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Polarization Dependent Azimuthal Scattering From Tilted Fibre Bragg Gratings

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Abstract

Polarization sensitive mode coupling characteristics of tilted fibre Bragg gratings (FBGs) have been exploited to develop a number of useful devices including fibre polarimeters, gain flattening filters, spectrum analyzers, polarization dependent loss (PDL) compensators, reconfigurable optical add / drop multiplexers (ROADM), as well as interferometric, and surface plasmon based sensors.

Recently it was demonstrated that a single grating structure could couple the light guided in a fibre to two azimuthally separated, polarization independent, radiated beams. However the reasons for such behaviour had not been fully explained, precluding the complete understanding, exploitation and optimization of this phenomenon.

This thesis explains the mechanisms underlying such behaviour through a thorough analytical examination of an existing equation formulated with the Volume Current Method (VCM), quantifying the degree to which a tilted FBG's radiation field is directionally dependent on the phase matching characteristics of a grating's three-dimensional structure as well as the polarization dependent dipole response of the medium itself.

Examination of the equation's parameter space, revealed the possibility of three-beam azimuthal responses as well, and resulted in some guidelines for the design and optimization of these devices.

Experimental measurements of the out-tapped field are also provided, clearly confirming these theoretical findings and reporting the fabrication of a three-beam azimuthal response grating for the first time.

Drawing upon these advances, an improved polarimeter design is proposed that samples more than four detected beams with only two tilted FBGs, theoretically resulting in average Stokes vector error reductions of roughly 20%, facilitating monitoring at lower signal to noise ratios (SNRs).

Finally, this thesis undertakes an analysis and re-derivation of the VCM formulation itself, designed to expand its applicability to FBGs written with ultrafast pulsed lasers, address some of the potentially limiting assumptions identified by Li et al, and provide users with computationally efficient formulae that are as accurate as possible.

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a	used for polarization state index and argument variable of exponential integral
a	grating radius, effectively limited to half mode field diameter since significant mode overlap is required to drive scattering.
A	instrument matrix
A	vector potential magnitude
$\vec{A}$	vector potential
$\vec{A}$	vector potential
$\vec{A}_{cl}$	vector potential induced by cladding light
$\vec{A}_{co}$	vector potential induced by core light
a_{clad}	cladding radius
a_{core}	core radius
a_k	summation coefficient defined in <i>F.4</i>
$\vec{A}_x$	vector potential induced by x polarization component
A_{xco}	magnitude of vector potential x component induced by core light
A_{xcl}	magnitude of vector potential x component induced by cladding light
$\vec{A}_y$	vector potential induced by y polarization component
A_{yco}	magnitude of vector potential y component induced by core light
A_{ycl}	magnitude of vector potential y component induced by cladding light
A_{zco}	magnitude of vector potential z component induced by core light
A_{zcl}	magnitude of vector potential z component induced by cladding light
a_1	electric field parameter, defined in <i>C.1.2</i>
a_2	electric field parameter, defined in <i>C.1.2</i>
$\mathbf{a}_1, \vec{a}_1$	primitive lattice vector
$\mathbf{a}_2, \vec{a}_2$	primitive lattice vector
$\mathbf{a}_3, \vec{a}_3$	primitive lattice vector
AC	‘alternating current’, referring to modulated index change components
ArF	Argon Fluoride
b	polarization state index
$\mathbf{B}, \vec{B}$	magnetic field vector
B_{f1}	magnetic field factor, defined by Eq. (E.2)
B_{f2}	magnetic field factor, defined by Eq. (E.3)
$\mathbf{B}_i$	incident magnetic field vector
$\vec{B}_{in}$	induced magnetic field vector
$\vec{B}_{in,x}$	induced magnetic field vector due to x polarization component
$\vec{B}_{in,y}$	induced magnetic field vector due to y polarization component
$\vec{B}_{incl}$	magnetic field induced by light in cladding
$\vec{B}_{inco}$	magnetic field induced by light in core
b_k	summation coefficient defined in <i>F.4</i>

$\mathbf{B}_r$	reflected magnetic field vector
$\mathbf{B}_t$	transmitted magnetic field vector
B_{xco}	magnitude of magnetic field x component induced by core light
B_{xcl}	magnitude of magnetic field x component induced by cladding light
B_{yco}	magnitude of magnetic field y component induced by core light
B_{ycl}	magnitude of magnetic field y component induced by cladding light
B_{zco}	magnitude of magnetic field z component induced by core light
B_{zcl}	magnitude of magnetic field z component induced by cladding light
$\vec{B}_0$	magnetic field amplitude vector
b_1	electric field parameter, defined in C.1.2
b_2	electric field parameter, defined in C.1.2
$\mathbf{b}_1, \vec{b}_1$	primitive reciprocal lattice vector
$\mathbf{b}_2, \vec{b}_2$	primitive reciprocal lattice vector
$\mathbf{b}_3, \vec{b}_3$	primitive reciprocal lattice vector
BiO ₂	Bismuth Oxide
c	speed of light
c	polarization state index
C	constant of proportionality used in Eq. (4.1)
C	parameter defined in 3.3.1
c_k	summation coefficient defined in F.4
C ₁ -C ₈	parameters defined in Table 3.1
CMT	coupled mode theory
d	polarization state index
d	radial distance between observation and scattering points
D	parameter defined in 3.3.1
d_k	summation coefficient defined in F.4
D ₁ -D ₉	parameters defined in Table 3.2
D_I - D_6	optimally detected polarization states
DC	‘direct current’, referring to uniform or unmodulated index change component
DOAP	division of amplitude photopolarimeter
DOP	degree of polarization
$\mathbf{E}, \vec{E}$	electric field vector
$\mathbf{E}_{circular}$	circularly polarized electric field vector
$\vec{E}_{clad}$	mode’s electric field in cladding
$\vec{E}_{core}$	mode’s electric field in core
$\mathbf{E}_{elliptical}$	elliptically polarized electric field vector
E_{exp}	total UV exposure energy
E_i	incident electric field amplitude
$\mathbf{E}_i$	incident electric field vector
$\vec{E}_{in}$	induced electric field vector
$\vec{E}_{in,x}$	induced electric field vector due to x polarization component

$\vec{E}_{in,y}$	induced electric field vector due to y polarization component
e_k	summation coefficient defined in <i>F.4</i>
$\mathbf{E}_{linear}$	linearly polarized electric field vector
E_r	reflected electric field amplitude
$\mathbf{E}_r$	reflected electric field vector
e_{rcl}	radial component electric field factor for cladding, defined by Eq. (C.17)
e_{rco}	radial component electric field factor for core, defined by Eq. (C.7)
E_{sat}	UV exposure saturation energy
E_t	transmitted electric field amplitude
$\mathbf{E}_t$	transmitted electric field vector
E_x	x component magnitude of induced electric field
$\mathbf{E}_x$	component of electric field in x direction
e_{xcl}	x component electric field factor for cladding, defined by Eq. (C.20)
E_{xcl}	magnitude of electric field x component induced by cladding light
e_{xco}	x component electric field factor for core, defined by Eq. (C.10)
E_{xco}	magnitude of electric field x component induced by core light
E_y	y component magnitude of induced electric field
$\mathbf{E}_y$	component of electric field in y direction
e_{ycl}	y component electric field factor for cladding, defined by Eq. (C.21)
E_{ycl}	magnitude of electric field y component induced by cladding light
e_{yco}	y component electric field factor for core, defined by Eq. (C.11)
E_{yco}	magnitude of electric field y component induced by core light
E_z	z component magnitude of induced electric field
e_{zcl}	z component electric field factor for cladding, defined by Eq. (C.19)
E_{zcl}	magnitude of electric field z component induced by cladding light
e_{zco}	z component electric field factor for core, defined by Eq. (C.9)
E_{zco}	magnitude of electric field z component induced by core light
E_0	electric field amplitude coefficient
$\vec{E}_0$	electric field amplitude vector
E_1	electric field amplitude coefficient
E_2	electric field amplitude coefficient
$e_{\phi cl}$	ϕ component electric field factor for cladding, defined by Eq. (C.18)
$e_{\phi co}$	ϕ component electric field factor for core, defined by Eq. (C.8)
Ei	exponential integral function
EDFA	Erbium doped fibre amplifier
F	phase factor, defined by Eq. (C.58)
f_k	summation coefficient defined in <i>F.4</i>
F_n	noise enhancement factor
f_1	dipole component factor of VCM Poynting vector expression, given by Eq. (3.3)
f_{1norm}	normalized dipole component factor of VCM Poynting vector expression
f_2	interference component factor of VCM Poynting vector, given by Eq. (3.5)
F_2	electric field parameter, defined in <i>C.1.2</i>
f_{2norm}	normalized interference component factor of VCM Poynting vector expression

F_g	special function defined by Eq. (C.34)
f_v	electric field parameter, defined in C.1.2
FBG	fibre Bragg grating
FWM	four wave mixing
g	finite length correction factor, defined by Eq. (D.17)
$\mathbf{G}$	reciprocal lattice vector
$\mathbf{G}_1$	reciprocal lattice vector
$\mathbf{G}_2$	reciprocal lattice vector
$\mathbf{G}_3$	reciprocal lattice vector
G_1 - G_{17}	parameters defined in Table 3.3
G_g	special function defined by Eq. (C.35)
g_v	electric field parameter, defined in C.1.2
h	Planck's constant
h	finite length correction factor, defined by Eq. (D.5)
H	horizontal polarization
$H_v^{(1)}$	Hankel function of the first kind (order ν)
$H_v^{(2)}$	Hankel function of the second kind (order ν)
HiBi	highly birefringent
$\hbar$	reduced Planck's constant, Dirac constant
i	index for tracking discrete detuning regions
i	imaginary unit
i_n	noise current vector
I	intensity
I	current vector
I_d	dark current
I_{opt}	detector current from optimally resonant wavelength
I_{p-p}	peak to peak intensity amplitude
$I_{\Delta\lambda}$	detector current correction for detuned wavelengths
$I(\phi)_{norm,polarized}$	detector current induced by polarized light
$I(\phi)_{norm,unpolarized}$	detector current induced by unpolarized light
I_p	detected photocurrent
IR	infrared
j	summation index
$\mathbf{J}$	polarization current density
$\vec{J}$	polarization current density
$\vec{J}_{core}$	polarization current density associated with core electric field
$\vec{J}_{clad}$	polarization current density associated with cladding electric field
J_v	Bessel function of the first kind (order ν)
k	term index number $A.1$
k	propagation constant / wavenumber / wavevector magnitude
$\mathbf{k}$	wavevector
K	electric dipole stiffness with units of N/m
$\mathbf{K}$	grating reciprocal lattice vector
$\vec{k}$	wavevector

k_B	Boltzmann's constant
K_g	longitudinal component of grating reciprocal lattice vector
$\mathbf{k}_i, \vec{k}_i$	incident wavevector
K_{new}	detuning vector magnitude, given by Eq. (3.4)
$\mathbf{K}_{new}, \vec{K}_{new}$	detuning vector
$(K_{new})_j$	detuning vector magnitude of j^{th} summation component, given by Eq. (C.50)
$\mathbf{k}_{out}, \vec{k}_{out}$	scattered wavevector
k_t	transverse component magnitude of scattered wavevector
K_t	transverse component magnitude of grating reciprocal lattice vector
$(k_t)_j$	magnitude of transverse scattered wavevector for j^{th} summation component
$(k_{t2})_j$	defined in terms of $(k_t)_j$ in C.2.2
k_{tmax}	upper limit of scattered wavevector transverse component magnitude
k_{tmin}	lower limit of scattered wavevector transverse component magnitude
k_0	freespace wavevector magnitude
K_ν	modified Bessel function of the second kind (order ν)
KrF	Krypton Fluoride
L	grating length
LC	left circular polarization
LP	linearly polarized waveguide mode
LPG	long period grating
m	diffraction order
m	particle mass
M_{QWP}	Mueller matrix of quarter wave plate
MFD	mode field diameter
n	summation term index number
n	refractive index
N	number of detectors or polarization states for calibration
n_{cl}	cladding refractive index
n_{clad}	cladding refractive index
n_{core}	core refractive index
n_{eff}	modal effective refractive index
n_i	refractive index of incident medium
n_t	refractive index of transmission medium
n_0	core refractive index
NA	numerical aperture
o	finite length correction factor, defined by Eq. (D.18)
p	degree of polarization
P	probability of a given error value (Fig. 5.5)
P_{in}	guided power incident grating
PCF	photonic crystal fibre
PDL	polarization dependent loss
PDS	polarization dependent scattering
$PD\lambda$	polarization dependent wavelength difference
PM	polarization maintaining
PMD	polarization mode dispersion

PSP	principal state of polarization
q	particle charge, electron charge
q	substitution variable
Q	polarization factor defined by Eq. (5.10)
r	radial cylindrical coordinate, coordinate of observation point
$\mathbf{r}$	position vector, displacement
$\hat{r}$	unit vector in radial direction
$\vec{r}$	position vector, displacement
$\vec{r}'$	position vector of scattering element
r'	radial cylindrical coordinate of scattering element
R_L	load resistance
r_p	reflection coefficient for p -polarization, parallel the plane of incidence
r_s	reflection coefficient for s -polarization, perpendicular the plane of incidence
R	radial dimension normalized with respect to core radius
R_1	series remainder
RC	right circular polarization
RMS	root-mean-square
ROADM	reconfigurable optical add / drop multiplexers
S	Stokes vector
S	Poynting vector magnitude
$\mathbf{S}, \vec{S}$	Poynting vector
$\mathbf{S}_i$	Poynting vector of incident wave
S_{noise}	calculated Stokes vector, noise included
S_{norm}	normalized Poynting vector magnitude
S_p	Poynting vector magnitude of scattered p -polarized light
$\mathbf{S}_r$	Poynting vector of reflected wave
S_s	Poynting vector magnitude of scattered s -polarized light
$\mathbf{S}_t$	Poynting vector of transmitted wave
S_{true}	true Stokes vector of SOP present in system
S_0	Stokes parameter
S_1	Stokes parameter
S_2	Stokes parameter
S_3	Stokes parameter
SMF	single mode fibre
SNR	signal to noise ratio
SOP	state of polarization
t	time
T	temperature
t_p	transmission coefficient for p -polarization, parallel the plane of incidence
t_s	transmission coefficient for s -polarization, perpendicular the plane of incidence
TIR	total internal reflection
u	normalized modal parameter, defined in C.1.2
U	modal parameter, defined in C.1.2
UV	ultraviolet
V	fibre normalized frequency
V	vertical polarization

VCM	volume current method
w	normalized modal parameter, defined in C.1.3
W	modal parameter, defined in C.1.2
WDM	wavelength division multiplexing
x	x Cartesian coordinate
X	substitution variable
$\hat{x}$	unit vector in x direction
x_j	substitution variable to facilitate integration
XPM	cross phase modulation
y	substitution variable to facilitate integration
y	y Cartesian coordinate
$\hat{y}$	unit vector in y direction
z	Bessel function argument, exponential integral argument
z	z coordinate
Z	VCM integral along z dimension
$\mathbb{Z}$	integer
$\hat{z}$	unit vector in z direction
z'	z coordinate of scattering element
$ZC_{L \neq \infty}$	finite length correction term, defined by Eq. (D.3)
α	auxiliary angle of polarization state
α	attenuation coefficient
β	modal propagation constant
χ	Poincaré sphere coordinate
χ_i	parameter defined in 3.4.2
δ	polarization reference angle of guided electric field, relative to $+x$ axis
Δ	longitudinal component magnitude of scattered wavevector
Δ_j	longitudinal component magnitude of j^{th} summation term's scattered wavevector
δ_{phase}	phase difference between x and y polarization components
δ_x	angle of guided field's x polarization component, relative to $+x$ axis
δ_y	angle of guided field's y polarization component, relative to $+x$ axis
δ_z	core-cladding index difference parameter
δ_ε	core-cladding index difference parameter
Δf	effective noise bandwidth of detector
δn	refractive index modulation amplitude
δn_{DC}	uniform or 'direct current' photoinduced change in refractive index
δn_{P-P}	peak to peak amplitude of refractive index modulation
δn_{sat}	saturated index change
Δn_{mod}	refractive index modulation
$\delta \varepsilon$	index perturbation
$\Delta \phi$	spread angle of azimuthally scattered beams
$\Delta \lambda$	wavelength detuning or bandwidth
ε	permittivity of medium
$\varepsilon_{\text{core}}$	permittivity of core
$\varepsilon_{\text{clad}}$	permittivity of cladding

$\varepsilon_{d=r}$	error generated by approximating d with r
$\varepsilon_{d=r-r'\cos(\phi-\phi')}$	error generated by approximating d with $r-r'\cos(\phi-\phi')$
ε_n	error in SOP due to random noise only
ε_{SOP}	error in SOP measurement
ε_0	permittivity of free space
ε_1	Real, positive number, used to facilitate integration
ε_2	Real, positive number, used to facilitate integration
$\varepsilon_{\Delta\lambda}$	error in SOP due to detuning only
ϕ	generally azimuthal cylindrical coordinate, also used for phase delay in 1.2.2
$\hat{\phi}$	unit vector in ϕ direction
ϕ	azimuthal cylindrical coordinate of scattering element
$(\phi_{new})_j$	substitution variable constrained by definition of Eq. (C.49)
ϕ_l	arbitrary phase offset
Γ	gamma function
γ	shape parameter for Maxwellian distribution (Fig. 5.5)
γ_{1i}	parameter defined in 3.4.3
γ_{2i}	parameter defined in 3.4.3
η	substitution variable
ϕ_{1i}	parameter defined in 3.4.2
ϕ_{2i}	parameter defined in 3.4.2
ϕ_{3i}	parameter defined in 3.4.2
$\mathcal{G}$	substitution variable
κ	constant defined in 3.2.2
κ_{1i}	parameter defined in 3.4.3
κ_{2i}	parameter defined in 3.4.3
λ	wavelength in free space
Λ	period of phase mask, lattice or grating
Λ_B	period of grating, perpendicular separation of grating planes
λ_{cutoff}	cutoff wavelength
μ_0	permeability of free space
ν	frequency
ν	mode index / Bessel function order
ν_{as}	anti-stokes frequency
ν_{phonon}	phonon frequency
ν_s	stokes frequency
ν_1	integer multiplier
ν_2	integer multiplier
ν_3	integer multiplier
ω	angular frequency
ω_0	natural angular frequency
Ω_1	parameter defined in 3.4.4
Ω_2	parameter defined in 3.4.4
Ω_3	parameter defined in 3.4.4

Ω_4	parameter defined in 3.4.4
Ω_5	parameter defined in 3.4.4
ρ	charge density
σ	standard deviation of noise
σ^2	root-mean-square noise
σ_s^2	root-mean-square shot noise
σ_{sd}^2	dark current component of shot noise
σ_{sp}^2	photocurrent component of shot noise
σ_T^2	root-mean-square thermal noise
θ	angle defined in Fig. 1.15
θ	phase mask azimuthal angle relative to fibre axis normal
θ_B	grating blaze angle relative to fibre axis
θ_i	angle of incidence
θ_m	diffraction angle of m th order phase mask beam
θ_r	angle of reflection
θ_t	angle of transmission
θ_1	angle defined in Fig. 1.15
θ_3	angle defined in Fig. 1.15
τ	offset used in probability distribution fit (Fig. 5.5)
ϖ	angle between two interfering beams
ς	number of photons required to bridge bandgap for photoinscription
ξ	grating blaze angle relative to transverse y axis
ξ_{out}	longitudinal tap angle of coupled radiation mode
ψ	Poincaré sphere coordinate
ψ	parameter defined in 3.4.2
Ψ_1	parameter defined in 3.4.3
Ψ_2	parameter defined in 3.4.3
ζ	damping coefficient
$\nabla \cdot$	divergence operator
$\nabla \times$	curl operator

Abstract

Polarization sensitive mode coupling characteristics of tilted fibre Bragg gratings (FBGs) have been exploited to develop a number of useful devices including fibre polarimeters [1-5], gain flattening filters [6], spectrum analyzers [7], polarization dependent loss (PDL) compensators [8, 9], reconfigurable optical add / drop multiplexers (ROADM) [10], as well as interferometric [11], and surface plasmon based sensors [12].

Recently it was demonstrated that a single grating structure could couple the light guided in a fibre to two azimuthally separated, polarization independent, radiated beams [5]. However the reasons for such behaviour had not been fully explained, precluding the complete understanding, exploitation and optimization of this phenomenon.

This thesis explains the mechanisms underlying such behaviour through a thorough analytical examination of an existing equation formulated with the Volume Current Method (VCM) [13], quantifying the degree to which a tilted FBG's radiation field is directionally dependent on the phase matching characteristics of a grating's three-dimensional structure as well as the polarization dependent dipole response of the medium itself.

Examination of the equation's parameter space, revealed the possibility of three-beam azimuthal responses as well, and resulted in some guidelines for the design and optimization of these devices [14-15].

Experimental measurements of the out-tapped field are also provided, clearly confirming these theoretical findings and reporting the fabrication of a three-beam azimuthal response grating for the first time [16].

Drawing upon these advances, an improved polarimeter design is proposed that samples more than four detected beams with only two tilted FBGs, theoretically resulting in average Stokes vector error reductions of roughly 20%, facilitating monitoring at lower signal to noise ratios (SNRs) [17].

Finally, this thesis undertakes an analysis and re-derivation of the VCM formulation itself, designed to expand its applicability to FBGs written with ultrafast pulsed lasers, address some of the potentially limiting assumptions identified by Li et al [13,18], and provide users with computationally efficient formulae that are as accurate as possible.

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Publications Related to the Thesis

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Introduction

Following the discovery of fibre photosensitivity [19], application of the side writing technique [20], and invention of the phase mask grating fabrication method [21], the use of fibre Bragg grating (FBG) optical components has become increasingly widespread due to their versatility, compact size, low loss and low cost characteristics.

FBGs are photonic crystals, comprised of a waveguide in which the refractive index varies in some sort of periodic manner. Most commonly this periodicity is treated in a one-dimensional fashion with planes of equivalent refractive index crossing the fibre axis at a 90° angle, however there is also a subset of these devices in which the grating planes are skewed with respect to the guided light. Often referred to as tilted, blazed or slanted FBGs, these devices enable significantly greater out-coupling to cladding and non-guided radiation modes, and are frequently observed to possess significant polarization sensitivity [3,22-28]. As a consequence, such technology is employed by a number of commercially available products including gain flattening filters [6], polarization dependent loss (PDL) compensators [8,9], in-line fibre polarimeters [1-5], spectrum analyzers [7], reconfigurable optical add / drop multiplexers (ROADM) [10], as well as interferometric [11], and surface plasmon based sensors [12].

Of particular note is the disclosure by Peupelmann et al [5], in which it was first reported that for a given wavelength, some tilted FBGs had been observed to possess multiple azimuthally distributed, polarization-dependent tap angles. Based on this interesting observation they proposed a unique fibre-polarimeter design that exploited this phenomenon. There was not however, a clear explanation of the physical mechanisms responsible for this effect, making optimization of such devices difficult and inhibiting the development of other components that might benefit from the precise control and exploitation of a tilted FBG's radiation field profile.

Following a brief review of fibre Bragg grating technology and some of the key physical concepts on which it is founded, this thesis answers these questions and proposes improvements, based on thorough theoretical analyses and experimental work. More specifically, *Chapter 1* reviews some concepts fundamental to the understanding of polarization, light-material interactions and Bragg scattering, with *Chapter 2* covering several important aspects of fibre Bragg grating fabrication, their characteristics and applications. The novel elements of this work then follow in *Chapters 3-6*.

In particular, *Chapter 3* presents original material published by the candidate [14,15], in which the work of [13] has been extended to characterize useful trends and explain the physical mechanisms responsible for the unique radiation mode coupling reported by [5].

In *Chapter 4*, based on another publication by this candidate [16], experimental validation of these mechanisms has been provided, along with the first reported fabrication of the 3-beam gratings predicted in [14,15].

Currently accepted for publication [17], *Chapter 5* provides a practical application of these findings, an improved FBG based polarimeter design in which components already integral to the present state of the art are utilized more effectively to reduce noise errors 20%.

In *Chapter 6*, motivation is presented for a more comprehensive analytical formulation of the VCM, along with a summary of the insights it provides. Among other things, this model, derived in *Appendix C*, is used to look more closely at the impact of the longitudinal guided field component [18], as well as gratings written in the cladding or by an ultrafast IR source.

Finally, a number of conclusions are provided with a view to future work.

Chapter 1

Understanding Light Scattering in FBGs

Light scattering phenomena are fundamental to the work presented in this thesis. Although the scope of available theory is much more extensive than the simple overview presented here, it is thought that a clear grasp of these concepts will aid the reader in developing a better understanding of the subject matter presented in later chapters. Beginning with light's electromagnetic wave nature, concepts including mathematical notation, the Poynting vector, polarization, interference and the Huygens-Fresnel principle are reviewed. Also, because of their relevance to wave guiding structures and photosensitivity, a summary of light-material interactions is presented in terms of refractive index, birefringence and dipole scattering. Finally this chapter concludes with a discussion of scattering effects from surfaces and periodic structures. For additional information the reader is referred to such texts as [29-32] in addition to the particular references specified.

1.1 Electromagnetic waves

1.1.1 Wave characteristics of light

In unbound vacuum, light behaves like a purely transverse wave, in which oscillating electric (E) and magnetic (B) fields, each orthogonal to the other as well as the direction of propagation, continually regenerate one another without need of any surrounding medium, as shown in Fig. 1.1.

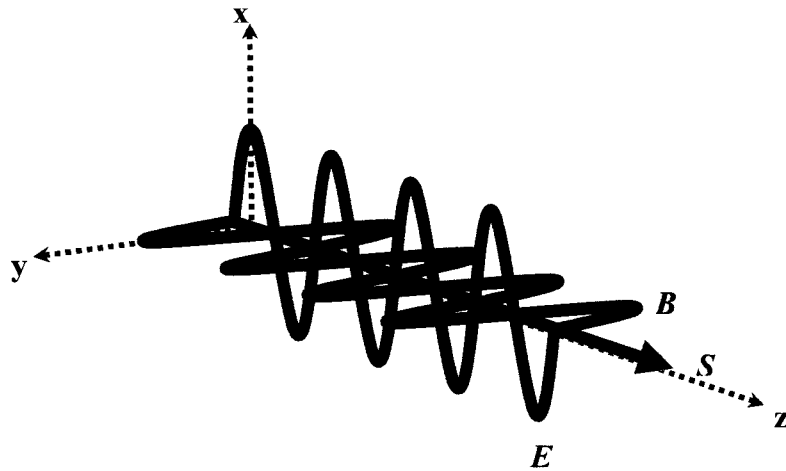


Fig. 1.1 – Light propagating in unbound vacuum.

The strength of such waves vary as a function of both position ($\vec{r}$) and time (t), with mathematical expressions for the electric ($\vec{E}$) and magnetic fields ($\vec{B}$) commonly taking the form:

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} + \omega t + \phi_1) \quad (1.1)$$

$$\vec{B} = \vec{B}_0 \cos(\vec{k} \cdot \vec{r} + \omega t + \phi_1) \quad (1.2)$$

where $\vec{E}_0$ and $\vec{B}_0$ are the unmodulated field amplitudes, $\vec{k}$ is the propagation constant, ω the angular frequency and ϕ_1 the phase offset of the field.

Since the electric and magnetic field components are related directly by Maxwell's equations Eqs. (1.3)-(1.6) [33], electromagnetic waves are often described solely in terms of their electric field, even though both components are always present.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (1.3)$$

$$\nabla \cdot \vec{B} = 0 \quad (1.4)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1.5)$$

$$\nabla \times \vec{B} = \mu_0 \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right) \quad (1.6)$$

where $\nabla \cdot$ is the divergence operator, $\nabla \times$ the curl operator, ρ is the charge density, $\vec{J}$ the current density, ϵ_0 the permittivity of free space and μ_0 its permeability.

Often for simplicity of notation, such waves are written in phasor form, solely as a function of their amplitude and phase arguments:

$$\vec{E} = \vec{E}_0 e^{-i(\vec{k} \cdot \vec{r} + \omega t + \phi_1)} \quad (1.7)$$

In terms of energy transfer and the directionality associated with an electromagnetic wavefront, this is quantifiable by means of the Poynting Vector (p. 47 of [29]):

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad (1.8)$$

1.1.2 Polarization

In examining the electric field, one notes that it is a vector quantity. This directionality of the fields composing an electromagnetic wave is termed polarization, a quality that is often exploited for the characterization and manipulation of light.

Depending on the phase and amplitude of various wave components, its polarization can be linear (where the direction of $\vec{E}$ is fixed as a function of time), circular or elliptical (where the direction of $\vec{E}$ varies with time), as depicted in Fig. 1.2.

Mathematically the various states of transverse polarization can be dealt with in matrix form, where two element Jones vectors (and 2×2 Jones matrices) are sufficient for dealing with purely polarized states, while four element Stokes vectors (and 4×4 Mueller matrices) are more generally suitable for calculations involving either purely or partially polarized light.

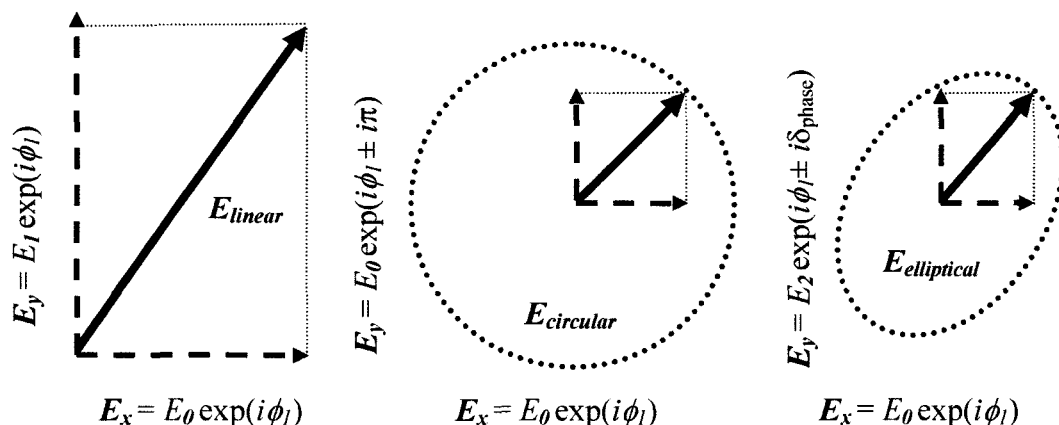


Fig. 1.2 – Examples of linear, circular and elliptical polarization, showing field components and phase differences. Note that orientation of circular and elliptical states varies with time and position as a result of the phase difference between x and y field components. E_l and E_2 are field amplitude coefficients with $E_2 \neq E_0 \neq 0$, while the components phase difference $\delta_{phase} \neq \pi \neq 0$.

In both cases, states can be displayed graphically using the Poincaré sphere, where linear states are confined to its equator, circular states to its poles, pure and partially polarized states to its surface and interior respectively, as depicted in Fig. 1.3.

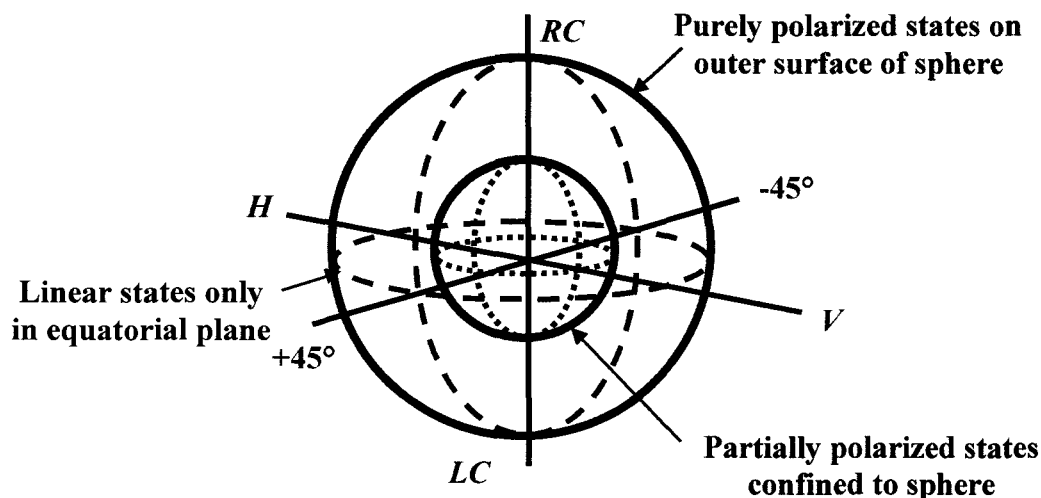


Fig. 1.3 – Poincaré sphere illustrating the locations of the horizontal (H), vertical (V) and $\pm 45^\circ$ linear states, as well as right (RC) and left circular (LC) states.

1.1.3 Interference and the Huygens-Fresnel Principle

Due to their wave nature, accounting for the superposition of multiple light particles or photons is not simply a matter of summing their individual amplitudes, the relative frequencies, phases and polarizations must also be taken into account.

As illustrated in Fig. 1.4, for two identically polarized, monochromatic waves, it is possible for this interference to be purely constructive, purely destructive, or somewhere in between, depending upon the relative phase difference between them.

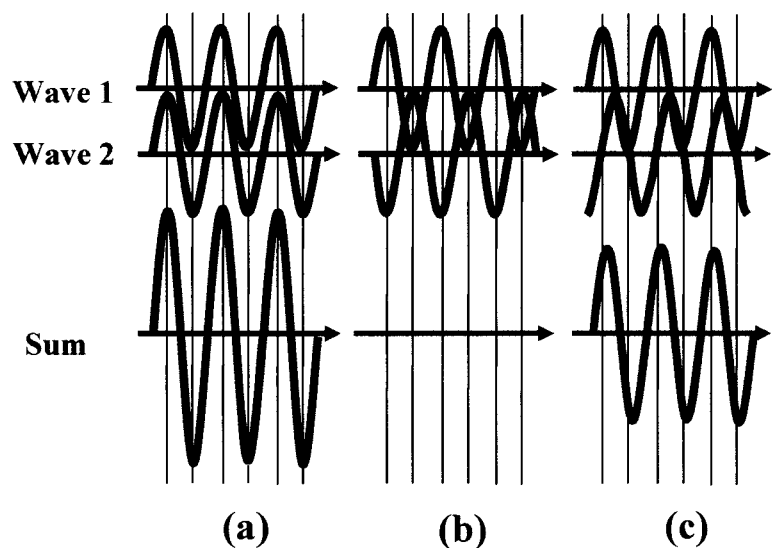


Fig. 1.4 – Examples of the superposition of two identically polarized monochromatic waves, including (a) purely constructive interference, (b) purely destructive interference and (c) a combination of the two.

Like the phenomenon of polarization, this effect has also been used to great advantage in the characterization and manipulation of light. As a practical example, consider the Huygens-Fresnel Principle which states in essence that every point on a wavefront acts as a secondary emitter, contributing to the wavefront formation at some later time, in proportion to the relative amplitudes and phases of each secondary wave.

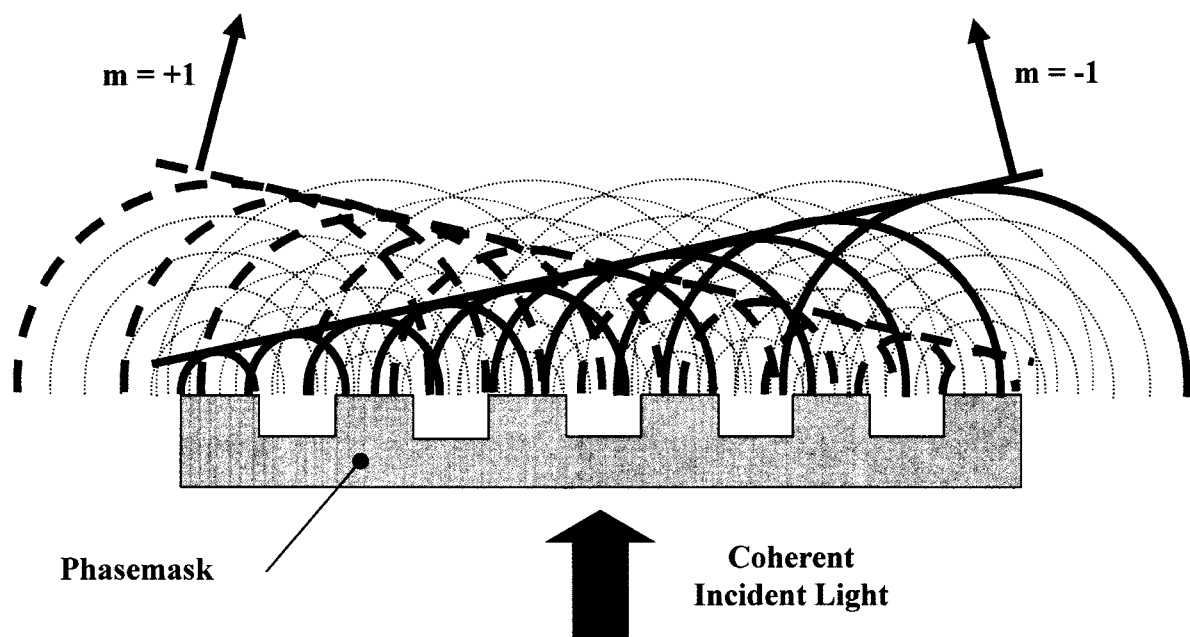


Fig. 1.5 – Huygens-Fresnel Principle applied to a zero-order nulled transmission phase mask. Note wavefronts passing through thicker glass sections (blue) are π out of phase with wavefronts passing through thinner glass sections (red), thereby nulling the $m=0$ response.

Among other things, as illustrated in Fig. 1.5 this principle can be used to visualize the splitting of coherent wavefronts incident a zero-order nulled phase mask, a type of

transmission diffraction grating often used for the fabrication of FBGs and discussed further in 2.2.1 and 2.3.2.

For light at normal incidence to the phase mask grating, the number m and angle θ_m of diffracted orders generated by the grating is dependent upon the grating period Λ and wavelength of light λ incident the mask such that:

$$\Lambda \sin \theta_m = m\lambda \quad (1.9)$$

In the special case of a zero-order nulled phase mask, the periodic structure of the diffraction grating is created by precisely etching a corrugation onto the surface of a glass plate. The etching is such that the base of the corrugation will produce light that is 180° out of phase with light that exits the ridges of the corrugation. In this way, coupling of light into the zero as well as even number diffracted orders is suppressed.

When regions of equal phase form planes as depicted by the $m = \pm 1$ wavefronts of Fig. 1.5, the light is considered to propagate as a plane wave. Waves having constant field amplitudes within each of these planes are called uniform plane waves, while those in which the field amplitudes vary are termed nonuniform plane waves (p. 101 of [34]).

1.2 Light / material interactions

Light propagation in a medium creates opportunities for interactions with constituent materials. This section will review some of the effects possible at the atomic and molecular levels as well as consequences that occur macroscopically as a result.

1.2.1 Processes involving photonic energy level transitions

Direct interactions between atoms and photons can occur in a number of ways. Two of the important factors involved, include the transitions between various atomic energy states and the energies of the photons themselves ($h\nu, \hbar\omega$), where h is Planck's constant, $\hbar$ the reduced Planck's constant, and ν the photon's frequency.

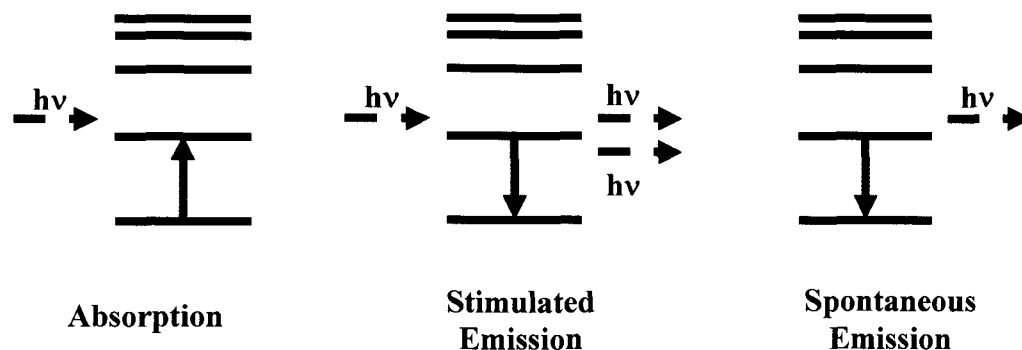


Fig. 1.6 – Some basic interactions between photons and valence electrons.

As shown in Fig. 1.6, when an incident photon has sufficient energy to bridge the gap between valence and conduction energy levels, the photon can be absorbed, elevating an electron from the valence state to the conduction band. Alternatively, if an electron is already present in the conduction band, such an incident photon can stimulate it to return to the valence state, emitting a coherent photon in the process. Since conduction states are relatively unstable, a return to the valence band can also occur spontaneously, usually after a delay characterized statistically by the state's relaxation time.

Beyond these simple effects, a number of nonlinear phenomena are also possible as illustrated in Fig. 1.7. Although the prevalence of such interactions is generally much less frequent, in the presence of high photon intensities such responses become increasingly likely and useful.

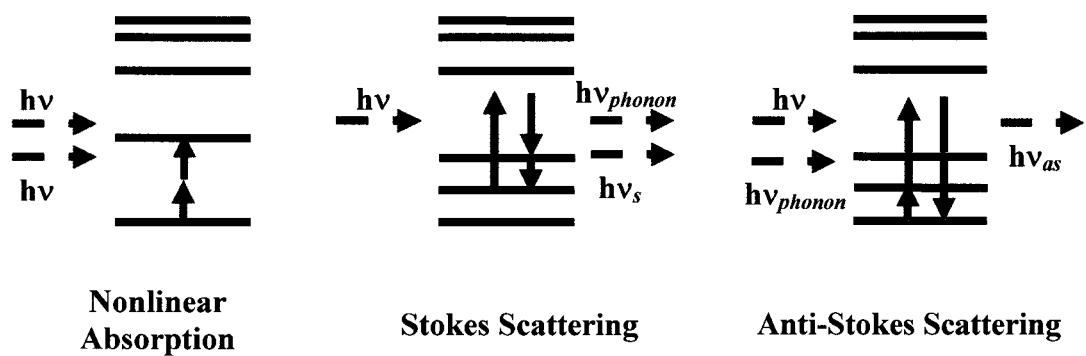


Fig. 1.7 – Examples of some nonlinear interactions between photons and atoms.

1.2.2 Forced vibration interactions

Often, photon energies are insufficient to bridge the band-gap between valence and conduction states, and intensities are too low to induce significant nonlinear effects. In these cases however, important interactions still occur at the atomic and molecular levels. Fundamentally, matter is composed of discrete electric charges and light is a fluctuating electromagnetic field that can dynamically excite these charges to oscillate. Because accelerating charges radiate electromagnetic waves the material affects the light field, just as the light field affects the material. Like the absorption and emission phenomena mentioned in 1.2.1 this cyclical process helps to explain how propagation in a medium differs from that within a vacuum.

A simple way to investigate the consequences of such situations is by studying the behaviour of a dipole harmonic oscillator, composed effectively of two particles, one having a positive charge and the other one negative, as shown in Fig. 1.8. These charges could be grouped in terms of protons and electrons in the case of an atom, or positive and negative ions in the case of some molecular structures.

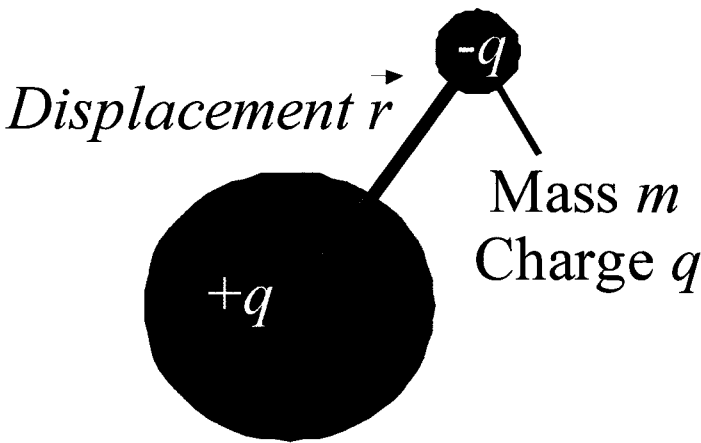


Fig. 1.8 – Simple harmonic oscillator.

A major class of problems in dynamics treats the oscillations of bodies subjected to perturbations in the presence of restoring tendencies. “Forced” vibration analysis deals with such motion, continuously reenergized by destabilizing forces [35]. This approach is rooted in the system’s equation of motion, given here:

$$q\vec{E} = m(\ddot{\vec{r}} + 2\zeta\omega_0\dot{\vec{r}} + \omega_0^2\vec{r}) \quad (1.10)$$

where ζ is the damping coefficient, $\omega_0 = \sqrt{K/m}$ the system’s natural frequency, K the bond stiffness (force per unit length displacement) and m the oscillating mass.

Note the left side of the equation represents the force applied by the field and the right side represents the restoring force produced by inertial tendencies of the charge’s mass ($m\ddot{\vec{r}}$), forces of attraction between the charges ($m\omega_0^2\vec{r}$) and damping or energy loss within the system ($2\zeta\omega_0\dot{\vec{r}}$).

When this system is excited by an oscillating electromagnetic wave, the relative position of the two charges will also oscillate in like manner:

$$\vec{E} = \vec{E}_0 e^{-i\omega t} \quad (1.11)$$

$$\vec{r} = \vec{r}_0 e^{-i\omega t} \quad (1.12)$$

However, of particular importance here is the presence of damping, or dissipative absorption, something that arises in most mechanical systems as a result of electron collisions and other loss mechanisms. Damping consumes energy and tends to diminish motion, resulting in a phase delay ϕ between the incident electromagnetic wave and an oscillator’s scattered response such that:

$$\phi = \tan^{-1} \left[\frac{2\zeta\omega / \omega_0}{1 - (\omega / \omega_0)^2} \right] \quad (1.13)$$

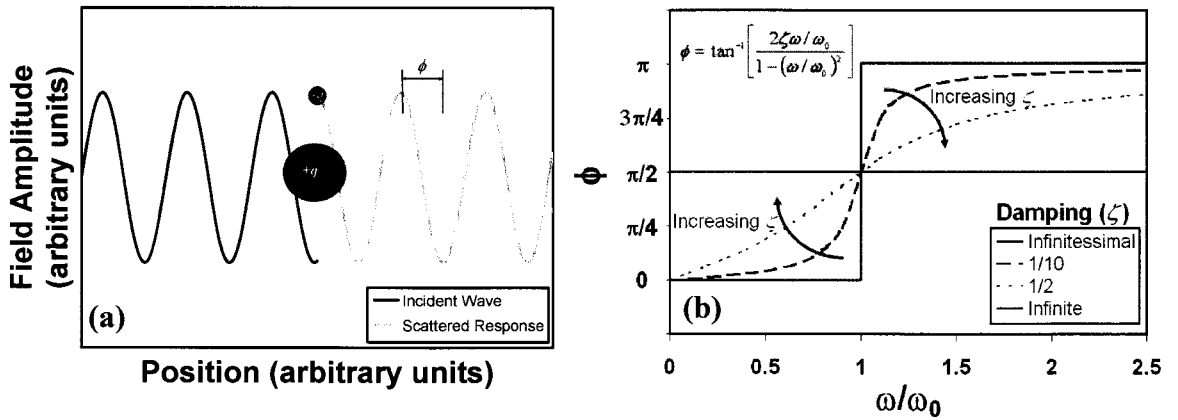


Fig. 1.9 – Frequency dependent phase delay created by a damped dynamic system subjected to forced vibrations, where ω is the oscillation frequency, ω_0 the natural frequency and ζ the damping / dissipative absorption of the system. (a) depicts the generation of phase delay between incident and scattered waves. (b) illustrates how this delay depends on damping, natural frequency and the frequency of the scattered field.

The generation of the phase delay between an incident and scattered wave is depicted in Fig. 1.9(a), with 1.9(b) illustrating how this delay depends on an oscillator’s damping,

natural frequency and the frequency of the scattered field. On a macroscopic level this phase delay and related attenuation are typically characterized by a material's complex refractive index, which not only depends upon the properties of each individual oscillator, but on the total delay accumulated over successive oscillator interactions, proportionate to material density as modeled by the Lorentz-Lorenz / Clausius-Mossotti formula, which relates the refractive index of a substance to the concentration of scatterers and their mean polarizability (p. 542 of [31]).

The frequency dependence of this delay explains the phenomenon of chromatic dispersion, with Eq. (1.13) bearing a strong resemblance to Sellmeier's formula [36], which gives an empirical relationship between refractive index and wavelength for a particular transparent medium. Thus, the very nature of chromatic dispersion supports the fact that there is a relationship between a material's absorption and refractive index characteristics.

Understanding the impact that individual oscillators have on a material's refractive index helps to develop an awareness of the manner in which changes to a medium's refractive index might be induced, and provides insight into other important macroscopic effects such as birefringence, trirefringence and the like, phenomena that arise due to asymmetries in media, which create refractive index differences as a function of polarization.

1.2.3 Polarizing effects of dipole scattering

In addition to birefringence, another polarization dependent material scattering phenomenon warrants consideration in dealing with the study at hand, specifically the directional dependence of polarized light scattered from a dipole.

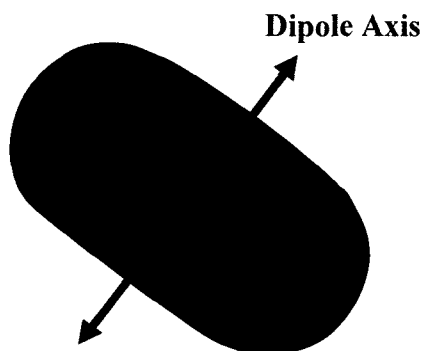


Fig. 1.10 – Toroidal intensity distribution of a radiating dipole.

As a consequence of the fact that light mainly propagates as a transverse electromagnetic wave, at radial distances much larger than the charge separation, the emission intensity of an oscillating dipole has a cosine dependence on radial angle. In three dimensions this results in a highly directional toroidal radiation pattern as shown in Fig. 1.10, with a maximum intensity at 90° to the dipole axis and with virtually no light being emitted along the axis.

1.3 Scattering from surfaces and periodic structures

Now that several concepts fundamental to the nature of light have been established, it is prudent to review some of the effects that can result from light propagating within discontinuous or non-uniform media.

1.3.1 Surface scattering

The simplest and most basic discontinuity one needs to account for is the surface or interface separating two different media, generally possessing distinct refractive indices. As a result

of this difference, two primary scattering phenomena arise, refraction and reflection. This point is well known, with the scattering angles of each beam, being respectively predicted by Snell's Law Eq. (1.14), and the Law of Reflection Eq. (1.15).

$$n_t \sin \theta_t = n_i \sin \theta_i \quad (1.14)$$

$$\theta_r = \theta_i \quad (1.15)$$

where n_t and n_i are refractive indices of the transmission and incident media, while θ_t , θ_i and θ_r are the angles of transmission, incidence and reflection illustrated in Fig. 1.11.

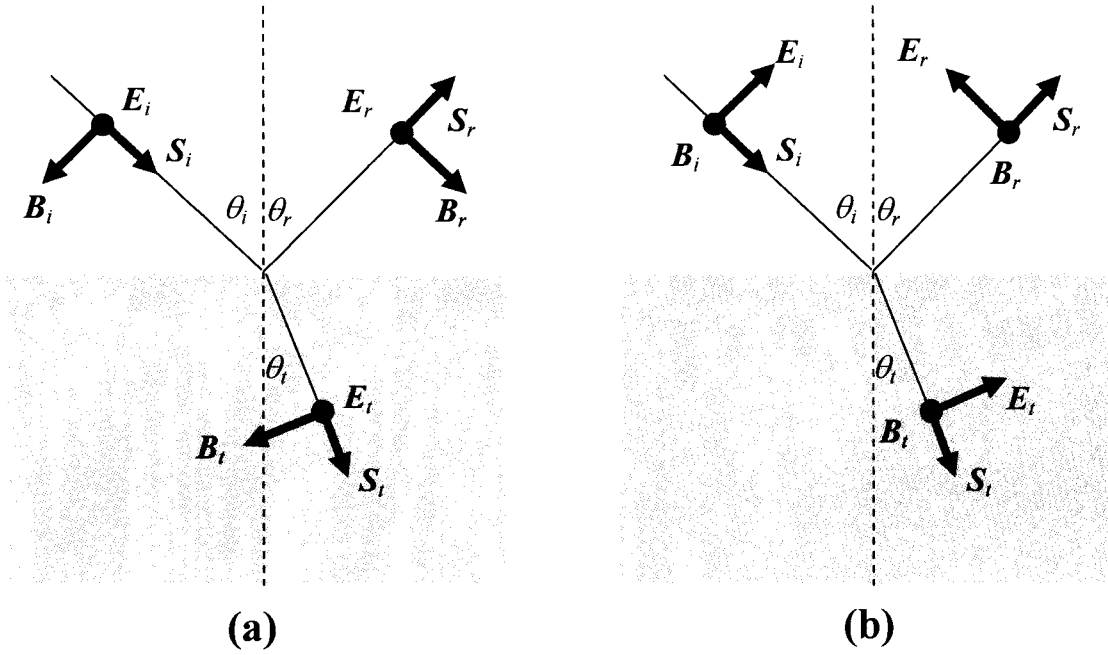


Fig. 1.11 – Schematic of incident, reflected and refracted electromagnetic waves for (a) *s*-polarized and (b) *p*-polarized light. Field components depicted with circles are assumed to point outwards towards the reader.

Beyond this simple overview however, there remain at least three consequences of importance to the study undertaken here: the possibility of total internal reflection, the phase shift sometimes experienced by scattered light, and the fact that scattering intensities from surfaces non-normal to the incident beam are polarization dependent.

Total internal reflection and the required critical angle of incidence are easily predicted by Snell's Law Eq. (1.14), whereas the other two phenomena are best quantified by the Fresnel equations Eqs. (1.15)-(1.18), which among other things lead to the identification of Brewster's angle at which only *s*-polarization (light polarized parallel to the surface) is reflected, and recognition of the fact that *s*-polarized waves experience a π -phase shift on reflection from higher index media.

$$r_s \equiv \left(\frac{E_r}{E_i} \right)_s = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \quad (1.16)$$

$$t_s \equiv \left(\frac{E_t}{E_i} \right)_s = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \quad (1.17)$$

$$r_p \equiv \left(\frac{E_r}{E_i} \right)_p = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \quad (1.18)$$

$$t_p \equiv \left(\frac{E_t}{E_i} \right)_p = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \quad (1.19)$$

1.3.2 Crystal structures

Sometimes the discontinuities in non-uniform media are arranged in a regular, predictable, periodic manner. Examples of such structures include one-dimensional thin film reflection and anti-reflection coatings, shown in Fig 1.12 (a) and (b) respectively. These coatings are formed from alternating layers of high and low refractive index materials. Utilizing elements of the optical theory presented thus far, one can show quite easily that when such layers have a thickness of a quarter or half wavelength respectively, purely constructive or destructive interference of the scattered light will result.

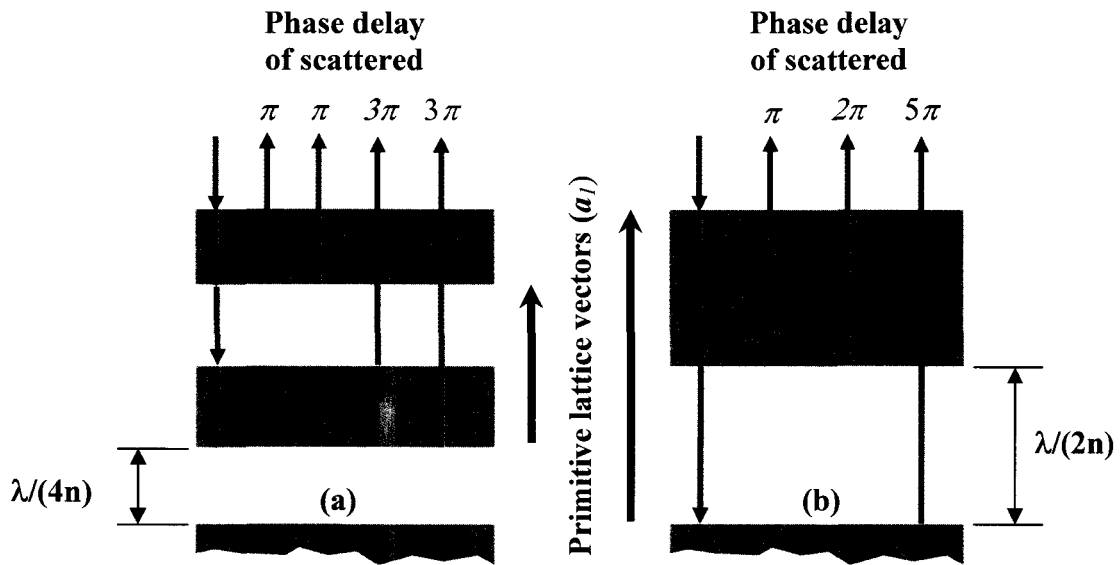


Fig. 1.12 – One-dimensional periodic thin-film (a) reflection and (b) anti-reflection coatings.

Of course, periodic order can also arise in multiple dimensions as well, like the two-dimensional periodicity associated with a typical photonic crystal fibre (PCF) shown in Fig. 1.13, and the three-dimensional periodicity inherent in many naturally occurring and man made crystals.

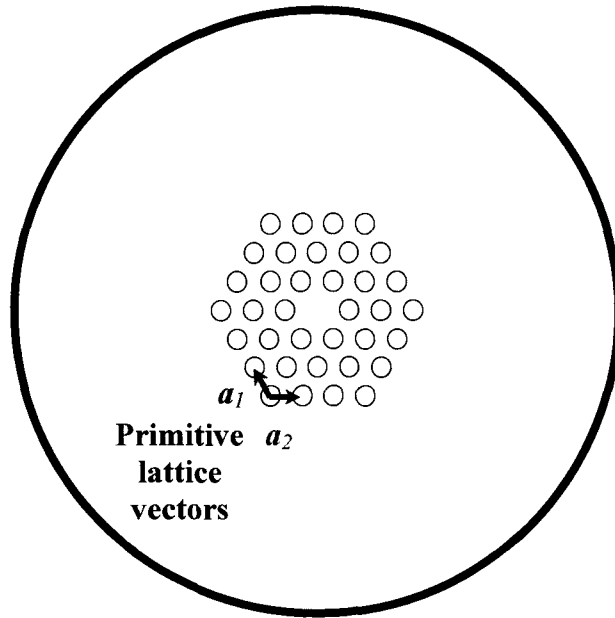


Fig. 1.13 – Two-dimensional periodicity of a PCF cross-section.

In the generalized three-dimensional case, such periodicity can be defined at the most basic level in space by three primitive lattice vectors $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$, as shown in Fig. 1.14.

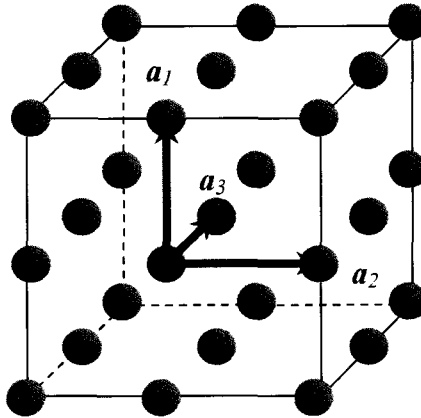


Fig. 1.14 – Three dimensional periodicity of a simple cubic crystal structure.

and in the frequency domain by three primitive vectors of the reciprocal lattice $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ (p. 86 of [31]).

$$\vec{b}_1 = 2\pi \frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3} \quad (1.20)$$

$$\vec{b}_2 = 2\pi \frac{\vec{a}_3 \times \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3} \quad (1.21)$$

$$\vec{b}_3 = 2\pi \frac{\vec{a}_1 \times \vec{a}_2}{\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3} \quad (1.22)$$

Once a crystal has been defined in terms of its primitive reciprocal lattice, the prediction of its elastic scattering response is relatively straight forward and may be accomplished vectorially using Conservation of Crystal Momentum as shown in Fig. 1.15. Specifically this figure illustrates three examples of elastic scattering from a simple cubic crystal structure: (1) orthogonal phase matched scattering in which purely constructive interference occurs at 90° to the incident beam, (2) resonant retro reflection where such interference occurs at 180° , and (3) quasi-phase matched or detuned scattering where a phase mismatch exists between scattered components of the new wavefront.

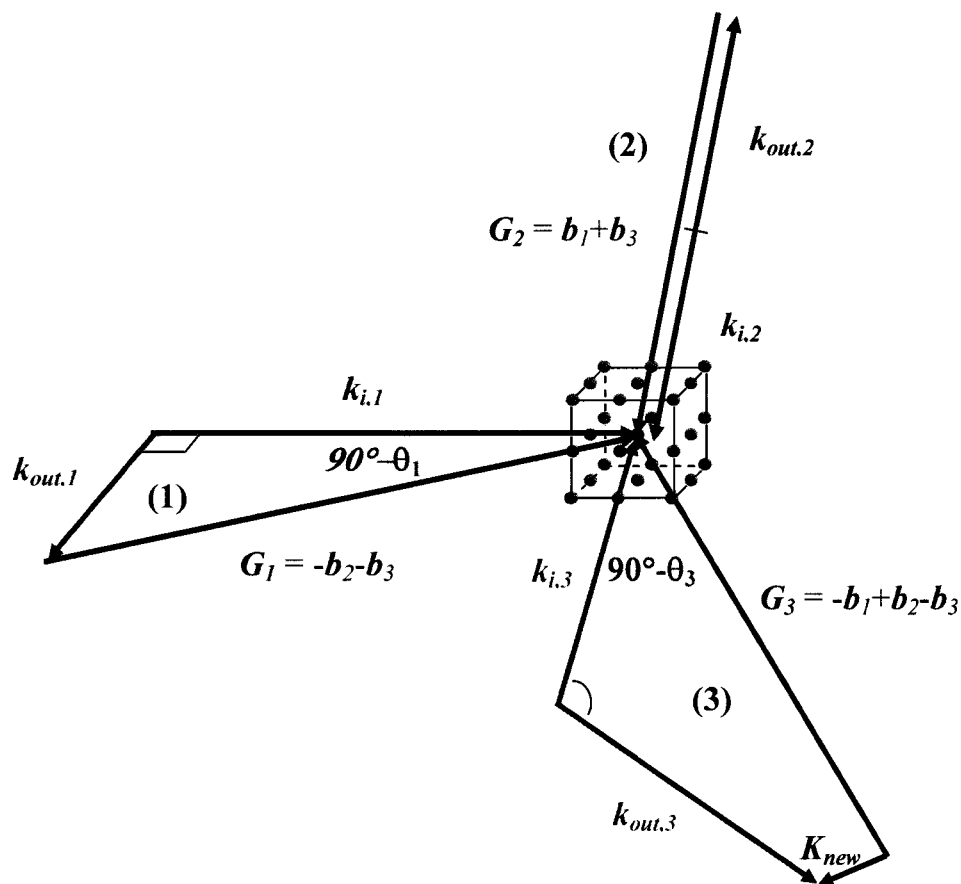


Fig. 1.15 – Three examples of elastic scattering from a simple cubic crystal structure, including (1) orthogonal phase matched scattering in which purely constructive interference occurs at 90° to the incident beam, (2) resonant retro reflection where such interference occurs at 180° , and (3) quasi-phase matched or detuned scattering where a phase mismatch exists between scattered components of the new wavefront. Note that the scattered wavevectors have been offset from their true positions to illustrate their vectorial dependencies.

For circumstances where the difference between scattered $(\vec{k}_{out})$ and incident $(\vec{k}_i)$ propagation vectors can be exactly represented by a linear combination of primitive reciprocal lattice vectors $(\vec{G} = v_1\vec{b}_1 + v_2\vec{b}_2 + v_3\vec{b}_3, v_i \in \mathbb{Z}, \vec{K}_{new} = 0)$, resonance or purely constructive interference is expected. Otherwise the interference is not purely constructive and detuning from resonance will be observed $(\vec{K}_{new} \neq 0)$, with weaker scattering intensities resulting from larger degrees of detuning.

For resonant scattering the well known Bragg condition takes the following form (p. 96 of [31]):

$$m\lambda = 2n_{eff}\Lambda \sin \theta \quad (1.23)$$

where the order number $m \in \mathbb{Z} \geq 1$, λ is the wavelength of scattered light in vacuum, n_{eff} is the effective refractive index of scattered light in the medium, Λ is the period of the lattice vector and θ is the angle between the incident light and the plane of atoms normal to the lattice vector.

The simple examples presented in Figs. 1.12-1.15 depict perfectly uniform structures, however in reality, for a variety of practical reasons periodic arrangements of scattering elements may incorporate continuous variations in optical periodicity (chirp), or scattering strength (apodization) from one element to another. If we consider a one-dimensional periodic structure such as the thin film filter shown in Fig. 1.12 (a) or a Bragg grating filter detailed in Chapter 2, the resonant interaction of electromagnetic radiation with that structure will depend on its periodicity. By varying that periodicity continuously, the structure is resonant with a broader range of wavelengths, as shown in Fig. 1.16 (b). With a uniform periodic structure of finite length, the discontinuity between the absence and presence of periodic scatterers results in a ‘ringing’ of the reflection response when the interacting wavelength is detuned from resonance, Fig. 1.16 (a). By reducing scattering strength near the structure’s edges and thereby easing the transition between absence and presence of scattering elements, ‘ringing’ of the detuned wavelength response is reduced as illustrated by Fig. 1.16 (c). With this in mind, chirp is often used to broaden the bandwidth of resonant wavelengths and induce chromatic dispersion, while apodization is applied to suppress detuned sidebands and improve contrast.

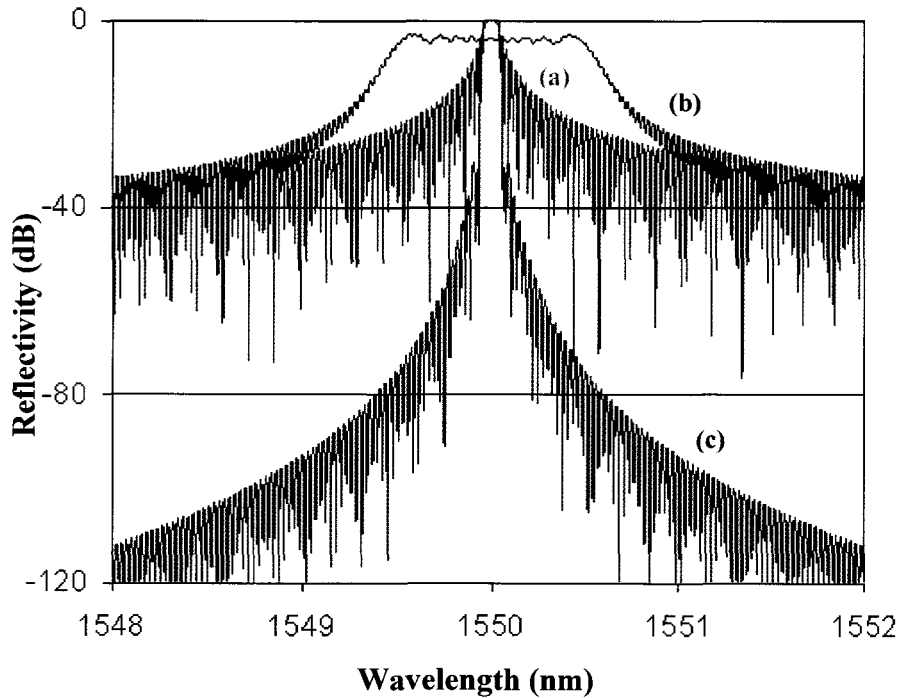


Fig. 1.16 – Simulated reflectivity responses of (a) uniform, (b) chirped and (c) apodized periodic structures.

Chapter 2

Fabrication Technology of FBGs and Their Characteristics

With some understanding of the basic physical concepts now firmly established, this chapter seeks to build upon that knowledge and provide the reader with a brief overview of fibre Bragg grating technology. This necessarily requires a summary of optical waveguide fundamentals and Bragg grating fabrication methods, but also benefits from an examination of photosensitivity mechanisms and characteristics, as well as a discussion of birefringence and lensing effects. Beyond the specific references provided, the reader is referred to such texts as [37-40] for a more extensive treatment of the material summarized here.

2.1 Optical waveguide fundamentals

Owing to numerous practical considerations, line of sight communication is not always possible or sufficient, and in the optical domain, efficient reflection of a signal from the Earth's ionosphere cannot be used to extend its range or bypass obstacles in its path. With such limitations in mind, having the ability to confine and guide light with relatively low loss in optical fibre waveguides, is one of the primary technologies responsible for the rapid growth and popularity of the vast majority of optical networks in use today.

