Grothendieck group decategorifications and derived abelian categories

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Abstract

The Grothendieck group is an interesting invariant of an exact category. It induces a decategorification from the category of essentially small exact categories (whose morphisms are exact functors) to the category of abelian groups. Similarly, the triangulated Grothendieck group induces a decategorification from the category of essentially small triangulated categories (whose morphisms are triangulated functors) to the category of abelian groups. In the case of an essentially small abelian category, its Grothendieck group and the triangulated Grothendieck group of its bounded derived category are isomorphic as groups via a natural map. Because of this, homological algebra and derived functors become useful in surprising ways. This thesis is an expository work that provides an overview of the theory of Grothendieck groups with respect to these decategorifications.

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I finally thank Canada for the interest and value we give to mathematics. None of this would be possible if we, as a country, did not respect and nourish the pursuit of knowledge.

—Aaron K. McBride

Contents

List of Symbols

$\mathcal{A}[S^{-1}]$	The localization of a category $\mathcal{A}$ by a localizing family of morphisms; see Definition 3.4.2.
$\mathbf{Add}(\mathcal{A}, \mathcal{B})$	The category of additive functors from an additive category $\mathcal{A}$ to an additive category $\mathcal{B}$; see Example 1.1.28.
$\mathcal{C}^{\mathrm{op}}$	The opposite category of a category $\mathcal{C}$; see Example 1.1.15.
$C(f)$	The mapping cone of a morphism of cochain complexes f; see Definition 3.2.9.
$\operatorname{cod} f$	The codomain of a morphism f in some category clear from the context.
$D^*(\mathcal{A})$	$(* \in \{\mathrm{b}, \mathrm{ub}, +, -\})$ The relevant subcategories of the derived category of an abelian category; see Definition 3.4.15.
$\operatorname{dom} f$	The domain of a morphism f in some category clear from the context.
$\mathbf{Fct}(\mathcal{C}, \mathcal{D})$	The functor category from a category $\mathcal{C}$ to a category $\mathcal{D}$; see Example 1.1.17.
$F_{\operatorname{Iso}\mathcal{C}}$	The free abelian group generated by the isomorphism classes of an essentially small category $\mathcal{C}$; see Definition 2.1.1.
$G_0(R)$	The Grothendieck group of R-$\mathbf{mod}^{\mathrm{fg}}$ for some ring with unity R; see Definition 2.2.7.

$\operatorname{Gr}^* \mathcal{A}$	$(* \in \{\mathrm{b}, \mathrm{ub}, +, -\})$ The relevant subcategories of the graded category of an abelian category $\mathcal{A}$; see Definition 3.1.1.
$\operatorname{Ho}^* \mathcal{A}$	$(* \in \{\mathrm{b}, \mathrm{ub}, +, -\})$ The relevant subcategories of the homotopy category of an abelian category $\mathcal{A}$; see Definition 3.2.5.
$\hom_{\mathcal{A}}$	The hom-functor of an additive category $\mathcal{A}$; see Example 1.1.23.
$\hom_{\mathcal{C}}(X, Y)$	The hom-set of homomorphisms from X to Y in a category $\mathcal{C}$.
$H^n(X)$	The n-th cohomology object of a cochain complex $(X^\bullet, d_X)$; see Definition 3.3.3.
id_X	The identity morphism of X in a category clear from the context.
$\operatorname{Inj} \mathcal{A}$	The collection of injective objects in an abelian category $\mathcal{A}$; see Definition 1.2.21.
$\operatorname{Iso} \mathcal{C}$	The set of isomorphism classes of an essentially small category $\mathcal{C}$; see Definition 1.1.4.
$K_0(\mathcal{A})$	The Grothendieck group of an essentially small exact category $\mathcal{A}$; see Definition 2.1.2.
$K_0^{\mathrm{split}}(\mathcal{A})$	The split Grothendieck group of an essentially small additive category $\mathcal{A}$; see Definition 2.1.4.
$K_0^{\triangle}(\mathcal{A})$	The triangulated Grothendieck group of an essentially small triangulated category $\mathcal{A}$.
$K_0(R)$	The Grothendieck group of R-$\mathbf{pmod}^{\mathrm{fg}}$ for some ring with unity R; see Definition 2.2.7.
$\operatorname{Kom}^* \mathcal{A}$	$(* \in \{\mathrm{b}, \mathrm{ub}, +, -\})$ The relevant subcategories of the category of cochain complexes in an abelian category $\mathcal{A}$; see Definition 3.1.7.
$\operatorname{Mor} \mathcal{C}$	The class of homomorphisms of a category $\mathcal{C}$.

$\operatorname{Ob}\mathcal{C}$	The class of objects of a category $\mathcal{C}$.
$\operatorname{Prj}\mathcal{A}$	The collection of projective objects in an abelian category $\mathcal{A}$; see Definition 1.2.21.
$\operatorname{Quis}\mathcal{A}$	The collection of quasi-isomorphisms in $\operatorname{Mor}\mathcal{A}$ where $\mathcal{A}$ is an abelian category.
RF	The total right derived functor of a left exact functor; see Definition 4.2.2.
R^nF	The n-th right derived functor of a left exact functor; see Definition 4.2.3.
$R\text{-}\mathbf{mod}$	The category of left R-modules for some ring with unity R; see Example 1.1.2.
$R\text{-}\mathbf{mod}^{\mathrm{fg}}$	The category of finitely generated left R-modules for some ring with unity R; see Example 1.3.12.
X^{mono}	The collection of all monomorphism with codomain X in some category that is clear from the context; see Definition 1.2.2 ($\sim$ on X^{mono} is defined there).
$X\langle k\rangle$	Tthe k-shift functor on an object X in the graded category of an abelian category; see Definition 3.1.5.
$(X^\bullet, d_X)\langle k\rangle$	The k-shift on an object $(X^\bullet, d_X)$ in $\operatorname{Kom}^*\mathcal{A}$; see Definition 3.1.11.

Introduction

Due to the all-encompassing nature of categories in modern mathematics, most categories are simply *unwieldy beasts*. For example, the task of classifying the indecomposable objects of an additive category up to isomorphism is a natural problem but typically a daunting feat. Possibly frustrated, mathematicians first look for manageable subcategories that preserve the properties of interest. After that, we turn to cleverly crafted *invariants* that hopefully help us understand and classify the more elusive categories. One always requires that invariants of two equivalent categories be "the same"; however, it is preferable that two categories with "the same" invariant share a new exciting property that is more interesting than equivalence.

In this vein, an area of study that has recently gained a wide interest is *decategorification*. In a nutshell, decategorification is the study of set-theoretic invariants for category-theoretic notions; its philosophy is that an invariant for an object, a morphism, a category, a functor, or a natural transformation should be respectively an element, a relation between elements, a set, a function, or a relation between functions. In particular, we would like this to be done such that the assignments of invariants are compatible and unique in some sense. For us, we will require that our invariants be, in addition, group-theoretic; hence, two equivalent categories should induce isomorphic groups, and functors should induce group homomorphisms between these groups in a compatible way. One can consult [Maz12] for a more in-depth introduction to this area of study.

Uninterested in the overly abstract, we opt to define a group-theoretic decategorification as a functor from a category into **Ab**, the category of abelian groups. We will be primarily concerned with the following:

- The decategorification $K_0\colon \mathbf{esExact} \to \mathbf{Ab}$ from the category of essentially small exact categories (whose morphisms are exact functors) that is given by the Grothendieck group.
- The decategorification $K_0^{\triangle}\colon \mathbf{esTri} \to \mathbf{Ab}$ from the category of essentially small triangulated categories (whose morphisms are triangulated functors) that is given by the triangulated Grothendieck group.

To define these decategorifications, a background in additive, abelian, exact and triangulated categories is required. To make the work accessible, Chapter 1 provides a quick and to-the-point overview of these types of categories; however, some undergraduate abstract algebra and basic category theory is required to understand the thesis.

In Chapter 2, the two above decategorifications are defined and basic properties about the invariants are explored. We shall explain, with proof or reference, common situations where the Grothendieck group of an exact category is known. This includes some cases where the Grothendieck group has a nice basis and an example where the Grothendieck group has torsion (Example 2.2.24). We also state some "resolution" theorems, with references to the literature, that greatly reduce the difficulty of computation.

The bounded derived category $D^{\mathrm{b}}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is a triangulated category constructed in Chapter 3. We consider this category because the natural inclusion $\mathcal{A} \to D^{\mathrm{b}}(\mathcal{A})$ induces an isomorphism of groups $K_0(\mathcal{A}) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$. This is an incredible link between the two decategorifications that provides us with more mathematical tools. To get there, we first define the bounded homotopy category

$\mathrm{Ho}^{\mathrm{b}}\,\mathcal{A}$ of an abelian category $\mathcal{A}$. As shown in [Ros11, Theorem 1.1], the natural inclusion induces a group isomorphism $K_0^{\mathrm{split}}(\mathcal{A}) \to K_0^{\triangle}(\mathrm{Ho}^{\mathrm{b}}\,\mathcal{A})$. The group $K_0(\mathcal{A})$ can be viewed as a quotient of $K_0^{\mathrm{split}}(\mathcal{A})$ where one quotients out by a subgroup built from the non-split short exact sequences. To impose these non-split short exact sequences on $\mathrm{Ho}^{\mathrm{b}}\,\mathcal{A}$ as triangles, we turn to an abstract categorical form of localization. We localize the quasi-isomorphisms and obtain the bounded derived category. This chapter ends with a proof that the homomorphism $K_0(\mathcal{A}) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$ is an isomorphism.

In Chapter 4, we see that the concept of derived functors is naturally encoded in the derived category. This plays an interesting role on the level of Grothendieck groups. Given a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, we have an exact functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Under certain conditions, we obtain a group homomorphism $[RF]\colon K_0(\mathcal{A}) \to K_0(\mathcal{B})$ via the $K_0^{\triangle}$ decategorification. Since many important functors are not exact, this can be a very useful tool for both understanding the category and classifying the Grothendieck group. For example, if a category has finite projective dimension (every object has a finite projective resolution of length bounded by some number), from every left exact functor F we should obtain a homomorphism $[RF]$. In Example 4.2.9, the category $\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}$ is shown to not have this property; therefore, we know it does not have finite projective dimension.

Chapter 1

Requisite Types of Categories

This chapter provides a brief introduction to additive, abelian, exact, and triangulated categories. Its purpose is to make the work as accessible as possible, and only some basic category theory and some undergraduate abstract algebra is assumed. Since references will be made to this chapter in the chapters that follow, readers with a background in category theory may choose to go directly to Chapter 2.

We will use the convention of writing $X \in \mathcal{C}$ in place of $X \in \operatorname{Ob}\mathcal{C}$. A category $\mathcal{C}$ is *essentially small* if there exists a small category $\mathcal{D}$ and an equivalence $\mathcal{C} \to \mathcal{D}$. To avoid set-theoretical issues in later chapters, all categories in the thesis will be assumed to be essentially small.

1.1 Additive Categories

An additive category has a trivial object, an addition on each hom-set compatible with composition, and a "direct sum" that has a commutative monoid structure up to isomorphism. Many commonly used categories in representation theory are additive, and, in fact, additive categories will be the basic building block for abelian, exact, and triangulated categories. Let us piece together the definition more precisely.

Definition 1.1.1 (Preadditive category)**.** A category is *preadditive* if all of its hom-sets are abelian groups such that composition is bilinear with respect to the group operations.

Throughout, let R be a ring with unity. It is straightforward to verify that left R-modules and R-linear maps form a category R-**mod** defined by the standard composition of functions. We shall eventually show that R-**mod** is additive.

Example 1.1.2 (R-**mod** is preadditive)**.** Let $M_1, M_2 \in R\text{-}\mathbf{mod}$. For all R-linear $f, g\colon M_1 \to M_2$, let $f + g$ be the R-linear map defined by pointwise addition (i.e. by $m \mapsto f(m) + g(m)$ for all $m \in M_1$). Checking the axioms at points and using the fact that M_2 is an abelian group, the pair $(\hom_{R\text{-}\mathbf{mod}}(M_1, M_2), +)$ is routinely verified to be an abelian group.

Now, let $M_1, M_2, M_3 \in R\text{-}\mathbf{mod}$ and let $f, g\colon M_1 \to M_2$ and $h, k\colon M_2 \to M_3$ be R-linear maps. Composition is bilinear with respect to the pointwise addition since $((h + k) \circ (f + g))(m) = ((h \circ f) + (h \circ g) + (k \circ f) + (k \circ g))(m)$ for all $m \in M_1$. Hence, the category R-**mod** is preadditive.

Definition 1.1.3 (Zero object)**.** Let $\mathcal{C}$ be a category. An object $X \in \mathcal{C}$ is a *zero object* of $\mathcal{C}$ if $\hom_{\mathcal{C}}(X, Y)$ and $\hom_{\mathcal{C}}(Y, X)$ each contain exactly one morphism for all $Y \in \mathcal{C}$.

Definition 1.1.4 (Isomorphism)**.** Let $\mathcal{C}$ be a category and $X, Y \in \mathcal{C}$. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called an *isomorphism* if there is another morphism $g\colon Y \to X$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Clearly, the morphism g is also an isomorphism, and it is called the *inverse* of f.

We write $X \cong Y$ in $\mathcal{C}$ if and only if there is an isomorphism $X \to Y$ in $\mathcal{C}$. Since the composition of two isomorphisms is an isomorphism, identity maps are isomorphisms, and the inverse of an isomorphism is also an isomorphism, it follows that $\cong$ is an equivalence relation on $\operatorname{Ob} \mathcal{C}$. The equivalence classes of this relation are called the *isomorphism classes* of $\mathcal{C}$. We denote by $\operatorname{Iso}\mathcal{C}$ the set (recall that we are assuming all categories are essentially small) of all isomorphism classes of $\mathcal{C}$.

If a collection of objects in $\mathcal{C}$ defined by some property belong to exactly one isomorphism class, then we say that such objects are *unique up to isomorphism.*

Example 1.1.5. Given two zero objects $0_{\mathcal{C}}$ and $0'_{\mathcal{C}}$ of a category $\mathcal{C}$, there is exactly one composite of the form $0_{\mathcal{C}} \to 0'_{\mathcal{C}} \to 0_{\mathcal{C}}$ and exactly one composite of the form $0'_{\mathcal{C}} \to 0_{\mathcal{C}} \to 0'_{\mathcal{C}}$, and, hence, these must coincide with $\mathrm{id}_{0_{\mathcal{C}}}$ and $\mathrm{id}_{0'_{\mathcal{C}}}$ respectively. Hence, zero objects are unique up to isomorphism.

Definition 1.1.6 (Zero morphism)**.** Let $\mathcal{C}$ be a category with a zero object $0_{\mathcal{C}}$, and let $X, Y \in \mathcal{C}$. There is precisely one composite of the form $X \to 0_{\mathcal{C}} \to Y$. We call it the *zero morphism* of $\hom_{\mathcal{C}}(X,Y)$ and denote it by $0_{X,Y}$.

Remark 1.1.7. Let $\mathcal{C}$ be a category, let $0_{\mathcal{C}}$ and $0_{\mathcal{C}'}$ be zero objects of $\mathcal{C}$, and let $X, Y \in \mathcal{C}$. It is straightforward to show that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathrm{id}_X"] & 0_{\mathcal{C}} \arrow[r] \arrow[d, "\cong"] & Y \arrow[d, "\mathrm{id}_Y"] \\
X \arrow[r] & 0'_{\mathcal{C}} \arrow[r] & Y.
\end{tikzcd}$$

This means that the morphism $0_{X,Y}$ is independent of the choice of zero object. Suppose further that $\mathcal{C}$ is preadditive. Then, we have the following in $\hom_{\mathcal{C}}(X,Y)$:

$$\begin{aligned}
0_{X,Y} + 0_{X,Y} &= 0_{0_{\mathcal{C}},Y} \circ 0_{X,0_{\mathcal{C}}} + 0_{0_{\mathcal{C}},Y} \circ 0_{X,0_{\mathcal{C}}} \\
&= 0_{0_{\mathcal{C}},Y} \circ (0_{X,0_{\mathcal{C}}} + 0_{X,0_{\mathcal{C}}}) = 0_{0_{C},Y} \circ 0_{X,0_{\mathcal{C}}} = 0_{X,Y}.
\end{aligned}$$

Hence, the additive identity of $\hom_{\mathcal{C}}(X,Y)$ is $0_{X,Y}$.

Example 1.1.8. In R-**mod**, the zero object is the trivial module and the zero morphisms are the zero R-linear maps.

Definition 1.1.9 (Biproduct)**.** Let $\mathcal{C}$ be a category with a zero object. Given $X_1, \dots, X_n \in \mathcal{C}$, their *biproduct* is an object X together with morphisms $p_{X_k} \colon X \to X_k$ and $i_{X_k} \colon X_k \to X$ for $k = 1, \dots, n$ such that the following conditions hold:

(BP1) For every $Y \in \mathcal{C}$ and morphisms $f_k \colon Y \to X_k$ for $k = 1, \dots, n$, there is a unique morphism $f \colon Y \to X$ such that $p_{X_k} \circ f = f_k$ for all $k = 1, \dots n$.

(BP2) For every $Y \in \mathcal{C}$ and morphisms $f_k \colon X_k \to Y$ for $k = 1, \dots, n$, there is a unique morphism $f \colon X \to Y$ such that $f \circ i_{X_k} = f_k$ for all $k = 1, \dots, n$.

(BP3) We have $p_{X_j} \circ i_{X_k} = \begin{cases} 0_{X_k, X_j} & \text{if } j \neq k \\ \mathrm{id}_{X_k} & \text{if } j = k \end{cases}$ for all $j, k = 1, \dots, n$.

Biproducts are unique up to isomorphism (see [ML13, Remark 2.2.7]). If a biproduct of $X_1, \dots, X_n$ exists, we denote a choice of it by $X_1 \oplus \cdots \oplus X_n$. We say that *all finitary biproducts exist* in $\mathcal{C}$ if $X_1 \oplus \cdots \oplus X_n$ exists for all $X_1, \dots, X_n \in \mathcal{C}$ and $n \in \mathbb{N}$.

Example 1.1.10. In R-**mod**, the biproduct of left R-modules is given by the direct sum. The canonical projection and injection R-linear maps are the p and i morphisms respectively.

With the terminology we have gathered, we can now define an additive category precisely in the following way. It is straightforward to verify that additivity is invariant under equivalence of categories.

Definition 1.1.11 (Additive category)**.** A category is *additive* if it is preadditive, has a zero object, and all finitary biproducts exist.

Example 1.1.12. The category R-**mod** is additive as shown in Example 1.1.2, Example 1.1.8, and Example 1.1.10.

Remark 1.1.13. We remark $\mathbb{Z}$-**mod** is equivalent to **Ab**, the category of abelian groups. Thus, the category **Ab** is additive. If k is a field, then k-**mod** is equivalent to $\mathbf{Vect}_k$, the category of vector spaces over k. Thus, the category $\mathbf{Vect}_k$ is additive.

Lemma 1.1.14. *Let $\mathcal{C}$ be a preadditive category with a zero object. The following are equivalent:*

1. *The category $\mathcal{C}$ is additive.*
2. *For any $X_1, \dots, X_n \in \mathcal{C}$, there exists an object $X \in \mathcal{C}$ and morphisms $p_{X_k} \colon X \to X_k$ for $k = 1, \dots, n$ satisfying (BP1) in Definition 1.1.9. The object X is then a biproduct of $X_1, \dots, X_n$.*
3. *For any $X_1, \dots, X_n \in \mathcal{C}$, there exists an object $X \in \mathcal{C}$ and morphisms $p_{X_k} \colon X \to X_k$ and $i_{X_k} \colon X_k \to X$ for $k = 1, \dots, n$ such that (BP3) in Definition 1.1.9 holds and*

$$i_{X_1} \circ p_{X_1} + \dots + i_{X_n} \circ p_{X_n} = \mathrm{id}_X \,. \tag{1.1.1}$$

The object X is then a biproduct of $X_1, \dots, X_n$.

Proof: The proof can be adapted from [KS06, Lemma 8.2.3 and Lemma 8.2.9]. □

Let $\mathcal{A}$ be a category. The *opposite category* of $\mathcal{A}$, denoted $\mathcal{A}^{\mathrm{op}}$, is given by $\mathrm{Ob}\,\mathcal{A}^{\mathrm{op}} := \mathrm{Ob}\,\mathcal{A}$, $\mathrm{hom}_{\mathcal{A}^{\mathrm{op}}}(X,Y) := \{f^{\mathrm{op}} \mid f \in \mathrm{hom}_{\mathcal{A}}(Y,X)\}$ for all $X, Y \in \mathcal{A}$, and $g^{\mathrm{op}} \circ f^{\mathrm{op}} := (f \circ g)^{\mathrm{op}}$ for all composable $f, g \in \mathrm{Mor}\,\mathcal{A}$.

Example 1.1.15 ($\mathcal{A}^{\mathrm{op}}$ additive if $\mathcal{A}$ is additive)**.** It is routine to show that the defined composition and $f^{\mathrm{op}} + g^{\mathrm{op}} = (f+g)^{\mathrm{op}}$ for all f, g in the same hom-set makes $\mathcal{A}^{\mathrm{op}}$ preadditive. Clearly, a zero object of $\mathcal{A}$ is a zero object for $\mathcal{A}^{\mathrm{op}}$. Biproducts are also the same except that the $(p_{X_j})^{\mathrm{op}}$ takes the place of i_{X_j} and $(i_{X_j})^{\mathrm{op}}$ takes the place of p_{X_j}. Hence, the category $\mathcal{A}^{\mathrm{op}}$ is additive.

The existence of biproducts in additive categories allows us to represent morphisms in the following more practical way.

Definition 1.1.16 (Biproduct matrix, biproduct morphism)**.** Let $f \colon X_1 \oplus \dots \oplus X_n \to Y_1 \oplus \dots \oplus Y_m$ be a morphism between biproducts in an additive category $\mathcal{A}$. Define

$f_{k,j} := p_{Y_k} \circ f \circ i_{X_j}$. We shall call the matrix

$$\begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & \ddots & \vdots \\ f_{m,1} & \cdots & f_{m,n} \end{pmatrix}$$

the *biproduct matrix* of f.

We identify morphisms between biproducts with their biproduct matrices. Under this identification, the addition of the morphisms corresponds to the matrix addition of biproduct matrices due to bilinearity of composition. Furthermore, the composition of the morphisms corresponds to the matrix multiplication of biproduct matrices (the product of entries being composition); this is due to bilinearity of composition and (1.1.1) of Lemma 1.1.14.

Let $f_i \colon X_i \to Y_i$ be a morphism in $\mathcal{A}$ for $i = 1, \dots, n$. The *biproduct morphism* $f_1 \oplus \cdots \oplus f_n \colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_n$ is given by the diagonal biproduct matrix such that $f_{i,i} = f_i$ for all $i = 1, \dots, n$.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Functors $\mathcal{A} \to \mathcal{B}$ and the natural transformations between them form a category $\mathbf{Fct}(\mathcal{A}, \mathcal{B})$ defined by the usual composition.

Example 1.1.17 ($\mathbf{Fct}(\mathcal{A}, \mathcal{B})$ is additive if $\mathcal{B}$ is additive)**.** <u>Preadditive:</u> Let $F, G \in \mathcal{C}$ and $\eta, \sigma \in \hom_{\mathcal{C}}(F, G)$. We define a natural transformation $\eta + \sigma$ by $(\eta + \sigma)^X := \eta^X + \sigma^X$ for all $X \in \mathcal{A}$. This is a well-defined because natural transformation since for any $f \colon X \to Y$ in $\mathcal{A}$ we have the following:

$$\begin{aligned}(\eta + \sigma)^Y \circ F(f) &= (\eta^Y + \sigma^Y) \circ F(f) = \eta^Y \circ F(f) + \sigma^Y \circ F(f) \\ &= G(f) \circ \eta^X + G(f) \circ \sigma^X = G(f) \circ (\eta^X + \sigma^X) = G(f) \circ (\eta + \sigma)^X.\end{aligned}$$

The verification that $(\hom_{\mathcal{C}}(F, G), +)$ is an abelian group is routine. The bilinearity of composition with respect to these group operations is also straightforward.

<u>Zero object:</u> Let $0_{\mathcal{B}}$ be a zero object of $\mathcal{B}$. There is a unique functor $Z \in \mathcal{C}$ given by $Z(X) = 0_{\mathcal{B}}$ for all $X \in \mathcal{B}$. Given natural transformations $\eta \in \hom_{\mathcal{C}}(Z, F)$ and

$\sigma \in \hom_{\mathcal{C}}(F, Z)$ for some $F \in \mathcal{C}$, we have that $\eta^X = 0_{0_{\mathcal{B}},F(X)}$ and $\sigma^X = 0_{F(X),0_{\mathcal{B}}}$ for all $X \in \mathcal{A}$, which defines precisely one morphism for each hom-set. Therefore, the functor Z is a zero object of $\mathcal{C}$.

