &= \frac{1}{v} [f(U, w) + \alpha I(\tau)], & U(0) &= \theta. \end{aligned} \quad (3.11)$$

System (3.11) is defined on the domain D consisting of the infinite strip

$$D = \{(\tau, U); -1 \leq \tau \leq +1, -\infty < U < +\infty\}. \quad (3.12)$$

It is easy to see from the properties of the kernel function $K(x)$ that $I(\tau)$ is monotone increasing, $I(0) = 1/2$, $I(1) = 1$, and $I(\tau) > 0$ for $-1 < \tau < 1$. Boundary conditions (3.4) now correspond to an orbit of equation (3.11) that approaches some fixed points when $z \rightarrow \pm\infty$. Thus, the traveling wave solution is a heteroclinic orbit of (3.11). Our aim is to find conditions that ensure that this heteroclinic orbit passes through the specified initial point $S = (\tau, U) = (0, \theta)$ and links the fixed points on the lines $\tau = \pm 1$, respectively.

We see that the fixed points of system (3.11) on D should satisfy $f(U, w) = 0$ on the line $\tau = -1$, and $f(U, w) + \alpha = 0$ on the line $\tau = +1$. In both cases, there is at least one and at most three fixed points, depending on the values of w (see Fig. 3.1). A careful case study is required to find the specified heteroclinic orbit.

Now we prove an existence and uniqueness theorem for heteroclinic orbits of system (3.11) in the case that $f(U, w) + \alpha = 0$ has only one solution.

Theorem 3.2.1. *Given the parameter $w \in (w_1^-, w_2^+)$ and the threshold $\theta \in (\rho_1^-, \rho_1^+)$, if $f(U, w) + \alpha/2 > 0$ for $U \leq \theta$, and the equation $f(U, w) + \alpha = 0$ has only one solution, $U = \phi_2(w)$, then there is a unique value $v = v(w, \theta) > 0$ such that equation (3.11) has a unique heteroclinic orbit passing through the point $S = (\tau, U) = (0, \theta)$, linking the fixed points $(\tau, U) = (-1, \phi_1(w))$ and $(\tau, U) = (1, \phi_2(w))$, and its U -coordinate is monotone increasing as z increases.*

Proof. We shall prove this theorem by proving four lemmas in two cases. For $w \in (w_1^-, w_2^+)$ the functions $\phi_1(w)$ and $\phi_2(w)$ are defined implicitly. The two fixed points $(\tau, U) = (-1, \phi_1(w))$ and $(\tau, U) = (1, \phi_2(w))$ exist with $\phi_1(w) < \phi_2(w)$. There are two cases to be considered.

Case 1. The case of the parameter $w > w_1^+$. Figure 3.2 shows the solutions of $f(U, w) = 0$ and $f(U, w) + \alpha = 0$ in this case. At $\tau = -1$, the derivative satisfies $U' = \frac{1}{v}f(U, w)$, because $I(-1) = 0$.

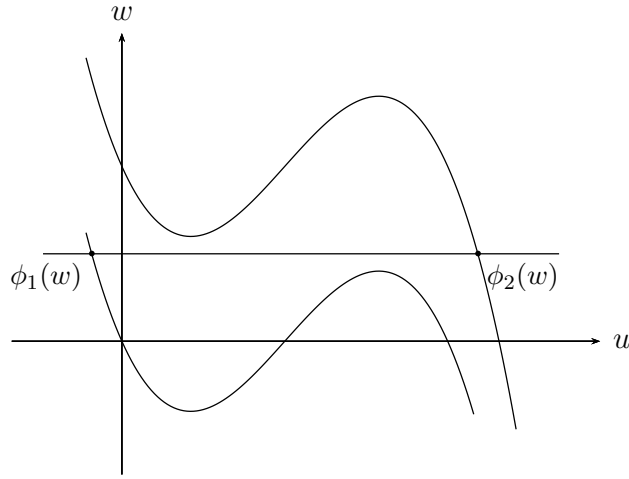


Figure 3.2: The case of $w > w_1^+$.

In this case, it is easy to see from Figure 3.2 that $f(U, w) = 0$ has only one solution, so $(\tau, U) = (-1, \phi_1(w))$ is the only fixed point of system (3.11) at $\tau = -1$. The domain D is divided into three parts by the horizontal lines

$$L_1 = \{(\tau, U); \tau \in [-1, 1], U = \phi_1(w)\}$$

and

$$L_2 = \{(\tau, U); \tau \in [-1, 1], U = \phi_2(w)\}.$$

The part where $U < \phi_1(w)$ is denoted by D_1 , and the one where $\phi_1(w) \leq U \leq \phi_2(w)$ by D_2 . (See Fig. 3.3.) We need to consider the geometric properties of the fixed points.

Lemma 3.2.2. *The fixed point $(\tau, U) = (-1, \phi_1(w))$ is a saddle point and its local unstable manifold is in D_2 and tangent to the line L_1 .*

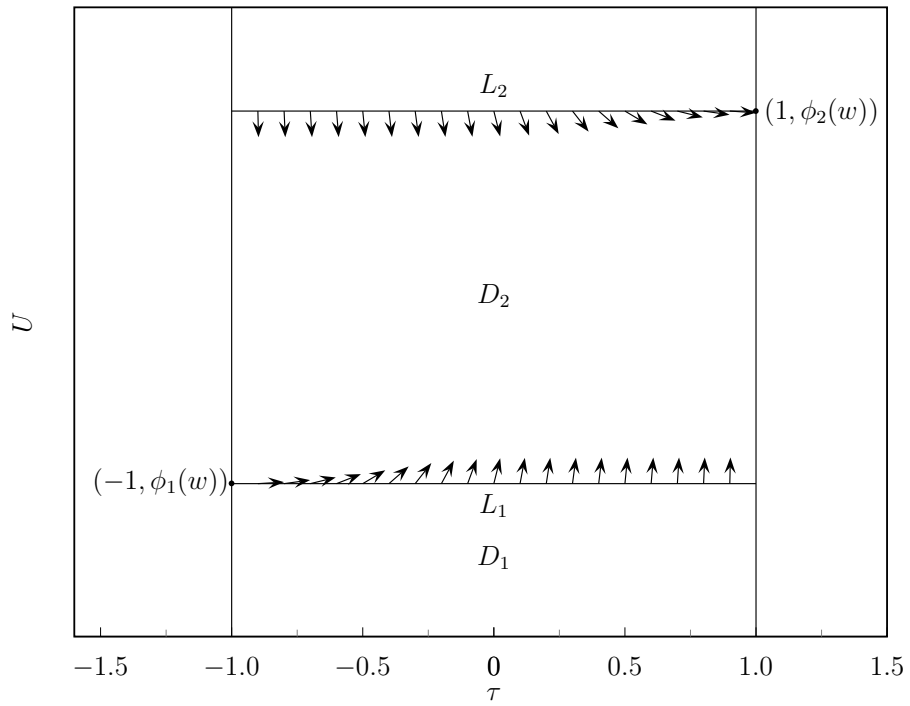


Figure 3.3: The domains D_1 and D_2 . Arrows show the direction of the vector fields on the lines L_1 and L_2 .

Proof. The Jacobian of the system (3.11) is

$$J = \begin{bmatrix} -2k\tau & 0 \\ \frac{1}{v}\alpha I'(\tau) & \frac{1}{v}f_u(\phi_1(w), w) \end{bmatrix}. \quad (3.13)$$

The derivative of $I(\tau)$ is

$$I'(\tau) = \frac{1}{k(1-\tau^2)} K \left(\frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right).$$

According to the properties of the kernel function $K(x)$,

$$0 \leq K \left(\frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right) \leq C \exp \left(-\gamma \left| \frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right| \right),$$

so

$$\begin{aligned} 0 \leq I'(\tau) &\leq \frac{C}{k(1-\tau^2)} \exp \left(-\gamma \left| \frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right| \right) \\ &= \frac{C(1+\tau)^{\frac{\gamma}{2k}-1}}{k(1-\tau)^{1+\frac{\gamma}{2k}}} \rightarrow 0. \end{aligned}$$

as $\tau \rightarrow -1$ and $k < \gamma/2$. The Jacobian (3.13) evaluated at the fixed point $(\tau, U) = (-1, \phi_1(w))$ is

$$\begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{v}f_u(\phi_1(w), w) \end{bmatrix}.$$

According to condition (b), the entry $\frac{1}{v}f_u(\phi_1(w), w) < 0$, hence this fixed point is a saddle point. From the Hartman–Grobman Theorem [10], there is a unique local unstable manifold, say Γ , tangent to the characteristic vector $(1, 0)$. We claim that Γ must be in the domain D_2 . In fact, in the interior of the domain

$$D_1 = \{(\tau, U); \tau \in [-1, +1], U < \phi_1(w)\},$$

we have

$$U' = \frac{1}{v}[f(U, w) + \alpha I(\tau)] > 0,$$

because $\alpha I(\tau) > 0$ and $f(U, w) > f(\phi_1(w), w) = 0$ for $U < \phi_1(w)$. Meanwhile, $\tau' = k(1 - \tau^2) > 0$. The vector field defined by system (3.11) points in a top-right direction. If Γ entered into D_1 , it had to go top-right (see Figure 3.4). According to the vector field, it then left the domain D_1 by passing through the boundary L_1 where $U' = f(\phi_1(w), w) + \alpha I(\tau) = \alpha I(\tau) > 0$ for $-1 < \tau$ (see Figure 3.4). Thus, there would be a point $P = (\tau_0, U_0) \in \Gamma \cap D_1$ such that $U' = f(U_0, w) + \alpha I(\tau_0) = 0$. But we have proved that $U' > 0$ in the interior of D_1 . This contradiction shows that the claim is true and the proof is complete.

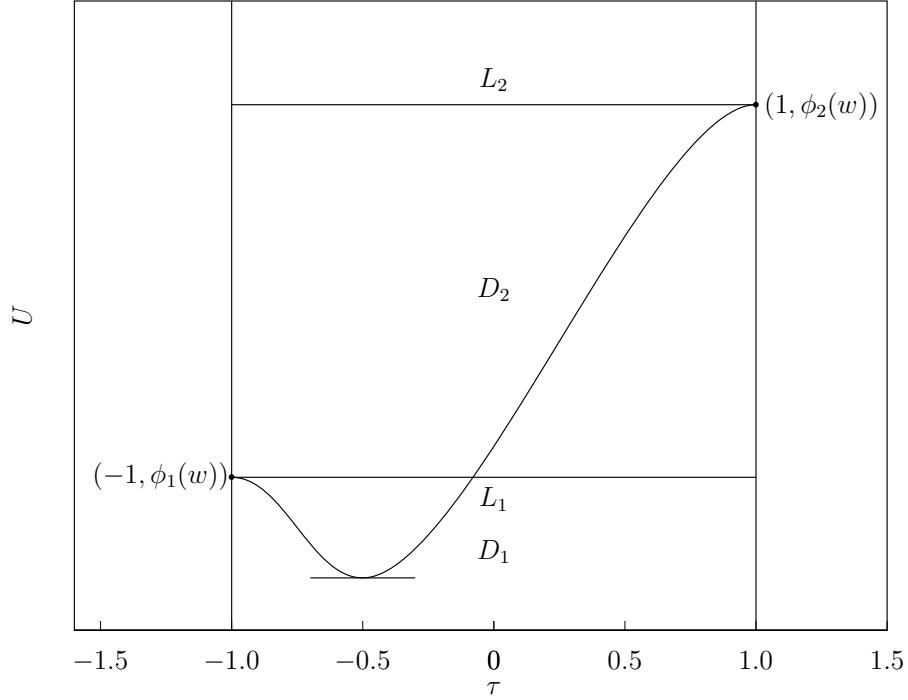


Figure 3.4: A schematic graph for Lemma 3.2.2.

□

Now we need to follow the evolution of the manifold Γ in D_2 as $z \rightarrow +\infty$.

Lemma 3.2.3. *The unstable manifold Γ of the saddle point $(\tau, U) = (-1, \phi_1(w))$ approaches the fixed point $(\tau, U) = (1, \phi_2(w))$ as $z \rightarrow +\infty$. It is the unique heteroclinic orbit linking the two fixed points.*

Proof. On the line L_1 , the vector field points in a top-right direction because

$$f(\phi_1(w), w) + \alpha I(\tau) = \alpha I(\tau) > 0 \quad \text{for } -1 < \tau.$$

On the line L_2 , the vector field points in a bottom-right direction because

$$f(\phi_2(w), w) + \alpha I(\tau) < f(\phi_2(w), w) + \alpha = 0 \quad \text{for } \tau < 1.$$

On the line $\tau = 1$, the vector field points vertically up since $f(U, w) + \alpha > 0$ for $U < \phi_2(w)$. On the line $\tau = -1$, the vector field points vertically down since $f(U, w) < 0$ for $U > \phi_1(w)$. Thus, D_2 is a positive invariant set. By Poincaré-Bendixson theorem, Γ has to go to a fixed point. Because $\tau'(z) \geq 0$, Γ must go to the fixed point $(1, \phi_2(w))$. Thus, Γ , as the unique local unstable manifold of the saddle point $(-1, \phi_1(w))$, is the unique heteroclinic orbit linking the fixed points $(\tau, U) = (-1, \phi_1(w))$ and $(\tau, U) = (1, \phi_2(w))$. \square

Lemma 3.2.4. *The U -coordinate, $U(z)$, of the unstable manifold Γ is a monotone increasing function of z .*

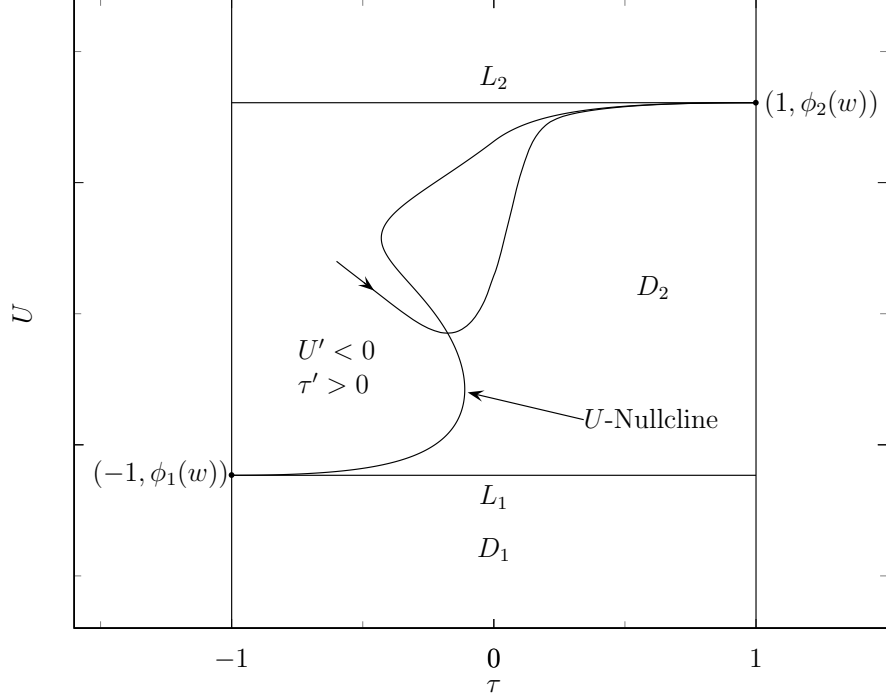


Figure 3.5: U -nullcline in Case 1.

Proof. The U -nullcline of (3.11), denoted by Σ , is defined as

$$\Sigma = \{(\tau, U) \in D; \tau \in (-1, 1), f(U, w) + \alpha I(\tau) = 0\}.$$

Figure 3.5 shows a U -nullcline of (3.11) in Case 1. Since the kernel function is positive, we have $I'(\tau) > 0$. Thus, Σ never has a horizontal tangent. As z decreases, the orbit starting at Σ will enter into the left side of Σ because the vector field on Σ is $(k(1 - \tau^2), 0)$. The U -nullcline divides the domain D_2 into two parts. We consider U' in the left part. Let (U_l, τ_l) be a point lying to the left of Σ , and (U_l, τ_n) be the point on Σ with the same U -coordinate.

Clearly, $\tau_l < \tau_n$. We then have

$$U'(U_l, \tau_l) = f(U_l, w) + \alpha I(\tau_l) < f(U_l, w) + \alpha I(\tau_n) = 0$$

by the properties of $I(\tau)$. If an orbit of (3.11) passes through a point lying to the left of Σ or a point on Σ , then it cannot approach the saddle point $(-1, \phi_1(w))$, because U increases when z decreases in the left part by $U' < 0$. (See Figure 3.5). Hence a heteroclinic orbit cannot pass the left part. This fact ensure that Γ must stay in the right part. We next consider the U' of points in the right part. Let (U_r, τ_r) be a point lying to the right of Σ , and (U_r, τ_c) be the point on Σ with the same U -coordinate. Clearly, $\tau_r > \tau_c$. We then have

$$U'(U_r, \tau_r) = f(U_r, w) + \alpha I(\tau_r) > f(U_r, w) + \alpha I(\tau_c) = 0$$

by the properties of $I(\tau)$. Since Γ stays in the right part, $U'(z)$ of Γ is always positive, which means that $U(z)$ is an increasing function of z . \square

Now, we prove that there exists a unique value v_0 such that the heteroclinic orbit Γ of (3.11) with $v = v_0$ passes through the initial point $S = (\tau(0), U(0)) = (0, \theta)$.

Lemma 3.2.5. *If $f(U, w) + \frac{1}{2}\alpha > 0$ for $U \leq \theta$, then there exists a unique $v_0 > 0$ such that the heteroclinic orbit of (3.11) with $v = v_0$ passes through the point S and links the fixed points $(\tau, U) = (-1, \phi_1(w))$ and $(\tau, U) = (1, \phi_2(w))$.*

Proof. In the proof of Lemmas 3.2.2 and 3.2.3, we have shown that system (3.11) has a unique orbit linking the fixed points for $v > 0$ under the conditions of Case 1. Moreover, Lemma 3.2.4 shows that its U -coordinate is

an increasing function of the τ -coordinate, because both $U(z)$ and $\tau(z)$ are increasing functions of the independent variable z . Denote this orbit as

$$\Gamma(v) = \{(\tau, U) = (\tau, U(\tau, v)); \tau \in (-1, 1)\}.$$

We prove this lemma in four steps.

Step 1. There exists $0 < v_1$ such that $U(0, v_1) > \theta$, and $0 < v_2$ such that $U(0, v_2) < \theta$.

By continuity of $f(U, w) + \alpha I(\tau)$, we can find a closed rectangle

$$R = \{(\tau, U); -\delta \leq \tau \leq 0, \phi_1(w) \leq U \leq \theta\} \subset D_2, \quad 0 < \delta < 1,$$

such that $f(U, w) + \alpha I(\tau) \geq \eta$ for some $\eta > 0$. Take a small $v_1 > 0$ such that

$$\frac{\eta}{v_1 k} > \frac{\theta - \phi_1(w)}{\delta}.$$

Then the orbit $O(z, -\delta, \phi_1(w))$ of (3.11), starting from the point $(\tau = -\delta, U = \phi_1(w))$ at $z = 0$, has the derivative

$$\frac{dU}{d\tau} = \frac{f(U, w) + \alpha I(\tau)}{v_1 k(1 - \tau^2)} \geq \frac{\eta}{v_1 k} > \frac{\theta - \phi_1(w)}{\delta}$$

on R . This means that the orbit $O(z, -\delta, \phi_1(w))$ will get to the position $(\tau, U) = (\tau(z_0) = 0, U(z_0) > \theta)$ for some $z_0 > 0$ as z increases. By the properties of phase flows of autonomous ODEs [1], the orbit $\Gamma(v_1)$ cannot intersect with the orbit $O(z, -\delta, \phi_1(w))$. Thus the unstable manifold $\Gamma(v_1)$ of the fixed point $(\tau, U) = (-1, \phi_1(w))$ will stay above the orbit $O(z, -\delta, \phi_1(w))$, and get to the point $(\tau, U) = (0, U(0, v_1))$ with $U(0, v_1) > \theta$. See Figure 3.6.

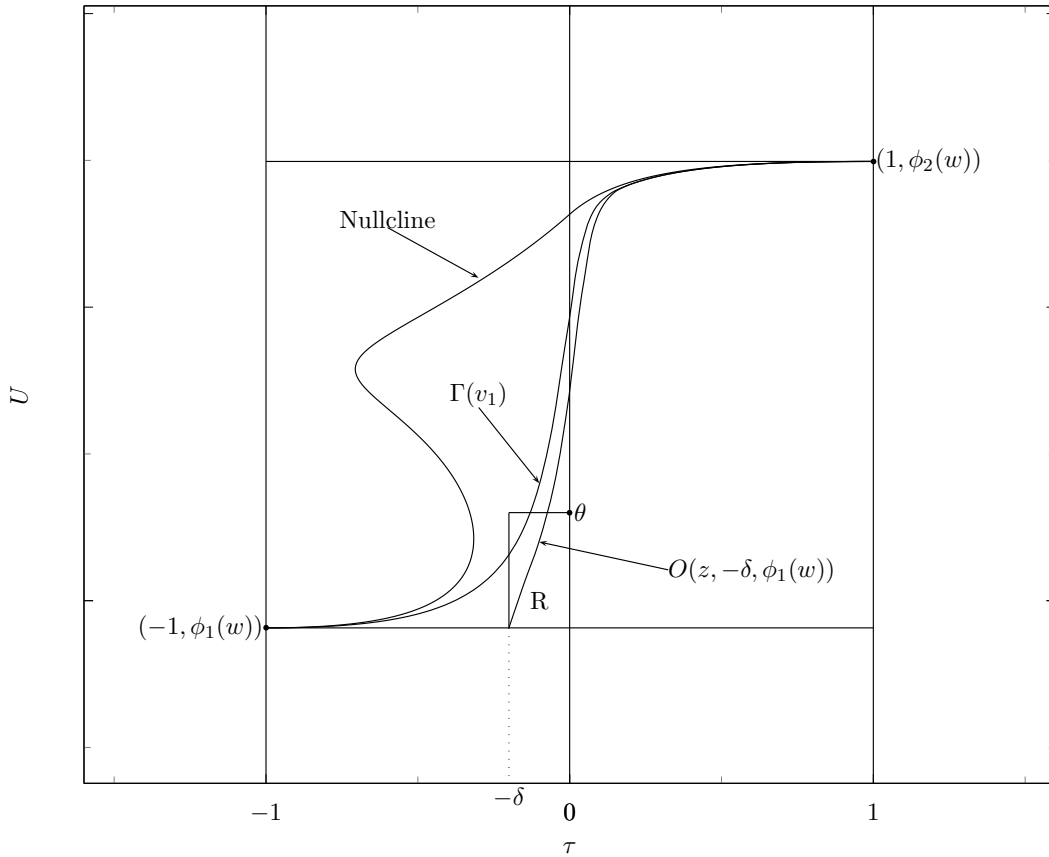


Figure 3.6: U -nullcline, $O(z, -\delta, \phi_1(w))$, and $\Gamma(v_1)$ in Case 1.

If we let $M = \max\{f(U, w) + \alpha I(\tau); -1 \leq \tau \leq 0 \text{ and } \phi_1(w) \leq U \leq \theta\}$, then $M > 0$. Consider the U -nullclenic curve

$$\Sigma = \{(\tau, U); \tau \in [-1, 1], f(U, w) + \alpha I(\tau) = 0\}.$$

Since $f_U(\phi_1(w), w) < 0$, $\alpha I(\tau) \geq 0$, and $I'(-1) = 0$, Σ is in a neighborhood of $(\tau, U) = (-1, \phi_1(w))$ for $\tau > -1$ and $U \geq \phi_1(w)$, and has the horizontal tangent line $U = \phi_1(w)$ at the point $(-1, \phi_1(w))$. Let (τ_0, U_0) be a point on the U -nullcline Σ near the fixed point $(-1, \phi_1(w))$ with $-1 < \tau_0 < -\frac{1}{2}$. Then the orbit $O(z, \tau_0, U_0)$, starting from $(\tau_0, U_0) \in \Sigma$ at $z = 0$, will stay in the

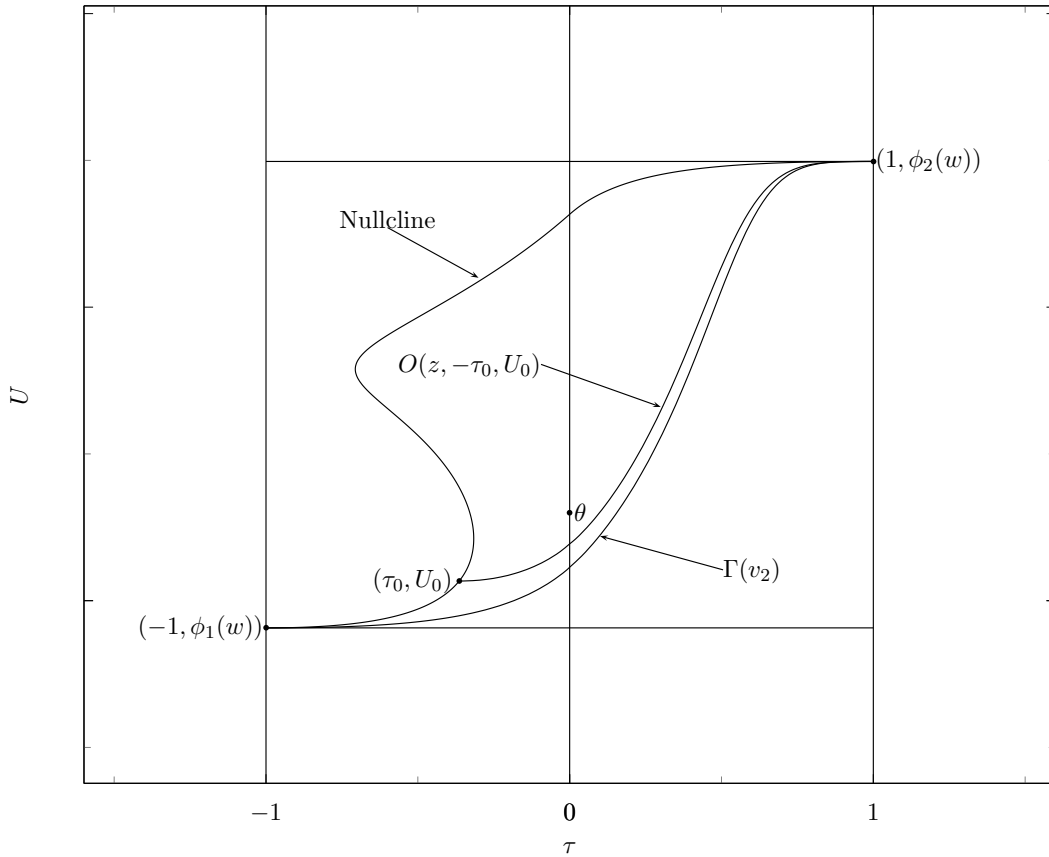


Figure 3.7: $O(z, \tau_0, U_0)$ and $\Gamma(v_2)$.

right side of Σ as z increases (See Figure 3.7). Take v_2 large enough such that $v_1 < v_2$ and

$$\frac{f(U, w) + \alpha I(\tau)}{v_2 k(1 - \tau^2)} \leq \frac{M}{v_2 k(1 - \tau_0^2)} < \frac{\theta - U_0}{-\tau_0}, \quad -1 \leq \tau_0 \leq \tau < 0,$$

then the orbit $O(z, \tau_0, U_0)$ of (3.11) will get to a point $(\tau(z_0) = 0, U(z_0) < \theta)$ for some $z_0 > 0$. By the properties of phase flows of autonomous ODEs [1], the unstable manifold $\Gamma(v_2)$ of the fixed point $(-1, \phi_1(w))$ will stay below $O(z, \tau_0, U_0)$ and get to a point $(\tau = 0, U(0, v_2) < \theta)$.

Step 2. For any $0 < v_1 < v_2$, $U(\tau, v_1) > U(\tau, v_2)$, $\tau \in (-1, 1)$.

If $U(\tau_0, v_1) = U(\tau_0, v_2)$, then

$$f(U(\tau_0, v_1), w) + \alpha I(\tau_0) = f(U(\tau_0, v_2), w) + \alpha I(\tau_0) = \mu_0 > 0.$$

Consider the tangents to $\Gamma(v_1)$ and $\Gamma(v_2)$ at the point $(\tau_0, U(\tau_0, v_1)) = (\tau_0, U(\tau_0, v_2))$; then the following inequality holds:

$$\frac{\mu_0}{v_1 k(1 - \tau_0^2)} > \frac{\mu_0}{v_2 k(1 - \tau_0^2)}. \quad (3.14)$$

On the left side of the intersecting point, $\Gamma(v_1)$ is below $\Gamma(v_2)$ for $\tau \in (\tau_0 - \delta, \tau_0)$, when δ is small enough. We claim that $\Gamma(v_1)$ cannot meet $\Gamma(v_2)$ again for $\tau < \tau_0$. In fact, if they met at some $\tau_1 < \tau_0$, then we would have

$$f(U(\tau_1, v_1), w) + \alpha I(\tau_1) = f(U(\tau_1, v_2), w) + \alpha I(\tau_1) = \mu_1 > 0. \quad (3.15)$$

Since $\Gamma(v_1)$ enters $\Gamma(v_2)$ from below, we would have

$$0 < \frac{\mu_1}{v_1 k(1 - \tau_0^2)} \leq \frac{\mu_1}{v_2 k(1 - \tau_0^2)}. \quad (3.16)$$

This cannot be true, because we assumed $v_1 < v_2$.