2.1.1 Types of optical fibre

Along these lines, optical fibres have proven themselves useful for a wide variety of applications. Such fibres are available in numerous forms each having their own unique characteristics, from total internal reflection (TIR) fibres to fibres based entirely on Bragg scattering, typically most fibres are manufactured with at least one core and cladding, whereby the majority of guided light is confined to a core, while cladding layers serve to minimize evanescent losses, tailor a fibre's mode characteristics and provide structural support for the core layers.

Most commonly in use today are cylindrical, step-index, single mode fibres (SMFs), having a single germanosilicate core surrounded by a single pure silica cladding as illustrated in Fig. 2.1. As a result of germanium doping in the core, this region of the fibre has a higher refractive index than the cladding, making guidance by total internal reflection possible.

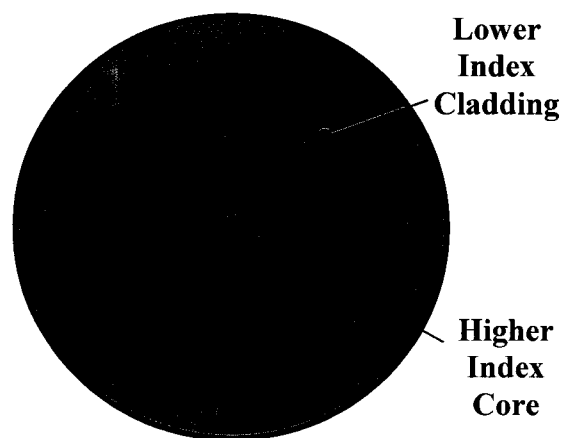


Fig. 2.1 – Standard cylindrical, step-index, single mode fibre.

Of course, the refractive index profile within a fibre is not limited to such simple step index designs. Fig. 2.2 presents several examples of different kinds of optical fibre. Many fibres obtain their unique dispersion and polarization characteristics from elliptical (Fig. 2.2 c) and graded index cores (Fig. 2.2 a), or multilayer claddings (Fig. 2.2 b), and more recently from a variety of photonic crystal fibre (PCF) structures (Fig. 2.2 e). Some of these PCF fibres still take advantage of TIR, reducing the effective cladding index below that of the core, through incorporation of air holes. Others however, particularly air-core fibres, rely entirely on Bragg scattering to achieve mode confinement.

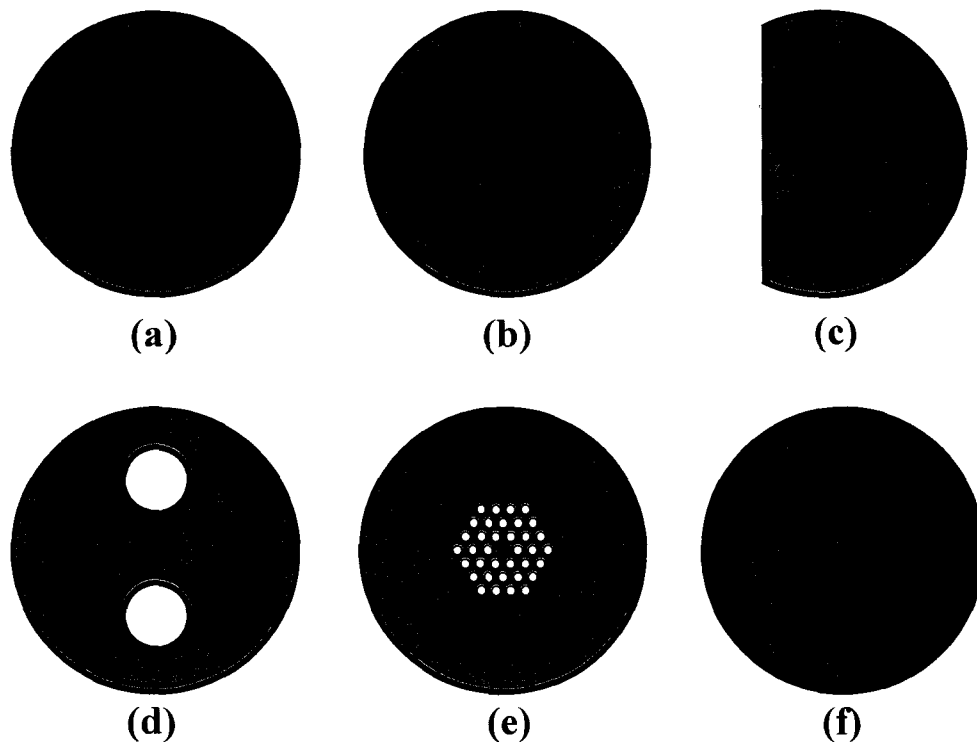


Fig. 2.2 – Some alternative fibre structures including (a) graded index, (b) dispersion shifted, (c) D-fibre, (d) PANDA, (e) TIR PCF and (f) 1D Bragg fibre.

2.1.1 Consequences of light confinement

As a consequence of light confinement, its interaction with the waveguide differs from that of bulk media and several effects are generated of which the reader should be aware. For one thing, light guided by a waveguide no longer regenerates a uniform plane wave but propagates in the form of modes, which each represent orthogonal solutions of Maxwell's equations, subject to the boundary conditions imposed by the waveguide structure.

Fibres are often classified as single mode or multimode waveguides. The normalized frequency of step index fibers $V = \frac{2\pi \cdot a}{\lambda} \sqrt{n_0^2 - n_{cl}^2}$ is defined at wavelength λ in terms of their core radius a , core index n_0 and cladding index n_{cl} . For such fibers to be single mode $V \leq 2.4$. Multimode waveguides can be very cost effective, reducing alignment tolerances and peak energy intensities when modal dispersion is negligible, however for most telecom systems, standard single mode fibre is used. Fig. 2.3 was generated using commercially available software and illustrates a few examples of different mode structures supported in the core and cladding regions of single mode optical fibres.

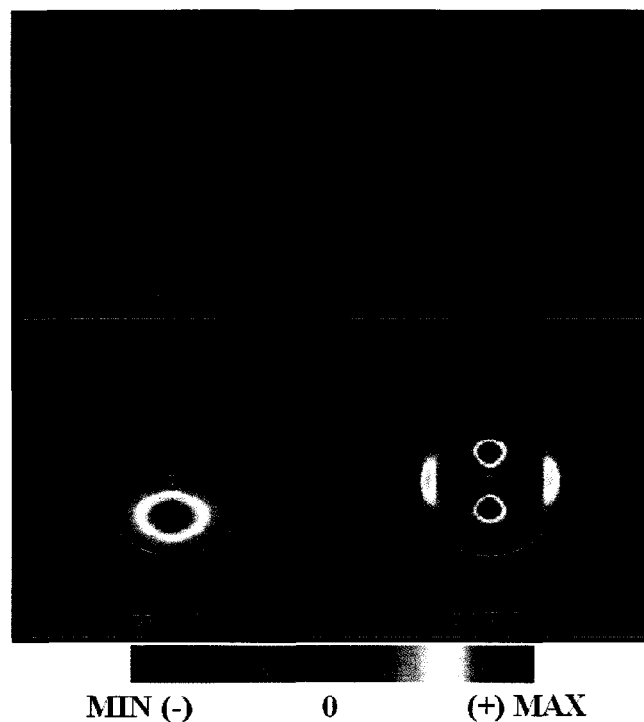


Fig. 2.3 – Several modes of standard SMF, superimposed on the fibre cross-section.

These modes each have their own effective refractive index, which ranges between that of the wave guiding material and the surrounding medium, depending on the mode’s particular energy distribution. For instance, the software used to generate Fig. 2.3 also enables plotting of Fig. 2.4, demonstrating that in standard single mode optical fibre, core modes have indices greater than the cladding index, while cladding modes have indices greater than that of the jacket or external medium.

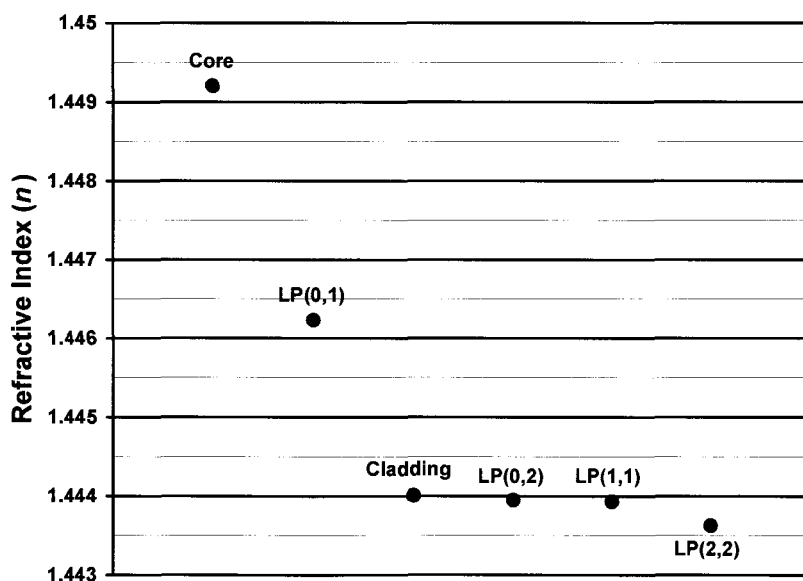


Fig. 2.4 – Refractive indices of core-cladding materials, and modes associated with Fig. 2.3.

As a result of chromatic dispersion in the core and cladding materials, the waveguide boundary conditions actually vary as a function of wavelength, leading to wavelength dependence of the modal energy distributions and a waveguide induced dispersion component that can be tailored to flatten or shift the chromatic dispersion characteristics associated with a particular fibre structure.

Similarly, though the substrate materials of a particular fibre may be ordinarily quite isotropic, by introducing asymmetries in the fibre structure itself, it is possible to alter the boundary conditions in such a way as to generate waveguide birefringence even when no underlying material birefringence is present, and to induce material birefringence through the generation of residual stress gradients. This is a principle advantage of the elliptical core and PANDA fibres illustrated in Fig. 2.2 (c) and (d), as well as other asymmetric PCF structures.

One final note is that because each mode essentially propagates as a nonuniform plane wave (see 1.1.3), their electric fields may have longitudinally polarized components oriented in the direction of propagation (p. 79 of [34]). Although in weakly guiding structures like standard fibre, such longitudinal components are significantly small relative to the transverse electric field (p. 291 of [38]); their presence cannot always be ignored [18].

2.2 Fibre Bragg gratings

2.2.1 Fabrication methods

Fibre gratings have been manufactured in any number of ways, including chemical etching [41], the application of mechanical strain [42] and melting of the glass substrate using a CO₂ laser [43]. In many cases, such processes are only suitable for the formation of long period structures, and so Bragg gratings are generally photoinduced, typically using the side-writing technique and a high-power ultraviolet (UV) laser [21].

The discovery of photosensitivity in optical fibre was first made at the Communications Research Centre Canada back in 1978 when Ken Hill et al, observed that 488 nm light from an Argon ion laser was being increasingly attenuated in transmission as a result of back reflection in the germanium doped fibre [19]. Later, it was discovered that this increasing reflection resulted from a permanent periodic index modulation, photoinduced through a two-photon process, and modulated by the standing wave interference pattern formed by transmitted and reflected waves. Such formation of a Hill type Bragg grating is shown schematically in Fig. 2.5.

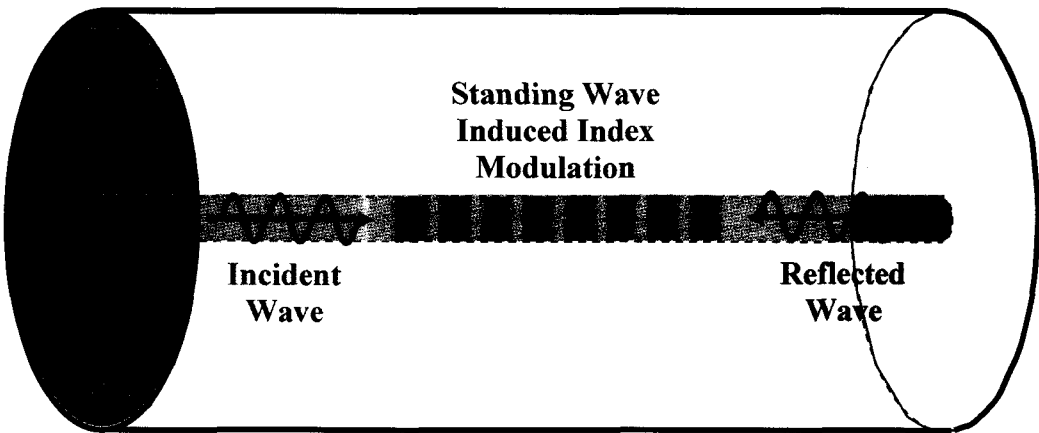


Fig. 2.5 – Illustration of Hill grating.

The benefits of these Hill gratings were apparent, natural fibre compatibility, easy insertion, low insertion loss and low fabrication costs. However a number of drawbacks were also noted, in particular the dependence of the photosensitivity mechanism and grating resonance on the writing wavelength and substrate material. The first limitation in particular was especially restrictive in that the principal transmission windows of standard optical fibres are in the infrared, near 1310 nm and 1550 nm. How could a grating resonant at 488 nm see any widespread use under these circumstances?

Although research on fibre photosensitivity continued, it was not until a decade later that one of these shortcomings was addressed, when researchers at United Technologies photoinduced gratings by using a bulk interferometer to project a periodic pattern of UV light through the side of a fibre and onto its photosensitive core [20]. The set-up for this interferometric side writing technique is shown in Fig. 2.6. By performing this side inscription, the period of the grating and hence its Bragg resonance, were no longer dependent on the wavelength of the writing beam, and changing periods became a simple matter of altering the angle between the two interfering wavefronts ϖ .

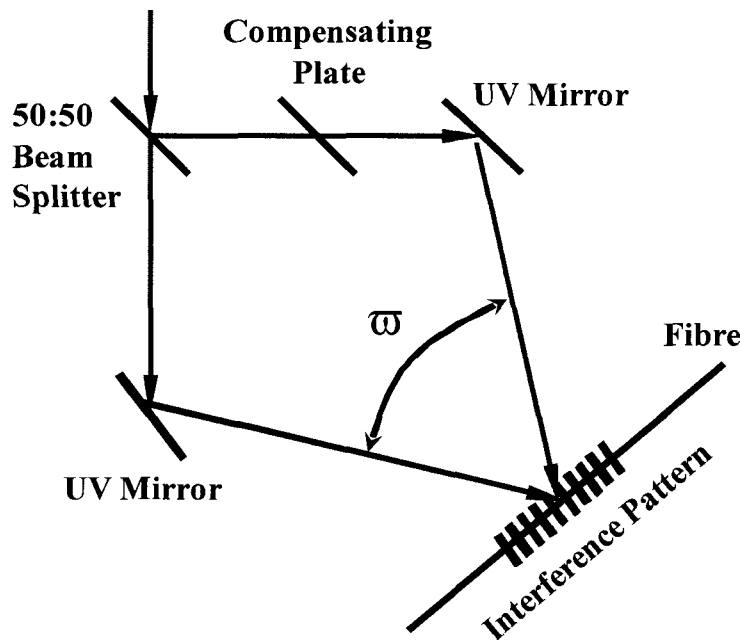


Fig. 2.6 – Bulk interferometer side-writing setup.

However this approach has its own shortfalls, which are rooted in the mechanical stability of the interferometer itself. The UV source must have high temporal coherence, so that the two path lengths are sufficiently matched to produce a fringe pattern when the interfering beams are overlapped at the fibre. If the path length difference of the two interfering beams is less than the laser’s coherence length, then a fringe pattern can be generated. For continuous wave (CW) sources such as frequency doubled argon ion lasers, the coherence length over which propagating light remains in phase with itself may be several centimetres. For high powered excimer lasers however, this length may be as small as a fraction of a millimeter. Combined with the sub-nanometre tolerances of optical systems, the ease and reproducibility of grating manufacture becomes sensitive to mechanical adjustments and variations in optical path length as a function of air currents and temperature fluctuations.

In order to address these issues, Hill et al proposed another option, side-writing using a specialized transmission diffraction grating called a zero-order nulled phase mask [21]. In this case the phase mask is used to generate the interference pattern directly. Fig. 1.5 of the previous chapter provides an illustration of this device and its working principle. To ensure that maximum energy is coupled to the ± 1 orders, the etch depth of the mask is precisely controlled and typically the $m = 0$ order is suppressed such that it transmits less than 5% of the incident light.

Reproducibility of fabricated gratings is increased because the phase mask is a relatively simple mechanical system with a set periodicity, fixing the angle of the scattered orders and in turn the period of the photoinduced grating. Unlike the bulk interferometer used in [20], the phase mask approach automatically matches path lengths of the interfering beams, making it ideally suited for grating inscription with excimer lasers, the most powerful UV laser sources available. These lasers, by their nature, are highly multimode UV sources with large spectral line widths of ~ 1 nm and very low coherence, with a spatial coherence length on the order of ~ 300 μm [44]. As the fibre is typically placed proximate the phase mask during exposure, to within a few microns or millimetres, optical path length variations arising as a result of air currents are also minimized.

In this work, the phase mask technique was used along with an excimer laser source. A photograph of a typical set up used to write retroreflective Bragg gratings is shown in Fig. 2.7. Often, a cylindrical lens is used to increase the writing beam intensity while maintaining normal incidence with respect to the phasemask, and micrometer resolution mechanical positioners are utilized in order to properly align the lens, phasemask and fibre with respect to one another.

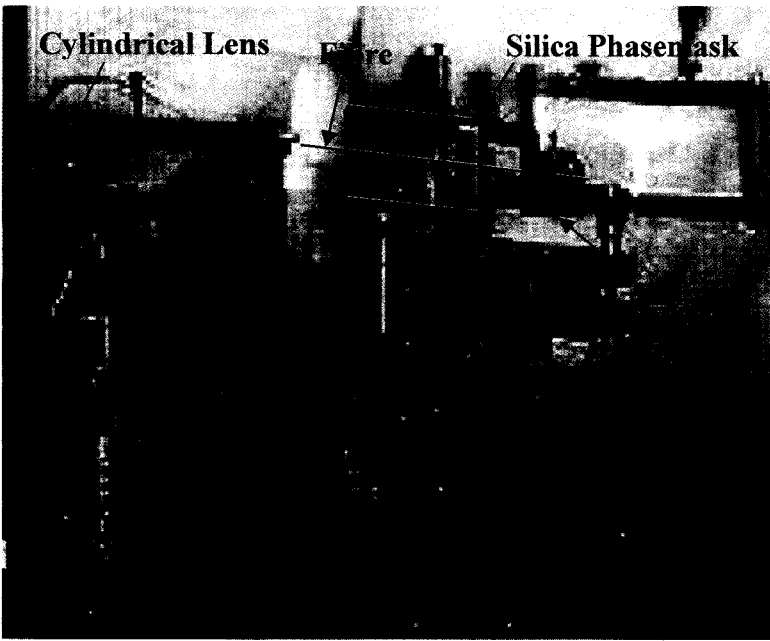


Fig. 2.7 – Photograph of typical side-writing setup, showing an incident UV beam, focused through a cylindrical lens and phase mask onto a fibre. The fluorescing glass plates behind the set up are used for alignment purposes.

2.2.2 Mode coupling characteristics

As a result of their periodicity and ease of insertion into waveguide structures, Bragg gratings are extremely useful mode coupling components. Originally, the primary mode coupling of interest was the retro-reflection of a single mode fibre's fundamental guided mode, but since their initial discovery, photoinduced FBGs and so-called long period gratings (LPGs) have shown themselves to be useful for coupling light between cladding and radiation modes as well.

Such capabilities are necessary in a wide variety of components, including dispersion compensators [45], optical add-drop filters for wavelength division multiplexing [46], cavity mirrors in fibre lasers [47], temperature, strain and refractive index sensors for environmental monitoring [48], and the list goes on.

In characterizing the performance of such components, typically one is interested in their spectral properties as measured by their transmitted and reflected power. Drawing upon the discussion of section 1.3.2, it seems apparent that the responses of these periodic structures should be maximized at resonant wavelengths for which the interference is purely constructive, while increasingly weaker responses are expected as the wavelength is detuned from resonance. In Fig. 2.8, the measured spectrum of a typical FBG is presented. As can be seen from the figure, the coupling of the grating into a reflected signal is reduced as the wavelength of the fibre's guided mode is detuned from the Bragg resonance, where the Bragg resonant wavelength satisfies Eq. 1.23.

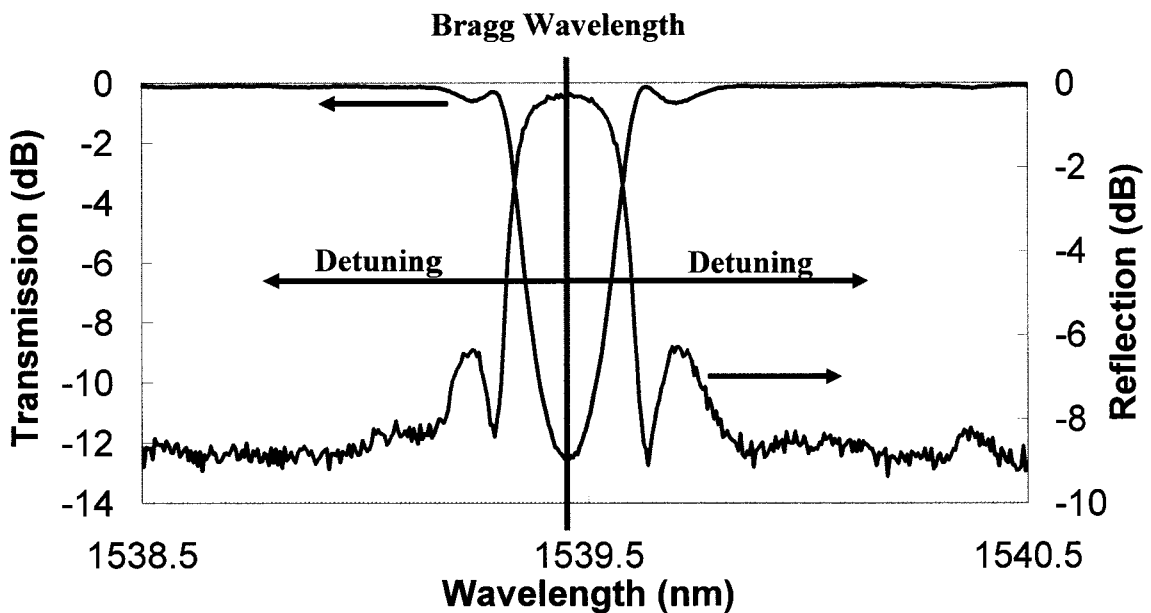


Fig. 2.8 – Example of transmission and reflection spectra for a retro-reflective FBG, showing the resonant wavelength peak, and the detuned response at non-Bragg wavelengths.

2.2.3 Tilted gratings

In addition to long period gratings, there is another subset of fibre gratings that is used to facilitate coupling to cladding and radiation modes. Tilted, blazed or slanted FBGs, are those devices in which the index modulation is skewed or tilted with respect to the fibre axis, so

that planes of equivalent refractive index change are no longer normal to the fibre axis [49]. A schematic of a tilted Bragg grating is shown in Fig. 2.9.

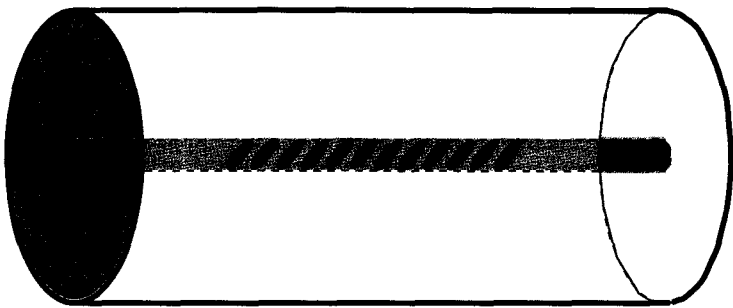


Fig. 2.9 – Schematic of a tilted FBG.

Unlike LPGs however, the radiated energy of tilted FBGs can be strongly directional and highly polarizing, as expected from conservation of momentum. As a result, these devices are quite useful as fibre polarizers and PDL compensators [8,9].

Additionally, because of their asymmetry about the fibre axis, these gratings are capable of coupling to non-symmetric fibre modes, and also generate an asymmetric response, with light incident from one direction scattered to one side, and light incident from the other direction scattered to the opposite side. Making use of this feature, an optical isolator has been proposed [50], with its operating principle illustrated in Fig. 2.10.

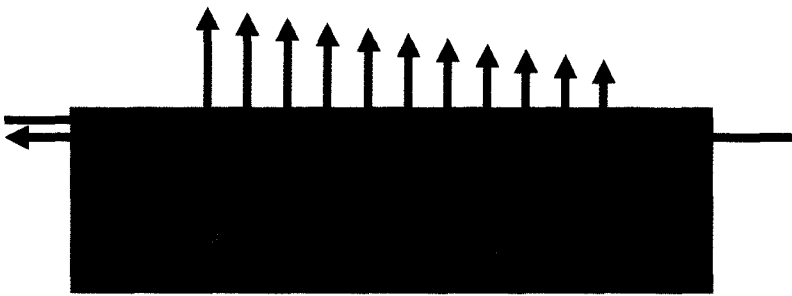


Fig. 2.10 – Schematic of a tilted Bragg grating optical isolator. Different claddings on top and bottom of waveguide frustrate coupling of light passing from right to left, but enable attenuation through radiation mode coupling of light passing from left to right.

2.3 Photosensitivity considerations

The study of photosensitivity mechanisms in glass is extensive and full of intricate detail. In germania-doped silicate glasses, Hill gratings were written with visible 488 nm light from an Argon ion laser.

The bulk of subsequent FBGs have been written in similar glasses using high-powered UV sources, such as KrF (248 nm) and ArF (193 nm) pulsed excimer lasers, and 244 nm continuous wave frequency doubled Argon ion systems.

More recently, ultrafast picosecond and femtosecond pulsed, infrared (IR) [51] and UV [52] sources have also been used, with IR sources proven useful for writing in a huge assortment of glasses and crystals, greatly expanding the possibilities for FBG components.

Photosensitization with Hydrogen or Deuterium is also commonly used in order to lower the threshold for grating formation and promote rapid, strong growth of the induced index change [53,54].

2.3.1 Type I and Type II index change

In any event, regardless of whether gratings are inscribed conventionally with a UV laser, or using an ultrafast source, there are two primary classifications for index change, one associated with subtle electrochemical changes, defect formation and compaction at the molecular level (Type I), and the other resulting from a more macroscopic restructuring of the material itself (Type II).

Examples of mechanisms responsible for Type I photosensitivity include the freeing and trapping of valence electrons, or the breakage and formation of relatively weak molecular bonds, whereas Type II restructuring results from such things as ionization induced dielectric breakdown, densification or compaction, residual stress relief and the formation of cracks or microvoids.

As a consequence, Type I index changes can be annealed out relatively easily at elevated temperatures, while Type II photosensitivity is stable with thermal loads approaching the glass transition temperature of the waveguide substrate itself [55]. In Fig. 2.11 (a) isochronal annealing curves are presented of grating index modulations produced using either an ArF UV laser with an operating wavelength of 193 nm or an ultrafast Ti:sapphire regenerative amplifier operating in the near-IR at a wavelength of 800 nm. For the UV case the grating is seen to erase as the temperature of the device is incremented. In the case of the ultrafast IR laser, the photoinduced index change has varying components that can be considered Type I and Type II. When 125 fs pulse durations are used to write the grating, the part of the index modulation that lies between the high intensity portions of the interference field results in a Type I index change that is annealable. Progressive increases in temperature result in an erasure of index change that was generated between the high intensity portions of the interference field, causing an increase in the index modulation. In the picosecond regime a larger component of the index change is due to Type I mechanisms, hence there is a larger erasure of the overall index modulation with temperature.

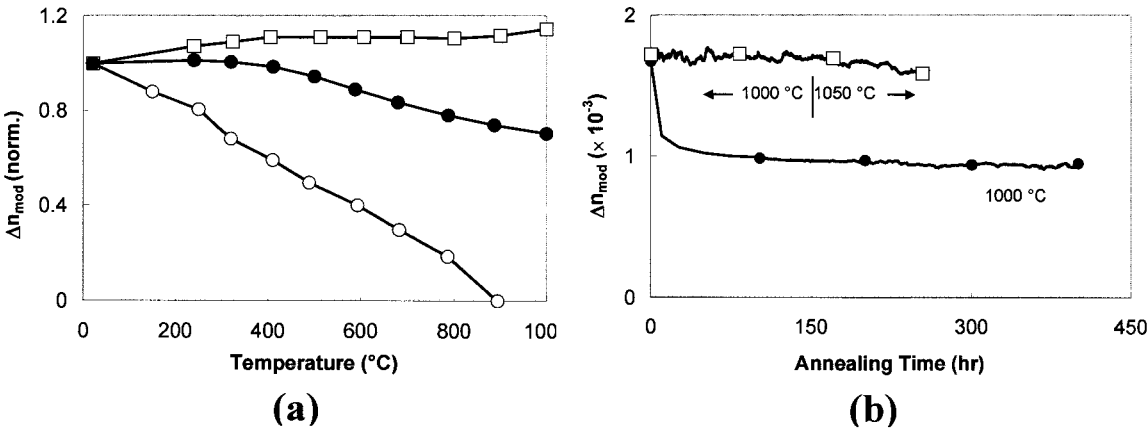


Fig. 2.11 – (a) Short term isochronal annealing of fs (white square) and ps (black circle) gratings written with ultrafast IR radiation and Type I UV (white circle) gratings. Each temperature increment was held for one hour. (b) Long term annealing at 1000 °C of fs IR (white square) and ps IR (black circle) gratings. After 150 hours, the annealing temperature for the fs grating was increased to 1050 °C [54].

In Fig. 2.11 (b) isothermal annealing curves of fs and ps –IR laser induced gratings are presented. With the gratings maintained at at least 1000 °C, in the case of gratings written with 125 fs pulses, there is no grating erasure. In the ps case, a component of the index change is annealed out, but the remaining index change is stable.

Although the annealing of Type I gratings will initially weaken their response, pre-annealing is often used as a means to stabilize their characteristics at temperatures below a chosen threshold [56]. An example of preannealing is demonstrated in Fig. 2.12 where gratings written in Bismuth Oxide fibres with an ultrafast laser in the Type I regime can be preannealed up to 250 °C (Fig. 2.12 a). The grating is then stable in temperature cycling up to 225 °C without any observable erasure (Fig. 2.12 b).

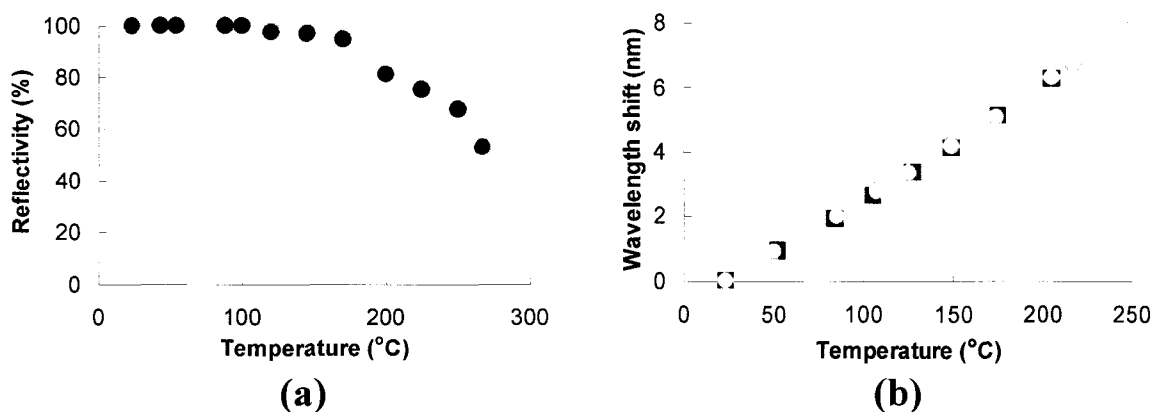


Fig. 2.12 – (a) Isochronal annealing profile of Type I, ultrafast IR FBG in BiO₂ fibre, and (b) λ_{Bragg} shift with temperature of same grating subject to pre-annealing. Increasing temperature cycle is denoted by circles, decreasing cycle denoted by squares [55].

2.3.2 Index modulation profiles

Because of the differences in photosensitivity mechanisms associated with various photoinscription processes, the index change or modulation induced within a fibre will vary depending on a number of factors.

Examining the side-writing process (discussed in 2.2.1) it becomes apparent that the first opportunity for variation lies in the intensity pattern projected onto the fibre axis. If the pattern is formed from the interference of two coherent beams it will be periodic in one dimension only. The resulting interference pattern has a sinusoidal intensity distribution. As three or more beams are added, two and three dimensional effects become possible.

A phase mask, even a zero-order nulled phase mask typically generates more than two diffracted beams with varying intensities. Usually, the ± 1 orders are the strongest, but even weak responses of the other diffracted orders can lead to non-uniformities of the projected pattern over the fibre cross section, manifesting in two dimensional periodicity known as the Talbot effect [57], and contributing to unintentional cladding mode losses and other undesirable effects.

This phenomenon can be avoided by writing with a bulk interferometer, or by using an ultrafast laser and employing the order walkoff technique [58,59], which takes advantage of the fact that the angular dispersion of the phase mask results in each $\pm$ order pair having a different velocity component perpendicular to the mask. Since the ultrafast pulses are extremely short in space as well as time, the different orders quickly separate from one

another. Placing the fibre a few millimetres from the phase mask, instead of directly beside it, then ensures that only two beam interference patterns are projected onto it.

In Fig. 2.13 the order walkoff effect of the diffracted femtosecond pulse duration beams from a phase mask irradiated by the ultrafast IR laser is presented. At different distances from the 4.28 μm pitched mask, either all diffracted orders, only 3 or 2 orders interfere resulting in the theoretical interference field shown in Fig. 2.13 (a), (b) and (c) respectively. Experimental results of corresponding gratings actually fabricated in SMF-28 optical fibre are presented in Fig. 2.13 (d), (e) and (f).

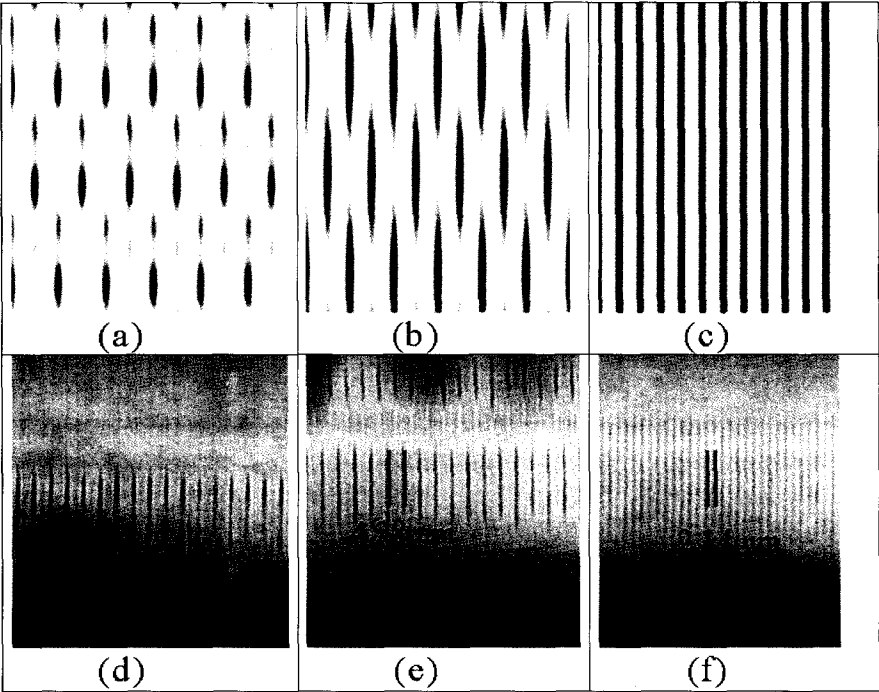


Fig. 2.13 – Simulation (a-c) and photographs (d-f), of five (a,d), three (b,e) and two beam (c,f) photoinduced index modulations [60].

A second cause for variation in index modulation patterns lies in the photosensitivity mechanism itself. Provided the process for index change depends on one photon (typical for Hydrogen loaded UV exposures), prior to saturation the induced index modification will resemble the projected $\cos^2$ intensity profile reasonably well. Keeping in mind that a $\cos^2$ index change consists of a constant term and a cosine modulation term, purely sinusoidal index modulations can be expected. Such a sinusoidal index modulation will not produce higher order retroreflective grating resonances. On the other hand for multiphoton processes entailing N photons, such as those required for ultrafast IR inscription, a one to one resemblance with intensity does not occur, and unsaturated index changes more closely resemble a $\cos^{2N}$ function [61]. Fig. 2.14 depicts how the variation in photonic order and grating saturation results in a physical difference to the index modulation.

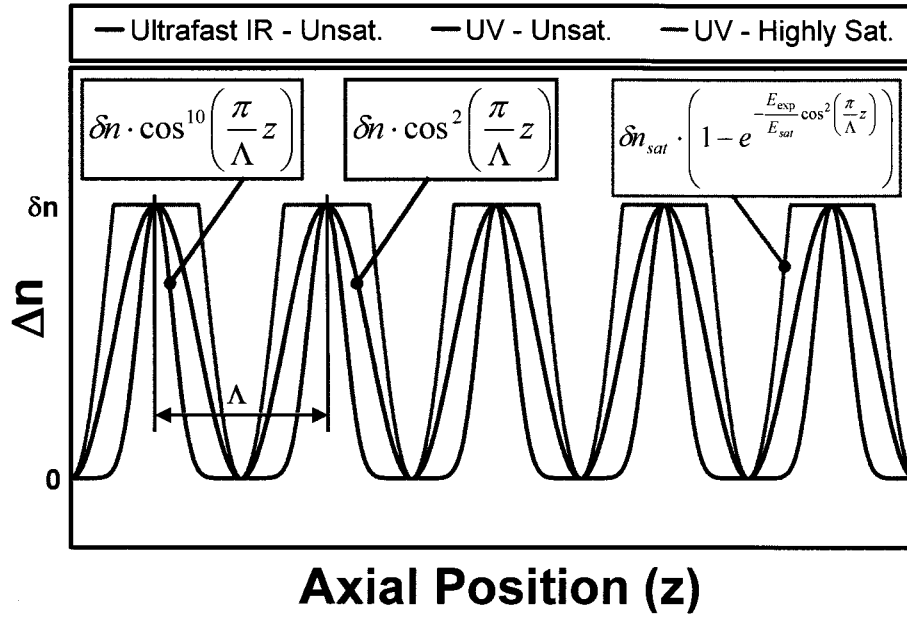


Fig. 2.14 – Unsaturated and saturated index modulation examples. E_{exp} is total energy of UV exposure, E_{sat} is saturation energy and δn_{sat} is saturated index change [61].

A third factor contributing to grating structure is the cladding's photosensitivity, which determines the extent of the grating in the radial direction. Although special fibres are available with Germanium doped claddings, Germanium doping is typically confined to the core of standard fibres, and hence the photosensitivity of their claddings to UV sources is generally significantly less than that possessed by their cores. On the other hand, with the advent of ultrafast IR photoinscription, gratings in the cladding of standard fibres are easy to fabricate, reducing undesirable coupling to cladding modes, creating birefringence in some cases, and potentially contributing to other effects.

Finally, transverse uniformity [62,63] and grating localization within the core are considerations that one may have need to examine depending on the fibre, writing laser and exposure conditions one is working with. These factors will be discussed further in 2.4.

2.4 Birefringence and lensing affects on FBG scattering

2.4.1 Birefringence

The affects of birefringence on a fibre Bragg grating's response are worthy on note, as the exploitation of such phenomena have proven useful in a variety of applications, including the simultaneous measurement and discrimination of temperature and strain from a single FBG [64].

Birefringent FBG responses can result from any number of different mechanisms. The strongest and most pronounced effects generally arise from grating inscription directly in highly birefringent (HiBi) or polarization maintaining (PM) fibres, which are based on highly non-axisymmetric structures. However, detectable birefringence can also be generated through the application of a non-axisymmetric stress gradient, or more directly as a consequence of the photoexposure itself [65,66].

So-called photoinduced birefringence can also be the product of multiple mechanisms, with asymmetry of the induced index change being one potential cause. Specifically, as a result of strong attenuation of the writing beam by the waveguide substrate, tight focusing

arrangements of the writing set up, or non-uniform relief or generation of residual stress gradients, the induced index changes may be non-uniform over the fibre cross section or non-concentric within the waveguide. This asymmetry of the UV-induced index change can be observed through a process of weak chemical etching in hydrofluoric acid (HF) of a fibre cleaved at the grating location, followed by atomic force microscopy (AFM) to measure the resulting topography [67]. Fig. 2.13 shows the crescent shaped index change ($> 1 \times 10^{-3}$) in the germania core of a single mode fibre after heavy exposure to ultraviolet radiation, as measured with an AFM.

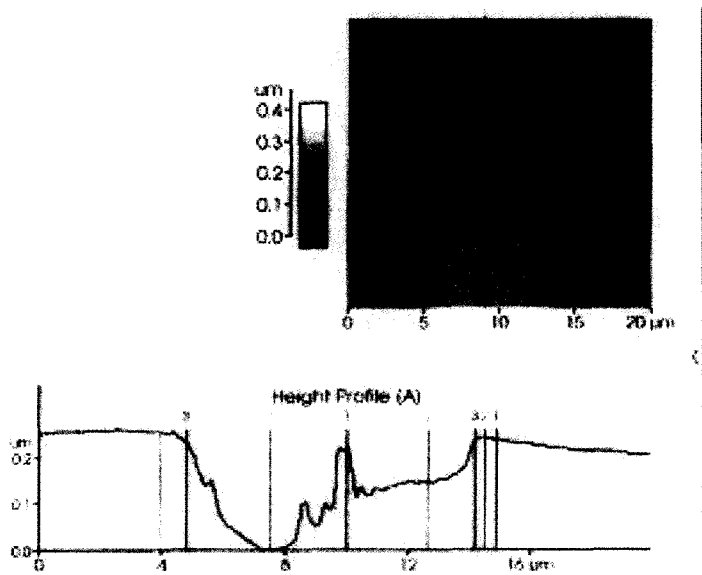


Fig. 2.15 – Atomic force microscope surface relief scan of FBG etched fibre end showing transverse asymmetry of induced index change [68].

Alternatively, even when the index change appears to be uniform on a macroscopic scale, photoinduced birefringence is possible as a result of dichroic absorption, whereby oscillators aligned with the electric field polarization of the writing beam, are modified preferentially, such that the refractive index in the direction of the writing beam polarization changes more than in other directions [69]. If the writing beam polarization is parallel to the fibre axis, then the impact of this effect on guided modes can be eliminated, on the other hand, if the polarization of the writing beam is orthogonal to the fibre axis, the impact of photoinduced birefringence on guided modes will be maximized [70], as shown schematically in Fig. 2.16.

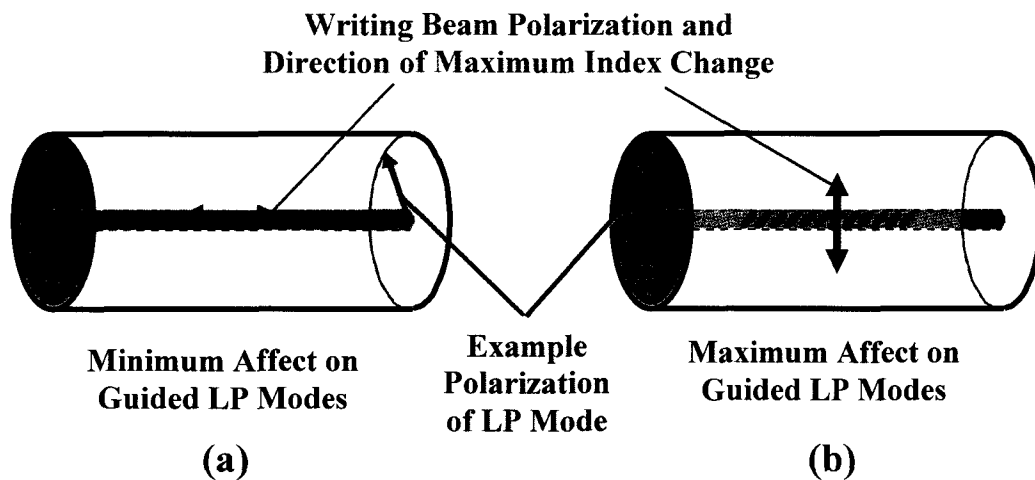


Fig. 2.16 – Illustration of dichroic index change, showing writing beam polarizations for (a) minimum and (b) maximum birefringence effects.

As a consequence of any of these mechanisms, orthogonal principal states of polarization (PSPs) will have slightly different effective mode indices (n_{eff}). It is then simple to see from the Bragg condition Eq. (1.23), that different wavelength resonances are expected for each PSP, the polarization dependent wavelength difference being termed $PD\lambda$ [71].

For tilted grating structures, using conservation of momentum as described in section 1.3.2, one can see that the effect of birefringence will primarily be a splitting in the scattered longitudinal angles combined with a slight degree of detuning as shown in Fig. 2.17, and indeed this is consistent with previously published work [49].

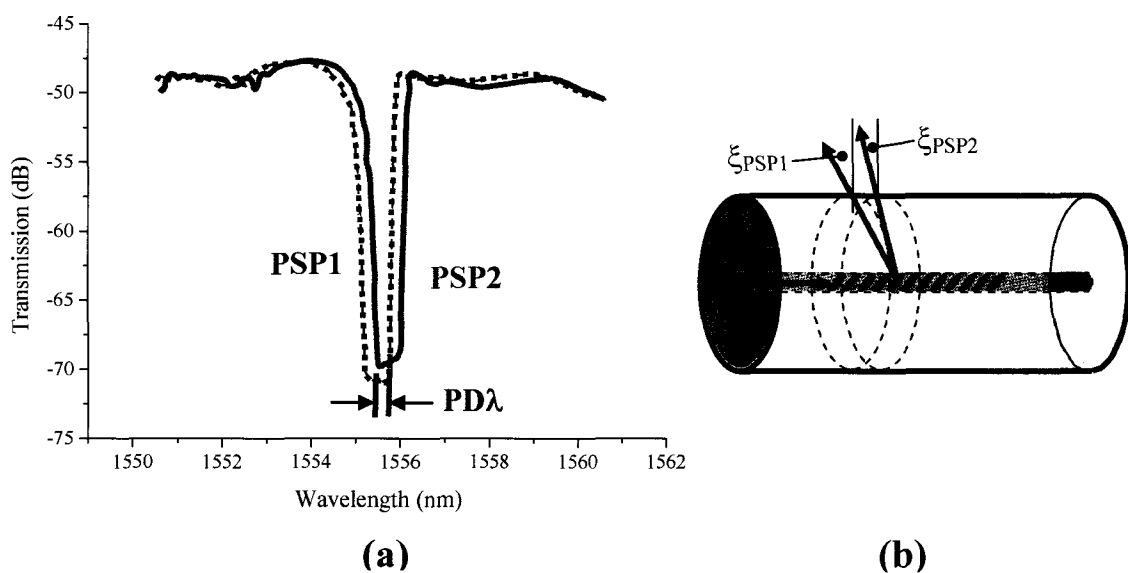


Fig. 2.17 – Birefringent grating responses including (a) untitled ridge waveguide grating spectra [68] and (b) schematic of tilted FBG radiation field.

2.4.2 Lensing distortion of writing fringes

There are a couple of different techniques for writing blazed FBGs, with a comprehensive discussion provided in [8]. It is important to understand that these methods each have their own limitations and can induce their own distortions, thereby altering a grating's geometry.

It is important to be aware of the possibilities for such distortions and the influences that they may exert over a grating's scattering characteristics. In particular, one case is especially important because the induced blaze angle within the fibre core can differ substantially from the angle of the writing fringes, yielding a result that can be quite different from what one might expect.

Specifically, when the fibre axis runs parallel to the phase mask, but is non-orthogonal to the writing beam fringes, the fringe pattern becomes distorted as shown in Fig. 2.18, altering the effective blaze angle of the grating and twisting the induced refractive index profile.

Not only is this phenomenon discussed at great length in [8], but several formula backed by theory and experiment, are provided in order to calculate a grating's blaze angle, based on a given set of writing equations that depends upon, writing beam, phase mask and fibre characteristics.

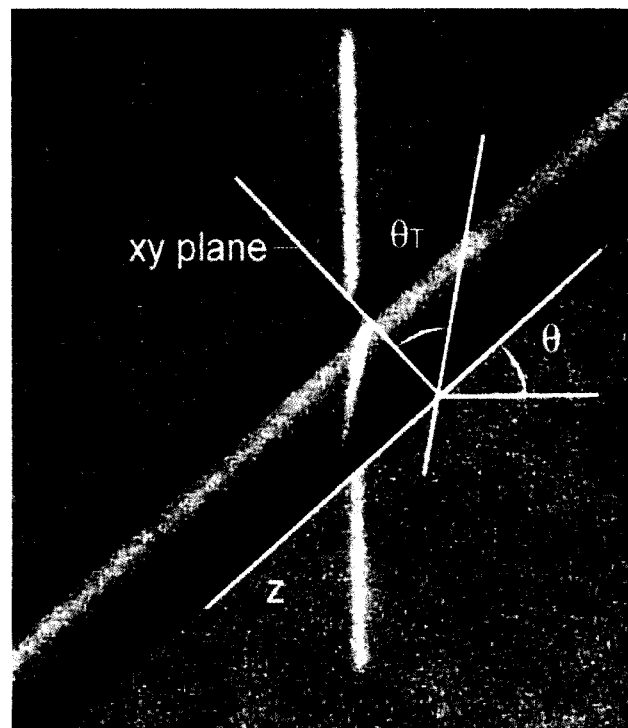


Fig. 2.18 – Real image of distortion photographed for a slit beam of white light directly incident the curved surface of a glass half cylinder, subsequently transmitted through a piece of paper at the planar surface. Notice that the blaze angle at the centre (core) is different from the angle of incident beam.

Chapter 3

Azimuthal Polarization Dependence of Fibre Bragg Gratings

Polarization sensitive mode coupling characteristics of tilted fibre Bragg gratings (FBG) have been exploited to develop a number of useful devices including fibre polarimeters, gain flattening filters, spectrum analyzers and polarization dependent loss (PDL) compensators [8,9]. Although a variety of tools are available to model blazed grating responses, a simplified explanation of the parametric dependencies and potential behaviour had never been fully presented, making the optimization of these components difficult and elusive at times. The contents of this chapter were published in [15], providing a thorough, intuitive discussion of these trends and possibilities as observed through an extensive theoretical and numerical analysis rooted in the Volume Current Method (VCM). In addition to reviewing the potential limitations and shortcomings of this formulation, some rough guidelines for the manufacture of various devices are also disclosed.

3.1 Introduction

As was mentioned in the general introduction, an FBG can be thought of as a photonic crystal, comprised of a waveguide in which the refractive index varies in some sort of periodic fashion. By controlling the distributed spacing of this periodicity (Λ) and the strength of the associated refractive index modulation (δn) a number of useful applications have been implemented including: add-drop devices for wavelength division multiplexing (WDM) [46], optical delay lines for chromatic dispersion compensation [45], and highly precise sensors for detecting strain, temperature and other environmental factors under potentially harsh conditions [48].

Most commonly this periodicity is one-dimensional with planes of equivalent refractive index crossing the fibre axis at a 90° angle. Typically for such structures in single mode waveguides, energy from forward-propagating guided modes is primarily coupled into retro-reflected guided modes as predicted by the Bragg law (p. 29 of [30]). This behaviour has been studied at great length and a number of useful methods have been developed in order to model the characteristics of these devices directly [72], as well as from an inverse scattering perspective [73].

With this in mind, it is also important to take note of a specific subset of these components in which the grating planes are skewed with respect to the fibre axis. Often referred to as tilted, blazed or slanted FBGs, these devices enable significantly greater coupling to cladding and non-guided radiation modes and are often observed to possess significant polarization sensitivity [3,22-28]. Making use of this characteristic a number of devices have been fabricated including in-line fibre polarimeters [2], gain flattening filters [6], spectrum analyzers [7], and PDL compensators [8].

Not long ago, Peupelmann et al. disclosed a unique fibre-polarimeter design [5], which exploits the fact that an FBG can have multiple polarization-dependent tap angles for a given wavelength [20]. Unfortunately Peupelmann et al. did not address the mechanisms responsible for this unique behaviour and most conventional models do not adequately predict the strength, orientation and polarization sensitivity of radiation modes emitted from tilted FBGs. As a result of these shortcomings, the ability to optimize and redesign such devices has been somewhat limited.

Recently an interest in addressing these issues has been presented [14,74], and this chapter expands upon that work, by including additional results and discussion which help to further clarify the physical relationships between a grating's structure and its radiation field characteristics.

Although there remain some higher order effects that warrant further investigation (longitudinal e-field component, fibre lensing distortions, non-uniformity of index change etc.), from a qualitative standpoint at least, these results are helpful for establishing a physical explanation of this phenomenon and emphasize the VCM's potential for optimizing and exploiting the puzzling observations of Peupelmann [5], and others.

3.2 Volume Current Method

The Volume Current Method is a useful perturbation analysis tool for solving numerous waveguide radiation problems that result from small refractive index fluctuations. Its derivation is rooted in the Huygens-Fresnel Principle and seeks to model Bragg gratings as a distribution of dipoles that essentially re-radiate the energy supplied by an incident electromagnetic wave.

Examples of its successful implementation can be found throughout the literature with relevant articles discussing its application to problems involving radiation losses from: fibres with irregular core surfaces [75,76], optical waveguide curves and bends [77-79], directional couplers and Y-branch structures [78,80], concentric-circle grating surface-emitting planar waveguides [81], surface-roughness loss and output coupling in microdisk resonators [82], as well as optical fibre sidetap filters [83]. Of particular interest here is a paper by Li et al [13], that provides several helpful formulations of this approach and reviews at great length its application to the study of tilted fibre Bragg gratings.

3.2.1 Assumptions

Before proceeding further, it should be noted that the formulation used here is subject to the following principal assumptions, and as a result the analysis and findings reported are equally constrained:

- 1) The fibre is single mode with a step index profile, having a small core-cladding index difference ($\sqrt{\delta_z} \ll 1$, $\delta_z = 1 - n_{clad}^2 / n_{core}^2$) and a cladding radius much greater than that of its circular core ($a_{clad} \gg a_{core}$).
- 2) The grating is confined within and uniform across the core, with a relatively weak ($\delta n \ll n$), perfectly sinusoidal, one-dimensional, uniform index modulation (i.e constant grating period) along an infinite length of the fibre axis.
- 3) The guided mode's longitudinal field component is neglected and the guided wave's polarization is taken to be normal to the fibre axis.
- 4) The out-coupled radiation is substantially transverse to the fibre axis, and as presented in *A.1* may be quantified as:

$$rk_t \gg \sqrt{\frac{9}{128}} \quad (3.1)$$

where r is the radial distance from the fibre axis and k_t is the transverse component magnitude of the radiation wavevector ($\mathbf{k}_{out}$) depicted in Fig. 3.1.

The coordinate system, various grating parameters and the resulting momentum vector diagrams used in this analysis of tilted gratings are presented in Fig. 3.1. A grating with a constant periodicity Λ is inclined at a blaze angle ξ with respect to the normal of the fibre axis y . The momentum vector of the grating, which is normal to the grating planes, is $K = 2\pi/\Lambda$. The incident guided mode, with effective index n_{eff} and freespace wavelength λ , propagates along the fibre core, with wavevector $k_i = 2\pi n_{eff}/\lambda$.

For specific values of Λ , ξ and λ , out-tapped radiation is generated with longitudinal tap angle ξ_{out} relative to the negative z -axis and azimuthal angle ϕ relative to the positive x -axis. The out-tapped radiation possesses the wavevector $k_{out} \approx 2\pi n_0/\lambda$ where n_0 is the core refractive index. The radiation wavevector k_{out} has transverse and longitudinal components $k_t = \sqrt{k_0^2 n_0^2 - \Delta^2}$ and $\Delta = k_0 n_{eff} - K_g$ that are normal and parallel to the fibre axis respectively. Correspondingly the longitudinal and transverse components of the grating vector K are $K_g = (2\pi/\Lambda)\cos\xi$ and $K_t = -(2\pi/\Lambda)\sin\xi$.

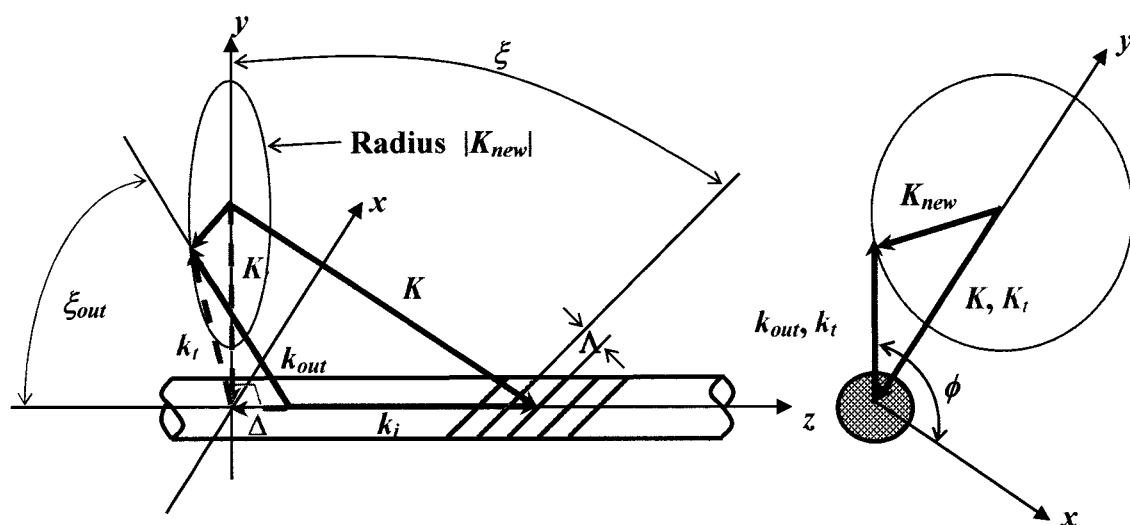


Fig. 3.1 - Schematic of selected grating parameters. r and ϕ represent polar coordinates about the fibre axis. $k_i = 2\pi n_{eff}/\lambda$ is the incident guided mode wavevector with effective index n_{eff} and freespace wavelength λ . k_t and Δ symbolize the transverse and longitudinal components of the radiation wavevector $k_{out} \approx 2\pi n_0/\lambda$ where n_0 is the core refractive index. $K_t = -2\pi \cdot \sin\xi / \Lambda$ is the transverse component of the grating vector $K = 2\pi/\Lambda$ where Λ is the grating period and ξ is the blaze angle. K_{new} is the detuning vector (phase mismatch) and ξ_{out} is the longitudinal tap angle of the radiated mode.

Despite these apparent limitations it is important to recognize that the resulting derivation provides useful insight into a very large percentage of practical problems, when azimuthal coupling and wavelength dependence characteristics are of primary concern.

3.2.2 Relevant insights by Li et al.

Subject to the stated assumptions, Li et al. have derived the Poynting vector $\vec{S}$ of the radiation field to be [13]:

$$\vec{S} \approx \left(\frac{\pi k_0 c K^2 E_0^2}{4\epsilon r} f_1 f_2 \right) \left(\hat{r} + \frac{\Delta}{k_t} \hat{z} \right) \quad (3.2)$$

where c is the speed of light, E_0 is the guided electric field amplitude, $\kappa = \epsilon_0 n_0 \delta n$ is the perturbation constant with ϵ_0 being the permittivity of free space and δn is the index modulation, k_0 is the freespace wavevector magnitude ($2\pi/\lambda$).

Subsequently, Li et al. also discuss the fact that polarization effects are expressed through the factor:

$$f_1 = \Delta^2 + k_i^2 \sin^2(\delta - \phi) \quad (3.3)$$

where δ is the polarization angle of the guided electric field, relative to the $+x$ axis. If the transverse observation angle ϕ is 90° to the guided mode's angle of polarization δ , then f_1 is simply the square of the radiation wavevector k_{out} . On the other hand, when the observation and polarization angles are parallel, then f_1 is reduced to Δ^2 .

The phase matching condition operates through the detuning parameter:

$$K_{new} = \sqrt{K_i^2 + k_i^2 + 2K_i k_i \sin \phi} \quad (3.4)$$

and

$$f_2 = \left(\frac{K_{new} a J_0(ua) J_1(K_{new} a) - ua J_1(ua) J_0(K_{new} a)}{K_{new}^2 - u^2} \right)^2 \quad (3.5)$$

where a represents the core radius, J_v denotes a Bessel function of the first kind (order v) and $u = k_0 \sqrt{n_0^2 - n_{eff}^2}$.

3.3. Phase matching affects on the radiation field profile

Expanding upon this prior work one is able to further understand blazed grating behaviour, explore trends in the Poynting vector's parameter space and develop some tools that could be useful for selecting a grating's physical characteristics (a , n_0 , n_{eff} , Λ/λ and ξ) based on a desired radiation field profile.

Restating Li's observations in a slightly different way, a detailed examination of the Poynting vector reveals that a grating's response consists of the product of two physical processes: A dipole interaction between the core material and incident light, coupled with the interference field characteristics generated by a grating's periodic structure.

In the case of a non-polarized guided mode, it is noted that the dipole component (f_1) becomes entirely independent of ϕ (as shown mathematically in the A.2) and as such the shape of the radiation field profile $S(\phi)$ is exclusively determined by a grating's interference properties (f_2). In Fig. 3.2, an example of a dual peak response is generated by solving Eq (3.5) assuming a 45° tilt angle for an 890 nm pitched grating in SMF-28 fibre assuming a non-polarized guided mode with a wavelength 1550 nm. With the parameters selected, the resultant radiated modes represent a detuning from resonance with the grating structure. The dependence of the radiation field profile on f_2 alone is illustrated in Fig. 3.2, where $S(\phi)$ is shown to be a scalar multiple of $f_2(\phi)$.

Combining this observation with an understanding of f_1 's polarization dependence one may conclude that phase matching effects (f_2) determine the potential shape characteristics of the radiation field, while a dipole response (f_1) serves to select and magnify that portion at 90° to the guided state of polarization (SOP). This finding is illustrated in Fig. 3.3, where the response of a grating, identical to that analyzed in Fig. 3.2, is modelled with four different linearly polarized states of the incident electric field. More specifically f_1 , f_2 and S

are plotted for the same grating geometry and hence same f_2 , subjected to four different SOPs ($\delta = 0^\circ, 30^\circ, 60^\circ$ and 90° in Fig. 3.3 a, b, c and d respectively). The final radiation field profile, described by the Poynting vector S , is clearly seen to depend on the state of polarization of the incident guided electric field.

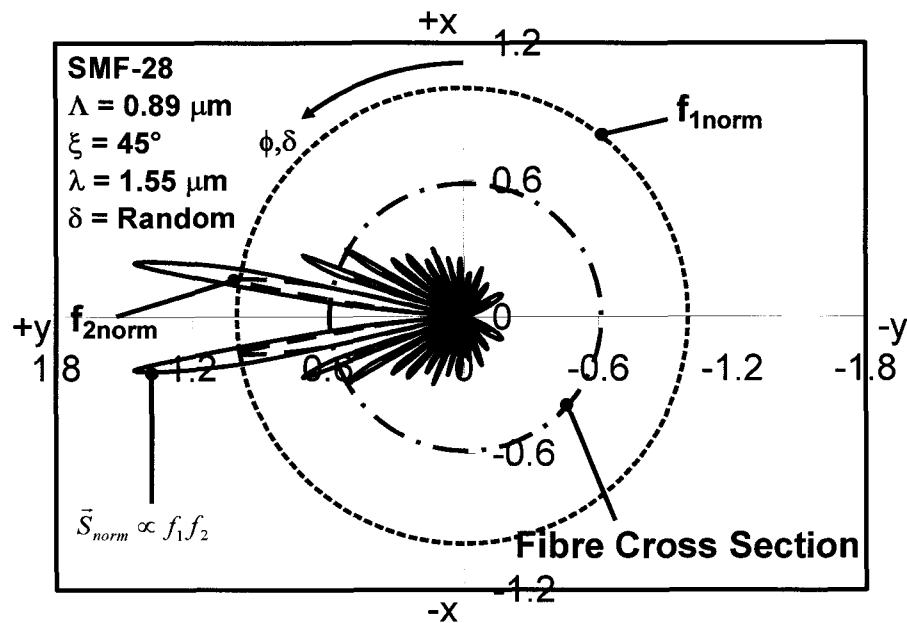


Fig. 3.2 - Normalized grating response for a non-polarized guided mode

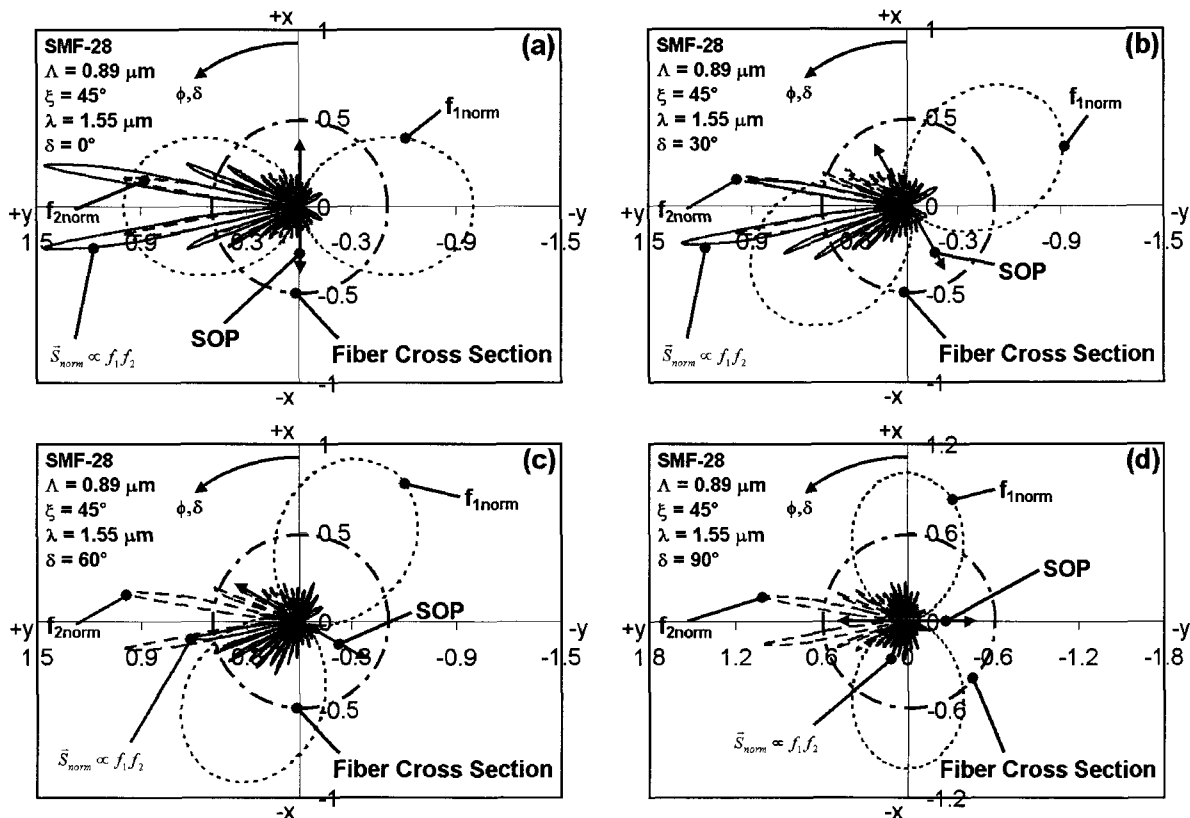


Fig. 3.3 - Normalized grating response to a polarized guided mode [15]. f_1, f_2 and S are plotted for a single grating (f_2) subjected to four different polarization states (δ).

3.3.1 General characteristics of the f_2 term

In order to more fully explore and address the potential variability of the radiation field profile, it is therefore necessary to examine the behaviour of f_2 more completely. To this end, one notes that this expression may be normalized and rewritten entirely in terms of three design variables: au , k_t/u and K_t/k_t . The product au describes the fibre parameters as a function of wavelength, a being the core radius and u being the product of the free space wavevector $2\pi/\lambda$ with the root of difference in the squares of the core index and the incident mode effective index. The grating period and tilt angles are incorporated into the parameters k_t and K_t .

$$f_{2norm} = \frac{f_2}{a^2} = \left(\frac{CJ_1(C)J_0(D) - DJ_1(D)J_0(C)}{C^2 - D^2} \right)^2 \quad (3.6)$$

$$C = K_{new}a = au \left(\frac{k_t}{u} \right) \sqrt{1 - 2 \left(\frac{K_t}{k_t} \right) \sin \phi + \left(\frac{K_t}{k_t} \right)^2} \quad \text{and} \quad D = au$$

Physically, au represents the waveguide characteristics (a , n_0 and n_{eff}) at a wavelength of interest (λ) and k_t/u relates the coupled mode's longitudinal tap angle (ξ_{out}) to the waveguide's core and effective refractive indices (n_0 and n_{eff}). K_t/k_t is a ratio between the transverse grating vector magnitude and a coupled mode's transverse wavenumber, making it a measure of the minimum transverse detuning. Ultimately, as illustrated by Eq. (3.4) this parameter affects the angular dependence of the detuning parameter $K_{new}(\phi)$. For perfect transverse phase matching, $|K_t/k_t| = 1$.