Biproducts: Given $F_1, \dots, F_n \in \mathcal{C}$, we define a functor $B \in \mathcal{C}$ by the following: $BX = F_1X \oplus \cdots \oplus F_nX$ for all $X \in \mathcal{A}$ and $B(f) = F_1(f) \oplus \cdots \oplus F_n(f)$ for all $f \in \operatorname{Mor}\mathcal{A}$. This gives rise to natural transformations $i_{F_j} \colon F_j \to B$ and $p_{F_j} \colon B \to F_j$ for $j = 1, \dots, n$ given by the following morphisms:

$$(i_{F_j})^X = \begin{pmatrix} 0_{F_jX,F_1X)} \\ \vdots \\ 0_{F_jX,F_{j-1}X} \\ \mathrm{id}_{F_j} \\ 0_{F_jX,F_{j+1}X} \\ \vdots \\ 0_{F_jX,F_nX} \end{pmatrix} \quad \text{and}$$

$$(p_{F_j})^X = \begin{pmatrix} 0_{F_1X,F_jX} & \cdots & 0_{F_{j-1}X,F_jX} & \mathrm{id}_{F_j} & 0_{F_{j+1}X,F_jX} & \cdots & 0_{F_nX,F_jX} \end{pmatrix}$$

for all $X \in \mathcal{A}$. One can show that B together with the i_{F_j} and p_{F_j} for $j = 1, \dots, n$ is a biproduct by noting that they satisfy (3) of Lemma 1.1.14.

Conclusion: If $\mathcal{B}$ is additive, the category $\mathcal{C} = \mathbf{Fct}(\mathcal{A}, \mathcal{B})$ is additive.

Lemma 1.1.18. *Let $\mathcal{A}$ be an additive category. The biproducts of $\mathcal{A}$ induce a commutative monoid structure on* $\operatorname{Iso}\mathcal{A}$ *(see Definition 1.1.4). That is, there is a binary operation on* $\operatorname{Iso}\mathcal{A}$ *given by $[X] \oplus [Y] = [X \oplus Y]$ for all $X, Y \in \mathcal{A}$, this operation is commutative and associative, and there is an identity in* $\operatorname{Iso}\mathcal{A}$ *under this operation.*

Proof: The following are inverse isomorphisms:

$$X \oplus Y \xrightarrow{\begin{pmatrix} 0_{X,Y} & \mathrm{id}_Y \\ \mathrm{id}_X & 0_{Y,X} \end{pmatrix}} Y \oplus X \quad \text{and} \quad Y \oplus X \xrightarrow{\begin{pmatrix} 0_{Y,X} & \mathrm{id}_X \\ \mathrm{id}_Y & 0_{X,Y} \end{pmatrix}} X \oplus Y.$$

Hence, we have $[X] \oplus [Y] = [X \oplus Y] = [Y \oplus X] = [Y] \oplus [X]$ for all $X, Y \in \mathcal{A}$. For all $X \in \mathcal{A}$, the following are inverse isomorphisms:

$$X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ 0_{X,0_{\mathcal{A}}} \end{pmatrix}} X \oplus 0_{\mathcal{A}} \quad \text{and} \quad X \oplus 0_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \mathrm{id}_X & 0_{0_{\mathcal{A}},X} \end{pmatrix}} X.$$

Hence, we have $[X] \oplus [0_{\mathcal{A}}] = [X \oplus 0_{\mathcal{A}}] = [X]$ for all $X \in \mathcal{A}$. The fact that the binary operation is well-defined and associative is proved similarly. □

The best kind of functors between additive categories are the following functors. We will see later that they induce group homomorphisms on the split Grothendieck groups of the additive categories.

Definition 1.1.19 (Additive functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *additive* if the induced map $F_{X,Y}\colon \hom_{\mathcal{A}}(X,Y) \to \hom_{\mathcal{B}}(FX, FY)$ is a group homomorphism for all $X, Y \in \mathcal{A}$.

Remark 1.1.20. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Since $0_{F(0_{\mathcal{A}}),F(0_{\mathcal{A}})} = F(0_{0_{\mathcal{A}},0_{\mathcal{A}}}) = F(\mathrm{id}_{0_{\mathcal{A}}}) = \mathrm{id}_{F(0_{\mathcal{A}})}$, we have $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$. The first equality makes use of Remark 1.1.7.

Lemma 1.1.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is additive if and only if $F(X \oplus_{\mathcal{A}} Y) \cong FX \oplus_{\mathcal{B}} FY$ for all $X, Y \in \mathcal{A}$.*

Proof: ⇒: The morphism

$$F(X \oplus_{\mathcal{A}} Y) \xrightarrow{\begin{pmatrix} F(p_X) \\ F(p_Y) \end{pmatrix}} FX \oplus_{\mathcal{B}} FY$$

is an isomorphism with inverse

$$FX \oplus_{\mathcal{B}} FY \xrightarrow{\begin{pmatrix} F(i_X) & F(i_Y) \end{pmatrix}} F(X \oplus_{\mathcal{A}} Y).$$

⇐: For this direction, see [Bor94, Proposition 1.3.4]. □

Definition 1.1.22 (Biadditive functor)**.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. A functor of the form $F\colon \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ is called a *bifunctor*. It induces the following functors:

- For each $X\in\mathcal{A}$, we define a functor $F(X,-)\colon\mathcal{B}\to\mathcal{C}$ by $F(X,-)(Z)=F(X,Z)$ for all $Z\in\mathcal{B}$ and $F(X,-)(f)=F(\mathrm{id}_X,f)$ for all $f\in\operatorname{Mor}\mathcal{B}$.

- For each $Y\in\mathcal{B}$, we define a functor $F(-,Y)\colon\mathcal{A}\to\mathcal{B}$ by $F(-,Y)(W)=F(W,Y)$ for all $W\in\mathcal{A}$ and $F(-,Y)(g)=F(g,\mathrm{id}_Y)$ for all $g\in\operatorname{Mor}\mathcal{A}$.

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are additive. We say that F is *biadditive* if it is *additive in both slots* (i.e. the functors $F(X,-)$ and $F(-,Y)$ are additive for all $X\in\mathcal{A}$ and $Y\in\mathcal{B}$).

The next example is a very important biadditive functor. It will show up quite a few times in this thesis and is used frequently in various areas of mathematics.

Example 1.1.23 ($\hom_{\mathcal{A}}$ is biadditive)**.** Let $f\colon X\to X'$ and $g\colon Y\to Y'$ be morphisms in an additive category $\mathcal{A}$. We define the following group homomorphism:

$$\begin{aligned} \hom_{\mathcal{A}}(f^{\mathrm{op}},g)\colon\ \hom_{\mathcal{A}}(X',Y) &\to \hom_{\mathcal{A}}(X,Y') && \text{given by} \\ h &\mapsto g\circ h\circ f && \text{for all } h\in\hom_{\mathcal{A}}(X',Y). \end{aligned}$$

The functor $\hom_{\mathcal{A}}\colon\mathcal{A}^{\mathrm{op}}\times\mathcal{A}\to\mathbf{Ab}$ given by $(X,Y)\mapsto\hom_{\mathcal{A}}(X,Y)$ for all $X,Y\in\mathcal{A}$ and $(f^{\mathrm{op}},g)\mapsto\hom_{\mathcal{A}}(f^{\mathrm{op}},g)$ for all $f,g\in\operatorname{Mor}\mathcal{A}$ is called the *hom-functor* of $\mathcal{A}$.

Recall that $\mathcal{A}^{\mathrm{op}}$ is additive since $\mathcal{A}$ is additive (Example 1.1.15), and recall that $\mathbf{Ab}$ is additive (Remark 1.1.13). Let $X\in\mathcal{A}$ and $f,g\in\hom_{\mathcal{A}}(Y,Z)$ for some $Y,Z\in\mathcal{A}$. We have the following:

- $(\hom_{\mathcal{A}}(X,-)(f+g))(h)=(f+g)\circ h\circ\mathrm{id}_X=f\circ h\circ\mathrm{id}_X+g\circ h\circ\mathrm{id}_X$
  $=(\hom_{\mathcal{A}}(X,-)(f)+\hom_{\mathcal{A}}(X,-)(g))(h)$ for all $h\in\hom_{\mathcal{A}}(X,Y)$.

- $(\hom_{\mathcal{A}}(-,X)(f^{\mathrm{op}}+g^{\mathrm{op}}))(h)=\mathrm{id}_X\circ h\circ(f+g)=\mathrm{id}_X\circ h\circ f+\mathrm{id}_X\circ h\circ g$
  $=(\hom_{\mathcal{A}}(-,X)(f^{\mathrm{op}})+\hom(-,X)_{\mathcal{A}}(g^{\mathrm{op}}))(h)$ for all $h\in\hom_{\mathcal{A}}(Z,X)$.

Hence, the hom-functor of $\mathcal{A}$ is biadditive.

Definition 1.1.24 (Strictly full subcategory)**.** Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *strictly full* if the following two conditions hold:

- The subcategory is *full*. That is, we have $\hom_{\mathcal{D}}(X,Y) = \hom_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathcal{D}$.
- The subcategory is *isomorphism-closed*. That is, if $f\colon X \to Y$ is an isomorphism in $\mathcal{C}$, then $X \in \mathcal{D}$ if and only if $Y \in \mathcal{D}$.

We will only be considering strictly full subcategories for practical reasons.

Definition 1.1.25 (Additive subcategory)**.** Let $\mathcal{A}$ be an additive category. A strictly full subcategory $\mathcal{D}$ of $\mathcal{A}$ is an *additive subcategory* if $0_{\mathcal{A}} \in \mathcal{D}$ and $X \oplus_{\mathcal{A}} Y \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$.

Remark 1.1.26. Let $\mathcal{D}$ be a strictly full additive subcategory of an additive category $\mathcal{A}$. Then, we have that $\mathcal{D}$ is additive with $0_{\mathcal{D}} \cong 0_{\mathcal{A}}$ and $X \oplus_{\mathcal{D}} Y \cong X \oplus_{\mathcal{A}} Y$ for all $X, Y \in \mathcal{D}$. Since the morphisms that define a biproduct in $\mathcal{A}$ are again in $\mathcal{D}$, the verification of this is routine.

Example 1.1.27 (R-$\mathbf{mod}^{\mathrm{fg}}$)**.** The category of finitely generated left R-modules, denoted R-$\mathbf{mod}^{\mathrm{fg}}$, is an additive subcategory of R-$\mathbf{mod}$.

Example 1.1.28 ($\mathbf{Add}(\mathcal{A},\mathcal{B})$)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and recall that $\mathbf{Fct}(\mathcal{A},\mathcal{B})$ is additive (Example 1.1.17). Let $\mathbf{Add}(\mathcal{A},\mathcal{B})$ be the strictly full subcategory of $\mathbf{Fct}(\mathcal{A},\mathcal{B})$ induced by additive functors. If $F, G \in \mathbf{Add}(\mathcal{A},\mathcal{B})$, then

$$\begin{aligned}
(F \oplus G)(X \oplus Y) &\cong F(X \oplus Y) \oplus G(X \oplus Y) && \text{(biproduct in Example 1.1.17)} \\
&\cong FX \oplus GX \oplus FY \oplus GY && \text{(Lemma 1.1.21)} \\
&\cong (F \oplus G)(X) \oplus (F \oplus G)(Y).
\end{aligned}$$

Hence, we have $F \oplus G \in \mathbf{Add}(\mathcal{A},\mathcal{B})$. The zero object of $\mathbf{Fct}(\mathcal{A},\mathcal{B})$ is clearly additive so $\mathbf{Add}(\mathcal{A},\mathcal{B})$ is an additive subcategory of $\mathbf{Fct}(\mathcal{A},\mathcal{B})$.

1.2 Abelian Categories

Abelian categories were motivated by the study of modules and later inspired homological algebra. An abelian category is an additive category where the notions of kernel and cokernel make sense for all the morphisms. The Freyd Mitchel Embedding Theorem (Theorem 3.3.5) tells us that all abelian categories are on some level simply a subcategory of some category of modules over a ring. Despite this, the definition of an abelian category still finds itself useful. Since objects in categories do not necessarily have "elements", we will need to generalize a few notions from abstract algebra.

Definition 1.2.1 (Monomorphism, epimorphism)**.** Let $\mathcal{C}$ be a category. A morphism $m\colon X \to Y$ in $\mathcal{C}$ is a *monomorphism* in $\mathcal{C}$ if for any two morphism $g_1, g_2\colon Z \to X$ in $\mathcal{C}$ such that $m \circ g_1 = m \circ g_2$, we have $g_1 = g_2$. A morphism $e\colon X \to Y$ in $\mathcal{C}$ is called an *epimorphism* in $\mathcal{C}$ if for any two morphism $g_1, g_2\colon Y \to Z$ in $\mathcal{C}$ such that $g_1 \circ e = g_2 \circ e$, we have $g_1 = g_2$.

Definition 1.2.2 (Subobject)**.** Let $\mathcal{C}$ be category and $X \in \mathcal{C}$. Let X^{mono} be the collection of all monomorphisms in $\mathcal{C}$ with codomain X. If $f, g \in X^{\text{mono}}$, we write $f \sim g$ if and only if there exists an isomorphism u in $\mathcal{C}$ such that $f \circ u = g$. The relation $\sim$ on X^{mono} can be routinely verified to be an equivalence relation, and the equivalence classes of this relation are called the *subobjects* of X in $\mathcal{C}$.

Remark 1.2.3. Note that two representatives of the same subobject have isomorphic domains. On the other hand, we often have representatives of subobjects with isomorphic domains that are not equivalent.

Definition 1.2.4 (Kernel)**.** Let $\mathcal{A}$ be an additive category, and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. A morphism $m\colon K \to X$ in $\mathcal{A}$ is a *kernel* of f if the following holds:

- We have $f \circ m = 0$.

- For each $m'\colon K' \to X$ in $\mathcal{A}$ such that $f \circ m' = 0$, there is a unique morphism $u\colon K' \to K$ in $\mathcal{A}$ such that $m \circ u = m'$.

Given a morphism $g\colon Z \to K$ in $\mathcal{A}$ for some $Z \in \mathcal{A}$, we have $f \circ m \circ g = 0$ and thus there is a unique u in $\mathcal{A}$ such that $m \circ u = m \circ g$; hence, a kernel is a monomorphism in $\mathcal{A}$. If a kernel of f exists, we denote one by $\operatorname{Ker} f$.

Lemma 1.2.5. *Let $\mathcal{A}$ be an additive category, and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ that has a kernel $\operatorname{Ker} f\colon K \to X$. Then, a monomorphism $m\colon Z \to X$ in $\mathcal{A}$ is a kernel of f if and only if $m \sim \operatorname{Ker} f$ in X^{mono}.*

Proof: $\Rightarrow$: By the definition of a kernel, there exists unique u and u' in $\mathcal{A}$ such that $m \circ u = \operatorname{Ker} f$ and $\operatorname{Ker} f \circ u' = m$. Since $\operatorname{Ker} f$ and m are monomorphisms, it follows that u is an isomorphism with inverse u'.

$\Leftarrow$: By definition, there is an isomorphism u such that $m \circ u = \operatorname{Ker} f$. Hence, $f \circ m \circ u = f \circ \operatorname{Ker} f = 0_{K,Y}$. Hence, $f \circ m = 0_{Z,Y}$. Suppose we have $m'\colon Z' \to X$ such that $f \circ m' = 0_{Z',Y}$. Then, there exists a unique $u'\colon Z' \to K$ such that $\operatorname{Ker} f \circ u' = m'$. Thus, we have $m \circ (u \circ u') = m'$, and the uniqueness of $u \circ u'$ follows from m being a monomorphism. ☐

Lemma 1.2.6. *Let $\mathcal{A}$ be an additive category. Let $f\colon X \to Y$ and $g\colon X' \to Y'$ be morphisms in $\mathcal{A}$ with kernels $\operatorname{Ker} f$ and $\operatorname{Ker} g$ respectively. Then, the morphism $\operatorname{Ker} f \oplus \operatorname{Ker} g$ is a kernel of $f \oplus g$ (see Definition 1.1.16).*

Proof: The proof is routine if one uses biproduct matrices. ☐

Definition 1.2.7 (Cokernel)**.** Let $\mathcal{A}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. A morphism $e\colon Y \to C$ in $\mathcal{A}$ is a *cokernel* of f if the following holds:

- We have $e \circ f = 0$.
- For each $e'\colon Y \to C'$ in $\mathcal{A}$ such that $e' \circ f = 0$, there is a unique morphism $u\colon C \to C'$ in $\mathcal{A}$ such that $u \circ e = e'$.

If a cokernel of f exists, we denote one by $\operatorname{Coker} f$.

Remark 1.2.8. Kernels and cokernels are *dual* notions. By this, we mean that $\operatorname{Ker} f$ and $(\operatorname{Coker} f^{\mathrm{op}})^{\mathrm{op}}$ represent the same subobject in $\mathcal{C}$ if f has a kernel and $(\operatorname{Coker} f)^{\mathrm{op}}$ and $\operatorname{Ker}(f^{\mathrm{op}})$ represent that same subobject in $\mathcal{C}^{\mathrm{op}}$ if f has a cokernel.

We can now give a precise definition of an abelian category. It is straightforward to note that being abelian is invariant under equivalence of categories.

Definition 1.2.9 (Abelian category)**.** An *abelian category* is an additive category $\mathcal{A}$ such that every morphism in $\mathcal{A}$ has a kernel and cokernel, every monomorphism in $\mathcal{A}$ is a kernel of another morphism in $\mathcal{A}$, and every epimorphism in $\mathcal{A}$ is a cokernel of another morphism in $\mathcal{A}$.

Lemma 1.2.10 ([Fre64, Theorem 2.12])**.** *Abelian categories are* balanced*. That is, if a morphism in an abelian category is both a monomorphism and an epimorphism, then it is an isomorphism.*

Definition 1.2.11 (Image)**.** Let $f\colon X \to Y$ be a morphism in some category $\mathcal{C}$. A monomorphism $m\colon Z \to Y$ in $\mathcal{C}$ is called an *image* of f if the following holds:

- We have $f = m \circ g$ for some morphism $g\colon X \to Z$ in $\mathcal{C}$.
- If $f = h \circ k$ for some monomorphism $h\colon Z' \to Y$ and morphism $k\colon X \to Z'$ in $\mathcal{C}$, then there is a unique morphism $u\colon Z \to Z'$ in $\mathcal{C}$ such that $m = h \circ u$.

If an image of f exists, we denote a choice of it by $\operatorname{Im} f$.

Proposition 1.2.12 ([Fre64, Theorem 2.19])**.** *Let $\mathcal{A}$ be an abelian category, and fix a choice of all kernels and cokernels. If $f\colon X \to Y$ is a morphism in $\mathcal{A}$, the morphism* $\operatorname{Ker}\operatorname{Coker} f$ *is an image of f. In addition, there is an epimorphism k in $\mathcal{A}$ such that $f = \operatorname{Im} f \circ k$. Furthermore, the subobject of Y that an image of f represents is independent of the choice of image.*

Example 1.2.13 (R-**mod** is abelian)**.** Let R be a ring with unity. Recall that R-**mod** is additive. In R-**mod**, the notions of kernel, image and cokernel correspond with their traditional definitions. That is, if $M, N \in R$-**mod** and $f\colon M \to N$ is a R-linear map, then the following holds:

- The subset $K = \{x \in M \mid f(x) = 0\}$ as a R-submodule of M yields a natural R-linear injection $K \to M$ that is a kernel of f.
- The traditional image $f(M) = \{f(m) \mid m \in M\}$ as a submodule of N yields a natural R-linear injection $f(M) \to N$ that is an image of f.
- The quotient R-module $C = N/f(M)$ yields a natural R-linear surjection $N \to C$ that is a cokernel of f.

The existence of kernels and cokernels in R-**mod** is clear from this correspondence. Furthermore, if f is a monomorphism, then f is a kernel of the natural map $N \to C$. If f is a epimorphism, then f is a cokernel of the natural map $K \to M$. In conclusion, the category R-**mod** is abelian.

Remark 1.2.14. Since $\mathbb{Z}$-**mod** is equivalent to **Ab**, we remark that **Ab** is abelian.

Example 1.2.15 ($\mathbf{Fct}(\mathcal{A}, \mathcal{B})$ is abelian if $\mathcal{B}$ is abelian)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Suppose $\mathcal{B}$ is abelian and let $\mathcal{C} = \mathbf{Fct}(\mathcal{A}, \mathcal{B})$.

Additive: Since $\mathcal{B}$ is additive, the category $\mathcal{C}$ is additive (Example 1.1.17).

Existence of kernels: Let $F, G \in \mathcal{C}$ and $\eta \in \hom_{\mathcal{C}}(F, G)$. For each $X \in \mathcal{A}$, fix a kernel $\operatorname{Ker}(\eta^X)$ of η^X and denote its domain by $H(X)$. Let $f\colon X \to Y$ in $\mathcal{A}$. By definition of a kernel, there is a unique morphism $H(f)$ such that $F(f) \circ \operatorname{Ker}(\eta^X) = \operatorname{Ker}(\eta^Y) \circ H(f)$ since $\eta^Y \circ F(f) \circ \operatorname{Ker}(\eta^X) = G(f) \circ \eta^X \circ \operatorname{Ker}(\eta^X) = 0$. One can routinely verify that this induces a functor H and a natural transformation $\sigma\colon H \to F$, defined by $\sigma^X = \operatorname{Ker}(\eta^X)$ for all $X \in \mathcal{A}$, that is a kernel of η. This means that every morphism has a kernel in $\mathcal{C}$.

Existence of cokernels: This follows by duality.

Monomorphisms are kernels: Let $F, G \in \mathcal{C}$. Let $\eta \in \hom_{\mathcal{C}}(F, G)$ be a monomorphism in $\mathcal{C}$. We have $(\operatorname{Coker} \eta) \circ \eta = 0$. Suppose there is $H \in \mathcal{C}$ and $\sigma \in \hom_{\mathcal{C}}(H, G)$ such that $(\operatorname{Coker}(\eta^X)) \circ \sigma^X = 0$. By [Fre64, Theorem 2.11], there is a unique $u^X \colon H(X) \to F(X)$ such that $\eta^X \circ u^X = \sigma^X$ for all $X \in \mathcal{A}$. It is routine to verify that this defines a unique natural transformation $u \colon H \to F$. Hence, the monomorphism η is a kernel of $\operatorname{Coker} \eta$.

Epimorphisms are cokernels: This follows by duality.

Conclusion: If $\mathcal{B}$ is abelian, then the category $\mathcal{C} = \mathbf{Fct}(\mathcal{A}, \mathcal{B})$ is abelian.

To make sense of abelian categories, we look at the following special sequences. They will play an fundamental role for the Grothendieck groups of abelian categories.

Definition 1.2.16 (Exact sequence)**.** Let $\mathcal{A}$ be an abelian category, and fix some kernels and images. We say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* at Y if $\operatorname{Im} f$ and $\operatorname{Ker} g$ represent the same subobject. A finite sequence of objects and morphisms in $\mathcal{A}$

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

is an *exact sequence* if it is exact at $X_1, \ldots, X_n$. A *short exact sequence* is an exact sequence of the form $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$. An infinite sequence of objects and morphisms in $\mathcal{A}$

$$\cdots \xrightarrow{f_{-2}} X_{-1} \xrightarrow{f_{-1}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

is an *exact sequence* if it is exact at all X_i for $i \in \mathbb{Z}$.

Lemma 1.2.17. *Let $\mathcal{A}$ be an abelian category. A sequence in $\mathcal{A}$*

$$\begin{tikzcd} 0_{\mathcal{A}} \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0_{\mathcal{A}} \end{tikzcd}$$

is a short exact sequence if and only if f is a monomorphism, g is an epimorphism, and f is a kernel of g.

Proof: The image of a zero morphism is a zero morphism. Hence, the kernel of f is a zero morphism, quickly implying that f is a monomorphism. Dually, we have that g is an epimorphism. By [Fre64, Theorem 2.11], both f and $\operatorname{Im} f$ represent the same subobject of Y so, by Lemma 1.2.5, we are done. ☐

Definition 1.2.18. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is *exact* if the following is a short exact sequence

$$0_{\mathcal{B}} \to FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \to 0_{\mathcal{B}}$$

for all short exact sequences $0_{\mathcal{A}} \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0_{\mathcal{A}}$ in $\mathcal{A}$.

