A METHOD FOR MULTI-THRESHOLD SYNTHESIS.

by

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ABSTRACT.

A systematic method, based on the sets of permissible and non-permissible expressions, is developed for the synthesis of multi-threshold threshold elements, and the corresponding realizability conditions are established. The set of non-permissible expressions, for a given Boolean function, are weight relations that are not allowable for any realization of the function. The set of permissible expressions, on the other hand, display all the properties of the function considered as well as the weight combinations that are allowable for the different ways of realization. Since the permissible expressions represent several different ways of realization, not all of them are compatible. For this reason, compatible subsets of these expressions have to be determined. By assigning the smallest allowable weight values, according to the expressions in the largest compatible subset, the resulting realization is minimal or at least near minimal.

The method is theoretically applicable to any Boolean function, regardless of the number of variables, and also provides a better insight into the properties of Boolean functions in general.

CONTENTS.

I. INTRODUCTION.

Recently, much attention has been given to the realization of Boolean functions by threshold logic elements, and in particular by multi-threshold elements.

The single-threshold threshold element is a generalization of the conventional logic gates (AND, OR, NAND, NOR) and it can realize linearly separable Boolean functions. However, a single-threshold threshold element can not realize linearly non-separable functions, and for this purpose networks of these elements have to be used.

The multi-threshold threshold element is a generalization of the single-threshold threshold element. It is a very powerful logic element, since it can realize any arbitrary Boolean function. Also, by studying the properties of the multi-threshold threshold element, one gains a better insight into the properties of the single-threshold threshold element and the realization of Boolean functions in general.

Presently, there are two main approaches to the synthesis of Boolean functions:

1./ Single-threshold threshold element realization.

   Several good algorithms have been developed for the synthesis of two subclasses of Boolean functions, the symmetric functions [1-5] and the threshold (or linearly separable) functions [6-17]. For these two subclasses of Boolean functions minimal realizations can be obtained. However, this approach has not yet been succesful to produce a practical synthesis method for the general Boolean function, although several attempts have been made.

2./ <u>Multi-threshold threshold element realization</u>.
This approach overcomes the above difficulty and provides a way to the synthesis of arbitrary Boolean functions. Also, the link between the two approaches has been established [18], so that multi-threshold realizations can readily be transformed into the presently more common single-threshold realizations, should this be economically more feasible. The algorithms available at present are, however, limited to a rather small number of variables.

This thesis follows the second line of approach for the synthesis of general Boolean functions. Hence, let us discuss some of the more important methods, that are presently available for multi-threshold synthesis.

Haring [18] developed two methods for the multi-threshold synthesis. The first method is based on the run-measure minimization, by choosing a weight vector that associates a unique excitation with each vertex of the function and then considering all permutations and complementations of the variables. Although the method is straightforward, $n!2^n$ permutations and complementations are required to minimize the run-measure of the given function. In the resulting weight-threshold vector the weight magnitudes and the number of thresholds required are generally not minimal. His second method is based on realizability conditions derived from single-threshold realizability. Theoretically a weight-threshold vector, for which the weight magnitudes and the number of thresholds required are minimal, can be obtained by this method. However, in the decompositions of the given Boolean function into component subfunctions, several incompletely defined functions have to be specified when the number of thresholds is greater than two. Also, with presently available tabulations, this method can only be used for functions having four or less variables.

The tabular method developed by Haring and Ohori [19] is based on the Rademacher-Walsh coefficients of the function, and the realizability conditions derived by Haring [18] for his second method. The method is relatively simple, but permutations and/or complementations are necessary to obtain the realization, and the method can only be used for functions having four or less variables.

Mow and Fu [20] proposed an algorithm based on resolving contradicting pairs of vertices by using incremental weights, then from the resulting positive weight-threshold vector the optimal solution is obtained by permutations and/or complementations of the variables. It is believed that incremental weights are related to weight ordering, which does not hold for linearly non-separable functions. Aside from that, $n!2^n$ permutations and/or complementations are needed, and the method becomes impractical when the number of variables is greater than six.

The algorithm proposed by Necula [21] is basically the same as Haring's run-measure minimization method [18], but by taking symmetry conditions into consideration more economical realizations are obtainable, when the given function is symmetric in a subset of its variables. It is a direct method, and less than $n!2^n$ permutations and/or complementations are required to minimize the number of thresholds. However, the method is only applicable to functions having not more than five variables, and in addition supplementary operations are needed to determine the symmetry conditions of the given function.

In this thesis, the main emphasis is placed on obtaining a set of relations that will ensure the separation of the true and false vertices of the given function, although minimality will be kept constantly in sight. A set of relations, the non-permissible expressions, will be obtained, and these will have to be satisfied in order to achieve separation. Then from the set of all possible expressions a smaller set of relations, the permissible expressions, will be deduced. The sets of permissible and non-permissible expressions are unique for a given Boolean function. The permissible expressions display all the symmetry relations among the variables, as well as, the weight combinations that are allowable, without affecting realizability. To obtain optimal realization, it is necessary to satisfy simultaneously as many of the permissible expressions as possible, without creating any non-permissible expressions. Therefore, compatible subsets of permissible expressions will be determined. By choosing the smallest allowable weight vector, according to the largest compatible subset of permissible expressions, the range of excitations for the given function will be minimum, and the resulting weight-threshold vector will always be minimal or at least near minimal.

It is believed, that the proposed method for the multi-threshold synthesis both differs from, and presents an improvement over, the already existing synthesis methods.

II. THE MULTI-THRESHOLD THRESHOLD ELEMENT.

1./ Definitions and basic properties.

It is well known that any ordered set of binary variables, $\vec{X}=\{x_1,\ldots,x_n\}$, can have 2^n distinct values. These 2^n distinct values are called vertices (also called minterms), and can be represented by their decimal equivalent. A particular vertex will be denoted by X_v, where v can take on the values from 0 to 2^n-1. A Boolean function can therefore be defined as a partition of the 2^n vertices into two subsets, the true and false vertices (considering completely defined Boolean functions only), and will be denoted by the decimal equivalent of its true vertices.

A threshold function is a Boolean function, whose true vertices can be separated from its false vertices by a single hyperplane. It is easy to see then, that a k-threshold function can be defined as a Boolean function, whose true vertices can be separated from its false vertices by k parallel hyperplanes.

A multi-threshold threshold element (MTTE) is a generalization of the conventional single-threshold threshold element (TE). An MTTE is defined by its weight-threshold vector $(\vec{W};\vec{T})$, where $\vec{W}=\{w_1,\ldots,w_n\}$ is an ordered set of n weights, corresponding to the n input variables $\vec{X}=\{x_1,\ldots,x_n\}$, and the ordered set of k thresholds

$\vec{T}=\{T_1,\ldots,T_k\}$ are required to separate the true and false vertices of a given function. The weight w_i, associated with the variable x_i, can be either positive or negative, and can also be zero if the given Boolean function is independent of the variable x_i. For the sake of consistancy, T_1 is defined to be the largest threshold and T_k to be the smallest. The representation of an MTTE is shown in Figure 1.

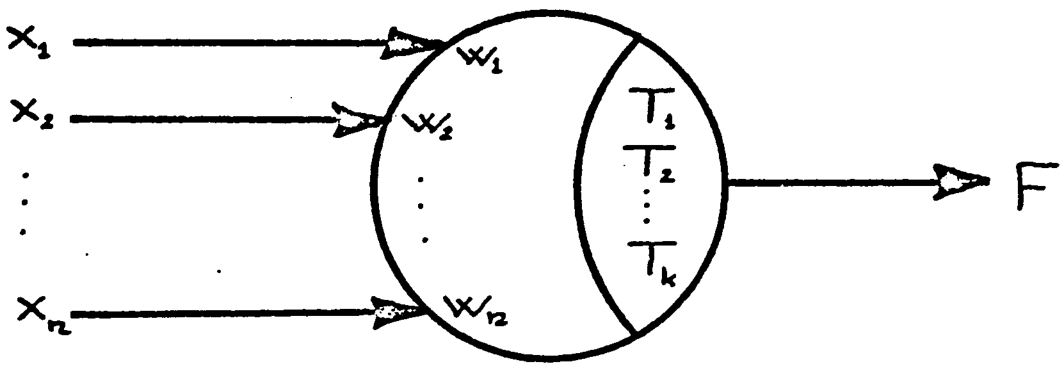


Figure 1. A k-threshold threshold element.

An MTTE is described by the following relations:

F=1, if $\quad E=\sum_{i=1}^{n} w_i x_i \geqslant T_1$, or if

$T_{2j} \geqslant E \geqslant T_{2j+1}$, and

F=0 otherwise, where $\quad j=1,2,3,\ldots$;
$T_j \in \{T_1,\ldots,T_k\}$;
$T_j > T_{j+1}$;

F is the logical output of the MTTE (0 or 1);

x_i is the i-th input variable (0 or 1);

w_i is the i-th weight (an integer);

T_j is the j-th threshold (a real number);

n is the total number of input variables;

k is the total number of thresholds; and

E is the excitation applied to the MTTE.

The excitation to an MTTE is defined as the linear weighted sum $E=\sum_{i=1}^{n} w_i x_i$. The weight vector $\vec{W}$ associates an excitation $E_v=\vec{W}.X_v$, where $v=(0,..,2^n-1)$, with each vertex X_v of the function.

In the following, the excitation associated with a true vertex of the given function will be called a <u>1-state</u>, and the excitation associated with a false vertex will be called a <u>0-state</u> of the MTTE. The state associated with a vertex X_v will be denoted by $E_v=\sum_{i=1}^{n} w_i x_i(X_v)$, where $x_i(X_v)$ represents the value of the variable x_i in vertex X_v. Some of these states may be the same, or they may all be distinct, depending on the values assigned to $\vec{W}$.

<u>Definition 1</u>: When two states are the same, for a given function, they will be called <u>equivalent states</u>.

The relevance of this definition will become clear in the next section of this chapter.

In the realization of a threshold function, the threshold divides the real line $(-\infty,\infty)$ into two distinct regions, one having only the 1-states and the other only the

0-states. Similarly, in the case of a k-threshold function the k thresholds partition the real line into (k+1) distinct regions, that alternately contain either 1-states or 0-states. The top region, i.e. above T_1, can be filled by either 1-states or 0-states, depending on the weight-threshold vector. However, the logical output of an MTTE is always 1 for this top region, and therefore when it is occupied by 0-states either of the following has to be done:

a./ consider the complement of the function, and modify the weight-threshold vector accordingly; or

b./ when writing the threshold vector, the largest threshold is chosen to be above the occupying states.

2./ Realizability conditions.

Realization consists of finding a weight-threshold vector $(\vec{W};\vec{T})$, that will achieve the separation of the true and false vertices of the given Boolean function. Therefore, $(\vec{W};\vec{T})$ has to satisfy certain conditions that will be defined below.

