Transformed Random Walks

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Abstract

We consider transformations of a given random walk on a countable group determined by Markov stopping times. We prove that these transformations preserve the Poisson boundary. Moreover, under some mild conditions, the asymptotic entropy (resp., rate of escape) of the transformed random walks is equal to the asymptotic entropy (resp., rate of escape) of the original random walk multiplied by the expectation of the corresponding stopping time. This is an analogue of the well-known Abramov's formula from ergodic theory.

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Dedication

To my parents whom can have another
reason to be proud of me beside being
the only kid who is not picky eater

Contents

Summary

The results of this thesis involve a combination of ergodic theory, group theory, measure theory and probability theory. We assume that the reader has some basic knowledge of these mathematical concepts. In Chapter 1, the reader is reminded of the main definitions and theorems that are employed in other parts of the this work. Chapter 2 includes the definition and some basic properties of the Poisson boundary of random walks on groups. In this chapter, we give more details in order to provide a better understanding of the Poisson boundary. Chapter 3 introduces the construction of transformed random walks via Markov stopping times and studies its Poisson boundaries. The entropy, asymptotic entropy, and the rate of escape of transformed random walks are investigated in Chapter 4. The results of Chapters 3 and 4 are adapted from the papers [23](which is a joint work with Kaimanovich) and [22], respectively.

Introduction

In the late 50's, Kesten introduced the notion of the random walk on a countable group G [49]. A random walk is determined by a probability measure μ on G which is interpreted as the transition probability from the group identity; the transition probabilities from other elements are obtained by group translation. In other words, a sample path of this Markov chain is a sequence of products $h_1 \cdots h_n$ of independent identically μ-distributed increments h_i. The *Poisson boundary* $(\partial_\mu G, \nu)$ is a probability space which describes the stochastically significant behavior of sample paths at infinity. It is endowed with a natural G-action, and the harmonic measure ν is μ–stationary under this action. The study of the Poisson boundary is closely related to algebraic properties of the underlying group. For example, the Poisson boundary of an abelian group (or, more generally, of a nilpotent group) is trivial (a singleton) for any random walk, see Theorem 2.4.1 below. On the other hand, non-amenable groups have non-trivial Poisson boundary, see Remark 2.4.8.

Most of the research in this area has been devoted to the identification of the Poisson boundary of a concrete probability measure on a concrete group. Much less attention was given to the situation when the group G is fixed and the measure μ is allowed to vary. Since the Poisson boundary is defined in terms of the path space of a given measure, a priori there is no way to identify the Poisson boundaries of different measures in the same group.

The only instance in which they may be compared is when they can be realized on the same measure G-space. In this case the corresponding harmonic measures have to be equivalent. This naturally leads to the question about the situations when the Poisson boundaries of the *different* probability measures on the same group actually coincide. Finding all probability measures with the same Poisson boundary is important for understanding the structure of a group G. The closely related problem of realizing a given measure G-space as the Poisson boundary of an appropriate random walk on G goes back to Furstenberg [25, 26] who used it for analyzing rigidity properties of lattices in semi-simple Lie groups. Furstenberg [25] proved that the Poisson boundary of $SL(n,\mathbb{R})$ is the same as the Poisson boundary of a lattice of $SL(n,\mathbb{R})$ endowed with an appropriate measure. By using this fact, he concluded that the lattices of $SL(n,\mathbb{R})$ for $n \geqslant 3$ are different from discrete subgroups of $SL(2,\mathbb{R})$.

In some special situations this problem was also addressed by Willis [72] and Muchnik [58]. However, the scopes of these constructions are all quite limited. Moreover, the arising measures are always infinitely supported.

A new technique to solve this problem was suggested in the joint paper [23] with Kaimanovich. This approach subsumes all previously known examples, and allows us to obtain *finitely supported* measures with the same Poisson boundary as well. The method is based on the observation that due to space homogeneity of random walks, any Markov stopping time can be iterated to produce the random walk on G governed by a new measure μ_τ (the distribution of the original random walk stopped at τ).

Theorem 0.0.1. *[23] Let τ be a Markov stopping time for a random walk (G,μ). Then the Poisson boundary of the random walk (G,μ_τ) coincides with the Poisson boundary of (G,μ).*

The proof relies on the fact that the group is countable. We prove it first for a *free semigroup* endowed with a probability measure concentrated on independent free generators. In this situation, the Poisson boundary is the space of all infinite words with the Bernoulli measure. Then by passing to the quotient, we prove it for an arbitrary countable group.

The drawback of Theorem 0.0.1 is that it does not subsume that the Poisson boundaries of a given measure and convex combinations of its convolution powers coincide, which has been shown by Kaimanovich [36]. In order to overcome to the aforementioned drawback, the construction can be further generalized by considering a *randomized Markov stopping time* under a weak condition, which allows us to construct more probability measures with the same Poisson boundary.

Theorem 0.0.2. *Let* τ *be a randomized Markov stopping time such that* $\boldsymbol{E}(\log\tau)$ *is finite. Then the Poisson boundary of the random walk* (G, μ_τ) *coincides with the Poisson boundary of* (G, μ).

The proof relies on the *entropy criterion*, i.e., the asymptotic entropy of conditional random walks are zero.

The notion of entropy of a countable probability space was introduced by Shannon in 1948. He used it to define the asymptotic entropy (entropy rate) in order to quantify the amount of information for a stationary stochastic process [66]. Later in the mid 1950's Kolmogorov developed the notion of entropy of a measure preserving dynamical system [52], and his work was completed by Sinai [67]. But, it was only in 1972, that Avez defined the asymptotic entropy of a random walk on a group [2]. Despite a formal similarity, the contexts of these definitions are different, and so far there is no common approach which would unify them.

The asymptotic entropy is an important quantity which describes the behavior of a random walk at infinity. For instance, the triviality of the Poisson boundary of a random walk is equivalent to vanishing of the asymptotic entropy [3], [11], [41].

There are various formulas for the asymptotic entropy of a random walk on a group:

(i) In terms of the entropy of convolution powers [41], see equality (4.1.1) below,

(ii) Shannon's formula [11, 41], see Theorem 4.1.2 below,

(iii) As the average of *Kullback–Liebler deviations* between the harmonic measure and its translates [36, 41], see equality (4.1.8) below,

(iv) As the exponential growth rate of the Radon–Nikodym derivatives of the translates of the harmonic measure along sample paths of the random walk [41], see equality (4.1.3) below.

In the last two formulas, the asymptotic entropy is expressed in terms of the Poisson boundary of the random walk, which suggests considering a possible relationship between asymptotic entropies for random walks on the same group which share a common Poisson boundary.

Earlier this relationship was studied in two particular situations:

(j) convex combinations of convolutions of a given probability measure [36],

(jj) the induced random walk on a recurrent subgroup [25, 43, 37, 33].

In case (j), the asymptotic entropy can be obtained by a direct calculation based on formula (iii) [36].

In case (jj), Furstenberg [25] introduced induced random walk on a recurrent subgroup and proved that its Poisson boundary is the same as the original Poisson boundary. Kaimanovich [37] used a similar model to study harmonic functions on a Riemannian manifold and to compare the asymptotic entropies in this context.

Although his setup was somewhat different, by the same approach one can also find the asymptotic entropy of the induced random walk on a recurrent subgroup [43]. Hartman, Lima and Tamuz [33] calculated the asymptotic entropy of the random walk induced on a finite index subgroup in an alternative way by using formula (iii) (although, apparently, they were not aware of [37] and [43]).

The probability measures arising in the above situations are examples of transformations of probability measures via (randomized) Markov stopping time, as mentioned earlier, which do not change the Poisson boundary.

In the solo paper [22], we show that the asymptotic entropy h' of the transformed random walk is the result of rescaling the asymptotic entropy h of the original random walk by the expectation $\boldsymbol{\tau}$ of the stopping time (Theorem 4.1.8):

$$h' = \boldsymbol{\tau} h. \tag{0.0.1}$$

The aforementioned examples (j) and (jj) are contained in this result as particular cases.

Equation (0.0.1), the rescaling of the asymptotic entropy under a "time change", is analogous to Abramov's formula [1] for the entropy of induced dynamical systems (Theorem 4.1.4). However, as we have already pointed out, we are not aware of any common context which would unify these two formulas.

Theorem 0.0.3. *Let μ be a probability measure with finite entropy. If τ is a randomized Markov stopping time with finite expectation, then*

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu).$$

Our proof consists of three steps. Firstly, by using the martingale theory, we prove that finiteness of the entropy of a probability measure is preserved after applying a Markov stopping time with a finite expectation (Lemma 4.1.6). Secondly, by taking into account formula (iv) for the asymptotic entropy, we will establish the main result. And finally, applying the same method, we will prove that this result holds for randomized Markov stopping times as well (Theorem 4.1.8).

We would like to emphasize that in our general setup finiteness of the expectation $\boldsymbol{\tau}$ is not related to finiteness of any associated space (which can already be observed in the case of convolution powers, see above (j)). On the other hand, the technique used by Hartman, Lima and Tamuz [33] crucially depends on the fact that for the induced random walks on recurrent subgroups $\boldsymbol{\tau}$ is finite if and only if the subgroup has finite index, in combination with a number of properties of finite state Markov chains (formulated in the Appendix to [33]).

The rate of escape is another quantity which describes behavior of a random walk at infinity. There are some interrelations between the rate of escape and asymptotic entropy [69, 47]. In the paper [22], we show that the rate of escape of a transformed random walk under a randomized Markov stopping time is also transformed according to formula (0.0.1).

Theorem 0.0.4. *Let $\mathcal{G}$ be a sub-additive gauge for group G and μ has a finite first moment. If τ is a randomized Markov stopping time with finite expectation, then*

$$\ell(\mu_\tau) = \boldsymbol{E}(\tau)\ell(\mu).$$

Chapter 1

Background

In this work, we will employ some definitions and results from Lebesgue spaces, ergodic theorems and martingale theory. In this chapter, we review these definitions and present the version of the theorems that will be used later. Also, we assume that the reader is familiar with the basic definitions and theorems in the context of group theory and measure theory.

1.1 Lebesgue spaces

A probability space $(X, \mathcal{F}, m)$ consists of a set X, a sigma algebra $\mathcal{F}$ and a probability measure m. If there is no confusion, we do not specify the sigma algebra related to a probability space.

The map $\varphi : (X, \mathcal{F}, m) \to (X', \mathcal{F}', m')$ between two probability spaces is called a *measurable map* if $\varphi^{-1}(A') \in \mathcal{F}$, when $A' \in \mathcal{F}'$. Moreover, φ is called *measure preserving* whenever $m'(A') = m(\varphi^{-1}(A'))$ for all $A' \in \mathcal{F}'$, and the probability measure $m' = \varphi(m)$ is called the image of the probability measure m under the map φ.

In the concept of measure spaces or *category of measure spaces*, the null sets (sets with a measure–zero) are negligible. We are interested in the properties after removing some null sets. We write the property $\boldsymbol{\Upsilon}$ holds *mod 0* or *almost everywhere (a.e.)* whenever the property $\boldsymbol{\Upsilon}$ is valid everywhere, except a set of a measure-zero. Thus, we can work with *complete sigma-algebras*, in which subsets of null sets are considered to be measurable sets.

A measurable map φ is called a *quotient map*, if it is measure–preserving after the removal of some null sets from spaces X and X'. More precisely, there are null sets N and N' such that $\varphi : X \backslash N \to X' \backslash N'$ is measurable and measure–preserving. In addition, if this map is invertible and its inverse is a quotient map, then φ is called an *isomorphism (mod 0)*.

All the “reasonable” measure spaces share the same property; namely each of them is a *Lebesgue space (standard probability space).* For example, a *Polish space*(separable completely metrizable) with the *Borel* sigma algebra (sigma algebra generated by the open sets) is a Lebesgue space. Rokhlin, in [62], introduced Lebesgue spaces based on an intrinsic definition related to separability and completeness properties and classified all of the Lebesgue spaces. We use this classification as the definition of Lebesgue spaces and refer the reader to [62] for more details.

Definition 1.1.1. Let X be the disjoint union of an interval $(0, a)$ and a countable set $\{a_1, a_2, \cdots, \}$. Endow X with the probability measure m such that the restriction of m to the interval is the Lebesgue measure and $m(a_1) + m(a_2) + \cdots = 1 - a$. A probability space is called a *Lebesgue space* if it is isomorphic to the probability space (X, m).

As corollary, all the *purely non-atomic* Lebesgue spaces (the probability measure is zero at each element) are isomorphic (mod 0) to the interval (0,1) equipped with the Lebesgue measure.

One of the fundamental features of Lebesgue spaces, to which we refer later, is the one-to-one correspondence between quotient maps and complete sub-sigma algebras:

Theorem 1.1.2 (Rokhlin). *For a Lebesgue space $(X, \mathcal{F}, m)$, there is a one-to-one correspondence between*

i) *quotient maps*

ii) *the complete sub-sigma algebras of $\mathcal{F}$.*

Another benefit of working with Lebesgue spaces is the existence of *conditional measures*, which provide a generalization of the *Fubini theorem* in the Lebesgue spaces.

Theorem 1.1.3 (Rokhlin). *Let $\varphi : (X, m) \to (Y, p)$ be a quotient map of Lebesgue spaces. There exists a system of probability measures $\{m_y\}$ for p-almost every point $y \in Y$ such that m_y is a probability measure on the fiber $\varphi^{-1}(y)$ such that*

$$m(A) = \int_Y m_y(A \cap \varphi^{-1}(y))dp(y)$$

for every measurable set A. The system of probability measure $\{m_y\}_{y \in Y}$ is called conditional measures associated with φ.

It is worth mentioning that if the set $C = \varphi^{-1}(y)$ has a positive probability in X, then the conditional probability m_y coincides with the classical conditional probability. More concretely,

$$m_y(A) = m(A|C) = \frac{m(A \cap C)}{m(C)} \, .$$

Lebesgue space properties imply the existence of a projective limit of a sequence of Lebesgue spaces, which is a generalization of *Kolmogorov's Consistency theorem* (existence of infinite products of Lebesgue spaces):

Theorem 1.1.4. *Let $\{(X_n, m_n)\}_{n \geqslant 1}$ be a sequence of Lebesgue spaces. If there are quotient maps $\varphi_{n+1} : (X_{n+1}, m_{n+1}) \to (X_n, m_n)$ for $n = 1, 2, \cdots$, then there exist a unique (mod 0)* projective limit *Lebesgue space (X, m) and quotient maps $\Psi_n : (X, m) \to (X_n, m_n)$ for $n = 1, 2, \cdots$ such that $\varphi_{n+1}\Psi_{n+1} = \Psi_n$, i.e., the diagram*

$$\begin{tikzcd}
(X, m) \arrow[r, "\Psi_{n+1}"] \arrow[d, "\Psi_n"] & (X_{n+1}, m_{n+1}) \arrow[d, "\varphi_{n+1}"] \\
(X_n, m_n) \arrow[r, "identity"] & (X_n, m_n)
\end{tikzcd}$$

is commutative.