Let Q be a point below $\Gamma(v_2)$ and above $\Gamma(v_1)$. See Figure 3.8. Consider the orbit $O(Q, v_1)$ of system (3.11) with $v = v_1$ which passes through Q . As $z \rightarrow -\infty$, $O(Q, v_1)$ must intersect $\Gamma(v_2)$ at some point $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$.

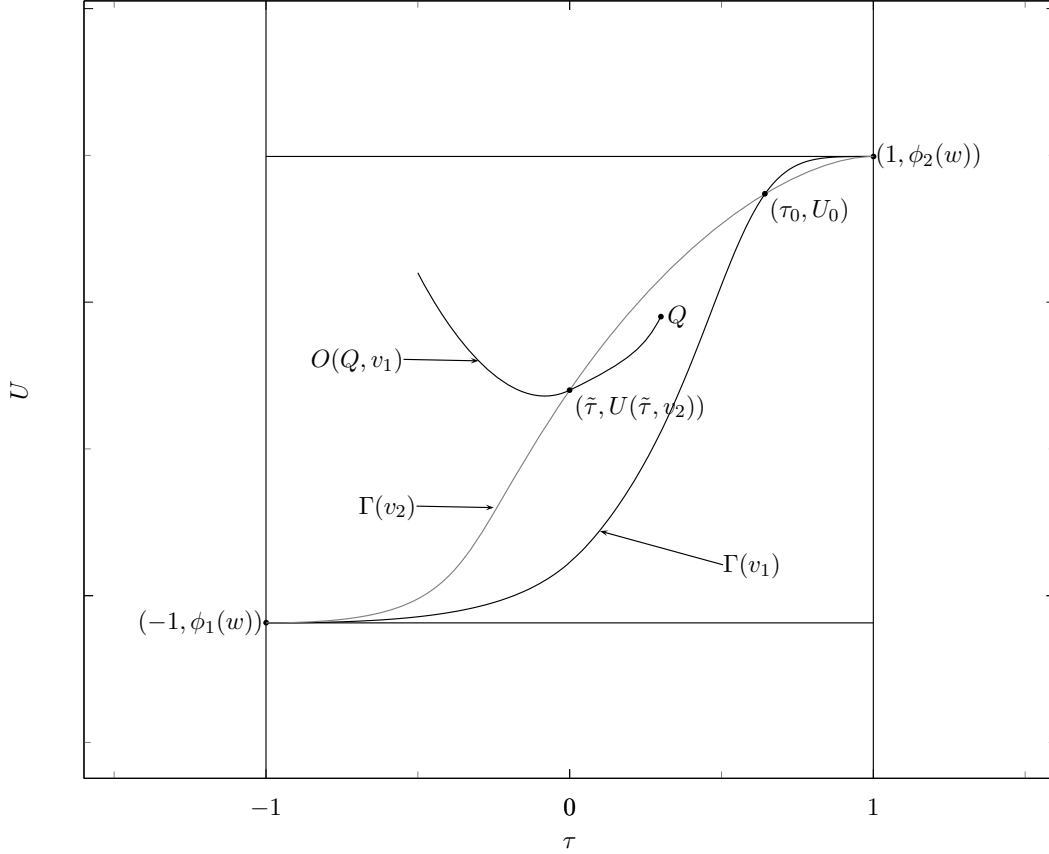


Figure 3.8: The schematic diagram of $\Gamma(v_1)$, $\Gamma(v_2)$, and $O(Q, v_1)$.

By comparing the tangents of $O(Q, v_1)$ and $\Gamma(v_2)$ at the point $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$, we also get

$$f(U(\tilde{\tau}, v_1), w) + \alpha I(\tilde{\tau}) = f(U(\tilde{\tau}, v_2), w) + \alpha I(\tilde{\tau}) = \tilde{\mu} > 0,$$

and the false inequality

$$0 < \frac{\tilde{\mu}}{v_1 k(1 - \tilde{\tau}^2)} \leq \frac{\tilde{\mu}}{v_2 k(1 - \tilde{\tau}^2)}.$$

This contradiction proves that $U(\tau, v_1) \neq U(\tau, v_2)$ for $\tau \in (-1, 1)$.

Next we prove that $U(\tau, v_1) > U(\tau, v_2)$ for $\tau \in (-1, 1)$. In fact, if $U(\tau, v_1) < U(\tau, v_2)$ for $\tau \in (-1, 1)$, we can take a point $Q = (\tau', U')$ below $\Gamma(v_2)$ and above $\Gamma(v_1)$. By repeating the reasoning for $O(Q, v_1)$ we get the same false inequality, and this contradiction proves that $U(\tau, v_1) > U(\tau, v_2)$ for $\tau \in (-1, 1)$. See Figure 3.9.

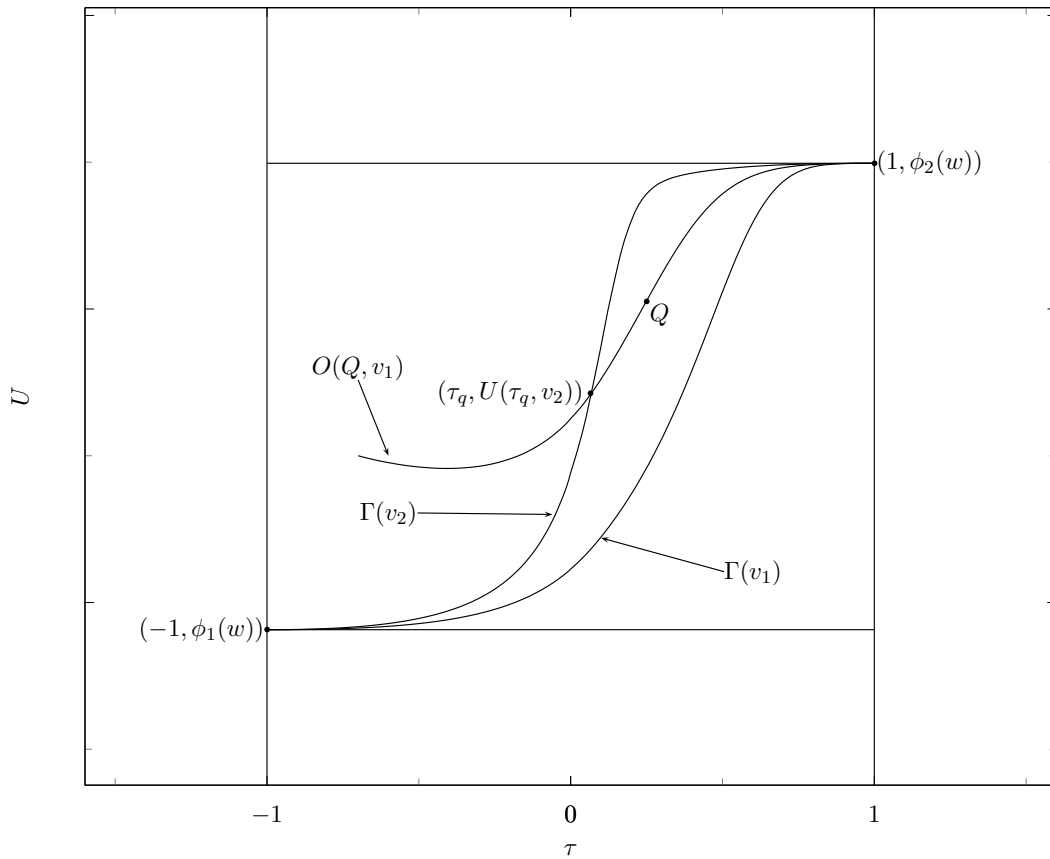


Figure 3.9: The schematic diagram of $\Gamma(v_1)$, $\Gamma(v_2)$, and $O(Q, v_1)$.

Step 3. If $\omega = \sup\{v; U(0, v) > \theta\}$ and $\Omega = \inf\{v, U(0, v) < \theta\}$, then $\omega = \Omega$.

From the result of step 2, it is easy to see that ω and Ω exist. If $\omega > \Omega$,

then there exists v such that $\omega > v > \Omega$. From $\omega > v$ we have $U(0, v) > \theta$. From $v > \Omega$ we have $U(0, v) < \theta$. This is a contradiction. If $\omega < \Omega$ then there exists v such that $\omega < v < \Omega$. Then, from $v < \Omega$ we have $U(0, v) \geq \theta$. From $\omega < v$ we have $U(0, v) \leq \theta$. Thus, we would have $U(0, v) = \theta$; but we can find a v'' such that $\omega < v' < v'' < \Omega$ and $U(0, v') = U(0, v'') = \theta$. This contradicts the result of step 2. Because of these two contradictions we must have $\omega = \Omega$, and $U(0, \omega) = U(0, \Omega)$.

Step 4. $U(0, \omega) = \theta = U(0, \Omega)$.

Suppose that $U(0, \omega) > \theta$. Consider the orbit $O(z, 0, \theta, \omega)$ of system (3.11) with $v = \omega$ and the initial value set to $(0, \theta)$ at $z = 0$. Then $O(z, 0, \theta, \omega)$ will stay below $\Gamma(\omega)$ and intersect the line segment

$$L_1 = \{(\tau, \phi_1(w)); -1 < \tau < 0\}$$

at some moment $z_0 < 0$. On the closed interval $[z_0 - 1, 0] \supset [z_0, 0]$, by the continuous dependence of the solution on parameters, there exists $v' > \omega$ such that the orbit $O(z, 0, \theta, v')$ of system (3.11) with $v = v'$, starting at the initial point $(0, \theta)$ at $z = 0$, sufficiently close to the orbit $O(z, 0, \theta, \omega)$ so that $O(z, 0, \theta, v')$ also enters the line segment L_1 at some moment $z' \in [z_0 - 1, 0]$. Thus, $\Gamma(v')$ will stay above $O(z, 0, \theta, v')$ and we will have $U(0, v') > \theta$. This contradicts the definition of ω . See Figure 3.10.

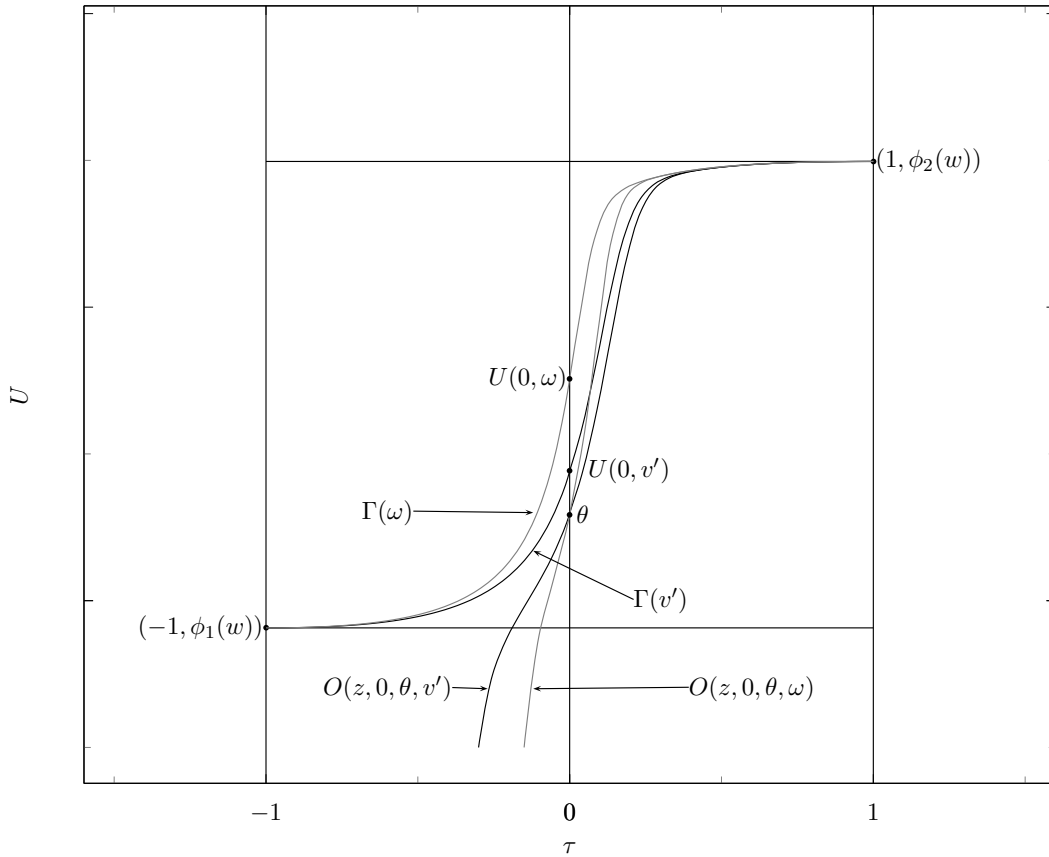


Figure 3.10: $\Gamma(\omega)$, $\Gamma(v')$, $O(z, 0, \theta, \omega)$, and $O(z, 0, \theta, v')$

Assume that $U(0, \Omega) < \theta$. We consider the orbit $O(z, 0, \theta, \Omega)$ of system (3.11) with $v = \Omega$ and initial value set to $(0, \theta)$ at $z = 0$. Then $O(z, 0, \theta, \Omega)$ will stay above $\Gamma(\Omega)$ and enter the U -nullcline Σ in a neighborhood of $(\tau, U) = (-1, \phi_1(w))$ at some moment $z_0 < 0$. By a similar reasoning as above, we can find a $v' < \Omega$ such that the orbit $O(z, 0, \theta, v')$ of system (3.11) with $v = v'$, starting at the initial point $(0, \theta)$ at $z = 0$, sufficiently close to the orbit $O(z, 0, \theta, \Omega)$ so that $O(z, 0, \theta, v')$ also enters the U -nullcline Σ at some moment $z' \in [z_0 - 1, 0]$. Thus, $\Gamma(v')$ will stay below $O(z, 0, \theta, v')$ and we

will have $U(0, v') < \theta$. This contradicts the definition of Ω . See Figure 3.11.

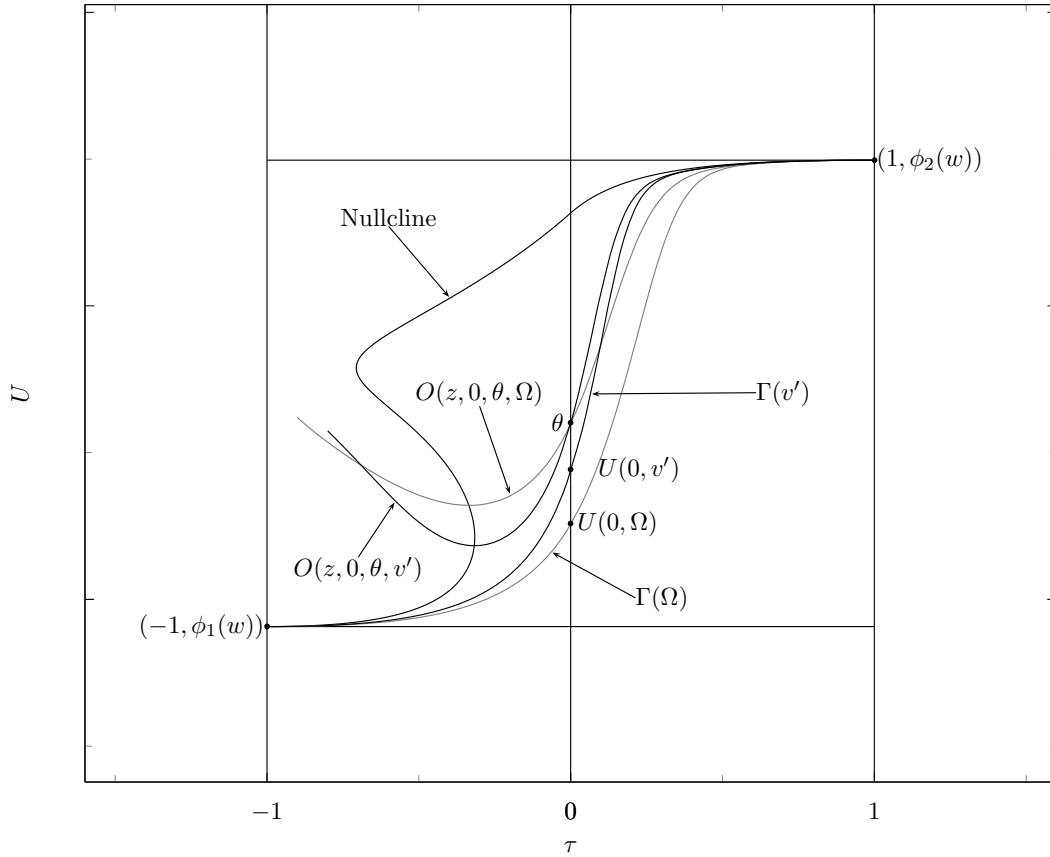


Figure 3.11: $\Gamma(\Omega)$, $\Gamma(v')$, $O(z, 0, \theta, \Omega)$, and $O(z, 0, \theta, v')$

Thus, there exists unique $\omega = \Omega = v$ such that $U(0, \omega) = \theta = U(0, \Omega)$. \square

At this stage, Theorem 3.2.1 has been proved for Case 1.

Case 2. Let $w_1^- < w < w_1^+$. Then there are three fixed points $(-1, \phi_1(w))$, $(-1, \phi_{11}(w))$, and $(-1, \phi_{12}(w))$ on the line $\tau = -1$.

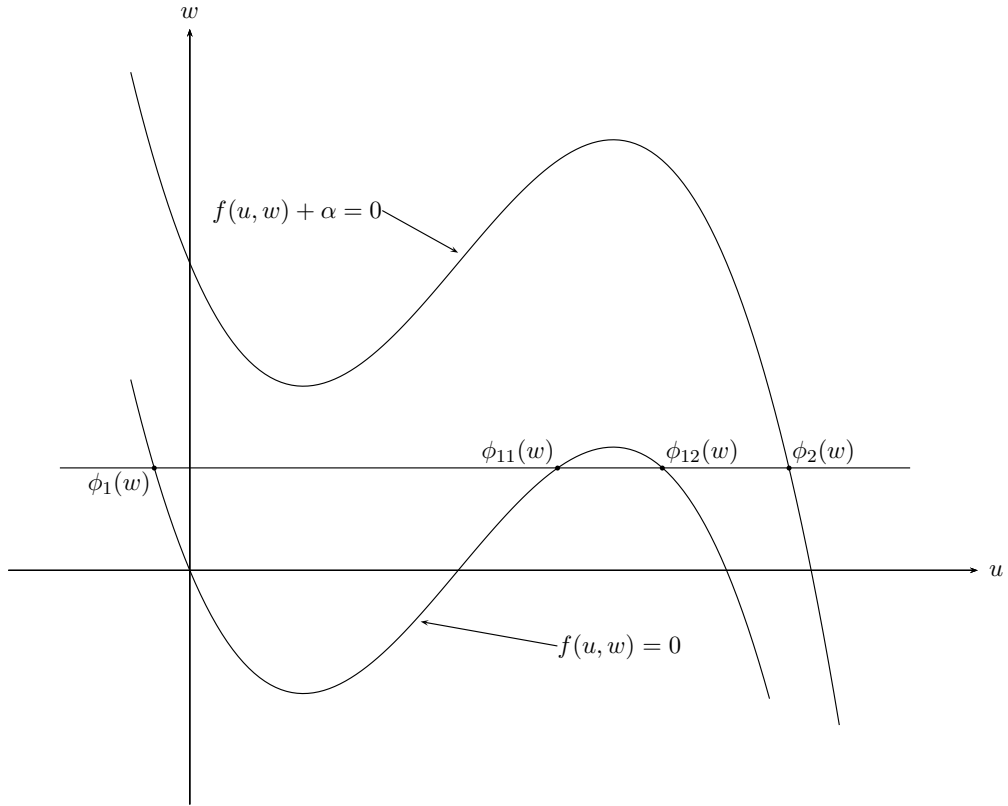


Figure 3.12: The solutions of $f(u, w) = 0$ and $f(u, w) + \alpha = 0$.

According to the assumptions on the curves C_1 and C_2 (see conditions (a), (b), and (c)), we have (see Figure 3.12)

$$\phi_1(w) < \phi_{11}(w) < \phi_{12}(w) < \phi_2(w).$$

It is easy to see that the fixed point $(-1, \phi_1(w))$ is a saddle point and $(-1, \phi_{11}(w))$ is an unstable node. On the other hand, $(-1, \phi_{12}(w))$ is a saddle point, and $(1, \phi_2(w))$ is a stable node. In this case, Lemmas 3.2.2, 3.2.3, 3.2.4, and 3.2.5 also hold. The difference with Case 1 is that the U -nullcline curve Σ has two branches; see Figure 3.13. One branch connects $(-1, \phi_1(w))$ and $(-1, \phi_{11}(w))$ and lies in D_2 with $\phi_1(w) < U < \phi_{11}(w)$ and $-1 < \tau < \tau_0 < 0$.

The other one connects $(-1, \phi_{12}(w))$ and $(1, \phi_2(w))$. The proof of Theorem 3.2.1 is complete.

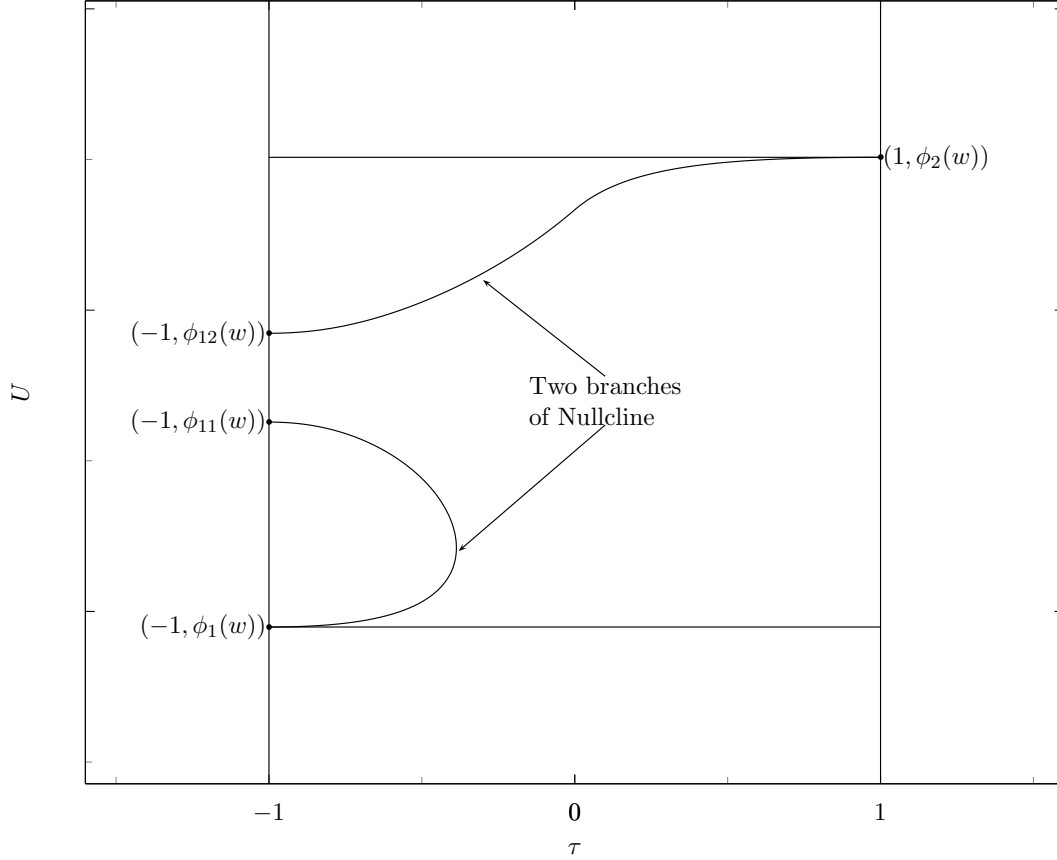


Figure 3.13: The two branches of Σ in the case of $w < w_1^+$.

□

Remark 3.2.6. The conclusion of Theorem 3.2.1 is still true even if the fixed points $(-1, \phi_{11}(w))$ and $(-1, \phi_{12}(w))$, or $(-1, \phi_{11}(w))$ and $(-1, \phi_1(w))$, overlap since Lemmas 3.2.2, 3.2.3, 3.2.4, and 3.2.5 are still true in these critical cases.

When $f(U, w) + \alpha = 0$ has three solutions, we have the following theorem.

Theorem 3.2.7. *Given the parameter $w \in (w_1^-, w_2^+)$, if the equation $f(U, w) + \alpha = 0$ has three solutions, then system (3.11) has no heteroclinic orbits linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$.*

Proof. Let $(1, \phi_{21}(w))$, $(1, \phi_{22}(w))$, and $(1, \phi_2(w))$ be three fixed points on the line $\tau = 1$, and $\phi_{21}(w) < \phi_{22}(w) < \phi_2(w)$. See Figure 3.14.

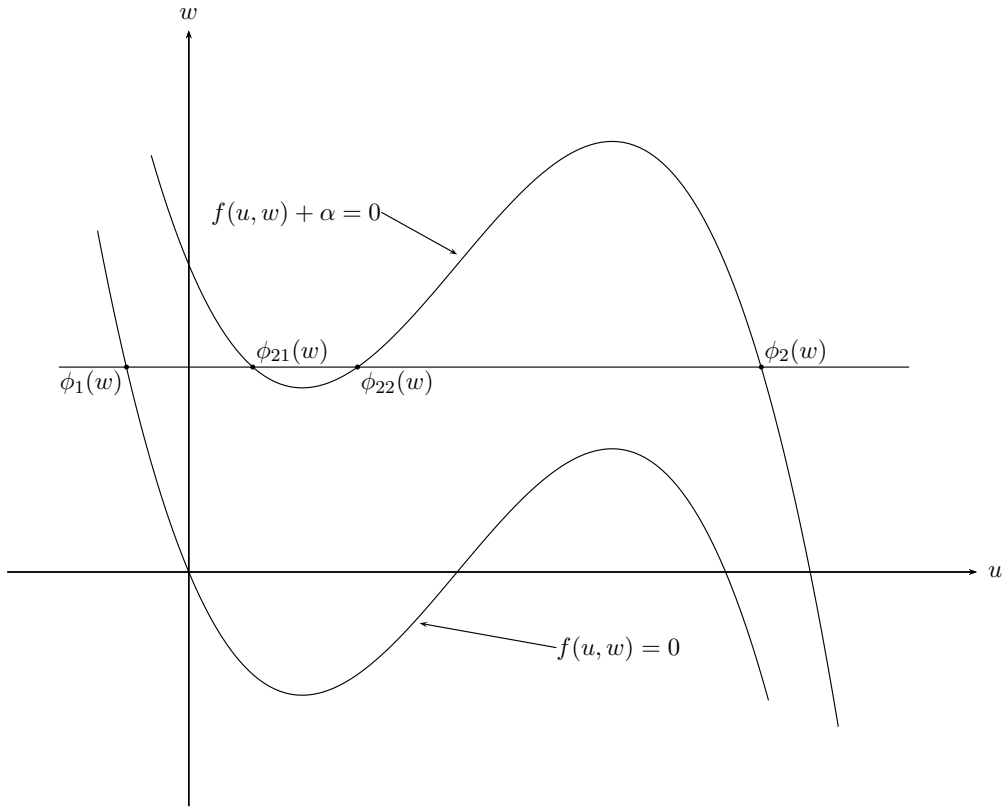


Figure 3.14: The solutions of $f(u, w) = 0$ and $f(u, w) + \alpha = 0$.

According to the assumptions on the curves C_1 and C_2 (see conditions (a), (b), and (c)), $\phi_1(w) < \phi_{21}(w)$, and we have $f(U, w) + \alpha < 0$ for $\phi_{21}(w) < U < \phi_{22}(w)$. It is easy to see that the points $(1, \phi_{21}(w))$ and $(1, \phi_2(w))$ are stable nodes and $(1, \phi_{22}(w))$ is a saddle point. On the segment of the horizontal

line

$$L_0 = \{(\tau, \phi_{22}(w)); -1 < \tau < 1\},$$

we have the inequality

$$f(\phi_{22}(w), w) + \alpha I(\tau) < f(\phi_{22}(w), w) + \alpha = 0.$$

This means that the vector field defined by system (3.11) cannot point up on this segment.