It is clear, from Definition 1 and the argument preceding it, that depending on the values assigned to $\vec{W}$ some of the states will be equivalent. This may be permitted in some cases, and in fact desirable, while it will be prohibited in others.

When one of the equivalent states is a 1-state and the other a 0-state, the equivalence will be called non-permissible. When the equivalent states are either both 1-states or both 0-states, the equivalence will be called permissible.

Therefore, it follows that, if E_m^1 is a 1-state and E_p^o is a 0-state, then to assure realizability a set of relations of the form

$$E_m^1 \not\equiv E_p^o, \text{ where } m \neq p \text{ and } m,p \in v=(0,\ldots,2^n-1),$$

that is, the set of non-permissible equivalences, have to be satisfied.

Some of these non-permissible equivalences, although they may appear to be different, contain the same information regarding realizability. For example, the two non-permissible equivalences $w_1 \not\equiv w_2$ and $w_1+w_3 \not\equiv w_2+w_3$ provide the same realizability condition. To avoid this, a non-permissible equivalence will be represented by an expression, or inequality, in which no term can appear on both sides. Such an expression may represent two or more non-permissible equivalences.

Definition 2: A non-redundant expression, representing one or more non-permissible equivalences, will be called a non-permissible expression.

Every Boolean function has a distinct set of these non-permissible expressions, which can be obtained by comparing each true vertex of the given function to all the false vertices, as follows

$$E_m=\sum_{i=1}^{n} w_i x_i(X_m^1) \neq \sum_{i=1}^{n} w_i x_i(X_p^0)=E_p, \text{ where } m\neq p. \qquad (1)$$

In (1) X_m^1 is a true vertex and X_p^o is a false vertex. The terms appearing on both sides of an expression are cancelled and the resulting redundant expressions are omitted.

Since an expression generally represents more than one equivalence, it is possible that one of those equivalences is permissible and the other is non-permissible. Therefore, an expression representing a permissible equivalence, that is, an equality of two equivalent 1-states or two equivalent 0-states, is not necessarily permissible. However, there do exist expressions in weights that are permissible.

<u>Definition 3</u>: An expression that is not non-permissible, will be called a <u>permissible expression</u>.

From the above definitions the next theorem follows directly.

<u>Theorem 1</u>: A Boolean function F is realizable by a weight-threshold vector $(\vec{W};\vec{T})$, if and only if there are no non-permissible expressions resulting from $\vec{W}$.

The total number of possible non-redundant expressions is equal to the sum of permissible and non-permissible expressions, and this total is constant for a given

number of n variables. To be more specific, let
N= number of non-permissible expressions;
P= number of permissible expressions; and
M(n)= number of possible expressions, for n variables;
then M(n)=N+P

$$M(n)=\left[\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}\right]+\binom{n}{1}\left[\tfrac{1}{2}\binom{n-1}{1}+\binom{n-1}{2}+\dots+\binom{n-1}{n-1}\right]$$

$$+\binom{n}{2}\left[\tfrac{1}{2}\binom{n-2}{2}+\binom{n-2}{3}+\dots+\binom{n-2}{n-2}\right]+\dots\dots+$$

$$+\binom{n}{r}\left[\tfrac{1}{2}\binom{n-r}{r}+\binom{n-r}{r+1}\right], \qquad (2)$$

where $r=\left[\frac{n}{2}\right]$, the largest integer $\leqslant \frac{n}{2}$.

It follows easily then, that those possible expressions that do not belong to the set of non-permissible expressions make up the set of permissible expressions, and they are of the form

$$E_m^1 = E_p^1 \text{ or } E_m^o = E_p^o, \text{ where } m\neq p \text{ and } m,p\in v=(0,..,2^n-1).$$

Hence, the set of permissible expressions can be obtained by listing all the possible expressions that do not appear in the non-permissible set. For example, for the function F(3)=(3,5,7) the set of non-permissible expressions is shown below.

$w_1 \neq 0$ $\quad$ $w_1 \neq w_3$
$w_2 \neq 0$ $\quad$ $w_2 \neq w_3$
$w_3 \neq 0$ $\quad$ $w_1 \neq w_2+w_3$
$w_1+w_2 \neq 0$ $\quad$ $w_2 \neq w_1+w_3$
$w_1+w_3 \neq 0$
$w_2+w_3 \neq 0$
$w_1+w_2+w_3 \neq 0$

Therefore, the set of permissible expressions is $w_1=w_2$ and $w_3=w_1+w_2$. The permissible expressions can be used to determine the desired weight-threshold vector, since they indicate the allowable weight combinations. In the simple example above, we have $w_1=w_2=1$ and $w_3=w_1+w_2=2$ and the resulting weight-threshold vector is $(\vec{W};\vec{T})=(1,1,2;2.5)$.

The set of permissible expressions is also unique for every Boolean function. However, it should be noted, that a given Boolean function F and its complement $\bar{F}$ both have the same set of permissible expressions.

The following lemmas, although well known, should be included here.

<u>Lemma 1</u>: If the function $F(x_1,..,x_i,..,x_n)$ is realized by the weight-threshold vector $(w_1,..,w_i,..,w_n;\vec{T})$, then the function $F(x_1,..,\bar{x}_i,..,x_n)$ is realized by the weight-threshold vector $(w_1,..,-w_i,..,w_n;T_1-w_i,..,T_k-w_i)$.

It is also clear from lemma 1, that $F(\bar{x}_1,..,\bar{x}_i,..,\bar{x}_n)$ is realized by $(-w_1,..,-w_n;T_1-\sum_{i=1}^{n} w_i,.., T_k-\sum_{i=1}^{n} w_i)$.

Lemma 2: If the function $F(x_1,..,x_i,..,x_j,..,x_n)$ is realized by the weight-threshold vector $(w_1,..,w_i,..,w_j,..,w_n;\vec{T})$, then the function $F(x_1,..,x_j,..,x_i,..,x_n)$ is realized by $(w_1,..,w_j,..,w_i,..,w_n;\vec{T})$.

3./ Independence and symmetry conditions.

The properties relevant to the realization of a given Boolean function are:

a./ the function's independence of a certain variable;

b./ the function's symmetry in a subset of its variables.

These relevant properties are clearly reflected in the set of permissible expressions, as it will be shown below.

Theorem 2: A Boolean function F is independent of the variable x_i, if and only if $w_i=0$ is a permissible expression.

Proof:

a./ Let F be independent of the variable x_i. Then, if $x_1^*..x_i...x_n^*$ is a true vertex of F, so must be $x_1^*..\bar{x}_i..x_n^*$, where x_j^* represents either the variable x_j or its complement $\bar{x}_j$. Also, if $x_1^*..x_i..x_n^*$ is a false vertex, so must be $x_1^*..\bar{x}_i..x_n^*$. Hence, $w_i=0$ is a permissible expression.

b./ Let $w_i=0$ be a permissible expression. Then $x_1^*..x_i..x_n^*$ and $x_1^*..\bar{x}_i..x_n^*$ are either both true vertices or both false vertices, since the corresponding states must be either both 1-states or both 0-states. Therefore, F is independent of x_i. Q.E.D.

When a given Boolean function has a subset of variables $\{x_i^*, x_j^*, \ldots\}$, where $i, j \in (1, \ldots, n)$, such that any permutation of these variables leaves the function unaltered, the function is said to be symmetric in those variables, or alternately it can be said that the function has a symmetric subset $S(x_i^*, x_j^*, \ldots)$ of those variables. When the symmetric subset contains all the variables of the given function, the function is called completely symmetric. Similarly, if the function has no symmetric subset, then the function is called non-symmetric. It is of course possible that a given function has two or more symmetric subsets.

It is important to note that some Boolean functions, as Necula [21] pointed out, may have two or more non-equivalent variants of the same symmetric subset. In other words, these symmetric subsets contain the same variables and differ only by the complementation of some of those variables. For example, F may have both the symmetric subsets of $S(x_1, x_2)$ and $S(\bar{x}_1, x_2)$. The symmetric variants $S(x_1, x_2)$ and $S(\bar{x}_1, \bar{x}_2)$, as well as $S(\bar{x}_1, x_2)$ and $S(x_1, \bar{x}_2)$, are considered to be equivalent. Necula called these functions <u>multiform symmetric</u>.

It will be seen, in the next chapter, that when a given Boolean function has these relevant properties, the assignment of suitable weight values will be considerably simpler in most of the cases.

Now, let us consider three theorems, related to symmetry, that will be important for the development of the synthesis procedure.

Theorem 3: A Boolean function F is symmetric in variables x_i and x_j, if and only if $w_i=w_j$ is a permissible expression.

Proof:

a./ Let $w_i=w_j$ be a permissible expression. Then $(w_1,..,w_i,..,w_j,..,w_n;\vec{T})$ and $(w_1,..,w_j,..,w_i,..,w_n;\vec{T})$ are equivalent realizations of F, and it follows from Lemma 2 that $F(x_1,..,x_i,..,x_j,..,x_n)=F(x_1,..,x_j,..,x_i,..,x_n)$. Therefore, F is symmetric in variables x_i and x_j.

b./ Let F be symmetric in variables x_i and x_j, and let $F(x_1,..,x_i,..,x_j,..,x_n)$ be realizable by the weight-threshold vector $(w_1,..,w_i,..,w_j,..,w_n;\vec{T})$. Then, since $F(x_1,..,x_i,..,x_j,..,x_n)=F(x_1,..,x_j,..,x_i,..,x_n)$ from Lemma 2 $(w_1,..,w_j,..,w_i,..,w_n;\vec{T})$ is an equivalent realization of F. Hence, $w_i=w_j$ can not be a non-permissible expression. Therefore, $w_i=w_j$ is a permissible expression.

Q.E.D.

It follows obviously from Lemma 1, that if w_i is the weight associated with variable x_i and $\bar{w}_i$ with $\bar{x}_i$, then $w_i=-\bar{w}_i$.

Theorem 4: A Boolean function F is symmetric in variables $\bar{x}_i$ and x_j, if and only if $w_i+w_j=0$ is a permissible expression.

Proof:

a./ Let $w_i+w_j=0$ be a permissible expression, i.e. $w_j=\bar{w}_i$. Then $(w_1,..,\bar{w}_i,..,w_j,..,w_n;\vec{T})$ and $(w_1,..,w_j,..,\bar{w}_i,..,w_n;\vec{T})$ are equivalent realizations of F, and it follows from Lemma 2 that $F(x_1,..,\bar{x}_i,..,x_j,..,x_n)=F(x_1,..,x_j,..,\bar{x}_i,..,x_n)$. Therefore, F is symmetric in variables $\bar{x}_i$ and x_j.

b./ Let F be symmetric in variables $\bar{x}_i$ and x_j, and let $F(x_1,..,\bar{x}_i,..,x_j,..,x_n)$ be realizable by the weight-threshold vector $(w_1,..,\bar{w}_i,..,w_j,..,w_n;\vec{T})$. Then, since $F(x_1,..,\bar{x}_i,..,x_j,..,x_n)=F(x_1,..,x_j,..,\bar{x}_i,..,x_n)$, from Lemma 2 $(w_1,..,w_j,..,\bar{w}_i,..,w_n;\vec{T})$ is an equivalent realization of F. Hence, $w_j=\bar{w}_i$ can not be a non-permissible expression. Therefore, $w_j=\bar{w}_i$, i.e. $w_i+w_j=0$, is a permissible expression. Q.E.D.

