

UNSTEADY SPHERICALLY SYMMETRIC EXPANSION
OF A FIXED MASS OF GAS INTO VACUUM: A GAS
FRONT ANALYSIS

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ABSTRACT

The problem of an unsteady spherically symmetric expansion of a fixed mass of gas into vacuum has been considered in the limit of small source Knudsen number. It is observed from previous works that an asymptotic solution of the governing B-G-K equation breaks down at the gas front.

In the present work an attempt has been made to overcome this problem of breakdown by introducing appropriate scaling of the gas parameters and thereby the governing B-G-K equation is reformed. Introducing suitable expansions of gas parameters contained in the governing equation, it becomes possible to form a closed set of moment equations for small scaled time and therefrom, solutions of the moment equations are obtained. Consequently it is possible to determine the gas front characteristic dimensions and predict the behavior of gas properties at the gas front-entrance region.

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List of Symbols

\bar{r}	dimensional radial distance from the gas cloud
\bar{t}	dimensional time
θ	meridian angle
φ	azimuthal angle
ξ, ζ, η	orthogonal components of particle velocity in spherical coordinate system
\bar{f}	distribution function in terms of dimensional parameters
\bar{N}	dimensional local number density
\bar{T}	dimensional local temperature
R	gas constant
$R(t)$	gas front radius
N_0	initial number density
T_0	initial temperature
r_0	initial radius of gas cloud
\bar{u}	dimensional mean velocity of expanding gas
N	non-dimensionalized number density
T	non-dimensionalized temperature

t	non-dimensionalized time
r	non-dimensionalized radius
f	non-dimensionalized distribution function
\bar{z}	$\bar{z}/(RT_0)^{\frac{1}{2}} - u(r, t)$
\bar{s}	$[(\eta^2 + \bar{z}^2)/RT_0]^{\frac{1}{2}}$
$\bar{\rho}$	specific density of gas
u	non-dimensionalized mean gas velocity
A, A'	varying parameter used in asymptotic expansion (see page 15)
μ_0, μ	coefficient of viscosity
\bar{F}	equilibrium distribution function
F	non-dimensionalized equilibrium distribution function
ν	collision frequency
λ	mean free path
λ	$\frac{r}{t} = \frac{r_1}{t_1} = \lambda$
m	mass of a molecule
k	Boltzmann constant
\bar{c}	mean gas velocity in the rectangular coordinate system

K_n Knudsen number

γ ratio of specific heat

\bar{P} dimensional pressure

P_0 initial pressure

P non-dimensionalized pressure

α dimensional number

$t_1, \nu_1, \phi, \psi, \pi, \tau$ see equation (2 - 26)

$\Phi, \bar{\Phi}, t_1$ see equation (2 - 27)

$K \quad K = 1 - \frac{\Delta^2}{3}$

$\Delta, \Phi', \bar{\Phi}', t_2, \pi', \tau', u'$ see equation (4 - 1)

$\Lambda \quad \Lambda = t_2^2 \Delta$

PART 1

Introduction

Venture of man into space has steadily increased in this decade, thereby creating new subjects for human endeavor. Accordingly, the expansion of gases into vacuum has been extensively studied because of its application to the problem of rocket plumes exhausting into space, and also to the explosion of gas sources, such as fuel tanks etc. in space.

These studies are a basic necessity for rocket development and performance evaluation in space and for the security of life in space as well. This work deals with the unsteady spherically symmetric expansion of a fixed mass of gas into vacuum. The relevance of this subject derives from its relation to gas source explosion in space.

Early attempts⁽¹⁰⁾ at the solution of this problem have been based on a continuum model of the gas, and have used the Navier-Stokes equations as governing equations for the problem. However, recently more emphasis has been placed on low density

regimes in which the Navier-Stokes equations do not adequately describe the flow. In this case one must consider the more fundamental equation of fluid flow, namely the Boltzmann equation. The full Boltzmann equation is an integro-differential form of great complexity, and no general solution has yet been found. Consequently, in this work the simpler B-G-K model of the Boltzmann equation is used. This results in a considerable simplification since in the B-G-K model the integral collision term is replaced by a linear relaxation term.

It has been shown that the B-G-K model correctly describes the near equilibrium or almost inviscid behavior of a gas.⁽⁹⁾ In addition it has been shown that the B-G-K model is capable of modeling the flow in a completely non-equilibrium regime, namely that of hypersonic low density flow into vacuum.⁽⁴⁾ In the present work the B-G-K model is assumed, with the view that its earlier successes should lead one to expect that it will provide a realistic model of the flow. A common method of utilizing the Boltzmann equation is to construct moment equations from it by multiplying through by functions of the molecular velocity and

integrating over velocity space. In this way equations for density, velocity, temperature, heat transfer and higher moments can be obtained. the basic difficulty encountered in this technique is that the moment equations do not form a closed set.

In the case of a steady expansion into vacuum, Edwards and Cheng⁽²⁾ as well as Hamel and Willis⁽⁸⁾ truncated the moment equations by means of the hypersonic^(*) approximation; in other words, they showed that the heat transfer terms in the energy equation were of the order of $1/M$ where M is Mach number, and hence vanished in the limit as $M \rightarrow \infty$. They were then able to solve the resulting set of equations.

Freeman⁽³⁾ showed that these same results could be obtained by asymptotically scaling the Boltzmann equation itself. This technique has been extended to study unsteady cylindrical expansion,⁽⁵⁾ the axi-symmetric jet⁽⁷⁾ and unsteady spherical expansion of a fixed mass of gas into vacuum.⁽⁶⁾ Freeman and Grundy's work⁽⁵⁾ on unsteady cylindrical expansion revealed several regions of interest.

(*): hypersonic speed in this case assumes flow speed is greater than Mach 5.

They demonstrated the existence of a region in which the flow behaved in a near-equilibrium fashion so that the thermodynamic variables were given by the solutions of the inviscid Navier-Stokes equations. This inviscid solution was shown to be the first term of an asymptotic series solution in powers of a small parameter depending on the initial conditions. They further demonstrated that this so-called inner solution is not uniformly valid, but breaks down for large values of radius and time.

These authors then constructed a scaled equation to describe the outer non-equilibrium region; consequently it was demonstrated that these equations were mathematically equivalent to the non-equilibrium equations found by Hamel and Willis⁽⁸⁾ and Edwards and Cheng⁽²⁾ for the steady spherical expansion. Their solution to this hypersonic flow problem is an example of non-equilibrium hypersonic similarity. A further breakdown of the solution was shown to exist. This was associated with the gas front, which is defined as that contour on which the density is zero. Obviously such a front is an approximation since there will always be some gas mol-

ecules present no matter how far away from the source.

Freeman and Grundy⁽⁵⁾ showed that the equations for this front region could be constructed, but that they were no longer of closed form so that a solution in the region of the front is as difficult as a solution of the full B-G-K equation. They were able to obtain asymptotic solutions to these front equations for small and large values of time and also demonstrated that these solutions matched asymptotically near-equilibrium and non-equilibrium solutions respectively.

Grundey and Thomas⁽⁶⁾ considered the related problem of the unsteady spherically symmetric expansion. They also were able to find a near equilibrium solution, but they showed that the breakdown of this solution differed from that encountered by Freeman and Grundy. They found that an outer region existed in which the leading term was again the equilibrium solution. Non-equilibrium effects arose only in the second term. They also pointed out that their solution become invalid near the gas

front, but gave no details of this aspect.

The main purpose of the present work is to examine the front problem corresponding to the work of Grundy and Thomas.⁽⁶⁾ It will be shown that a region similar to that found by Freeman and Grundy exists, and that again an asymptotic solution can be constructed. It is unfortunate that the difficulty regarding the non-closure of the moment equations is again encountered, and no full solution to the front region can be obtained.

In order to present this work in a comprehensive form, the work of Grundy and Thomas is extensively discussed in Part II.

PART II

Previous Theoretical Development

The general expression for the Boltzmann equation in a spherical coordinate system and excluding external force effects is

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} + \bar{\xi} \frac{\partial \bar{f}}{\partial \bar{r}} + \frac{\eta}{\bar{r} \sin \theta} \frac{\partial \bar{f}}{\partial \varphi} + \frac{\zeta}{\bar{r}} \frac{\partial \bar{f}}{\partial \theta} \\ + \left(\frac{\eta^2 + \zeta^2}{\bar{r}} \right) \frac{\partial \bar{f}}{\partial \bar{\xi}} - \left(\frac{\eta \zeta}{\bar{r}} \cot \theta + \frac{\eta \bar{\xi}}{\bar{r}} \right) \frac{\partial \bar{f}}{\partial \eta} \\ + \left(\frac{\eta^2}{\bar{r}} \cot \theta - \frac{\zeta \bar{\xi}}{\bar{r}} \right) \frac{\partial \bar{f}}{\partial \zeta} = \left(\frac{\partial \bar{f}}{\partial t} \right)_{coll} \quad (2 - 1) \end{aligned}$$

where \bar{r} is the distance from the origin of the coordinate system, \bar{t} the time, θ and φ the meridian and azimuthal angles respectively, and $\bar{\xi}$, ζ and η the orthogonal components of particle velocity.

Here the symbol bar(-) is used to represent dimensional quantities. The distribution function in six-dimensional phase space, although itself dimensionless, is represented by \bar{f} since it is expressed in terms of dimensional variables.

The right hand side of equation (2 - 1) represents the collision integral term. Fig. 1 shows graphical representation of the phase space coordinate system and its variables.

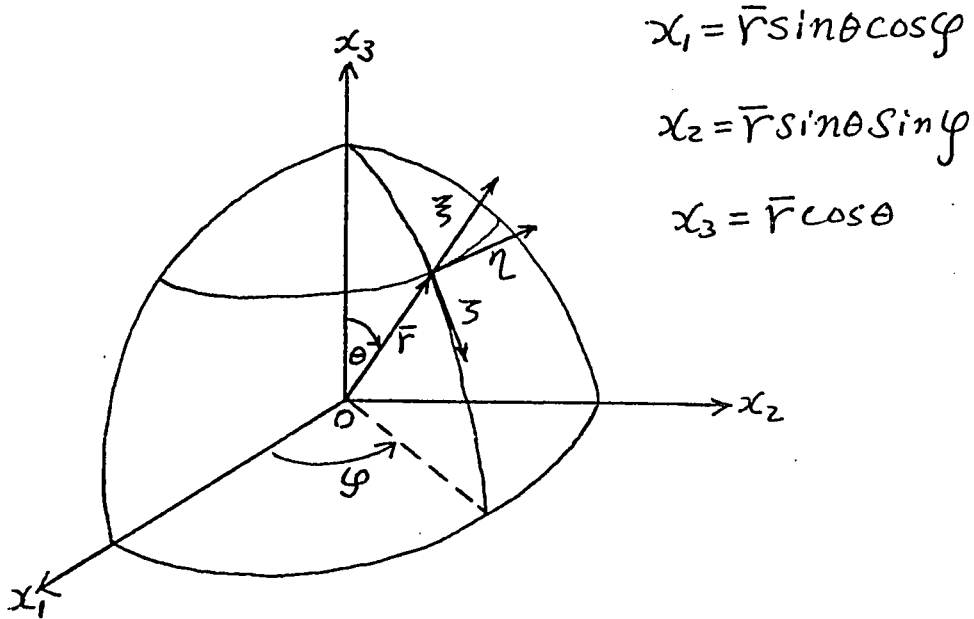


Fig. 1. Graphical Representation of Phase Space.

Since we are concerned with the unsteady, spherically symmetric form of gas expansion, the effects of gradients of the distribution function with respect to the angles θ and φ vanish and the resulting Boltzmann equation becomes,

$$\frac{\partial \bar{f}}{\partial t} + \frac{\xi}{\bar{r}} \frac{\partial \bar{f}}{\partial \bar{r}} + \frac{(\eta^2 + \zeta^2)}{\bar{r}} \frac{\partial \bar{f}}{\partial \xi} - \frac{\xi \zeta}{\bar{r}} \frac{\partial \bar{f}}{\partial \zeta} - \frac{\xi \eta}{\bar{r}} \frac{\partial \bar{f}}{\partial \eta} + \frac{\cot \theta}{\bar{r}} \left(\eta^2 \frac{\partial \bar{f}}{\partial \zeta} - \eta \zeta \frac{\partial \bar{f}}{\partial \eta} \right) = \left(\frac{\partial \bar{f}}{\partial t} \right)_{\text{coll}} \quad (2 - 2)$$

Local number density \bar{N} and temperature \bar{T} are defined as

$$\bar{N} = \iiint_{-\infty-\infty-\infty}^{\infty\infty\infty} \bar{f} d\bar{\zeta} d\bar{\zeta} d\eta \quad (2 - 3)$$

$$3\bar{N}R\bar{T} = \iiint \bar{f} [(c\bar{\zeta} - \bar{u})^2 + \eta^2 + \bar{\zeta}^2] d\bar{\zeta} d\bar{\zeta} d\eta \quad (2 - 4)$$

Where \bar{u} denotes the mean radial velocity of the expanding gas cloud. Now, all variables are non-dimensionalized in equation (2 - 2) with respect to the initial conditions, denoted by the subscript 0 ,

$$\left. \begin{aligned} N &= \bar{N}/N_0 \\ T &= \bar{T}/T_0 \\ u &= \bar{u}/(RT_0)^{1/2} \\ t &= \bar{t}(RT_0)^{1/2}/r_0 \\ r &= \bar{r}/r_0 \\ f &= \bar{f}(RT_0)^{3/2}/N_0 \end{aligned} \right\} \quad (2 - 5)$$

Here r_0 is the initial radius of the gas cloud, N_0 the initial number density, and T_0 the initial gas temperature.

In order to simplify the equation, a new set of velocity components are defined as

$$\left. \begin{aligned} \xi_1 &= \xi / (RT_0)^{\frac{1}{2}} - u(r, t) \\ \rho &= [(\eta^2 + \zeta^2) / (RT_0)]^{\frac{1}{2}} \end{aligned} \right\} \quad (2 - 6)$$

Where ξ_1 represent the dimensionless component of the thermal velocity which is radially directed, and ρ the dimensionless component of velocity perpendicular to ξ_1 . The radially directed particle velocity ξ is non-dimensionalized by dividing it by $(RT_0)^{\frac{1}{2}}$ whereas $u(r, t)$ initially represents a dimensionless quantity. The same division by $(RT_0)^{\frac{1}{2}}$ has been applied to the velocity components η and ζ where η , ζ and ξ are orthogonal to each other.

With the new variables of equations (2 - 6), equation (2 - 2) can be transformed into a non-dimensionalized Boltzmann equation. Now the distribution function \bar{f} which was a function of the dimensional variables r , t , ξ , ζ and η becomes a function of new variables r , t , ξ_1 and

ρ ; namely $\bar{f} = f(r , t , \bar{z}_1 , \rho)$,
 where $f(r , t , \bar{z}_1 , \rho)$ indicates the
 distribution function in terms of non-dimensionalized
 variables.

Each term of equation (2 - 2) is transformed
 as follows. Since $f = f(r , t , \bar{z}_1 , \rho)$,
 $d\bar{f}$ can be written as

$$d\bar{f} = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial f}{\partial \rho} d\rho$$

Dividing by $d\bar{t}$

$$\frac{d\bar{f}}{d\bar{t}} = \frac{\partial f}{\partial r} \frac{dr}{d\bar{t}} + \frac{\partial f}{\partial t} \frac{dt}{d\bar{t}} + \frac{\partial f}{\partial \bar{z}_1} \frac{d\bar{z}_1}{d\bar{t}} + \frac{\partial f}{\partial \rho} \frac{d\rho}{d\bar{t}}$$

The partial differential form of the above expression
 becomes

$$\begin{aligned} \left(\frac{\partial \bar{f}}{\partial \bar{t}} \right)_{r, \bar{z}_1, \rho} &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial \bar{t}} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial \bar{t}} + \frac{\partial f}{\partial \bar{z}_1} \frac{\partial \bar{z}_1}{\partial \bar{t}} + \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial \bar{t}} \\ &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial \bar{t}} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial \bar{t}} + \frac{\partial f}{\partial \bar{z}_1} \frac{\partial \bar{z}_1}{\partial \bar{t}} + \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial \bar{t}} \end{aligned}$$

Utilizing equations (2 - 5) and (2 - 6), the partial
 differential terms $\frac{\partial t}{\partial \bar{t}}$, $\frac{\partial r}{\partial \bar{t}}$, $\frac{\partial \rho}{\partial \bar{t}}$ and $\frac{\partial \bar{z}_1}{\partial \bar{t}}$
 become

$$\frac{\partial t}{\partial \bar{t}} = \frac{(RT_0)^{\frac{1}{2}}}{r_0}$$

$$\frac{\partial r}{\partial t} = 0$$

$$\frac{\partial p}{\partial t} = 0$$

$$\frac{\partial \xi_1}{\partial t_1} = -\frac{\partial u}{\partial t} \left(\because \xi_1 = \frac{r}{r_0} (RT_0)^{\frac{1}{2}} - u(r, t) \right)$$

hence $\left(\frac{\partial f}{\partial \bar{t}}\right)_{\bar{r}, \bar{\xi}_1, \bar{p}, \bar{t}}$ becomes

$$\begin{aligned} \left(\frac{\partial f}{\partial \bar{t}}\right)_{\bar{r}, \bar{\xi}_1, \bar{p}, \bar{t}} &= \frac{\partial f}{\partial t} \frac{(RT_0)^{\frac{1}{2}}}{r_0} + \frac{\partial f}{\partial \xi_1} (-) \frac{\partial u}{\partial t} \frac{(RT_0)^{\frac{1}{2}}}{r_0} \\ &= \frac{(RT_0)^{\frac{1}{2}}}{r_0} \left(\frac{\partial f}{\partial t} - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \xi_1} \right) \end{aligned}$$

Similarly the rest of the differential terms in equation (2 - 2) are transformed as follows:

$$\begin{aligned} \frac{\partial f}{\partial \bar{r}} &= \frac{\partial f}{\partial r} \frac{\partial r}{\partial \bar{r}} + \frac{\partial f}{\partial \xi_1} \frac{\partial \xi_1}{\partial r} \frac{\partial r}{\partial \bar{r}} \\ &= \frac{\partial f}{\partial r} \frac{1}{r_0} + \frac{\partial f}{\partial \xi_1} (-) \frac{\partial u}{\partial r} \frac{1}{r_0} \\ &= \frac{1}{r_0} \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \xi_1} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial \bar{\xi}} &= \frac{\partial f}{\partial \xi_1} \frac{\partial \xi_1}{\partial \bar{\xi}} = \frac{\partial f}{\partial \xi_1} \frac{1}{(RT_0)^{\frac{1}{2}}} \\ \frac{\partial f}{\partial \bar{p}} &= \frac{\partial f}{\partial p} \frac{\partial p}{\partial \bar{p}} = \frac{\partial f}{\partial p} \frac{1}{RT_0} \end{aligned}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial \eta} = \frac{\partial f}{\partial \rho} \frac{1}{RT_0} \frac{\eta}{\rho}$$

Again utilizing equations (2 - 5) and (2 - 6) the second term of equation (2 - 2) becomes

$$\frac{\bar{z}}{r} \frac{\partial \bar{f}}{\partial r} = (\bar{z}_1 + u) \frac{(RT_0)^{\frac{1}{2}}}{r_0} \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right)$$

Similarly third, fourth and fifth terms become

$$\frac{(\eta^2 + z^2)}{\bar{r}} \frac{\partial \bar{f}}{\partial \bar{z}} = \frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{\rho^2}{r} \frac{\partial f}{\partial \bar{z}_1}$$

$$-\frac{\bar{z}z}{\bar{r}} \frac{\partial \bar{f}}{\partial z} = -\frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{(\bar{z}_1 + u)}{r} \frac{z^2}{\rho} \frac{\partial f}{\partial \rho}$$

$$-\frac{\bar{z}\eta}{\bar{r}} \frac{\partial \bar{f}}{\partial \eta} = -\frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{(\bar{z}_1 + u)}{r} \frac{\eta^2}{\rho} \frac{\partial f}{\partial \rho}$$

Combining fourth and fifth terms, there results

$$-\left(\frac{\bar{z}z}{\bar{r}} \frac{\partial \bar{f}}{\partial z} + \frac{\bar{z}\eta}{\bar{r}} \frac{\partial \bar{f}}{\partial \eta} \right) = -\frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{(\bar{z}_1 + u)}{r} \rho \frac{\partial f}{\partial \rho}$$

The last term of the left hand side of equation (2 - 2) becomes zero since two terms inside of the bracket become equal, namely,

$$\eta^2 \frac{\partial \bar{f}}{\partial \xi} = \eta^2 \frac{1}{RT_0} \frac{\xi}{\rho} \frac{\partial f}{\partial \rho}$$

and

$$\eta \xi \frac{\partial \bar{f}}{\partial \eta} = \eta^2 \frac{1}{RT_0} \frac{\xi}{\rho} \frac{\partial f}{\partial \rho}$$

from which, the last term becomes

$$\frac{\cot \theta}{r} \left(\eta^2 \frac{1}{RT_0} \frac{\xi}{\rho} \frac{\partial f}{\partial \rho} - \eta^2 \frac{1}{RT_0} \frac{\xi}{\rho} \frac{\partial f}{\partial \rho} \right) = 0$$

Finally, the right hand side of equation (2 - 2) becomes

$$\left(\frac{\partial \bar{f}}{\partial t} \right)_{coll} = \left[\frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{\partial f}{\partial t} \right]_{coll}$$

Consequently, equation (2 - 2) in non - dimensionalized form becomes

$$\begin{aligned} & \frac{(RT_0)^{\frac{1}{2}}}{r_0} \left(\frac{\partial f}{\partial t} - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \xi_1} \right) + \frac{(RT_0)^{\frac{1}{2}}}{r_0} (\xi_1 + u) \cdot \\ & \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \xi_1} \right) + \frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{\rho^2}{r} \frac{\partial f}{\partial \xi_1} \\ & - \frac{(RT_0)^{\frac{1}{2}}}{r_0} (\xi_1 + u) \frac{\rho}{r} \frac{\partial f}{\partial \rho} = \left[\frac{(RT_0)^{\frac{1}{2}}}{r_0} \frac{\partial f}{\partial t} \right]_{coll} \end{aligned}$$

Rewriting the above expression

$$\frac{\partial f}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} + \frac{p^2}{r} \frac{\partial f}{\partial \bar{z}_1}$$

$$- \int \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f}{\partial p} = - \frac{\gamma_0}{(RT_0)^{\frac{1}{2}}} \left[\frac{(RT_0)^{\frac{1}{2}}}{\gamma_0} \frac{\partial f}{\partial t} \right]_{coll}$$

The reasons for using the B-G-K model of the Boltzmann equation have been discussed previously in Part I.

Accordingly, the right hand side of the above expression, the Boltzmann collision integral term, is replaced by the simpler non-dimensionalized B-G-K model, $AN(F-f)$.

$$\frac{\partial f}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} + \frac{p^2}{r} \frac{\partial f}{\partial \bar{z}_1}$$

$$- \int \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f}{\partial p} = AN(F-f) \quad (2-7)$$

Here A is defined by Grundy and Thomas⁽⁶⁾

$$A = \frac{N_0 \gamma_0 \Omega}{(RT_0)^{\frac{1}{2}}}, \quad \text{where } \Omega = \frac{T_0}{M_0} \quad (2-8)$$

and μ_0 is the coefficient of viscosity.

Q. viscosity

The non - dimensional form of the equilibrium distribution function F which appears in the B-G-K model is derived from the dimensional form of the equilibrium distribution function \bar{F} , which is given by

$$\bar{F} = \frac{\bar{N}}{(2\pi R\bar{T})^{3/2}} \exp^{-\frac{1}{2R\bar{T}}[(\bar{z}-\bar{u})^2 + \eta^2 + \bar{z}^2]}$$

Upon replacing dimensional variables in \bar{F} by the non - dimensionalized variables defined in equations (2 - 5) and (2 - 6), \bar{F} becomes

$$\bar{F} = F \frac{N_0}{(RT_0)^{3/2}} = \frac{N_0}{(2\pi RT_0)^{3/2}} \frac{N}{T^{3/2}} \exp^{-\frac{1}{2T}(\bar{z}_1^2 + \rho^2)}$$

Thus

$$F = \frac{N}{(2\pi T)^{3/2}} \exp^{-\frac{1}{2T}(\bar{z}_1^2 + \rho^2)} \quad (2 - 9)$$

The original dimensional B-G-K model⁽¹⁾ is defined as $\nu(\bar{F} - \bar{f})$. The collision frequency ν is generally taken as a function of the phase velocity except in the special case of Maxwellian molecules.

For Maxwellian molecules it can be shown that the collision frequency becomes independent of molecular velocity. Liepmann, Narashimha and Chahine⁽⁹⁾, defined the B-G-K model as $A'N(F-f)$ and showed that A' is in fact proportional to $\frac{T}{\mu}$ for Maxwellian molecules where T and μ are the temperature and the coefficient of viscosity of the gas respectively. Proof of this follows; from kinetic theory, they showed that $\mu = \frac{R}{m}(\frac{T}{\mu})$, and since R and m are constants,

$$A' \propto \frac{T}{\mu}$$

It can be seen that Grundy and Thomas⁽⁶⁾ have defined A in equation (2 - 8) in a similar manner to that defined by Liepmann, Narashimha and Chahine⁽⁹⁾. It must be noted however that A is independent of the molecular velocity. From equation (2 - 8), it can be seen that A is proportional to $\frac{T_0}{\mu_0}$ corresponding to the proportionality of A' to $\frac{T}{\mu}$ as obtained by Liepmann, Narashimha and Chahine. From equation (2 - 8), it can be shown that A becomes inversely proportional to Knudsen number which is defined as

$$K_n = \frac{\lambda_0}{r_0}$$

where λ_0 is the initial mean free path of a molecule. Rewriting equation (2 - 8), A becomes

$$A = \frac{N_0 T_0}{(R T_0)^2} \frac{1}{\mu_0}$$

Viscosity μ_0 can be expressed as a function of mean free path λ_0 . From the theory of momentum transport Chapman and Cowling⁽¹⁾ and Vincenti and Kruger⁽¹⁵⁾ showed that viscosity μ is proportional to mean free path λ , i.e.

$$\mu = \frac{1}{2} u \rho \bar{c} \lambda$$

where $u = 0.998$, $\rho = m N$, $\bar{c} = \sqrt{\bar{c}^2} / 1.086$,

$$\bar{c}^2 = \frac{3kT}{m} \text{ and } k \text{ is the Boltzmann constant.}$$

Thus

$$\mu_0 = \frac{1}{2} (0.998) m N_0 (\sqrt{\bar{c}^2} / 1.086) \cdot \lambda_0$$

Substituting μ_0 into A , A becomes inversely proportional to Knudsen number,

$$A \propto \frac{1}{Kn}$$

Physically we are concerned with a gas cloud initially bounded by a sphere of radius γ_0 within which exist uniform gas properties. Expansion takes place with spherical symmetry resulting in decreased gas cloud density with increased particle mean free path. This may be seen from the following relationship between mean free path and number density, ⁽¹⁵⁾

$$\lambda = \frac{1}{\sqrt{2} \pi d^2 N}$$

where d is the diameter of a molecule.

Hence, during expansion Knudsen number increases and A decreases, i.e.

$$Kn. \longrightarrow \infty$$

$$A \longrightarrow 0$$

Compression could lead to opposite limits.

Hence, the parameter A becomes a measure of the degree of departure from equilibrium state of the gas cloud.

As A approaches infinity, the degree of departure from the equilibrium state decreases. In this work the method of asymptotic solution has been adapted as a means of solving B-G-K equation (2 - 7). In this method, the distribution function f is expanded in terms of increasing value of A .

Before going into the solution of equation (2 - 7), a further interpretation of the expanding gas phenomena becomes necessary.

There exist three mutually distinctive regions of the expanding gas cloud with regard to the number of molecular collisions. These three regions are designated as the collision dominated region, the transitional region and the free-molecule flow region respectively. Naturally the collision dominated region is that near the original gas cloud before much of the original gas has expanded.

The free-molecule region is that in which it is assumed no collisions take place, hence this region is generally far from the initial gas cloud. The intermediate region is called the transitional region. The collision dominated region is also known as the continuum flow region in which the somewhat simpler Navier-Stokes equations may be applied. Mirels and Mullen⁽¹⁰⁾ as well as others worked on the problem in this region. In the other regions, the Navier-Stokes equation can not be applied and hence the Boltzmann equation must be used. The asymptotic method of solution has been utilized to solve the governing equation in these regions.

The general technique used here is to extend the solution from the known equilibrium value to the unknown non-equilibrium region by successive perturbation. In order to solve the B-G-K equation (2 - 7) the distribution function f is expanded in terms of A as follows

$$f = f_0 + A^{-1} f_1 + O(A^{-2}) \quad (2 - 10)$$

where $O(A^{-2})$ designates that the approximation to f is extended up to the first order in A^{-1} . Here f_0 is the zeroth order or equilibrium solution of f , and f_1 is the first order solution of f . It should be indicated here that the nature of this asymptotic expansion is such that only a few terms are required to approximate f since higher order terms are indeed very small when divided by powers of A and A approaches infinity.

Substituting equation (2 - 10) into equation (2 - 7)

$$\begin{aligned} & \frac{\partial f_0}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_0}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_0}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f_0}{\partial \bar{z}_1} \\ & + \frac{p^2}{r} \frac{\partial f_0}{\partial \bar{z}_1} - \rho \left(\frac{u + \bar{z}_1}{r} \right) \frac{\partial f_0}{\partial p} \\ & + \left[\frac{\partial f_1}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_1}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_1}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f_1}{\partial \bar{z}_1} \right] \end{aligned}$$

$$+ \frac{\rho^2}{r} \frac{\partial f_1}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_1}{\partial \rho} \Big] A^{-1} = AN (F - f_0 - A^{-1} f_1)$$

Dividing the above expression by A ,

$$\begin{aligned} & \frac{1}{A} \left[\frac{\partial f_0}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_0}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_0}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f_0}{\partial \bar{z}_1} \right. \\ & + \frac{\rho^2}{r} \frac{\partial f_0}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_0}{\partial \rho} \\ & + \frac{1}{A^2} \left[\frac{\partial f_1}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_1}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_1}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f_1}{\partial \bar{z}_1} \right. \\ & \left. + \frac{\rho^2}{r} \frac{\partial f_1}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_1}{\partial \rho} \right] = N (F - f_0 - A^{-1} f_1) \end{aligned}$$

Letting A approach infinity,

$$0 = N (F - f_0)$$

$$\therefore f_0 = F \quad (2 - 11)$$

Equation (2 - 11) is the zeroth order solution for the distribution function of equation (2 - 10).

In order to obtain the first order solution of f , it is necessary to make A approach infinity in such a way that the A^{-1} term remains but higher terms go to zero. This means that the region of solution may include not only the gas cloud but also a region extending slightly from the original gas cloud.

To obtain the first order solution, equation (2 - 7) is rewritten

$$f = F - \frac{A^{-1}}{N} \left[\frac{\partial f}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} \right. \\ \left. + \frac{\rho^2}{r} \frac{\partial f}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f}{\partial \rho} \right]$$

substituting $f = f_0 + A^{-1} f_1 + O(A^{-2})$ into the above equation

$$f_0 + A^{-1} f_1 = F - \frac{A^{-1}}{N} \left[\frac{\partial f_0}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_0}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_0}{\partial \bar{z}_1} \right) \right. \\ \left. - \frac{\partial u}{\partial t} \frac{\partial f_0}{\partial \bar{z}_1} + \frac{\rho^2}{r} \frac{\partial f_0}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_0}{\partial \rho} \right] - \frac{A^{-2}}{N} \left[\frac{\partial f_1}{\partial t} \right. \\ \left. + (\bar{z}_1 + u) \left(\frac{\partial f_1}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_1}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f_1}{\partial \bar{z}_1} \right]$$

$$+ \frac{\rho^2}{r} \frac{\partial f_1}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_1}{\partial \rho}]$$

Neglecting terms involving A^{-2}

$$f_0 + A^{-1} f_1 = F - \frac{A^{-1}}{N} \left[\frac{\partial f_0}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f_0}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f_0}{\partial \bar{z}_1} \right) \right. \\ \left. - \frac{\partial u}{\partial t} \frac{\partial f_0}{\partial \bar{z}_1} + \frac{\rho^2}{r} \frac{\partial f_0}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f_0}{\partial \rho} \right]$$

Since $f_0 = F$, f_1 becomes

$$f_1 = - \frac{1}{N} \left[\frac{\partial F}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial F}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial F}{\partial \bar{z}_1} \right) \right. \\ \left. - \frac{\partial u}{\partial t} \frac{\partial F}{\partial \bar{z}_1} + \frac{\rho^2}{r} \frac{\partial F}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial F}{\partial \rho} \right]$$

where F is given by equation (2 - 9).