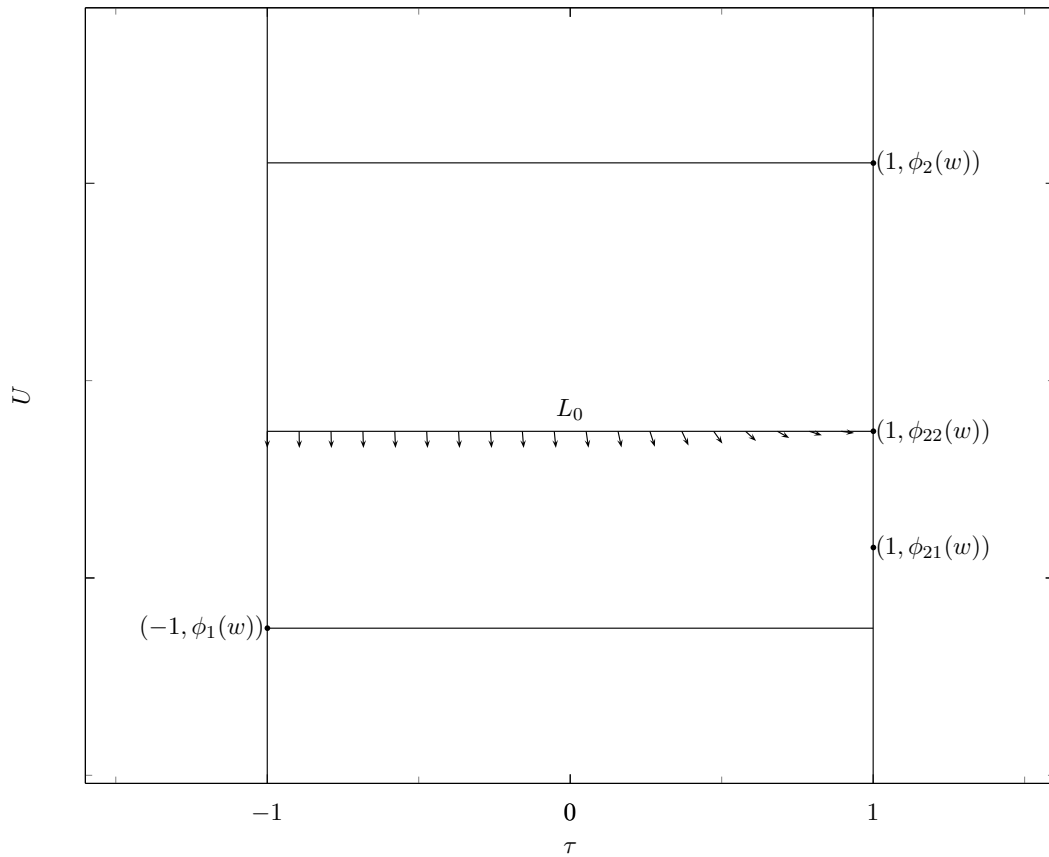


Figure 3.15: Segment L_0 and its vector field.

Thus, there exists no orbit, in the domain D_2 , entering through the line

L_0 in the top direction as z increases. Moreover, there is no heteroclinic orbit, linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, in this domain. \square

Remark 3.2.8. The conclusion of Theorem 3.2.7 is still true even in the critical case where $(1, \phi_{21}(w))$ and $(1, \phi_{22}(w))$ overlap since the vector field on the line L_0 still points in the bottom-right direction and there is no heteroclinic orbit intersecting this line and linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$.

Remark 3.2.9. If $w_2^- < w < w_2^+$, then the equation $f(U, w) + \alpha = 0$ certainly has three solutions. If $w_1^- < w < w_2^-$, then the function $f(U, w) + \alpha = 0$ has a unique solution, $U = \phi_2(w)$, and $f(U, w) = 0$ admits the solution $U = \phi_1(w)$.

The condition $f(U, w) + \alpha/2 > 0$ for $U \leq \theta$ is necessary by the next theorem.

Theorem 3.2.10. *If $f(U_0, w) + \frac{1}{2}\alpha \leq 0$ for some $\phi_1(w) < U_0 \leq \theta$, then there is no heteroclinic orbit passing through the point $(\tau = 0, U = \theta)$, linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, and having its U -coordinate increasing as z increases.*

Proof. In fact, if there is some $\phi_1(w) < U_0 < \theta$ such that $f(U_0, w) + \frac{1}{2}\alpha < 0$, then $f(U_0, w) + \alpha I(\tau) < 0$ on the segment $L = \{(U_0, \tau); -1 < \tau < 0\}$ because of the inequality $I(\tau) < 1/2$ for $-1 < \tau < 0$. The vector field on L points in a bottom-right direction. The orbit starting at the point S could not penetrate this segment from above in a bottom-left direction as $z \rightarrow -\infty$. See Figure 3.16.

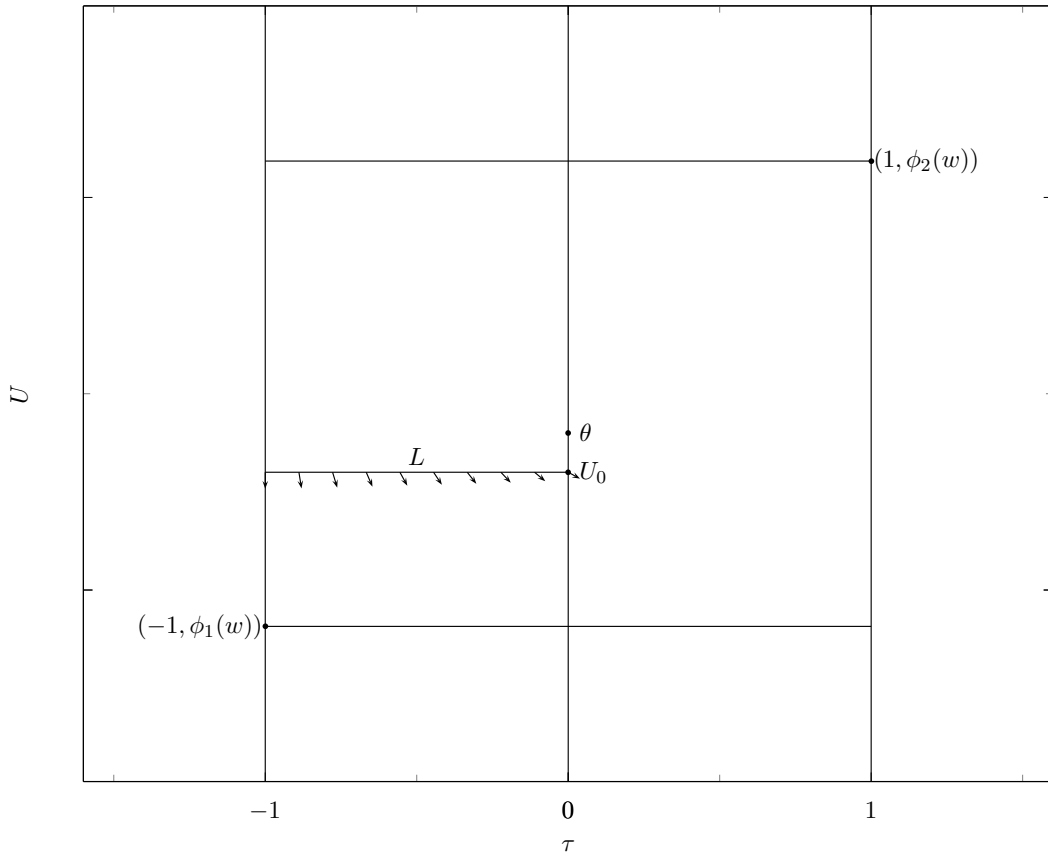


Figure 3.16: Segment L and its vector field.

If $f(U_0, w) + \alpha/2 = 0$ and $f(U, w) + \frac{1}{2}\alpha > 0$ for $\phi_1(w) < U < U_0$, then at the point $P = (\tau, U) = (0, U_0) \in \Sigma$ the vector field defined by system (3.11) is $(k(1 - \tau_0^2), 0)$. By what has been proved in Lemma 3.2.4, the orbit starting at P will go into the left side of Σ and its U -coordinate will increase as z decreases.

If $f(\theta, w) + \alpha/2 < 0$, then the U -coordinate of the orbit starting at $S = (0, \theta)$ also increases as z decreases. Thus, in all these cases, there is no heteroclinic orbit passing through S and having its U -coordinate increasing as z increases, and linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. \square

Remark 3.2.11. From conditions (a), (b), and (c), the inequality $f(\phi_1(w), w) + \alpha/2 > 0$ holds. Moreover, $f(U, w) + \alpha/2 > 0$ for $U < \phi_1(w)$. It is clear that the condition $f(U, w) + \alpha/2 > 0$ for $U \leq \theta$ is equivalent to the following condition:

The function $f(U, w) + \alpha/2$ has no zero point on the interval $\phi_1(w) \leq U \leq \theta$.

From Theorems 3.2.1, 3.2.7, and 3.2.10 we have a necessary and sufficient corollary for monotone increasing traveling wave solutions of equation (3.1).

Corollary 3.2.12. *Suppose equation (3.1) satisfies conditions (3.2) and (a)–(d). Then, there is a unique $v = v(w, \theta) > 0$ such that equation (3.11) has a unique heteroclinic orbit*

$$\{(\tau(z, w), U(z, w)); \tau(0, w) = 0, U(0, w) = \theta\},$$

if and only if $w_1^- < w < w_2^-$ and the function $f(U, w) + \alpha/2$ has no zero for $\phi_1(w) \leq U \leq \theta$. The function $U(z, w)$ is monotone increasing in z . The orbit satisfies the boundary conditions

$$\begin{aligned} \lim_{z \rightarrow +\infty} U(z, w) &= \phi_2(w), & \lim_{z \rightarrow -\infty} U(z, w) &= \phi_1(w), \\ \lim_{z \rightarrow +\infty} \tau(z, w) &= 1, & \lim_{z \rightarrow -\infty} \tau(z, w) &= -1, \end{aligned} \tag{3.17}$$

linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. The function

$$u(t, x) = U(x + vt, w) = U(z, w), \text{ where } z = x + vt,$$

with some abuse of notation, is the unique traveling wave solution of equation (3.1) such that

(α) $U(z, w)$ is a monotone increasing function of z ,

(β) $U(0, w) = \theta$,

(γ) the limits (3.17) hold and $\lim_{z \rightarrow \pm\infty} U_z(z, w) = 0$.

Remark 3.2.13. Let

$$C_0 = \left\{ (U, w); f(U, w) + \frac{1}{2}\alpha = 0 \right\}.$$

It is a curve in the U - w plane, lying between C_1 and C_2 . Denote its w -minimum and w -maximum points by (ρ_0^-, w_0^-) and (ρ_0^+, w_0^+) , respectively. If $\rho_0^- \leq \rho_1^-$, then the conditions $w_1^- < w < w_2^-$ and $f(U, w) + \frac{1}{2}\alpha > 0$ for $\phi_1(w) \leq U \leq \theta$ and $\rho_1^- < \theta < \rho_1^+$ are equivalent to $w_1^- < w < w_0^-$. In fact, when $w_1^- < w < w_0^-$, we can obtain $w_1^- < w < w_2^-$ and $f(U, w) + \frac{1}{2}\alpha > 0$ for $\phi_1(w) \leq U \leq \theta$ and $\rho_1^- < \theta < \rho_1^+$. Conversely, $w_1^- < w < w_0^-$ is the necessary condition of $f(U, w) + \frac{1}{2}\alpha > 0$ for $\phi_1(w) \leq U \leq \theta$ and $\rho_1^- < \theta < \rho_1^+$. Thus, provided $\rho_0^- \leq \rho_1^-$, Corollary 3.2.12 holds if and only if $w_1^- < w < w_0^-$.

Next, we consider the relationship between the wave speed $v = v(w, \theta)$ and the threshold θ .

Theorem 3.2.14. *Given $w_1^- < w < w_2^-$, if there is $\rho_1^- < \theta_0 < \rho_1^+$ such that $f(\theta, w) + \alpha/2 > 0$ for $\theta < \theta_0$ then $v = v(w, \theta)$ is a monotone decreasing function of the threshold $\theta \in (\rho_1^-, \theta_0)$.*

Proof. The conditions in this theorem guarantee that the monotone increasing traveling wave solution for equation (3.1) exists, and the function $v = v(w, \theta)$ is well defined for the threshold values $\theta \in (\rho_1^-, \theta_0)$. Let $v_i = v(w, \theta_i)$, $i = 1, 2$, and $\theta_1 < \theta_2$. At first $v_1 \neq v_2$. Otherwise, there would be two

unstable manifolds for system (3.11) at the fixed point $(-1, \phi_1(w))$. This contradicts the uniqueness of the unstable manifold of saddle points. Secondly, we assume $v_1 < v_2$. Let $\Gamma(v_1)$ and $\Gamma(v_2)$ be heteroclinic orbits of systems (3.11) with $v = v_1$ and $v = v_2$ and passing through the initial points $(0, \theta_1)$ and $(0, \theta_2)$, respectively. In Step 2 of the proof of Lemma 3.2.5 we have the result $U(\tau, v_1) > U(\tau, v_2)$ if $v_1 < v_2$ for the heteroclinic orbits. From this inequality it is easy to see that $\theta_1 = U(0, v_1) > U(0, v_2) = \theta_2$. This means that $v(w, \theta)$ is a monotone decreasing function of θ . \square

We also consider the relationship between v and w .

Theorem 3.2.15. *Given $\theta \in (\rho_1^-, \rho_1^+)$. If $f(U, w) + \frac{1}{2}\alpha > 0$ for $w < w^* < w_2^-$ and $U \leq \theta$, then $v(w, \theta)$ is a monotonic decreasing function of $w \in (w_1^-, w^*)$.*

Lemma 2.5 in [38], claims that v is monotonic decreasing with respect to w (see the end of its proof). In that proof, U is differentiated with respect to v . This is not reasonable since v is uniquely determined by fixed θ and w , so there is no traveling wave solution for any v , which passes through $(0, \theta)$. We here give a correct proof of this theorem.

Proof. For a fixed $\theta \in (\rho_1^-, \rho_1^+)$, the conditions in the theorem guarantee that the traveling wave solutions exist for every $w \in (w_1^-, w^*)$. Thus, $v(w, \theta)$, as function of w , is well defined. Let $w', w'' \in (w_1^-, w^*)$ and $w' < w''$; then $f(U, w') > f(U, w'')$, because $f_w < 0$. We claim that

$$v(w', \theta) > v(w'', \theta).$$

In fact, if $v(w', \theta) \leq v(w'', \theta)$, noting that both of them are positive, we would

have

$$\frac{f(\theta, w') + \frac{1}{2}\alpha}{v(w', \theta)} > \frac{f(\theta, w'') + \frac{1}{2}\alpha}{v(w'', \theta)} > 0.$$

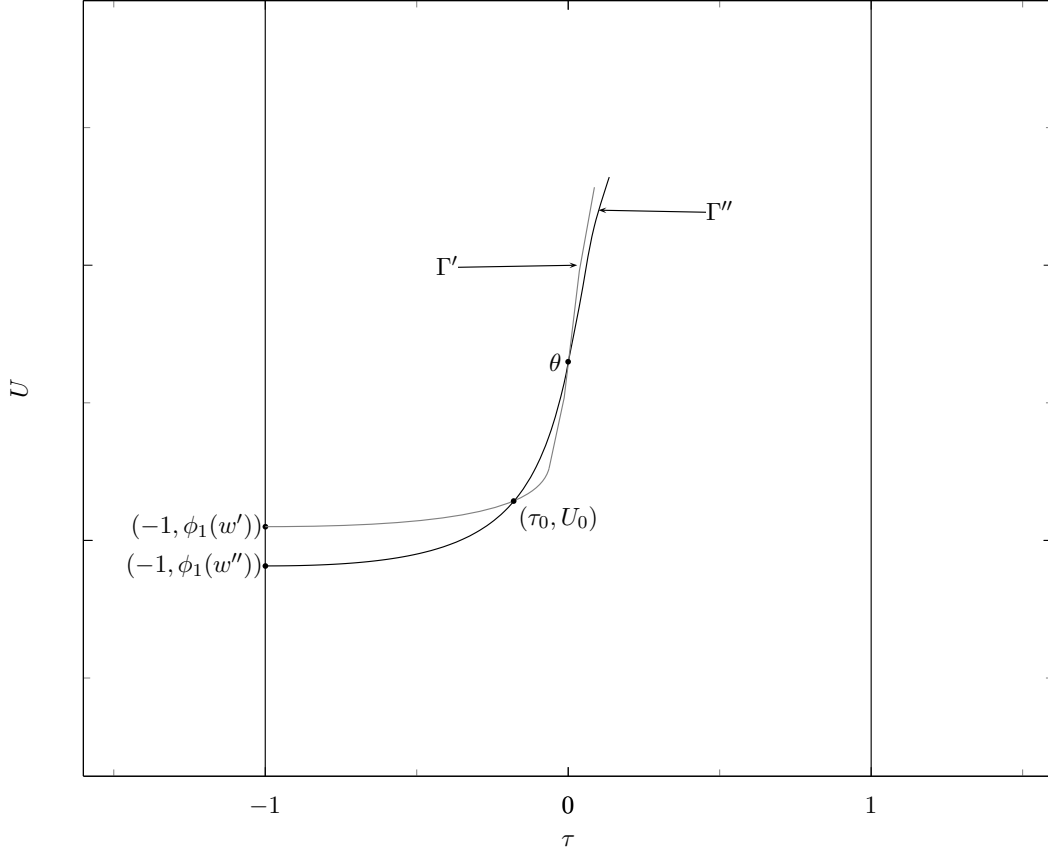


Figure 3.17: The schematic diagram of Γ' and Γ''

Let Γ' be the heteroclinic orbit of system (3.11) with $w = w'$ and Γ'' be the orbit with $w = w''$. At time $z = 0$, Γ' and Γ'' would meet at the point $(\tau, U) = (0, \theta)$. On the left side of this point, where $-\delta < z < 0$ for some $\delta > 0$ small enough, Γ' was below Γ'' . Both of them would approach their limit points $(\tau, U) = (-1, \phi_1(w'))$ and $(\tau, U) = (-1, \phi_1(w''))$, respectively. According to the condition on the curve C_1 , we have $\phi_1(w') > \phi_1(w'')$. Thus,

Γ'' had to penetrate Γ' with bigger slope from above at some point, say (τ_0, U_0) , $-1 < \tau_0 < 0$; but at this point

$$\frac{f(U_0, w') + \alpha I(\tau_0)}{v(w', \theta)k(1 - \tau_0^2)} > \frac{f(U_0, w'') + \alpha I(\tau_0)}{v(w'', \theta)k(1 - \tau_0^2)} > 0.$$

We have a contradiction. □

3.3 Monotone decreasing traveling wave solutions

Finding monotone decreasing traveling wave solution of equation (3.1) is equivalent to finding the solution that satisfies equation (3.6), the boundary condition (3.4), and $U_z < 0$. We are concerned with solutions with initial condition $U(0, w) = \theta$. For this class of solutions, we must have $U(z - \xi) < \theta$ for $\xi < z$ and $U(z - \xi) > \theta$ for $\xi \geq z$, hence the equation

$$H(U(z - \xi, w) - \theta) = 1$$

holds only for $z \leq \xi$. Thus we obtain

$$\int_{\mathbb{R}} K(\xi) H(U(z - \xi, w) - \theta) d\xi = \int_z^{\infty} K(\xi) d\xi,$$

and equation (3.6) becomes

$$vU_z = f(U(z, w), w) + \alpha \int_z^{\infty} K(\xi) d\xi, \quad U_z < 0, \quad U(0, w) = \theta. \quad (3.18)$$

Let

$$I(z) = \int_z^{+\infty} K(\xi) d\xi.$$

If we substitute z in (3.9), then $I(z)$ is transformed into

$$I(\tau) = \int_{\tau}^1 \frac{1}{k(1-\eta^2)} K\left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta}\right) d\eta.$$

It is easy to see from the properties of the kernel $K(x)$ that $I(\tau)$ is monotone decreasing, $I(0) = 1/2$, $I(-1) = 1$, and $I(\tau) > 0$ for $-1 < \tau < 1$. With $I(\tau)$ and transformation (3.9), equation (3.18) can be transformed into an equivalent autonomous system in the phase space (τ, U) . A monotone decreasing traveling wave solution of equation (3.1) must satisfy the system

$$\begin{aligned} \frac{d\tau}{dz} &= k(1-\tau^2), & \tau(0) &= 0, \\ \frac{dU}{dz} &= \frac{1}{v}[f(U, w) + \alpha I(\tau)], & U(0) &= \theta. \end{aligned} \tag{3.19}$$

System (3.19) can be studied in the same way as system (3.11). We consider the fixed points defined by the equations $\tau = -1$, $f(U, w) + \alpha = 0$, and $\tau = 1$, $f(U, w) = 0$, respectively, and search for heteroclinic orbits linking the fixed points with decreasing U -coordinate as z increases. For $w_1^- < w < w_2^+$, the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ certainly exist.

Theorem 3.3.1. *Given the parameter $w \in (w_1^-, w_2^+)$ and the threshold $\theta \in (\rho_1^-, \rho_1^+)$, if $f(U, w) + \alpha/2 < 0$ for $U \geq \theta$ and the equation $f(U, w) = 0$ has only one solution $U = \phi_1(w)$, then there is a unique value $v = v(w, \theta) > 0$ such that system (3.19) has a unique heteroclinic orbit passing through the point $S = (\tau, U) = (0, \theta)$ linking the fixed points $(\tau, U) = (-1, \phi_2(w))$ and $(\tau, U) = (1, \phi_1(w))$, and its U -coordinate is monotone decreasing as z increases.*

Proof. We shall prove this theorem by proving four lemmas in two cases. Similar to system (3.11), system (3.19) has at least two fixed points $(\tau, U) =$

$(-1, \phi_2(w))$ and $(\tau, U) = (1, \phi_1(w))$ with $\phi_1(w) < \phi_2(w)$. There are two cases to be considered.

Case 1. $w < w_2^-$. At $\tau = -1$, the derivative $U' = \frac{1}{v}[f(U, w) + \alpha]$, because $I(-1) = 1$. In this case, we can observe from Figure 3.18 that $f(U, w) + \alpha = 0$ has only one solution, so $(\tau, U) = (-1, \phi_2(w))$ is the only fixed point of system (3.19) at $\tau = -1$. We then consider the geometric properties of the fixed points.

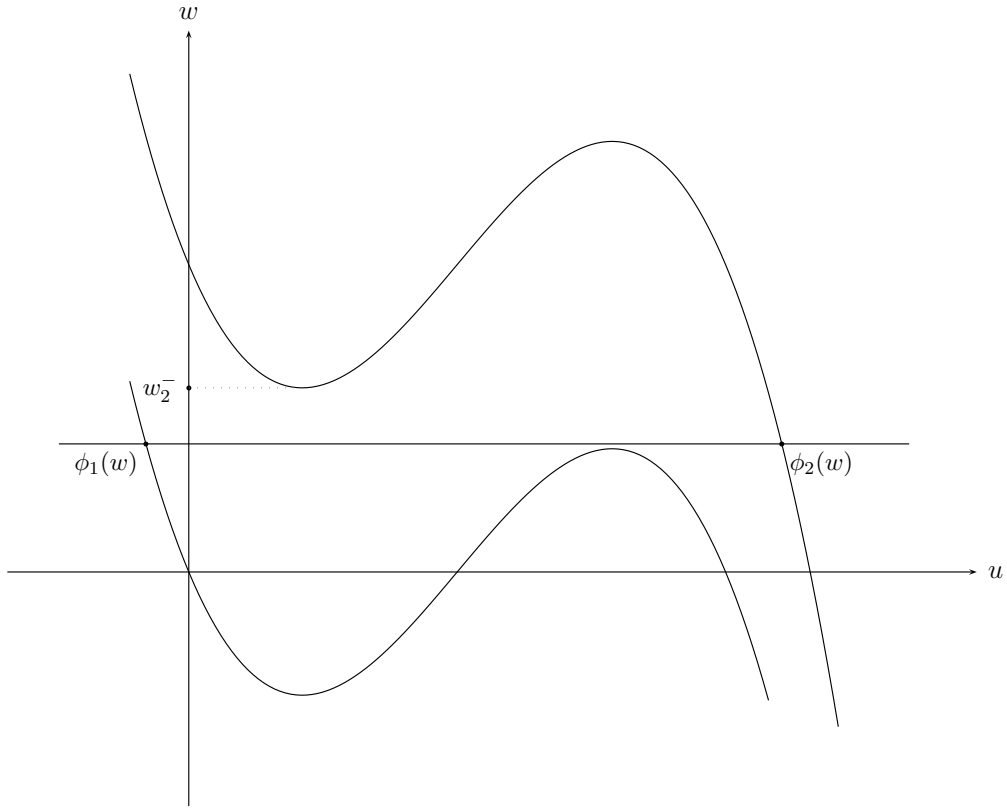


Figure 3.18: The solutions of $f(u, w) = 0$ and $f(u, w) + \alpha = 0$ when $w_1^+ < w < w_2^-$.

Lemma 3.3.2. *The fixed point $(\tau, U) = (-1, \phi_2(w))$ is a saddle point and*

its local unstable manifold is in D_2 and tangent to the line L_2 .

Proof. The Jacobian of system (3.19) is

$$J = \begin{bmatrix} -2k\tau & 0 \\ \frac{1}{v}\alpha I'(\tau) & \frac{1}{v}f_u(U, w) \end{bmatrix}. \quad (3.20)$$

We have the derivative

$$I'(\tau) = -\frac{1}{k(1-\tau^2)} K \left(\frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right).$$

According to properties of the kernel function $K(x)$,

$$0 \leq K \left(\frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right) \leq C \exp \left(-\gamma \left| \frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right| \right).$$

As $\tau \rightarrow -1$ and $k < \gamma/2$,

$$\begin{aligned} 0 \leq \lim_{\tau \rightarrow -1} -I'(\tau) &\leq \lim_{\tau \rightarrow -1} \frac{C}{k(1-\tau^2)} \exp \left(-\gamma \left| \frac{1}{2k} \ln \frac{1+\tau}{1-\tau} \right| \right) \\ &= \lim_{\tau \rightarrow -1} \frac{C(1+\tau)^{\frac{\gamma}{2k}-1}}{k(1-\tau)^{1+\frac{\gamma}{2k}}} = 0. \end{aligned}$$

Thus $I(-1) = 0$. The Jacobian (3.13) evaluated at the fixed point $(\tau, U) = (-1, \phi_2(w))$ is

$$\begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{v}f_u(\phi_2(w), w) \end{bmatrix}.$$

According to condition (b), the entry $\frac{1}{v}f_u(\phi_1(w), w)$ is negative, hence this fixed point is a saddle point. From the Hartman–Grobman Theorem [10], there is a unique local unstable manifold, say Γ , tangent to the characteristic vector $(1, 0)$. Just like system (3.11), system (3.19) is defined in the same

domain D which is divided into three parts by the same horizontal lines L_1 and L_2 , and we denote the part where $U > \phi_2(w)$ by

$$D_3 = \{(\tau, U); \tau \in (-1, +1), U > \phi_2(w)\}.$$

We claim that Γ must be in the domain D_2 . We consider the direction of vector field in D_3 . In the domain D_3 , we have $f(U, w) < f(\phi_2(w), w)$, because $U > \phi_2(w)$ and $f_u(\phi_2(w), w) < 0$. According to the properties of $I(\tau)$, $I(\tau) < I(-1)$ for $-1 < \tau < 1$, we obtain

$$f(U, w) + \alpha I(\tau) < f(\phi_2(w), w) + \alpha I(-1) = 0,$$

hence, $U' < 0$ in D_3 . Noticed that $\tau' > 0$ for $-1 < \tau < 1$, so vector fields in D_3 point in the bottom-right direction. If an orbit of system (3.19) passes through a point in D_3 , then it cannot approach $(-1, \phi_2(w))$ as z goes to $-\infty$. Thus, the heteroclinic orbit Γ cannot enter D_3 . \square

Lemma 3.3.3. *The unstable manifold Γ of the saddle point $(\tau, U) = (-1, \phi_2(w))$ approaches the fixed point $(\tau, U) = (1, \phi_1(w))$ as $z \rightarrow +\infty$. It is the unique heteroclinic orbit linking the two fixed points.*

Proof. On the line L_2 , the vector field points in the bottom-right direction because

$$f(\phi_2(w), w) + \alpha I(\tau) < f(\phi_2(w), w) + \alpha = 0.$$

On the line L_1 , the vector field points to top-right direction because

$$f(\phi_1(w), w) + \alpha I(\tau) = \alpha I(\tau) > 0.$$

On the line $\tau = -1$, the vector field points vertically up since $f(U, w) + \alpha > 0$ for $U < \phi_2(w)$. On the line $\tau = 1$, the vector field points vertically down

since $f(U, w) < 0$ for $U > \phi_1(w)$. Thus, D_2 is a positive invariant set, and by Poincaré-Bendixson theorem, Γ must go to the fixed point $(1, \phi_1(w))$ because τ is monotone increasing in D_2 as z increases.

Thus, Γ , as the unique local unstable manifold of the saddle point $(-1, \phi_2(w))$, is the unique heteroclinic orbit linking the fixed points $(\tau, U) = (-1, \phi_2(w))$ and $(\tau, U) = (1, \phi_1(w))$. \square

Lemma 3.3.4. *The heteroclinic orbit Γ must stay on the right side of the U -nullcline curve of system (3.19), and the U -coordinate, $U(z)$, of Γ is a monotone decreasing function of z .*

Proof. The U -nullcline of (3.19), denoted by Σ , is defined as

$$\Sigma = \{(\tau, U) \in D; \tau \in (-1, 1), f(U, w) + \alpha I(\tau) = 0\}.$$

Figure 3.19 shows a U -nullcline of (3.19) in Case 1.