Theorem 5: A Boolean function F is multiform symmetric in variables x_i and x_j, that is, F has both symmetric subsets $S(x_i,x_j)$ and $S(\bar{x}_i,x_j)$, if and only if both $w_i=w_j$ and $w_i+w_j=0$ are permissible expressions.

Proof: Follows from Theorems 3 and 4.

It is clear from the above theorems, that the permissible expressions automatically provide information on whether the function is symmetric in a subset (or subsets) of its variables, and if so which of the variables are to be complemented. In other words, information is gained about which of the weights can be assigned equal magnitudes. It also follows, that if the given Boolean function is non-symmetric, the condition

$$|w_i| \neq |w_j|, \text{ where } i \neq j \text{ and } i,j \in (1,\ldots,n),$$

must be satisfied, otherwise at least one non-permissible weight relation will result and realization can not be attained.

4./ Minimality.

There are infinitely many weight-threshold vectors that will partition the 1 and 0 states of any given Boolean function. It is also well known, that any Boolean function can be realized by a positive weight-threshold vector, if k is chosen to be sufficiently large. However, from the point of view of reliability and economy only the minimal realizations are of interest, and for some functions this is not possible without the use of some non-positive weights.

More specifically, there are three factors affecting the minimality of an MTTE realization:

a./ k, the number of thresholds required;

b./ $\omega = \sum_{i=1}^{n} |w_i|$; and

c./ $\tau = \sum_{j=1}^{k} |T_j|$.

The relevant properties and the permissible expressions of a given Boolean function influence the size of these factors. Therefore, let us examine them in their relation to minimality.

Clearly, when a given function is independent of a variable x_i, we can set $w_i=0$. This reduces ω, and obviously any other weight assignment will make minimal realization impossible.

By assigning the same weight magnitude to the variables contained in a symmetric subset of the given Boolean function, ω is reduced. In fact, a minimal weight-threshold vector can not be obtained without considering the symmetry conditions. In particular, if the Boolean function considered is completely symmetric, then the assignment of $|w_i|=1$, where $i=(1,\ldots,n)$, will result in a minimal weight-threshold vector.

The permissible expressions, in addition to displaying the symmetry conditions, also indicate what state equivalences will result if the weight values are assigned accordingly. For example, let $w_1=w_2+w_3$ be a permissible expression, then we have $E(x_1\bar{x}_2\bar{x}_3\bar{x}_4)=E(\bar{x}_1x_2x_3\bar{x}_4)$ and also $E(x_1\bar{x}_2\bar{x}_3x_4)=E(\bar{x}_1x_2x_3x_4)$. Therefore, to obtain a minimal realization, weight values should be assigned to satisfy simultaneously as many permissible expressions as possible, without creating any non-permissible weight relations.

Since the factors affecting the minimality of an MTTE realization are not independent of each other, the criterion for minimality can be defined in several different ways.

Mow and Fu [20] defined the optimal solution as a weight-threshold vector possessing the following ordered set of properties:

a./ minimum number of thresholds, k_m;

b./ for k_m solutions, $\min(\omega)$; and

c./ for k_m and $\min(\omega)$, $\min\left[\max(|T_1|,|T_k|)\right]$.

Necula [21], on the other hand, obtained the following criteria for the minimality of an MTTE realization:

a./ k minimum and ω minimum for the obtained k;

b./ ω minimum and k minimum for the obtained ω;

c./ $(k+\omega)$ minimum.

It should be noted, that the optimal solution, as defined by Mow and Fu, is just one of the several ways it can be defined, and also that $\min(\tau) \leq \min\left[\max(|T_1|,|T_k|)\right]$. The criteria obtained by Necula are also incomplete, since τ is not considered for the minimality at all. Therefore, for the sake of completeness, we list here all the possible ways of defining the minimality criterion, although some of them may be impractical.

(1) minimize k and for min(k) minimize ω, with these conditions satisfied choose the $(\vec{W};\vec{T})$ with min(τ);

(2) minimize k and for min(k) minimize τ, with these conditions satisfied choose $(\vec{W};\vec{T})$ that has min(ω);

(3) minimize k and for min(k) choose $(\vec{W};\vec{T})$ with min($\omega+\tau$);

(4) minimize $(k+\omega)$, then choose $(\vec{W};\vec{T})$ that has min(τ);

(5) minimize $(k+\tau)$, then choose $(\vec{W};\vec{T})$ with min(ω);

(6) minimize ω and for min(ω) find min(k), with these conditions satisfied choose $(\vec{W};\vec{T})$ with min(τ);

(7) minimize ω and for min(ω) minimize τ, with these conditions satisfied find $(\vec{W};\vec{T})$ that has min(k);

(8) minimize ω, then select $(\vec{W};\vec{T})$ that has min($k+\tau$);

(9) minimize $(\omega+\tau)$, and then find $(\vec{W};\vec{T})$ with min(k);

(10) minimize τ and for min(τ) minimize ω, with these conditions satisfied select $(\vec{W};\vec{T})$ that has min(k);

(11) minimize τ and for min(τ) minimize k, with these conditions satisfied select $(\vec{W};\vec{T})$ with min(ω);

(12) minimize τ and then select $(\vec{W};\vec{T})$ with min($k+\omega$);

(13) minimize $(k+\omega+\tau)$.

The minimization of k will generally result in the reduction of cost, while the minimization of both k, ω, and τ will increase the reliability of the MTTE. However, it was found that rather than minimizing k first and then the remaining factors, it is more advantageous to assign the smallest allowable weight values, according to the permissible expressions, and then select a weight-threshold vector for realization using criterion (9) above. By minimizing ω, the range of excitations will be as narrow as possible, without affecting realizability, and the resulting weight-threshold vector will always be minimal or at least near minimal.

III. MULTI-THRESHOLD SYNTHESIS.

The basic properties of Boolean functions and their realizability conditions, with respect to the proposed method, have been established. These provide the foundation for the multi-threshold synthesis procedure, which will be described and discussed here in detail. Illustrative examples will be used to bring into focus the important features of the method.

The first step of the synthesis procedure is to list the true and false vertices, of the given function, into two separate columns.

Then, we compare each true vertex to all the false vertices, in each case writing an expression

$$E_m=\sum_{i=1}^{n} w_i x_i(X_m^1) \neq \sum_{i=1}^{n} w_i x_i(X_p^o)=E_p \text{ , where } m \neq p,$$

and cancelling the terms that appear on both sides. If the resulting expression has been obtained before, then the redundant expression is omitted. By exhausting all the true vertices, in the manner described above, the set of N non-permissible expressions is obtained.

Next, the set of P=M(n)-N permissible expressions is deduced, by listing all the non-redundant possible expressions that do not occur in the non-permissible set.

These two sets of expressions form the very essence of the synthesis procedure, as can be illustrated by the following simple example.

Example 1: Consider the function F(4)=(1,4,5,7,9,12,13,15). The true and false vertices of the function are shown in the table below.

True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 0 1	0 0 0 0
0 1 0 0	0 0 1 0
0 1 0 1	0 0 1 1
0 1 1 1	0 1 1 0
1 0 0 1	1 0 0 0
1 1 0 0	1 0 1 0
1 1 0 1	1 0 1 1
1 1 1 1	1 1 1 0

Comparing the first true vertex to the first false vertex, we get $w_4 \nless 0$; comparing it to the second false vertex, we get $w_3 \nless w_4$, and so on. Proceeding the same way, and rearranging the resulting expressions, the set of non-permissible expressions is

$w_2 \neq 0$	$w_1 \neq w_2$	$w_3 \neq w_2+w_4$
$w_3 \neq 0$	$w_1 \neq w_3$	$w_4 \neq w_1+w_3$
$w_4 \neq 0$	$w_1 \neq w_4$	$w_4 \neq w_2+w_3$
$w_1+w_2 \neq 0$	$w_2 \neq w_3$	$w_1 \neq w_2+w_3+w_4$
$w_1+w_3 \neq 0$	$w_3 \neq w_4$	$w_2 \neq w_1+w_3+w_4$
$w_1+w_4 \neq 0$	$w_1 \neq w_2+w_4$	$w_3 \neq w_1+w_2+w_4$
$w_2+w_4 \neq 0$	$w_2 \neq w_1+w_3$	$w_4 \neq w_1+w_2+w_3$
$w_1+w_2+w_4 \neq 0$	$w_2 \neq w_3+w_4$	$w_1+w_2 \neq w_3+w_4$
$w_2+w_3+w_4 \neq 0$	$w_3 \neq w_1+w_2$	$w_1+w_3 \neq w_2+w_4$
$w_1+w_2+w_3+w_4 \neq 0$	$w_3 \neq w_1+w_4$	$w_1+w_4 \neq w_2+w_3$

Here N=30 and from equation (2) M(4)=40, therefore we have P=40-30=10 permissible expressions. These are the ten possible expressions that are not in the non-permissible set above, and they are as follows:

$w_1=0$	$w_2=w_4$
$w_2+w_3=0$	$w_1=w_2+w_3$
$w_3+w_4=0$	$w_1=w_3+w_4$
$w_1+w_2+w_3=0$	$w_2=w_1+w_4$
$w_1+w_3+w_4=0$	$w_4=w_1+w_2$

It can readily be seen from the permissible set that F is independent of the variable x_1, since $w_1=0$ is a permissible expression. Also, since $w_2+w_3=0$, $w_3+w_4=0$ and $w_2=w_4$ are permissible expressions, we have $w_2=-w_3=w_4$. Thus, F has the symmetric subset of variables $S(x_2,\bar{x}_3,x_4)$, by Theorems 3 and 4. Therefore, the realization, easily deduced in this case, is $(\vec{W};\vec{T})=(0,1,-1,1;0.5)$. However, in the majority of cases, the realization can not be obtained so simply.

In the set of permissible expressions some of the expressions may be dependent, that is, they can be obtained by combining two or more of the other expressions. Since these dependent expressions do not contribute any additional information, they can be removed from the permissible set. Thus, the <u>set of independent permissible expressions</u> is formed.

The next step in the synthesis procedure is to determine the relevant properties of the given Boolean function, from the set of independent permissible expressions.

We further classify the permissible expressions into three different types, according to the relevant properties, namely

a./ the permissible independence expressions;

b./ the permissible symmetry expressions; and

c./ the permissible non-symmetry expressions.

<u>Definition 4</u>: Any permissible expression of the form $w_i=0$, where $i\in(1,..,n)$, will be called a <u>permissible independence expression</u>.