In any event, after exploring the azimuthal characteristics of f_{2norm} over a relatively large scope of these variables ($0 \leq au \leq 2.4$, $2 \leq k_t/u \leq 2^{12}$, $0 \leq |K_t/k_t| \leq 10$, full scope defined in 3.4.1) certain behavioural tendencies become apparent. For example, in the case of gratings written in a given type of fibre, operating with a fixed incident wavelength λ , and axial out-tap angle ξ_{outs} , au and k_t/u will be constant. Under these circumstances Figs. 3.4 and 3.5 illustrate the predicted azimuthal dependence on K_t/k_t , which is fixed by one's choice of grating period, blaze angle and operating wavelength.

By studying many such plots generated using the VCM it is observed that in cases where the radiation is highly focused and asymmetric relative to the x -axis (i.e. significant grating blaze, transverse detuning or alternative asymmetry) there are several discrete regions (i) within which changes in K_t/k_t will either affect the peak magnitude or the dominant shape of this term.

More specifically, as the detuning increases from zero, one departs from transverse phase matching and $|K_t/k_t| \neq 1$, revealing two characteristic behaviours:

Regime A) As illustrated in Fig. 3.4, when f_2 is primarily focused along the y -axis and s-polarized light is coupled predominantly into a single lobe, additional detuning will cause the magnitude of this central peak to decrease. So long as the center peak is dominant this decrease will result in a corresponding reduction of f_2 's peak magnitude.

Regime B) On the other hand, as shown in Fig. 3.5 the potential for multiple lobed radiation is also predicted, and when the central peak becomes smaller than any side peaks, the maximum magnitude of f_2 will remain constant while the side peak spread will tend towards the center.

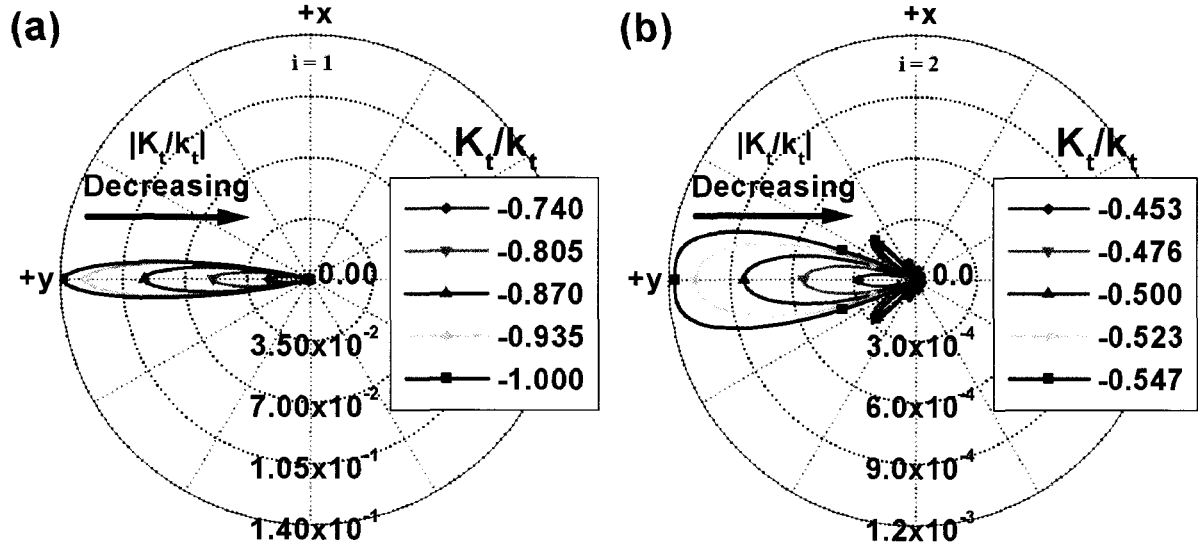


Fig. 3.4 - Regime A: Variation of $f_{2\text{norm}}$ azimuthal distribution with K_t/k_t ($au=1.5$, $k_t/u=8$)

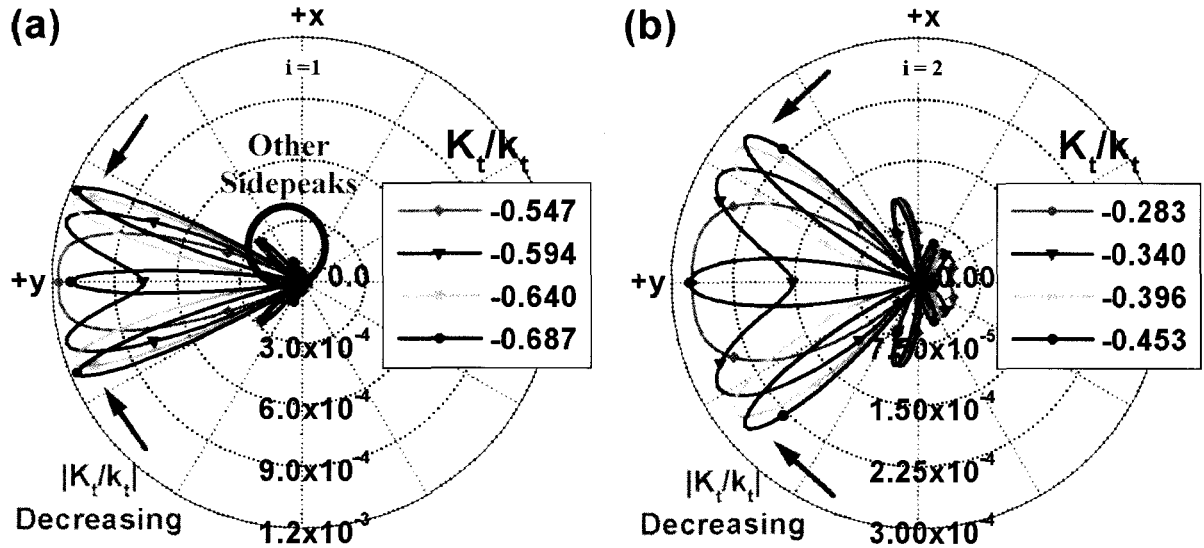


Fig. 3.5 - Regime B: Variation of $f_{2\text{norm}}$ azimuthal distribution with K_t/k_t ($au=1.5$, $k_t/u=8$)

As $|K_t/k_t|$ moves away from unity, the response will alternate between these two regimes with each cycle representing a single region of interest ($i = 1, 2, \dots$).

Initially this behaviour may seem surprising and physically unintuitive, but upon further reflection it becomes apparent that it is a direct result of quasi phase matching in the radiation field, originating from the detuning parameter's angular dependence $K_{\text{new}}(\phi)$ (see Eq. (3.4)) and illustrated in Fig. 3.1 as the phase mismatch (K_{new}) between the two wavevectors ($\mathbf{k}_i$, $\mathbf{k}_{\text{out}}$) and grating reciprocal lattice vector ($\mathbf{K}$).

Mathematically the significance of detuning is evident from the fact that K_{new} is the only component of f_2 directly dependent upon ϕ . Hence the angular variation of f_2 relies upon K_{new} . Nevertheless it is also important to recall that unlike the simplified diagram pictured in Fig. 3.1, the total field is actually a superposition of wavefronts re-radiated by numerous dipoles throughout the grating's active volume. As a result, the complicated structure of f_2 is

also an important factor in determining the ultimate azimuthal dependence. Based on these observations, the cross sectional uniformity of a grating and shape of the guided mode are also expected to affect the field profile by respectively varying the significance of transverse detuning (K_{new}) and exciting individual dipoles to scatter with different intensities over the fibre cross section.

All of these observations emphasize the importance of thorough grating design and are of particular interest to those wishing to fabricate multiple peak polarimeter gratings similar to those proposed by Peupelmann et al [5]. They illustrate the fact that only certain combinations of these parameters (and hence a , n_0 , n_{eff} , Λ/λ , and ξ) are expected to allow for such behaviour and therefore highlight the potential benefit of understanding these effects in more detail.

3.3.2 Detuning: Explanation and design considerations

As mentioned in 3.3.1 and indicated by Li et al [13], the directionality and magnitude of f_2 and hence the grating response are highly dependent on the degree of detuning (K_{new}) present in a system. Physically, detuning represents a mismatch between the grating's periodicity and a coupled wavevector, a difference between the resonant wavelength peak (100% constructive interference) and the quasi phase matching associated with the remainder of a grating's bandwidth.

Once this point is clearly understood, it is easy to see that the coupling efficiency is expected to decrease as K_{new} grows in magnitude. It is also worth noting that while this decrease in efficiency warrants some consideration, it is not necessarily something to be completely avoided. In fact, as alluded to in 3.3.1 and illustrated in Fig. 3.1, from an interference point of view it is detuning alone (K_{new}) that makes coupling off the y axis possible (conservation of momentum requires that vectors sum to zero).

Additionally it is important to recognize that although the degree of detuning present is governed by the phase matching condition, its impact on the field intensity is related to the number of periodic scattering elements aligned in any particular direction. More specifically, as the number of active scatterers increases, quasi phase matching effects tend to cancel each other out, resulting in an ever increasing adherence to the ideal phase matching condition ($K_{new}=0$). This is evidenced by the well-known fact that a retro-reflective grating's spectral bandwidth will narrow as its effective length is increased. Indeed then, it is not only the presence of detuning but also the spatial variability in its impact (due to a grating's three-dimensional structure) that results in the complicated interference profiles described by f_2 .

Combining this knowledge with the performance characteristics desired for a particular type of device, appropriate tradeoffs must be made between coupling efficiency and angular spread. In the case of in-fibre polarizers for instance, optimum coupling is desirable and achievable by minimizing detuning ($K_{new}=0$) and maximizing a grating's active scattering volume. On the other hand gratings for in-line polarimeters are only meant to sample an extremely small portion of the guided signal, and as such are more accommodating to quasi resonances, particularly when they allow for a device's polarization sensitivity to be maximized.

3.4. Parametric relationships and empirical recipes

Making practical use of section 3.3's observations it is noted that the response of f_2 can often be classified in one of three ways:

- a) A centered response that is primarily focused along the y axis.
- b) A split response that is mainly skewed with respect to the y axis, or
- c) A mixed response in which both behaviours are simultaneously present.

These three cases, illustrated in Fig. 3.6, are easily created when significant detuning is present, ie. K_t/k_t is equal to -0.56 , -0.64 and -0.67 for a, b and c respectively. The fibre parameter constant au is chosen to be 1.6, while the grating period and angle are incorporated into the constant k_t/u , which is set to 12.5 for this simulation.

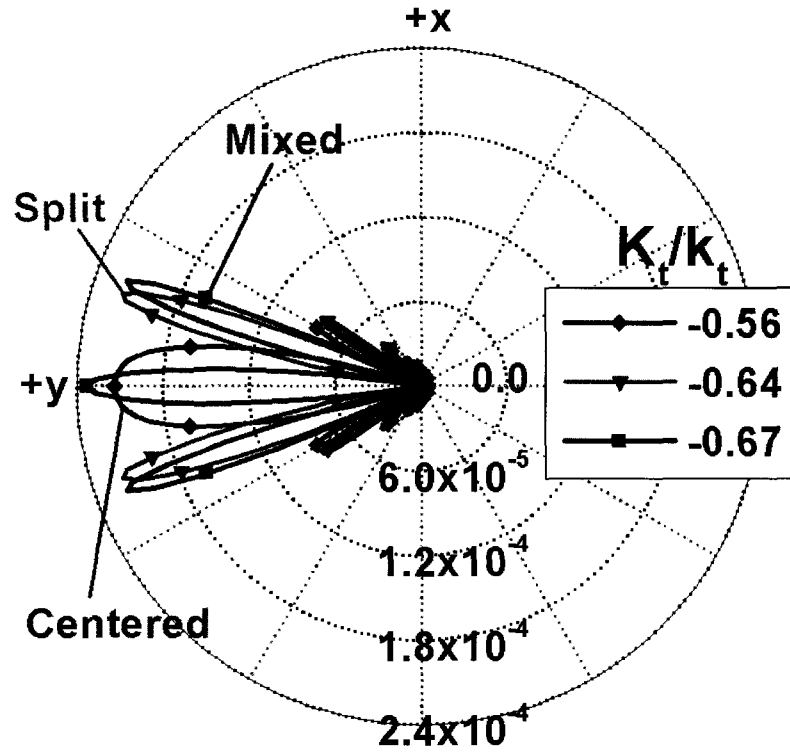


Fig. 3.6 - General classification of $f_{2\text{norm}}$ response types ($au=1.6$, $k_t/u=12.5$)

Each of these distinct regimes may be applicable to the design of certain devices in those cases when having a properly shaped radiation field is particularly important. With this in mind, the following subsections quantify parametric relationships governing this behaviour and disclose specific recipes which can serve to simplify a component's preliminary design process, making it significantly less iterative and much more methodical.

3.4.1 Theoretical bounds on design variables

In deriving and applying these empirical relationships, it is important to recognize the impact of the stated assumptions (section 3.2.1) on the permitted scope of f_2 's parameter space.

The term au

Beginning with the variable au , one notes that in its expanded form ($au = ak_0 \sqrt{n_0^2 - n_{\text{eff}}^2}$) it resembles the expression for a fibre's normalized frequency ($v = ak_0 \sqrt{n_0^2 - n_{\text{clad}}^2}$). By also recognizing that $n_{\text{clad}} \leq n_{\text{eff}} \leq n_0$ it becomes evident that au must be less than or equal to V , and because single mode operation has been assumed:

$$0 \leq au \leq V \leq 2.4 \quad (3.7)$$

The term k_t/u

For k_t/u it is observed that the minimum value of k_t may be derived from assumption 4, where the out-tapped radiation is significantly transverse to the fibre axis. From Eq. A.1 of section A.1 in Appendix A, if the remainder term $|R_1|$ of the Bessel function expansion J_v is allowed to be 1% of the $n = 1$ expansion term $|Term_{n=1}|$ one obtains:

$$k_{t\min} = \frac{100}{r_{\max}} \sqrt{\frac{9}{128}} = \frac{75}{2\sqrt{2}} \left(\frac{1}{a_{\text{clad}}} \right)$$

Which can then be used to establish $(k_t/u)_{\min}$ in terms of au :

$$\left(\frac{k_t}{u} \right)_{\min} = \frac{ak_{t\min}}{au} \approx \frac{26.517}{au} \left(\frac{a}{a_{\text{clad}}} \right)$$

To calculate an upper limit, notice from Fig. 3.1 that $k_{t\max} = k_{\text{out}} = k_0 n_0$ and as a result:

$$\left(\frac{k_t}{u} \right)_{\max} = \frac{k_{t\max}}{k_0 \sqrt{n_0^2 - n_{\text{eff}}^2}} = \frac{1}{\sqrt{1 - n_{\text{eff}}^2 / n_0^2}}$$

Combining these two limits into a single expression yields:

$$\frac{26.517}{au} \left(\frac{a}{a_{\text{clad}}} \right) \leq \frac{k_t}{u} \leq \frac{1}{\sqrt{1 - n_{\text{eff}}^2 / n_0^2}} \quad (3.8)$$

And so like Eq. (3.7), the limits on k_t/u are quantified exclusively in terms of the selected waveguide properties, with large values increasingly likely for transverse scattering from fibres where the mode effective index approaches the refractive index of the core material.

The term K_t/k_t

Where K_t/k_t is concerned, theoretically its magnitude can lie anywhere within the following range:

$$0 \leq |K_t/k_t| \leq \infty \quad (3.9)$$

When there is resonance between the incident mode and the grating structure such that a strong out-tapped beam is formed, $|K_t/k_t| = 1$. In practice however Li points out the fact that as K_t and k_t become vastly different quantities, the directionality of the detuning (K_{new}) and interference (f_2) terms is lost, yielding a nearly uniform response in all directions [13].

3.4.2 Dependence of response type on design variables

With the constraints of section 3.4.1 in mind, if one desires that the radiation field profile be centered, split or some combination of the two, then having a design tool to facilitate the choice of necessary grating geometry is expected to be of some value.

Fig. 3.7 provides an example of how the VCM was used to empirically determine equations that characterize the dependencies of a split response type on the three design variables introduced in 3.3.1 ($|K_t/k_t|$, k_t/u and au). Specifically Fig. 3.7 (a) plots the relationship between $|K_t/k_t| \geq 1$ and k_t/u for a first order resonance ($i = 1$) and various values of au . Fig. 3.7 (b) shows the dependence of fitting parameter ψ on au and i , where a vertical line at $au = 0.85$ serves to delineate two regions each characterized by a different mathematical relationship. Finally, Figs. 3.7 (c) and (d) demonstrate how the remaining fitting parameters (χ_i , ϕ_{1i} , ϕ_{2i} and ϕ_{3i}) depend on i .

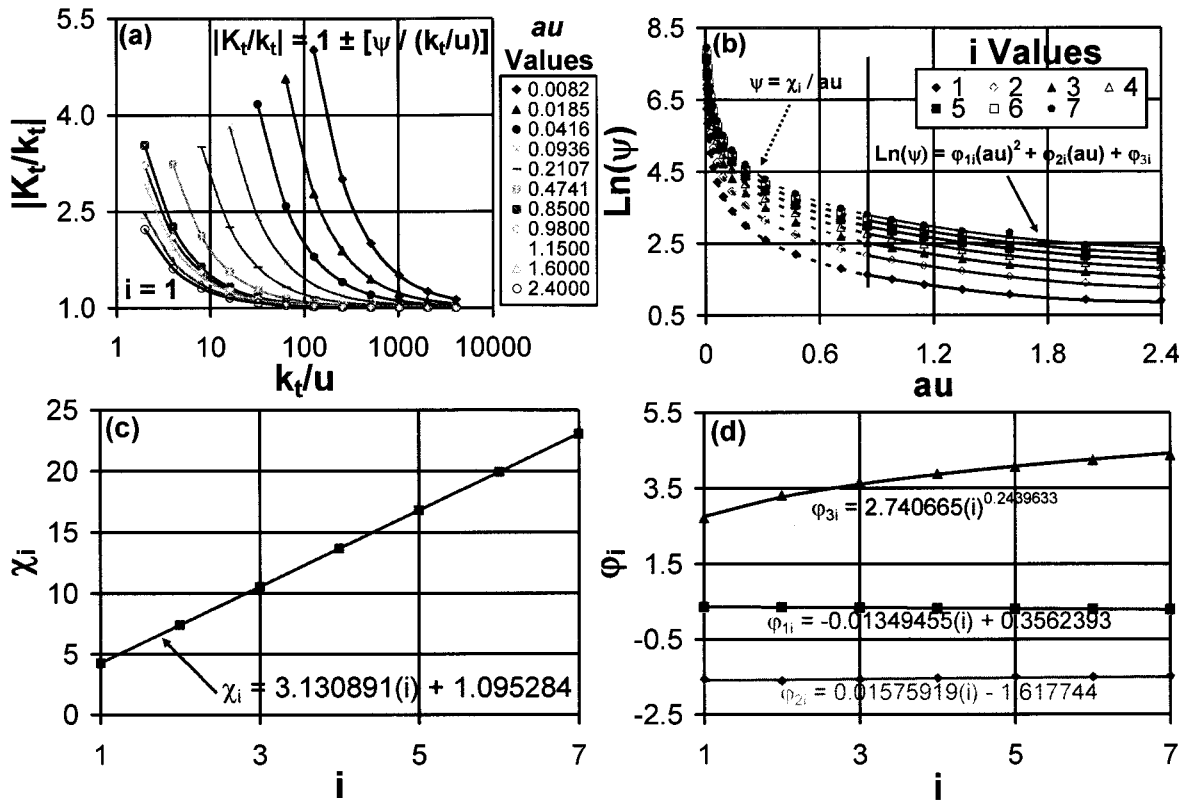


Fig. 3.7 - Typical example of VCM data and empirical fits: Split response type

Given the complexity of Eq. (3.2), data calculated numerically with the VCM have been fitted with several empirical expressions that quantify the relationships between K_t/k_t , k_t/u and au required in order to generate a particular type of behaviour. The generalized trends are illustrated in Fig. 3.7, where split response data have been plotted for various values of au .

In Fig. 3.7(a), combinations of $|K_t/k_t|$ and k_t/u capable of generating a split response are shown for different values of au . 3.7(b) illustrates the fact that multiple constants (ψ) exist for a given value of au , depending on the degree of detuning present (phase mismatch increases with i), while 3.7(c) and 3.7(d) plot the dependence of constants χ_i , ϕ_{1i} , ϕ_{2i} and ϕ_{3i} on i .

From these plots it is evident that the parametric relationships are relatively well ordered in multiple regions (i) centered about the indicated trend lines. Physically the integer i , is analogous to a grating's harmonic order of resonance (p. 128 of [39]), with its relationship to the radiation field profile described in section 3.3.1 and subsequently illustrated in Fig. 3.8. Note that grating recipes based on smaller values of i not only exhibit stronger peak responses, but also generate additional side peaks of similar magnitude and direction to those associated with larger values of i .

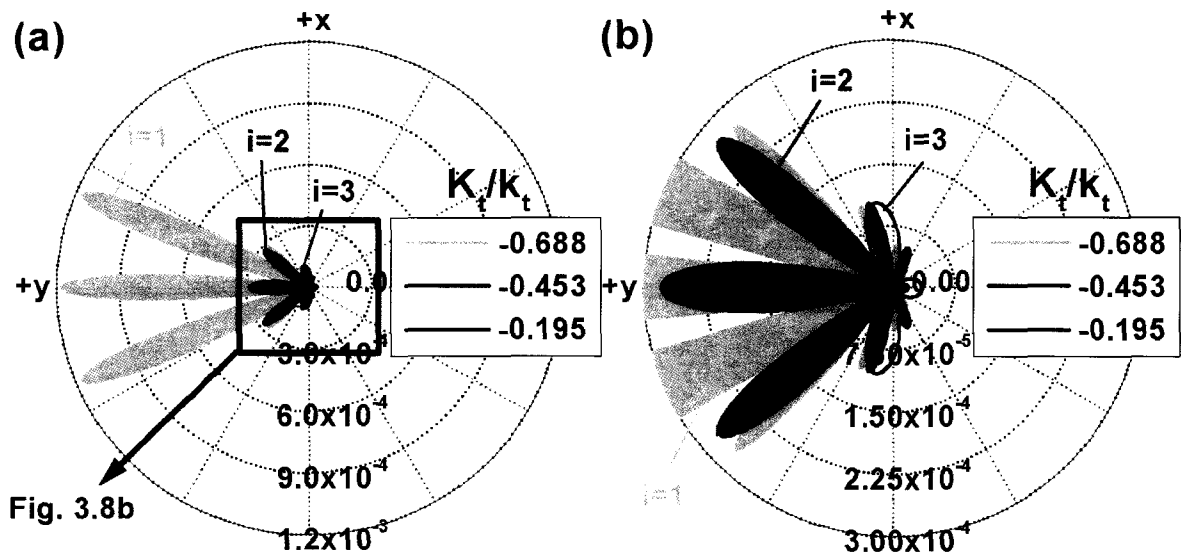


Fig. 3.8 - Physical significance of grating order i on magnitude and directionality of out-tapped radiation ($au = 1.5$, $k_t/u = 8$)

Quantitatively these relationships are well represented by the following approximation, with behavioural dependent coefficients provided in Table 3.1 ($|K_t/k_t|$ fit error: average < 1%, maximum < 7%):

$$\left| \frac{K_t}{k_t} \right| = 1 \pm \frac{\psi}{k_t/u} \quad \text{where } 2 \leq k_t/u \leq 2^{12}, \quad 0.01 \leq au \leq 2.4 \quad (3.10)$$

$$\psi = \frac{\chi_i}{au} \quad \text{where } 0.01 \leq au \leq 0.85 \quad (3.11)$$

$$\psi = \exp[\varphi_{1i}(au)^2 + \varphi_{2i}(au) + \varphi_{3i}] \quad \text{where } 0.85 \leq au \leq 2.4 \quad (3.12)$$

$$\chi_i = C_1(i) + C_2 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.13)$$

$$\varphi_{1i} = C_3(i) + C_4 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.14)$$

$$\varphi_{2i} = C_5(i) + C_6 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.15)$$

$$\varphi_{3i} = C_7(i)^{C_8} \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.16)$$

Table 3.1 - Empirical recipe coefficients for various response types

Response	C_1	C_2	C_3	C_4
Centered	3.140987	2.596592	-0.01365171	0.3475737
Split	3.130891	1.095284	-0.01349455	0.3562393
Mixed	3.076742	0.1751473	-0.007438972	0.3662597
Response	C_5	C_6	C_7	C_8
Centered	0.02176955	-1.631147	3.027064	0.1956879
Split	0.01575919	-1.617744	2.740665	0.2439633
Mixed	-0.01454217	-1.606962	2.553396	0.2937874

Note also that centered responses are generally predicted at resonance when $|K_t/k_t| = 1$, and be aware that the limits provided are only intended to indicate the scope of variables examined during this study.

By satisfying these relations it is now possible to select a set of design parameters (K_t/k_t , k_t/u and au) that will generate a chosen type of behaviour (centered, split or mixed).

3.4.3 Dependence of angular spread on design variables

Following a similar approach, empirical relationships have also been formulated in order to quantify the parametric dependence of the spread angle $\Delta\phi$ between the two strongest side peaks. Fig. 3.9 provides an example how this curve fitting was performed. Specifically Fig. 3.9 (a) plots the relationship between $\Delta\phi$ and k_t/u for a first order resonance ($i = 1$) and various values of au . Figs. 3.9 (b) and (c) show the dependence of fitting parameters Ψ_1 and Ψ_2 on au and i , while Fig. 3.9 (d) demonstrates how the remaining fitting parameters (κ_{1i} , κ_{2i} , γ_{1i} and γ_{2i}) vary with i .

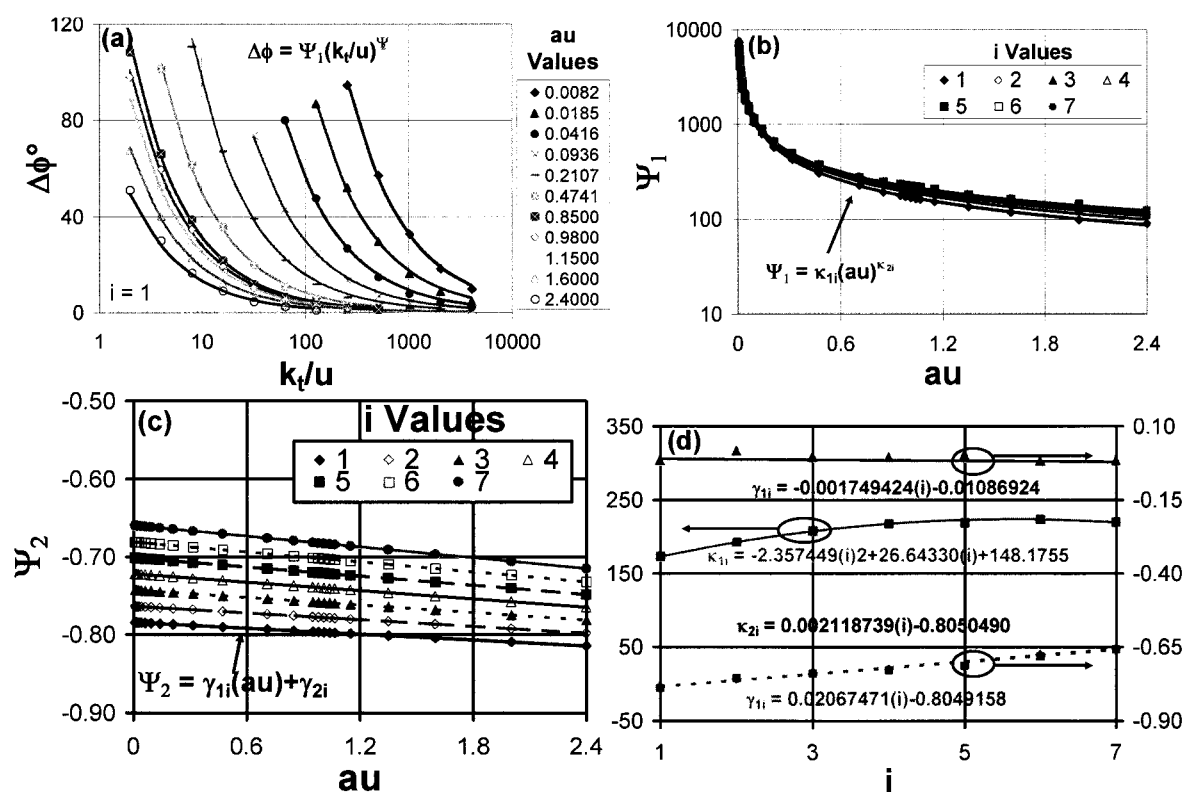


Fig. 3.9 - Typical example of VCM data and empirical fits: Split response angular spread

As in 3.4.2, the relationships are well ordered and can be represented with the following approximation, having behavioural dependent coefficients provided in Table 3.2 ($\Delta\phi$ fit error: average $< 2^\circ$, maximum $< 5^\circ$):

$$\Delta\phi = \Psi_1 \left(\frac{k_t}{u} \right)^{\Psi_2} \quad \text{where } 2 \leq k_t/u \leq 2^{12}, \quad 0.01 \leq au \leq 2.4 \quad (3.17)$$

$$\Psi_1 = \kappa_{1i} (au)^{\kappa_{2i}} \quad \text{where } 0.01 \leq au \leq 2.4 \quad (3.18)$$

$$\Psi_2 = \gamma_{1i} (au) + \gamma_{2i} \quad \text{where } 0.01 \leq au \leq 2.4 \quad (3.19)$$

$$\kappa_{1i} = D_1(i)^2 + D_2(i) + D_3 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.20)$$

$$\kappa_{2i} = D_4(i) + D_5 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.21)$$

$$\gamma_{1i} = D_6(i) + D_7 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.22)$$

$$\gamma_{2i} = D_8(i) + D_9 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.23)$$

Table 3.2 - Empirical angular spread coefficients for split and mixed responses types

Response	D ₁	D ₂	D ₃
Split	-2.357449	26.64330	148.1755
Mixed	-4.549978	44.78548	224.8580
Response	D ₄	D ₅	D ₆
Split	0.02118739	-0.8050490	-0.001749424
Mixed	0.02538035	-0.8520713	0.0009550463
Response	D ₇	D ₈	D ₉
Split	-0.01086924	0.02067471	-0.8049158
Mixed	-0.003717130	0.02483495	-0.8487707

3.4.4 Dependence of peak coupling efficiency ($f_{2\text{-peak}}$) on design variables

For the centered, split and mixed response recipes provided in 3.4.2, the peak coupling efficiency (i.e. maximum interference term value predicted from $0 \leq \phi \leq 360^\circ$) is exclusively dependent on au and i as shown in Fig. 3.10. Physically, the dependence on au is rooted in the fact that stronger scattering is expected as grating-mode overlap improves due to tighter mode field confinement, while higher order responses exhibit weaker scattering due to weaker corresponding Fourier components in the induced index change. Fig. 3.10 (a) illustrates the relationship between the peak coupling strength and au for different order resonances (i), while Figs. 3.10 (b), (c) and (d) demonstrate how the fitting parameters ($\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and Ω_5) depend on i .

Once again, quantifiable trends are distinctly observed with the relationships being well represented by the following approximation and behavioural dependent coefficients provided in Table 3.3 ($f_{2\text{norm(peak)}}$ fit error: average $\approx 3\%$, maximum $\approx 8\%$):

Table 3.3 - Empirical coefficients for peak coupling efficiency of all response types

Constant	Value	Constant	Value
G ₁	0.003914322	G ₁₀	-0.001416267
G ₂	-0.1323257	G ₁₁	0.01535387
G ₃	0.2207577	G ₁₂	0.07575297
G ₄	-0.009618348	G ₁₃	-0.2988538
G ₅	0.4273341	G ₁₄	-0.01478773
G ₆	-0.9439067	G ₁₅	0.2684945
G ₇	0.007570827	G ₁₆	-2.079955
G ₈	-0.4289765	G ₁₇	-3.612419
G ₉	0.5410415	-	-

$$f_{2\text{norm(peak)}} = a^2 \exp \left[\frac{\Omega_1(au)^4 + \Omega_2(au)^3 + \Omega_3(au)^2 + \Omega_4(au) + \Omega_5}{\Omega_3(au)^2 + \Omega_4(au) + \Omega_5} \right] \quad \text{where } 2 \leq k_t/u \leq 2^{12}, 0.01 \leq au \leq 2.4 \quad (3.24)$$

$$\Omega_1 = G_1 i^2 + G_2 i + G_3 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.25)$$

$$\Omega_2 = G_4 i^2 + G_5 i + G_6 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.26)$$

$$\Omega_3 = G_7 i^2 + G_8 i + G_9 \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.27)$$

$$\Omega_4 = G_{10} i^3 + G_{11} i^2 + G_{12} i + G_{13} \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.28)$$

$$\Omega_5 = G_{14} i^3 + G_{15} i^2 + G_{16} i + G_{17} \quad \text{where } i \in \mathbb{Z}, 1 \leq i \leq 7 \quad (3.29)$$

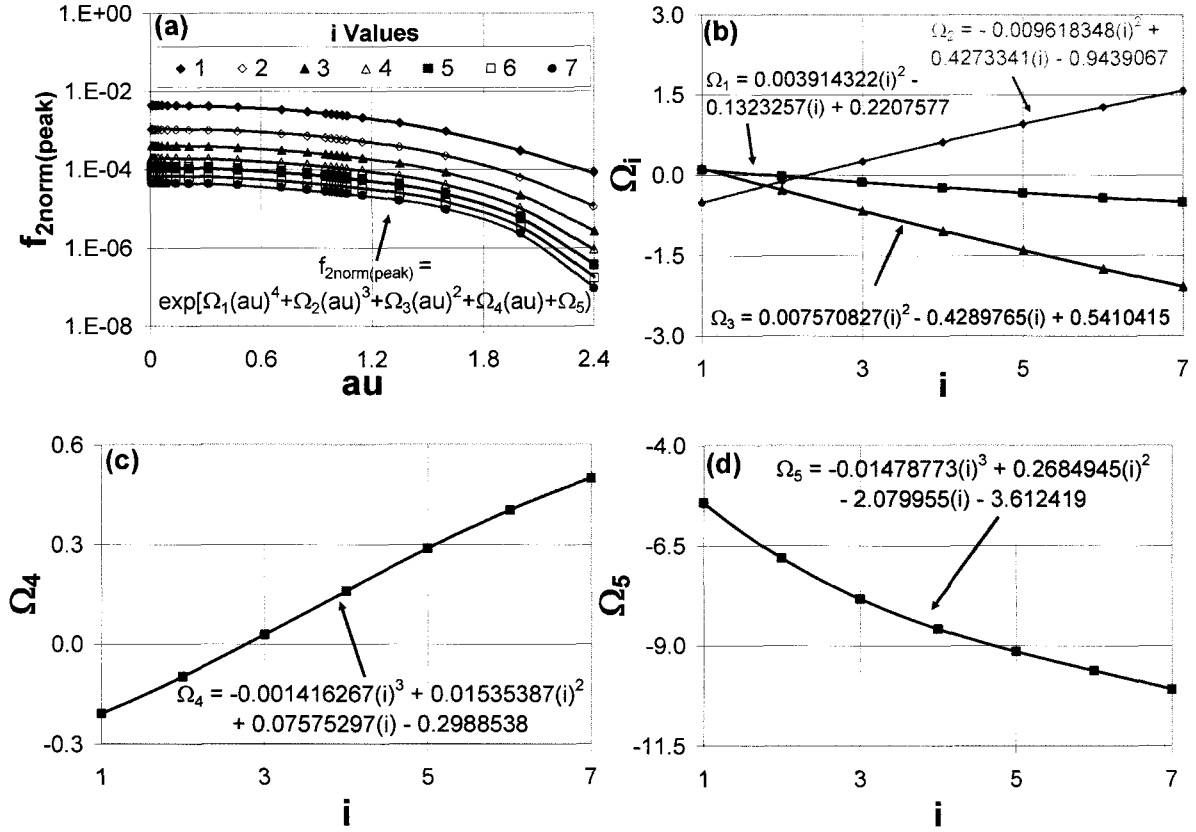


Fig. 3.10 - Typical example of VCM data and empirical fits: Peak coupling efficiency

3.5. Application of recipes to the design process

As a practical example of how these trends might be applied to the design process, take the common case of SMF-28 fibre for which au is approximately equal to 1.6 @ 1550 nm.

3.5.1 Calculate waveguide dependent constants

In order to design for a particular response type (centered, split, mixed), angular spread ($\Delta\phi$) or coupling efficiency ($f_{2\text{peak}}$) under these conditions, one must first calculate appropriate constants (ψ , Ψ_1 , Ψ_2 , $f_{2\text{norm(peak)}}$) using the coefficients and equations provided in sections 3.4.2 through 3.4.4. Figs. 3.11 (a) through (d) respectively illustrate how each of these constants might be obtained graphically, in the case of SMF-28 fibre for which $au \approx 1.6$. Notice in each figure that the points of intersection marked by solid circles represent fitting constants that characterize the split responses of different grating orders in SMF-28 fibre.

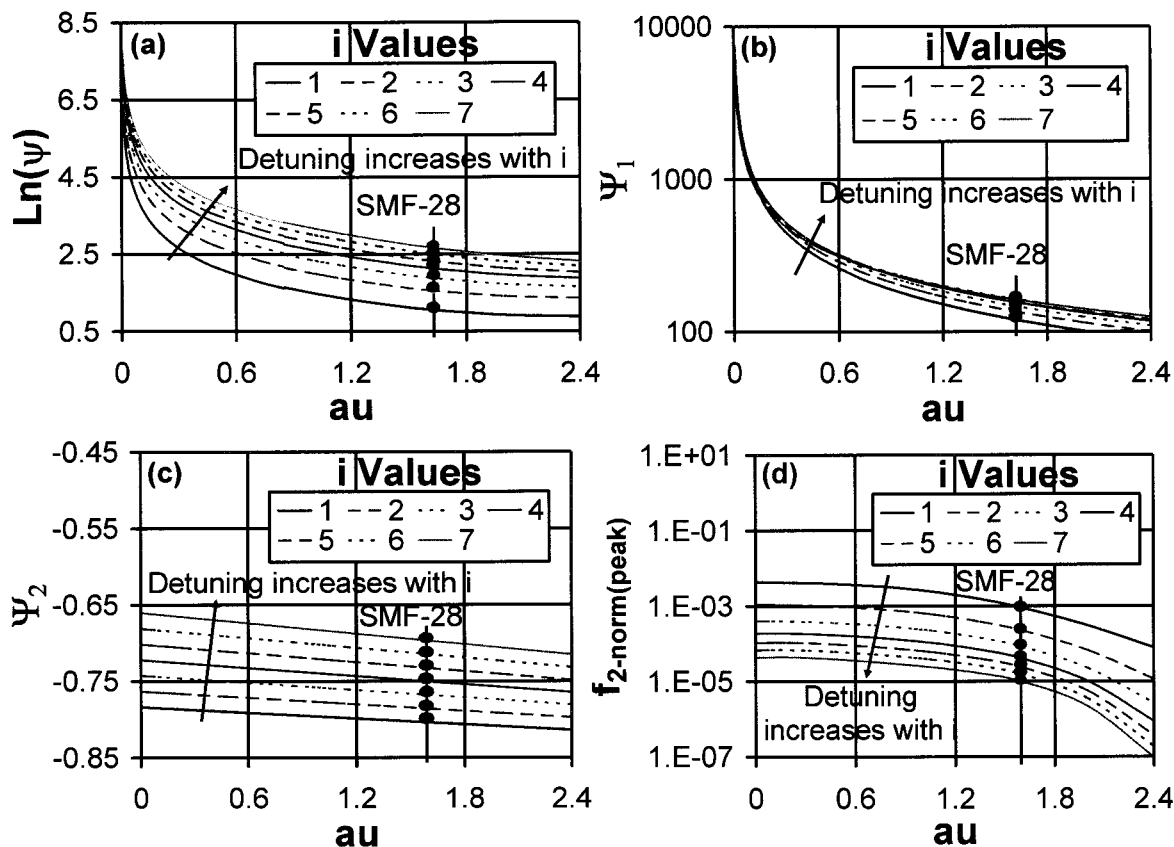


Fig. 3.11 - Constants dependent on au : Split response

3.5.2 Select grating dependent constants

One may then make the necessary tradeoffs and engineer the optimum response characteristics by selecting values of i , k_t/u and $|K_t/k_t|$ that are consistent with these relationships (Fig. 3.12).

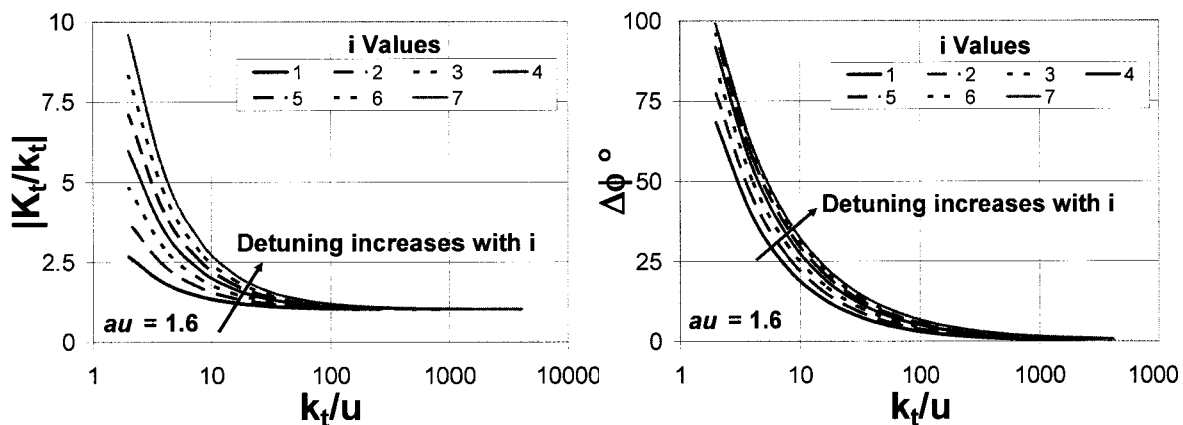


Fig. 3.12 - Dependencies of $|K_t/k_t|$ and $\Delta\phi$ on k_t/u : Split response

For instance by overlaying Figs. 3.12(a) and 3.12(b), and adjusting the scale of $\Delta\phi$ one may predict the parameters $|K_t/k_t|$ and k_t/u necessary in order to generate a split response having a particular azimuthal spread between the peaks (i.e. 25° as illustrated in Fig. 3.13):

Based on the available choices (potential parameter combinations, phase masks on hand etc.) and accounting for the impact of detuning, the selection of design variables may now be finalized. For example the coupling efficiency will decrease as detuning increases, hence the left most point in Fig. 3.13 is expected to produce the strongest response.

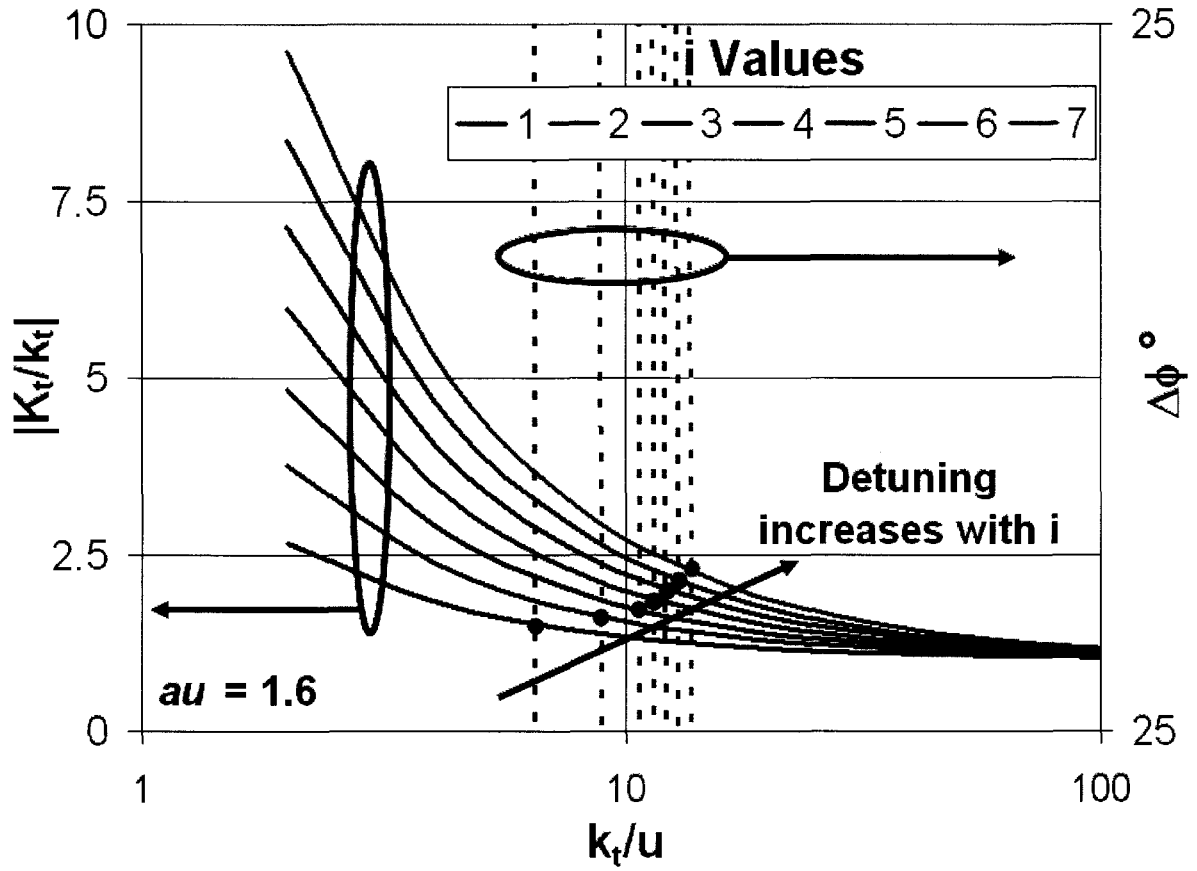


Fig. 3.13 - Combinations of $|K_t/k_t|$ and k_t/u yielding a split response and 25° azimuthal spread ($au=1.6$)

3.5.3 Translate results into common specifications for grating geometry

With the selection of these parameters concluded, the required blaze angle (ξ) and grating period (Λ) may be calculated using Eqs. (3.30) and (3.31).

$$\xi = \arctan \left(\frac{\left| \frac{K_t}{k_t} \right| \left(\frac{k_t}{u} \right) \sqrt{1 - \left(\frac{n_{eff}}{n_0} \right)^2}}{\left(\frac{n_{eff}}{n_0} \right) \mp \sqrt{1 - \left(\frac{k_t}{u} \right)^2 \left[1 - \left(\frac{n_{eff}}{n_0} \right)^2 \right]}} \right) \quad (3.30)$$

$$\Lambda = \frac{\lambda}{n_0 \sqrt{\left(\left(\frac{K_t}{k_t} \right)^2 \left(\frac{k_t}{u} \right)^2 \left[1 - \left(\frac{n_{eff}}{n_0} \right)^2 \right] + \left[\left(\frac{n_{eff}}{n_0} \right) \mp \sqrt{1 - \left(\frac{k_t}{u} \right)^2 \left[1 - \left(\frac{n_{eff}}{n_0} \right)^2 \right]} \right]^2}} \quad (3.31)$$

Eqs. (3.30) and (3.31) are derived in A.3 and A.4 respectively.

3.6. Summary

Based on Li's original formulation of the Volume Current Method [13], it has been shown graphically how the phase matching condition (expressed through f_2) determines the potential shape characteristics of the radiation field, while a dipole driven response to the guided mode's SOP (expressed through f_1) serves to select and magnify that portion at 90° to it.

Making use of this premise the parametric space of the Poynting vector has been examined, showing that three types of radiation profiles are predicted, a centered response, a split response and some combination of the two. This finding offers a well-founded explanation for the multiple tap angles experimentally observed by Peupelmann [5], and others.

Additionally, using the VCM some general guidelines were formulated in order to facilitate the manufacture and optimization of real-world devices.

As a final note, one should recall that these insights are subject to the assumptions stated in 3.2.1. Recently Li et al. have published another paper [18], in which they discuss the potential importance of a guided mode's longitudinal field component. The significance of this finding as it pertains to the work presented here is somewhat more complicated and is evaluated further in *Chapters 4* and *6*. Nevertheless while this factor and others (distortions in the uniformity of the grating etc.) may affect the accuracy of the quantitative findings, the general phenomena and tendencies predicted are expected to remain unchanged.

Chapter 4

Experimental Validation of Azimuthal Radiation Mode Coupling Mechanisms

A number of useful fibre optic devices depend on being able to predict and manipulate the radiation field emitted by tilted fibre Bragg gratings. The previous chapter demonstrated analytically the manner in which this radiation field is directionally dependent on the phase matching characteristics of a grating's three-dimensional structure as well as the polarization dependent dipole response of the medium itself. The contents of this chapter were published in [16], and present experimental measurements of the out-tapped field that clearly illustrate and confirm the existence of the predicted trends associated with each of these physical mechanisms. Using an infrared camera and commercially available beam profiling software, these findings were gathered from a number of tilted fibre Bragg gratings written with an ultraviolet excimer laser at a variety of blaze angles.

4.1 Introduction

In the last chapter, this thesis addressed outstanding questions arising from the work of Peupelmann et al. [5], through a thorough theoretical analysis [14,15], based on a formulation of the Volume Current Method (VCM) by Li et al [13]. As a result of this study it became clear that the radiation field emitted by a tilted Bragg grating is directionally dependent on the phase matching characteristics of the grating's three-dimensional structure as well as the polarization dependent dipole response of the medium in which it is written. The manner in which these two quantifiable contributions are manifested was also clearly illustrated along with the calculated trends in azimuthal behaviour for various grating geometries.

This chapter provides experimental evidence that directly supports these theoretical conclusions, and highlights the first reported fabrication of a three-peak device analogous to Peupelmann's two-peak gratings. Although tilted gratings have been studied quite extensively in recent years, there are few published measurements of their radiation fields, this, despite the importance of such data for understanding and optimizing any number of devices dependent on the predictability and control of this emitted light [2,5,7,10-12].

4.2 Theory

Over the years a number of modeling methods have been proposed for analyzing the responses of fibre Bragg gratings. In the case of tilted gratings three of the most relevant examples have included Coupled Mode Theory (CMT), Conservation of Crystal Momentum and the Volume Current Method.

Coupled Mode Theory is by far one of the most popular and well established methods for modeling fibre Bragg gratings, but although it has been used to examine radiation mode coupling problems [24,85], its prediction of the scattered field outside the fibre is not so straightforward. This is because there is an infinite continuum of unbounded radiation modes, many of which have to be included in the calculation, making the solution cumbersome [85].

As discussed in 1.3.2 Conservation of Crystal Momentum is responsible for a vector relationship between light incident a periodic structure and light scattered by it. Using Fourier analysis, it has been shown that the periodicity of a crystal can be described in terms

of reciprocal lattice vectors (denoted $\mathbf{G}$ or $\mathbf{K}$ in this thesis) [30,31]. Photons exhibit momentum equal to $\hbar\mathbf{k}$, and periodic structures impart an effective momentum $\hbar\mathbf{K}$ when these photons are scattered. In order for all of the scattered light to be in phase, the difference between incident and scattered wavevectors must exactly equal $\hbar\mathbf{K}$, if it does not, detuning will be present and at least some destructive interference will result. Since $\mathbf{K}$ is composed of integer multiples of primitive reciprocal lattice vectors ($\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$) the sense of $\mathbf{K}$, positive or negative is unimportant, and both have been used in the literature [30,31].

One of the benefits of Crystal Momentum Conservation is its simplicity. It can be used to easily predict the longitudinal out-tap angle ξ_{out} of tilted and untilted fibre Bragg gratings, by noticing that along the fibre axis, destructive interference produced from very large numbers of scattering elements, prohibits vector mismatch in this direction. Fig. 4.1 is presented as an example, illustrating that a second order untilted grating can radiate light at 90° to the fibre axis. Although phase matching is satisfied along the fibre axis, in this case as a consequence of large transverse detuning the radiated response will be weak.

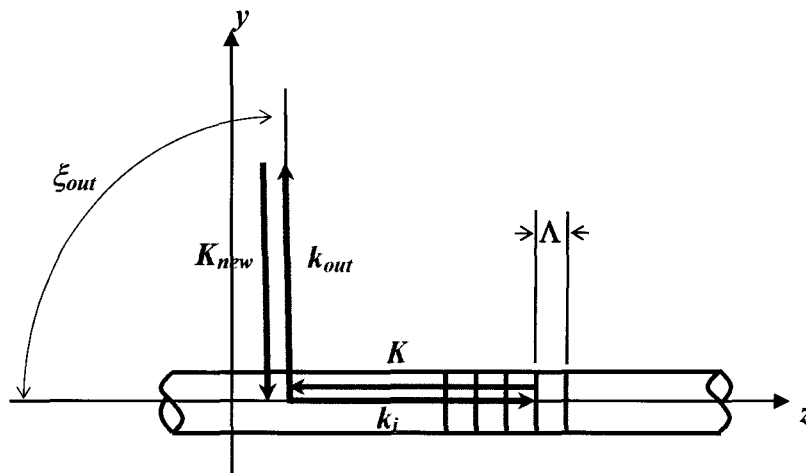


Fig. 4.1 - Crystal Momentum Conservation applied to a 2nd order untilted grating. Destructive interference generated by large numbers of scattering elements along the fibre axis, enforces strict longitudinal phase matching; while less extensive periodicity in the transverse plane allows for significant transverse detuning and weak radiation coupling at 90° to the fibre axis.

In any event, despite the usefulness of Crystal Momentum Conservation, this approach is not suited for modelling the azimuthal distribution of radiated light that results from transverse detuning, because in this plane a grating's periodicity is not very extensive and destructive interference fails to suppress the directional tendencies of the detuned response.

The VCM overcomes this problem, with its principal advantages and features discussed at length in the previous chapter. Its original formulation is attributed to Snyder [78,75], with its extension to tilted FBGs being first made by Holmes et al [83]. Its application to tilted FBG radiation field distributions is well explained in a series of papers by Li et al [13,18,85,86], that provide several helpful formulations of this approach.

Recalling *Chapter 3*'s most important concepts, Li's implementation of the VCM results in parametric expressions for the Poynting vectors associated with a grating's radiation mode coupling, which subject to the formulation's assumptions, fully describe the light scattering from such devices. These expressions take the following form:

$$\bar{S} \approx C \cdot f_1 \cdot f_2 \cdot \left(\hat{r} + \frac{\Delta}{k_t} \hat{z} \right) \quad (4.1)$$

where C is a proportionality factor expanded in Eq. (3.2).

A detailed examination of this expression's component factors reveals the manner in which a grating's response is generated by two physical processes: a dipole interaction between the guided light and the medium (proportional to f_1), coupled with the interference characteristics of a grating's three dimensional structure (proportional to f_2) [15].

As illustrated in Fig. 3.3, the phase matching effects (f_2) determine the potential shape characteristics of a grating's radiation field, while the dipole response of the medium (f_1) serves to select and magnify that portion at 90° to the guided mode's state of polarization (SOP). This polarization dependent directionality does not require birefringence, just as the PDL of such devices does not require it.

The f_2 term fully determines the potential field pattern associated with a given device configuration, including such features as the number of azimuthally distributed peaks and the angular separation of any two peaks. As the transverse detuning increases ($|K_t/k_t|$ departing from unity) f_2 's response has a tendency to cycle from one peak to three to two peaks to one to three peaks and so on. As well, with increased detuning, the magnitude of the peak response is seen to decrease. Fig. 4.2, which is a resequencing of the plots presented in Figs. 3.4 and 3.5, demonstrates these effects as detuning is increased [15].

Although these findings are based on a well established model, it was difficult to assess the applicability of these insights to actual device fabrication given the limited availability of published radiation field measurements combined with the expected discrepancies between real world devices and the idealized structures analyzed. In particular fibre lensing distortions [8], Talbot fringe generation [57], and transverse non-uniformity of the induced index change [62,63], are three considerations that are not necessarily insignificant.

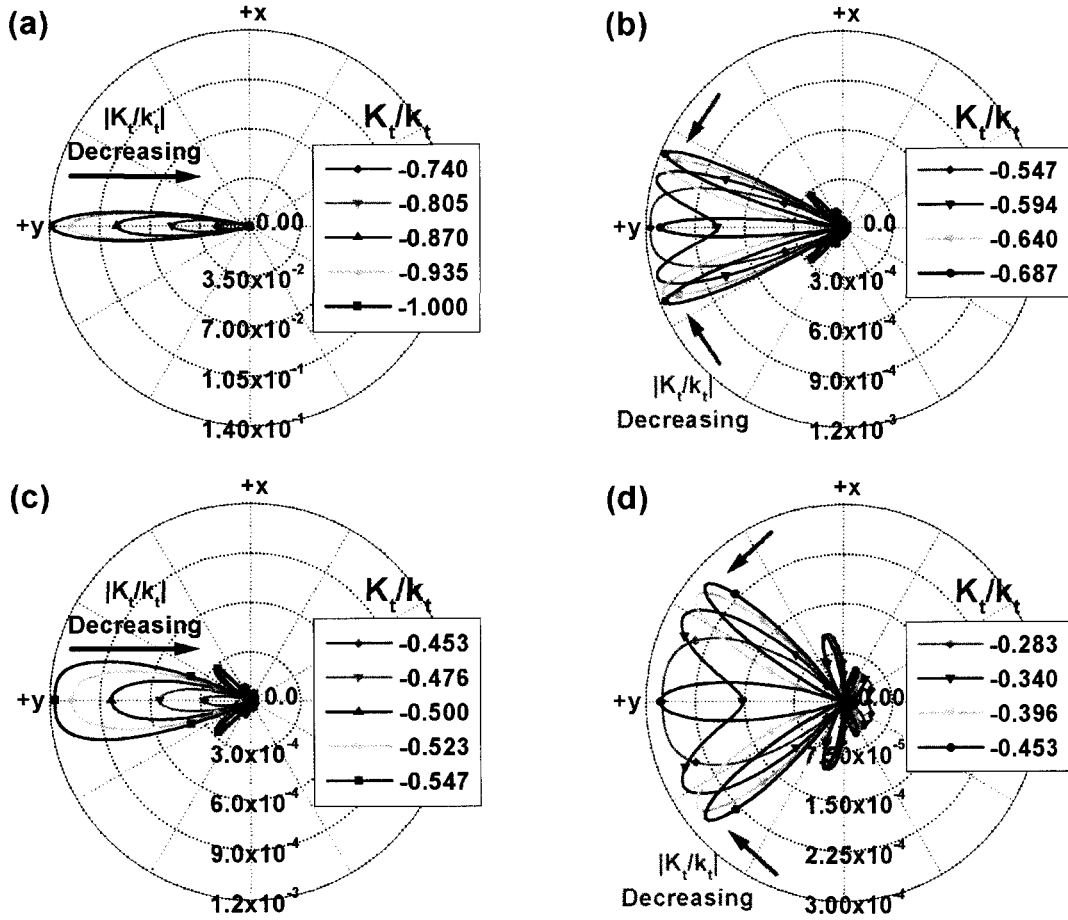


Fig. 4.2 - Variation of $f_{2\text{norm}}$ azimuthal distribution ($au=1.5$, $k_t/u=8$) [15].
Decreasing $|K_t/k_t|$ corresponds with increased detuning.

Interestingly, independent assessment of tilted FBG radiation coupling was presented by Kotačka et al [74], through an extension of the formulation provided in [83]. Strong similarities exist between Kotačka's and Li's formulations since they are both ultimately derived from the original work of Snyder. The main difference between the two is that Kotačka's derivation relies on the introduction of a polarization factor, whereas such polarization dependencies are intrinsically predicted by Li's derivation process.

4.3 Experiment

In this study several tilted FBGs having gradually increased levels of detuning were fabricated and examined. Direct measurements of the radiation fields were then compared with VCM calculations.

4.3.1 Grating Fabrication

To strengthen the out-coupled signal, the induced index change and guided mode overlap with the grating, were maximized by writing the devices in a commercially available, highly germanium doped, high NA , step index, single mode fibre having the properties listed in Table 4.1. The fibre was hydrogen loaded at 2600 psi and room temperature for two weeks to further enhance its photosensitivity to the ultraviolet (UV) laser, then kept at -55°C in a cryogenic freezer unit prior to use.

Table 4.1 - Relevant Fibre Properties

Property	Value	Reference
Name	Fibercore SM1500(4.2/125)	Manufacturer specified [87].
Cutoff Wavelength (λ_{cutoff})	1370 nm	Manufacturer delivered.
Mode Field Diameter (MFD)	4.3 μ m	Manufacturer delivered.
Numerical Aperture (NA)	0.29	Manufacturer delivered.
Fibre Radius (a_{clad})	62.4 μ m	Manufacturer delivered.
n_{clad} @ 1550 nm	1.4440	Pure silica assumed.
n_{core} @ 1550 nm	1.4734	Calculated from NA (Eq. 2.1.3 of [37]).
Core Radius (a)	1.8 μ m	Calculated from λ_{cutoff} and MFD (Eqs. 2.2.35 and 2.2.45 of [37]).

The fabrication setup for these components is presented in detail elsewhere [8]. The inscription process for the tilted gratings utilized a fused silica phase mask having a period of 1500 nm. A stationary, uniform, collimated beam of 248 nm, UV light from a KrF excimer laser was used to expose a 20 mm length of each fibre through the tilted phase mask, to an average fluence of 72 mJ/cm²/pulse at a 100 Hz repetition rate for 10 minutes. 248 nm illumination of the phase mask produced coupling of 25%, 24% and 8% of the incident energy into the +1, -1 and 0 diffractive orders respectively.

Identical exposures of non-tilted devices using a 1069 nm mask resulted in index modulations on the order of 2×10^{-3} . Prior to writing the back-reflection of the silica mask was used to ensure that it was normal to the excimer beam path and the azimuthal rotation of the mask relative to the fibre axis was found to be accurate and easily repeatable to within $\pm 0.2^\circ$. Following rotation of the mask, the fibre was gently placed in parallel and near contact with it.

Since excimer laser beams generally tend to be highly unpolarized and given the induced index change was relatively modest, no additional precautions were taken to avoid the generation of photoinduced birefringence [64,66,69,70,88].

The properties associated with each fabricated device are listed in Table 4.2, where Ideal and Talbot respectively refer to grating structures based on Ideal or Talbot grating periods. The Ideal period is that expected for gratings generated by an ideal zero-order nulled phase mask (Λ), while the Talbot period is 2Λ , with both arising in the case of poor zero-order nulling [57].

Table 4.2 - Fabricated Grating Properties

Grating	θ (°)	ξ^a, θ_B^b (°)	Λ^a, Λ_g^b (nm)		k_t/u		$ K_t/k_t $		PDL (dB)
			Ideal	Talbot	Ideal	Talbot	Ideal	Talbot	
A	27.0	37.5	667	1340	8.53	8.20	0.994	0.517	2.2
B	23.0	32.6	686	1370	8.42	8.27	0.866	0.441	2.9
C	20.5	29.4	697	1390	8.36	8.31	0.782	0.394	1.8
D	18.5	26.8	706	1410	8.31	8.33	0.713	0.355	0.2
E	17.5	25.4	710	1420	8.28	8.34	0.678	0.336	0.7
F	16.0	23.4	716	1430	8.25	8.36	0.624	0.308	0.2
G	15.0	22.0	720	1440	8.22	8.37	0.587	0.289	0.2
H	14.0	20.6	723	1450	8.20	8.38	0.550	0.270	0.3
I	13.0	19.2	727	1450	8.18	8.39	0.513	0.250	0.4
J	11.0	16.3	733	1470	8.15	8.40	0.437	0.212	0.4
K	8.0	12.0	741	1480	8.10	8.42	0.321	0.154	0.1
L	7.0	10.5	743	1490	8.09	8.42	0.281	0.135	0.2
M	6.0	9.0	745	1490	8.08	8.42	0.242	0.116	0.1
N	5.5	8.3	746	1490	8.07	8.43	0.222	0.106	0.1
O	5.0	7.5	746	1490	8.07	8.43	0.202	0.0965	0.1
P	4.0	6.0	748	1500	8.06	8.43	0.162	0.0772	0.1

^a Notation used by Walker [14,15], and Li [13].

^b Notation used by Mihailov[8].

θ - Phase mask azimuthal angle relative to fibre axis normal.

ξ, θ_B - Grating blaze relative to fibre axis.

Λ, Λ_g - Perpendicular separation between grating planes.

k_t/u - Measure of transverse scattering @ 1550 nm.

$|K_t/k_t|$ - Measure of azimuthal dependence @ 1550 nm.

PDL - Polarization dependent loss measured @ 1550 nm.

4.3.2 Measurement Setup

Measurements of the relative radiated intensity were taken using an infrared (IR) video camera (Model ITC-52 by Ikegami Tsushinki Co. Ltd.) and commercially available beam profiling software.

As shown in Fig. 4.3(a), to measure the radiation field profiles each FBG was placed in a special rotation jig allowing controllable rotations about the fibre axis in increments of 2.5°. This jig was mounted on top of a turntable enabling the longitudinal tap angles to be measured with a repeatability of ±1.3°. The camera was then placed behind a rectangular aperture (1.1 cm high × 7 cm wide) located 25 cm from the fibre to ensure that only 2.5° of azimuthal arc was measured at any given time.

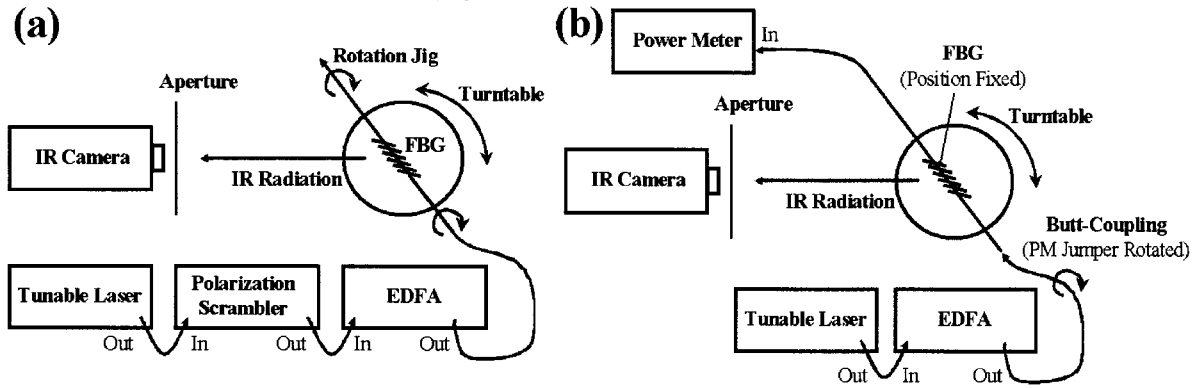


Fig. 4.3 - Measurement setups used to quantify (a) the azimuthal distribution of radiated power and (b) the polarization dependence of radiation model coupling.

In order to ensure that the guided light was monochromatic and randomly polarized, an Agilent 81640A high power tuneable laser source and an Adaptif Photonics polarization

scrambler were used. To ensure that the scattered light was sufficiently intense at a radial distance of 25 cm from the fibre an Erbium Doped Fibre Amplifier (EDFA) was also employed to boost the guided power incident on the grating.

For each device, a video monitor was used to visually align the grating’s longitudinal tap direction with the camera by rotating the turntable until a maximum intensity was observed to be incident the camera’s centre. The turntable position was then locked and the beam profiling software used to measure the relative power variation at 2.5° increments as the fibre was rotated azimuthally using the rotation jig. This process was repeated for each longitudinal tap angle observed for a given device.

To confirm that the polarization dependence of the radiation out-tap [5], was also present in these devices, a similar setup was employed. In this case however, as illustrated in Fig. 4.3(b), the polarization scrambler was not used and linearly polarized incident light was butt-coupled to the fibre by a polarization maintaining (PM) fibre jumper 100 mm from the grating. With the grating held stationary, the PM fibre was rotated through 360° in 15° increments. After each rotation of the PM fibre the butt-coupling was realigned as necessary to maintain the coupled power. Using the camera and beam profiling software, the radiated power was measured as a function of incident polarization for each azimuthal maximum.