When the above expression is further simplified by the use of the moment equations generated from equation (2 - 7), one obtains a Chapman-Enskog⁽¹⁾ type solution for the B-G-K model of the Boltzmann equation.

Since F is given as

$$F = \frac{N}{(2\pi T)^{3/2}} \exp^{-\frac{1}{2T} (\bar{z}_1^2 + \rho^2)}$$

it is possible to calculate

$$\left(\frac{\partial F}{\partial r}\right)_{t, \bar{z}, \bar{s}}, \left(\frac{\partial F}{\partial t}\right)_{r, \bar{z}, \bar{s}}, \left(\frac{\partial F}{\partial \bar{z}_1}\right)_{r, t, \bar{s}} \text{ and } \left(\frac{\partial F}{\partial \bar{s}}\right)_{r, t, \bar{z}_1}$$

Before going into the actual calculation it is necessary to know the relationship between N and T . This may be established using the following assumptions made by Mirels and Mullen⁽¹⁰⁾ which states that the expanding gas from the initial gas cloud state follows an isentropic process within the continuum flow region. The isentropic relation is

$$\bar{p} \bar{v}^\gamma = \text{const.}$$

Where \bar{p} , the pressure, \bar{v} , the specific volume and γ , the ratio of specific heat of gas respectively. Consequently

$$\frac{\bar{p}}{\bar{\rho}^\gamma} = \frac{P_0}{\rho_0^\gamma}$$

where the subscript "0" indicates initial values and here the primed ρ is meant to indicate density.

$\gamma = \frac{5}{3}$ for a monatomic gas. Since $\bar{p}' = m\bar{N}$ and $\bar{\rho}' = m\bar{N}_0$

$$\bar{p} \bar{N}_0^\gamma = P_0 N^{-\gamma}$$

$$\therefore \frac{\bar{p}}{P_0} = \left(\frac{\bar{N}}{N_0}\right)^\gamma$$

$$\therefore P = N^{5/3} \quad (2 - 12)$$

From the definition of temperature

$$3NRT = \iiint_{-\infty}^{\infty} \bar{f} [(\bar{z} - \bar{u})^2 + \eta^2 + z^2] d\bar{z} d\eta dz$$

The non-dimensional expression for temperature becomes

$$3NN_0RTT_0 = \int_0^\pi \int_0^\infty \int_0^\infty \frac{N_0}{(RT_0)^{3/2}} \bar{f} [(RT_0)\bar{z}_1^2 + (RT_0)\rho^2] [J] d\bar{z}_1 d\rho d\theta'$$

where $[J]$ is the Jacobian of the transformation

between the two coordinate systems, namely the \bar{z} ,

z, η and \bar{z}_1, ρ, θ' coordinate systems.

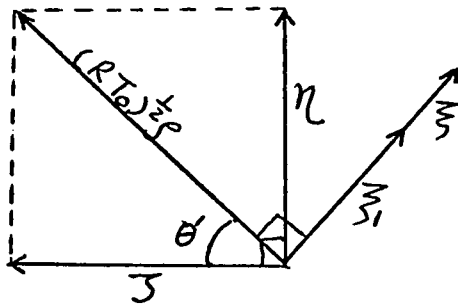


Fig. 2. Two Velocity Space Coordinate Systems; \bar{z} ,

z, η and \bar{z}_1, ρ .

From Fig. 2 ξ and η can be written

$$\xi = (RT_0)^{\frac{1}{2}} \rho \cos \theta'$$

$$\eta = (RT_0)^{\frac{1}{2}} \rho \sin \theta'$$

and from the definition of ξ_1 ,

$$\xi = (RT_0)^{\frac{1}{2}} U(\eta, t) = (RT_0)^{\frac{1}{2}} \xi_1$$

Hence the Jacobian $[J]$ becomes

$$[J] = \begin{vmatrix} \frac{\partial \xi}{\partial \xi_1} & \frac{\partial \xi}{\partial \rho} & \frac{\partial \eta}{\partial \xi_1} \\ \frac{\partial \xi}{\partial \rho} & \frac{\partial \xi}{\partial \rho} & \frac{\partial \eta}{\partial \rho} \\ \frac{\partial \xi}{\partial \theta'} & \frac{\partial \xi}{\partial \theta'} & \frac{\partial \eta}{\partial \theta'} \end{vmatrix}$$

$$= \begin{vmatrix} (RT_0)^{\frac{1}{2}} & 0 & 0 \\ 0 & \frac{(RT_0)^{\frac{1}{2}}}{\cos \theta'} & \frac{(RT_0)^{\frac{1}{2}}}{\sin \theta'} \\ 0 & -(RT_0)^{\frac{1}{2}} \rho \sin \theta' & (RT_0)^{\frac{1}{2}} \rho \cos \theta' \end{vmatrix}$$

$$= (RT_0)^{\frac{1}{2}} (RT_0 \rho + RT_0 \rho) = 2(RT_0)^{\frac{3}{2}} \rho$$

Substituting [J] into the previous expression

$$3NT = 2 \iiint f [\bar{z}_1^2 + \rho^2] \rho d\rho d\bar{z}_1 d\theta'$$

Integrating θ' from 0 to π (refer to Fig. 2)

$$3NT = 2\pi \iint f (\bar{z}_1^2 + \rho^2) \rho d\rho d\bar{z}_1$$

Using the kinetic theory definition of pressure⁽¹⁵⁾
expressions for P_r and P_θ are

$$P_r = 2\pi \iint f \bar{z}_1^2 \rho d\rho d\bar{z}_1$$

$$P_\theta = 2\pi \iint f \rho^2 \rho d\rho d\bar{z}_1$$

where pressure P_r is radially directed and P_θ is directed perpendicular to P_r .

Hence, non-dimensional temperature becomes⁽¹⁵⁾

$$3NT = P_r + P_\theta$$

hence $NT = \frac{1}{3}(P_r + P_\theta) = P$

Thus, the non-dimensional total pressure P

$$P = NT$$

Combining equations (2 - 12) and (2 - 13), the relation between N and T becomes

$$N = T^{3/2} \quad (2 - 14)$$

Substituting equation (2 - 14) into F

$$F = \frac{1}{(2\pi)^{3/2}} \exp^{-\frac{1}{2T}(\xi_1^2 + p^2)}$$

It is possible now to calculate gradients of the distribution function F with respect to the new phase space variables,

$$\begin{aligned} \left(\frac{\partial F}{\partial t}\right)_{r, \xi_1, p} &= \frac{1}{(2\pi)^{3/2}} \left[\left(\exp^{-\frac{\xi_1^2}{2}}\right) \frac{\xi_1^2}{2} T^{-2} \frac{\partial T}{\partial t} \exp^{-\frac{p^2}{2T}} \right. \\ &\quad \left. + \left(\exp^{-\frac{\xi_1^2}{2T}}\right) \left(\exp^{-\frac{p^2}{2T}}\right) \frac{p^2}{2} T^{-2} \frac{\partial T}{\partial t} \right] \\ &= \frac{1}{(2\pi)^{3/2}} \exp^{-\frac{\xi_1^2 + p^2}{2T}} \left[\frac{\xi_1^2}{2} T^{-2} \frac{\partial T}{\partial t} + \frac{p^2}{2} T^{-2} \frac{\partial T}{\partial t} \right] \\ &= F \frac{T^{-2} \partial T}{\partial t} (\xi_1^2 + p^2) \end{aligned}$$

$$\left(\frac{\partial F}{\partial r}\right)_{t, \xi_1, p} = F \frac{T^{-2} \partial T}{\partial r} (\xi_1^2 + p^2)$$

$$\left(\frac{\partial F}{\partial \xi_1}\right)_{r, t, p} = \frac{1}{(2\pi)^{3/2}} \exp^{-\frac{p^2 T^{-1}}{2}} \left(\exp^{-\frac{\xi_1^2 T^{-1}}{2}}\right) (-)\xi_1 T^{-1} = (-)F \xi_1 T^{-1}$$

$$\left(\frac{\partial F}{\partial \beta}\right)_{r,t,\bar{z}_1} = \frac{1}{(2\pi)^{3/2}} \exp^{-\frac{\bar{z}_1^2 T^{-1}}{2}} \left(\exp^{-\frac{\beta T^{-1}}{2}}\right) \psi \beta T^{-1} = \psi F \beta T^{-1}$$

substituting these expressions into f_1

$$f_1 = -\frac{F}{N} \left\{ \left[-\frac{T^{-2}}{2} \frac{\partial T}{\partial t} (\bar{z}_1^2 + \beta^2) \right] + (\bar{z}_1 + u) \left[\frac{T^{-2}}{2} \frac{\partial T}{\partial r} (\bar{z}_1^2 + \beta^2) \right] \right. \\ \left. + (\bar{z}_1 + u) \frac{\partial u}{\partial r} \bar{z}_1 T^{-1} + \frac{\partial u}{\partial t} \bar{z}_1 T^{-1} - \frac{p^2}{r} \bar{z}_1 T^{-1} + \frac{p^2}{r} (\bar{z}_1 + u) T^{-1} \right\}$$

A further simplification is possible when use is made of equations (2 - 3) and (2 - 7).

Recalling equation (2 - 3)

$$\bar{N} = \iiint_{-\infty}^{\infty} \bar{f} d\bar{z} d\beta d\eta$$

non-dimensionalizing \bar{N} and transforming it into a function of the new velocity space variables

$$N N_0 = \iiint_{-\infty}^{\infty} \frac{N_0}{(R T_0)^{3/2}} f[J] d\bar{z}_1 d\beta d\theta'$$

$$= 2\pi \frac{(RT_0)^{3/2}}{N_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N_0}{(RT_0)^{3/2}} f d\bar{z}_1 d\rho$$

hence

$$N = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \rho d\rho d\bar{z}_1 \quad (2 - 15)$$

From equation (2 - 7), the general expression for the moment equations becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma \left[\frac{\partial f}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} + \frac{\rho^2}{r} \frac{\partial f}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f}{\partial \rho} \right] \rho d\rho d\bar{z}_1 = AN \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma (F - f) \rho d\rho d\bar{z}_1 \quad (2 - 15')$$

Where Γ takes an values of 1 and \bar{z}_1 .

For $\Gamma = 1$, the moment equation becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{\partial f}{\partial t} + (\bar{z}_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} + \frac{\rho^2}{r} \frac{\partial f}{\partial \bar{z}_1} - \rho \left(\frac{\bar{z}_1 + u}{r} \right) \frac{\partial f}{\partial \rho} \right] \rho d\rho d\bar{z}_1 = AN \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F - f) \rho d\rho d\bar{z}_1 \quad (2 - 15'')$$

where the range of integration for \bar{z}_1 is $-\infty$ to ∞ and for ρ , 0 to ∞ since ρ can not take negative values according to the definition of ρ .

The right hand side of the moment equation, multiplied by molecular mass m is

$$\int_{-\infty}^{\infty} \int_0^{\infty} m(F - f) f^2 df d\zeta_1$$

The two terms in the integral arise since the outgoing mass of gas is subtracted from the incoming mass of gas and then integrated over all velocity space around a central volume defined for the expanding gas.⁽¹⁵⁾

This expression must become zero according to the conservation of mass principle. Consequently, the convective terms on the left hand side of moment equation (2 - 15'') lead to the continuity equation. Multiplying the right hand side of the moment equation by $m \zeta_1$ and following the similar reasoning as it did to equation (2 - 15''), the left hand side of moment equation (2 - 15') leads to momentum equation ($\Gamma = \zeta_1$), since the right hand side becomes zero due to the conservation of momentum.

$\Gamma = 1$ and $\Gamma = \zeta_1$ are examples of collision invariants.

The integrals of equation (2 - 15'') can be evaluated term by term as follows.

$$\iint \frac{\partial f}{\partial t} f d^3p d^3z_1 = \frac{\partial}{\partial t} \iint f f d^3p d^3z_1 = \frac{\partial}{\partial t} \left(\frac{N}{2\pi} \right)$$

$$\iint z_1 \frac{\partial f}{\partial r} f d^3p d^3z_1 = \frac{\partial}{\partial r} \iint f z_1 d^3z_1 f d^3p = 0$$

since $\int_{-\infty}^{\infty} f z_1 d^3z_1 = 0$

$$\iint u \frac{\partial f}{\partial r} f d^3p d^3z_1 = u \frac{\partial}{\partial r} \iint f f d^3p d^3z_1 = u \frac{\partial}{\partial r} \left(\frac{N}{2\pi} \right)$$

$$\iint (-) z_1 \frac{\partial u}{\partial r} \frac{\partial f}{\partial z_1} f d^3p d^3z_1 = (-) \frac{\partial u}{\partial r} \iint z_1 \frac{\partial f}{\partial z_1} d^3z_1 f d^3p$$

$$= (-) \frac{\partial u}{\partial r} \left[z_1 f \Big|_{-\infty}^{\infty} - \int f d^3z_1 \right] f d^3p$$

$$= \frac{\partial u}{\partial r} \iint f f d^3p d^3z_1 = \frac{\partial u}{\partial r} \left(\frac{N}{2\pi} \right)$$

since $\left[z_1 f \right] \Big|_{-\infty}^{\infty} = 0$

$$\iint (-) u \frac{\partial u}{\partial r} \frac{\partial f}{\partial z_1} f d^3p d^3z_1 = (-) u \frac{\partial u}{\partial r} \iint \frac{\partial f}{\partial z_1} d^3z_1 f d^3p = 0$$

since $\int_{-\infty}^{\infty} \frac{\partial f}{\partial z_1} d^3z_1 = f \Big|_{-\infty}^{\infty} = 0$

$$\iint (-) \frac{\partial u}{\partial t} \frac{\partial f}{\partial \xi_1} d\xi_1 \rho d\rho = 0$$

$$\iint \frac{1}{r} \rho^2 \frac{\partial f}{\partial \xi_1} d\xi_1 \rho d\rho = \frac{1}{r} \iint \frac{\partial f}{\partial \xi_1} d\xi_1 \rho^3 d\rho = 0$$

$$\iint (+) \frac{\rho \xi_1}{r} \frac{\partial f}{\partial \rho} \rho d\rho d\xi_1 = (-) \frac{1}{r} \iint \frac{\partial f}{\partial \rho} \rho^2 d\rho \xi_1 d\xi_1$$

$$= (+) \frac{1}{r} \left[\rho^2 f \right]_0^{\infty} - 2 \int f \rho d\rho \xi_1 d\xi_1$$

$$= -\frac{2}{r} \iint f \xi_1 d\xi_1 \rho d\rho = 0$$

since $\rho^2 f \Big|_0^{\infty} = 0$ and $\iint f \xi_1 \rho d\rho d\xi_1 = 0$

$$\begin{aligned} \iint (+) \frac{\rho u}{r} \frac{\partial f}{\partial \rho} \rho d\rho d\xi_1 &= (-) \frac{u}{r} \iint \frac{\partial f}{\partial \rho} \rho^2 d\rho d\xi_1 = 2 \frac{u}{r} \iint f \rho d\rho d\xi_1 \\ &= 2 \frac{u}{r} \left(\frac{N}{2\pi} \right) \end{aligned}$$

Summing up these terms, the continuity equation for the continuum region is obtained. Here $\bar{\rho} \bar{v}^{\delta} = \text{const.}$ applies.

$$\frac{\partial}{\partial t} \left(\frac{N}{2\pi} \right) + u \frac{\partial}{\partial r} \left(\frac{N}{2\pi} \right) + \frac{\partial u}{\partial r} \left(\frac{N}{2\pi} \right) + 2 \frac{u}{r} \left(\frac{N}{2\pi} \right) = 0$$

multiplying by 2π and substituting for $N = T^{3/2}$

$$\frac{\partial}{\partial t}(T^{3/2}) + u \frac{\partial}{\partial r}(T^{3/2}) + \frac{\partial u}{\partial r} T^{3/2} + 2 \frac{u}{r} T^{3/2} = 0$$

rewriting

$$\frac{3}{2} T^{1/2} \frac{\partial T}{\partial t} + u \frac{3}{2} T^{1/2} \frac{\partial T}{\partial r} + \frac{\partial u}{\partial r} T^{3/2} + 2 \frac{u}{r} T^{3/2} = 0$$

The final form of the continuity equation becomes

$$\frac{3}{2} \frac{\partial T}{\partial t} + \frac{3}{2} u \frac{\partial T}{\partial r} + \frac{\partial u}{\partial r} T + 2 \frac{u}{r} T = 0 \quad (2 - 16)$$

For $r = \xi_1$, the moment equation becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 \left[\frac{\partial f}{\partial t} + (\xi_1 + u) \left(\frac{\partial f}{\partial r} - \frac{\partial u}{\partial r} \frac{\partial f}{\partial \xi_1} \right) - \frac{\partial u}{\partial t} \frac{\partial f}{\partial \xi_1} + \frac{p^2 \partial f}{r \partial \xi_1} - p \left(\frac{\xi_1 + u}{r} \right) \frac{\partial f}{\partial p} \right] p dp d\xi_1 = AN \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 (F - f) p dp d\xi_1$$

Again working term by term,

$$\int \int \xi_1 \frac{\partial f}{\partial t} p dp d\xi_1 = \frac{\partial}{\partial t} \int \int \xi_1 f p dp d\xi_1 = 0$$

$$\text{since } \int_{-\infty}^{\infty} f \xi_1 d\xi_1 = 0$$

$$\int \int \xi_1 \xi_1 \frac{\partial f}{\partial r} p dp d\xi_1 = \frac{\partial}{\partial r} \int \int f \xi_1^2 p dp d\xi_1 = \frac{\partial}{\partial r} \left(\frac{P_r}{2\pi} \right)$$

where P_r has been defined previously.

$$\iint \xi_1 u \frac{\partial f}{\partial r} p dp d\xi_1 = u \frac{\partial}{\partial r} \iint f \xi_1 d\xi_1 p dp = 0$$

$$\text{since } \int_{-\infty}^{\infty} f \xi_1 d\xi_1 = 0$$

$$\iint \xi_1 (-) \xi_1 \frac{\partial u}{\partial r} \frac{\partial f}{\partial \xi_1} p dp d\xi_1 = (-) \frac{\partial u}{\partial r} \iint \xi_1^2 \frac{\partial f}{\partial \xi_1} p dp d\xi_1$$

$$= (-) \frac{\partial u}{\partial r} \left[\xi_1^2 f \Big|_{-\infty}^{\infty} - 2 \int f \xi_1 d\xi_1 \right] p dp$$

$$= 2 \frac{\partial u}{\partial r} \iint f \xi_1 d\xi_1 p dp = 0$$

$$\text{where } \xi_1^2 f \Big|_{-\infty}^{\infty} = 0 \text{ and } \int_{-\infty}^{\infty} f \xi_1 d\xi_1 = 0$$

$$\iint (-) u \frac{\partial u}{\partial r} \frac{\partial f}{\partial \xi_1} p dp d\xi_1 = (-) u \frac{\partial u}{\partial r} \iint \xi_1 \frac{\partial f}{\partial \xi_1} p dp d\xi_1$$

$$= (-) u \frac{\partial u}{\partial r} \left[\xi_1 f \Big|_{-\infty}^{\infty} - \int f d\xi_1 \right] p dp$$

$$= u \frac{\partial u}{\partial r} \iint f p dp d\xi_1 = u \frac{\partial u}{\partial r} \left(\frac{N}{2\pi} \right)$$

$$\iint \xi_1 (-) \frac{\partial u}{\partial t} \frac{\partial f}{\partial \xi_1} p dp d\xi_1 = (-) \frac{\partial u}{\partial t} \iint \xi_1 \frac{\partial f}{\partial \xi_1} d\xi_1 p dp$$

$$= (-) \frac{\partial u}{\partial t} \left[\xi_1 f \Big|_{-\infty}^{\infty} - \int f d\xi_1 \right] p dp$$

$$\begin{aligned}
&= \frac{\partial u}{\partial t} \iint f p d f d \bar{z}_1 = \frac{\partial u}{\partial t} \left(\frac{N}{2\pi} \right) \\
&\iint \bar{z}_1 \frac{p^2}{r} \frac{\partial f}{\partial \bar{z}_1} p d f d \bar{z}_1 = \frac{1}{r} \iint \bar{z}_1 \frac{\partial f}{\partial \bar{z}_1} d \bar{z}_1 p^3 d f \\
&\Leftrightarrow \frac{1}{r} \iint f p^2 p d f d \bar{z}_1 = -\frac{1}{2\pi} \frac{P_0}{r}
\end{aligned}$$

where P_0 has been defined previously.

$$\begin{aligned}
&\iint \bar{z}_1 (-) \frac{p \bar{z}_1}{r} \frac{\partial f}{\partial p} = (-) \frac{1}{r} \iint \frac{\partial f}{\partial p} p^2 d f \bar{z}_1^2 d \bar{z}_1 \\
&\Leftrightarrow \frac{1}{r} \int [p^2 f]_0^\infty - 2 \int f p d f \bar{z}_1^2 d \bar{z}_1 \\
&= \frac{2}{r} \iint f \bar{z}_1^2 p d f d \bar{z}_1 = \frac{1}{2\pi} \frac{2}{r} P_r
\end{aligned}$$

$$\text{since } p^2 f \Big|_0^\infty = 0$$

$$\begin{aligned}
&\iint (-) \bar{z}_1 \frac{p u}{r} \frac{\partial f}{\partial p} p d f d \bar{z}_1 = -\frac{u}{r} \iint p^2 \frac{\partial f}{\partial p} d f \bar{z}_1 d \bar{z}_1 \\
&= -\frac{u}{r} \int [p^2 f]_0^\infty - 2 \int f p d f \bar{z}_1 d \bar{z}_1
\end{aligned}$$

$$= \frac{2u}{r} \iint f \bar{z}_1 d\bar{z}_1 p dp = 0$$

$$\text{Since } \int_{-\infty}^{\infty} f \bar{z}_1 d\bar{z}_1 = 0$$

Summing up these terms the momentum equation for the continuum region is obtained.

$$\frac{1}{2\pi} \frac{\partial}{\partial r} (P_r) + u \frac{\partial u}{\partial r} \left(\frac{N}{2\pi} \right) + \frac{\partial u}{\partial t} \left(\frac{N}{2\pi} \right) - \frac{1}{2\pi} \frac{P_\theta}{r} + \frac{1}{2\pi} \frac{2P_r}{r} = 0 \quad (2 - 16')$$

The relation between P_θ and P_r becomes $P_\theta = 2P_r$ when the distribution function $f = F$.

This can be proved easily as follows:

$$P_r = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N}{(2\pi T)^{3/2}} \exp^{-\frac{1}{2T}(\bar{z}_1^2 + p^2)} \bar{z}_1^2 p dp d\bar{z}_1$$

$$= 2\pi \frac{N}{(2\pi T)^{3/2}} \int_0^{\infty} \bar{z}_1^2 \exp^{-\frac{\bar{z}_1^2}{2T}} d\bar{z}_1 \cdot \int_{-\infty}^{\infty} p \exp^{-\frac{p^2}{2T}} dp$$

$$= 2\pi \frac{N}{(2\pi T)^{3/2}} \left[\frac{1}{2} \left(\frac{\pi}{a^3} \right)^{1/2} \left(\frac{1}{2a} \right) \right]$$

$$\text{where } a = \frac{1}{2T}$$

Similarly for P_θ

$$P_\theta = 2\pi \iint \frac{N}{(2\pi T)^{3/2}} \exp^{-\frac{1}{2T}(\bar{z}_1^2 + \rho^2)} \rho^2 d\rho d\bar{z}_1$$

$$= 2\pi \frac{N}{(2\pi T)^{3/2}} \left[\left(\frac{\pi}{a^3} \right)^{1/2} \left(\frac{1}{2a} \right) \right]$$

Comparing P_r with P_θ it may be seen that

$$P_\theta = 2P_r$$

From equations (2 - 12), (2 - 13) and (2 - 14)

$$P_r = P = T^{5/2}$$

From equation (2 - 14),

$$N = T^{3/2}$$

Substituting expressions for P_θ , P_r and N into the momentum equation (2 - 16'),

$$\frac{\partial}{\partial r} (T^{5/2}) + u \frac{\partial u}{\partial r} T^{3/2} + \frac{\partial u}{\partial t} T^{3/2} = 0$$

Rewriting, the final form of the momentum equation becomes

$$\frac{5}{2} \frac{\partial T}{\partial r} + u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} = 0 \quad (2 - 17)$$

From the continuity and momentum equations, the temperature and mean velocity gradients with respect to time become

$$\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial r} - \frac{2}{3} \frac{\partial u}{\partial r} T - \frac{4}{3} \frac{u}{r} T$$

$$\frac{\partial u}{\partial t} = -\frac{5}{2} \frac{\partial T}{\partial r} - u \frac{\partial u}{\partial r}$$

Substituting these expressions into f_1 , thereby eliminating time-derivative terms

$$\begin{aligned} f_1 &= -\frac{F}{N} \left[\frac{T^2}{2} \left(-u \frac{\partial T}{\partial r} - \frac{2}{3} \frac{\partial u}{\partial r} T - \frac{4}{3} \frac{u}{r} T \right) (\bar{z}_1^2 + \rho^2) \right. \\ &\quad + \frac{T^2}{2} \bar{z}_1 \frac{\partial T}{\partial r} (\bar{z}_1^2 + \rho^2) + \frac{T^2}{2} u \frac{\partial T}{\partial r} (\bar{z}_1^2 + \rho^2) \\ &\quad + T^{-1} \frac{\partial u}{\partial r} \bar{z}_1^2 + u \frac{\partial u}{\partial r} T^{-1} \bar{z}_1 + \bar{z}_1 T^{-1} \left(-\frac{5}{2} \frac{\partial T}{\partial r} - u \frac{\partial u}{\partial r} \right) \\ &\quad \left. + \frac{\rho^2}{r} u T^{-1} \right] \\ &= -\frac{F}{N} \left[-\frac{T^2}{2} u \frac{\partial T}{\partial r} \bar{z}_1^2 - \frac{T^2}{3} \frac{\partial u}{\partial r} \bar{z}_1^2 - \frac{2}{3} T^{-1} \frac{u}{r} \bar{z}_1^2 \right. \\ &\quad \left. - \frac{T^2}{2} u \frac{\partial T}{\partial r} \rho^2 - \frac{T^2}{3} \frac{\partial u}{\partial r} \rho^2 - \frac{2}{3} T^{-1} \frac{u}{r} \rho^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{T^2}{2} \bar{z}_1 \frac{\partial T}{\partial r} (\bar{z}_1^2 + f^2) + \frac{U}{2} T^{-2} \frac{\partial T}{\partial r} \bar{z}_1^2 + \frac{U}{2} T^{-2} \frac{\partial T}{\partial r} f^2 \\
& + T^{-1} \frac{\partial U}{\partial r} \bar{z}_1^2 + U \frac{\partial U}{\partial r} T^{-1} \bar{z}_1 - \frac{5}{2} T^{-1} \frac{\partial T}{\partial r} \bar{z}_1 - T^{-1} U \frac{\partial U}{\partial r} \bar{z}_1 + \frac{f^2}{r} U T^{-1} \\
& = -\frac{F}{N} \left[\frac{T^2}{2} \frac{\partial T}{\partial r} (\bar{z}_1^2 + f^2) \bar{z}_1 - \frac{5}{2} T^{-1} \frac{\partial T}{\partial r} \bar{z}_1 \right. \\
& \left. + \frac{2}{3} T^{-1} \left(\frac{\partial U}{\partial r} - \frac{U}{r} \right) \bar{z}_1^2 + \frac{T^{-1}}{3} \left(\frac{U}{r} - \frac{\partial U}{\partial r} \right) f^2 \right]
\end{aligned}$$

Collecting terms

$$\begin{aligned}
f_1 & = -\frac{F}{N} \left[\frac{\partial \ln T}{\partial r} \left(\frac{(\bar{z}_1^2 + f^2)}{2T} - \frac{5}{2} \right) \bar{z}_1 + \frac{2}{3} T^{-1} \left(\frac{\partial U}{\partial r} - \frac{U}{r} \right) \bar{z}_1^2 \right. \\
& \left. + \frac{T^{-1}}{3} \left(\frac{U}{r} - \frac{\partial U}{\partial r} \right) f^2 \right]
\end{aligned}$$

Further simplification of f_1 is possible.

To do this the results of Mirels and Mullen,⁽¹⁰⁾ which shall be outlined now, are used. They give the solutions for number density N and velocity U of self-similar flows^(*) in the continuum region as

(*) Self-similar flows are those for which the governing flow equations depend only on one independent parameter and time. Examples are cylindrically symmetric flow, spherically symmetric flow, plain wave motion, etc.

$$N = R^{-(\alpha+1)} [1 - \eta^2]^{\frac{1}{\gamma-1}} \quad (2 - 18)$$

$$u = \left(\frac{dR}{dt} \right) \eta \quad (2 - 19)$$

here

$$\frac{dR}{dt} = \frac{1}{(\alpha+1)} \left(\frac{2}{\gamma-1} \right) \left[1 - R^{-(\alpha+1)(\gamma-1)} \right]^{\frac{1}{2}}$$

$$\eta = \frac{r}{R(t)}$$

Where $R(t)$, being the leading edge of a gas cloud, is given by

$$\frac{1}{(\alpha+1)^{\frac{1}{2}} \frac{2}{\gamma-1}} t = (R^2 - 1)^{\frac{1}{2}} \text{ for } N_0 = 1 \quad (2 - 19')$$

The initial number density N_0 is defined as

$$N_0 = 2 / (\alpha+1)(\gamma-1)$$

where $\gamma = 5/3$ and $\alpha = 2$ for spherically symmetric expansions.

Hence, N_0 becomes

$$N_0 = 2 / (2+1) \left(\frac{5}{3} - \frac{3}{3} \right) = 1$$

Rewriting equation (2 - 19')

$$\frac{1}{3^{1/2}} \frac{2}{\frac{5}{3} - \frac{3}{3}} t = (R^2 - 1)^{\frac{1}{2}}$$

$$\therefore \sqrt{3}t = (R^2 + 1)^{\frac{1}{2}}$$

$$\therefore R^2 = 1 + 3t^2 \quad (2 - 20)$$

Substituting $\alpha = 2$ and $\gamma = \frac{5}{3}$ into equation (2 - 18) and (2 - 19)

$$N = \frac{[1 - (\frac{r}{R})^2]^{\frac{3}{2}}}{R^3} \quad (2 - 20')$$

$$U = \frac{\sqrt{3}r(R^2 - 1)^{\frac{1}{2}}}{R^2} \quad (2 - 20'')$$

From equation (2 - 13)

$$T = \frac{P}{N} = \frac{N^{\frac{5}{3}}}{N} = N^{\frac{2}{3}}$$

$$\therefore T = \frac{[1 - (\frac{r}{R})^2]^2}{R^2} \quad (2 - 20''')$$

From equation (2 - 20), it may be seen that as t becomes very large, say, $t \rightarrow \infty$, then

$$R \sim \sqrt{3}t$$

At the same time

$$\left(\frac{r}{R}\right)^2 = \frac{r^2}{(1 + 3t^2)} \sim \frac{r^2}{3t^2}$$

and

$$(R^2 - 1)^{\frac{1}{2}} \sim R$$

Thus, substituting these expressions into equations (2 - 20'), (2 - 20'') and (2 - 20''')

$$N \sim \frac{\left[1 - \frac{r^2}{3t^2}\right]^{3/2}}{3\sqrt{3}t^3} \quad (2 - 21)$$

$$T \sim \frac{\left[1 - \frac{r^2}{3t^2}\right]}{3t^2} \quad (2 - 22)$$

$$u \sim \frac{r}{t} \quad (2 - 23)$$

The ratio between the two variables, $\frac{r}{t}$ is necessarily restricted for the purpose of asymptotic analysis. When the ratio $\frac{r}{t}$ approaches infinity the number density N also approaches infinity. In practice this case is not acceptable since it is impossible to have the case of infinite gas number density. Hence, the ratio $\frac{r}{t}$ is assumed to be of the order of 1, that is, $\frac{r}{t} = O(1)$. This implies that in an asymptotic analysis, there exists an arbitrary constant K_1 for some value A_1 , however large the arbitrary constant K_1 may be, such that $\frac{r}{t} < K_1$ for all values of $A > A_1$. In this case the ratio $\frac{r}{t}$ is said to be bounded, that is $\frac{r}{t} < \infty$.