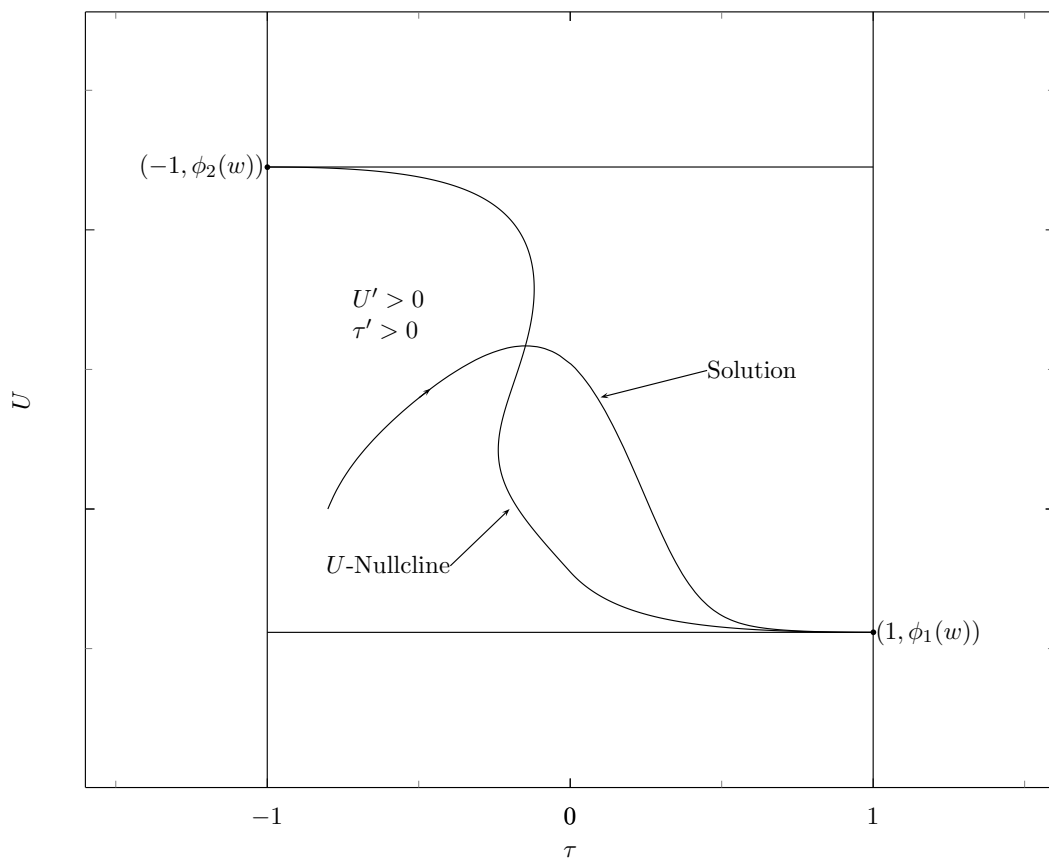


Figure 3.19: The U -nullcline of system (3.19) divides the domain D_2 into two parts. $U' > 0$ in the left part, and $U' < 0$ in the right part.

The U -nullcline divides the domain D_2 into two parts. If a point P is on Σ , then the orbit $O(z, P)$, starting at P when $z = 0$, will enter the left side of Σ as z decreases. We consider the set U' of points in the left part. Let (U_l, τ_l) be a point lying to the left of Σ , and (U_l, τ_n) be the point on Σ with the same U -coordinate. Clearly, $\tau_l < \tau_n$. We then have

$$U'(U_l, \tau_l) = f(U_l, w) + \alpha I(\tau_l) > f(U_l, w) + \alpha I(\tau_n) = 0$$

by the properties of $I(\tau)$. If an orbit of (3.19) passes through a point lying to

the left of Σ , then it cannot approach the saddle point $(-1, \phi_2(w))$, because U decreases when z decreases in the left part by $U' > 0$ (See Figure 3.19). Hence a heteroclinic orbit cannot pass into the left part of Σ . This fact ensures that the heteroclinic orbit Γ must stay in the right part. Next, we consider the set U' of points in the right part. Let (U_r, τ_r) be a point lying to the right of Σ , and (U_r, τ_c) be the point on Σ with the same U -coordinate. Clearly, $\tau_r > \tau_c$. We then have

$$U'(U_r, \tau_r) = f(U_r, w) + \alpha I(\tau_r) < f(U_r, w) + \alpha I(\tau_c) = 0$$

due to the decreasing properties of $I(\tau)$. Since Γ stay in the right part, $U'(z)$ of Γ is always negative, which means that $U(z)$ is a decreasing function of z . \square

Lemma 3.3.5. *If $f(U, w) + \frac{1}{2}\alpha < 0$ for $\theta \leq U \leq \phi_2(w)$, then there exists a unique $v_0 > 0$ such that the heteroclinic orbit of (3.19) with $v = v_0$ passes through the point S and links the fixed points $(\tau, U) = (-1, \phi_2(w))$ and $(\tau, U) = (1, \phi_1(w))$.*

Proof. In the proof of Lemmas 3.3.2 and 3.3.3, we have shown that system (3.19) has a unique orbit linking the fixed points for $v > 0$ under the conditions of Case 1. Moreover, Lemma 3.3.4 shows that this orbit can define its U -coordinate as a decreasing function of the τ -coordinate, because $U(z)$ is a decreasing function of the independent variable z , and $\tau(z)$ is an increasing function of the independent variable z . Denote this orbit as

$$\Gamma(v) = \{(\tau, U) = (\tau, U(\tau, v)); \tau \in (-1, 1)\}.$$

We prove this lemma in four steps.

Step 1. There exists $0 < v_1$ such that $U(0, v_1) < \theta$, and $v_2 > 0$ such that $U(0, v_2) > \theta$.

On the segment

$$\{(\tau, U); \tau = 0, \theta \leq U \leq \phi_2(w)\},$$

by assumption we have

$$f(U, w) + \frac{1}{2}\alpha < 0.$$

By the continuity of $f(U, w) + \alpha I(\tau)$ we can find a closed rectangle

$$R = \{(\tau, U); -\delta \leq \tau \leq 0, \theta \leq U \leq \phi_2(w)\} \subset D_2, \quad \delta > 0,$$

such that $f(U, w) + \alpha I(\tau) \leq \eta < 0$ for $(\tau, U) \in R$. Take a small $v_1 > 0$ such that

$$\frac{\eta}{v_1 k} < \frac{\theta - \phi_2(w)}{\delta}.$$

Then the orbit $O(z, -\delta, \phi_2(w))$ of (3.19), starting from the point $(\tau = -\delta, U = \phi_2(w))$ at $z = 0$, has the derivative

$$\frac{dU}{d\tau} = \frac{f(U, w) + \alpha I(\tau)}{v_1 k(1 - \tau^2)} \leq \frac{\eta}{v_1 k} < \frac{\theta - \phi_2(w)}{\delta}$$

on R . This means that the orbit $O(z, -\delta, \phi_2(w))$ will get to the position $(\tau, U) = (\tau(z_0) = 0, U(z_0) < \theta)$ for some $z_0 > 0$. By the properties of phase flows of autonomous ODEs [1], the orbit $\Gamma(v_1)$ cannot intersect with the orbit $O(z, -\delta, \phi_2(w))$. Thus the unstable manifold $\Gamma(v_1)$ of the fixed point $(\tau, U) = (-1, \phi_2(w))$ will stay below the orbit $O(z, -\delta, \phi_2(w))$, and get to the point $(\tau, U) = (0, U(0, v_1))$ with $U(0, v_1) < \theta$. (See Figure 3.20.)

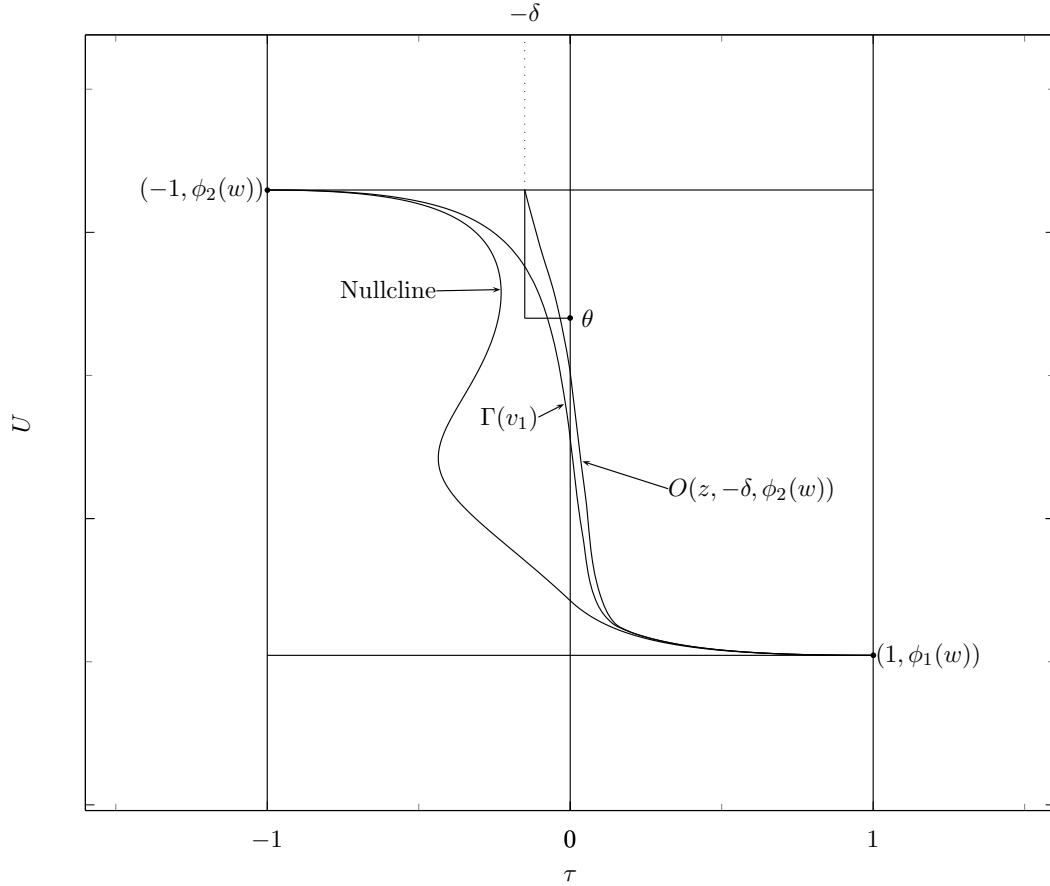


Figure 3.20: The heteroclinic orbit $\Gamma(v_1)$ must stay under the orbit $O(z, -\delta, \phi_2(w))$.

Let $M = \min\{f(U, w) + \alpha I(\tau)\}$ for $-1 \leq \tau \leq 0$ and $\theta \leq U \leq \phi_2(w)$. Then $M < 0$. Consider the U -nullclenic curve

$$\Sigma = \{(\tau, U); \tau \in [-1, 1], f(U, w) + \alpha I(\tau) = 0\}.$$

Since $f_U(\phi_2(w), w) < 0$, $\alpha I(\tau) \geq 0$, and $I'(-1) = 0$, Σ is in a neighborhood of $(\tau, U) = (-1, \phi_2(w))$ for $\tau > -1$ and $U < \phi_2(w)$, and has the horizontal tangent line $U = \phi_2(w)$ at the point $(-1, \phi_2(w))$. Let (τ_0, U_0) be a point in the U -nullclenic curve Σ near the fixed point $(\tau = -1, U = \phi_2(w))$ with

$-1 < \tau_0 < -\zeta < 0$ for some $\zeta > 0$, then the orbit $O(z, \tau_0, U_0)$, starting from $(\tau_0, U_0) \in \Sigma$ at $z = 0$, will stay on the right side of Σ as z increases. Take v_2 large enough such that $v_1 < v_2$ and

$$\frac{f(U, w) + \alpha I(\tau)}{v_2 k(1 - \tau^2)} \geq \frac{M}{v_2 k(1 - \tau_0^2)} > \frac{\theta - U_0}{-\tau_0}, \quad -1 \leq \tau_0 \leq \tau < 0,$$

then the orbit $O(z, \tau_0, U_0)$ of (3.19) will get to a point $(\tau(z_0) = 0, U(z_0) > \theta)$ for some $z_0 > 0$. By the properties of phase flows of autonomous ODEs [1], the unstable manifold $\Gamma(v_2)$ of the fixed point $(-1, \phi_2(w))$ will stay above $O(z, \tau_0, U_0)$ and get to a point $(\tau = 0, U(0, v_2) > \theta)$.

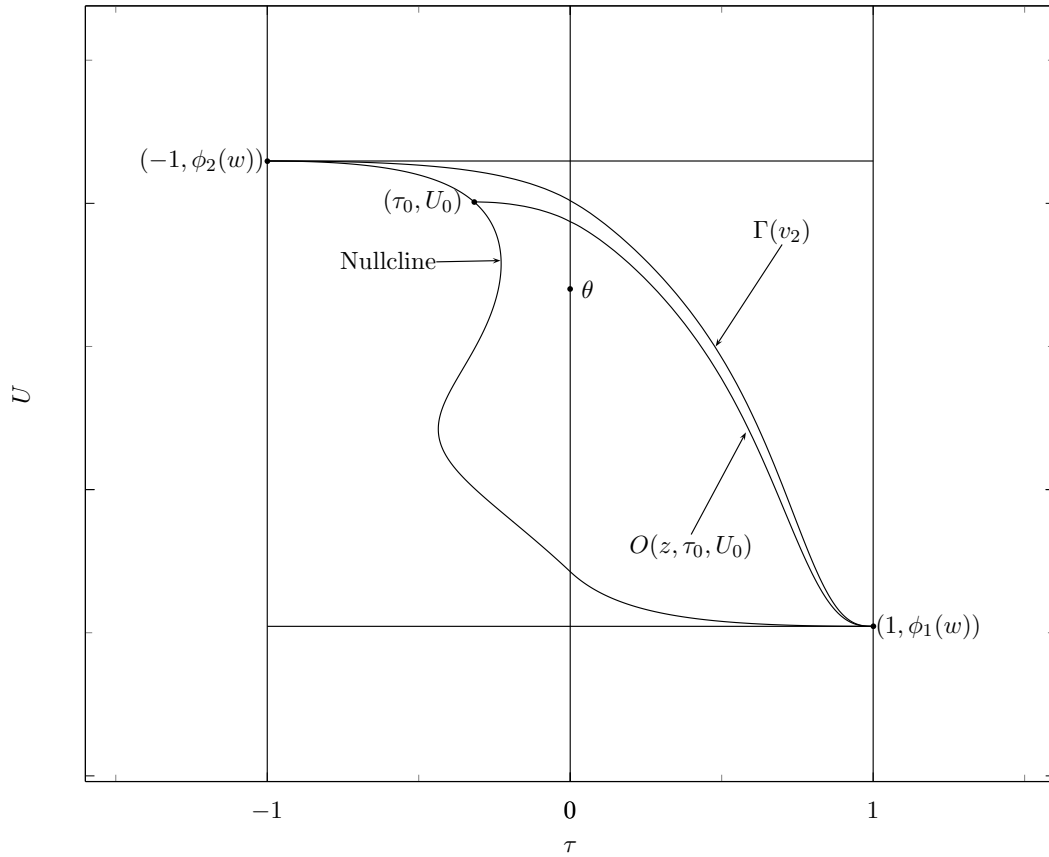


Figure 3.21: The heteroclinic orbit $\Gamma(v_2)$ must stay above the orbit $O(z, -\tau_0, \phi_2(w))$.

Step 2. For any $0 < v_1 < v_2$, $U(\tau, v_1) < U(\tau, v_2)$, $\tau \in (-1, 1)$.

Here we prove

Lemma 3.3.6. *Let $\Gamma(v_1)$ and $\Gamma(v_2)$ be two heteroclinic orbits of system (3.19) with speed v_1 and v_2 , respectively. If $0 < v_1 < v_2$, then $U(\tau, v_1) < U(\tau, v_2)$ for $\tau \in (-1, 1)$.*

Proof. We assume that $U(\tau_0, v_1) = U(\tau_0, v_2)$ at some $-1 < \tau_0 < 1$, which

means that $\Gamma(v_1)$ and $\Gamma(v_2)$ intersect at $\tau = \tau_0$. Let

$$\mu_0 = f(U(\tau_0, v_1), w) + \alpha I(\tau_0) = f(U(\tau_0, v_2), w) + \alpha I(\tau_0).$$

Since heteroclinic orbits are monotone decreasing, $\mu_0 < 0$. Consider the tangents to $\Gamma(v_1)$ and $\Gamma(v_2)$ at the point $(\tau_0, U(\tau_0, v_1)) = (\tau_0, U(\tau_0, v_2))$; then the following inequality holds:

$$\frac{\mu_0}{v_1 k(1 - \tau_0^2)} < \frac{\mu_0}{v_2 k(1 - \tau_0^2)}. \quad (3.21)$$

On the left side of the intersecting point, $\Gamma(v_1)$ is above $\Gamma(v_2)$ for $\tau \in (\tau_0 - \delta, \tau_0)$, when δ is small enough, hence we can find a point Q above $\Gamma(v_2)$ and below $\Gamma(v_1)$. See Figure 3.22.

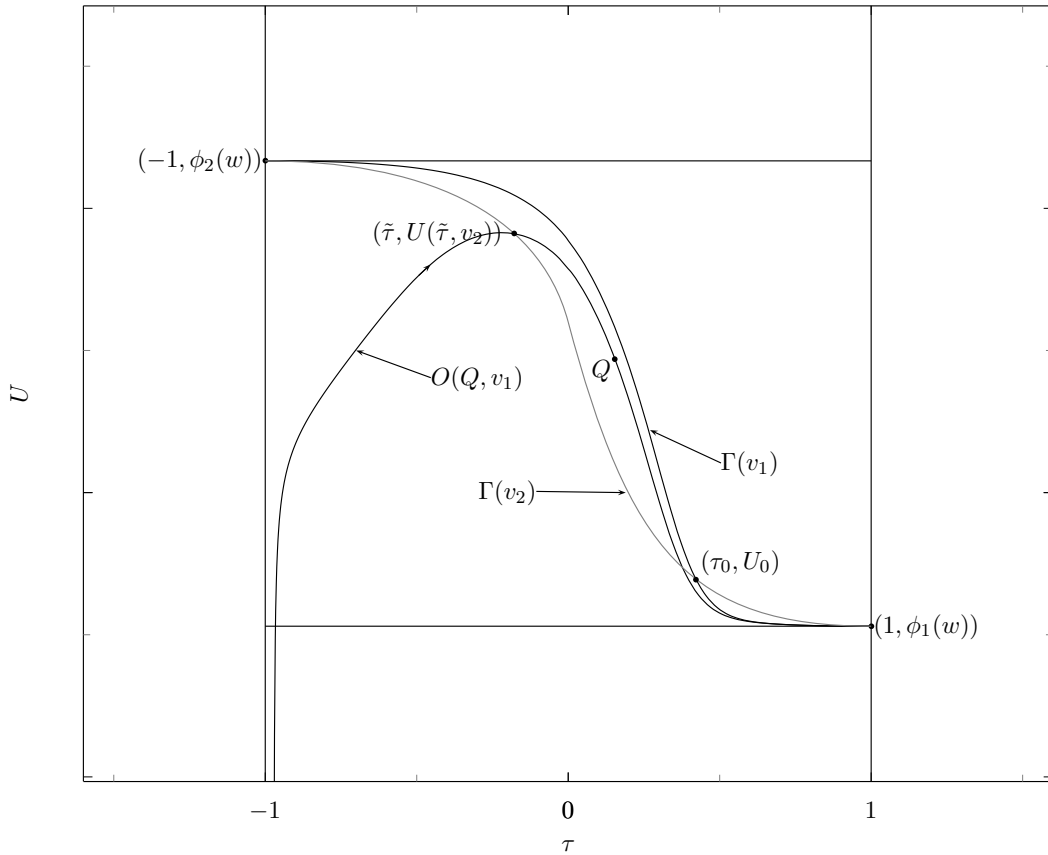


Figure 3.22: If $\Gamma(v_1)$ intersect with $\Gamma(v_2)$ at some $-1 < \tau_0 < 1$, then we can find an orbit $O(Q, v_1)$ intersecting with $\Gamma(v_2)$ at $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$.

Consider the orbit $O(Q, v_1)$ of system (3.19) with $v = v_1$ which passes through Q . As $z \rightarrow -\infty$, the orbit $O(Q, v_1)$ cannot approach the fixed point $(-1, \phi_2(w))$ because the heteroclinic orbit $\Gamma(v_1)$ is the only unstable manifold. Because of the properties of phase flows of autonomous ODEs, the orbit $O(Q, v_1)$ cannot cross $\Gamma(v_1)$. Since $\tau' > 0$ for $-1 < \tau < 1$, $O(Q, v_1)$ must enter $\Gamma(v_2)$ at some point $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$ as $z \rightarrow -\infty$.

At the intersecting point $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$, we have

$$f(U(\tilde{\tau}, v_1), w) + \alpha I(\tilde{\tau}) = f(U(\tilde{\tau}, v_2), w) + \alpha I(\tilde{\tau}) = \tilde{\mu} < 0.$$

By comparing the tangents of $O(Q, v_1)$ and $\Gamma(v_2)$ at this point, we obtain the following false inequality

$$\frac{\tilde{\mu}}{v_2 k(1 - \tilde{\tau}^2)} < \frac{\tilde{\mu}}{v_1 k(1 - \tilde{\tau}^2)}.$$

This contradiction proves that $U(\tau, v_1) \neq U(\tau, v_2)$ for $\tau \in (-1, 1)$.

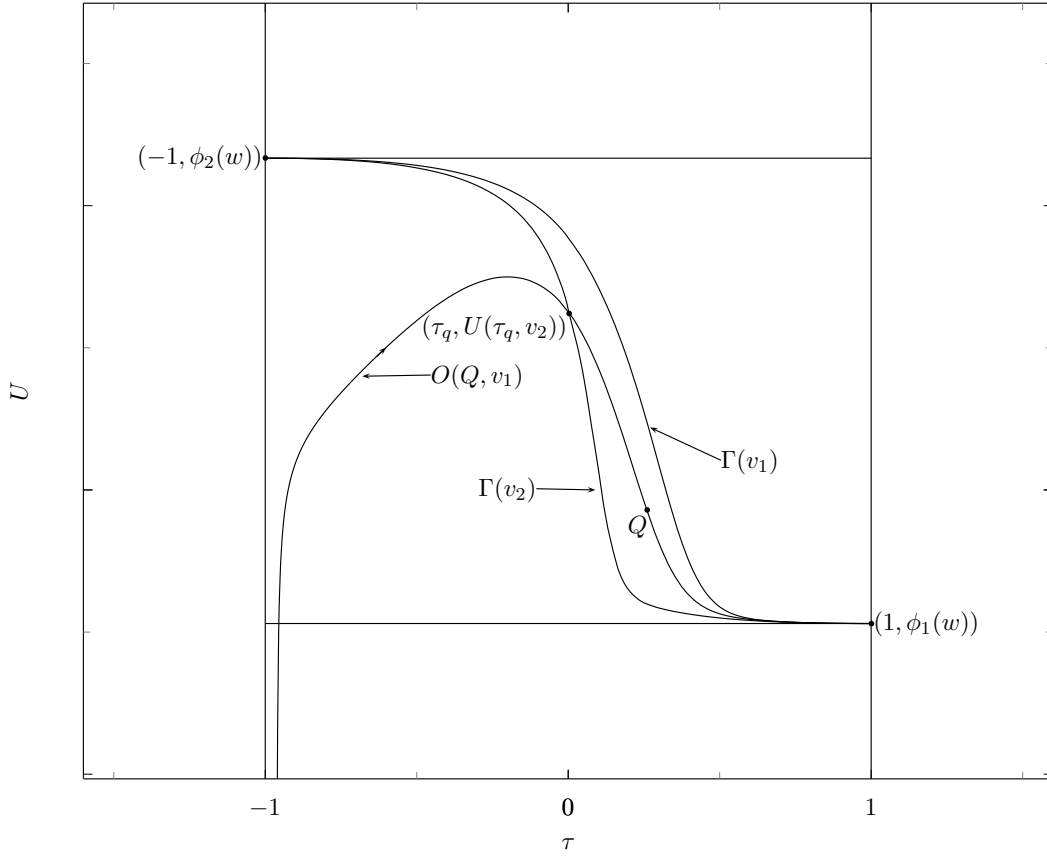


Figure 3.23: The orbit $O(Q, v_1)$ intersects with $\Gamma(v_2)$ at $(\tilde{\tau}, U(\tilde{\tau}, v_2)) \in \Gamma(v_2)$.

Next we prove that $U(\tau, v_1) < U(\tau, v_2)$ for $\tau \in (-1, 1)$. In fact, if

$U(\tau, v_1) > U(\tau, v_2)$ for $\tau \in (-1, 1)$, we can take a point $Q = (\tau', U')$, below $\Gamma(v_1)$ and above $\Gamma(v_2)$. See Figure 3.23. By repeating the reasoning for $O(Q, v_1)$ we get the same false inequality again, and this contradiction proves that $U(\tau, v_1) < U(\tau, v_2)$ for $\tau \in (-1, 1)$. \square

Step 3. If $\omega = \sup\{v; U(0, v) < \theta\}$ and $\Omega = \inf\{v, U(0, v) > \theta\}$, then $\omega = \Omega$.

From Lemma 3.3.6, it is easy to see that ω and Ω exist. If $\omega > \Omega$, then there exists v such that $\omega > v > \Omega$. From $\omega > v$ we have $U(0, v) < \theta$. From $v > \Omega$ we have $U(0, v) > \theta$. This is a contradiction. If $\omega < \Omega$ then there exist v_1 such that $\omega < v_1 < \Omega$. Then from $v_1 < \Omega$ we have $U(0, v_1) \leq \theta$. From $\omega < v_1$ we have $U(0, v_1) \geq \theta$. Thus, we would have $U(0, v_1) = \theta$; but we can find a v_2 such that $\omega < v_1 < v_2 < \Omega$ and $U(0, v_1) = U(0, v_2) = \theta$. This contradicts Lemma 3.3.6. Because of these two contradictions we must have $\omega = \Omega$.

Step 4. $U(0, \omega) = \theta = U(0, \Omega)$.

Suppose that $U(0, \omega) < \theta$. Consider the orbit $O(z, 0, \theta, \omega)$ of system (3.19) with $v = \omega$ and the initial value set to $(0, \theta)$ at $z = 0$. Then $O(z, 0, \theta, \omega)$ will stay above $\Gamma(\omega)$ and intersect the line segment

$$L_2 = \{(\tau, \phi_2(\omega)); -1 < \tau < 0\}$$

at some moment $z_0 < 0$. On the closed interval $[z_0 - 1, 0] \supset [z_0, 0]$, by the continuous dependence of the solution on parameters, there exists $v_1 > \omega$ such that the orbit $O(z, 0, \theta, v_1)$ of system (3.19) with $v = v_1$, starting at the initial point $(0, \theta)$ at $z = 0$, sufficiently close to the orbit $O(z, 0, \theta, \omega)$, will also cross the line segment L_2 at some moment $z' \in [z_0 - 1, 0]$. Thus, $\Gamma(v_1)$

will stay below $O(z, 0, \theta, v_1)$ and we will have $U(0, v_1) > \theta$. This contradicts the definition of ω . See Figure 3.24.

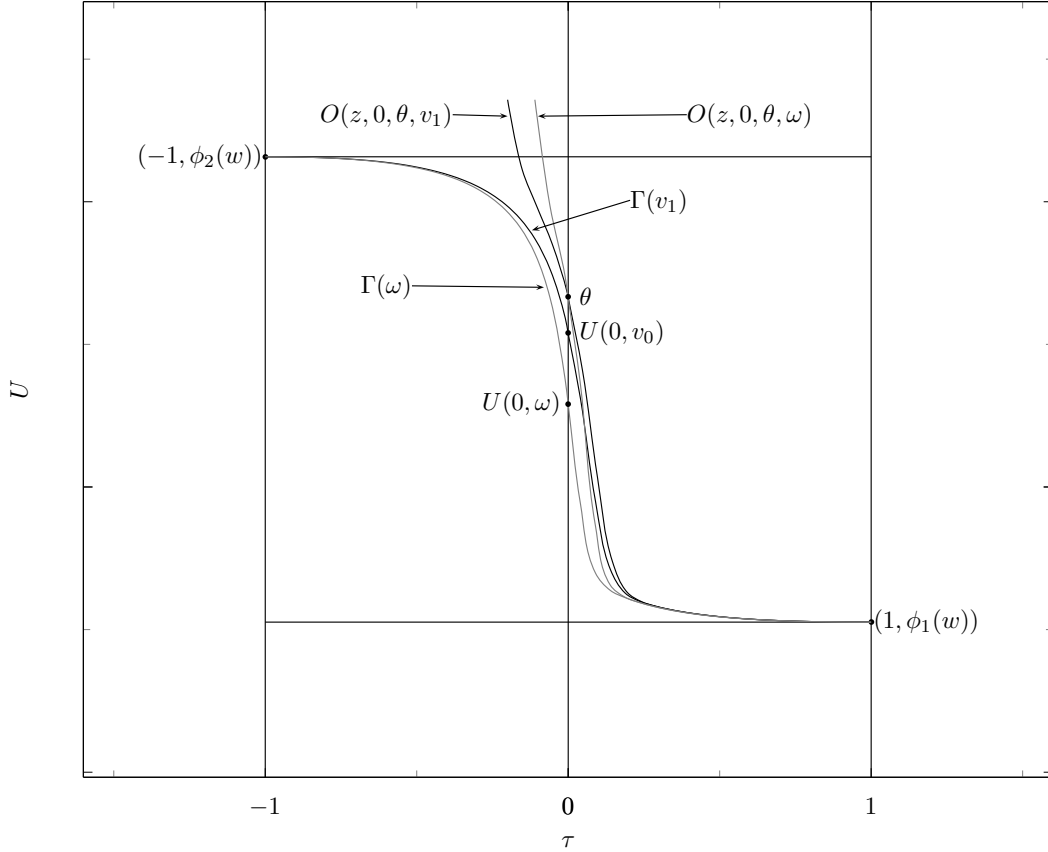


Figure 3.24: $U(0, \omega) < \theta$, orbit $O(z, 0, \theta, v_1)$, $\Gamma(v_1)$ and $\Gamma(\omega)$.