4.4 Results

Beginning with the longitudinal tap angle measurements (defined relative to the negative fibre axis), it is evident from Fig. 4.4 that both the Ideal and Talbot responses of the fabricated gratings are consistent with the simulated results, within the measurement error ($\pm 1.3^\circ$).

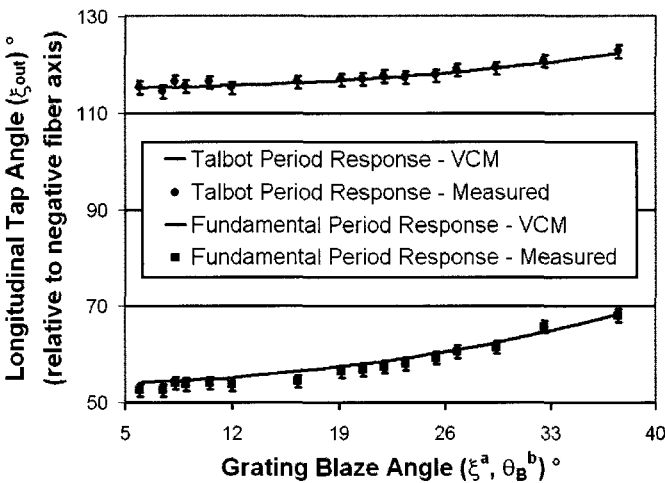


Fig. 4.4 - Longitudinal tap angle as a function of grating blaze. Comparison between experimental measurements and VCM predictions.

As discussed in 4.2, it is possible and quite common to calculate the longitudinal tap angles vectorially without need of the VCM. Fig. 4.5 uses Conservation of Crystal Momentum to analyze Grating P from Table 4.2, and serves as a specific example of how this method may be applied to determine the longitudinal tap angles of tilted gratings. With this in mind, beyond validating the predictions of the VCM, the results presented in Fig. 4.4 also confirm that the devices fabricated do in fact resemble the grating structures intended.

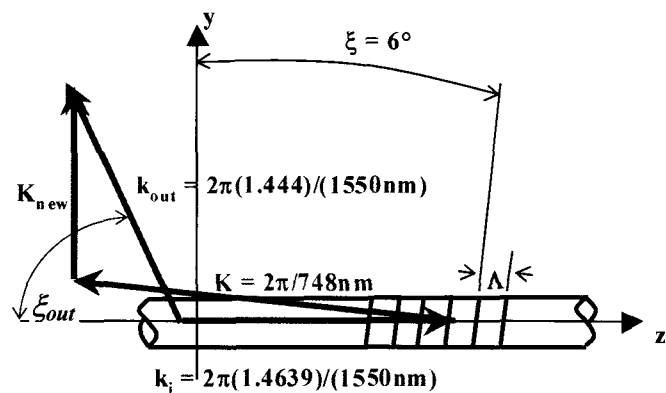


Fig. 4.5 - Crystal Momentum Conservation applied to Grating P of Table 4.2. Enforcing conservation of the longitudinal vector components, this method predicts $\xi_{out}=65.6^\circ$ in the fibre, with Snell's Law yielding $\xi_{out}=53.3^\circ$ in air.

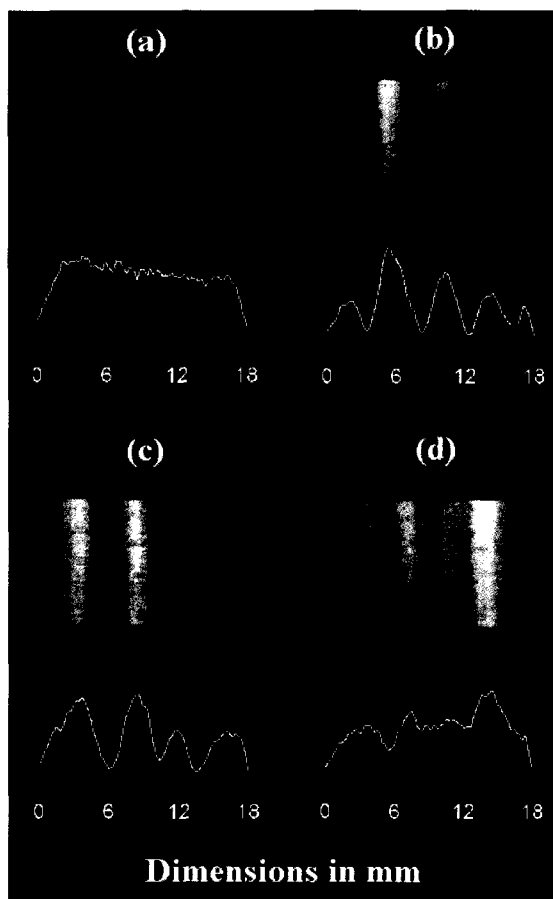


Fig. 4.6 - Photographic examples of longitudinal variations in the radiated power. (a) Uniform Talbot response from device K and (b) significantly periodic Talbot response from device G. (c) Higher contrast and (d) lower contrast periodic Ideal responses measured at different azimuthal angles of device G.

When measuring the radiation field, it was noted that in addition to varying directionally, the intensities also differed with position along the fibre axis. Although some fluctuation might be expected due to non-uniformities in the index change it was interesting to observe

photographically that such differences could be significant, with both highly periodic and nearly uniform responses possible as illustrated in Fig. 4.6(a) and (b). The degree and measure of this periodicity varied from device to device and even from one azimuthal angle to another as shown by Fig. 4.6(c) and (d). It should be noted that Li's unapodized model does not predict this variation, because successful integration of its analytical result necessarily assumes the observation point lies in the center of the grating's length.

When apodization is present, [13] predicts that longitudinal variations in the near field will resemble the grating's apodization profile, with variations in the far field resembling the apodization's Fourier transform. For a 20 cm long, uniformly apodized structure this results in a sinc function having a 10 mm period. In Figs. 4.6 (b) and (c) the period is less than half this amount and the drastic amplitude attenuation of a sinc function is not observed.

Although the cause of this periodicity remains somewhat unclear, as illustrated in Fig. 4.7 (a), the Bragg condition Eq. 1.23 predicts that a 4.5 mm period can be generated by two interfering beams of 1550 nm light having angular separation 2θ , on the order of 0.02° . As seen in Fig. 4.7 (b), for the grating pictured in 4.6 (c) such variations of the longitudinal tap angle ξ_{out} are consistent with differences in n_{eff} of approximately 5×10^{-4} , which is roughly 25% of the total induced index change. Non-uniformities in the excimer beam could potentially result in such effective index variations along the length of the grating.

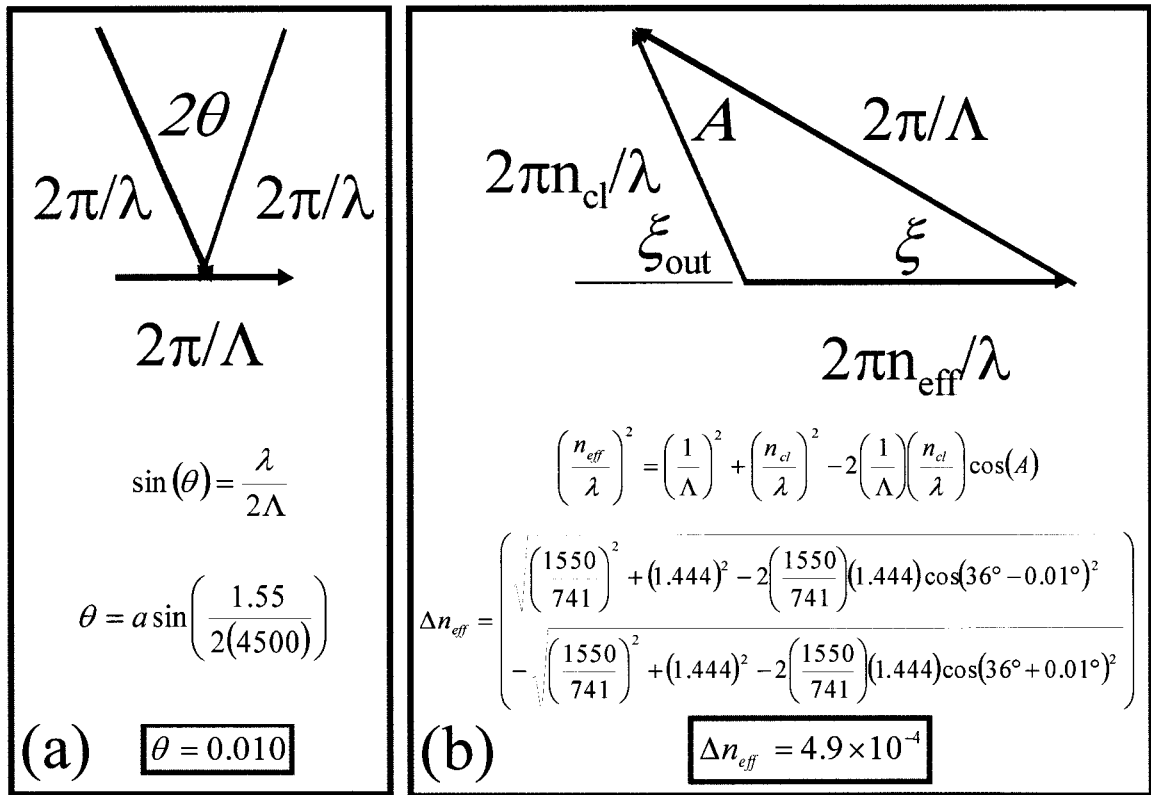


Fig. 4.7 - (a) Bragg condition results in 4.5 mm period interference for two 1550 nm beams having angular separation $2\theta = 0.02^\circ$. (b) In standard fibre, at 1550 nm, a mean longitudinal tap angle of 36° requires an effective index difference of roughly 5×10^{-4} to generate this angular spread.

Nevertheless, since the beam profiling software effectively integrated all of the light sampled for each azimuthal measurement, it was still possible to collect the data presented in

Fig. 4.8 and Fig. 4.9 and contrast it with values calculated using the VCM. For the sake of comparison all calculated values were averaged over an azimuthal angle of 2.5° to account for the azimuthal integration inherent in the measurements.

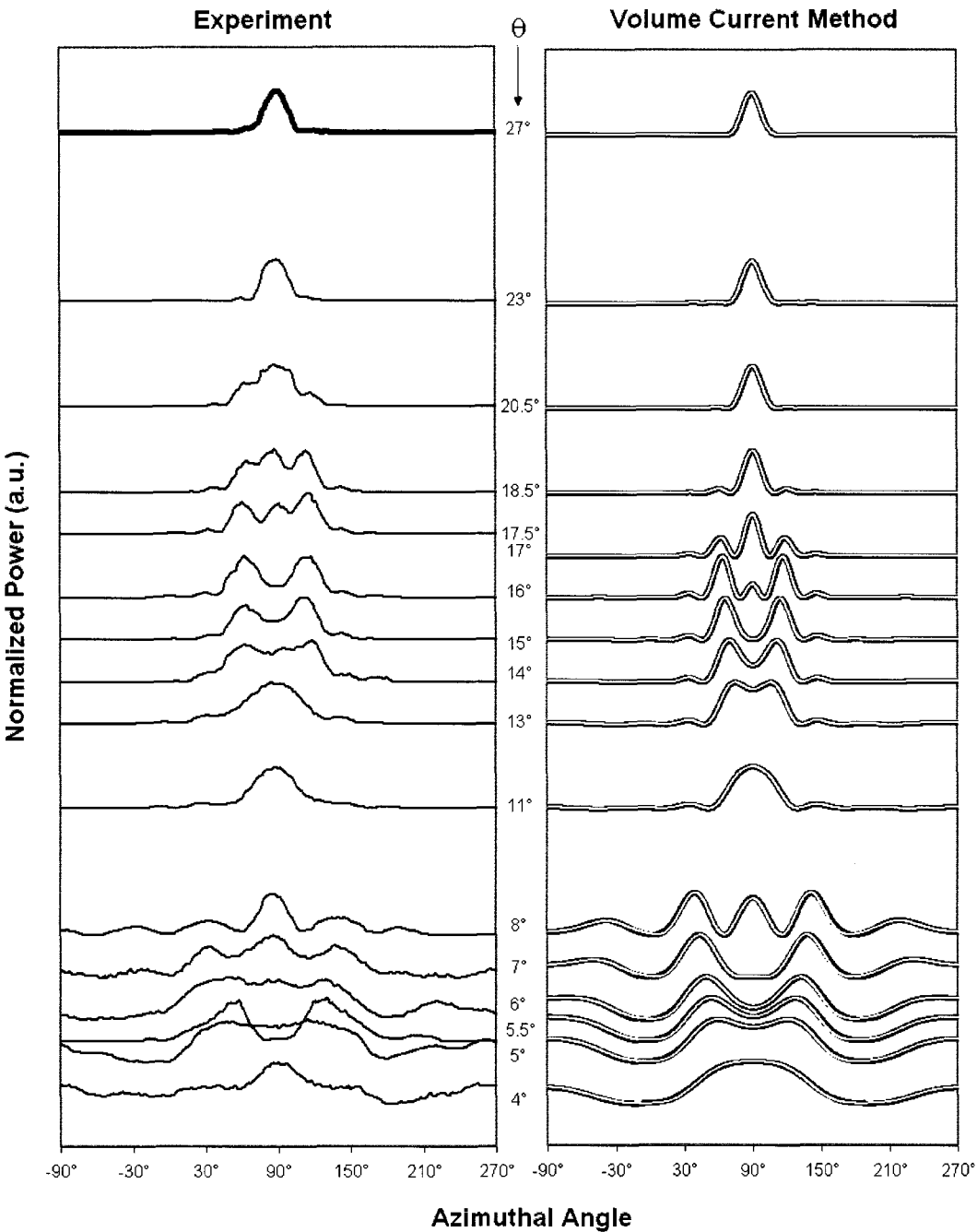


Fig. 4.8 - Radiated Power vs. Fibre Azimuthal Angle (Ideal Resonances). Responses offset proportionally to phase mask tilt θ used to write each device. VCM plots display two results for each device, black curves [13] and white curves [18] which are almost completely superposed.

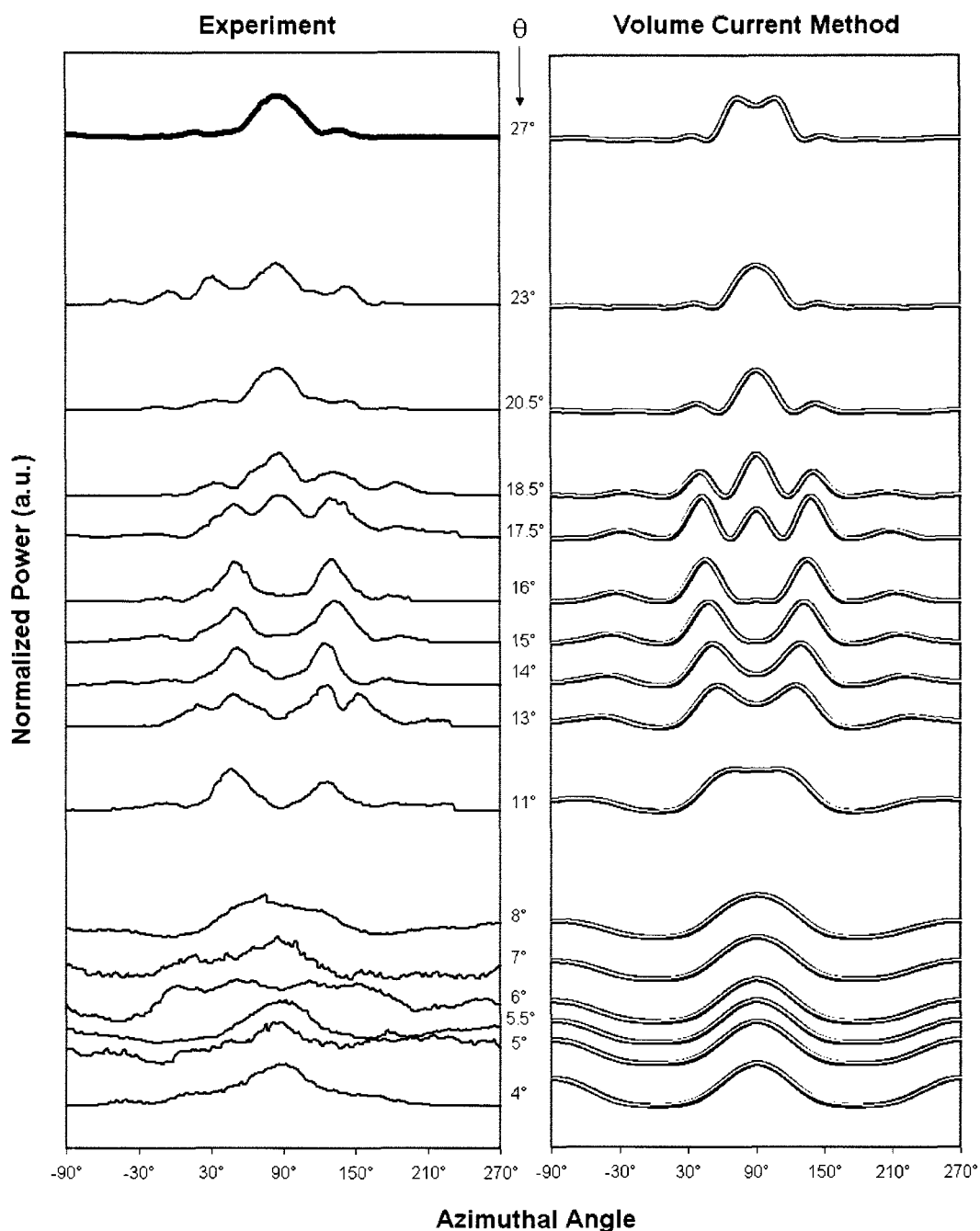


Fig. 4.9 - Radiated Power vs. Fibre Azimuthal Angle (Talbot Resonances). Responses offset proportionally to phase mask tilt θ used to write each device. VCM plots display two results for each device, black curves [13] and white curves [18] which are almost completely superposed.

In Fig. 4.8 and Fig. 4.9, each curve represents the normalized radiated power (from 0 to 1) measured for a given device, and all curves are offset in proportion with the phase mask tilt angle used to write or model each grating. The theoretical results were calculated using two different VCM formulations [13,18]. Although some minor differences are apparent, these results may be treated as essentially identical.

In examining Figs. 4.8 and 4.9, there are obvious differences between the measured and calculated profiles. The origins of these inconsistencies lie in a number of areas. The model assumes that the grating is strictly one-dimensional with a purely sinusoidal index modulation, although it has been clearly demonstrated that the gratings fabricated here incorporate a slightly two-dimensional Talbot structure. Other physical effects not accounted for by the model include distortions arising from fibre lensing as well as transverse non-uniformities of the photoinduced index change. The fact that many of the model inputs were necessarily based on estimated fibre properties (i.e. core size, core/cladding index etc.) must also be considered as a potential source of discrepancy.

Despite these minor differences however it is apparent that the model has performed quite well. The qualitative characteristic tendencies resulting from increased detuning are present in both cases (one peak to three to two peaks to one etc.), occurring with similar blaze angles and azimuthal spreads to those predicted by the model. Also evident from Figs. 4.8 and 4.9, are the first reported examples of the three-peak devices predicted by [14,15].

In terms of the polarization dependence of these devices, the measured values of PDL included in Table 4.2 appear to be in-line with what is expected from a review of [8], with larger values attributable to gratings having stronger phase matching and blaze angles closer to 45°. The results displayed in Fig. 4.10 for device F of Table 4.2, indicate that the radiation coupling is also consistent with the dipole behaviour described previously, with Fig. 4.10(b) resembling other published figures such as the PDL response of a tilted FBG probed with linearly polarized light [89]. This likeness is of no surprise since the PDL in such devices results primarily from radiation mode coupling.

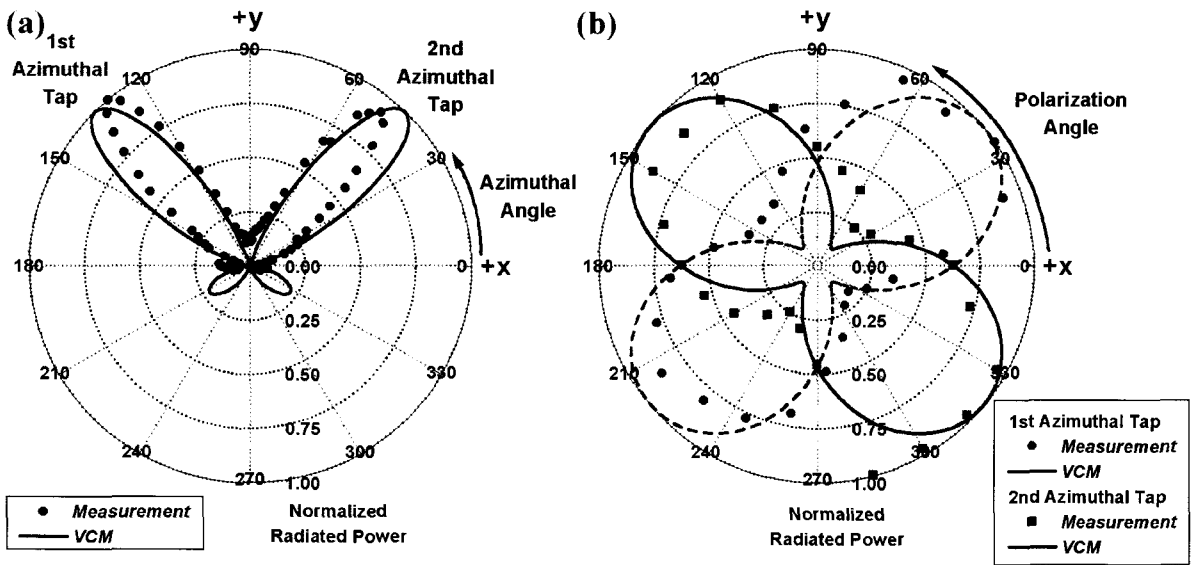


Fig. 4.10 - Normalized radiated power for Device F - Talbot response, measured as a function of (a) azimuthal angle for random polarization and (b) incident polarization angle for 1st and 2nd azimuthal tap locations.

Unfortunately, because these gratings have such weakly resonant broadband responses, it was not possible to measure their photoinduced birefringence directly. However, untilted gratings fabricated with identical exposure conditions, and subject to several weeks of residual hydrogen out-diffusion at room temperature did not exhibit any measurable

polarization dependent wavelength (PD λ) shift of their resonances. This indicates that the induced birefringence does not exceed the intrinsic birefringence of the fibres themselves ($\sim 10^{-6}$).

4.5 Summary

These experiments support the conclusion that the radiation field emitted by tilted Bragg gratings is directionally dependent on two physical mechanisms: phase matching associated with a grating's three-dimensional structure combined with the polarization dependent dipole response of the medium itself.

In terms of azimuthal scattering these results clearly demonstrate that three-peak devices can be fabricated in addition to the two-peak structures originally reported by Peupelmann. The fact that these gratings were fabricated by design as a result of their theoretical discovery is a testament to the accuracy of the VCM. They support its usefulness as a tool for analyzing the radiation field of tilted FBGs, and for optimizing their structures to achieve the design requirements of applications dependent on such guided to radiation mode coupling.

Chapter 5

Application to Improved FBG Polarimeter Design

Polarimeters for measuring the Stokes parameters have evolved significantly over the years from a variety of bulk-optic configurations to a number of elegant in-fibre designs. In this chapter, the advances in tilted fibre Bragg grating (FBG) technology presented earlier are exploited to design an improved in-line polarimeter that samples more than four detected beams with only two tilted FBGs. Based on theoretical analysis and simulation utilizing an extended formulation of the Volume Current Method (VCM), it is demonstrated that without increasing component complexity, this design reduces average Stokes vector errors by 20% for every state of polarization (SOP), facilitating monitoring at lower signal to noise ratios (SNRs). These findings have been accepted for publication in [17].

5.1 Introduction

Demand for higher capacity optical networks inspires the development of faster bitrate systems having more optical channels. As a result, tolerance to errors is reduced requiring greater control of effects such as polarization mode dispersion (PMD), polarization dependent loss (PDL), four-wave mixing (FWM), cross-phase modulation (XPM) and other nonlinear phenomena.

Over the years several solutions have been proposed to help manage and mitigate these factors, including devices for PMD [90-92], and PDL compensation [8,9,93], as well as various strategies for increasing reliability and channel densification, through polarization interleaving [94-96], and multiplexing [97-99].

In light of these examples, reliable polarization monitoring is becoming increasingly important, with polarimeters serving as useful stand-alone instruments as well as components within more complicated devices. Indeed, having the ability to measure a signal's state of polarization (SOP) and degree of polarization (DOP), is an essential prerequisite to determining the PMD and PDL of optical waveguides and components, characterizing birefringent and polarizing materials, evaluating sensors on a polarimetric basis or extracting control signals for the polarization compensators and controllers already referenced.

In describing a light beam, the most general state of partial polarization is conveniently represented by four Stokes parameters, together named a Stokes vector (S_0, S_1, S_2, S_3). A large number of polarimeters have been proposed to measure these values with a thorough review of the early work provided by [100].

Also relevant is a significant body of research, comprising nearly 300 publications by R. M. A. Azzam et al. Although there is only room for a fraction of these citations here [1,101-107], this mass of literature includes extensive analyses and a wealth of innovations, in particular the division of amplitude photopolarimeter (DOAP) [101], in-line polarimeter [102], fibre optic polarimeter [102], sixteen-beam DOAP [106,107], and a fibre Bragg grating (FBG) based polarimeter [1] utilizing tilted FBGs [49].

Subsequently, Westbrook et al reported the first working tilted FBG based polarimeter after incorporating a birefringent element to enable discrimination between elliptical SOPs [2-4]. Enhancing these devices further, Peupelmann et al first disclosed that some gratings exhibit a unique azimuthal scattering behaviour, exploiting this insight to halve the required number of gratings by utilizing only two specialized FBGs, each radiating two azimuthally separate polarization dependent beams [5].

Although four measurands are sufficient to uniquely determine four unknown Stokes parameters, the inclusion and assessment of additional data can often be helpful in averaging out noise and other disturbances that may perturb a system and adversely influence its measurement accuracy. Examples of such polarimeter designs are provided by [105-107], but the concept of utilizing an over-specified or over-constrained system of redundant measurement data, has also been used to advantage in other polarimetry applications and scientific endeavours [108-114].

Having recently characterized the mechanisms responsible for tilted grating two-beam responses and demonstrating that structures eliciting three-beams can be fabricated [14-16], a six-beam polarimeter is proposed and simulated, in which two tilted FBGs each producing three-beams are substituted for the two-beam gratings used in [5]. In doing so, an overall improvement in the statistical nature of noise generated errors is observed, with average error reductions of roughly 20% possible for every polarization state as a result of the new design.

In order to facilitate the calculations, the Volume Current Method (VCM) was extended to enable its handling of elliptical polarization states. This generalized formulation is also presented here, and will be of interest to those characterizing radiation-mode coupling from tilted FBG structures, for any number of applications.

Finally, it is noted that although this work is motivated by advances in fibre optic components, its principle findings and methods remain true and applicable for polarimeters in general.

5.2 Theory

5.2.1 Polarization measurement

Given the instrument matrix (A) of a polarimeter, the Stokes vector (S) associated with any measured set of photodetector currents (I) is calculated by Eq. (5.1) [103]:

$$S = A^{-1}I \quad (5.1)$$

In a four-detector system this equation can be expanded as follows:

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \quad (5.2)$$

where the 4×4 instrument matrix for such a polarimeter is obtained for a real-world device through the measurement of four known polarization states (a,b,c,d) and Eq. (5.3).

$$A^{-1} = \begin{bmatrix} S_{0,a} & S_{0,b} & S_{0,c} & S_{0,d} \\ S_{1,a} & S_{1,b} & S_{1,c} & S_{1,d} \\ S_{2,a} & S_{2,b} & S_{2,c} & S_{2,d} \\ S_{3,a} & S_{3,b} & S_{3,c} & S_{3,d} \end{bmatrix} \begin{bmatrix} I_{1,a} & I_{1,b} & I_{1,c} & I_{1,d} \\ I_{2,a} & I_{2,b} & I_{2,c} & I_{2,d} \\ I_{3,a} & I_{3,b} & I_{3,c} & I_{3,d} \\ I_{4,a} & I_{4,b} & I_{4,c} & I_{4,d} \end{bmatrix}^{-1} \quad (5.3)$$

For maximum stability and accuracy, the determinant of the instrument matrix should be as large as possible [103]. This is accomplished by choosing analyzer positions that maximize the spread of optimally detected states on the Poincaré sphere. For a four-detector system, this requires that the optimally detected incident states represent vertices of a tetrahedron occupying the largest possible volume [103], with one possibility depicted in Fig. 5.1(a). An example of a polarimeter design based on four tilted FBGs that maximize the

tetrahedral volume is presented in [4], whereas this chapter proposes an in-line device based on two tilted FBGs, producing a total of 6 beams, optimized by maximizing the octahedral volume within the Poincaré sphere as shown in Fig. 5.1(b).

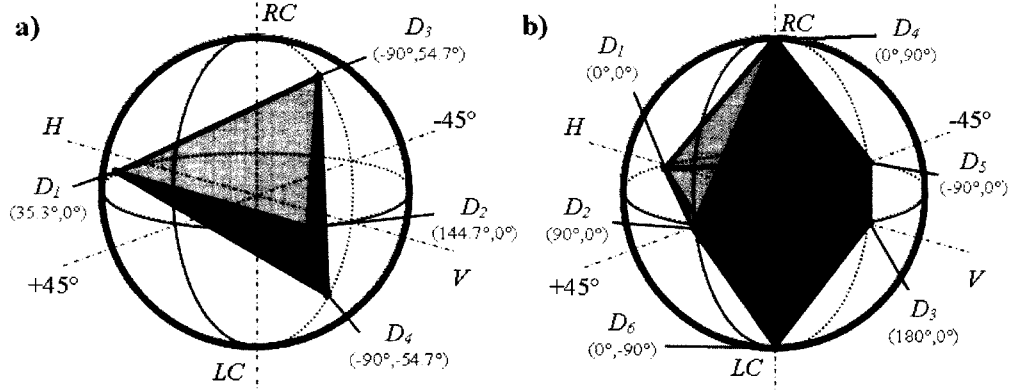


Fig. 5.1 - Examples of optimally detected incident polarization states ($D_1, D_2, D_3, D_4, D_5, D_6$) for (a) four and (b) six-detector polarimeters. Including, right (RC) and left-circular (LC), horizontal (H), vertical (V) and $\pm 45^\circ$ linear states.

5.2.2 Extension to an over-constrained system

A number of references have provided means by which such over-defined systems may be evaluated [105-107,114,115]. Specifically, for a device with more than four detectors (N) the procedure resembles that described in 5.2.1, except that the instrument matrix is no longer square but $4 \times N$. As a result, calibration now requires the measurement of N known polarization states (a, b, c, ... N) as illustrated by Eq. (5.4).

$$A^{-1} = \begin{bmatrix} S_{0,a} & S_{0,b} & S_{0,c} & \dots & S_{0,N} \\ S_{1,a} & S_{1,b} & S_{1,c} & \dots & S_{1,N} \\ S_{2,a} & S_{2,b} & S_{2,c} & \dots & S_{2,N} \\ S_{3,a} & S_{3,b} & S_{3,c} & \dots & S_{3,N} \end{bmatrix} \begin{bmatrix} I_{1,a} & I_{1,b} & I_{1,c} & \dots & I_{1,N} \\ I_{2,a} & I_{2,b} & I_{2,c} & \dots & I_{2,N} \\ I_{3,a} & I_{3,b} & I_{3,c} & \dots & I_{3,N} \\ I_{4,a} & I_{4,b} & I_{4,c} & \dots & I_{4,N} \\ \dots & \dots & \dots & \dots & \dots \\ I_{N,a} & I_{N,b} & I_{N,c} & \dots & I_{N,N} \end{bmatrix}^{-1} \quad (5.4)$$

In this case, maximizing the spread of optimally detected states on the Poincaré sphere requires that these states represent the vertices of an octahedron occupying the largest possible volume. One example of such a distribution is depicted in Fig. 5.1(b).

Direct inversion of the $N \times N$ current matrix ($N > 4$) is difficult. Its determinant was found to be zero or virtually so ($\sim 10^{-15}$) in every case analyzed, due to redundancies in the over-constrained system. As a result, the Current Matrix was perpetually singular, creating numerical instability during inversion. In order to resolve this issue Eq. (5.4) was reformulated as Eq. (5.5) using the Moore-Penrose pseudoinverse for non-square matrices [116,117]. Following this process, one is able to calibrate the instrument matrix of an over-specified polarimeter without difficulty.

$$\mathbf{A}^{-1} = \left(\begin{bmatrix} I_{1,a} & I_{1,b} & I_{1,c} & \dots & I_{1,N} \\ I_{2,a} & I_{2,b} & I_{2,c} & \dots & I_{2,N} \\ I_{3,a} & I_{3,b} & I_{3,c} & \dots & I_{3,N} \\ I_{4,a} & I_{4,b} & I_{4,c} & \dots & I_{4,N} \\ \dots & \dots & \dots & \dots & \dots \\ I_{N,a} & I_{N,b} & I_{N,c} & \dots & I_{N,N} \end{bmatrix} \begin{bmatrix} S_{0,a} & S_{0,b} & S_{0,c} & \dots & S_{0,N} \\ S_{1,a} & S_{1,b} & S_{1,c} & \dots & S_{1,N} \\ S_{2,a} & S_{2,b} & S_{2,c} & \dots & S_{2,N} \\ S_{3,a} & S_{3,b} & S_{3,c} & \dots & S_{3,N} \end{bmatrix}^{-1} \right)^{-1} \quad (5.5)$$

5.2.3 Noise mechanisms of primary interest

Polarimeters are diagnostic instruments, often designed to track variations in a signal's polarization state. Hence, the impact of optical noise sources is not as relevant to its operation as the contribution of detector or receiver related noise factors. Shot noise (σ_s^2) and thermal noise (σ_T^2) are two fundamental mechanisms producing fluctuations in the receiver currents generated by photodetectors, with the total root-mean-square (RMS) noise current (σ^2) given by Eq. (5.6) [118].

$$\sigma^2 = \sigma_s^2 + \sigma_T^2 \quad (5.6)$$

Shot noise results because the detector current is composed of electrons generated at somewhat random times. From Eq. (5.7) below [118], it is evident that the RMS shot noise can be described as a summation of two components, one proportional to the detected current I_p , and one proportional to the dark current I_d , a property of the receiver.

$$\sigma_s^2 = 2q(I_p + I_d)\Delta f = \sigma_{sp}^2 + \sigma_{sd}^2 \quad (5.7)$$

where q and Δf are the electron charge and effective noise bandwidth of the detector.

Thermal noise (also called Johnson or Nyquist noise), results from the random thermal motion of electrons. From Eq. (5.8) below [118], it is evident that the RMS thermal noise is entirely a function of receiver design, with k_B representing Boltzmann's constant, T temperature, R_L the load resistance and F_n a noise enhancement factor.

$$\sigma_T^2 = (4k_B T / R_L) F_n \Delta f \quad (5.8)$$

Thermal noise dominates receiver performance ($\sigma_T^2 \gg \sigma_s^2$) in most cases of practical interest [118], especially for the weak detected signals of in-line devices that characteristically sample only a tiny fraction of the guided energy. To generalize, the detector current term (σ_{sp}^2) of Eq. (5.7) is neglected, and the RMS noise taken to be exclusively dependent on detector and receiver design characteristics.

5.2.4 Extended VCM formulation

For the simulation, it is necessary to estimate the detected currents generated by each SOP. The Volume Current Method predicts the polarization dependent radiated power seen by each detector, allowing calculation of the expected current values, since they are proportional to the incident optical power.

The VCM is a useful perturbation analysis tool for solving numerous waveguide radiation problems that result from small refractive index fluctuations. Of particular relevance to tilted FBGs is a series of papers by Li et al [13,18,85,86], that present parametric expressions fully describing the optical power radiated from such devices.

Unfortunately present formulations assume that the incident guided wave is linearly polarized and do not account for the radiation generated by circular or elliptically polarized

guided modes. To address this, the following formula for elliptical polarization states is used, derived in *Appendix B*:

$$\begin{aligned}\vec{S} &\approx \frac{1}{2\mu_0} \vec{E}_{in} \times \vec{B}_{in}^* \\ &\approx \frac{|\vec{A}|^2}{2\omega\mu_0^2\epsilon} \left[\Delta^2 + k_t^2 \left(\sin^2(\alpha + \phi) \sin^2(\delta_{phase}/2) + \sin^2(\alpha - \phi) \cos^2(\delta_{phase}/2) \right) \right] (k_t \hat{r} + \Delta \hat{z})\end{aligned}\quad (5.9)$$

where $|\vec{A}|$ is the vector potential magnitude, Δ and k_t are longitudinal and traverse components of the radiated wavevector, α is the polarization state's auxiliary angle and δ_{phase} its phase difference between x and y components [119], ω is the mode's angular frequency, μ_0 is the magnetic permeability and ϵ the electric permittivity of the medium.

Recognizing that Eq. 3.20 of [13] contains a typographical error in the vector portion ($k_t \hat{\phi}$ should read $k_t \hat{r}$ according to the calculation), this result is almost identical, with the $\sin^2(\delta - \phi)$ term of [13] replaced by the more complicated factor specified in Eq. (5.10).

$$Q = \sin^2(\alpha + \phi) \sin^2(\delta_{phase}/2) + \sin^2(\alpha - \phi) \cos^2(\delta_{phase}/2) \quad (5.10)$$

For light coupled to purely forward or backward propagating modes, $k_t = 0$ and Eq. (5.9) contains no polarization dependence. For light coupled to purely transverse radiation modes $\Delta = 0$ and the influence of this polarization factor Eq. (5.10), is maximized, as in the optimal case. More generally, Δ and k_t may be determined by phase matching arguments [13,15], enabling impact assessment of wavelength changes as guided modes detune ($\Delta\lambda$) from the grating / polarimeter design resonance (λ).

5.2.5 Detector currents

Since a detector azimuthally located at ϕ will receive incident optical power $|\vec{S}(\phi)|$, using Eq. (5.11) it is possible to calculate detector currents normalized to the optimal case ($\Delta=0$), which reduces to Eq. (5.12) for randomly polarized light.

$$I(\phi)_{norm, polarized} \approx Q + \frac{\Delta^2}{k_0^2 n_0^2} [1 - Q] \quad (5.11)$$

$$I(\phi)_{norm, unpolarized} \approx \frac{\int_0^{\pi/2} \int_{-\pi}^{\pi} I(\phi)_{norm, polarized} \cdot d\delta_{phase} \cdot d\alpha}{\int_0^{\pi/2} d\alpha \cdot \int_{-\pi}^{\pi} d\delta_{phase}} = \frac{1}{2} \left(1 + \frac{\Delta^2}{k_0^2 n_0^2} \right) \quad (5.12)$$

where k_0 and n_0 are the freespace wavevector magnitude and core refractive index, that depend on the wavelength of the sampled light.

In either case, the normalized noise-free currents may be expressed using Eq. (5.13), as the summation of an optimally resonant component (I_{opt}) and one arising from wavelength detuning ($I_{\Delta\lambda}$):

$$I = I_{opt} + I_{\Delta\lambda} \quad (5.13)$$

It is interesting but not surprising to note that the result of Eq. (5.10) is the same when two polarization states differ only in the sign of δ_{phase} . An example of this is right-hand and left-hand circular polarizations. Both will generate the same radiated power, which is why a birefringent element is required, to create a difference in the auxiliary angles of such paired states and allow for distinction between them. This requirement is not unique to FBG polarimeters.

Eq. (5.10) is already normalized with a minimum of 0 and a maximum of 1. Analysis of this expression confirms that maximum radiated intensities of linearly polarized guided modes result when the azimuthal angle of phase matching or quasi-phase matching is at 90° to the polarization angle [13-16]. It also reveals that equal-power, elliptically polarized modes cannot achieve the same maximum radiated intensity, which is not entirely unexpected since only a fraction of the guided field is contained in each orthogonal polarization component.

To maximize an FBG polarimeter’s sensitivity, these insights imply that the waveplate elements should linearize each photodiode’s optimally detectable state prior to detection. Furthermore, the detectors should be azimuthally located at 90° to these states.

5.2.6 Optimal FBG polarimeter designs

Optimal four-detector FBG polarimeter designs have been disclosed and discussed in detail [2-4]. Building upon this prior art and making use of the discoveries in [14-16], it is possible to construct a two-grating, six-detector polarimeter by simply substituting appropriate three-beam gratings for the two-beam ones used in [5].

If detector arrays are used to monitor all the beams emanating from a single grating, as suggested in [5], then no additional components are required, and the device’s size and assembly process should remain virtually unaffected by incorporation of this improvement.

An example of specific analyzer properties and orientations for such an optimized device, are provided in Fig. 5.2, along with a corresponding illustration of the evolving SOPs. For such a device, detectors should be placed at a 90° azimuthal angle to the optimal linearized states.

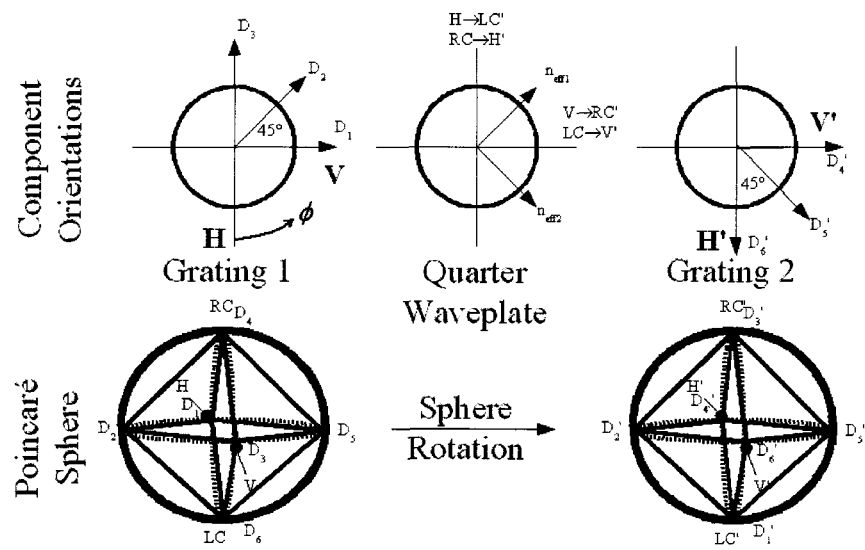


Fig. 5.2 - Example of optimal analyzer properties and orientations for a six-detector polarimeter, with corresponding illustrations of the evolving SOPs. Detectors azimuthally located at 90° to corresponding linearized optimal states.

In particular Fig. 5.2 shows that states forming vertices of an octahedron with the largest possible volume, can be measured if the three-beam gratings are designed to induce an angular spread of 45° between each radiated beam. Furthermore it illustrates the relative orientation that is required for each of the various elements, with a 90° rotational displacement between the two gratings and a quarter-waveplate oriented to transform the polarization states as shown, with horizontal and vertical polarization states before the waveplate respectively becoming left and right-circular polarization states after. As a result, this example not only ensures maximum coverage of the Poincaré sphere, but also that the optimal state corresponding with each detector is linearized prior to detection as discussed in 5.2.5.

5.3 Modeling and simulation

The physical system and modeling process are summarized in Fig. 5.3, with detailed explanations provided in the following four subsections. Essentially, polarization states present in the optical system generate detector currents, which are then multiplied by an instrument matrix in order to determine the measured Stokes parameters associated with those states. Potential sources of error include the noise mechanisms discussed in 5.2.3, wavelength detuning discussed in 5.2.4, or inaccuracies in the matrix calculation itself. By comparing the resulting SOPs with the initial states, it is possible to obtain an estimate of the error.

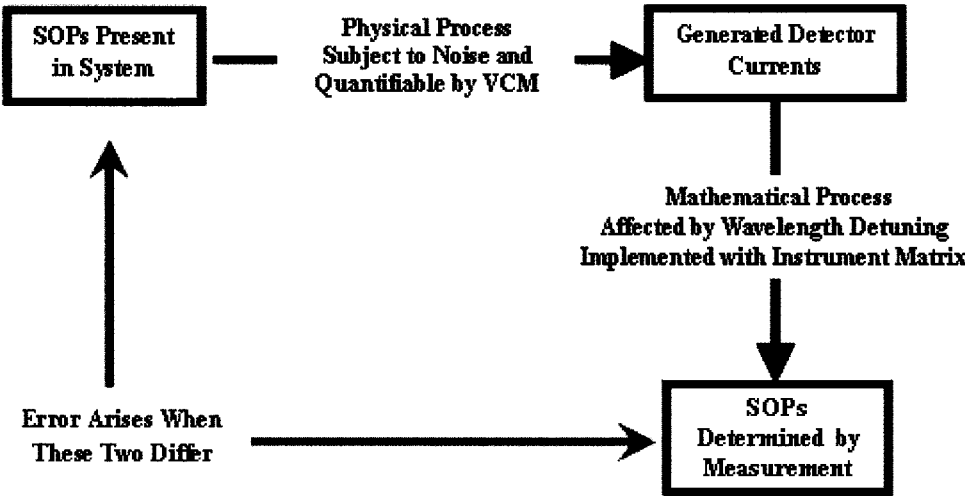


Fig. 5.3 - Block diagram of modeled system, summarizing means of operation and sources of potential error.

5.3.1 Calculation of instrument matrices

To evaluate and compare the relative impact of errors introduced through fluctuations in the detector currents, it was first necessary to calculate instrument matrices for optimal four and six-detector polarimeters.

As illustrated in Fig. 5.1, optimal polarization states for detection and calibration of these devices are known in terms of their Poincaré sphere coordinates $(2\psi, 2\chi)$, and may be expressed as Stokes vectors using Eqs. (5.14) through (5.16), adapted for states of partial polarization [119]:

$$S_1 = p \cdot S_0 \cos(2\chi) \cos(2\psi) \tag{5.14}$$

$$S_2 = p \cdot S_0 \cos(2\chi) \sin(2\psi) \quad (5.15)$$

$$S_3 = p \cdot S_0 \sin(2\chi) \quad (5.16)$$

where p is used to represent the DOP.

By example, for the six-detector states depicted in Fig. 5.1(b), one obtains the following normalized Stokes vectors:

$$D_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, D_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, D_5 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, D_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

To calculate the rotated polarization states after the polarimeter's birefringent element, one has simply to multiply the incident Stokes vectors by the appropriate Mueller matrix (M_{QWP}).

$$M_{QWP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (5.17)$$

For the incident states shown in Fig. 5.1(b), this yields:

$$D_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, D_2' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, D_3' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, D_4' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, D_5' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, D_6' = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Next the new Poincaré sphere coordinates can be determined for each transformed state, by utilizing Eqs. (5.18) and (5.19) adapted for partially polarized light [119].

$$\psi = \frac{1}{2} \arctan\left(\frac{S_2}{S_1}\right) \quad (5.18)$$

$$\chi = \frac{1}{2} \arcsin\left(\frac{S_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}}\right) \quad (5.19)$$

For the transformed states calculated above, one thereby obtains:

$$D_1' = (0, -90^\circ), D_2' = (90^\circ, 0), D_3' = (0, 90^\circ), D_4' = (0, 0), D_5' = (-90^\circ, 0), D_6' = (180^\circ, 0)$$

With the aid of Eqs. (5.11) and (5.12), normalized detector currents can now be computed after each polarization state incident a given detector is expressed in terms of its auxiliary angle (α), and phase delay (δ_{phase} , denoted δ in [119]). To this end Eqs. (5.20) and (5.21) are provided, by combining $\tan(2\psi) = \tan(2\alpha)\cos(\delta_{phase})$ and $\sin(2\chi) = \sin(2\alpha)\sin(\delta_{phase})$ of [119]:

$$\begin{aligned}
\cos^2(\delta_{phase}) + \sin^2(\delta_{phase}) &= \frac{\tan^2(2\psi)}{\tan^2(2\alpha)} + \frac{\sin^2(2\chi)}{\sin^2(2\alpha)} \\
\tan^2(2\psi)\cos^2(2\alpha) &= \sin^2(2\alpha) - \sin^2(2\chi) \\
[1 + \tan^2(2\psi)]\cos^2(2\alpha) &= \cos^2(2\chi) \\
\alpha &= \frac{1}{2} \arccos[\cos(2\chi)\cos(2\psi)] \quad 0 \leq \alpha \leq \frac{\pi}{2}
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\frac{\sin(\delta_{phase})}{\cos(\delta_{phase})} &= \frac{\tan(2\alpha)\sin(2\chi)}{\sin(2\alpha)\tan(2\psi)} \\
\frac{\sin(\delta_{phase})}{\cos(\delta_{phase})} &= \frac{\sin(2\chi)}{\cos(2\chi)\cos(2\psi)\tan(2\psi)} \\
\delta_{phase} &= \arctan\left(\frac{\tan(2\chi)}{\sin(2\psi)}\right) \quad -\pi \leq \delta_{phase} \leq \pi
\end{aligned} \tag{5.21}$$

For the various states of the example, this results in the following (α, δ_{phase}) values:

$$\begin{aligned}
D_1 &= (0,0), D_2 = (45^\circ,0), D_3 = (90^\circ,0), D_4 = (45^\circ,90^\circ), D_5 = (45^\circ,180^\circ), D_6 = (45^\circ,-90^\circ) \\
D_1' &= (45^\circ,-90^\circ), D_2' = (45^\circ,0), D_3' = (45^\circ,90^\circ), D_4' = (0,0), D_5' = (45^\circ,180^\circ), D_6' = (90^\circ,0)
\end{aligned}$$

Having determined the Stokes (S) and current (I) vectors associated with each calibration state (D_i) and using the detector locations (ϕ_i, ϕ_i') pictured in Fig. 5.2, instrument matrices (A^{-1}) of optimized four and six-detector polarimeters were obtained by way of Eqs. (5.3) and (5.5), with the matrix for the six-detector example, calculated below:

$$\begin{aligned}
S &= \begin{bmatrix} & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\ S_0 & 1 & 1 & 1 & 1 & 1 & 1 \\ S_1 & 1 & 0 & -1 & 0 & 0 & 0 \\ S_2 & 0 & 1 & 0 & 0 & -1 & 0 \\ S_3 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow S^{-1} = \begin{bmatrix} \frac{1}{6} & 0.5 & 0 & 0 \\ \frac{1}{6} & 0 & 0.5 & 0 \\ \frac{1}{6} & -0.5 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0.5 \\ \frac{1}{6} & 0 & -0.5 & 0 \\ \frac{1}{6} & 0 & 0 & -0.5 \end{bmatrix} \\
I &= \begin{bmatrix} & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\ \phi_1 & 1 & 0.5 & 0 & 0.5 & 0.5 & 0.5 \\ \phi_2 & 0.5 & 1 & 0.5 & 0.5 & 0 & 0.5 \\ \phi_3 & 0 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ \phi_4' & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0 \\ \phi_5' & 0.5 & 0 & 0.5 & 0.5 & 1 & 0.5 \\ \phi_6' & 0.5 & 0.5 & 0.5 & 0 & 0.5 & 1 \end{bmatrix} \\
A^{-1} &= \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & -0.5 & 0 \\ 0.5 & 0 & 0 & -0.5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}
\end{aligned}$$

5.3.2 Calculation of “true” Stokes vectors and noise-free detector currents

After discretizing the Poincaré sphere in $1^\circ \times 1^\circ$ increments, Eqs. (5.14) to (5.17) were used to calculate “true” Stokes vectors associated with each defined polarization state. Following calculation of the corresponding waveplate rotated Stokes vectors, normalized noise-free detector currents were also computed for each state, using the same procedure described in 5.3.1.

For the analysis of partially polarized light, results from Eq. (5.11) were multiplied by the degree of polarization, with the randomly polarized contribution $[(1-p) \cdot I(\phi)_{norm,unpolarized}]$ subsequently added to each detector.

As an example, for $DOP = 0.5$, the state $(2\psi, 2\chi) = (45^\circ, 45^\circ)$ has the following Stokes vectors and Poincaré sphere coordinates, before and after the waveplate:

$$S_{(45^\circ, 45^\circ)} = \begin{bmatrix} 1 \\ 0.25 \\ 0.25 \\ \frac{1}{2\sqrt{2}} \end{bmatrix}, S'_{(35.3^\circ, -30^\circ)} = \begin{bmatrix} 1 \\ \frac{1}{2\sqrt{2}} \\ 0.25 \\ -0.25 \end{bmatrix}$$

Making use of the resulting (α, δ_{phase}) values, the detector positions shown in Fig. 5.2, and assuming $\Delta=0$, the transpose of the predicted current vector is:

$$\begin{aligned} (\alpha, \delta_{phase})_{(45^\circ, 45^\circ)} &= (30^\circ, 54.7^\circ), (\alpha', \delta'_{phase})_{(35.3^\circ, -30^\circ)} = (22.5^\circ, -45^\circ) \\ I_{DOP=1}^T &= [0.75 \quad 0.75 \quad 0.25 \quad 0.85 \quad 0.25 \quad 0.15] \\ I_{DOP=0}^T &= [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5] \\ I_{DOP=0.5}^T &= 0.5 \cdot I_{DOP=1}^T + 0.5 \cdot I_{DOP=0}^T = [0.63 \quad 0.63 \quad 0.38 \quad 0.68 \quad 0.38 \quad 0.32] \end{aligned}$$

5.3.3 Calculation of noise currents and “measured” Stokes vectors

Given that the noise contributions discussed in 5.2.3 are random processes with approximately Gaussian characteristics [118], noise currents (i_n) were randomly determined for each polarization state using a normally distributed random number generator (*randn*) multiplied by defined standard deviation values σ .

$$i_{n,SOP} = \sigma \cdot \begin{bmatrix} randn \\ randn \\ randn \\ \dots \\ randn \end{bmatrix} \quad (5.22)$$

In order to assess the magnitude of polarization measurement errors, arising from the noise mechanisms discussed, Eq. (5.22) was added to the noise-free detector currents calculated in 5.3.2. The resulting noise-added current vectors, were then multiplied by the instrument matrices determined in 5.3.1, to obtain noise-added or “measured” Stokes vectors.

$$S_{noise} = A^{-1} (I + i_{n,SOP}) \quad (5.23)$$

For the example state $(2\psi, 2\chi) = (45^\circ, 45^\circ)$, and $\sigma = 0.1$, one set of randomly generated noise currents yields:

$$S_{\text{noise}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.63 + 0.084 \\ 0.63 - 0.009 \\ 0.38 + 0.144 \\ 0.68 - 0.039 \\ 0.38 + 0.109 \\ 0.32 - 0.110 \end{bmatrix} = \begin{bmatrix} 1.0663 \\ 0.1900 \\ 0.1320 \\ 0.4310 \end{bmatrix}$$

5.3.4 Calculation and comparison of polarimeter errors

Subtraction from these results of the “true” Stokes vectors defined in 5.3.2, yielded vector errors for each state, with a scalar estimate of their magnitude provided by their vector norms.

$$\mathcal{E}_{SOP} = S_{\text{noise}} - S_{\text{true}} = A^{-1}(I + i_{n,SOP}) - S_{\text{true}} \quad (5.24)$$

$$\|\mathcal{E}_{SOP}\| = \sqrt{(S_0 - S_{0,\text{noise}})^2 + (S_1 - S_{1,\text{noise}})^2 + (S_2 - S_{2,\text{noise}})^2 + (S_3 - S_{3,\text{noise}})^2} \quad (5.25)$$

For the example state already discussed the resulting error is:

$$\|S_{\text{noise}} - S_{(45^\circ, 45^\circ)}\| = \left\| \begin{bmatrix} 1.0663 \\ 0.1900 \\ 0.1320 \\ 0.4310 \end{bmatrix} - \begin{bmatrix} 1 \\ 0.25 \\ 0.25 \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0.0663 \\ -0.0600 \\ -0.1180 \\ 0.0774 \end{bmatrix} \right\| = 0.167$$

In fact, since the noise currents are randomly varying, this process was repeated a number of times for each state in order to gain some insight into the average error expected for each configuration of the polarimeter.

5.4 Results

Combining Eqs. (5.13) and (5.23), it is apparent that the calculated SOP is a summation of three factors, each a potential contributor to measurement error and each individually examined in the following three subsections.

$$S_{\text{noise}} \approx A^{-1}(I_{\text{opt}} + I_{\Delta\lambda} + i_{n,SOP}) \approx A^{-1}I_{\text{opt}} + A^{-1}I_{\Delta\lambda} + A^{-1}i_{n,SOP} \quad (5.26)$$

5.4.1 Validity of linear matrix system

Provided the FBG scattering mechanisms are linearly dependent on polarization, at the design wavelength and in the absence of noise, the linear mathematical system described in Eq. (5.1) should exactly model the devices in question. Although this is the case, Eqs. (5.9) and (5.10) have a tendency to obscure the linear polarization dependence of the radiated beam and resulting detector currents.

Examining the errors calculated for noise-free four and six-detector polarimeters, with DOPs of 1, 0.7, 0.4 and 0.1, revealed identical results and confirmed that in the absence of noise and at the design wavelength, SOPs across the entire Poincaré sphere ($1^\circ \times 1^\circ$ discretization) can be modeled with virtually no error using Eq. (5.1). Hence $S_{\text{true}} \approx A^{-1}I_{\text{opt}}$, and Eq. (5.24) reduces to Eq. (5.27), from which it becomes apparent that the error is limited to two terms, a constant factor dependent on wavelength detuning ($\mathcal{E}_{\Delta\lambda}$), as well as a random noise component ($\mathcal{E}_n$).

$$\mathcal{E}_{SOP} \approx A^{-1}(I_{\text{opt}} + I_{\Delta\lambda} + i_{n,SOP}) - A^{-1}I_{\text{opt}} \approx A^{-1}(I_{\Delta\lambda} + i_{n,SOP}) \approx \mathcal{E}_{\Delta\lambda} + \mathcal{E}_n \quad (5.27)$$

5.4.2 Effect of wavelength detuning

In order to assess the significance and behaviour of this term, for a variety of $(\Delta/k_0 n_0)$ values, Eq. (5.28) was calculated for the same SOPs and DOPs used in 5.4.1.

$$\varepsilon_{\Delta\lambda} = A^{-1} I_{\Delta\lambda} \quad (5.28)$$

Despite the polarization variables inherent in $I_{\Delta\lambda}$, the error arising as a result of detuning was observed to be uniform over the entire domain of the Poincaré sphere. This error was also found to be unaffected by DOP. In Fig. 5.4, the relative error of the Stokes vectors $\|\varepsilon_{\Delta\lambda}\|/\|I_{opt}\|$ calculated for each polarimeter are plotted as a function of $(\Delta/k_0 n_0)$. From this figure it is apparent that the error generated by detuning has remained virtually unaffected by the proposed polarimeter design change.

As no additional error of this kind is predicted to result from the proposed changes, and since present devices can effectively tolerate or compensate for this systematic error mechanism, functioning over bandwidths of 70 nm to 130 nm [3,120], there is no further need to discuss this issue here.

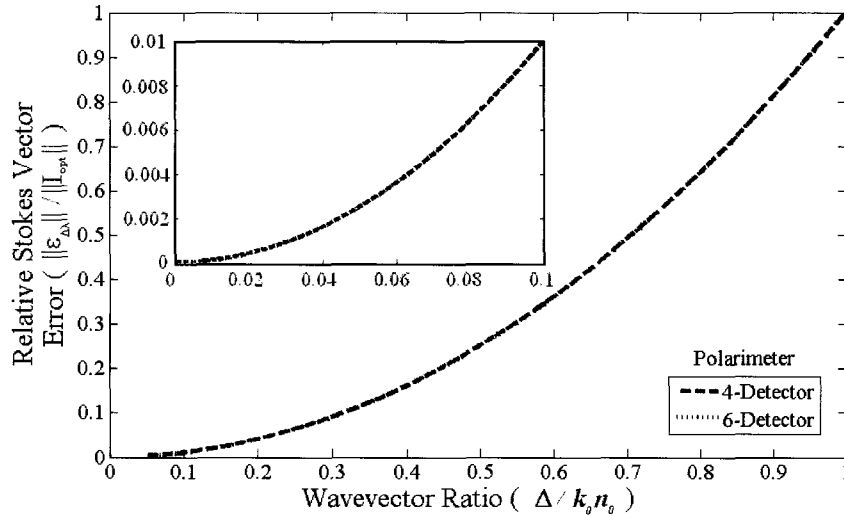


Fig. 5.4 - Relative error arising from wavelength detuning.

5.4.3 Effect of random noise currents

Investigating the effects generated by random noise currents, Eq. (5.29) is utilized, noticing that at the design wavelength this equation is a direct calculation of the total error, with magnitudes proportional to the square root of the receiver's RMS noise value (σ).

$$\varepsilon_n = A^{-1} i_{n,SOP} \quad (5.29)$$

Assuming identical detector and receiver characteristics are used, at the design wavelength the relative improvement between six and four-detector devices ($\|\varepsilon_{SOP,6}\|/\|\varepsilon_{SOP,4}\|$), is therefore independent of the specific component related RMS noise values.

Subject to these conditions, in Fig. 5.5 calculated probability data are provided for each polarimeter, based on one million random noise current vectors. These data have been fitted with Maxwellian probability distributions having fitting parameter γ , and offset along the horizontal axis by τ . The good agreement obtained is consistent with what is expected for thermal noise dominated components, and naturally results from the product of three

orthogonal normally distributed variables. On average, it was found that error norms for the modeled six-detector polarimeter were only 81.6% of those determined for the four-detector device. Beyond this reduction in mean error, Fig. 5.5 also reveals that the statistical variation of the six-detector measurement is more tightly confined, with a similar ratio (~81.4%) found between the full width half maxima (FWHM). Another improvement illustrated in Fig. 5.5, is that for the six-detector design, the probability of occurrence drops much more rapidly for larger errors ($>3\sigma$).

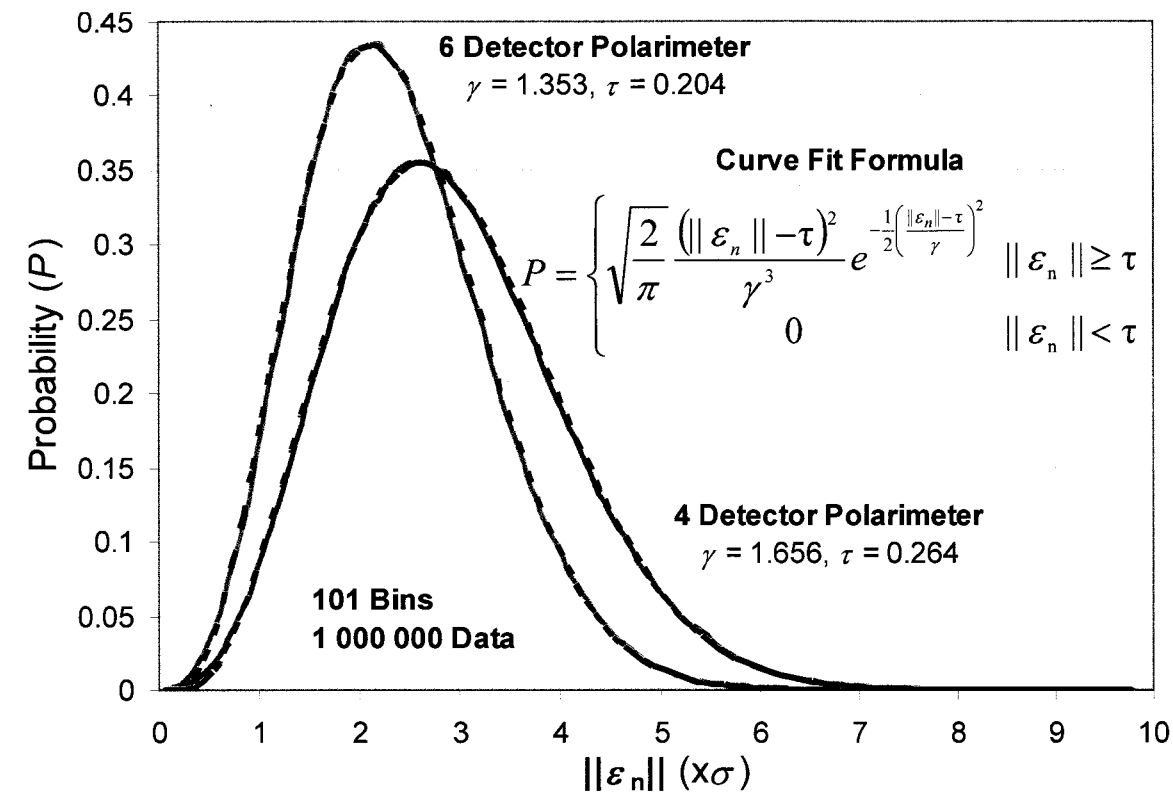


Fig. 5.5 - Probability plot of Stokes vector error norm data, indicating lower mean error and standard deviation for six-detector polarimeters subjected to noise. Overlaid dashed black lines illustrate fitted Maxwell distributions.

5.5 Summary

Capitalizing on recent advances in tilted FBG technology an improved FBG polarimeter has been proposed.

Utilizing, an extended formulation of the VCM, which calculates the radiated Poynting vector for any polarization state, the performance of this polarimeter has been simulated and compared with optimal configurations of previously published designs.

The results demonstrate that linearly monitoring six optimally placed detectors instead of four, can reduce noise errors in the Stokes vector by roughly 20% for every polarization state. This result is especially important for the design of in-line devices because it enables weaker signals to be sampled and reliably analyzed, maximizing device accuracy while minimizing residual impacts on the signals being monitored.

The proposed design was chosen because its improvement can be realized with very little alteration to existing manufacturing processes and no additional FBGs, providing significant benefit for little cost.

Chapter 6

Summary and Application of Reformulated Volume Current Method

The Volume Current Method has already been introduced in *Chapter 3* [13]. This chapter summarizes an extensive analytical reformulation designed to expand the VCM's applicability to ultrafast written FBGs, to investigate and address some of the potentially limiting assumptions identified by Li et al [18], and to provide users with a computationally efficient formula that is as accurate as possible. It successfully demonstrates that this new formulation compares well with [13,18,85] when identical assumptions are imposed, and is used to provide some general insights into the differences predicted for higher order and saturated index modulations as well as gratings extending into the cladding.

6.1 Introduction

A number of assumptions have been utilized in the derivation of the VCM formulation presented in [13], as a result, inaccuracies may arise and the degree with which this solution is applicable to real-world problems will be somewhat limited.

One example of this is the subject of another paper by Li et al [18], which explores the fact that the derivation of [13] is based on an approximation that neglects the contribution of guided longitudinal field components, to the radiated mode.

With ultrafast laser inscription of gratings becoming increasingly prevalent and advantageous, additional limitations also appear to be in need of assessment. Specifically, as discussed in 2.3.2, index modulations induced by such processes consist of higher order sinusoids instead of the pure sine wave forms assumed in [13]. Additionally, because ultrafast photosensitivity does not depend upon the presence of germanium doping, it is much easier to induce index changes in the fibre cladding as well as its core, changing the distribution of scattering elements driven to re-radiate by the incident guided mode.

In order to gain some insight into the potential significance of these effects and others, a reformulation of the model was undertaken with details provided in *Appendix C*. In this chapter comparisons are made with [13,18], and the resulting improvements and remaining limitations are summarized, along with some general trends and characteristics of such influences on tilted FBG scattering.

6.1.1 Principle improvements

This formulation closely follows the derivation of [13], but incorporates the following principle advances:

- 1) Exact field expressions are utilized for the guided modes.
 - Enabling accurate assessment of effects generated by longitudinal guided field components, and allowing one to study the grating scattering of higher order and asymmetric fibre modes.
- 2) Index modulations are generalized to include higher order sinusoidal terms.
 - For more accurate modeling of ultrafast and nonsinusoidal grating structures.
- 3) Fields in the cladding are added to the integration.
 - Allowing modeling of gratings that extend into the cladding.

As well, several other improvements have also been added for completeness:

- 4) The asymptotic expansion utilized by [13], is replaced with a new approximation.
 - Enabling application to small tap angles, since the new expression is valid for all k_d .
- 5) Introduction of an attenuation coefficient α .
 - Allows one to account for the effects of non-negligible radiation and fibre loss.
- 6) A new solution for the integration over z (Z), that is valid for all values of Δ .
- 7) Explicitly provide a finite length uniform apodization correction.
- 8) Higher order $1/r$ terms are retained.

6.1.2. Remaining assumptions

Ultimately then, this formulation remains subject to the following assumptions:

- 1) The fibre has a step index profile, a small core-cladding index difference ($\sqrt{\delta_\epsilon} \ll 1$, $\delta_\epsilon = 1 - \epsilon_{\text{clad}} / \epsilon_{\text{core}}$) and a minimum cladding radius much greater than the maximum grating radius that significantly contributes to scattering ($a_{\text{clad}} \gg a$).
- 2) The effective grating elements are uniform across the fibre and free of significant distortion, with a relatively weak ($\delta n \ll n$), one-dimensional, uniform index modulation along the fibre axis.
- 3) $d \approx r - r' \cos(\phi - \phi')$ when present in exponential terms, and $d \approx r$ otherwise.
- 4) The predicted field is strictly accurate only at the mid-length point of the grating's active scattering elements ($z = 0$).