<u>Definition 5</u>: Any permissible expression of the form $w_i=w_j$ or $w_i+w_j=0$, where $i\neq j$ and $i,j\in(1,..,n)$, will be called a <u>permissible symmetry expression</u>.

<u>Definition 6</u>: Any permissible expression of the form other than those indicated in Definitions 4 and 5, will be called a <u>permissible non-symmetry expression</u>.

When the given function has a permissible independence expression $w_i=0$, the function is independent of the variable x_i, by Theorem 2. Hence, we can set $w_i=0$, and eliminate w_i from both the permissible and non-permissible sets of expressions. These sets will thereby be reduced, since some of the expressions will become redundant, as a result of eliminating w_i. When the given function has no such expressions, we directly proceed to the following step of the procedure, which takes the symmetry conditions into consideration.

If the given function has any permissible symmetry expressions, then the function is symmetric in at least one subset of its variables, according to Theorems 3,4 and 5.

It should be noted, that when the given function is not multiform symmetric, none of the permissible symmetry expressions are contradictory to one another, that is, all of them can be satisfied simultaneously. However, if the function is multiform symmetric, then some of the permissible symmetry expressions will be incompatible, that is, they can not all be satisfied simultaneously. Therefore, in order to have a unified synthesis procedure for any Boolean function, and at the same time take all the possible ways of realization into consideration, it is necessary to obtain all the distinct compatible symmetry subsets.

Definition 7: A subset of m permissible symmetry expressions, $m \geqslant 2$, will be called a compatible symmetry subset, if no combination of those expressions will result in a non-permissible expression.

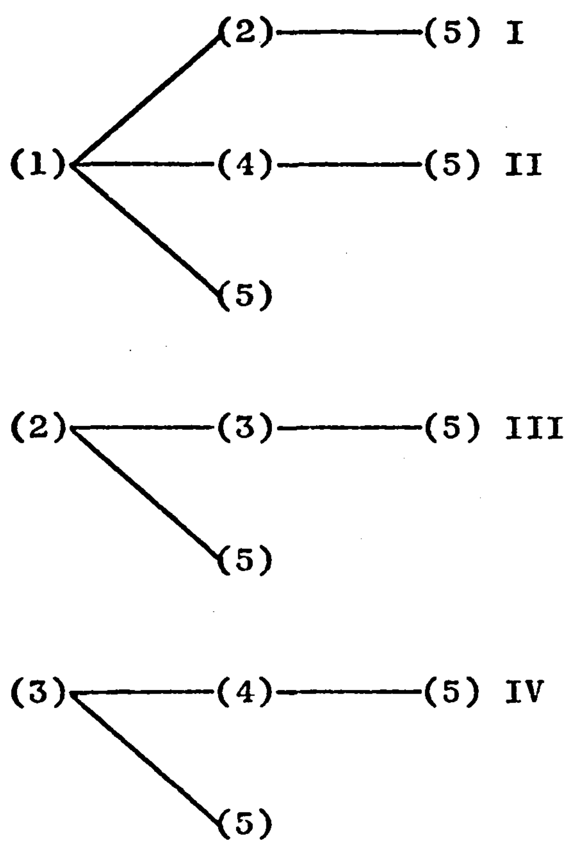
To be more specific, two permissible symmetry expressions are compatible either if they have no common terms, or if the expression resulting from the substitution of one expression into the other, through their common terms, is not non-permissible.

If the given function is not multiform symmetric, then there will be only one distinct compatible symmetry subset. However, if the function is multiform symmetric, then there will be several distinct compatible symmetry subsets, one for each of the permissible combinations of the non-equivalent symmetry variants, that are created by multiform symmetry.

The compatible symmetry subset (or subsets) can be obtained, using tree diagrams, by the following procedure. All the independent symmetry expressions are listed and numbered. The first expression is chosen as the origin of a tree diagram. Then each successive expression is compared to the first expression, and the compatible ones are joined to the origin by a straight line, on the diagram. These expressions will make up the first level of the tree. When all the permissible symmetry expressions, with the exception of the expression at the origin, are included in the first level, the function is not multiform symmetric and the process can be stopped. Then all the permissible symmetry expressions are included in the only distinct compatible symmetry subset. Otherwise, the function is multiform symmetric, and all the branches of the tree have to be completed. For further branching, at the j-th level, where $j \geqslant 2$, only the expressions at the j-1-th level need to be considered, the others being already incompatible with previous expressions. To each first-level expression on the diagram we connect, by a straight line, the expressions that are compatible to both expressions in the branch being considered, and so on, until all the branches have been completed. Next, the second, third, etc expressions are chosen as the origin of a tree, and the same procedure is repeated in each case. It should be noted, that each expression, in a given branch of the compatibility tree, is followed only by higher numbered expressions. Now, to determine the compatible symmetry subsets only the branches containing the highest number of expressions are considered, and the expressions in each such branch form a compatible symmetry subset. Each of the compatible symmetry subsets will have to be considered separately, to obtain all the possible ways of realization. For example, consider the following set of independent permissible symmetry expressions, and assume that $w_i \neq 0$ for $i=(1,\ldots,n)$.

(1)	$w_1+w_2=0$
(2)	$w_3+w_4=0$
(3)	$w_1=w_2$
(4)	$w_3=w_4$
(5)	$w_5=w_6$

The resulting compatibility tree diagrams and the corresponding compatible symmetry subsets are shown below.



$$(4) \text{———} (5) \quad \text{and} \quad (5)$$

$$\begin{pmatrix} w_1+w_2=0 \\ w_3+w_4=0 \\ w_5=w_6 \end{pmatrix} \text{I}; \quad \begin{pmatrix} w_1+w_2=0 \\ w_3=w_4 \\ w_5=w_6 \end{pmatrix} \text{II}; \quad \begin{pmatrix} w_3+w_4=0 \\ w_1=w_2 \\ w_5=w_6 \end{pmatrix} \text{III}; \quad \begin{pmatrix} w_1=w_2 \\ w_3=w_4 \\ w_5=w_6 \end{pmatrix} \text{IV}.$$

Next, one of the compatible symmetry subsets is selected, and according to the permissible symmetry expressions contained therein, all the weights associated with the same symmetry subset of variables are expressed in terms of one of those weights, which will be called a <u>representative weight</u>.

These representative weights are substituted, for each of the weights they represent, into the set of non-permissible expressions and the redundant expressions are omitted. The resulting expressions will be called a <u>reduced set of non-permissible expressions</u>.

Similarly, the representative weights are then substituted into the set of permissible non-symmetry expressions. The redundant and non-permissible expressions, resulting from the substitution, are omitted. The set of expressions so obtained will be called a <u>reduced set of permissible expressions</u>.

From the reduced set of permissible expressions, we form distinct compatible realizability subsets, following the same procedure as that described for the formation of compatible symmetry subsets.

Definition 8: A subset of m permissible non-symmetry expressions, $m \geqslant 2$, will be called a compatible realizability subset, if no combination of those expressions will result in a non-permissible expression.

The same procedure is then repeated for each of the compatible symmetry subsets. However, only the compatible realizability subset (or subsets) containing the largest number of expressions need to be considered for minimal realization.

When there are no permissible symmetry expressions, for the given Boolean function, the function is non-symmetric. Hence, we directly proceed to form compatible realizability subsets, from the set of permissible non-symmetry expressions.

It is clear, therefore, that by following the procedure described, for any Boolean function, we can obtain a number of compatible realizability subsets, each of which is distinct and contains the largest number of independent permissible expressions.

It is a well known fact, that r independent variables can have at most (r-1) independent relations. Therefore, if r is the number of independent variables, and m is the number of independent expressions in a compatible realizability subset, then $m \leqslant (r-1)$.

When $m=(r-1)$, for a given compatible realizability subset, any weight assignment made, satisfying the m permissible expressions therein, will be a realization of the given function. This was assured by eliminating all the non-permissible expressions, that resulted from the restrictions on the weight values, by the relevant properties of the function, prior to obtaining the compatible realizability subsets. In this case, all the weights associated with the independent variables can be expressed in terms of one of those weights, by substituting the m independent expressions into one another. Then assigning the smallest weight magnitude to that weight, all other weights can be evaluated as well. Hence, we obtain a weight-threshold vector for each of the selected compatible realizability subsets. From these weight-threshold vectors one of them is chosen for realization, according to the minimality criterion (9).

If $m<(r-1)$, as sometimes is the case for multiform symmetric functions, then at least one restricting relation among the weights is missing. In such case, all combination of the smallest allowable weight values, that satisfy the m independent expressions, has to be obtained and checked against the reduced set of non-permissible expressions. From the weight-threshold vectors so obtained, one of them is chosen for realization, according to the minimality criterion (9). The minimality of the realization in these cases can not be ascertained, because of the heuristic assignment of weight values.

Now, depending on the symmetry conditions of the given Boolean function, the following four possibilities can arise:

a./ the function is completely symmetric;

b./ the function is not completely symmetric, but each variable is included in some symmetry subset of the variables;

c./ the function has one or more symmetric subset of the variables, but some of the variables are not included in any of them;

d./ the function has no symmetric subset of variables, that is, the function is non-symmetric.

Let us consider some simple examples, for each of these four possibilities.

Example 2: Let F(4)=(0,3,4,5,6,9,10,11,12,15).

True vertices	False vertices	True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 0 0	0 0 0 1	1 0 0 1	1 1 1 0
0 0 1 1	0 0 1 0	1 0 1 0	
0 1 0 0	0 1 1 1	1 0 1 1	
0 1 0 1	1 0 0 0	1 1 0 0	
0 1 1 0	1 1 0 1	1 1 1 1	

The non-permissible expressions are

$$\begin{array}{ll}
w_1 \neq 0 & w_1 \neq w_2 \\
w_2 \neq 0 & w_2 \neq w_3 \\
w_3 \neq 0 & w_2 \neq w_4 \\
w_4 \neq 0 & w_1 \neq w_2 + w_3 \\
w_1 + w_3 \neq 0 & w_1 \neq w_2 + w_4 \\
w_1 + w_4 \neq 0 & w_1 \neq w_3 + w_4 \\
w_3 + w_4 \neq 0 & w_3 \neq w_1 + w_2 \\
w_1 + w_2 + w_3 \neq 0 & w_3 \neq w_1 + w_4 \\
w_1 + w_2 + w_4 \neq 0 & w_3 \neq w_2 + w_4 \\
w_2 + w_3 + w_4 \neq 0 & w_4 \neq w_1 + w_2 \\
 & w_4 \neq w_1 + w_3 \\
 & w_4 \neq w_2 + w_3
\end{array}$$

Here, N=22 and the P=40-22=18 permissible expressions are

$$\begin{array}{ll}
w_1 + w_2 = 0 & w_2 = w_1 + w_4 \\
w_2 + w_3 = 0 & w_2 = w_3 + w_4 \\
w_2 + w_4 = 0 & w_1 = w_2 + w_3 + w_4 \\
w_1 + w_2 + w_4 = 0 & w_2 = w_1 + w_3 + w_4 \\
w_1 + w_2 + w_3 + w_4 = 0 & w_3 = w_1 + w_2 + w_4 \\
w_1 = w_3 & w_4 = w_1 + w_2 + w_3 \\
w_1 = w_4 & w_1 + w_2 = w_3 + w_4 \\
w_3 = w_4 & w_1 + w_3 = w_2 + w_4 \\
w_2 = w_1 + w_3 & w_1 + w_4 = w_2 + w_3
\end{array}$$

Eliminating the dependent expressions, we obtain the following set of independent permissible expressions.

$$w_1+w_2=0 \qquad w_2=w_1+w_3$$
$$w_1+w_2+w_4=0 \qquad w_2=w_1+w_3+w_4$$
$$w_1+w_2+w_3+w_4=0 \qquad w_1+w_2=w_3+w_4$$
$$w_1=w_3 \qquad w_1+w_3=w_2+w_4$$
$$w_1=w_4 \qquad w_1+w_4=w_2+w_3$$

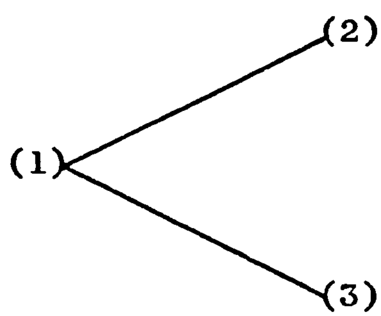
It can be seen, from the set of independent permissible expressions, that F is not independent of any variable, but has the following permissible symmetry expressions.