1.2 Ergodic theory

Ergodic theory is a powerful tool that can be employed in different areas of mathematics. Thus, we will also take advantage of ergodic theorems.

Definition 1.2.1. Let T be a measurable and measure–preserving map from the probability space (X, m) to itself. The triple (X, m, T) is called *ergodic* if the only *invariant* sets are either null sets or full measure sets, where a measurable set A is called an invariant if $A = T^{-1}(A)$ almost everywhere.

One of the most important tools in ergodic theory is the *Kingman subadditive ergodic theorem* due to Kingman 1968:

Theorem 1.2.2. *[51] Let (X, m, T) be ergodic. If $\{f_n\}_{n\geqslant 1}$ is a sequence of positive integrable functions such that*

$$f_{n+k} \leqslant f_n + f_k(T^n)$$

m-almost everywhere, then $\frac{f_n}{n}$ converges to a constant in $L^1(X, m)$ and m-almost everywhere.

The Kingman subadditive ergodic theorem is a generalization of *Fekete's lemma* [20] and *Birkhoff's ergodic theorem*[5]. Although the original proof of the Kingman subadditive ergodic theorem relied on both of aforemention theorems, there are some proofs that do not employ Birkhof's ergodic theorem (see [46] for a recent proof).

Ergodicity can be studied in the context of group actions. Let G be a countable group that acts on the set X. The group G acts naturally on the space of measures by translations. More precisely, if (X, m) is a Lebesgue space, then

$$gm(A) = m(g^{-1}A)$$

for every measurable set A. If the null sets are preserved under the action of G, or in other words, for every $g \in G$ the probability measures gm and m are equivalent, $gm \sim m$, , then the measure m is called *quasi-invariant* under G.

A measurable subset A of X is called an invariant set under the action of G whenever $gA = A$ almost everywhere for every g in G. The action G on X is called ergodic if the invariant sets under the action of G are null sets or sets with full measure. Let the group $G = \mathbb{Z}$. Define the action $(n, x) = T^n(x)$ for a measure–preserving invertible map $T : X \to X$, then ergodicity of the action is the same as the ergodicity of (X, m, T).

It is possible to decompose any group action into "smaller" ergodic actions, which are called *ergodic components*, owing to the existence of conditional measures!:

Theorem 1.2.3. *[29] Let G act on the Lebesgue space (X, m) and m be a quasi-invariant under G. If φ is the quotient map associated with the completion of the sigma algebra generated with invariant sets, then almost every conditional measure associated with the quotient map φ is quasi-invariant and ergodic.*

Once again if $G = \mathbb{Z}$, then decomposition of this action implies the classical ergodic decomposition for a measure preserving map T, see [50] and [65].

1.3 Martingales

The theory of *martingales* was developed by Doob in [14]. Two of the useful theorems in this concept are:

- the *martingale convergence theorem* (Theorem 1.3.2), which shows that the limit of a certain sequence of functions exists,

- *Doob's optional stopping theorem* (Theorem 1.3.4), which provides the integrability of a certain measurable map with some special features.

In order to formulate these two useful theorems, we must first recall the definitions of martingales and *conditional expectation.* For additional details, we refer the reader to the notable book [16] by Doob.

Let $\mathcal{G}$ be a complete sub-sigma algebra of the Lebesgue space $(X, \mathcal{F}, m)$. Let the associated quotient map and the quotient space be φ and Y, respectably. Hence, there exist conditional measures $\{m_y\}_{y \in Y}$. The *conditional expectation*, with respect to the sub-sigma algebra $\mathcal{G}$, is defined as

$$E(f|\mathcal{G})(y) = \int_y f|_{\varphi^{-1}(y)} dm_y.$$

In other words, $E(f|\mathcal{G})$ is the unique (mod 0) $\mathcal{G}$–measurable function such that

$$\int_A E(f|\mathcal{G}) dm = \int_A f dm$$

for every measurable set A in $\mathcal{G}$.

Definition 1.3.1. A *filtration* is an increasing sequence of sub-sigma algebras $\{\mathcal{F}_n\}_{n \geqslant 0}$ of the Lebesgue space $(X, \mathcal{F}, m)$ such that $\bigcup_n \mathcal{F}_n = \mathcal{F}$. The sequence $\{(M_n, \mathcal{F}_n)\}_{n \geqslant 0}$ is called a *submartingale* whenever $\{\mathcal{F}_n\}_{n \geqslant 0}$ is a filtration and each M_n is an integrable function and measurable with respect to the sigma-algebra $\mathcal{F}_n$ such that

$$E(M_{n+1}|\mathcal{F}_n) \geqslant M_n \quad \text{a.e.}$$

Moreover, if $\{(-M_n, \mathcal{F}_n)\}_{n \geqslant 0}$ is also a submartingale, then $\{(M_n, \mathcal{F}_n)\}_{n \geqslant 0}$ is called a *martingale.*

Theorem 1.3.2 (Martingale convergence theorem). *Let the sequence* $\{(M_n, \mathcal{F}_n)\}_{n \geqslant 0}$ *be a martingale. If M_n's are uniformly bounded, then*

$$\lim_n M_n = M_\infty$$

exists in L^1 *and almost everywhere. Moreover,* $E(M_\infty|\mathcal{F}_n) = M_n$ *for each* $n = 0, 1, 2, \cdots$.

Definition 1.3.3. Let τ be a measurable map from a Lebesgue space $(X, \mathcal{F}, m)$ to natural numbers and $\{\mathcal{F}_n\}_{n\geqslant 0}$ be a filtration. We call τ a *Markov stopping time* if the set $\tau^{-1}(n)$ is measurable with respect to the sub-sigma algebra $\mathcal{F}_n$.

Theorem 1.3.4 (Doob's optional stopping theorem)**.** *Let* $\{(M_n, F_n)\}_{n\geqslant 0}$ *be a submartingale. If* τ *is a bounded Markov stopping time, then* $\int M_0 dm \leqslant \int M_\tau dm$*, where* $M_\tau(x) = M_{\tau(x)}(x)$ *for* x *in* X*.*

We remind the reader that these theorems are valid under weaker conditions, however these simple versions can satisfy our needs later.

Chapter 2

Random Walks

In 1963, Furstenberg [24] defined the Poisson boundary of a locally compact group G. His definition is based on continuous bounded harmonic functions on G. In these terms, the triviality of the Poisson boundary is equivalent to the absence of nonconstant bounded harmonic functions (the Liouville property). Later, an equivalent definition of the Poisson boundary appeared in the context of random walks on groups. The first studies of random walks on groups are attributed to [49], [17], [28] and [48]. It was in the early 1980s, however, that great progress in the study of random walks on groups and their Poisson boundaries was made via, for example, the work of Derriennic [11], Rosenblatt [63], and Kaimanovich-Vershik [41].

This chapter is devoted to establishing the definition of a random walk on a countable group and its Poisson boundary, while also describing some of the properties of the Poisson boundaries. For additional studies in this field, we refer the reader to [41], [39], Chapter 12 of [34], and [19], as well as the references therein.

2.1 Random walk and its path space

The *random walk* (G, μ) on a countable group G determined by a probability measure μ is the Markov chain on G with the transitions

$$g \overset{\mu(h)}{\rightsquigarrow} gh \ ,$$

i.e., the transition probabilities π_g are the translates

$$\pi_g = g\mu \ , \qquad g \in G \ , \tag{2.1.1}$$

of the measure μ. In other words, the probability of moving from element g to gh is $\mu(h)$.

We shall denote by $G^{\mathbb{Z}_+}$ the space of sample paths $\boldsymbol{g} = (g_0, g_1, \dots)$, and by

$$\vartriangle : \boldsymbol{g} = (g_n)_{n=0}^{\infty} \ \mapsto \ \vartriangle\boldsymbol{g} = (\vartriangle g_n)_{n=1}^{\infty} \ , \qquad \vartriangle g_n = g_{n-1}^{-1} g_n \ , \tag{2.1.2}$$

the *increment map* from $G^{\mathbb{Z}_+}$ to the *space of increments* $G^{\mathbb{N}}$. Conversely, any sample path (g_n) can be recovered from its initial position g_0 and the sequence of increments $h_n = \vartriangle g_n$ as

$$g_n = g_0 h_1 h_2 \dots h_n \ . \tag{2.1.3}$$

Any initial distribution θ on G determines the associated measure $\mathbf{P}_\theta$ on $G^{\mathbb{Z}_+}$ described as follows: We shall denote by C_h^n the set of the sample paths whose positions at time n is h,

$$C_h^n = \{\boldsymbol{g} \in G^{\mathbb{Z}_+} \ : \ g_n = h \ \},$$

and $C_{g_0,g_1,\cdots,g_n} = \bigcap_{i=0}^n C_{g_i}^i$. The sets $C_{g_0,g_1,\cdots,g_n}$ provide a basis for the probability space $(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ with

$$\mathbf{P}_\theta(C_{g_0,g_1,\cdots,g_n}) = \theta(g_0)\pi_{g_0}(g_1)\pi_{g_1}(g_2)\cdots\pi_{g_{n-1}}(g_n) \ .$$

In other words, the probability space $(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ is the projective limit of the Lebesgue spaces $(G^{n+1}, \mathbf{P}_\theta^{[n]})$, where $\mathbf{P}_\theta^{[n]}(g_0, g_1, \cdots, g_n) = \mathbf{P}_\theta(C_{g_0,g_1,\cdots,g_n})$. Therefore, the space of sample paths with the probability measure $\mathbf{P}_\theta$ is a Lebesgue space.

We shall denote by $\mathbf{P} = \mathbf{P}_{\delta_e}$ the probability measure on the path space corresponding to the initial distribution δ_e concentrated on the identity e of the group G. In the latter case the position of the random walk at time n is, by (2.1.3), just the product of n i.i.d. μ-distributed increments h_i, so that the increment map $\vartriangle$ (2.1.2) establishes an isomorphism between the space $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ and the space of increments $G^{\mathbb{N}}$ endowed with the Bernoulli measure $\mu^{\otimes\mathbb{N}}$.

Let us remind the reader that the convolution of the two probability measures μ and θ, denoted $\mu * \theta$, is the image of the product measure $\mu \otimes \theta$ under the map $(g, h) \to gh$, hence

$$\mu * \theta = \sum_h \mu(h) h\theta \ .$$

The *one-dimensional distribution* of the random walk $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ at the time n is the distribution of the random walk after n steps, which are the *n-fold convolution* of μ denoted by μ^{*n}:

$$\mathbf{P}(C_g^n) = \mathbf{P}\{\boldsymbol{g} \ : \ g_n = g\} = \sum_{h_1 \cdots h_n = g} \mu(h_1) \cdots \mu(h_n) = \mu^{*n}(g) \ .$$

We shall denote by $\operatorname{supp} \mu$ the support of the probability measure μ, the set of elements in G whose μ–probability is positive:

$$\operatorname{supp} \mu = \{g \in G \ : \ \mu(g) > 0\} \ .$$

We can observe that the the support of one-dimensional distributions of the random walk $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ at time n is $\operatorname{supp} \mu^{*n} = (\operatorname{supp} \mu)^n$. Also, we can see that $\bigcup_n \operatorname{supp} \mu^{*n}$ is the semigroup generated by the support of μ denoted by $\operatorname{sgr} \mu$. Some authors assume that μ is a *non-degenerate or generate* probability measure, which means that the semigroup generated by the probability measure μ coincides with the group G. Here, we do not assume the non-degeneracy condition, unless otherwise specified.

There are many examples of random walks on groups. One of the classical example is the *simple random walk* on $\mathbb{Z}^n$, which is due to Pólya [61]. For instance, in the two-dimensional case, the measure can be described as

$$\mu = \frac{1}{4}\big(\delta_{(0,-1)} + \delta_{(0,1)} + \delta_{(-1,0)} + \delta_{(1,0)}\big) \ .$$

A random walker has four possibilities, each with an equal probability 1/4, of walking on the $\mathbb{Z}^2$ lattice. This random walk is *recurrent*, which means that for almost every sample path, the random walker returns to the point zero.

Another classical example is the simple random walk on the free group F_2 with two generators a and b. More concretely, $\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$. Like the simple random walk on $\mathbb{Z}^2$, there are four possible options at each step with probability 1/4 for the random walker. Unlike the simple random walk on $\mathbb{Z}^2$, this random walk is not recurrent (the heuristics behind it is that at each step the random walker moves forward with probability 3/4). Kesten [48] posed a question regarding the classification of finitely generated groups that admit a recurrent random walk. This problem was solved by Varopoulos [70]:

Theorem 2.1.1 (Varopoulos)**.** *A finitely generated group G admits a recurrent random walk with respect to a non-degenerate probability measure if and only if G has either $\mathbb{Z}$ or $\mathbb{Z}^2$ as a finite index subgroup.*

Example 2.1.2 (Lamplighter groups)**.** Let

$$\mathrm{fun}(\mathbb{Z}^d, \mathbb{Z}_2) = \{f : \mathbb{Z}^d \to \mathbb{Z}_2 \ : \ f^{-1}(1) \text{ is at most finite}\} \ .$$

It is an additive group with the operation of pointwise addition mod 2. The element of the group $\mathrm{fun}(\mathbb{Z}^d, \mathbb{Z}_2)$ is called the *finite configurations on* $\mathbb{Z}^d$. One can imagine that there is a lamp attached to each element of the lattice $\mathbb{Z}^d$, and a configuration shows which lamps are on (at most finitely many of them) on the lattice of $\mathbb{Z}^d$. The group $\mathbb{Z}^d$ acts on $\mathrm{fun}(\mathbb{Z}^d, \mathbb{Z}_2)$ by shifts, i.e., for every $x, y \in \mathbb{Z}^d$ and finite configuration f,

$$xf(y) = f(y - x) \ .$$

The *lamplighter group* G_d is the *semidirect product* of $\mathbb{Z}^d$ with the group $\mathrm{funn}(\mathbb{Z}^d, \mathbb{Z}_2)$ with respect to shifts. In other words, (x, f) are elements of $G_d = \mathbb{Z}^d \leftthreetimes \mathrm{fun}(\mathbb{Z}^d, \mathbb{Z}_2)$ for each $x \in \mathbb{Z}^d$ and $f \in \mathrm{fun}(\mathbb{Z}^d, \mathbb{Z}_2)$ with the following group operation:

$$(x, f)(x', f') = (x + x', f + xf') \ .$$

Let z_i be the standard generators of $\mathbb{Z}^d$ for $i = \pm 1, \cdots, \pm d$. Define the configuration $\mathbf{0}(x) = 0$, i.e., all the lamps are off. It is easy to see that the elements (e_d, δ_{e_d})

and $(z_i, \mathbf{0})$ for $i = \pm 1, \cdots, \pm d$, where e_d is the identity element of the lattice $\mathbb{Z}^d$, are generators of the lamplighter group G_d. Let μ be equally distributed on the generators of G_d. The simple random walk (G_d, μ) can be interpreted as a random walker at the position x on the lattice with the configuration f moves to either one of the neighbours of x without changing the current configuration or it stays at the same position and changes the configuration only at the position of x.