Although these suppositions still provide some limits with respect to applicability, this modified formulation can serve much more generally, helping to quantify the significance of limitations identified by Li et al., and improving one's understanding of the scattering characteristics of grating's written in step index fibre.

It should be noted that this model does not automatically account for interference effects that may accumulate as a result of coupling to non-radiation modes, which may subsequently be scattered to the radiation field by another part of the grating structure further along the fibre axis. In order to completely assess such effects as well as those generated by transverse asymmetry, multi-dimensional periodicity (2D / 3D multi-beam interference patterns), air holes, stress rods and core non-circularity, additional reformulation or the application of numerical methods remains necessary.

6.2 Model implementation and comparison

The model was implemented using Matlab in order to facilitate calculation and visualization of the radiation fields associated with a given grating geometry.

6.2.1 Validation against Li's models

Examination of the HE mode expressions presented in C.1.2, reveals that substitution of $a_1 \approx -1$, $a_2 \approx 0$ and $E_0 = E_1 / J_1(ua_{\text{core}})$, yields the guided electric field amplitudes used by Li et al (Eq. 3.2 of [13], and Eq. 1 of [18]).

Assuming both derivations have been carried out correctly, utilization of these parameters in the final solution should provide a good comparison and indeed this is the case.

Utilizing these substitutions, assuming the grating length is infinite, neglecting higher order $1/r$ terms and taking only the summation component with strongest phase matching, it is shown in Eq. (E.5) of E.1 and Eq. (E.9) of E.3 that the predicted magnetic and electric fields are virtually identical to those given by Eqs. 3.18 and 3.19 of [13].

Running the Matlab simulation, subject to the same substitutions, for several different grating geometries and guided mode conditions ($\delta n = 10^{-3}$, $P_{\text{in}} = 1$ mW, $L = \infty$, $\alpha = 0$), also

provides good comparison with Li's models and helps to demonstrate that the code is functioning correctly. Three examples of this are provided by Figs. 6.1-6.3, for a variety of core sizes and blaze angles. Fig. 6.1 presents the scattered intensity of a near-resonant out-tapped beam as a function of azimuthal angle about the fibre axis. In Figs. 6.2 and 6.3, responses to much larger detuning are presented. Parameters used in the simulation are included in each figure. Generally, very good agreement is observed, with larger differences in Fig. 6.3 resulting from additional terms in Eqs. (E.4) and (E.8) becoming non-negligible.

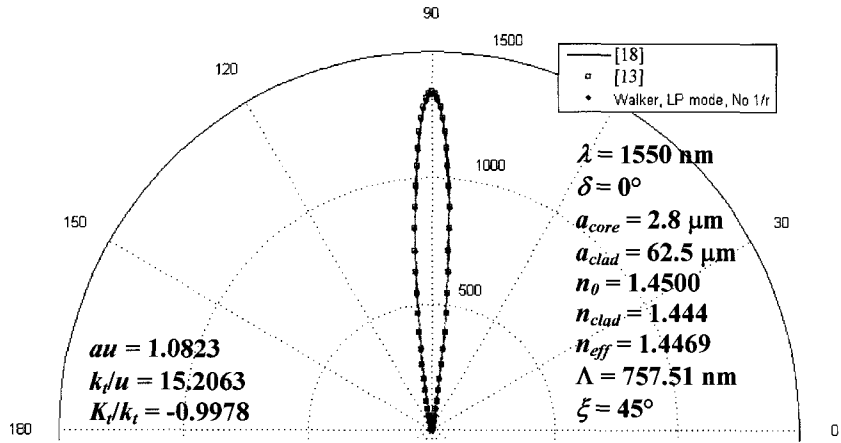


Fig. 6.1 – Comparison 1 of simplified formulation with Li's models.

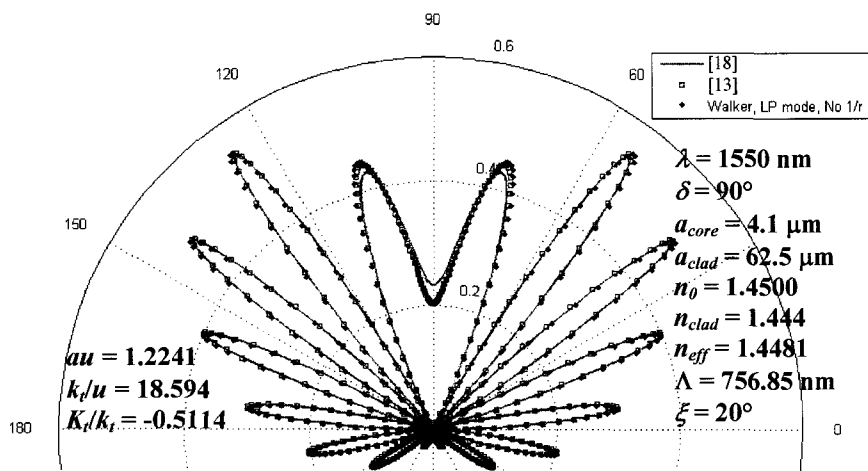


Fig. 6.2 – Comparison 2 of simplified formulation with Li's models.

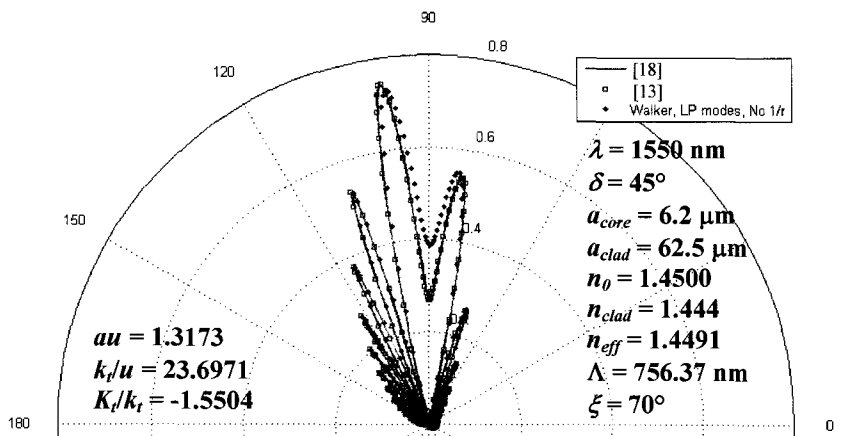


Fig. 6.3 – Comparison 3 of simplified formulation with Li's models.

6.2.2 Effects of formulation improvements

The comparisons with Li's approach of 6.2.1 help to provide some level of confidence in the implementation of this model, but there are obviously some differences as well, especially when run with all of the modifications included. To assess whether or not these variations lie within expected levels for standard gratings, this model was used to predict the response of devices already analyzed by Li et al using both their VCM approach and CMT [85].

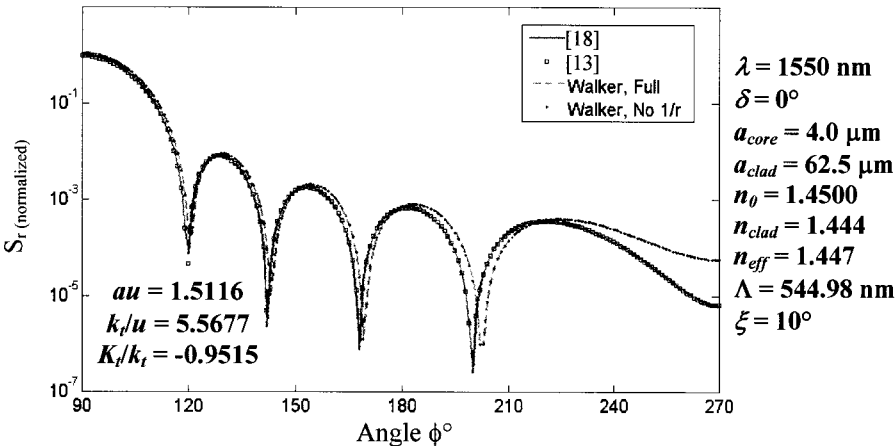


Fig. 6.4 – Results for direct comparison with Fig. 5(c) of [85].

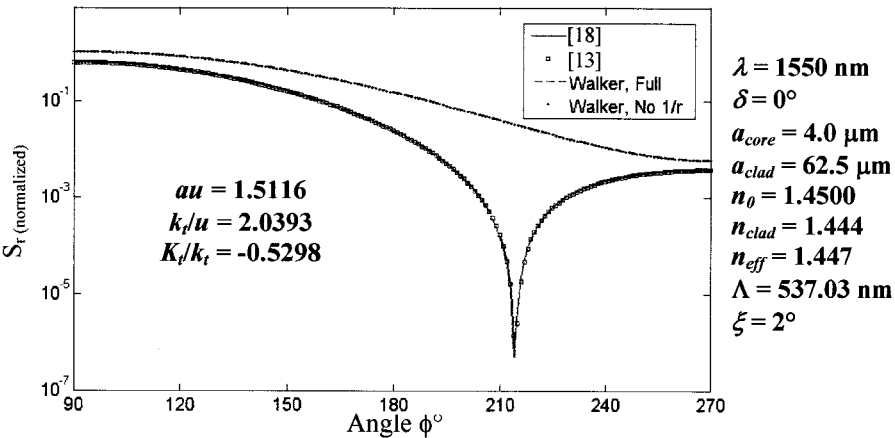


Fig. 6.5 – Results for direct comparison with Fig. 6(c) of [85].

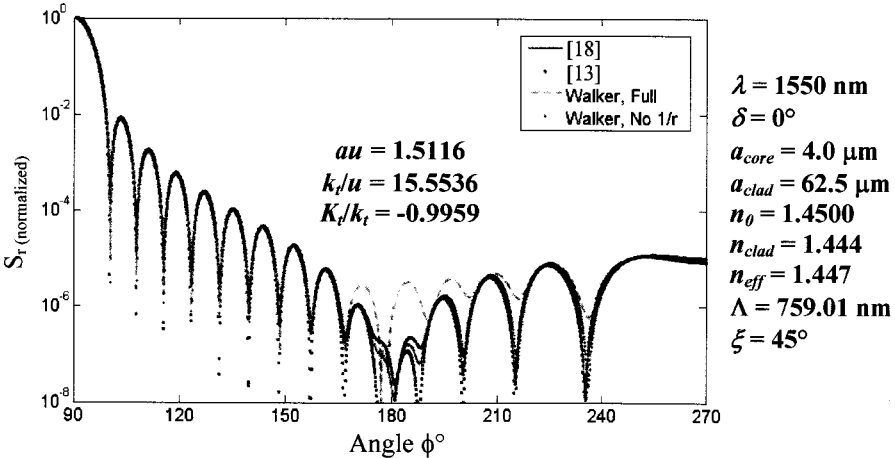


Fig. 6.6 – Results for direct comparison with Fig. 7(c) of [85].

In order to compare directly with the CMT results presented by Li [85], the Poynting vector results as a function of azimuthal angle ϕ are presented on an x-y plot rather than a polar plot as in Fig 6.1. For angles $\phi < 180^\circ$, the two solutions agree almost exactly.

As expected some differences are apparent, with variations on the same order as those presented in the comparison figures of [85]. Interestingly, Fig. 6.6 in particular appears to more closely match the results of Li's CMT analysis, especially at larger azimuthal angles ($>170^\circ$) which may be taken as a sign of improvement.

Running this model for the same cases analyzed in 6.2.1 yields Figs. 6.7-6.9, which illustrate that incorporation of the complete HE field expressions produces a significant difference in calculated magnitudes as well as slight variations in shape under some circumstances.

In particular predictions for the near-resonant tap modelled in Fig. 6.7 are a factor of 2 higher, while calculations for the detuned gratings associated with Figs. 6.8 and 6.9, result in much larger differences in the Poynting vector amplitudes. In Fig 6.9 for example, use of the complete HE field expressions results in a Poynting vector solution that is an order of magnitude higher than those generated by Li's model, which is based on an LP mode approximation rather than the full mode field.

Generally however, the behavioural characteristics and design guidelines discussed in Chapter 3 still provide good guidance for understanding and estimating the radiation patterns associated with a given set of tilted FBG conditions.

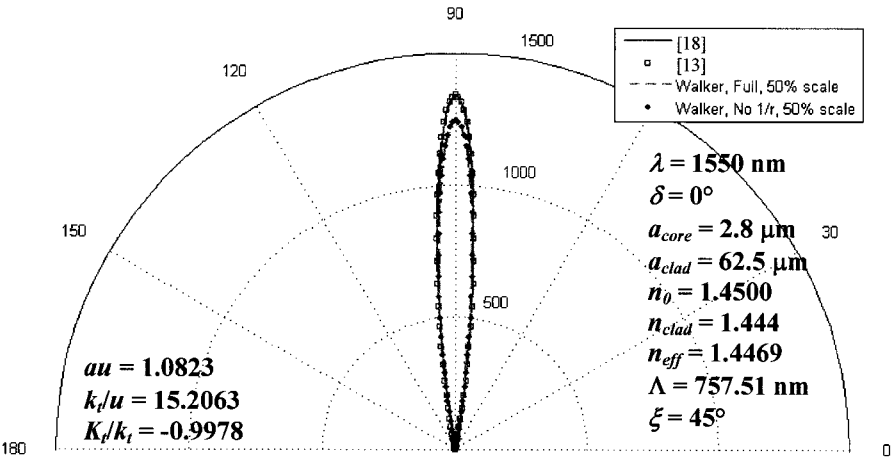


Fig. 6.7 – Comparison 1 of comprehensive formulation with Li's models.

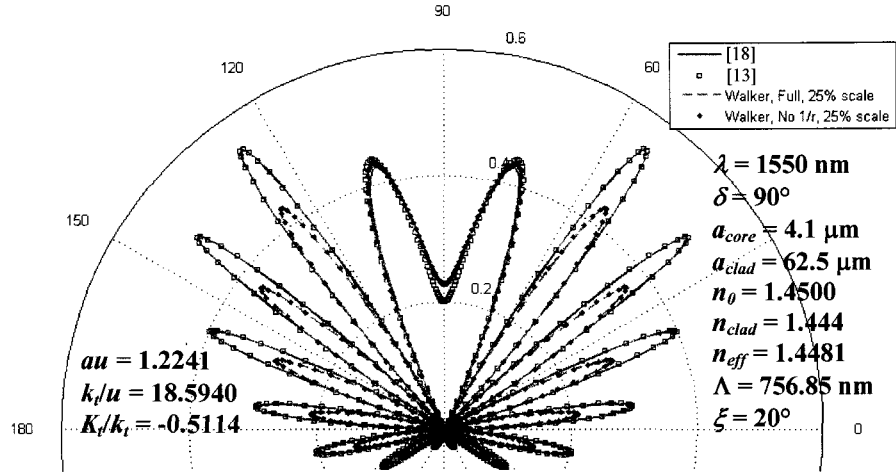


Fig. 6.8 – Comparison 2 of comprehensive formulation with Li's models.

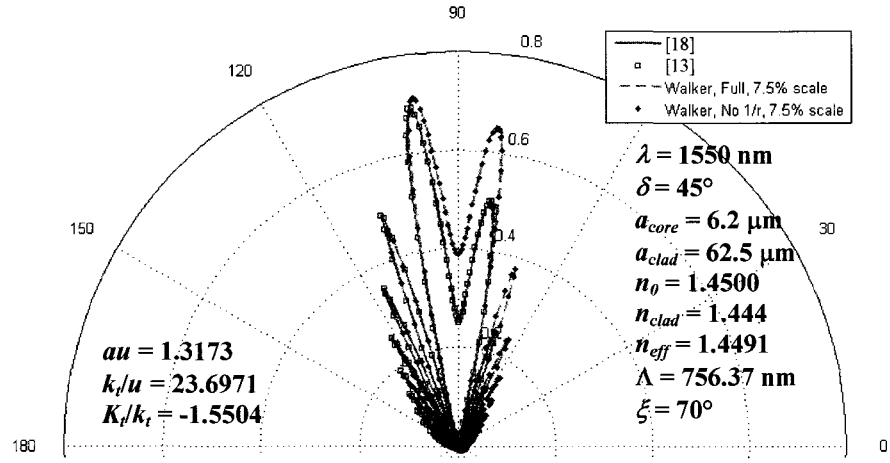


Fig. 6.9 – Comparison 3 of comprehensive formulation with Li's models.

6.2.3 Significance of LP mode approximation

In order to gain further insight using a broad scope of results and to obtain some idea of how these calculations compare with [18], Fig. 6.10 was generated using dispersion based fibre indices from [121], for direct contrast with Li's previous work, where the polarization dependent scattering (PDS) is given by Eq. (6.1).

$$PDS = 10 \cdot \log_{10} \left| \frac{S_s}{S_p} \right| \quad (6.1)$$

with S_s and S_p representing Poynting vector magnitudes of s -polarized and p -polarized light, (respectively perpendicular and parallel to the direction of scattering).

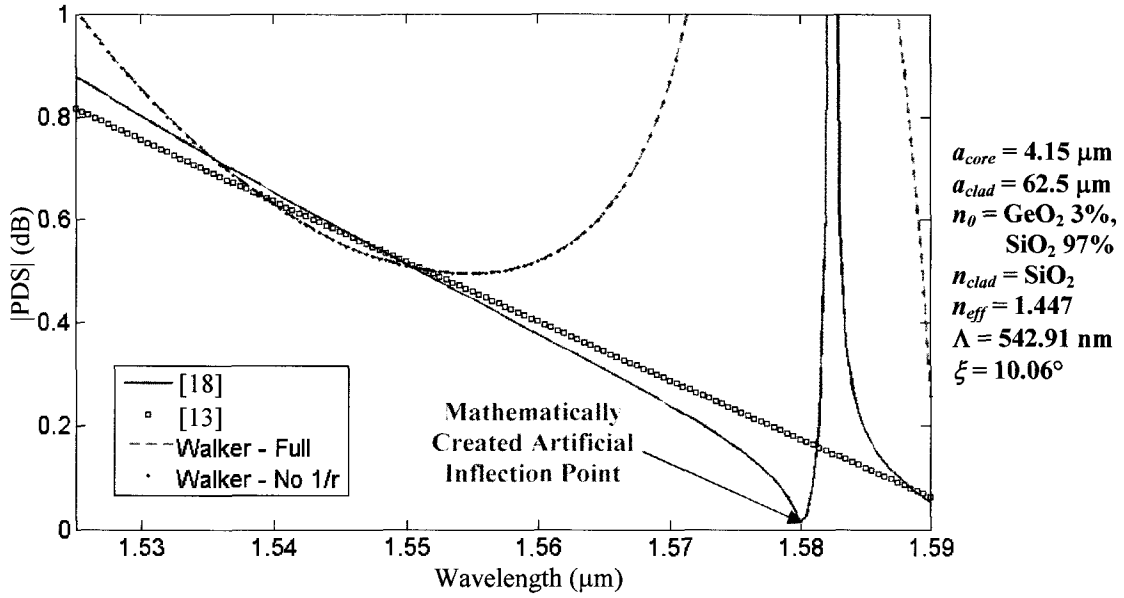


Fig. 6.10 – Results for direct comparison with Fig. 2 of [18].

In this figure, it is apparent that relatively large differences are present in the three model predictions, especially in the region centered near 1558 nm. It is somewhat disappointing that the comprehensive model derived in *Appendix C* does not appear to fit the available data points presented in [18], but after all this simulation has been based on available measurement information only, which is limited in this paper.

On the other hand the modified model presented by Li et al [18] is not a perfect fit either, and seems to depend upon mathematics (the absolute value of PDS) to turn the simulated curve upwards (near 1.58 μm) and away from negative values that do not correspond with the measured data points. This would seem to be an artificial effect that requires more than just a single test case to validate. At least with the comprehensive model presented here, it is mildly encouraging to note the prediction of a smoothly increasing upward sloping tendency at higher wavelengths, as supported by Li's data points.

In any event, it is apparent from [18] and Fig. 6.10 above that even small changes in the longitudinal guided field component can alter the expected radiation profile in a noticeable way, and it is therefore not surprising that similar corrections to the transverse field components also produce noticeable differences.

6.2.4 Significance of higher order $1/r$ terms

As illustrated in Figs. 6.4 - 6.10, in many cases the higher order $1/r$ terms can be neglected and the simplified relations of *Appendix E* used to calculate the distribution of radiated light. However the $1/r$ terms become significant and should not be ignored when the grating is at or near resonance ($|K_t/k_t| \rightarrow 1$) and the observation angle ϕ , approaches that of the guided polarization state δ as shown in Figs. 6.6, 6.11 and 6.12, hence all remaining analysis is conducted using the full formulation of *Appendix C*.

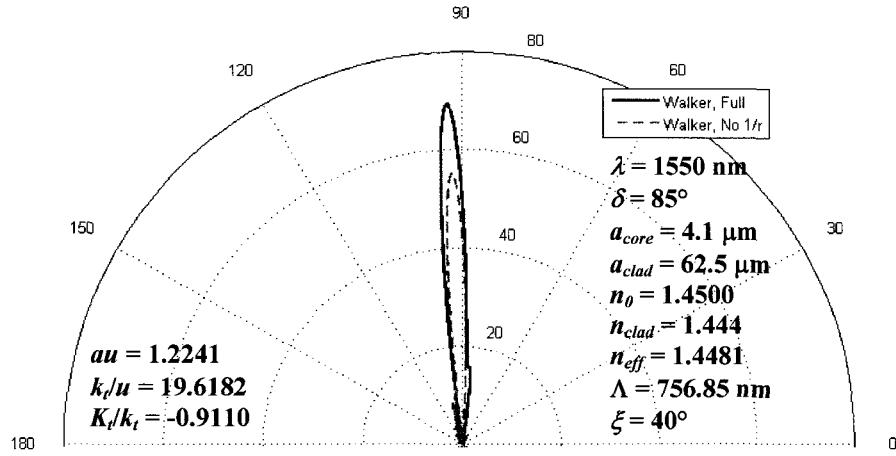


Fig. 6.11 – Example 1 of significant contributions from higher order 1/r terms.

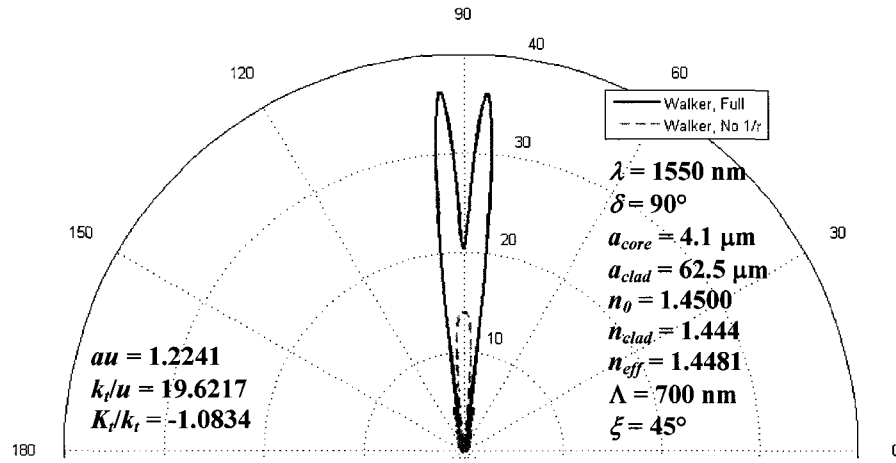


Fig. 6.12 – Example 2 of significant contributions from higher order 1/r terms.

6.2.5 Effects of multiphoton index change

As discussed in 2.3.2, in the case of multiphoton induced index changes the unsaturated photoinduced index modulation is no longer linearly proportional to the writing beam intensity. Instead, the index change now depends on intensity raised to the number of required photons (ζ). A detailed explanation for this is provided in [122], but qualitatively this effect stems from the fact that higher intensities are increasingly more likely to concentrate enough photons in time and space to overcome the bandgap needed to induce the index change or ionize the material.

Recently, index modulations induced by ultrafast IR lasers have been shown to offer several advantages over conventional UV inscription techniques. However such photoexposures are particularly susceptible to the nonlinear index changes mentioned here, and it is somewhat unclear as to how such variations might affect the VCM's performance.

Remodelling the grating structures analyzed in 6.2.1 assuming fifth and sixth order intensity dependence, comparisons with purely sinusoidal 1-photon structures are provided in Figs. 6.13-6.15, with some additional higher order examples provided by Figs. 6.16 and 6.17.

These figures illustrate that the primary influence of the index modulation order, is on the strength of a grating's response (Figs. 6.13-6.17), with changes in grating shape being mainly restricted to instances with large detuning (Figs. 6.16-6.17). In some cases, the scattered

response was observed to weaken with increasing dependence on multiphoton absorption (Figs. 6.13-6.15), while in others strengthening was observed. This is especially obvious when instead of evaluating a fundamental order resonance to radiation modes, a higher order resonance is considered, analogous to higher order resonance retroreflective gratings. In Fig. 6.13 for example, the effect of a multiphoton process to induce index change, results in a reduction of the radiated field magnitude, as denoted by the Poynting vector in the figure for the case where the radiated mode is at resonance with the tilted grating.

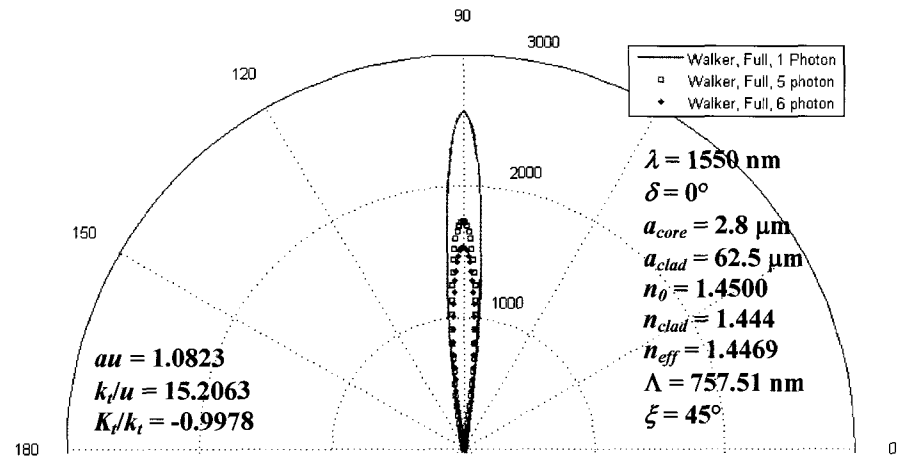


Fig. 6.13 – Multiphoton index change affects on 1st order grating response 1.

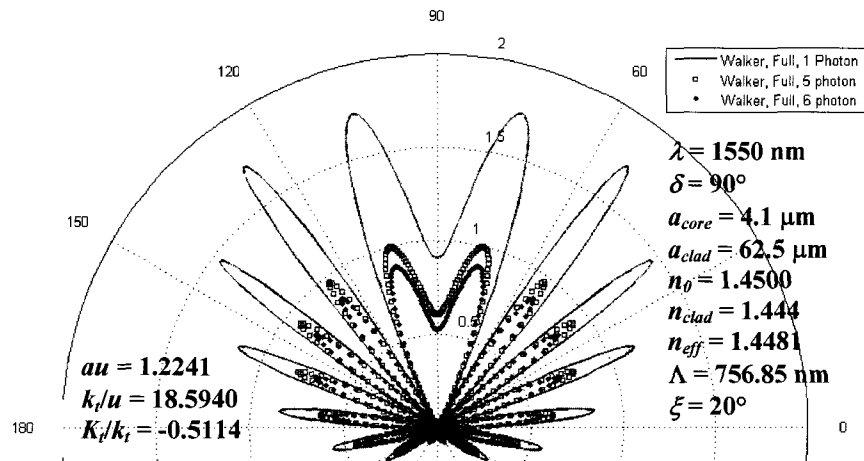


Fig. 6.14 – Multiphoton index change affects on 1st order grating response 2.

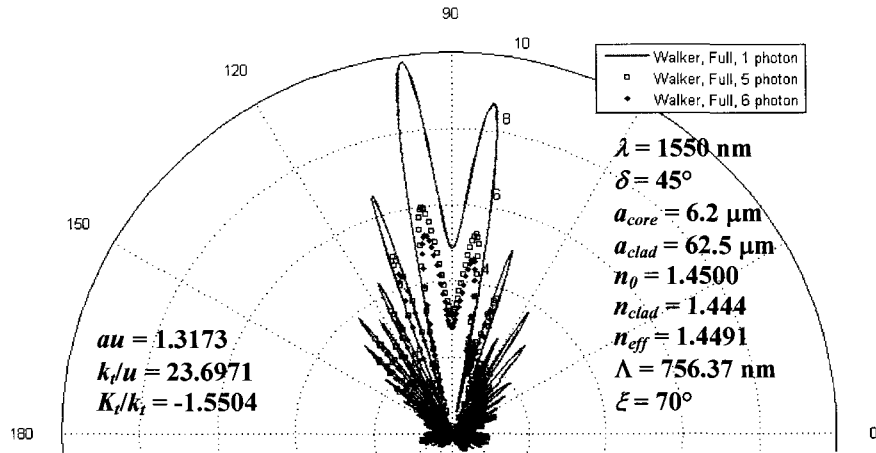


Fig. 6.15 – Multiphoton index change affects on 1st order grating response 3.

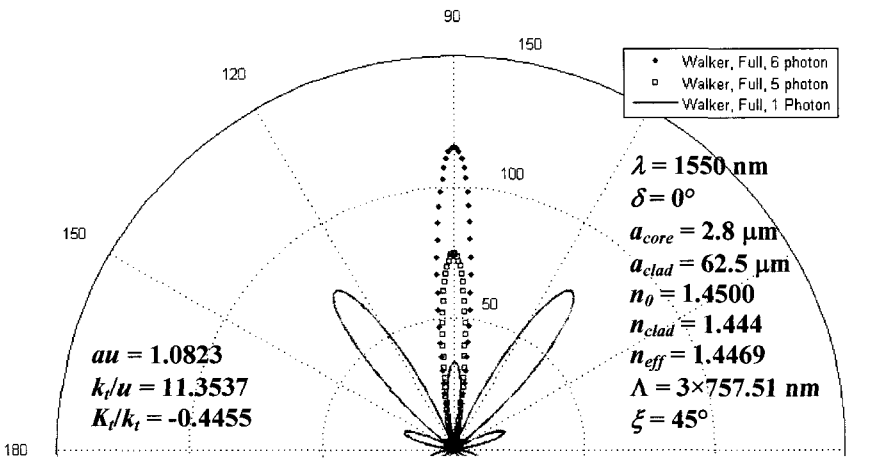


Fig. 6.16 – Example of multiphoton index change affects on 3rd order grating.

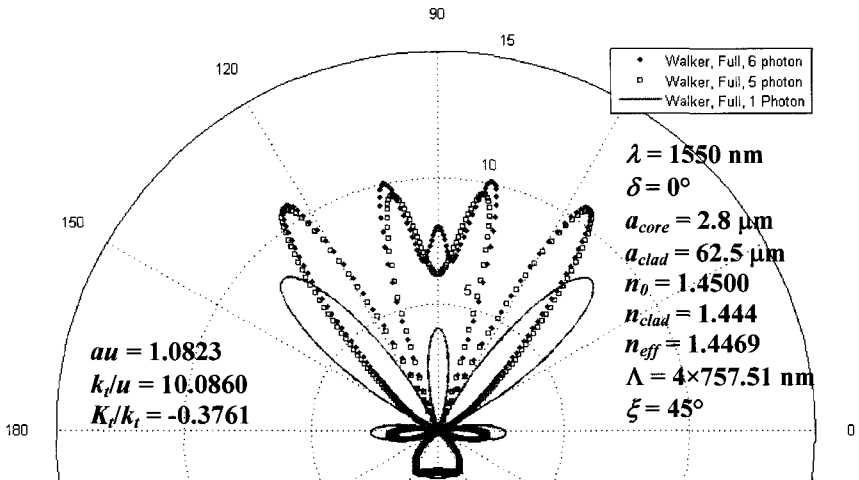


Fig. 6.17 – Example of multiphoton index change affects on 4th order grating.

From the derivation itself (*Appendix C*), it is apparent that increasing this nonlinearity (ζ) results in additional field summation terms (j) as the scattering from various grating harmonics is included. More specifically this summation results in $2\zeta+1$ terms, where term

ς arises from the uniform ‘DC’ component and the remaining terms result from the $|\varsigma-j|$ modulated ‘AC’ components of the induced index change, each having an effective lattice vector length of $|\varsigma-j|K$.

These observations confirm that an already well established technique for untilted gratings is also applicable to the modeling of tilted FBG structures, specifically that an arbitrary index modulation can be characterized as a summation of Fourier sinusoids, with the scattered fields being a superposition of the responses generated by each Fourier component [123-125].

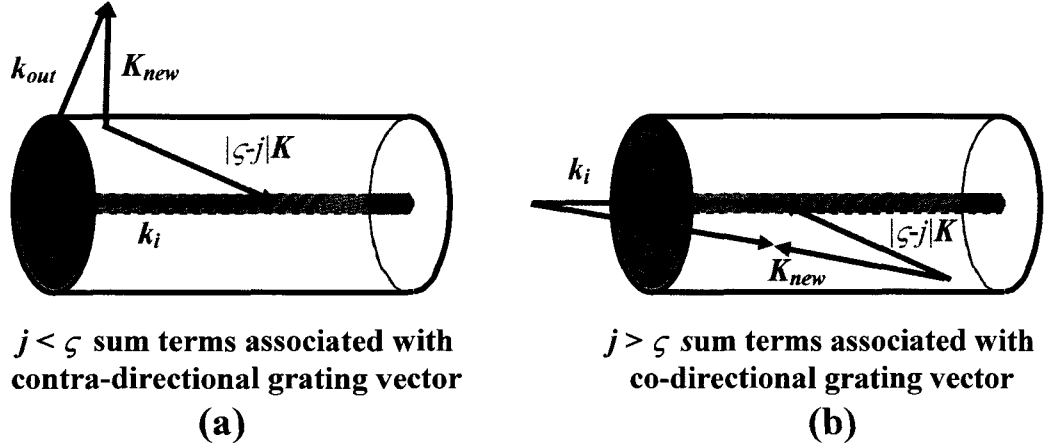


Fig. 6.18 – Phase matching associated with various summation components. Terms with $j > \varsigma$ are easily suppressed because large longitudinal detuning cannot be avoided.

With respect to the modulated ‘AC’ components, Fig. 6.18 presents Crystal Momentum vector diagrams associated with various field summation components j from Appendix C, illustrating that terms $j < \varsigma$ generally contribute more to the grating’s response than terms $j > \varsigma$, which are easily suppressed because large longitudinal detuning cannot be avoided. The opposite is true for backward propagating modes. Because each ‘AC’ component effectively has its own lattice vector, the strength of each term’s phase matching will depend on wavelength, with the strongest transverse scattering component (smallest Δ_j) varying as a function of the harmonic order associated with the grating’s response. This can be determined from the Bragg condition (Eq. 1.23), where $\sin \theta = \cos \theta_B$, and the grating’s harmonic order m , is the positive integer that most closely matches that predicted by Eq. (6.2).

$$m \approx \frac{2n_{eff} \Lambda \cos(\theta_B)}{\lambda}, m \in \mathbb{Z} \quad (6.2)$$

For example, in the case of gratings written with a phasemask that induces a first order resonance, the $j = (\varsigma-1)$ term will dominate, with all harmonic terms less $(\varsigma-1)$ being significantly detuned and extremely weak, since their effective grating vectors are integer multiples of that required for a Bragg resonance. On the other hand, as illustrated in Fig. 6.19(a), subject to the requirement that $j \geq 0$, m^{th} order grating periods will be strongly resonant with the $j = (\varsigma - m)$ component, and some components in the range $(\varsigma - m) < j \leq (\varsigma - 1)$ as shown by Fig. 6.19(c). From Fig. 6.19(b) it is apparent that harmonics

less than $(\zeta - m)$ will be significantly detuned with lesser detuning possible for the remaining components, represented by Fig. 6.19 (d).

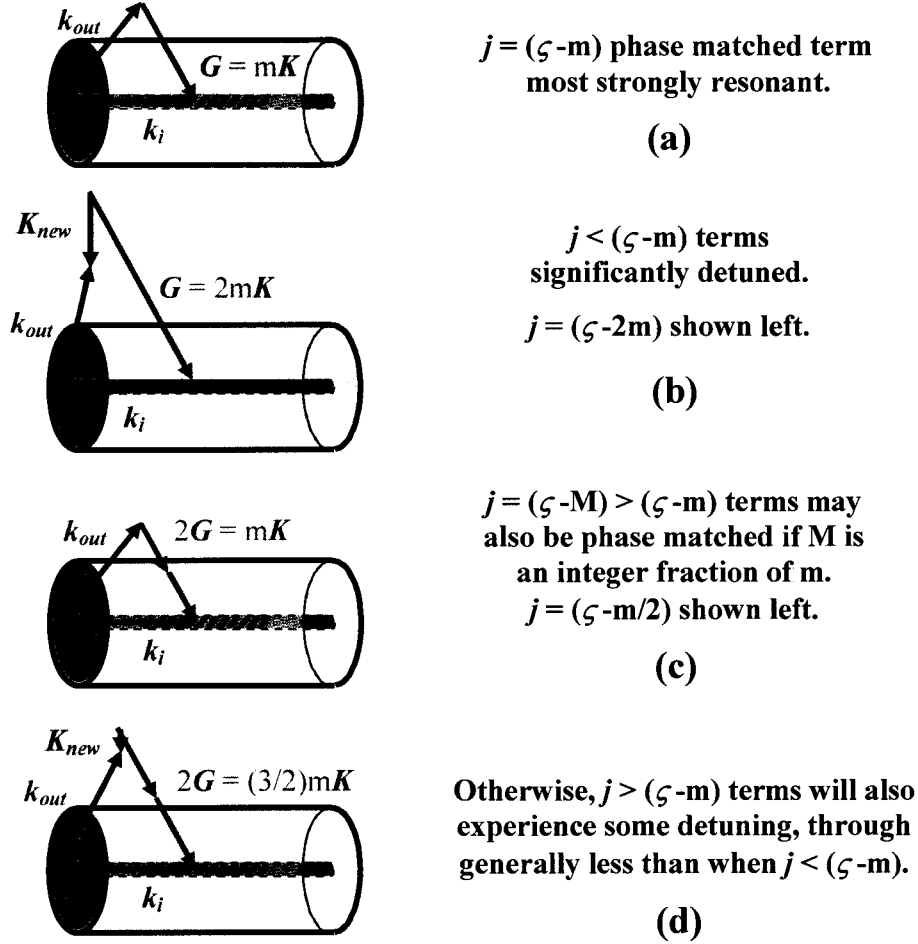


Fig. 6.19 – Phase matching associated with various phasemask harmonic orders. Examples all possible in the case of $m = 4, 8, 12$ etc. Terms with $j < (\zeta - m)$ are easily suppressed because of larger detuning.

These guidelines can be used to help simplify the comprehensive field equations by limiting one's analysis only to the most significant summation terms, however, for structures exhibiting large detuning, one should take care not to dismiss the impact of lesser summation terms without due consideration, since their relative contribution will be higher in such cases.

Generalizing, for devices that are not significantly detuned, assuming that the $0 \leq j = (\zeta - m)$ term dominates, and noting that the scattered electric and magnetic field formulae are both proportional to $\frac{1}{2^{2\zeta-1}} \binom{2\zeta}{j}$, one finds that:

$$S \propto \left[\frac{1}{2^{2\zeta-1}} \binom{2\zeta}{\zeta - m} \right]^2, m \leq \zeta \quad (6.3)$$

Based on Eq. 6.3, Table 6.1 presents normalized values of S , calculated for different index modulation orders ζ , and harmonic orders of the grating period m .

Table 6.1 – Eq. (6.3) Tabulated For Various ζ and m Values

ζ	m									
	1	2	3	4	5	6	7	8	9	10
1	2.5E-01									
2	2.5E-01	1.6E-02								
3	2.2E-01	3.5E-02	9.8E-04							
4	1.9E-01	4.8E-02	3.9E-03	6.1E-05						
5	1.7E-01	5.5E-02	7.7E-03	3.8E-04	3.8E-06					
6	1.5E-01	5.8E-02	1.2E-02	1.0E-03	3.4E-05	2.4E-07				
7	1.3E-01	6.0E-02	1.5E-02	2.0E-03	1.2E-04	2.9E-06	1.5E-08			
8	1.2E-01	6.0E-02	1.8E-02	3.1E-03	2.9E-04	1.3E-05	2.4E-07	9.3E-10		
9	1.1E-01	5.9E-02	2.0E-02	4.3E-03	5.5E-04	3.9E-05	1.4E-06	1.9E-08	5.8E-11	
10	1.0E-01	5.8E-02	2.2E-02	5.5E-03	8.7E-04	8.5E-05	4.7E-06	1.3E-07	1.5E-09	3.6E-12

So that within the range of parameters tabulated in Table 6.1, phase matched first order gratings will generally exhibit a reduction in scattering strength with increasing modulation order, while higher order gratings will strengthen as the modulation order is increased. In cases where $m > \zeta$, phase matching is not possible and weaker detuned responses can be expected.

Again, this is consistent with what is known about untilted gratings [123-125] and is physically explained by the fact that the grating response is proportional to the resonant Fourier component of a particular index modulation profile. As illustrated by Eqs. (6.4) through (6.9), each index modulation $\cos^{2\zeta}(\Lambda/2)$, may be decomposed into purely sinusoidal components, each having their own spatial frequencies and amplitudes. When the first order period Λ is resonant, the coefficients generally decrease with order size ($1/2$, $1/8$, $15/32$, $7/16$, $105/256$, $99/256$), alternatively when higher order periods are resonant, 2Λ for example, the coefficients generally increase (0 , $1/8$, $3/16$, 732 , $15/64$, $495/2048$).

$$\cos^2(\Lambda/2) = \frac{1}{2} + \frac{1}{2}\cos(\Lambda) \quad (6.4)$$

$$\cos^4(\Lambda/2) = \frac{3}{8} + \frac{1}{2}\cos(\Lambda) + \frac{1}{8}\cos(2\Lambda) \quad (6.5)$$

$$\cos^6(\Lambda/2) = \frac{5}{16} + \frac{15}{32}\cos(\Lambda) + \frac{3}{16}\cos(2\Lambda) + \frac{1}{32}\cos(3\Lambda) \quad (6.6)$$

$$\cos^8(\Lambda/2) = \frac{35}{128} + \frac{7}{16}\cos(\Lambda) + \frac{7}{32}\cos(2\Lambda) + \frac{1}{16}\cos(3\Lambda) + \frac{1}{228}\cos(4\Lambda) \quad (6.7)$$

$$\cos^{10}(\Lambda/2) = \frac{63}{256} + \frac{105}{256}\cos(\Lambda) + \frac{15}{64}\cos(2\Lambda) + \frac{45}{512}\cos(3\Lambda) + \frac{5}{256}\cos(4\Lambda) + \frac{1}{512}\cos(5\Lambda) \quad (6.8)$$

$$\cos^{12}(\Lambda/2) = \left(\begin{aligned} &\frac{231}{1024} + \frac{99}{256}\cos(\Lambda) + \frac{495}{2048}\cos(2\Lambda) + \frac{55}{512}\cos(3\Lambda) \\ &+ \frac{33}{1024}\cos(4\Lambda) + \frac{3}{512}\cos(5\Lambda) + \frac{1}{2048}\cos(6\Lambda) \end{aligned} \right) \quad (6.9)$$

Finally, including all summation terms, Figs. 6.20 and 6.21 illustrate an example of these trends for first and second order grating periods constructed of various index modulation profiles. As with Fig. 6.13, Fig. 6.20 shows that for gratings having a first order harmonic

periodicity ($m = 1$), increased photon order ($\zeta > 1$) results in a reduction of the radiation strength, while Fig. 6.21 illustrates that higher order index modulations ($\zeta > 1$), substantially increase the coupling strength associated with second order grating periods, with the 1 photon index modulation requiring $20\times$ scaling to make it visible on the plot.

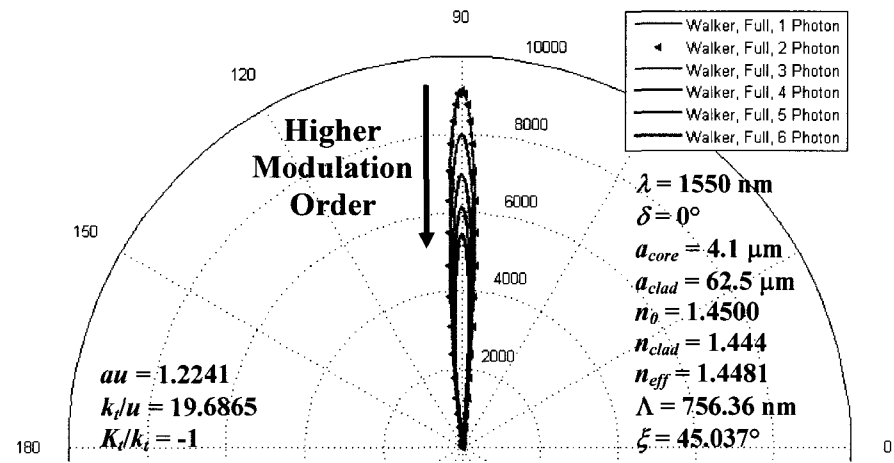


Fig. 6.20 – Example of single and multiphoton, 1st order grating responses.

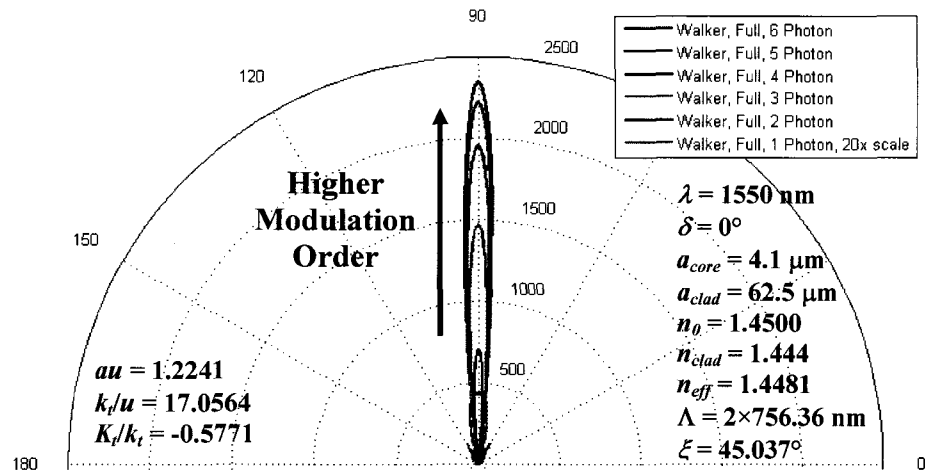


Fig. 6.21 – Example of single and multiphoton, 2nd order grating responses.

6.2.6 Influences of gratings in the cladding

Gratings extending into the cladding are nothing new and have been fabricated in specialty fibres without ultrafast sources for some time as a means of improving grating-mode overlap and reducing inadvertent coupling to cladding modes [126-129]. However Li’s VCM model is limited to core confined structures, and given the possibility of extending gratings into the cladding, it seemed prudent to generalize the equations and explore the affects of such variations on a grating’s radiation characteristics.

Fig. 6.22 illustrates the case of a resonant radiation mode out-tap at 90° to the fibre axis of a standard telecom fibre. As a grating’s radius increases, the strength of outcoupled radiation increases until the grating diameter is roughly 4.3 times that of the core. This results from the fact that the LP₀₁ guided mode is tightly confined to the core, and is consistent with what is already known by makers of photosensitive cladding fibre. When the grating structure

completely subtends the entire field of this guided mode, maximum enhancement of the outcoupled radiation is achieved and further increases in the grating diameter have no effect.

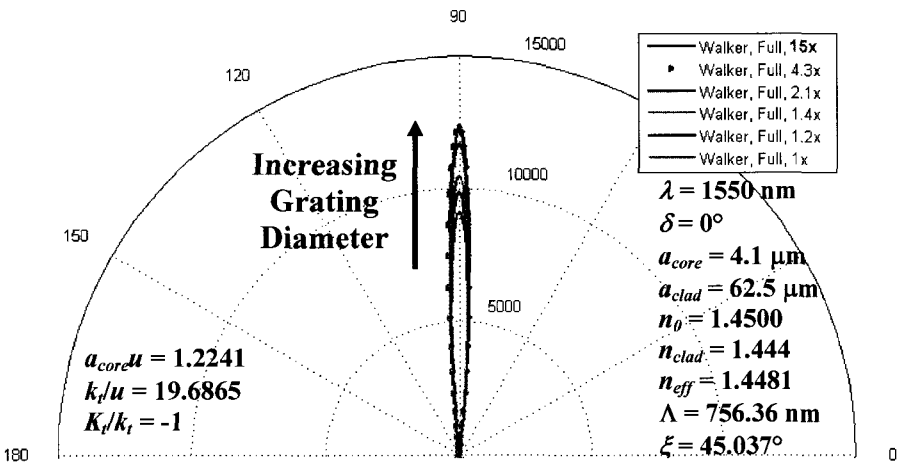


Fig. 6.22 – Phase matched response example 1 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 1-Photon, $\xi_{out} = 90^\circ$.

In Fig.6.23, the case of a resonant acute angled out-tap in standard fibre is considered. Although not as strong as the 90° resonant out-tap shown in Fig. 6.22, it is observed that the amount of coupling to the radiation mode is also maximized when the grating diameter fully subtends the LP_{01} guided mode.

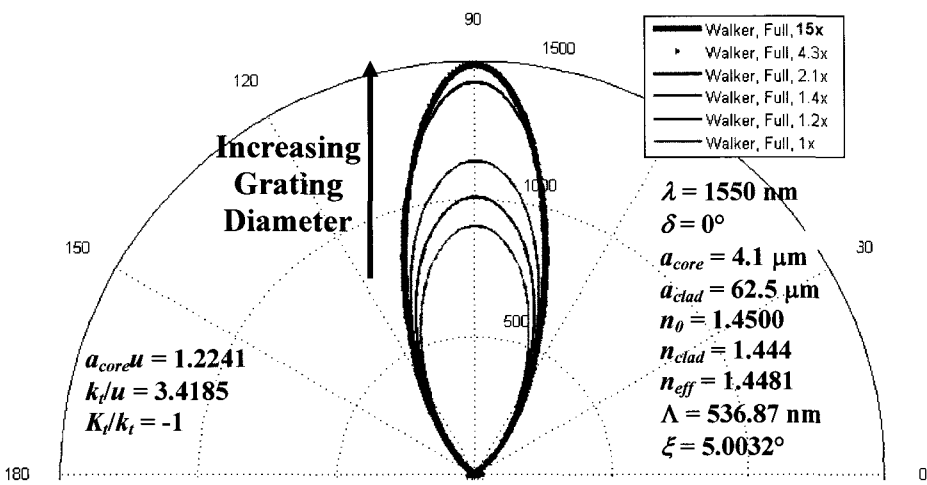


Fig. 6.23 – Phase matched response example 2 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 6-Photon, $\xi_{out} = 10^\circ$.

Similarly for the phase matched response to an obtuse angled radiation mode in Fig. 6.24, in standard fibre the observed radiation mode coupling is maximized when the grating diameter fully subtends the LP_{01} guided mode.

Using the model to simulate numerous grating configurations, having various diffraction orders, longitudinal tap angles and index modulation profiles, it was found that in the case of phase matched structures, enlarging the grating diameter created an increase in radiation mode coupling and a narrowing of the radiated bandwidth (Figs. 6.22-6.25 for example).

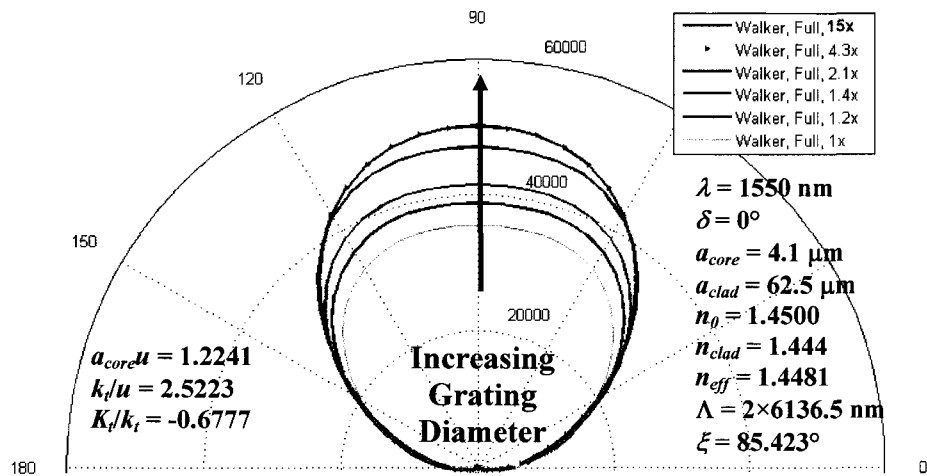


Fig. 6.24 – Phase matched response example 3 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 5-Photon, $\xi_{out} = 170^\circ$.

Increasing the grating diameter results in an increase in the number of grating planes that produce the radiation out-tap. In the case of retro-reflective gratings, increasing the effective length (i.e. the number of interacting grating planes), results in a narrowing of the reflection bandwidth. Considering the wavelength dependence of the phase matched response shown in Fig. 6.22, it is apparent from Fig. 6.25 that the wavelength bandwidth of light radiated by the tap similarly narrows as the number of participating scattering elements increases. As noted earlier, this effect is limited by the grating's overlap with the guided mode, with changes in bandwidth ceasing once the grating diameter completely subtends the LP_{01} mode field.

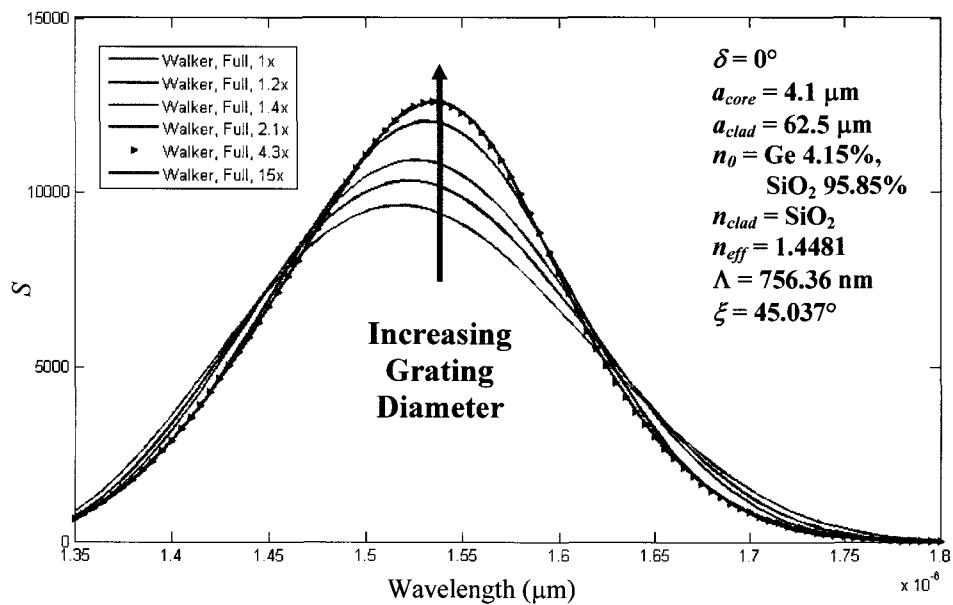


Fig. 6.25 – Wavelength dependence of $\xi_{out} = 90^\circ$, $\lambda = 1550$ nm phase matched response. Radii in legend are multiples of a_{core} . 1-photon index modulation.

Since extending a blazed structure into the cladding increases the mode overlap for each grating plane and generates more transverse scattering elements this makes good physical sense. Under phase matched conditions, the improved modal overlap and increased number

of scattering elements contribute to a stronger response, and more transverse scattering elements will create additional destructive interference, narrowing the response bandwidth.

For detuned structures on the other hand, it was expected that larger grating diameters and more transverse scattering elements, would generate additional destructive interference and weaker responses. The modeled result partially supports this, as extending gratings into the cladding under such circumstances always created weaker scattering than core confined structures. However, as illustrated in Figs. 6.27, 6.29 and 6.31, which plot peak radiated intensity as a function of grating radius for three different detuned structures in standard fibre, the influence of an expanding grating diameter appears to jump back and forth between weakening and strengthening regimes (Figs. 6.26-6.31 for example), with higher periodicity for longitudinal tap angles ξ_{out} approaching 90° . As with the resonant responses, increases in grating diameter beyond the mode field will not impact the out tapped response.

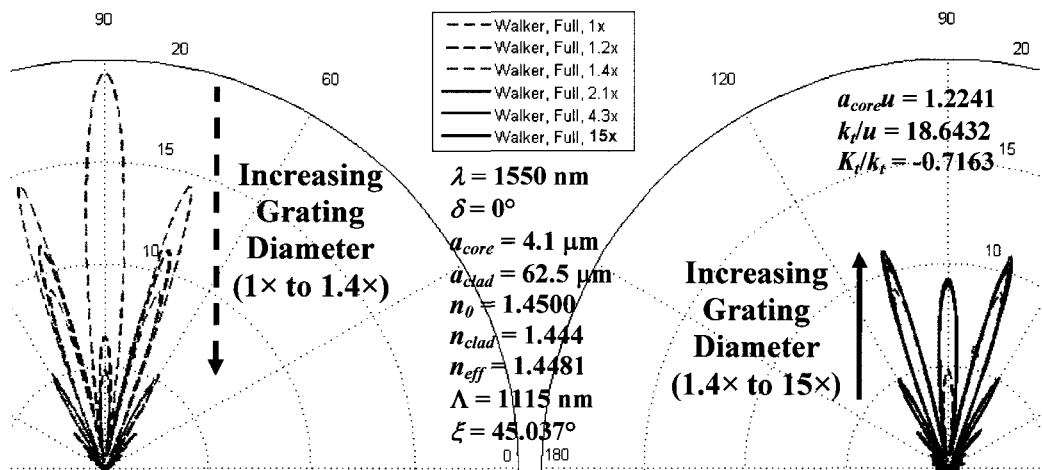


Fig. 6.26 – Detuned response example 1 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 6-photon index modulation.

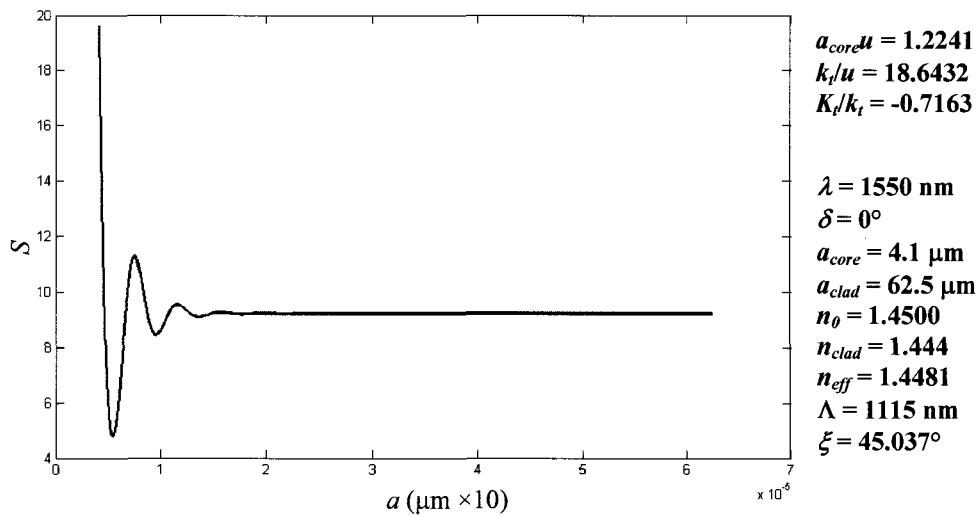


Fig. 6.27 – Variation of central peak radiated intensity with grating radius in standard fibre. 6-photon index modulation. $\xi_{out} \approx 110^\circ$

Although the exact reasoning for such behaviour remains unclear, this may be the result of two competing effects. While the mode overlap is continuously increasing with grating diameter, an integer number of transverse scattering elements or grating planes cannot. Thus the weakening destructive interference generated by additional grating planes occurs only in discrete steps, between which strengthening may result from a higher grating-mode overlap.

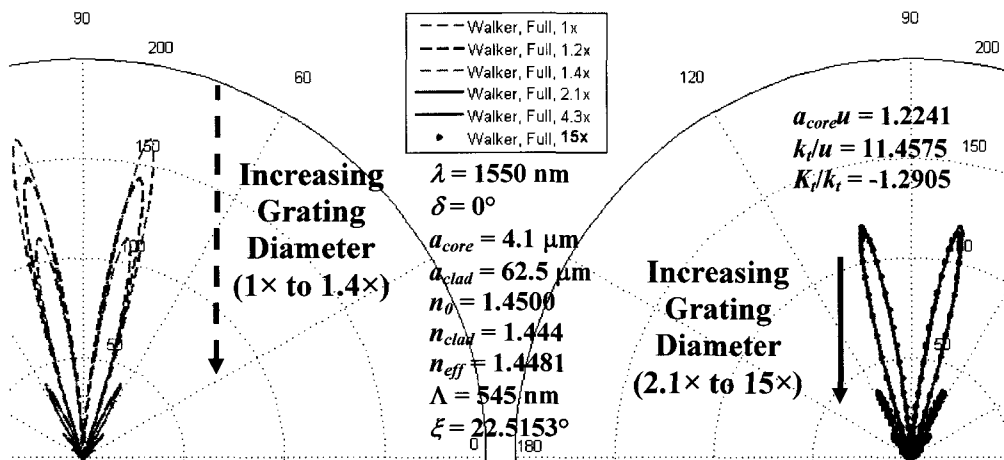


Fig. 6.28 – Detuned response example 2 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 1-photon index modulation.

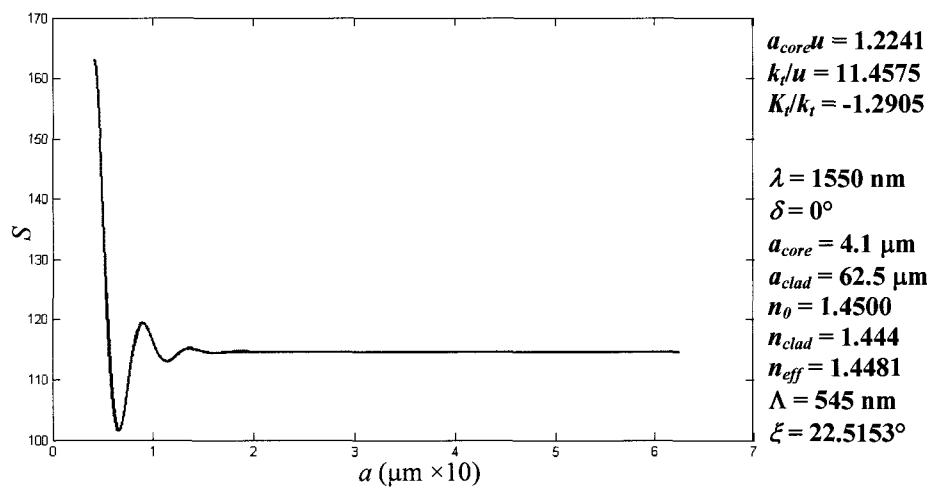


Fig. 6.29 – Variation of peak radiated intensity with grating radius in standard fibre. 1-photon index modulation. $\xi_{out} \approx 34^\circ$

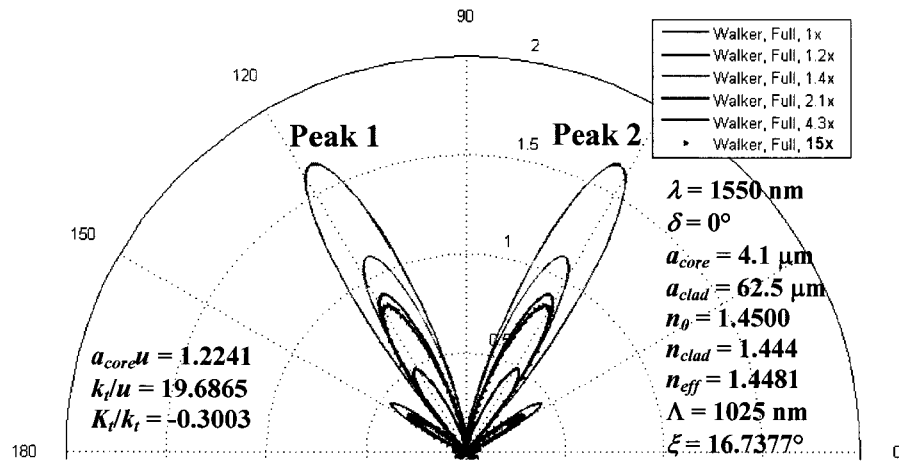


Fig. 6.30 – Detuned response example 3 of various grating radii in standard fibre. Radii in legend are multiples of a_{core} . 1-Photon, $\xi_{out} = 90^\circ$.

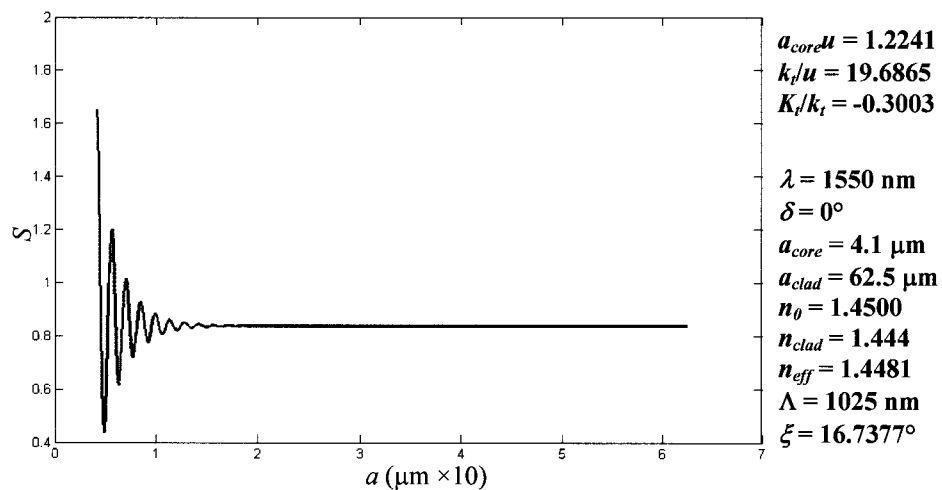


Fig. 6.31 – Variation of peak radiated intensity with grating radius in standard fibre. 1-photon index modulation. $\xi_{out} = 90^\circ$.

From a polarization perspective there are also some interesting differences. Plotting peak intensity as a function of the polarization angle δ for multiple peak structures illustrates that different grating radii preferentially couple slightly different polarization states. For instance, in the case of the two peak structure characterized in Fig. 6.30, this intensity dependence on polarization is plotted in Fig. 6.32 for two different grating diameters. Specifically, this figure plots the scattered intensity S of each peak, as a function of polarization angle δ , making it apparent that the azimuthal spread between radiation peaks is reduced from 58° to 40° in this case, when the grating diameter is expanded by 40% beyond the core. The simulations presented utilize the full model, including the higher order $1/r$ terms, described in Appendix C.