(1) $w_1+w_2=0$

(2) $w_1=w_3$

(3) $w_1=w_4$

From these permissible symmetry expressions, we obtain the following compatibility tree and corresponding compatible symmetry subset.



$$\begin{pmatrix} w_1+w_2=0 \\ w_1=w_3 \\ w_1=w_4 \end{pmatrix}$$

The above compatibility tree has not been completed, since both expressions (2) and (3) are included in the first level, and therefore the function is not multiform symmetric. It is obvious, from the compatible symmetry subset, that F is completely symmetric and $w_1=-w_2=w_3=w_4$. When the representative weight w_4 is substituted into the set of non-permissible expressions, the only expression remaining in the reduced set of non-permissible expressions is $w_4 \neq 0$. Also, when the representative weight w_4 is substituted into the set of permissible non-symmetry expressions, they all become redundant or non-permissible, since all the possible independent expressions, for this realization, are already included in the compatible symmetry subset. Assigning the smallest weight values, we obtain the weight-threshold vector (1,-1,1,1;1.5,0.5), which is a minimal realization.

Example 3: Let F(4)=(3,4,7,8,11,12,13,14,15).

True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 1 1	0 0 0 0
0 1 0 0	0 0 0 1
0 1 1 1	0 0 1 0
1 0 0 0	0 1 0 1
1 0 1 1	0 1 1 0
1 1 0 0	1 0 0 1
1 1 0 1	1 0 1 0
1 1 1 0	
1 1 1 1	

The non-permissible expressions resulting are

$w_1 \neq 0$

$w_2 \neq 0$

$w_3 \neq 0$

$w_4 \neq 0$

$w_1 + w_2 \neq 0$

$w_1 + w_3 \neq 0$

$w_1 + w_4 \neq 0$

$w_2 + w_3 \neq 0$

$w_2 + w_4 \neq 0$

$w_3 + w_4 \neq 0$

$w_1 + w_2 + w_3 \neq 0$

$w_1 + w_2 + w_4 \neq 0$

$w_1 + w_3 + w_4 \neq 0$

$w_2 + w_3 + w_4 \neq 0$

$w_1 + w_2 + w_3 + w_4 \neq 0$

$w_1 \neq w_3$

$w_1 \neq w_4$

$w_2 \neq w_3$

$w_2 \neq w_4$

$w_1 \neq w_2 + w_3$

$w_1 \neq w_2 + w_4$

$w_2 \neq w_1 + w_3$

$w_2 \neq w_1 + w_4$

$w_3 \neq w_1 + w_2$

$w_3 \neq w_1 + w_4$

$w_3 \neq w_2 + w_4$

$w_4 \neq w_1 + w_2$

$w_4 \neq w_1 + w_3$

$w_4 \neq w_2 + w_3$

$w_3 \neq w_1 + w_2 + w_4$

$w_4 \neq w_1 + w_2 + w_3$

The set of independent permissible expressions is

$w_1 = w_2$

$w_3 = w_4$

$w_1 = w_3 + w_4$

$w_1 = w_2 + w_3 + w_4$

$w_2 = w_1 + w_3 + w_4$

$w_1 + w_2 = w_3 + w_4$

It is clear, from the set of independent permissible expressions, that F is not independent of any variable and has a single compatible symmetry subset

$$\begin{pmatrix} w_1 = w_2 \\ w_3 = w_4 \end{pmatrix}$$

It follows, from Theorem 3, that F has the symmetric subset of variables $S(x_1,x_2)$ and $S(x_3,x_4)$. Substituting the representative weights w_2 and w_4, into the sets of non-permissible and permissible non-symmetry expressions, we obtain the following reduced sets of non-permissible and permissible expressions, respectively.

$w_2 \neq 0$	$w_2 = 2w_4$
$w_4 \neq 0$	
$w_2 + w_4 \neq 0$	
$2w_2 + w_4 \neq 0$	
$w_2 + 2w_4 \neq 0$	
$w_2 \neq w_4$	
$w_4 \neq 2w_2$	

Here, this single permissible expression forms the only compatible realizability subset. Therefore,assigning the smallest weight values accordingly, we obtain the weight-threshold vector (2,2,1,1;3.5,2.5,1.5), which is believed to be minimal. Mow and Fu [20] obtained the same realization for this function.

<u>Example 4</u>: Let F(4)=(1,2,4,7,8,11,12,13,14,15).

True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 0 1	0 0 0 0
0 0 1 0	0 0 1 1
0 1 0 0	0 1 0 1
0 1 1 1	0 1 1 0
1 0 0 0	1 0 0 1
1 0 1 1	1 0 1 0
1 1 0 0	
1 1 0 1	
1 1 1 0	
1 1 1 1	

The non-permissible expressions are

$w_1 \neq 0$

$w_2 \neq 0$

$w_3 \neq 0$

$w_4 \neq 0$

$w_1 + w_2 \neq 0$

$w_1 + w_3 \neq 0$

$w_1 + w_4 \neq 0$

$w_2 + w_3 \neq 0$

$w_2 + w_4 \neq 0$

$w_1 + w_2 + w_3 \neq 0$

$w_1 + w_2 + w_4 \neq 0$

$w_1 + w_3 + w_4 \neq 0$

$w_2 + w_3 + w_4 \neq 0$

$w_1 + w_2 + w_3 + w_4 \neq 0$

$w_1 \neq w_3$

$w_1 \neq w_4$

$w_2 \neq w_3$

$w_2 \neq w_4$

$w_1 \neq w_2 + w_3$

$w_1 \neq w_2 + w_4$

$w_1 \neq w_3 + w_4$

$w_2 \neq w_1 + w_3$

$w_2 \neq w_1 + w_4$

$w_2 \neq w_3 + w_4$

$w_3 \neq w_1 + w_2$

$w_3 \neq w_1 + w_4$

$w_3 \neq w_2 + w_4$

$w_4 \neq w_1 + w_2$

$w_4 \neq w_1 + w_3$

$w_4 \neq w_2 + w_3$

$w_1 + w_2 \neq w_3 + w_4$

The independent permissible expressions are as follows.

(1) $w_3+w_4=0$

(2) $w_1=w_2$

(3) $w_3=w_4$

$w_3=w_1+w_2+w_4$

$w_4=w_1+w_2+w_3$

The numbered expressions in the above list are the permissible symmetry expressions. The compatibility trees and the corresponding compatible symmetry subsets are shown below.

(1)————(2) I

(2)————(3) II

$$\begin{pmatrix} w_3+w_4=0 \\ w_1=w_2 \end{pmatrix} \text{ I}; \qquad \begin{pmatrix} w_1=w_2 \\ w_3=w_4 \end{pmatrix} \text{ II}.$$

Substituting the representative weights w_2 and $-w_4$, from compatible symmetry subset I, and then the representative weights w_2 and w_4, from compatible symmetry subset II, into the set of non-permissible expressions, we obtain the following reduced set of non-permissible expressions in both cases.

$w_2\neq 0$	$w_2+2w_4\neq 0$
$w_4\neq 0$	$w_2\neq w_4$
$w_2+w_4\neq 0$	$w_2\neq 2w_4$
$2w_2+w_4\neq 0$	$w_4\neq 2w_2$

Substituting the representative weights for the compatible symmetry subsets I and II, into the set of permissible non-symmetry expressions, it is found that in both cases all the resulting expressions become non-permissible. Therefore, the compatible realizability subsets are empty, for this multiform symmetric function. From the reduced set of non-permissible expressions it can be seen, that the smallest allowable weight magnitudes are 1 and 3, and from their combinations the following weight-threshold vectors are obtained.

$(\vec{W};\vec{T})_{I/1}=(1,1,-3,3;4.5,3.5,0.5,-0.5,-1.5,-2.5)$, $(\omega+\tau)=21.0$;

$(\vec{W};\vec{T})_{I/2}=(1,1,3,-3;4.5,3.5,0.5,-0.5,-1.5,-2.5)$, $(\omega+\tau)=21.0$;

$(\vec{W};\vec{T})_{I/3}=(3,3,-1,1;4.5,3.5,2.5,1.5,0.5,-0.5)$, $(\omega+\tau)=21.0$;

$(\vec{W};\vec{T})_{I/4}=(3,3,1,-1;4.5,3.5,2.5,1.5,0.5,-0.5)$, $(\omega+\tau)=21.0$;

$(\vec{W};\vec{T})_{II/1}=(1,1,3,3;6.5,5.5,4.5,3.5,0.5)$, $(\omega+\tau)=28.5$;

$(\vec{W};\vec{T})_{II/2}=(1,1,-3,-3;0.5,-0.5,-1.5,-2.5,-5.5)$, $(\omega+\tau)=18.5$;

$(\vec{W};\vec{T})_{II/3}=(3,3,1,1;4.5,3.5,2.5,1.5,0.5)$, $(\omega+\tau)=20.5$;

$(\vec{W};\vec{T})_{II/4}=(3,3,-1,-1;2.5,1.5,0.5,-0.5,-1.5)$, $(\omega+\tau)=14.5$.

Therefore, $(\vec{W};\vec{T})_{II/4}$ is chosen for the realization.

It should be noted, that if $m<(r-1)$ and a restricting relation among the weights is missing, then the assignment of the smallest weight magnitudes may not always result in a minimal realization. This is the situation for the above multiform symmetric function, and although $(\omega+\tau)$=14.5 is minimal, the minimal weight-threshold vector is believed to be (3,3,-2,-2;1.5,0.5,-2.5). Mow and Fu obtained this minimal realization for the function. However, even in this special case, the realization obtained, by using the smallest allowable weight magnitudes, is very close to minimal.

