Université d'Ottawa | University of Ottawa

A New Distance Measure Including the Weak Preference Relation: Application to the Multiple Criteria Aggregation Procedure for Mixed Evaluations

Sarah Ben Amor
Jean-Marc Martel

WORKING PAPER
WP.2011.08

October 2011
ISSN 0701-3086

École de gestion TELFER School of Management

Accredited by Association of MBAs

AACSB ACCREDITED EST. 1916

A NEW DISTANCE MEASURE INCLUDING THE WEAK PREFERENCE RELATION: APPLICATION TO THE MULTIPLE CRITERIA AGGREGATION PROCEDURE FOR MIXED EVALUATIONS

Sarah Ben Amor [a*], Jean-Marc Martel [b]

[a] Telfer School of Management, University of Ottawa, 55 Laurier East, Ottawa, Ontario, K1R 6N5 Canada

[b] Faculté des Sciences Administratives, Université Laval, Ste-Foy, Québec, G1K 7P4 Canada

In this paper we propose a new distance measure between two preorders where the indifference, strict preference, weak preference and incomparability relations are considered. This distance measure is then used in the aggregation stage of a multiple criteria aggregation procedure for mixed evaluations designed to deal with different kinds of information imperfections. Introducing the weak preference relation allows for more nuance and flexibility to the decision- maker in this context.

Keywords: Multiple criteria analysis; Uncertainty modeling; Preference relations; Stochastic dominance; Distance measure.

[*] Corresponding author. Tel.: +16135625800(4909); Fax: +16135625164.
E-mail address: benamor@telfer.uottawa.ca (S. Ben Amor).

1. Introduction

In the literature, one can find several works addressing the issue of a distance measure between two preorders. Most of these works were developed in a context of group decision making where individual preorders have to be aggregated into a collective or compromise preorder. In fact, Kemeney and Snell (1962) were the first authors to use a distance-based model for this purpose. They presented a set of conditions that a distance measure must satisfy. Many authors such as Cook and Seiford (1978), Cook and Kress (1985) among others, have proposed similar conditions to those of Kemeney and Snell (1962) to build different distance measures between two total preorders (including the indifference and the preference relations). However, fewer works have studied the case of partial preorders (including the preference, indifference and incomparability relations). Bogart (1975) generalized the model of Kemeney and Snell (1962) to accommodate partial preorders. Nevertheless, his definition to partial preorder excludes the indifference relation as an alternative a is either preferred or incomparable to an alternative b. For Cook et al. (1986a, 1986b), a partial preorder is represented by a pair of binary matrices. An information matrix that indicates whether the pairs of alternatives are comparable or not, and a (transitive) preference matrix that contains only pairs of alternatives related by a preference relation. In both papers, the authors presented a set of axioms showing the existence and uniqueness of a distance measure between two individual preorders but the aggregation problem for determining a collective preorder has not been addressed. In a multicriteria decision aiding context on water supply systems, Roy and Slowinski (1993) have proposed two sets of conditions (logical conditions and significance conditions) for a criterion of distance between partial preorders. Their approach was adapted by Ben Khélifa and Martel (2001) to tackle the aggregation problem and an algorithm was proposed to determine a total collective preorder from partial individual preorders. Jabeur et al. (2004) presented a new distance measure that is thought to bring some improvements to these previous works.

As an extension to the work of Jabeur et al. (2004), we propose a new distance measure between preorders including the weak preference relation. The weak preference relation (Q) is an intermediate relation between preference and indifference. It has been introduced by Roy (1978) with the ELECTRE III method. It represents a zone of ambiguity and/or uncertainty where it is difficult to make a distinction between preference and indifference. Including the weak preference relation allows for more flexibility and nuance to the decision maker while expressing his preferences.

In this paper, we present the new distance measure D including the weak preference relation (section 2). In section 3, the distance measure D will be used in the multiple criteria aggregation procedure for mixed evaluations (Ben Amor et al. 2007) to aggregate n unicriterion (local) preorders into a multicriterion (global) preorder. This procedure was designed to deal with different types of information imperfections (uncertainty, imprecision, ambiguity, …). It leads to non-valued global preference relations where the p.r.s. (preference relational system) consists of the strict preference, the indifference and the incomparability relations (Ben Amor *et al.*, 2007). It is composed of two main phases. The first phase consists of constructing local (unicriterion) preference relations while the second phase consists of aggregating them into global (multicriterion) binary relations. Extending this procedure to include the weak preference relation in its p.r.s. will give more freedom to the decision-maker and allow him to qualify his preferences. Several changes were required for this purpose. The general framework and notation used for this procedure are presented in section 3.1, while changes operated on both phases are presented in sections 3.2 and 3.3 respectively. The exploitation of the resulting global preference relations is discussed in section 3.4. Finally, a numerical example is discussed in section 4. Conclusions and future work follow in section 5.

2. Extended Distance Measure *D*

Let A be a finite set of objects (alternatives, options or actions in the context of this paper) and (a_i, a_k) an ordered pair of objects belonging to A. In order to set a preference between a_i and a_j, four binary relations are considered. The strict preference (P), the weak preference (Q), the indifference (I) and the incomparability (?). For the sake of convenience, we use $a_i\ P^{-1}\ a_k$ for $a_k\ P\ a_i$ and $a_i\ Q^{-1}\ a_k$ for $a_k\ Q\ a_i$, P^{-1} and Q^{-1} stand for the inverse strict preference and the inverse weak preference respectively. Based on the works of Jabeur *et al.* (2004), we propose a set of seven "logical" conditions (an axiomatic) allowing to comparing the distance between each pair of binary relations $\{P, P^{-1}, Q, Q^{-1}, I, ?\}$. If distance D satisfies these conditions, it can be easily proven that D is a metric, i.e. it verifies the non-negativity, the symmetry and the triangular inequality properties.