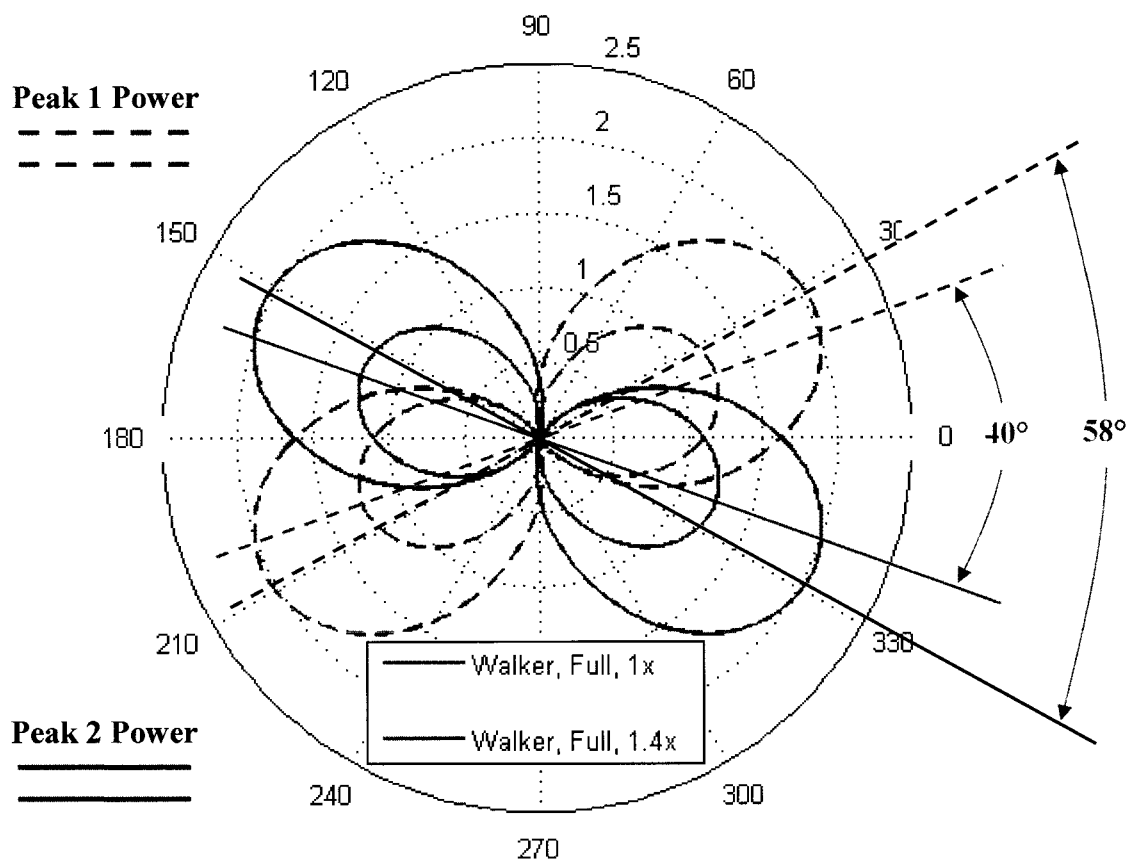


Fig. 6.32 – Polarization dependence of grating radii in standard fibre. Radii in legend are multiples of a_{core} . 1-Photon, $\xi_{\text{out}} = 90^\circ$.

It has already been shown in section 3.3 that the azimuthal dependence of a grating's scattered response depends on core size through the parameter au , whereby relationships have been provided in 3.4.3 to optimize the selection of this parameter based on the angular spread desired for a given two-peak or three-peak response. However the practical variability of this parameter depends upon the availability of the desired fibre substrate.

On the other hand, it is apparent from the work here that varying a grating's diameter, through focusing or beam sweeping techniques for example, could be used to provide additional control over this parameter, making the optimum 54.7° or 90° angular spreads required for in-line polarimeters possible, even if not initially supported for core confined gratings of a particular fibre substrate.

6.3 Summary

This chapter highlights some of the reasoning, principle improvements and insights related to the development of a more comprehensive analytical VCM formulation. It successfully demonstrates that the model derived in *Appendix C* yields the same values as Li's when identical assumptions are imposed on the resulting scattered field equations, and compares its predictions directly with Li's analysis of longitudinal guided field effects and CMT results.

While the general equation's structure is more complicated than the formula provided by Li et al, it can be simplified as appropriate for the circumstances, enabling greater accuracy and versatility when necessary. This formulation is applicable for all values of $k_d d$ and provides an estimation of the scattered fields for any guided mode incident a grating in any

step index fibre, single mode or otherwise. It also addresses the effects of higher order index modulations and provides some capability for analyzing gratings in the cladding, both of which are especially relevant to ultrafast written gratings.

Further improvements in accuracy and understanding have also been provided through explicit discussion of the formulation's errors and retention of higher order terms.

Generally, comparison of the various models indicates only minor variations in terms of the behavioural trends identified by [15], however differences on the same order as those predicted in [18] have also been shown, indicating that they may not necessarily be considered insignificant.

In the course of this analysis it has been shown that the fields scattered from tilted gratings of arbitrary index modulation (higher order period, higher order profile, saturated etc.) may be modeled as a summation of their most significant Fourier components, and that extending gratings into the cladding may be useful for altering their azimuthal polarization dependencies when suitably designed fibres are not available.

Conclusions

As detailed in *Chapter 3* and published in [14,15], this thesis has extended the work of [13] in order to quantitatively explain the physical mechanisms responsible for the polarization dependent, azimuthally distributed, two beam radiation mode coupling first reported by [5].

Utilizing this analysis, parametrically dependent behavioural trends have been illustrated and several empirical design tools generated for facilitating the optimization and fabrication of such Bragg grating structures; revealing the possibility of comparable three beam devices which have subsequently been observed experimentally. These results have been reported in *Chapter 4* and published in [16].

In *Chapter 5*, accepted for publication in [17], a useful application of these novel Bragg grating responses has been proposed, offering a 20% reduction in noise errors for inline fibre optic polarimeters, without a requirement for any additional components or drastic changes in fabrication costs and procedures.

Finally as summarized in *Chapter 6*, a more comprehensive analytical model has been developed in order to examine the impact of potential limitations identified by [18] and inherent in the simplifying assumptions utilized in [13]. Aside from enabling an exploration of these issues, this formulation makes it possible for one to evaluate the scattering of non-fundamental guided modes and to analyze the impact of gratings in the cladding as well as non-sinusoidal index modulations that may result from the use of ultrafast writing sources or saturated exposures. This model has been shown to compare well with [13,18,85], with predicted variations in grating responses having reasonable physical explanations.

In order to build upon the findings presented here, it is expected that future work may include construction and testing of the proposed polarimeter design, as well as the fabrication and characterization of tilted ultrafast written FBGs.

Appendix A

Miscellaneous Calculations for Chapter 3

A.1 Substantially transverse radiation mode requirement

Eq. (3.1) is a consequence of Li's derivation, and although it had not previously been published it simply follows from the Bessel function expansion he used (Eqs. 8.451 of [84]):

$$|Term_{n=1}| \gg |R_1| \quad (A.1)$$

$$\left| \frac{(-1)^k \Gamma(v+2k+\frac{1}{2})}{(2z)^{2k} (2k)! \Gamma(v-2k+\frac{1}{2})} \right|_{k=(n-1), v=0, z=k_t r} \gg \left| \frac{\Gamma(v+2n+\frac{1}{2})}{(2z)^{2n} (2n)! \Gamma(v-2n+\frac{1}{2})} \right|_{n=1, v=0, z=k_t r} \quad (A.2)$$

$$1 \gg \left| \frac{\Gamma(\frac{1}{2}+2)}{2(2k_t r)^2 \Gamma(\frac{1}{2}-2)} \right| \quad (A.3)$$

$$rk_t \gg \sqrt{\left| \frac{\Gamma(\frac{1}{2}+2)}{8\Gamma(\frac{1}{2}-2)} \right|} \quad (A.4)$$

$$rk_t \gg \sqrt{\frac{9}{128}} \quad (A.5)$$

Rewriting this allowing $|R_1|$ to be 1% of $|Term_{n=1}|$ yields:

$$\cos \xi \geq \frac{\Lambda}{\lambda} \left(n_{eff} - \sqrt{n_0^2 - \frac{5625\lambda^2}{32\pi^2 r^2}} \right) \quad (A.6)$$

$$\text{i.e. } |\xi| \leq 90^\circ \text{ (SMF-28, } \lambda=1.55\mu\text{m, } \Lambda=2\mu\text{m)}$$

A.2 Non-polarized guided mode: Azimuthal independence of f_l term

$$\begin{aligned} f_{1-NonPolarized}(\phi) &= \frac{1}{\pi} \int_0^\pi f_1(\delta, \phi) d\delta \\ &= \frac{1}{\pi} \int_0^\pi [\Delta^2 + k_t^2 \sin^2(\delta - \phi)] d\delta \\ &= \Delta^2 + \frac{k_t^2}{2} \end{aligned} \quad (A.7)$$

A.3 Blaze angle calculation

From the expressions for k_t and u given in 3.2.2:

$$\frac{k_t}{u} = \frac{\sqrt{k_0^2 n_0^2 - \Delta^2}}{k_0 \sqrt{n_0^2 - n_{eff}^2}} \quad (A.8)$$

$$\left(\frac{k_t}{u} \right)^2 (n_0^2 - n_{eff}^2) = n_0^2 - \left[n_{eff} - \left(\frac{\lambda}{\Lambda} \right) \cos \xi \right]^2 \quad (A.9)$$

$$\frac{\lambda}{\Lambda} = \frac{n_{eff} \mp \sqrt{n_0^2 - \left(\frac{k_t}{u}\right)^2 (n_0^2 - n_{eff}^2)}}{\cos \xi} \quad (\text{A.10})$$

Similarly using the equations for K_t and k_t provided in the same section:

$$\left| \frac{K_t}{k_t} \right| = \frac{2\pi \sin \xi}{\Lambda \sqrt{k_0^2 n_0^2 - \Delta^2}} \quad (\text{A.11})$$

$$= \frac{2\pi \sin \xi}{\Lambda k_0 \sqrt{n_0^2 - \left(n_{eff} - \frac{K_g}{k_0}\right)^2}} \quad (\text{A.12})$$

$$= \frac{\left(\frac{\lambda}{\Lambda}\right) \sin \xi}{\sqrt{n_0^2 - \left[n_{eff} - \left(\frac{\lambda}{\Lambda}\right) \cos \xi\right]^2}} \quad (\text{A.13})$$

ξ can then be isolated by combining these two expressions:

$$\left| \frac{K_t}{k_t} \right| = \frac{\left(n_{eff} \mp \sqrt{n_0^2 - \left(\frac{k_t}{u}\right)^2 (n_0^2 - n_{eff}^2)} \right) \tan \xi}{\sqrt{n_0^2 - \left[n_{eff} - \left(n_{eff} \mp \sqrt{n_0^2 - \left(\frac{k_t}{u}\right)^2 (n_0^2 - n_{eff}^2)} \right) \right]^2}} \quad (\text{A.14})$$

$$\xi = \arctan \left(\frac{\left| \frac{K_t}{k_t} \right| \left(\frac{k_t}{u} \right) \sqrt{1 - \left(\frac{n_{eff}}{n_0} \right)^2}}{\left(\frac{n_{eff}}{n_0} \right) \mp \sqrt{1 - \left(\frac{k_t}{u} \right)^2 \left[1 - \left(\frac{n_{eff}}{n_0} \right)^2 \right]}} \right) \quad (\text{A.15})$$

A.4 Grating period formula

Re-using the expressions for k_t/u and $|K_t/k_t|$ derived in the A.3:

$$\left(\frac{k_t}{u} \right) \left| \frac{K_t}{k_t} \right| = \left(\frac{2\pi \sin \xi}{\Lambda \sqrt{k_0^2 n_0^2 - \Delta^2}} \right) \left(\frac{\sqrt{k_0^2 n_0^2 - \Delta^2}}{k_0 \sqrt{n_0^2 - n_{eff}^2}} \right) \quad (\text{A.16})$$

$$\Lambda = \frac{\lambda}{\left(\frac{k_t}{u} \right) \left| \frac{K_t}{k_t} \right| \left[n_0 \sqrt{1 - \left(\frac{n_{eff}}{n_0} \right)^2} \right]} \sin \xi \quad (\text{A.17})$$

Substituting Eq. (3.30) into this formula yields an expression for the grating period:

$$\Lambda = \frac{\lambda}{n_0 \sqrt{\left(\frac{K_t}{k_t}\right)^2 \left(\frac{k_t}{u}\right)^2 \left[1 - \left(\frac{n_{eff}}{n_0}\right)^2\right] + \left[\left(\frac{n_{eff}}{n_0}\right) \mp \sqrt{1 - \left(\frac{k_t}{u}\right)^2 \left[1 - \left(\frac{n_{eff}}{n_0}\right)^2\right]}\right]^2}} \quad (\text{A.18})$$

Appendix B

Generalization of VCM formulation to elliptical polarization states

In simulating a polarimeter it is important to accurately calculate the relative magnitudes of the various detector currents that will be generated by any given state of polarization. Extending the VCM formulation to handle generalized elliptically polarized states was necessary in order to predict the detector currents accurately and without introducing unnecessary errors that would impact the simulation. The expressions obtained are presented as a convenient, accessible reference for those who are interested.

Begin by recognizing that the incident electric field can generally be defined as two orthogonal components with phase difference δ_{phase} . This allows one to rewrite Eq. 3.2 of [13] as Eq. (B.1):

$$\begin{aligned}\vec{E}_0(\vec{r}) &= \vec{E}_0(r, \phi, z) = \vec{E}_x(\vec{r}) + \vec{E}_y(\vec{r}) \\ &= E_x J_0(ur) e^{-i(\beta z + \delta_{phase}/2)} \hat{x} + E_y J_0(ur) e^{-i(\beta z - \delta_{phase}/2)} \hat{y}\end{aligned}\quad (B.1)$$

where $\vec{E}_x(\vec{r})$ and $\vec{E}_y(\vec{r})$ are electric field components in the $\hat{x}$ and $\hat{y}$ directions; E_0 , $E_x = E_0 \cdot \cos \alpha \cdot e^{-i\delta_{phase}/2}$ and $E_y = E_0 \cdot \sin \alpha \cdot e^{i\delta_{phase}/2}$ are electric field amplitude factors, $J_0(ur)$ is a Bessel function of the first kind, $u = \sqrt{k_0^2 n_0^2 - \beta^2}$, $k_0 = 2\pi / \lambda$, n_0 is the core refractive index, and β is the propagation constant.

Integrating these two components separately, term by term, enables one to rewrite the vector potential as Eq. (B.2), a summation of the original solution given in Eq. 3.17 of [13]:

$$\begin{aligned}\vec{A} &= \vec{A}_x + \vec{A}_y \\ &= \left(\begin{aligned} &|\vec{A}| \cdot \left[\hat{r} \cos(\delta_x - \phi) + \hat{\phi} \sin(\delta_x - \phi) \right] \cos \alpha \cdot e^{-i\delta_{phase}/2} \\ &+ |\vec{A}| \cdot \left[\hat{r} \cos(\delta_y - \phi) + \hat{\phi} \sin(\delta_y - \phi) \right] \sin \alpha \cdot e^{i\delta_{phase}/2} \end{aligned} \right) \\ &= |\vec{A}| \cdot \left(\begin{aligned} &\left[\hat{r} \cos(\phi) - \hat{\phi} \sin(\phi) \right] \cos \alpha \cdot e^{-i\delta_{phase}/2} \\ &+ \left[\hat{r} \sin(\phi) + \hat{\phi} \cos(\phi) \right] \sin \alpha \cdot e^{i\delta_{phase}/2} \end{aligned} \right)\end{aligned}\quad (B.2)$$

where $\vec{E}_x(\vec{r})$ and $\vec{E}_y(\vec{r})$ induce vector potential components $\vec{A}_x$ and $\vec{A}_y$, $\delta_x = 0$ and $\delta_y = \pi/2$.

Next the distributive property of the curl operator is used to determine the induced magnetic and electric fields in terms of similar summations of x and y components ($\vec{B}_{m,x}, \vec{B}_{m,y}, \vec{E}_{m,x}, \vec{E}_{m,y}$). Rewriting Eqs. 3.18 and 3.19 of [13], then yields Eqs. (B.3) and (B.4):

$$\begin{aligned}
\vec{B}_m(r, \phi, z) &= \nabla \times \vec{A} = \nabla \times (\vec{A}_x + \vec{A}_y) \\
&= (\nabla \times \vec{A}_x) + (\nabla \times \vec{A}_y) = \vec{B}_{m,x}(r, \phi, z) + \vec{B}_{m,y}(r, \phi, z) \\
&\approx iA \left[\Delta \sin(\delta_x - \phi) \hat{r} - \Delta \cos(\delta_x - \phi) \hat{\phi} - k_t \sin(\delta_x - \phi) \hat{z} \right] \cos \alpha \cdot e^{-i\delta_{phase}/2} \\
&\quad + iA \left[\Delta \sin(\delta_y - \phi) \hat{r} - \Delta \cos(\delta_y - \phi) \hat{\phi} - k_t \sin(\delta_y - \phi) \hat{z} \right] \sin \alpha \cdot e^{i\delta_{phase}/2} \\
&\approx A \begin{pmatrix} -\Delta [\sin(\phi + \alpha) \sin(\delta_{phase}/2) + i \sin(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{r} \\ -\Delta [\cos(\phi + \alpha) \sin(\delta_{phase}/2) + i \cos(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{\phi} \\ + k_t [\sin(\phi + \alpha) \sin(\delta_{phase}/2) + i \sin(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{z} \end{pmatrix} \quad (B.3) \\
\vec{E}_m(r, \phi, z) &= \frac{\nabla \times \vec{B}_m(r, \phi, z)}{i\omega\mu_0\epsilon} = \frac{\nabla \times \vec{B}_{m,x}(r, \phi, z) + \nabla \times \vec{B}_{m,y}(r, \phi, z)}{i\omega\mu_0\epsilon} \\
&= \vec{E}_{m,x}(r, \phi, z) + \vec{E}_{m,y}(r, \phi, z) \\
&\approx \begin{pmatrix} \frac{A}{i\omega\mu_0\epsilon} \left[\Delta^2 \cos(\delta_x - \phi) \hat{r} + k_0^2 n_0^2 \sin(\delta_x - \phi) \hat{\phi} \right] \cos \alpha \cdot e^{-i\delta_{phase}/2} \\ - k_t \Delta \cos(\delta_x - \phi) \hat{z} \\ + \frac{A}{i\omega\mu_0\epsilon} \left[\Delta^2 \cos(\delta_y - \phi) \hat{r} + k_0^2 n_0^2 \sin(\delta_y - \phi) \hat{\phi} \right] \sin \alpha \cdot e^{i\delta_{phase}/2} \\ - k_t \Delta \cos(\delta_y - \phi) \hat{z} \end{pmatrix} \\
&\approx \frac{A}{\omega\mu_0\epsilon} \begin{pmatrix} -\Delta^2 [\cos(\phi + \alpha) \sin(\delta_{phase}/2) + i \cos(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{r} \\ + k_0^2 n_0^2 [\sin(\phi + \alpha) \sin(\delta_{phase}/2) + i \sin(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{\phi} \\ + k_t \Delta [\cos(\phi + \alpha) \sin(\delta_{phase}/2) + i \cos(\phi - \alpha) \cos(\delta_{phase}/2)] \cdot \hat{z} \end{pmatrix} \quad (B.4)
\end{aligned}$$

With these equations in hand it is now possible to derive a new Poynting vector expression for non-linear polarization states, rewriting Eq. 3.20 of [13], one then obtains Eq. (5.9).

Appendix C

Robust Analytical Reformulation of the Volume Current Method

As summarized in *Chapter 6*, in an effort to study and understand the impact of various physical factors and limiting assumptions, and to expand the scope of applicability for this method, this appendix contains a step by step analytical reformulation of the VCM following the framework established in [13]. The result is a number of parametric equations describing a grating's scattered electric and magnetic fields (C.3.3-C.3.6), from which the Poynting vector can be obtained. Simplification of these expressions and reconciliation with Li's formulation [13] is undertaken in *Appendix E*, while *Appendix F* evaluates the impact of various approximations and simplifying assumptions within the derivation.

C.1 Step 1 - Find the equivalent polarization current density

To begin, formulate an expression for the index perturbation $\delta\epsilon$, and calculate an equivalent polarization current density $\mathbf{J}$, using the guided wave's electric field $\mathbf{E}$.

C.1.1 Index perturbation

Noting that in the absence of significant fibre lensing distortion, the intensity modulation of the writing fringes may be written as follows:

$$I(\vec{r}) = I_{p-p} \cos^2 \left(\frac{K_g z + K_t r \sin \phi}{2} \right) \quad (C.1)$$

And since multiphoton index changes have been shown to scale in proportion to a power (ζ) of the writing fringe intensity (see 6.2.5), it is straightforward to rewrite Eq. 3.1 of [13] as:

$$n(\vec{r}) = n(r, \phi, z) = n_0 + \frac{\partial n}{\partial \vec{r}} = n_0 + \delta n_{p-p} \cos^{2\zeta} \left(\frac{K_g z + K_t r \sin \phi}{2} \right) \quad (C.2)$$

For a standard UV written grating ($\zeta = 1$) this reduces to:

$$n(\vec{r}) = n_0 + \delta n_{p-p} \left[\frac{1 + \cos(K_g z + K_t r \sin \phi)}{2} \right] = (n_0 + \delta n_{DC}) + \delta n \cos(K_g z + K_t r \sin \phi)$$

with further simplification for small 'DC' index changes, revealing Eq. 3.1 of [13]:

$$n(\vec{r}) \approx n_0 + \delta n \cos(K_g z + K_t r \sin \phi)$$

where the fibre's core refractive index n_0 , modulated by a positive index change having total peak to peak amplitude δn_{p-p} , modulation amplitude δn , 'DC' index change δn_{DC} , sinusoidal order 2ζ , and longitudinal and transverse grating vectors K_g and K_t . If the grating period is Λ and the tilt angle is ξ ($-90^\circ < \xi < 90^\circ$) relative to the transverse y axis, then $K_g = (2\pi/\Lambda)\cos\xi$ and $K_t = -(2\pi/\Lambda)\sin\xi$ [13].

In any case, because $\epsilon(\vec{r}) = \epsilon_0 n(\vec{r})^2$, Eq. (C.2) can be used to write:

$$\frac{\partial \epsilon(\vec{r})}{\partial n} = 2\epsilon_0 n(\vec{r})$$

$$\partial \varepsilon(\vec{r}) = 2\varepsilon_0 \left[n_0 + \delta n_{p-p} \cos^{2\zeta} \left(\frac{K_g z + K_l r \sin \phi}{2} \right) \right] \delta n_{p-p} \cos^{2\zeta} \left(\frac{K_g z + K_l r \sin \phi}{2} \right)$$

Assuming $\delta n \ll n$ the δn^2 term is negligible, omitting the $e^{i\omega t}$ term for simplicity, the expression for the perturbation (Eq. 3.3 of [13]) can be rewritten as:

$$\partial \varepsilon(\vec{r}) \approx 4\kappa \cos^{2\zeta} \left(\frac{K_g z + K_l r \sin \phi}{2} \right) = \frac{\kappa}{2^{2(\zeta-1)}} \left(e^{-i \frac{K_g z + K_l r \sin \phi}{2}} + e^{i \frac{K_g z + K_l r \sin \phi}{2}} \right)^{2\zeta} \quad (C.3)$$

In Li's original work, just one exponential term is kept "Because only the phase-matched term contributes significantly to radiation" [13]. However in cases where the detuning is very large, these terms may not be insignificant. In the interest of generality and completeness all terms are retained for now and it is left to the user to simplify the final analytical solution as desired.

Using the binomial theorem one next obtains:

$$\partial \varepsilon(\vec{r}) \approx \frac{\kappa}{2^{2(\zeta-1)}} \sum_{j=0}^{\infty} \left[\binom{2\zeta}{j} \cdot e^{-i \cdot j \frac{K_g z + K_l r \sin \phi}{2}} e^{i \cdot (2\zeta-j) \frac{K_g z + K_l r \sin \phi}{2}} \right] \quad (C.4)$$

If 2ζ is a positive integer, the series terminates at 2ζ yielding [130]:

$$\partial \varepsilon(\vec{r}) \approx \frac{\kappa}{2^{2(\zeta-1)}} \sum_{j=0}^{2\zeta} \left[\binom{2\zeta}{j} \cdot e^{i(\zeta-j)(K_g z + K_l r \sin \phi)} \right] \quad (C.5)$$

where the perturbation constant is defined to be $\kappa = \varepsilon_0 n_0 \delta n = \frac{1}{2} \varepsilon_0 n_0 \delta n_{p-p}$.

The summation term represents the various Fourier contributions associated with the index modulation. In particular, the $(\zeta - j) = 0$ term represents the 'DC' component of the photoinduced index change, effectively modifying the material indices n_0 and n_{cl} depending on grating location. This alters the waveguide's mode characteristics (refractive indices and energy distributions) but contributes little else to the grating's scattering properties.

C.1.2 Core guided electric field

From p. 227 of [38]:

$$U = a_{core} \sqrt{k_0^2 n_0^2 - \beta^2}, \quad V = a_{core} k_0 \sqrt{n_0^2 - n_{cl}^2}, \quad W = a_{core} \sqrt{\beta^2 - k_0^2 n_{cl}^2}$$

where β is the incident guided mode's propagation constant.

Using Table 12.3 on p. 250 of [38] the field in the core (Eq. 3.2 of [13]) may be rewritten as:

$$\vec{E}_{core}(\vec{r}) = \vec{E}_{core}(r, \phi, z) = E_1 (e_{rco} \hat{r} + e_{\phi co} \hat{\phi} + e_{zco} \hat{z}) e^{-(i\beta + \alpha)z} \quad (C.6)$$

with various components of the expression provided as follows:

$$e_{rco} = -\frac{[a_1 J_{\nu-1}(UR) + a_2 J_{\nu+1}(UR)]}{J_{\nu}(U)} f_{\nu}(\phi) \quad (C.7)$$

$$e_{\phi co} = -\frac{[a_1 J_{\nu-1}(UR) - a_2 J_{\nu+1}(UR)]}{J_{\nu}(U)} g_{\nu}(\phi) \quad (C.8)$$

$$e_{zco} = i \frac{U \cdot J_\nu(UR)}{a_{core} \beta J_\nu(U)} f_\nu(\phi) \quad (C.9)$$

$$R = \frac{r}{a_{core}}, \quad a_1 = \frac{F_2 - 1}{2}, \quad a_2 = \frac{F_2 + 1}{2}, \quad F_2 = \left(\frac{V}{UW} \right)^2 \frac{\nu}{b_1 + b_2}$$

$$b_1 = \frac{J_{\nu-1}(U) - J_{\nu+1}(U)}{2UJ_\nu(U)}, \quad b_2 = -\frac{K_{\nu-1}(W) + K_{\nu+1}(W)}{2WK_\nu(W)}$$

$$f_\nu(\phi) = \begin{cases} \cos \left[\nu \left(\phi - \delta + \frac{\pi}{2} \right) \right] & \nu \in \text{even} \\ \sin \left[\nu \left(\phi - \delta + \frac{\pi}{2} \right) \right] & \nu \in \text{odd} \end{cases}, \quad g_\nu(\phi) = \begin{cases} -\sin \left[\nu \left(\phi - \delta + \frac{\pi}{2} \right) \right] & \nu \in \text{even} \\ \cos \left[\nu \left(\phi - \delta + \frac{\pi}{2} \right) \right] & \nu \in \text{odd} \end{cases}$$

It is noted that a sign error initially present in the z component has been corrected so that it is consistent with [18] and several other references [131-134].

Also, it is noted that the parameter α has been added to account for the loss per unit length arising from radiation mode coupling, back reflection and other sources of attenuation. For gratings designed to have significant backscattering, or to tap a large fraction of light from the fibre, inclusion of this term is important for accurate scaling of the scattered field magnitude, and for uniformly apodized gratings this value can be taken as a constant per unit length [89].

In order to simplify the mathematics and in keeping with Li's notation as much as possible the normalized modal parameter $u = \frac{U}{a_{core}} = \sqrt{k_0^2 n_0^2 - \beta^2}$ is used from this point forward.

To facilitate the required integration the fields are rewritten in terms of x , y and z components.

$$e_{xco} = e_{rco} \cos \phi - e_{\phi co} \sin \phi$$

$$= -\frac{[a_1 J_{\nu-1}(ur) + a_2 J_{\nu+1}(ur)]}{J_\nu(ua_{core})} f_\nu(\phi) \cos \phi + \frac{[a_1 J_{\nu-1}(ur) - a_2 J_{\nu+1}(ur)]}{J_\nu(ua_{core})} g_\nu(\phi) \sin \phi$$

$$= \frac{a_1 J_{\nu-1}(ur) [g_\nu(\phi) \sin \phi - f_\nu(\phi) \cos \phi] - a_2 J_{\nu+1}(ur) [g_\nu(\phi) \sin \phi + f_\nu(\phi) \cos \phi]}{J_\nu(ua_{core})} \quad (C.10)$$

$$e_{yco} = e_{rco} \sin \phi + e_{\phi co} \cos \phi$$

$$= -\frac{[a_1 J_{\nu-1}(ur) + a_2 J_{\nu+1}(ur)]}{J_\nu(ua_{core})} f_\nu(\phi) \sin \phi - \frac{[a_1 J_{\nu-1}(ur) - a_2 J_{\nu+1}(ur)]}{J_\nu(ua_{core})} g_\nu(\phi) \cos \phi$$

$$= -\frac{a_1 J_{\nu-1}(ur) [g_\nu(\phi) \cos \phi + f_\nu(\phi) \sin \phi] - a_2 J_{\nu+1}(ur) [g_\nu(\phi) \cos \phi - f_\nu(\phi) \sin \phi]}{J_\nu(ua_{core})} \quad (C.11)$$

For $\nu = \text{odd modes}$:

$$e_{xco} = -\frac{a_1 J_{\nu-1}(ur) \sin \left[(\nu - 1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] + a_2 J_{\nu+1}(ur) \sin \left[(\nu + 1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{J_\nu(ua_{core})} \quad (C.12)$$

$$e_{yco} = - \frac{a_1 J_{\nu-1}(ur) \cos \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] - a_2 J_{\nu+1}(ur) \cos \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{J_\nu(ua_{core})} \quad (C.13)$$

For $\nu = \text{even modes}$:

$$e_{xco} = - \frac{a_1 J_{\nu-1}(ur) \cos \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] + a_2 J_{\nu+1}(ur) \cos \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{J_\nu(ua_{core})} \quad (C.14)$$

$$e_{yco} = \frac{a_1 J_{\nu-1}(ur) \sin \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] - a_2 J_{\nu+1}(ur) \cos \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{J_\nu(ua_{core})} \quad (C.15)$$

where the notation has been modified as required to maintain consistency with [13,15], and the functions $f_\nu(\phi)$, $g_\nu(\phi)$ have been modified to align the polarization reference with the unit-length polarization vector $\hat{\delta} = \cos(\delta)\hat{x} + \sin(\delta)\hat{y} = \cos(\delta - \phi)\hat{r} + \sin(\delta - \phi)\hat{\phi}$, and where δ is the polarization reference angle of incident light measured in the transverse plane with respect to the +x axis.

C.1.3 Cladding guided electric field

Similarly, using Table 12.3 on p. 250 of [38] the field in the cladding may be represented as:

$$\vec{E}_{clad}(\vec{r}) = \vec{E}_{clad}(r, \phi, z) = E_1 (e_{rcl}\hat{r} + e_{\phi cl}\hat{\phi} + e_{zcl}\hat{z}) e^{-(i\beta + \alpha)z} \quad (C.16)$$

$$e_{rcl} = - \frac{U}{W} \frac{[a_1 K_{\nu-1}(WR) - a_2 K_{\nu+1}(WR)]}{K_\nu(W)} f_\nu(\phi) \quad (C.17)$$

$$e_{\phi cl} = - \frac{U}{W} \frac{[a_1 K_{\nu-1}(WR) + a_2 K_{\nu+1}(WR)]}{K_\nu(W)} g_\nu(\phi) \quad (C.18)$$

$$e_{zcl} = i \frac{U \cdot K_\nu(WR)}{a_{core} \beta K_\nu(W)} f_\nu(\phi) \quad (C.19)$$

In order to simplify the mathematics, the normalized modal parameter $w = \frac{W}{a_{core}} = \sqrt{\beta^2 - k_0^2 n_{cl}^2}$ is used from this point forward. To facilitate the required integrations these fields are also rewritten in terms of x, y and z components.

$$\begin{aligned} e_{xcl} &= e_{rcl} \cos \phi - e_{\phi cl} \sin \phi \\ &= - \frac{U}{W} \frac{[a_1 K_{\nu-1}(wr) - a_2 K_{\nu+1}(wr)]}{K_\nu(wa_{core})} f_\nu(\phi) \cos \phi + \frac{U}{W} \frac{[a_1 K_{\nu-1}(wr) + a_2 K_{\nu+1}(wr)]}{K_\nu(wa_{core})} g_\nu(\phi) \sin \phi \\ &= \frac{U}{W} \frac{a_1 K_{\nu-1}(wr) [g_\nu(\phi) \sin \phi - f_\nu(\phi) \cos \phi] + a_2 K_{\nu+1}(wr) [g_\nu(\phi) \sin \phi + f_\nu(\phi) \cos \phi]}{K_\nu(wa_{core})} \end{aligned} \quad (C.20)$$

$$\begin{aligned}
e_{ycl} &= e_{rel} \sin \phi + e_{\phi cl} \cos \phi \\
&= -\frac{U}{W} \frac{[a_1 K_{\nu-1}(wr) - a_2 K_{\nu+1}(wr)]}{K_\nu(wa_{core})} f_\nu(\phi) \sin \phi - \frac{U}{W} \frac{[a_1 K_{\nu-1}(wr) + a_2 K_{\nu+1}(wr)]}{K_\nu(wa_{core})} g_\nu(\phi) \cos \phi \\
&= -\frac{U}{W} \frac{a_1 K_{\nu-1}(wr) [g_\nu(\phi) \cos \phi + f_\nu(\phi) \sin \phi] + a_2 K_{\nu+1}(wr) [g_\nu(\phi) \cos \phi - f_\nu(\phi) \sin \phi]}{K_\nu(wa_{core})} \quad (C.21)
\end{aligned}$$

For $\nu = \text{odd modes}$:

$$e_{xcl} = -\frac{U}{W} \frac{a_1 K_{\nu-1}(wr) \sin \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] - a_2 K_{\nu+1}(wr) \sin \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{K_\nu(wa_{core})} \quad (C.22)$$

$$e_{ycl} = -\frac{U}{W} \frac{a_1 K_{\nu-1}(wr) \cos \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] + a_2 K_{\nu+1}(wr) \cos \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{K_\nu(wa_{core})} \quad (C.23)$$

For $\nu = \text{even modes}$:

$$e_{xcl} = -\frac{U}{W} \frac{a_1 K_{\nu-1}(wr) \cos \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] - a_2 K_{\nu+1}(wr) \cos \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{K_\nu(wa_{core})} \quad (C.24)$$

$$e_{ycl} = \frac{U}{W} \frac{a_1 K_{\nu-1}(wr) \sin \left[(\nu-1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right] + a_2 K_{\nu+1}(wr) \sin \left[(\nu+1)\phi - \nu \left(\delta - \frac{\pi}{2} \right) \right]}{K_\nu(wa_{core})} \quad (C.25)$$

Comparison reveals that the cladding fields are extremely similar in form to the core fields. Specifically for the cladding fields U is replaced with W , Bessel functions of the first kind J_ν , are replaced with modified Bessel functions of the second kind K_ν , an additional factor of U/W is applied, and there is a sign change in front of the a_2 terms. As a result, the cladding field contribution to the radiation fields can be easily determined directly from substitution into the core field results.

C.1.4 Equivalent polarization current density

With the expressions for $\delta\mathcal{E}(\mathbf{r})$ and the complete electric field $\mathbf{E}$, the equivalent polarization current density (Eqs. 3.4 and 3.5 of [13]) may be rewritten as follows:

$$\vec{J}(\vec{r}) = \vec{J}_{core}(\vec{r}) + \vec{J}_{clad}(\vec{r}) \quad (C.26)$$

$$\begin{aligned}
\vec{J}_{core/clad}(\vec{r}) &= i\omega\partial\mathcal{E}(\vec{r})\vec{E}_{core/clad}(\vec{r}) \\
&\approx i\omega\kappa E_1 (e_x \hat{x} + e_y \hat{y} + e_z \hat{z}) \frac{e^{-(i\beta+\alpha)z}}{2^{2(\zeta-1)}} \sum_{j=0}^{2\zeta} \left[\begin{pmatrix} 2\zeta \\ j \end{pmatrix} \cdot e^{i(\zeta-j)(K_g z + K_r r \sin \phi)} \right] \quad (C.27)
\end{aligned}$$

$$\begin{aligned}
\vec{J}_{core/clad}(r, \phi, z) &\approx \frac{i\omega\kappa E_1 (e_x \hat{x} + e_y \hat{y} + e_z \hat{z})}{2^{2(\zeta-1)}} \sum_{j=0}^{2\zeta} \left[\begin{pmatrix} 2\zeta \\ j \end{pmatrix} \cdot e^{i[(\zeta-j)K_g - \beta + i\alpha]z} e^{i(\zeta-j)K_r r \sin \phi} \right] \\
&\approx \frac{i\omega\kappa E_1 (e_x \hat{x} + e_y \hat{y} + e_z \hat{z})}{2^{2(\zeta-1)}} \sum_{j=0}^{2\zeta} \left[\begin{pmatrix} 2\zeta \\ j \end{pmatrix} \cdot e^{-i\Delta_j z} e^{i(\zeta-j)K_r r \sin \phi} \right] \quad (C.28)
\end{aligned}$$

where $\binom{2\zeta}{j} = \frac{(2\zeta)!}{j!(2\zeta-j)!}$, and $\Delta_j = \beta - i\alpha - (\zeta - j) \cdot K_g$. The real part of Δ_j represents the phase mismatch between the incident mode and the grating's $|\zeta - j|^{\text{th}}$ order Fourier component [123-125]. Physically this is equal to the z component of the scattered wavevector plus any longitudinal detuning that may be present.

C.2 Step 2 - Determine the vector potential

The field everywhere can now be found by solving the wave equation in terms of the vector potential A [13].

C.2.1 Formulate initial expressions for the vector potential

Often the core-cladding refractive index difference is very small and a relatively large ratio between the cladding and grating diameters can be assumed. As such, the accumulated phase difference between a perturbation point $\mathbf{r}' = (r', \phi', z')$ and an observation point $\mathbf{r} = (r, \phi, z)$ may be approximated by:

$$k|\vec{r} - \vec{r}'| \approx k_0 n_{cl} \sqrt{r^2 - 2rr' \cos(\phi - \phi') + (z - z')^2 + r'^2} = k_0 n_{cl} \sqrt{(z - z')^2 + d^2} \quad (\text{C.29})$$

Here, d is defined to be $d|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rr' \cos(\phi - \phi') + r'^2}$ and represents the transverse projection of the distance between these two points.

This equation is slightly different than Eq. 3.6 of [13], in that n_{cl} is used instead of n_0 , however such a substitution is consistent with the approach taken in other references (i.e. Eq. 34-17 on p. 660 of [38]). Although the difference may be negligible when $n_{cl} \sim n_0$, given that the solutions sought are in the transverse farfield, the phase accumulated along a given path is dominated by propagation in the cladding, making the use of n_{cl} a more accurate choice.

It is also noted that for the time dependence $e^{i\omega t}$, the solution to Maxwell's equations is given by $\vec{B}_m = \nabla \times \vec{A}$ and $\vec{E}_m = \frac{\nabla \times \vec{B}_m}{i\omega\mu_0\epsilon}$. Accordingly one may rewrite Eq. 2.6 of [13], $\vec{A}$ outside the core as:

$$\vec{A}(\vec{r}, t) = \vec{A}_{co}(\vec{r}) + \vec{A}_{cl}(\vec{r}) \approx \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t) e^{-ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} dV' \quad (\text{C.30})$$

Neglecting the time dependence this becomes:

$$\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') e^{-ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} dV' \quad (\text{C.31})$$

And given the similarity between the core and cladding field equations of C.2.3, the core component calculations of the vector potential may be used as a template for the cladding component results. Expanding Eq. (C.31) using Eqs. (C.9)-(C.11) and Eq. (C.28) produces:

$$\bar{A}_{co}(r, \phi, z) \approx -i \frac{\omega \kappa \mu_0 E_l}{2^{2\zeta} \pi J_{\nu'}(u a_{core})} \sum_{j=0}^{2\zeta} \binom{2\zeta}{j} \cdot \left[\begin{aligned} & \left(\begin{aligned} & a_1 \int_0^a J_{\nu'-1}(ur') r'^{\nu'} \Phi(j, r, \phi, z, r', F_{\nu'-1}(\phi')) \cdot dr' \\ & + a_2 \int_0^a J_{\nu'+1}(ur') r'^{\nu'} \Phi(j, r, \phi, z, r', F_{\nu'+1}(\phi')) \cdot dr' \end{aligned} \right) \hat{x} \\ & - (-1)^{\nu'} \left(\begin{aligned} & a_1 \int_0^a J_{\nu'-1}(ur') r'^{\nu'} \Phi(j, r, \phi, z, r', G_{\nu'-1}(\phi')) \cdot dr' \\ & - a_2 \int_0^a J_{\nu'+1}(ur') r'^{\nu'} \Phi(j, r, \phi, z, r', G_{\nu'+1}(\phi')) \cdot dr' \end{aligned} \right) \hat{y} \\ & - i \frac{u}{\beta} \int_0^a J_{\nu'}(ur') r'^{\nu'} \Phi(j, r, \phi, z, r', F_{\nu'}(\phi')) \cdot dr' \hat{z} \end{aligned} \right] \quad (C.32)$$

where:

$$\Phi(j, r, \phi, z, r', X(\phi')) = \int_{-\pi}^{\pi} [X(\phi') e^{i(\zeta-j)K_r r' \sin \phi'} Z(j, r, \phi, z, r', \phi')] d\phi' \quad (C.33)$$

$$F_g(\phi') = \begin{cases} \sin \left[\mathcal{G} \cdot \phi' - \nu \left(\delta - \frac{\pi}{2} \right) \right] & \nu \in \text{odd} \\ \cos \left[\mathcal{G} \cdot \phi' - \nu \left(\delta - \frac{\pi}{2} \right) \right] & \nu \in \text{even} \end{cases} \quad (C.34)$$

$$G_g(\phi') = \begin{cases} \cos \left[\mathcal{G} \cdot \phi' - \nu \left(\delta - \frac{\pi}{2} \right) \right] & \nu \in \text{odd} \\ \sin \left[\mathcal{G} \cdot \phi' - \nu \left(\delta - \frac{\pi}{2} \right) \right] & \nu \in \text{even} \end{cases} \quad (C.35)$$

$$Z(j, r, \phi, z, r', \phi') = \int_{-L/2}^{L/2} \frac{e^{-i \left(k_0 n_{cl} \sqrt{(z-z')^2 + d^2} + \Delta_j z' \right)}}{\sqrt{(z-z')^2 + d^2}} \partial z' \quad (C.36)$$

C.2.2 Sum scattered wavefronts over grating length

Drawing one's attention first to the integral over z' , it is observed that by substituting $x = (z' - z)$ and $\partial x = \partial z'$, this integral can be simplified:

$$Z(j, r, \phi, z, r', \phi') = e^{-i\Delta_j z} \int_{-L/2-z}^{L/2-z} \frac{e^{-ik_0 n_{cl} \sqrt{x^2 + d^2}} e^{-i\Delta_j x}}{\sqrt{x^2 + d^2}} \partial x \quad (C.37)$$

If the observation point lies midway between the ends ($z = 0$), an analytical solution exists for infinitely long gratings:

$$\begin{aligned} Z(j, r, \phi, z = 0, r', \phi') &= e^{-i\Delta_j z} \int_{-L/2}^{L/2} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} e^{-i\Delta_j z'}}{\sqrt{z'^2 + d^2}} \partial z' \\ &= \int_{-L/2}^{L/2} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} [\cos(\Delta_j z') - i \sin(\Delta_j z')]}{\sqrt{z'^2 + d^2}} \partial z' \end{aligned} \quad (C.38)$$

Since the sine term in the numerator is an odd function, integration symmetrically about the origin produces a null result. Hence only the cosine term contributes to the solution and one can write:

$$\begin{aligned}
Z(j, r, \phi, z=0, r', \phi') &= 2e^{-i\Delta_j z} \int_0^{L/2} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z' \\
&= 2e^{-i\Delta_j z} \left(\int_0^\infty \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z' - \int_{L/2}^\infty \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z' \right) \\
&= Z_{L=\infty}(j, r, \phi, z=0, r', \phi') + ZC_{L \neq \infty}(j, r, \phi, z=0, r', \phi')
\end{aligned} \tag{C.39}$$

Eq. (C.39) contains two integrals, the first representing an infinitely long grating $Z_{L=\infty}(j, r, \phi, z=0, r', \phi')$ and the second a finite length correction term $ZC_{L \neq \infty}(j, r, \phi, z=0, r', \phi')$. An explicit expression for $ZC_{L \neq \infty}(j, r, \phi, z=0, r', \phi') = -ie^{-i\Delta_j z} \pi h(k_0 n_{cl}, \Delta_j, L, d)$ is provided in D.1. Alternative methods for analyzing non-infinite grating structures are also provided by [13].

To solve the first integral, [13] expanded the exponential and applied Eqs. 3.876-1 and 3.876-2 of [84]. Alternatively and more generally, taking $\varepsilon_1 > 0$, $d > 0$ and $|\Delta_j| + \varepsilon_2 > 0$, Eq. 3.914-4 of [84] can be used to provide a single expression for all Δ_j .

$$\begin{aligned}
Z_{L=\infty}(j, r, \phi, z=0, r', \phi') &= 2e^{-i\Delta_j z} \int_0^\infty \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j |z'|)}{\sqrt{z'^2 + d^2}} \partial z' \\
&= 2e^{-i\Delta_j z} \lim_{\varepsilon_1 \rightarrow 0^+} \left(\lim_{\varepsilon_2 \rightarrow 0^+} \int_0^\infty \frac{e^{-(\varepsilon_1 + ik_0 n_{cl}) \sqrt{z'^2 + d^2}} \cos((|\Delta_j| + \varepsilon_2) z')}{\sqrt{z'^2 + d^2}} \partial z' \right) \\
&= 2e^{-i\Delta_j z} \lim_{\varepsilon_1 \rightarrow 0^+} \left(\lim_{\varepsilon_2 \rightarrow 0^+} K_0 \left(d \sqrt{(\varepsilon_1 + ik_0 n_{cl})^2 + (|\Delta_j| + \varepsilon_2)^2} \right) \right) \\
&= 2e^{-i\Delta_j z} K_0 \left(d \sqrt{\Delta_j^2 - (k_0 n_{cl})^2} \right)
\end{aligned} \tag{C.40}$$

where K_0 is a modified Bessel function of the second kind, $(k_{t2})_j = \sqrt{\Delta_j^2 - k_0^2 n_{cl}^2} = i(k_t)_j$ and $(k_t)_j$ is a measure of the transverse component wave vector for the radiated light emitted by the grating's $(\zeta - j)^{th}$ order Fourier component.

Assuming that the product $(k_t)_j d$ is sufficiently large one can facilitate the integration process by replacing the Bessel function with its asymptotic expansion Eq. 8.451-6 of [84] which is very similar to the asymptotic expansion of the Hankel function given by 8.451-4 of [84].

$$K_0((k_{t2})_j d) \approx \sqrt{\frac{\pi}{2(k_{t2})_j d}} e^{-(k_{t2})_j d} \sum_{k=0}^{n-1} \left[\frac{1}{(2(k_{t2})_j d)^k} \frac{\Gamma\left(\frac{1}{2} + k\right)}{k! \Gamma\left(\frac{1}{2} - k\right)} \right] \tag{C.41}$$

For $0 \leq \Delta_j \leq k_0 n_{cl}$ and $(k_{t2})_j = i(k_t)_j$, it is also noted that:

$$K_0((k_{t2})_j d) = H_0^{(2)}((k_t)_j d) \approx \sqrt{\frac{\pi}{2(k_t)_j d}} e^{-i[(k_t)_j d - \frac{\pi}{4}]} \sum_{k=0}^{n-1} \left[\frac{1}{(2i(k_t)_j d)^k} \frac{\Gamma(\frac{1}{2} + k)}{k! \Gamma(\frac{1}{2} - k)} \right] \quad (C.42)$$

With the two asymptotic series Eqs. 8.451-6 and 8.451-4 of [84] becoming identical under such conditions. In any event, the associated error has also been discussed at length in *Appendix F* wherein it has been demonstrated that when $|(k_{t2})_j d| \geq 2$, the error is on the order of 1% or less.

As taught by Li et al., provided (r'/r) is sufficiently small, the solution can be greatly simplified by taking $d \approx r$ when it is not in the exponential phase term and $d \approx r - r' \cos(\phi - \phi')$ when it is [13]. Where the errors introduced by these approximations are demonstrated in *F.1* and given by Eqs. (C.43) and (C.44):

$$\varepsilon_{d=r} \leq \frac{1}{\left(\frac{a}{a_{clad}}\right)^{-1} - 1} \quad (C.43)$$

$$\varepsilon_{d=r-r'\cos(\phi-\phi')} \leq \left(\frac{a}{a_{clad}}\right) \quad (C.44)$$

Taking advantage of these simplifications and following the analysis of *D.1* yields:

$$K_0((k_{t2})_j d) \approx K_0((k_{t2})_j r) e^{(k_{t2})_j r' \cos(\phi - \phi')} \quad (C.45)$$

and:

$$h(k_0 n_{cl}, \Delta_j, L, d) \approx h(k_0 n_{cl}, \Delta_j, L, r) \quad (C.46)$$

where in *F.2* the error introduced by this relation is demonstrated to be less than that associated with the approximations for d . Thus:

$$Z(j, r, \phi, z=0, r', \phi') \approx \left[2K_0((k_{t2})_j r) e^{(k_{t2})_j r' \cos(\phi - \phi')} - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \right] \cdot e^{-i\Delta_j z} \quad (C.47)$$

Although the permanent introduction of an asymptotic series requires that $(k_{t2})_j d$ is “sufficiently large” [13], its transitional use in the development of the preceding relationship has eliminated any need for this requirement. Hence, significant reductions in error for small $(k_{t2})_j d$ values are realized through the use of this relationship, extending the VCM’s applicability in such instances.

C.2.3 Sum scattered wavefronts over azimuthal dimension

Rewriting the integral function of Φ yields:

$$\Phi(j, r, \phi, z=0, r', X(\phi')) \approx \left(\begin{aligned} & 2K_0((k_{t2})_j r) \int_{-\pi}^{\pi} X(\phi') e^{i[(\zeta-j)K_t \sin \phi' - i(k_{t2})_j \cos(\phi - \phi')]r} d\phi' \\ & - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_t r' \sin \phi'} d\phi' \end{aligned} \right) e^{-i\Delta_j z} \quad (C.48)$$

In the same manner as Eq. 3.13 of [13], substitute the following variables:

$$\begin{cases} (K_{new})_j \cos(\phi_{new})_j = -i(k_{t2})_j \cos \phi \\ (K_{new})_j \sin(\phi_{new})_j = (\varsigma - j)K_t - i(k_{t2})_j \sin \phi \end{cases} \quad (C.49)$$

where $(K_{new})_j$ is derived from these equations to be:

$$(K_{new})_j = \sqrt{(\varsigma - j)^2 K_t^2 - (k_{t2})_j^2 - i2(\varsigma - j)K_t(k_{t2})_j \sin \phi} \quad (C.50)$$

This leads to:

$$\Phi(j, r, \phi, z = 0, r', X(\phi')) \approx \begin{pmatrix} 2K_0((k_{t2})_j, r) \int_{-\pi}^{\pi} X(\phi') e^{i(K_{new})_j r' \cos(\phi' - (\phi_{new})_j)} d\phi' \\ -i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\varsigma - j)K_t r' \sin \phi'} d\phi' \end{pmatrix} e^{-i\Delta_j z} \quad (C.51)$$

Next substitute $x_j = \phi' - (\phi_{new})_j$, $dx_j = d\phi'$ in the first integral and $\phi' = y + \pi/2$, $d\phi' = dy$ in the second so:

$$\Phi(j, r, \phi, z = 0, r', X(\phi')) \approx \begin{pmatrix} 2K_0((k_{t2})_j, r) \int_{-\pi - (\phi_{new})_j}^{\pi - (\phi_{new})_j} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \\ -i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi - \pi/2}^{\pi - \pi/2} X\left(y + \frac{\pi}{2}\right) e^{i(\varsigma - j)K_t r' \cos(y)} dy \end{pmatrix} e^{-i\Delta_j z} \quad (C.52)$$

If $X(\phi') = \sin(\eta) = \sin\left[\mathcal{G} \cdot \phi' - \nu\left(\delta - \frac{\pi}{2}\right)\right]$ one has:

$$\begin{aligned} \Phi(j, r, \phi, z = 0, r', \sin(\eta)) &\approx \begin{pmatrix} 2K_0((k_{t2})_j, r) \int_{-\pi}^{\pi} \sin\left[\mathcal{G} \cdot (x_j + (\phi_{new})_j) - \nu\left(\delta - \frac{\pi}{2}\right)\right] e^{i(K_{new})_j r' \cos(x_j)} dx_j \\ -i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} \sin\left[\mathcal{G} \cdot \left(y + \frac{\pi}{2}\right) - \nu\left(\delta - \frac{\pi}{2}\right)\right] e^{i(\varsigma - j)K_t r' \cos(y)} dy \end{pmatrix} e^{-i\Delta_j z} \\ &\approx \begin{pmatrix} 2K_0((k_{t2})_j, r) \int_{-\pi}^{\pi} \sin\left[\mathcal{G} \cdot x_j + \mathcal{G} \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] e^{i(K_{new})_j r' \cos(x_j)} dx_j \\ -i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} \sin\left[\mathcal{G} \cdot y - \nu\delta + (\mathcal{G} + \nu)\frac{\pi}{2}\right] e^{i(\varsigma - j)K_t r' \cos(y)} dy \end{pmatrix} e^{-i\Delta_j z} \\ &\approx \begin{pmatrix} 2K_0((k_{t2})_j, r) \int_{-\pi}^{\pi} \left(\sin(\mathcal{G} \cdot x_j) \cos\left[\mathcal{G} \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] + \cos(\mathcal{G} \cdot x_j) \sin\left[\mathcal{G} \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] \right) e^{i(K_{new})_j r' \cos(x_j)} dx_j \\ -i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} \left(\sin(\mathcal{G} \cdot y) \cos\left[-\nu\delta + (\mathcal{G} + \nu)\frac{\pi}{2}\right] + \cos(\mathcal{G} \cdot y) \sin\left[-\nu\delta + (\mathcal{G} + \nu)\frac{\pi}{2}\right] \right) e^{i(\varsigma - j)K_t r' \cos(y)} dy \end{pmatrix} e^{-i\Delta_j z} \end{aligned}$$

$$\approx 2 \left(2K_0((k_{i2})_j r) \sin \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] \int_0^\pi \cos(\mathcal{G} \cdot x_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \sin \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] \int_0^\pi \cos(\mathcal{G} \cdot y) e^{i(\zeta-j)K_i r' \cos(y)} dy \right) e^{-i\Delta_j z} \quad (C.53)$$

which from Eq. 3.915-2 of [84] becomes:

$$\Phi(j, r, \phi, z = 0, r', \sin(\eta)) \approx i^\mathcal{G} 2\pi \left(2K_0((k_{i2})_j r) \sin \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] J_\mathcal{G}((K_{new})_j r') \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \sin \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] J_\mathcal{G}((\zeta - j)K_i r') \right) e^{-i\Delta_j z} \quad (C.54)$$

where $J_\mathcal{G}$ are Bessel functions of the first kind.

Similarly if $X(\phi') = \cos(\eta) = \cos \left[\mathcal{G} \cdot \phi' - \nu \left(\delta - \frac{\pi}{2} \right) \right]$ one has:

$$\Phi(j, r, \phi, z = 0, r', \cos(\eta)) \approx \left(2K_0((k_{i2})_j r) \int_{-\pi}^\pi \cos \left[\mathcal{G} \cdot (x_j + (\phi_{new})_j) - \nu \left(\delta - \frac{\pi}{2} \right) \right] e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^\pi \cos \left[\mathcal{G} \cdot \left(y + \frac{\pi}{2} \right) - \nu \left(\delta - \frac{\pi}{2} \right) \right] e^{i(\zeta-j)K_i r' \cos(y)} dy \right) e^{-i\Delta_j z} \\ \approx \left(2K_0((k_{i2})_j r) \int_{-\pi}^\pi \cos \left[\mathcal{G} \cdot x_j + \mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^\pi \cos \left[\mathcal{G} \cdot y - \nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] e^{i(\zeta-j)K_i r' \cos(y)} dy \right) e^{-i\Delta_j z} \\ \approx \left(2K_0((k_{i2})_j r) \int_{-\pi}^\pi \left(\cos(\mathcal{G} \cdot x_j) \cos \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] \right. \right. \\ \left. \left. - \sin(\mathcal{G} \cdot x_j) \sin \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] \right) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^\pi \left(\cos(\mathcal{G} \cdot y) \cos \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] \right. \right. \\ \left. \left. - \sin(\mathcal{G} \cdot y) \sin \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] \right) e^{i(\zeta-j)K_i r' \cos(y)} dy \right) e^{-i\Delta_j z} \\ \approx 2 \left(2K_0((k_{i2})_j r) \cos \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] \int_0^\pi \cos(\mathcal{G} \cdot x_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \cos \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] \int_0^\pi \cos(\mathcal{G} \cdot y) e^{i(\zeta-j)K_i r' \cos(y)} dy \right) e^{-i\Delta_j z} \quad (C.55)$$

And from Eq. 3.915-2 of [84] this becomes:

$$\Phi(j, r, \phi, z = 0, r', \cos(\eta)) \approx i^\mathcal{G} 2\pi e^{-i\Delta_j z} \left(2K_0((k_{i2})_j r) \cos \left[\mathcal{G} \cdot (\phi_{new})_j - \nu \left(\delta - \frac{\pi}{2} \right) \right] J_\mathcal{G}((K_{new})_j r') \right. \\ \left. - i\pi h(k_0 n_{cl}, \Delta_j, L, r) \cos \left[-\nu \delta + (\mathcal{G} + \nu) \frac{\pi}{2} \right] J_\mathcal{G}((\zeta - j)K_i r') \right) \quad (C.56)$$

Using the relations in D.2 the vector potential can then be rewritten as:

$$\vec{A}_{co}(r, \phi, z=0) \approx -F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_{\nu'}(ua_{core})} \times \sum_{j=0}^{2\zeta} \left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} \left[\begin{aligned} & 2K_0((k_{t2})_j, r) \left(\begin{aligned} & \left(\begin{aligned} & a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \\ & - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \end{aligned} \right) \hat{x} \\ & \left(\begin{aligned} & a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \\ & + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \end{aligned} \right) \hat{y} + \\ & + \frac{u}{\beta} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu'}(ur') J_{\nu'}((K_{new})_j r') r' dr' \hat{z} \end{aligned} \right) \\ & (-i)^{\nu} \pi h(k_0 n_{cl}, \Delta_j, L, r) \left(\begin{aligned} & \left(\frac{e^{i\nu\delta} - (-1)^{\nu} e^{-i\nu\delta}}{2} \right) \left(\begin{aligned} & a_1 \int_0^a J_{\nu-1}(ur') J_{\nu-1}((\zeta-j)K_i r') r' dr' \\ & + a_2 \int_0^a J_{\nu+1}(ur') J_{\nu+1}((\zeta-j)K_i r') r' dr' \end{aligned} \right) \hat{x} - \\ & i \left(\frac{e^{i\nu\delta} + (-1)^{\nu} e^{-i\nu\delta}}{2} \right) \left(\begin{aligned} & \left(\begin{aligned} & a_1 \int_0^a J_{\nu-1}(ur') J_{\nu-1}((\zeta-j)K_i r') r' dr' \\ & - a_2 \int_0^a J_{\nu+1}(ur') J_{\nu+1}((\zeta-j)K_i r') r' dr' \end{aligned} \right) \hat{y} \\ & + \frac{u}{\beta} \int_0^a J_{\nu'}(ur') J_{\nu'}((\zeta-j)K_i r') r' dr' \hat{z} \end{aligned} \right) \end{aligned} \right) \end{aligned} \right] \quad (C.57)$$

$$F = \begin{cases} i & \nu \in \text{odd} \\ 1 & \nu \in \text{even} \end{cases} \quad (C.58)$$

C.3 Step 3 - Calculate the Poynting vector expression

C.3.1 Calculation summary

Keeping higher order $1/r$ terms, the Poynting vector can be calculated using the following formulas. First the magnetic field:

$$\vec{B}_{in}(r, \phi, z) = \nabla \times \vec{A} = \nabla \times (\vec{A}_{co} + \vec{A}_{cl}) = \nabla \times \vec{A}_{co} + \nabla \times \vec{A}_{cl} = \vec{B}_{inco}(r, \phi, z) + \vec{B}_{incl}(r, \phi, z) \quad (C.59)$$

Treating only the core field contribution for now:

$$\begin{aligned}
\vec{B}_{inco}(r, \phi, z) &= \left(\frac{\partial A_{zco}}{\partial y} - \frac{\partial A_{yco}}{\partial z} \right) \hat{x} + \left(\frac{\partial A_{xco}}{\partial z} - \frac{\partial A_{zco}}{\partial x} \right) \hat{y} + \left(\frac{\partial A_{yco}}{\partial x} - \frac{\partial A_{xco}}{\partial y} \right) \hat{z} \\
&= \left[\left(\frac{\partial r}{\partial y} \frac{\partial A_{zco}}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) \hat{x} + \left(\frac{\partial A_{xco}}{\partial z} - \frac{\partial r}{\partial x} \frac{\partial A_{zco}}{\partial r} - \frac{\partial \phi}{\partial x} \frac{\partial A_{zco}}{\partial \phi} \right) \hat{y} \right. \\
&\quad \left. + \left(\frac{\partial r}{\partial x} \frac{\partial A_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial A_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial A_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial A_{xco}}{\partial \phi} \right) \hat{z} \right] \quad (C.60)
\end{aligned}$$

Using $r = \sqrt{x^2 + y^2}$, $x = r \cos \phi$, $y = r \sin \phi$ one can obtain:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos \phi, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \phi, \quad \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}$$

That yield:

$$\vec{B}_{inco}(r, \phi, z) = \left[\left(\sin \phi \frac{\partial A_{zco}}{\partial r} + \frac{\cos \phi}{r} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) \hat{x} + \left(\frac{\partial A_{xco}}{\partial z} - \cos \phi \frac{\partial A_{zco}}{\partial r} + \frac{\sin \phi}{r} \frac{\partial A_{zco}}{\partial \phi} \right) \hat{y} \right. \\
\left. + \left(\cos \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{yco}}{\partial \phi} - \sin \phi \frac{\partial A_{xco}}{\partial r} - \frac{\cos \phi}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \hat{z} \right] \quad (C.61)$$

Transforming the result to cylindrical vector components one obtains:

$$\begin{aligned}
\vec{B}_{inco}(r, \phi, z) &= \left[(\cos \phi \cdot B_{xco} + \sin \phi \cdot B_{yco}) \hat{r} + (-\sin \phi \cdot B_{xco} + \cos \phi \cdot B_{yco}) \hat{\phi} + B_{zco} \hat{z} \right] \\
&= \left[\left(\frac{1}{r} \frac{\partial A_{zco}}{\partial \phi} - \cos \phi \frac{\partial A_{yco}}{\partial z} + \sin \phi \frac{\partial A_{xco}}{\partial z} \right) \hat{r} + \left(-\frac{\partial A_{zco}}{\partial r} + \sin \phi \frac{\partial A_{yco}}{\partial z} + \cos \phi \frac{\partial A_{xco}}{\partial z} \right) \hat{\phi} \right. \\
&\quad \left. + \left(\cos \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{yco}}{\partial \phi} - \sin \phi \frac{\partial A_{xco}}{\partial r} - \frac{\cos \phi}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \hat{z} \right] \quad (C.62)
\end{aligned}$$

Next the electric field is calculated by:

$$\vec{E}_{in}(r, \phi, z) = \frac{\nabla \times \vec{B}_{in}}{i\omega\mu_0\epsilon} = \frac{\nabla \times (\vec{B}_{inco} + \vec{B}_{incl})}{i\omega\mu_0\epsilon} = \frac{\nabla \times \vec{B}_{inco} + \nabla \times \vec{B}_{incl}}{i\omega\mu_0\epsilon} = \vec{E}_{inco} + \vec{E}_{incl} \quad (C.63)$$

Again, treating only the contribution of the core field for now.

$$\begin{aligned}
\vec{E}_{inco}(r, \phi, z) &= \frac{1}{i\omega\mu_0\epsilon} \left[\left(\frac{\partial B_{zco}}{\partial y} - \frac{\partial B_{yco}}{\partial z} \right) \hat{x} + \left(\frac{\partial B_{xco}}{\partial z} - \frac{\partial B_{zco}}{\partial x} \right) \hat{y} + \left(\frac{\partial B_{yco}}{\partial x} - \frac{\partial B_{xco}}{\partial y} \right) \hat{z} \right] \\
&= \frac{1}{i\omega\mu_0\epsilon} \left[\left(\frac{\partial r}{\partial y} \frac{\partial B_{zco}}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial B_{zco}}{\partial \phi} - \frac{\partial B_{yco}}{\partial z} \right) \hat{x} + \left(\frac{\partial B_{xco}}{\partial z} - \frac{\partial r}{\partial x} \frac{\partial B_{zco}}{\partial r} - \frac{\partial \phi}{\partial x} \frac{\partial B_{zco}}{\partial \phi} \right) \hat{y} \right. \\
&\quad \left. + \left(\frac{\partial r}{\partial x} \frac{\partial B_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial B_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial B_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial B_{xco}}{\partial \phi} \right) \hat{z} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{i\omega\mu_0\varepsilon} + \left[\begin{aligned} &\left(\frac{\partial r}{\partial y} \frac{\partial}{\partial r} \left(\frac{\partial r}{\partial x} \frac{\partial A_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial A_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial A_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial A_{xco}}{\partial \phi} \right) \right. \\ &+ \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \left(\frac{\partial r}{\partial x} \frac{\partial A_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial A_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial A_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial A_{xco}}{\partial \phi} \right) \\ &\left. - \frac{\partial}{\partial z} \left(\frac{\partial A_{xco}}{\partial z} - \frac{\partial r}{\partial x} \frac{\partial A_{zco}}{\partial r} - \frac{\partial \phi}{\partial x} \frac{\partial A_{zco}}{\partial \phi} \right) \right) \hat{x} \\ &\left(\frac{\partial}{\partial z} \left(\frac{\partial r}{\partial y} \frac{\partial A_{zco}}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) \right. \\ &- \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \left(\frac{\partial r}{\partial x} \frac{\partial A_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial A_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial A_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial A_{xco}}{\partial \phi} \right) \\ &\left. - \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \left(\frac{\partial r}{\partial x} \frac{\partial A_{yco}}{\partial r} + \frac{\partial \phi}{\partial x} \frac{\partial A_{yco}}{\partial \phi} - \frac{\partial r}{\partial y} \frac{\partial A_{xco}}{\partial r} - \frac{\partial \phi}{\partial y} \frac{\partial A_{xco}}{\partial \phi} \right) \right) \hat{y} \\ &+ \left(\frac{\partial r}{\partial x} \frac{\partial}{\partial r} \left(\frac{\partial A_{xco}}{\partial z} - \frac{\partial r}{\partial x} \frac{\partial A_{zco}}{\partial r} - \frac{\partial \phi}{\partial x} \frac{\partial A_{zco}}{\partial \phi} \right) + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \left(\frac{\partial A_{xco}}{\partial z} - \frac{\partial r}{\partial x} \frac{\partial A_{zco}}{\partial r} - \frac{\partial \phi}{\partial x} \frac{\partial A_{zco}}{\partial \phi} \right) \right. \\ &\left. - \frac{\partial r}{\partial y} \frac{\partial}{\partial r} \left(\frac{\partial r}{\partial y} \frac{\partial A_{zco}}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) - \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \left(\frac{\partial r}{\partial y} \frac{\partial A_{zco}}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) \right) \hat{z} \end{aligned} \right] \quad (C.64)
\end{aligned}$$