The above statement simply means that the value of $\frac{r}{t}$ is restricted in such a fashion that $\frac{r}{t}$ does not become infinite. Further instructions on $O(1)$ can be found in appendix A. The substitution of equations (2 - 22) and (2 - 23) into equations (2 - 16) and (2 - 17) yields, for continuity;

$$-\frac{2}{3}t^{-3} + \left(\frac{r}{3}\right)^2 4t^{-5} + \frac{r}{3}t^{-3} - \left(\frac{r}{3}\right)^2 2t^{-5} \\ + \frac{2}{3}t^{-3} - 2\left(\frac{r}{3}\right)^2 t^{-5} = 0$$

For the large time this expression becomes

$$\frac{1}{3}\left(\frac{r}{t}\right)\frac{1}{t^2} \sim 0$$

Noting the $\frac{r}{t}$ does not become infinite but $\frac{1}{t^2}$ approaches zero, thus the continuity equation is approximately satisfied. Similarly equations (2 - 22) and (2 - 23) satisfies the momentum equation: i.e.

$$\frac{5(-)}{2}\frac{2r}{(3t^2)^2} + \frac{r}{t^2} - \frac{r}{t^2} \sim 0$$

noting $\left(\frac{5r}{9t}\right)\frac{1}{t^3}$ approaches 0.

Therefore, solutions for the continuity and momentum equations (2 - 16) and (2 - 17) are equations (2 - 22) and (2 - 23) respectively.

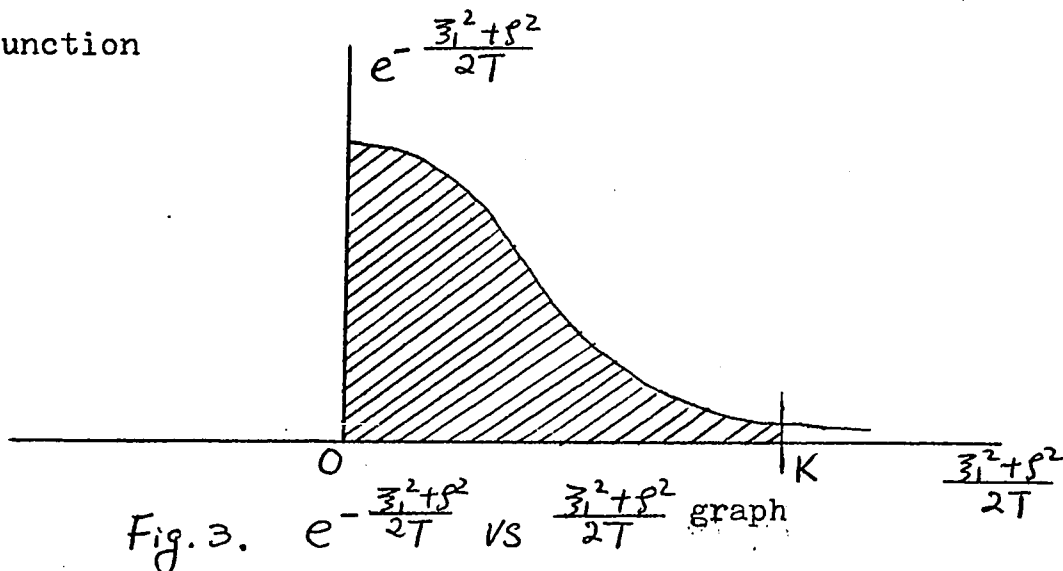
Substituting equation (2 - 23) into f_1 , we obtain a simplified Chapman and Enskog type solution of f_1 as

$$f_1 \sim -\frac{E}{N} \left\{ \frac{\partial \ln I}{\partial r} \left[\frac{(\bar{z}_1^2 + \rho^2)}{2T} - \frac{5}{2} \right] \bar{z}_1 \right\} \quad (2 - 24)$$

Consequently f becomes

$$f = F \left[1 - \frac{A^{-1}}{N} \left\{ \frac{\partial \ln I}{\partial r} \left[\frac{(\bar{z}_1^2 + \rho^2)}{2T} - \frac{5}{2} \right] \bar{z}_1 \right\} + O(A^{-2}) \right] \quad (2 - 25)$$

The equation (2 - 25) does not have validity over the whole range of r and t . An order of magnitude analysis will show the magnitudes of r and t , above which the asymptotic solution breaks down. The break down of an asymptotic solution occurs when any two consecutive terms in the series become of the same order of magnitude. In order to proceed with an order of magnitude analysis of equation (2 - 25), it is necessary to determine the order of magnitude of the term $\frac{\bar{z}_1^2 + \rho^2}{2T}$. This may be done by first analyzing the distribution function



From equation (2 - 9), it may be seen that the distribution function F is proportional to $\exp - \frac{\bar{x}_1^2 + s^2}{2T}$, i.e.

$$F \propto \exp - \frac{\bar{x}_1^2 + s^2}{2T}$$

The graphical representation of the function $\exp - \frac{\bar{x}_1^2 + s^2}{2T}$ is shown on Fig. 3. If K is selected appropriately, most of the area under the curve is contained in the shaded portion from 0 to K on the $\frac{\bar{x}_1^2 + s^2}{2T}$ axis. In practice, the number density, proportional to area under the distribution curve, is very well approximated by the area in this finite region. Hence $\frac{\bar{x}_1^2 + s^2}{2T} \leq K$ may be considered to be bounded and may be satisfactorily defined as $\frac{\bar{x}_1^2 + s^2}{2T} = O(1)$.

Taking r and $t = O(A^n)$ and as found previously $\frac{r}{t} = O(1)$, the order of temperature T in equation (2 - 22) becomes

$$T \sim O(A^{-2n})$$

Differentiating equation (2 - 22) with respect to r

$$\frac{\partial T}{\partial r} \sim (-) \frac{2r}{9t^4} \sim O(A^{-3n})$$

Similarly equation (2 - 21) becomes

$$N \sim O(A^{-3n})$$

From $\frac{\xi_1^2 + \eta^2}{2T} = O(1)$

$$\xi_1 \sim O(A^{-n})$$

Upon substituting the above expressions into the second term of equation (2 - 25), the order becomes

$$O\left(A^{-1} \frac{A^{3n} A^{2n}}{A^{3n}} \frac{1}{A^n}\right) = O(A^{n-1})$$

When $n=1$, the second term becomes of zero order, i.e.

$$O(A^0)$$

Consequently the first and second terms in equation (2- 25) become of like order. Hence this asymptotic series (2 - 25) breaks down. However when new variables are scaled at the point of breakdown and substituted into the B-G-K equation a collisionless B-G-K equation results.⁽⁶⁾ This new scaling is therefore incorrect since a near equilibrium solution can not be matched to a near free molecular solution, this does not make physical sense, and one must therefore conclude that breakdown occurs before γ and t become $O(A^1)$. In fact, when $\gamma, t = O(A^{\frac{1}{2}})$, the B-G-K equation, after being suitably scaled, becomes its convective and collision term becomes same order in magnitude.⁽⁶⁾ Grundy and Thomas⁽⁶⁾ found that this $O(A^{\frac{1}{2}})$ scaling results due to the

fact that the second and third terms in the series (2 - 25) become of like order before the first and second terms. In order to extend the solution (2 - 25) further, a new scaling should be introduced at the point of breakdown namely $r, t = O(A^{\frac{1}{2}})$. Grundy and Thomas⁽⁶⁾ introduced a new set of scaled variables at the point of breakdown. These are;

$$\left. \begin{aligned} t_1 &= t A^{-\frac{1}{2}} \\ r_1 &= r A^{-\frac{1}{2}} \\ \phi &= \bar{\phi}, A^{\frac{1}{2}} \\ \psi &= \bar{\psi} A^{\frac{1}{2}} \\ n &= \bar{n} A^{\frac{3}{2}} \\ \zeta &= \bar{\zeta} A \end{aligned} \right\} \quad (2 - 26)$$

Since the breakdown occurs where r and $t = O(A^{\frac{1}{2}})$, the new scaling for r and t are defined as

$$r = A^{\frac{1}{2}} r_1$$

$$t = A^{\frac{1}{2}} t_1$$

hence

$$r_1 = r A^{-\frac{1}{2}}$$

$$t_1 = t A^{-\frac{1}{2}}$$

The rest of the scaled variables in equation (2 - 26) are obtained from equations (2 - 21) and (2 - 22).

The orders of T and N become $O(A^{-1})$ and $O(A^{-\frac{3}{2}})$ respectively. Hence new variables for T and N are defined as

$$\tau = AT$$

$$n = A^{\frac{3}{2}}N$$

Since by definition

$$\frac{\bar{\xi}_1^2 + \bar{\rho}^2}{2T} = O(1)$$

the order of $\bar{\xi}_1$ and $\bar{\rho}$ become $O(A^{-\frac{1}{2}})$.

The scaled variables for $\bar{\xi}_1$ and $\bar{\rho}$ become

$$\phi = A^{\frac{1}{2}}\bar{\xi}_1$$

$$\psi = A^{\frac{1}{2}}\bar{\rho}$$

For convenience, a new set of transformed variables are introduced;

$$\left. \begin{aligned} \bar{\Phi} &= \phi t_1 \\ \bar{\Psi} &= \psi t_1 \\ \lambda &= \frac{n}{t_1} \\ t_1 & \end{aligned} \right\} \quad (2 - 27)$$

With these front variables, the B-G-K equation (2 - 7)

is transformed in the following way:

Since the distribution function f becomes a function of $\bar{\Phi}$, $\bar{\varphi}$, λ and t_1 , thus

$$df = \frac{\partial f}{\partial \bar{\Phi}} d\bar{\Phi} + \frac{\partial f}{\partial \bar{\varphi}} d\bar{\varphi} + \frac{\partial f}{\partial \lambda} d\lambda + \frac{\partial f}{\partial t_1} dt_1$$

Dividing by dt

$$\frac{df}{dt} = \frac{\partial f}{\partial \bar{\Phi}} \frac{d\bar{\Phi}}{dt} + \frac{\partial f}{\partial \bar{\varphi}} \frac{d\bar{\varphi}}{dt} + \frac{\partial f}{\partial \lambda} \frac{d\lambda}{dt} + \frac{\partial f}{\partial t_1} \frac{dt_1}{dt}$$

The partial differential form of the above expression becomes

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \bar{\Phi}} \frac{\partial \bar{\Phi}}{\partial t} + \frac{\partial f}{\partial \bar{\varphi}} \frac{\partial \bar{\varphi}}{\partial t} + \frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial t} + \frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial t}$$

By making use of equations (2 - 26) and (2 - 27) each term of above expression becomes

$$\frac{\partial f}{\partial \bar{\Phi}} \frac{\partial \bar{\Phi}}{\partial t} = \frac{\partial f}{\partial \bar{\Phi}} \left(\frac{\bar{\Phi}}{t_1} A^{-\frac{1}{2}} - \frac{\partial \lambda}{\partial t} t_1 \right)$$

$$\frac{\partial f}{\partial \bar{\varphi}} \frac{\partial \bar{\varphi}}{\partial t} = A^{-\frac{1}{2}} \frac{\bar{\varphi}}{t_1} \frac{\partial f}{\partial \bar{\varphi}}$$

$$\frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial t} = -\frac{\partial f}{\partial \lambda} \frac{\lambda}{t_1} A^{-\frac{1}{2}}$$

$$\frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial t} = A^{-\frac{1}{2}} \frac{\partial f}{\partial t_1}$$

Summing these terms

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \Phi} \left(\frac{\Phi}{t_1} A^{-\frac{1}{2}} - \frac{\partial \Phi}{\partial t} t_1 \right) + A^{-\frac{1}{2}} \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi}$$

$$- \frac{\partial f}{\partial \lambda} \frac{\lambda}{t_1} A^{-\frac{1}{2}} + A^{-\frac{1}{2}} \frac{\partial f}{\partial t_1}$$

Since

$$\left(\frac{\Phi}{t_1} + u \right) = \left(A^{-\frac{1}{2}} \frac{\Phi}{t_1} + u \right)$$

and

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial r} + \frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial r} + \frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial r} + \frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial r}$$

where

$$\frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial r} = 0$$

$$\frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial r} = 0$$

$$\frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial r} = \frac{\partial f}{\partial \lambda} \frac{A^{-\frac{1}{2}}}{t_1}$$

$$\frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial r} = 0$$

the second term of equation (2 - 7) becomes

$$\left(\frac{\Phi}{t_1} + u \right) \frac{\partial f}{\partial r} = \left(A^{-\frac{1}{2}} \frac{\Phi}{t_1} + u \right) \frac{\partial f}{\partial \lambda} \frac{A^{-\frac{1}{2}}}{t_1}$$

From

$$(\bar{z}_1 + u) = (A^{-\frac{1}{2}} \frac{\Phi}{t_1} + u)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \lambda} \frac{\partial \lambda}{\partial r} \frac{\partial r}{\partial r} = \frac{\partial u}{\partial \lambda} A^{-\frac{1}{2}} \frac{1}{t_1}$$

$$\frac{\partial f}{\partial \bar{z}_1} = \frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial \bar{z}_1} + \frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial \bar{z}_1} + \frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial \bar{z}_1}$$

where

$$\frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial \bar{z}_1} = \frac{\partial f}{\partial \Phi} A^{\frac{1}{2}} t_1$$

$$\frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial \bar{z}_1} = 0$$

$$\frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial \bar{z}_1} = 0$$

$$\frac{\partial f}{\partial t_1} \frac{\partial t_1}{\partial \bar{z}_1} = 0$$

the third term becomes

$$-(\bar{z}_1 + u) \frac{\partial u}{\partial r} \frac{\partial f}{\partial \bar{z}_1} = -(A^{-\frac{1}{2}} \frac{\Phi}{t_1} + u) \frac{\partial u}{\partial \lambda} \frac{\partial f}{\partial \Phi}$$

The fourth term

$$-\frac{\partial u}{\partial t} \frac{\partial f}{\partial \bar{z}_1} = \lambda \frac{\partial f}{\partial \Phi} \frac{\partial u}{\partial \lambda}$$

The fifth term

$$\frac{r^2 \partial f}{r \partial \bar{z}_1} = \frac{\partial f}{\partial \Phi} \frac{\Phi^2}{A \lambda t_1^2}$$

Since

$$\frac{r(\bar{z}_1 + u)}{r} = \frac{\Phi (u t_1 A^{\frac{1}{2}} t_1 \Phi)}{\lambda t_1^3 A^{3/2}}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial \Phi} A^{\frac{1}{2}} t_1$$

the sixth term becomes

$$-\frac{r(u + \bar{z}_1) \partial f}{r \partial r} = -\frac{(u t_1 A^{\frac{1}{2}} + \Phi) \partial f}{\lambda A t_1^2 \partial \Phi}$$

Substituting these terms into equation (2 - 7),

$$\frac{\partial f}{\partial \Phi} \frac{\Phi}{t_1} A^{-\frac{1}{2}} - \frac{\partial f}{\partial \Phi} \frac{\partial u}{\partial t_1} t_1 + A^{-\frac{1}{2}} \frac{\Phi \partial f}{t_1 \partial \Phi} - \frac{\partial f}{\partial \lambda} \frac{\lambda}{t_1} A^{-\frac{1}{2}}$$

$$+ \frac{\partial f}{\partial t_1} A^{-\frac{1}{2}} + \frac{\partial f}{\partial \lambda} \frac{\Phi}{t_1^2} A^{-1} + \frac{\partial f}{\partial \lambda} u \frac{A^{-\frac{1}{2}}}{t_1}$$

$$- \frac{\partial f}{\partial \Phi} \frac{\partial u}{\partial \lambda} \frac{\Phi}{t_1} A^{-\frac{1}{2}} - \frac{\partial f}{\partial \Phi} \frac{\partial u}{\partial \lambda} u + \frac{\partial f}{\partial \Phi} \lambda \frac{\partial u}{\partial \lambda} + \frac{\partial f}{\partial \Phi} \frac{\Phi^2}{A \lambda t_1^2}$$

$$- \frac{\partial f}{\partial \Phi} \frac{\Phi u}{\lambda t_1} A^{-\frac{1}{2}} - \frac{\partial f}{\partial \Phi} \frac{\Phi \Phi}{\lambda t_1^2} A^{-1} = n A^{-\frac{3}{2}} A (F - f)$$

multiplying both sides by $A^{\frac{1}{2}}$, the scaled B-G-K equation becomes

$$\begin{aligned}
 & -A^{\frac{1}{2}} \frac{\partial f}{\partial \Phi} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) + \frac{\partial f}{\partial t_1} \\
 & + \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi} \left(1 - \frac{\partial u}{\partial \lambda} \right) + \frac{\Psi}{t_1} \frac{\partial f}{\partial \Psi} \left(1 - \frac{u}{\lambda} \right) + \frac{(u - \lambda)}{t_1} \frac{\partial f}{\partial \lambda} \\
 & + A^{\frac{1}{2}} \left\{ \frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Psi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Psi}{\lambda t_1^2} \frac{\partial f}{\partial \Psi} \right\} = n (F - f)
 \end{aligned}$$

(2 - 28)

Distribution function F , number density n and temperature τ are expressed in terms of scaled variables as follows:

$$\begin{aligned}
 F &= \frac{N}{(2\pi T)^{3/2}} \exp - \frac{(\bar{z}_1^2 + \beta^2)}{2T} \\
 &= \frac{n A^{-3/2}}{(2\pi \tau A^{-1})^{3/2}} \exp - \frac{(\Phi A^{-\frac{1}{2}})^2 + (\Psi A^{-\frac{1}{2}})^2}{2\tau A^{-1}} \\
 &= \frac{n}{(2\pi \tau)^{3/2}} \exp - \frac{\Phi^2 + \Psi^2}{2\tau t_1^2}
 \end{aligned}$$

$$\therefore F = \frac{n}{(2\pi\gamma)^{3/2}} \exp - \frac{\bar{\Phi}^2 + \bar{\Psi}^2}{2t_1^2\gamma}$$

and from

$$N = 2\pi \iint f \rho d\rho d\bar{\zeta}_1$$

$$n = NA^{3/2} = \frac{2\pi}{t_1^3} \iint f \bar{\Phi} d\bar{\Phi} d\bar{\Psi} \quad (2 - 30)$$

where $\bar{\zeta}$ and η range from $-\infty$ to ∞ and from 0 to ∞ respectively.

From the relation

$$3NT = 2\pi \iint f(\bar{\zeta}_1^2 + \rho^2) \rho d\rho d\bar{\zeta}_1$$

γ becomes

$$3n\gamma = \frac{2\pi}{t_1^3} \iint f(\bar{\Phi}^2 + \bar{\Psi}^2) \bar{\Phi} d\bar{\Phi} d\bar{\Psi} \quad (2 - 31)$$

In general if a molecular property is expressed by

$\Gamma(\bar{\Phi}, \bar{\Psi})$ the mean value of this property $\bar{\Gamma}$ is given by

$$n\bar{\Gamma} = \frac{2\pi}{t_1^3} \iint f \Gamma \bar{\Phi} d\bar{\Phi} d\bar{\Psi} \quad (2 - 32)$$

With the aid of equation(2 - 32) the moment equations are derived from equation(2 - 28):

for $\Gamma = 1$, the zeroth moment becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ -A^{\frac{1}{2}} \frac{\partial f}{\partial \Phi} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \right. \\ & + \left[\frac{\partial f}{\partial t_1} + \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi} \left(1 - \frac{\partial u}{\partial \lambda} \right) \right. \\ & + \left. \left. \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi} \left(1 - \frac{u}{\lambda} \right) + \frac{1}{t_1} \frac{\partial f}{\partial \lambda} (u - \lambda) \right] \right. \\ & + \left. A^{-\frac{1}{2}} \left[\frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Phi}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} \right] \right\} \Phi d\Phi d\Phi \\ & = n \int_{-\infty}^{\infty} \int_0^{\infty} (F - f) \Phi d\Phi d\Phi \end{aligned}$$

Noting
$$\begin{cases} -\infty \leq \Phi \leq \infty \\ 0 \leq \Phi \leq \infty \end{cases}$$

the first term of the above expression becomes

$$\begin{aligned} & \int \int (-A^{\frac{1}{2}} \frac{\partial f}{\partial \Phi} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \Phi d\Phi d\Phi \\ & = -A^{\frac{1}{2}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \int \int \frac{\partial f}{\partial \Phi} \Phi d\Phi d\Phi \end{aligned}$$

$$= -A^{\frac{1}{2}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \int f \int_{-\infty}^{\infty} \bar{\psi} d\bar{\psi} = 0$$

$$\text{since } f \Big|_{-\infty}^{\infty} = 0$$

And the rest of the terms are:

$$\iiint \left(\frac{\partial f}{\partial t_1} \right) \bar{\psi} d\bar{\psi} d\psi$$

$$= \frac{\partial}{\partial t_1} \iiint f \bar{\psi} d\bar{\psi} d\psi = \frac{\partial}{\partial t} \left(\frac{n t_1^3}{2\pi} \right)$$

$$\iiint \left(1 - \frac{\partial u}{\partial \lambda} \right) \frac{\bar{\psi}}{t_1} \frac{\partial f}{\partial \bar{\psi}} \bar{\psi} d\bar{\psi} d\psi$$

$$= \frac{\left(1 - \frac{\partial u}{\partial \lambda} \right)}{t_1} \iiint \frac{\partial f}{\partial \bar{\psi}} \bar{\psi} d\bar{\psi} d\psi$$

$$= \frac{\left(1 - \frac{\partial u}{\partial \lambda} \right)}{t_1} \left[\bar{\psi} f \Big|_{-\infty}^{\infty} - \int f d\bar{\psi} \right] \bar{\psi} d\bar{\psi}$$

$$\Rightarrow \frac{\left(1 - \frac{\partial u}{\partial \lambda} \right)}{t_1} \iiint f \bar{\psi} d\bar{\psi} d\psi$$

$$= (-) \left(1 - \frac{\partial u}{\partial \lambda}\right) \frac{\eta t_1^3}{2\pi t_1}$$

$$\iint \left(\frac{\Phi}{t_1} \left(1 - \frac{\partial u}{\partial \lambda}\right) \frac{\partial f}{\partial \Phi} \Phi d\Phi d\Phi\right)$$

$$= \frac{(1 - \frac{u}{\lambda})}{t_1} \iint \frac{\partial f}{\partial \Phi} \Phi^2 d\Phi d\Phi$$

$$= \frac{(1 - \frac{u}{\lambda})}{t_1} \left[\int_{-\infty}^{\infty} \Phi^2 f - \int f \cdot 2 \cdot \Phi d\Phi \right] d\Phi$$

$$= \frac{(1 - \frac{u}{\lambda})}{t_1} 2 \iint f \Phi d\Phi d\Phi = - \frac{(1 - \frac{u}{\lambda})}{t_1} \left(\frac{2\eta t_1^3}{2\pi} \right)$$

$$\iint \left(\frac{u - \lambda}{t_1} \frac{\partial f}{\partial \lambda} \Phi d\Phi d\Phi \right) = \frac{(u - \lambda)}{t_1} \frac{\partial}{\partial \lambda} \iint f \Phi d\Phi d\Phi$$

$$= \frac{(u - \lambda)}{t_1} \frac{\partial}{\partial \lambda} \left(\frac{\eta t_1^3}{2\pi} \right)$$

$$\iint A^{-\frac{1}{2}} \left\{ \frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi}{\lambda} \frac{\Phi}{t_1^2} \frac{\partial f}{\partial \Phi} \right\} \Phi d\Phi d\Phi$$

$$= A^{-\frac{1}{2}} \left[\frac{\partial}{\partial \lambda} \iint f \Phi \Phi d\Phi d\Phi + \frac{1}{\lambda t_1^2} \iint \frac{\partial f}{\partial \Phi} \Phi^2 \Phi d\Phi d\Phi \right]$$

$$-\frac{1}{\lambda t_1^2} \iint \left[\bar{\Phi} \bar{\Phi}^2 \frac{\partial f}{\partial \bar{\Phi}} d\bar{\Phi} d\bar{\Phi} \right] = A^{-\frac{1}{2}} [0 + 0 + 0] = 0$$

Since $\iint f \bar{\Phi} \bar{\Phi} d\bar{\Phi} d\bar{\Phi} = 0$

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial \bar{\Phi}} d\bar{\Phi} = 0$$

and

$$\int \bar{\Phi}^2 \frac{\partial f}{\partial \bar{\Phi}} d\bar{\Phi} = -2 \int f \bar{\Phi} d\bar{\Phi}$$

The right hand side of the zeroth moment equation reduces to

$$n \iint (F - f) \bar{\Phi} d\bar{\Phi} d\bar{\Phi} = n \left(\frac{n t_1^3}{2\pi} - \frac{n t_1^3}{2\pi} \right) = 0$$

Summing up all terms and multiplying by 2π , the resulting continuity equation becomes

$$\frac{\partial}{\partial t_1} (n t_1^3) - \frac{n t_1^3}{t_1} \left(1 - \frac{\partial u}{\partial \lambda} \right) - \frac{2 n t_1^3}{t_1} \left(1 - \frac{u}{\lambda} \right)$$

$$+ \frac{1}{t_1} \frac{\partial n t_1^3}{\partial \lambda} (u - \lambda) = 0 \quad (2 - 33)$$

For $\Gamma = \bar{\Phi}$, the momentum equation becomes

$$\iint \left\{ -A^{\frac{1}{2}} \frac{\partial f}{\partial \bar{\Phi}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) + \frac{\partial f}{\partial t_1} \right. \\ \left. + \frac{\bar{\Phi}}{t_1} \frac{\partial f}{\partial \bar{\Phi}} \left(1 - \frac{\partial u}{\partial \lambda} \right) + \frac{\bar{\Phi}}{t_1} \frac{\partial f}{\partial \bar{\Phi}} \left(1 - \frac{u}{\lambda} \right) + \frac{u - \lambda}{t_1} \frac{\partial f}{\partial \lambda} \right.$$

$$\begin{aligned}
& + A^{-\frac{1}{2}} \left[\frac{\bar{\Phi}}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\bar{\Phi}^2}{\lambda t_1^2} \frac{\partial f}{\partial \bar{\Phi}} - \frac{\bar{\Phi} \Phi}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} \right] \bar{\Phi} \Phi d\bar{\Phi} d\Phi \\
& = n \int_{-\infty}^{\infty} \int_0^{\infty} \bar{\Phi} (F - f) \Phi d\bar{\Phi} d\Phi \quad (2 - 33')
\end{aligned}$$

The first term of above expression becomes

$$\begin{aligned}
& \iint (-) A^{\frac{1}{2}} \frac{\partial f}{\partial \bar{\Phi}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \bar{\Phi} \Phi d\bar{\Phi} d\Phi \\
& = (-) A^{\frac{1}{2}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \iint \frac{\partial f}{\partial \bar{\Phi}} \bar{\Phi} \Phi d\bar{\Phi} d\Phi \\
& = A^{\frac{1}{2}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) \iint f \bar{\Phi} \Phi d\bar{\Phi} d\Phi \\
& = A^{\frac{1}{2}} \left[(u - \lambda) \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right] \frac{n t_1^3}{2\pi}
\end{aligned}$$

and the rest of the terms are

$$\begin{aligned}
& \iint \frac{\partial f}{\partial t_1} \bar{\Phi} \Phi d\bar{\Phi} d\Phi = \frac{\partial}{\partial t_1} \iint f \bar{\Phi} \Phi d\bar{\Phi} d\Phi = 0 \\
& (-) \left(1 - \frac{\partial u}{\partial \lambda} \right) \frac{1}{t_1} \iint \frac{\partial f}{\partial \bar{\Phi}} \bar{\Phi}^2 \Phi d\bar{\Phi} d\Phi = 0
\end{aligned}$$

$$(1 - \frac{u}{\lambda}) \frac{1}{t_1} \iint \left(\frac{\partial f}{\partial \Phi} \Phi \Psi d\Phi d\Psi \right) = 0$$

$$\frac{(u - \lambda)}{t_1} \frac{\partial}{\partial \lambda} \iint f \Phi \Psi d\Phi d\Psi = 0$$

$$\iint A^{-\frac{1}{2}} \left[\frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Psi}{\lambda t_1^2} \frac{\partial f}{\partial \Psi} \right] \Phi \Psi d\Phi d\Psi$$

$$= \frac{A^{-\frac{1}{2}}}{t_1^2} \left\{ \frac{\partial}{\partial \lambda} \iint f \Phi^2 \Psi d\Phi d\Psi + \frac{1}{\lambda} \iint \frac{\partial f}{\partial \Phi} \Phi^2 \Psi d\Phi d\Psi \right.$$

$$\left. - \frac{1}{\lambda} \iint \frac{\partial f}{\partial \Psi} \Phi^2 \Psi d\Phi d\Psi \right\}$$

$$= \frac{A^{-\frac{1}{2}}}{t_1^2} \left\{ \frac{\partial}{\partial \lambda} \iint f \Phi^2 \Psi d\Phi d\Psi - \frac{1}{\lambda} \iint f \Phi^2 \Psi d\Phi d\Psi \right.$$

$$\left. + \frac{2}{\lambda} \iint f \Phi^2 \Psi d\Phi d\Psi \right\}$$

$$= \frac{A^{-\frac{1}{2}}}{t_1^2} \left\{ \frac{\partial}{\partial \lambda} \left(\frac{n t_1^3}{2\pi} \bar{\Phi}^2 \right) - \frac{n t_1^3}{2\pi \lambda} \bar{\Phi}^2 + \frac{2 n t_1^3}{2\pi \lambda} \bar{\Phi}^2 \right\}$$

The expression

$$n \iint (F - f) \Phi \bar{\Psi} d\bar{\Psi} d\Phi$$

becomes zero due to the conservation of momentum.