Assume that $U(0, \Omega) > \theta$ and consider the orbit $O(z, 0, \theta, \Omega)$ of system (3.19) with $v = \Omega$ and initial value set to $(0, \theta)$ at $z = 0$. Then $O(z, 0, \theta, \Omega)$ will stay below $\Gamma(\Omega)$ and enter the U -nullcline Σ in a neighborhood of $(\tau, U) = (-1, \phi_2(w))$ at some moment $z_0 < 0$. By a similar reasoning as above, we can find a $v_0 < \Omega$ such that the orbit $O(z, 0, \theta, v_0)$ of system (3.19) with $v = v_0$, starting at the initial point $(0, \theta)$ at $z = 0$, sufficiently close

to the orbit $O(z, 0, \theta, \Omega)$, will also enter the U -nullcline Σ at some moment $z' \in [z_0 - 1, 0]$. Thus, $\Gamma(v_0)$ will stay above $O(z, 0, \theta, v_0)$ and we will have $U(0, v_0) > \theta$. See Figure 3.25. This contradicts the definition of Ω .

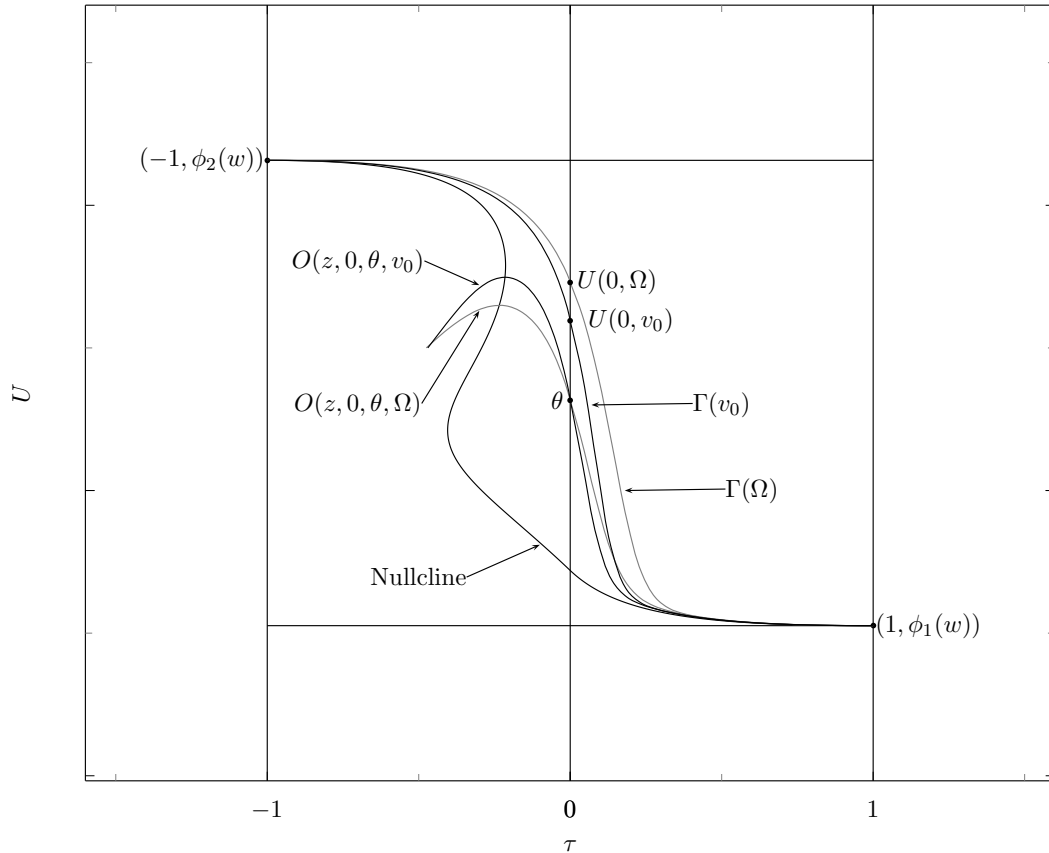


Figure 3.25: $U(0, \Omega) > \theta$, the orbit $O(z, 0, \theta, \Omega)$, $O(z, 0, \theta, v_0)$, $\Gamma(v_0)$, and $\Gamma(\Omega)$.

Thus, there exists a unique $\omega = \Omega = v$ such that $U(0, \omega) = \theta = U(0, \Omega)$.

□

Case 2. $w_2^- < w < w_2^+$.

Then there are three fixed points, $(-1, \phi_2(w))$, $(-1, \phi_{21}(w))$, and $(-1, \phi_{22}(w))$,

on the line $\tau = -1$. According to the assumptions on the curves C_1 and C_2 (see conditions (a), (b), and (c)), we have

$$\phi_1(w) < \phi_{21}(w) < \phi_{22}(w) < \phi_2(w).$$

In the proof of Lemma 3.3.2, we proved that the fixed point $(-1, \phi_2(w))$ is a saddle point. Evaluating the Jacobian of (3.19) at $(-1, \phi_{21}(w))$ gives

$$\begin{bmatrix} 2k & 0 \\ \frac{1}{v}\alpha I'(-1) & \frac{1}{v}f_u(\phi_{21}(w), w) \end{bmatrix};$$

thus, we have a saddle point. On the other hand, $(-1, \phi_{22}(w))$ is an unstable node, and $(1, \phi_1(w))$ is a stable node. In this case, Lemmas 3.3.2, 3.3.3, 3.3.4, and 3.3.5 also hold. The difference with Case 1 is that the U -nullclinic curve Σ has two branches.

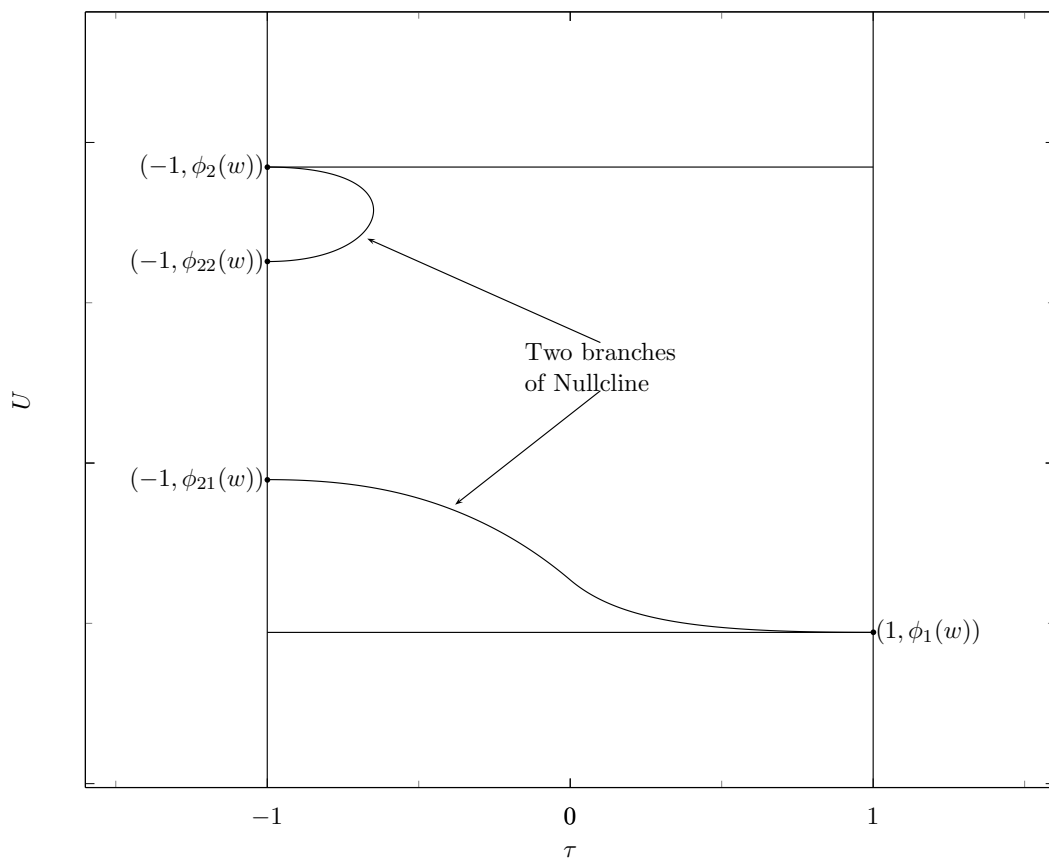


Figure 3.26: In Case 2, the U -nullcline curve of (3.19) has two branches.

One branch connects $(-1, \phi_2(w))$ and $(-1, \phi_{22}(w))$ and lies in D_2 with $\phi_{21}(w) < U < \phi_2(w)$ and $-1 < \tau < 0$. The other one connects $(-1, \phi_{21}(w))$ and $(1, \phi_1(w))$. The proof of Theorem 3.2.1 is complete. \square

Theorem 3.3.7. For $w_1^- < w < w_1^+$, system (3.19) has three fixed points on the line $\tau = 1$, i. e.

$$(1, \phi_1(w)), \quad (1, \phi_{11}(w)), \quad (1, \phi_{12}(w)),$$

where

$$\phi_1(w) < \phi_{11}(w) < \phi_{12}(w) < \phi_2(w),$$

and has no heteroclinic orbit linking $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ with the U -coordinate, $U(z, w)$, which is a decreasing function of z .

Proof. From conditions (a), (b), and (c), system (3.19) obviously has three fixed points on the line $\tau = 1$. On the segment

$$L_3 = \{(\tau, U); -1 < \tau < 1, U = \phi_{11}(w)\}$$

the vector field points in a top-right direction because of the inequalities

$$k(1 - \tau^2) > 0 \quad \text{and} \quad \frac{1}{v} [f(\phi_{11}(w), w) + \alpha I(\tau)] = \frac{\alpha}{v} I(\tau) > 0.$$

Figure 3.27 shows the pattern of the vector field in this case.

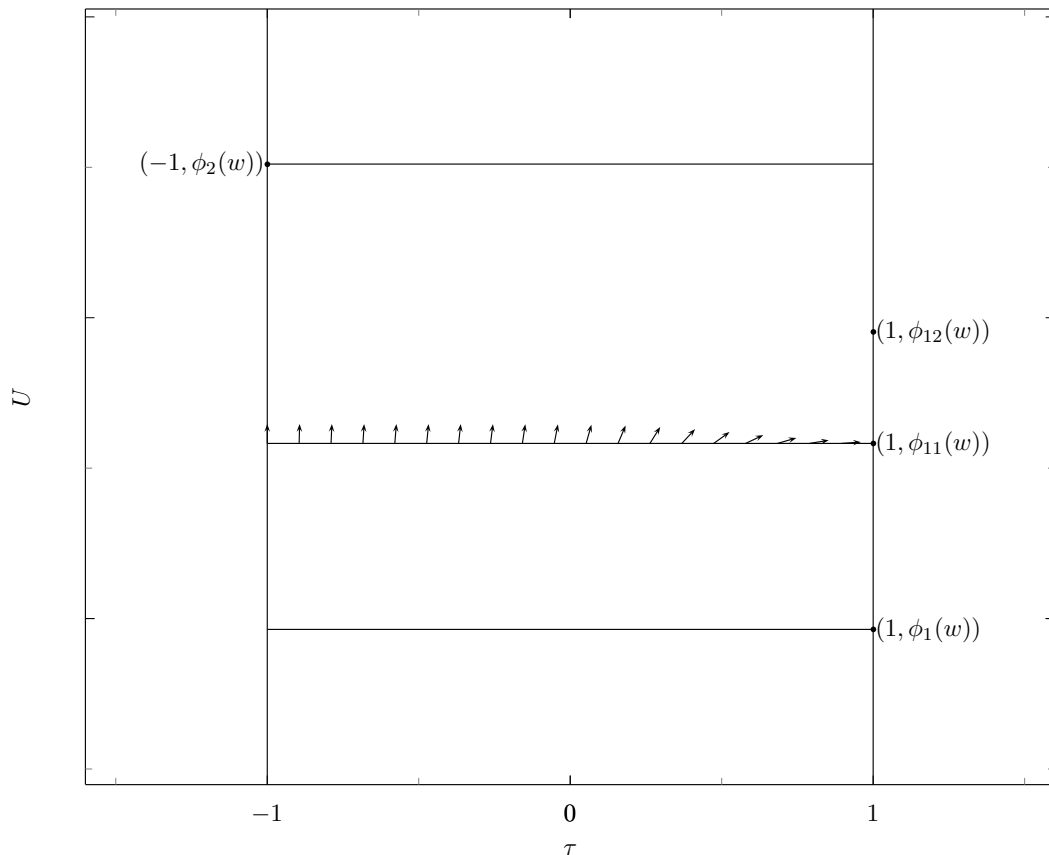


Figure 3.27: Vector field on L_3 points in a top-right direction.

Thus, there is no orbit passing through L_3 in the bottom-right direction and linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ with the U -coordinate, $U(z, w)$, being a decreasing function of z . \square

Theorem 3.3.8. *If $f(U_0, w) + \frac{1}{2}\alpha \geq 0$ for some $\phi_2(w) > U_0 \geq \theta$, then there is no heteroclinic orbit passing through the point $(\tau = 0, U = \theta)$, linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$, and having its U -coordinate decreasing as z increases.*

Proof. In fact, if there are some $\phi_2(w) > U_0 > \theta$ such that $f(U_0, w) + \frac{1}{2}\alpha \geq 0$, then $f(U_0, w) + \alpha I(\tau) > 0$ on the segment

$$L = \{(U_0, \tau); -1 < \tau < 0\}$$

because of the inequality $I(\tau) > 1/2$ for $-1 < \tau < 0$. The vector field on L points in a top-right direction. The orbit starting at the point S could not cross this segment from below in a bottom-left direction as $z \rightarrow -\infty$.

Summing up what has been proved by Theorems 3.3.1, 3.3.7, and 3.3.8, we obtain

Corollary 3.3.9. *Suppose equation (3.1) satisfies conditions (3.2) and (a)–(d). Then, there is a unique $v = v(w, \theta) > 0$ such that equation (3.19) has a unique heteroclinic orbit*

$$\{(\tau(z, w), U(z, w)); \tau(0, w) = 0, U(0, w) = \theta\},$$

if and only if $w_1^+ < w < w_2^+$, and $f(U, w) + \alpha/2 < 0$ for $\theta \leq U \leq \phi_2(w)$. The function $U(z, w)$ is monotone decreasing in z . The orbit satisfies the boundary conditions

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tau(z, w) &= -1, & \lim_{z \rightarrow +\infty} \tau(z, w) &= 1, \\ \lim_{z \rightarrow -\infty} U(z, w) &= \phi_2(w), & \lim_{z \rightarrow +\infty} U(z, w) &= \phi_1(w), \end{aligned} \tag{3.22}$$

linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$. The function

$$u(t, x) = U(x + vt, w) = U(z, w), \quad \text{where } z = vt + x,$$

with some abuse of notation, is the unique traveling wave solution of equation (3.1) with the properties

(α) $U(z, w)$ is a monotone decreasing function of z ,

(β) $U(0, w) = \theta$,

(γ) the limits (3.22) hold and $\lim_{z \rightarrow \mp\infty} U_z(z, w) = 0$.

Remark 3.3.10. Similar to Remark 3.2.13, if $\rho_0^+ \geq \rho_1^+$, the conditions $w_1^+ < w < w_2^+$ and $f(U, w) + \frac{1}{2}\alpha < 0$ for $U > \theta$ and $\theta \in (\rho_1^-, \rho_1^+)$ are equivalent to $w_2^+ > w > w_0^+$.

For decreasing solutions, the relationship between the wave speed $v = v(w, \theta)$ and the threshold θ is in the following theorem.

Theorem 3.3.11. *Let $w_1^+ < w < w_2^+$. If there is $\theta_0 \in (\rho_1^-, \rho_1^+)$ such that $f(\theta, w) + \alpha/2 < 0$ for $\theta > \theta_0$, then $v = v(w, \theta)$ is a monotone increasing function of the threshold $\theta \in (\theta_0, \rho_1^+)$.*

Proof. The conditions in this theorem guarantee that the monotone decreasing traveling wave solution for equation (3.1) exists, and the function $v = v(w, \theta)$ is well defined for the threshold values $\theta \in (\theta_0, \rho_1^+)$. Let $v_i = v(w, \theta_i)$, $i = 1, 2$, and $\theta_1 < \theta_2$. At first $v_1 \neq v_2$. Otherwise, there would be two unstable manifolds for system (3.19) at the fixed point $(-1, \phi_2(w))$. This contradicts the uniqueness of the unstable manifold of saddle points. Secondly, we assume $v_1 > v_2$. Let $\Gamma(v_1)$ and $\Gamma(v_2)$ be heteroclinic orbits of systems (3.19) with $v = v_1$ and $v = v_2$ and passing through the initial points $(0, \theta_1)$ and $(0, \theta_2)$, respectively. By Lemma 3.3.6 we must have $\theta_1 = U(0, v_1) > U(0, v_2) = \theta_2$, which is a contradiction. \square

Theorem 3.3.12. *Let $\rho_1^- < \theta < \rho_1^+$. If there is $w^* \in (w_1^+, w_2^+)$ such that $f(U, w) + \frac{1}{2}\alpha < 0$ for $w^* < w < w_2^+$ and $U > \theta$, then $v(w, \theta)$ is a monotone increasing function of $w \in (w^*, w_2^+)$.*

Proof. For a fixed $\theta \in (\rho_1^-, \rho_1^+)$, the conditions in the theorem guarantee that the traveling wave solutions exist for every $w \in (w^*, w_2^+)$, thus $v(w, \theta)$, as a function of w , is well defined. Let $w_1, w_2 \in (w^*, w_2^+)$ and $w_1 < w_2$; then $f(U, w_1) > f(U, w_2)$, because $f_w < 0$. Correspondingly, we have $v(w_1, \theta)$ and $v(w_2, \theta)$. We claim that

$$v(w_1, \theta) < v(w_2, \theta).$$

In fact, if $v(w_1, \theta) \geq v(w_2, \theta)$, noting that both of them are positive, we would have

$$0 > \frac{f(\theta, w_1) + \frac{1}{2}\alpha}{v(w_1, \theta)} > \frac{f(\theta, w_2) + \frac{1}{2}\alpha}{v(w_2, \theta)}.$$

Let Γ_1 be the heteroclinic orbit of system (3.19) with $w = w_1$ and Γ_2 be the orbit with $w = w_2$. At $\tau = 0$, Γ_1 and Γ_2 meet at the point $(\tau, U) = (0, \theta)$. In the left side of this point, where $-\delta < z < 0$ for some $\delta > 0$ small enough, Γ_1 was below Γ_2 . As $z \rightarrow -\infty$, both of them would approach their limit points $(\tau, U) = (-1, \phi_2(w_1))$ and $(\tau, U) = (-1, \phi_2(w_2))$, respectively. According to the condition on curve C_2 , we had $\phi_2(w_1) > \phi_2(w_2)$. Thus, Γ_2 had to penetrate Γ_1 from above at some point, say (τ_0, U_0) , $-1 < \tau_0 < 0$. If we compare the tangents to Γ_1 and Γ at this point we have

$$\frac{f(U_0, w_1) + \alpha I(\tau_0)}{v(w_1, \theta)} < \frac{f(U_0, w_2) + \alpha I(\tau_0)}{v(w_2, \theta)} < 0,$$

which is a false inequality.

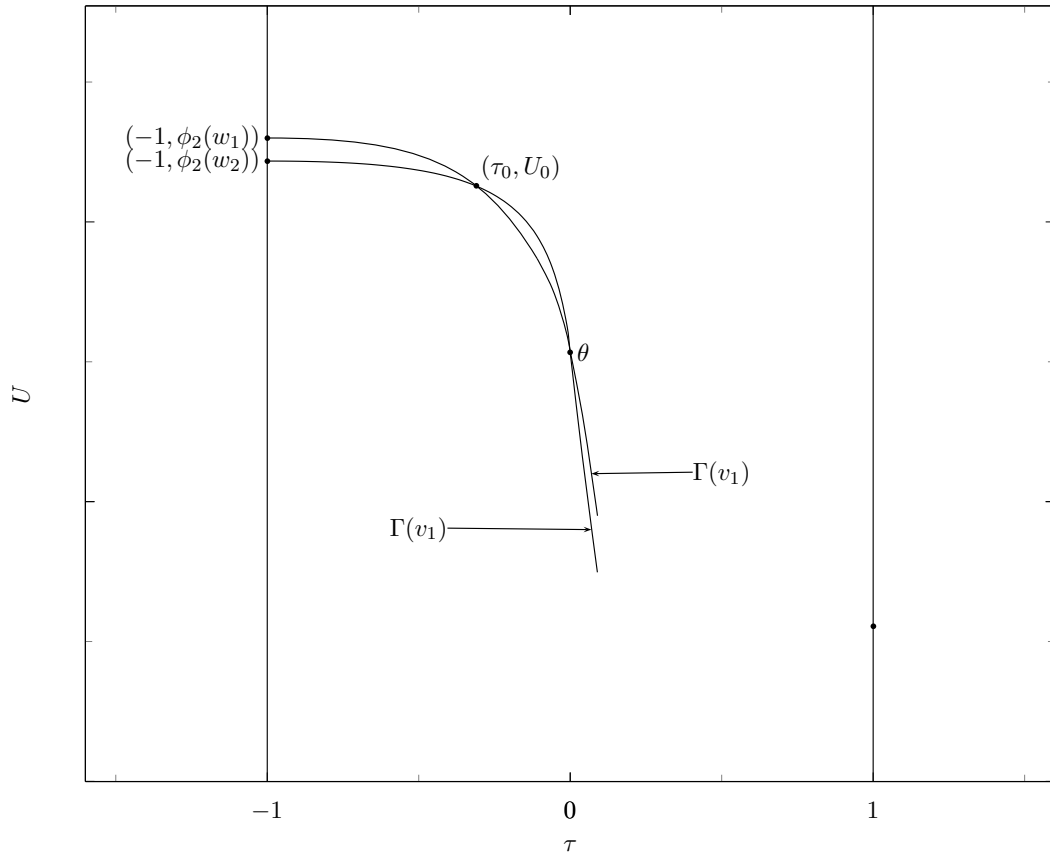


Figure 3.29: Γ_1 and Γ_2 intersect at right of $\tau = 0$.

□

3.4 Symmetric traveling wave solutions

Now we consider traveling wave solutions of (3.1) in the positive x -direction with positive speed $v > 0$. For the wave front, the function

$$u(t, x, w) = U(x - v(w, \theta)t, w) = U(z, w), \quad z = x - v(w, \theta)t, \quad (3.23)$$

is monotone decreasing, satisfies the initial condition $U(0, w) = \theta$ and the boundary condition

$$\lim_{z \rightarrow -\infty} U(z, w) = \phi_2(w), \quad \lim_{z \rightarrow +\infty} U(z, w) = \phi_1(w).$$

Plugging (3.23) into (3.1) we have

$$-vU_z = f(U(z, w), w) + \alpha \int_z^{+\infty} K(\xi) d\xi, \quad U_z < 0. \quad (3.24)$$

The corresponding autonomous system is

$$\begin{aligned} \frac{d\tau}{dz} &= \tau' = k(1 - \tau^2), & \tau(0) &= 0, \\ \frac{dU}{dz} &= -\frac{1}{v} [f(U, w) + \alpha I(\tau)], & U(0) &= \theta, \end{aligned} \quad (3.25)$$

where

$$I(\tau) = \int_{\tau}^1 \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta.$$

We search for the heteroclinic orbit of system (3.25) that has a monotone decreasing U -coordinate as z increases and links the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$.

Let $\sigma = -\tau$ and $s = -z$. Then system (3.25) is transformed into the system

$$\begin{aligned} \frac{d\sigma}{ds} &= \tau' = k(1 - \sigma^2), & \sigma(0) &= 0, \\ \frac{dU}{ds} &= \frac{1}{v} [f(U, w) + \alpha I(\sigma)], & U(0) &= \theta, \end{aligned} \quad (3.26)$$

where

$$\begin{aligned} I(\sigma) &= \int_{-1}^{\sigma} \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta \\ &= \int_{\tau}^1 \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta, \end{aligned}$$

because the integrand is an even function on the interval $[-1, 1]$. The fixed points are transformed into $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. We are searching for the heteroclinic orbit of system (3.26) that has a monotone increasing U -coordinate as s increases and links the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. Now we face the same problem as with system (3.11). This means that system (3.25) is the symmetric form of system (3.11), and similar results on system (3.11) are also correct for traveling wave solutions with wave speed $v > 0$.

If system (3.26) has orbit $\{\sigma(s), U(s)\}$ satisfying the conditions

$$\begin{aligned} \sigma(0) &= 0, & \lim_{s \rightarrow -\infty} (\sigma(s), U(s)) &= (-1, \phi_1(w)), \\ U(0) &= \theta, & \lim_{s \rightarrow +\infty} (\sigma(s), U(s)) &= (1, \phi_2(w)), \\ \lim_{s \rightarrow \pm\infty} U_s(s) &= 0, \end{aligned}$$

and $U(s)$ is an increasing function of s , then $\tilde{\tau}(z) = -\sigma(-s)$, $\tilde{U}(z) = U(-s)$ is the orbit of system (3.25) satisfying the conditions

$$\begin{aligned} \tilde{U}(0) &= \theta, & \lim_{z \rightarrow -\infty} (\tilde{\tau}(z), \tilde{U}(z)) &= (-1, \phi_2(w)), \\ \tilde{\tau}(0) &= 0, & \lim_{z \rightarrow +\infty} (\tilde{\tau}(z), \tilde{U}(z)) &= (1, \phi_1(w)), \\ \lim_{z \rightarrow \pm\infty} \tilde{U}_z(z) &= 0, \end{aligned}$$

and $\tilde{U}(z)$ is an decreasing function of z .

We sum up all results in the following theorems.

Theorem 3.4.1. *Suppose equation (3.1) satisfies conditions (3.2) and (a)–(d). Then there exists a unique $v = v(w, \theta) > 0$, such that equation (3.11) has a unique heteroclinic orbit*

$$\{(\tau(z, w), U(z, w)); \tau(0, w) = 0, U(0, w) = \theta\}$$

if and only if $w_1^- < w < w_2^-$ and the function $f(U, w) + \frac{1}{2}\alpha$ has no zero for $\phi_1(w) \leq U \leq \theta$. The function $U(z, w)$ is monotone increasing in z . The orbit satisfies the boundary conditions

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tau(z, w) &= -1, & \lim_{z \rightarrow -\infty} U(z, w) &= \phi_1(w), \\ \lim_{z \rightarrow +\infty} \tau(z, w) &= 1, & \lim_{z \rightarrow +\infty} U(z, w) &= \phi_2(w), \\ \lim_{z \rightarrow \pm\infty} U_z(z, w) &= 0, \end{aligned}$$

linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. The function

$$u(t, x) = U(x + vt, w) = U(z, w), \text{ where } z = x + vt,$$

with some abuse of notation, is the unique traveling wave solution of equation (3.1) such that

(α) $U(z, w)$ is a monotone increasing function of z ,

(β) $U(0, w) = \theta$,

(γ) the following limits hold

$$\lim_{z \rightarrow -\infty} U(z, w) = \phi_1(w), \quad \lim_{z \rightarrow +\infty} U(z, w) = \phi_2(w), \quad \lim_{z \rightarrow \pm\infty} U_z(z, w) = 0.$$

Meanwhile, system (3.25) has a unique heteroclinic orbit

$$\{(\tilde{\tau}(s, w), \tilde{U}(s, w)); \tilde{\tau}(0, w) = 0, \tilde{U}(0, w) = 0\}.$$

The function $\tilde{U}(s, w)$ is monotone decreasing in s . The orbit satisfies the

boundary conditions:

$$\begin{aligned}\lim_{s \rightarrow -\infty} \tilde{\tau}(s, w) &= -1, & \lim_{s \rightarrow -\infty} U(s, w) &= \phi_2(w), \\ \lim_{s \rightarrow +\infty} \tilde{\tau}(s, w) &= 1, & \lim_{s \rightarrow +\infty} U(s, w) &= \phi_1(w), \\ \lim_{s \rightarrow \pm\infty} U_z(s, w) &= 0,\end{aligned}$$

linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$. The function

$$\tilde{u}(x, t) = \tilde{U}(x - vt, w) = \tilde{U}(s, w), \quad s = x - vt,$$

is the unique traveling wave solution of equation (3.1) such that

- (a) $\tilde{U}(s, w)$ is a monotone decreasing function of s ,
- (b) $\tilde{U}(0, w) = \theta$,
- (c) the following limits hold:

$$\lim_{s \rightarrow -\infty} \tilde{U}(s, w) = \phi_2(w), \quad \lim_{s \rightarrow +\infty} \tilde{U}(s, w) = \phi_1(w), \quad \lim_{s \rightarrow \mp\infty} U_s(s, w) = 0.$$

The wave speed $v = v(w, \theta)$ is a decreasing function the threshold θ and the parameter w .