Example 5: Let F(4)=(1,2,3,5,7,9,15).

True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 0 1	0 0 0 0
0 0 1 0	0 1 0 0
0 0 1 1	0 1 1 0
0 1 0 1	1 0 0 0
0 1 1 1	1 0 1 0
1 0 0 1	1 0 1 1
1 1 1 1	1 1 0 0
	1 1 0 1
	1 1 1 0

The non-permissible expressions are as follows.

$w_1 \neq 0$

$w_2 \neq 0$

$w_3 \neq 0$

$w_4 \neq 0$

$w_1 + w_2 \neq 0$

$w_1 + w_3 \neq 0$

$w_1 + w_4 \neq 0$

$w_2 + w_4 \neq 0$

$w_3 + w_4 \neq 0$

$w_1 + w_3 + w_4 \neq 0$

$w_2 + w_3 + w_4 \neq 0$

$w_1 + w_2 + w_3 + w_4 \neq 0$

$w_1 \neq w_2$

$w_1 \neq w_3$

$w_1 \neq w_4$

$w_2 \neq w_3$

$w_2 \neq w_4$

$w_3 \neq w_4$

$w_1 \neq w_2 + w_4$

$w_1 \neq w_3 + w_4$

$w_2 \neq w_1 + w_3$

$w_2 \neq w_1 + w_4$

$w_2 \neq w_3 + w_4$

$w_3 \neq w_1 + w_2$

$w_4 \neq w_1 + w_2$

$w_4 \neq w_1 + w_3$

$w_4 \neq w_2 + w_3$

$w_1 \neq w_2 + w_3 + w_4$

$w_3 \neq w_1 + w_2 + w_4$

$w_4 \neq w_1 + w_2 + w_3$

$w_1 + w_2 \neq w_3 + w_4$

$w_1 + w_3 \neq w_2 + w_4$

$w_1 + w_4 \neq w_2 + w_3$

The set of independent permissible expressions is

(1) $w_2 + w_3 = 0$

$w_1 + w_2 + w_3 = 0$

$w_1 = w_2 + w_3$

$w_3 = w_1 + w_4$

$w_3 = w_2 + w_4$

$w_2 = w_1 + w_3 + w_4$

Here, (1) is the only permissible symmetry expression, and therefore by Theorem 4, F has the symmetric subset of variables $S(\bar{x}_2,x_3)$. Substituting the representative weight $-w_3$, into the sets of non-permissible and permissible expressions, we obtain the following reduced sets of non-permissible and permissible expressions, respectively.

$w_1 \neq 0$
$w_3 \neq 0$
$w_4 \neq 0$
$w_1+w_3 \neq 0$
$w_1+w_4 \neq 0$
$w_3+w_4 \neq 0$
$w_1+2w_3 \neq 0$
$2w_3+w_4 \neq 0$
$w_1+w_3+w_4 \neq 0$
$w_1 \neq w_3$
$w_1 \neq w_4$
$w_1 \neq 2w_3$
$w_3 \neq w_4$
$w_1 \neq w_3+w_4$
$w_1 \neq 2w_3+w_4$
$2w_3 \neq w_1+w_4$
$w_4 \neq w_1+w_3$
$w_4 \neq w_1+2w_3$

(1) $w_1+2w_3+w_4=0$
(2) $w_3=w_1+w_4$
(3) $w_4=2w_3$

The compatibility trees and the corresponding compatible realizability subset are shown below.

(1)————(3)

(2)

$$\begin{pmatrix} w_1+2w_3+w_4=0 \\ w_4=2w_3 \end{pmatrix}$$

Substituting the second expression, of the compatible realizability subset, into the first expression, we have $w_1=-4w_3$, and since $w_4=2w_3$, all the weights are expressed in terms of one weight. Assigning the smallest weight values, we obtain the weight-threshold vector (-4,-1,1,2; 0.5,-1.5,-2.5), which is believed to be minimal.

Example 6: Let F(4)=(1,2,6,7,9,11,12,13,14,15).

True vertices	False vertices
$x_1x_2x_3x_4$	$x_1x_2x_3x_4$
0 0 0 1	0 0 0 0
0 0 1 0	0 0 1 1
0 1 1 0	0 1 0 0
0 1 1 1	0 1 0 1
1 0 0 1	1 0 0 0
1 0 1 1	1 0 1 0
1 1 0 0	
1 1 0 1	
1 1 1 0	
1 1 1 1	

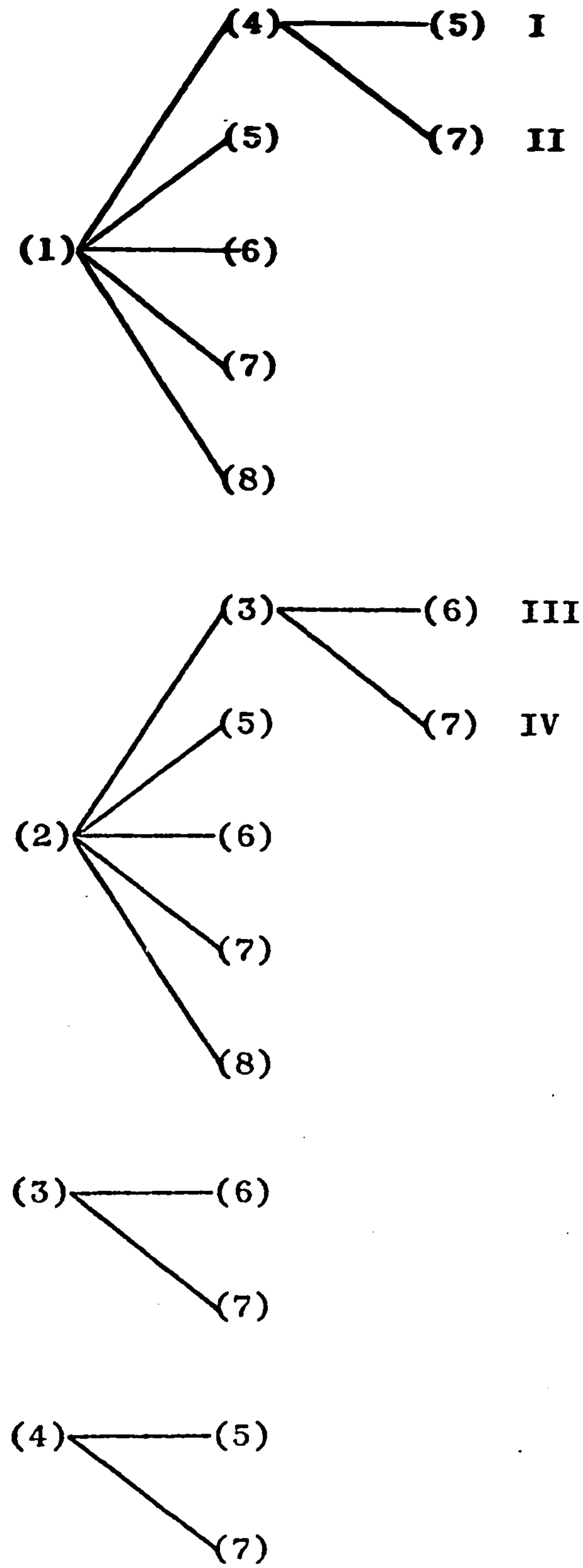
The non-permissible expressions are as follows.

$w_1 \neq 0$	$w_1 \neq w_2$
$w_2 \neq 0$	$w_1 \neq w_3$
$w_3 \neq 0$	$w_1 \neq w_4$
$w_4 \neq 0$	$w_2 \neq w_3$
$w_1 + w_2 \neq 0$	$w_2 \neq w_4$
$w_1 + w_3 \neq 0$	$w_3 \neq w_4$
$w_1 + w_4 \neq 0$	$w_1 \neq w_2 + w_3$
$w_2 + w_3 \neq 0$	$w_1 \neq w_2 + w_4$
$w_2 + w_4 \neq 0$	$w_2 \neq w_1 + w_3$
$w_3 + w_4 \neq 0$	$w_2 \neq w_1 + w_4$
$w_1 + w_2 + w_3 \neq 0$	$w_3 \neq w_1 + w_2$
$w_1 + w_2 + w_4 \neq 0$	$w_3 \neq w_2 + w_4$
$w_1 + w_3 + w_4 \neq 0$	$w_4 \neq w_1 + w_2$
$w_2 + w_3 + w_4 \neq 0$	$w_4 \neq w_1 + w_3$
$w_1 + w_2 + w_3 + w_4 \neq 0$	$w_1 \neq w_2 + w_3 + w_4$
	$w_2 \neq w_1 + w_3 + w_4$
	$w_1 + w_2 \neq w_3 + w_4$

The set of independent permissible expressions is

(1)	$w_1 = w_3 + w_4$	(5)	$w_3 = w_1 + w_2 + w_4$
(2)	$w_2 = w_3 + w_4$	(6)	$w_4 = w_1 + w_2 + w_3$
(3)	$w_3 = w_1 + w_4$	(7)	$w_1 + w_3 = w_2 + w_4$
(4)	$w_4 = w_2 + w_3$	(8)	$w_1 + w_4 = w_2 + w_3$

It can be seen, that F is neither independent of any variable, nor symmetric in any subset of its variables. Hence, we directly proceed to form compatible realizability subsets. The compatibility trees and the corresponding compatible realizability subsets are shown below.



(5); (6); (7); (8).

$$\begin{pmatrix} w_1=w_3+w_4 \\ w_4=w_2+w_3 \\ w_3=w_1+w_2+w_4 \end{pmatrix} \mathrm{I} \qquad \begin{pmatrix} w_1=w_3+w_4 \\ w_4=w_2+w_3 \\ w_1+w_3=w_2+w_4 \end{pmatrix} \mathrm{II}$$

$$\begin{pmatrix} w_2=w_3+w_4 \\ w_3=w_1+w_4 \\ w_4=w_1+w_2+w_3 \end{pmatrix} \mathrm{III} \qquad \begin{pmatrix} w_2=w_3+w_4 \\ w_3=w_1+w_4 \\ w_1+w_3=w_2+w_4 \end{pmatrix} \mathrm{IV}$$

After expressing all the weights in terms of one weight, and assigning the smallest allowable weight values, the following weight-threshold vectors are obtained.

$(\vec{W};\vec{T})_{I}=(4,-2,3,1;7.5,6.5,4.5,3.5,0.5)$, $(\omega+\tau)=32.5$;

$(\vec{W};\vec{T})_{II}=(4,2,1,3;5.5,3.5,2.5,1.5,0.5)$, $(\omega+\tau)=23.5$;

$(\vec{W};\vec{T})_{III}=(-2,4,1,3;7.5,6.5,4.5,3.5,0.5)$, $(\omega+\tau)=32.5$;

$(\vec{W};\vec{T})_{IV}=(2,4,3,1;5.5,3.5,2.5,1.5,0.5)$, $(\omega+\tau)=23.5$.