Upon substituting these terms into equation (2 - 33')

and multiplying by 2π , the momentum equation becomes

$$n t_1^3 \left[(u - \lambda) \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right] + A^{-1} \frac{1}{t_1^2} \left[\frac{\partial}{\partial \lambda} (n t_1^3 \bar{\Phi}^2) - \frac{n t_1^3 \bar{\Psi}^2}{\lambda} + \frac{2}{\lambda} n t_1^3 \bar{\Phi}^2 \right] = 0 \quad (2 - 34)$$

In order to solve the momentum and continuity equations (2 - 33) and (2 - 34), u and $n t_1^3$ are expanded in the following forms.

$$\left. \begin{aligned} u &= u_0 + A^{-1} u_1 + O(A^{-2}) \\ n t_1^3 &= B_0 + A^{-1} B_1 + O(A^{-2}) \end{aligned} \right\} \quad (2 - 35)$$

Substituting equations (2 - 35) into (2 - 33) and (2 - 34), the continuity equation is obtained.

$$\frac{\partial B_0}{\partial t_1} - \frac{B_0}{t_1} + \frac{B_0}{t_1} \frac{\partial u_0}{\partial \lambda} - \frac{2 B_0}{t_1} + \frac{2 B_0}{t_1} \frac{u_0}{\lambda} + \frac{u_0}{t_1} \frac{\partial B_0}{\partial \lambda} - \frac{1}{t_1} \frac{\partial B_0}{\partial \lambda}$$

$$\begin{aligned}
& +A^{-1} \left[\frac{\partial B_1}{\partial t_1} + \frac{B_1}{t_1} + \frac{B_0 \partial u_1}{t_1 \partial \lambda} + \frac{B_1 \partial u_0}{t_1 \partial \lambda} + \frac{2B_0 u_1}{t_1 \lambda} - \frac{2B_1}{t_1} \right. \\
& \left. + \frac{2B_1 u_0}{t_1 \lambda} + \frac{1}{t_1} \frac{\partial B_0}{\partial \lambda} u_1 + \frac{u_0 \partial B_1}{t_1 \partial \lambda} - \frac{\lambda \partial B_1}{t_1 \partial \lambda} \right] \\
& +A^{-2} \left[\frac{B_1 \partial u_1}{t_1 \partial \lambda} + \frac{2B_1 u_1}{t_1 \lambda} + \frac{u_1 \partial B_1}{t_1 \partial \lambda} \right] = 0
\end{aligned}$$

Similarly the momentum equation is as follows:

$$\begin{aligned}
& B_0 \left[(u_0 - \lambda) \frac{\partial u_0}{\partial \lambda} + t_1 \frac{\partial u_0}{\partial t_1} \right] + A^{-1} \left[B_0 u_1 \frac{\partial u_0}{\partial \lambda} + B_0 u_0 \frac{\partial u_1}{\partial \lambda} \right. \\
& \left. - B_0 \lambda \frac{\partial u_1}{\partial \lambda} + B_0 t_1 \frac{\partial u_1}{\partial t_1} + B_1 u_0 \frac{\partial u_0}{\partial \lambda} - B_1 \lambda \frac{\partial u_0}{\partial \lambda} + B_1 t_1 \frac{\partial u_0}{\partial t_1} \right. \\
& \left. + \frac{1}{t_1^2} \frac{\partial}{\partial \lambda} (B_0 \bar{\Phi}^2) - \frac{1}{t_1^2} \frac{B_0}{\lambda} \bar{\Phi}^2 + \frac{2B_0}{t_1^2} \frac{\bar{\Phi}^2}{\lambda} \right] \\
& +A^{-2} \left[B_0 u_1 \frac{\partial u_1}{\partial \lambda} + B_1 u_0 \frac{\partial u_1}{\partial \lambda} + B_1 u_1 \frac{\partial u_0}{\partial \lambda} \right. \\
& \left. - B_1 \lambda \frac{\partial u_1}{\partial \lambda} + B_1 t_1 \frac{\partial u_1}{\partial t_1} + \frac{1}{t_1^2} \frac{\partial}{\partial \lambda} (B_1 \bar{\Phi}^2) + \frac{1}{t_1^2} \frac{B_1}{\lambda} \bar{\Phi}^2 \right. \\
& \left. + \frac{1}{t_1^2} \frac{2B_1}{\lambda} \bar{\Phi}^2 \right] + A^{-3} B_1 u_1 \frac{\partial u_1}{\partial \lambda} = 0
\end{aligned}$$

Since only the order of expansion up to $O(A^{-1})$ is of concern, terms higher than $O(A^{-1})$ can be neglected. Hence, the continuity equation becomes

$$\begin{aligned} & \frac{\partial B_0}{\partial t_1} - \frac{B_0}{t_1} \left(1 - \frac{\partial u_0}{\partial \lambda}\right) - \frac{2B_0}{t_1} \left(1 - \frac{u_0}{\lambda}\right) + \frac{1}{t_1} (u_0 - \lambda) \frac{\partial B_0}{\partial \lambda} \\ & + A^{-1} \left[\frac{\partial B_1}{\partial t_1} + \frac{B_1}{t_1} \left(1 + \frac{\partial u_0}{\partial \lambda}\right) + \frac{B_0}{t_1} \left(\frac{\partial u_1}{\partial \lambda} + 2 \frac{u_1}{\lambda}\right) \right. \\ & \left. - \frac{2B_1}{t_1} \left(1 - \frac{u_0}{\lambda}\right) + \left(\frac{u_1}{t_1} + \frac{u_0}{t_1} - \frac{\lambda}{t_1}\right) \frac{\partial B_1}{\partial \lambda} \right] = 0 \quad (2 - 36) \end{aligned}$$

The momentum equation becomes

$$\begin{aligned} & B_0 \left[(u_0 - \lambda) \frac{\partial u_0}{\partial \lambda} + t_1 \frac{\partial u_0}{\partial \lambda} + t_1 \frac{\partial u_0}{\partial t_1} \right] + A^{-1} \left\{ B_0 t_1 \frac{\partial u_1}{\partial t_1} \right. \\ & + B_1 t_1 \frac{\partial u_0}{\partial t_1} + (B_0 u_1 + B_1 u_0 - B_1 \lambda) \frac{\partial u_0}{\partial \lambda} + (B_0 u_0 - B_0 \lambda) \frac{\partial u_1}{\partial \lambda} \\ & \left. + \frac{1}{t_1} \left[\frac{\partial}{\partial \lambda} (B_0 \bar{\Phi}^2) - \frac{B_0}{\lambda} \bar{\Phi}^2 + 2B_0 \frac{\bar{\Phi}^2}{\lambda} \right] \right\} = 0 \end{aligned}$$

The terms involving $\bar{\Phi}^2$ and $\bar{\Phi}^2$ will again produce A^{-1} order terms when f is expanded as $f = f^0 + A^{-1} f^1 + O(A^{-2})$.

From

$$\bar{\Phi}^2 \frac{nt_1^3}{2\pi} = \iiint f \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} \quad \text{and} \quad \bar{\Psi}^2 \frac{nt_1^3}{2\pi} = \iiint f \bar{\Psi}^2 \bar{\Phi} d\bar{\Psi} d\bar{\Phi}$$

$$\bar{\Phi}^2 = \frac{2\pi}{nt_1^3} \iiint (f^0 + A^{-1}f') \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi}$$

$$= \frac{2\pi}{B_0} \iiint f^0 \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} + A^{-1} \frac{2\pi}{B_0} \iiint f' \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi}$$

Similarly

$$\bar{\Psi}^2 = \frac{2\pi}{B_0} \iiint f^0 \bar{\Psi}^2 \bar{\Phi} d\bar{\Psi} d\bar{\Phi} + A^{-1} \frac{2\pi}{B_0} \iiint f' \bar{\Psi}^2 \bar{\Phi} d\bar{\Psi} d\bar{\Phi}$$

substituting these expressions into the momentum equation, it can be seen naturally that there again appears A^{-2} order terms. Hence, neglecting those terms and rewriting the momentum equation,

$$B_0[(t_0 u_0 - \lambda) \frac{\partial u_0}{\partial \lambda} + t_1 \frac{\partial u_0}{\partial t_1}] + A^{-1} \{ B_0 t_1 \frac{\partial u_1}{\partial t_1} + B_1 t_1 \frac{\partial u_0}{\partial t_1}$$

$$+ (B_0 u_1 + B_1 u_0 - B_1 \lambda) \frac{\partial u_0}{\partial \lambda} + (B_0 u_0 - B_0 \lambda) \frac{\partial u_1}{\partial \lambda}$$

$$+ \frac{1}{t_1^2} \left[\frac{\partial}{\partial \lambda} (2\pi \iiint f^0 \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi}) - \frac{2\pi}{\lambda} \iiint f^0 \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} \right]$$

$$+ 2 \frac{2\pi}{\lambda} \left\{ \int \int f^0 \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} \right\} = 0 \quad (2 - 37)$$

The zeroth order continuity and momentum equations become

$$\frac{\partial B_0}{\partial t_1} - \frac{B_0}{t_1} \left(1 - \frac{\partial u_0}{\partial \lambda} \right) - \frac{2B_0}{t_1} \left(1 - \frac{u_0}{\lambda} \right) + \frac{1}{t_1} (u_0 - \lambda) \frac{\partial B_0}{\partial \lambda} = 0$$

$$B_0 \left[(u_0 - \lambda) \frac{\partial u_0}{\partial \lambda} + t_1 \frac{\partial u_0}{\partial t_1} \right] = 0$$

The solutions of these equations after matching with the equilibrium solutions (2 - 23) and (2 - 21) for large time are

$$u_0 = \lambda \quad (2 - 38)$$

$$B_0 = \left(\frac{K}{3} \right)^{3/2} \quad (2 - 39)$$

where

$$K = 1 - \frac{\lambda^2}{3} \quad (2 - 40)$$

On multiplying the B-G-K equation (2 - 28) by $\bar{\Phi}^2 + \bar{\Psi}^2$ and integrating

$$\int \int (\bar{\Phi}^2 + \bar{\Psi}^2) \left[(-) A^{\frac{1}{2}} \frac{\partial f}{\partial \bar{\Phi}} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) + \frac{\partial f}{\partial t_1} \right.$$

$$\left. + \frac{\bar{\Phi}}{t_1} \frac{\partial f}{\partial \bar{\Phi}} \left(1 - \frac{\partial u}{\partial \lambda} \right) + \frac{\bar{\Psi}}{t_1} \frac{\partial f}{\partial \bar{\Psi}} \left(1 - \frac{u}{\lambda} \right) + \frac{(u - \lambda)}{t_1} \frac{\partial f}{\partial \lambda} \right]$$

$$+ A^{-\frac{1}{2}} \left(\frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Phi}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} \right) \Phi d\Phi d\Phi$$

$$= n \iint (\Phi^2 + \Phi^2) (F - f) \Phi d\Phi d\Phi \quad (2 - 28')$$

Working again term by term, the first term of equation (2 - 28') becomes

$$\begin{aligned} & \hookrightarrow A^{\frac{1}{2}} \left(\lambda \frac{\partial \psi}{\partial \lambda} - \lambda \frac{\partial \psi}{\partial \lambda} + t_1 \frac{\partial \psi}{\partial t_1} \right) \iint (\Phi^2 + \Phi^2) \frac{\partial f}{\partial \Phi} \Phi d\Phi d\Phi \\ & = \hookrightarrow A^{\frac{1}{2}} \left(\lambda \frac{\partial \psi}{\partial \lambda} - \lambda \frac{\partial \psi}{\partial \lambda} + t_1 \frac{\partial \psi}{\partial t_1} \right) \left\{ \int [\Phi^2 f]_{-\infty}^{\infty} - 2 \int f \Phi d\Phi \right\} \Phi d\Phi \\ & + \int f \int_{-\infty}^{\infty} \Phi^3 d\Phi \} = 0 \end{aligned}$$

The second term becomes

$$\frac{\partial}{\partial t_1} \iint f (\Phi^2 + \Phi^2) \Phi d\Phi d\Phi = \frac{3}{2\pi} \frac{\partial}{\partial t_1} (n \gamma t_1^5)$$

Since

$$\iint f (\Phi^2 + \Phi^2) \Phi d\Phi d\Phi = \frac{3n\gamma}{2\pi} t_1^5$$

The rest of the terms becomes zero when the zeroth order solution of velocity U is substituted.

The right hand side of equation (2 - 28) becomes zero through conservation of energy.⁽¹⁵⁾

Consequently, equation (2 - 28') becomes

$$\frac{\partial}{\partial t_1} (\gamma t_1^2) = 0 \quad (2 - 41)$$

The general solution of equation (2 - 41) is

$$\gamma = \frac{C}{t_1^2} \quad (2 - 41')$$

Since $\gamma = TA$ and $t_1 = tA^{-\frac{1}{2}}$, equation (2 - 41') becomes

$$T = \frac{C}{t^2} \quad (2 - 41'')$$

Matching (2 - 41'') with the outer limit equilibrium solution (2 - 22), i.e.

$$T \sim \frac{K/3}{t^2} \quad (2 - 22')$$

where $K = 1 - \frac{\lambda^2}{3}$

The constant C becomes $K/3$, hence (2 - 41') becomes

$$\gamma = \frac{K/3}{t_1^2} \quad (2 - 42)$$

Equation (2 - 42) is the zeroth order solution for temperature.

With the aid of equations (2 - 38), (2 - 39) and (2 - 42), the zeroth order B-G-K equation (2 - 28) becomes, upon replacing $B_0 = n t_1^3$ and $u_0 = \lambda = u$,

$$\frac{\partial f}{\partial t_1} = \frac{(K/3)^{3/2}}{t_1^3} \left[\frac{1}{(2\pi)^{3/2}} \exp - \frac{3(\bar{\Phi}^2 + \bar{\Psi}^2)}{2K} - f \right] \quad (2 - 43)$$

The solution of this equation is

$$f = f^0 = \frac{1}{(2\pi)^{3/2}} \exp - \frac{3(\bar{\Phi}^2 + \bar{\Psi}^2)}{2K} \quad (2 - 44)$$

This distribution function in the outer region of Grundy and Thomas is of the near equilibrium type.

The next first order solution of f , that is f' , is obtained as follows. Using Equations (2 - 38) and (2 - 39) equations (2 - 36) and (2 - 37) become

$$\frac{\partial B_1}{\partial t_1} + \frac{2B_1}{t_1} + \frac{B_0}{t_1} \left(\frac{\partial u_1}{\partial \lambda} + 2 \frac{u_1}{\lambda} \right) + \frac{u_1}{t_1} \frac{\partial B_1}{\partial \lambda} = 0 \quad (2 - 45)$$

$$-B_0 t_1 \frac{\partial u_1}{\partial t_1} + B_0 u_1 + \frac{1}{t_1^2} \left[\frac{\partial}{\partial \lambda} (B_0 \bar{\Phi}^2) - \frac{B_0 \bar{\Phi}^2}{\lambda} + 2B_0 \frac{\bar{\Phi}^2}{\lambda} \right] = 0 \quad (2 - 46)$$

Using equation (2 - 44) $\bar{\Phi}^2$ and $\bar{\Psi}^2$ become

$$\bar{\Phi}^2 = \frac{2\pi}{B_0} \iint f^0 \bar{\Phi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} = \frac{(K/3)^{5/2}}{B_0}$$

and

$$\bar{\Psi}^2 = \frac{2\pi}{B_0} \iint f^0 \bar{\Psi}^2 \bar{\Psi} d\bar{\Psi} d\bar{\Phi} = \frac{2 \left(\frac{K}{3}\right)^{5/2}}{B_0}$$

substituting these expressions into equation (2 - 46)

$$B_0 t_1 \frac{\partial u_1}{\partial t_1} + B_0 u_1 + \frac{1}{t_1^2} \frac{\partial}{\partial \lambda} \left(\frac{K}{3}\right)^{5/2} - \frac{2}{\lambda} \left(\frac{K}{3}\right)^{5/2} + \frac{2}{\lambda} \left(\frac{K}{3}\right)^{5/2} = 0$$

Rewriting

$$B_0 t_1 \frac{\partial u_1}{\partial t_1} + B_0 u_1 + \frac{1}{t_1^2} \frac{\partial}{\partial \lambda} \left(\frac{K}{3}\right)^{5/2} = 0 \quad (2 - 47)$$

Now, the first order solution u_1 can be obtained from equation (2 - 47).

The integrating factor of equation (2 - 47) is

$$e^{\int \frac{1}{t_1} dt} = e^{\log t_1} = t_1$$

Hence

$$\begin{aligned} t_1 u_1 &= \int t_1 \frac{5\lambda}{3^2 t_1^3} dt_1 + \text{const.} \\ &= (-) \frac{5}{3^2} \lambda t_1^{-1} + \text{const.} \end{aligned}$$

When $\lambda = 0$, $u_1 = 0$, thus, $\text{const} = 0$

Consequently u_1 becomes

$$u_1 = - \frac{5\lambda}{3^2 t_1^2} \quad (2 - 48)$$

Therefore, equation (2 - 35)

$$u = \lambda - A^{-1} \frac{5\lambda}{3^2 t_1^2} + O(A^{-2}) \quad (2 - 49)$$

$$n t_1^3 = \left(\frac{K}{3}\right)^{3/2} + A^{-1} B_1 + O(A^{-2}) \quad (2 - 50)$$

Substituting equation (2 - 49) into the B-G-K equation (2 - 28) and collecting terms upto A^{-1} order,

$$\begin{aligned} \frac{\partial f}{\partial t_1} + \frac{A^{-1/2}}{t_1^2} \left[-\frac{5\lambda}{3^2} \frac{\partial f}{\partial \Phi} + \Phi \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Psi}{\lambda} \frac{\partial f}{\partial \Psi} \right] \\ + \frac{A^{-1}}{t_1^3} \left[\frac{5}{3^2} \Phi \frac{\partial f}{\partial \Phi} + \frac{5\Psi}{3^2} \frac{\partial f}{\partial \Psi} - \frac{5\lambda}{3^2} \frac{\partial f}{\partial \lambda} \right] = \eta(F - f) \end{aligned} \quad (2 - 51)$$

An asymptotic solution of equation (2 - 51) can be sought by writing

$$f = f^0 + A^{-1/2} f^1 + O(A^{-1}) \quad (2 - 52)$$

Where f^0 is given by equation (2 - 44)

On substituting equations (2 - 49) and (2 - 50) into equation (2 - 51), the B-G-K equation (2 - 51), valid upto the order of $A^{-1/2}$, can be written as

$$\begin{aligned} \frac{\partial f}{\partial t_1} + \frac{A^{-1/2}}{t_1^2} \left[-\frac{5\lambda}{3^2} \frac{\partial f}{\partial \Phi} + \Phi \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Psi}{\lambda} \frac{\partial f}{\partial \Psi} \right] \\ = \frac{\left(\frac{K}{3}\right)^{3/2}}{t_1^3} (f^0 - f) \end{aligned} \quad (2 - 53)$$

Substituting equation (2 - 52) into equation (2 - 53)

$$\begin{aligned} & \frac{\partial f^0}{\partial t_1} + \frac{A^{-\frac{1}{2}}}{t_1^2} \left[-\frac{5\lambda}{3^2} \frac{\partial f^0}{\partial \Phi} + \Phi \frac{\partial f^0}{\partial \lambda} + \frac{\Phi^2}{\lambda} \frac{\partial f^0}{\partial \Phi} - \frac{\Phi \Phi}{\lambda} \frac{\partial f^0}{\partial \Phi} \right] \\ & + A^{-\frac{1}{2}} \frac{\partial f^1}{\partial t_1} + A^{-1} \frac{1}{t_1^2} \left[-\frac{5\lambda}{3^2} \frac{\partial f^1}{\partial \Phi} + \Phi \frac{\partial f^1}{\partial \lambda} + \frac{\Phi^2}{\lambda} \frac{\partial f^1}{\partial \Phi} - \frac{\Phi \Phi}{\lambda} \frac{\partial f^1}{\partial \Phi} \right] \\ & = -\frac{(K/3)^{3/2}}{t_1^3} \left[f^0 - f^0 - A^{-\frac{1}{2}} f^1 \right] \quad (2-53') \end{aligned}$$

Since $f^0 = \frac{1}{(2\lambda)^{3/2}} \exp - \frac{3}{2K} (\Phi^2 + \Phi^2)$

$$\frac{\partial f^0}{\partial t_1} = 0$$

$$-\frac{5\lambda}{3^2} \frac{\partial f^0}{\partial \Phi} = \frac{5}{3} \lambda \Phi f^0$$

$$\Phi \frac{\partial f^0}{\partial \lambda} = -\Phi \frac{\lambda}{K^2} (\Phi^2 + \Phi^2) f^0$$

$$\frac{\Phi^2}{\lambda} \frac{\partial f^0}{\partial \Phi} = -\frac{\Phi^2}{\lambda} f^0 \frac{3\Phi}{K}$$

$$-\frac{\Phi \Phi}{\lambda} \frac{\partial f^0}{\partial \Phi} = \frac{\Phi \Phi^2}{\lambda} \frac{3}{K} f^0$$

Upon substituting these quantities into equation (2 - 53') and collecting terms containing the order of A upto $A^{-\frac{1}{2}}$,

$$\frac{\partial f'}{\partial t_1} + \frac{(K/3)^{3/2}}{t_1^3} f' = \frac{\lambda f^0}{K t_1^2} \left[\frac{5}{3} \Phi - \Phi \frac{(\Phi^2 + \Psi^2)}{K} \right] \quad (2 - 54)$$

This is a first order linear differential equation and its solution is easily obtained:

The integrating factor is

$$e^{\int \frac{(K/3)^{3/2}}{t_1^3} dt_1} = e^{-\frac{(K/3)^{3/2}}{2t_1^2}}$$

Hence, the solution becomes

$$e^{-\frac{(K/3)^{3/2}}{2t_1^2}} f' = \int e^{-\frac{(K/3)^{3/2}}{2t_1^2}} \frac{\lambda f^0}{K t_1^2} \left[\frac{5}{3} \Phi - \Phi \frac{(\Phi^2 + \Psi^2)}{K} \right] dt_1$$

Defining t_2 as

$$t_2 = \frac{t_1}{(K/3)^{3/4}} \quad (2 - 55)$$

The final form of f' becomes

$$f' = e^{\frac{1}{2t_2^2}} g(\lambda, \Phi, \Psi) \int_0^{t_2} \frac{e^{-\frac{1}{2t_2^2}}}{-t_2} dt_2 \quad (2 - 56)$$

where

$$g(\lambda, \Phi, \Psi) = \frac{\lambda f^0}{3(K/3)^{3/4}} \left[\frac{5}{3} \Phi - \Phi \frac{(\Phi^2 + \Psi^2)}{K} \right] \quad (2 - 57)$$

Consequently, the distribution function f

$$f = f^0 \left\{ 1 + A^{-\frac{1}{2}} e^{\frac{1}{2t_2}} g(\lambda, \bar{\Phi}, \bar{\Psi}) \int_0^{t_2} \frac{e^{-\frac{1}{2t_2}}}{t_2} dt_2 + O(A^{-1}) \right\} \quad (2 - 58)$$

Defining $H(t_2)$ as

$$H(t_2) = e^{\frac{1}{2t_2}} \int_0^{t_2} \frac{e^{-\frac{1}{2t_2}}}{t_2} dt_2$$

equation (2 - 58) becomes

$$f = f^0 \left\{ 1 + A^{-\frac{1}{2}} \frac{\lambda}{3(K/3)^{3/4}} \left[\frac{5}{3} \bar{\Phi} - \frac{(\bar{\Phi}^2 + \bar{\Psi}^2)}{K} \bar{\Phi} \right] H(t_2) + O(A^{-1}) \right\} \quad (2 - 59)$$

Equation (2 - 59) is the asymptotic solution of the distribution function f valid in the outer region as defined by Grundy and Thomas⁽⁶⁾.

PART III

Development of Front Parameters

In order to determine the gas front parameter, similar to the variables used in equations (2 - 26) and (2 - 27), the order of magnitude of terms contained in equation (2 - 59) should be examined. A breakdown of equation (2 - 59) occurs when the first and second terms on the right hand side become of like order. The first term has the order of magnitude $O(1)$ and is bounded.^(*) The second term must be $O(1)$ in order that equation (2 - 59) breaks down, that is

$$A^{-\frac{1}{2}} \frac{\lambda \bar{\Phi}}{3(K/3)^{3/4}} \left[1 - \frac{(\bar{\Phi}^2 + \bar{\Psi}^2)}{K} \right] H(t_2) = O(1) \quad (3 - 1a)$$

here $H(t_2) = O(1)$ since t_2 is bounded.

The quantity $\frac{\bar{\Phi}^2 + \bar{\Psi}^2}{K}$, which is the power of the exponential term in the expression for the equilibrium distribution function f^0 , becomes $O(1)$ as a result of the same reason previously given on page 46, namely $\frac{\bar{\Phi}^2 + \bar{\Psi}^2}{K} = O(1)$.

Hence, from this relation the order of $\bar{\Phi}$ and $\bar{\Psi}$ become

$$\bar{\Phi} = O(K^{\frac{1}{2}}) \text{ and } \bar{\Psi} = O(K^{\frac{1}{2}}) \text{ respectively.}$$

Upon substituting the orders of $\frac{\bar{\Phi}^2 + \bar{\Psi}^2}{K}$, $\bar{\Phi}$ and $\bar{\Psi}$ into (3 - 1a), there results

$$A^{-\frac{1}{2}} \frac{\lambda K^{\frac{1}{2}}}{K^{3/4}} = O(1)$$

(*) see appendix A.

At the gas front number density $n \rightarrow 0$; hence, from equation (2 - 39) $K \rightarrow 0$. Then, from the relation (2 - 40),

$$\text{i.e. } K = 1 - \frac{\lambda^2}{3}, \quad \lambda \rightarrow \sqrt{3} .$$

Consequently, at the gas front

$$A^{-\frac{1}{2}} \frac{K^{\frac{1}{2}}}{K^{\frac{1}{4}}} = O(1)$$

$$\therefore K = O(A^{-\frac{2}{5}})$$

From

$$K = 1 - \frac{\lambda^2}{3} = O(A^{-\frac{2}{5}})$$

$$\lambda^2 = 3[1 - O(A^{-\frac{2}{5}})]$$

$$\therefore \lambda = \sqrt{3} [1 - O(A^{-\frac{2}{5}})]^{\frac{1}{2}}$$

Expanding the expression $[1 - O(A^{-\frac{2}{5}})]^{\frac{1}{2}}$

$$[1 - O(A^{-\frac{2}{5}})]^{\frac{1}{2}} = 1 + \frac{1}{2}(-) O(A^{-\frac{2}{5}})$$

$$+ \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} [O(A^{-\frac{2}{5}})]^2 + \dots$$

For first order approximation, collecting the first two terms, λ becomes

$$\lambda = \sqrt{3} + \sqrt{3} \frac{1}{2}(-) O(A^{-\frac{2}{5}}) \quad (3 - 1b)$$

As explained earlier it is needed to develop a new front scale at the point of breakdown. The formal way of setting a new scale is to set a scale in such a way that either the scale become stretched or shrunken⁽¹⁶⁾. From equation (3 - 1b) a front scale Δ with respect to the old scale λ is defined as

$$\lambda = \sqrt{3} + A^{-\frac{2}{5}} \Delta \quad (3 - 1)$$

It should be noted that the front scale Δ , as seen from the stand point of Grundy and Thomas' scale λ , appears very much shrunken because Δ is divided by $A^{2/5}$ where $A \rightarrow \infty$.

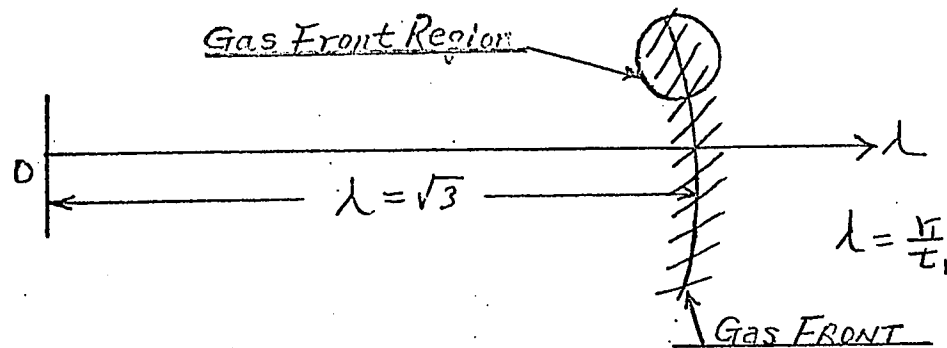


Fig. 4. Schematic Diagram of Gas Front Region.

Since the orders of Φ and Ψ are $\Phi = O(K^{\frac{1}{2}})$ and $\Psi = O(K^{\frac{1}{2}})$, the corresponding front variables Φ' and Ψ' with regard to Φ and Ψ are defined in a similar manner as λ ,

$$\left. \begin{aligned} \Phi &= K^{\frac{1}{2}} \Phi' = A^{-\frac{1}{5}} \Phi' \\ \Psi &= K^{\frac{1}{2}} \Psi' = A^{-\frac{1}{5}} \Psi' \end{aligned} \right\} \quad (3-2)$$

From equation (2-55), t_1 becomes

$$t_1 = 3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2 \quad (3-3)$$

The order of t_1 is $O(A^{-\frac{3}{10}})$.

Since $n \sim \frac{K^{\frac{3}{2}}}{t_1^3}$, (i.e. equation (2-21)) the order of n becomes $O(A^{\frac{3}{10}})$.

Front number density n' should be defined as

$$n = A^{\frac{3}{10}} n' \quad (3-4)$$

Likewise,

$$\tau = \frac{K^{\frac{1}{2}}}{t_1^2} = O(A^{\frac{1}{5}})$$

τ' is defined

$$\tau = A^{\frac{1}{5}} \tau' \quad (3-5)$$

Substituting equation (3-1) into equation (2-49),

U becomes

$$U = (\sqrt{3} + A^{-\frac{2}{5}} \Delta) \left(1 - \frac{5}{3^2} A^{-1} \frac{A^{\frac{3}{5}}}{t_2^2} \right)$$

$$= (\sqrt{3} + A^{-\frac{2}{5}} \Delta) \left(1 - \frac{5}{3^2} \frac{A^{-\frac{2}{5}}}{t_2^2} \right)$$

From the relation (3 - 3), the term $\frac{5 A^{-\frac{2}{5}}}{3^2 t_2^2}$ becomes

$$\frac{5 A^{-\frac{2}{5}}}{3^2 t_2^2} = \frac{5}{3^{\frac{11}{4}} A^{\frac{2}{5}} A^{\frac{2}{5}} t_1^2} = \frac{5}{3^{\frac{11}{4}} A t_1^2}$$

Since $t_1 \sim O(A^{-\frac{3}{10}})$, this term becomes very small, hence U for large t_1 can be approximated by

$$U \sim (\sqrt{3} + A^{-\frac{2}{5}} \Delta)$$

Hence, the corresponding front variable U' with regard to U should be defined as

$$U = \sqrt{3} + A^{-\frac{2}{5}} U' \quad (3 - 6)$$

Definitions (3 - 1) to (3 - 6) form a new set of front variables.