Random walks on lamplighter groups, introduced by Kaimanovich and Vershik [41], have interesting and surprising behaviors. Lamplighter groups are special case of *wreath products* and much research has been devoted to these groups, especially the behavior of their random walks, see [60], [18], [64].

These are all examples of simple random walks. A simple random walk is a random walk on a finitely generated group associated to the probability measure that is equally distributed on the generators (including their inverses). A *Cayley graph* can be associated with a finitely generated group. The vertices of this graph are the elements of the group and there is an edge between the vertices v and w whenever vw^{-1} is in the generating set (equivalently, wv^{-1}). The simple random walk on a finitely generated group means that a random walker moves from one vertex to its neighbours on the Cayley graph with the same probability. Additional details regarding random walks on graphs are found in the book [73] by Woess.

2.2 The Poisson boundary

We shall denote by U and T the time shifts on the spaces $G^{\mathbb{N}}$ and $G^{\mathbb{Z}_+}$, respectively:

$$(U\boldsymbol{h})_n = h_{n+1} \ , \qquad (T\boldsymbol{g})_n = g_{n+1} \ . \tag{2.2.1}$$

Obviously, the diagram

$$\begin{tikzcd}
G^{\mathbb{Z}_+} \arrow[r, "T"] \arrow[d, "\Delta"] & G^{\mathbb{Z}_+} \arrow[d, "\Delta"] \\
G^{\mathbb{N}} \arrow[r, "U"] & G^{\mathbb{N}}
\end{tikzcd}$$

is commutative. Note that U is a *Bernoulli shift* that preserves the measure $\mu^{\otimes \mathbb{N}}$ and is ergodic with respect to this measure (by Kolmogorov 0-1 law), whereas T does *not* preserve the measure $\mathbf{P}$, nor its class. However, the shift T preserves the σ-finite measure $\mathbf{P}_{\#}$ on the path space whose initial distribution is the counting measure $\#$ on G.

Proposition 2.2.1. *Let θ be a probability measure on group G. Then $T\mathbf{P}_\theta = \mathbf{P}_{\theta * \mu}$.*

Proof: Let θ be a probability measure on group G. For the measurable set $C_{g_0,g_1,\cdot,g_n}$, we have $T^{-1}(C_{g_0,g_1,\cdot,g_n}) = \bigcup_{g \in G} C_{g,g_0,g_1,\cdot,g_n}$. Hence,

$$\mathbf{P}_\theta(T^{-1}(C_{g_0,g_1,\cdot,g_n})) = \sum_{g \in G} \theta(g)\mu(g^{-1}g_0)\mu(g_0^{-1}g_1)\cdots\mu(g_{n-1}^{-1}g_n).$$

The definition of convolution implies that

$$\mathbf{P}_\theta(T^{-1}(C_{g_0,g_1,\cdot,g_n})) = (\theta * \mu)(g_0)\mu(g_0^{-1}g_1)\cdots\mu(g_{n-1}^{-1}g_n) = \mathbf{P}_{\theta*\mu}(C_{g_0,g_1,\cdot,g_n})\ .$$

Because $C_{g_0,g_1,\cdots,g_n}$ are elements of a basis for the $\left(G^{\mathbb{Z}_+}, \mathbf{P}\right)$, we have $T\mathbf{P}_\theta = \mathbf{P}_{\theta*\mu}$. ▮

Proposition 2.2.2. *If there is a probability measure θ such that $\mathbf{P}_\theta$ is preserved under the time shift T, then the support of μ generates a finite semigroup, i.e.,* $\operatorname{sgr}\mu$ *is a finite set.*

Proof: Assume there exists θ such that $T\mathbf{P}_\theta = \mathbf{P}_\theta$. By Proposition 2.2.1, $\theta = \theta * \mu$. Since $\sum_g \theta(g) = 1$, the probability measure θ must take its maximum at some h_1 in

G. We have

$$\theta(h_1) = (\theta * \mu)(h_1) = \sum_g \theta(h_1 g^{-1})\mu(g) \ .$$

If there exists g in $\operatorname{supp}\mu$ such that $\theta(h_1 g^{-1}) < \theta(h_1)$, then

$$\theta(h_1) < \sum_g \theta(h_1 g^{-1})\mu(g) \leqslant \theta(h_1) \ .$$

Therefore, $\theta(h_1) = \theta(h_1 g^{-1})$ for every g in the support of μ. We can conclude that

$$\theta(h_1) = \theta(h_1 g^{-1}), \ \forall g \in \bigcup_n \operatorname{supp}\mu^{*n},$$

because $\theta = \theta * \mu^{*n}$. On the other hand, θ is a probability measure; therefore, $\bigcup_n \operatorname{supp}\mu^{*n}$ must be finite.

∎

Definition 2.2.3. The *Poisson boundary* $\partial_\mu G$ of the random walk (G, μ) is the space of ergodic components of the time shift T on the space $\big(G^{\mathbb{Z}_+}, \mathbf{P}_\#\big)$, and any initial distribution θ on G determines the corresponding *harmonic measure* ν_θ on $\partial_\mu G$.

In other words, let $\mathcal{A}_T$ be the sigma-algebra of all T-invariant sets (mod 0) in the path space $\big(G^{\mathbb{Z}_+}, \mathbf{P}_\#\big)$, which is the sigma algebra generated with the *orbit equivalence relation* $\sim$ of the shift T on the path space $G^{\mathbb{Z}_+}$,

$$\boldsymbol{g} \sim \boldsymbol{g}' \Leftrightarrow \exists \ m, n \ s.t \ T^n \boldsymbol{g} = T^m \boldsymbol{g}'.$$

According to the Rokhlin correspondence theorem (Theorem 1.1.3), there exist the unique (up to an isomorphism) quotient space $\partial_\mu G$ and the quotient map

$$\mathbf{bnd} : G^{\mathbb{Z}_+} \to \partial_\mu G$$

such that $\mathcal{A}_T$ coincides with the preimage of the measurable sets of $\partial_\mu G$ under the quotient map **bnd**. For an arbitrary initial distribution θ, the harmonic measure is $\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta)$. We denote that $\nu = \nu_{\delta_e}$.

Let us remind the reader that the equivalence between two measures means that both measures have the same null sets. The class of all measures such that ν_θ is equivalent to the measure $\nu_\#$ is called *harmonic measure type.*

Since $\partial_\mu G$ is the space of ergodic components, we have

$$\nu_\theta = \mathbf{bnd}(T\mathbf{P}_\theta) = \mathbf{bnd}(\mathbf{P}_{\theta*\mu}) = \nu_{\theta*\mu} \ . \tag{2.2.2}$$

Consider the coordinate-wise action $g(g_n) = (gg_n)$ on the path space. Because this action commutes with the shift T and preserves the $\sim$-equivalence classes, it induces an action on the Poisson boundary and

$$\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(\theta * \mathbf{P}) = \theta * \mathbf{bnd}(\mathbf{P}) = \theta * \nu \ . \tag{2.2.3}$$

Combining the equations (2.2.2) and (2.2.3) for $\theta = \delta_e$, implies that the harmonic measure ν is *μ-stationary* with respect to this action:

$$\nu = \nu_\mu = \mu * \nu = \sum_g \mu(g)\, g\nu = \nu \ .$$

The stationarity of the harmonic measure ν implies that whenever ν is zero, its translation $g\nu$ is also zero for every g in the support of μ. Iterating the stationarity formula yields to $\mu^{*n} * \nu = \nu$, provided us with the following corollary:

Corollary 2.2.4. *The harmonic measure $g\nu$ is absolutely continuous with respect to the harmonic measure ν for every $g \in \operatorname{sgr} \mu$.*

Although the harmonic measure ν need not be quasi-invariant(if the semigroup generated by $\operatorname{supp} \mu$ is smaller than G; see Remark 2.4.6), it is the case whenever μ is a non-degenerate probability measure:

Proposition 2.2.5. *Let μ be a non-degenerate probability measure, i.e.,* $\operatorname{sgr}\mu = G$. *Then, the harmonic measure ν is quasi-invariant and belongs to the class of harmonic measure types.*

Proof: Let $g\nu(A) = \nu(g^{-1}A) = 0$ for a measurable subset of the Poisson boundary. Since $\operatorname{sgr}\mu = G$, we can conclude by Corollary 2.2.4 $\nu(A) = g^{-1}\nu(g^{-1}A) = 0$, which means that ν is absolutely continuous with respect to $g\nu$ as well. We have

$$\mathbf{P}_{\mu^{*n}} = \mu^{*n} * \mathbf{P} = \sum_{g \in \operatorname{supp}\mu^{*n}} \mu^{*n}(g)\delta_g * \mathbf{P} \,. \tag{2.2.4}$$

Let A be a measurable subset of $\partial_\mu G$ such that $\nu(A) = 0$. The stationarity of the probability measure ν implies that $\nu(A) = \nu_{\mu^{*n}}(A) = 0$ for every $n = 1, 2, \cdots$. The combination of the equation 2.2.4 and $\nu_{\mu^{*n}} = \mathbf{bnd}\,\mathbf{P}_{\mu^{*n}}$ implies that $\delta_g * \mathbf{P}(\mathbf{bnd}^{-1} A) = 0$ for every $g \in \operatorname{supp}\mu^{*n}$. Since $\bigcup_{n \in \operatorname{supp}\mu^{*n}} = G$, we have $\delta_g * \mathbf{P}(\mathbf{bnd}^{-1} A) = 0$ for every g in G; therefore, $\nu_{\#}(A) = \mathbf{P}_{\#}(\mathbf{bnd}^{-1} A) = 0$. Consequently, $g\nu$ is a harmonic measure type for any g in G. Hence, ν is quasi-invariant under the action. ∎

Below, when talking about the Poisson boundary $\partial_\mu G$, we (unless otherwise specified) shall endow it with the harmonic measure ν of the initial distribution δ_e. In other words, ν is the image of the measure $\mathbf{P} \prec \mathbf{P}_{\#}$ on the path space under the quotient map.

Proposition 2.2.6. *The action of* $\operatorname{sgr}\mu$ *on the Poisson boundary is ergodic.*

Proof: Let A be an invariant subset of the Poisson boundary under the action $\operatorname{sgr}\mu$ with positive probability measure. Markov property implies that for every set $C_{e,g_1,\cdots,g_n}$,

$$\mathbf{P}(C_{e,g_1,\cdots,g_n} | \,\mathbf{bnd}(A^{-1})) = \mathbf{P}(C_{e,g_1,\cdots,g_n})\mathbf{P}(g_n^{-1}\,\mathbf{bnd}(A^{-1})) = \mathbf{P}(C_{e,g_1,\cdots,g_n})g_n\nu(A) \,. \tag{2.2.5}$$

Since A is an invariant set under the action, we have

$$\mathbf{P}(C_{e,g_1,\cdots,g_n} | \overset{-1}{\mathbf{bnd}}(A)) = \mathbf{P}(C_{e,g_1,\cdots,g_n})\nu(A) \ ,$$

which implies that the sets $\mathbf{bnd}(A^{-1})$ and $C_{e,g_1,\cdots,g_n}$ are independent. Since $C_{e,g_1,\cdots,g_n}$ are bases for the sigma-algebra on the path space, $\mathbf{bnd}(A^{-1})$ and any measurable set of path space is independent. Hence, the set A must be independent of itself, which implies that the probability measure A is either zero or one. ▮

Remark 2.2.7. Define two sample paths $\boldsymbol{g}$ and $\boldsymbol{g}'$ to be $\simeq$–equivalent, whenever there is an integer n such that $T^n(\boldsymbol{g}) = T^n(\boldsymbol{g}')$. If we replace $\sim$ by $\simeq$ in the construction of the Poisson boundary, the Lebesgue space associated with this equivalence relation is called the *tail boundary*. By employing *the Foguel 0-2 law*[21], Derriennic [13] showed that the tail boundary coincides with the Poisson boundary of random walks issued from the identity. See [38] for the relation of the Poisson boundary and the tail boundary for other initial distributions.

Definition 2.2.8. A function f on G is called *μ-harmonic*, if it is preserved by the *Markov operator of the random walk* (G, μ)

$$P_\mu f(g) = \langle f, \pi_g \rangle = \sum_h \mu(h) f(gh) \ .$$

The classic harmonic function on the Euclidean plane satisfies the Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ .$$

The classic harmonic functions have several special properties, such as *the maximum principle* and *the mean value property.* These properties have their analog in the framework of random walks on groups.

Proposition 2.2.9 (Maximum principle)**.** *Assume* $\operatorname{sgr}\mu = G$. *Let f be a μ–harmonic function. If f takes its maximum on G, then f is constant.*

Proof: Let f take its maximum at h. If $f(h) > f(hg)$ for some $g \in \operatorname{supp}\mu$, then

$$f(h) = \sum_{g \in \operatorname{supp}\mu} f(hg)\mu(g) < f(h).$$

Therefore, $f(h) = f(hg)$. Since every μ–harmonic function is a μ^{*n}–harmonic function, we can conclude that $f(h) = f(hg)$ for every $g \in \operatorname{sgr}\mu$. ∎

Theorem 2.2.10 (*Poisson formula*)**.**

$$f(g) = \langle \widehat{f}, g\nu \rangle \,, \qquad g \in G \,, \tag{2.2.6}$$

establishes an isometric isomorphism between the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions f on the group G and the L^∞ space on the boundary $\partial_\mu G$ with respect to the quotient measure class determined by the measure $\mathbf{P}_\#$, where $\lim_n f(g_n) = \widehat{f}(\mathbf{bnd}\,\boldsymbol{g})$ *for almost every sample path $\boldsymbol{g} = (g_n)$.*

Proof: The proof relies on the martingale convergence theorem, Theorem 1.3.2. Let $\mathcal{A}_n$ be the sigma algebra generated by the first n position of the random walk (G,μ). Then, it is easy to see that for a bounded function f, $M_n(\boldsymbol{g}) = f(g_n)$ is a martingale with respect to the filteration $\mathcal{A}_n$ if and only if f is a μ-harmonic. Now the martingale convergence theorem (Theorem 1.3.2) implies the Poisson formula. ∎

If g in formula (2.2.6) is only allowed to take values in the semigroup $\operatorname{sgr}\mu$, then (2.2.6) becomes an isometric isomorphism between the space of bounded μ-harmonic functions on $\operatorname{sgr}\mu$ and $L^\infty(\partial_\mu G, \nu)$. This property uniquely characterizes the Poisson boundary, namely, any G-space (B, λ), for which formula (2.2.6) is an isometric isomorphism between $H^\infty(\operatorname{sgr}\mu, \mu)$ and $L^\infty(B, \lambda)$, is isomorphic to the Poisson boundary $(\partial_\mu G, \nu)$.