As above, transforming the result to cylindrical vector components yields:

$$\begin{aligned} \vec{E}_{inco}(r, \phi, z) = & \left[(\cos \phi \cdot E_{xco} + \sin \phi \cdot E_{yco}) \hat{r} + (-\sin \phi \cdot E_{xco} + \cos \phi \cdot E_{yco}) \hat{\phi} + E_{zco} \hat{z} \right] \\ & \left[\frac{1}{r} \frac{\partial}{\partial \phi} \left(\cos \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{yco}}{\partial \phi} - \sin \phi \frac{\partial A_{xco}}{\partial r} - \frac{\cos \phi}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \right. \\ & \left. - \frac{\partial}{\partial z} \left(\cos \phi \frac{\partial A_{xco}}{\partial z} + \sin \phi \frac{\partial A_{yco}}{\partial z} - \frac{\partial A_{zco}}{\partial r} \right) \right] \hat{r} \\ & - \left[\frac{\partial}{\partial r} \left(\cos \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{yco}}{\partial \phi} - \sin \phi \frac{\partial A_{xco}}{\partial r} - \frac{\cos \phi}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \right. \\ & \left. - \frac{\partial}{\partial z} \left(\sin \phi \frac{\partial A_{xco}}{\partial z} - \cos \phi \frac{\partial A_{yco}}{\partial z} + \frac{1}{r} \frac{\partial A_{zco}}{\partial \phi} \right) \right] \hat{\phi} \\ & + \left[\frac{\partial}{\partial r} \left(\cos \phi \frac{\partial A_{xco}}{\partial z} + \sin \phi \frac{\partial A_{yco}}{\partial z} - \frac{\partial A_{zco}}{\partial r} \right) \right. \\ & \left. - \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} \left(\frac{\partial A_{xco}}{\partial z} - \cos \phi \frac{\partial A_{zco}}{\partial r} + \frac{\sin \phi}{r} \frac{\partial A_{zco}}{\partial \phi} \right) - \frac{\cos \phi}{r} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial A_{zco}}{\partial r} + \frac{\cos \phi}{r} \frac{\partial A_{zco}}{\partial \phi} - \frac{\partial A_{yco}}{\partial z} \right) \right] \hat{z} \end{aligned}$$

$$\begin{aligned}
& \left[\begin{aligned} & \left(\frac{1}{r} \left(\sin \phi \left(\frac{\partial A_{yco}}{\partial r} + \frac{1}{r} \frac{\partial^2 A_{yco}}{\partial \phi^2} + \frac{\partial^2 A_{xco}}{\partial \phi \partial r} - \frac{1}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \right. \right. \\ & \left. \left. + \cos \phi \left(\frac{\partial A_{xco}}{\partial r} + \frac{1}{r} \frac{\partial^2 A_{xco}}{\partial \phi^2} - \frac{\partial A_{yco}}{\partial \phi \partial r} + \frac{1}{r} \frac{\partial A_{yco}}{\partial \phi} \right) \right) \right. \\ & \left. + \frac{\partial}{\partial z} \left(\cos \phi \frac{\partial A_{xco}}{\partial z} + \sin \phi \frac{\partial A_{yco}}{\partial z} - \frac{\partial A_{zco}}{\partial r} \right) \right) \hat{r} \\ & \left(\sin \phi \left(\frac{1}{r^2} \frac{\partial A_{yco}}{\partial \phi} - \frac{1}{r} \frac{\partial^2 A_{yco}}{\partial \phi \partial r} - \frac{\partial^2 A_{xco}}{\partial r^2} \right) \right. \\ & \left. + \cos \phi \left(\frac{1}{r^2} \frac{\partial A_{xco}}{\partial \phi} - \frac{1}{r} \frac{\partial^2 A_{xco}}{\partial \phi \partial r} + \frac{\partial^2 A_{yco}}{\partial r^2} \right) \right) \hat{\phi} \\ & \left. - \frac{\partial}{\partial z} \left(\sin \phi \frac{\partial A_{xco}}{\partial z} - \cos \phi \frac{\partial A_{yco}}{\partial z} + \frac{1}{r} \frac{\partial A_{zco}}{\partial \phi} \right) \right) \hat{\phi} \\ & + \left(\frac{\partial^2 A_{zco}}{\partial r^2} + \frac{1}{r} \frac{\partial A_{zco}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_{zco}}{\partial \phi^2} \right. \\ & \left. - \frac{\partial}{\partial z} \left(\cos \phi \frac{\partial A_{xco}}{\partial r} + \sin \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{xco}}{\partial \phi} + \frac{\cos \phi}{r} \frac{\partial A_{yco}}{\partial \phi} \right) \right) \hat{z} \end{aligned} \right] \\
& = -\frac{1}{i\omega\mu_0\varepsilon} + \dots \tag{C.65}
\end{aligned}$$

Finally the Poynting vector can be calculated as follows:

$$\begin{aligned}
\vec{S} & \approx \frac{1}{2\mu_0} \vec{E}_m \times \vec{B}_m^* \\
& \approx \frac{1}{2\mu_0} \left[(E_y B_z^* - B_y^* E_z) \hat{x} + (B_x^* E_z - E_x B_z^*) \hat{y} + (E_x B_y^* - B_x^* E_y) \hat{z} \right] \tag{C.66}
\end{aligned}$$

C.3.2 Vector potential derivatives

It is evident from the preceding equations that the field formulae all depend upon first and second order differentials of the vector potential A , making it necessary to calculate the following quantities:

$$\frac{\partial A(r, \phi, z=0)}{\partial r}, \frac{\partial A(r, \phi, z=0)}{\partial \phi}, \frac{\partial^2 A(r, \phi, z=0)}{\partial r^2}, \frac{\partial^2 A(r, \phi, z=0)}{\partial r \partial \phi}, \frac{\partial^2 A(r, \phi, z=0)}{\partial \phi^2}$$

Since “differentiation of an asymptotic series is, in general, not permissible” 0.331 [84], and because solution accuracy is improved by delaying the introduction of approximations for as long as possible, it is desirable to calculate the $\partial/\partial r$ terms within the integrals prior to making any asymptotic approximations or other substitutions for the Bessel functions $K_n((k_{t2})_j d)$. To this end, as detailed in D.3 one obtains the following quantities:

$$\begin{aligned}
\frac{\partial \bar{A}_{co}(r, \phi, z=0)}{\partial \phi} &\approx F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_{\nu}(ua_{core})} \times \\
&\sum_{j=0}^{2\zeta} \left[\left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} 2K_0((k_{t2})_j r) \times \right. \\
&\left. \begin{aligned}
&\left(\frac{\partial(\phi_{new})_j}{\partial \phi} \left((\nu-1)a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \right. \\
&\quad \left. \left. - (\nu+1)a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \right. \\
&\quad \left. - \frac{\partial(K_{new})_j}{\partial \phi} \left(a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \right. \\
&\quad \left. \left. - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right) \right) \hat{x} \\
&\quad + \left(\frac{\partial(\phi_{new})_j}{\partial \phi} \left((\nu-1)a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \right. \\
&\quad \left. \left. + (\nu+1)a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \right. \\
&\quad \left. + \frac{\partial(K_{new})_j}{\partial \phi} \left(a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \right. \\
&\quad \left. \left. + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right) \right) \hat{y} \\
&\quad + \frac{u}{\beta} \left(\frac{\partial(\phi_{new})_j}{\partial \phi} \nu \sin[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_j r') r' dr' \right. \\
&\quad \left. - \frac{\partial(K_{new})_j}{\partial \phi} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu}((K_{new})_j r') \right) r' dr' \right) \hat{z}
\end{aligned} \right] \quad (C.68)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \vec{A}_{co}(r, \phi, z=0)}{\partial r^2} \approx & -F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_\nu(ua_{core})} \times \\
& \sum_{j=0}^{2\zeta} \left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} \left[\begin{aligned} & \left(2(k_{i2})_j^2 \left(\frac{K_1((k_{i2})_j r)}{(k_{i2})_j r} + K_0((k_{i2})_j r) \right) \times \right. \\ & \left(\begin{aligned} & \left(a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \\ & \left. - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \hat{x} \\ & - \left(a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \\ & \left. + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \hat{y} \\ & \left. + \frac{u}{\beta} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_\nu(ur') J_\nu((K_{new})_j r') r' dr' \hat{z} \right) \end{aligned} \right. \\ & + (-i)^\nu \pi \omega(k_0 n_{cl}, \Delta_j, L, r) \times \\ & \left(\begin{aligned} & \left(\frac{e^{i\nu\delta} - (-1)^\nu e^{-i\nu\delta}}{2} \right) \left(a_1 \int_0^a J_{\nu-1}(ur') J_{\nu-1}((\zeta-j)K_i r') r' dr' \right. \\ & \left. + a_2 \int_0^a J_{\nu+1}(ur') J_{\nu+1}((\zeta-j)K_i r') r' dr' \right) \hat{x} \\ & - i \left(\frac{e^{i\nu\delta} + (-1)^\nu e^{-i\nu\delta}}{2} \right) \left(a_1 \int_0^a J_{\nu-1}(ur') J_{\nu-1}((\zeta-j)K_i r') r' dr' \right. \\ & \left. - a_2 \int_0^a J_{\nu+1}(ur') J_{\nu+1}((\zeta-j)K_i r') r' dr' \right) \hat{y} \\ & \left. + \frac{u}{\beta} \int_0^a J_\nu(ur') J_\nu((\zeta-j)K_i r') r' dr' \hat{z} \right) \end{aligned} \right) \end{aligned} \right] \quad (C.69)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{A}_{co}(r, \phi, z=0)}{\partial \phi \partial r} &\approx -F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_\nu(ua_{core})} \times \\
&\left[\left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} 2(k_{i2})_j K_1((k_{i2})_j r) \times \right. \\
&\left. \left(\frac{\partial(\phi_{new})_j}{\partial \phi} \left(\begin{aligned} &(\nu-1)a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \\ & - (\nu+1)a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \end{aligned} \right) \right. \right. \\
&\left. \left. - \frac{\partial(K_{new})_j}{\partial \phi} \left(\begin{aligned} &a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \\ & - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \end{aligned} \right) \right) \right] \hat{x} \\
&+ \left(\frac{\partial(\phi_{new})_j}{\partial \phi} \left(\begin{aligned} &(\nu-1)a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \\ & + (\nu+1)a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \end{aligned} \right) \right. \\
&\left. + \frac{\partial(K_{new})_j}{\partial \phi} \left(\begin{aligned} &a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \\ & + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \end{aligned} \right) \right) \hat{y} \\
&\left. + \frac{u}{\beta} \left(\begin{aligned} &\frac{\partial(\phi_{new})_j}{\partial \phi} \nu \sin[\nu((\phi_{new})_j - \delta)] \int_0^a J_\nu(ur') J_\nu((K_{new})_j r') r' dr' \\ & - \frac{\partial(K_{new})_j}{\partial \phi} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_\nu(ur') \left(\frac{\partial}{\partial(K_{new})_j} J_\nu((K_{new})_j r') \right) r' dr' \end{aligned} \right) \hat{z} \right] \quad (C.70)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \vec{A}_{co-x}(r, \phi, z=0)}{\partial \phi^2} &\approx F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_\nu(u a_{core})} \hat{x} \times \\
&\left[\begin{aligned} &\left(\frac{2\zeta}{j} \right) e^{-i a_j z} 2K_0((k_{t2})_j r) \times \\ &\left(\frac{\partial^2 (\phi_{new})_j}{\partial \phi^2} \left((\nu-1) a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \right. \\ &\quad \left. \left. - (\nu+1) a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \right. \\ &- \frac{\partial^2 (K_{new})_j}{\partial \phi^2} \left(a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \\ &\quad \left. - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right) \\ &+ \left(\frac{\partial (\phi_{new})_j}{\partial \phi} \right)^2 \left((\nu-1)^2 a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \\ &\quad \left. - (\nu+1)^2 a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \\ &+ 2 \frac{\partial (\phi_{new})_j}{\partial \phi} \frac{\partial (K_{new})_j}{\partial \phi} \left((\nu-1) a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial J_{\nu-1}((K_{new})_j r')}{\partial (K_{new})_j} \right) r' dr' \right. \\ &\quad \left. - (\nu+1) a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial J_{\nu+1}((K_{new})_j r')}{\partial (K_{new})_j} \right) r' dr' \right) \\ &- \left(\frac{\partial (K_{new})_j}{\partial \phi} \right)^2 \left(a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial^2}{\partial (K_{new})_j^2} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \\ &\quad \left. - a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial^2}{\partial (K_{new})_j^2} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right) \end{aligned} \right] \quad (C.71)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \vec{A}_{co-y}(r, \phi, z=0)}{\partial \phi^2} &\approx F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_\nu(u a_{core})} \hat{y} \times \\
&\left[\left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} 2K_0((k_{t2})_j r) \times \right. \\
&\left[\begin{aligned}
&\frac{\partial^2 (\phi_{new})_j}{\partial \phi^2} \left((\nu-1) a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \\
&\quad \left. + (\nu+1) a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \\
&+ \frac{\partial^2 (K_{new})_j}{\partial \phi^2} \left(a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \\
&\quad \left. + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right) \\
&- \left(\frac{\partial (\phi_{new})_j}{\partial \phi} \right)^2 \left((\nu-1)^2 a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_j r') r' dr' \right. \\
&\quad \left. + (\nu+1)^2 a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_j r') r' dr' \right) \\
&+ 2 \frac{\partial (\phi_{new})_j}{\partial \phi} \frac{\partial (K_{new})_j}{\partial \phi} \left((\nu-1) a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial J_{\nu-1}((K_{new})_j r')}{\partial (K_{new})_j} \right) r' dr' \right. \\
&\quad \left. + (\nu+1) a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial J_{\nu+1}((K_{new})_j r')}{\partial (K_{new})_j} \right) r' dr' \right) \\
&+ \left(\frac{\partial (K_{new})_j}{\partial \phi} \right)^2 \left(a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] \int_0^a J_{\nu-1}(ur') \left(\frac{\partial^2}{\partial (K_{new})_j^2} J_{\nu-1}((K_{new})_j r') \right) r' dr' \right. \\
&\quad \left. + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] \int_0^a J_{\nu+1}(ur') \left(\frac{\partial^2}{\partial (K_{new})_j^2} J_{\nu+1}((K_{new})_j r') \right) r' dr' \right)
\end{aligned} \right] \quad (C.72)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \vec{A}_{co-z}(r, \phi, z=0)}{\partial \phi^2} &\approx F \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_{\nu}(ua_{core})} \hat{z} \times \\
&\sum_{j=0}^{2\zeta} \left\{ \begin{aligned}
&\left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} 2K_0((k_{t2})_j r) \frac{u}{\beta} \times \\
&\frac{\partial^2 (\phi_{new})_j}{\partial \phi^2} \nu \sin[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_j r') r' dr' \\
&- \frac{\partial^2 (K_{new})_j}{\partial \phi^2} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu}((K_{new})_j r') \right) r' dr' \\
&+ \left(\frac{\partial (\phi_{new})_j}{\partial \phi} \right)^2 \nu^2 \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_j r') r' dr' \\
&+ 2 \frac{\partial (\phi_{new})_j}{\partial \phi} \frac{\partial (K_{new})_j}{\partial \phi} \nu \sin[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') \left(\frac{\partial}{\partial (K_{new})_j} J_{\nu}((K_{new})_j r') \right) r' dr' \\
&- \left(\frac{\partial (K_{new})_j}{\partial \phi} \right)^2 \cos[\nu((\phi_{new})_j - \delta)] \int_0^a J_{\nu}(ur') \left(\frac{\partial^2}{\partial (K_{new})_j^2} J_{\nu}((K_{new})_j r') \right) r' dr'
\end{aligned} \right\} \quad (C.73)
\end{aligned}$$

The derivative subcomponents of these equations may be solved with the aid of Eq. 8.471-2 of [84] and relations provided in D.3.4.

C.3.3 Calculation of core radiated magnetic field

$$\vec{B}_{inco}(r, \phi, z) = \left[\begin{aligned} & \left(\frac{1}{r} \frac{\partial A_{xco}}{\partial \phi} - \cos \phi \frac{\partial A_{yco}}{\partial z} + \sin \phi \frac{\partial A_{xco}}{\partial z} \right) \hat{r} + \left(-\frac{\partial A_{xco}}{\partial r} + \sin \phi \frac{\partial A_{yco}}{\partial z} + \cos \phi \frac{\partial A_{xco}}{\partial z} \right) \hat{\phi} \\ & + \left(\cos \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi}{r} \frac{\partial A_{xco}}{\partial \phi} - \sin \phi \frac{\partial A_{yco}}{\partial r} - \frac{\cos \phi}{r} \frac{\partial A_{xco}}{\partial \phi} \right) \hat{z} \end{aligned} \right] \quad (C.74)$$

$$\vec{B}_{inco-r}(r, \phi, z=0) \approx iF \frac{\omega \kappa H_0 E_1}{2^{2\zeta-1} J_v(ua_{core})} \sum_{j=0}^{2\zeta} \left(\begin{array}{c} 2\zeta \\ j \end{array} \right) e^{-i\Delta_j z} \left[\begin{array}{c} K_0((k_2)_j r) \left[2\Delta_j \left(\begin{array}{c} a_1 \sin[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^a \int_{t-1}^t J_{t-1}(ur') J_{t-1}((K_{new})_j r') \nu' dr' \\ + a_2 \sin[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] - \phi \int_0^a \int_{t+1}^t J_{t+1}(ur') J_{t+1}((K_{new})_j r') \nu' dr' \end{array} \right) \right. \\ \left. - i \frac{1}{r} \frac{u}{\beta} \left(\begin{array}{c} 2 \frac{\partial(\phi_{new})_j}{\partial \phi} \nu \sin[\nu((\phi_{new})_j - \delta)] \int_0^a \int_t^t J_t(ur') J_t((K_{new})_j r') \nu' dr' \\ - \frac{\partial(K_{new})_j}{\partial \phi} \cos[\nu((\phi_{new})_j - \delta)] \int_0^a \int_{t-1}^t J_{t-1}(ur') J_{t-1}((K_{new})_j r') - J_{t+1}((K_{new})_j r') \nu'^2 dr' \end{array} \right) \right. \\ \left. - (-i)^{j+1} \pi h(k_0, \Delta_j, L, r) \Delta_j \left(\begin{array}{c} a_1 \left(\frac{e^{i(\nu\delta-\phi)} + (-1)^j e^{-i(\nu\delta-\phi)}}{2} \right) \int_{t-1}^t J_{t-1}(ur') J_{t-1}((\zeta-j)K_t r') \nu' dr' \\ - a_2 \left(\frac{e^{i(\nu\delta+\phi)} + (-1)^j e^{-i(\nu\delta+\phi)}}{2} \right) \int_0^a \int_{t+1}^t J_{t+1}(ur') J_{t+1}((\zeta-j)K_t r') \nu' dr' \end{array} \right) \right] \hat{r} \end{array} \right] \quad (C.75)$$

$$\begin{aligned}
\vec{B}_{micro-\phi}(r, \phi, z=0) \approx iF \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} J_{\nu'}(ua_{core})} \sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i\Delta_j z} & \left[\begin{aligned} & 2K_0((k_{12})_j r) \Delta_j \left(\begin{aligned} & a_1 \cos[\nu(\phi_{new_j} - \delta) - (\phi_{new_j})_{i-1}] \int_0^\sigma J_{\nu'-1}(ur') J_{\nu'-1}((K_{new})_j r') r' dr' \\ & - a_2 \cos[\nu(\phi_{new_j} - \delta) + (\phi_{new_j})_{i+1}] \int_0^a J_{\nu'+1}(ur') J_{\nu'+1}((K_{new})_j r') r' dr' \end{aligned} \right) \\ & + i \frac{u}{\beta} \left(\begin{aligned} & 2K_1((k_{12})_j r) \cos[\nu(\phi_{new_j} - \delta)] \int_0^\sigma J_{\nu'}(ur') J_{\nu'}((K_{new})_j r') r' dr' - \\ & (-i)^{\nu'+1} \pi g(k_0 n_{cl}, \Delta_j, L, r) \left(\frac{e^{i\nu\delta} + (-1)^\nu e^{-i\nu\delta}}{2} \right) \int_0^a J_{\nu'}(ur') J_{\nu'}((\zeta-j)K_{\nu'} r') r' dr' \end{aligned} \right) + \\ & (-i)^\nu \pi h(k_0 n_{cl}, \Delta_j, L, r) \Delta_j \left(\begin{aligned} & a_1 \left(\frac{e^{i(\nu\delta-\phi)} - (-1)^\nu e^{-i(\nu\delta-\phi)}}{2} \right) \int_0^{a_{i-1}} J_{\nu-1}(ur') J_{\nu-1}((\zeta-j)K_{\nu'} r') r' dr' \\ & + a_2 \left(\frac{e^{i(\nu\delta+\phi)} - (-1)^\nu e^{-i(\nu\delta+\phi)}}{2} \right) \int_0^a J_{\nu+1}(ur') J_{\nu+1}((\zeta-j)K_{\nu'} r') r' dr' \end{aligned} \right) \end{aligned} \right] \hat{\phi} \quad (C.76)
\end{aligned}$$

$$\begin{aligned}
\vec{B}_{inc \rightarrow z}(r, \phi, z=0) \approx iF \frac{\omega K \mu_0 E_1}{2^{2\zeta-1} J_v(ua_{core})} \hat{z} \times \\
\left[\sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i\Delta_j z} \left(-\frac{K_0((k_{12})_j r)}{r} \right. \right. \\
\left. \left. \begin{aligned} & \left[2i(k_{12})_j K_1((k_{12})_j r) \left(\begin{aligned} & a_1 \sin[\nu(\phi_{new})_j - \delta] - (\phi_{new})_j + \phi \int_0^{\nu} \int_{v-1}^J (ur') J_{v-1}((K_{new})_j r') r' dr' \\ & + a_2 \sin[\nu(\phi_{new})_j - \delta] + (\phi_{new})_j - \phi \int_0^{\nu} \int_{v+1}^J (ur') J_{v+1}((K_{new})_j r') r' dr' \end{aligned} \right) \right. \\ & \left. \left(\begin{aligned} & 2i \frac{\partial(\phi_{new})_j}{\partial \phi} \left(\begin{aligned} & (\nu-1)a_1 \sin[\nu(\phi_{new})_j - \delta] - (\phi_{new})_j + \phi \int_0^{\nu} \int_{v-1}^J (ur') J_{v-1}((K_{new})_j r') r' dr' \\ & - (\nu+1)a_2 \sin[\nu(\phi_{new})_j - \delta] + (\phi_{new})_j - \phi \int_0^{\nu} \int_{v+1}^J (ur') J_{v+1}((K_{new})_j r') r' dr' \end{aligned} \right) \right. \\ & \left. -i \frac{\partial(K_{new})_j}{\partial \phi} \left(\begin{aligned} & a_1 \cos[\nu(\phi_{new})_j - \delta] - (\phi_{new})_j + \phi \int_0^{\nu} \int_{v-1}^J (ur') J_{v-2}((K_{new})_j r') - J_v((K_{new})_j r') r'^2 dr' \\ & - a_2 \cos[\nu(\phi_{new})_j - \delta] + (\phi_{new})_j - \phi \int_0^{\nu} \int_{v+1}^J (ur') J_{v+1}((K_{new})_j r') - J_{v+2}((K_{new})_j r') r'^2 dr' \end{aligned} \right) \right) \right. \\ & \left. + (-i)^\nu \pi g(k_0 n_{cl}, \Delta_j, L, r) \left(\begin{aligned} & a_1 \left(\frac{e^{i(\nu\delta-\phi)} + (-1)^\nu e^{-i(\nu\delta-\phi)}}{2} \right) \int_0^{\nu} \int_{v-1}^J (ur') J_{v-1}((\zeta-j) K_i r') r' dr' \\ & - a_2 \left(\frac{e^{i(\nu\delta+\phi)} + (-1)^\nu e^{-i(\nu\delta+\phi)}}{2} \right) \int_0^{\nu} \int_{v+1}^J (ur') J_{v+1}((\zeta-j) K_i r') r' dr' \end{aligned} \right) \right) \right] \right) \right] \quad (C.77)
\end{aligned}$$

It is noted that most of these integrals may be solved analytically using 5.54 of [84], while a few need to be solved numerically. Using 8.471-1 of [84] the number of different integrals requiring a numerical solution can be reduced in order to minimize calculation time. As an example:

$$J_{v-2}((K_{new})_j r') = \left[\frac{2(\nu-1)}{(K_{new})_j r'} J_{v-1}((K_{new})_j r') \right], J_{v-1}((K_{new})_j r') = \left[\frac{2\nu}{(K_{new})_j r'} J_v((K_{new})_j r') \right], J_{v+2}((K_{new})_j r') = \left[\frac{2(\nu+1)}{(K_{new})_j r'} J_{v+1}((K_{new})_j r') \right] \quad (C.78)$$

C.3.4 Calculation of cladding radiated magnetic field

As discussed in C.1.3, the field contributions from the fibre cladding can now be calculated as well.

$$\begin{aligned}
 \vec{B}_{incl-r}(r, \phi, z=0) \approx & \\
 & iF \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} K_1(wa_{core}) W} \sum_{j=0}^{2\zeta} \left(\frac{2\zeta}{j} \right) e^{-i\Delta_j z} \left[\begin{aligned} & K_0((k_{12})_j r) \left[\begin{aligned} & 2\Delta_j \left[\begin{aligned} & a_1 \sin[v((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^a K_{i-1}(wr') J_{i-1}((K_{new})_j r') r' dr' \\ & - a_2 \sin[v((\phi_{new})_j - \delta) + (\phi_{new})_j - \phi] \int_0^a K_{i+1}(wr') J_{i+1}((K_{new})_j r') r' dr' \end{aligned} \right] \\ & -i \frac{1}{r} \frac{w}{\beta} \left[\begin{aligned} & \left(2 \frac{\partial(\phi_{new})_j}{\partial \phi} v \sin[v((\phi_{new})_j - \delta)] \int_0^a K_i(wr') J_i((K_{new})_j r') r' dr' \right. \\ & \left. - \frac{\partial(K_{new})_j}{\partial \phi} \cos[v((\phi_{new})_j - \delta)] \int_0^a K_{i-1}(wr') J_{i-1}((K_{new})_j r') - J_{i+1}((K_{new})_j r') r'^2 dr' \right] \end{aligned} \right] \end{aligned} \right] \end{aligned} \right] \hat{r} \quad (C.79)
 \end{aligned}$$

$$\begin{aligned}
& \bar{B}_{incl-\phi}(r, \phi, z=0) \approx iF \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} K_{\nu} (w a_{core})} \frac{U}{W} \sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i\Delta_j z} \left[\begin{aligned} & \left(2K_0((k_{i2})_j r) \Delta_j \left(\begin{aligned} & a_1 \cos[\nu((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^\sigma K_{\nu-1}(wr') J_{\nu-1}((K_{new})_j r') \nu' dr' \\ & + a_2 \cos[\nu((\phi_{new})_j - \delta) + (\phi_{new})_j] - \phi \int_0^\sigma K_{\nu+1}(wr') J_{\nu+1}((K_{new})_j r') \nu' dr' \end{aligned} \right) \right. \\ & \left. + i \frac{w}{\beta} \left(\begin{aligned} & 2K_1((k_{i2})_j r) \cos[\nu((\phi_{new})_j - \delta)] \int_0^\sigma K_{\nu}(wr') J_{\nu}((K_{new})_j r') \nu' dr' \\ & - (-i)^{\nu+1} \pi g(k_0 n_{cl}, \Delta_j, L, r) \left(\frac{e^{i\nu\delta} + (-1)^\nu e^{-i\nu\delta}}{2} \int_0^\sigma K_{\nu}(wr') J_{\nu}((\zeta - j) K_{\nu} r') \nu' dr' \right. \right. \\ & \left. \left. + (-i)^\nu \pi h(k_0 n_{cl}, \Delta_j, L, r) \Delta_j \left(\begin{aligned} & a_1 \left(\frac{e^{i(\nu\delta-\phi)} - (-1)^\nu e^{-i(\nu\delta-\phi)}}{2} \int_0^\sigma K_{\nu-1}(wr') J_{\nu-1}((\zeta - j) K_{\nu} r') \nu' dr' \right. \right. \right. \\ & \left. \left. - a_2 \left(\frac{e^{i(\nu\delta+\phi)} - (-1)^\nu e^{-i(\nu\delta+\phi)}}{2} \int_0^\sigma K_{\nu+1}(wr') J_{\nu+1}((\zeta - j) K_{\nu} r') \nu' dr' \right) \right) \right) \right) \right] \end{aligned} \right] \hat{\phi} \quad (C.80)
\end{aligned}$$

$$\begin{aligned}
\bar{B}_{\text{ncd}-z}(r, \phi, z=0) \approx iF \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta-1} K_{\nu} (w a_{\text{core}})} \frac{U}{W} \hat{z} \times \\
\left[\sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i(\Delta_j/z} \right. \\
\left. -i \frac{K_0((k_{12})_j r)}{r} \right. \\
\left. + (-i)^\nu \pi \mathcal{G}(k_0 n_d, \Delta_j, L, r) \right. \\
\left. \times \left[\begin{aligned} & 2i(k_{12})_j K_1((k_{12})_j r) \left(\begin{aligned} & a_1 \sin[\nu((\phi_{\text{new}})_j - \delta) - (\phi_{\text{new}})_j + \phi] \int_0^\sigma K_{\nu-1}(wr') J_{\nu-1}((K_{\text{new}})_j r') \nu' dr' \\ & - a_2 \sin[\nu((\phi_{\text{new}})_j - \delta) + (\phi_{\text{new}})_j - \phi] \int_0^\sigma K_{\nu+1}(wr') J_{\nu+1}((K_{\text{new}})_j r') \nu' dr' \end{aligned} \right) \\ & \left(\begin{aligned} & 2 \frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \left(\begin{aligned} & (\nu-1)a_1 \sin[\nu((\phi_{\text{new}})_j - \delta) - (\phi_{\text{new}})_j + \phi] \int_0^\sigma K_{\nu-1}(wr') J_{\nu-1}((K_{\text{new}})_j r') \nu' dr' \\ & + (\nu+1)a_2 \sin[\nu((\phi_{\text{new}})_j - \delta) + (\phi_{\text{new}})_j - \phi] \int_0^\sigma K_{\nu+1}(wr') J_{\nu+1}((K_{\text{new}})_j r') \nu' dr' \end{aligned} \right) \\ & - \frac{\partial(K_{\text{new}})_j}{\partial \phi} \left(\begin{aligned} & a_1 \cos[\nu((\phi_{\text{new}})_j - \delta) - (\phi_{\text{new}})_j + \phi] \int_0^\sigma K_{\nu-1}(wr') J_{\nu-2}((K_{\text{new}})_j r') - J_{\nu}((K_{\text{new}})_j r') \nu'^2 dr' \\ & + a_2 \cos[\nu((\phi_{\text{new}})_j - \delta) + (\phi_{\text{new}})_j - \phi] \int_0^\sigma K_{\nu+1}(wr') J_{\nu-1}((K_{\text{new}})_j r') - J_{\nu+2}((K_{\text{new}})_j r') \nu'^2 dr' \end{aligned} \right) \end{aligned} \right) \\ & + (-i)^\nu \pi \mathcal{G}(k_0 n_d, \Delta_j, L, r) \left(\begin{aligned} & a_1 \left(\frac{e^{i(\nu\delta-\phi)} + (-1)^\nu e^{-i(\nu\delta-\phi)}}{2} \int_0^\sigma K_{\nu-1}(wr') J_{\nu-1}((\zeta-j)K_{\nu'} r') \nu' dr' \right. \\ & \left. + a_2 \left(\frac{e^{i(\nu\delta+\phi)} + (-1)^\nu e^{-i(\nu\delta+\phi)}}{2} \int_0^\sigma K_{\nu+1}(wr') J_{\nu+1}((\zeta-j)K_{\nu'} r') \nu' dr' \right) \end{aligned} \right) \end{aligned} \right) \\
\end{aligned}
\end{aligned}
\tag{C.81}$$

C.3.5 Calculation of core radiated electric field

$$\begin{aligned}
 \vec{E}_{inc0}(r, \phi, z) = & -\frac{1}{i\omega\mu_0} \times \\
 & \left[\left\{ \frac{1}{r} \left[\sin \phi \left(\frac{\partial A_{yco}}{\partial r} + \frac{1}{r} \frac{\partial^2 A_{yco}}{\partial \phi^2} + \frac{\partial^2 A_{xco}}{\partial \phi \partial r} - \frac{1}{r} \frac{\partial A_{xco}}{\partial \phi} \right) + \cos \phi \left(\frac{\partial A_{xco}}{\partial r} + \frac{1}{r} \frac{\partial^2 A_{xco}}{\partial \phi^2} - \frac{\partial A_{yco}}{\partial \phi \partial r} + \frac{1}{r} \frac{\partial A_{yco}}{\partial \phi} \right) \right] + \frac{\partial}{\partial z} \left(\cos \phi \frac{\partial A_{xco}}{\partial z} + \sin \phi \frac{\partial A_{yco}}{\partial z} - \frac{\partial A_{zco}}{\partial r} \right) \right\} \hat{r} \right. \\
 & + \left[\sin \phi \left(\frac{1}{r^2} \frac{\partial A_{yco}}{\partial \phi} - \frac{1}{r} \frac{\partial^2 A_{yco}}{\partial \phi \partial r} - \frac{\partial^2 A_{xco}}{\partial r^2} \right) + \cos \phi \left(\frac{1}{r^2} \frac{\partial A_{xco}}{\partial \phi} - \frac{1}{r} \frac{\partial^2 A_{xco}}{\partial \phi \partial r} + \frac{\partial^2 A_{yco}}{\partial r^2} \right) - \frac{\partial}{\partial z} \left(\sin \phi \frac{\partial A_{yco}}{\partial z} - \cos \phi \frac{1}{r} \frac{\partial A_{zco}}{\partial \phi} \right) \right] \hat{\phi} \\
 & \left. + \left[\frac{\partial^2 A_{zco}}{\partial r^2} + \frac{1}{r} \frac{\partial A_{zco}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_{zco}}{\partial \phi^2} - \frac{\partial}{\partial z} \left(\cos \phi \frac{\partial A_{xco}}{\partial r} + \sin \phi \frac{\partial A_{yco}}{\partial r} - \frac{\sin \phi \partial A_{xco}}{r} + \frac{\cos \phi \partial A_{yco}}{r} \right) \right] \hat{z} \right]
 \end{aligned}
 \tag{C.82}$$

$$\begin{aligned}
\vec{E}_{\text{inc} \rightarrow r}(r, \phi, z=0) \approx i\Gamma \frac{\kappa L_i}{2^{2k-1} \mathcal{E}_i(u_{\text{opt}})} \hat{r} \times \\
\left[\begin{aligned} & \left\{ (\nu-1) \left[\frac{\partial^2(\phi_{\text{new}})_j}{\partial \phi^2} \sin[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \right] + \left(\frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \right)^2 \left[(\nu-1) \cos[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \right] a_1 \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right. \\ & \left. - (\nu+1) \left[\frac{\partial^2(\phi_{\text{new}})_j}{\partial \phi^2} \sin[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \right] + \left(\frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \right)^2 \left[(\nu+1) \cos[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \right] a_2 \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right\} \\ & - \frac{2K_0((k_{12})_j r)}{r^2} \left[\frac{1}{4} \left(\frac{\partial(K_{\text{new}})_j}{\partial \phi} \right)^2 \left\{ a_1 \cos[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \right\} \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* - 2J_{\nu-1}((K_{\text{new}})_j r^i) + J_{\nu+1}((K_{\text{new}})_j r^i) \right\}^{13} dr^* \\ & - a_2 \cos[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* - 2J_{\nu+1}((K_{\text{new}})_j r^i) + J_{\nu+3}((K_{\text{new}})_j r^i) \right\}^{12} dr^* \right] \\ & - \left(\frac{1}{2} \frac{\partial^2(K_{\text{new}})_j}{\partial \phi^2} \cos[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \right] - \frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \frac{\partial(K_{\text{new}})_j}{\partial \phi} (\nu-1) \sin[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* - J_{\nu-2}((K_{\text{new}})_j r^i) \right\}^{\nu^2} dr^* \\ & + \left(\frac{1}{2} \frac{\partial^2(K_{\text{new}})_j}{\partial \phi^2} \cos[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \right] - \frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \frac{\partial(K_{\text{new}})_j}{\partial \phi} (\nu+1) \sin[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* - J_{\nu+2}((K_{\text{new}})_j r^i) \right\}^{12} dr^* \end{aligned} \right] \\
+ 2i(k_{12})_j K_1((k_{12})_j r) \Delta_j \frac{u}{\beta} \cos[\nu(\phi_{\text{new}})_j - \delta] \int_0^a [J_{\nu}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* + 2 \left(\frac{K_0((k_{12})_j r)^2}{(k_{12})_j K_1((k_{12})_j r)} - a_1 \cos[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right. \\ \left. - a_2 \cos[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right) \\
\left[\begin{aligned} & \left\{ 2 \frac{\partial(\phi_{\text{new}})_j}{\partial \phi} \left[(\nu-1) a_1 \cos[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right. \right. \\ & \left. \left. + (\nu+1) a_2 \cos[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right\} \right. \\ & \left. + \frac{\partial(K_{\text{new}})_j}{\partial \phi} \left[a_1 \sin[\nu(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [J_{\nu-1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right. \right. \\ & \left. \left. + a_2 \sin[\nu(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [J_{\nu+1}(ur)]_j ((K_{\text{new}})_j)^{\nu^2} dr^* \right] \right\} \end{aligned} \right] \\
- (-i) \pi g(k_0 n_d, \Delta_j, L, r) \Delta_j \frac{u}{\beta} \left(\frac{e^{i\nu\delta} + (-1)^\nu e^{-i\nu\delta}}{2} \int_0^a [J_{\nu}(ur)]_j ((\zeta - j) K_{\nu} r^i) dr^* - \left(\frac{h(k_0 n_d, \Delta_j, L, r) \Delta_j^2}{g(k_0 n_d, \Delta_j, L, r)} \right. \right. \\ \left. \left. + a_2 \left[\frac{e^{i(\nu\delta - \theta)} - (-1)^\nu e^{-i(\nu\delta - \theta)}}{2} \int_0^a [J_{\nu-1}(ur)]_j ((\zeta - j) K_{\nu} r^i) dr^* \right] \right) \right] \\
\left. \right] \sum_{j=1}^{2\zeta} \binom{2\zeta}{j} e^{-i(\Delta_j) \cdot z}
\end{aligned}
\tag{C.83}$$

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$$\vec{E}_{incor-\phi}(r, \phi, z=0) \approx iF \frac{\kappa \vec{E}_1}{2^{2\zeta-1} \varepsilon J_v(u a_{core})} \hat{\phi} \times$$

$$\left[\sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i\Delta_j z} \left(\begin{aligned} & 2 \left(\frac{(k_{12})_j K_1((k_{12})_j r)}{r} - k_0^2 n_{cl}^2 ((k_{12})_j r) \right) \left(\begin{aligned} & a_1 \sin[v((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^a J_{\nu-1}(ur) J_{\nu-1}((K_{new})_j r) r^{\nu} dr' \\ & + a_2 \sin[v((\phi_{new})_j - \delta) + (\phi_{new})_j - \phi] \int_0^a J_{\nu+1}(ur) J_{\nu+1}((K_{new})_j r) r^{\nu} dr' \end{aligned} \right) \\ & \left(\frac{(k_{12})_j K_1((k_{12})_j r)}{r} + \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right) \left(\begin{aligned} & 2 \frac{\partial(\phi_{new})_j}{\partial \phi} \left(\begin{aligned} & (\nu-1) a_1 \sin[v((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^a J_{\nu-1}(ur) J_{\nu-1}((K_{new})_j r) r^{\nu} dr' \\ & - (\nu+1) a_2 \sin[v((\phi_{new})_j - \delta) + (\phi_{new})_j - \phi] \int_0^a J_{\nu+1}(ur) J_{\nu+1}((K_{new})_j r) r^{\nu} dr' \end{aligned} \right) \\ & \left(\frac{(k_{12})_j K_1((k_{12})_j r)}{r} + \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right) \left(\begin{aligned} & a_1 \cos[v((\phi_{new})_j - \delta) - (\phi_{new})_j] + \phi \int_0^a J_{\nu-2}(ur) J_{\nu-2}((K_{new})_j r) r^{\nu} dr' \\ & - a_2 \cos[v((\phi_{new})_j - \delta) + (\phi_{new})_j - \phi] \int_0^a J_{\nu+2}(ur) J_{\nu+2}((K_{new})_j r) r^{\nu} dr' \end{aligned} \right) \end{aligned} \right) \\ & + i \frac{K_0((k_{12})_j r)}{r} \frac{u}{\Delta_j} \frac{\beta}{\partial \phi} \left(\begin{aligned} & 2 \frac{\partial(\phi_{new})_j}{\partial \phi} \nu \sin[v((\phi_{new})_j - \delta) \int_0^a J_{\nu}(ur) J_{\nu}((K_{new})_j r) r^{\nu} dr' \\ & \frac{\partial(K_{new})_j}{\partial \phi} \cos[v((\phi_{new})_j - \delta) \int_0^a J_{\nu}(ur) J_{\nu}((K_{new})_j r) r^{\nu} dr' - J_{\nu+1}((K_{new})_j r) r^{\nu} dr' \end{aligned} \right) \\ & - (-i)^{\nu+1} \pi \left(\frac{\alpha k_0 n_{cl} \Delta_j L, r}{h(k_0 n_{cl} \Delta_j L, r) \Delta_j^2} - \frac{e^{i(\nu\delta-\phi)} + (-1)^{\nu} e^{-i(\nu\delta-\phi)}}{2} \int_0^a J_{\nu-1}(ur) J_{\nu-1}((\zeta-j) K_{\nu} r) r^{\nu} dr' \right. \\ & \left. - a_2 \left(\frac{e^{i(\nu\delta+\phi)} + (-1)^{\nu} e^{-i(\nu\delta+\phi)}}{2} \int_0^a J_{\nu+1}(ur) J_{\nu+1}((\zeta-j) K_{\nu} r) r^{\nu} dr' \right) \right) \end{aligned} \right) \right] \quad (C.84)$$

$$\begin{aligned}
\bar{E}_{\text{iso-z}}(r, \phi, z) \approx -iF \frac{\kappa E_i}{2^{2\zeta-1} \mathcal{E} J_i(u a_{\text{core}})} \hat{z} \times \\
\left[\begin{aligned} & 2(k_{i2})^2 K_0((k_{i2})_i r) \frac{u}{\beta} \cos[\nu(\phi_{\text{new}})_i - \delta] \int_0^a \int_{i-1} J_i(ur') J_i((K_{\text{new}})_i r') r' dr' \\ & - 2i\Delta_j K_i((k_{i2})_i r) \left(a_1 \cos[\nu(\phi_{\text{new}})_i - \delta] - (\phi_{\text{new}})_i + \phi \int_0^a \int_{i-1} J_{i-1}(ur') J_i((K_{\text{new}})_i r') r' dr' - a_2 \cos[\nu(\phi_{\text{new}})_i - \delta] + (\phi_{\text{new}})_i - \phi \int_0^a \int_{i+1} J_{i+1}(ur') J_i((K_{\text{new}})_i r') r' dr' \right) \\ & \left[\begin{aligned} & \left(2 \left[\frac{\partial^2(\phi_{\text{new}})_i}{\partial \phi^2} \sin[\nu(\phi_{\text{new}})_i - \delta] + \left(\frac{\partial(\phi_{\text{new}})_i}{\partial \phi} \right)^2 \nu \cos[\nu(\phi_{\text{new}})_i - \delta] \right] \int_0^a \int_{i-1} J_{i-1}(ur') J_i((K_{\text{new}})_i r') r' dr' \right. \\ & \left. - \left(\frac{\partial^2(K_{\text{new}})_i}{\partial \phi^2} \cos[\nu(\phi_{\text{new}})_i - \delta] - 2 \frac{\partial(\phi_{\text{new}})_i}{\partial \phi} \frac{\partial(K_{\text{new}})_i}{\partial \phi} \nu \sin[\nu(\phi_{\text{new}})_i - \delta] \right) \int_0^a \int_{i-1} J_{i-1}(ur') J_i((K_{\text{new}})_i r') r' dr' \right. \\ & \left. - \left(\frac{\partial(K_{\text{new}})_i}{\partial \phi} \right)^2 \cos[\nu(\phi_{\text{new}})_i - \delta] \right] \int_0^a \int_{i-2} J_{i-2}((K_{\text{new}})_i r') - 2J_i((K_{\text{new}})_i r') + J_{i+2}((K_{\text{new}})_i r') r' dr' \right) \\ & \left[\begin{aligned} & 2i \frac{\partial(\phi_{\text{new}})_i}{\partial \phi} \left((\nu-1) a_1 \cos[\nu(\phi_{\text{new}})_i - \delta] - (\phi_{\text{new}})_i + \phi \int_0^a \int_{i-1} J_{i-1}(ur') J_i((K_{\text{new}})_i r') r' dr' \right. \\ & \left. + (\nu+1) a_2 \cos[\nu(\phi_{\text{new}})_i - \delta] + (\phi_{\text{new}})_i - \phi \int_0^a \int_{i+1} J_{i+1}(ur') J_i((K_{\text{new}})_i r') r' dr' \right) \\ & + \Delta_j \left(\begin{aligned} & a_1 \sin[\nu(\phi_{\text{new}})_i - \delta] - (\phi_{\text{new}})_i + \phi \int_0^a \int_{i-2} J_{i-2}((K_{\text{new}})_i r') - J_i((K_{\text{new}})_i r') r' dr' \\ & + a_2 \sin[\nu(\phi_{\text{new}})_i - \delta] + (\phi_{\text{new}})_i - \phi \int_0^a \int_{i+1} J_{i+1}(ur') J_i((K_{\text{new}})_i r') r' dr' \end{aligned} \right) \end{aligned} \right) \\ & + (-i)^{i+1} \pi \left(o(k_0 n_c, \Delta, L, r) + \frac{g(k_0 n_c, \Delta, L, r)}{r} \right) u \left(\frac{e^{i\nu\delta} + (-1)^i e^{-i\nu\delta}}{2} \right) \int_0^a \int_{i-1} J_{i-1}(ur') J_i((\zeta-j) K_i r') r' dr' \\ & - (-i)^{i+1} \pi g(k_0 n_c, \Delta, L, r) \Delta_j \left[a_1 \left(\frac{e^{i(\nu\delta-\phi)} - (-1)^i e^{-i(\nu\delta-\phi)}}{2} \right) \int_0^a \int_{i-1} J_{i-1}(ur') J_i((\zeta-j) K_i r') r' dr' + a_2 \left(\frac{e^{i(\nu\delta+\phi)} - (-1)^i e^{-i(\nu\delta+\phi)}}{2} \right) \int_0^a \int_{i+1} J_{i+1}(ur') J_i((\zeta-j) K_i r') r' dr' \right] \end{aligned} \right] \quad (\text{C.85})
\end{aligned}$$

It is again noted that most of these integrals may be solved analytically using 5.54 of [84], while a few must be solved numerically. Using 8.471-1 of [84] the number of different integrals requiring a numerical solution can be reduced in order to minimize calculation time.

C.3.6 Calculation of cladding radiated electric field

$$\begin{aligned}
 \vec{E}_{\text{inc} \rightarrow r}(r, \phi, z=0) &\approx i\Gamma \frac{\kappa \mathcal{E}_1}{2^{2s-1} \delta K_v(wa_{\text{core}})} \hat{r} \times \\
 &\left[\sum_{j=0}^{2s} \binom{2s}{j} e^{-i\Lambda_j z} \left(\begin{aligned} &2K_0((k_{12})_j r) \left[\frac{1}{r^2} \frac{\partial(K_{\text{res}})_j}{\partial \phi} \right] \left(\begin{aligned} &(\nu-1) \left[\frac{\partial^2(\phi_{\text{res}})_j}{\partial \phi^2} \sin[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \right] + \left(\frac{\partial(\phi_{\text{res}})_j}{\partial \phi} \right)^2 (\nu-1) \cos[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \right] a_1 \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \\ &+ (\nu+1) \left[\frac{\partial^2(\phi_{\text{res}})_j}{\partial \phi^2} \sin[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \right] + \left(\frac{\partial(\phi_{\text{res}})_j}{\partial \phi} \right)^2 (\nu+1) \cos[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \right] a_2 \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \end{aligned} \right) \\ &+ \frac{2K_0((k_{12})_j r)}{r^2} \left(\begin{aligned} &\frac{1}{4} \left(\frac{\partial(K_{\text{res}})_j}{\partial \phi} \right)^2 \left(\begin{aligned} &a_1 \cos[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{\text{res}})_j r') - 2J_{\nu-1}((K_{\text{res}})_j r') + J_{\nu+1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \\ &+ a_2 \cos[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{\text{res}})_j r') - 2J_{\nu+1}((K_{\text{res}})_j r') + J_{\nu+3}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \end{aligned} \right) \\ &- \left(\frac{1}{2} \frac{\partial^2(K_{\text{res}})_j}{\partial \phi^2} \cos[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \right] - \frac{\partial(\phi_{\text{res}})_j}{\partial \phi} \frac{\partial(K_{\text{res}})_j}{\partial \phi} (\nu-1) \sin[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \int_0^a K_{\nu-1}(wr') J_{\nu-2}((K_{\text{res}})_j r') - J_{\nu}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \\ &- \left(\frac{1}{2} \frac{\partial^2(K_{\text{res}})_j}{\partial \phi^2} \cos[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \right] - \frac{\partial(\phi_{\text{res}})_j}{\partial \phi} \frac{\partial(K_{\text{res}})_j}{\partial \phi} (\nu+1) \sin[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \int_0^a K_{\nu+1}(wr') J_{\nu}((K_{\text{res}})_j r') - J_{\nu+2}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \end{aligned} \right) \end{aligned} \right) \\ &+ 2i(k_{12})_j K_1((k_{12})_j r) \Lambda_j \frac{w}{\beta} \cos[\nu(\phi_{\text{res}})_j - \delta] \int_0^a K_{\nu}(wr') J_{\nu}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' + 2 \left(\begin{aligned} &\left(K_0((k_{12})_j r) \Delta_j^2 + \left((k_{12})_j K_1((k_{12})_j r) \right. \right. \\ &\left. \left. + a_2 \cos[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \right) \end{aligned} \right) \\ &+ \frac{1}{r} \left((k_{12})_j K_1((k_{12})_j r) + \frac{K_0((k_{12})_j r)}{r} \right) \left(\begin{aligned} &2 \frac{\partial(\phi_{\text{res}})_j}{\partial \phi} \left(\begin{aligned} &(\nu-1) \mathcal{K}_1 \cos[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \\ &-(\nu+1) \mathcal{K}_2 \cos[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \end{aligned} \right) \\ &+ \frac{\partial(K_{\text{res}})_j}{\partial \phi} \left(\begin{aligned} &a_1 \sin[\nu(\phi_{\text{res}})_j - \delta] - (\phi_{\text{res}})_j + \phi \int_0^a K_{\nu-1}(wr') J_{\nu-2}((K_{\text{res}})_j r') - J_{\nu}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \\ &- a_2 \sin[\nu(\phi_{\text{res}})_j - \delta] + (\phi_{\text{res}})_j - \phi \int_0^a K_{\nu+1}(wr') J_{\nu}((K_{\text{res}})_j r') - J_{\nu+2}((K_{\text{res}})_j r') \mathcal{P}^{\nu} dr' \end{aligned} \right) \end{aligned} \right) \\ &- (-i)^{\nu} \pi g(k_0 n_{cl}, \Delta_j, L, r) \Lambda_j \frac{w}{\beta} \left(\frac{e^{i\nu\delta} + (-1)^{\nu} e^{-i\nu\delta}}{2} \int_0^a K_{\nu}(wr') J_{\nu}((\zeta-j)K_{\text{res}}) \mathcal{P}^{\nu} dr' - \left(\begin{aligned} &\frac{h(k_0 n_{cl}, \Delta_j, L, r) \Delta_j^2}{g(k_0 n_{cl}, \Delta_j, L, r)} \right. \\ &\left. - a_2 \left(\frac{e^{i(\nu\delta+\theta)} - (-1)^{\nu} e^{-i(\nu\delta+\theta)}}{2} \right) \int_0^a K_{\nu+1}(wr') J_{\nu+1}((\zeta-j)K_{\text{res}}) \mathcal{P}^{\nu} dr' \right) \end{aligned} \right) \end{aligned} \right)
 \end{aligned}
 \tag{C.86}$$

$$\vec{E}_{incl-\phi}(r, \phi, z=0) \approx iF \frac{\kappa E_1}{2^{2s-1} \varepsilon K_v (wa_{core})} \frac{U}{W} \hat{\phi} \times$$

$$\left[\sum_{j=0}^{2\zeta} \binom{2\zeta}{j} e^{-i\Delta_j z} \left(\begin{aligned} & \left(\frac{(k_{t2})_j K_1 ((k_{t2})_j r)}{-k_0^2 n_{cl}^2 K_0 ((k_{t2})_j r)} \right) \left(\begin{aligned} & a_1 \sin[\nu(\phi_{new_j} - \delta) - (\phi_{new})_j] + \phi \int_0^a K_{i-1}(wr) J_{i-1}((K_{new})_j r) \nu^i dr \\ & - a_2 \sin[\nu(\phi_{new_j} - \delta) + (\phi_{new})_j] - \phi \int_0^a K_{i+1}(wr) J_{i+1}((K_{new})_j r) \nu^i dr \end{aligned} \right) \\ & + \frac{1}{r} \left(\frac{(k_{t2})_j K_1 ((k_{t2})_j r)}{K_0 ((k_{t2})_j r)} \right) \left(\begin{aligned} & 2 \frac{\partial(\phi_{new})_j}{\partial \phi} \left(\begin{aligned} & (\nu-1) a_1 \sin[\nu(\phi_{new_j} - \delta) - (\phi_{new})_j] + \phi \int_0^a K_{i-1}(wr) J_{i-1}((K_{new})_j r) \nu^i dr \\ & + (\nu+1) a_2 \sin[\nu(\phi_{new_j} - \delta) + (\phi_{new})_j] - \phi \int_0^a K_{i+1}(wr) J_{i+1}((K_{new})_j r) \nu^i dr \end{aligned} \right) \\ & - \frac{\partial(K_{new})_j}{\partial \phi} \left(\begin{aligned} & a_1 \cos[\nu(\phi_{new_j} - \delta) - (\phi_{new})_j] + \phi \int_0^a K_{i-1}(wr) J_{i-2}((K_{new})_j r) - J_{i-1}((K_{new})_j r) \nu^{i+2} dr \\ & + a_2 \cos[\nu(\phi_{new_j} - \delta) + (\phi_{new})_j] - \phi \int_0^a K_{i+1}(wr) J_{i-1}((K_{new})_j r) - J_{i+2}((K_{new})_j r) \nu^{i+2} dr \end{aligned} \right) \end{aligned} \right) \\ & + i \frac{K_0((k_{t2})_j r)}{r} \Delta_j \frac{w}{\beta} \left(\begin{aligned} & 2 \frac{\partial(\phi_{new})_j}{\partial \phi} \nu \sin[\nu(\phi_{new_j} - \delta) \int_0^a K_i(wr) J_i((K_{new})_j r) \nu^i dr \\ & - \frac{\partial(K_{new})_j}{\partial \phi} \cos[\nu(\phi_{new_j} - \delta) \int_0^a K_i(wr) J_{i-1}((K_{new})_j r) - J_{i+1}((K_{new})_j r) \nu^{i+2} dr \end{aligned} \right) \\ & - (-i)^{i+1} \pi \left(\begin{aligned} & o(k_0 n_{cl}, \Delta_j, L, r) - \\ & h(k_0 n_{cl}, \Delta_j, L, r) \Delta_j^2 \end{aligned} \right) \left(\begin{aligned} & a_1 \left(\frac{e^{i(\nu\delta-\phi)} + (-1)^i e^{-i(\nu\delta-\phi)}}{2} \right)^a \int_0^a K_{i-1}(wr) J_{i-1}((\zeta-j)K_i r) \nu^i dr \\ & + a_2 \left(\frac{e^{i(\nu\delta+\phi)} + (-1)^i e^{-i(\nu\delta+\phi)}}{2} \right)^a \int_0^a K_{i+1}(wr) J_{i+1}((\zeta-j)K_i r) \nu^i dr \end{aligned} \right) \end{aligned} \right) \right] \quad (C.87)$$

$$\begin{aligned}
\vec{E}_{\text{incl}-z}(r, \phi, z) \approx -i\Gamma \frac{\kappa E_1}{2^{2\zeta-1} \varepsilon K_1(wa_{\text{cov}})} \frac{U}{W} \hat{z} \times \\
\left[\begin{aligned} & 2(k_{z2})_j^2 K_0((k_{z2})_j r) \frac{w}{\beta} \cos[v(\phi_{\text{new}})_j - \delta] \int_0^a [K_v(wr') J_{v-1}((K_{\text{new}})_j r') r'] dr' \\ & - 2i\Delta_j (k_{z2})_j K_1((k_{z2})_j r) \left\{ a_1 \cos[v(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [K_v(wr') J_{v-1}((K_{\text{new}})_j r') r'] dr' + a_2 \cos[v(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [K_{v+1}(wr') J_{v+1}((K_{\text{new}})_j r') r'] dr' \right\} \\ & \left[\begin{aligned} & 2 \left[\frac{\partial^2 (\phi_{\text{new}})_j}{\partial \phi^2} \sin[v(\phi_{\text{new}})_j - \delta] \right] + \left(\frac{\partial (\phi_{\text{new}})_j}{\partial \phi} \right)^2 v \cos[v(\phi_{\text{new}})_j - \delta] \int_0^a [v \int_0^a [K_v(wr') J_v((K_{\text{new}})_j r') r'] dr' \\ & \frac{1}{r} \frac{w}{\beta} - \left(\frac{\partial^2 (K_{\text{new}})_j}{\partial \phi^2} \cos[v(\phi_{\text{new}})_j - \delta] \right] - 2 \frac{\partial (\phi_{\text{new}})_j}{\partial \phi} \frac{\partial (K_{\text{new}})_j}{\partial \phi} v \sin[v(\phi_{\text{new}})_j - \delta] \int_0^a [K_v(wr') J_{v-1}((K_{\text{new}})_j r') - J_{v+1}((K_{\text{new}})_j r')] r'^2 dr' \\ & - \left(\frac{\partial (K_{\text{new}})_j}{\partial \phi} \right)^2 \cos[v(\phi_{\text{new}})_j - \delta] \int_0^a [K_v(wr') J_{v-2}((K_{\text{new}})_j r') - 2J_v((K_{\text{new}})_j r') + J_{v+2}((K_{\text{new}})_j r')] r'^3 dr' \end{aligned} \right] \\ & \frac{K_0((k_{z2})_j r)}{r} - \left[\begin{aligned} & 2 \frac{\partial (\phi_{\text{new}})_j}{\partial \phi} \left\{ (\nu-1) a_1 \cos[v(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [K_{v-1}(wr') J_{v-1}((K_{\text{new}})_j r') r'] dr' \right. \\ & \left. - (\nu+1) a_2 \cos[v(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [K_{v+1}(wr') J_{v+1}((K_{\text{new}})_j r') r'] dr' \right\} \\ & + \frac{\partial (K_{\text{new}})_j}{\partial \phi} \left\{ a_1 \sin[v(\phi_{\text{new}})_j - \delta] - (\phi_{\text{new}})_j + \phi \int_0^a [K_{v-1}(wr') J_{v-2}((K_{\text{new}})_j r') - J_v((K_{\text{new}})_j r')] r'^2 dr' \right. \\ & \left. - a_2 \sin[v(\phi_{\text{new}})_j - \delta] + (\phi_{\text{new}})_j - \phi \int_0^a [K_{v+1}(wr') J_v((K_{\text{new}})_j r') - J_{v+2}((K_{\text{new}})_j r')] r'^2 dr' \right\} \end{aligned} \right] \\ & + (-i)^{j+1} \pi \left(k_0 n_{\text{cl}} \Delta_j L, r \right) \frac{g(k_0 n_{\text{cl}} \Delta_j L, r)}{r} \beta \left(\frac{e^{i\phi} + (-1)^j e^{-i\phi}}{2} \right) \int_0^a [K_v(wr') J_v((\zeta-j) K_1 r') r'] dr' \\ & - (-i)^{j+1} \pi g(k_0 n_{\text{cl}} \Delta_j L, r) \Delta_j \left[a_1 \left(\frac{e^{i(\phi-\delta)} - (-1)^j e^{-i(\phi-\delta)}}{2} \right) \int_0^a [K_{v-1}(wr') J_{v-1}((\zeta-j) K_1 r') r'] dr' - a_2 \left(\frac{e^{i(\phi+\delta)} - (-1)^j e^{-i(\phi+\delta)}}{2} \right) \int_0^a [K_{v+1}(wr') J_{v+1}((\zeta-j) K_1 r') r'] dr' \right] \end{aligned} \right] \quad (\text{C.88})
\end{aligned}$$

It should be noted that 5.5 of [84] only applies to the solution of integrals involving Bessel functions of the first, second and third kinds (J_ν , N_ν , $H_\nu^{(1)}$, $H_\nu^{(2)}$). In order to use it to solve the cladding contribution, having modified Bessel functions of the second kind

(K_ν) , one must first convert them using 8.407-1 or 8.407-2 of [84]. As z is always real in the case of wr' , either relation will do.

$$K_\nu(z)=\begin{cases} i^{\nu+1}\frac{\pi}{2}H^{(1)}_\nu(iz) & -\pi<\arg(z)\leq\pi/2 \\ i^{\nu+1}\frac{\pi}{2}H^{(2)}_{-\nu}(-iz) & -\pi/2<\arg(z)\leq\pi \end{cases} \tag{C.89}$$

Appendix D

Miscellaneous Calculations For Appendix C

D.1 Finite length correction factor

The finite length correction term $ZC_{L \neq \infty}(j, r, \phi, z = 0, r', \phi')$ of Eq. (C.39) is detailed explicitly here, as an alternative to the generalized apodization methods presented in [13].

$$ZC_{L \neq \infty}(j, r, \phi, z = 0, r', \phi') = -2e^{-i\Delta_j z} \int_{L/2}^{\infty} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z' \quad (D.1)$$

As demonstrated in F.4 one can make the following approximation provided $q \geq \frac{L}{2d}$ is sufficiently large.

$$\frac{e^{-ia\sqrt{1+q^2}}}{\sqrt{1+q^2}} \approx \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi} k!} \frac{e^{-iaq}}{q^{2k+1}} \quad (D.2)$$

The integrands of the second term functions can then be rewritten as follows:

$$\frac{e^{-ik_0 n_{cl} d \sqrt{1 + \left(\frac{z'}{d}\right)^2}}}{d \sqrt{1 + \left(\frac{z'}{d}\right)^2}} \cos(\Delta_j z') \approx \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi} k! z'^{2k+1}} e^{-ik_0 n_{cl} z'} d^{2k} \right] \cos(\Delta_j z') \quad (D.3)$$

As indicated by 0.330 of [84] the error associated with such an alternating series is less than the magnitude of the first discarded term, so providing z'/d is large this expression should yield high accuracy with only a small number of terms.

To solve the integral of Eq. (D.1), Eq. 3.944-4 of [84] is useful:

$$\begin{aligned} ZC_{L \neq \infty}(j, r, \phi, z = 0, r', \phi') &\approx -2e^{-i\Delta_j z} \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right) \cdot d^{2k}}{\sqrt{\pi} k!} \lim_{\varepsilon \rightarrow 0^+} \left(\int_{L/2}^{\infty} \frac{e^{(-ik_0 n_{cl} + \varepsilon)z'} \cos(\Delta_j z')}{z'^{2k+1}} \partial z' \right) \right] \\ &\approx -e^{-i\Delta_j z} \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right) \cdot d^{2k}}{\sqrt{\pi} k!} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{(ik_0 n_{cl} - \varepsilon + i\Delta_j)^{2k} \cdot \Gamma[-2k, (ik_0 n_{cl} - \varepsilon + i\Delta_j)L/2] +}{(ik_0 n_{cl} - \varepsilon - i\Delta_j)^{2k} \cdot \Gamma[-2k, (ik_0 n_{cl} - \varepsilon - i\Delta_j)L/2]} \right) \right] \\ &\approx -e^{-i\Delta_j z} \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right) \cdot d^{2k}}{\sqrt{\pi} k!} \left(\frac{[i(k_0 n_{cl} + \Delta_j)]^{2k} \cdot \frac{Ei[2k+1, i(k_0 n_{cl} + \Delta_j)L/2]}{[i(k_0 n_{cl} + \Delta_j)L/2]^{2k}} +}{[i(k_0 n_{cl} - \Delta_j)]^{2k} \cdot \frac{Ei[2k+1, i(k_0 n_{cl} - \Delta_j)L/2]}{[i(k_0 n_{cl} - \Delta_j)L/2]^{2k}}} \right) \right] \end{aligned}$$

$$\approx -e^{-i\Delta_j z} \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right) \cdot \left(\frac{2d}{L}\right)^{2k}}{\sqrt{\pi} k!} \left(\frac{Ei[2k+1, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+1, i(k_0 n_{cl} - \Delta_j)L/2]} \right)^+ \right]$$

$$= -ie^{-i\Delta_j z} \pi h(k_0 n_{cl}, \Delta_j, L, d) \quad (D.4)$$

where

$$h(k_0 n_{cl}, \Delta_j, L, d) = -\frac{2i}{\pi} \int_{L/2}^{\infty} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z'$$

$$\approx -i \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right) \cdot \left(\frac{2d}{L}\right)^{2k}}{\pi^{3/2} k!} \left(\frac{Ei[2k+1, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+1, i(k_0 n_{cl} - \Delta_j)L/2]} \right)^+ \right] \quad (D.5)$$

$Ei(a, z) = z^{a-1} \cdot \Gamma(1-a, z)$ is the Exponential Integral (Maple, Mathematica and Mathcad all handle this 2 parameter form) and $\Gamma(1-a, z)$ is the Incomplete Gamma Function.