Definition 1.2.19 (Abelian subcategory)**.** Let $\mathcal{A}$ be an abelian category and fix kernels and cokernels. A strictly full additive subcategory $\mathcal{D}$ of $\mathcal{A}$ is an *abelian subcategory* if for every morphism $f \in \operatorname{Mor} \mathcal{D}$ we have $\operatorname{Ker}_{\mathcal{A}} f \in \operatorname{Mor} \mathcal{D}$ and $\operatorname{Coker}_{\mathcal{A}} f \in \operatorname{Mor} \mathcal{D}$.

Remark 1.2.20. Let $\mathcal{D}$ be an strictly full abelian subcategory of an abelian category $\mathcal{A}$. Then, it is easy to check that $\mathcal{D}$ is abelian. Kernels and cokernels are the same as in $\mathcal{A}$. Furthermore, since the image and the kernel of the cokernel of a morphism represent the same subobject in abelian categories, it is clear that a sequence in $\mathcal{D}$ is exact in $\mathcal{D}$ if and only if it is exact in $\mathcal{A}$.

Definition 1.2.21 (Injective/projective objects)**.** Let $\mathcal{A}$ be an abelian category. We say that an object $I \in \mathcal{A}$ is *injective* if $\hom_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is an exact functor (see Example 1.1.23). Denote by $\operatorname{Inj} \mathcal{A}$ the strictly full subcategory induced by injective objects of $\mathcal{A}$. Dually, an object $P \in \mathcal{A}$ is *projective* if $\hom_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$ is an exact functor. Denote by $\operatorname{Prj} \mathcal{A}$ the strictly full subcategory induced by projective objects of $\mathcal{A}$.

Due to the duality, we will typically only look only at projective objects in this chapter. Identical dual arguments could be made for injective objects.

Lemma 1.2.22. *Let $\mathcal{A}$ be an abelian category. If we have a short exact sequence*

$$0_{\mathcal{A}} \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} P \longrightarrow 0_{\mathcal{A}}$$

such that $X, Y \in \mathcal{A}$ and $P \in \operatorname{Prj} \mathcal{A}$, then $Y \cong X \oplus P$ in $\mathcal{A}$. Furthermore, we have $X \oplus Y \in \operatorname{Prj} \mathcal{A}$ if and only if $X, Y \in \operatorname{Prj} \mathcal{A}$.

Proof: By definition, the functor $\hom_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$ is exact. In particular, the homomorphism $\hom_{\mathcal{A}}(P, g)$ is surjective by Lemma 1.2.17. Hence, there is a morphism $h \in \hom_{\mathcal{A}}(P, Y)$ such that $g \circ h = \mathrm{id}_P$. Therefore, we have $Y \cong X \oplus P$ by [KS06, Proposition 8.3.14]. For the second statement, it suffices to note that we have a natural isomorphism $\hom_{\mathcal{A}}(X \oplus Y, -) \to \hom_{\mathcal{A}}(X, -) \oplus \hom_{\mathcal{A}}(Y, -)$ and that the former functor is exact if and only if each summand of the latter is exact. ☐

1.3 Exact Categories

One motivation for exact categories is the fact that some important strictly full subcategories of abelian categories do not have kernels/cokernels for all their morphisms. In particular, given an abelian category $\mathcal{A}$, the subcategories $\operatorname{Inj} \mathcal{A}$ and $\operatorname{Prj} \mathcal{A}$ are not necessarily abelian.

Definition 1.3.1 (Exact category)**.** Let $\mathcal{A}$ be an additive category. Let D be a family of sequences in $\mathcal{A}$ of the form

$$0_{\mathcal{A}} \to A \xrightarrow{f} B \xrightarrow{g} C \to 0_{\mathcal{A}}. \tag{1.3.1}$$

We say that $(\mathcal{A}, D)$ is an *exact category* if there is an abelian category $\mathcal{B}$ and a full additive embedding $F\colon \mathcal{A} \to \mathcal{B}$ such that the following hold:

(1) A sequence of the form (1.3.1) is in D if and only if $0_{\mathcal{B}} \to FA \xrightarrow{F(f)} FB \xrightarrow{F(g)} FC \to 0_{\mathcal{B}}$ is a short exact sequence in $\mathcal{B}$.

(2) Given a short exact sequence of the form $0_{\mathcal{B}} \to FX \to C \to FY \to 0_{\mathcal{B}}$ in $\mathcal{B}$ for some $X, Y \in \mathcal{A}$ and $C \in \mathcal{B}$, then there exists $Z \in \mathcal{A}$ such that $FZ \cong C$ in $\mathcal{B}$.

We shall call the sequences in D *conflations*. If the conflations are clear, then we shall denote an exact category simply by $\mathcal{A}$.

Example 1.3.2 (Abelian categories)**.** Let $\mathcal{A}$ be an abelian category. The pair $(\mathcal{A}, D)$ is an exact category where D is simply the collection of short exact sequences. This is clearly exact as $\mathcal{A}$ fully embeds into itself.

Remark 1.3.3. Given an exact category $(\mathcal{A}, D)$, all split short exact sequences are conflations. Let $F\colon \mathcal{A} \to \mathcal{B}$ be its full additive embedding into an abelian category. An arbitrary split short exact sequence is isomorphic to $0_{\mathcal{A}} \to X \to X \oplus Y \to Y \to 0_{\mathcal{A}}$ for some $X, Y \in \mathcal{A}$. Since the corresponding sequence $0_{\mathcal{B}} \to FX \to F(X \oplus Y) \to FY \to 0_{\mathcal{A}}$ is isomorphic to $0_{\mathcal{B}} \to FX \to FX \oplus FY \to FY \to 0_{\mathcal{B}}$ by Lemma 1.1.21 and the latter sequence is a short exact sequence, a split short exact sequence is a conflation by Definition 1.3.1(1).

Example 1.3.4 (Additive categories)**.** Let $\mathcal{A}$ be an additive category. The pair $(\mathcal{A}, \oplus)$ is an exact category where $\oplus$ is the collection of all split exact sequences. By split exact sequences, we mean sequences of the form (1.3.1) such that there exists some $X, Y \in \mathcal{A}$ and isomorphisms u, v and w making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "\begin{pmatrix} \mathrm{id}_X \\ 0_{X,Y} \end{pmatrix}"] \arrow[d, "u"] & X \oplus Y \arrow[r, "\begin{pmatrix} 0_{X,Y} & \mathrm{id}_Y \end{pmatrix}"] \arrow[d, "v"] & Y \arrow[d, "w"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C.
\end{tikzcd}$$

This is an exact category since $\mathcal{A}$ fully embeds into $\mathbf{Fct}(\mathcal{A}, \mathbf{Ab})$ by the Yoneda Lemma. Let us denote $\mathbf{Fct}(\mathcal{A}, \mathbf{Ab})$ by $\mathcal{C}$, and let us denote the functor $\hom_{\mathcal{A}}(P, -)$ (an element in $\mathcal{C}$) by h^P for all $P \in \mathcal{A}$. The embedding $h^-\colon \mathcal{A} \to \mathcal{C}$ is full by the

Yoneda lemma. Furthermore, given an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, we have a commutative diagram of the following form:

$$\begin{tikzcd}
0 \arrow[r] & \hom_{\mathcal{C}}(h^P, X) \arrow[r] \arrow[d, "\cong"] & \hom_{\mathcal{C}}(h^P, Y) \arrow[r] \arrow[d, "\cong"] & \hom_{\mathcal{C}}(h^P, Z) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & X(P) \arrow[r] & Y(P) \arrow[r] & Z(P) \arrow[r] & 0
\end{tikzcd}$$

for all $P \in \mathcal{A}$. Thus, the object h^P is projective in $\mathcal{C}$ for all $P \in \mathcal{A}$, and it follows by Lemma 1.2.22 that $(\mathcal{A}, \oplus)$ is exact. See [Awo10, Lemma 8.2] for a proof of the Yoneda lemma.

Remark 1.3.5. Since not all additive categories are abelian, it follows that not all exact categories are abelian.

Lemma 1.3.6. *Let $\mathcal{A}$ be an exact category. Conflations are closed under isomorphisms. That is, given a conflation*

$$0_{\mathcal{A}} \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0_{\mathcal{A}}$$

and isomorphisms $s\colon X \to X'$, $t\colon Y \to Y'$, and $u\colon Z \to Z'$, then

$$0_{\mathcal{A}} \longrightarrow X' \xrightarrow{t \circ f \circ s^{-1}} Y' \xrightarrow{u \circ g \circ t^{-1}} Z' \longrightarrow 0_{\mathcal{A}}$$

is a conflation.

Proof: First fix a full embedding into an abelian category. Then, using Lemma 1.2.17, the proof is routine. □

Lemma 1.3.7. *Let $\mathcal{A}$ be an exact category. Let*

$$0_{\mathcal{A}} \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0_{\mathcal{A}} \quad \text{and} \quad 0_{\mathcal{A}} \longrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \longrightarrow 0_{\mathcal{A}}$$

be conflations in $\mathcal{A}$. Then, the sequence

$$0_{\mathcal{A}} \longrightarrow X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \longrightarrow 0_{\mathcal{A}}$$

is a conflation.

Proof: Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full embedding into an abelian category. It is routine to show that $F(f) \oplus F(f')$ is a monomorphism and $F(g) \oplus F(g')$ is an epimorphism. In $(FY \oplus FY')^{\mathrm{mono}}$, we have the following equivalences (see Definition 1.2.2):

$$\operatorname{Ker}(F(g) \oplus F(g')) \sim \operatorname{Ker} F(g) \oplus \operatorname{Ker} F(g') \sim F(f) \oplus F(f').$$

The first equivalence is by Lemma 1.2.6 and Lemma 1.2.5. To show the second equivalence, note that we have isomorphisms u and u' such that $\operatorname{Ker} F(g) \circ u = F(f)$ and $\operatorname{Ker} F(g') \circ u' = F(f')$ by our assumptions and Lemma 1.2.17. Hence, the morphism $u \oplus u'$ is an isomorphism and $(F(g) \oplus F(g')) \circ (u \oplus u') = F(f) \oplus F(f')$. Note that functor F is an equivalence on its image, and thus is additive meaning that Lemma 1.1.21 applies. Hence, by Lemma 1.2.17 and Lemma 1.3.6 (for the abelian category), the following sequence is exact in $\mathcal{B}$:

$$0_{\mathcal{B}} \longrightarrow F(X \oplus X') \xrightarrow{F(f \oplus f')} F(Y \oplus Y') \xrightarrow{F(g \oplus g')} F(Z \oplus Z') \longrightarrow 0_{\mathcal{B}}.$$

We then get our conclusion by the definition of an exact category. □

Corollary 1.3.8. *Let $\mathcal{A}$ be an exact category. For all $X, Y \in \mathcal{A}$, we have that*

$$0_{\mathcal{A}} \to X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ 0_{X,Y} \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} 0_{X,Y} & \mathrm{id}_Y \end{pmatrix}} Y \to 0_{\mathcal{A}}$$

is a conflation.

Proof: This is an immediate consequence of Lemma 1.3.6 and Lemma 1.3.7. □

Definition 1.3.9 (Exact functors)**.** Let $(\mathcal{A}, D_{\mathcal{A}})$ and $(\mathcal{B}, D_{\mathcal{B}})$ be exact categories. An additve functor $F\colon \mathcal{A} \to \mathcal{B}$ is *exact* if it *preserves conflations.* That is, for every sequence

$$0_{\mathcal{A}} \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0_{\mathcal{A}}$$

in $D_{\mathcal{A}}$, the sequence

$$0_{\mathcal{B}} \longrightarrow FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \longrightarrow 0_{\mathcal{B}}$$

belongs to $D_{\mathcal{B}}$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be exact categories. A biadditive $H\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *biexact* if $H(X, -)$ and $H(-, Y)$ are exact for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

Remark 1.3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories viewed as exact categories (Example 1.3.4). Then, the exact functors between them are precisely the additive functors.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories viewed as exact categories (Example 1.3.2). By definition, an exact functor preserves short exact sequences, which coincides with the standard notion of an exact functor between abelian categories (Definition 1.2.18).

Definition 1.3.11 (Exact subcategory, thick)**.** Let $(\mathcal{A}, D)$ be an exact category. A strictly full subcategory $\mathcal{B}$ of $\mathcal{A}$ together with the subcollection $D|_{\mathcal{B}}$ of conflations in $\mathcal{A}$ whose objects belong to $\mathcal{B}$ is an *exact subcategory* if $(\mathcal{B}, D|_{\mathcal{B}})$ is an exact category. An exact subcategory of an abelian category is called *thick* if it is also an abelian subcategory.

Example 1.3.12 (R-$\mathbf{mod}^{\text{fg}}$ is an exact subcategory of R-$\mathbf{mod}$)**.** Let R be a ring with unity. Finitely generated left R-modules induce a strictly full subcategory of R-$\mathbf{mod}$ denoted by R-$\mathbf{mod}^{\text{fg}}$.

The zero R-module is finitely generated. If M and M' are left R-modules finitely generated left by X and Y respectively, then the product $M \times M'$ is a biproduct of M and M' and is finitely generated by $X \times Y$. Hence, we have that R-$\mathbf{mod}^{\text{fg}}$ is a (strictly full) additive subcategory of R-$\mathbf{mod}$.

Now, suppose that

$$0 \to M \xrightarrow{f} N \xrightarrow{g} M' \to 0$$

is a short R-$\mathbf{mod}$-exact sequence where M is finitely generated by $\{x_1, \ldots, x_n\}$ and M' is finitely generated by $\{x'_1, \ldots, x'_m\}$. An epimorphism in R-$\mathbf{mod}$ is surjective.

So, there exists $y_1, \dots, y_m \in X$ such that $g(y_i) = x'_i$ for $i = 1, \dots, m$. Let $z \in N$ and note that $g(z) = \sum_{i=1}^{m} r_i x'_i$ for some $r_1, \dots, r_m \in R$. We can use the traditional kernels and images for R-**mod**. We have $z - \sum_{i=1}^{m} r_i y_i \in \operatorname{Ker} g = \operatorname{Im} f$. Hence, there exists some $s_1, \dots, s_n$ such that $z = \sum_{i=1}^{n} s_i f(x_i) + \sum_{i=1}^{m} r_i y_i$ and we see that N is finitely generated. Hence, we have that $R\text{-}\mathbf{mod}^{\mathrm{fg}}$ is exact via the exact sequences in R-**mod**.

Example 1.3.13 ($R\text{-}\mathbf{mod}^{\mathrm{fg}}$ is thick in R-**mod** if R is left-Noetherian)**.** A commutative ring R with unity is *left-Noetherian* if every left ideal is finitely generated. If R is a left-Noetherian ring with unity, then $R\text{-}\mathbf{mod}^{\mathrm{fg}}$ is thick in R-**mod** because finitely generated modules over left-Noetherian rings are Noetherian and so are their submodules and quotients.

On the other hand, consider the ring $R = \mathbb{R}[x_1, x_2, \dots]$ of polynomials over the reals with countably many indeterminants, a non-Noetherian ring. Note that it is finitely generated over itself. The R-linear map $R \to R$ given by evaluation at zero has a cokernel whose codomain is isomorphic to the ideal $I = \langle x_1, x_2, \dots \rangle$. Since this ideal is not finitely generated over R, we have that $R\text{-}\mathbf{mod}^{\mathrm{fg}}$ is not thick in R-**mod**.

Example 1.3.14 ($\operatorname{Prj} \mathcal{A}$ is an exact subcategory of $\mathcal{A}$)**.** Let $\mathcal{A}$ be an abelian category. It is clear that $\hom_{\mathcal{A}}(0_{\mathcal{A}}, -)$ sends a short exact sequence to a short exact sequence of trivial groups. Hence, we have $0_{\mathcal{A}} \in \operatorname{Prj} \mathcal{A}$. Now, let $X, Y \in \operatorname{Prj} \mathcal{A}$ and let

$$0_{\mathcal{A}} \to A \xrightarrow{f} B \xrightarrow{g} C \to 0_{\mathcal{A}}$$

be a short exact sequence with $A, B, C \in \mathcal{A}$. Then, we note that we have the following equivalences in $\hom_{\mathcal{A}}(X \oplus Y, B)^{\mathrm{mono}}$ (see Definition 1.2.2):

$$\begin{aligned}
\operatorname{Ker} \hom(X \oplus Y, g) &\sim \operatorname{Ker}(\hom(X, g_{1,1}) \oplus \hom(Y, g_{1,2})) \\
&\sim \operatorname{Ker} \hom(X, g_{1,1}) \oplus \operatorname{Ker} \hom(Y, g_{1,2})) \qquad \text{(by Lemma 1.2.6)} \\
&\sim \hom(X, f_{1,1}) \oplus \hom(X, f_{1,2}) \qquad \text{(since } X, Y \in \operatorname{Prj} \mathcal{A}) \\
&\sim \hom(X \oplus Y, f).
\end{aligned}$$

It is routine to show that $\hom(X \oplus Y, f)$ is a monomorphism and $\hom(X \oplus Y, g)$ is an epimorphism. Hence, by Lemma 1.2.17, we have $X \oplus Y \in \operatorname{Prj} \mathcal{A}$ since we have thus shown that $\hom(X \oplus Y, -)$ is exact. Therefore, the subcategory $\operatorname{Prj} \mathcal{A}$ is a strictly full additive subcategory of $\mathcal{A}$.

If $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$ is a short exact sequence in $\mathcal{A}$ and $X, Z \in \operatorname{Prj} \mathcal{A}$, then we have $Y \cong X \oplus Z$ by Lemma 1.2.22 and, hence, $Y \in \operatorname{Prj} \mathcal{A}$. Thus, the additive category $\operatorname{Prj} \mathcal{A}$ is an exact subcategory of $\mathcal{A}$ and in that sense is precisely $(\operatorname{Prj} \mathcal{A}, \oplus)$.

1.4 Triangulated Categories

If one has not encountered a triangulated category before, its definition might appear strange. Its axioms are motivated by the not necessarily abelian structure of the derived category of an abelian category. The role of short exact sequences is replaced in a nice way by distinguished triangles.

That said, many experts admit that the axioms are slightly unfinished/incorrect, but they work well for what we intend to do. In fact, the reader may choose to not worry too much about (TR4) in Definition 1.4.2 as it will contribute nothing on the level of Grothendieck groups.

Definition 1.4.1 (Triangle)**.** Let $\mathcal{A}$ be an additive category and $T\colon \mathcal{A} \to \mathcal{A}$ be an equivalence of categories. A *triangle* in $(\mathcal{A}, T)$ is a sequence of morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX.$$

Definition 1.4.2 (Triangulated category)**.** A *triangulated category* $(\mathcal{A}, T, D)$ consists of an additive category $\mathcal{A}$, an equivalence of categories $T\colon \mathcal{A} \to \mathcal{A}$, and a collection D of triangles in $(\mathcal{A}, T)$, which we call *distinguished triangles*, such that the following axioms hold:

(TR1a) For all $X \in \mathcal{A}$, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0_{\mathcal{A}} \longrightarrow TX$$

is a distinguished triangle.

(TR1b) If $f\colon X \to Y$ is an arbitrary morphism of $\mathcal{A}$, there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

for some $Z \in \mathcal{A}$.

(TR1c) If f, g and h are isomorphisms making a diagram of the following form commute

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & TX \arrow[d, "T(f)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX',
\end{tikzcd}$$

then the first row is a distinguished triangle if and only if the second row is a distinguished triangle.

(TR2) A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$

is a distinguished triangle if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-T(f)} TY$$

is a distinguished triangle. (Recall that hom-sets in additive categories are abelian groups and $-T(f)$ is the additive inverse of $T(f)$.)

(TR3) Given a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

and two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \qquad \text{and} \qquad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX',$$

there exists a morphism $w\colon Z \to Z'$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & TX \arrow[d, "T(u)"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & TX'.
\end{tikzcd}$$

(TR4) Given distinguished triangles

$$X \xrightarrow{n} Y \xrightarrow{g} Z \xrightarrow{h} TX, \qquad Y \xrightarrow{m} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TY,$$

and a distinguished triangle of the form

$$X \xrightarrow{m \circ n} Y' \xrightarrow{g''} Z'' \xrightarrow{h''} TX\ ,$$

we have a distinguished triangle of the form

$$Z \xrightarrow{u} Z'' \xrightarrow{v} Z' \xrightarrow{w} TZ$$

such that

$$g' = v \circ g'', \quad h = h'' \circ u, \quad w = T(g) \circ h',$$

$$h' \circ v = T(n) \circ h'', \quad \text{and} \quad u \circ g = g'' \circ m.$$

We will sometimes write simply $\mathcal{A}$ for $(\mathcal{A}, T, D)$ if there is no ambiguity.

Lemma 1.4.3 ([Nee01, Proposition 1.2.1])**.** *Suppose that the following are distinguished triangles in a triangulated category $\mathcal{A}$:*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \quad \textit{and } X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'.$$

Then, the componentwise biproduct

$$X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} TX$$

is also a distinguished triangle in $\mathcal{A}$.

Corollary 1.4.4. *Let $\mathcal{A}$ be a triangulated category. Then, the following is a distinguished triangle for all $X, Y \in \mathcal{A}$:*

$$X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ 0_{X,Y} \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} 0_{Y,X} & \mathrm{id}_Y \end{pmatrix}} Y \xrightarrow{0_{Y,TX}} TX.$$

Proof: This follows immediately from (TR1a), (TR2), and Lemma 1.4.3. ☐

Definition 1.4.5 (Triangulated functor)**.** Let $(\mathcal{A}, T, D)$ and $(\mathcal{B}, T', D')$ be triangulated categories. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *triangulated* if there is a natural transformation $\eta\colon F \circ T \to T' \circ F$ such that

$$FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{\eta^X \circ F(h)} T'(FX)$$

is a triangle in D' for every triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ in D.

Definition 1.4.6 (Triangulated subcategory)**.** Let $(\mathcal{A}, T, D)$ be a triangulated category. Suppose a non-empty full subcategory $\mathcal{D}$ of $\mathcal{A}$ satisfies the following:

- For every $X \in \mathcal{E}$, we have $T(X) \in \mathcal{E}$.
- If D has a triangle in $(\mathcal{A}, T)$ of the form $X \to Y \to Z \to T(X)$ and two objects of $\{X, Y, Z\}$ belongs to $\mathcal{E}$, then the third object belongs to $\mathcal{E}$.

By (TR1a) in Definition 1.4.2, we have a distinguished triangle of the form $X \xrightarrow{\mathrm{id}_X} X \to 0_{\mathcal{A}} \to T(X)$ for any $X \in \mathcal{E}$. Hence, we have $0_{\mathcal{A}} \in \mathcal{E}$. By (TR1a) and (TR1c), we see that $\mathcal{E}$ is isomorphism closed in $\mathcal{A}$. By Corollary 1.4.4, we have that $\mathcal{E}$ is closed under biproducts in $\mathcal{A}$. Thus, the category $\mathcal{E}$ is a strictly full additive subcategory of $\mathcal{A}$.

It is routine to verify that the induced functor $T|_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E}$ is an equivalence of categories. Furthermore, it is clear that $D|_{\mathcal{E}}$, the collection of distinguished triangles in $\mathcal{A}$ with objects in $\mathcal{E}$, satisfies the axioms making $(\mathcal{E}, T|_{\mathcal{E}}, D|_{\mathcal{E}})$ a triangulated category. We call this category a *triangulated subcategory* of $\mathcal{A}$.

Definition 1.4.7 (Dense)**.** A strictly full triangulated subcategory $\mathcal{E}$ of a triangulated category $\mathcal{A}$ is *dense* if for all $X \in \mathcal{E}$ the following holds: if $X \cong Y$ in $\mathcal{A}$ for some $Y \in \mathcal{A}$, then $Y \in \mathcal{E}$.

Chapter 2

Grothendieck Groups

We now are ready to define the Grothendieck group of an exact category and the triangulated Grothendieck group of a triangulated category. Recall that we are assuming that all categories are essentially small (i.e. equivalent to a small category).

2.1 Basic Properties

Definition 2.1.1 ($F_{\mathrm{Iso}\,\mathcal{A}}$)**.** Let $\mathcal{A}$ be a category. We define $F_{\mathrm{Iso}\,\mathcal{A}}$ to be the free abelian group whose basis is the set $\mathrm{Iso}\,\mathcal{A}$ of the isomorphism classes of $\mathcal{A}$ (see Definition 1.1.4).

Definition 2.1.2 (Grothendieck group of an exact category)**.** Let $\mathcal{A}$ be an exact category. Let N be the (normal) subgroup of $F_{\mathrm{Iso}\,\mathcal{A}}$ generated by the following set:

$$\{[Y]-[X]-[Z] \mid \text{there exists a conflation of the form } 0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}\}$$

The *Grothendieck group* of $\mathcal{A}$ is the quotient group $F_{\mathrm{Iso}\,\mathcal{A}}/N$ and is denoted by $K_0(\mathcal{A})$. We shall use a standard abuse of notation and denote the cosets of N in $F_{\mathrm{Iso}\,\mathcal{A}}$ by representatives in $F_{\mathrm{Iso}\,\mathcal{A}}$. Context will ensure no ambiguity.