PART IV

Scaled B-G-K model of Boltzmann Equation

For convenience equation (2 - 28) and the front variables are rewritten,

$$-A^{\frac{1}{2}} \frac{\partial f}{\partial \Phi} \left(u \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial u}{\partial \lambda} + t_1 \frac{\partial u}{\partial t_1} \right) + \frac{\partial f}{\partial t_1} + \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi} \times$$

$$\left(1 - \frac{\partial u}{\partial \lambda} \right) + \frac{\Phi}{t_1} \frac{\partial f}{\partial \Phi} \left(1 - \frac{u}{\lambda} \right) + \frac{(u - \lambda)}{t_1} \frac{\partial f}{\partial \lambda}$$

$$+ A^{-\frac{1}{2}} \left[\frac{\Phi}{t_1^2} \frac{\partial f}{\partial \lambda} + \frac{\Phi^2}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} - \frac{\Phi \Phi}{\lambda t_1^2} \frac{\partial f}{\partial \Phi} \right] = n(F - f)$$

(2 - 28)

$$\lambda = \sqrt{3} + A^{-\frac{2}{5}} \Delta$$

$$\Phi = A^{-\frac{1}{5}} \Phi'$$

$$\bar{\Phi} = A^{-\frac{1}{5}} \bar{\Phi}'$$

$$t_1 = 3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2$$

$$n = A^{\frac{3}{10}} n'$$

$$\gamma = A^{\frac{1}{5}} \gamma'$$

$$u = \sqrt{3} + A^{-\frac{2}{5}} u'(\Delta, t_2)$$

(4 - 1)

With the aid of front variables (4 - 1), equation (2 - 28) is transformed as follows:

since the distribution function f becomes a function of the front variables Δ , t_2 , $\bar{\Phi}'$ and $\bar{\Psi}'$, i.e.

$$f = f(\Delta, t_2, \bar{\Phi}', \bar{\Psi}')$$

the total differential for f becomes

$$df = \frac{\partial f}{\partial \Delta} d\Delta + \frac{\partial f}{\partial t_2} dt_2 + \frac{\partial f}{\partial \bar{\Phi}'} d\bar{\Phi}' + \frac{\partial f}{\partial \bar{\Psi}'} d\bar{\Psi}'$$

Dividing by $d\bar{\Phi}$

$$\frac{df}{d\bar{\Phi}} = \frac{\partial f}{\partial \Delta} \frac{d\Delta}{d\bar{\Phi}} + \frac{\partial f}{\partial t_2} \frac{dt_2}{d\bar{\Phi}} + \frac{\partial f}{\partial \bar{\Phi}'} \frac{d\bar{\Phi}'}{d\bar{\Phi}} + \frac{\partial f}{\partial \bar{\Psi}'} \frac{d\bar{\Psi}'}{d\bar{\Phi}}$$

the partial differential form of the above expression becomes

$$\left(\frac{\partial f}{\partial \bar{\Phi}}\right)_{\lambda, t_1, \bar{\Psi}} = \frac{\partial f}{\partial \Delta} \frac{\partial \Delta}{\partial \bar{\Phi}} + \frac{\partial f}{\partial t_2} \frac{\partial t_2}{\partial \bar{\Phi}} + \frac{\partial f}{\partial \bar{\Phi}'} \frac{\partial \bar{\Phi}'}{\partial \bar{\Phi}} + \frac{\partial f}{\partial \bar{\Psi}'} \frac{\partial \bar{\Psi}'}{\partial \bar{\Phi}}$$

From equations (4 - 1), it can be seen that $\frac{\partial \Delta}{\partial \bar{\Phi}}$, $\frac{\partial t_2}{\partial \bar{\Phi}}$

and $\frac{\partial \bar{\Psi}'}{\partial \bar{\Phi}}$ become zero leaving $\frac{\partial \bar{\Phi}'}{\partial \bar{\Phi}} = A^{1/5}$.

Hence

$$\left(\frac{\partial f}{\partial \bar{\Phi}}\right)_{\lambda, t_1, \bar{\Psi}} = \frac{\partial f}{\partial \bar{\Phi}'} \frac{\partial \bar{\Phi}'}{\partial \bar{\Phi}} = A^{1/5} \frac{\partial f}{\partial \bar{\Phi}'}$$

The mean velocity u is a function of Δ and t_2 .

Thus,

$$\frac{\partial u}{\partial \lambda} = \frac{\partial u}{\partial \Delta} \frac{\partial \Delta}{\partial \lambda} + \frac{\partial u}{\partial t_2} \frac{\partial t_2}{\partial \lambda}$$

where $\frac{\partial \Delta}{\partial \lambda} = A^{3/5}$ and $\frac{\partial t_2}{\partial \lambda} = 0$.

Consequently $\frac{\partial u}{\partial \lambda}$ becomes

$$\frac{\partial u}{\partial \lambda} = \frac{\partial u}{\partial \Delta} A^{2/5}$$

where $\frac{\partial u}{\partial \Delta} = A^{-3/5} \frac{\partial u'}{\partial \Delta}$

Finally $\frac{\partial u}{\partial \lambda}$ becomes

$$\frac{\partial u}{\partial \lambda} = \left(\frac{\partial u}{\partial \lambda} \right)_{t_1} = \frac{\partial u'}{\partial \Delta}$$

Similarly,

$$\frac{\partial f}{\partial \Phi} = \left(\frac{\partial f}{\partial \Phi} \right)_{\lambda, t_1, \Xi} = A^{1/5} \frac{\partial f}{\partial \Phi'}$$

$$\frac{\partial u}{\partial t_1} = \left(\frac{\partial u}{\partial t_1} \right)_{\lambda} = 3^{3/4} A^{-1/10} \frac{\partial u'}{\partial t_2}$$

$$\frac{\partial f}{\partial t_1} = \left(\frac{\partial f}{\partial t_1} \right)_{\lambda, \Xi, \Phi} = 3^{3/4} A^{3/10} \frac{\partial f}{\partial t_2}$$

$$\frac{\partial f}{\partial \lambda} = \left(\frac{\partial f}{\partial \lambda} \right)_{t, \Xi, \Phi} = A^{3/5} \frac{\partial f}{\partial \Delta}$$

Substituting these expressions into equation (2 - 28),

we obtain

$$\begin{aligned}
& -A^{\frac{1}{2}} A^{\frac{1}{5}} \frac{\partial f}{\partial \Phi'} \left[(\sqrt{3} + A^{-\frac{2}{5}} U') \frac{\partial U'}{\partial \Delta} - (\sqrt{3} + A^{-\frac{2}{5}} \Delta) \frac{\partial U'}{\partial \Delta} \right. \\
& \left. + 3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2 3^{\frac{3}{4}} A^{-\frac{1}{10}} \frac{\partial U'}{\partial t_2} \right] + \left\{ 3^{-\frac{3}{4}} A^{\frac{3}{10}} \frac{\partial f}{\partial t_2} + \frac{A^{-\frac{1}{5}} \Phi'}{3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2} A^{\frac{1}{5}} \frac{\partial f}{\partial \Phi'} \right\} \times \\
& \left(1 - \frac{\partial U'}{\partial \Delta} \right) + \frac{A^{-\frac{1}{5}} \Phi'}{3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2} A^{\frac{1}{5}} \frac{\partial f}{\partial \Phi'} \left(1 - \frac{\sqrt{3} + A^{-\frac{2}{5}} U'}{\sqrt{3} + A^{-\frac{2}{5}} \Delta} \right) \\
& \left. + \frac{1}{3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2} A^{\frac{2}{5}} \frac{\partial f}{\partial \Delta} \left[\sqrt{3} + A^{-\frac{2}{5}} U' - (\sqrt{3} + A^{-\frac{2}{5}} \Delta) \right] \right\} \\
& + A^{\frac{1}{2}} \left[\frac{A^{-\frac{1}{5}} \Phi'}{(3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2)^2} A^{\frac{2}{5}} \frac{\partial f}{\partial \Delta} + \frac{(A^{-\frac{1}{5}} \Phi')^2}{(\sqrt{3} + A^{-\frac{2}{5}} \Delta) (3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2)^2} A^{\frac{1}{5}} \frac{\partial f}{\partial \Phi'} \right. \\
& \left. - \frac{A^{-\frac{1}{5}} \Phi' A^{-\frac{1}{5}} \Phi'}{(\sqrt{3} + A^{-\frac{2}{5}} \Delta) (3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2)^2} \left(\frac{\partial f}{\partial \Phi'} \right) A^{\frac{1}{5}} \right] \\
& = -A^{\frac{3}{10}} \frac{\partial f}{\partial \Phi'} \left[(U' - \Delta) \frac{\partial U'}{\partial \Delta} + t_2 \frac{\partial U'}{\partial t_2} \right] + A^{\frac{3}{10}} 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial U'}{\partial \Delta} \right) \times \right. \\
& \left. \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \left(1 - \frac{\sqrt{3} + A^{-\frac{2}{5}} U'}{\sqrt{3} + A^{-\frac{2}{5}} \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \frac{U' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right. \\
& \left. + A^{\frac{3}{10}} (3^{\frac{3}{4}})^2 \frac{\Phi'}{(t_2)^2} \left(\frac{\partial f}{\partial \Delta} \right) + A^{-\frac{1}{10}} (3^{\frac{3}{4}})^2 \frac{1}{(\sqrt{3} + A^{-\frac{2}{5}} \Delta)} \left(\frac{\Phi'^2 \partial f}{t_2^2 \partial \Phi'} - \frac{\Phi' \Phi' \partial f}{t_2^2 \partial \Phi'} \right) \right] \\
& = n' A^{\frac{3}{10}} (F - f)
\end{aligned}$$

Rewriting

$$\begin{aligned}
& -\frac{\partial f}{\partial \Phi'} \left[(U' - \Delta) \frac{\partial U'}{\partial \Delta} + t_2 \frac{\partial U'}{\partial t_2} \right] + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial U'}{\partial \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) \right. \\
& \left. + \left(1 - \frac{\sqrt{3} + A^{-\frac{2}{5}} U'}{\sqrt{3} + A^{-\frac{2}{5}} \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \frac{U' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right] + (3^{\frac{3}{4}})^2 \frac{\Phi'}{(t_2)^2} \left(\frac{\partial f}{\partial \Delta} \right)
\end{aligned}$$

$$A^{-\frac{2}{5}} \frac{(3^{\frac{3}{4}})^2}{(\sqrt{3} + A^{-\frac{2}{5}} \Delta)} \left(\frac{\Phi'^2 \partial f}{t_2^2 \partial \Phi'} - \frac{\Phi' \Phi' \partial f}{t_2^2 \partial \Phi'} \right) = n' (F - f) \quad (4 - 2)$$

Neglecting terms containing $A^{-\frac{2}{5}}$ in equation (4 - 2)

$$\begin{aligned} & -\frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) \right. \\ & \left. + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right] + (3^{\frac{3}{4}})^2 \frac{\Phi'}{t_2^2} \left(\frac{\partial f}{\partial \Delta} \right) = n' (F - f) \quad (4 - 3) \end{aligned}$$

Equation (4 - 3) is the governing equation of the gas front region.

Now, substituting equation (4 - 1) into equation (2 - 29)

the equilibrium distribution function becomes

$$F = \frac{n'}{(2\pi \gamma')^{3/2}} \exp - \frac{(\Phi'^2 + \Psi'^2)}{2(3^{-\frac{3}{4}} t_2)^2 \gamma'} \quad (4 - 4)$$

PART V

Solution of Front Equation

Front equation (4 - 3) is rewritten

$$\begin{aligned}
 & -\frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \\
 & + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right] \\
 & + \left(3^{\frac{3}{4}} \right)^2 \frac{\Phi'}{t_2^2} \left(\frac{\partial f}{\partial \Delta} \right) = n' (F - f) \quad (4 - 3)
 \end{aligned}$$

As indicated previously in Part I, a method utilized here to solve the B-G-K equation (4 - 3) is to construct moment equations from it by multiplying through by functions of the molecular velocity and integrating over velocity space. In this way equations for density, velocity, temperature, heat transfer and higher moments can be obtained.

In order to develop moment equations from equation (4 - 3) it is necessary to transform equation (2 - 24) into a form incorporating new front variables, so that it may be used in developing moment equations.

Equation (2 - 24) is given as

$$\frac{n t_1^3}{2\pi} = \int_{-\infty}^{\infty} \int_0^{\infty} f \bar{\Psi} d\bar{\Psi} d\bar{\Phi}$$

From equation (4 - 1)

$$\left. \begin{aligned} \bar{\Psi} &= A^{-1/5} \bar{\Psi}' \\ d\bar{\Psi} &= A^{-1/5} d\bar{\Psi}' \\ d\bar{\Phi} &= A^{-1/5} d\bar{\Phi}' \\ n &= A^{3/10} n' \\ t_1 &= 3^{-3/4} A^{-3/10} t_2 \end{aligned} \right\} (4 - 1')$$

Substituting equations (4 - 1') into equation (2 - 24)

$$(3^{-9/4}) \frac{n' t_2^3}{2\pi} = \int \int f \bar{\Psi}' d\bar{\Psi}' d\bar{\Phi}' \quad (5 - 1)$$

Now, from equation (4 - 3) the general expression for moment equations is given by

$$\int \int \Gamma \left\{ t_1 \frac{\partial f}{\partial \bar{\Psi}'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3^{3/4} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial u'}{\partial \Delta} \right) x \right] \right.$$

$$\left. \frac{\bar{\Phi}'}{t_2} \left(\frac{\partial f}{\partial \bar{\Phi}'} \right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) + (3^{3/4})^2 \frac{\bar{\Phi}'}{t_2^2} \left(\frac{\partial f}{\partial \Delta} \right) \right\} \bar{\Psi}' d\bar{\Psi}' d\bar{\Phi}'$$

$$= \iint \Gamma n' (F - f) \Psi' d\Psi' d\Phi' \quad (5 - 2)$$

where $\Gamma = 1$, Φ' , Φ'^2 and Φ'^2 , Φ'^3 respectively give continuity, momentum, energy equations and heat transfer terms.

The zeroth order moment equation ($\Gamma = 1$) becomes

$$\iint \left\{ \left(1 - \frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3 \frac{\partial^3 f}{\partial t_2^3} + \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right\} \Psi' d\Psi' d\Phi' = n' \iint (F - f) \Psi' d\Psi' d\Phi'$$

working term by term, the first term

$$\begin{aligned} & \int_0^{\infty} \int_{-\infty}^{\infty} \left[\left(1 - \frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \right) \Psi' d\Psi' d\Phi' \right. \\ & \left. = \left(1 - \frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \right) \int_0^{\infty} f \int_{-\infty}^{\infty} \Psi' d\Psi' d\Phi' = 0 \right. \end{aligned}$$

as $f \int_{-\infty}^{\infty} = 0$. Also it must be noted that the range of integration for Ψ' is 0 to ∞ since ρ can never take negative values.

The second term,

$$\iint 3 \frac{\partial^3 f}{\partial t_2^3} \Psi' d\Psi' d\Phi' = 3 \frac{\partial^3}{\partial t_2^3} \iint f \Psi' d\Psi' d\Phi'$$

$$3^{\frac{3}{4}} \frac{\partial}{\partial t_2} (3^{-\frac{3}{4}})^3 \frac{n' t_2^3}{2\pi} = \frac{(3^{-\frac{3}{4}})^2}{2\pi} \frac{\partial (n' t_2^3)}{\partial t_2}$$

The third term,

$$\begin{aligned} & \iint 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'}\right) \Psi' d\Psi' d\Phi' \\ &= \frac{3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right)}{t_2} \iint \left(\frac{\partial f}{\partial \Phi'}\right) \Phi' \Psi' d\Psi' d\Phi' \\ &= \frac{3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right)}{t_2} \left[\int_{\Phi'=-\infty}^{\Phi'=\infty} \Phi' f \right] \Psi' d\Psi' \\ &= (-) \frac{3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right)}{t_2} \iint f \Psi' d\Psi' d\Phi' \\ &= (+) \frac{3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{n' t_2^3}{2\pi} (3^{-\frac{3}{4}})^3}{t_2} = (-) \frac{(3^{-\frac{3}{4}})^2 \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{n' t_2^3}{2\pi}}{t_2} \end{aligned}$$

The fourth term,

$$\iint 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta}\right) \Psi' d\Psi' d\Phi' = \frac{(3^{-\frac{3}{4}})^2}{2\pi} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} (n' t_2^3)$$

And fifth term,

$$\iint (3^{\frac{3}{4}})^2 \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Delta}\right) \Psi' d\Psi' d\Phi' = (3^{\frac{3}{4}})^2 \frac{1}{t_2} \frac{\partial}{\partial \Delta} \iint \Phi' f \Psi' d\Psi' d\Phi' = 0$$

Now, the right hand side of the zeroth order momentum equation is

$$\begin{aligned} & \iint n' (F - f) \Phi' d\Phi' d\Phi' \\ &= n' \left[\iint F \Phi' d\Phi' d\Phi' - \iint f \Phi' d\Phi' d\Phi' \right] = 0 \end{aligned}$$

for, $\iint F \Phi' d\Phi' d\Phi' = \iint f \Phi' d\Phi' d\Phi'$

Summing up these terms there results

$$\frac{(3^{-\frac{3}{4}})^2}{2\pi} \frac{\partial(n't_2^3)}{\partial t_2} - \frac{(3^{-\frac{3}{4}})^2}{2\pi} n't_2^3 \frac{(1 - \frac{\partial u'}{\partial \Delta})}{t_2} + \frac{(3^{-\frac{3}{4}})^2}{2\pi} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} (n't_2^3) = 0$$

Rewriting this expression,

$$\frac{\partial(n't_2^3)}{\partial t_2} - n't_2^3 \frac{(1 - \frac{\partial u'}{\partial \Delta})}{t_2} + \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} (n't_2^3) = 0 \quad (5-3)$$

Which is the continuity equation⁽¹⁵⁾.

The first order moment equation ($\Gamma = \Phi'$) becomes

$$\iint \Phi' \left\{ t_2 \frac{\partial f}{\partial \Phi'} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + (1 - \frac{\partial u'}{\partial \Delta}) \right] \right\} \Phi' d\Phi' d\Phi'$$

$$\frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) + \left(3^{\frac{3}{4}} \right)^2 \frac{\Phi'}{t_2^2} \left(\frac{\partial f}{\partial \Delta} \right) \left\} \Phi' d\Phi' d\Phi'$$

$$= n' \iint \Phi' (F - f) \Phi' d\Phi' d\Phi'$$

Working term by term, the first term,

$$\begin{aligned}
 & \iint (-) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \frac{\partial f}{\partial \Phi'} \Phi' \Psi' d\Phi' d\Psi' \\
 &= (-) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \iint \frac{\partial f}{\partial \Phi'} \Phi' \Psi' d\Phi' d\Psi' \\
 &= \frac{(3^{-\frac{3}{4}})^3}{2\pi} (n't_2^3) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right]
 \end{aligned}$$

Since

$$\begin{aligned}
 & \iint \frac{\partial f}{\partial \Phi'} \Phi' \Psi' d\Phi' d\Psi' = \int \left[\Phi' f \right]_{-\infty}^{\infty} - \int f d\Phi' \Psi' \\
 &= (-) \iint f \Psi' d\Phi' d\Psi' = (-) (3^{-\frac{3}{4}})^3 \frac{n't_2^3}{2\pi}
 \end{aligned}$$

the second term,

$$\iint 3^{\frac{3}{4}} \frac{\partial f}{\partial t_2} \Phi' \Psi' d\Phi' d\Psi' = 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} \iint f \Phi' \Psi' d\Phi' d\Psi' = 0$$

the third term,

$$\begin{aligned}
 & \iint 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{1}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) \Phi'^2 \Psi' d\Phi' d\Psi' \\
 &= 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{1}{t_2} \iint \left(\frac{\partial f}{\partial \Phi'} \right) \Phi'^2 \Psi' d\Phi' d\Psi' = 0
 \end{aligned}$$

since

$$\begin{aligned} & \iint \left(\frac{\partial f}{\partial \Phi'} \right) \Phi'^2 d\Phi' d\Phi' \\ &= \int \left[\Phi'^2 f \right]_{-\infty}^{\infty} - 2 \int f \Phi' d\Phi' \Big|_{-\infty}^{\infty} = 0 \end{aligned}$$

the fourth term

$$\begin{aligned} & \iint 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \Phi' d\Phi' d\Phi' \\ &= 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} \iint f \Phi' d\Phi' d\Phi' = 0 \end{aligned}$$

and the fifth term,

$$\iint \left(3^{\frac{3}{4}} \right)^2 \frac{1}{t_2} \frac{\partial}{\partial \Delta} \iint f \Phi'^2 d\Phi' d\Phi' = \left(3^{\frac{3}{4}} \right)^2 \frac{1}{t_2} \frac{\partial}{\partial \Delta} P_{rr}$$

where
$$P_{rr} = \iint f \Phi'^2 d\Phi' d\Phi'$$

Now, the right hand side of the first order moment equation is

$$\begin{aligned} & n' \iint \Phi' (F - f) d\Phi' d\Phi' \\ &= n' \iint F \Phi' d\Phi' d\Phi' - n' \iint f \Phi' d\Phi' d\Phi' \\ &= 0 - 0 = 0 \end{aligned}$$

Since momentum integrated over the whole velocity space remains unchanged.

Summing up these terms, there results

$$(3^{-\frac{3}{4}})^3 \frac{n' t_2^3}{2\pi} \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + \frac{(3^{\frac{3}{4}})^2}{t_2^2} \frac{\partial}{\partial \Delta} P_{rr} = 0 \quad (5-4)$$

which is the momentum equation ⁽¹⁵⁾.

The first of the two second order moment equations ($\Gamma = \bar{\Phi}^2$),

$$\left\{ \int \bar{\Phi}^2 \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \frac{\partial f}{\partial \bar{\Phi}'} + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{\bar{\Phi}'}{t_2} \left(\frac{\partial f}{\partial \bar{\Phi}'} \right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right] + \left(3^{\frac{3}{4}} \right)^2 \frac{\bar{\Phi}'}{t_2^2} \frac{\partial f}{\partial \Delta} \right\} \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = \int \left(n'(F-f) \bar{\Phi}^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' \right)$$

Now again working term by term, the first term,

$$\int \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \frac{\partial f}{\partial \bar{\Phi}'} \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}'$$

$$= \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \int \left[\frac{\partial f}{\partial \bar{\Phi}'} \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' \right] = 0$$

the second term,

$$\int \left[3^{\frac{3}{4}} \frac{\partial f}{\partial t_2} \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' \right] = 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} \int \left[f \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' \right] = 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} P_{rr}$$

the third term,

$$\begin{aligned} & \iint 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) \left(\frac{\partial f}{\partial \bar{\Phi}'}\right) \bar{\Phi}' \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' \\ &= 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{1}{t_2} \iint \left(\frac{\partial f}{\partial \bar{\Phi}'}\right) \bar{\Phi}'^3 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = \Leftrightarrow 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{3P_{rr}}{t_2} \end{aligned}$$

since, $\iint \left(\frac{\partial f}{\partial \bar{\Phi}'}\right) \bar{\Phi}'^3 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = \int \left[\bar{\Phi}'^3 f \Big|_{-\infty}^{\infty} - 3 \int f \bar{\Phi}'^3 d\bar{\Phi}'\right] \bar{\Phi}' d\bar{\Phi}'$

$$= \Leftrightarrow 3 \iint f \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = \Leftrightarrow 3P_{rr}$$

The fourth term,

$$\iint 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial f}{\partial \Delta} \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2}$$

$$\times \frac{\partial}{\partial \Delta} \iint f \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} P_{rr}$$

and the fifth term,

$$\iint \left(3^{\frac{3}{4}}\right)^2 \frac{1}{t_2^2} \frac{\partial f}{\partial \Delta} \bar{\Phi}' \bar{\Phi}'^2 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}'$$

$$= \left(3^{\frac{3}{4}}\right)^2 \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} \iint f \bar{\Phi}'^3 \bar{\Phi}' d\bar{\Phi}' d\bar{\Phi}' = \left(3^{\frac{3}{4}}\right)^2 \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} Q_{rrr}$$

where

$$Q_{rrr} = \iiint f \Phi'^3 \Psi' d\Psi' d\Phi'$$

Summing up these terms

$$\begin{aligned} & 3^{\frac{3}{4}} \frac{\partial P_{rr}}{\partial t_2} - 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) 3 P_{rr} + 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial P_{rr}}{\partial \Delta} \\ & + \left(3^{\frac{3}{4}}\right)^2 \frac{1}{t_2^2} \frac{\partial Q_{rrr}}{\partial \Delta} = \iiint n'(F-f) \Phi'^2 \Psi' d\Psi' d\Phi' \quad (5-5) \end{aligned}$$

The other second order moment equation ($\Gamma = \Psi'$)

becomes

$$\begin{aligned} & \iiint \Psi'^2 \left\{ \left(\frac{\partial f}{\partial \Phi'}\right) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} \right. \right. \\ & \left. \left. + \left(1 - \frac{\partial u'}{\partial \Delta}\right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'}\right) + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta}\right) \right] \right. \\ & \left. + \left(3^{\frac{3}{4}}\right)^2 \frac{\Phi'}{t_2^2} \left(\frac{\partial f}{\partial \Delta}\right) \right\} \Psi' d\Psi' d\Phi' = \iiint n'(F-f) \Psi'^2 \Psi' d\Psi' d\Phi' \end{aligned}$$

Again, working term by term, the first term,

$$\begin{aligned} & \iiint \left(\frac{\partial f}{\partial \Phi'}\right) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \Psi'^2 \Psi' d\Psi' d\Phi' \\ & = \left(\frac{\partial f}{\partial \Phi'}\right) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] \iiint \Psi'^2 \Psi' d\Psi' d\Phi' = 0 \end{aligned}$$

since

$$\iint \frac{\partial f}{\partial \Phi'} d\Phi' \Psi'^3 d\Psi' = 0$$

the second term

$$\begin{aligned} \iint 3^{\frac{3}{4}} \frac{\partial f}{\partial t_2} \Psi'^2 \Psi' d\Psi' d\Phi' &= 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} \iint f \Psi'^2 \Psi' d\Psi' d\Phi' \\ &= 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} P_{00} \end{aligned}$$

where

$$P_{00} = \iint f \Psi'^2 \Psi' d\Psi' d\Phi'$$

the third term

$$\begin{aligned} \iint 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) \left(\frac{\partial f}{\partial \Phi'}\right) \Psi' \Psi'^2 \Psi' d\Psi' d\Phi' \\ &= 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) \iint \left(\frac{\partial f}{\partial \Phi'}\right) \Psi' d\Phi' \Psi'^2 \Psi' d\Psi' \\ &= 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) \iint f \Psi'^2 \Psi' d\Psi' d\Phi' \\ &= \left(-1\right) 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) P_{00} \end{aligned}$$

since

$$\begin{aligned} \iint \left(\frac{\partial f}{\partial \Phi'}\right) \Psi' d\Phi' \Psi'^2 \Psi' d\Psi' &= \left[\Psi' f\right]_{-\infty}^{\infty} - \int f d\Psi' \Psi'^2 \Psi' \\ &= \iint f \Psi'^2 \Psi' d\Psi' d\Phi' \end{aligned}$$

the fourth term

$$\begin{aligned} \iint 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial f}{\partial \Delta} \Phi^2 \Phi' d\Phi d\Phi' &= 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} \iint f \Phi^2 \Phi' d\Phi d\Phi' \\ &= 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} P_{00} \end{aligned}$$

and the fifth term,

$$\begin{aligned} \iint (3^{\frac{3}{4}})^2 \frac{1}{t_2^2} \frac{\partial f}{\partial \Delta} \Phi' \Phi^2 \Phi' d\Phi' d\Phi &= (3^{\frac{3}{4}})^2 \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} \iint f \Phi' \Phi^2 \Phi' d\Phi d\Phi' \\ &= (3^{\frac{3}{4}})^2 \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} Q_{r00} \end{aligned}$$

where

$$Q_{r00} = \iint f \Phi' \Phi^2 \Phi' d\Phi' d\Phi$$

Summing up these terms, there results

$$\begin{aligned} 3^{\frac{3}{4}} \frac{\partial}{\partial t_2} P_{00} - 3^{\frac{3}{4}} \left(1 - \frac{\partial u'}{\partial \Delta}\right) \left(\frac{1}{t_2}\right) P_{00} + 3^{\frac{3}{4}} \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} P_{00} \\ + (3^{\frac{3}{4}})^2 \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} Q_{r00} = \iint n' (F - f) \Phi^2 \Phi' d\Phi d\Phi' \end{aligned} \quad (5 - 6)$$

The temperature γ has been expressed as

$$3n\gamma = \frac{2\pi}{t_1^5} \iint f (\Phi^2 + \Phi'^2) \Phi d\Phi d\Phi' \quad (2-31)$$

Substituting equations (4 - 1') and $\tau = A^{1/2} \tau'$ into equation (2 - 31),

we obtain the expression for τ' as

$$n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5 = \iint f [\Phi'^2 + \Phi'^2] \Phi' d\Phi' d\Phi' \quad (5 - 7)$$

Equation (5 - 7) can be expressed as

$$n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5 = P_{rr} + P_{\theta\theta} \quad (5 - 8)$$

In order to make equation (5 - 8) consistent with the actual definition of pressure tensors P_{rr} and $P_{\theta\theta}$, we must redefine them as,

$$\left. \begin{aligned} P_{rr}' &= \frac{2\pi (3^{\frac{3}{4}})^5}{t_2^5} P_{rr} \\ P_{\theta\theta}' &= \frac{2\pi (3^{\frac{3}{4}})^5}{t_2^5} P_{\theta\theta} \end{aligned} \right\} \quad (5 - 9)$$

So that equation (5 - 8) yields

$$n' \tau' = \frac{1}{3} (P_{rr}' + P_{\theta\theta}') \quad (5 - 10)$$

It should be noted that equation (5 - 10) holds for equilibrium as well as non-equilibrium cases.

Now, heat flux tensor is defined⁽¹⁾ as

$$q_{rrr} = n \overline{\xi_1^3} = 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} \xi_1^3 f p d p d \xi_1$$

$$q_{r\theta\theta} = n \overline{\xi_1 p^2} = 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} f \xi_1 p^3 d p d \xi_1$$

From equations (2 - 26) and (4 - 1)

$$\left. \begin{aligned}
 \bar{z}_1 &= A^{-\frac{1}{2}} \frac{\Phi}{t_1} = A^{-\frac{7}{10}} \frac{\Phi'}{t_1} \\
 \rho &= A^{-\frac{1}{2}} \frac{\Psi}{t_1} = A^{-\frac{7}{10}} \frac{\Psi'}{t_1} \\
 d\bar{z}_1 &= A^{-\frac{7}{10}} \frac{d\Phi'}{t_1} \\
 d\rho &= A^{-\frac{7}{10}} \frac{d\Psi'}{t_1} \\
 t_1 &= 3^{-\frac{3}{4}} A^{-\frac{3}{10}} t_2
 \end{aligned} \right\} (4 - 1a)$$

Substituting (4 - 1a) into g_{rrrr}

$$g_{rrrr} = 2\pi A^{-\frac{12}{5}} \frac{(3^{\frac{3}{4}})^6}{t_2^6} \iint f \Phi^3 \Psi^2 d\Phi' d\Psi'$$

Introducing another heat flux tensors Q'_{rrr} and $Q'_{r\theta\theta}$.

$$Q'_{rrr} = A^{\frac{12}{5}} g_{rrrr} \quad (5 - 11)$$

$$= 2\pi \frac{(3^{\frac{3}{4}})^6}{t_2^6} \iint f \Phi^3 \Psi^2 d\Phi' d\Psi'$$

$$Q'_{r\theta\theta} = A^{\frac{12}{5}} g_{r\theta\theta} = 2\pi \frac{(3^{\frac{3}{4}})^6}{t_2^6} \iint f \Phi^1 \Psi^2 \Psi^2 d\Phi' d\Psi' \quad (5 - 12)$$

The moment equations (5 - 3) to (5 - 6) should be reformed in such a way that pressure and heat flux tensor terms appearing in those equation become consistent with the original definitions of those tensors.

Reformed moment equations are

$$\frac{\partial(n't_2^3)}{\partial t_2} - n't_2^3 \frac{(1 - \frac{\partial u'}{\partial \Delta})}{t_2} + \frac{u' - \Delta}{t_2} \frac{\partial}{\partial \Delta} (n't_2^3) = 0 \quad (5 - 13)$$

$$n't_2^3 \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + t_2^3 \frac{\partial P_{rr}'}{\partial \Delta} = 0 \quad (5 - 14)$$

$$\begin{aligned} \frac{\partial}{\partial t_2} (t_2^5 P_{rr}') - (1 - \frac{\partial u'}{\partial \Delta}) \frac{3}{t_2} (t_2^5 P_{rr}') + \left(\frac{u' - \Delta}{t_2} \right) \frac{\partial}{\partial \Delta} (t_2^5 P_{rr}') \\ + \frac{1}{t_2} \frac{\partial}{\partial \Delta} (t_2^6 Q_{rr}') = 2\pi (3^{\frac{3}{4}})^4 \iint n'(F-f) \Phi'^2 \Phi' d\Phi' d\Phi' \end{aligned} \quad (5 - 15)$$

$$\begin{aligned} \frac{\partial}{\partial t_2} (t_2^5 P_{\theta\theta}') - (1 - \frac{\partial u'}{\partial \Delta}) \frac{1}{t_2} (t_2^5 P_{\theta\theta}') + \left(\frac{u' - \Delta}{t_2} \right) \frac{\partial}{\partial \Delta} (t_2^5 P_{\theta\theta}') \\ + \frac{1}{t_2} \frac{\partial}{\partial \Delta} (t_2^6 Q_{r\theta\theta}') = 2\pi (3^{\frac{3}{4}})^4 \iint n'(F-f) \Phi'^2 \Phi' d\Phi' d\Phi' \end{aligned} \quad (5 - 16)$$

The moment equations do not form a closed set because the last term on the left hand side of equation (4 - 3) causes a second order term to appear in the first order moment equation and a third order term in the second order moment equation and so on. Therefore, a closed set of moment equations will never be formed no matter how many higher order moment equations are taken in. Moment equations (5 - 13) to (5- 16) are of non-linear, integro-partial differential form of great complexity. We have four moment equations and six unknowns, namely n' , u' , P_{rr}' , $P_{\theta\theta}'$, Q_{rrr}' and $Q_{r\theta\theta}'$. In order to solve this set of equations, one must find some way to form them into a closed set.