Corollary 2.2.11. *The Poisson boundary of the random walk* (G, μ) *is trivial if and only if the* μ*–harmonic functions on* $\operatorname{sgr}\mu$ *are constant.*

Proof: Let the Poisson boundary be trivial, i.e., $\partial_\mu G = \{x\}$ and $\nu = \delta_x$. The Poisson formula implies that $f(g) = \langle \hat{f}, g\delta_x\rangle = \hat{f}(x)$, so f is a constant. Now, let every μ–harmonic function be constant. Let A be a measurable subset of the Poisson boundary of (G, μ). By the Poisson formula, $f(g) = \langle 1_A, g\nu\rangle = g\nu(A)$ is a μ–harmonic function, so it must be constant and $\nu(A) = g\nu(A)$ for every $g \in \operatorname{sgr}\mu$. By equation (2.2.5), for every set $C_{e,g_1,\cdots,g_n}$, we have

$$\mathbf{P}(C_{e,g_1,\cdots,g_n} | \overset{-1}{\mathbf{bnd}} A) = \mathbf{P}(C_{e,g_1,\cdots,g_n})\nu(A) .$$

Like the argument of ergodicity of the action on the Poisson boundary, $\mathbf{bnd}^{-1} A$ is independent of itself; therefore, $\nu(A)$ is either zero or one. Hence, the Poisson boundary is trivial.

∎

Corollary 2.2.12. *The Poisson boundary of the random walk* $(G\,\mu)$ *is either purely non-atomic or trivial.*

Proof: Let the Poisson boundary have some atoms. Assume ν takes its maximum at atom x. By the Poisson formula, $f(g) = \langle 1_x, g\nu\rangle = g\nu(x)$ is μ–harmonic and takes its maximum at the identity element of the group G. By the maximum principle, the μ–harmonic function f must be constant, which implies that the set

$$A = \{y \in \partial_\mu G \;:\; \nu(x) = \nu(y)\}$$

is invariant. By the ergodicity of the action, $\nu(A)$ is either zero or one. Let $\nu(A)$ be one, hence the Poisson boundary is the finite set

$$A = \partial_\mu G = \{x_1, \cdots, x_n\},$$

and $\nu(x_1) = \cdots \nu(x_n) = \frac{1}{n}$. Because

$$f(g) = \langle \hat{f}, g\nu \rangle = \frac{\hat{f}(x_1) + \cdots + \hat{f}(x_n)}{n}$$

is constant for the bounded μ–harmonic function f. Hence, the Poisson boundary must be trivial. ▮

We have shown that a non-trivial Poisson boundary must be isomorphic (as measure spaces) to the interval $(0, 1)$ equipped with the Lebesgue measure.

2.3 Conditional Random Walks

As mentioned in the first chapter, there exists the system of conditional measures with respect to a quotient map, and we will find these conditional measures with respect to the **bnd** , see [39]. Conditional random walks are one of the tools that help us to identify the Poisson boundary of a random walk.

In order to find conditional measures, we need to employ the concept of conditional random walks introduced by Doob under the name of *h–process* [15]. In current literature, this method is referred to as *Doob's transformations*.

Consider the random walk (G, μ) with a positive function f on $\operatorname{sgr}\mu$. The aim is to rescale the transition probabilities by using f. Define

$$\pi_g^f(h) = \pi_g(h)\frac{f(h)}{f(g)} \, .$$

We have

$$\sum_h \pi_g^f(h) = \frac{1}{f(g)} \sum_h \pi_g(h) f(h) = \frac{1}{f(g)} P_\mu f(h)$$

are transition probabilities if and only if f is a μ–harmonic function as well. Denote by $\mathbf{P}^f$ the probability measure on the path space of the transition probabilities π^f with the initial distribution δ_e, thus we can write

$$\mathbf{P}^f(C_{e,g_1,g_2\cdots,g_n}) = \mathbf{P}(C_{e,g_1,g_2\cdots,g_n}) \frac{f(g_n)}{f(e)} \ .$$

We would like conditioning the random walk on a measurable subset A of the Poisson boundary with positive ν–probability. By the equation 2.2.5,

$$\mathbf{P}(C_{e,g_1,g_2\cdots,g_n} | \overset{-1}{\mathbf{bnd}} A) = \mathbf{P}(C_{e,g_1,g_2\cdots,g_n}) \frac{g_n\nu(A)}{\nu(A)}$$

which means that the conditional random walk by the set A is Doob's transformation for the μ–harmonic function $f(g) = \langle f, g\nu \rangle$. By Corollary 2.2.4, for each g in the $\operatorname{sgr}\mu$ the probability measure $g\nu$ is absolutely continuous to the probability measure ν, the Radon-Nykodim derivative $\frac{dg\nu}{d\nu}$ exists. For x in the Poisson boundary of (G, μ), define

$$\mathbf{P}^x(C_{e,g_1,g_2\cdots,g_n}) = \mathbf{P}(C_{e,g_1,g_2\cdots,g_n}) \frac{dg_n\nu}{d\nu}(x) \ .$$

By combining the preceding equation and the fact that

$$g\nu(A) = \int_A \frac{dg\nu}{d\nu}(x) d\nu(x) \ ,$$

we can write that

$$\mathbf{P}^A = \frac{1}{\nu(A)} \int_A \mathbf{P}^x d\nu(x).$$

It means that $\{\mathbf{P}^x\}_{x\in A}$ are the conditional measures of $\mathbf{P}^A$. By choosing $A = \partial_\mu G$, we conclude that

$$\mathbf{P} = \int \mathbf{P}^x d\nu(x)$$

and $\{\mathbf{P}^x\}$ are the conditional measures of the probability measure $\mathbf{P}$ with respect to the Poisson boundary.

2.4 Examples

Here, we give some examples of the Poisson boundaries of different groups. We will see how the algebraic structures of a group are related to the Poisson boundary.

One of the first studies to describe the Poisson boundary of a random walk is ascribed to Blackwell [8], who showed that the Poisson boundary of an abelian group is trivial. This result was also discovered independently by Choquet and Deny [10]. In classical harmonic analysis, the analog of this result is called *Liouville property in* $\mathbb{R}^n$ that was shown by Nelson[59].

Theorem 2.4.1. *[Blackwell, Choquet-Deny] The Poisson boundary of an abelian group is trivial.*

Proof: Let $\mu(h) > 0$, and $E_n = \{(h_1, h_2, \cdots) \in G^{\mathbb{N}} \ : \ h_n = h\}$. Then we have $\mu^{\bigotimes \mathbb{N}}(E_n) = \mu(h)$. Because the sets of E_n are independent,

$$\mu^{\bigotimes \mathbb{N}}(\bigcap_{n=1}^{k} E_n^c) = (1 - \mu(h))^k \ .$$

It implies that $\lim_k \mu^{\bigotimes \mathbb{N}}(\bigcap_{n=1}^{k} E_n^c) = 0$, and consequently,

$$\mu^{\bigotimes \mathbb{N}}\{\text{infnitly many of } E_n \text{ occur}\} = 1 \ .$$

This means that the property of $g_{n+1} = g_n h$ occurs infinitely many times for almost every sample path (g_n). We can conclude that there exists a subsequence of (g_n) such that $\boldsymbol{g} = (g_{n_i})$, we have $g_{n_i+1} = g_{n_i} h$. Let f be a bounded harmonic function on $\operatorname{sgr} \mu$. By the Poisson formula and the fact that the group G is abelian, we can write

$$\hat{f}(\mathbf{bnd}\, \boldsymbol{g}) = \lim f(g_{n_{i+1}}) = \lim f(g_{n_i} h) = \lim f(h g_{n_i}) = h^{-1} \hat{f}(\mathbf{bnd}\, \boldsymbol{g})$$

for every $\hat{f} \in L^\infty(\partial_\mu G, \nu)$ and every $h \in \operatorname{supp} \mu$. We have

$$f(h) = \langle \hat{f}, h\nu \rangle = \langle h^{-1} \hat{f}, \nu \rangle = f(e) \ ,$$

for every $h \in \operatorname{supp} \mu$. Since f is a μ^{*n}–harmonic function as well, $f(h)$ is constant for every $h \in \operatorname{sgr} \mu$. By Corollary 2.2.11, the Poisson boundary is trivial. ∎

The same result can be extended to the *nilpotent groups*. First, we need to show a relation between a random walk on a group and its induced random walk on a quotient group.

Theorem 2.4.2. *Let N be a normal subgroup of G and $\check{\mu}$ be the image measure of μ with respect to the quotient map $G \to G/N$. Then the $H^\infty(G/N, \check{\mu})$ is isomorphic to the space of all μ–harmonic functions that are N–invariant. In other words, the Poisson boundary of $(G/N, \check{\mu})$ is the space of N-ergodic components of $(\partial_\mu G, \nu)$.*

Proof: Notice that

$$\check{\mu}(gN) = \sum_{n \in N} \mu(gn), \ \check{\pi}_{gN}(hN) = \sum_{n \in N} \pi_g(hn) \ .$$

Function f is an N-invariant function (which means $f(gn) = f(g)$ for every $n \in N$) if and only if

$$\check{f}(gN) = f(g)$$

is well defined on the quotient group G/N . The partition $G = \bigsqcup_i a_i N$ allows us to change the order of summations: firstly, we take the summations over the elements of $a_i N$, and then we add up all the summations from the first part. So, we can have

$$\sum_h f(h)\pi_{g_j}(h) = \sum_{a_i N} \sum_{n \in N} f(a_i n)\pi_{a_j}(a_i n) = \sum_{a_i N} f(a_i) \sum_{n \in N} \pi_{a_j}(a_i n) = \sum_{a_i N} \check{f}(a_i N)\check{\pi}_{a_i N}(a_j N) \ .$$

We have shown that $\langle f, \pi_g \rangle = \langle \check{f}, \check{\pi}_{gN} \rangle$ whenever f is an N-invariant function, thereby proving our claim. ∎

Remark 2.4.3. Bowen studied induced random walks and their Poisson boundaries for the cases where N is a more general subgroup [9].

By combining Theorem 2.4.1 and Theorem 2.4.2, we can show that the Poisson boundary of any nilpotent group is trivial due to Dynkin and Maljutov in [17].

Theorem 2.4.4 (Dynkin-Maljutov)**.** *[17] Any random walk on a nilpotent group has trivial Poisson boundary. Or equivalently, any bounded harmonic function is constant.*

Note that the preceding theorem does not hold if we drop the "boundedness condition". For example, consider the simple random walk on $\mathbb{Z}$. Then, the function f is harmonic if and only if

$$2f(n) = f(n-1) + f(n+1) \ .$$

Let $f(n) = n$, then obviously, f is a non-constant harmonic function with respect to the simple random walk, but it is unbounded. A generalization of this result regarding positive harmonic functions (not necessarily bounded) was discussed by Margulis in [56].

Lemma 2.4.5. *Let $\mathcal{F} =< w_i : i = 1, 2, \cdots >$ be a free group where $W = \{w_1, w_2, \cdots\}$ is the set of free independent generators of $\mathcal{F}$. More precisely, for each $w \in W$, the intersection of the group generated by w (which is isomorphic as a group to the group $\mathbb{Z}$) with W is only w. Let μ be a probability measure distributed on the free independent generators of $\mathcal{F}$, i.e.,*

$$\mu = \sum_i a_i \delta_{w_i} \ .$$

Then the Poisson boundary of the random walk $(\mathcal{F}, \mu)$ is the space of increments of the random walk $(\mathcal{F}, \mu)$.

Proof: Let $\boldsymbol{g} = (e, g_1, g_2, \cdots)$ be a sample path. Obviously, $g_1 = w_{i1}$ for some integer $i1$. We have $g_2 = w_{i1}w_{i2}$ for a natural number $i2$. Since both w_{i1} and w_{i2} are generators of the free group and they are not the inverse of each other, g_2 is a reduced word whose length is two. By induction, the length of g_n is n and, moreover, $g_n = w_{i1} \cdots w_{in}$ if and only if $g_{n-1} = w_{i1} \cdots w_{i(n-1)}$. In other words, the position at time n uniquely describes all the steps of the random walk before time n. Therefore, two sample paths $\boldsymbol{g}$ and $\boldsymbol{g}'$ are $\sim$–equivalent, if and only if $\boldsymbol{g} = \boldsymbol{g}'$. ▮

Remark 2.4.6. In the pervious Lemma, let $\mathcal{F}_{w_1}$ be the set of all infinite reduced words starting with w_1:

$$\mathcal{F}_{w_1} = \{(w_1, h_2, h_3, \cdots) \ : \ h_i \in W \ i = 1, 2, \cdots\} .$$

We have $\nu(\mathcal{F}_{w_1}) = \mu(w_1) > 0$, however, $w_2\nu(\mathcal{F}_{w_1}) = 0$. It shows that ν is not absolutely continuous with respect to $w_2\nu$; consequently, ν is not a harmonic measure type.

Example 2.4.7. The Poisson boundary of the simple random walk on $\mathcal{F}_2$ is the space of all infinite reduced words equipped with the *uniform measure* m_2. More concretely, let $\mathcal{F}_w$ be the set of all infinite reduced words starting with the finite reduced word. If $w = w_1 \cdots w_{n+1}$, then

$$m_2(\mathcal{F}_w) = \frac{1}{(4)(3)^n} .$$

The identification of the Poisson boundary of an arbitrary random walk on a free group is an unsolved problem. If we restrict the random walks with some "moment" conditions, then the Poisson boundary is well studied. For example, if the probability measure μ has finite support, then the Poisson boundary associated with μ is the space of all reduced infinite words and the harmonic measure is the unique stationary measure, for more details see [54] and [39].

Remark 2.4.8. Free groups are examples of *non-amenable groups.* It has been shown that every random walk on non-amenable group associated with a non-degenerate probability measure has non-trivial Poisson boundary. In contrast to the amenability of the lamplighter groups, the simple random walk on the lamplighter group G_3 has non-trivial Poisson boundary. Every amenable group admits a non-degenerating probability measure whose random walk has trivial Poisson boundary, see for example [41] and [63].