Lemma 2.1.3. *Let $\mathcal{A}$ be an exact category. The following hold:*

1. *We have* $[X \oplus Y] = [X] + [Y]$ *in* $K_0(\mathcal{A})$ *for all* $X, Y \in \mathcal{A}$.

2. *The class of a zero object represents the additive identity of* $K_0(\mathcal{A})$. *That is, we have* $[0_\mathcal{A}] = 0_{K_0(\mathcal{A})}$.

3. *Every element in* $K_0(\mathcal{A})$ *can be written in the form* $[X]-[Y]$ *for some* $X, Y \in \mathcal{A}$.

Proof: The first statement follows from Corollary 1.3.8. The second statement is an immediate consequence since $0_\mathcal{A} \oplus 0_\mathcal{A} \cong 0_\mathcal{A}$. For the third statement, we note that we can write an arbitrary element of $K_0(\mathcal{A})$ in the form $\sum_{i=1}^n [X_i] - \sum_{i=1}^m [Y_i]$ for some $X_1, \ldots, X_n, Y_1, \ldots, Y_m \in \mathcal{A}$ since $K_0(\mathcal{A})$ is abelian. By the first statement, this element is equal to $[\bigoplus_{i=1}^n X_i] - [\bigoplus_{i=1}^m Y_i]$. □

Definition 2.1.4 (Split Grothendieck group)**.** Let $\mathcal{A}$ be an additive category. Recall that $(\mathcal{A}, \oplus)$ is an exact category (Example 1.3.4). The Grothendieck group of $(\mathcal{A}, \oplus)$ is called the *split Grothendieck group* of $\mathcal{A}$ and is denoted $K_0^{\text{split}}(\mathcal{A})$.

Lemma 2.1.5. *Let* $\mathcal{A}$ *be an exact category and let* $X, Y \in \mathcal{A}$. *Then, the following are equivalent:*

1. $[X] = [Y]$ *in* $K_0(\mathcal{A})$.

2. *There are conflations of the form*

$$0_\mathcal{A} \to U \to X \oplus V \to W \to 0_\mathcal{A} \quad \textit{and} \quad 0_\mathcal{A} \to U \to Y \oplus V \to W \to 0_\mathcal{A}$$

   for some $U, V, W \in \mathcal{A}$.

Proof: (2)⇒(1) is straightforward. We shall show (1)⇒(2). Suppose $[X] = [Y]$ in $K_0(\mathcal{A})$. Since $[X] - [Y]$ thus belongs to N of Definition 2.1.2, there are conflations of the form

$$0_\mathcal{A} \to A_i \to B_i \to C_i \to 0_\mathcal{A} \quad \text{and} \quad 0_\mathcal{A} \to A'_j \to B'_j \to C'_j \to 0_\mathcal{A}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that the following equality holds in $F_{\mathrm{Iso}\,\mathcal{A}}$:

$$[X] - [Y] = \sum_{i=1}^{n}([B_i] - [A_i] - [C_i]) - \sum_{j=1}^{m}([B'_j] - [A'_j] - [C'_j]).$$

After some rearrangement, we have the following:

$$[X] + \sum_{i=1}^{n}([A_i] + [C_i]) + \sum_{j=1}^{m}[B'_j] = [Y] + \sum_{i=1}^{n}[B_i] + \sum_{j=1}^{m}([A'_j] + [C'_j]).$$

Since this equality is in $F_{\mathrm{Iso}\,\mathcal{A}}$, the terms on the left are a permutation of the terms on the right (an immediate consequence of the definition of a free abelian group). Thus, by Lemma 1.1.18, we have the following isomorphism in $\mathcal{A}$:

$$X \oplus \bigoplus_{i=1}^{n}(A_i \oplus C_i) \oplus \bigoplus_{j=1}^{m} B'_j \cong Y \oplus \bigoplus_{i=1}^{n} B_i \oplus \bigoplus_{j=1}^{m}(A'_j \oplus C'_j). \tag{2.1.1}$$

Set the following:

$$A = \bigoplus_{i=1}^{n} A_i, \qquad B = \bigoplus_{i=1}^{n} B_i, \qquad C = \bigoplus_{i=1}^{n} C_i,$$
$$A' = \bigoplus_{j=1}^{m} A'_i, \qquad B' = \bigoplus_{j=1}^{m} B'_i, \qquad C' = \bigoplus_{j=1}^{m} C'_i.$$

One can easily verify that there are conflations of the form $0_{\mathcal{A}} \to A' \to A' \to 0_{\mathcal{A}} \to 0_{\mathcal{A}}$ and $0_{\mathcal{A}} \to 0_{\mathcal{A}} \to X \oplus Y \oplus C' \to X \oplus Y \oplus C' \to 0_{\mathcal{A}}$. By Lemma 1.3.7, there is a conflation of the form $0_{\mathcal{A}} \to A \to B \to C \to 0_{\mathcal{A}}$ and thus a conflation of the form

$$0_{\mathcal{A}} \longrightarrow A \oplus A' \longrightarrow X \oplus Y \oplus B \oplus A' \oplus C' \longrightarrow X \oplus Y \oplus C \oplus C' \longrightarrow 0_{\mathcal{A}}.$$

By a similar argument, we have a conflation of the form

$$0_{\mathcal{A}} \longrightarrow A \oplus A' \longrightarrow Y \oplus A \oplus B' \oplus X \oplus C \longrightarrow X \oplus Y \oplus C \oplus C' \longrightarrow 0_{\mathcal{A}}. \tag{2.1.2}$$

The isomorphism in (2.1.1) can be written as $X \oplus A \oplus C \oplus B' \cong Y \oplus B \oplus A' \oplus C'$. By Lemma 1.3.6, there is a conflation of the form

$$0_{\mathcal{A}} \longrightarrow A \oplus A' \longrightarrow Y \oplus Y \oplus B \oplus A' \oplus C' \longrightarrow X \oplus Y \oplus C \oplus C' \longrightarrow 0_{\mathcal{A}}.$$

since $X \oplus A \oplus C \oplus B'$ is a summand of the middle term of (2.1.2). We set $U = A \oplus A'$, $V = Y \oplus B \oplus A' \oplus C'$, and $W = X \oplus Y \oplus C \oplus C'$ and note that we are done. □

Corollary 2.1.6. *Let $\mathcal{A}$ be an additive category and $X, Y \in \mathcal{A}$. We have $[X] = [Y]$ in $K_0^{\text{split}}(\mathcal{A})$ if and only if $X \oplus Z \cong Y \oplus Z$ for some $Z \in \mathcal{A}$.*

Proof: Since the conflations are split, this is a special case of Lemma 2.1.5. ☐

Proposition 2.1.7 (Universal property)**.** *Let $\mathcal{A}$ be an exact category. Let H be an abelian group, and let $f\colon \operatorname{Iso}\mathcal{A} \to H$ be a map that is* additive over conflations*. That is, if there exists a conflation of the form $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$, then $f[Y] = f[X] + f[Z]$. Then, there exists a unique group homomorphism $g\colon K_0(\mathcal{A}) \to H$ such that $g \circ i = f$, where $i\colon \operatorname{Iso}\mathcal{A} \to K_0(\mathcal{A})$ is the natural map (i.e. $i[X] = [X]$ for all $X \in \mathcal{A}$).*

Proof: By the universal property of free abelian groups, there exists a unique group homomorphism $\tilde{f}\colon F_{\operatorname{Iso}\mathcal{A}} \to H$ such that $\tilde{f} \circ j = f$ where $j\colon \operatorname{Iso}\mathcal{A} \to F_{\operatorname{Iso}\mathcal{A}}$ is the canonical inclusion. Since $\tilde{f}([X] - [Y] + [Z]) = \tilde{f}[X] - \tilde{f}[Y] + \tilde{f}[Z] = f[X] - f[Y] + f[Z] = 0$ for all conflations $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$ in $\mathcal{A}$, the subgroup N of Definition 2.1.2 is contained in the kernel of $\tilde{f}$. Therefore, by the factorization theorem (see [Gri07, Theorem I.5.1]), there exists a unique group homomorphism $g\colon K_0(\mathcal{A}) \to H$ making the following diagram commute:

$$\begin{tikzcd}
\operatorname{Iso}\mathcal{A} \arrow[r, "f"] \arrow[d, "j"] \arrow[dd, bend right, "i"'] & H \\
F_{\operatorname{Iso}\mathcal{A}} \arrow[ur, "\exists!\tilde{f}"] \arrow[d, "\pi"] & \\
K_0(\mathcal{A}) \arrow[uur, "\exists!g"'] &
\end{tikzcd}$$

where π is the quotient map. ☐

Proposition 2.1.8 (Homomorphisms induced from exact functors)**.** *Let $(\mathcal{A}, D_{\mathcal{A}})$ and $(\mathcal{B}, D_{\mathcal{B}})$ be exact categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor. There is a group homomorphism*

$$[F]\colon K_0(\mathcal{A}) \to K_0(\mathcal{B})$$

such that $[F]([X]-[Y]) = [FX]-[FY]$ *for all* $X, Y \in \mathcal{A}$.

Proof: Define a map $f\colon \operatorname{Iso}\mathcal{A} \to K_0(\mathcal{B})$ by $f[X] = [FX]$ for all $X \in \mathcal{A}$. If a sequence of the form $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$ is in $D_{\mathcal{A}}$, then a sequence of the form $0_{\mathcal{B}} \to FX \to FY \to FZ \to 0_{\mathcal{B}}$ is in $D_{\mathcal{B}}$ by exactness of F. Hence, we have $f[Y] = [FY] = [FX] + [FZ] = f[X] + f[Z]$ in $K_0(\mathcal{B})$. Thus, the universal property of Proposition 2.1.7 defines a homomorphism $[F]$ with the desired property. □

Example 2.1.9 ($K_0(\mathcal{A})$ is quotient of $K_0^{\text{split}}(\mathcal{A})$)**.** Let $(\mathcal{A}, D)$ be an exact category. The identity functor $I\colon \mathcal{A} \to \mathcal{A}$ defines an exact functor from $(\mathcal{A}, \oplus)$ to $(\mathcal{A}, D)$ (see Remark 1.3.3). Hence, we have a homomorphism $[I]\colon K_0^{\text{split}}(\mathcal{A}) \to K_0(\mathcal{A})$ by Proposition 2.1.8. Clearly, this group homomorphism is surjective. By the first isomorphism theorem, we have that $K_0(\mathcal{A})$ is a quotient of $K_0^{\text{split}}(\mathcal{A})$ by the elements in the kernel of $[I]$, which is generated by the $[X]-[Y]+[Z]$ for the non-split conflations $0_{\mathcal{A}} \to X \to Y \to Z \to 0_{\mathcal{A}}$ in D.

Lemma 2.1.10. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be exact categories. The following hold:*

(1) If $F, G\colon \mathcal{A} \to \mathcal{B}$ *are exact functors and there exists a natural isomorphism* $\eta\colon F \to G$, *then the induced group homomorphisms satisfy* $[F] = [G]$.

(2) If $F\colon \mathcal{A} \to \mathcal{B}$ *is an equivalence of categories, then the induced group homomorphism* $[F]\colon K_0(\mathcal{A}) \to K_0(\mathcal{B})$ *is an isomorphism.*

Proof: (1) Let $X, Y \in \mathcal{A}$. Since $\eta^X\colon FX \to GX$ and $\eta^Y\colon FY \to GY$ are isomorphisms, we have $[F]([X]-[Y]) = [FX]-[FY] = [GX]-[GY] = [G]([X]-[Y])$. Thus, we have $[F] = [G]$ by Lemma 2.1.3(3)

(2) By definition, there exists a functor $G\colon \mathcal{B} \to \mathcal{A}$ and two natural isomorphisms $\eta\colon F \circ G \to \mathrm{id}_{\mathcal{B}}$ and $\sigma\colon \mathrm{id}_{\mathcal{A}} \to G \circ F$. Thus, we have $(G \circ F)([X]-[Y]) = [(G \circ F)X] - [(G \circ F)Y] = [X]-[Y]$ for all $X, Y \in \mathcal{A}$, and, therefore, $G \circ F = \mathrm{id}_{K_0(\mathcal{A})}$. Similarly, $F \circ G = \mathrm{id}_{K_0(\mathcal{B})}$. □

Lemma 2.1.11. *Let $\mathcal{A}$ be an abelian category. Let*

$$0_{\mathcal{A}} \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \to 0_{\mathcal{A}}$$

be an $\mathcal{A}$-exact sequence. We have $\sum_{i=0}^{n}(-1)^i[X_i] = 0$ in $K_0(\mathcal{A})$.

Proof: Fix kernels and cokernels. Let $e_i \colon X_i \to Z_i$ be the epimorphism such that $\operatorname{Im} f_i \circ e_i = f_i$ for $i = 0, 1, \ldots, n-1$. We have the following exact sequence:

$$0_{\mathcal{A}} \to K_i \xrightarrow{\operatorname{Ker} f_i} X_i \xrightarrow{e_i} Z_i \to 0_{\mathcal{A}}$$

for all $i = 0, 1, \ldots, n-1$. Note that $K_0 \cong 0_{\mathcal{A}}$ and $Z_{n-1} \cong X_n$. Since $\operatorname{Ker} f_i$ and $\operatorname{Im} f_{i-1}$ represent the same subobject we have $Z_{i-1} \cong K_i$ for all $i = 1, 2, \ldots, n-1$. Thus, we have the following:

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i[X_i] &= (-1)^n[X_n] + \sum_{i=1}^{n-1}(-1)^i([K_i] + [Z_i]) + [K_0] + [Z_0] \\
&= (-1)^n[X_n] + (-1)^{n-1}[Z_{n-1}] - [Z_0] + [K_0] + [Z_0] \\
&= (-1)^n[X_n] + (-1)^{n-1}[X_n] + [0_{\mathcal{A}}] \\
&= 0.
\end{aligned}$$

The equalities taking place in $K_0(\mathcal{A})$. □

Definition 2.1.12 (Triangulated Grothendieck group)**.** Let $(\mathcal{A}, T, D)$ be a triangulated category. The quotient group

$$F_{\operatorname{Iso}(\mathcal{A})}/\langle [Y] - [X] - [Z] \mid X \to Y \to Z \to T(X) \text{ is a member of } D \rangle$$

is called the *triangulated Grothendieck group* of $(\mathcal{A}, T, D)$ and is denoted by $K_0^{\triangle}(\mathcal{A})$.

Lemma 2.1.13. *Let $(\mathcal{A}, T, D)$ be a triangulated category. Then, the following holds:*

1. *For all $X, Y \in \mathcal{A}$, we have $[X \oplus Y] = [X] + [Y]$ in $K_0^{\triangle}(\mathcal{A})$.*

2. *For all $X \in \mathcal{A}$, we have $[T(X)] = -[X]$ in $K_0^{\triangle}(\mathcal{A})$.*

3. *Every element in* $K_0^{\triangle}(\mathcal{A})$ *is of the form* $[X]$ *for some* $X \in \mathcal{A}$.

Proof: The first statement holds by Corollary 1.4.4. The second statement holds by (TR1a) and (TR2) of Definition 1.4.2. The third statement is proven in similar way as Lemma 2.1.3. □

Lemma 2.1.14. *Let* $(\mathcal{A}, T, D)$ *be a triangulated category. Then, for any objects* $X, Y \in \mathcal{A}$, *the following are equivalent:*

1. $[X] = [Y]$ *in* $K_0^{\triangle}(\mathcal{A})$.

2. *There are distinguished triangles*

$$X \oplus C_1 \to C_2 \to C_3 \to T(X \oplus C_1) \quad \textit{and} \quad Y \oplus C_1 \to C_2 \to C_3 \to T(Y \oplus C_1)$$

for some objects $C_1, C_2, C_3 \in \mathcal{A}$.

Proof: See [ML13, Corollary 2.6.16] for a proof of this statement. □

Lemma 2.1.14 is very reminiscent of Lemma 2.1.5. We can also give a similar universal property (see [ML13, Proposition 2.6.14]), which we employ in the following proposition:

Proposition 2.1.15. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be triangulated categories and let* $F\colon \mathcal{A} \to \mathcal{B}$ *be a triangulated functor. We have a group homomorphism* $[F]\colon K_0^{\triangle}(\mathcal{A}) \to K_0^{\triangle}(\mathcal{B})$ *such that* $[F][X] = [FX]$ *for all* $X \in \mathcal{A}$.

Proof: We define a map $f\colon \operatorname{Iso}\mathcal{A} \to K_0^{\triangle}(\mathcal{B})$ by $f[X] = [FX]$ for all $X \in \mathcal{A}$. It is well defined and $f[Y] = f[X] + f[Z]$ for all distinguished triangles $X \to Y \to Z \to TX$ in $D_{\mathcal{A}}$ since F is triangulated. By [ML13, Proposition 2.6.14]), we get the group homomorphism $[F]$. □

Definition 2.1.16. Let $\mathcal{A}$ be a triangulated category. If $\mathcal{B}$ is a strictly full dense triangulated subcategory of $\mathcal{A}$, then the natural inclusion is a triangulated functor and induces a homomorphism $K_0^{\triangle}(\mathcal{B}) \to K_0^{\triangle}(\mathcal{A})$. Denote the image of this map by $H_{\mathcal{B}}$.

Let H be a subgroup of $K_0^{\triangle}(\mathcal{A})$. The set $\{X \mid [X] \in H\}$ induces a strictly full dense triangulated subcategory of $\mathcal{A}$. Denote this category by $\mathcal{A}_H$.

Theorem 2.1.17 ([Tho97, Theorem 2.1])**.** *Let $\mathcal{A}$ be a triangulated category. Let X be the collection of all strictly full dense triangulated subcategories of $\mathcal{A}$ and Y be the collection of all subgroups of $K_0^{\triangle}(\mathcal{A})$. The map $X \to Y$ given by $\mathcal{B} \mapsto H_{\mathcal{B}}$ for all $\mathcal{B} \in X$ and the map $Y \to X$ given by $H \mapsto \mathcal{A}_H$ for all $H \in Y$ yield a one-to-one correspondence.*

Corollary 2.1.18. *Let $(\mathcal{A}, T, D)$ be a triangulated category. If the smallest strictly full dense triangulated subcategory of $\mathcal{A}$ whose object class contains some set $S \subseteq \operatorname{Ob}\mathcal{A}$ is $\mathcal{A}$ itself, then we have $K_0^{\triangle}(\mathcal{A}) = \langle [X] \in K_0^{\triangle}(\mathcal{A}) \mid X \in S \rangle$.*

Proof: Since $\langle [X] \in K_0^{\triangle}(\mathcal{A}) \mid X \in S \rangle$ is a subgroup of $K_0^{\triangle}(\mathcal{A})$, it is assigned a strictly full dense triangulated subcategory of $(\mathcal{A}, T, D)$ by Theorem 2.1.17, which contains S. Since $(\mathcal{A}, T, D)$ is the smallest subcategory with this property, we have $K_0^{\triangle}(\mathcal{A}) = \langle [X] \in K_0^{\triangle}(\mathcal{A}) \mid X \in S \rangle$. □

2.2 Some Computations and Theorems

Given an essentially small exact category $\mathcal{A}$, the group homomorphism

$$[\mathrm{id}_{\mathcal{A}}] \colon K_0(\mathcal{A}) \to K_0(\mathcal{A})$$

induced by the identity functor (Proposition 2.1.8) is equal to the identity group homomorphism $\mathrm{id}_{K_0(\mathcal{A})}$. If $\mathcal{B}$ and $\mathcal{C}$ are also essentially small exact categories and $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{C}$ are exact functors, then the composite induces a group homomorphism $[G \circ F] \colon K_0(\mathcal{A}) \to K_0(\mathcal{C})$ that is routinely verified to be equal to $[G] \circ [F]$. This leads us to the following proposition:

Proposition 2.2.1 (Decategorification)**.** *Let* **esExact** *be the category of all essentially small exact categories whose morphisms are exact functors. The Grothendieck group and induced group homomorphisms yield a functor* $K_0\colon \mathbf{esExact} \to \mathbf{Ab}$*.*

Similarly, the triangulated Grothendieck group induces a functor $K_0^{\triangle}\colon \mathbf{esTri} \to \mathbf{Ab}$ *from the category of essentially small triangulated categories whose morphisms are triangulated functors.*

Despite Proposition 2.2.1, the isomorphism class of a particular Grothendieck group can be difficult to determine. In this section, we will consider various examples where the answer is known.

Definition 2.2.2 (Krull-Schmidt property)**.** Let $\mathcal{A}$ be an additive category. An object $X \in \mathcal{A}$ is *indecomposable* if whenever $X \cong \bigoplus_{i\in I} Y_i$ for some index set I, there is some index $j \in I$ such that $X \cong Y_j$ and $Y_i \cong 0$ for all indexes $i \neq j$. We say that $\mathcal{A}$ has the *Krull-Schmidt property* if the following hold for all $X \in \mathcal{A}$:

- We have $X \cong \bigoplus_{i=1}^{n} Y_i$ for some indecomposable $Y_1, \ldots, Y_n \in \mathcal{A}$.
- If, in addition, $X \cong \bigoplus_{i=1}^{m} Z_i$ for some indecomposable $Z_1, \ldots, Z_m \in \mathcal{A}$, then $n = m$ and there is some permutation ρ on $\{1, \ldots, n\}$ such that $Y_i \cong Z_{\rho(i)}$ for all $i = 1, \ldots, n$.

A zero object of $\mathcal{A}$ satisfies these conditions by convention.

Lemma 2.2.3. *Let* $\mathcal{A}$ *be an essentially small additive category with the Krull-Schmidt property. If* $X \oplus Z \cong Y \oplus Z$ *for some* $X, Y, Z \in \mathcal{A}$*, then* $X \cong Y$*. In particular, by Corollary 2.1.6, we have* $[X] = [Y]$ *in* $K_0^{\text{split}}(\mathcal{A})$ *if and only if* $X \cong Y$*.*

Proof: By Definition 2.2.2, we have $X \cong \bigoplus_{i=1}^{m} X_i$, we have $Y \cong \bigoplus_{i=1}^{n} Y_i$, and we have $Z \cong \bigoplus_{i=1}^{k} Z_i$ for some indecomposables $X_1, \ldots, X_m, Y_1, \ldots, Y_n, Z_1, \ldots, Z_k \in \mathcal{A}$. For convenience, we denote these objects by the following:

$$U_i := \begin{cases} X_i & \text{for } i = 1, \ldots, m \\ Z_{i-m} & \text{for } i = m+1, \ldots, m+k \end{cases} \qquad V_i := \begin{cases} Y_i & \text{for } i = 1, \ldots, n \\ Z_{i-n} & \text{for } i = n+1, \ldots, n+k. \end{cases}$$

Since $\bigoplus_{i=1}^{m+k} U_i \cong X \oplus Z \cong Y \oplus Z \cong \bigoplus_{i=1}^{n+k} V_i$, we have $m = n$ and a permutation ρ on $\{1, \ldots, m+k\}$ such that $U_i \cong V_{\rho(i)}$ for all $i = 1, \ldots, m+k$ by Definition 2.2.2. Note that $V_i = Z_{i-m} = U_i \cong V_{\rho(i)}$ for $i = m+1, \ldots, m+k$. Since ρ is a permutation, let r_i be the smallest positive integer such that $\rho^{r_i}(i) \in \{1, \ldots, n\}$ for all $i = 1, \ldots, n$. First, we note that $X_i = U_i \cong V_{\rho(i)} \cong V_{\rho^2(i)} \cong \cdots \cong V_{\rho^{r_i}(i)} = Y_{\rho^{r_i}(i)}$ for all $i = 1, \ldots, m$. Second, we note that if $\rho^{r_i}(i) = \rho^{r_j}(j)$ and $r_i > r_j$ for some $i, j \in \{1, \ldots, m\}$, then $\rho^{r_i - r_j}(i) = j$ contradicting that fact that r_i is smallest. Hence, we have $\rho^{r_i}(i) \neq \rho^{r_j}(j)$ for all distinct $i, j = 1, \ldots, m$. From these two facts, we conclude that $X \cong Y$. □

Proposition 2.2.4. *Let $\mathcal{A}$ be an additive category with the Krull-Schmidt property. The split Grothendieck group of $\mathcal{A}$ is freely generated by the set*

$$S = \{[X] \in K_0^{\text{split}}(\mathcal{A}) \mid X \text{ is indecomposable}\}.$$

Proof: Generated: Let $X, Y \in \mathcal{A}$. By the Krull-Schmidt property, there are indecomposable objects $X_1, \ldots, X_n, Y_1, \ldots, Y_m \in \mathcal{A}$ such that $X \cong \bigoplus_{i=1}^{n} X_i$ and $Y \cong \bigoplus_{j=1}^{m} Y_j$. Then, by (1) of Lemma 2.1.3, we have $[X] - [Y] = \sum_{i=1}^{n} [X_i] - \sum_{j=1} [Y_j]$ and thus $K_0^{\text{split}}(\mathcal{A})$ is generated by S by (3) of Lemma 2.1.3.