Therefore, either $(\vec{W};\vec{T})_{II}$ or $(\vec{W};\vec{T})_{IV}$ can be chosen for the realization, the two of them being equivalent with respect to minimality. The realization obtained by Mow and Fu [20] for the same function is (2,3,4,5;8.5,7.5, 6.5,3.5,1.5), with $(\omega+\tau)=41.5$.

The main features of the synthesis method have been enlarged upon, through the examples of different types of

Boolean functions. Therefore, the complete synthesis procedure can be established as follows.

Step 1: List all the true and false vertices, of the given function, into two separate columns.

Step 2: Determine the M(n) possible expressions (for hand computation this step is not necessary).

Step 3: Determine the N non-permissible expressions and list them, so that the same type of expressions follow each other in the order of increasing i of the terms, where $i=(1,\ldots,n)$.

Step 4: Determine the $P=M(n)-N$ permissible expressions, by listing all the possible expressions that are not included in the non-permissible set.

Step 5: Eliminate dependent expressions, to obtain the set of independent permissible expressions.

Step 6: Does the function have any permissible independence expressions $w_i=0$, where $i\in(1,\ldots,n)$?

a./ Yes, go to Step 7.

b./ No, go to Step 8.

Step 7: Set $w_i=0$, and eliminate w_i from all permissible and non-permissible expressions.

Step 8: Does the function have any permissible symmetry expressions $w_i+w_j=0$ and/or $w_i=w_j$, where $i\neq j$ and $i,j\in(1,\ldots,n)$?

a./ Yes, go to Step 9.

b./ No, go to Step 16.

<u>Step 9:</u> List and number all permissible symmetry expressions.

<u>Step 10:</u> Determine the compatible symmetry subsets, by using compatibility trees.

<u>Step 11:</u> Select a compatible symmetry subset.

<u>Step 12:</u> Determine the representative weights.

<u>Step 13:</u> Substitute the representative weights, for all the weights they represent, into the set of non-permissible expressions, to obtain the reduced set of non-permissible expressions.

<u>Step 14:</u> Substitute the representative weights, into the set of permissible non-symmetry expressions.

<u>Step 15:</u> Cancel terms appearing on both sides of an expression, and omit the resulting redundant and/or non-permissible expressions, to obtain the reduced set of permissible expressions.

<u>Step 16:</u> Determine the compatible realizability subset (or subsets), by using compatibility trees.

<u>Step 17:</u> Is there another compatible symmetry subset?

a./ Yes, go to Step 11.

b./ No, stop.

Since a general method has not been found at the present time, for the assignment of actual weight values, the corresponding steps have not been included in the synthesis procedure.

So far in this thesis only relatively simple functions were used, to illustrate certain features of the proposed method. To show the power of the method, the synthesis of a six-variable function will be presented. A multiform symmetric function was chosen for this purpose, since this type of function is generally the most troublesome.

Example 7: Let F(6)=(3,5,9,16,17,19,21,22,23,25,26, 27,28,29,31,32,33,35,37,38,39,41,42,43,44,45,47,51,53,57).

True vertices	False vertices	True vertices	False vertices
$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$
0 0 0 0 1 1	0 0 0 0 0 0	0 1 1 0 1 0	0 0 1 1 0 1
0 0 0 1 0 1	0 0 0 0 0 1	0 1 1 0 1 1	0 0 1 1 1 0
0 0 1 0 0 1	0 0 0 0 1 0	0 1 1 1 0 0	0 0 1 1 1 1
0 1 0 0 0 0	0 0 0 1 0 0	0 1 1 1 0 1	0 1 0 0 1 0
0 1 0 0 0 1	0 0 0 1 1 0	0 1 1 1 1 1	0 1 0 1 0 0
0 1 0 0 1 1	0 0 0 1 1 1	1 0 0 0 0 0	0 1 1 0 0 0
0 1 0 1 0 1	0 0 1 0 0 0	1 0 0 0 0 1	0 1 1 1 1 0
0 1 0 1 1 0	0 0 1 0 1 0	1 0 0 0 1 1	1 0 0 0 1 0
0 1 0 1 1 1	0 0 1 0 1 1	1 0 0 1 0 1	1 0 0 1 0 0
0 1 1 0 0 1	0 0 1 1 0 0	1 0 0 1 1 0	1 0 1 0 0 0

True vertices	False vertices	True vertices	False vertices
$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$	$x_1x_2x_3x_4x_5x_6$
1 0 0 1 1 1	1 0 1 1 1 0	1 1 0 0 1 1	1 1 1 0 0 0
1 0 1 0 0 1	1 1 0 0 0 0	1 1 0 1 0 1	1 1 1 0 1 0
1 0 1 0 1 0	1 1 0 0 0 1	1 1 1 0 0 1	1 1 1 0 1 1
1 0 1 0 1 1	1 1 0 0 1 0		1 1 1 1 0 0
1 0 1 1 0 0	1 1 0 1 0 0		1 1 1 1 0 1
1 0 1 1 0 1	1 1 0 1 1 0		1 1 1 1 1 0
1 0 1 1 1 1	1 1 0 1 1 1		1 1 1 1 1 1

The non-permissible expressions are not shown, for this function, because of their large number. The set of independent permissible expressions is shown below.

(1) $w_1+w_2=0$

$w_1+w_2+w_3+w_4+w_5=0$

$w_1+w_2+w_3+w_4+w_6=0$

$w_1+w_2+w_3+w_4+w_5+w_6=0$

(2) $w_1=w_2$

(3) $w_3=w_4$

(4) $w_3=w_5$

$w_1=w_2+w_3+w_4$

$w_3=w_1+w_2+w_4$

$w_1=w_2+w_3+w_4+w_5+w_6$

$w_3=w_1+w_2+w_4+w_5+w_6$

$w_6=w_1+w_2+w_3+w_4+w_5$

$w_1+w_2=w_3+w_4+w_5$

$w_1+w_2=w_3+w_4+w_6$

$w_1+w_2=w_3+w_4+w_5+w_6$

$w_1+w_2+w_3=w_4+w_5+w_6$

$w_1+w_2+w_6=w_3+w_4+w_5$

$w_1+w_3+w_4=w_2+w_5+w_6$

$w_1+w_3+w_6=w_2+w_4+w_5$

The compatibility trees and the corresponding compatible symmetry subsets are as follows.

$$\begin{tikzcd}[arrows=dash]
(1) \arrow[r] \arrow[dr] & (3) \arrow[r] & (4) & \mathrm{I} \\
 & (4) & &
\end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash]
(2) \arrow[r] \arrow[dr] & (3) \arrow[r] & (4) & \mathrm{II} \\
 & (4) & &
\end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash]
(3) \arrow[r] & (4)
\end{tikzcd}$$

$$\begin{pmatrix} w_1+w_2=0 \\ w_3=w_4 \\ w_3=w_5 \end{pmatrix} \mathrm{I} \qquad \begin{pmatrix} w_1=w_2 \\ w_3=w_4 \\ w_3=w_5 \end{pmatrix} \mathrm{II}$$

Substituting the representative weights, for compatible symmetry subset I, into the set of non-permissible expressions, the following reduced set of non-permissible expressions is obtained.

$$\begin{array}{ll}
w_2\neq 0 & w_2+3w_5\neq 0 \\
w_5\neq 0 & 2w_5+w_6\neq 0 \\
w_6\neq 0 & 3w_5+w_6\neq 0 \\
w_2+w_5\neq 0 & 2w_2+w_5\neq 0 \\
w_2+w_6\neq 0 & 2w_2+w_6\neq 0 \\
w_5+w_6\neq 0 & 2w_2+3w_5\neq 0 \\
w_2+2w_5\neq 0 & w_2+w_5+w_6\neq 0
\end{array}$$

$w_2+2w_5+w_6\neq 0$	$w_6\neq w_2+w_5$
$w_2+3w_5+w_6\neq 0$	$w_2\neq 2w_5+w_6$
$2w_2+w_5+w_6\neq 0$	$w_2\neq 3w_5+w_6$
$2w_2+2w_5+w_6\neq 0$	$w_5\neq 2w_2+w_6$
$w_2\neq w_5$	$w_6\neq w_2+2w_5$
$w_2\neq w_6$	$w_6\neq w_2+3w_5$
$w_5\neq w_6$	$w_6\neq 2w_2+w_5$
$w_2\neq 2w_5$	$w_6\neq 2w_2+2w_5$
$w_2\neq 3w_5$	$w_6\neq 2w_2+3w_5$
$w_5\neq 2w_2$	$2w_2\neq w_5+w_6$
$w_6\neq 2w_2$	$2w_2\neq 2w_5+w_6$
$w_6\neq 2w_5$	$2w_5\neq w_2+w_6$
$w_6\neq 3w_5$	$2w_5\neq 2w_2+w_6$
$2w_2\neq 3w_5$	$3w_5\neq w_2+w_6$
$w_2\neq w_5+w_6$	$3w_5\neq 2w_2+w_6$
$w_5\neq w_2+w_6$	

Substituting the representative weights, for compatible symmetry subset I, into the set of permissible non-symmetry expressions, the following reduced set of permissible expressions is obtained.

(1) $2w_2+3w_5+w_6=0$

(2) $2w_2=3w_5+w_6$

It is easy to see that these two permissible expressions are incompatible, and therefore each of them is a compatible realizability subset by itself.

Substituting the representative weights of compatible symmetry subset II, into the set of non-permissible expressions, the following reduced set of non-permissible expressions is obtained.

$w_2 \neq 0$	$w_2 \neq w_5 + w_6$
$w_5 \neq 0$	$w_5 \neq 2w_2$
$w_6 \neq 0$	$w_5 \neq w_2 + w_6$
$w_2 + w_5 \neq 0$	$w_6 \neq 2w_2$
$w_2 + w_6 \neq 0$	$w_6 \neq w_2 + w_5$
$w_5 + w_6 \neq 0$	$w_6 \neq 2w_5$
$2w_2 + w_5 \neq 0$	$w_2 \neq 3w_5$
$2w_2 + w_6 \neq 0$	$w_2 \neq 2w_5 + w_6$
$w_2 + 2w_5 \neq 0$	$w_5 \neq 2w_2 + w_6$
$w_2 + w_5 + w_6 \neq 0$	$w_6 \neq 2w_2 + w_5$
$2w_5 + w_6 \neq 0$	$w_6 \neq w_2 + 2w_5$
$2w_2 + w_5 + w_6 \neq 0$	$w_6 \neq 3w_5$
$w_2 + 3w_5 \neq 0$	$w_2 \neq 3w_5 + w_6$
$w_2 + 2w_5 + w_6 \neq 0$	$w_6 \neq 2w_2 + 2w_5$
$3w_5 + w_6 \neq 0$	$w_6 \neq w_2 + 3w_5$
$w_2 + 3w_5 + w_6 \neq 0$	$2w_2 \neq w_5 + w_6$
$w_2 \neq w_5$	$2w_5 \neq w_2 + w_6$
$w_2 \neq w_6$	$3w_5 \neq w_2 + w_6$
$w_5 \neq w_6$	$2w_5 \neq 2w_2 + w_6$
$w_2 \neq 2w_5$	

Substituting the representative weights of compatible symmetry subset II, into the set of permissible non-symmetry expressions, the following reduced set of permissible expressions is obtained.