Chapter 3

Transformed Random Walks

This chapter is devoted to study of transformation of random walks. For a given random walk, we apply a Markov stopping time to construct a new random walk in such a way that both random walks share the same Poisson boundaries. The results of this section are a joint work with V. Kaimanovich in [23].

3.1 Comparison of Poisson boundaries

Let first clarify what it means two random walks have the same Poisson boundaries.

Since the Poisson boundary of a random walk is a measure category object defined as a quotient of the corresponding path space, *a priori* there is no way to identify Poisson boundaries of two different measures μ, μ' on the same group G other than saying that both of them can be realized on the same measure G-space.

Definition 3.1.1. The Poisson boundaries of two measures μ, μ' on the same group G are *equivalent* if there exists a G-space B endowed with an ergodic quasi-invariant measure θ, and two measures λ, λ' absolutely continuous with respect to θ such that the spaces (B, λ) and (B, λ') are, respectively, isomorphic to the Poisson boundaries of the random walks (G, μ) and (G, μ'). If $\lambda = \lambda'$, i.e., both Poisson boundaries can be realized on the same measure G-space, we shall say that the Poisson boundaries of the measures μ and μ' *coincide*.

Example 3.1.2. For any probability measure μ on a group G and any $g \in G$ the Poisson boundaries of the measures μ and $\mu' = g\mu g^{-1}$ are equivalent. The heuristic behind this claim is that the harmonic measure ν of the random walk (G, μ) should be interpreted as a "limit" of the sequence of convolution powers μ^{*n} endowed with the left action of the group G. Therefore, the harmonic measure ν' of the random walk (G, μ') should be equal to $g\nu$. On a more formal level it follows, for instance, from the fact that a function f on G is μ-harmonic if and only if the function $f'(x) = f(xg)$

is μ'-harmonic. If f is bounded, then by the Poisson formula for the random walk (G, μ) it corresponds to a function $\widehat{f} \in L^\infty(\partial_\mu G, \nu)$, whence

$$f'(x) = f(xg) = \langle \widehat{f}, xg\nu \rangle = \langle \widehat{f}, x\nu' \rangle ,$$

so that the space $(\partial_\mu G, \nu')$ provides the Poisson representation for the random walk (G, μ').

Remark 3.1.3. The above example shows that *inner automorphisms of a group G do not change the class of the Poisson boundary of any measure μ.* On the other hand, this is no longer true for *outer* automorphisms of G. The simplest example is provided by the free group $\mathcal{F}_d$ with d generators endowed with the measure μ equidistributed on the set K of generators and their inverses. In this case the Poisson boundary is classically known to be the topological boundary of the free group endowed with the probability measure m_K *uniform* with respect to K (e.g., see Example 2.4.7 and [54]). Now, the measures $m_K, m_{K'}$ corresponding to two different generating sets are mutually singular unless K' can be obtained by K by an inner automorphism of $\mathcal{F}_d$ (e.g., see [45]).

Problem 3.1.1. It would be interesting to investigate the behaviour of the Poisson boundary under outer automorphisms for other measures and groups.

The situation described in Example 3.1.2 is a particular case of a more general phenomenon.

Theorem 3.1.4. *If two measures μ, μ' on a group G can be decomposed as convolution products*

$$\mu = \mu_1 * \mu_2 \qquad \text{and} \qquad \mu' = \mu_2 * \mu_1 , \tag{3.1.2}$$

then the Poisson boundaries of the measures μ and μ' are equivalent.

Proof: This is essentially the argument outlined in Example 3.1.2. By (3.1.2) the Markov operators associated to the measures μ and μ' decompose as

$$P_\mu = P_{\mu_1} P_{\mu_2} \qquad \text{and} \qquad P_{\mu'} = P_{\mu_2} P_{\mu_1} \ ,$$

whence the operators P_{μ_1}, P_{μ_2} establish an isometric isomorphism of the spaces of bounded harmonic functions $H^\infty(G,\mu)$ and $H^\infty(G,\mu')$ as

$$P_{\mu_2} : H^\infty(G,\mu) \to H^\infty(G,\mu') \ , \qquad P_{\mu_1} : H^\infty(G,\mu') \to H^\infty(G,\mu) \ .$$

Therefore, if $f' = P_{\mu_2} f \in H^\infty(G,\mu')$, then by the Poisson formula for the random walk (G,μ)

$$f'(g) = P_{\mu_2} f(g) = \sum_h \mu_2(h)\, f(gh) = \sum_h \mu_2(h) \langle \widehat{f}, gh\nu \rangle = \langle \widehat{f}, g\mu_2 * \nu \rangle \ ,$$

so that the Poisson boundary $\partial_\mu G$ endowed with the measure $\nu' = \mu_2 * \nu$ provides the Poisson representation for the space $H^\infty(G,\mu')$, whence the claim. ∎

We are not aware of any other constructions or examples which would provide equivalent but not coinciding Poisson boundaries (however, see Introduction for all previously known examples of coincidence). We shall now proceed with describing our general method for obtaining random walks on the same group with the same Poisson boundary.

3.2 The random walk determined by a Markov stopping time

3.2.1 Markov stopping times

Let us first remind that, according to a classical definition (e.g., see [57]), a *Markov stopping time* τ is a $\mathbb{Z}_+$-valued function τ on the path space of a Markov chain with the property that any preimage set $\tau^{-1}(n)\,, n \in \mathbb{Z}_+$, belongs to the sigma-algebra $\mathcal{A}_0^n$ generated by the positions of the chain between times 0 and n. [Unless otherwise specified, we only consider Markov stopping times which are almost surely finite.] Alternatively, a Markov stopping time τ is determined by a set T of finite length paths on the state space such that for a.e. sample path $(\xi_0, \xi_1, \dots)$ there exists a unique $\tau \geqslant 0$ (the Markov stopping time) with $(\xi_0, \xi_1, \dots, \xi_\tau) \in \mathsf{T}$.

Definition 3.2.1. For a Markov stopping time τ on the path space $\left(G^{\mathbb{Z}_+}, \mathbf{P}\right)$ of a random walk (G, μ) we shall denote by μ_τ the distribution of g_τ, i.e.,

$$\mu_\tau(g) = \mathbf{P}\{\boldsymbol{g} : g_\tau = g\}\ .$$

The random walk (G, μ_τ) is *the transformation of the random walk (G, μ) determined by the Markov stopping time τ.* By

$$\mathcal{T}(\mu) = \left\{\mu_\tau : \tau \text{ is a Markov stopping time on } \left(G^{\mathbb{Z}_+}, \mathbf{P}\right)\right\}$$

we denote the class of all probability measures on G which can be obtained in this way.

3.2.2 Composition and iteration of Markov stopping times

As we shall see in a moment, the random walk (G, μ_τ) is obtained from the original random walk (G, μ) by restricting it to the sequence of *iterations* of the Markov stopping time τ.

Indeed, by using the increment map (2.1.2) one can identify τ with the Markov stopping time $\tau^{\vartriangle}$ on the Bernoulli space of increments $\big(G^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}\big)$ defined as

$$\tau^{\vartriangle}(\vartriangle\boldsymbol{g}) = \tau(\boldsymbol{g}) \ .$$

By using space homogeneity of random walks on groups, τ can then be uniquely extended to a G-invariant Markov stopping time $\widetilde{\tau}$ on the space of sample paths with an arbitrary starting point by the formula

$$\widetilde{\tau}(\boldsymbol{g}) = \tau(g_0^{-1}\boldsymbol{g}) = \tau^{\vartriangle}(\vartriangle\boldsymbol{g}) \ .$$

Now, one can iterate $\widetilde{\tau}$ as

$$\widetilde{\tau}_{n+1} = \widetilde{\tau}_n + \widetilde{\tau} \circ T^{\widetilde{\tau}_n} \quad \text{with} \quad \widetilde{\tau}_1 \equiv \widetilde{\tau} \ ,$$

where T is the time shift (2.2.1) on the path space. In other words, after having stopped a sample path $\boldsymbol{g}$ at the moment $\widetilde{\tau}_n(\boldsymbol{g})$, we then apply the time $\widetilde{\tau}$ to the portion of the path $\boldsymbol{g}$ starting at time $\widetilde{\tau}_n$ from the point $g_{\widetilde{\tau}_n}$. Obviously, $\widetilde{\tau}_n$ are G-invariant Markov stopping times as well. Denote by τ_n the restriction of $\widetilde{\tau}_n$ to $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$, and by $\tau_n^{\vartriangle}$ the associated Markov stopping time on the space of increments. Then

$$\tau_{n+1}^{\vartriangle} = \tau_n^{\vartriangle} + \tau^{\vartriangle} \circ U^{\tau_n^{\vartriangle}} \ ,$$

where U is the shift (2.2.1) on the space of increments $\big(G^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}\big)$.

Clearly, the sequence $\{g_{\tau_n}\}$ is the random walk (G, μ_τ). [More rigorously, the image of the measure $\mathbf{P} = \mathbf{P}^{\mu}$ on the space of sample paths of the random walk (G, μ) under the map $\boldsymbol{g} \mapsto \{g_{\tau_n}\}$ is the measure $\mathbf{P}^{\mu_\tau}$ on the space of sample paths of the random walk (G, μ_τ).] In particular, the distribution of g_{τ_n} is the n-fold convolution of the measure μ_τ.

Remark 3.2.2. In the same way as above one can define the composition $\tau_1 \circ \tau_2$ of any two Markov stopping times τ_1, τ_2, so that the set of all Markov stopping times becomes a (non-commutative) semigroup with the identity $\mathbf{0}$. The map $\tau \mapsto \mu_\tau$ is then a homomorphism from the semigroup of Markov stopping times to the convolution semigroup of the group G. In particular, *the class $\mathcal{T}(\mu)$ is closed under convolution.*

Remark 3.2.3. The following minimal example shows that, generally speaking, *the class $\mathcal{T}(\mu)$ is not closed under taking convex combinations.* Let $G = \mathbb{Z}_2 = \{0, 1\}$ and $\mu = \delta_1$, then the random walk (G, μ) is deterministic, so that the space $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ consists just of the single path $(0, 1, 0, 1, 0, \cdots)$. Therefore, in this situation any Markov stopping time must be constant. Consequently, the only probability measures that can be obtained by using Markov stopping times on $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ are convolution powers of δ_1, which are just δ_0 and δ_1.

3.3 Markov stopping times and covering random walks

For yet another interpretation of the transformed random walk (G, μ_τ) from Definition 3.2.1 let us recall that a *covering Markov chain* with the deck group G [42] (in somewhat different contexts also known under the names of a *random walk with internal degrees of freedom* [53] and a *semi-Markov chain* [4]) is a Markov chain with G-invariant transition probabilities on the product $G \times X$ for a certain set X.

In our situation let

$$G^* = \bigcup_{n \geqslant 0} G^n$$

be the space of all finite strings of elements of G (including the empty one $\varnothing$ for $n = 0$) considered as increments of paths on G issued from the group identity. By

$|x|$ we denote the length of a string x, i.e., $|x| = n$ whenever $x \in G^n$, and by $[x]_k$ the k-truncation of x (i.e., its initial segment of length k).

Then the random walk (G, μ) naturally drives the covering Markov chain on $G \times G^*$ with the transitions

$$(g, x) \overset{\mu(h)}{\rightsquigarrow} (gh, x.h) \;, \tag{3.3.1}$$

where $x.h$ denotes the concatenation of the string x with h (i.e., the string of the length $|x| + 1$ obtained from x by adding h on the right).

Let now τ be a Markov stopping time on the path space $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$. In view of the identification $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big) \cong \big(G^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}\big)$ described in Chapter 2, we can consider the associated set T (see Section 3.2.1) as a subset of G^*. In these terms τ is just the moment when the covering chain (3.3.1) attains the set $G \times \mathsf{T}$ (provided it starts from $(e, \varnothing)$ at time 0), and μ_τ is the distribution of the G component of the covering chain at time τ.

Ir order to interpret the iterated Markov stopping times in these terms one just has to reset the G^* component of the chain (3.3.1) to $\varnothing$ whenever it attains T, which gives rise to the covering chain on the product of G and the set

$$[\varnothing, \mathsf{T}) = \{[x]_k : x \in \mathsf{T}, 0 \leqslant k < |x|\} \;.$$

with the transitions

$$(g, x) \overset{\mu(h)}{\rightsquigarrow} \begin{cases} (gh, x.h) \;, & \text{if } x.h \in [\varnothing, \mathsf{T}) \\ (gh, \varnothing) \;, & \text{if } x.h \in \mathsf{T} \end{cases} . \tag{3.3.2}$$

The chain (3.3.2) is also driven by the random walk (G, μ). If it starts at the point $(e, \varnothing)$ at time 0, then its trace on the set $G \times \{\varnothing\}$ is precisely the random walk (G, μ_τ) run from the group identity.

3.4 Examples

Below are several straightforward examples of transformations of random walks determined by Markov stopping times. Of course, the variety of possible Markov stopping times is much greater, and with just a little imagination one can produce much more exotic illustrations of this notion.

Example 3.4.1. Let $\tau(\boldsymbol{g}) = n$, $n \in \mathbb{Z}_+$, be a constant function. Then μ_τ is just the n-fold convolution μ^{*n} of the measure μ.

Example 3.4.2. Given a subset $A \subset G$ with $\mu(A) > 0$, let $\tau^\vartriangle = \tau^\vartriangle_A$ be the corresponding first hitting time for the Bernoulli increments chain (h_n), and let

$$\tau(\boldsymbol{g}) = \tau^\vartriangle(\vartriangle\boldsymbol{g}) = \min\{n \in \mathbb{N} : \vartriangle g_n \in A\}$$

be the associated Markov stopping time on the path space $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$. Then

$$\mu_\tau = \sum_{n=0}^{\infty} \beta^{*n} * \alpha = (1-\beta)^{-1} * \alpha \ ,$$

where α is the restriction of μ to A, and $\beta = \mu - \alpha$.

Remark 3.4.3. The transformation $\alpha + \beta \mapsto (1-\beta)^{-1} * \alpha$ was first introduced by Willis [72] who showed that it does not change the Poisson boundary of the associated random walk by using methods of functional analysis. Since this is a transformation determined by a Markov stopping time, this fact will also follow from our general Theorem 3.6.1 below.