To do this the unknown parameters contained in the moment equations are expanded in terms of positive powers of time t_2 . Substituting these expansions back into the moment equations and collecting only the terms containing the lowest powers of t_2 we obtain a closed set of moment equations for small time t_2 . This becomes possible because of the elimination of two heat transfer terms in the process. We observe that the second terms in the expansions of unknown parameters other than heat transfer parameters (Q_{rrr}' and $Q_{r\theta\theta}'$) and the first terms in the expan-

sions of Q_{rrr}' and $Q_{rr\theta}'$ produce, on substitution into the moment equations, terms containing the same powers of time t_2 . The reason for collecting only the terms containing the lowest powers of t_2 is that the contributions of higher order terms in t_2 becomes negligibly small for small time t_2 . More precisely, terms contained in the closed set of moment equations are the first terms in the expansions except in the expansions of heat transfer parameters. It must be noted however that small time t_2 does not necessarily imply small real time. This becomes obvious from the relation given in equation (4 - 1), i.e.

$$t_2 = 3^{\frac{3}{4}} A^{\frac{3}{10}} t_1$$

Since $t_1 = t A^{-\frac{1}{2}}$, real dimensionless time t becomes

$$t = 3^{-\frac{3}{4}} A^{\frac{1}{5}} t_2 \quad (4 - 1'')$$

This means real time t is of order $A^{\frac{1}{5}}$ times greater than t_2 .

The expansion of the unknown parameters in the moment equations for small values of time t_2 is, by no means, arbitrary. As was indicated by Grundy and Thomas⁽⁶⁾, the validity of the equilibrium solution obtained by Mirels and Mullen⁽¹⁰⁾ extends up to real time $t \sim O(A^{\frac{1}{2}})$ and

beyond this value of time the non-equilibrium solution obtained by Grundy and Thomas⁽⁶⁾ applies. This is shown schematically on Fig. 5.

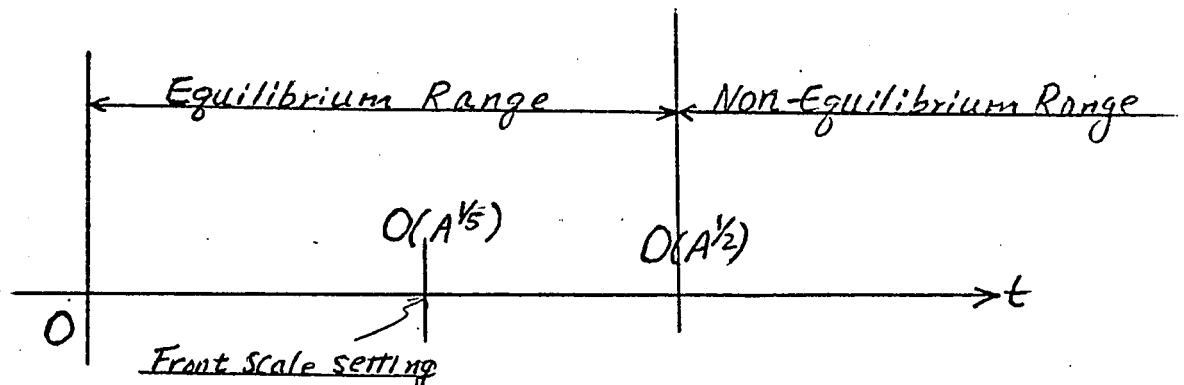


Fig. 5. Expansion of Gas Cloud into Vacuum with respect to Time t

As has been already established in equation (4 - 1) the gas front scale t_2 is set at the real time $t \sim O(A^{1/5})$. Further referring to equation (4 - 1'') a small change in the scaled time t_2 does not constitute a significant change of real time t , therefore, the entire gas front will remain within the equilibrium region for all small values of scaled time t_2 . Consequently one can expect that the outcoming solution of the front parameters shall be in some way at least,

similar to those obtained by Mirels and Mullen for the equilibrium region. More precisely, the approach to the problem would be to convert the results of Mirels and Mullen from the real time t into the scaled time t_2 and then match them with the results to be obtained for the front region solution which will be in terms of front time t_2 . In this way we can achieve asymptotic matching and avoid discontinuity of the solution between the two regions.

In carrying out the above approach we need not actually substitute the front parameter expansions into the moment equations, obtain final results, and then compare with Mirels and Mullens' results, instead, comparison is possible simply by comparing the first terms in the expansions with those results. Since we have expanded the front parameters in terms of the powers of t_2 , the powers will not be altered in the final solution; rather, the other variable functions contained in the expansions with powers of t_2 will be affected by the final solution. Therefore it is possible to determine the powers of t_2 contained in the expansion directly comparing with those results of Mirels and Mullen (or inner solutions). Conversion of inner solutions (Mirels and Mullen) from real time t to front time t_2 is accomplished as follows:

utilizing equation (2 - 18) for N

$$N \sim \frac{[1 - \frac{v^2}{3t^2}]^{3/2}}{3\sqrt{3} t^3}$$

$$= \frac{[1 - \frac{\lambda^2}{3}]^{3/2}}{3\sqrt{3} t^3} \quad \text{since } \lambda = \frac{v_1}{t_1} \text{ and } v_1 = vA^{-1/2}, t_1 = tA^{-1/2}$$

$$= \frac{[3 - \lambda^2]^{3/2}}{3^3 t^3} = \frac{[\sqrt{3} + \lambda]^{3/2} [\sqrt{3} - \lambda]^{3/2}}{3^3 t^3}$$

bearing the fact that near to the front, $K \rightarrow 0$ and $\lambda \rightarrow \sqrt{3}$, the order of magnitude analysis for N may be introduced in the following manner;

$$N \sim \frac{(2\sqrt{3})^{3/2} [\sqrt{3} - \lambda]^{3/2}}{3^2 t^3}$$

from (4 - 1)

$$\lambda = \sqrt{3} + A^{-2/5} \Delta$$

$$\Delta = (-)(\sqrt{3} - \lambda) A^{2/5}$$

$$\therefore (\sqrt{3} - \lambda) = (-) \Delta A^{-2/5}$$

substituting $(-) \Delta A^{-2/5}$ for $(\sqrt{3} - \lambda)$, N becomes

$$N \sim \frac{(2\sqrt{3})^{3/2} [(-)\Delta]^{3/2} [A^{-2/5}]^{3/2}}{3^3 t^3}$$

Considering only the absolute value order of magnitude for N , N becomes

$$N \sim \frac{A^{-\frac{3}{5}} \Delta^{3/2}}{t^3}$$

Since $t_1 = t A^{-\frac{1}{2}}$ and $t_1 = 3^{-\frac{3}{4}} A^{\frac{3}{10}} t_2$, thus $t = 3^{-\frac{3}{4}} A^{1/5} t_2$

and introducing a new parameter which combines both Δ and t_2 as $\Lambda = t_2^n \Delta$ and rewriting for N

$$N \sim \frac{A^{-\frac{3}{5}} A^{-\frac{3}{5}} [\Lambda t_2^{-n}]^{3/2}}{t^3} = A^{-\frac{6}{5}} \frac{\Lambda^{3/2}}{t_2^{\frac{3}{2}n+3}}$$

Since

$$N = n A^{-\frac{3}{2}}, \quad n = n' A^{\frac{3}{10}} \quad \therefore N = n' A^{-\frac{6}{5}}$$

Thus

$$N = n' A^{-\frac{6}{5}} \sim A^{-\frac{6}{5}} \frac{\Lambda^{3/2}}{t_2^{\frac{3}{2}n+3}}$$

$$\therefore n'_1 \sim \frac{\Lambda^{3/2}}{t_2^{\frac{3}{2}n+3}} \quad (5-17)$$

Subscript 1 in n'_1 is used to indicate the equilibrium value of n' . Again using equation (2-22) for T

$$T \sim \frac{[1 - \frac{v^2}{3t^2}]}{3t^2}$$

since $\lambda = \frac{v}{t_1}$ and $v_1 = \gamma A^{-\frac{1}{2}} \therefore t_1 = t A^{-\frac{1}{2}}$, hence T becomes

$$T \sim \frac{[3 - \lambda^2]}{3^2 t^2} = \frac{[\sqrt{3} + \lambda][\sqrt{3} - \lambda]}{3^2 t^2}$$

and following exactly the same method of order of magnitude analysis for T as N , one obtains for T ,

$$T \sim A^{-\frac{4}{5}} \frac{\Delta}{t_2} = A^{-\frac{4}{5}} \frac{\Lambda t_2^{-n}}{t_2} = A^{-\frac{4}{5}} \frac{\Lambda}{t_2^{2+n}}$$

since $T = \gamma A^{-1}$, $\gamma = \gamma' A^{1/5} \therefore T = A^{-\frac{4}{5}} \gamma'$, thus

$$\gamma' A^{-\frac{4}{5}} \sim A^{-\frac{4}{5}} \frac{\Lambda}{t_2^{2+n}}$$

$$\therefore \gamma' \sim \frac{\Lambda}{t_2^{2+n}} \quad (5 - 18)$$

again subscript 1 in γ'_1 is used to indicate the equilibrium value of γ' . Recalling equation (5 - 10)

$$n' \gamma' = \frac{1}{3} (P_{rr}' + P_{\theta\theta}')$$

The relationship between P_{rr}' and $P_{\theta\theta}'$ under equilibrium flow may be established by solving for P_{rr}' and $P_{\theta\theta}'$. As before subscript 1 in P_{rr}' and $P_{\theta\theta}'$ is used to indicate equilibrium values of P_{rr}' and $P_{\theta\theta}'$. From equation (5 - 9) we can write

$$P_{rr}' = \frac{2\pi (3^{\frac{3}{4}})^5}{t_2^5} P_{rr1}$$

$$P_{\theta\theta}' = \frac{2\pi (3^{\frac{3}{4}})^5}{t_2^5} P_{\theta\theta1}$$

where

$$P_{rr1} = \iint F \Phi'^2 \Psi' d\Phi' d\Psi'$$

and

$$P_{\theta\theta 1} = \iint F \Psi'^2 \Phi' d\Phi' d\Psi'$$

where F is given by equation (4 - 4).

Substituting for F

$$P_{rr1} = \int_0^{\infty} \int_0^{\infty} \frac{n'}{(2\pi r')^{3/2}} \exp\left[-\frac{(\Phi'^2 + \Psi'^2)}{2(3^{-\frac{3}{4}} t_2)^2 r'}\right] \Phi'^2 \Psi' d\Phi' d\Psi'$$

$$= \frac{n'}{(2\pi r')^{3/2}} \frac{1}{2} \left(\frac{\pi}{a^3}\right)^{\frac{1}{2}} \frac{1}{2a}$$

$$P_{\theta\theta 1} = \int_0^{\infty} \int_0^{\infty} \frac{n'}{(2\pi r')^{3/2}} \exp\left[-\frac{(\Phi'^2 + \Psi'^2)}{2(3^{-\frac{3}{4}} t_2)^2 r'}\right] \Psi'^2 \Phi' d\Phi' d\Psi'$$

$$= \frac{n'}{(2\pi r')^{3/2}} \left(\frac{\pi}{a^3}\right)^{\frac{1}{2}} \frac{1}{2a}$$

where

$$a = \frac{1}{2(3^{-\frac{3}{4}} t_2)^2 r'}$$

thus,

$$P_{\theta\theta 1} = 2 P_{rr1} \quad (5 - 10')$$

This result enables us to write

$$n' r' = \frac{1}{3} (P_{rr1} + 2P_{\theta\theta 1}) = P_{rr1}, \text{ and rewriting}$$

$$n' r' = P_{rr1} \quad (5 - 19)$$

Now substituting equations (5 - 17) and (5 - 18) into (5 - 19), P_{rr}' becomes

$$P_{rr}' \sim \frac{\Lambda^{5/2}}{t_2^{5/2n+5}} \quad (5 - 20)$$

Writing $n' \omega$ for the first term of expansion of n' , as

$$n' \omega = \frac{F_1(\Lambda)}{t_2^m}$$

where

$$\Lambda = t_2^n \Delta$$

Now, in order to determine the powers of t_2 , i.e.

m and n , the powers of t_2 between $n' \omega$ and equation (5 - 17) should be brought to match each other, that is

$$n' \omega \sim \frac{\Lambda^{3/2}}{t_2^{3/2n+3}} \quad \text{and} \quad n' \omega = \frac{F_1(\Lambda)}{t_2^m}$$

Thence,

$$\frac{3}{2}n + 3 = m \quad (5 - 21)$$

In order to obtain one more equation relating m and

n , the first term of expansion of P_{rr}' which is the first order moment term must be considered.

The first term of the expansion of P_{rr}' , $P_{rr}' \omega$, is given by

$$P_{rr}' \omega = \frac{P_{r1}(\Lambda)}{t_2^l}$$

Again matching P_{rr}' with the equilibrium value P_{rr}' , given by equation (5 - 20)

$$P_{rr}' \sim \frac{\Lambda^{5/2}}{t_2^{5/2n+5}} \quad \text{and} \quad P_{rr}' = \frac{P_r(\Lambda)}{t_2^l}$$

thus,

$$\frac{5}{2}n + 5 = l \quad (5 - 22)$$

Since a new unknown power of t_2 , which is l , appears in equation (5 - 22), we need again one equation relating m , n and l . This is obtained as follows: the expansions of the front parameters are designed in such a way that the first terms in the expansions contain only the parameters contained in the first order moment equation and the second terms in the expansions contain only the parameters contained in the second order moment equation and so on. The parameters contained in the first order moment equations are n' , u' and P_{rr}' . Now, in order that only these parameters be contained in the first terms in the expansions, the powers of t_2 to be contained in these first terms should be such that when these terms are substituted into the moment equations produce the same powers of t_2 for each parameter appearing in the resulting expression. Now the last two terms on the left hand side of equation (5 - 14) are

$$n't_2^3 t_2 \frac{\partial u'}{\partial t_2} + t_2^3 \frac{\partial P_{rr}'}{\partial \Delta}$$

Substituting for $n(\lambda) = \frac{F_1(\lambda)}{t_2^m}$, $P_{r_1}(\lambda) = \frac{P_{r_1}(\lambda)}{t_2^l}$ and

$U(\lambda) = \frac{U_1(\lambda)}{t_2^n}$, and comparing only the last two terms on the left hand side,

$$\begin{aligned} & F_1(\lambda) t_2^{-m+4} \frac{\partial}{\partial t_2} (t_2^{-n} U_1(\lambda)) + t_2^3 \frac{\partial}{\partial \lambda} (t_2^l P_{r_1}(\lambda)) \\ &= F_1(\lambda) t_2^{-m+4} \left[(-n) t_2^{-n-1} U_1(\lambda) + t_2^{-n} \frac{dU_1(\lambda)}{d\lambda} \frac{\partial \lambda}{\partial t_2} \right] + t_2^3 \left[t_2^{-l} \frac{dP_{r_1}(\lambda)}{d\lambda} \frac{\partial \lambda}{\partial \lambda} \right] \\ &= F_1(\lambda) t_2^{-m+4} \left[(-n) t_2^{-n-1} U_1(\lambda) + (n) t_2^{-n-1} \lambda \frac{dU_1(\lambda)}{d\lambda} \right] + t_2^3 \left[t_2^{-l} \frac{dP_{r_1}(\lambda)}{d\lambda} t_2^n \right] \\ &= F_1(\lambda) (-n) U_1(\lambda) t_2^{-m+4-n-1} + F_1(\lambda) (n) \lambda \frac{dU_1(\lambda)}{d\lambda} t_2^{-m+4-n-1} + t_2^{3-l+n} \frac{dP_{r_1}(\lambda)}{d\lambda} \end{aligned}$$

collecting the powers of t_2 to become equal for each term, we obtain the relation

$$-m+4-n-1 = 3-l+n$$

thus

$$m+2n = l \quad (5-23)$$

which gives the third relationship between m , n and l .

Rewriting (5-21), (2-22) and (5-23)

$$\frac{3}{2}n + 3 = m$$

$$\frac{5}{2}n + 5 = l$$

$$m + 2n = l$$

Solving these simultaneous equations, we obtain

$$n=2, \quad m=6, \quad l=10$$

substituting these values into $n'(u)$, $P_{rr}'(u)$ and $U'(u)$ the first terms of n' , P_{rr}' and U' are obtained as

$$n'(u) = \frac{F_1(u)}{t_2^6}$$

$$U'(u) = \frac{U_1(u)}{t_2^2}$$

$$P_{rr}'(u) = \frac{P_{r1}(u)}{t_2^{10}}$$

applying a similar approach to $P_{\theta\theta}'(u)$,

$$P_{\theta\theta}'(u) = \frac{P_{\theta 1}(u)}{t_2^{10}}$$

It should be noted that the first terms in the expansions are the terms involved in the first order moment equation (5 - 14). The unknown variables involved in this equation are n' , U' and P_{rr}' and no heat flux terms are involved. Heat flux terms may be included in the second term of the expansion. The second term of the expansion may include second term of n' , U' , P_{rr}' , $P_{\theta\theta}'$ and first terms of Q_{rr}' and $Q_{\theta\theta}'$, the third term of expansion may include third terms of n' , U' , P_{rr}' , $P_{\theta\theta}'$ and second terms of Q_{rr}' and $Q_{\theta\theta}'$ and so on.

Next step is to obtain second terms in the expansions and this is done by obtaining the power of t_2 in the heat transfer term Q_{rr}' :

$$Q_{rr}' = n \bar{\xi}_1^3 = 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} \xi_1^3 f p d\xi_1 dp$$

where f is given by equation (2 - 18)

$$f = F \left[1 - \frac{A^{-1}}{N} \left\{ \frac{\partial \ln T}{\partial r} \left(\frac{5}{2} - \frac{(\xi_1^2 + p^2)}{2T} \right) \xi_1 \right\} \right]$$

on substitution for f

$$\begin{aligned} Q_{rr}' &= 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^3 p d\xi_1 dp - \frac{A^{-1}}{N} \frac{5}{2} \frac{\partial \ln T}{\partial r} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^4 p d\xi_1 dp \\ &+ \frac{A^{-1}}{N} \frac{\partial \ln T}{\partial r} \frac{1}{2T} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^6 p d\xi_1 dp + \frac{A^{-1}}{N} \frac{\partial \ln T}{\partial r} \frac{1}{2T} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^4 p^3 d\xi_1 dp \end{aligned}$$

Solving term by term the first term,

$$2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^3 p d\xi_1 dp = \frac{2\pi N}{(2\pi T)^{3/2}} \int_0^{\infty} \int_{-\infty}^{\infty} \xi_1^3 \exp\left(-\frac{\xi_1^2}{2T}\right) d\xi_1 \exp\left(-\frac{p^2}{2T}\right) p dp = 0$$

since

$$\int_{-\infty}^{\infty} \xi_1^3 \exp\left(-\frac{\xi_1^2}{2T}\right) d\xi_1 = 0$$

where F is the equilibrium distribution function.

The second term,

$$-\frac{A^{-1}}{N} \frac{5}{2} \frac{\partial \ln T}{\partial r} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^4 p d\xi_1 dp$$

$$\begin{aligned}
&= -\frac{A^{-1} 5 \partial \ln T}{N^2 \partial r} 2\pi \frac{N}{(2\pi T)^{3/2}} \int_0^{\infty} \int_{-\infty}^{\infty} \exp^{-\frac{\xi_1^2}{2T}} \xi_1^4 d\xi_1 \exp^{-\frac{p^2}{2T}} p dp \\
&= -\frac{A^{-1} (\frac{5}{2}) \partial \ln T}{N \partial r} 2\pi \frac{N}{(2\pi T)^{3/2}} \frac{3}{4} [\pi(2T)^5]^{1/2} T
\end{aligned}$$

the third term,

$$\begin{aligned}
&\frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^6 p d\xi_1 dp \\
&= \frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \frac{N}{(2\pi T)^{3/2}} \int_0^{\infty} \int_{-\infty}^{\infty} \xi_1^6 e^{-\frac{\xi_1^2}{2T}} d\xi_1 p e^{-\frac{p^2}{2T}} dp \\
&= \frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \frac{N}{(2\pi T)^{3/2}} \frac{15}{8} [\pi(2T)^7]^{1/2} T
\end{aligned}$$

the fourth term

$$\begin{aligned}
&\frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} F \xi_1^4 p^3 d\xi_1 dp \\
&= \frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \frac{N}{(2\pi T)^{3/2}} \int_0^{\infty} \int_{-\infty}^{\infty} \xi_1^4 e^{-\frac{\xi_1^2}{2T}} d\xi_1 p^3 e^{-\frac{p^2}{2T}} dp \\
&= \frac{A^{-1} \partial \ln T}{N \partial r} \frac{1}{2T} 2\pi \frac{N}{(2\pi T)^{3/2}} \frac{3}{4} [\pi(2T)^5]^{1/2} \frac{(2T)^2}{2}
\end{aligned}$$

summing up all these terms

$$Q_{rr}' = A^{-1} 3 T^2 \frac{\partial \ln T}{\partial r}$$

$$= A^{-1} 3 T \frac{\partial T}{\partial r}$$

From $T = A^{-\frac{4}{5}} \tau'$ ($\because \tau = TA, \tau = \tau' A^{1/5} \therefore T = A^{-\frac{4}{5}} \tau'$)
 and $r = r_i A^{1/2}, \lambda = \frac{r_i}{t_i}, \lambda = \sqrt{3} + A^{-\frac{2}{5}} \Delta, \lambda = t_2^2 \Delta$
 $\frac{dT}{dr}$ becomes

$$\frac{dT}{dr} = A^{-\frac{4}{5}} \frac{d\tau'}{d\lambda} \frac{d\lambda}{dA} \frac{dA}{dr} \left(\frac{\partial \lambda}{\partial r_i} \frac{dr_i}{dr} + \frac{\partial \lambda}{\partial t_i} \frac{dt_i}{dr} \right) \frac{dr_i}{dr}$$

$$= A^{-\frac{4}{5}} \frac{d\tau'}{d\lambda} t_2^2 A^{\frac{2}{5}} t_i^{-1} A^{-\frac{1}{2}} \sim A^{-\frac{11}{10}} \frac{d\tau'}{d\lambda} t_2$$

thus

$$Q_{rr}' \sim A^{-\frac{29}{10}} 3 \tau' \frac{d\tau'}{d\lambda} t_2$$

from

$$\tau_i' \tau_i = P_{rr}'$$

$$\tau_i' = \frac{P_{rr}'}{\tau_i} = \frac{\frac{P_{rr}(\lambda)}{t_i^{16}}}{\frac{F(\lambda)}{t_i^6}} = \frac{P_{rr}(\lambda)}{F(\lambda)} t_2^{-4} = S_1 t_2^{-4}$$

where

$$S_1 = \frac{P_{rr}(\lambda)}{F(\lambda)} = S_1(\lambda)$$

$$\therefore Q_{rr}' \sim A^{-\frac{29}{10}} 3 S_1 t_2^{-4} \frac{d\tau'}{d\lambda} t_2$$

using equation (5 - 18) and replacing $\frac{d\tau'}{d\lambda}$ into $\frac{d\tau'}{d\lambda}$ one

obtains

$$\frac{d\gamma_i}{d\Lambda} \sim \frac{1}{t_2^4}$$

Thus

$$Q_{rrr'} \sim A^{-\frac{29}{10}} 3 S_1 t_2^{-3} t_2^{-4}$$

As far as the power of t_2 in $Q_{rrr'}$ is concerned the power of t_2 is t_2^{-7} , therefore, the first term of $Q_{rrr'}$ becomes

$$Q_{rrr'}(z) = \frac{Q_{rrr2}(z)}{t_2^7}$$

here subscript (2) has been used because this term corresponds to the second terms in the expansions of the other parameters. Second terms of expansions include second terms of n' , u' , P_{rr}' , $P_{\theta\theta}'$ and first terms of Q_{rrr}' and $Q_{\theta\theta\theta}'$.

Now arranging the powers of t_2 for each term in such a way that when these terms are substituted into the moment equations, the powers of t_2 in each term of the moment equation become identical. These are obtained in the following manner:

selecting the first and the last terms on the left hand side of equation (5 - 15), that is

$$\frac{\partial}{\partial t_2} (t_2^5 P_{rr}') + \frac{1}{t_2^2} \frac{\partial}{\partial \Lambda} (t_2^6 Q_{rrr}')$$

P_{rr}' and Q_{rrr}' are expanded as

$$Pr' = \frac{Pr_1(\Lambda)}{t_2^{10}} + t_2^x Pr_2(\Lambda) + \dots$$

$$Q_{rrr}' = 0 + \frac{Q_{rrr2}(\Lambda)}{t_2^7} + \dots$$

where x is an unknown power of t_2 .

Now substituting Pr' and Q_{rrr}' and working only up to the second term,

$$\begin{aligned} & \frac{\partial}{\partial t_2} [t_2^{-5} Pr_1(\Lambda) + t_2^{5+x} Pr_2(\Lambda)] + \frac{1}{t_2^2} \frac{\partial}{\partial \Delta} [t_2^{-1} Q_{rrr2}(\Lambda)] \\ &= \frac{\partial}{\partial t_2} [t_2^{-5} Pr_1(\Lambda)] + (5+x) t_2^{5+x-1} Pr_2(\Lambda) + t_2^{5+x} \frac{dPr_2(\Lambda)}{d\Lambda} \frac{d\Lambda}{dt_2} \\ &+ t_2^{-2} [t_2^{-1} \frac{dQ_{rrr2}(\Lambda)}{d\Lambda} \frac{d\Lambda}{d\Delta}] \end{aligned}$$

where $\frac{d\Lambda}{d\Delta} = t_2^2$ since $\Lambda = t_2^2 \Delta$ and $\frac{d\Lambda}{dt_2} = 2\Lambda t_2^{-1}$

thus

$$\begin{aligned} & \frac{\partial}{\partial t_2} [t_2^{-5} Pr_1(\Lambda)] + (5+x) t_2^{5+x-1} Pr_2(\Lambda) + 2\Lambda t_2^{5+x-1} \frac{dPr_2(\Lambda)}{d\Lambda} \\ &+ t_2^{-1} \frac{dQ_{rrr2}(\Lambda)}{d\Lambda} \end{aligned}$$

Collecting the power of t_2 in these last two terms and equating

$$5+x-1 = -1 \quad \therefore x = -5$$

therefore the second term of P_{rr}' becomes $\frac{Pr_2(\mathcal{L})}{t_2^5}$

hence

$$P_{rr}' = \frac{1}{t_2^{10}} [Pr_1(\mathcal{L}) + t_2^5 Pr_2(\mathcal{L}) + \dots]$$

applying similar method to these of n' , u' and $P_{\theta\theta}'$, we obtain consequently final form of expansion

$$n' = \frac{1}{t_2^6} (F_1(\mathcal{L}) + t_2^5 F_2(\mathcal{L}) + \dots) \quad (5 - 24)$$

$$u' = \frac{1}{t_2^2} (U_1(\mathcal{L}) + t_2^5 U_2(\mathcal{L}) + \dots) \quad (5 - 25)$$

$$P_{rr}' = \frac{1}{t_2^{10}} (Pr_1(\mathcal{L}) + t_2^5 Pr_2(\mathcal{L}) + \dots) \quad (5 - 26)$$

$$P_{\theta\theta}' = \frac{1}{t_2^{10}} (P_{\theta\theta 1}(\mathcal{L}) + t_2^5 P_{\theta\theta 2}(\mathcal{L}) + \dots) \quad (5 - 27)$$

$$Q_{rrr}' = \frac{1}{t_2^7} (Q_{rrr 1}(\mathcal{L}) + t_2^5 Q_{rrr 2}(\mathcal{L}) + \dots) \quad (5 - 28)$$

$$Q_{\theta\theta\theta}' = \frac{1}{t_2^7} (Q_{\theta\theta\theta 1}(\mathcal{L}) + t_2^5 Q_{\theta\theta\theta 2}(\mathcal{L}) + \dots) \quad (5 - 29)$$

since $Q_{r\theta\theta}$ must have the same form of expansion as Q_{rrr} due to the similarity between the equations (5 - 15) and (5 - 16). Substituting from equations (5 - 24) through (5 - 29) into the moment equations (5 - 13) through (5 - 16) the continuity equation (5 - 13) becomes,

$$\begin{aligned} & \frac{\partial}{\partial t_2} \left\{ [t_2^{-6} F_1(\lambda) + t_2^{-7} F_2(\lambda) + \dots] t_2^3 \right\} \\ & - [t_2^{-6} F_1(\lambda) + t_2^{-7} F_2(\lambda) + \dots] t_2^2 \left\{ 1 - \frac{\partial}{\partial \Delta} [t_2^{-2} U_1(\lambda) + \right. \\ & \left. t_2^3 U_2(\lambda) + \dots] \right\} + t_2^{-1} \left\{ t_2^{-2} U_1(\lambda) + t_2^3 U_2(\lambda) + \dots - t_2^{-2} \Lambda \right\} \\ & \times \frac{\partial}{\partial \Delta} \left\{ [t_2^{-6} F_1(\lambda) + t_2^{-7} F_2(\lambda) + \dots] t_2^3 \right\} = 0 \end{aligned}$$