Linear Independence: Let indecomposable $X_1, \ldots, X_n \in \mathcal{A}$ be such that $[X_1], \ldots, [X_n]$ are distinct elements of S. Suppose $\sum_{i=1}^{n} a_i [X_i] = 0$ in $K_0^{\text{split}}(\mathcal{A})$ for $a_1, \ldots, a_n \in \mathbb{Z}$. Without loss of generality, suppose $a_1, \ldots, a_k \geq 0$ and $a_{k+1}, \ldots, a_n \leq 0$ for some k. Thus, we have the following equality in $K_0^{\text{split}}(\mathcal{A})$:

$$\left[\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{a_i} X_i \right] = \left[\bigoplus_{i=1}^{n-k} \bigoplus_{j=1}^{-a_{k+i}} X_{k+i} \right].$$

By Lemma 2.2.3, we have $\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{a_i} X_i \cong \bigoplus_{i=1}^{n-k} \bigoplus_{j=1}^{-a_{k+i}} X_{k+i}$. Since none of the summands on the left side of the equality can be isomorphic to a summand on the right side (this would contradict our choice of distinct elements of S) and the summands are all indecomposable, any nonzero a_i would contradict the Krull-Schmidt property. Hence, we have $a_1 = \cdots = a_n = 0$. □

One can use the split Grothendieck group in an interesting way when the conflations of $\mathcal{D}$ are all split:

Proposition 2.2.5 (Category with finite projective resolutions)**.** *Let $\mathcal{A}$ be an abelian category with the Krull-Schmidt property. Suppose that for each $X \in \mathcal{A}$ there exists an exact sequence of the form*

$$0_{\mathcal{A}} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0_{\mathcal{A}}$$

with $P_0, P_1, \ldots, P_n \in \operatorname{Prj} \mathcal{A}$ for some $n \in \mathbb{N}$. Then, the Grothendieck group of $\mathcal{A}$ is freely generated by the set

$$S = \{[X] \in K_0(\mathcal{A}) \mid X \text{ is indecomposable projective}\}.$$

The proof of Proposition 2.2.5 makes use of the following theorem:

Theorem 2.2.6 ([Wei13, Theorem 7.6])**.** *Let $\mathcal{A}$ be an abelian category and let $\mathcal{C}$ be a strictly full exact subcategory of $\mathcal{A}$. Let $\mathcal{D}$ be an strictly full exact subcategory of $\mathcal{C}$ and let $I \colon \mathcal{D} \to \mathcal{C}$ be the inclusion. Suppose the following hold:*

- *For each morphism $e \in \operatorname{Mor} \mathcal{C}$ that is an epimorphism in $\mathcal{A}$, we have $\operatorname{Ker} e \in \mathcal{C}$.*
- *If $C \in \mathcal{C}$, then there is an exact sequence in $\mathcal{A}$ of the form*

  $$0_{\mathcal{A}} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to C \to 0_{\mathcal{A}}$$

  for some $n \in \mathbb{N}$ and $X_0, X_1, \ldots, X_n \in \mathcal{D}$.

Then, the homomorphism $[I] \colon K_0(\mathcal{D}) \to K_0(\mathcal{C})$ is an isomorphism.

Proof of Proposition 2.2.5: Consider $\mathcal{A}$ as a strictly full exact subcategory of itself. Then, by Theorem 2.2.6, we have $K_0(\operatorname{Prj} \mathcal{A}) \cong K_0(\mathcal{A})$ as groups. Furthermore, we have $K_0(\operatorname{Prj} \mathcal{A}) = K_0^{\text{split}}(\operatorname{Prj} \mathcal{A})$ as all the conflations in $\operatorname{Prj} \mathcal{A}$ are split by Example 1.3.14. Finally, since $\mathcal{A}$ has the Krull-Schmidt property so does $\operatorname{Inj} \mathcal{A}$ (its an additive subcategory) and thus, by Proposition 2.2.4, we are done. □

Definition 2.2.7 (The G_0-theory of a ring)**.** Let R be a ring with unity. Recall that R-$\mathbf{mod}^{\mathrm{fg}}$ denotes the exact category of finitely generated left R-modules whose morphisms are R-linear maps. We define $G_0(R)$ to be $K_0(R\text{-}\mathbf{mod}^{\mathrm{fg}})$. The exact subcategory of finitely generated projective left R-modules is denoted by R-$\mathbf{pmod}^{\mathrm{fg}}$. We define $K_0(R)$ to be $K_0(R\text{-}\mathbf{pmod}^{\mathrm{fg}})$ and note that it is equal to $K_0^{\mathrm{split}}(R\text{-}\mathbf{pmod}^{\mathrm{fg}})$ by Example 1.3.14.

Remark 2.2.8. In some texts, the category of left-Noetherian modules over R is used instead to define $G_0(R)$, but for the rings that we will consider the left-Noetherian modules are precisely the finitely generated modules.

Example 2.2.9. Let R be a principal ideal domain. The Fundamental Theorem (see [Nic07, §7.2 Theorem 6]) tells us that R-$\mathbf{mod}^{\mathrm{fg}}$ is Krull-Schmidt. Since we have the following short exact sequence of R-modules

$$0 \to (d) \to R \to R/(d) \to 0$$

for all (non-unit) $d \in R$, every invariant factor has a finite projective resolution as (d) is free or zero. Hence, Proposition 2.2.5 applies and since R is the only projective indecomposable up to isomorphism, we note that $K_0(R)$ and $G_0(R)$ are freely generated by $\{[R]\}$.

Definition 2.2.10 (Simple object, semisimple)**.** Let $\mathcal{A}$ be an abelian category and fix a zero object $0_{\mathcal{A}}$. An object $X \in \mathcal{A}$ is *simple* if it has precisely two subobjects (these two subobjects are represented by id_X and $0_{0_{\mathcal{A}},X}$). An object X is *semisimple* if it is isomorphic to a direct sum of simple objects.

Definition 2.2.11 (Artinian, Noetherian)**.** Let $\mathcal{A}$ be an abelian category. An object $X \in \mathcal{A}$ is *Artinian* provided the following descending chain condition holds: if a sequence $m_0, m_1, m_2 \dots$ of monomorphisms with codomain X admits a sequence $u_0, u_1, u_2, \dots$ making (2.2.1) commute for all $i \in \mathbb{N}$, then the u_i are all isomorphisms for sufficiently large i.

$$\begin{tikzcd}
X & \\
X_i \arrow[u, "m_i"] & X_{i+1} \arrow[ul, "m_{i+1}"'] \arrow[l, "u_i"']
\end{tikzcd} \tag{2.2.1}$$

An *Artinian category* is an (essentially small) category whose objects are all Artinian. An object $X \in \mathcal{A}$ is *Noetherian* provided the following ascending chain condition holds: if a sequence $m_0, m_1, m_2 \dots$ of monomorphisms with codomain X admit a sequence $u_0, u_1, u_2, \dots$ making (2.2.2) commute for all $i \in \mathbb{N}$, then the u_i are all isomorphisms for sufficiently large i.

$$\begin{tikzcd}
X & \\
X_i \arrow[u, "m_i"] \arrow[r, "u_i"] & X_{i+1} \arrow[ul, "m_{i+1}"']
\end{tikzcd} \tag{2.2.2}$$

A *Noetherian category* is an (essentially small) category whose objects are all Noetherian.

Remark 2.2.12. Let X be an Artinian object in an abelian category and fix kernels. Let $e_0, e_1, e_2, \dots$ be a sequence of epimorphisms with domain X such that there is a sequence of morphisms $v_0, v_1, v_2, \dots$ making (2.2.3) commute for all $i \in \mathbb{N}$.

$$\begin{tikzcd}
X_i & X_{i+1} \arrow[l, "v_i"'] \\
X \arrow[u, "e_i"] \arrow[ur, "e_{i+1}"'] &
\end{tikzcd} \tag{2.2.3}$$

By definition of a kernel, there exists a unique sequence of morphisms $u_0, u_1, u_2, \dots$ such that (2.2.4) commutes for all $i \in \mathbb{N}$. Since X is Artinian, the u_i are all isomorphisms for sufficiently large i. By [Fre64, Theorem 2.11], one notes that both e_i and e_{i+1} are cokernels for $\operatorname{Ker} e_i$ and $\operatorname{Ker} e_{i+1}$ respectively and, thus, since u_i is an isomorphism for sufficiently large i so is v_i for all sufficiently large i.

$$\begin{tikzcd}
X_i & X_{i+1} \arrow[l, "v_i"'] \\
X \arrow[u, "e_i"] \arrow[ur, "e_{i+1}"'] & \\
K_i \arrow[u, "\operatorname{Ker} e_i"] & K_{i+1} \arrow[l, "u_i"] \arrow[ul, "\operatorname{Ker} e_{i+1}"']
\end{tikzcd} \tag{2.2.4}$$

In other words, an Artinian object in $\mathcal{A}$ is Noetherian in $\mathcal{A}^{\mathrm{op}}$, and, by a similar argument, a Noetherian object in $\mathcal{A}$ is Artinian in $\mathcal{A}^{\mathrm{op}}$. In a similar fashion, one can show that an object is simple in $\mathcal{A}$ if and only if it is simple in $\mathcal{A}^{\mathrm{op}}$.

Proposition 2.2.13. *Let $\mathcal{A}$ be a Noetherian and Artinian abelian category. Then, the Grothendieck group $K_0(\mathcal{A})$ is freely generated by the set*

$$\{[X] \in K_0(\mathcal{A}) \mid X \textit{ is simple}\}.$$

Our proof of Proposition 2.2.13 will use the following theorem:

Theorem 2.2.14 (Heller, [Wei13, II. Theorem 6.3])**.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a strictly full abelian subcategory. Suppose that $\mathcal{D}$ is closed under subobjects and quotient objects in $\mathcal{A}$. Furthermore, suppose that for all $X \in \mathcal{A}$ there exists $n \in \mathbb{N}$ and a sequence of monomorphisms*

$$0_{\mathcal{A}} \xrightarrow{m_{n+1}} X_n \xrightarrow{m_n} X_{n-1} \xrightarrow{m_{n-1}} \cdots \xrightarrow{m_1} X_0 \xrightarrow{m_0} X$$

such that $\operatorname{cod}(\operatorname{Coker} m_i) \in \mathcal{D}$ for all $i = 0, \ldots, n+1$. Then, the inclusion induces a group isomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{A})$.

Proof of Proposition 2.2.13: Let $\mathcal{A}^{\mathrm{ss}}$ be the strictly full abelian subcategory of $\mathcal{A}$ consisting of its semisimple objects. Note that it is closed under subobjects and quotient objects (a monomorphism with codomain in $\mathcal{A}^{\mathrm{ss}}$ also has its domain in $\mathcal{A}^{\mathrm{ss}}$ and an epimorphism with domain in $\mathcal{A}^{\mathrm{ss}}$ also has its codomain in $\mathcal{A}^{\mathrm{ss}}$).

Let $X \in \mathcal{A}$. Let $e_0, e_1, e_2, \dots$ be a sequence of epimorphisms together with a sequence $u_0, u_1, u_2, \dots$ satisfying (2.2.5) for all $i \in \mathbb{N}$. A pair of sequences clearly exists and let us take one with the greatest n such that u_i is not an isomorphism nor a zero morphism for $i = 0, 1, \dots, n$.

$$\begin{tikzcd} X_i \arrow[r, "u_i"] & X_{i+1} \\ X \arrow[u, "e_i"] \arrow[ur, "e_{i+1}"'] & \end{tikzcd} \tag{2.2.5}$$

Since $\mathcal{A}^{\mathrm{op}}$ is Noetherian, one with this property (and finite n) must exist and X_{n+1} is a simple object (so it is in $\mathcal{A}^{\mathrm{ss}}$). Set $m_0 \colon K_0 \to X$ to be a kernel of e_{i+1}. We repeat this process for K_0 and get another monomorphism $m_1 \colon K_1 \to K_0$ whose codomain of its cokernel is in $\mathcal{A}^{\mathrm{ss}}$. This process must terminate (K_i must be simple for some $i \in \mathbb{N}$) or it would contradict the fact that $\mathcal{A}$ is Artinian. We have thus constructed a sequence as in Theorem 2.2.14 and thus the inclusion induces a group isomorphism $K_0(\mathcal{A}^{\mathrm{ss}}) \to K_0(\mathcal{A})$.

Since every conflation in $\mathcal{A}^{\mathrm{ss}}$ is split, we have $K_0^{\mathrm{split}}(\mathcal{A}^{\mathrm{ss}}) = K_0(\mathcal{A}^{\mathrm{ss}})$ and, since the indecomposable objects of $\mathcal{A}^{\mathrm{ss}}$ are the simple ones, the conclusion follows. ☐

Example 2.2.15 (Fields)**.** Let $\mathbb{F}$ be a field. The category $\mathbb{F}$-$\mathbf{mod}^{\mathrm{fg}}$ is equivalent to the abelian category of finite dimensional vector spaces whose morphisms are linear transformations. The Rank-Nullity Theorem for finite dimensional vector spaces implies that this category has finite length. The simple objects are the vectors spaces of dimension one. Since a vector space of a certain dimension is unique up to isomorphism, all vector spaces are projective and, furthermore, $G_0(\mathbb{F}) \cong K_0(\mathbb{F}) \cong \mathbb{Z}$ as groups by Proposition 2.2.13.

More explicitly, we can consider the following map:

$$\begin{aligned} \dim \colon \operatorname{Iso} \mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}} &\to \mathbb{Z}, && \text{given by} \\ [V] &\mapsto \dim_{\mathbb{F}} V && \text{for all } V \in \mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}}. \end{aligned}$$

Since a vector space of a certain dimension is unique up to isomorphism, this map is well defined. By the Rank-Nullity Theorem for finite dimensional vector spaces, it is also additive over exact sequences. By Proposition 2.1.7, this yields a group homomorphism $[\dim]\colon G_0(\mathbb{F}) \to \mathbb{Z}$ that must be an isomorphism (since its clearly surjective and $G_0(\mathbb{F}) \cong \mathbb{Z}$ as groups). Note that $\mathbb{F}$ is principal ideal domain (PID). In Example 2.2.25, we shall show that all PIDs have this property.

Example 2.2.16 ($\mathbb{Z}/4\mathbb{Z}$)**.** One can easily show that $\mathbb{Z}/4\mathbb{Z}$-$\mathbf{mod}^{\mathrm{fg}}$ is Krull-Schmidt. In particular, every object in $\mathbb{Z}/4\mathbb{Z}$-$\mathbf{mod}^{\mathrm{fg}}$ is isomorphic to a direct sum of copies of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Since the inclusion of $\mathbb{Z}/2\mathbb{Z}$ into $\mathbb{Z}/4\mathbb{Z}$ is a monomorphism in $\mathbb{Z}/4\mathbb{Z}$-$\mathbf{mod}^{\mathrm{fg}}$ that is not equivalent to a zero morphism or an identity morphism, we note that $\mathbb{Z}/2\mathbb{Z}$ is the only simple object up to isomorphism. Since Krull-Schmidt categories have finite length, we have that $G_0(\mathbb{Z}/4\mathbb{Z})$ is freely generated by $\{[\mathbb{Z}/2\mathbb{Z}]\}$ by Proposition 2.2.13.

Definition 2.2.17 (Generic rank)**.** Let R be an integral domain and let M be a finitely generated R-module. The *generic rank* $\operatorname{Rank}_R M$ of M over R is the number of basis elements in a maximal free R-submodule.

Lemma 2.2.18. *Let R be an integral domain and let M be a finitely generated R-module. Let Q be the field of fractions of R. We have* $\operatorname{Rank}_R M = \dim_Q(M \otimes_R Q)$.

Proof: Let $\{b_1, \ldots, b_n\}$ be a basis for a maximal free R-submodule of M. The subset $S = \{b_1 \otimes 1, \ldots, b_n \otimes 1\}$ of $M \otimes_R Q$ is clearly linearly independent over R. After some straightforward rearrangement, one notes that an arbitrary element in $M \otimes_R Q$ is of the form $q(m \otimes 1)$ for some $q \in Q$ and $m \in M$. Hence, we can fix a Q-basis of $M \otimes_R Q$ of the form $\{m_i \otimes 1\}_{i=1}^{k}$. We have the following:

$$\sum_{i=1}^{k} r_i m_i = 0 \implies \sum_{i=1}^{k} r_i(m_i \otimes 1) = 0 \implies r_1 = \cdots = r_k = 0.$$

This tells us that $k \leq n$ and the linear independence of S tells us that $k \geq n$. Thus, we have $\dim_Q(M \otimes_R Q) = n = \operatorname{Rank}_R M$. □

Lemma 2.2.19. *Let R be an integral domain. Generic rank induces a surjective group homomorphism* $[\mathrm{Rank}]\colon G_0(R) \to \mathbb{Z}$ *such that* $[\mathrm{Rank}][M] = \mathrm{Rank}_R\, M$ *for all* $M \in R\text{-}\mathbf{mod}^{\mathrm{fg}}$*.*

Proof: Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a short exact sequence of R-modules. Since an arbitrary element in $M_1 \otimes_R Q$ can be written in the form $q(m \otimes 1)$ for some $q \in Q$ and $m \in M_1$, the Q-linear map $f \otimes \mathrm{id}_Q\colon M_1 \otimes_R Q \to M_2 \otimes_R Q$ is easily shown to be injective:

$$\begin{aligned}(f \otimes \mathrm{id}_Q)(q(m \otimes 1)) = 0 &\implies q(f(m) \otimes 1) = 0 \\ &\implies q = 0 \text{ or } f(m) = 0 \implies q = 0 \text{ or } m = 0 \implies q(m \otimes 1) = 0.\end{aligned}$$

Similarly, it is clear that the Q-linear map $g \otimes \mathrm{id}_Q\colon M_2 \otimes_R Q \to M_3 \otimes_R Q$ is surjective. Now, let $m \in M_2$ and $q \in Q$. We have the following:

$$\begin{aligned}q(m \otimes 1) \in \mathrm{Ker}(g \otimes \mathrm{id}_Q) &\iff q = 0 \text{ or } m \in \mathrm{Ker}\, g \\ &\iff q = 0 \text{ or } m \in \mathrm{Im}\, f \iff q(m \otimes 1) \in \mathrm{Im}(f \otimes \mathrm{id}_Q).\end{aligned}$$

Hence, the functor $(- \otimes_R Q)\colon R\text{-}\mathbf{mod}^{\mathrm{fg}} \to Q\text{-}\mathbf{mod}^{\mathrm{fg}}$ is exact. By Proposition 2.1.8, we have a (surjective) group homomorphism $[-_R \otimes Q]\colon G_0(R) \to G_0(Q)$. Our conclusion follows from Example 2.2.15 and Lemma 2.2.18 by defining the group homomorphism $[\mathrm{Rank}]$ to be the composite $[\dim] \circ [- \otimes_R Q]$. □

Definition 2.2.20 (Regular ring, fractional ideal, invertible ideal, Dedekind domain)**.** Let R be an integral domain. We say that R is *regular* if it is Noetherian and every finitely generated left R-module has a finite resolution by finitely generated projective left R-modules. Let Q be the field of fractions of R. An R-submodule M of Q is a *fractional ideal* if there is an $r \in R$ such that $rM \subseteq R$. A fractional ideal M is *invertible* if there is another fractional ideal N such that $MN = R$. A *Dedekind domain* is an integral domain such that every fractional ideal is invertible. A Dedekind domain is a regular ring (see [Sin11, Proposition 20.4.1]).

Lemma 2.2.21 ([Swa70, Theorem 1.1])**.** *The natural exact inclusion* $R\text{-}\mathbf{pmod}^{\mathrm{fg}} \to R\text{-}\mathbf{mod}^{\mathrm{fg}}$ *induces a group homomorphism* $K_0(R) \to G_0(R)$ *called the* Cartan homomorphism*. If* R *is a regular integral domain, then the Cartan homomorphism is an isomorphism.*

Definition 2.2.22 (Steinitz class group)**.** Let R be a Dedekind domain. We define the *Steinitz class group* $\operatorname{Cl} R$ of R to be the collection of isomorphism classes of the fractional ideals under the group operation induced by the ideal product.

Proposition 2.2.23. *We have* $G_0(R) \cong K_0(R) \cong \mathbb{Z} \oplus \operatorname{Cl} R$ *as groups.*

Proof: The first isomorphism is by Lemma 2.2.21. The second isomorphism is by [Kos10, Lemmas 5.5 and 5.6] and Corollary 2.1.6 since $K_0(R)$ is the split Grothendieck group of $R\text{-}\mathbf{pmod}^{\mathrm{fg}}$. Explicitly, the isomorphism is the group homomorphism

$$[(\mathrm{Rank}, \mathrm{Cl})] \colon K_0(R) \to \mathbb{Z} \oplus \operatorname{Cl} R$$

such that $[(\mathrm{Rank}, \mathrm{Cl})][R^{n-1} \oplus I] = (n, [I])$ for all $[R^{n-1} \oplus I] \in K_0(R)$ (everything in $R\text{-}\mathbf{pmod}^{\mathrm{fg}}$ can be written in the form $R^{n-1} \oplus I$) where I is a fractional ideal. ☐

Example 2.2.24 ([Dug, Example 4.2])**.** The ring $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain. We have $G_0(R) \cong K_0(R) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ as groups.

Example 2.2.25. Let R be a principal ideal domain (which is a Dedekind domain). Since R is Noetherian, a fractional ideal is finitely generated. We can write all the generators with the same denominator and then the span of the numerators is an ideal of R. Hence, every fractional ideal is in fact principal and thus isomorphic to R as an R-module (by the integral domain property). So, we have that $\operatorname{Cl} R = \{[R]\}$ is a trivial group. Thus, we have $G_0(R) \cong K_0(R) \cong \mathbb{Z}$ as groups.

Definition 2.2.26 (R-linear, hom-finite)**.** Let R be an integral domain. A category is *R-linear* if its hom-sets are R-modules and composition is R-bilinear. A R-linear category is *hom-finite* if the hom-sets are finitely generated as R-modules.

Example 2.2.27. Let R be an integral domain and let $\mathcal{A}$ be an R-linear hom-finite abelian category. Consider the biexact bifunctor

$$\begin{aligned} &\hom_{\mathcal{A}}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \operatorname{Inj}\mathcal{A} \to R\text{-}\mathbf{mod}^{\mathrm{fg}}, \\ &\qquad\qquad (X, I) \mapsto \hom(X, I) \quad \text{for all } X \in \mathcal{A} \text{ and } I \in \operatorname{Inj}\mathcal{A}, \\ &\qquad\qquad (f, g) \mapsto g \circ - \circ f \qquad \text{for all } f, g \in \operatorname{Mor}\mathcal{A}, \text{composition in } \mathcal{A}. \end{aligned}$$

This functor is exact in the first coordinate by the definition of an injective object and exact in the second coordinate since every sequence of injective objects splits and the hom-functor is biadditive. This bifunctor induces a group homomorphism defined by

$$\begin{aligned} &K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\operatorname{Inj}\mathcal{A}) \to G_0(R), \\ &\qquad\qquad [X] \otimes [I] \mapsto [\hom(X, I)] \qquad \text{for all } X \in \mathcal{A} \text{ and } I \in \operatorname{Inj}\mathcal{A}, \end{aligned}$$

and extended by linearity.

Chapter 3

Bounded Derived Categories

From an abelian category $\mathcal{A}$, we define a triangulated category called its *bounded derived category* $D^{\mathrm{b}}(\mathcal{A})$. The importance of bounded derived categories in this thesis comes from an isomorphism $K_0(\mathcal{A}) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$ that arises in a natural way. We construct the bounded derived category by first passing to the bounded homotopy category $\mathrm{Ho}^{\mathrm{b}}\,\mathcal{A}$. This is an illustrative way to understand the bounded derived category, and it is a natural step on the level of Grothendieck groups because $K_0^{\mathrm{split}}(\mathcal{A})$ is isomorphic as a group to $K_0^{\triangle}(\mathrm{Ho}^{\mathrm{b}}\,\mathcal{A})$.

3.1 Category of Cochain Complexes

We build the bounded derived category in several steps. In this section, we look at the first step in its construction and that is the category of cochain complexes, whose objects are sequences of morphisms that are "almost exact" in the sense that the composite of any two subsequent morphisms is a zero morphism.