Example 3.4.4. A subset $A \subset G$ is *recurrent* for the random walk (G, μ) if its first hitting time

$$\tau(\boldsymbol{g}) = \tau_A(\boldsymbol{g}) = \min\{n \in \mathbb{N} : g_n \in A\}$$

is almost surely finite. Note that, in general, the iterations of τ are *not* hitting times of A, as, for instance, g_{τ_2} need not belong to A. However, if A is a *subgroup* of G, then the random walk (G, μ_τ) is the *trace* of the random walk (G, μ) on A, see [25].

3.5 Randomized Markov stopping times

As we have seen in Remark 3.2.3, the class of measures $\mathcal{T}(\mu)$ obtained from Markov stopping times of the random walk (G, μ) is not, generally speaking, closed with respect to taking convex combinations. We shall now modify this procedure in order to have more freedom in transforming the original random walk, and, in particular, to repair this drawback.

The idea consists in adding to the original random walk a sequence of i.i.d. random variables from an auxiliary probability space. It goes back to Kaimanovich [37] who used it in order to describe Furstenberg's discretization of Brownian motion on Riemannian manifolds [25] (also see [55]) in terms of appropriate randomized Markov stopping times and to prove coincidence of the corresponding Poisson boundaries.

More precisely, let (Ω, m) be a probability space. We shall define the associated *extended random walk* as (g_n, ω_n), where $(g_n) = \boldsymbol{g}$ are sample paths of the original random walk (G, μ), and $(\omega_n) = \boldsymbol{\omega}$ is a sequence of i.i.d. m-distributed random variables on Ω independent of $\boldsymbol{g}$. In other words, the extended random walk is a Markov chain with the state space $G \times \Omega$ and transition probabilities

$$\overline{\pi}_{g,\omega} = \pi_g \otimes m = g\mu \otimes m ,$$

where $\pi_g = g\mu$ are the transition probabilities (2.1.1) of the original random walk. The measure $\mathbf{P}^m = \mathbf{P} \otimes m^{\otimes \mathbb{Z}_+}$ on the path space $(G \times \Omega)^{\mathbb{Z}_+}$ of the extended random walk corresponds to the initial distribution $\delta_e \otimes m$.

Definition 3.5.1. A *randomized Markov stopping time* of the random walk (G, μ) is a Markov stopping time τ on the path space $\big((G \times \Omega)^{\mathbb{Z}_+}, \mathbf{P}^m\big)$ of the extended random walk. As for ordinary Markov stopping times we shall denote the distribution of the G component of the extended random walk at time τ by μ_τ:

$$\mu_\tau(g) = \mathbf{P}^m\{(\boldsymbol{g}, \boldsymbol{\omega}) : g_\tau = g\} \ .$$

By

$$\mathcal{R}(\mu) = \{\mu_\tau : \tau \text{ is a randomized Markov stopping time of the random walk } (G, \mu)\}$$

we denote the class of all probability measures on G which can be obtained in this way.

Remark 3.5.2. In the particular case when the space (Ω, m) is just a singleton the above definition obviously coincides with Definition 3.2.1, so that $\mathcal{T}(\mu) \subset \mathcal{R}(\mu)$. On the other hand, the example from Remark 3.2.3 in combination with Theorem 3.5.3 (or Example 3.5.4) below show that, generally speaking, this inclusion is strict.

The definition of iterated Markov stopping times and their interpretation from Section 3.2.2 and Section 3.3 almost *verbatim* carry over to this randomized setup as well.

Theorem 3.5.3. *The class $\mathcal{R}(\mu)$ is closed with respect to taking convolutions and convex combinations.*

Proof: Let us first notice that any two randomized Markov stopping times τ_1 and τ_2 can always be realized by using the same auxiliary probability space. Indeed, if (Ω_i, m_i) are the spaces used for defining τ_i, $i = 1, 2$, then both τ_1 and τ_2 can also be defined by using the product space $(\Omega, m) = (\Omega_1, m_1) \times (\Omega_2, m_2)$. [Alternatively, one can just notice that since $\big(G^{\mathbb{Z}_+}, \mathbf{P}\big)$ is a Lebesgue space, it is enough to consider Lebesgue auxiliary spaces, for which there exists a universal one.]

Now, as for ordinary Markov stopping times (see Remark 3.2.2), the measure $\mu_{\tau_1} * \mu_{\tau_2}$ corresponds to the composition of the Markov stopping times τ_1 and τ_2.

Let us now consider a convex combination $p_1\mu_1 + p_2\mu_2$. In order to realize it as μ_τ for a random Markov stopping time τ we consider the auxiliary probability space $(\Omega', m') = (\Omega, m) \times (\{1,2\}, p)$, where (Ω, m) is the common auxiliary space of τ_1 and τ_2 (see above), and p is the distribution on the set $\{1,2\}$ with the weights p_i. Paths of the extended random walk on $G \times \Omega'$ sampled from the measure $\mathbf{P}^{m'} = \mathbf{P} \otimes m^{\otimes \mathbb{Z}_+} \otimes p^{\otimes \mathbb{Z}_+}$ are then triples $(\boldsymbol{g}, \boldsymbol{\omega}, \boldsymbol{\varepsilon})$ consisting of 3 independent components: a path $\boldsymbol{g}$ of the random walk (G, μ), a Bernoulli $m^{\otimes \mathbb{Z}_+}$-distributed sequence $\boldsymbol{\omega} = (\omega_n)$, and a Bernoulli $p^{\otimes \mathbb{Z}_+}$-distributed sequence $\boldsymbol{\varepsilon} = (\varepsilon_n)$. If we now put

$$\tau(\boldsymbol{g}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}) = \begin{cases} \tau_1(\boldsymbol{g}, \boldsymbol{\omega}) \ , & \text{if } \varepsilon_0 = 1 \\ \tau_2(\boldsymbol{g}, \boldsymbol{\omega}) \ , & \text{if } \varepsilon_0 = 2 \end{cases} ,$$

then clearly $\mu_\tau = p_1\mu_1 + p_2\mu_2$. Note that the same argument obviously works for countable convex combinations as well. ▮

3.5.1 Examples of randomized Markov stopping times

The first example is a specialization of Theorem 3.5.3.

Example 3.5.4. Let $\mu' = \sum_n p_n \mu^{*n}$, where $p_n \geqslant 0$ and $\sum_n p_n = 1$, be a (possibly infinite) convex combination of convolution powers of a measure μ. Then there exists a randomized Markov stopping time τ such that $\mu' = \mu_\tau$.

In this situation it is the set $\mathbb{Z}_+$ endowed with the distribution $p = (p_n)$ which can serve as an auxiliary probability space. Then paths of the extended random walk on $G \times \mathbb{Z}_+$ sampled from the measure $\mathbf{P}^p = \mathbf{P} \otimes p^{\otimes \mathbb{Z}_+}$ are couples $(\boldsymbol{g}, \boldsymbol{\varepsilon})$ consisting of two independent components: a path $\boldsymbol{g}$ of the random walk (G, μ) and a Bernoulli $p^{\otimes \mathbb{Z}_+}$-distributed sequence $\boldsymbol{\varepsilon} = (\varepsilon_n)$. If we put

$$\tau(\boldsymbol{g}, \boldsymbol{\varepsilon}) = \varepsilon_0 \ ,$$

then obviously $\mu_\tau = \mu'$.

The next example is a generalization of 3.4.2, in which we considered a decomposition of a probability measure into a sum of two mutually singular sub-probability ones. Randomization allows one to eliminate the singularity assumption.

Example 3.5.5. Let $\mu = \alpha + \beta$ be a decomposition of a probability measure μ into a sum of two sub-probability measures α and β. Then there exists a randomized Markov stopping time τ such that

$$\mu_\tau = (1 - \beta)^{-1} * \alpha \ . \tag{3.5.1}$$

In this situation we take for (Ω, m) the unit interval endowed with the standard Lebesgue measure. Let

$$A = \{(g, \omega) \in G \times \Omega : 0 \leqslant \omega \leqslant \alpha(g)\} \ ,$$

so that the image of the restriction $\mu \otimes m|_A$ under the map $(g, \omega) \mapsto g$ is precisely α, and let

$$\tau(\boldsymbol{g}, \boldsymbol{\omega}) = \min\{n \geqslant 1 : (h_n, \omega_n) \in A\} \ ,$$

where $\boldsymbol{h} = \vartriangle \boldsymbol{g}$ is the sequence of increments of a sample path $\boldsymbol{g}$. Then the measure μ_t is given by formula (3.5.1) in precisely the same way as in Example 3.4.2.

3.6 Poisson boundary of transformed random walks

This section is devoted to study of the Poisson boundary of transformed random walks. Namely, we will show the Poisson boundary is preserved under transformation of the random walk by a Markov stopping time.

Theorem 3.6.1. *Let τ be a Markov stopping time for the random walk (G, μ). Then the Poisson boundary of the random walk (G, μ) is the same as the Poisson boundary of the random walk (G, μ_τ).*

In [25], Furstenberg showed that the Poisson boundary of a group G is the same as the Poisson boundary of a recurrent subgroup H. His proof involves direct calculations and is mostly based on the fact that hitting measures are harmonic function. The proof can be generalized just to a Markov stopping τ_A for a recurrent subset A of G. However, our approach which is more general relies on the following lemma:

Lemma 3.6.2. *Let μ be a probability measure on a free semigroup $\mathcal{F}$ which is distributed on the independent generators of the free semigroup $\mathcal{F}$ (see Lemma 2.4.5 for the precise definition). If τ is a Markov stopping time, then $\partial_\mu \mathcal{F} = \partial_{\mu_\tau} \mathcal{F}$ is the space of all infinite words $\mathcal{F}^\infty$ endowed with the corresponding measure or, in other words, the space of increments of the random walk $(\mathcal{F}, \mu)$.*

Proof: By Lemma 2.4.5, the Poisson boundary of $(\mathcal{F}, \mu)$ is the space of its increments. On the other hand, let $g = g_1 g_2 \dots g_m$ for g_i in support of μ. Once again the condition Lemma 2.4.5 yields to $g_\tau = h_1 h_2 \dots h_m$ if and only if $g_m = h_1 h_2 \dots h_m$ for a sample path $\boldsymbol{g} = (g_n)$ with increment $\vartriangle\boldsymbol{g} = (h_1, h_2, \cdots)$ and consequently, $\mu_\tau(g) = \mu(g_1)\mu(g_2)\dots\mu(g_m)$. With the same argument as Lemma 2.4.5 the Poisson boundary of $(\mathcal{F}, \mu_\tau)$ is the space of increments of $(\mathcal{F}, \mu)$. ∎

Proof: (**Proof of 3.6.1**)

Let $\mathcal{F}$ be the free semigroup generated by the support of μ in such a way that each element in the support of μ is considered as an independent generator of $\mathcal{F}$. In other words, the condition of Lemma 3.6.2 holds for $(\mathcal{F}, \hat{\mu})$ where $\hat{\mu}(g) = \mu(g)$ for every $g \in \mathcal{F}$.

Let $\phi : \mathcal{F} \to G$ be a semigroup homeomorphisms such that $\phi(g)$ is the reduced word in the group G for $g \in F$. A Markov stopping time τ on the random walk (G, μ) gives rise to a Markov stopping $\hat{\tau}$ of $(\mathcal{F}, \mu)$ as follows: for every sample path $\hat{\boldsymbol{g}} = (\hat{g}_n)$ of the random walk $(\mathcal{F}, \hat{\mu})$, define the

$$\hat{\tau}(\hat{g}_n) = \tau(\phi(g_n)).$$

By Lemma (3.6.2),

$$\partial_{\hat{\mu}}\mathcal{F} = \partial_{\hat{\mu}_{\hat{\tau}}}\mathcal{F}. \tag{3.6.1}$$

Since the group homeomorphism ϕ is onto, $G = \mathcal{F}/\text{Ker}\phi$. By applying Lemma 2.4.2 we have $\partial_{\mu}G = \partial_{\mu_\tau}G$. ∎

Establishing the same result for the case of randomized Markov stopping times needs more tools, namely, entropy and rate of escape. We provide the proof here, however, for the background related to entropy and rate of escape we refer the reader to the next chapter.

In the case of randomized Markov stopping times, the measure μ_τ is not necessarily distributed on the independent generators of the free semigroup $\mathcal{F}$ in Lemma 3.6.2. Even in this quite degenerate situation, we do not know whether $(\mathcal{F}^{\mathbb{N}}, \mu^{\otimes\mathbb{N}})$ is the Poisson boundary of the random walk $(\mathcal{F}, \mu_\tau)$. The entropy technique provides the positive answer for measures μ_τ with finite entropy and first logarithmic moment if the free semigroup $\mathcal{F}$ is finitely generated (cf. the more general cases of finitely generated free groups or of word hyperbolic groups [39]). We shall now show that in the free semigroup case these assumptions can be further relaxed, so that neither finiteness of the generators $\mathcal{F}$ nor finiteness of the entropy $H(\mu_\tau)$ are no longer necessary.

Denote by $d(g)$ the length of word g in the free semigroup $\mathcal{F}$. More precisely, if $g = g_1 g_2 \cdots g_n$ for independent generators $g_1, g_2, \cdots$ of $\mathcal{F}$, then $d(g) = n$.

Theorem 3.6.3. *If μ is a probability measure on a free semigroup $\mathcal{F}$ with a finite first logarithmic moment*

$$\sum_{g \in \mathcal{F}} \mu(g) \log^+ d(g) \ ,$$

then the Poisson boundary of the random walk $(\mathcal{F}, \mu)$ is the space of infinite words $\mathcal{F}^\infty$ endowed with the associated hitting measure λ.

The proof is still based on the entropy criterion, but here we apply it in a somewhat different setup: instead of the original random walk we consider just the conditional random walks determined by points from $\mathcal{F}^\infty$. Coincidence of the Poisson boundary of the random walk $(\mathcal{F}, \mu)$ with the space $(\mathcal{F}^\infty, \lambda)$ means precisely that the Poisson boundaries of these conditional random walks are trivial. However, due to the absence of cancellations, in the free semigroup case the conditional random walks can be considered just as random walks on $\mathbb{Z}$ (actually, even $\mathbb{Z}_+$) in a stationary random environment (or, in the language of equivalence relations, as a random walk along the orbits of the time shift in $\mathcal{F}^\infty$). The entropy criterion for triviality of the Poisson boundary for random walks along classes of equivalence relations (Theorem A.0.2) was established in [44]. It is due to the aforementioned special structure of the conditional random walks in the free semigroup case that the entropy of transition probabilities of these walks can be finite in spite of the entropy of the original measure μ being infinite.

Let us remind the reader that $\mathbb{R}$-valued sequence $(x_n)_{n\geqslant 1}$ of measurable functions from a Lebesgue space is a *stationary sequence* if the joint probability of $(x_{n_1}, \cdots, x_{n_k})$ is the same as that of $(x_{n_1+n}, \cdots, x_{n_k+n})$ for any natural numbers n_i and n.