In a similar way, the problem for wave back with speed $v > 0$ is given by

$$\begin{aligned}\frac{d\tilde{\tau}}{ds} &= \tau' = k(1 - \tilde{\tau}^2), & \tilde{\tau}(0) &= 0, \\ \frac{d\tilde{U}}{ds} &= -\frac{1}{v} [f(\tilde{U}, w) + \alpha I(\tilde{\tau})], & \tilde{U}(0) &= \theta,\end{aligned}\tag{3.27}$$

where

$$I(\tilde{\tau}) = \int_{-1}^{\tilde{\tau}} \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta.$$

We search for the heteroclinic orbit of system (3.27) that has a monotone increasing \tilde{U} -coordinate as s increases and links the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. It can be transformed into the same problem as system (3.19). We can obtain the following corresponding result.

Theorem 3.4.2. *Suppose equation (3.1) satisfies conditions (3.2) and (a)–(d). Then there exists a unique $v = v(w, \theta) > 0$, such that equation (3.19) has a unique heteroclinic orbit*

$$\{(\tau(z, w), U(z, w)); \tau(0, w) = 0, U(0, w) = \theta\},$$

if and only if $w_1^+ < w < w_2^+$ and the function $f(U, w) + \frac{1}{2}\alpha$ has no zero for $\theta \leq U \leq \phi_2(w)$. The function $U(z, w)$ is monotone decreasing in z . The orbit satisfies the boundary conditions:

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tau(z, w) &= -1, & \lim_{z \rightarrow -\infty} U(z, w) &= \phi_2(w), \\ \lim_{z \rightarrow +\infty} \tau(z, w) &= 1, & \lim_{z \rightarrow +\infty} U(z, w) &= \phi_1(w), \\ \lim_{z \rightarrow \pm\infty} U_z(z, w) &= 0, \end{aligned}$$

linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$. The function

$$u(t, x) = U(x + vt, w) = U(z, w), \text{ where } z = x + vt,$$

with some abuse of notation, is the unique traveling wave solution of equation (3.1) such that

(α) $U(z, w)$ is a monotone decreasing function of z ,

(β) $U(0, w) = \theta$,

(γ) the following limits hold:

$$\lim_{z \rightarrow -\infty} U(z, w) = \phi_2(w), \quad \lim_{z \rightarrow +\infty} U(z, w) = \phi_1(w), \quad \lim_{z \rightarrow \pm\infty} U_z(z, w) = 0.$$

Meanwhile, the system (3.27) has a unique heteroclinic orbit

$$\{(\tilde{\tau}(s, w), \tilde{U}(s, w)); \tilde{\tau}(0, w) = 0, \tilde{U}(0, w) = 0\}.$$

The function $\tilde{U}(s, w)$ is monotone increasing in s . The orbit satisfies the boundary conditions:

$$\begin{aligned} \lim_{s \rightarrow -\infty} \tilde{\tau}(s, w) &= -1, & \lim_{s \rightarrow -\infty} \tilde{U}(s, w) &= \phi_1(w), \\ \lim_{s \rightarrow +\infty} \tilde{\tau}(s, w) &= 1, & \lim_{s \rightarrow +\infty} \tilde{U}(s, w) &= \phi_2(w), \\ \lim_{s \rightarrow \pm\infty} \tilde{U}_z(s, w) &= 0, \end{aligned}$$

linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. The function

$$\tilde{u}(x, t) = \tilde{U}(x - vt, w) = \tilde{U}(s, w), \quad s = x - vt,$$

is the unique traveling wave solution of equation (3.1) such that

(a) $\tilde{U}(s, w)$ is a monotone increasing function of s ,

(b) $\tilde{U}(0, w) = \theta$,

(c) the following limits hold:

$$\lim_{s \rightarrow -\infty} \tilde{U}(s, w) = \phi_1(w), \quad \lim_{s \rightarrow +\infty} \tilde{U}(s, w) = \phi_2(w), \quad \lim_{s \rightarrow \mp\infty} \tilde{U}_s(s, w) = 0.$$

The wave speed $v = v(w, \theta)$ is an increasing function of the threshold θ and the parameter w .

These two theorems show that both the wave front and the wave back are traveling symmetrically, with the same speeds and forms, along the two directions of the one-dimensional net. This can be verified by numerical simulation.

3.5 Discussions

3.5.1 What is the difference from the results of [38]?

Now we compare our results with those of [38]. The problem considered in both cases is the same. Conditions (3.2) and (a)–(d) are the same as in [38]. Our results are different from Theorem 1 of [38] in the following points. In order to ensure the existence of a monotone increasing traveling wave solution for equation (3.1) we set more conditions on the function $f(u, w)$, the synaptic number α , and the parameter w , such that $f(u, w) + \alpha = 0$ has only one solution $\phi_2(w)$, and the function f satisfies $f(u, w) + \frac{1}{2}\alpha > 0$ for $u < \theta$. We have proved these two conditions are necessary and sufficient. For monotone decreasing traveling wave solutions, the corresponding conditions are that the equation $f(u, w) = 0$ has only one solution and that $f(u, w) + \frac{1}{2}\alpha < 0$ for $u > \theta$.

In [38], by its first condition, α is required to be large enough so that $f(u, 0) + \alpha = 0$ has only one solution $\beta > 1$ and $f_u(\beta, 0) < 0$. By its second condition, it is supposed that the threshold θ satisfies $\rho_1^- < \theta < \rho_1^+$ and there exists a unique number w_0 , such that $2f(\theta, w_0) + \alpha = 0$, $2f(\theta, w) + \alpha < 0$ for $w > w_0$, and $2f(\theta, w) + \alpha > 0$ for $w < w_0$. Under these two conditions, Theorem 1 of [38] claims that a monotone increasing traveling wave solution

exists for $w_1^- < w < w_0$ and a monotone decreasing traveling wave solution exists for $w_0 < w < w_2^+$. Apparently, our conditions for the existence of traveling wave solution are stronger than those of [38], but our conclusions are stronger than those of [38]. We set necessary and sufficient conditions for the existence and uniqueness of traveling wave solutions, while [38] gives only sufficient conditions. In fact, the conditions of [38] are too weak to ensure the existence of traveling wave solutions.

3.5.2 A counterexample

We give the following counterexample to the result of [38] as follows. Let $f(u, w) = u(u - 0.5)(1 - u) - w$ and $\alpha = 0.07$. We calculate that

$$(\rho_1^-, w_1^-) = \left(\frac{3 - \sqrt{3}}{6}, -\frac{\sqrt{3}}{36} \right), \quad (\rho_1^+, w_1^+) = \left(\frac{3 + \sqrt{3}}{6}, \frac{\sqrt{3}}{36} \right)$$

and $f(u, 0) + \alpha = u(u - 0.5)(1 - u) + 0.07 = 0$ has a unique positive solution $\beta > 1$, because at its extreme point the function $w = u(u - 0.5)(1 - u) + 0.07$ attains its minimum value

$$0.07 - \frac{1}{36}\sqrt{3} \approx 0.07 - 0.0481 = 0.0219 > 0.$$

The equation $f(\theta, w) + \frac{1}{2}\alpha = 0$ is equivalent to $w = \theta(\theta - 0.5)(1 - \theta) + 0.035$. In the interval $\frac{3 - \sqrt{3}}{6} < \theta < \frac{3 + \sqrt{3}}{6}$, it is an increasing function. For given θ , there is a unique w_0 such that $f(\theta, w_0) + \frac{1}{2}\alpha = \theta(\theta - 0.5)(1 - \theta) + 0.035 - w_0 = 0$, $f(\theta, w) + \frac{1}{2}\alpha > 0$ for $w < w_0$, and $f(\theta, w) + \frac{1}{2}\alpha < 0$ for $w > w_0$. Thus, the requirements of Theorem 1 in [38] are fulfilled.

We set $\theta = 0.6$ and $w_0 = 0.059$. Then, for any $0.0219- < w < w_0$, $f(U, w) + \alpha = 0$ has three solutions. By Theorem 3.2.7, there is no heteroclinic orbit linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$.

For $w = 0$, $f(u, 0) + \alpha = u(u - 0.5)(1 - u) + 0.07 = 0$ has a unique positive solution $\beta = \phi_2(0) > 1$. On the other hand $f(\rho_1^-, 0) + \frac{1}{2}\alpha \approx -0.0481 + 0.035 < 0$. According to Theorem 3.2.14 there is still no heteroclinic orbit passing through $(0, \theta > \rho_1^-)$ linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$. See Figure 3.30.

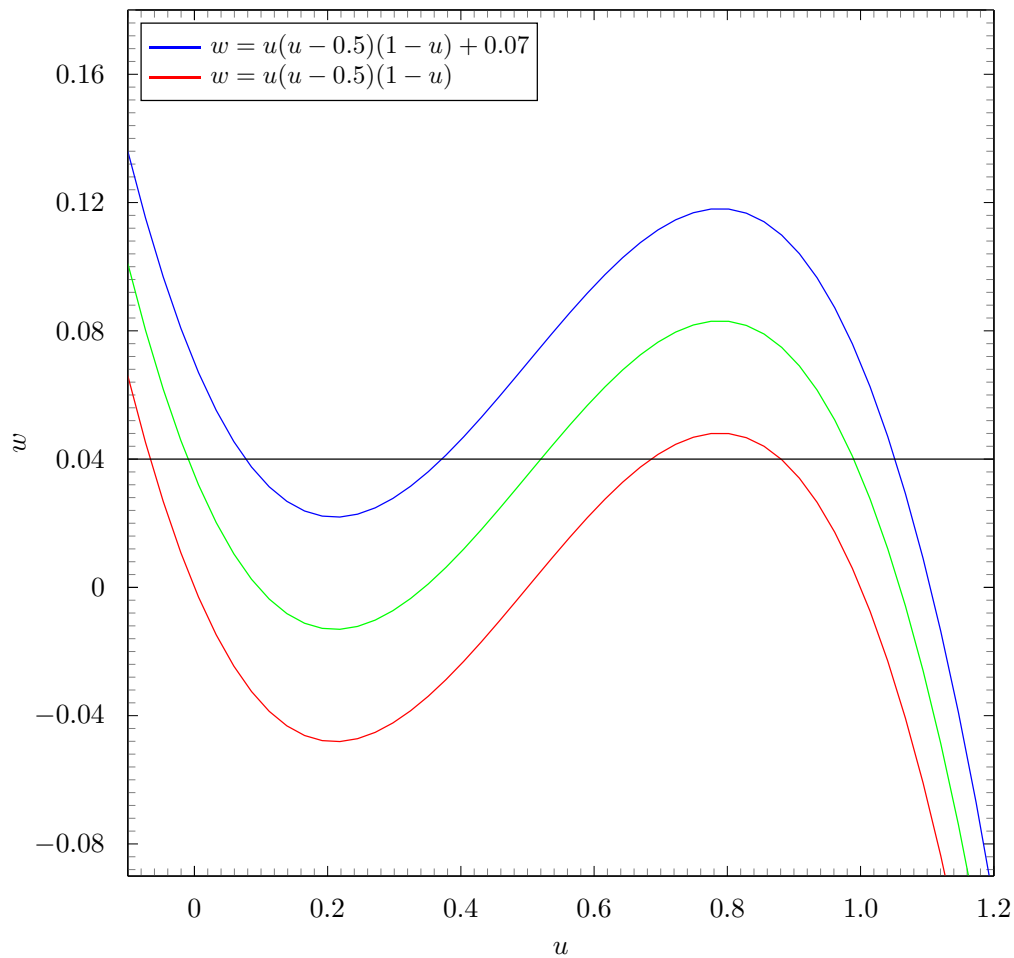


Figure 3.30: Figure of $w = u(u - 0.5)(1 - u) + 0.07$ and $w = u(u - 0.5)(1 - u)$.

The fault of [38] is in its Lemma 2 where the existence of the global solution for equation (3.7) was proved in the following way. The existence of

the solution defined on the interval $(-\infty, -M)$ having the property

$$\lim_{z \rightarrow -\infty} U(z) = \phi_1(w),$$

and the existence of the solution defined on $(M, +\infty)$ having the property

$$\lim_{z \rightarrow +\infty} U(z) = \phi_2(w)$$

were proved by using the Banach contraction mapping theorem. Then the author assumes that these two local solutions can be extended step by step to a global solution. In view of the equivalent system of ordinary differential equations (3.11), these two local solutions are orbits which approach the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, respectively. It is not always possible that these two orbits extend to a heteroclinic orbit. Our results clarify the conditions under which a heteroclinic orbit uniquely exists.

3.5.3 On the kernel function

To simplify the reasoning procedure we assumed that $0 < K(x)$ is continuous and satisfies condition (3.2). Now we assume that $K(x)$ is a nonnegative and piecewise continuous function and satisfies condition (3.2). Thus, $K(x)$ may have the form

$$K(x) \begin{cases} = 0, & M < |x| \text{ and } a_i < x < b_i, \quad -M < a_i, b_i < M, \quad i = 1, 2, \dots, N, \\ > 0, & \text{otherwise,} \end{cases}$$

which is piecewise continuous. The shape of this kernel function will cause changes in the properties of the function $I(\tau)$. In this generalized case, $I(\tau)$ is nonnegative, continuous, nondecreasing, and piecewise differentiable. At

a point of non differentiability, the left and right derivatives are continuous. Here we need to carefully consider the points to be addressed in proving the theorems of this chapter. We only discuss this issue for system (3.11).

On the invariant set D_2 . On the line

$$\{(\tau, U); -1 < \tau < 1, U = \phi_1(w)\},$$

$I(\tau) = 0$ for some $\tau \leq \tau_0 < 0$ and $I(\tau) > 0$ for $\tau > \tau_0$.

The vector field defined by system (3.11) is

$$(k(1 - \tau^2), f(\phi_1(w), w) + \alpha I(\tau) = \alpha I(\tau)).$$

The vector field points in a top-right direction for $\tau > \tau_0$ because $I(\tau) > 0$ for $\tau > \tau_0$. The segment of L_1 for $\tau \leq \tau_0$ is a piece of the orbit of system (3.11), which is a piece of the unstable manifold, Γ , of the saddle $(-1, \phi_1(w))$ because the vector field on this segment is

$$(k(1 - \tau^2), f(\phi_1(w), w) + \alpha I(\tau) = \alpha I(\tau) = 0)$$

for $\tau \leq \tau_0$.

On the line

$$\{(\tau, U); -1 < \tau < 1, U = \phi_2(w)\},$$

$I(\tau) = 1$ for some $0 < \tau_1 \leq \tau$ and $I(\tau) < 1$ for $\tau < \tau_1$.

The vector field defined by system (3.11) is

$$(k(1 - \tau^2), f(\phi_2(w), w) + \alpha I(\tau)).$$

The vector field points in a bottom-right direction for $\tau < \tau_1$ because $I(\tau) < 1$ for $\tau < \tau_1$. The segment of L_2 for $\tau \geq \tau_0$ is a piece of the orbit of system

(3.11), which approaches the stable node $(1, \phi_2(w))$ because the vector field on this segment is

$$(k(1 - \tau^2), f(\phi_2(w), w) + I(\tau) = I(\tau) = 0)$$

for $\tau \geq \tau_1$.

Thus, D_2 is also a positive invariant set for system (3.11) in this situation.

On the nullcline Σ . The U -nullcline is defined by

$$\Sigma = \{(\tau, U); f(U, w) + \alpha I(\tau) = 0\}.$$

It is possible that Σ contains a horizontal line segment

$$L_\Sigma = \{(\tau, U_0); \tau_2 < \tau < \tau_3\}.$$

In this case,

$$f(U_0, w) + \alpha I(\tau) \begin{cases} = 0, & \text{for } \tau_2 \leq \tau \leq \tau_3, \\ < 0, & \text{for } \tau < \tau_2, \end{cases}$$

and

$$f(U_0, w) + \alpha I(\tau) > 0, \quad \text{for } \tau_3 < \tau.$$

The line segment L_Σ is a piece of the orbit of system (3.11) because the vector field on this line segment is

$$(k(1 - \tau^2), f(U_0, w) + \alpha I(\tau) = 0), \quad \text{for } \tau_2 < \tau < \tau_3.$$

At the point (τ_2, U_0) , the orbit goes to the left side of Σ because the orbit has the tangent vector

$$(k(1 - \tau_2^2) > 0, f(U_0, w) + \alpha I(\tau) = 0).$$

On the left side of Σ , the orbit will go in a top-left direction as z decreases because the vector field is

$$(k(1 - \tau^2) > 0, f(U, w) + \alpha I(\tau) < 0),$$

which points in a bottom-right direction. The orbit has derivative

$$\frac{dU}{d\tau} = \frac{f(U, w) + \alpha I(\tau)}{kv(1 - \tau^2)} < 0, \quad \text{for } \tau < \tau_2.$$

Similarly, on the right side of Σ , the orbit will go in a top-right direction as z increases. Σ consists of the minimum points of the orbits. Thus, the orbit starting from a point on Σ cannot approach to the fixed point $(-1, \phi_1(w))$.

On the heteroclinic orbit Γ . In this case, for any $v > 0$, if the heteroclinic orbit $(\tau(z), U(z))$ contains the piece of the line segment

$$L_0 = \{(\tau, U); -1 < \tau < \tau_0, U = \phi_1(w)\},$$

then the U -nullcline Σ also contains L_0 as its left piece. The function $U(\tau, v)$ which appeared in the proof of Theorem 3.2.1 and step 2 of Lemma 3.2.5 can be defined by letting $z = z^{-1}(\tau)$ and $U = U(z^{-1}(\tau))$, where $z^{-1}(\tau)$ is the inverse function of $\tau(z)$. All the statements on heteroclinic orbits hold on the interval $\tau \in (\tau_0, 1)$.

In this case, we cannot use Poincaré-Bendixson theorem to prove that the unstable manifold of $(-1, \phi_1(w))$ is a heteroclinic orbit, because $I(\tau)$ is only piecewise differentiable. However, we can prove it by monotonicity of the manifold. Both $\frac{dU}{dz}$ and $\frac{d\tau}{dz}$ of the manifold are monotone increasing, so the manifold can only approach the stable fixed point $(1, \phi_2(w))$ as z goes to infinity.

Thus, all the theorems in this chapter are also true for the general form of the kernel function $K(x)$.

Chapter 4

Improved Model

4.1 Improved model

The model for excitable neuronal network is

$$\begin{aligned}\frac{\partial u}{\partial t} &= f(u, w) - g_{\text{syn}}(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x - y) H(u(t, y) - \theta) s(t, y) dy, \\ \frac{\partial w}{\partial t} &= \epsilon g(u, w), \\ \frac{\partial s}{\partial t} &= \alpha H(u(t, x) - \theta)(1 - s) - \beta s.\end{aligned}$$

Here $H(x)$ is the Heaviside function, $u = u(t, x)$, $w = w(t, x)$, $s = s(t, x)$, and $-\infty < x < +\infty$.

In real neuronal networks, the commonly known excitatory synapses have AMPA receptors. The postsynaptic conductance rises rapidly with time of the order of one millisecond [6]. We may take $s = \frac{\alpha}{\alpha + \beta}$, $A = g_{\text{syn}} \frac{\alpha}{\alpha + \beta}$, and $\epsilon = 0$ to simplify the equations of the system to

$$\frac{\partial u}{\partial t} = f(u, w) - A(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x - y) H(u(t, y) - \theta) dy. \quad (4.1)$$

In this equation w is constant. We have analogous results for system (4.1) to those for system (3.1) of Chapter 3.

We propose the following assumptions as basic conditions for the improved model.

- A1. $0 \leq K(x) = K(|x|) \leq Ce^{-\gamma|x|}$ is a piecewise continuous function and $\int_{-\infty}^{+\infty} K(x)dx = 1$. In the following, we suppose that $0 < K(x)$ and continuous. The general form of $K(x)$ will be discussed at the end of this chapter.
- A2. $f(u, w)$ satisfies the requirements listed in Chapter 3. The curve $C_1 = \{(u, w); f(u, w) = 0\}$ is a cubic-like curve. $f_w(u, w) < 0$; the w -minimum point (u_1^-, w_1^-) , and the w -maximum point (u_1^+, w_1^+) , satisfy $u_1^- < u_1^+ < u_{\text{syn}}$. The unique implicit function $u = \phi_1(w)$ is defined by $f(\phi_1(w), w) = 0$ for $u < u_1^-, w > w_1^-$. Moreover $f_u(\phi_1(w), w) < 0$.
- A3. The curve $C_2 = \{(u, w); f(u, w) - A(u - u_{\text{syn}}) = 0\}$ is a cubic-like curve. The w -minimum point (u_2^-, w_2^-) and the w -maximum point (u_2^+, w_2^+) satisfy $u_2^- < u_2^+ < u_{\text{syn}}$. The unique implicit function $\phi_2(w)$ is defined by $f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}}) = 0$ for $u > u_2^+, w < w_2^+$. Moreover, $f_u(\phi_2(w), w) - A < 0$.
- A4. θ is a threshold which satisfies $u_1^- < \theta < u_1^+$.
- A5. The synaptic reversal (equilibrium) potential u_{syn} is sufficiently large so that $u_{\text{syn}} > u_1^* > u_1^+$, where $f(u_1^*, w_1^-) = 0$.

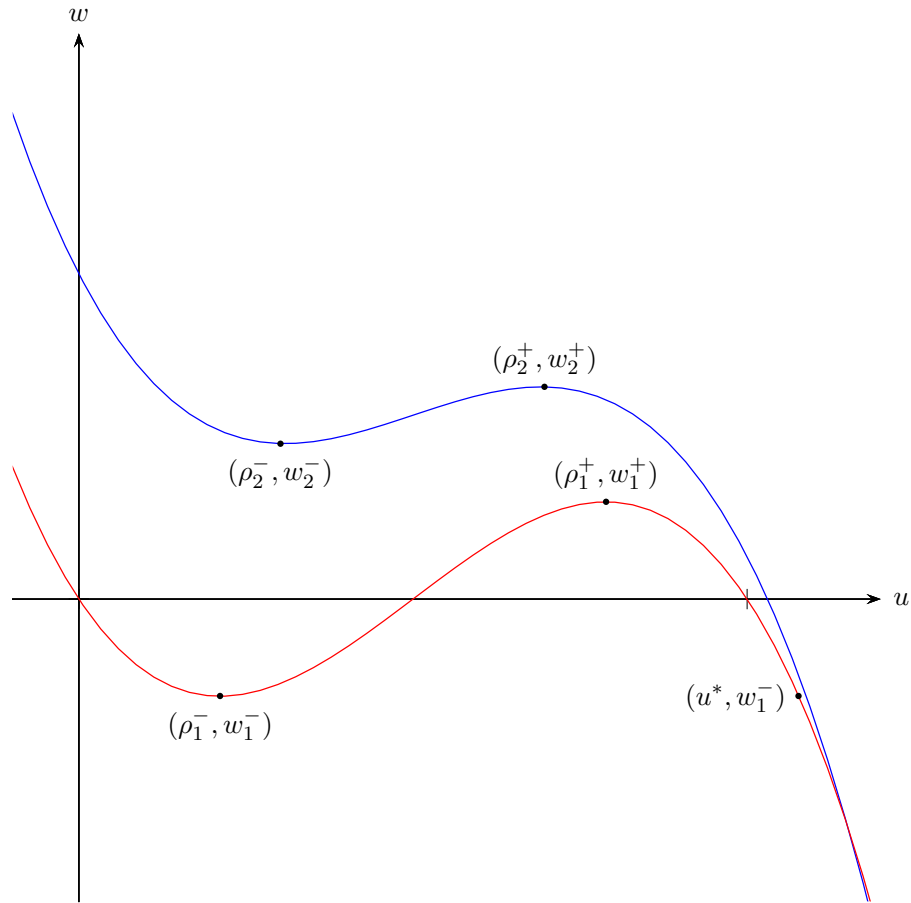


Figure 4.1: Typical pattern of C_1 , C_2 , and C_0

The last assumption is reasonable since, in excitatory synapses, the reversal potential is above the rest potential by 100mV [6] and it is bigger than the amplitude of action potentials. This model is more realistic and more complicated than the one discussed in Chapter 3, because the coupling term is the ion current determined by the membrane potential $(u - u_{\text{syn}})$. By the geometrical singular perturbation theory, we may first study equation (4.1) to find whether a traveling wave solution exists when the network is excited.

4.2 Monotone increasing traveling wave solutions

Suppose $u(x, t) = U(x + vt) = U(z)$, $z = x + vt$, where $v > 0$ is a constant. The monotone increasing traveling wave solution of (4.1) should satisfy the equation

$$v \frac{\partial U}{\partial z} = f(U, w) - A(U - u_{\text{syn}}) \int_{-\infty}^z K(\xi) d\xi \quad (4.2)$$

and the conditions

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = l_-(w), \quad \lim_{z \rightarrow \infty} U(z) = l_+(w), \quad l_-(w) < l_+(w).$$

Here, we suppose $U(0) = \theta$ and $U(z) > \theta$ for $z > 0$. This solution describes the wave front traveling along the x -axis in the negative (left) direction.

Letting

$$z = \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau}, \quad 0 < k < \frac{\gamma}{2}, \quad (4.3)$$

from (4.2), we obtain the autonomous system:

$$\begin{aligned} \frac{d\tau}{dz} &= k(1 - \tau^2), & \tau(0) &= 0, \\ v \frac{dU}{dz} &= f(U, w) - A(U - u_{\text{syn}})I(\tau), & U(0) &= \theta, \end{aligned} \quad (4.4)$$

where

$$I(\tau) = \int_{-1}^{\tau} \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta.$$

It is easy to see from the properties of $K(x)$ that $I(\tau)$ is monotone increasing and differentiable, $I(-1) = 0$, $I(0) = 1/2$, and $I(1) = 1$. System (4.4) is defined on the strip

$$D = \{(\tau, U); -1 \leq \tau \leq 1, -\infty < U < +\infty\}.$$

A monotone increasing traveling wave solution of (4.2) corresponds to a heteroclinic orbit of system (4.4) which links the fixed points located on the border lines $\tau = -1$ and $\tau = 1$, respectively, and has the U -coordinate being a monotone increasing function of z .

Now we study the existence of traveling wave solutions. Under the above conditions, the following facts hold.

Proposition 4.2.1. *If $u < u_{\text{syn}}$, then the curve C_2 is located above C_1 .*

Proof. Take $u_0 < u_{\text{syn}}$. If $f(u_0, w_0) = 0$, then $f(u_0, w_0) - A(u_0 - u_{\text{syn}}) > 0$. If there is w_1 such that $f(u_0, w_1) - A(u_0 - u_{\text{syn}}) = 0$ then $w_1 > w_0$ because $f_w < 0$. \square

Proposition 4.2.2. *If $f(u_1^*, w_1^-) = f(u_0, w_1^-) - A(u_0 - u_{\text{syn}}) = 0$, then $u_1^* < u_0$.*

Proof.

$$f(u_0, w_1^-) = A(u_0 - u_{\text{syn}}) < 0 = f(u_1^*, w_1^-).$$

Then $u_1^* < u_0$ because, in the right-hand side of the right branch of C_1 , $f(u, w) < 0$. \square

From these two propositions, we get the following inequalities:

$$w_1^- < w_1^+ < w_2^+.$$

If $\phi_1(w)$, $\phi_{11}(w)$, and $\phi_{12}(w)$ are the solutions of $f(u, w) = 0$, and $\phi_1(w) < \phi_{11}(w) < \phi_{12}(w)$, then $\phi_{12}(w) < \phi_2(w)$.

Theorem 4.2.3. *For given $w_1^- < w < w_2^+$ and $u_1^- < \theta < u_1^+$, there is a unique $v > 0$ such that system (4.4) has a unique heteroclinic orbit linking*

the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, passing through the initial point $\tau(0) = 0$, $U(0) = \theta$, and its U -coordinate is an increasing function of z , if $\phi_2(w) = U$ is the unique solution of equation $f(U, w) - A(U - u_{\text{syn}}) = 0$ and $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) > 0$ for $U \leq \theta$.

Proof. The fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$ exist because $\phi_1(w)$ and $\phi_2(w)$ are defined such that $f(\phi_1(w), w) = 0$ and $f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}}) = 0$.

At the fixed point $(-1, \phi_1(w))$ the Jacobian of (4.4) is

$$\begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{v}f_u(\phi_1(w), w) \end{bmatrix}.$$

Since $2k > 0$ and $\frac{1}{v}f_u(\phi_1(w), w) < 0$, the fixed point $(-1, \phi_1(w))$ is a saddle point. For the eigenvalue $2k$, the eigenvector is $(1, 0)$. The unstable manifold, denoted by Γ , of $(-1, \phi_1(w))$ is tangent to the vector $(1, 0)$. The manifold Γ must stay above the horizontal line $U = \phi_1(w)$ because the vector field defined by (4.4) points in the top-right direction for $-1 < \tau < 1$, $U \leq \phi_1(w)$.

Consider the rectangle

$$R = \{(\tau, U); -1 \leq \tau \leq 1, \phi_1(w) \leq U \leq \phi_2(w)\}. \quad (4.5)$$

On the lines $\tau = -1$ and $\tau = 1$, $\tau' = k(1 - \tau^2) = 0$; hence the vector field defined by (4.4) is straight up or straight down, respectively.