Definition 3.1.1 (Graded category)**.** Let $\mathcal{C}$ be a category. Since context will ensure no ambiguity, denote by $\mathbb{Z}$ the discrete category (i.e. only having identity morphisms) such that $\mathrm{Ob}\,\mathbb{Z} = \mathbb{Z}$. The category $\mathrm{Gr}^{\mathrm{ub}}\,\mathcal{C} := \mathbf{Fct}(\mathbb{Z}, \mathcal{C})$ is called the *graded category* of $\mathcal{C}$. Its objects are called *graded* $\mathcal{C}$*-objects* and its morphisms are called *graded* $\mathcal{C}$*-morphisms* (see Example 1.1.17 if unfamiliar with the functor category notation). If $\mathcal{A}$ is an abelian category, then $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$ is abelian (see Example 1.2.15).

Definition 3.1.2 ($\operatorname{Gr}^* \mathcal{C}$)**.** Let $\mathcal{A}$ be an abelian category. We define three strictly full abelian subcategories of $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$ denoted by $\operatorname{Gr}^{+} \mathcal{A}$, $\operatorname{Gr}^{-} \mathcal{A}$, and $\operatorname{Gr}^{\mathrm{b}} \mathcal{A}$ by requiring the following:

- For each $X \in \operatorname{Gr}^{+} \mathcal{A}$, we require that there exists some integer N such that $X(n) = 0$ for all $n \leq N$.
- For each $X \in \operatorname{Gr}^{-} \mathcal{A}$, we require that there exists some integer N such that $X(n) = 0$ for all $n \geq N$.
- For each $X \in \operatorname{Gr}^{\mathrm{b}} \mathcal{A}$, we require that there exists some integer N such that $X(n) = 0$ for all $|n| \geq N$.

These categories form the following Hasse diagram under inclusion:

$$\begin{tikzcd}
 & \operatorname{Gr}^{\mathrm{ub}} \mathcal{A} & \\
\operatorname{Gr}^{-} \mathcal{A} \arrow[ur, hook] & & \operatorname{Gr}^{+} \mathcal{A} \arrow[ul, hook'] \\
 & \operatorname{Gr}^{\mathrm{b}} \mathcal{A} \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

When we want to make a statement that holds for all four categories, we simply use the notation $\operatorname{Gr}^* \mathcal{A}$. We will be primarily interested in the *bounded* version $\operatorname{Gr}^{\mathrm{b}} \mathcal{A}$.

Proposition 3.1.3. *Let* $\mathcal{A}$ *be an abelian category. We have*

$$K_0(\operatorname{Gr}^{\mathrm{b}} \mathcal{A}) \cong \bigoplus_{z \in \mathbb{Z}} K_0(\mathcal{A})$$

as groups.

Proof: Note that a conflation $0 \to F \xrightarrow{\eta} G \xrightarrow{\sigma} H \to 0$ in $\operatorname{Gr}^{\mathrm{b}} \mathcal{A}$ yields conflations

$0_{\mathcal{A}} \to Fz \xrightarrow{\eta^z} Gz \xrightarrow{\sigma^z} Hz \to 0_{\mathcal{A}}$ for all $z \in \mathbb{Z}$ (the morphisms in $\operatorname{Gr}^{\mathrm{b}} \mathcal{A}$ are precisely bounded products of the morphisms in $\mathcal{A}$).

Hence, we can define a map that is additive over conflations:

$$\begin{aligned} \operatorname{Iso} \operatorname{Gr}^{\mathrm{b}} \mathcal{A} &\to \bigoplus_{z \in \mathbb{Z}} K_0(\mathcal{A}), && \text{given by} \\ [F] &\mapsto ([Fz])_{z \in \mathbb{Z}} && \text{for all } F \in \operatorname{Gr}^{\mathrm{b}} \mathcal{A}. \end{aligned}$$

By Proposition 2.1.7, we have a group homomorphism that one can easily check is a group isomorphism. ☐

Remark 3.1.4. Using a similar argument, one has that $K_0(\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}) \cong \prod_{z \in \mathbb{Z}} K_0(\mathcal{A})$ and $K_0(\operatorname{Gr}^{+} \mathcal{A}) \cong K_0(\operatorname{Gr}^{-} \mathcal{A}) \cong (\prod_{n \in \mathbb{N}} K_0(\mathcal{A})) \oplus (\bigoplus_{n \in \mathbb{N}} K_0(\mathcal{A}))$ as groups. The latter isomorphisms can be seen by considering the nonnegative integers and negative integers separately.

Definition 3.1.5 (Shift functor)**.** Let $\mathcal{A}$ be an abelian category and let $k \in \mathbb{Z}$. For each graded $\mathcal{A}$-object X, we define an graded $\mathcal{A}$-object $X\langle k \rangle$ by $(X\langle k \rangle)(n) = X(n+k)$ for all $n \in \mathbb{Z}$. Similarly, for each graded $\mathcal{A}$-morphism $f \colon X \to Y$, we define a graded A-morphism $f\langle k \rangle \colon X\langle k \rangle \to Y\langle k \rangle$ given by $(f\langle k \rangle)^n = f^{n+k}$ for all $n \in \mathbb{Z}$. This defines four exact equivalences of categories ($* \in \{\mathrm{ub}, -, +, \mathrm{b}\}$):

$$\begin{aligned} \langle k \rangle \colon \operatorname{Gr}^* \mathcal{A} &\to \operatorname{Gr}^* \mathcal{A}, \\ X &\mapsto X\langle k \rangle && \text{for all } X \in \operatorname{Gr}^* \mathcal{A}, \\ f &\mapsto f\langle k \rangle && \text{for all } f \in \operatorname{Mor} \operatorname{Gr}^* \mathcal{A}. \end{aligned}$$

We call this the *k-shift functor* on $\operatorname{Gr}^* \mathcal{A}$. In particular, the functor denoted by $\langle 1 \rangle$ is simply called the *shift functor* on $\operatorname{Gr}^* \mathcal{A}$.

Remark 3.1.6. Most references use [1] instead of $\langle 1 \rangle$ as the notation for the shift functor, but we will not due to the notation used in Grothendieck groups. Furthermore, some references call $\langle -1 \rangle$ the shift functor and denote our $\langle k \rangle$ by $\langle -k \rangle$ for all $k \in \mathbb{Z}$. We follow the convention of recent references that study cochain complexes.

Definition 3.1.7 (Category of cochain complexes)**.** Let $\mathcal{A}$ be an abelian category. A *cochain complex* in $\mathcal{A}$ is a pair $(X^\bullet, d_X)$ where $X^\bullet$ is a graded $\mathcal{A}$-object and $d_X\colon X^\bullet \to X^\bullet\langle 1\rangle$ is a graded $\mathcal{A}$-morphism, called the *differential*, that satisfies

$$d_X\langle 1\rangle \circ d_X = 0_{\mathrm{Gr}^{\mathrm{ub}}\mathcal{A}}.$$

A *morphism of cochain complexes* $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ is a graded $\mathcal{A}$-morphism $X^\bullet \to Y^\bullet$ that intertwines with the differentials: that is,

$$f\langle 1\rangle \circ d_X = d_Y \circ f.$$

Cochain complexes in $\mathcal{A}$ and such morphisms between them form a category $\mathrm{Kom}^{\mathrm{ub}}\mathcal{A}$ called the *category of cochain complexes* in $\mathcal{A}$, where composition is as in $\mathrm{Gr}^{\mathrm{ub}}\mathcal{A}$.

The strictly full subcategory of $\mathrm{Kom}^{\mathrm{ub}}\mathcal{A}$ containing precisely objects $(X^\bullet, d_X)$ such that $X^\bullet$ belongs to $\mathrm{Gr}^{+}\mathcal{A}$, (resp. $\mathrm{Gr}^{-}\mathcal{A}$, resp. $\mathrm{Gr}^{\mathrm{b}}\mathcal{A}$) is denoted by $\mathrm{Kom}^{+}\mathcal{A}$ (resp. $\mathrm{Kom}^{-}\mathcal{A}$, resp. $\mathrm{Kom}^{\mathrm{b}}\mathcal{A}$). As before, we use $\mathrm{Kom}^{*}\mathcal{A}$ when a statement could be made for any of the four categories.

Remark 3.1.8. Let $(X^\bullet, d_X)$ be a cochain complex. We will follow the convention of denoting $X^\bullet(n)$ by X^n. We can visualize $(X^\bullet, d_X)$ as a diagram

$$\cdots \xrightarrow{d_X^{n-3}} X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^{n} \xrightarrow{d_X^{n}} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \xrightarrow{d_X^{n+2}} \cdots$$

and a morphism of cochain complexes $(X^\bullet, d_X) \to (Y^\bullet, d_Y)$ as a graded $\mathcal{A}$-morphism $f\colon X^\bullet \to Y^\bullet$ making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r, "d_X^{n-3}"] & X^{n-2} \arrow[r, "d_X^{n-2}"] \arrow[d, "f^{n-2}"] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^{n} \arrow[r, "d_X^{n}"] \arrow[d, "f^{n}"] & X^{n+1} \arrow[r, "d_X^{n+1}"] \arrow[d, "f^{n+1}"] & X^{n+2} \arrow[r, "d_X^{n+2}"] \arrow[d, "f^{n+2}"] & \cdots \\
\cdots \arrow[r, "d_Y^{n-3}"] & Y^{n-2} \arrow[r, "d_Y^{n-2}"] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^{n} \arrow[r, "d_Y^{n}"] & Y^{n+1} \arrow[r, "d_Y^{n+1}"] & Y^{n+2} \arrow[r, "d_Y^{n+2}"] & \cdots
\end{tikzcd}$$

We will not define the *category of chain complexes* in $\mathcal{A}$ since it can be identified with $\mathrm{Kom}^{\mathrm{ub}}(\mathcal{A}^{\mathrm{op}})$.

Lemma 3.1.9. *Let $\mathcal{A}$ be an abelian category. The category $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$ is abelian, and the categories $\operatorname{Kom}^{+} \mathcal{A}$, $\operatorname{Kom}^{-} \mathcal{A}$, and $\operatorname{Kom}^{\mathrm{b}} \mathcal{A}$ are strictly full abelian subcategories of $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$.*

Proof: The zero object of $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$ has only one possible differential, and together they are the zero object of $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$. By bilinearity in $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$, it is clear that the sum of two graded $\mathcal{A}$-morphisms from the same hom-set is a graded $\mathcal{A}$-morphism of that hom-set. Hence, the category $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$ is preadditive as all its hom-sets are subgroups of the corresponding hom-sets in $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$. Furthermore, given objects $(X_1^\bullet, d_{X_1}), \ldots, (X_n^\bullet, d_{X_n})$ in $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$, we note that $(X_1^\bullet \oplus \cdots \oplus X_n^\bullet, d_{X_1} \oplus \cdots \oplus d_{X_n})$ is their biproduct. This is because the graded $\mathcal{A}$-morphisms i_{X_j} and p_{X_j} given by the biproduct in $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$ intertwine with the differentials for all $j = 1, \ldots, n$. Thus, the category $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$ is additive.

Let $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be a morphism of cochain complexes. Let $\operatorname{Ker} f\colon Z \to X^\bullet$ be a kernel of f in $\operatorname{Gr}^{\mathrm{ub}} \mathcal{A}$. We have

$$f\langle 1\rangle \circ d_X \circ \operatorname{Ker} f = d_X \circ f \circ \operatorname{Ker} f = 0.$$

Hence, there is a graded $\mathcal{A}$-morphism $u\colon Z \to Z\langle 1\rangle$ such that $\operatorname{Ker} f\langle 1\rangle \circ u = d_X \circ \operatorname{Ker} f$. We have $\operatorname{Ker}\langle 2\rangle \circ u\langle 1\rangle \circ u = d_X\langle 1\rangle \circ \operatorname{Ker}\langle 1\rangle \circ u = d_X\langle 1\rangle \circ d_X \circ \operatorname{Ker} f = 0$ and thus $u\langle 1\rangle \circ u = 0$. Thus, the morphism $\operatorname{Ker} f$ is a morphism of cochain complexes $(Z, u) \to (X^\bullet, d_X)$ that is a kernel of f in $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$. The existence of cokernels is dual. It quickly follows that $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}$ is abelian.

The proof for the subcategories is straightforward. □

Remark 3.1.10. A short exact sequence in $\operatorname{Kom}^{*} \mathcal{A}$ is thus a sequence

$$0 \to (X, d_X) \xrightarrow{f} (Y, d_Y) \xrightarrow{g} (Z, d_Z) \to 0$$

(where f and g are morphisms of cochain complexes) such that

$$0_{\mathcal{A}} \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0_{\mathcal{A}}$$

is a short exact sequence for all $n \in \mathbb{Z}$.

Definition 3.1.11 (Shift functor)**.** For each $k \in \mathbb{Z}$, we abuse notation and define a functor $\langle k\rangle\colon \operatorname{Kom}^* \mathcal{A} \to \operatorname{Kom}^* \mathcal{A}$ (where $* \in \{\mathrm{ub}, -, +, \mathrm{b}\}$) by

$$\begin{aligned} (X^\bullet, d_X) &\mapsto (X^\bullet\langle k\rangle, (-1)^k d_X\langle k\rangle) && \text{for all } (X^\bullet, d_X) \in \operatorname{Kom}^* \mathcal{A}, \text{ and} \\ f &\mapsto f\langle k\rangle && \text{for all } f \in \operatorname{Mor} \operatorname{Kom}^* \mathcal{A}. \end{aligned}$$

This functor is called the *k-shift functor* on $\operatorname{Kom}^* \mathcal{A}$. The functor denoted $\langle 1\rangle$ is called simply the *shift functor* on $\operatorname{Kom}^* \mathcal{A}$.

Definition 3.1.12 (Stalk complex, stalk morphism)**.** Let $X \in \mathcal{A}$ and $k \in \mathbb{Z}$. We define a cochain complex $(X\langle k\rangle, 0)$ where $(X\langle k\rangle)^n$ is X if $n = -k$ and $0_{\mathcal{A}}$ otherwise, and the differential 0^n is a zero morphism for all n. This complex is called the *stalk complex* of X concentrated in degree k.

Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. We define a morphism of complexes $f\langle k\rangle\colon (X\langle k\rangle, 0) \to (Y\langle k\rangle, 0)$ where $(f\langle k\rangle)^n$ is f if $n = -k$ and 0 otherwise. This morphism is called the *stalk morphism* of f concentrated in degree k.

For each $k \in \mathbb{Z}$, we have a functor $\langle k\rangle\colon \mathcal{A} \to \operatorname{Kom}^{\mathrm{b}} \mathcal{A}$ given by

$$\begin{aligned} X &\mapsto (X\langle k\rangle, 0) && \text{for all } X \in \mathcal{A}, \\ f &\mapsto f\langle k\rangle && \text{for all } f \in \operatorname{Mor} \mathcal{A}. \end{aligned}$$

Context will ensure that this notation will not be confused with the shift functor. Furthermore, the notation behaves well in combination with the shift functor.

Lemma 3.1.13. *The shift functor induces a module structure over the integral Laurent polynomials $\mathbb{Z}[x, x^{-1}]$ on $K_0(\operatorname{Gr}^* \mathcal{A})$ and $K_0(\operatorname{Kom}^* \mathcal{A})$.*

Proof: The integral *Laurent polynomials* are the quotient ring $\mathbb{Z}[x, y]/(xy - 1)$, and we write y as x^{-1} to arrive at the above notation. In both settings, the k-shift functor $\langle k\rangle$ is an exact functor for all $k \in \mathbb{Z}$. Thus, we can define the action by $x^n \cdot [X] = [X\langle n\rangle]$ for all X in $\operatorname{Gr}^* \mathcal{A}$ (resp. $\operatorname{Kom}^* \mathcal{A}$) and extend by linearity. It is routine to check that this is well-defined. □

Proposition 3.1.14. *Let $\mathcal{A}$ be an abelian category. We have*

$$K_0(\operatorname{Gr}^{\mathrm{b}} \mathcal{A}) \cong K_0(\operatorname{Kom}^{\mathrm{b}} \mathcal{A})$$

as modules over the Laurent polynomials.

Proof: Homomorphism: the functor $\operatorname{Gr}^{\mathrm{b}} \mathcal{A} \to \operatorname{Kom}^{\mathrm{b}} \mathcal{A}$ given by $[X] \mapsto [(X, 0)]$ for all $X \in \operatorname{Gr}^{\mathrm{b}} \mathcal{A}$ and $f \mapsto f$ for all $f \in \operatorname{Mor} \operatorname{Gr}^{\mathrm{b}} \mathcal{A}$ is an exact functor (where 0 is the zero differential given by $0^n = 0_{\mathcal{A}}$ for all $n \in \mathbb{Z}$). By Proposition 2.1.8, this functor yields a group homomorphism. It is clear that is also a module homomorphism.

Injectivity: suppose that $[(X, 0)] = [(Y, 0)]$ in $K_0(\operatorname{Kom}^{\mathrm{b}} \mathcal{A})$. Then, we have conflations as in Lemma 2.1.5 of cochain complexes, but these are simply conflations of graded $\mathcal{A}$-objects. So, we have $[X] = [Y]$ in $K_0(\operatorname{Gr}^{\mathrm{b}} \mathcal{A})$. By Lemma 2.1.3(3), the group homomorphism is injective.

Surjectivity: let $(X, d_X) \in \operatorname{Kom}^{\mathrm{b}} \mathcal{A}$. Fix some $N \in \mathbb{Z}$ such that $X^n \cong 0_{\mathcal{A}}$ for $n \leq N$. Define a cochain complex $(X_{>n}, d_{X_{>n}})$ by $X^m_{>n} = X^m$ for $m > n$ and zero elsewhere and $d^m_{X_{>n}} = d^m_X$ for $m > n$ and zero elsewhere. For all $n \geq N$, we have an obvious (the morphisms of cochain complexes are made up of identity and zero morphisms) short exact sequence of the form:

$$0 \to (X_{>n+1}, d_{X_{>n+1}}) \to (X_{>n}, d_{X_{>n}}) \to X^n\langle -n \rangle \to 0.$$

We can visualize this short exact sequence by the following commutative diagram:

$$\begin{tikzcd}
 & 0_{\mathcal{A}} \arrow[d] & 0_{\mathcal{A}} \arrow[d] & 0_{\mathcal{A}} \arrow[d] & 0_{\mathcal{A}} \arrow[d] & \\
\cdots \arrow[r] & 0_{\mathcal{A}} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_{n+2} \arrow[r] \arrow[d] & X_{n+3} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_{n+2} \arrow[r] \arrow[d] & X_{n+3} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_n \arrow[r] \arrow[d] & 0_{\mathcal{A}} \arrow[r] \arrow[d] & 0_{\mathcal{A}} \arrow[r] \arrow[d] & 0_{\mathcal{A}} \arrow[r] \arrow[d] & \cdots \\
 & 0_{\mathcal{A}} & 0_{\mathcal{A}} & 0_{\mathcal{A}} & 0_{\mathcal{A}} &
\end{tikzcd}$$

We thus have $[(X_{>n}, d_{X_{>n}})] = [(X_{>n+1}, d_{X_{>n+1}})] + [X^n\langle -n\rangle] = [(X_{>n+1}, d_{X_{>n+1}})] + (-1)^n[X^n\langle 0\rangle]$ for all $n \geq N$. In the case of $n = N$ we have $[(X_{>n}, d_{X_{>n}})] = [(X, d_X)]$. Since cochain complexes are bounded, we have $[(X_{>n+1}, d_{X_{>n+1}})] = 0$ for sufficiently large n. Hence, we have $[(X, d_X)] = \sum_{n\geq N}(-1)^n[X^n\langle 0\rangle]$, which is in the image of the homomorphism. □

3.2 Homotopy Category

Our next step in defining the derived category is to mod-out homotopy equivalence from the category of cochain complexes of an abelian category. This gets us a triangulated category called the homotopy category of an abelian category.

Definition 3.2.1 (Congruence, quotient category)**.** A *congruence* R on a category $\mathcal{C}$ is a collection of equivalence relations $R_{X,Y}$ on $\hom_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$ such that the relations are *compatible with respect to composition*. That is, if $f \sim f'$ in $R_{X,Y}$ and $g \sim g'$ in $R_{Y,Z}$, then we have $g \circ f \sim g' \circ f \sim g \circ f' \sim g' \circ f'$ in $R_{X,Z}$.

Given a congruence R on $\mathcal{C}$, one can form the *quotient category* $\mathcal{C}/R$ whose objects are that of $\mathcal{C}$, whose morphisms are the equivalence classes determined by R, and whose composition is inherited from $\mathcal{C}$. The compatibility of the equivalence relations with respect to composition ensures that this category is properly defined.

Lemma 3.2.2. *Let $\mathcal{A}$ be an abelian category. Let R be a congruence. The quotient category $\mathcal{A}/R$ is additive.*

Proof: Let $X, Y \in \mathcal{A}$. Let $H_{X,Y}$ be the subgroup $\langle f - g \mid (f, g) \in R_{X,Y}\rangle$ of $\hom_{\mathcal{C}}(X, Y)$. It is well known that we can identify $\hom_{\mathcal{C}}(X, Y)/R_{X,Y}$ with the group $\hom(X, Y)/H_{X,Y}$. Hence, we can conclude that the quotient category $\mathcal{A}/R$ is preadditive and has a zero object. Since the zero morphisms of $\mathcal{A}$ represent the zero morphisms of $\mathcal{A}/R$ and the identity morphisms of $\mathcal{A}$ represent the identity morphisms of $\mathcal{A}/R$, we note that Lemma 1.1.14(3) is clear. □

To define the homotopy category, we make use of a concept from topology.

Definition 3.2.3 (Null-homotopic morphism)**.** A morphism of cochain complexes $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ is called *null-homotopic* if there exists a graded $\mathcal{A}$-morphism $h\colon X^\bullet \to Y^\bullet\langle -1\rangle$ such that

$$f = d_Y\langle -1\rangle \circ h + h\langle 1\rangle \circ d_X. \tag{3.2.1}$$

We say two morphisms in $\operatorname{Kom}^* \mathcal{A}$ are *chain homotopic* if their difference is null homotopic.

Lemma 3.2.4. *We have the following:*

1. *Chain homotopy defines an equivalence relation* $R_{X,Y}$ *on* $\hom_{\operatorname{Kom}^* \mathcal{A}}(X^\bullet, Y^\bullet)$*,*
2. *The null-homotopic morphisms of a hom-set form a subgroup.*
3. *If* g *or* f *is null-homotopic, then* $g \circ f$ *is null-homotopic,*
4. *Chain homotopy defines a congruence* R *on* $\operatorname{Kom}^* \mathcal{A}$*.*

Proof: The first statement is straightforward. The second statement is proven in [Mil, Lemma 1.3.1]. The third statement is proven in [Mil, Lemma 1.3.2] and, in more generality, in [KS06, Lemma 11.2.3]. The third statement implies that the equivalence relations of the first statement are compatible with respect to composition; hence, we have the fourth statement. □

Definition 3.2.5 (Homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The *homotopy category* of cochain complexes in $\mathcal{A}$, which we shall denote $\operatorname{Ho}^{\mathrm{ub}} \mathcal{A}$, is the quotient category $\operatorname{Kom}^{\mathrm{ub}} \mathcal{A}/R$ where R is the congruence formed by chain homotopic morphisms. Similarly, the quotient category $\operatorname{Kom}^{\mathrm{b}} \mathcal{A}/R$ (resp. $\operatorname{Kom}^{+} \mathcal{A}/R$, resp. $\operatorname{Kom}^{-} \mathcal{A}/R$) is denoted $\operatorname{Ho}^{\mathrm{b}} \mathcal{A}$ (resp. $\operatorname{Ho}^{+} \mathcal{A}$, resp. $\operatorname{Ho}^{-} \mathcal{A}$) and is called the *bounded* (resp. *bounded below*, resp. *bounded above*) homotopy category. As usual, we write $\operatorname{Ho}^* \mathcal{A}$ for a statement that applies to any of the four categories.

We will often denote morphisms of $\mathrm{Ho}^* \mathcal{A}$ by their equivalence class representatives as long as it will not cause confusion. The functor $\langle k\rangle \colon \mathrm{Kom}^* \mathcal{A} \to \mathrm{Kom}^* \mathcal{A}$ induces an additive functor $\mathrm{Ho}^* \mathcal{A} \to \mathrm{Ho}^* \mathcal{A}$ that, by further abusing notation, we again denote by $\langle k\rangle$ and we call $\langle 1\rangle$ the *shift functor* on $\mathrm{Ho}^* \mathcal{A}$.

Remark 3.2.6. Let $\mathcal{A}$ be an abelian category. Note that $\mathrm{Ho}^* \mathcal{A}$ is *not necessarily* abelian. For example, **Ab** is abelian but Ho^{b} **Ab** is not. See [HJR10, §1, Example 2.6] for the details.

Definition 3.2.7 (Chain equivalent)**.** A cochain complex $(X^\bullet, d_X)$ is *chain equivalent* to a cochain complex $(Y^\bullet, d_Y)$ if they are isomorphic in $\mathrm{Ho}^{\mathrm{ub}} \mathcal{A}$.