From $\Lambda = t_2^2 \Delta$, $\frac{d\Lambda}{d\Delta} = t_2^2$ and $\frac{d\Lambda}{dt_2} = 2\Lambda t_2^{-1}$, hence

$$\begin{aligned} & \frac{\partial}{\partial t_2} [t_2^{-3} F_1(\lambda) + t_2^2 F_2(\lambda) + \dots] \\ & - [t_2^{-4} F_1(\lambda) + t_2^1 F_2(\lambda) + \dots] \left[1 - \frac{dU_1(\lambda)}{d\lambda} - t_2^5 \frac{dU_2(\lambda)}{d\lambda} - \dots \right] \end{aligned}$$

$$+ [t_2^{-3} U_1(\lambda) + t_2^2 U_2(\lambda) + \dots - t_2^{-3} \lambda]$$

$$\times [t_2^{-1} \frac{dF_1(\lambda)}{d\lambda} + t_2^4 \frac{dF_2(\lambda)}{d\lambda} + \dots] = 0$$

rewriting

$$[(t_2^{-3}) t_2^{-4} F_1(\lambda) + t_2^{-3} 2\lambda t_2^{-1} \frac{dF_1(\lambda)}{d\lambda} + t_2^{-1} 2\lambda \frac{dF_2(\lambda)}{d\lambda} + \dots]$$

$$- t_2^{-4} F_1(\lambda) - t_2^{-1} F_2(\lambda) + t_2^{-4} F_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} + t_2^{-1} F_1(\lambda) \frac{dU_2(\lambda)}{d\lambda}$$

$$+ t_2^{-1} F_2(\lambda) \frac{dU_1(\lambda)}{d\lambda} + t_2^{-6} F_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} + \dots]$$

$$+ t_2^{-4} U_1(\lambda) \frac{dF_1(\lambda)}{d\lambda} + t_2^{-1} U_1(\lambda) \frac{dF_2(\lambda)}{d\lambda}$$

$$+ t_2^{-1} U_2(\lambda) \frac{dF_1(\lambda)}{d\lambda} + t_2^{-6} U_2(\lambda) \frac{dF_2(\lambda)}{d\lambda} + \dots$$

$$- t_2^{-4} \lambda \frac{dF_1(\lambda)}{d\lambda} - t_2^{-1} \lambda \frac{dF_2(\lambda)}{d\lambda} + \dots = 0$$

rearranging terms in successive order of the powers of t_2

$$-4F_1(\omega)t_2^{-4} + 2\lambda \frac{dF_1(\omega)}{d\lambda} t_2^{-4} + F_1(\omega) \frac{dU_1(\omega)}{d\lambda} t_2^{-4}$$

$$+ U_1(\omega) \frac{dF_1(\omega)}{d\lambda} t_2^{-4} - \lambda \frac{dF_1(\omega)}{d\lambda} t_2^{-4}$$

$$+ 2\lambda \frac{dF_2(\omega)}{d\lambda} t_2 - F_2(\omega)t_2 + F_1(\omega) \frac{dU_2(\omega)}{d\lambda} t_2$$

$$+ F_2(\omega) \frac{dU_1(\omega)}{d\lambda} t_2 + U_1(\omega) \frac{dF_2(\omega)}{d\lambda} t_2 + U_2(\omega) \frac{dF_1(\omega)}{d\lambda} t_2 - \lambda \frac{dF_2(\omega)}{d\lambda} t_2$$

$$+ F_2(\omega) \frac{dU_2(\omega)}{d\lambda} t_2^6 + U_2(\omega) \frac{dF_2(\omega)}{d\lambda} t_2^6 + \dots = 0$$

multiplying by t_2^4

$$-4F_1(\omega) + \lambda \frac{dF_1(\omega)}{d\lambda} + F_1(\omega) \frac{dU_1(\omega)}{d\lambda} + U_1(\omega) \frac{dF_1(\omega)}{d\lambda}$$

$$+ t_2^5 \left[\lambda \frac{dF_2(\omega)}{d\lambda} - F_2(\omega) + F_1(\omega) \frac{dU_2(\omega)}{d\lambda} + F_2(\omega) \frac{dU_1(\omega)}{d\lambda} \right.$$

$$\left. + U_1(\omega) \frac{dF_2(\omega)}{d\lambda} + U_2(\omega) \frac{dF_1(\omega)}{d\lambda} \right]$$

$$+t_2^{10} \left[F_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} + U_2(\lambda) \frac{dF_2(\lambda)}{d\lambda} + \dots \right] + \dots = 0$$

when t_2 is small ($t_2 \ll 1$), then t_2^5 and t_2^{10} become still smaller and the contribution of these terms becomes negligible, so that

$$[U_1(\lambda) + \lambda] \frac{dF_1(\lambda)}{d\lambda} + \left[\frac{dU_1(\lambda)}{d\lambda} - 4 \right] F_1(\lambda) = 0 \quad (5 - 30)$$

For momentum equation (5 - 14)

$$[t_2^{-6} F_1(\lambda) + t_2^{-1} F_2(\lambda) + \dots] t_2^3 \left\{ [t_2^{-2} U_1(\lambda) + t_2^3 U_2(\lambda) \right.$$

$$+ \dots - t_2^{-2} \lambda] \times \frac{\partial}{\partial \lambda} [t_2^{-2} U_1(\lambda) + t_2^3 U_2(\lambda) + \dots]$$

$$\left. + t_2 \frac{\partial}{\partial t_2} [t_2^{-2} U_1(\lambda) + t_2^3 U_2(\lambda) + \dots] \right\}$$

$$+ t_2^3 \frac{\partial}{\partial \lambda} [-t_2^{-10} P_{r1}(\lambda) + t_2^{-5} P_{r2}(\lambda) + \dots] = 0$$

rewriting

$$[t_2^{-3} F_1(\lambda) + t_2^2 F_2(\lambda) + \dots] \left\{ [t_2^{-2} U_1(\lambda) + t_2^3 U_2(\lambda) + \dots - t_2^{-2} \lambda] \right.$$

$$\left. \times \left[\frac{dU_1(\lambda)}{d\lambda} + t_2^5 \frac{dU_2(\lambda)}{d\lambda} + \dots \right] + t_2 [t_2^{-2} \lambda t_2^{-1} \frac{dU_1(\lambda)}{d\lambda} \right.$$

$$-2t_2^{-3}U_1(\lambda) + 3t_2^2U_2(\lambda) + t_2^3t_2^{-1}2\lambda \frac{dU_2(\lambda)}{d\lambda} + \dots -] \}$$

$$+ t_2^3 \left[t_2^{-8} \frac{dPr_1(\lambda)}{d\lambda} + t_2^{-3} \frac{dPr_2(\lambda)}{d\lambda} + \dots \right] = 0$$

rewriting

$$t_2^{-5}F_1(\lambda)U_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} + F_1(\lambda)U_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} + F_1(\lambda)U_2(\lambda) \frac{dU_1(\lambda)}{d\lambda}$$

$$+ t_2^5F_1(\lambda)U_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} - t_2^{-5}F_1(\lambda)\lambda \frac{dU_1(\lambda)}{d\lambda} - F_1(\lambda)\lambda \frac{dU_2(\lambda)}{d\lambda}$$

$$+ F_2(\lambda)U_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} + t_2^5F_2(\lambda)U_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} + t_2^5F_2(\lambda)U_2(\lambda) \frac{dU_1(\lambda)}{d\lambda}$$

$$+ t_2^{10}F_2(\lambda)U_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} - F_2(\lambda)\lambda \frac{dU_1(\lambda)}{d\lambda} - t_2^5F_2(\lambda)\lambda \frac{dU_2(\lambda)}{d\lambda}$$

$$+ t_2^{-5}2\lambda F_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} - t_2^{-5}2F_1(\lambda)U_1(\lambda) + 3F_1(\lambda)U_2(\lambda)$$

$$+ 2\lambda F_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} + 2\lambda F_2(\lambda) \frac{dU_1(\lambda)}{d\lambda} - 2U_1(\lambda)F_2(\lambda) + t_2^53F_2(\lambda)U_2(\lambda)$$

$$+t_2^5 2\Lambda F_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} + t_2^{-5} \frac{dP_1(\lambda)}{d\lambda} + \frac{dP_2(\lambda)}{d\lambda} + \dots = 0$$

arranging terms in the order of powers of t_2 and multiplying the equation by t_2^5

$$F_1(\lambda)U_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} + \Lambda F_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} - 2F_1(\lambda)U_1(\lambda) + \frac{dP_1(\lambda)}{d\lambda}$$

$$+ t_2^5 \left[F_1(\lambda)U_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} + F_1(\lambda)U_2(\lambda) \frac{dU_1(\lambda)}{d\lambda} - F_1(\lambda)\Lambda \frac{dU_2(\lambda)}{d\lambda} \right.$$

$$\left. + F_2(\lambda)U_1(\lambda) \frac{dU_1(\lambda)}{d\lambda} - F_2(\lambda)\Lambda \frac{dU_1(\lambda)}{d\lambda} + 3F_1(\lambda)U_2(\lambda) \right.$$

$$\left. + 2\Lambda F_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} + 2\Lambda F_2(\lambda) \frac{dU_1(\lambda)}{d\lambda} - 2F_2(\lambda)U_1(\lambda) \right.$$

$$\left. + \frac{dP_2(\lambda)}{d\lambda} \right] + t_2^{10} \left[F_1(\lambda)U_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} + F_2(\lambda)U_1(\lambda) \frac{dU_2(\lambda)}{d\lambda} \right.$$

$$\left. + F_2(\lambda)U_2(\lambda) \frac{dU_1(\lambda)}{d\lambda} + 3F_2(\lambda)U_2(\lambda) + \Lambda F_2(\lambda) \frac{dU_2(\lambda)}{d\lambda} \right]$$

$$+t_2^{15} \left[F_2(\omega) U_2(\omega) \frac{dU_2(\omega)}{d\omega} \right] + \dots = 0$$

Since $t_2 \ll 1$, all the terms involving t_2^5 , t_2^{10} and t_2^{15} become negligible, thus we obtain

$$[U_1(\omega) + \lambda] \frac{dU_1(\omega)}{d\omega} - 2U_1(\omega) + \frac{1}{F_1(\omega)} \frac{dP_1(\omega)}{d\omega} = 0 \quad (5-31)$$

For energy equations (5-15) and (5-16), and working first with equation (5-15),

$$\frac{\partial}{\partial t_2} \left\{ t_2^5 [t_2^{-10} P_{n1}(\omega) + t_2^{-5} P_{r2}(\omega) + \dots] \right\}$$

$$- \left\{ 1 - \frac{\partial}{\partial \Delta} [t_2^{-2} U_1(\omega) + t_2^3 U_2(\omega) + \dots] \right\}$$

$$\times \frac{\partial}{\partial t_2} \left\{ t_2^5 [t_2^{-10} P_{n1}(\omega) + t_2^{-5} P_{r2}(\omega) + \dots] \right\}$$

$$+ \frac{1}{t_2} [t_2^{-2} U_1(\omega) + t_2^3 U_2(\omega) + \dots - t_2^{-2} \omega]$$

$$\times \frac{\partial}{\partial \Delta} \left\{ t_2^5 [t_2^{-10} P_{n1}(\omega) + t_2^{-5} P_{r2}(\omega) + \dots] \right\}$$

$$+\frac{1}{t_2^2} \frac{\partial}{\partial \Lambda} \left\{ t_2^6 [t_2^{-7} Q_{rrr2}(\Lambda) + \dots] \right\} = 2\pi (3^{\frac{3}{4}})^4 \iint n'(F-f) \Phi'^2 \Phi' d\Phi' d\Phi'$$

rewriting

$$t_2^{-6} 2\Lambda \frac{dP_{r1}(\Lambda)}{d\Lambda} + t_2^{-6} (-5) P_{r1}(\Lambda) + t_2^{-1} 2\Lambda \frac{dP_{r2}(\Lambda)}{d\Lambda}$$

$$-3 [t_2^{-6} P_{r1}(\Lambda) + t_2^{-1} P_{r2}(\Lambda) - t_2^{-6} P_{r1}(\Lambda) \frac{dU_1(\Lambda)}{d\Lambda} - t_2^{-1} P_{r2}(\Lambda) \frac{dU_1(\Lambda)}{d\Lambda}]$$

$$-t_2^{-1} P_{r1}(\Lambda) \frac{dU_2(\Lambda)}{d\Lambda} - t_2^{-4} P_{r2}(\Lambda) \frac{dU_2(\Lambda)}{d\Lambda} + \dots]$$

$$+ t_2^{-6} U_1(\Lambda) \frac{dP_{r1}(\Lambda)}{d\Lambda} + t_2^{-1} U_1(\Lambda) \frac{dP_{r2}(\Lambda)}{d\Lambda} + t_2^{-1} U_2(\Lambda) \frac{dP_{r1}(\Lambda)}{d\Lambda}$$

$$+ t_2^{-4} U_2(\Lambda) \frac{dP_{r2}(\Lambda)}{d\Lambda} - t_2^{-6} \Lambda \frac{dP_{r1}(\Lambda)}{d\Lambda} - t_2^{-1} \Lambda \frac{dP_{r2}(\Lambda)}{d\Lambda}$$

$$+ t_2^{-1} \frac{dQ_{rrr2}(\Lambda)}{d\Lambda} + \dots$$

$$= 2\pi (3^{\frac{3}{4}})^4 \iint n'(F-f) \Phi'^2 \Phi' d\Phi' d\Phi'$$

rearranging and multiplying both sides by t_2^6 ,

$$\Lambda \frac{dP_{11}(A)}{d\Lambda} - 8P_{11}(A) + 3P_{11}(A) \frac{dU_1(A)}{d\Lambda} + U_1(A) \frac{dP_{11}(A)}{d\Lambda}$$

$$+ t_2^5 \left[\Lambda \frac{dP_{22}(A)}{d\Lambda} - 3P_{22}(A) - 3P_{22}(A) \frac{dU_1(A)}{d\Lambda} - P_{11}(A) \frac{dU_2(A)}{d\Lambda} \right.$$

$$\left. + U_1(A) \frac{dP_{22}(A)}{d\Lambda} + U_2(A) \frac{dP_{11}(A)}{d\Lambda} + \frac{dQ_{1122}(A)}{d\Lambda} \right]$$

$$+ t_2^{10} \left[3P_{22}(A) \frac{dU_2(A)}{d\Lambda} + U_2(A) \frac{dP_{22}(A)}{d\Lambda} + \dots \right] + \dots$$

$$= 2\pi t_2^6 (3^{\frac{3}{4}})^4 \iint n'(F-f) \Phi^2 \Phi' d\Phi' d\Phi' \quad (5-15')$$

working similarly for equation (5-16),

$$t_2^{-6} 2\Lambda \frac{dP_{01}(A)}{d\Lambda} + t_2^{-6} (-5)P_{01}(A) + t_2^{-7} 2\Lambda \frac{dP_{02}(A)}{d\Lambda}$$

$$- t_2^{-6} P_{01}(A) - t_2^{-7} P_{02}(A) + t_2^{-6} P_{01}(A) \frac{dU_1(A)}{d\Lambda} + t_2^{-7} P_{02}(A) \frac{dU_1(A)}{d\Lambda}$$

$$+ t_2^{-7} P_{01}(A) \frac{dU_2(A)}{d\Lambda} + t_2^{-4} P_{02}(A) \frac{dU_2(A)}{d\Lambda}$$

$$+t_2^{-6} U_1(\lambda) \frac{dP_{01}(\lambda)}{d\lambda} + t_2^{-1} U_1(\lambda) \frac{dP_{02}(\lambda)}{d\lambda} + t_2^{-1} U_2(\lambda) \frac{dP_{01}(\lambda)}{d\lambda}$$

$$+t_2^4 U_2(\lambda) \frac{dP_{02}(\lambda)}{d\lambda} - t_2^{-6} \lambda \frac{dP_{01}(\lambda)}{d\lambda} - t_2^{-1} \lambda \frac{dP_{02}(\lambda)}{d\lambda}$$

$$+t_2^{-1} \frac{dQ_{\text{res2}}(\lambda)}{d\lambda} + \dots = 2\pi (3^{\frac{3}{4}})^4 \int n'(F-f) \Psi^2 \Psi' d\Psi' d\Psi'$$

rearranging and multiplying it by t_2^6 ,

$$\lambda \frac{dP_{01}(\lambda)}{d\lambda} - 6P_{01}(\lambda) + P_{01}(\lambda) \frac{dU_1(\lambda)}{d\lambda} + U_1(\lambda) \frac{dP_{01}(\lambda)}{d\lambda}$$

$$+t_2^5 \left[\lambda \frac{dP_{02}(\lambda)}{d\lambda} - P_{02}(\lambda) + P_{02}(\lambda) \frac{dU_1(\lambda)}{d\lambda} + P_{01}(\lambda) \frac{dU_2(\lambda)}{d\lambda} \right.$$

$$\left. + U_1(\lambda) \frac{dP_{02}(\lambda)}{d\lambda} + U_2(\lambda) \frac{dP_{01}(\lambda)}{d\lambda} + \frac{dQ_{\text{res2}}(\lambda)}{d\lambda} \right]$$

$$+t_2^{10} \left[P_{02}(\lambda) \frac{dU_2(\lambda)}{d\lambda} + U_2(\lambda) \frac{dP_{02}(\lambda)}{d\lambda} \right] + \dots$$

$$= 2\pi t_2^6 (3^{\frac{3}{4}})^4 \int n'(F-f) \Psi^2 \Psi' d\Psi' d\Psi'$$

eliminating terms involving t_2^5 and t_2^{10} from equations (5 - 15) and (5 - 16') and adding the resulting equations together, the left hand side of the expression becomes,

$$[u_1(\lambda) + 1] \frac{d}{d\lambda} [P_{r1}(\lambda) + P_{e1}(\lambda)] + [3 \frac{dU_1(\lambda)}{d\lambda} - 8] P_{r1}(\lambda) + [\frac{dU_1(\lambda)}{d\lambda} - 6] P_{e1}(\lambda)$$

and right hand side becomes

$$\begin{aligned} & 2\pi t_2^6 (3^{\frac{3}{4}})^4 \left[\iint n'(F-f) \Phi'^2 \Phi' d\Phi' d\Phi' + \iint n'(F-f) \Phi'^2 \Phi d\Phi' d\Phi' \right] \\ & = 2\pi n' t_2^6 (3^{\frac{3}{4}})^4 \left[\iint F[\Phi'^2 + \Phi'^2] \Phi' d\Phi' d\Phi' - \iint f[\Phi'^2 + \Phi'^2] \Phi d\Phi' d\Phi' \right] \end{aligned}$$

where

$$\iint F[\Phi'^2 + \Phi'^2] \Phi' d\Phi' d\Phi' = P_{rr1} + P_{ee1}$$

$$= \frac{n'}{(2\pi \tau')^{3/2}} \left[\frac{\pi}{a^3} \right]^{1/2} \frac{1}{2a} \left[\frac{1}{2} + 1 \right], \quad a = \frac{1}{2(3^{-\frac{3}{4}})^2 \tau'}$$

$$= n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5$$

From equation (5 - 7)

$$\iint f[\Phi'^2 + \Phi'^2] \Phi d\Phi' d\Phi' = n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5$$

thus, the right hand side becomes

$$2\pi n' t_2^6 (3^{\frac{3}{4}})^4 \left[n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5 - n' \tau' \frac{3}{2\pi} (3^{-\frac{3}{4}} t_2)^5 \right] = 0$$

consequently

$$[U_1(\lambda) + \lambda] \frac{d}{d\lambda} [P_{r1}(\lambda) + P_{\theta 1}(\lambda)] \\ + \left[3 \frac{dU_1(\lambda)}{d\lambda} - 8 \right] P_{r1}(\lambda) + \left[\frac{dU_1(\lambda)}{d\lambda} - 6 \right] P_{\theta 1}(\lambda) = 0 \quad (5 - 32)$$

Since the general form of equation (5 - 10) is given as

$$n' r' = \frac{1}{3} (P_{rr'} + P_{\theta \theta'}) \quad (5 - 10)$$

$$\therefore r' = \frac{1}{3} \frac{(P_{rr'} + P_{\theta \theta'})}{n'} \quad (5 - 10'')$$

Referring to (5 - 10'') the form of expansion for r' takes the following form

$$r' = \frac{1}{t_2^4} [S_1(\lambda) + t_2^5 S_2(\lambda) + \dots] \quad (5 - 33)$$

From (5 - 24) and (5 - 33) $n' r'$ become

$$n' r' = [t_2^{-6} F_1(\lambda) + t_2^{-1} F_2(\lambda) + \dots] [t_2^{-4} S_1(\lambda) + t_2^5 S_2(\lambda) + \dots]$$

$$= t_2^{-10} F_1(\lambda) S_1(\lambda) + t_2^{-5} F_1(\lambda) S_2(\lambda) + t_2^{-5} F_2(\lambda) S_1(\lambda)$$

$$+ F_2(\lambda) S_2(\lambda) + \dots$$

From (5 - 26) and (5 - 27)

$$\frac{1}{3}(P_{r1}' + P_{e1}') = \frac{1}{3} [t_2^{-10} P_{r1}(A) + t_2^{-5} P_{r2}(A) + t_2^{-10} P_{e1}(A) + t_2^{-5} P_{e2}(A) + \text{-----}]$$

Substituting from these two expressions into (5 - 10) and multiplying both sides by t_2^{10} ,

$$F_1(A)S_1(A) + t_2^5 [F_1(A)S_2(A) + F_2(A)S_1(A)] + t_2^{10} [F_2(A)S_2(A)] + \text{-----} = \frac{1}{3} \{ [P_{r1}(A) + P_{e1}(A)] + t_2^5 [P_{r2}(A) + P_{e2}(A)] + \text{-----} \}$$

neglecting terms with higher powers of t_2 , for reasons explained earlier,

$$F_1(A)S_1(A) = \frac{1}{3} [P_{r1}(A) + P_{e1}(A)] \quad (5 - 33')$$

It is rather important to investigate whether the first terms of the expansions of front parameters refer to equilibrium or non-equilibrium solution. This can be done by examining the associated distribution functions with regard to those first terms. Substituting the first term of n' into equation (4 - 3) and multiplying by t_2^6

$$t_2 \left\{ \left(\frac{\partial f}{\partial \Phi'} \right) \left[(u' - \Delta) \frac{\partial u'}{\partial \Delta} + t_2 \frac{\partial u'}{\partial t_2} \right] + 3^{\frac{3}{4}} \left[\frac{\partial f}{\partial t_2} + \left(1 - \frac{\partial u'}{\partial \Delta} \right) \frac{\Phi'}{t_2} \left(\frac{\partial f}{\partial \Phi'} \right) \right. \right. \\ \left. \left. + \frac{u' - \Delta}{t_2} \left(\frac{\partial f}{\partial \Delta} \right) \right] + \left(3^{\frac{3}{4}} \right)^2 \frac{\Phi'}{t_2^2} \left(\frac{\partial f}{\partial \Delta} \right) \right\} = F_1(\lambda) (F - f)$$

Since t_2 is very small, the terms on the left hand side of this equation can be neglected, and hence

$$0 = F - f$$

$$\therefore f = F$$

This shows that the distribution function associated with the first terms of the expansions is the equilibrium distribution. Hence, in a way similar to equation (5 - 10') which is applicable to the equilibrium solution, $P_{01}(\lambda)$ can be written as

$$P_{01}(\lambda) = 2P_n(\lambda)$$

Consequently equation (5 - 33') becomes

$$F_1(\lambda) S_1(\lambda) = P_{r1}(\lambda) \quad (5 - 34)$$

Substituting this expression into equation (5 - 32)

and dividing by $P_n(\lambda) [U_1(\lambda) + \lambda]$,

$$\frac{1}{P_n(\lambda)} \frac{dP_{r1}(\lambda)}{d\lambda} + \frac{5}{3} \frac{1}{[U_1(\lambda) + \lambda]} \left[\frac{dU_1(\lambda)}{d\lambda} - 4 \right] = 0 \quad (5 - 35)$$

Rewriting equations (5 - 30), (5 - 31) and (5 - 35)

$$(U_i + 1) \frac{dF_i}{d\lambda} + \left(\frac{dU_i}{d\lambda} - 4 \right) F_i = 0$$

$$(U_i + 1) \frac{dU_i}{d\lambda} - 2U_i + \frac{1}{F_i} \frac{dP_{ri}}{d\lambda} = 0$$

$$\frac{1}{P_{ri}} \frac{dP_{ri}}{d\lambda} + \frac{5}{3} \frac{1}{(U_i + 1)} \left[\frac{dU_i}{d\lambda} - 4 \right] = 0$$

These equations (5 - 30), (5 - 31) and (5 - 35) form a closed set of transformed moment equations having three transformed unknown parameters $U_i(\lambda)$, $F_i(\lambda)$ and $P_{ri}(\lambda)$.

The solution of these equations is attempted below: rewriting equation (5 - 35)

$$\left[\frac{dU_i}{d\lambda} - 4 \right] = -\frac{3}{5} [U_i + 1] \frac{1}{P_{ri}} \frac{dP_{ri}}{d\lambda}$$

substituting this expression into (5 - 30),

$$[U_i + 1] \frac{dF_i}{d\lambda} - \frac{3}{5} (U_i + 1) \frac{1}{P_{ri}} \frac{dP_{ri}}{d\lambda} F_i = 0 \quad (5 - 36)$$

hence

$$\frac{1}{P_{ri}} \frac{dP_{ri}}{d\lambda} = \frac{5}{3} \frac{1}{F_i} \frac{dF_i}{d\lambda}$$

which should have the solution of the form

$$P_{ri} = C' F_i^{\frac{5}{3}} \quad (5 - 37)$$

where C' is an arbitrary constant.

Differentiating P_{F_1} with respect to \mathcal{L} and substituting into (5 - 31)

$$(U_1 + 1) \frac{dU_1}{d\mathcal{L}} - 2U_1 + \frac{5}{2} \frac{dF_1^{2/3}}{d\mathcal{L}} = 0 \quad (5 - 38)$$

where the last term in (5 - 31) has been converted as

$$\begin{aligned} \frac{1}{F_1} \frac{dP_1}{d\mathcal{L}} &= \frac{1}{F_1} C' \frac{5}{3} F_1^{2/3} \frac{dF_1}{d\mathcal{L}} \\ &= \frac{5}{3} \frac{1}{(F_1)^{1/3}} \frac{dF_1}{d\mathcal{L}} \quad \text{when } C'=1 \\ &= \frac{5}{2} \frac{dF_1^{2/3}}{d\mathcal{L}} \end{aligned}$$

and rearranging (5 - 30)

$$\frac{1}{F_1} \frac{dF_1}{d\mathcal{L}} = -\frac{1}{(U_1 + 1)} \left(\frac{dU_1}{d\mathcal{L}} - 4 \right) \quad (5 - 39)$$

Now, let us assume a general form of solution for $U_1(\mathcal{L})$ as

$$U_1(\mathcal{L}) = \mathcal{L} + C \quad (5 - 40)$$

Substituting (5 - 40) into equation (5 - 38),

$$(2\mathcal{L} + C) - 2(\mathcal{L} + C) + \frac{5}{2} \frac{dF_1^{2/3}}{d\mathcal{L}} = 0$$

hence

$$F_1^{2/3} = \frac{2}{5} C\mathcal{L} + d$$

where C and d are arbitrary constants and consequently the solution for F_1 becomes

$$F_1 = \left(\frac{2}{5} C\mathcal{L} + d \right)^{3/2} \quad (5 - 41)$$

These general outer solutions (5 - 40) and (5 - 41) should be matched with the inner equilibrium solutions of Mirels and Mullen, i.e. the constants C and d should be determined such that equations (5 - 40) and (5 - 41) would match with those of the equilibrium solutions.

It should be noted however that the constant in both equations (5 - 40) and (5 - 41) may not be the same when matching is established. This happens because several steps of approximation to those inner solutions are needed to establish matching. Thus, one must rewrite the equations (5 - 40) and (5 - 41) more appropriately

$$U_1 = \Lambda + C_1 \quad (5 - 42)$$

$$F_1 = \left(\frac{2}{5} C_2 + d \right)^{3/2} \quad (5 - 42')$$

where C_1 and C_2 are constants.

The matching is established in the following manner: starting with the velocity given by equation (2 - 19) rewritten below,

$$U = \left(\frac{dR}{dt} \right) \eta$$

where

$$\frac{dR}{dt} = \sqrt{3} (1 - R^{-2})^{1/2} \text{ and } \eta = \frac{r}{R(t)} \quad (2 - 19')$$

$$\therefore U = \sqrt{3} (1 - R^{-2})^{1/2} \frac{r}{R} \quad (5 - 43)$$

In equation (2 - 19'), the expansion of the term $(1 - R^{-2})^{\frac{1}{2}}$ is to be restricted to second approximation. This is justified in the light of the fact that for large values of time ^(*), the radius R becomes large, and therefore the terms containg higher negative powers of R contribute very little to the expansion. Therefore

$$(1 - R^{-2})^{\frac{1}{2}} \approx (1 - \frac{1}{2}R^{-2})$$

thus

$$\frac{dR}{dt} \approx \sqrt{3} (1 - \frac{1}{2}R^{-2}) \quad (2 - 19'')$$

Since we are dealing with the case of large time t and corresponding large value of $R(t)$, the above approximation is close to real value.

Taking first term of (2 - 19'')

$$\frac{dR}{dt} \sim \sqrt{3}, \quad dR \sim \sqrt{3} dt \quad \therefore R \sim \sqrt{3} t$$

substituting this back into $\frac{dR}{dt}$, we obtain

$$\frac{dR}{dt} \sim \sqrt{3} [1 - \frac{1}{2}(\sqrt{3}t)^{-2}] = \sqrt{3} - \frac{t^{-2}}{\sqrt{3}}$$

$$dR \sim \sqrt{3} dt - \frac{t^{-2}}{\sqrt{3}} dt$$

$$\therefore R \sim \sqrt{3} t + \frac{t^{-1}}{\sqrt{3}} \quad (5 - 44)$$

(*): It should be noted that large time here implies in context of inner solutions. Further explanation in this regard is provided in the succeeding pages.

Substituting equation (5 - 44) into equation (5 - 43)

$$\begin{aligned}
 U &\sim \sqrt{3} \frac{r}{[\sqrt{3}t + (\sqrt{3}t)^{-1}]} \left[1 - (\sqrt{3}t + \frac{1}{\sqrt{3}t})^{-2} \right]^{\frac{1}{2}} \\
 &\sim \sqrt{3} r [\sqrt{3}t + (\sqrt{3}t)^{-1}]^{-1} \left[1 - (\sqrt{3}t + \frac{1}{\sqrt{3}t})^{-2} \right]^{\frac{1}{2}} \\
 &\sim \frac{r}{t} \left[1 + \frac{1}{(\sqrt{3}t)^2} \right]^{-1} \left[1 - (\sqrt{3}t)^{-2} \left(1 + \frac{1}{(\sqrt{3}t)^2} \right)^{-2} \right]^{\frac{1}{2}}
 \end{aligned}$$

applying the same method of approximation again for U

$$\begin{aligned}
 U &\sim \lambda \left[1 - \frac{1}{(\sqrt{3}t)^2} \right] \left[1 - \frac{1}{2} (\sqrt{3}t)^{-2} \left(1 - \frac{2}{(\sqrt{3}t)^2} \right) \right] \\
 &\sim \lambda \left[1 - \frac{3}{2} (\sqrt{3}t)^{-2} + \frac{3}{2} \frac{1}{(\sqrt{3}t)^4} - \frac{1}{(\sqrt{3}t)^6} \right]
 \end{aligned}$$

For large time, taking up to 2nd term

$$U \sim \lambda \left[1 - \frac{3}{2} (\sqrt{3}t)^{-2} \right] \quad (5 - 43')$$

Recall

$$\begin{aligned}
 U &= \sqrt{3} + A^{-\frac{2}{5}} U' \\
 \lambda &= \sqrt{3} + A^{-\frac{2}{5}} \Delta
 \end{aligned}$$

Substituting from above in equation (5 - 43')

$$\sqrt{3} + A^{-\frac{2}{5}}U' \sim (\sqrt{3} + A^{-\frac{2}{5}}\Delta) \left[1 - \frac{3}{2}(\sqrt{3}t)^{-2} \right]$$

$$\sim \sqrt{3} - \frac{\sqrt{3}}{2} \frac{1}{t^2} + A^{-\frac{2}{5}}\Delta - A^{-\frac{2}{5}} \frac{\Delta}{2} \frac{1}{t^2}$$

$$\therefore A^{\frac{2}{5}}U' \sim A^{-\frac{2}{5}}\Delta - \frac{\sqrt{3}}{2} \frac{1}{t^2} - A^{-\frac{2}{5}} \frac{\Lambda}{2.t_2^2 t^2} \quad \text{where } \Lambda = t_2^2 \Delta$$

As t is large, neglecting third term

$$A^{-\frac{2}{5}}U' \sim A^{-\frac{2}{5}}\Delta - \frac{\sqrt{3}}{2} \frac{1}{t^2}$$

from $t = (3^{-\frac{3}{4}})A^{\frac{3}{10}}t_2$

$$A^{-\frac{2}{5}}U' \sim A^{-\frac{2}{5}}\Delta - \frac{\sqrt{3}}{2} (3^{\frac{3}{4}})^2 \frac{A^{-\frac{2}{5}}}{t_2^2}$$

thus, using $\Lambda = t_2^2 \Delta$

$$U' \sim \frac{\Lambda}{t_2^2} - \frac{\sqrt{3}}{2} (3^{\frac{3}{4}})^2 \frac{1}{t_2^2} \quad (5 - 45)$$

taking U' up to first approximation from (5 - 25), that is $\frac{U(\omega)}{t_2^2}$, and matching this with equation (5 - 45)

$$\frac{U(\omega)}{t_2^2} = \frac{\Lambda}{t_2^2} - \frac{\sqrt{3}}{2} (3^{\frac{3}{4}})^2 \frac{1}{t_2^2}$$

Hence, the constant C_1 in equation (5 - 40) is found to be

$$C_1 = -\left(\frac{3}{r_2}\right)^2$$

thus finally we obtain U_1 as

$$U_1(\lambda) = \lambda - \frac{9}{2} \quad (5 - 46)$$

Equation (5 - 46) is a solution of equations (5 - 38) and (5 - 39) which is essentially the first order equilibrium solution of moment equations for small time t_2 .