On the line $U = \phi_1(w)$, the vector field defined by system (4.4) points in the top-right direction because

$$\begin{aligned} \frac{d\tau}{dz} &= k(1 - \tau^2) \geq 0, \\ \frac{dU}{dz} &= \frac{1}{v}[f(U, w) - A(U - u_{\text{syn}})I(\tau)] = -\frac{1}{v}A(\phi_1(w) - u_{\text{syn}})I(\tau) \geq 0. \end{aligned}$$

The equality in the first line holds for $\tau = \pm 1$, but the equality in the second line holds only for $\tau = -1$. On the line $U = \phi_2(w)$, the vector field defined by system (4.4) points in the bottom-right direction because

$$\begin{aligned}\frac{d\tau}{dz} &= k(1 - \tau^2) \geq 0, \\ \frac{dU}{dz} &= \frac{1}{v}[f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})I(\tau)] \\ &\leq \frac{1}{v}[f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})] = 0.\end{aligned}$$

The first equality holds for $\tau = \pm 1$, but the second one holds only for $\tau = -1$. Thus, the rectangle R is a positive invariant set for system (4.4). It is easily seen that $(1, \phi_2(w))$ is a stable fixed point. The unstable manifold Γ must approach the fixed point $(1, \phi_2(w))$. This heteroclinic orbit exists uniquely for any $v > 0$ because the unstable manifold of the fixed point $(-1, \phi_1(w))$ is unique.

Consider the U -nullcline of system (4.4)

$$\Sigma = \{(\tau, U) \in D; f(U, w) - A(U - u_{\text{syn}})I(\tau) = 0\}.$$

The orbit starting from any point on Σ will go into the left side as z decreases because, at this point, the tangent vector is $(k(1 - \tau^2) > 0, 0)$. In the left side of Σ

$$f(U, w) - A(U - u_{\text{syn}})I(\tau) < 0.$$

The orbits on the left side of Σ go in the top-left direction as z decreases and cannot approach the fixed point $(-1, \phi_1(w))$. If the unstable manifold Γ contacts with Σ , then it must go into the left side of Σ as z decreases. Thus, Γ must be on the right side of Σ . On the right side of Σ ,

$$f(U, w) - A(U - u_{\text{syn}})I(\tau) > 0,$$

which implies that the U -coordinate of Γ is a monotone increasing function of z .

Lemma 4.2.4. *The heteroclinic orbit $\Gamma(v) = \{\tau(z), U(z)\}$ defines $U = U(\tau, v)$ as a monotone increasing function of τ . If $v_1 < v_2$, then sufficiently and necessarily $U(\tau; v_2) < U(\tau; v_1)$ for $-1 < \tau < 1$.*

Proof. For arbitrary $v > 0$, we denote the heteroclinic orbit of (4.4) as $\Gamma(v) = \{\tau(z, v), U(z, v)\}$. Since

$$\frac{dU}{d\tau} = \frac{1}{v} \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{k(1 - \tau^2)} > 0,$$

$\Gamma(v)$ defines U as an increasing function of τ , denoted by $U(\tau, v)$. We claim that $\Gamma(v_1)$ and $\Gamma(v_2)$ cannot meet at any point for any $v_1 < v_2$. If $U(\tau_0, v_2) = U(\tau_0, v_1)$, then

$$\frac{\frac{dU}{d\tau}(\tau_0, v_1)}{\frac{dU}{d\tau}(\tau_0, v_2)} = \frac{v_2}{v_1} > 1,$$

which means that $U(\tau, v_1) < U(\tau, v_2)$ and $\Gamma(v_1)$ is below $\Gamma(v_2)$ in a neighborhood of τ_0 with $\tau < \tau_0$. Both $\Gamma(v_1)$ and $\Gamma(v_2)$ approach the fixed point $(-1, \phi_1(w))$ as $z \rightarrow -\infty$. Let (τ_1, U_1) be a point satisfying the inequalities

$$\tau_1 < \tau_0, \quad U(\tau_1, v_1) < U_1 < U(\tau_1, v_2).$$

Let $O(z, \tau_1, U_1, v_1)$ denote the orbit of (4.4) with $v = v_1$ and the initial value set to (τ_1, U_1) at $z = 0$. As $z \rightarrow -\infty$, $O(z, \tau_1, U_1, v_1)$ will penetrate through $\Gamma(v_2)$ with a smaller slope (from below) at some point (τ_2, U_2) . If we compare slopes at this point, we will have

$$\frac{\frac{dU}{d\tau}(\tau_2, v_1)}{\frac{dU}{d\tau}(\tau_2, v_2)} = \frac{v_2}{v_1} < 1,$$

which is a contradiction. At this stage, we have proved that, provided $v_1 < v_2$ and $U(\tau_0, v_1) \leq U(\tau_0, v_2)$ for only one value of τ_0 , we shall have a contradiction. Thus, the inequality $U(\tau, v_2) < U(\tau, v_1)$ must hold for $-1 < \tau < 1$ provided $v_1 < v_2$.

Conversely, if $U(\tau, v_2) < U(\tau, v_1)$ for $-1 < \tau < 1$ and $v_1 = v_2$, then the uniqueness of an unstable manifold is violated. If $v_1 > v_2$, then, by what we just proved, we have $U(\tau, v_1) < U(\tau, v_2)$. This contradicts the precondition. Thus $v_1 < v_2$. \square

The uniqueness of v such that $U(0, v) = \theta$ is obvious because, if there are $v_1 < v_2$ such that $U(0, v_1) = \theta = U(0, v_2)$, then this is a contradiction to Lemma 4.2.4.

We prove the existence of a v such that $U(0, v) = \theta$ in four steps.

Step 1. $P = \{v > 0; U(0, v) > \theta\} \neq \emptyset$, $Q = \{v > 0; U(0, v) < \theta\} \neq \emptyset$, and $P \cap Q = \emptyset$.

Since $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) > 0$ for $U \leq \theta$, there is some $\delta > 0$ such that

$$f(U, w) - A(U - u_{\text{syn}})I(\tau) > 0$$

on the closed rectangle

$$R_1 = \{(\tau, U); -\delta \leq \tau \leq 0, \phi_1(w) \leq U \leq \theta\}.$$

Let

$$m = \min_{R_1} \{f(U, w) - A(U - u_{\text{syn}})I(\tau)\} > 0.$$

Take $v_0 > 0$ such that

$$\frac{\theta - \phi_1(w)}{\delta} < \frac{m}{kv_0} \leq \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0k(1 - \tau^2)},$$

for $(\tau, U) \in R_1$. Let $O(z, -\delta, \phi_1(w), v_0)$ be the orbit of (4.4) with $v = v_0$ starting at point $(-\delta, \phi_1(w))$ when $z = 0$. This orbit intersects the vertical line $\tau = 0$ at a point with $U(z_0) > \theta$ because

$$\begin{aligned} U(\tau = 0) &= \phi_1(w) + \int_{-\delta}^0 \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0 k(1 - \tau^2)} d\tau \\ &> \phi_1(w) + \frac{\theta - \phi_1(w)}{\delta} \delta = \theta. \end{aligned}$$

The heteroclinic orbit $\Gamma(v_0)$ goes above $O(z, -\delta, \phi_1(w), v_0)$ as z increases, hence we have $U(0, v_0) > \theta$ and $v_0 \in P$.

Let $\delta > 0$ be small enough such that the point $(-\sqrt{1 - \delta}, U_0)$ be on the U -nullcline Σ and $U_0 < \frac{\theta + \phi_1(w)}{2}$. This point certainly exists because Σ is defined in the neighborhood of $(-1, \phi_1(w))$. Let

$$R_2 = \{(\tau, U); -1 \leq \tau \leq 0, \phi_1(w) \leq U \leq \theta\},$$

and

$$M = \max_{R_2} \{f(U, w) - A(U - u_{\text{syn}})I(\tau)\}.$$

Take $v_1 > 0$ such that

$$\frac{\theta - \phi_1(w)}{2} > \frac{M}{kv_1\delta} \geq \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_1 k(1 - \tau^2)},$$

for $(\tau, U) \in R_2$ and $\tau > -\sqrt{1 - \delta}$. The orbit $O(z, -\sqrt{1 - \delta}, U_0, v_1)$ of system (4.4) with $v = v_1$, starting at $(-\sqrt{1 - \delta}, U_0)$ when $z = 0$, will intersect the vertical line $\tau = 0$ at a point with $U(z_0) < \theta$ because

$$\begin{aligned} U(\tau = 0) &= U_0 + \int_{-\sqrt{1 - \delta}}^0 \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_1 k(1 - \tau^2)} d\tau \\ &< \frac{\theta + \phi_1(w)}{2} + \frac{\theta - \phi_1(w)}{2} = \theta. \end{aligned}$$

As z increases, the heteroclinic orbit $\Gamma(v_1)$ will stay below $O(z, -\sqrt{1-\delta}, U_0, v_1)$. Thus $U(0, v_1) < \theta$ and $v_1 \in Q$. From the definitions of P and Q it is clear that $P \cap Q = \emptyset$.

Step 2 Any $p \in P$ is a lower bound of the set Q . Any $q \in Q$ is an upper bound of P .

We have $p \neq q$ because $P \cap Q = \emptyset$. By definition, $U(0, p) > \theta$ and $U(0, q) < \theta$. If $p = q$ then $U(0, p) = U(0, q)$ by the uniqueness of unstable manifold. This is also a contradiction. Thus, $p < q$ for any $p \in P$ and $q \in Q$.

Step 3 Let $\omega = \sup(P)$ and $\Omega = \inf(Q)$, then $\omega = \Omega$.

According to Step 2, we have $\omega = \sup(P) \leq \Omega = \inf(Q)$. If $\omega < \Omega$, then there exist v_1 and v_2 such that $\omega < v_1 < v_2 < \Omega$. If $U(0, v_1) > \theta$ then this contradicts the definition of ω . If $U(0, v_1) < \theta$, then this contradicts the definition of Ω . So the only possibility is $U(0, v_1) = U(0, v_2) = \theta$, but this is a contradiction to Lemma 4.2.4. Thus, we proved $\omega = \Omega$.

Step 4 $U(0, \omega) = \theta$.

If $U(0, \omega) < \theta$, then the orbit $O(z, 0, \theta, \omega)$ of system (4.4) with $v = \omega$, starting at $(0, \theta)$ when $z = 0$, will intersect the U -nullcline Σ at some point $(\tau(z_0), U(z_0)) \in \Sigma$, $z_0 < 0$ as $z \rightarrow -\infty$. By the continuous dependence of solution on the parameters of ordinary differential equations, there is $v_1 < \omega$ such that the orbit $O(z, 0, \theta, v_1)$ of system (4.4) with $v = v_1$, starting at $(0, \theta)$ when $z = 0$, will intersect Σ at some point $(\tau(z_1), U(z_1)) \in \Sigma$, $z_1 \in [z_0 - 1, 0]$. Then the heteroclinic orbit $\Gamma(v_1)$ will stay below $O(z, 0, \theta, v_1)$, and we shall have $U(0, v_1) < \theta$ and $v_1 \in Q$. This contradicts the definition of $\omega = \inf(Q)$.

If $U(0, \omega) > \theta$, then the orbit $O(z, 0, \theta, \omega)$ of system (4.4) with $v = \omega$, starting at $(0, \theta)$ when $z = 0$, will intersect the horizontal line $U = \phi_1(w)$ at some point $(\tau(z_0), \phi_1(w))$ for $z_0 < 0$. We can find $v_2 > \omega$ such that the orbit $O(z, 0, \theta, v_2)$ of system (4.4) with $v = v_2$, starting at $(0, \theta)$ when $z = 0$, will intersect the horizontal line $U = \phi_1(w)$ at some point $(\tau(z_1), \phi_1(w))$, $z_1 \in [z_0 - 1, 0]$. The heteroclinic orbit $\Gamma(v_2)$ will stay above $O(z, 0, \theta, v_2)$, and we shall have $U(0, v_2) > \theta$ and $v_2 \in P$. This contradicts the definition $\omega = \sup(P)$.

At this stage we have proved that $U(0, \omega) = \theta$. □

Remark 4.2.5. The function v depends on w and θ . We denote $v = v(w, \theta)$.

Theorem 4.2.6. *If equation $f(U, w) - A(U - u_{\text{syn}}) = 0$ has three real solutions $\phi_{21}(w) \leq \phi_{22}(w) < \phi_2(w)$, then there is no heteroclinic orbit linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$ for any $v > 0$.*

Proof. On the horizontal line

$$L = \{(\tau, U); \tau \in (-1, 1), U = \phi_{22}(w)\},$$

the vector field points in the bottom-right direction because

$$k(1 - \tau) > 0$$

and

$$f(\phi_{22}(w), w) - A(\phi_{22}(w) - u_{\text{syn}})I(\tau) < f(\phi_{22}(w), w) - A(\phi_{22}(w) - u_{\text{syn}}) = 0.$$

Hence no orbit of system (4.4), for any $v > 0$, can penetrate L from below into the top and get to the fixed point $(1, \phi_2(w))$ as z increases. □

Theorem 4.2.7. *If $f(U_0, w) - \frac{1}{2}A(U_0 - u_{\text{syn}}) \leq 0$ for $U_0 \leq \theta$, then there is no heteroclinic orbit passing through the point $(0, \theta)$ and linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$ with increasing U -coordinate as z increases for any $v > 0$.*

Proof. Let $U_0 \leq \theta$ and $f(U_0, w) + \frac{1}{2}A(U_0, u_{\text{syn}}) = 0$. We consider the orbit $O(z, 0, U_0) = (\tau(z), U(z))$ starting from $(0, U_0)$ at $z = 0$. $O(z, 0, U_0)$ has tangent vector $(k, 0)$ at point $(0, U_0)$. Σ has tangent vector $(f_u(U_0, w) - \frac{1}{2}A, A(U_0 - u_{\text{syn}})I'(0) \neq 0)$. $O(z, 0, U_0)$ will enter the left side of Σ and its U -coordinate will increase as z decreases. Thus, if $U_0 < \theta$, then the orbit passing through $(0, \theta)$ cannot penetrate $O(z, 0, U_0)$ and approach $(-1, \phi_1(w))$.

If $U_0 = \theta$, then $O(z, 0, \theta)$ will increase as z decreases, and will not approach $(-1, \phi_1(w))$.

If $U_0 \leq \theta$ and $f(U_0, w) + \frac{1}{2}A(U_0 - u_{\text{syn}}) < 0$, then on the line

$$L_0 = \{(\tau_0, U_0); -1 < \tau \leq 0\}$$

the vector field is in the bottom-right direction. No orbit can penetrate L_0 into the top-right direction and approach $(-1, \phi_1(w))$ as z decreases. \square

Summing up what have been proved by Theorems 4.2.3, 4.2.6, and 4.2.7 we get

Corollary 4.2.8. *For given $w_1^- < w < w_2^+$ and $u_1^- < \theta < u_1^+$, there is a unique $v > 0$ such that system (4.4) has a unique heteroclinic orbit, linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, passing through the initial point $\tau(0) = 0$, $U(0) = \theta$, and with increasing U -coordinate as a function of z , if and only if $U = \phi_2(w)$ is the unique solution of the equation*

$$f(U, w) - A(U - u_{\text{syn}}) = 0,$$

and

$$f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) > 0 \quad \text{for } U \leq \theta.$$

Theorem 4.2.9. *If $\theta_1 < \theta_2$ and $v_1 = v(w, \theta_1)$, $v_2 = v(w, \theta_2)$ exist, then $v_1 > v_2$.*

Proof. By Lemma 4.2.4, the functions $U(\tau, v_1)$ and $U(\tau, v_2)$ are well defined. If $v_1 = v_2$ then the uniqueness of the heteroclinic orbit is violated. If $v_1 < v_2$ then $U(\tau, v_2) < U(\tau, v_1)$ by Lemma 4.2.4. This means that $\theta_2 = U(0, v_2) < U(0, v_1) = \theta_1$, which is a contradiction. Thus, $v_1 > v_2$. \square

Theorem 4.2.10. *Given $w_1 < w_2$, if $v_1 = v(w_1, \theta)$ and $v_2 = v(w_2, \theta)$ exist, then $v_1 > v_2$.*

Proof. At the point $(0, \theta)$ the heteroclinic orbit $\Gamma(v_1)$ has the slope

$$s_1 = \frac{f(\theta, w_1) - \frac{1}{2}(\theta - u_{\text{syn}})}{v_1 k},$$

and the heteroclinic orbit $\Gamma(v_2)$ has the slope

$$s_2 = \frac{f(\theta, w_2) - \frac{1}{2}(\theta - u_{\text{syn}})}{v_2 k}.$$

If $v_1 \leq v_2$, then $s_1 > s_2 > 0$ because

$$f(\theta, w_1) - \frac{1}{2}A(\theta - u_{\text{syn}}) > f(\theta, w_2) - \frac{1}{2}A(\theta - u_{\text{syn}}) > 0$$

due to $f_w < 0$. For $\tau < 0$ and close to 0, $\Gamma(v_1)$ is below $\Gamma(v_2)$. As τ goes to -1 , $\Gamma(v_1)$ will approach the point $(-1, \phi_1(w_1))$ and $\Gamma(v_2)$ will approach the point $(-1, \phi_1(w_2))$. Since $\phi_1(w_1) > \phi_1(w_2)$, $\Gamma(v_1)$ must penetrate $\Gamma(v_2)$ from below at some point, say (τ_0, U_0) . If the two orbits intersected like that, at that point their slopes would satisfy the inequality

$$\frac{f(U_0, w_1) - (U_0 - u_{\text{syn}})I(\tau_0)}{v_1 k(1 - \tau_0^2)} < \frac{f(U_0, w_2) - (U_0 - u_{\text{syn}})I(\tau_0)}{v_2 k(1 - \tau_0^2)},$$

which is a false inequality. Thus, we have $v_1 > v_2$. \square

4.3 Monotone decreasing traveling wave solutions

Now we consider monotone decreasing traveling wave solutions. Let $u(t, x) = u(x + vt) = U(z)$, $z = x + vt$, $v > 0$ be a monotone decreasing function of z , and $U(0) = \theta$. If $u(x + vt)$ is a solution of (4.2) then it should satisfy the equation

$$v \frac{\partial U}{\partial z} = f(U, w) - A(U - u_{\text{syn}}) \int_z^{+\infty} K(\xi) d\xi \quad (4.6)$$

and the boundary conditions

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = l_-(w), \quad \lim_{z \rightarrow +\infty} U(z) = l_+(w), \quad l_-(w) > l_+(w).$$

Transformation (4.3) transforms equation (4.6) into the system

$$\begin{aligned} \frac{d\tau}{dz} &= k(1 - \tau^2), & \tau(0) &= 0, \\ v \frac{dU}{dz} &= f(U, w) - A(U - u_{\text{syn}})I(\tau), & U(0) &= \theta. \end{aligned} \quad (4.7)$$

Here $I(\tau) = \int_{\tau}^1 \frac{1}{k(1-\eta^2)} K\left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta}\right) d\eta$, and $I(\tau)$ satisfies $I(\tau) \geq 0$, $I(-1) = 1$, $I(0) = 1/2$, $I(1) = 0$ and $I(\tau)$ is monotone decreasing. System (4.7) is defined in the same domain as system (4.4). We search for the heteroclinic orbit of this system which links the fixed points located on the border lines $\tau = 1$ and $\tau = -1$, respectively, and has the U -coordinate being a monotone decreasing function of z .

Theorem 4.3.1. *Given $u_1^- < \theta < u_1^+$ and $w_1^- < w < w_2^+$, if the equation $f(U, w) = 0$ has only one solution $U = \phi_1(w)$ and $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) < 0$*

for $U > \theta$, then there is unique $v = v(\theta, w) > 0$ such that system (4.7) has a unique heteroclinic orbit $\Gamma = (\tau(z), U(z))$, which links the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$, goes through the initial point $(0, \theta)$, and $U(z)$ is a monotone decreasing function of z .

Proof. Although the proof of this theorem is similar to the proof of Theorem 3.3.1, for completeness we write down the proof in detail.

(a) The fixed points.

Since $w_1^- < w < w_2^+$, the points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ are well defined. $(1, \phi_1(w))$ is the unique fixed point on the line $\tau = 1$. At $(-1, \phi_2(w))$ the Jacobian is

$$\begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{v} f_U(\phi_2(w), w) \end{bmatrix}$$

This fixed point is a saddle point. Its unstable manifold $\Gamma = \{\tau(z), U(z)\}$ has tangent vector $(1, 0)$. Γ is inside the domain D below line $U = \phi_2(w)$ because

$$f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})I(\tau) \leq f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})I(-1) = 0$$

$$\text{and } f(U, w) + A(U - u_{\text{syn}})I(\tau) \leq f(U, w) + A(U - u_{\text{syn}}) < 0 \text{ for } U > \phi_2(w).$$

The equality holds only for $\tau = -1$.

The fixed point $(1, \phi_1(w))$ is a node since its Jacobian

$$\begin{bmatrix} -2k & 0 \\ 0 & \frac{1}{v} f_U(\phi_1(w), w) \end{bmatrix}$$

has two negative eigenvalues.

(b) The invariant set.

The rectangle R defined by (4.5) is a positive invariant set. On the vertical lines $\tau = 1$ the vector field defined by system (4.7) is parallel to the vector $(0, 1)$. On the line $U = \phi_1(w)$, the vector field points in the top-right direction because

$$\begin{aligned}\frac{d\tau}{dz} &= k(1 - \tau^2) \geq 0, \\ \frac{dU}{dz} &= \frac{1}{v}[f(\phi_1(w), w) - A(\phi_1(w) - u_{\text{syn}})I(\tau)] \\ &= \frac{1}{v}[-A(\phi_1(w) - u_{\text{syn}})I(\tau)] \geq 0.\end{aligned}$$

The first equality holds for $\tau = \pm 1$ and the second equality holds only for $\tau = -1$. On the line $U = \phi_2(w)$, the vector field points in the bottom-right direction because

$$\begin{aligned}\frac{d\tau}{dz} &= k(1 - \tau^2) \geq 0, \\ \frac{dU}{dz} &= \frac{1}{v}[f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})I(\tau)] \\ &\leq \frac{1}{v}[f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}})I(-1)] = 0.\end{aligned}$$

The first equality holds for $\tau = \pm 1$ and the second equality holds only for $\tau = -1$. Thus, the unstable manifold Γ must go to the right border of R and approach the unique fixed point $(1, \phi_1(w))$. This heteroclinic orbit exists for any $v > 0$ and is unique because the unstable manifold of $(-1, \phi_2(w))$ is the unique orbit approaching it.

(c) $U(z)$ is a decreasing function of z .

Consider the U -nullcline of system (4.7)

$$\Sigma = \{(\tau, U) \in R; f(U, w) - A(U - u_{\text{syn}})I(\tau) = 0\}.$$

It has no horizontal tangent for $\tau > -1$ because $I'(\tau) \neq 0$. On the left side of Σ , $f(U, w) - A(U - u_{\text{syn}})I(\tau) > 0$, the orbits of system (4.7) go in the bottom-left direction as z decreases. If the heteroclinic Γ contacts with Σ then Γ must go into the left side of Σ as z decreases. Thus, Γ must be on the right side of Σ and approach the fixed point $(1, \phi_1(w))$ as $z \rightarrow \infty$. On the right side of Σ

$$f(U, w) - A(U - u_{\text{syn}})I(\tau) < 0,$$

and $U(z)$ is a monotone decreasing function of z .

(d) The heteroclinic orbit $\Gamma(v) = \{\tau(z), U(z)\}$.

Lemma 4.3.2. *If the heteroclinic orbit $\Gamma(v) = \{\tau(z), U(z)\}$ exists, then it defines $U = U(\tau, v)$ as a monotone decreasing function of τ for $v > 0$. If $v_1 < v_2$, then sufficiently and necessarily, $U(\tau, v_1) < U(\tau, v_2)$, $-1 < \tau < 1$.*

Proof. For given $v > 0$, we denote the heteroclinic Γ as $\Gamma(v) = \{\tau(z, v), U(z, v)\}$.

Since

$$\frac{dU}{d\tau} = \frac{1}{v} \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{k(1 - \tau^2)} < 0,$$

$\Gamma(v)$ defines U as a decreasing function of τ , denoted by $U(\tau, v)$. We claim that $\Gamma(v_1)$ and $\Gamma(v_2)$ cannot meet at any point for any $v_1 < v_2$. If $U(\tau_0, v_2) = U(\tau_0, v_1)$ then

$$\frac{\frac{dU(\tau_0, v_1)}{d\tau}}{\frac{dU(\tau_0, v_2)}{d\tau}} = \frac{v_2}{v_1} > 1,$$

which means that $U(\tau, v_1) > U(\tau, v_2)$ for $\tau < \tau_0$, and $\Gamma(v_1)$ is above $\Gamma(v_2)$ in the vicinity of τ_0 with $\tau < \tau_0$. Both $\Gamma(v_1)$ and $\Gamma(v_2)$ approach the fixed point $(-1, \phi_2(w))$ as $z \rightarrow -\infty$. Let (τ_1, U_1) be a point satisfying the inequalities

$$\tau_1 < \tau_0, \quad U(\tau_1, v_1) > U_1 > U(\tau_1, v_2).$$

The orbit $O(z, \tau_1, U_1, v_1)$, starting at (τ_1, U_1) when $z = 0$, of system (4.7) with $v = v_1$ will penetrate through $\Gamma(v_2)$ with bigger (negative) slope (from above) at some point (τ_2, U_2) as $z \rightarrow -\infty$. At this point we still have

$$\frac{\frac{dU(\tau_2, v_1)}{d\tau}}{\frac{dU(\tau_2, v_2)}{d\tau}} = \frac{v_2}{v_1} > 1,$$

This is a contradiction. At this stage, we have proved that, provided $v_1 < v_2$ and $U(\tau_1, v_1) \geq U(\tau_1, v_2)$ for only one value of τ_1 we shall have a contradiction.. Thus, we must have

$$U(\tau, v_2) > U(\tau, v_1)$$

for $-1 < \tau < 1$, provided $v_2 > v_1$. □

(e) The uniqueness of v such that $U(0, v) = \theta$.

If there are $v_1 < v_2$ such that $U(0, v_1) = \theta = U(0, v_2)$ then this is a contradiction to Lemma 4.3.2.

(f) The existence of v such that $U(0, v) = \theta$.

We prove this statement in four steps.

Step 1. Suppose $P = \{v > 0; U(0, v) > \theta\} \neq \emptyset$, $Q = \{v > 0; U(0, V) < \theta\} \neq \emptyset$, and $P \cap Q = \emptyset$. Since $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) < 0$ for $U > \theta$, there is $\delta > 0$ such that

$$M = \max_R \{f(U, w) - A(U - u_{\text{syn}})I(\tau)\} < 0$$

on the closed rectangle

$$R = \{(\tau, U); -\delta \leq \tau \leq 0, \theta \leq U \leq \phi_2(w)\}.$$

Take $q = v_0 > 0$ small enough such that

$$0 > \frac{\theta - \phi_2(w)}{\delta} > \frac{M}{kv_0} \geq \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0k(1 - \tau^2)}$$

for $(\tau, U) \in R$. Let $O(z, -\delta, \phi_2(w), v_0)$ be the orbit of (4.7) with $v = v_0$ starting at the point $(-\delta, \phi_2(w))$ when $z = 0$. This orbit intersects the vertical line $\tau = 0$ at a point with $U(z_0) < \theta$ since

$$\begin{aligned} U(\tau = 0) &= \phi_2(w) + \int_{-\delta}^0 \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0k(1 - \tau^2)} d\tau \\ &< \phi_2(w) + \frac{\theta - \phi_2(w)}{\delta} \delta = \theta. \end{aligned}$$

The heteroclinic orbit, $\Gamma(v_0)$, of system (4.7) decreases below $O(z, -\delta, \phi_2(w), v_0)$ as z increases. So we must have $U(0, v_0) < \theta$ and $v_0 \in Q$.

Let $\delta > 0$ be small enough such that the point $(-\sqrt{1 - \delta}, U_0)$ be on the U -nullcline Σ and $U_0 > \frac{\theta + \phi_2(w)}{2}$. This point certainly exists because Σ is defined in the neighborhood of $(-1, \phi_2(w))$. Let

$$\Lambda = \{(\tau, U); -1 \leq \tau \leq 0, \theta \leq U \leq \phi_2(w)\}$$

and

$$m = \min_{\Lambda} \{f(U, w) - A(U - u_{\text{syn}})I(\tau)\} < 0.$$

Take $v_0 > 0$ large enough such that

$$\frac{\theta - \phi_2(w)}{2} < \frac{m}{kv_0\delta} \leq \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0k(1 - \tau^2)}$$

for $(\tau, U) \in \Lambda$ and $\tau > -\sqrt{1 - \delta}$. The orbit $O(z, -\sqrt{1 - \delta}, U_0, v_0)$ of system (4.7) with $v = v_0$, starting at $(-\sqrt{1 - \delta}, U_0)$ when $z = 0$, will intersect the vertical line $\tau = 0$ at a point with $U(z_0) > \theta$, $z_0 > 0$, since

$$\begin{aligned} U(\tau = 0) &= U_0 + \int_{-\sqrt{1 - \delta}}^0 \frac{f(U, w) - A(U - u_{\text{syn}})I(\tau)}{v_0k(1 - \tau^2)} d\tau \\ &\geq U_0 + \frac{\theta - \phi_2(w)}{2} \sqrt{1 - \delta} > U_0 + \frac{\theta - \phi_2(w)}{2} > \theta. \end{aligned}$$

The heteroclinic orbit of system (4.7) with $v = v_0$ goes above $O(z, -\sqrt{1-\delta}, U_0, v_0)$ as z increases. Thus, $U(0, v_0) > \theta$ and $v_0 \in P$. From the definitions of P and Q it is clear that $P \cap Q = \emptyset$.