Lemma 3.2.8. *The homotopy category of cochain complexes satisfies the following universal property. For every functor* $F \colon \mathrm{Kom}^* \mathcal{A} \to \mathcal{C}$ *that sends chain equivalences to isomorphisms, there is a unique functor* $G \colon \mathrm{Ho}^* \mathcal{A} \to \mathcal{C}$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Kom}^* \mathcal{A} \arrow[r, "F"] \arrow[d] & \mathcal{C} \\
\mathrm{Ho}^* \mathcal{A} \arrow[ur, dashed, "G"] &
\end{tikzcd}$$

Proof: The following proof was influenced by [Wei94, Proposition 10.1.2].

Let $f, g \colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be chain homotopic morphisms. Denote by Z the cochain complex

$$\left(Y^\bullet \oplus Y^\bullet\langle -1\rangle \oplus Y^\bullet, \begin{pmatrix} d_Y & 0 & 0 \\ \mathrm{id}_Y & -d_Y\langle -1\rangle & -\mathrm{id}_Y \\ 0 & 0 & d_Y \end{pmatrix} \right)$$

and consider the following maps of cochain complexes:

$$Z \xrightarrow{k=\begin{pmatrix} \mathrm{id}_Y & 0 & 0 \end{pmatrix}} (Y^\bullet, d_Y) \quad \text{and} \quad (Y^\bullet, d_Y) \xrightarrow{r=\begin{pmatrix} \mathrm{id}_Y \\ 0 \\ \mathrm{id}_Y \end{pmatrix}} Z.$$

Note that $k \circ r = \mathrm{id}_Y$ and that $r \circ k - \mathrm{id}_Z$ is null homotopic via the graded $\mathcal{A}$-morphism

$$Y \oplus Y\langle -1\rangle \oplus Y \xrightarrow{h=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{id}_Y\langle -1\rangle & 0 \end{pmatrix}} Y\langle -1\rangle \oplus Y\langle -2\rangle \oplus Y\langle -1\rangle.$$

Let $F\colon \operatorname{Kom}^* \mathcal{A} \to \mathcal{C}$ be a functor that sends chain equivalences to isomorphisms. Then, we have $F(r \circ k) = \mathrm{id}_{FZ}$. Hence,

$$F(f) = F\left(\begin{pmatrix} 0 & 0 & \mathrm{id}_Y \end{pmatrix} \circ r \circ k \circ \begin{pmatrix} f \\ t \\ g \end{pmatrix}\right) = F\begin{pmatrix} 0 & 0 & \mathrm{id}_Y \end{pmatrix} \circ F\begin{pmatrix} f \\ t \\ g \end{pmatrix} = F(g),$$

where t is the graded $\mathcal{A}$-morphism satisfying (3.2.1) for f and g. The lemma quickly follows. □

In order to define distinguished triangles for the homotopy category, we make use of the concept of a mapping cone.

Definition 3.2.9 (Mapping cone)**.** Let $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be a morphism of cochain complexes. The *mapping cone* of f is the cochain complex $(C(f), d_{C(f)})$ given by the following:

$$C(f) = X^\bullet\langle 1\rangle \oplus Y^\bullet, \qquad d_{C(f)} = \begin{pmatrix} -d_X\langle 1\rangle & 0 \\ f\langle 1\rangle & d_Y \end{pmatrix}.$$

Definition 3.2.10. We say a sequence in $\operatorname{Ho}^* \mathcal{A}$

$$(X^\bullet, d_X) \longrightarrow (Y^\bullet, d_Y) \longrightarrow (Z^\bullet, d_Z) \longrightarrow (X^\bullet, d_X)\langle 1\rangle$$

is a *distinguished triangle* if it is isomorphic to a sequence of the form

$$(\tilde{X}^\bullet, d_{\tilde{X}}) \xrightarrow{f} (\tilde{Y}^\bullet, d_{\tilde{Y}}) \xrightarrow{i_{\tilde{Y}^\bullet}} (C(f), d_{C(f)}) \xrightarrow{p_{\tilde{X}^\bullet\langle 1\rangle}} (\tilde{X}^\bullet, d_{\tilde{X}})\langle 1\rangle \tag{3.2.2}$$

for some $f \in \operatorname{Mor} \operatorname{Ho}^* \mathcal{A}$ (by isomorphic, we mean there are four vertical chain equivalences that make the diagrams commute).

Theorem 3.2.11. *The triple* $(\operatorname{Ho}^* \mathcal{A}, \langle 1\rangle, D)$ *where* D *is the class of distinguished triangles is a triangulated category.*

Proof: We verify the axiom (TR1a) of Definition 1.4.2. Let $(X^\bullet, d_X) \in \operatorname{Ho}^* \mathcal{A}$. Set $U = C(\mathrm{id}_{X^\bullet})$. We have $\mathrm{id}_U = d_U\langle -1\rangle \circ h + h\langle 1\rangle \circ d_U$, where $h\colon C \to C\langle -1\rangle$ is the graded $\mathcal{A}$-morphism given by

$$X^{n+1} \oplus X^n \xrightarrow{h^n = \begin{pmatrix} 0 & \mathrm{id}_{X^n} \\ 0 & 0 \end{pmatrix}} X^n \oplus X^{n-1} \qquad \text{for all } n \in \mathbb{Z}.$$

Hence, we have that id_U is null-homotopic and thus U is chain equivalent to 0. Hence, we have the following isomorphism of sequences:

$$\begin{tikzcd}
X^\bullet \arrow[r, "\mathrm{id}_{X^\bullet}"] \arrow[d] & X^\bullet \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X^\bullet\langle 1\rangle \arrow[d] \\
X^\bullet \arrow[r, "\mathrm{id}_{X^\bullet}"] & X^\bullet \arrow[r] & U \arrow[r] & X^\bullet\langle 1\rangle
\end{tikzcd}$$

Therefore, the upper row is a distinguished triangle, verifying the axiom. Note that (TR1b) and (TR1c) are obvious. The interested reader can consult [KS06, Theorem 11.2.6] for the verification of the rest of the axioms. □

Lemma 3.2.12. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories. An additive functor* $F\colon \mathcal{A} \to \mathcal{B}$ *induces a triangulated functor*

$$\begin{aligned}
\operatorname{Ho}^* F\colon\ \operatorname{Ho}^* \mathcal{A} &\to \operatorname{Ho}^* \mathcal{B}, \\
(X^\bullet, d_X) &\mapsto (F(X), F(d_X)) && \textit{for all } (X^\bullet, d_X) \in \operatorname{Ho}^* \mathcal{A}, \\
f &\mapsto F(f) && \textit{for all } f \in \operatorname{Mor} \operatorname{Ho}^* \mathcal{A},
\end{aligned}$$

where $F(X)^n = F(X^n)$, $F(d_X)^n = F(d_X^n)$, *and* $F(f)^n = F(f^n)$ *for all* $n \in \mathbb{Z}$.

Proof: Every pair $(F(X), F(d_X))$ is a cochain complex since

$$\begin{aligned}
F(d_X)^{n+1} \circ F(d_X)^n = F(d_X^{n+1} \circ d_X^n) \\
= F(0_{X^n, X^{n+2}}) = 0_{F(X^n), F(X^{n+2})} = 0_{F(X)^n, F(X)^{n+2}}
\end{aligned}$$

for all $n \in \mathbb{Z}$. Let $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be a morphism in $\operatorname{Kom}^* \mathcal{A}$; then, the morphism $F(f)$ in $\mathcal{B}$ is a morphism in $\operatorname{Kom}^* \mathcal{B}$ since

$$F(f)^{n+1} \circ F(d_X)^n = F(f^{n+1} \circ d_X^n) = F(d_X^n \circ f^n) = F(d_X)^n \circ F(f)^n$$

for all $n \in \mathbb{Z}$. If f is null homotopic in $\mathcal{A}$, then $F(f)$ is null homotopic in $\mathcal{B}$ because F preserves (3.2.1). Hence, the functor is well defined.

It is straightforward to verify that $\operatorname{Ho}^* F(C(f)) = C(\operatorname{Ho}^* F(f))$ for every morphism f in $\operatorname{Kom}^* \mathcal{A}$. Using the canonical natural isomorphism $\eta\colon \operatorname{Ho}^* F \circ \langle 1 \rangle \cong \langle 1 \rangle \circ \operatorname{Ho}^* F$, this quickly implies that $\operatorname{Ho}^* F$ is a triangulated functor. ☐

Proposition 3.2.13 ([Ros11, Theorem 1.1])**.** *Let $\mathcal{A}$ be an essentially small abelian category. The triangulated Grothendieck group $K_0^{\triangle}(\operatorname{Ho}^{\mathrm{b}} \mathcal{A})$ of the bounded homotopy category of $\mathcal{A}$ is isomorphic as a group to the split Grothendieck group $K_0^{\mathrm{split}}(\mathcal{A})$ of $\mathcal{A}$. The isomorphism is induced by the stalk cochain complex.*

3.3 Cohomology

Cohomology is a very important notion in algebra. Here we consider cohomology in the general context of abelian categories. We will use this to define derived categories, and it will play an important role later with derived functors.

Lemma 3.3.1. *Let $\mathcal{A}$ be an abelian category. Let $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ be elements of $\operatorname{Kom}^* \mathcal{A}$. Fix kernels and cokernels for all morphisms in $\mathcal{A}$. Fix $n \in \mathbb{Z}$. There are unique morphisms u_X^n and v_X^n making the following diagrams commute:*

$$\begin{tikzcd}
Z_X^n \arrow[d, "v_X^n"'] \arrow[dr, "\operatorname{Im} d_X^{n-1}"] & \\
K_X^n \arrow[r, "\operatorname{Ker} d_X^n"] & X^n
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
Y^{n+1} & \\
Y^n \arrow[u, "d_Y^n"] \arrow[r, "\operatorname{Coker} d_Y^{n-1}"] & C_Y^n. \arrow[ul, "u_Y^n"']
\end{tikzcd}$$

Furthermore, there is a unique morphism h_f^n *such that the following diagram commutes:*

$$\begin{tikzcd}
K_X^n \arrow[r, "\operatorname{Ker} d_X^n"] \arrow[d, "\operatorname{Coker} v_X^n"] & X^n \arrow[r, "f^n"] & Y^n \arrow[r, "\operatorname{Coker} d_Y^{n-1}"] & C_Y^n \\
H^n(X)' \arrow[rrr, "h_f^n"] & & & H^n(Y). \arrow[u, "\operatorname{Ker} u_Y^n"']
\end{tikzcd}$$

In addition, we have the following:

- *The morphism* $h_{X^\bullet}^n := h_{\mathrm{id}_{X^\bullet}}^n$ *is an isomorphism for all* $(X^\bullet, d_X) \in \operatorname{Kom}^* \mathcal{A}$.
- *We have* $h_g^n \circ (h_{\operatorname{dom} g}^n)^{-1} \circ h_f^n = h_{g \circ f}^n$ *for all composable* $f, g \in \operatorname{Mor} \operatorname{Kom}^* \mathcal{A}$.

Thus, we have a functor $H^n \colon \operatorname{Kom}^* \mathcal{A} \to \mathcal{A}$ *such that* $H^n(X^\bullet, d_X) = H^n(X)$ *for all* $(X^\bullet, d_X) \in \operatorname{Kom}^* \mathcal{A}$ *and* $H^n(f) = h_f^n \circ (h_{\operatorname{dom} f}^n)^{-1}$ *for all* $f \in \operatorname{Mor} \operatorname{Kom}^* \mathcal{A}$.

Proof: Although it is presented differently here, see [GM03, II.6.3]. □

Definition 3.3.2 (Cohomological functors)**.** Let $(\mathcal{A}, T, D)$ be a triangulated category and $\mathcal{B}$ be an abelian category. An additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is a *cohomological functor* if $FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ$ is exact (i.e. $\operatorname{Im} F(f) \sim \operatorname{Ker} F(g)$ in $(FY)^{\text{mono}}$) for every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ in D.

Definition 3.3.3 (H^n)**.** Let $(X^\bullet, d_X) \in \operatorname{Kom}^* \mathcal{A}$ and $n \in \mathbb{Z}$. The object $H^n(X)$ in Lemma 3.3.1 is called the n-th *cohomology object* of $(X^\bullet, d_X)$. If f is null-homotopic, then $H^n(f) = 0$ for all $n \in \mathbb{Z}$ (see Lemma 12.2.2 in [KS06]). Hence, the functor H^n induces a functor $\operatorname{Ho}^* \mathcal{A} \to \mathcal{A}$ whose notation we shall abuse and also denote by H^n. This functor is cohomological (see [KS06, Corollary 12.2.5]).

Remark 3.3.4. If $\mathcal{A}$ is some subcategory of R-**mod** for some ring with unity R, then, using traditional kernels and cokernel, we have $H^n(X) = \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1}$ for all $(X^\bullet, d_X) \in \operatorname{Kom}^* \mathcal{A}$. Given a morphism of cochain complexes $f \colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$, the morphism $H^n(f) \colon \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1} \to \operatorname{Ker} d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the natural morphism. That is, we have $H^n(f)(x + \operatorname{Im} d_X^{n-1}) = f(x) + \operatorname{Im} d_Y^{n-1}$ for all $x \in \operatorname{Ker} d_X^n$. Because of this, many references make use of the following theorem.

Theorem 3.3.5 (Freyd Mitchel Embedding Theorem)**.** *Let $\mathcal{A}$ be an essentially small abelian category. There exists a fully faithful exact functor $\mathcal{A} \to R$-**mod** for some (not necessarily commutative) ring with unity R.*

Proof: See [KS06, Theorem 9.6.10] for a proof for small categories. Since $\mathcal{A}$ is essentially small, there is an equivalence of categories $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is small. This is automatically fully faithful exact and, thus, the composition with a functor $\mathcal{B} \to R$-**mod** for some ring with unity R is fully faithful exact. □

Example 3.3.6. Let $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be the following the morphism of cochain complexes in $\mathbb{Z}$-**mod**$^{\text{fg}}$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2\cdot"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\operatorname{mod} 2"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

If $\mathbb{Z}/2\mathbb{Z}$ is in degree zero, then $H^iX \cong H^iY \cong 0$ for all $i \neq 0$ and $H^0X \cong H^0Y \cong \mathbb{Z}/2\mathbb{Z}$ and $H^i(f)$ is an isomorphism for all i.

Lemma 3.3.7. *Let $\mathcal{A}$ be an abelian category. Let $(X, d_X) \xrightarrow{f} (Y, d_X) \xrightarrow{g} (Z, d_Z) \xrightarrow{h} (X, d_X)\langle 1 \rangle$ be a distinguished triangle in $\operatorname{Ho}^* \mathcal{A}$. Then, there is an exact sequence in $\mathcal{A}$ of the following form:*

$$\cdots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to \cdots$$

Proof: Using the fact that H^n is cohomological for all $n \in \mathbb{Z}$, that $H^n(W\langle 1 \rangle) \cong H^{n+1}(W)$ for all $W \in \mathcal{A}$ and $n \in \mathbb{Z}$, and that (TR2) of Definition 1.4.2 holds for $\operatorname{Ho}^* \mathcal{A}$, the conclusion follows. □

3.4 Bounded Derived Category

In this section, we reach the definition of the derived category of an abelian category. We do this by formally inverting morphisms that induce isomorphisms on cohomology via a process called localization.

Definition 3.4.1 (Localizing family of morphisms)**.** Let $\mathcal{C}$ be a category and S a family of morphisms in $\mathcal{C}$. The family S is called *localizing* (or a *multiplicative system*) if the following conditions hold:

(L1) S is closed under composition and $\mathrm{id}_X \in S$ for all $X \in \mathcal{C}$.

(L2) Given morphisms $f\colon X \to Z$ and $s\colon X \to Y$ with $s \in S$, there exist morphisms $g\colon Y \to W$ and $t\colon Z \to W$ with $t \in S$ making the following diagram commute:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "g"'] & X \arrow[l, "s"'] \arrow[r, "f"] & Z \arrow[dl, dashed, "t"] \\
& W &
\end{tikzcd}$$

(L3) Given morphisms $f\colon Y \to X$ and $s\colon Z \to X$ with $s \in S$, there exist morphisms $g\colon W \to Z$ and $t\colon W \to Y$ with $t \in S$ making the following diagram commute:

$$\begin{tikzcd}
Y \arrow[r, "f"] & X & Z \arrow[l, "s"'] \\
& W \arrow[ul, dashed, "t"] \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

(L4) Let $f, g\colon Y \to Z$ be morphisms in $\mathcal{C}$. There exists $s\colon X \to Y$ in S such that $f \circ s = g \circ s$ if and only if there exists $t\colon Z \to W$ in S such that $t \circ f = t \circ g$.

Some references require that every isomorphism in $\mathcal{C}$ belong to S, but we will not.

Definition 3.4.2 (Localization)**.** Let $\mathcal{C}$ be a category and S a family of morphisms in $\mathcal{C}$. A *localization* of $\mathcal{C}$ by S consists of a category $\mathcal{C}[S^{-1}]$ and a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that the following hold:

- For all $s \in S$, $Q(s)$ is an isomorphism.
- For any category $\mathcal{D}$ and functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$, there exists a functor $F_S\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F \cong F_S \circ Q$.
- For any category $\mathcal{D}$, the functor
$$(- \circ Q)\colon \mathbf{Fct}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathbf{Fct}(\mathcal{C}, \mathcal{D})$$
is fully faithful. A localization is unique up to an equivalence of categories.

Definition 3.4.3 (Roof)**.** Let S be a localizing family of morphisms in a category $\mathcal{C}$. Fix some $X, Y \in \mathcal{C}$. We will be concerned with diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

such that $s \in S$. We define an equivalence relation on such diagrams declaring that

$$\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & & \text{is equivalent to} & & W' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y & & X & & Y \end{tikzcd}$$

if and only if there exists and object E and morphisms t and g with $t \in S$ such that the following diagram commutes:

$$\begin{tikzcd} & & E \arrow[dl, "t"'] \arrow[dr, "g"] & & \\ & W \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & W' \arrow[dlll, "s'" near end] \arrow[dr, "f'"] & \\ X & & & & Y \end{tikzcd}$$

Reflexivity and symmetry are obvious. For transitivity, see [GM03, III.2 Lemma 8]. For our purposes, a *roof* from X to Y will be an equivalence class of this relation.

Lemma 3.4.4. *Let $\mathcal{C}$ be a category and S a localizing class of morphisms in $\mathcal{C}$. A localization $(\mathcal{C}[S^{-1}], Q)$ of $\mathcal{C}$ by S exists and is constructed as follows. We set*

$$\operatorname{Ob}\mathcal{C}[S^{-1}] = \operatorname{Ob}\mathcal{C},$$
$$\hom_{\mathcal{C}[S^{-1}]}(X,Y) = \{\text{all roofs from } X \text{ to } Y\}.$$

Composition of roofs represented by

$$\begin{tikzcd} & W' \arrow[dl, "t"'] \arrow[dr, "g"] & & \text{and} & & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ Y & & Z & & X & & Y \end{tikzcd}$$

is the roof represented by

$$\begin{tikzcd} & E \arrow[dl, "s \circ r"'] \arrow[dr, "g \circ h"] & \\ X & & Z \end{tikzcd}$$

where $r \in S$ and the morphisms r and h make the following diagram commute:

$$\begin{tikzcd} & & E \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & W \arrow[dl, "s"'] \arrow[dr, "f"] & & W' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

Such morphisms r and h must exist because S is localizing (see axiom (L3)). Finally, the functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ acts as the identity on objects and sends a morphism $f\colon X \to Y$ to the roof represented by

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

Proof: See III.2 Lemma 8 in [GM03]. (Composition being well defined is left as an exercise there; it follows from the final axiom of a localizing class of morphisms.) □

Corollary 3.4.5. *Let S be a localizing family of morphisms in an additive category $\mathcal{A}$. Let $(A[S^{-1}], Q)$ be a localization of $\mathcal{A}$ by S. The category $A[S^{-1}]$ is additive and the functor Q is additive.*

Proof: Let $\mathcal{A}[S^{-1}]$ be the localization described in Lemma 3.4.4. Fix an $X \in \mathcal{A}$ and some morphism $f\colon W \to X$ in $\mathcal{A}$. The following diagram commutes:

$$\begin{tikzcd}
& & W' \arrow[dl, "\mathrm{id}_{W'}"'] \arrow[dr, "0_{W',W}"] & & \\
& W' \arrow[dl] \arrow[drrr, "0_{W',X}"' near end] & & W \arrow[dlll] \arrow[dr, "f"] & \\
0_{\mathcal{A}} & & & & X
\end{tikzcd}$$

Hence, all the roofs from $0_{\mathcal{A}}$ to X are equivalent. The fact that all roofs from X to $0_{\mathcal{A}}$ are equivalent follows from (L3) of Definition 3.4.1. The following are roofs by (L1):

$$\begin{tikzcd}
& 0_{\mathcal{A}} \arrow[dl, "\mathrm{id}_{0_{\mathcal{A}}}"'] \arrow[dr] & \\
0_{\mathcal{A}} & & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 0_{\mathcal{A}} \arrow[dl, "\mathrm{id}_{X}"'] \arrow[dr] & \\
X & & 0_{\mathcal{A}}
\end{tikzcd}$$

Hence, it is clear that $0_{\mathcal{A}}$ is a zero object of $\mathcal{A}[S^{-1}]$. Let $X, Y \in \mathcal{A}$. Given two roofs from X to Y represented by ($s, s' \in S$ and $f, f' \in \mathcal{A}$)

$$\begin{tikzcd}
& W \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& W' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\
X & & Y,
\end{tikzcd}$$

we have, by (L3), a commutative diagram of the following form ($t \in S$ and $t' \in \mathcal{A}$):

$$\begin{tikzcd}
W \arrow[r, "s"] & X & W' \arrow[l, "s'"'] \\
& Z \arrow[ul, "t"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

Hence, we shall define the sum of these two roofs to be the roof represented by

$$\begin{tikzcd}
 & Z \arrow[dl, "s \circ t"'] \arrow[dr, "f \circ t + f' \circ t'"] & \\
X & & Y.
\end{tikzcd}$$

Using the fact S is localizing, one can show that this is well defined. For $X_1, \ldots, X_n$, let $i_{Q(X_i)} = Q(i_{X_i})$ and $p_{Q(X_j)} = Q(p_{X_j})$ and note that the biproducts of $\operatorname{Ho}^* \mathcal{A}$ with these morphisms satisfy Lemma 1.1.14(3). Hence, since localizations are unique up to equivalence, $\mathcal{A}[S^{-1}]$ is additive.

As for the additivity of Q, we simply note that $Q(f+g) = Q(f) + Q(g)$ for all $f, g \in \hom_{\mathcal{A}}(X, Y)$ and $X, Y \in \mathcal{A}$. $\square$

Definition 3.4.6 (Compatible with triangulation)**.** Let S be a localizing system in a triangulated category $(\mathcal{A}, T, D)$. We say that S is *compatible with triangulation* if the following holds:

- We have $s \in S$ if and only if $T(s) \in S$ for all $s \in \operatorname{Mor} \mathcal{A}$.
- Given a commutative diagram of the form

  $$\begin{tikzcd}
  X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\
  X' \arrow[r, "f'"] & Y'
  \end{tikzcd}$$

  such that $u, v \in S$ and two distinguished triangles

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \qquad \text{and} \qquad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X'),$$

  there exists a morphism $w \colon Z \to Z'$ in S such that the following diagram commutes:

  $$\begin{tikzcd}
  X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & T(X) \arrow[d, "T(u)"] \\
  X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X').
  \end{tikzcd}$$

Proposition 3.4.7. *Let S be a localizing family of morphisms in a triangulated category $(\mathcal{A}, T, D)$ that is compatible with triangulation. Let $(\mathcal{A}[S^{-1}], Q)$ be a localization of $\mathcal{A}$ by S (which exists by Lemma 3.4.4, and $\mathcal{A}[S^{-1}]$ and Q are additive by Lemma 3.4.5). We have the following:*

1. *There exists a morphism $T' \colon A[S^{-1}] \to A[S^{-1}]$ such that $T' \circ Q \cong Q \circ T$.*
2. *Define D' to be the collection of triangles in $(A[S^{-1}], T')$ isomorphic to*

$$Q(X) \xrightarrow{Q(f)} Q(Y) \xrightarrow{Q(g)} Q(Z) \xrightarrow{Q(h)} Q(T(X))$$

for some triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ in D. The triple $(\mathcal{A}[S^{-1}], T', D')$ is a triangulated category.