(1) $2w_2+3w_5=0$

(2) $2w_2+2w_5+w_6=0$

(3) $2w_2+3w_5+w_6=0$

(4) $w_6=2w_2+3w_5$

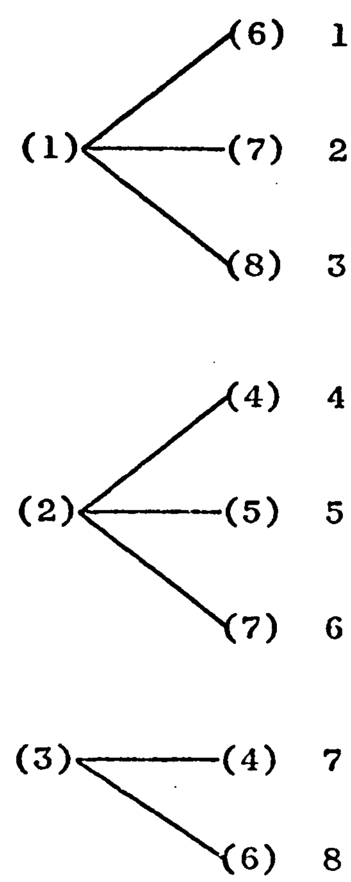
(5) $2w_2=3w_5$

(6) $2w_2=2w_5+w_6$

(7) $2w_2=3w_5+w_6$

(8) $3w_5=2w_2+w_6$

The compatibility trees and the corresponding compatible realizability subsets are shown below.



(4)————(5) 9

(6)————(8) . 10

(5); (7); (8).

$$\begin{pmatrix} 2w_2+3w_5=0 \\ 2w_2=2w_5+w_6 \end{pmatrix} \text{II/1} \quad \begin{pmatrix} 2w_2+3w_5=0 \\ 2w_2=3w_5+w_6 \end{pmatrix} \text{II/2} \quad \begin{pmatrix} 2w_2+3w_5=0 \\ 3w_5=2w_2+w_6 \end{pmatrix} \text{II/3}$$

$$\begin{pmatrix} 2w_2+2w_5+w_6=0 \\ 2w_2=3w_5 \end{pmatrix} \text{II/4} \quad \begin{pmatrix} 2w_2+2w_5+w_6=0 \\ w_6=2w_2+3w_5 \end{pmatrix} \text{II/5} \quad \begin{pmatrix} 2w_2+2w_5+w_6=0 \\ 2w_2=3w_5+w_6 \end{pmatrix} \text{II/6}$$

$$\begin{pmatrix} 2w_2+3w_5+w_6=0 \\ 2w_2=3w_5 \end{pmatrix} \text{II/7} \quad \begin{pmatrix} 2w_2+3w_5+w_6=0 \\ 2w_2=2w_5+w_6 \end{pmatrix} \text{II/8} \quad \begin{pmatrix} 2w_2=3w_5 \\ w_6=2w_2+3w_5 \end{pmatrix} \text{II/9}$$

$$\begin{pmatrix} 2w_2=2w_5+w_6 \\ 3w_5=2w_2+w_6 \end{pmatrix} \text{II/10}$$

Expressing all the weights in terms of one weight, for each compatible realizability subset, and assigning the smallest allowable weight values, the following weight-threshold vectors are obtained.

$(\vec{W};\vec{T})_{II/1}$=(3,3,-2,-2,-2,10;14.5,12.5,11.5,10.5,9.5,6.5, 3.5,2.5,-0.5,-1.5), $(\omega+\tau)$=95.0;

$(\vec{W};\vec{T})_{II/2}$=(3,3,-2,-2,-2,12;16.5,14.5,13.5,12.5,11.5,8.5, 3.5,2.5,-0.5,-1.5), $(\omega+\tau)$=109.0;

$(\vec{W};\vec{T})_{II/3}$=(-3,-3,2,2,2,12;15.5,12.5,11.5,10.5,9.5,6.5,1.5, -0.5,-2.5,-3.5), $(\omega+\tau)$=98.0;

$(\vec{W};\vec{T})_{II/4}$=(-3,-3,-2,-2,-2,10;8.5,6.5,5.5,4.5,3.5,0.5,-2.5, -3.5,-6.5,-7.5), $(\omega+\tau)$=71.0;

$(\vec{W};\vec{T})_{II/5}$=(-5,-5,4,4,4,2;9.5,8.5,6.5,4.5,3.5,2.5,1.5,0.5, -2.5,-5.5), $(\omega+\tau)$=69.0;

$(\vec{W};\vec{T})_{II/6}$=(1,1,4,4,4,-10;9.5,8.5,3.5,2.5,1.5,0.5,-0.5,-1.5, -2.5,-6.5,-8.5,-9.5), $(\omega+\tau)$=79.0;

$(\vec{W};\vec{T})_{II/7}$=(3,3,2,2,2,-12;10.5,8.5,7.5,6.5,5.5,2.5,-2.5, -3.5,-6.5,-7.5), $(\omega+\tau)$=85.0;

$(\vec{W};\vec{T})_{II/8}$=(1,1,-4,-4,-4,10;11.5,10.5,8.5,4.5,3.5,2.5,1.5, 0.5,-0.5,-1.5,-6.5,-7.5), $(\omega+\tau)$=83.0;

$(\vec{W};\vec{T})_{II/9}$= (3,3,2,2,2,12;21.5,18.5,17.5,16.5,15.5,12.5, 7.5,6.5,3.5,2.5), $(\omega+\tau)$=146.0;

$(\vec{W};\vec{T})_{II/10}$=(5,5,4,4,4,2;19.5,18.5,16.5,14.5,13.5,12.5,11.5, 10.5,7.5,4.5), $(\omega+\tau)$=153.0 .

From the above weight-threshold vectors, $(\vec{W};\vec{T})_{II/5}$ is chosen for the realization, and it is believed to be minimal.

IV. CONCLUDING REMARKS.

A new approach to multi-threshold synthesis, based on the sets of permissible and non-permissible expressions, has been presented and the corresponding realizability conditions have been established.

The method is relatively simple and has the advantage of avoiding any permutation and/or complementation of the variables, while automatically taking all the relevant properties of the given function into consideration. It also provides a better insight, into the properties of Boolean functions in general.

The method is theoretically applicable to any Boolean function, regardless of the number of variables, and the realizations obtained are always minimal or at least near minimal.

The method is systematic and can be programmed up to and including the determination of the largest compatible realizability subsets, from which the smallest allowable weight vector can be easily obtained, by hand computation. Unfortunately, for certain Boolean functions some of the restricting relations among the weights are not obtainable, in which case the determination of the smallest allowable weight vectors is somewhat heuristic. However, the number of these functions is very small.

The potential of the proposed synthesis method is great, and the results obtained compare very favorably with those obtained by presently existing methods. However, a general procedure for the assignment of weight values should be found, to make possible the full utilization of computers, and to use the method to its best advantage.

REFERENCES.

[1] R.C. Minnick: "Linear input logic", IRE Trans. Electronic Computers, vol.EC-10, pp 6-16, March 1961.

[2] W.H. Kautz: "The realization of symmetric switching functions with linear-input logical elements", IRE Trans. Electronic Computers, vol.EC-10, pp 371-378, September 1961.

[3] C.L. Sheng: "A graphical interpretation of realization of symmetric Boolean functions with threshold logic element", IEEE Trans. Electronic Computers, vol.EC-14, pp 8-18, February 1965.

[4] P.K.S. Roy: "Synthesis of symmetric switching functions using threshold logic elements", IEEE Trans. Electronic Computers, vol.EC-16, pp 359-364, June 1967.

[5] M.A. Fischler and M. Tannembaum: "Assumptions in the threshold synthesis of symmetric switching functions", IEEE Trans. Electronic Computers, vol.EC-17, pp 273-279, March 1968.

[6] S. Muroga, I. Toda, and S. Takasu: "Theory of majority decision elements", J. of the Franklin Institute, vol. 271, pp 376-418, May 1961.

[7] S.B. Akers: "Threshold logic and two-person zero-sum games", Switching Circuit Theory and Logical Design, pp 27-33, September 1961.

[8] C.L. Coates and P.M. Lewis,II: "Linearly separable switching functions", J. of the Franklin Institute, vol. 272, pp 360-410, November 1961.

[9] S.N. Einhorn: "The use of the simplex algorithm in the mechanization of Boolean switching functions by means of magnetic cores", IRE Trans. Electronic Computers, vol.EC-10, pp 615-622, December 1961.

[10] R.O. Winder: "Threshold logic", PhD. dissertation, Princeton University, Princeton, N.J., May 1962.

[11] C.L. Coates, R.B. Kirchner, and P.M. Lewis,II: "A simplified procedure for the realization of linearly separable switching functions", IRE Trans. Electronic Computers, vol.EC-11, pp 447-458, August 1962.

[12] I.J. Gabelman: "The synthesis of Boolean functions using a single threshold element", IEEE Trans. Electronic Computers, vol.EC-11, pp 639-642, October 1962.

[13] C.L. Sheng: "A method for testing and realization of threshold functions", IEEE Trans. Electronic Computers, vol.EC-13, pp 232-239, June 1964.

[14] M.L. Dertouzos: "Threshold logic:A synthesis approach", MIT Press, Cambridge, Mass., 1965.

[15] S.T. Hu: "Threshold logic", University of California Press, 1965.

[16] H.C. Torng: "An approach for the realization of linearly separable switching functions", IEEE Trans. Electronic Computers, vol.EC-15, pp 14-20, February 1966.

[17] C.L. Sheng and H.R. Hwa: "Testing and realization of threshold functions by successive higher ordering of incremental weights", IEEE Trans. Electronic Computers, vol.EC-15, pp 212-220, April 1966.

[18] D.R. Haring: "Multi-threshold threshold elements", IEEE Trans. Electronic Computers, vol.EC-15, pp 45-65, February 1966.

[19] D.R. Haring and D. Ohori: "A tabular method for the synthesis of multi-threshold threshold elements", IEEE Trans. Electronic Computers, vol.EC-16, pp 216-220, April 1967.

[20] C.W. Mow and K.S. Fu: "An approach for the realization of multi-threshold threshold elements", IEEE Trans. Electronic Computers, vol.EC-17, pp 32-46, January 1968.

[21] N.N. Necula: "An algorithm for multi-threshold threshold element synthesis", IEEE Trans. Electronic Computers, vol.EC-17, pp 978-985, October 1968.

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