Step 2 Any $p \in P$ is an upper bound of the set Q . Any $q \in Q$ is a lower bound of P .

By the definitions of P and Q we have $U(0, p) > \theta$ and $U(0, q) < \theta$. If $q > p$ then $U(0, p) < U(0, q)$ by Lemma 4.3.2. This is a contradiction. If $p = q$ then $P \cap Q \neq \emptyset$. This is also a contradiction. Thus, $p > q$ for any $p \in P$ and $q \in Q$.

Step 3 If $\Omega = \inf(P)$ and $\omega = \sup(Q)$, then $\omega = \Omega$.

By Step 2, we have $\omega = \sup(Q) \leq \Omega = \inf(P)$. If $\omega < \Omega$, then $\omega < v_1 < v_2 < \Omega$. If $U(0, v_1) > \theta$ then this contradicts the definition of ω . If $U(0, v_1) < \theta$, this is a contradiction to the definition of Ω . The only possibility is $U(0, v_1) = U(0, v_2) = \theta$. This is a contradiction to Lemma 4.3.2. Thus, we have proved that $\omega = \Omega$.

Step 4 $U(0, \omega) = \theta$.

If $U(0, \omega) > \theta$, then the orbit $O(z, 0, \theta, \omega)$ of system (4.7) with $v = \omega$, starting at $(0, \theta)$ when $z = 0$ will intersect Σ at some point $(\tau(z_0), U(z_0)) \in \Sigma$, $z_0 < 0$, as $z \rightarrow -\infty$. By the continuous dependence of solutions on the parameters of ordinary differential equations, there is $v' < \omega$ such that the orbit $O(z, 0, \theta, v')$ of system (4.7) with $v = v'$, starting at $(0, \theta)$ when $z = 0$, will intersect Σ at some point $(\tau(z_1), U(z_1)) \in \Sigma$, $z_1 \in [z_0 - 1, 0]$. The heteroclinic orbit $\Gamma(v')$ must satisfy $U(0, v') > \theta$ and $v' \in P$. This contradicts

the definition of $\omega = \Omega = \inf(P)$.

If $U(0, \omega) < \theta$, then the orbit $O(z, 0, \theta, \omega)$ of system (4.7) with $v = \omega$, starting at $(0, \theta)$ when $z = 0$ will intersect the horizontal line $U = \phi_2(w)$ at some point $(\tau(z_0), \phi_2(w))$, $z_0 < 0$ as $z \rightarrow -\infty$. We can find $v'' > \omega$ such that the orbit $O(z, 0, \theta, v'')$ of system (4.7) with $v = v''$, starting at $(0, \theta)$ when $z = 0$ will intersect the horizontal line $U = \phi_2(w)$ at some point $(\tau(z_1), \phi_2(w))$, $z_1 \in [z_0 - 1, 0]$. The heteroclinic orbit $\Gamma(v'')$ must have $U(0, v'') < \theta$ and $v'' \in Q$. This contradicts the definition $\omega = \sup(Q)$.

At this stage we have proved $U(0, \omega) = \theta$. □

Theorem 4.3.3. *If equation $f(U, w) = 0$ has three solutions $\phi_1(w) < \phi_{11}(w) \leq \phi_{12}(w)$, then there is no heteroclinic orbit linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ for any $v > 0$.*

Proof. Consider the line segment

$$L = \{(\tau, \phi_{11}(w)); -1 < \tau < 1\}.$$

On L , the vector field is in the top-right direction because

$$f(\phi_{11}(w), w) - A(\phi_{11}(w) - u_{\text{syn}})I(\tau) = -A(\phi_{11}(w) - u_{\text{syn}})I(\tau) > 0.$$

Thus, no orbit can penetrate L in the bottom-right direction to approach $(1, \phi_1(w))$ as z increases. □

Theorem 4.3.4. *If $f(U_0, w) - \frac{1}{2}A(U_0 - u_{\text{syn}}) \geq 0$ and $U_0 \geq \theta$, then there are no heteroclinic orbits passing through the point $(0, \theta)$ and linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ for any $v > 0$.*

Proof. Let $U_0 \geq \theta$ and $f(U_0, w) - \frac{1}{2}A(U_0, u_{\text{syn}}) = 0$. We consider the orbit $O(z, 0, U_0) = (\tau(z), U(z))$ starting from $(0, U_0)$ at $z = 0$. Then the vector $(k, 0)$ is tangent to $O(z, 0, U_0)$ at the point $(0, U_0)$. Σ has tangent vector

$$(f_u(U_0, w) - \frac{1}{2}A, A(U_0 - u_{\text{syn}})I'(0) \neq 0).$$

$O(z, 0, U_0)$ will go into the left side of Σ and its U -coordinate will decrease as z decreases. Thus, if $U_0 > \theta$, then the orbit passing through $(0, \theta)$ cannot penetrate $O(z, 0, U_0)$ to approach $(-1, \phi_2(w))$. If $U_0 = \theta$, then $O(z, 0, \theta)$ will decrease as z decreases, and will not approach $(-1, \phi_2(w))$.

If $U_0 \leq \theta$ and $f(U_0, w) + \frac{1}{2}A(U_0 - u_{\text{syn}}) > 0$, then, on the line

$$L_0 = \{(\tau_0, U_0); -1 < \tau \leq 0\},$$

the vector field is in the top-right direction. No orbit can penetrate L_0 in the top-left direction and approach $(-1, \phi_2(w))$ as z decreases. \square

Corollary 4.3.5. *For given $w_1^- < w < w_2^+$, and $u_1^- < \theta < u_1^+$, there is a unique $v > 0$ such that system (4.7) has a unique heteroclinic orbit, linking the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$, passing through the initial point $\tau(0) = 0$, $U(0) = \theta$, and its U -coordinate being a decreasing function of z , if and only if $U = \phi_1(w)$ is the unique solution of the equation*

$$f(U, w) = 0$$

and

$$f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) < 0 \quad \text{for } U \geq \theta.$$

Theorem 4.3.6. *Given $\theta_1 < \theta_2$, if $v_1 = v(w, \theta_1)$ and $v_2 = v(w, \theta_2)$ exist, then $v_1 < v_2$.*

Proof. By Lemma 4.3.2, the functions $U(\tau, v_1)$ and $U(\tau, v_2)$ are well defined. If $v_1 = v_2$ then the uniqueness of the heteroclinic orbit will be violated. If $v_1 > v_2$, then $U(\tau, v_2) < U(\tau, v_1)$ again by Lemma 4.3.2. This means that $\theta_2 = U(0, v_2) < U(0, v_1) = \theta_1$, which is a contradiction. Thus, $v_1 < v_2$. \square

Theorem 4.3.7. *Given $w_1 < w_2$, if $v_1 = v(w_1, \theta)$ and $v_2 = v(w_2, \theta)$ exist, then $v_1 < v_2$.*

Proof. At the point $(0, \theta)$ the heteroclinic orbit $\Gamma(v_1)$ has slope

$$s_1 = \frac{f(\theta, w_1) - \frac{1}{2}(\theta - u_{\text{syn}})}{v_1 k},$$

and the heteroclinic orbit $\Gamma(v_2)$ has slope

$$s_2 = \frac{f(\theta, w_2) - \frac{1}{2}(\theta - u_{\text{syn}})}{v_2 k}.$$

If $v_1 \geq v_2$, then $0 > s_1 > s_2$ because

$$0 > f(\theta, w_1) - \frac{1}{2}A(\theta - u_{\text{syn}}) > f(\theta, w_2) - \frac{1}{2}A(\theta - u_{\text{syn}}).$$

For $\tau < 0$ and close to 0, $\Gamma(v_1)$ is below $\Gamma(v_2)$. As τ goes to -1 , $\Gamma(v_1)$ will approach the point $(-1, \phi_2(w_1))$, and $\Gamma(v_2)$ will approach the point $(-1, \phi_2(w_2))$. Since $f_u(\phi_2(w)) < 0$, we must have $\phi_2(w_1) > \phi_2(w_2)$, so $\Gamma(v_1)$ must penetrate $\Gamma(v_2)$ from below at some point, say (τ_0, U_0) . If the two orbits intersected like that, at that point the slope of the orbits would satisfy

$$\frac{f(\tau_0, w_1) - (U_0 - u_{\text{syn}})I(\tau)}{v_1 k(1 - \tau_0^2)} < \frac{f(\tau_0, w_2) - (U_0 - u_{\text{syn}})I(\tau)}{v_2 k(1 - \tau_0^2)},$$

which implies $v_1 > v_2$. This contradicts our assumption $v_1 < v_2$. \square

4.4 Symmetric traveling wave solutions

Now we consider the traveling wave solutions of system (4.1) propagating along the x -axis in the positive (right) direction. We guess this kind of solution exists since the architecture of the network is symmetric in space. The traveling wave solution, which describes the wave front, is a monotone decreasing function of the form

$$u(x, t) = U(x - vt) = U(z), \quad z = x - vt, \quad v > 0,$$

with boundary conditions

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = l_-(w), \quad \lim_{z \rightarrow \infty} U(z) = l_+(w), \quad l_-(w) > l_+(w).$$

Plugging $U(z)$ into (4.4) and noting that $U(z) > \theta$ for $z < 0$, we get

$$-v \frac{\partial U}{\partial z} = f(U, w) - A(U - u_{\text{syn}}) \int_z^{+\infty} K(\xi) d\xi.$$

This form of solution describes the wave front traveling along the x -axis in the positive (right) direction. By the transformation of variable (4.3) we obtain the autonomous system

$$\begin{aligned} \frac{d\tau}{dz} &= k(1 - \tau^2), & \tau(0) &= 0, \\ -v \frac{dU}{dz} &= f(U, w) - A(U - u_{\text{syn}}) \tilde{I}(\tau), & U(0) &= \theta. \end{aligned} \tag{4.8}$$

Here

$$\tilde{I}(\tau) = \int_{\tau}^1 K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) \frac{1}{k(1 - \eta^2)} d\eta.$$

It is easy to see from the properties of $K(x)$ that $\tilde{I}(\tau)$ is monotone decreasing, $\tilde{I}(\tau) \geq 0$, $\tilde{I}(-1) = 1$, $\tilde{I}(0) = 1/2$, and $\tilde{I}(1) = 0$. System (4.8) is defined in

the same domain as system (4.4). The traveling wave solution corresponds to a heteroclinic orbit of system (4.8) which links the fixed points located on the boundary lines $\tau = -1$ and $\tau = 1$, respectively, and has the U -coordinate being a monotone decreasing function of z . The fixed points of system (4.8) are the solutions of the equations

$$1 - \tau^2 = 0,$$

$$f(U, w) - A(U - u_{\text{syn}})\tilde{I}(\tau) = 0.$$

We search for a heteroclinic orbit of system (4.8) which is symmetric with respect to that of system (4.4) with the following properties:

- (1) linking $(-1, \phi_2(w))$ and $(1, \phi_1(w))$ as z goes from $-\infty$ to $+\infty$,
- (2) passing through the point $(0, \theta)$ when $z = 0$,
- (3) $U(z)$ is a monotone decreasing function of z .

Let

$$\sigma = -\tau, \quad s = -z.$$

The function $\tilde{I}(\tau)$ is transformed into

$$\begin{aligned} \tilde{I}(\tau) = \tilde{I}(-\sigma) &= \int_{-\sigma}^1 K \left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta} \right) \frac{1}{k(1-\eta^2)} d\eta \\ &= \int_{-1}^{\sigma} K \left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta} \right) \frac{1}{k(1-\eta^2)} d\eta \\ &= I(\sigma). \end{aligned}$$

Here the symmetry $K(x) = K(-x)$ and the definition of $I(\sigma)$ as in system (4.4) were employed. System (4.8) is transformed into

$$\begin{aligned} \frac{d\sigma}{ds} &= k(1 - \sigma^2), & \tau(0) &= 0, \\ v \frac{dU}{ds} &= f(U, w) - A(U - u_{\text{syn}})I(\sigma), & U(0) &= \theta. \end{aligned} \tag{4.9}$$

The fixed point $(-1, \phi_2(w))$ of system (4.8) now corresponds to $(1, \phi_2(w))$ and the fixed point $(1, \phi_1(w))$ corresponds to $(-1, \phi_1(w))$. The heteroclinic orbit of system (4.8) corresponds to the heteroclinic orbit of system (4.9) with the properties:

- (a) Linking the fixed point $(-1, \phi_1(w))$ and $(1, \phi_2(w))$.
- (b) Passing through the point $(\sigma = 0, U = \theta)$ when $s = 0$.
- (c) $U(s)$ being a monotone increasing function of s .

Now we face the same problem as that for system (4.4). All the results for system (4.4) can be adapted to system (4.8) in symmetric form. For example, in 4.2.3 the conditions for the existence of the heteroclinic orbit $\Gamma = \{\tau(z), U(z)\}$ with properties (a), (b), (c), now for system (4.4) are that the equation $f(U, w) - A(U - u_{\text{syn}}) = 0$ has a unique solution and $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) > 0$ for $U \leq \theta$. These conditions also ensure that there exists a heteroclinic orbit with properties (1), (2), (3) for system (4.8). We propose the following theorem, as the symmetric form of Theorem 4.2.3, for system (4.8) without proof.

Theorem 4.4.1. *For given $w_1^- < w < w_2^+$ and $u_1^- < \theta < u_1^+$, there is a unique $v > 0$ such that system (4.8) has a unique heteroclinic orbit which links the fixed points $(-1, \phi_2(w))$ and $(1, \phi_1(w))$, and passes through the initial point $\tau(0) = 0$, $U(0) = \theta$, and its U -coordinate is a decreasing function of z , if $\phi_2(w) = U$ is the unique solution of equation $f(U, w) - A(U - v_{\text{syn}}) = 0$ and inequation $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) > 0$ for $U \leq \theta$.*

Theorems 4.2.6, 4.2.7, 4.2.9, and 4.2.10 have their versions for system (4.8), respectively. We omit them.

The back-wave described by the traveling wave propagating along the positive (right) direction of the x -axis is in the form

$$u(x, t) = U(x - vt) = U(z), \quad z = x - vt, \quad v > 0,$$

where $U(z)$ is a monotone increasing function satisfying the boundary conditions

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = l_-(w), \quad \lim_{z \rightarrow \infty} U(z) = l_+(w), \quad l_-(w) < l_+(w).$$

If such solution exists, then it is the solution of the equation

$$-v \frac{\partial U}{\partial z} = f(U, w) - A(U - u_{\text{syn}}) \int_{-\infty}^z K(r) dr, \quad U(0) = \theta.$$

By transformation (4.3) we obtain the autonomous system

$$\begin{aligned} \frac{d\tau}{dz} &= k(1 - \tau^2), & \tau(0) &= 0, \\ -v \frac{dU}{dz} &= f(U, w) - A(U - u_{\text{syn}}) \tilde{I}(\tau), & U(0) &= \theta. \end{aligned} \tag{4.10}$$

Here

$$\tilde{I}(\tau) = \int_{-1}^{\tau} \frac{1}{k(1 - \eta^2)} K \left(\frac{1}{2k} \ln \frac{1 + \eta}{1 - \eta} \right) d\eta,$$

where $\tilde{I}(-1) = 0$, $\tilde{I}(0) = 1/2$, and $\tilde{I}(1) = 1$. System (4.10) is defined on the same domain as system (4.4). We search for the heteroclinic orbit of (4.10) which is symmetric to that of system (4.7) with the following properties:

(4) linking $(-1, \phi_1(w))$ and $(1, \phi_2(w))$ as z goes from $-\infty$ to $+\infty$, where

$$f(\phi_2(w), w) - A(\phi_2(w) - u_{\text{syn}}) = 0, \text{ and } f(\phi_1(w), w) = 0.$$

(5) passing through the point $(0, \theta)$ when $z = 0$.

(6) $U(z)$ being a monotone increasing function of z .

Let

$$\sigma = -\tau, s = -z.$$

The function $\tilde{I}(\tau)$ is transformed into

$$\begin{aligned} \tilde{I}(\tau) = \tilde{I}(-\sigma) &= \int_{-1}^{-\sigma} K \left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta} \right) \frac{1}{k(1-\eta^2)} d\eta \\ &= \int_{\sigma}^1 K \left(\frac{1}{2k} \ln \frac{1+\eta}{1-\eta} \right) \frac{1}{k(1-\eta^2)} d\eta \\ &= I(\sigma). \end{aligned}$$

Here the symmetry $K(x) = K(-x)$ and the definition of $I(\tau)$ as in system (4.7) were employed. System (4.10) is transformed into

$$\begin{aligned} \frac{d\sigma}{ds} &= k(1 - \sigma^2), & \sigma(0) &= 0 \\ v \frac{dU}{ds} &= f(U, w) - A(U - u_{\text{syn}})I(\sigma), & U(0) &= \theta \end{aligned} \quad (4.11)$$

The fixed point $(-1, \phi_1(w))$ of system (4.10) now corresponds to $(1, \phi_1(w))$ and the fixed point $(1, \phi_2(w))$ to $(-1, \phi_2(w))$. The heteroclinic orbit of system (4.10) corresponds to the heteroclinic orbit of system (4.11) with the following properties:

- (d) Linking the fixed point $(-1, \phi_2(w))$ and $(1, \phi_1(w))$.
- (e) Passing through the point $(\sigma = 0, U = \theta)$ when $s = 0$.
- (f) $U(s)$ being a monotone decreasing function of s .

Now we face the same problem as with system (4.7). All the results for system (4.7) can be adapted for system (4.10) in the symmetry form. For example, for the symmetric form of 4.3.1 we have the following theorem.

Theorem 4.4.2. *Given $u_1^- < \theta < u_1^+$ and $w_1^- < w < w_2^+$, if the equation $f(U, w) = 0$ has only one solution $U = \phi_1(w)$ and $f(U, w) - \frac{1}{2}A(U - u_{\text{syn}}) < 0$ for $U > \theta$, then there is a unique $v = v(\theta, w) > 0$ such that system (4.10) has a unique heteroclinic orbit $\Gamma = (\tau(z), U(z))$, linking the fixed points $(-1, \phi_1(w))$ and $(1, \phi_2(w))$, and passing through the initial point $(\tau(0), U(0))$, and $U(z)$ is a monotone increasing function of z .*

4.5 Summary

Equation (4.2) was studied under the assumptions A1–A6 . We sum up the main results briefly in the following theorems

Theorem 4.5.1. *For given $w_1^- < w < w_2^+$ and $u_1^- < \theta < u_1^+$, there is a unique $v = v(w, \theta) > 0$ such that system (4.4) of ordinary differential equations has a unique heteroclinic orbit $\Gamma(z) = (\tau(z), U(z))$, where $U(z)$ is a monotone increasing function of z , $\tau(0) = 0$, $U(0) = \theta$, and*

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tau(z) &= -1, & \lim_{z \rightarrow -\infty} U(z) &= \phi_1(w), \\ \lim_{z \rightarrow \infty} \tau(z) &= 1, & \lim_{z \rightarrow \infty} U(z) &= \phi_2(w), \end{aligned}$$

if and only if

$$f(U, w) - A(U - u_{\text{syn}}) = 0$$

has a unique solution $U = \phi_2(w)$, and

$$f(U, w) + \frac{1}{2}A(U - u_{\text{syn}}) > 0$$

for $U \leq \theta$. Then the traveling wave solution, $u(t, x) = U(x + vt) = U(z)$, $z = x + vt$, of equation (4.2) propagates along the negative (left) direction of

the x -axis such that

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = \phi_1(w), \quad \lim_{z \rightarrow \infty} U(z) = \phi_2(w), \quad \lim_{z \rightarrow \mp\infty} U'(z) = 0.$$

Meanwhile, system (4.8) has a unique heteroclinic orbit $\tilde{\Gamma}(z) = (\tilde{\tau}(z), \tilde{U}(z))$, where $\tilde{U}(z)$ is a monotone decreasing function of z , $\tilde{\tau}(0) = 0$, $\tilde{U}(0) = \theta$, and

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tilde{\tau}(z) &= -1, & \lim_{z \rightarrow -\infty} \tilde{U}(z) &= \phi_2(w), \\ \lim_{z \rightarrow \infty} \tilde{\tau}(z) &= 1, & \lim_{z \rightarrow \infty} \tilde{U}(z) &= \phi_1(w). \end{aligned}$$

The traveling wave solution, $\tilde{u}(t, x) = \tilde{U}(x - vt) = \tilde{U}(s)$, $s = x - vt$, of equation (4.2) propagates along the positive (right) direction of the x -axis, such that

$$\tilde{U}(0) = \theta, \quad \lim_{z \rightarrow -\infty} \tilde{U}(s) = \phi_2(w), \quad \lim_{s \rightarrow \infty} \tilde{U}(s) = \phi_1(w), \quad \lim_{s \rightarrow \mp\infty} \tilde{U}'(z) = 0.$$

The parameter $v = v(w, \theta) > 0$ is decreasing and depends on both w and θ .

Theorem 4.5.2. For given $w_1^- < w < w_2^+$ and $u_1^- < \theta < u_1^+$, there is a unique $v = v(w, \theta) > 0$ such that system (4.7) of ordinary differential equations has a unique heteroclinic orbit $\Gamma(z) = (\tau(z), U(z))$, where $U(z)$ is a monotone decreasing function of z , $\tau(0) = 0$, $U(0) = \theta$, and

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tau(z) &= -1, & \lim_{z \rightarrow -\infty} U(z) &= \phi_2(w), \\ \lim_{z \rightarrow \infty} \tau(z) &= 1, & \lim_{z \rightarrow \infty} U(z) &= \phi_1(w), \end{aligned}$$

if and only if

$$f(U, w) = 0$$

has the unique solution $U = \phi_1(w)$, and

$$f(U, w) + \frac{1}{2}A(U - u_{\text{syn}}) < 0$$

for $U \geq \theta$. The traveling wave solution $u(t, x) = U(x+vt) = U(z)$, $z = x+vt$, of equation (4.2) propagates along the negative (left) direction of the x -axis such that

$$U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = \phi_2(w), \quad \lim_{z \rightarrow \infty} U(z) = \phi_1(w), \quad \lim_{z \rightarrow \mp\infty} U'(z) = 0.$$

Meanwhile, system (4.10) has a unique heteroclinic orbit $\tilde{\Gamma}(z) = (\tilde{\tau}(z), \tilde{U}(z))$, where $\tilde{U}(z)$ is a monotone increasing function of z , $\tilde{\tau}(0) = 0$, $\tilde{U}(0) = \theta$, and

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tilde{\tau}(z) &= -1, & \lim_{z \rightarrow -\infty} \tilde{U}(z) &= \phi_1(w), \\ \lim_{z \rightarrow \infty} \tilde{\tau}(z) &= 1, & \lim_{z \rightarrow \infty} \tilde{U}(z) &= \phi_2(w). \end{aligned}$$

The traveling wave solution, $\tilde{u}(t, x) = \tilde{U}(x - vt) = \tilde{U}(s)$, $s = x - vt$, of equation (4.2) propagates along the positive (right) direction of the x -axis such that

$$\tilde{U}(0) = \theta, \quad \lim_{s \rightarrow -\infty} \tilde{U}(s) = \phi_1(w), \quad \lim_{s \rightarrow \infty} \tilde{U}(s) = \phi_2(w), \quad \lim_{s \rightarrow \mp\infty} \tilde{U}'(z) = 0.$$

The parameter $v = v(w, \theta) > 0$ is increasing and depends on both w and θ .

In assumption A5 we assume that the kernel function $K(x)$ is positive and continuous. For a general form, such as non-negative piecewise continuous functions, all the results proved in this chapter are also true. We have discussed this issue in section 3.5.3 of Chapter 3.

Chapter 5

Conclusions and Future Work

In this thesis, we establish the necessary and sufficient conditions for the existence and uniqueness of monotone traveling wave solutions of the two integro-differential equations (1.2) and (1.3). These two equations are used for modeling one-dimensional neuronal networks. The monotone traveling wave solutions can describe inhibitory processes and fire processes of neuronal networks. With these results we can predict if neuronal networks will fire or inhibit under given conditions. This thesis also shows how the wave speed v of traveling waves rely on the threshold θ and on the parameter w for these two models, and these relationships agree with theories and observations in biology. Another result of this thesis is the symmetry of the monotone traveling wave solutions in opposite directions in a one-dimensional network. With this symmetry the results in both directions are obtained.

5.1 Conclusions and future work on equation

(1.2)

Equation (1.2) is a singular perturbation of the system

$$\begin{aligned}u_t &= f(u, w) + \alpha \int_{\mathbb{R}} K(x - y) H(u(t, y) - \theta) dy, \\w_t &= \epsilon g(u, w).\end{aligned}\tag{5.1}$$

The mathematical results on equation (1.2) are summarized in Theorems 3.4.1 and 3.4.2. In view of neuron physiology, Theorem 3.4.1 gives the following results. When a neuronal network is in the silent phase, the channel variable satisfies the inequality $w > w_1^-$. Provided the excitatory coupling effect is strong, that is, $\alpha > 0$ is big enough such that $f(u, w) + \alpha = 0$ has only one solution and $f(u, w) + \frac{1}{2}\alpha > 0$ for $u \leq \theta$, then the whole neuronal network will go into the active phase if a local part of the network is stimulated and stays in active phase. The spatial time dynamical behavior of the network can be described by symmetric traveling waves going along the one-dimensional network in both (left and right) directions. The forms of the traveling waves are uniquely determined, and the wave speed is a decreasing function of the threshold θ and the parameter w . Similarly, Theorem 3.4.2 describes the dynamics of the network going to its rest phase. When the network is in the active phase, the channel variable satisfies the inequality $w < w_2^+$. If $f(u, w) = 0$ has only one solution and $f(u, w) + \frac{1}{2}\alpha < 0$ for $u \geq \theta$, then the whole neuronal network will go to the rest phase if a local part of the network goes to the silent phase. A similar statement holds for the traveling wave.

Future work on problem (1.2) could be the study of the evolution of the network, especially, to find a traveling wave of w when the neuronal network is in the rest phase or the active phase, respectively. If we let $\tau = \epsilon t$ and set $\epsilon = 0$, then the singular perturbation of system (5.1) becomes

$$\begin{aligned} 0 &= f(u, w) + \alpha, \\ w_\tau &= g(u, w), \end{aligned}$$

when the neuronal network is in the active phase and

$$\begin{aligned} 0 &= f(u, w), \\ w_\tau &= g(u, w), \end{aligned}$$

when the neuronal network is in the rest phase. These equation can be used to model ion channel permeabilities in the network. They are differential-algebraic equations [21]. Searching for their traveling wave solutions is an interesting and challenging problem.

5.2 Conclusions and future work on equation

(1.3)

Model (1.3) is a singular perturbation of the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= f(u, w) - g_{\text{syn}}(u - u_{\text{syn}}) \int_{\mathbb{R}} K(x - y) s(t, y) dy, \\ \frac{\partial w}{\partial t} &= \epsilon g(u, w), \\ \frac{\partial s}{\partial t} &= \alpha H(u(t, x) - \theta)(1 - s) - \beta s. \end{aligned} \tag{5.2}$$

Our results on model (1.3) are summarized in Theorems 4.5.1 and 4.5.2. In view of neuron physiology, Theorem 4.5.1 describes the dynamics of a

neuronal network that goes from the silent phase to the active phase. In the silent phase, the channel variable satisfies the inequality $w > w_1^-$. If $f(u, w) - A(u - u_{\text{syn}}) = 0$ has only one solution and $f(u, w) + \frac{1}{2}A(u - u_{\text{syn}}) > 0$ for $u \leq \theta$, then the whole neuronal network will go into the active phase when a local part of the network is stimulated and stays in active phase. The spatial time dynamical behavior of the network can be described by symmetric traveling waves going along the one-dimensional network in both (left and right) directions. The forms of the traveling waves are uniquely determined and the wave speed v is a decreasing function of the threshold θ and the parameter w . Theorem 4.5.2 describes the dynamics of the network going from active phase to silent phase. When a network is in the active phase, the channel variable w satisfies the inequality $w < w_2^+$. If $f(u, w) = 0$ has only one solution and $f(u, w) + \frac{1}{2}A(u - u_{\text{syn}}) < 0$ for $u \geq \theta$, then the whole neuronal network will go into the silent phase when a local part of the network arrives in silent phase. A similar statement on the traveling wave holds.

Future works on problem (1.3) could be to study the evolution of the network, especially, to find the traveling wave of w when the neuronal network in the rest phase or the active phase, respectively. Let $\tau = \epsilon t$ and set $\epsilon = 0$, then the singular perturbation of system (5.2) becomes

$$\begin{aligned} 0 &= f(u, w) - A(u - u_{\text{syn}}), \\ w_\tau &= g(u, w), \end{aligned}$$

when the neuronal network is in active phase, and

$$\begin{aligned} 0 &= f(u, w), \\ w_\tau &= g(u, w). \end{aligned}$$

when it is in rest phase. Thus the study of equations (1.2) and (1.3) is similar.

The steady state (u_0, w_0) of system (5.1) satisfies $f(u_0, w_0) = 0$ and $g(u_0, w_0) = 0$. It stands for the silent phase of the network. The steady state (u_0, w_0, s_0) of system (5.2) satisfies $f(u_0, w_0) = 0$, $g(u_0, w_0) = 0$, and $s_0 = 0$. It stands for the silent phase of the network. For each model, the ultimate objective is to put the four pieces of traveling wave solutions together to construct one solution with the steady state as its boundary condition. Thus, this solution could describe the whole dynamic procedure of the network: get stimulated, go into active phase, evolve in active phase, and return to silent phase.

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