Lemma 3.6.4. *Let $(x_n)_{n\geqslant 1}$ be a stationary sequence with non-negative entries. If*

$$\mathbf{E}\log(1+x_1) < \infty \ ,$$

then

$$\lim_{n\to\infty} \frac{\log(1+x_1+x_2+\cdots+x_n)}{n} = 0 \tag{3.6.2}$$

a.e. and in L^1.

Proof: Let us first notice that, obviously,

$$\log(1+x+y) \leqslant \log(1+x) + \log(1+y) \ ,$$

so that the sequence $\log(1+x_1+x_2+\cdots+x_n)$ satisfies the conditions of Kingman's subadditive ergodic theorem. Therefore the limit (3.6.2) exists, and it only remains to show that its value L actually vanishes. Indeed, convergence in (3.6.2) also holds in probability, so that for any $\varepsilon > 0$

$$\mathbf{P}\left\{ \left| \frac{\log(1+x_1+x_2+\cdots+x_n)}{n} - L \right| \leqslant \varepsilon \right\} \to 1 \ ,$$

and by stationarity

$$\mathbf{P}\left\{ \left| \frac{\log(1+x_{n+1}+x_2+\cdots+x_{2n})}{n} - L \right| \leqslant \varepsilon \right\} \to 1 \ .$$

On the other hand, also

$$\mathbf{P}\left\{ \left| \frac{\log(1+x_1+x_2+\cdots+x_{2n})}{2n} - L \right| \leqslant \varepsilon \right\} \to 1 \ ,$$

which is only possible if $L = 0$. ▮

Proof: (**Proof of Theorem 3.6.3**)

Finiteness of the first logarithmic moment of the measure μ is the same as finiteness of its first moment with respect to the logarithmic gauge

$$\mathcal{G}_n = \{g \in \mathcal{F} \ : \ \log(1 + d(g)) \leqslant n\} \ .$$

When restricted to the geodesic ray in $\mathcal{F}^\infty$ determined by an infinite word from $\mathcal{F}^\infty$, this gauge obviously has a finite exponential growth rate, hence Theorem 4.2.4 implies that for conditional random walks the average entropy of the one-step transition probabilities is finite. Lemma 3.6.4 implies that the rate of escape with respect to the logarithmic gauge vanishes, whence the asymptotic entropy of conditional random walks also vanishes, so that by Theorem A.0.2 the conditional random walks have trivial Poisson boundary, whence the claim. ▮

Let us look at the difference between non-randomized and randomized Markov stopping times in the sense of conditional random walks. Let τ be an ordinary (i.e., non-randomized) Markov stopping time of the random walk $(\mathcal{F}, \mu)$ in Lemma 3.6.2. Since τ is a function of the path space, the corresponding conditional chain on $\mathcal{F}^\infty$ is deterministic, as we concluded in Lemma 3.6.2 the Poisson boundaries of the random walks $(\mathcal{F}, \mu)$ and $(\mathcal{F}, \widetilde{\mu}_\tau)$ are isomorphic. In the randomized case, the only difference is that now the conditional random walks are not necessarily deterministic. However, under conditions of Theorem 3.6.3 their Poisson boundaries are still trivial and Lemma 2.4.2 combined, which imply

Theorem 3.6.5. *Let* τ *be a randomized Markov stopping times. If* $\boldsymbol{E}(\log \tau)$ *is finite, then the Poisson boundary of* (G, μ) *and* (G, μ_τ) *are isomorphic.*

Chapter 4

Entropy, Rate of Change and Transformed Random Walks

The notion of entropy of a countable probability space was introduced by Shannon in 1948. He used it to define the asymptotic entropy (entropy rate) in order to quantify the amount of information for a stationary stochastic process [66]. Later in the mid 1950's Kolmogorov developed the notion of entropy of a measure preserving dynamical system [52], and his work was completed by Sinai [67]. But, it was only in 1972, that Avez defined the asymptotic entropy of a random walk on a group [2]. Despite a formal similarity, the contexts of these definitions are different, and so far there is no common approach which would unify them.

4.1 Asymptotic entropy

The quantity

$$H(\mu) = -\sum_{g} \mu(g) \log \mu(g)$$

is called the *entropy* of the probability measure μ.

Note that the entropy of an arbitrary probability measure is not necessarily finite.

Proposition 4.1.1. *Let $G = \{a_1, a_2, \cdots\}$ be an infinite group. There exists probability measure μ with infinite entropy.*

Proof: Notice that with integral test, the series $\sum_n \frac{1}{n \log n (\log\log n)^2}$ is convergent to a finite number c. On the other hand, $\sum_n \frac{\log(n \log n(\log\log n)^2)}{n \log n (\log\log n)^2}$ is divergent that once again can be verified by the integral test. Hence, $\mu(a_n) = \frac{1}{cn \log n (\log\log n)^2}$ has infinite entropy. ∎

The definition of convolution implies that

$$\mu^{*2}(gh) \geqslant \mu(g)\mu(h),$$

and consequently $-\log \mu^{*2}(gh)) \leqslant -\log \mu(g) - \log \mu(h)$. Multiplying both sides by $\mu(g)\mu(h)$ and taking summation all over g and h in G, we obtain $H(\mu^{*2}) \leqslant 2H(\mu)$. In the same way, we can show that the sequence $\{H_n\}$ is sub-additive, i.e., $H_{n+m} \leqslant H_n + H_m$, where $H_n = H(\mu^{*n})$. Therefore, the limit H_n/n, called the *asymptotic entropy* of (G, μ), exists and is denoted by

$$h(\mu) = \lim_n \frac{H_n}{n}. \tag{4.1.1}$$

If a random walk has finite entropy, then its Poisson boundary is trivial if and only if its asymptotic entropy is zero (for more details see, [41]). Obviously, finiteness of the entropy is necessary. For instance consider a countable nilpotent group with the probability measure as in Proposition 4.1.1. Then, the Poisson boundary is trivial but the asymptotic entropy is infinite. More interestingly, there exists an amenable group such that a random walk with finite entropy on this group has non-trivial Poisson boundary [18].

Throughout the rest of this chapter, we always assume that $H(\mu) = H_1$ **is finite**, in which case $h(\mu) \leqslant H_1 < \infty$.

For a sample path $\boldsymbol{g} = (g_n)$, let $F_n(\boldsymbol{g}) = -\log \mu^{*n}(g_n)$. We can write

$$g_{n+m} = g_n h_{n+1} \cdots h_{n+m} \ ,$$

where h_i's are the increment of $\boldsymbol{g}$, again by the definition of the convolution,

$$\mu^{*(n+m)}(g_{n+m}) \geqslant \mu^{*n}(g_n)\mu^{*m}(h_{n+1} \cdots h_{n+m}).$$

By using U, the transformation of the path space induced by the time shift in the space of increments,

$$F_{n+m}(\boldsymbol{g}) \leqslant F_n(\boldsymbol{g}) + F_m(U^n(\boldsymbol{g})) \ .$$

Applying Kingman subadditive ergodic theorem, the asymptotic entropy can also be obtained by Shannon's formula:

Theorem 4.1.2. *[41, 11] For* $\mathbf{P}$*-almost every sample path* (g_n)*,*

$$-h(\mu) = \lim \frac{1}{n} \log \mu^{*n}(g_n) \ .$$

Moreover, the convergence holds in $L^1(\mathbf{P})$*.*

Shannon's formula leads to the following description of the asymptotic entropy of (G, μ) as well:

Theorem 4.1.3. *[41] For* $\mathbf{P}$*-almost every sample path* $\boldsymbol{g} = (g_n)$*,*

$$h(\mu) = \lim_n \frac{1}{n} \log \frac{dg_n\nu}{d\nu} \mathbf{bnd}(\boldsymbol{g}) \ .$$

4.1.1 Transformed Entropy

As a motivation, consider the asymptotic entropy of the random walk determined by the k-fold convolution of μ, i.e., (G, μ^{*k}). By definition of the asymptotic entropy

$$h(\mu^{*k}) = \lim_n \frac{1}{n} H_{kn} = \lim_n k \frac{H_{kn}}{kn} = kh(\mu) \ . \tag{4.1.2}$$

In the language of Markov stopping times, we can write the equation (4.1.2) as following:

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu) \ , \tag{4.1.3}$$

where τ is constant function k (see Example 3.4.1).

The aim of the next theorem is to show that equality (4.1.3) can be generalized to all Markov stopping times (and to all randomized Markov stopping times as we shall do later, see Theorem 4.1.8) with finite expectation. This is analogous to Abramov's theorem on the entropy of induced dynamical systems:

Theorem 4.1.4. *[1] Let* (X, ϕ, μ) *be an ergodic measure preserving dynamical system. If* A *is a measurable subset of* X *with* $\mu(A) > 0$*, then*

$$h(\mu_A, \phi_A) = \frac{1}{\mu(A)} h(\mu, \phi) \ .$$

In order to complete our analogy, note that the proportion $\dfrac{1}{\mu(A)}$ is equal to the expectation of the return time to the set A (Kac formula) [35].

Since the sequence $\{\mu_{\tau_n}\}$ is not, generally speaking, a subsequence of convolution powers of μ, the generalization of the equality (4.1.3) cannot be done by the same trick as in (4.1.2).

Theorem 4.1.5. *Let* τ *be a Markov stopping time with a finite expectation. Then* $H(\mu_\tau)$ *is also finite and*

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu) \ .$$

The proof is based on the fact that each sample path of the random walk transformed by a Markov stopping time is a subsequence of a sample path of the original random walk, and the description of the asymptotic entropy as the exponential growth rate of the Radon–Nikodym derivatives of the translates of the harmonic measure along the sample path (Theorem 4.1.3). In order to apply Theorem 4.1.3, we need the Poisson boundaries of the random walks (G, μ) and (G, μ_τ) to be the same (which follows from Theorem 3.6.1), and the entropy of the random walk (G, μ_τ) to be finite.

First, we will show that the entropy of μ_τ is finite.

Lemma 4.1.6. *If the expectation of Markov stopping time* τ *is finite, then the entropy of* μ_τ *is also finite, and*

$$H(\mu_\tau) \leqslant \boldsymbol{E}(\tau) H_1 \ .$$

Proof: Let

$$M_n(\boldsymbol{g}) = nH_1 + \log \mathbf{P}(C^n_{g_n}) = nH_1 + \log \mu^{*n}(g_n) \ .$$

Because H_1 is finite, M_n's are integrable. If $\mathcal{A}^n_0$ is the σ-algebra generated between time 0 and n of the random walk (G, μ), then it is obvious that

$$\boldsymbol{E}(M_{n+1}|\mathcal{A}^n_0)(\boldsymbol{g}) = (n+1)H_1 + \sum_h \mu(h) \log \mu^{*(n+1)}(g_n h) \ .$$

The sequence $\{M_n, \mathcal{A}^n_0\}$ is a sub-martingale, i.e.,

$$\boldsymbol{E}(M_{n+1}|\mathcal{A}^n_0)(\boldsymbol{g}) \geqslant (n+1)H_1 + \sum_h \mu(h)(\log \mu^{*n}(g_n) + \log \mu(h)) = M_n(\boldsymbol{g}) \ ,$$

because

$$\mu^{*(n+1)}(g_n h) \geqslant \mu^{*n}(g_n)\mu(h) \ . \tag{4.1.4}$$

Let $\tau \wedge n = \min\{\tau, n\}$. Then, Doob's optional theorem, Theorem 1.3.4, implies that

$$\boldsymbol{E}(M_{\tau \wedge n}) \ \geqslant \boldsymbol{E}(M_0) = 0 \ ,$$

and consequently,

$$\boldsymbol{E}(\tau \wedge n)H_1 + \sum_g \mathbf{P}(g_{\tau \wedge n} = g) \log \mathbf{P}(g_{\tau \wedge n} = g) \geqslant 0 \ .$$

Since $\boldsymbol{E}(\tau \wedge n) \leqslant \boldsymbol{E}(\tau)$, we can write

$$-\sum_h \mathbf{P}(g_{\tau \wedge n} = h) \log \mathbf{P}(g_{\tau \wedge n} = h) \leqslant \boldsymbol{E}(\tau)H_1 \ . \tag{4.1.5}$$

Applying Fatou's lemma to the inequality (4.1.5) gives

$$-\sum_g \liminf_n \mathbf{P}(g_{\tau \wedge n} = h) \log \mathbf{P}(g_{\tau \wedge n} = h) \leqslant \boldsymbol{E}(\tau)H_1 \ . \tag{4.1.6}$$

On the other hand, the combination of $\lim_n \mathbf{P}(g_{\tau \wedge n} = h) = \mu_\tau(h)$ and the continuity of the function $x \log x$ implies that

$$\lim_n \mathbf{P}(g_{\tau \wedge n} = h) \log \mathbf{P}(g_{\tau \wedge n} = h) = \mu_\tau(h) \log \mu_\tau(h) \ . \tag{4.1.7}$$

Now, by (4.1.6) and (4.1.7) we obtain

$$H(\mu_\tau) = -\sum_h \lim_n \mathbf{P}(g_{\tau\wedge n} = h) \log \mathbf{P}(g_{\tau\wedge n} = h) \leqslant \boldsymbol{E}(\tau) H_1 \ .$$

Therefore, we have proved that,

$$\lim_{n\to\infty} H(\mu_{\tau\wedge n}) = H(\mu_\tau) \ .$$

∎

Remark 4.1.7. In a more general setup, the convergence of a sequence of probability measures does not imply the convergence of the (asymptotic) entropy associated with these probability measures. However, these questions have been studied for some special cases by Erschler and Kaimanovich in [40].