3. *Clearly, the functor $Q \colon (\mathcal{A}, T, D) \to (\mathcal{A}[S^{-1}], T', D')$ is then triangulated.*

Proof: Since S is compatible with triangulation, we have $T(s) \in S$ for all $s \in S$. Thus, $(Q \circ T)(s)$ is an isomorphism for all $s \in S$. By Definition 3.4.2, there exists $T' \colon \mathcal{A}[S^{-1}] \to \mathcal{A}[S^{-1}]$ such that $Q \circ T \cong T' \circ Q$. If $U \colon \mathcal{A} \to \mathcal{A}$ satisfies $T \circ U \cong U \circ T \cong \mathrm{id}_{\mathcal{A}}$, then U also yields a functor $U' \colon \mathcal{A}[S^{-1}] \to \mathcal{A}[S^{-1}]$. Therefore,

$$U' \circ T' \circ Q \cong U' \circ Q \circ T \cong Q \circ U \circ T \cong Q \circ \mathrm{id}_{\mathcal{A}}.$$

Thus, we have $U' \circ T' \cong \mathrm{id}_{\mathcal{A}}$ by Definition 3.4.2. After a symmetric argument, we see that T' is an equivalence of categories. For the proof of the other two statements see [Mil, Theorem 1.6.1]. ☐

Definition 3.4.8 (Null System)**.** Let $(\mathcal{A}, T, D)$ be a triangulated category. A *null system* is a strictly full subcategory $\mathcal{N}$ such that the following holds:

- $0 \in \mathcal{N}$.
- $X \in \mathcal{N}$ if and only if $TX \in \mathcal{N}$.

- If $X \to Y \to Z \to TX$ is in D and $X, Z \in \mathcal{N}$, then $Y \in \mathcal{N}$.

We associate to a null system a family of morphisms:

$$S(\mathcal{N}) := \{f\colon X \to Y \mid \text{ there exists } X \xrightarrow{f} Y \to Z \to TX \in D \text{ with } Z \in \mathcal{N}\}.$$

Lemma 3.4.9 ([KS06, Theorem 10.2.3(i)])**.** *If $\mathcal{N}$ is a null system of a triangulated category $\mathcal{A}$, then $S(\mathcal{N})$ is a localizing class that is compatible with triangulation.*

Definition 3.4.10 (Quasi-isomorphism)**.** A morphism f in $\operatorname{Kom}^* \mathcal{A}$ is called a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. Denote by $\operatorname{Quis} \mathcal{A}$ the collection of quasi-isomorphisms in $\operatorname{Kom}^* \mathcal{A}$ modulo chain homotopy. A quasi-isomorphism $A \to B$ does not necessarily imply the existence of a quasi-isomorphism $B \to A$ (see Example 3.3.6 for a quasi-isomorphism and note that no non-zero arrow exists in reverse).

Definition 3.4.11. A cochain complex $(X^\bullet, d_X)$ is *acyclic* if $H^n(X) = 0$ for all $n \in \mathbb{Z}$. Note that a cochain complex is acyclic if and only if the differential gives an exact sequence.

Lemma 3.4.12. *Let $\mathcal{A}$ be an abelian category. Let $N_H(\mathcal{A})$ be the (strictly) full subcategory of $\operatorname{Ho}^* \mathcal{A}$ consisting of all objects who are homotopic to an acyclic cochain complex. We have that $N_H(\mathcal{A})$ is a triangulated subcategory of $\operatorname{Ho}^* \mathcal{A}$.*

Proof: Note that if $(X^\bullet, d_X) \in N_H(\mathcal{A})$, then $H^n(X^\bullet\langle k\rangle) = 0$ for all $n, k \in \mathbb{Z}$. If $(X^\bullet, d_X) \xrightarrow{f} (Y^\bullet, d_Y) \to (Z^\bullet, d_Z) \to (X^\bullet, d_X)\langle 1\rangle$ is a distinguished triangle, then $0 \to H^n(X) \to H^n(Y) \to H^n(Z) \to 0$ is a short exact sequence. Hence, if any two of $(X^\bullet, d_X)$, $(Y^\bullet, d_Y)$, and $(Z^\bullet, d_Z)$ are in $N_H(\mathcal{A})$, then the third is as well. □

Lemma 3.4.13 ([KS06, 10.2.1(a)])**.** *If $\mathcal{N}$ is a strictly full triangulated subcategory of a triangulated category $\mathcal{A}$, then $\mathcal{N}$ is a null system.*

Remark 3.4.14. In particular, the category $N_H(\mathcal{A})$ is a null system and $S(N_H\mathcal{A})$ is a localizing class compatible with triangulation.

Definition 3.4.15 (Derived category)**.** Let $\mathcal{A}$ be an abelian category. We denote by $D^*(\mathcal{A})$ the localization of $\operatorname{Ho}^* \mathcal{A}$ by $S(N_H(\mathcal{A}))$ as per Lemma 3.4.4. The category $D^*(\mathcal{A})$ is called the *derived category* and $D^{\mathrm{b}}(\mathcal{A})$ is called that *bounded derived category*.

Remark 3.4.16. $S(N_H(\mathcal{A}))$ is simply $\operatorname{Quis} \mathcal{A}$.

Proposition 3.4.17. *Let $\mathcal{A}$ be an abelian category. Since $D^*(\mathcal{A})$ is a localization we have a functor $Q\colon \operatorname{Ho}^* \mathcal{A} \to D^*(\mathcal{A})$. Distinguished triangles in $D^*(\mathcal{A})$ are sequences isomorphic to those of the form*

$$Q(X^\bullet) \xrightarrow{Q(f)} Q(Y^\bullet) \xrightarrow{Q(g)} Q(Z^\bullet) \xrightarrow{Q(h)} Q(X^\bullet\langle 1\rangle)$$

where

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet\langle 1\rangle$$

is a distinguished triangle in $\operatorname{Ho}^* \mathcal{A}$. These distinguished triangles together with the shift functor form a triangulated category. The functor Q is thus a triangulated functor.

Proof: This is a consequence of Proposition 3.4.7. ☐

Proposition 3.4.18. *Let $\mathcal{A}$ be an abelian category. The derived category satisfies the following universal property. For every functor $F\colon \operatorname{Kom}^* \mathcal{A} \to \mathcal{C}$ that sends quasi-isomorphisms to isomorphisms, there is a unique functor $G\colon D^*(\mathcal{A}) \to \mathcal{C}$ making the following diagram commute:*

$$\begin{tikzcd}
\operatorname{Kom}^* \mathcal{A} \arrow[r, "F"] \arrow[d] & \mathcal{C} \\
D^*(\mathcal{A}) \arrow[ur, dashed, "G"] &
\end{tikzcd}$$

Proof: This is a consequence of the fact that $D^*(\mathcal{A})$ localization by $\operatorname{Quis} \mathcal{A}$. ☐

Proposition 3.4.19. *Let $\mathcal{A}$ be an essentially small abelian category and $D^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. The natural functor $\mathcal{A} \to D^{\mathrm{b}}(\mathcal{A})$ induces an isomorphism of groups $K_0(\mathcal{A}) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$.*

Proof: This proof is heavily influenced by the proof in [Vir]. First, we show that a group homomorphism exists. Define a map $\operatorname{Iso}\mathcal{A} \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$ by $[X] \mapsto [X\langle 0\rangle]$ for all $X \in \mathcal{A}$. Clearly, if $X \cong Y$ in $\mathcal{A}$, then $X\langle 0\rangle \cong Y\langle 0\rangle$ in $D^{\mathrm{b}}(\mathcal{A})$ so $[X\langle 0\rangle] = [Y\langle 0\rangle]$ in $K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$. So, the map is well defined. Suppose that

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence in $\mathcal{A}$. It is straightforward to show that $C(f\langle 0\rangle)$ (the cone of $f\langle 0\rangle$) is quasi-isomorphic to $Z\langle 0\rangle$. Hence, we have $[Y\langle 0\rangle] = [X\langle 0\rangle] + [Z\langle 0\rangle]$ in $K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$. By Propositon 2.1.7, we have a group homomorphism $F\colon K_0(\mathcal{A}) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A}))$ such that $F([X]-[Y]) = [X\langle 0\rangle] - [Y\langle 0\rangle]$ for all $X, Y \in \mathcal{A}$.

Now, we construct an inverse for this homomorphism. Define a map $\operatorname{Iso} D^{\mathrm{b}}(\mathcal{A}) \to K_0(\mathcal{A})$ by $(X^\bullet, d_X) \mapsto \sum_{n\in\mathbb{Z}}(-1)^n[H^n(X)]$ for all $(X^\bullet, d_X) \in D^{\mathrm{b}}(\mathcal{A})$. Note that the sum is finite since the complexes are bounded. If $(X^\bullet, d_X) \cong (Y^\bullet, d_Y)$ in $D^{\mathrm{b}}(\mathcal{A})$, then $H^n(X) \cong H^n(Y)$ for all $n \in \mathbb{Z}$ and thus the map is well defined. Suppose that

$$(X^\bullet, d_X) \to (Y^\bullet, d_Y) \to (Z^\bullet, d_Z) \to (X^\bullet, d_X)\langle 1\rangle$$

is a distinguished triangle in $D^{\mathrm{b}}(\mathcal{A})$. Then, it is a distinguished triangle in $\operatorname{Ho}^{\mathrm{b}}\mathcal{A}$. By Lemma 3.3.7, we have the following exact sequence in $\mathcal{A}$:

$$\cdots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to H^1(Y) \to \cdots$$

It follows from Lemma 2.1.11 via the fact that the cochain complexes are bounded that

$$\sum_{n\in\mathbb{Z}}(-1)^n[H^n(Y)] = \sum_{n\in\mathbb{Z}}(-1)^n[H^n(X)] + \sum_{n\in\mathbb{Z}}(-1)^n[H^n(Z)].$$

Thus, we have a group homomorphism $G\colon K_0^{\triangle}(D^{\mathrm{b}}(\mathcal{A})) \to K_0(\mathcal{A})$.

In order to show that F and G are inverse morphisms, one needs the fact that $\sum_{n\in\mathbb{Z}}(-1)^n[H^n(X)\langle 0\rangle] = [(X^\bullet, d_X)]$ in $K_0^\triangle(D^{\mathrm{b}}(\mathcal{A}))$ for all $(X^\bullet, d_X) \in D^{\mathrm{b}}(\mathcal{A})$. To show this let $n \in \mathbb{Z}$ and let C_X^n be as in Lemma 3.3.1. The cone of the morphism of cochain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & H^n(X) \arrow[r] \arrow[d, "\operatorname{Ker} u_X^n"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C_X^n \arrow[r, "u_X^n"] & X^{n+1} \arrow[r, "d_X^{n+1}"] & X^{n+2} \arrow[r, "d_X^{n+2}"] & \cdots
\end{tikzcd} \tag{3.4.1}$$

is quasi-isomorphic to

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & C_X^{n+1} \arrow[r] & X^{n+2} \arrow[r, "d_X^{n+2}"] & X^{n+3} \arrow[r, "d_X^{n+3}"] & \cdots
\end{tikzcd}$$

Since $(X^\bullet, d_X)$ is bounded below, the second row in (3.4.1) is quasi-isomorphic to $(X^\bullet, d_X)$ for sufficiently small n. Using a similar argument as Proposition 3.1.14, we have

$$[(X^\bullet, d_X)] = \sum_{n\in\mathbb{Z}}[H^n(X)\langle -n\rangle] = \sum_{n\in\mathbb{Z}}(-1)^n[H^n(X)\langle 0\rangle]$$

and the sums are finite since $(X^\bullet, d_X)$ is bounded. □

Remark 3.4.20. Even though the $K_0(\mathcal{A})$ and $K_0^\triangle(D^{\mathrm{b}}(\mathcal{A}))$ are isomorphic groups in a natural way for all abelian categories $\mathcal{A}$, there is more structure in the bounded derived category available for use on the invariant. In Example 3.4.21, we show that multiplication by negative one is decategorified by $K_0^\triangle(D^{\mathrm{b}}(-))$ but not $K_0(-)$ for $\mathbb{F}$-$\mathbf{mod}^{\mathrm{fg}}$ where $\mathbb{F}$ is a field.

Example 3.4.21. Consider the group $G_0(\mathbb{F})$ where $\mathbb{F}$ is field. Suppose there is an exact functor $G\colon \mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}} \to \mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}}$ such that $[G][\mathbb{F}] = -[\mathbb{F}]$, where $[G]$ is the homomorphism defined in Proposition 2.1.8. Thus, we have $[G\mathbb{F} \oplus \mathbb{F}] = 0$ in $G_0(\mathbb{F})$ but this is a contradiction as $[\dim][G\mathbb{F} \oplus \mathbb{F}] \neq 0$. Hence, there is no such functor.

Let $\alpha\colon G_0(\mathbb{F}) \to K_0^\triangle(D^{\mathrm{b}}(\mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}}))$ be the isomorphism given in Proposition 3.4.19. The shift functor $\langle 1\rangle\colon D^{\mathrm{b}}(\mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}}) \to D^{\mathrm{b}}(\mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}})$ is triangulated and induces a homomorphism $K_0^{\triangle}(\langle 1\rangle)\colon K_0^{\triangle}(D^{\mathrm{b}}(\mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}})) \to K_0^{\triangle}(D^{\mathrm{b}}(\mathbb{F}\text{-}\mathbf{mod}^{\mathrm{fg}}))$. In contrast to the preceding paragraph, we have $(\alpha^{-1} \circ K_0^{\triangle}(\langle 1\rangle) \circ \alpha)[X] = -[X]$ for all $[X] \in G_0(\mathbb{F})$.

Chapter 4

Derived Functors

Derived functors are the main motivation for derived categories. We look at derived functors because they induce interesting group homomorphisms between Grothendieck groups.

4.1 Injective and Projective Resolutions

Recall the definition of injective and projective objects (see Definition 1.2.21).

Definition 4.1.1 (Enough injectives/projectives, resolution)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *enough injectives* if for every $X \in \mathcal{A}$ there exists a monomorphism $m\colon X \to I$ for some $I \in \operatorname{Inj}\mathcal{A}$. Dually, we say that $\mathcal{A}$ has *enough projectives* if for every $X \in \mathcal{A}$ there exists an epimorphism $e\colon P \to X$ for some $P \in \operatorname{Prj}\mathcal{A}$.

Let $(X^\bullet, d_X) \in \operatorname{Kom}^+ \mathcal{A}$. An *injective resolution* of $(X^\bullet, d_X)$ is a complex $(I^\bullet, d_I) \in \operatorname{Kom}^+ \operatorname{Inj}\mathcal{A}$ such that there exists a quasi-isomorphism $(X^\bullet, d_X) \to (I^\bullet, d_I)$. Dually, a *projective resolution* of $(X^\bullet, d_X) \in \operatorname{Kom}^- \mathcal{A}$ is a complex $(P^\bullet, d_P) \in \operatorname{Kom}^- \operatorname{Prj}\mathcal{A}$ such that there exists a quasi-isomorphism $(P^\bullet, d_P) \to (X^\bullet, d_X)$.

Lemma 4.1.2. *Suppose $\mathcal{A}$ has enough injectives (resp. projectives). Then, every bounded below (resp. bounded above) cochain complex has a injective (resp. projective) resolution.*

Proof: The proof is given, but in more generality, in [KS06, Lemma 13.2.1]. □

Lemma 4.1.3 ([Mil, §5.2])**.** *A injective (or projective) resolution is unique up to chain equivalence.*

Corollary 4.1.4. *If $\mathcal{A}$ has enough injectives (resp. projectives), then every bounded below (resp. bounded above) cochain complex is quasi-isomorphic to an injective resolution (resp. projective resolution).*

Example 4.1.5 (Standard injective resolution)**.** Let $\mathcal{A}$ be an abelian category with enough injectives. Let $X \in \mathcal{A}$. We have a monomorphism $m_0\colon X \to I_0$ for some $I_0 \in \operatorname{Inj}\mathcal{A}$ because $\mathcal{A}$ has enough injectives. Fix $\operatorname{Coker} m_0\colon I_0 \to C_0$. By induction, we have monomorphisms $m_i\colon C_{i-1} \to I_i$ with $I_i \in \operatorname{Inj}\mathcal{A}$ for all $i \in \mathbb{N}$ where we fix $\operatorname{Coker} m_i\colon I_i \to C_i$ for all $i \in \mathbb{N}$. This yields a cochain complex

$$\cdots \to 0_{\mathcal{A}} \to I_0 \xrightarrow{m_1 \circ \operatorname{Coker} m_0} I_1 \xrightarrow{m_2 \circ \operatorname{Coker} m_1} I_2 \xrightarrow{m_3 \circ \operatorname{Coker} m_2} \cdots$$

The image of $m_i \circ \operatorname{Coker} m_{i-1}$ is m_i for all $i \in \mathbb{N}$ and the kernel of $m_{i+1} \circ \operatorname{Coker} m_i$ is m_i for all $i \in \mathbb{N}$. It follows that this is an injective resolution of X , the *standard construction*. We have a dual construction if $\mathcal{A}$ has enough projectives.

Proposition 4.1.6 ([GM03, III.5 Theorem 21])**.** *If $\mathcal{A}$ has enough injectives, then there is an triangulated equivalence of categories*

$$\operatorname{Inj}\colon D^+(\mathcal{A}) \xrightarrow{\sim} \operatorname{Ho}^+ \operatorname{Inj}\mathcal{A}$$

induced by injective resolution. This functor is unique up to natural isomorphism. Dually, if $\mathcal{A}$ has enough projectives, then there is an triangulated equivalence of categories

$$\operatorname{Prj}\colon D^-(\mathcal{A}) \xrightarrow{\sim} \operatorname{Ho}^- \operatorname{Prj}\mathcal{A}$$

induced by projective resolution. Again, this functor is unique up to natural isomorphism.

4.2 Right Derived Functors

Definition 4.2.1 (Left exact functor)**.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is a *left exact functor* if the sequence $0_{\mathcal{B}} \to FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ$ in $\mathcal{B}$ is exact for every exact sequence $0_{\mathcal{A}} \to X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

Definition 4.2.2 (Total right derived functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Since F is additive, it naturally induces a triangulated functor $\operatorname{Ho}^+(F)\colon \operatorname{Ho}^+ \operatorname{Inj} \mathcal{A} \to \operatorname{Ho}^+ \mathcal{B}$ (see Proposition 3.2.12). Let $Q\colon \operatorname{Ho}^+ \mathcal{B} \to D^+(\mathcal{B})$ be the functor associated to the localization. Let $\operatorname{Inj}\colon D^+(\mathcal{A}) \to \operatorname{Ho}^+ \operatorname{Inj} \mathcal{A}$ be an equivalence of categories as in Proposition 4.1.6. The composite functor

$$RF = Q \circ \operatorname{Ho}^+(F) \circ \operatorname{Inj}\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

is called a *total right derived functor* of F. As with Inj, this functor is unique up to natural isomorphism. Since all three functors in the composition are triangulated, the functor RF is triangulated.

Definition 4.2.3 (Right derived functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let RF be a total right derived functor and let H^n be the functor as in Definition 3.3.3. We define a functor $R^nF\colon \mathcal{A} \to \mathcal{A}$, called the *$n$-th right derived functor*, by $(R^nF)(X) = H^n(RF(X\langle 0\rangle))$ for all $X \in \mathcal{A}$ and $(R^nF)(f) = H^n(RF(f\langle 0\rangle))$ for all $f \in \operatorname{Mor} \mathcal{A}$.

Proposition 4.2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be short*

$\mathcal{A}$-exact sequence. There is an exact sequence of the following form:

$$0 \to R^0F(X) \xrightarrow{R^0F(f)} R^0F(Y) \xrightarrow{R^0F(g)} R^0F(Z) \to R^1F(X) \xrightarrow{R^1F(f)} \cdots$$

Proof: This is a consequence of Lemma 3.3.7. $\square$

Example 4.2.5 (hom$_{\mathcal{A}}$)**.** The hom-functor of an abelian category $\mathcal{A}$ is left exact in both slots. If $\mathcal{A}$ has enough injectives, then we can construct the total right derived functor $R\hom_{\mathcal{A}}(X,-)\colon D^+(\mathcal{A}) \to D^+(\mathbf{Ab})$ for all $X \in \mathcal{A}$. If $\mathcal{A}$ has enough projectives, then we can construct $R\hom_{\mathcal{A}}(-,X)\colon D^+(\mathcal{A}^{\mathrm{op}}) \to D^+(\mathbf{Ab})$ for all $X \in \mathcal{A}$ (recall that projectives in $\mathcal{A}$ are injectives in $\mathcal{A}^{\mathrm{op}}$). For all $X, Y \in \mathcal{A}$, we have that $R\hom_{\mathcal{A}}(X,-)(Y\langle 0\rangle)$ is isomorphic to $R\hom_{\mathcal{A}}(-,Y)(X\langle 0\rangle)$ in $D^+(\mathcal{A})$. If this is the case, then we can write

$$R^i\hom_{\mathcal{A}}(X,Y) := R^i\hom_{\mathcal{A}}(X,-)(Y) \cong R^i\hom_{\mathcal{A}}(-,Y)(X)$$

for all $i \in \mathbb{Z}$ and $X, Y \in \mathcal{A}$.

Example 4.2.6 ($\otimes_R$)**.** Let R be a commutative Noetherian ring. Let $\mathcal{A} = R\text{-}\mathbf{mod}^{\mathrm{fg}}$. The tensor product of modules yields a biadditive functor $(- \otimes_R -)\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. The functor is right exact in both slots. Thus, for each $X \in \mathcal{A}$ we have two total derived functors:

$$L(X \otimes_R -)\colon D^+(\mathcal{A}) \to D^+(\mathcal{A}) \quad \text{and} \quad L(- \otimes_R X)\colon D^+(\mathcal{A}) \to D^+(\mathcal{A}),$$

where $L-$ is the total left derived functor constructed in a dual way to $R-$.

Proposition 4.2.7. *Let $\mathcal{A}$ be an abelian category with enough injectives and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor, where $\mathcal{B}$ is another abelian category. Suppose that for every $X \in \mathcal{A}$ there exists $N \in \mathbb{N}$ such that $[(R^nF)X] = 0$ in $K_0(\mathcal{B})$ for all $n \geq N$. Then, we can define a group homomorphism $[RF]\colon K_0(\mathcal{A}) \to K_0(\mathcal{B})$ such that $[RF][X] = \sum_{n\in\mathbb{Z}}[(R^nF)X]$.*

Proof: This is a consequence of Proposition 4.2.4, Lemma 2.1.11, and Proposition 2.1.7. $\square$

Example 4.2.8. Let $\mathcal{A}$ be an abelian category with enough injectives and let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor, where $\mathcal{B}$ is another abelian category. Let $X \in \mathcal{A}$ and let $(I^\bullet, d_I)$ be an injective resolution of X. Since the sequence

$$0_{\mathcal{A}} \to F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \xrightarrow{F(d^2)} \cdots$$

is exact, it is clear that we have $(R^nF)X \cong FX$ if $n = 0$ and $(R^nF)X \cong 0_{\mathcal{A}}$ if $n \neq 0$. It follows that $[RF] = [F]$, where $[F]$ is the homomorphism defined in Proposition 2.1.8.

Example 4.2.9. If A is an integral domain, then the hom-sets of $A\text{-}\mathbf{mod}^{\mathrm{fg}}$ have naturally an A-linear structure. We get a left exact endofunctor $\hom_{A\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, X)$ on $A\text{-}\mathbf{mod}^{\mathrm{fg}}$ for all $X \in A\text{-}\mathbf{mod}^{\mathrm{fg}}$. Suppose A is a principal ideal domain. Then, the homomorphism

$$[R\hom_{A\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, A)]\colon G_0(A) \to G_0(A)$$

is the identity map. On the other hand, the functor $\hom_{\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, \mathbb{Z}/2\mathbb{Z})$ does not satisfy the hypothesis of Proposition 4.2.7. If it did, then we would have

$$[R\hom_{\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, \mathbb{Z}/2\mathbb{Z})][\mathbb{Z}/4\mathbb{Z}] = [\hom_{\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})] = [\mathbb{Z}/2\mathbb{Z}],$$

due to the fact that $\mathbb{Z}/4\mathbb{Z}$ is projective and the conflation $0_{\mathcal{A}} \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0_{\mathcal{A}}$, and this would mean that

$$[R\hom_{\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, \mathbb{Z}/2\mathbb{Z})][\mathbb{Z}/4\mathbb{Z}] = [R\hom_{\mathbb{Z}/4\mathbb{Z}\text{-}\mathbf{mod}^{\mathrm{fg}}}(-, \mathbb{Z}/2\mathbb{Z})](2[\mathbb{Z}/2\mathbb{Z}]) = 2[\mathbb{Z}/2\mathbb{Z}],$$

which contradicts the fact $\{[\mathbb{Z}/2\mathbb{Z}]\}$ freely generates $G_0(\mathbb{Z}/4\mathbb{Z})$ (Example 2.2.16).

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Index