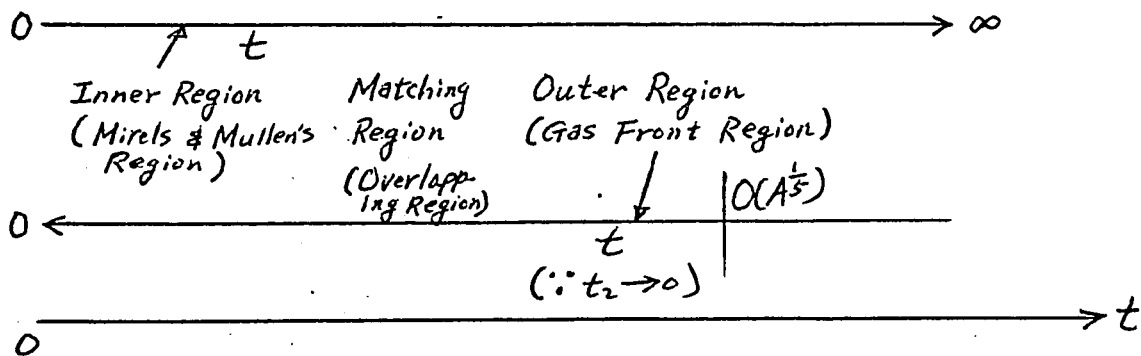


Fig.6. Schematic Representation of Matching between Inner and Outer Regions.

From the relation $t \sim A^{1/2} t_2$,

$$t = O(A^{1/2}) \text{ when } t_2 = 1$$

The origin of the front scaling has been set at $t = O(A^{1/2})$.

Obviously, as t_2 tends to zero the value of t will decrease due to the relation $t \sim A^{1/2} t_2$.

This decrease in time t is expressed in Fig. 6, by the indication of an arrow directed toward the inner region from the outer region whereas, time t is also increasing toward the outer region from the inner region and hence, there must be the intermediate overlapping region where the asymptotic matching is established by adjusting the constants c and d . However the overlapping time is not a definite quantity since the inner solution has been obtained for large time t without specifying any particular value of time. Therefore, the matching point on a real time scale can not be exactly specified.

Now, the asymptotic matching for the other variable λ must be considered. At the gas front, λ approaches $\sqrt{3}$, and the rest of the region excluding the gas front must assume the value of λ less than $\sqrt{3}$, therefore, when the asymptotic matching is established with the inner region, the value of λ must decrease inwardly and hence from the relation $\lambda = \sqrt{3} + A^{-\frac{2}{3}} \Delta$, Δ must assume negative value decreasing from 0^(*).

(*): while analyzing the order of magnitude of \mathcal{N} on page(105), Δ was arbitrarily assumed positive, it can now be seen that it will always be negative.

Λ is given as $\Lambda = t_2^2 \Delta$, thus Λ must take negative values decreasing from 0. The covering range of gas front region by extends from 0 to $-O(1)$ since

t_2 varies from 1 to $O(A^{-\frac{1}{5}})$ and Δ from 0 to $-O(A^{\frac{2}{5}})$, i.e. from $\Lambda = t_2^2 \Delta$

$$\Lambda \sim O(A^{-\frac{1}{5}})^2 O(A^{\frac{2}{5}}) = O(1)$$

In order to solve for $F_1(\Lambda)$ in equation (5 - 42') one must start with the density distribution given by Mirels and Mullen in equation (2 - 18)

$$N = R^{-(\alpha+1)} [1 - \eta^2]^{\frac{1}{\alpha-1}}$$

where $\alpha = 2$ and $\delta = \frac{5}{3}$

Substituting these values into the above expression

$$\begin{aligned} N &= R^{-3} [1 - (\frac{r}{R})^2]^{\frac{3}{2}} \\ &= \left\{ R^{-2} [1 - (\frac{r}{R})^2] \right\}^{\frac{3}{2}} \\ &= \left(R^{-2} - \frac{r^2}{R^4} \right)^{\frac{3}{2}} \end{aligned}$$

substituting into the above expression for R given by equation (5 - 44)

$$N \sim \left[\left(\sqrt{3}t + \frac{1}{\sqrt{3}t} \right)^{-2} - r^2 \left(\sqrt{3}t + \frac{1}{\sqrt{3}t} \right)^{-4} \right]^{\frac{3}{2}}$$

$$\sim \left[(\sqrt{3}t)^{-2} \left(1 + \frac{1}{(\sqrt{3}t)^2} \right)^{-2} - r^2 (\sqrt{3}t)^{-4} \left(1 + \frac{1}{(\sqrt{3}t)^2} \right)^{-4} \right]^{3/2}$$

$$\sim \left[(\sqrt{3}t)^{-2} \left(1 - \frac{2}{(\sqrt{3}t)^2} - r^2 (\sqrt{3}t)^{-4} \left(1 - \frac{4}{(\sqrt{3}t)^2} \right) \right) \right]^{3/2}$$

From equation (4 - 1), $\lambda = \sqrt{3} + A^{-\frac{2}{5}} \Delta$

$$\lambda^2 = 3 \left(1 + \frac{\Delta}{\sqrt{3}} A^{-\frac{2}{5}} \right)^2 \quad \text{where } \lambda = \frac{r}{t}$$

approximating

$$\lambda^2 \sim 3 \left(1 + 2 \frac{\Delta}{\sqrt{3}} A^{-\frac{2}{5}} \right)$$

thus

$$N \sim \left[(\sqrt{3}t)^{-2} \left(1 - \frac{2}{(\sqrt{3}t)^2} \right) - (\sqrt{3})^{-4} t^{-2} \lambda^2 \left(1 - \frac{4}{(\sqrt{3}t)^2} \right) \right]^{3/2}$$

$$\sim \left[\frac{1}{(\sqrt{3}t)^2} \left(1 - \frac{2}{(\sqrt{3}t)^2} \right) - \frac{\left(1 + 2 \frac{\Delta}{\sqrt{3}} A^{-\frac{2}{5}} \right)}{(\sqrt{3}t)^2} \left(1 - \frac{4}{(\sqrt{3}t)^2} \right) \right]^{3/2}$$

$$\sim \left[\frac{1}{(\sqrt{3}t)^2} - \frac{1}{(\sqrt{3}t)^2} - \frac{2}{(\sqrt{3}t)^4} - \frac{2}{(\sqrt{3}t)^2} \frac{\Delta}{\sqrt{3}} A^{-\frac{2}{5}} \right.$$

$$\left. + \frac{4}{(\sqrt{3}t)^2 (\sqrt{3}t)^2} + \frac{4}{(\sqrt{3}t)^2 (\sqrt{3}t)^2} \frac{2}{\sqrt{3}} \Delta A^{-\frac{2}{5}} \right]^{3/2}$$

neglecting the last term which will contain t_2^{-6} terms when Δ and t are converted into t_2 . Maintaining up to t^{-4} term which will eventually produce up to 2nd approximation of N .

$$N \sim \left[\frac{3}{(\sqrt{3}t)^4} - \frac{1}{(\sqrt{3}t)^2} \frac{2}{\sqrt{3}} \Delta A^{-\frac{2}{5}} \right]^{3/2}$$

$$\sim \frac{2^{3/2}}{(\sqrt{3}t)^4} \left[1 - \sqrt{3} t^2 \Delta A^{-\frac{2}{5}} \right]^{3/2}$$

from equation (4 - 1)

$$t = (3^{-\frac{3}{4}}) t_2 A^{1/5}, \quad \Delta = t_2^2 \Delta$$

$$N \sim \left(\frac{\sqrt{2}}{3} \right)^2 \frac{A^{-\frac{6}{5}}}{(3^{-\frac{3}{4}})^6 t_2^6} \left[1 - \sqrt{3} (3^{-\frac{3}{4}})^2 \Delta \right]$$

N is now converted into n' using equations (4 - 1),

$$N = n A^{-3/2}, \quad n = n' A^{3/10} \quad \therefore N = n' A^{-\frac{6}{5}}$$

therefore

$$n' A^{-\frac{6}{5}} \sim \left(\frac{\sqrt{2}}{3} \right)^2 \frac{A^{-\frac{6}{5}}}{(3^{-\frac{3}{4}})^6 t_2^6} \left[1 - \sqrt{3} (3^{-\frac{3}{4}})^2 \Delta \right]^{3/2}$$

$$\therefore n' \sim \left(\frac{\sqrt{2}}{3} \right)^2 \frac{1}{(3^{-\frac{3}{4}})^6 t_2^6} \left[1 - \sqrt{3} (3^{-\frac{3}{4}})^2 \Delta \right]^{3/2}$$

Matching this with the first order term of n' from (5 - 24)

$$\frac{F(\lambda)}{t_2^6} = \left(\frac{\sqrt{2}}{3}\right)^3 \frac{1}{(3^{-\frac{3}{4}})^6 t_2^6} \left[1 - \sqrt{3} (3^{-\frac{3}{4}})^2 \lambda\right]^{\frac{3}{2}}$$

consequently

$$F_1(\lambda) = (\sqrt{2})^3 3^{\frac{3}{2}} (1 - 3^{-1} \lambda)^{\frac{3}{2}} \quad (5 - 47)$$

Now, equation (5 - 37) gives the relationship between

P_{r1} and $F(\lambda)$, that is

$$P_{r1} = F_1^{\frac{5}{3}} \quad \text{for } C' = 1$$

substituting from (5 - 47)

$$P_{r1} = \left\{ \left(\frac{\sqrt{2}}{3}\right)^3 (3^{\frac{3}{4}})^6 \left[1 - \sqrt{3} (3^{-\frac{3}{4}})^2 \lambda\right]^{\frac{3}{2}} \right\}^{\frac{5}{3}}$$

$$\therefore P_{r1}(\lambda) = (\sqrt{2})^5 3^{\frac{25}{4}} \left[1 - 3^{-1} \lambda\right]^{\frac{5}{2}} \quad (5 - 48)$$

From equation (5 - 34), $S_1(\lambda)$ is given as

$$S_1(\lambda) = \frac{P_{r1}(\lambda)}{F_1(\lambda)}$$

therefore from (4 - 47) and (5 - 48)

$$S_1(\lambda) = 6 \left(1 - \frac{\lambda}{3}\right) \quad (5 - 49)$$

Equations (5 - 46), (5 - 47), (5 - 48) and (5 - 49) are the first order equilibrium solutions of moment equations (5 - 13), (5 - 14), (5 - 15) and (5 - 16) for small time t_2 . The covering range of t_2 associated with \mathcal{A} extends from 1 to $O(\mathcal{A}^{-\frac{1}{2}})$, but the small time t_2 naturally restricts the range of t_2 and consequently that of t , though not to the same extent.

Throughout this work, parameter \mathcal{A} has been used in the asymptotic expansion as a variable approaching infinity. The specific information on the applicable range of magnitude of \mathcal{A} in the asymptotic expansion has been given in appendix B.

The next step has been to investigate the validity of the solutions with respect to the front scale Δ .

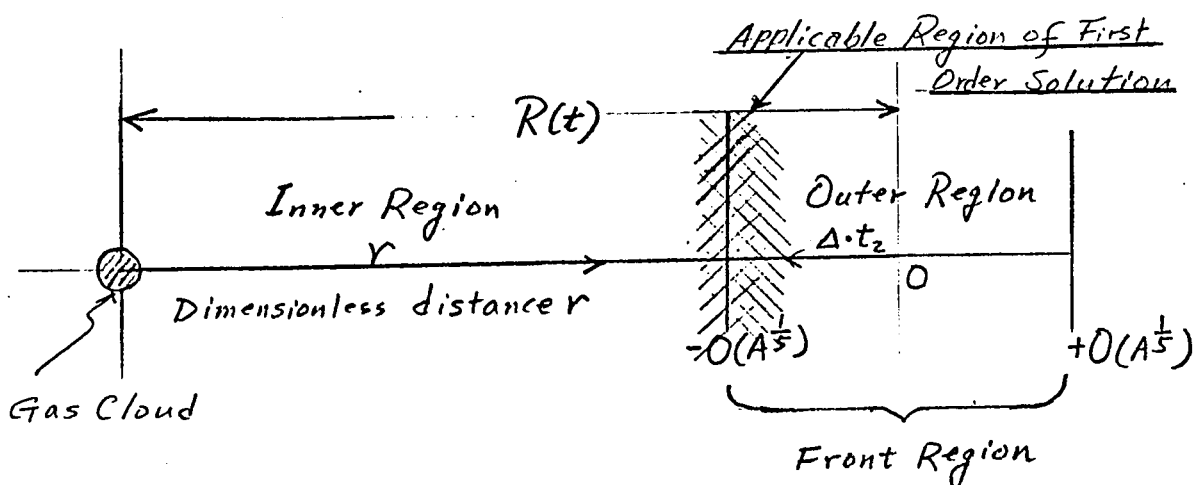


Fig. 10. Schematic Diagram of Front Region with respect to Dimensionless Distance γ .

The origin of the front coordinate system has been set at as indicated in Fig. 10. and also the directions of variation of the dimensionless distance $\Delta \cdot t_2$ are indicated in the same figure. From relation $t = (\beta^{-\frac{2}{7}}) t_2 A^{1/5}$ it can be seen that the front scale t_2 stretches from 1 to $O(A^{-\frac{1}{5}})$ whereas Δ stretches from 0 to $(-) O(A^{\frac{2}{5}})$. The dimensionless distance $\Delta \cdot t_2$ stretches therefore from 0 to $(-) O(A^{1/5})$. As previously discussed on page 141 the order of magnitude of Λ is given as $O(1)$, and the variable range of time t_2 and Δ are given from 1 to $O(A^{-\frac{1}{5}})$ and 0 to $-O(A^{\frac{2}{5}})$ respectively. Since the value of t_2 has been chosen to be very small, in fact as small as $t_2 \sim O(A^{-\frac{1}{5}})$, $\Delta \cdot t_2$ (which is the dimensionless distance from the origin of front coordinate system 0 as indicated on Fig. 10) becomes $(-) O(A^{1/5})$,

i.e.

$$\Delta \cdot t_2 \sim (-) O(A^{-\frac{1}{5}}) \cdot O(A^{\frac{2}{5}}) = (-) O(A^{1/5})$$

where order of Δ has been chosen as $(-) O(A^{\frac{2}{5}})$ since the matching has been established with the inner equilibrium region as indicated on Fig. 10 which corresponds to the value of $\Delta \cdot t_2 \sim (-) O(A^{1/5})$. In order that $\Delta \cdot t_2 \sim (-) O(A^{1/5})$, Δ has to be $(-) O(A^{2/5})$. Therefore, the front first order solution of the expansion of front parameters apply from the entering region of the front region toward the center and yet it is not known however how far these solutions validate

toward the central region since no higher order solutions in the expansion have been obtained.

Figures 8 and 9 show plots of the values of $F(\lambda)$, $S(\lambda)$, $U(\lambda)$ and $P(\lambda)$ versus the values of λ which ranges from 0 to $O(1)$, whereas Δ ranges from 0 to $O(\lambda^{\frac{2}{3}})$. As we see from Fig. 8 and Fig. 9, the curves tend to become straight lines as λ increases in the negative direction. Fig. 7 shows the functional relationship between gas front radius R and time t .

PART VI

Discussion of Results

It was shown in PART II that Grundy and Thomas' solution breaks down at the gas front, that is where $K \rightarrow 0$ and $\lambda \rightarrow \sqrt{3}$. In order to overcome this discontinuity of their solution the B-G-K equation has been rescaled at the point of breakdown. As was previously encountered by Freeman and Grundy⁽⁵⁾ in their paper on the unsteady cylindrical expansion, the same problem of non-closure of the moment equations again was encountered in this work. Introducing however a suitable expansion of physical parameters with respect to front time t_2 , a closed set of transformed moment equations for small time t_2 were formed; consequently solutions to these moment equations were obtained. As already discussed in PART V, these solutions are restricted to small values of time t_2 . The general form of the expansions is such that the range of validity with respect to t_2 can be extended some what since perturbations are of order t_2^5 . However these higher order perturbations have not been obtained due to the fact that they depend on the evaluation of higher moments which is an extremely tedious undertaking and, although the calculations are tedious they should not present any great difficulty in principle.

The validity of the solutions with regard to the front scale Δ has also been discussed in PART V. The significant implications with regard to the first order front solutions are such that it indicates the smooth transition of gas properties entering from the inner region into the front region as Fig. 8 and 9 indicate.

In conclusion therefore, this work has shown that the breakdown of the Grundy and Thomas' solution can be resolved by means of suitably scaled variables in an asymptotic front region, and the characteristic dimensions of this front region have been determined. The front equations are unfortunately unsolvable in general by present techniques, but an asymptotic solution for small time has been given. Therefore, this work has explored the completely unknown front region covering after the end of the inner region and thereby extended its knowledge further into the center of the gas front by breaking through the barriers imposed by the previous inner solutions.

Expansion of gas front with respect to time

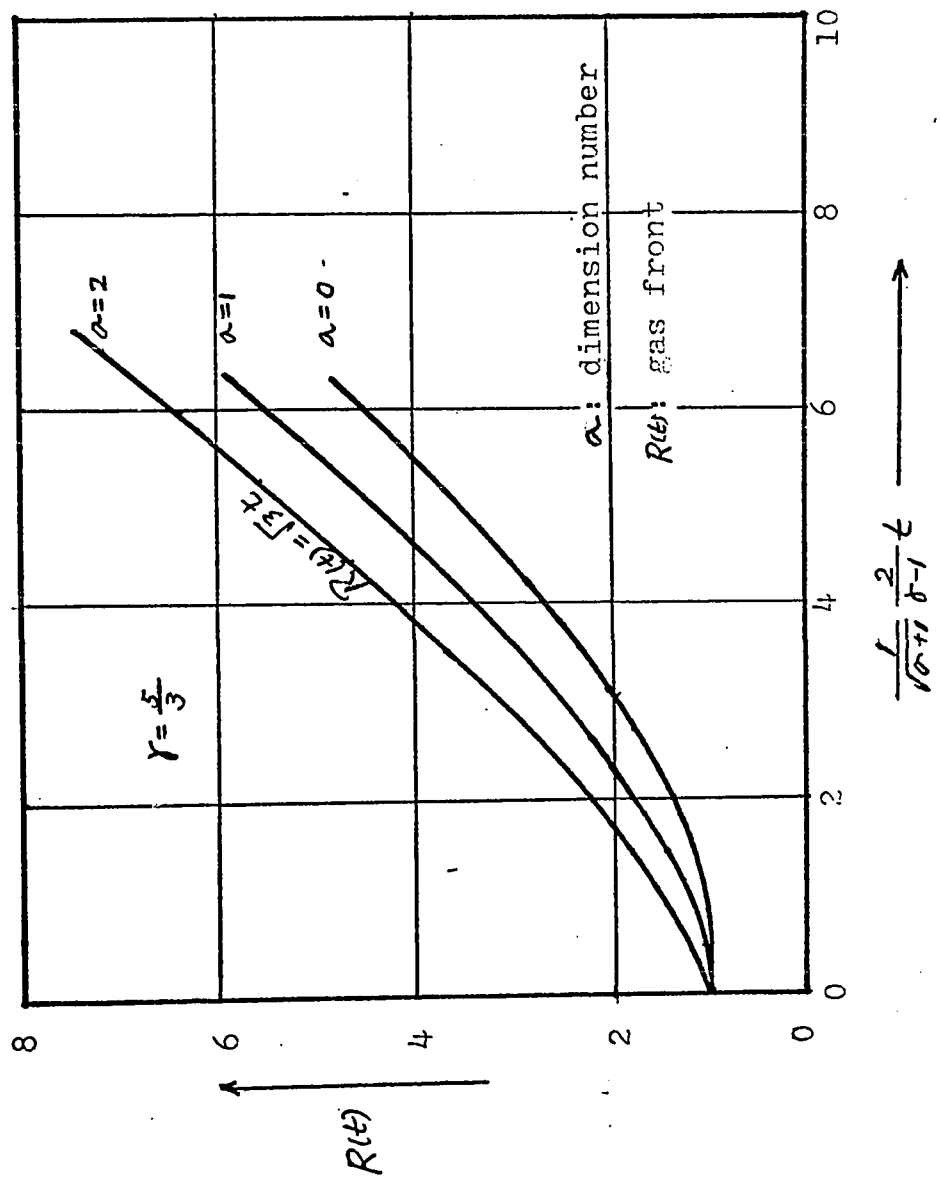


Fig. 7. Gas Fronts for Expansion of Self-Similar Flows

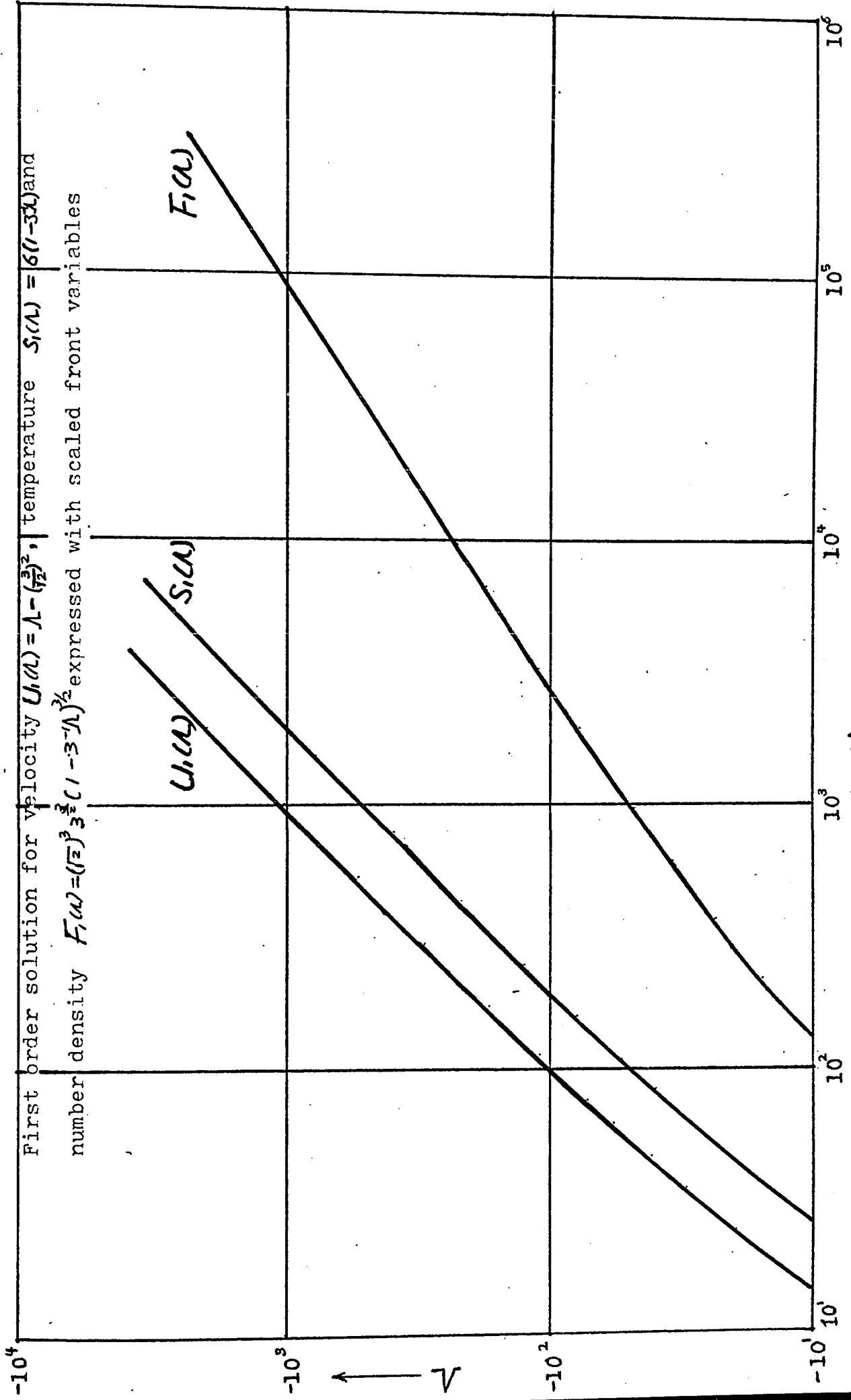


Fig. 8. $F_1(\lambda)$, $S_1(\lambda)$ and $U(\lambda)$ vs λ

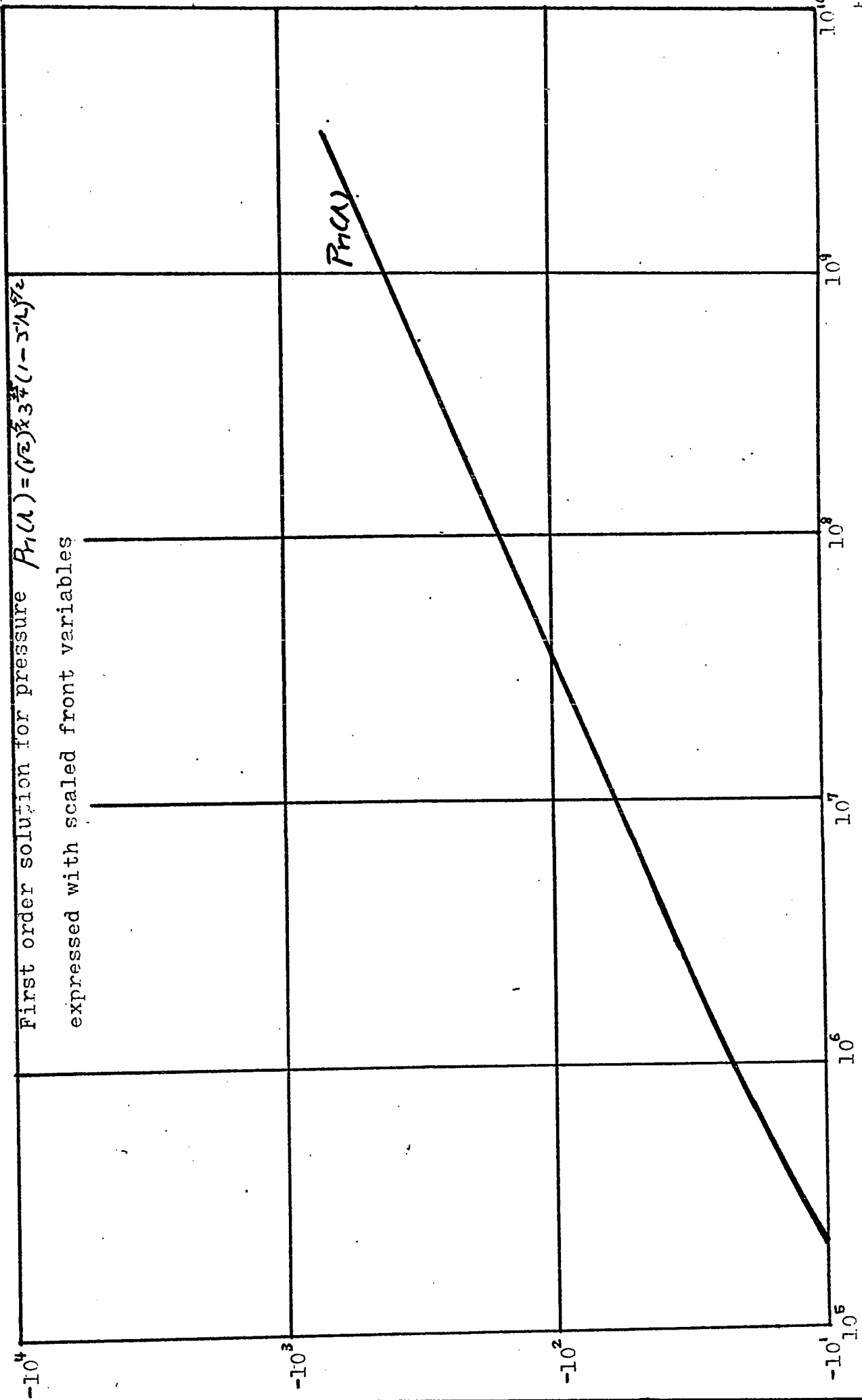


Fig. 9. $P_1(u)$ vs u/L

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Appendix A

There are a number of ways to describe the limiting value of a function. There are three possibilities describing the limiting value qualitatively: the function may be

$$\left. \begin{array}{ll} \text{(a) vanishing:} & f(\varepsilon) \rightarrow 0 \\ \text{(b) bounded :} & f(\varepsilon) < \infty \\ \text{(c) infinite :} & f(\varepsilon) \rightarrow \infty \end{array} \right\} \text{ as } \varepsilon \rightarrow 0$$

The one of the formal ways of describing limits of the function is to describe qualitatively the rate at which the limit is approached. This can be done by comparing with a set of gauge functions. These are functions that are so familiar that their limiting behavior can be regarded as known intuitively. The comparison is made using the order symbols O ("big oh") and o ("little oh"). They provide an indispensable means for keeping account of the degree of approximation in a perturbation solution. The symbol O is used if comparing $f(\varepsilon)$ with some gauge function $d(\varepsilon)$ shows that the ratio $\frac{f(\varepsilon)}{d(\varepsilon)}$ remains bounded as $\varepsilon \rightarrow 0$.

One writes

$$f(\varepsilon) = O[d(\varepsilon)] \text{ as } \varepsilon \rightarrow 0 \text{ if } \lim_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{d(\varepsilon)} < \infty$$

The symbol o is used instead if the ratio tends to zero.
One writes

$$f(\varepsilon) = O[\delta(\varepsilon)] \text{ as } \varepsilon \rightarrow 0 \text{ if } \lim_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{\delta(\varepsilon)} = 0$$

some examples are

$$\sin 2\varepsilon = O(\varepsilon), \quad 1 - \cos \varepsilon = O(\varepsilon^2) = o(\varepsilon)$$

$$\sec^{-1}(1+\varepsilon) = O(\varepsilon^{1/2}) = O(1)$$

Neither order symbol necessarily describes the actual rate of approach to the limit, but provides only an upper bound. Thus it is formally correct to replace the first example by

$$\sin 2\varepsilon = O(1), O(\varepsilon), O(\varepsilon^{1/2}), O(\varepsilon^{\pm}), \text{ etc.}$$

The functions not containing ε can be thought of containing ε by multiplying $\frac{\varepsilon}{\varepsilon}$ to the function; for example $f_0 = f_0 \frac{\varepsilon}{\varepsilon}$.

Since f_0 is bounded, that is $f_0 < \infty$, the upper bound of f_0 can be set as $O(1)$, i.e.

$$f_0 = O(1)$$

Appendix B

The general form of asymptotic series is

$$C_0 + \frac{C_1}{\chi} + \frac{C_2}{\chi^2} + \dots \dots \dots (C_0, C_1, C_2, \dots \text{const.})$$

The series need not converge for any value of χ .

When χ is smaller than constants, the series may diverge and vice versa. Now, with this in mind, the choice of the value of A is by no means arbitrary. In our case A corresponds to χ , then, the choice of the value of A must be such that the series does not diverge. There is a limiting value of A for which the series converges and for any value of A less than that the series starts to diverge. That value of A is expressed as $A_{\text{conv-div}}$ and any A greater than $A_{\text{conv-div}}$ becomes sufficient for the series to be convergent. In this work, however the value of $A_{\text{conv-div}}$ is not available since the full asymptotic series is not obtained. In order to obtain higher order terms in the expansion the higher order analysis is needed.