Proof of Theorem 4.1.5

Now, we can find the asymptotic entropy of the random walk (G, μ_τ). Since the expectation of the Markov stopping time τ is finite, Lemma 4.1.6 implies that $H(\tau)$ is also finite. Therefore, Theorem 4.1.3 implies that

$$h(\mu_\tau) = \lim_n \frac{1}{n} \log \frac{dg_{\tau_n}\nu}{d\nu} \mathbf{bnd}(\boldsymbol{g})$$

for almost every sample path $\boldsymbol{g} = (g_n)$. Since U is a measure preserving transformation, $\boldsymbol{E}(\tau_n) = n\boldsymbol{E}(\tau)$, and, moreover, $\lim_n \frac{\tau_n(\boldsymbol{g})}{n} = \boldsymbol{E}(\tau)$ for almost every sample path $\boldsymbol{g}$. Therefore, it is obvious that

$$h(\mu_\tau) = \lim_n \frac{\tau_n(\boldsymbol{g})}{n} \frac{1}{\tau_n(\boldsymbol{g})} \log \frac{dg_{\tau_n}\nu}{d\nu} \mathbf{bnd}(\boldsymbol{g}) \ .$$

By applying Theorem 4.1.3 to the random walk (G, μ), we will have

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu) \ .$$

4.1.2 Entropy and randomized Markov stopping times

Let τ be a randomized Markov stopping time with finite expectation. If we replace the sub-martingale in Lemma 4.1.6 with

$$M_n(\boldsymbol{g},\boldsymbol{\omega}) = nH_1 + \log \mathbf{P}^m(C^n_{g_n}) = nH_1 + \log \mu^{*n}(g_n) \ ,$$

then finiteness of the entropy of μ_τ can be obtained by mimicking the proof of Lemma 4.1.6. Since the Poisson boundary of (G,μ_τ) is the same as the Poisson boundary (G,μ), the proof of Theorem 4.1.3 applies. Hence, we have the following theorem:

Theorem 4.1.8. *Let τ be a randomized Markov stopping time for the random walk (G,μ). If $\boldsymbol{E}(\tau)$ is finite, then*

$$h(\mu_\tau) = \boldsymbol{E}(\tau)h(\mu) \ .$$

The combination of Example 3.5.4 and the preceding theorem implies the following result of Kaimanovich [36].

Example 4.1.9. Let $\mu' = \sum\limits_{k\geqslant 0} a_k\mu_k$, where $\sum\limits_{k\geqslant 0} a_k = 1$ and $a_k \geqslant 0$. Then

$$h(\mu') = (\sum_{k\geqslant 0} k a_k)h(\mu) \ .$$

Example 4.1.10. Let $\mu = \alpha + \beta$ be as in Example 3.5.5. Let $\mu' = \beta + \sum\limits_{i\geqslant 1} \beta^{*i} * \alpha$, then

$$h(\mu') = \frac{1}{\|\alpha\|}h(\mu) \ ,$$

where $\|\alpha\|$ is the total mass of α which is equal with $\|\alpha\|_{l_1}$.

4.1.3 μ-boundary

A μ-boundary is a quotient of the Poisson boundary with respect to a G-invariant measurable partition (see, [25, 39]). Let $(\partial_\mu G_\xi, \nu_\xi)$ be a μ-boundary for G-invariant measurable partition ξ. Denote by $\underset{\xi}{\mathbf{bnd}}$ be the canonical projection

$$\underset{\xi}{\mathbf{bnd}} : \left(G^{\mathbb{Z}_+}, \mathbf{P}\right) \to (\partial_\mu G, \nu) \to (\partial_\mu G_\xi, \nu_\xi) \ .$$

Then, the *differential entropy* of μ-boundary is defined as follows

$$E_\mu(\partial_\mu G_\xi, \nu_\xi) = \sum_g \mu(g) \int \log \frac{dg\nu_\xi}{d\nu_\xi}(gx_\xi) d\nu_\xi(x_\xi) \ . \tag{4.1.8}$$

In [36] Kaimanovich has shown that the asymptotic entropy $h(\mu)$ is the upper bound for the asymptotic entropies of μ-boundaries, i.e.,

$$E_\mu(\partial_\mu G_\xi, \nu_\xi) \leqslant h(\mu) \ .$$

Moreover, in [39] Kaimanovich proved an analogue of Theorem 4.1.3 for μ-boundaries.

Theorem 4.1.11. *[39] Let $(\partial_\mu G_\xi, \nu_\xi)$ be a μ–boundary. Then for ν_ξ– almost every x,*

$$E_\mu(\partial_\mu G_\xi, \nu_\xi) = \lim_n \frac{1}{n} \log \frac{dg_n \nu}{d\nu}(\underset{\xi}{\mathbf{bnd}}\, \boldsymbol{g})$$

for almost every $\mathbf{P}^{x_\xi}$–sample path $\boldsymbol{g}$.

Theorem 4.1.12. *[39] Let $(\partial_\mu G_\xi, \nu_\xi)$ be a μ–boundary. Then for ν_ξ– almost every x,*

$$h(\mathbf{P}^{x_\xi}) = E_\mu(\partial_\mu G_\xi, \nu_\xi) - h(\mu) \ .$$

Moreover, $(\partial G_\xi, \nu_\xi)$ is the Poisson boundary if and only if $h(\mathbf{P}^{x_\xi})$ is zero for ν_ξ–almost every x_ξ.

Therefore, the same theorem as in Theorem 4.1.8 holds for the asymptotic entropy of G-invariant measurable partition of the Poisson boundary of (G, μ).

Theorem 4.1.13. *Let ξ be a measurable G-invariant partition of the Poisson boundary (G, μ). If τ is a randomized Markov stopping time with finite expectation, then*

$$E_{\mu_\tau}((\partial_\mu G_\xi, \nu_\xi)) = \boldsymbol{E}(\tau) E_\mu((\partial_\mu G_\xi, \nu_\xi)) \ .$$

4.2 Rate of escape

In this section, we will recall the definition of the rate of escape of a random walk. Then we will show the relation between the escape rate of a random walk transformed via a randomized Markov stopping time and of the original one.

Definition 4.2.1. A gauge $\mathcal{G} = (\mathcal{G}_n)$ on a group G is an increasing sequence of subsets of G such that $G = \bigcup\limits_n \mathcal{G}_n$. A gauge function $|.|_\mathcal{G}$ on G is defined as follows:

$$|g| = |g|_\mathcal{G} = \min\{n \ : \ g \in \mathcal{G}_n\} \ .$$

A gauge $\mathcal{G}$ is called sub-additive, whenever its gauge function is sub-additive.

A measure μ has a finite *first moment* with respect to a gauge $\mathcal{G}$, if

$$|\mu| = \sum_g |g| \mu(g)$$

is finite. If $\mathcal{G}$ is a sub-additive gauge, then

$$|g_{n+m}| \leqslant |g_n| + |(U^n \boldsymbol{g})_m|,$$

so that Kingman's sub-additive theorem implies

Theorem 4.2.2. *[11, 32] Let $\mathcal{G}$ be a sub-additive gauge and μ have a finite first moment with respect to $\mathcal{G}$. Then*

$$\ell(G, \mu, \mathcal{G}) = \ell(\mu) = \lim_{,} \frac{|g_n|}{n} \tag{4.2.1}$$

exists for $\mathbf{P}$-almost every sample path $\boldsymbol{g} = (g_n)$, and also in $L^1(\mathbf{P})$.

The quantity $\ell(\mu)$ is called the rate of escape (drift) of the random walk (G, μ) with respect to the gauge $\mathcal{G}$.

Example 4.2.3. Let G be a group generated with a finite set $S = S^{-1}$. Define $\mathcal{G}_n = S^n$, then $\mathcal{G} = (\mathcal{G}_n)$ is a sub-additive gauge.

The following theorem shows the relation between first moment and entropy.

Theorem 4.2.4. *[12] Let $\mathcal{G} = (\mathcal{G}_n)$ be a gauge such that each $\mathcal{G}_n$ is finite and the gauge sets grow at most exponentially, i.e.,*

$$\sup_k \frac{1}{k} \log card\mathcal{G}_n < \infty \ .$$

If the first moment $|\mu|$ with respect to $\mathcal{G}$ is finite, then the entropy $H(\mu)$ is also finite.

Another quantity that can be related to a finite generated group is *growth volume* with respect to a generating set. We can define

$$V(G) = V = \lim_n \frac{\log |S^n|}{n} \ .$$

This limit exists because the gauge $\mathcal{G}_n = S^n$ is subadditive. Also, we have $|S^n| \leqslant |S|^n$, hence the growth is exponential at most and $V \leqslant |S|$. For example, for the case of a free group with d generators, $V = \log(2d-1)$. Therefore, if $d > 1$ the growth is exponential and if $d = 1$, the growth is polynomial. Gromov [31] proved that the growth of a group is polynomial if and only if it has a nilpotent subgroup with finite index. Moreover, Milnor questioned whether there exist groups with *intermediate growth*. A positive answer was given by Grigorchuk in [30]. Vershik in [71] showed the relation between three quantities, that is, asymptotic entropy, the rate of escape and volume growth of a finite generated group, as follows:

$$h(\mu) \leqslant V(G)\ell(\mu) \ .$$

This inequality is called the *fundamental inequality.* Therefore, if the rate of escape is zero, then the asymptotic entropy is zero; consequently, the Poisson boundary is trivial. Under some conditions, Karlsson and Ledrappier [47] proved that the Poisson boundary is trivial if and only if the rate of escape is zero.

There are many studies that investigate the point at which the fundamental inequality is indeed equality. There are several examples which show that the fundamental inequality could either be a strict inequality or an equality. We invite the reader to see the recent works of [27], [68] and the references therein.

Example 4.2.5. Let $\tau_h = \min\{n : g_n = h\}$. Assume $F(h) = \mathbf{P}[\boldsymbol{g} : \tau_h < \infty]$ is finite for $\mathbf{P}$-almost every h. Let $\mathcal{G}_n = \{g \in G : -\ln F(g) \leqslant n\}$, then $\mathcal{G} = (\mathcal{G}_n)_{n\geqslant 0}$ is a sub-additive gauge. Indeed, $-\ln F(g)$ is the "distance" of g from the identity element in the *Green metric* [6]. The rate of escape with respect to the Green metric is equal to the asymptotic entropy [7].

Theorem 4.2.6. *Let $\mathcal{G}$ be a sub-additive gauge for a group G and suppose that μ has a finite first moment. Let τ be a randomized Markov stopping time with finite expectation. Then, the probability μ_τ has a finite first moment with respect to the gauge $\mathcal{G}$, and*

$$\ell(\mu_\tau) = \boldsymbol{E}(\tau)\ell(\mu) \ . \tag{4.2.2}$$

Proof: Let $L_n(\boldsymbol{g}, \boldsymbol{\omega}) = |g_n| - n\boldsymbol{E}(|g_1|)$. Then

$$\boldsymbol{E}(L_{n+1}(\boldsymbol{g}, \boldsymbol{\omega})|\mathcal{A}_0^n) = (n+1)\sum_h |g_n h|\mu(h) \ .$$

The sub-additivity of the gauge $\mathcal{G}$ implies that $\{L_n, \mathcal{A}_0^n\}$ is a super-martingale. Now applying the same trick as in Lemma 4.1.6 and Theorem 4.1.5, we get the equality (4.2.2). ∎

Theorem 4.1.8 and Theorem 4.2.2 imply that randomized stopping times with finite expectations preserve the fundamental equality.

Corollary 4.2.7. *Let G be a finitely generated group. If $h(\mu) = V(G)\ell(\mu)$ for some probability measure μ, then $h(\mu_\tau) = V(G)\ell(\mu_\tau)$ for any randomized stopping time with finite expectation.*

Remark 4.2.8. If A is a recurrent subgroup of G, then $\boldsymbol{E}(\tau_A) = [G : A]$, see [33]. Therefore, by Theorem 4.1.8 and Theorem 4.2.2, we have

$$h(\mu_{\tau_A}) = [G : A]h(\mu) \text{ and } \quad \ell(\mu_{\tau_A}) = [G : A]\ell(\mu) \; .$$

4.3 Open problems

There are some interesting questions related to transformed random walks via stopping times which can be considered as potential future projects.

Problem 4.3.1. It seems that the logarithmic moment condition of a randomized Markov stopping time could be eliminated from Theorem 3.6.5.

Problem 4.3.2. The construction of transformed random walks via (randomized) Markov stopping times can be naturally generalized to continuous groups. The proof related to both qualitative and quantitative properties of transformed random walks rely on the fact that the group is countable. Could the same results be obtained for continuous groups?

Problem 4.3.3. As mentioned earlier, the identification of the Poisson boundaries of random walks on free groups associated with measures with finite supports are well-studied. However, it is not known which of these Poisson boundaries coincide. Find probability measures with finite support on $\mathcal{F}_2$ whose Poisson boundaries coincide.

Problem 4.3.4. The other quantity related to a random walk is the *Green function*. Finding the relation between the Green functions after and before transformations via Markov stopping times can lead to study of *Martin boundary* of a random walk. We refer the reader to [73] for definitions of the Green functions and Martin boundary. What would be the relation of transformed Green functions with the original ones?

Appendix A

Random walks on equivalence classes

In this appendix, we provide the definition of random walks on equivalence classes and state the entropy criteria in this setup. All materials here are adapted from [44].

Let $(\mathcal{X}, \theta)$ be a Lebesgue space and R an equivalence relation that is a Borel subset of the product space $\mathcal{X} \times \mathcal{X}$. Assume that each equivalence class

$$[x] = R(x) = \{y \in \mathcal{X} \ : \ (y, x) \in R\}$$

is at most countable for each x in $\mathcal{X}$. The triple $(\mathcal{X}, \theta, R)$ is called *discrete measured equivalence* whenever $\theta(A) = 0$ implies that the set $[A] = \bigcup_{x \in A}[x]$ has also measure zero. In other words, the measure class of θ is preserved by the equivalence relation R.

Definition A.0.1. A *random walk along equivalence classes* of a discrete measured equivalence relation $(\mathcal{X}, \theta, R)$ is determined by a measurable family of leafwise transition probabilities $\{\pi_x\}_{x \in \mathcal{X}}$, so that any π_x is concentrated on the equivalence class $[x]$, and

$$(x, y) \to \pi_x(y)$$

is a measurable function on R. By $\pi_x^n(y)$, we shall denote the corresponding n–step transition probabilities, which are then also measurable as a function on R.

Like random walks on groups, a Markov operator P associated with transition probabilities $\{\pi_x\}_{x \in \mathcal{X}}$ acts on $L^\infty(\mathcal{X}, \theta)$ and its dual acts on the space of measures λ that is absolutely continuous with respect to θ.

The concept of entropy in this setup can be defined the same as in the group case.

Let $H(\pi_x^n)$ be the *entropies* of n–step transition probabilities. For the measure $\lambda \prec \theta$, define

$$H_n = \int_{\mathcal{X}} H(\pi_x^n) d\lambda(x) \ .$$

If H_1 is finite, then

$$\mathcal{H} = \lim_n \frac{H_n}{n}$$

exists. The quantity $\mathcal{H}$ is called asymptotic entropy.

Theorem A.0.2. *Let λ be a stationary measure with respect to the Markov operator P. If H_1 is finite, then $\mathcal{H} = 0$ if and only if for λ–almost everywhere x in $\mathcal{X}$, the Poisson boundary of leafwise Markov chain is trivial with respect to measure $\mathbf{P}_x$. The probability measure $\mathbf{P}_x$ is the measure in the space of sample paths of the associated leafwise Markov chain issued from a point x in $\mathcal{X}$.*

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