D.2 Useful mathematical relations

$$\sin\left(X + \nu \frac{\pi}{2}\right) = \begin{cases} i^{\nu-1} \cos(X) & \nu \in \text{odd} \\ i^{\nu} \sin(X) & \nu \in \text{even} \end{cases} \quad (D.6)$$

$$\cos\left(X + \nu \frac{\pi}{2}\right) = \begin{cases} i^{\nu+1} \sin(X) & \nu \in \text{odd} \\ i^{\nu} \cos(X) & \nu \in \text{even} \end{cases} \quad (D.7)$$

$$\sin\left[-\nu\delta + (2\nu-1)\frac{\pi}{2}\right] = \begin{cases} \cos(\nu\delta) & \nu \in \text{odd} \\ -\cos(\nu\delta) & \nu \in \text{even} \end{cases} \quad (D.8)$$

$$\sin\left[-\nu\delta + (2\nu+1)\frac{\pi}{2}\right] = \begin{cases} -\cos(\nu\delta) & \nu \in \text{odd} \\ \cos(\nu\delta) & \nu \in \text{even} \end{cases} \quad (D.9)$$

$$\cos\left[-\nu\delta + (2\nu-1)\frac{\pi}{2}\right] = \begin{cases} \sin(\nu\delta) & \nu \in \text{odd} \\ -\sin(\nu\delta) & \nu \in \text{even} \end{cases} \quad (D.10)$$

$$\cos\left[-\nu\delta + (2\nu+1)\frac{\pi}{2}\right] = \begin{cases} -\sin(\nu\delta) & \nu \in \text{odd} \\ \sin(\nu\delta) & \nu \in \text{even} \end{cases} \quad (D.11)$$

$$\sin[\nu(-\delta + \pi)] = \sin(\nu\delta) \quad \nu \in \text{odd} \quad (D.12)$$

$$\cos[\nu(-\delta + \pi)] = \cos(\nu\delta) \quad \nu \in \text{even} \quad (D.13)$$

D.3 Vector potential derivatives

D.3.1 Intermediary calculations for the differential integrals over z'

In this section the following terms are calculated:

$$\frac{\partial Z(j, r, \phi, z=0, r', \phi')}{\partial r} \quad \text{and} \quad \frac{\partial^2 Z(j, r, \phi, z=0, r', \phi')}{\partial r^2}$$

To accomplish this it is noted that:

$$\frac{\partial d}{\partial r} = \frac{\partial}{\partial r} \sqrt{r^2 - 2rr' \cos(\phi - \phi') + r'^2} = \frac{r - r' \cos(\phi - \phi')}{d} \approx 1 \text{ when } r \gg r' \quad (\text{D.14})$$

$$\frac{\partial}{\partial d} K_0((k_{t2})_j d) = -(k_{t2})_j K_1((k_{t2})_j d) \quad (\text{D.15})$$

$$\frac{\partial^2}{\partial d^2} K_0((k_{t2})_j d) = -(k_{t2})_j \frac{\partial}{\partial d} K_1((k_{t2})_j d) = (k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j d)}{(k_{t2})_j d} + K_0((k_{t2})_j d) \right] \quad (\text{D.16})$$

$$\begin{aligned} g(k_0 n_{cl}, \Delta_j, L, d) &= \frac{\partial}{\partial d} h(k_0 n_{cl}, \Delta_j, L, d) = -\frac{2i}{\pi} \frac{\partial}{\partial d} \int_{L/2}^{\infty} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{\sqrt{z'^2 + d^2}} \partial z' \\ &= \frac{2d}{\pi} \int_{L/2}^{\infty} \left[i \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{(z'^2 + d^2)^{3/2}} - k_0 n_{cl} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{z'^2 + d^2} \right] \partial z' \end{aligned} \quad (\text{D.17})$$

$$\begin{aligned} o(k_0 n_{cl}, \Delta_j, L, d) &= \frac{\partial^2}{\partial d^2} h(k_0 n_{cl}, \Delta_j, L, d) \\ &= \frac{2}{\pi} \int_{L/2}^{\infty} \left[3d^2 \left[k_0 n_{cl} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{(z'^2 + d^2)^2} - i \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{(z'^2 + d^2)^{5/2}} \right] \right. \\ &\quad \left. + i[(k_0 n_{cl} d)^2 + 1] \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{(z'^2 + d^2)^{3/2}} - k_0 n_{cl} \frac{e^{-ik_0 n_{cl} \sqrt{z'^2 + d^2}} \cos(\Delta_j z')}{z'^2 + d^2} \right] \partial z' \end{aligned} \quad (\text{D.18})$$

Approximating these integrands using the Maclaurin series in *F.4* one obtains:

$$\begin{aligned} \frac{\partial}{\partial d} h(k_0 n_{cl}, \Delta_j, L, d) &\approx \frac{2d}{\pi} \int_{L/2}^{\infty} \left[i \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k! z'^{2k+3}} e^{-ik_0 n_{cl} z'} d^{2k} \right. \\ &\quad \left. - k_0 n_{cl} \sum_{k=0}^{\infty} \frac{(-1)^k}{z'^{2k+2}} e^{-ik_0 n_{cl} z'} d^{2k} \right] \cos(\Delta_j |z'|) \partial z' \\ &\approx \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k d^{2k+1} \left[i \frac{2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \int_{L/2}^{\infty} \left[\frac{e^{-ik_0 n_{cl} z'} \cos(\Delta_j z')}{z'^{2k+3}} \right] \partial z' \right. \\ &\quad \left. - k_0 n_{cl} \int_{L/2}^{\infty} \left[\frac{e^{-ik_0 n_{cl} z'} \cos(\Delta_j z')}{z'^{2k+2}} \right] \partial z' \right] \\ &\approx \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k d^{2k+1} \left[i \frac{2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \lim_{\epsilon_1 \rightarrow 0^+} \left(\int_{L/2}^{\infty} \left[\frac{e^{-(ik_0 n_{cl} + \epsilon_1) z'} \cos(\Delta_j z')}{z'^{2k+3}} \right] \partial z' \right) \right. \\ &\quad \left. - k_0 n_{cl} \lim_{\epsilon_1 \rightarrow 0^+} \left(\int_{L/2}^{\infty} \left[\frac{e^{-(ik_0 n_{cl} + \epsilon_1) z'} \cos(\Delta_j z')}{z'^{2k+2}} \right] \partial z' \right) \right] \end{aligned} \quad (\text{D.19})$$

$$\begin{aligned}
& \left[\begin{aligned} & k_0 n_{cl} \sum_{k=0}^{\infty} \left[\frac{(-1)^k (k+1)}{z^{12k+4}} e^{-ik_0 n_{cl} z'} d^{2k} \right] \cos(\Delta_j z') \\ & 3d^2 \left[-i \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 4 \cdot \Gamma\left(\frac{2k+5}{2}\right)}{3\sqrt{\pi} k! z^{12k+5}} e^{-ik_0 n_{cl} z'} d^{2k} \right] \cos(\Delta_j z') \end{aligned} \right] + \\
& \frac{\partial^2}{\partial d^2} h(k_0 n_{cl}, \Delta_j, L, d) \approx \frac{2}{\pi} \int_{L/2}^{\infty} \left[\begin{aligned} & i \left[(k_0 n_{cl} d)^2 + 1 \right] \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k! z^{12k+3}} e^{-ik_0 n_{cl} z'} d^{2k} \cos(\Delta_j z') \partial z' \\ & - k_0 n_{cl} \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{z^{12k+2}} e^{-ik_0 n_{cl} z'} d^{2k} \right] \cos(\Delta_j z') \end{aligned} \right] \partial z' \\
& \approx \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k d^{2k} \left[\begin{aligned} & d^2 \left(\begin{aligned} & 3k_0 n_{cl} (k+1) \lim_{\varepsilon_1 \rightarrow 0^+} \left[\int_{L/2}^{\infty} \frac{e^{-(ik_0 n_{cl} + \varepsilon_1) z'} \cos(\Delta_j z')}{z^{12k+4}} \partial z' \right] - \\ & \frac{4i \Gamma\left(\frac{2k+5}{2}\right)}{\sqrt{\pi} k!} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\int_{L/2}^{\infty} \frac{e^{-(ik_0 n_{cl} + \varepsilon_1) z'} \cos(\Delta_j z')}{z^{12k+5}} \partial z' \right] \end{aligned} \right) + \\ & \frac{2i \left[(k_0 n_{cl} d)^2 + 1 \right] \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\int_{L/2}^{\infty} \frac{e^{-(ik_0 n_{cl} + \varepsilon_1) z'} \cos(\Delta_j z')}{z^{12k+3}} \partial z' \right] \\ & - k_0 n_{cl} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\int_{L/2}^{\infty} \frac{e^{-(ik_0 n_{cl} + \varepsilon_1) z'} \cos(\Delta_j z')}{z^{12k+2}} \partial z' \right] \end{aligned} \right] \quad (D.20)
\end{aligned}$$

Using Eq. 3.944-4 of [84]:

$$\begin{aligned}
g(k_0 n_{cl}, \Delta_j, L, d) &= \frac{\partial}{\partial d} h(k_0 n_{cl}, \Delta_j, L, d) \approx \\
&\frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k d^{2k+1} \left[\begin{aligned} &i \frac{2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+2} \times}{\Gamma[-2k-2, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +} \\ &\frac{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+2} \times}{\Gamma[-2k-2, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right] \\ &- k_0 n_{cl} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+1} \times}{\Gamma[-2k-1, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +} \\ &\frac{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+1} \times}{\Gamma[-2k-1, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right] \end{aligned} \right] \\
&\approx \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{2d}{L} \right)^{2k+1} \left[\begin{aligned} &i \frac{2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k! (L/2)} \left[\begin{aligned} &Ei[2k+3, i(k_0 n_{cl} + \Delta_j)L/2] + \\ &Ei[2k+3, i(k_0 n_{cl} - \Delta_j)L/2] \end{aligned} \right] \\ &- k_0 n_{cl} \left[\begin{aligned} &Ei[2k+2, i(k_0 n_{cl} + \Delta_j)L/2] + \\ &Ei[2k+2, i(k_0 n_{cl} - \Delta_j)L/2] \end{aligned} \right] \end{aligned} \right] \quad (D.21)
\end{aligned}$$

$Ei(a, z) = z^{a-1} \cdot \Gamma(1-a, z)$ is the Exponential Integral (Maple, Mathematica and Mathcad all handle this 2 parameter form) and $\Gamma(1-a, z)$ is the Incomplete Gamma Function.

$$\begin{aligned}
o(k_0 n_{cl}, \Delta_j, L, d) &= \frac{\partial^2}{\partial d^2} h(k_0 n_{cl}, \Delta_j, L, d) \approx \\
&\frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k d^{2k} \left[\begin{aligned} &d^2 \left[\begin{aligned} &\frac{(3k_0 n_{cl} (k+1) \times}{\lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+3} \cdot \Gamma[-2k-3, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +}{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+3} \cdot \Gamma[-2k-3, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right]} \\ &4i\Gamma\left(\frac{2k+5}{2}\right)}{\sqrt{\pi} k!} \times \\ &\lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+4} \cdot \Gamma[-2k-4, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +}{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+4} \cdot \Gamma[-2k-4, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right] \end{aligned} \right] \\ &+ \frac{2i[(k_0 n_{cl} d)^2 + 1]\Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \times \\ &\lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+2} \cdot \Gamma[-2k-2, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +}{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+2} \cdot \Gamma[-2k-2, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right] - \\ &k_0 n_{cl} \lim_{\varepsilon_1 \rightarrow 0^+} \left[\begin{aligned} &\frac{(ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)^{2k+1} \cdot \Gamma[-2k-1, (ik_0 n_{cl} + \varepsilon_1 + i\Delta_j)L/2] +}{(ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)^{2k+1} \cdot \Gamma[-2k-1, (ik_0 n_{cl} + \varepsilon_1 - i\Delta_j)L/2]} \end{aligned} \right] \end{aligned} \right]
\end{aligned}$$

$$\approx \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{2d}{L} \right)^{2k} \left[\begin{aligned} & \left(\frac{2d}{L} \right)^2 \left[\frac{3k_0 n_{cl} (k+1)}{(L/2)} \left[\frac{Ei[2k+4, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+4, i(k_0 n_{cl} - \Delta_j)L/2]} \right] - \right. \\ & \left. \frac{4i\Gamma\left(\frac{2k+5}{2}\right)}{\sqrt{\pi}k!(L/2)^2} \left[\frac{Ei[2k+5, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+5, i(k_0 n_{cl} - \Delta_j)L/2]} \right] \right] \\ & + \frac{2i \left[(k_0 n_{cl})^2 \left(\frac{2d}{L} \right)^2 + \frac{1}{(L/2)^2} \right] \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi}k!} \times \\ & \left[\frac{Ei[2k+3, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+3, i(k_0 n_{cl} - \Delta_j)L/2]} \right] \\ & - \frac{k_0 n_{cl}}{(L/2)} \left[\frac{Ei[2k+2, i(k_0 n_{cl} + \Delta_j)L/2]}{Ei[2k+2, i(k_0 n_{cl} - \Delta_j)L/2]} \right] \end{aligned} \right] \quad (D.22)$$

$$\begin{aligned} \frac{\partial Z(j, r, \phi, z=0, r', \phi')}{\partial r} &= e^{-i\Delta_j z} \left[2 \frac{\partial}{\partial d} K_0((k_{i2})_j d) - i\pi \frac{\partial}{\partial d} h(k_0 n_{cl}, \Delta_j, L, d) \right] \cdot \frac{\partial d}{\partial r} \\ &= -e^{-i\Delta_j z} \left[2(k_{i2})_j K_1((k_{i2})_j d) + i\pi g(k_0 n_{cl}, \Delta_j, L, d) \right] \end{aligned} \quad (D.23)$$

At this point one can facilitate the integration process by replacing the Bessel function with its asymptotic expansion Eq. 8.451-6 of [84].

$$K_1((k_{i2})_j d) \approx \sqrt{\frac{\pi}{2(k_{i2})_j d}} e^{-(k_{i2})_j d} \sum_{k=0}^{n-1} \left[\frac{1}{(2(k_{i2})_j d)^k} \frac{\Gamma\left(\frac{3}{2} + k\right)}{k! \Gamma\left(\frac{3}{2} - k\right)} \right] \quad (D.24)$$

For $0 \leq \Delta_j \leq k_0 n_{cl}$ and $(k_{i2})_j = i(k_i)_j$, it is also noted that:

$$K_1((k_{i2})_j d) = H_1^{(2)}((k_i)_j d) \approx \sqrt{\frac{\pi}{2(k_i)_j d}} e^{-[(k_i)_j d + \frac{\pi}{4}]} \sum_{k=0}^{n-1} \left[\frac{1}{(2i(k_i)_j d)^k} \frac{\Gamma\left(\frac{3}{2} + k\right)}{k! \Gamma\left(\frac{3}{2} - k\right)} \right] \quad (D.25)$$

with the two asymptotic series 8.451-6 and 8.451-4 becoming identical.

In any event, as taught by Li et al. and discussed in *F.1*, provided (r'/r) is sufficiently small, the solution can be greatly simplified by taking $d \approx r$ when it is not in the exponential phase term and $d \approx r - r' \cos(\phi - \phi')$ when it is [13].

Making use of these simplifications yields:

$$g(k_0 n_{cl}, \Delta_j, L, d) \approx g(k_0 n_{cl}, \Delta_j, L, r) \quad (D.26)$$

And:

$$K_1((k_{i2})_j d) \approx K_1((k_{i2})_j r) e^{(k_{i2})_j r' \cos(\phi - \phi')} \quad (D.27)$$

Where it has been demonstrated in *F.2* that the error introduced by the latter relation is less than that associated with the approximations for d . Furthermore as specified by Li et al.

although the permanent introduction of an asymptotic series requires that $(k_{i2})_j d$ be “sufficiently large”, as has also demonstrated in *F.2*, its transitional use in the development of the preceding relationship has eliminated any need for this requirement. Thus:

$$\frac{\partial Z(j, r, \phi, z = 0, r', \phi')}{\partial r} \approx -e^{-i\Delta_j z} \left[2(k_{i2})_j K_1((k_{i2})_j r) e^{(k_{i2})_j r' \cos(\phi - \phi')} + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \right] \quad (D.28)$$

Likewise:

$$\begin{aligned} \frac{\partial^2 Z(j, r, \phi, z = 0, r', \phi')}{\partial r^2} &= e^{-i\Delta_j z} \frac{\partial}{\partial d} \left[\left(2 \frac{\partial}{\partial d} K_0((k_{i2})_j d) - i\pi \frac{\partial}{\partial d} h(k_0 n_{cl}, \Delta_j, L, d) \right) \cdot \frac{\partial d}{\partial r} \right] \cdot \frac{\partial d}{\partial r} \\ &= e^{-i\Delta_j z} \left[2 \frac{\partial^2}{\partial d^2} K_0((k_{i2})_j d) - i\pi \frac{\partial^2}{\partial d^2} h(k_0 n_{cl}, \Delta_j, L, d) \right] \cdot \left(\frac{\partial d}{\partial r} \right)^2 \\ &= e^{-i\Delta_j z} \left[2(k_{i2})_j^2 \left(\frac{K_1((k_{i2})_j d)}{(k_{i2})_j d} + K_0((k_{i2})_j d) \right) - i\pi o(k_0 n_{cl}, \Delta_j, L, d) \right] \\ &\approx e^{-i\Delta_j z} \left[2(k_{i2})_j^2 \left(\frac{K_1((k_{i2})_j r)}{(k_{i2})_j r} + K_0((k_{i2})_j r) \right) e^{(k_{i2})_j r' \cos(\phi - \phi')} - i\pi o(k_0 n_{cl}, \Delta_j, L, r) \right] \end{aligned} \quad (D.29)$$

where:

$$o(k_0 n_{cl}, \Delta_j, L, d) \approx o(k_0 n_{cl}, \Delta_j, L, r) \quad (D.30)$$

D.3.2 Intermediary calculations for the differential integrals over ϕ'

In this section, using the results of *D.3.1* the following terms are calculated:

$$\frac{\partial \Phi(j, r, \phi, z = 0, r', X(\phi'))}{\partial r} \quad \text{and} \quad \frac{\partial^2 \Phi(j, r, \phi, z = 0, r', X(\phi'))}{\partial r^2}$$

Beginning with the first:

$$\begin{aligned} \frac{\partial \Phi(j, r, \phi, z = 0, r', X(\phi'))}{\partial r} &= \frac{\partial}{\partial r} \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_i r' \sin \phi'} Z(j, r, \phi, z = 0, r', \phi') \cdot d\phi' \\ &= \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_i r' \sin \phi'} \frac{\partial Z(j, r, \phi, z = 0, r', \phi')}{\partial r} \cdot d\phi' \\ &\approx -e^{-i\Delta_j z} \left[2(k_{i2})_j K_1((k_{i2})_j r) \int_{-\pi}^{\pi} X(\phi') e^{i[(\zeta-j)K_i \sin \phi' - i(k_{i2})_j \cos(\phi - \phi')]r'} d\phi' \right. \\ &\quad \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_i r' \sin \phi'} d\phi' \right] \end{aligned} \quad (D.31)$$

with substitution of Eq. (C.50) leading to:

$$\frac{\partial \Phi(j, r, \phi, z = 0, r', X(\phi'))}{\partial r} \approx -e^{-i\Delta_j z} \left[2(k_{i2})_j K_1((k_{i2})_j r) \int_{-\pi}^{\pi} X(\phi') e^{i(K_{new})_j r' \cos(\phi' - (\phi_{new})_j)} d\phi' \right. \\ \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_i r' \sin \phi'} d\phi' \right]$$

$$\approx -e^{-i\Delta_j z} \left[2(k_{t2})_j K_1((k_{t2})_j r) \cdot \int_{-\pi-(\phi_{new})_j}^{\pi-(\phi_{new})_j} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi-\pi/2}^{\pi-\pi/2} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_l r' \cos(y)} dy \right] \quad (D.32)$$

where substitutions of $x_j = \phi' - (\phi_{new})_j$, $dx_j = d\phi'$ in the first integral and $\phi' = y + \pi/2$, $d\phi' = dy$ in the second integral have also been made.

$$\frac{\partial^2 \Phi(j, r, \phi, z=0, r', X(\phi'))}{\partial r^2} = \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_l r' \sin \phi'} \frac{\partial^2 Z(j, r, \phi, z=0, r', \phi')}{\partial r^2} \cdot d\phi' \\ \approx e^{-i\Delta_j z} \left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \int_{-\pi}^{\pi} X(\phi') e^{i[(\zeta-j)K_l \sin \phi' - i(k_{t2})_j \cos(\phi - \phi')]r'} \cdot d\phi' \right. \\ \left. - i\pi o(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_l r' \sin \phi'} \cdot d\phi' \right] \\ \approx e^{-i\Delta_j z} \left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \int_{-\pi}^{\pi} X(\phi') e^{i(K_{new})_j r' \cos(\phi' - (\phi_{new})_j)} \cdot d\phi' \right. \\ \left. - i\pi o(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X(\phi') e^{i(\zeta-j)K_l r' \sin \phi'} \cdot d\phi' \right] \\ \approx e^{-i\Delta_j z} \left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \int_{-\pi-(\phi_{new})_j}^{\pi-(\phi_{new})_j} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. - i\pi o(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi-\pi/2}^{\pi-\pi/2} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_l r' \cos(y)} dy \right] \quad (D.33)$$

Where substitutions of $x_j = \phi' - (\phi_{new})_j$, $dx_j = d\phi'$ in the first integral and $\phi' = y + \pi/2$, $d\phi' = dy$ in the second integral have also been made.

If $X(\phi') = \sin(\eta) = \sin\left[\mathcal{G} \cdot \phi' - \nu\left(\delta - \frac{\pi}{2}\right)\right]$ one then obtains:

$$\frac{\partial \Phi(j, r, \phi, z=0, r', \sin(\eta))}{\partial r} \approx -e^{-i\Delta_j z} \left[2(k_{t2})_j K_1((k_{t2})_j r) \int_{-\pi}^{\pi} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\ \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_l r' \cos(y)} dy \right] \\ \approx -i^g 2\pi e^{-i\Delta_j z} \left[2(k_{t2})_j K_1((k_{t2})_j r) \sin\left[\mathcal{G} \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] J_g((K_{new})_j r') \right. \\ \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \sin\left[(\mathcal{G} + \nu)\frac{\pi}{2} - \nu\delta\right] J_g((\zeta-j)K_l r') \right] \quad (D.34)$$

The last two integrals were solved previously in detail when calculating $\Phi(j, r, \phi, z=0, r', \sin(\eta))$ in C.2.3.

$$\begin{aligned}
\frac{\partial^2 \Phi(j, r, \phi, z=0, r', X(\phi'))}{\partial r^2} &\approx \\
e^{-i\Delta_j z} &\left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \int_{-\pi}^{\pi} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\
&\quad \left. - i\pi \omega(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_t r' \cos(y)} dy \right] \\
&\approx i^g 2\pi e^{-i\Delta_j z} \left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \sin\left[\vartheta \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] J_{\vartheta}((K_{new})_j r') \right. \\
&\quad \left. - i\pi \omega(k_0 n_{cl}, \Delta_j, L, r) \sin\left[(\vartheta + \nu)\frac{\pi}{2} - \nu\delta\right] J_{\vartheta}((\zeta - j)K_t r') \right]
\end{aligned} \tag{D.35}$$

If $X(\phi') = \cos(\eta) = \cos\left[\vartheta \cdot \phi' - \nu\left(\delta - \frac{\pi}{2}\right)\right]$ one obtains:

$$\begin{aligned}
\frac{\partial \Phi(j, r, \phi, z=0, r', \cos(\eta))}{\partial r} &\approx -e^{-i\Delta_j z} \left[2(k_{t2})_j K_1((k_{t2})_j r) \int_{-\pi}^{\pi} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\
&\quad \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_t r' \cos(y)} dy \right] \\
&\approx -i^g 2\pi e^{-i\Delta_j z} \left[2(k_{t2})_j K_1((k_{t2})_j r) \cos\left[\vartheta \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] J_{\vartheta}((K_{new})_j r') \right. \\
&\quad \left. + i\pi g(k_0 n_{cl}, \Delta_j, L, r) \cos\left[(\vartheta + \nu)\frac{\pi}{2} - \nu\delta\right] J_{\vartheta}((\zeta - j)K_t r') \right]
\end{aligned} \tag{D.36}$$

The last two integrals were solved previously in detail when calculating $\Phi(j, r, \phi, z=0, r', \cos(\eta))$ in C.2.3.

$$\begin{aligned}
\frac{\partial^2 \Phi(j, r, \phi, z=0, r', \cos(\eta))}{\partial r^2} &\approx \\
e^{-i\Delta_j z} &\left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \int_{-\pi}^{\pi} X(x_j + (\phi_{new})_j) e^{i(K_{new})_j r' \cos(x_j)} dx_j \right. \\
&\quad \left. - i\pi \omega(k_0 n_{cl}, \Delta_j, L, r) \int_{-\pi}^{\pi} X\left(y + \frac{\pi}{2}\right) e^{i(\zeta-j)K_t r' \cos(y)} dy \right] \\
&\approx i^g 2\pi e^{-i\Delta_j z} \left[2(k_{t2})_j^2 \left[\frac{K_1((k_{t2})_j r)}{(k_{t2})_j r} + K_0((k_{t2})_j r) \right] \cos\left[\vartheta \cdot (\phi_{new})_j - \nu\left(\delta - \frac{\pi}{2}\right)\right] J_{\vartheta}((K_{new})_j r') \right. \\
&\quad \left. - i\pi \omega(k_0 n_{cl}, \Delta_j, L, r) \cos\left[-\nu\delta + (\vartheta + \nu)\frac{\pi}{2}\right] J_{\vartheta}((\zeta - j)K_t r') \right]
\end{aligned} \tag{D.37}$$

Where the last two integrals have again been solved in the manner previously detailed during the calculation of $\Phi(j, r, \phi, z=0, r', \cos(\eta))$ in C.2.3.

D.3.3 Final calculations of the vector potential derivatives

As specified in C.3.2 because of limitations on asymptotic series it is desirable to calculate the $\partial/\partial r$ terms prior to integration. To this end, this section calculates $\frac{\partial A(r, \phi, z=0)}{\partial r}$ and $\frac{\partial^2 A(r, \phi, z=0)}{\partial r^2}$ as follows:

$$\frac{\partial \vec{A}_{co}(r, \phi, z=0)}{\partial r} \approx -i \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta} \pi J_\nu(u a_{core})} \sum_{j=0}^{2\zeta} \left(\begin{matrix} 2\zeta \\ j \end{matrix} \right) \left[\begin{matrix} \left(\begin{matrix} a_1 \int_0^a J_{\nu-1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_{\nu-1}(\phi'))}{\partial r} \cdot dr' \\ + a_2 \int_0^a J_{\nu+1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_{\nu+1}(\phi'))}{\partial r} \cdot dr' \end{matrix} \right) \hat{x} \\ - (-1)^\nu \left(\begin{matrix} a_1 \int_0^a J_{\nu-1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', G_{\nu-1}(\phi'))}{\partial r} \cdot dr' \\ - a_2 \int_0^a J_{\nu+1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', G_{\nu+1}(\phi'))}{\partial r} \cdot dr' \end{matrix} \right) \hat{y} \\ - \frac{i u}{\beta} \int_0^a J_\nu(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_\nu(\phi'))}{\partial r} \cdot dr' \hat{z} \end{matrix} \right] \quad (D.38)$$

$$\frac{\partial^2 \vec{A}_{co}(r, \phi, z=0)}{\partial r^2} \approx -i \frac{\omega \kappa \mu_0 E_1}{2^{2\zeta} \pi J_\nu(u a_{core})} \sum_{j=0}^{2\zeta} \left(\begin{matrix} 2\zeta \\ j \end{matrix} \right) \left[\begin{matrix} \left(\begin{matrix} a_1 \int_0^a J_{\nu-1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_{\nu-1}(\phi'))}{\partial r^2} \cdot dr' \\ + a_2 \int_0^a J_{\nu+1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_{\nu+1}(\phi'))}{\partial r^2} \cdot dr' \end{matrix} \right) \hat{x} \\ - (-1)^\nu \left(\begin{matrix} a_1 \int_0^a J_{\nu-1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', G_{\nu-1}(\phi'))}{\partial r^2} \cdot dr' \\ - a_2 \int_0^a J_{\nu+1}(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', G_{\nu+1}(\phi'))}{\partial r^2} \cdot dr' \end{matrix} \right) \hat{y} \\ - \frac{i u}{\beta} \int_0^a J_\nu(ur') r' \cdot \frac{\partial \Phi(j, r, \phi, z, r', F_\nu(\phi'))}{\partial r^2} \cdot dr' \hat{z} \end{matrix} \right] \quad (D.39)$$

Differentiating Eqs. (C.57) and (D.36) also allows one to also calculate:

$$\frac{\partial A(r, \phi, z=0)}{\partial \phi}, \quad \frac{\partial^2 A(r, \phi, z=0)}{\partial \phi^2}, \quad \frac{\partial^2 A(r, \phi, z=0)}{\partial r \partial \phi}$$

Expanding these formulas and making use of the relations of D.2 one obtains Eqs. (C.67)-(C.73).

D.3.4 Solutions for differential subcomponents

In the special case:

$$(\zeta - j) = 0 \quad (D.40)$$

$$(\phi_{new})_j = \phi \quad (D.41)$$

$$\frac{\partial(\phi_{new})_j}{\partial\phi} = 1 \quad (D.42)$$

$$\frac{\partial^2(\phi_{new})_j}{\partial\phi^2} = 0 \quad (D.43)$$

$$\frac{\partial(K_{new})_j}{\partial\phi} = 0 \quad (D.44)$$

$$\frac{\partial^2(K_{new})_j}{\partial\phi^2} = 0 \quad (D.45)$$

Otherwise in the general case one obtains:

$$\tan(\phi_{new})_j = \frac{(\varsigma - j)K_t - i(k_{t2})_j \sin \phi}{-i(k_{t2})_j \cos \phi} \text{ provides:} \quad (D.46)$$

$$\frac{\partial(\phi_{new})_j}{\partial\phi} \frac{\partial \tan(\phi_{new})_j}{\partial(\phi_{new})_j} = \frac{\partial}{\partial\phi} \left(\frac{(\varsigma - j)K_t - i(k_{t2})_j \sin \phi}{-i(k_{t2})_j \cos \phi} \right) \quad (D.47)$$

$$\begin{aligned} \frac{\partial(\phi_{new})_j}{\partial\phi} &= \cos^2(\phi_{new})_j \left(1 + \frac{(\varsigma - j)K_t \sin \phi - i(k_{t2})_j \sin^2 \phi}{-i(k_{t2})_j \cos^2 \phi} \right) \\ &= - \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \left(1 + \frac{(\varsigma - j)K_t \sin \phi}{-i(k_{t2})_j} \right) \end{aligned} \quad (D.48)$$

$$\begin{aligned} \frac{\partial^2(\phi_{new})_j}{\partial\phi^2} &= - \left[\frac{\partial(K_{new})_j}{\partial\phi} \left[\frac{\partial}{\partial(K_{new})_j} \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \right] \left(1 + \frac{(\varsigma - j)K_t \sin \phi}{-i(k_{t2})_j} \right) \right. \\ &\quad \left. + \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \frac{\partial}{\partial\phi} \left(1 + \frac{(\varsigma - j)K_t \sin \phi}{-i(k_{t2})_j} \right) \right] \\ &= - \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \frac{(\varsigma - j)K_t \cos \phi}{-i(k_{t2})_j} \left[1 + 2 \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \left(1 + \frac{(\varsigma - j)K_t \sin \phi}{-i(k_{t2})_j} \right) \right] \end{aligned} \quad (D.49)$$

And similarly from $(K_{new})_j = \sqrt{(\varsigma - j)^2 K_t^2 - (k_{t2})_j^2 - i2(\varsigma - j)K_t(k_{t2})_j \sin \phi}$ one obtains:

$$\begin{aligned} \frac{\partial(K_{new})_j}{\partial\phi} &= \frac{-i(\varsigma - j)K_t(k_{t2})_j \cos \phi}{(K_{new})_j} \\ \frac{\partial^2(K_{new})_j}{\partial\phi^2} &= - \left[\frac{\partial(K_{new})_j}{\partial\phi} \left[\frac{\partial}{\partial(K_{new})_j} \frac{i(\varsigma - j)K_t(k_{t2})_j}{(K_{new})_j} \right] \cos \phi + \frac{i(\varsigma - j)K_t(k_{t2})_j}{(K_{new})_j} \frac{\partial}{\partial\phi} \cos \phi \right] \\ &= \left(\frac{(k_{t2})_j}{(K_{new})_j} \right) i(\varsigma - j)K_t \left[\sin \phi - \left(\frac{(k_{t2})_j}{(K_{new})_j} \right)^2 \left[\frac{(\varsigma - j)K_t}{-i(k_{t2})_j} \right] \cos^2 \phi \right] \end{aligned} \quad (D.50)$$

If $(K_{new})_j = 0$, from phase matching arguments it is known that either $K_t = 0$ and $(k_{t2})_j = 0$, $\phi = \frac{\pi}{2}$ and $(\zeta - j)K_t = i(k_{t2})_j$, or $\phi = -\frac{\pi}{2}$ and $(\zeta - j)K_t = -i(k_{t2})_j$.

The first case applies only to non-blazed gratings with no radiation, there is no fixed relationship between $(\phi_{new})_j$ and ϕ , so $\frac{\partial(\phi_{new})_j}{\partial\phi}$ and $\frac{\partial^2(\phi_{new})_j}{\partial\phi^2}$ are not defined, although since $(K_{new})_j$ is constant, it is clear that $\frac{\partial(K_{new})_j}{\partial\phi} = 0$ and $\frac{\partial^2(K_{new})_j}{\partial\phi^2} = 0$. This point is mute anyway since $K_0((k_{t2})_j r)$ becomes infinite when $(k_{t2})_j = 0$. To deal with this numerically $(k_{t2})_j$ is allowed to approach 0 but not equal it.

In the second case, one can apply l'Hôpital's rule once:

$$\frac{\partial(K_{new})_j}{\partial\phi} = \frac{i(\zeta - j)K_t(k_{t2})_j \sin\phi}{\frac{\partial(K_{new})_j}{\partial\phi}} \rightarrow \frac{\partial(K_{new})_j}{\partial\phi} = (k_{t2})_j \sqrt{\sin\phi} = (k_{t2})_j \quad (D.51)$$

And then again twice consecutively:

$$\begin{aligned} \frac{\partial(\phi_{new})_j}{\partial\phi} &= - \left(\frac{(k_{t2})_j^2}{2(K_{new})_j \frac{\partial(K_{new})_j}{\partial\phi}} \right) \left(\frac{(\zeta - j)K_t \cos\phi}{-i(k_{t2})_j} \right) = - \left(\frac{(k_{t2})_j}{2(K_{new})_j} \right) \left(\frac{(\zeta - j)K_t \cos\phi}{-i(k_{t2})_j} \right) \\ &= \left(\frac{(k_{t2})_j}{2 \frac{\partial(K_{new})_j}{\partial\phi}} \right) \left(\frac{(\zeta - j)K_t \sin\phi}{-i(k_{t2})_j} \right) = -\frac{1}{2} \end{aligned} \quad (D.52)$$

From which one also obtains:

$$\frac{\partial^2(K_{new})_j}{\partial\phi^2} = 0 \quad (D.53)$$

$$\frac{\partial^2(\phi_{new})_j}{\partial\phi^2} = 0 \quad (D.54)$$

Appendix E

Simplified Field Expressions For The Fundamental Mode

E.1 Core scattered magnetic field

Assuming $L = \infty$, taking only the summation component with strongest phase matching $j = (\zeta - m)$ (see 6.2.5), neglecting the higher order $1/r$ terms, one can simplify the equations of C.3.3 as follows:

$$\begin{aligned} \vec{B}_{inco}(r, \phi, z=0) \approx & \\ & \left[\begin{aligned} & \left(a_1 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. + a_2 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{r} \\ & \left(a_1 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. - a_2 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{\phi} \\ & + i \frac{u}{\beta} B_{f2} \cos[\nu((\phi_{new})_{\zeta-m} - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_{\zeta-m} r') r' dr' \\ & + i B_{f2} \cdot \left(a_1 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. + a_2 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{z} \end{aligned} \right] \quad (E.1) \end{aligned}$$

$$B_{f1} \equiv i \left(\frac{2\zeta}{\zeta - m} \right) \frac{\omega \kappa \mu_0 E_1 K_0 ((k_{t2})_{\zeta-m} r)}{2^{2(\zeta-1)} J_{\nu}(ua_{core})} e^{-i\Delta_{\zeta-m} z} \quad (E.2)$$

$$B_{f2} \equiv \frac{(k_{t2})_{\zeta-m} K_1((k_{t2})_{\zeta-m} r)}{\Delta_{\zeta-m} K_0((k_{t2})_{\zeta-m} r)} \quad (E.3)$$

where the integrals remain directly solvable using 5.54-1 of [84].

Assuming $\nu=1$, $\zeta=1$, $m=1$, and recalling that substitution of $a_1 \approx -1$, $a_2 \approx 0$ and $E_0 = E_1 / J_1(ua_{core})$, into the HE field expressions of C.1.2 yields the guided electric field amplitudes used by Li (Eq. 3.2 of [13] and Eq. 1 of [18]), one obtains:

$$\vec{B}_{inco}(r, \phi, z=0) \approx -F \cdot B_{f1} \cdot \Delta_{\zeta-m} \cdot \left[\begin{aligned} & \sin(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \hat{r} \\ & \left(\cos(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \right. \\ & \left. - i \frac{u}{\beta} B_{f2} \cos(\phi_{new} - \delta) \int_0^a J_1(ur') J_1(K_{new} r') r' dr' \right) \hat{\phi} \\ & + i B_{f2} \sin(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \hat{z} \end{aligned} \right] \quad (E.4)$$

Provided the $\int_0^a J_1(ur')J_1(K_{new}r')r' dr'$ term is negligible, solving the integral and replacing the Bessel functions with single term asymptotic expansions yields the same result as Eq. 3.18 of [13], noting that $(k_{t2})_0 = ik_t$ when $\Delta_0 < k_0 n_{cl}$, and assuming $n_0 \approx n_{cl}$.

$$\begin{aligned} \bar{B}_{inco}(r, \phi, z=0) \approx & \left[-\frac{\omega\mu_0 KE_0}{2} e^{i\frac{\pi}{4}} \sqrt{\frac{2\pi}{k_t r}} e^{-ik_t r} \cdot \frac{K_{new} a J_0(ua) J_1(K_{new} a) - ua J_1(ua) J_0(K_{new} a)}{K_{new}^2 - u^2} \right] \\ & \cdot e^{-i\Delta z} \cdot [i\Delta \sin(\delta - \phi) \hat{r} - i\Delta \cos(\phi - \delta) \hat{\phi} - ik_t \sin(\delta - \phi) \hat{z}] \\ \approx & A \cdot [i\Delta \sin(\delta - \phi) \hat{r} - i\Delta \cos(\phi - \delta) \hat{\phi} - ik_t \sin(\delta - \phi) \hat{z}] \end{aligned} \quad (E.5)$$

For the fundamental mode, $n_{eff} \rightarrow n_0$ and $\frac{u}{\beta} = \sqrt{\frac{n_0^2}{n_{eff}^2} - 1} \rightarrow 0$, so there is an argument to be made for this in many cases, however when the longitudinal tap angle ξ_{out} approaches 90° , $\Delta_{\zeta-m} \rightarrow 0$ and the $\int_0^a J_1(ur')J_1(K_{new}r')r' dr'$ term becomes much more significant.

E.2 Cladding scattered magnetic field

Assuming $L = \infty$, taking only the summation component with strongest phase matching $j = (\zeta-m)$ (see 6.2.5), neglecting the higher order $1/r$ terms, one can simplify the equations of C.3.4 as follows:

$$\begin{aligned} \bar{B}_{incl}(r, \phi, z=0) \approx & \frac{U}{W} \frac{J_{\nu}(ua_{core})}{K_{\nu}(wa_{core})} \times \\ & \left[\begin{aligned} & \left(a_1 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. - a_2 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{r} \\ & \left(a_1 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. + a_2 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{\phi} \\ & + i \frac{w}{\beta} B_{f2} \cos[\nu((\phi_{new})_{\zeta-m} - \delta)] \int_0^a K_{\nu}(wr') J_{\nu}((K_{new})_{\zeta-m} r') r' dr' \\ & + i B_{f2} \left(a_1 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \right. \\ & \left. - a_2 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \right) \hat{z} \end{aligned} \right] \end{aligned} \quad (E.6)$$

where the integrals remain directly solvable using 5.54-1 of [84].

E.3 Core scattered electric field

Assuming $L = \infty$, taking only the summation component with strongest phase matching $j = (\varsigma-m)$ (see 6.2.5), neglecting the higher order $1/r$ terms, one can simplify the equations of C.3.5 as follows:

$$\vec{E}_{inco}(r, \phi, z=0) \approx \frac{F \cdot B_{f1}}{\omega \epsilon \mu_0} \cdot \left[\Delta_{\varsigma-1}^2 \begin{pmatrix} a_1 \cos[\nu((\phi_{new})_{\varsigma-1} - \delta) - (\phi_{new})_{\varsigma-1} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\varsigma-1} r') r' dr' \\ -a_2 \cos[\nu((\phi_{new})_{\varsigma-1} - \delta) + (\phi_{new})_{\varsigma-1} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\varsigma-1} r') r' dr' \\ + i \frac{u}{\beta} B_{f2} \cos[\nu((\phi_{new})_{\varsigma-1} - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_{\varsigma-1} r') r' dr' \end{pmatrix} \hat{r} \right. \\ \left. - k_0^2 n_{cl}^2 \begin{pmatrix} a_1 \sin[\nu((\phi_{new})_{\varsigma-m} - \delta) - (\phi_{new})_{\varsigma-m} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\varsigma-m} r') r' dr' \\ + a_2 \sin[\nu((\phi_{new})_{\varsigma-m} - \delta) + (\phi_{new})_{\varsigma-m} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\varsigma-m} r') r' dr' \end{pmatrix} \hat{\phi} \right. \\ \left. + \begin{pmatrix} i \Delta_{\varsigma-1}^2 B_{f2} \cdot \begin{pmatrix} a_1 \cos[\nu((\phi_{new})_{\varsigma-m} - \delta) - (\phi_{new})_{\varsigma-m} + \phi] \int_0^a J_{\nu-1}(ur') J_{\nu-1}((K_{new})_{\varsigma-m} r') r' dr' \\ -a_2 \cos[\nu((\phi_{new})_{\varsigma-m} - \delta) + (\phi_{new})_{\varsigma-m} - \phi] \int_0^a J_{\nu+1}(ur') J_{\nu+1}((K_{new})_{\varsigma-m} r') r' dr' \end{pmatrix} \hat{z} \right. \\ \left. - (k_{t2})_{\varsigma-m}^2 \frac{u}{\beta} \cos[\nu((\phi_{new})_{\varsigma-m} - \delta)] \int_0^a J_{\nu}(ur') J_{\nu}((K_{new})_{\varsigma-m} r') r' dr' \right] \quad (E.7)$$

where the integrals remain directly solvable using 5.54-1 of [84].

Assuming $\nu=1$, $\varsigma=1$, $m=1$, and recalling that substitution of $a_1 \approx -1$, $a_2 \approx 0$ and $E_0 = E_1 / J_1(ua_{core})$, into the HE field expressions of C.1.2 yields the guided electric field amplitudes used by Li (Eq. 3.2 of [13] and Eq. 1 of [18]), one obtains:

$$\vec{E}_{inco}(r, \phi, z=0) \approx \frac{F \cdot B_{f1}}{\omega \epsilon \mu_0} \cdot \left[\Delta_{\varsigma-1}^2 \begin{pmatrix} -\cos(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \\ + i \frac{u}{\beta} B_{f2} \cos(\phi_{new} - \delta) \int_0^a J_1(ur') J_1(K_{new} r') r' dr' \end{pmatrix} \hat{r} \right. \\ \left. + k_0^2 n_{cl}^2 \sin(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \hat{\phi} \right. \\ \left. - \begin{pmatrix} i \Delta_{\varsigma-1}^2 B_{f2} \cdot \cos(\phi - \delta) \int_0^a J_0(ur') J_0(K_{new} r') r' dr' \\ + (k_{t2})_{\varsigma-m}^2 \frac{u}{\beta} \cos(\phi_{new} - \delta) \int_0^a J_1(ur') J_1(K_{new} r') r' dr' \end{pmatrix} \hat{z} \right] \quad (E.8)$$

Provided the $\int_0^a J_1(ur') J_1(K_{new} r') r' dr'$ terms are negligible, solving the integral and replacing the Bessel functions with single term asymptotic expansions yields the same result as Eq. 3.19 of [13], noting that $(k_{t2})_0 = ik_t$ when $\Delta_0 < k_0 n_{cl}$, and assuming $n_0 \approx n_{cl}$.

$$\begin{aligned}
\vec{E}_{inco}(r, \phi, z) &\approx \left(i \frac{\kappa E_0}{2\varepsilon} e^{\frac{i\pi}{4}} \sqrt{\frac{2\pi}{k_r}} e^{-ik_r r} \cdot \frac{K_{new} a J_0(ua) J_1(K_{new} a) - ua J_1(ua) J_0(K_{new} a)}{K_{new}^2 - u^2} \right) \\
&\cdot e^{-i\Delta z} \cdot \left[\Delta_0^2 \cos(\phi - \delta) \hat{r} - k_0^2 n_0^2 \sin(\phi - \delta) \hat{\phi} + i(k_{t2})_0 \Delta_0 \cos(\phi - \delta) \hat{z} \right] \\
&\approx \frac{A}{i\omega\mu_0\varepsilon} \cdot \left[\Delta_0^2 \cos(\phi - \delta) \hat{r} - k_0^2 n_0^2 \sin(\phi - \delta) \hat{\phi} + i(k_{t2})_0 \Delta \cos(\phi - \delta) \hat{z} \right] \\
&\approx \frac{A}{i\omega\mu_0\varepsilon} \cdot \left[\Delta^2 \cos(\delta - \phi) \hat{r} + k_0^2 n_0^2 \sin(\delta - \phi) \hat{\phi} - k_t \Delta \cos(\delta - \phi) \hat{z} \right] \quad (E.9)
\end{aligned}$$

For the fundamental mode, $n_{eff} \rightarrow n_0$ and $\frac{u}{\beta} = \sqrt{\frac{n_0^2}{n_{eff}^2} - 1} \rightarrow 0$, so there is an argument for this in many cases, however when the longitudinal tap angle ξ_{out} approaches 90° , $\Delta_{\zeta-m} \rightarrow 0$ and the $\int_0^a J_1(ur') J_1(K_{new} r') r' dr'$ terms become much more significant.

E.4 Cladding scattered electric field

Assuming $L = \infty$, taking only the summation component with strongest phase matching $j = (\zeta-m)$ (see 6.2.5), neglecting the higher order $1/r$ terms, one can simplify the equations of C.3.6 as follows:

$$\begin{aligned}
\vec{E}_{incl}(r, \phi, z=0) &\approx \frac{U}{W} \frac{J_v(wa_{core})}{K_v(wa_{core})} \times \\
&\left[\Delta_{\zeta-1}^2 \cdot \left(\begin{aligned} &a_1 \cos[\nu((\phi_{new})_{\zeta-1} - \delta) - (\phi_{new})_{\zeta-1} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-1} r') r' dr' \\ &+ a_2 \cos[\nu((\phi_{new})_{\zeta-1} - \delta) + (\phi_{new})_{\zeta-1} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-1} r') r' dr' \\ &+ i \frac{W}{\beta} B_{f2} \cos[\nu((\phi_{new})_{\zeta-1} - \delta)] \int_0^a K_{\nu}(wr') J_{\nu}((K_{new})_{\zeta-1} r') r' dr' \end{aligned} \right) \hat{r} \right. \\
&\frac{F \cdot B_{f1}}{\omega \varepsilon \mu_0} \cdot \left(\begin{aligned} &a_1 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \\ &- a_2 \sin[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \end{aligned} \right) \hat{\phi} \\
&+ \left(\begin{aligned} &i \Delta_{\zeta-1}^2 B_{f2} \cdot \left(\begin{aligned} &a_1 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) - (\phi_{new})_{\zeta-m} + \phi] \int_0^a K_{\nu-1}(wr') J_{\nu-1}((K_{new})_{\zeta-m} r') r' dr' \\ &+ a_2 \cos[\nu((\phi_{new})_{\zeta-m} - \delta) + (\phi_{new})_{\zeta-m} - \phi] \int_0^a K_{\nu+1}(wr') J_{\nu+1}((K_{new})_{\zeta-m} r') r' dr' \end{aligned} \right) \hat{z} \\ &- (k_{t2})_{\zeta-m}^2 \frac{W}{\beta} \cos[\nu((\phi_{new})_{\zeta-m} - \delta)] \int_0^a K_{\nu}(wr') J_{\nu}((K_{new})_{\zeta-m} r') r' dr' \end{aligned} \right) \hat{z} \right] \quad (E.10)
\end{aligned}$$

where the integrals remain directly solvable using 5.54-1 of [84].

Appendix F

Analysis of Approximation Errors

F.1 Errors due to approximations for d

Just what exactly is meant by “sufficiently small” when speaking about r'/r ? The relative errors associated with the approximations for d are given by:

$$\varepsilon_{d=r} = \left(\frac{r - r' \sqrt{1 - 2 \left(\frac{r'}{r} \right) \cos(\phi - \phi') + \left(\frac{r'}{r} \right)^2}}{r \sqrt{1 - 2 \left(\frac{r'}{r} \right) \cos(\phi - \phi') + \left(\frac{r'}{r} \right)^2}} \right)_{\max} \leq \frac{1}{\left(\frac{a}{a_{clad}} \right)^{-1} - 1} \quad (\text{F.1})$$

and:

$$\varepsilon_{d=r-r'\cos(\phi-\phi')} = \left| \frac{[r - r' \cos(\phi - \phi')] - d}{d} \right|_{\max} \leq 1 - \sqrt{1 - \left(\frac{a}{a_{clad}} \right)^2} \quad (\text{F.2})$$

For gratings confined to the core of SMF-28 one then obtains:

$$\varepsilon_{d=r} \leq \frac{1}{\left(\frac{4.1}{62.5} \right)^{-1} - 1} \approx 7\% \quad (\text{F.3})$$

and:

$$\varepsilon_{d=r-r'\cos(\phi-\phi')} \leq 1 - \sqrt{1 - \left(\frac{4.1}{62.5} \right)^2} \approx 0.2\% \quad (\text{F.4})$$

Even though these quantities can appear to be relatively large, particularly for larger grating radii (a), it is important to remember that they represent maximum values, and over the range of integration should average out to a much lower percentage of the total result.

As well, for modes guided primarily within the fibre core, the strength of scattering sources further from the fibre axis drops off quickly, making their contributions to the radiation field relatively weak. In such cases, larger errors may be more tolerable.

F.2 Errors introduced by the approximations for $K_\nu((k_{t2})_j d)$

F.2.1 Direct evaluation of the relative error

The relative error between the left and right sides of the expression can be written as:

$$\varepsilon_{K_\nu((k_{t2})_j d)} = \left| 1 - \frac{K_\nu((k_{t2})_j r) e^{(k_{t2})_j r' \cos(\phi - \phi')}}{K_\nu((k_{t2})_j d)} \right| \quad (\text{F.5})$$

Since $d \approx r - r' \cos(\phi - \phi')$,

$$\begin{aligned}\varepsilon_{K_\nu((k_{i2})_j d)} &\approx \left| 1 - \frac{K_\nu((k_{i2})_j r) e^{(k_{i2})_j (r-d)}}{K_\nu((k_{i2})_j d)} \right| \\ &\approx \left| 1 - \frac{K_\nu((k_{i2})_j d \frac{r}{d}) e^{(k_{i2})_j d (\frac{r}{d}-1)}}{K_\nu((k_{i2})_j d)} \right|\end{aligned}\quad (\text{F.6})$$

It has already been established that the greatest relative error in the approximations for d is given by $\varepsilon_{d=r} = \left(\frac{r}{d} - 1\right)$, hence:

$$\varepsilon_{K_\nu((k_{i2})_j d)} \approx \left| 1 - \frac{K_\nu((k_{i2})_j d (\varepsilon_{d=r} + 1)) e^{(k_{i2})_j d \varepsilon_{d=r}}}{K_\nu((k_{i2})_j d)} \right| \quad (\text{F.7})$$

F.2.2 When $(k_{i2})_j$ is Imaginary

Plotting this for $\nu = 0$ and $\nu = 1$ shows that $\varepsilon_{K_\nu((k_{i2})_j d)}$ increases with increasing $\varepsilon_{d=r}$.

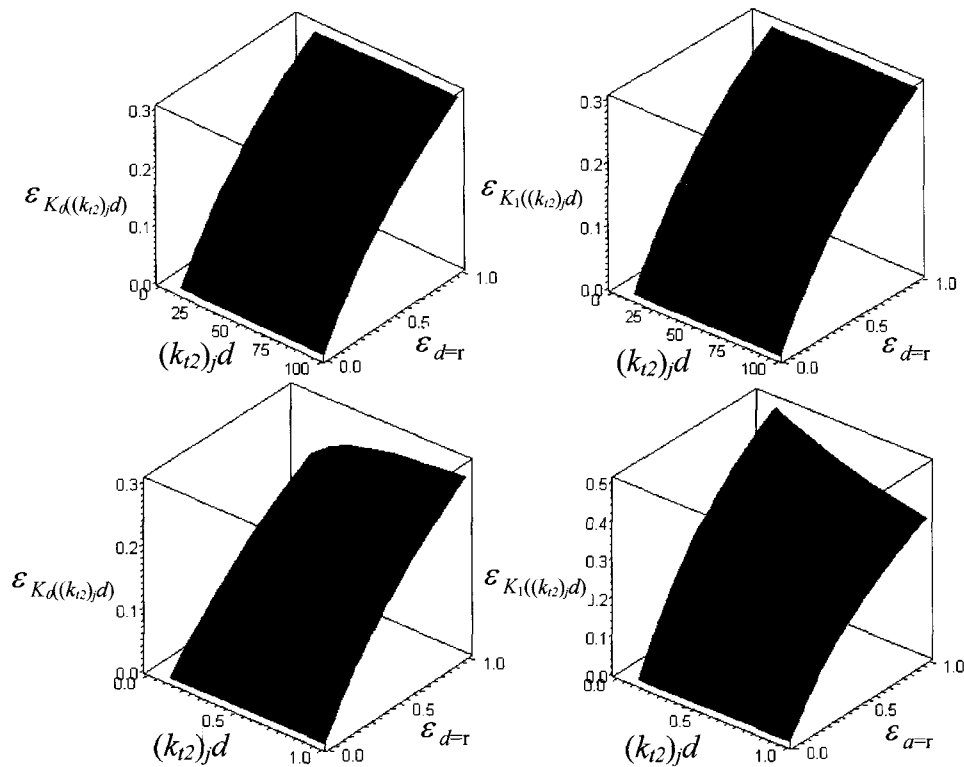


Fig. F.1 – Relative error in approximation for $K_\nu((k_{i2})_j d)$. Imaginary $(k_{i2})_j$.

One can also observe from the right side of each plot, that the choice of $(k_{i2})_j d$ has relatively little effect when it is large, although for smaller values decreasing $(k_{i2})_j d$ magnifies the error for $\nu = 1$ and reduces it for $\nu = 0$.

In order to get some idea of the extremes possible the following relationships are plotted:

$$\varepsilon_{K_0((k_{i2})_j d)} \approx \left| 1 - \frac{K_0(10^{-6} i(\varepsilon_{d=r} + 1)) e^{i 10^{-6} \varepsilon_{d=r}}}{K_0(10^{-6} i)} \right|, \quad \varepsilon_{K_1((k_{i2})_j d)} \approx \left| 1 - \frac{K_1(10^{-6} i(\varepsilon_{d=r} + 1)) e^{i 10^{-6} \varepsilon_{d=r}}}{K_1(i 10^{-6})} \right|$$

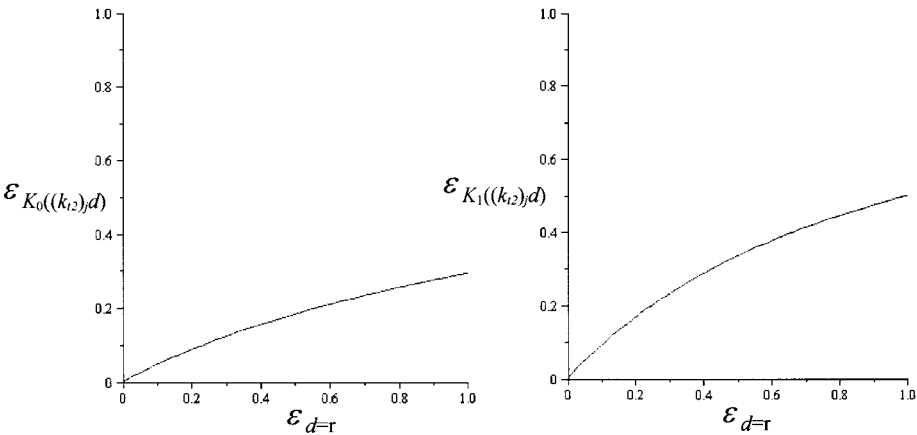


Fig. F.2 – Error in $K_\nu((k_{i2})_j d)$ versus the error in d . Imaginary $(k_{i2})_j$.

Hence, even for very small r (large r'/r and $\varepsilon_{d=r}$) the error in the Bessel function approximations will be less than any errors in the approximations for d .

F.2.3 When $(k_{i2})_j$ is Real

One observes almost identical results.

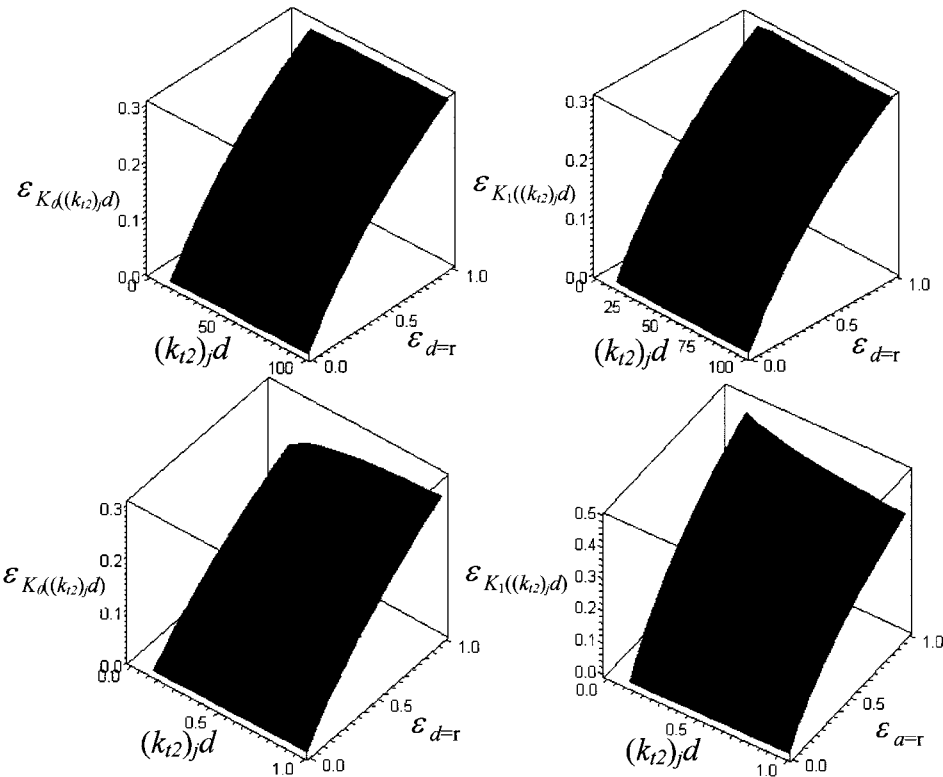


Fig. F.3 – Relative error in approximation for $K_\nu((k_{i2})_j d)$. Real $(k_{i2})_j$.

And for the following relationships:

$$\varepsilon_{K_0((k_{i2})_j d)} \approx \left| 1 - \frac{K_0(10^{-6}(\varepsilon_{d=r} + 1))e^{10^{-6}\varepsilon_{d=r}}}{K_0(10^{-6})} \right|, \quad \varepsilon_{K_1((k_{i2})_j d)} \approx \left| 1 - \frac{K_1(10^{-6}(\varepsilon_{d=r} + 1))e^{10^{-6}\varepsilon_{d=r}}}{K_1(10^{-6})} \right|$$

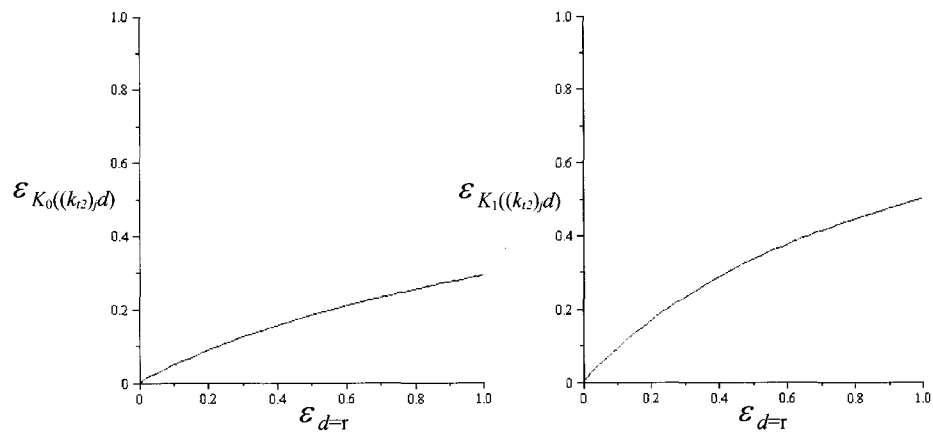


Fig. F.4 – Error in $K_\nu((k_{i2})_j d)$ versus the error in d . Real $(k_{i2})_j$.

Thus the former restrictions on $(k_{i2})_j r$ ’s magnitude, that existed because of the presence of asymptotic Bessel function approximations in the final result, have effectively been removed.

F.3 Errors associated with asymptotic series

The asymptotic expansion of the Bessel function K_ν is an alternating series, and as discussed in 0.330 of [84], the greatest accuracy of such a series is obtained if truncation occurs beginning with the term of lowest absolute value. This yields a maximum error equal to the magnitude of the first discarded term.

Upon careful examination it becomes apparent that the optimum number of terms is dependent on the value of $(k_{i2})_j d$, with the error being reduced as the magnitude of $|(k_{i2})_j d|$ increases. More specifically for $K_0((k_{i2})_j d)$ one finds:

Table F.1 - Error in asymptotic expansion of $H_0^{(2)}((k_{i2})_j d)$ as a function of $(k_{i2})_j d$.

$ (k_{i2})_j d $	$(k_{i2})_j d$ is Imaginary		$(k_{i2})_j d$ is Real	
	Optimum # of Terms	Maximum Relative Error	Optimum # of Terms	Maximum Relative Error
>1	2	10.1 %	2	8.04 %
>1.25	2	5.83 %	3	3.97 %
>1.5	3	3.19 %	3	2.39 %
>1.75	3	1.85 %	4	1.28 %
>2	3	1.01 %	4	0.741 %
>3	5	0.112 %	6	0.0815 %
>4	7	0.0132 %	8	0.00953 %
>5	10	0.00160 %	10	0.00115 %

It is noted that Li’s derivation is only based on a single term expansion, so some significant improvement is possible especially for lower values of $k_i d$.

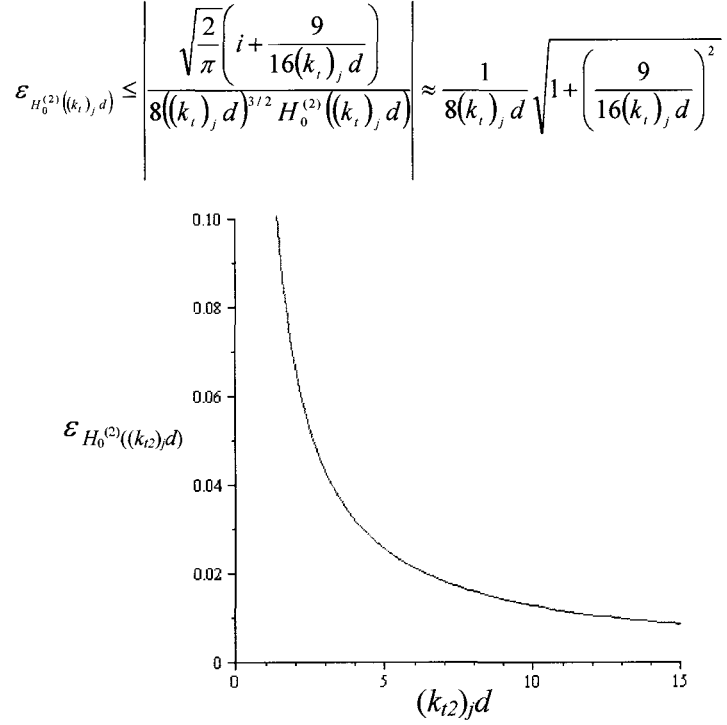


Fig. F.5 – Error in asymptotic expansion of $H_0^{(2)}((k_{i2})_j d)$ as a function of $(k_i)_j d$.

Error in the approximation for d will have a greater affect on the accuracy of these series when $k_i d$ is small and the error slope is large. Thus, when $k_i d$ is small it is even more important to minimize such errors by ensuring that (r'/r) is also small.

As demonstrated in F.2, by converting the asymptotic series in d back into a multiple of the asymptotic series in r , the restriction on $(k_{i2})_j d$'s magnitude can be eliminated and the error reduced to less than that introduced by the approximation for d .

F.4 Approximations made in calculation of finite length integral

It is desirable to rewrite the integrands of the second term functions into more workable forms. To this end one can make use of the following Maclaurin series expansions in the variable $(1/q)$:

$$\begin{aligned} \sqrt{1+q^2} &\approx q + \left(\frac{1}{2q} - \frac{1}{8q^3} + \frac{1}{16q^5} - \frac{5}{128q^7} + \frac{7}{256q^9} + \dots \right) \\ &\approx q + \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right)}{2\sqrt{\pi}(k+1)!} \frac{1}{q^{2k+1}} = q + \sum_{k=0}^{\infty} a_k \end{aligned} \quad (F.8)$$

$$\begin{aligned} \frac{1}{\sqrt{1+q^2}} &\approx \frac{1}{q} \left(1 - \frac{1}{2q^2} + \frac{3}{8q^4} - \frac{5}{16q^6} + \frac{35}{128q^8} - \frac{63}{256q^{10}} + \dots \right) \\ &\approx \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi} k!} \frac{1}{q^{2k+1}} = \sum_{k=0}^{\infty} b_k \end{aligned} \quad (F.9)$$

$$\begin{aligned}\frac{1}{1+q^2} &\approx \frac{1}{q^2} \left(1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6} + \frac{1}{q^8} - \frac{1}{q^{10}} + \dots \right) \\ &\approx \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{q^{2k+2}} \right] = \sum_{k=0}^{\infty} c_k\end{aligned}\quad (\text{F.10})$$

$$\begin{aligned}\frac{1}{(1+q^2)^{3/2}} &\approx \frac{1}{q^3} \left(1 - \frac{3}{2q^2} + \frac{15}{8q^4} - \frac{35}{16q^6} + \frac{315}{128q^8} - \frac{693}{256q^{10}} + \dots \right) \\ &\approx \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot 2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \frac{1}{q^{2k+3}} \right] = \sum_{k=0}^{\infty} d_k\end{aligned}\quad (\text{F.11})$$

$$\begin{aligned}\frac{1}{(1+q^2)^2} &\approx \frac{1}{q^4} \left(1 - \frac{2}{q^2} + \frac{3}{q^4} - \frac{4}{q^6} + \frac{5}{q^8} - \frac{6}{q^{10}} + \dots \right) \\ &\approx \sum_{k=0}^{\infty} \left[\frac{(-1)^k (k+1)}{q^{2k+4}} \right] = \sum_{k=0}^{\infty} e_k\end{aligned}\quad (\text{F.12})$$

$$\begin{aligned}\frac{1}{(1+q^2)^{5/2}} &\approx \frac{1}{q^5} \left(1 - \frac{5}{2q^2} + \frac{35}{8q^4} - \frac{105}{16q^6} + \frac{1155}{128q^8} - \frac{3003}{256q^{10}} + \dots \right) \\ &\approx \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cdot 4 \cdot \Gamma\left(\frac{2k+5}{2}\right)}{3\sqrt{\pi} k! q^{2k+5}} \right] = \sum_{k=0}^{\infty} f_k\end{aligned}\quad (\text{F.13})$$

Note that these series converge absolutely provided:

$$\begin{aligned}\lim_{k \rightarrow \infty} a_k = 0 &= \lim_{k \rightarrow \infty} \left[\frac{(-1)^k \Gamma\left(\frac{2k+1}{2}\right)}{2\sqrt{\pi} (k+1)! q^{2k+1}} \right] \approx \pm \frac{1}{q^\infty}, \quad \lim_{k \rightarrow \infty} b_k = 0 = \lim_{k \rightarrow \infty} \left[\frac{(-1)^k \Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi} k! q^{2k+1}} \right] \approx \pm \frac{1}{q^\infty} \\ \lim_{k \rightarrow \infty} c_k = 0 &= \lim_{k \rightarrow \infty} \left[\frac{(-1)^k}{q^{2k+2}} \right] \approx \pm \frac{1}{q^\infty}, \quad \lim_{k \rightarrow \infty} d_k = 0 = \lim_{k \rightarrow \infty} \left[\frac{(-1)^k \cdot 2 \cdot \Gamma\left(\frac{2k+3}{2}\right)}{\sqrt{\pi} k!} \frac{1}{q^{2k+3}} \right] \approx \pm \frac{1}{q^\infty} \\ \lim_{k \rightarrow \infty} e_k = 0 &= \lim_{k \rightarrow \infty} \left[\frac{(-1)^k (k+1)}{q^{2k+4}} \right] \approx \pm \frac{\infty}{q^\infty}, \quad \lim_{k \rightarrow \infty} f_k = 0 = \lim_{k \rightarrow \infty} \left[\frac{(-1)^k \cdot 4 \cdot \Gamma\left(\frac{2k+5}{2}\right)}{3\sqrt{\pi} k! q^{2k+5}} \right] \approx \pm \frac{1}{q^\infty}\end{aligned}$$

As indicated by 0.330 of [84], the error for an alternating series is less than the first truncated term provided the term magnitudes are perpetually decreasing, so:

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{2k+1}{2k+4} \frac{1}{q^2} < 1, \quad \left| \frac{b_{k+1}}{b_k} \right| = \frac{2k+1}{2k+2} \frac{1}{q^2} < 1, \quad \left| \frac{c_{k+1}}{c_k} \right| = \frac{1}{q^2} < 1$$

$$\left| \frac{d_{k+1}}{d_k} \right| = \frac{2k+3}{2k+2} \frac{1}{q^2} < 1, \quad \left| \frac{b_{k+1}}{b_k} \right| = \frac{k+2}{k+1} \frac{1}{q^2} < 1, \quad \left| \frac{c_{k+1}}{c_k} \right| = \frac{2k+5}{2k+2} \frac{1}{q^2} < 1$$

All of these expressions are satisfied when $q > \left(\sqrt{\frac{2k+5}{2k+2}} \right)_{\max} = \sqrt{\frac{5}{2}}$ and since $q = \frac{z'}{d} \geq \frac{L}{2d}$ for the integrals in question one can write:

$$e^{-ia\sqrt{1+q^2}} \approx e^{-iaq}$$

Where the relative error in the exponent is given by:

$$\varepsilon_{\sqrt{1+q^2}} \leq \frac{1}{2q\sqrt{1+q^2}}$$

For an error of less than 1% then:

$$\begin{aligned} 0 &\leq q^4 + q^2 - 2500 \\ q^2 &\geq \frac{-1 + \sqrt{1+10000}}{2} \approx 49.503 \\ q &\geq \frac{L}{2d_{\max}} \geq 7.04 \\ L &\geq 14.08(a_{\text{clad}} + a_{\text{core}}) \\ L_{\text{SMF-28}} &\geq 1 \text{ mm} \end{aligned}$$

Relaxing the maximum error tolerance to 10% the criteria becomes:

$$\begin{aligned} 0 &\leq q^4 + q^2 - 25 \\ q^2 &\geq \frac{-1 + \sqrt{1+100}}{2} \approx 4.525 \\ q &\geq \frac{L}{2d_{\max}} \geq 2.13 \\ L &\geq 4.26(a_{\text{clad}} + a_{\text{core}}) \\ L_{\text{SMF-28}} &\geq 0.3 \text{ mm} \end{aligned}$$

In order to simplify the integration process one is limited to a single term for the exponential expansion, however the same limitation is not necessary for the other series, and as such all of their terms will be retained for now, allowing a numerical solver to select the appropriate number for error minimization. For extremely short gratings it may become desirable to solve the finite length correction terms directly by numerical means.

Nonetheless, combining the first two series expansions yields:

$$\frac{e^{-ia\sqrt{1+q^2}}}{\sqrt{1+q^2}} \approx \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi} k!} \frac{e^{-iaq}}{q^{2k+1}} \quad (\text{F.14})$$

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