Condition 1: (C1).

$$D(P, ?) = D(P^{-1}, ?) \quad \text{and} \quad D(P, I) = D(P^{-1}, I) \tag{1}$$

$$D(Q, ?) = D(Q^{-1}, ?) \quad \text{and} \quad D(Q, I) = D(Q^{-1}, I) \tag{2}$$

$$D(P, Q^{-1}) = D(Q, P^{-1}) \tag{3}$$

This condition is natural since the strict preference P and the inverse strict preference P^{-1}, as the weak preference Q and the inverse weak preference Q^{-1} are symmetrical to each other ($a_i P a_k \Leftrightarrow a_k P^{-1} a_i$ and $a_i Q a_k \Leftrightarrow a_k Q^{-1} a_i$).

Condition 2: (C2).

$$D(P, P^{-1}) = Max\{D(O,U) / O,U \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}\} \tag{4}$$

Equation (4) indicates that the strict preference and the inverse strict preference relations are the most discordant relations.

Condition 3: (C3).

$$D(O,U) > 0 \text{ if } O \neq U \text{ and } D(O,U) = 0 \text{ if } O \equiv U \text{ when } O,U \in \{P, P^{-1}, Q, Q^{-1}, I, ?\} \tag{5}$$

This condition states that the minimum distance between two distinct relations is positive. It is null in the opposite case.

Condition 4: (C4).

$$D(P,?) = D(Q, ?) = D(I, ?) \tag{6}$$

Following the interpretation of several authors such as Roy and Bouyssou (1993), Shärlig (1996) the incomparability is seen as the affirmation of the incapacity to establish the relation type: there is no indifference, no weak preference and no strict preference between the two alternatives under consideration. Thus, in the absence of any additional information, the incomparability relation should be considered as equidistant from the other preference relations, in this case $\{P, P^{-1}, Q, Q^{-1}, I\}$. In fact, this consideration relies on the insufficient reason principle of Laplace.

This argument was also followed by Jabeur *et al.* (2004), as they considered that the incomparability relation is equidistant from the indifference, the strict preference and the inverse strict preference relations. Whereas, Roy and Slowinski (1993) as Ben Khélifa and Martel (2001) stipulated that the passage from indifference to incomparability is less demanding than the passage from preference to incomparability, i.e. $D(P,?) \geq D(I, ?)$.

Condition 5: (C5).

$$D(Q, P) = D(Q, I) \tag{7}$$

This condition states that the weak preference is at an equal distance from the indifference and the strict preference. One can easily state that the weak preference relation lies between the indifference and the strict preference relation. The hypothesis about equidistance seems reasonable if one thinks of the weak preference relation as a state of mind expressing hesitation between indifference and strict preference (Roy, 1978). There is no reason to place the weak preference relation in a closer position to one or the other of the two preference relations considered here. Consequently, Laplace's principle of insufficient reason can be invoked to justify equidistance.

Condition 6: (C6).

$$D(P, ?) \geq D(P, I) \tag{8}$$

$$D(?, I) \geq D(P, I) \tag{9}$$

$$D(P, I) \geq D(Q, I) \tag{10}$$

$$D(Q^{-1}, P) \geq D(Q, P) \tag{11}$$

$$D(P, Q^{-1}) \geq D(Q, Q^{-1}) \tag{12}$$

$$D(I, ?) \geq D(Q, Q^{-1}) \tag{13}$$

Inequality (8) was justified by Jabeur *et al.* (2004). Equation (9) follows considering condition 4, while inequalities (10) to (12) are consistent conditions considering the preference relations at hand. For instance, the passage from strict preference to indifference is more demanding than the passage from weak preference to indifference. For sake of consistency, inequality (13) is assumed.

Graphically, the previous conditions are illustrated in figure 1 where:

- $D(P,?) = D(Q, ?) = D(I, ?) = D(?, Q^{-1}) = D(?, P^{-1}) = x$ (by C4)
- $D(P, Q) = D(Q, I) = D(\mathrm{I}, Q^{-1}) = D(Q^{-1}, P^{-1}) = a$ (by C1 and C5)
- $D(P, I) = D(P^{-1}, I) = b$ (by C1)
- $D(P, Q^{-1}) = D(Q, P^{-1}) = c$ (by C1)
- $D(P^{-1}, P^{-1}) = y$
- $D(Q, Q^{-1}) = b'$

The conditions C1 to C6 induce the following partial ordering:

$$y \geq \begin{bmatrix} c \\ x \end{bmatrix} \geq \begin{bmatrix} b \\ b' \end{bmatrix} \geq a \geq 0, \tag{14}$$

Moreover, the triangular inequality requirement led us to consider all relevant triangles. After eliminating redundant and trivial relations, we obtained inequalities listed in condition 7.

Condition 7: (C7).

$$y \leq 2b, \tag{15}$$

$$b \leq 2a, \tag{16}$$

$$b' \leq 2a, \tag{17}$$

$$b \leq a + c, \tag{18}$$

$$b' \leq a + c, \tag{19}$$

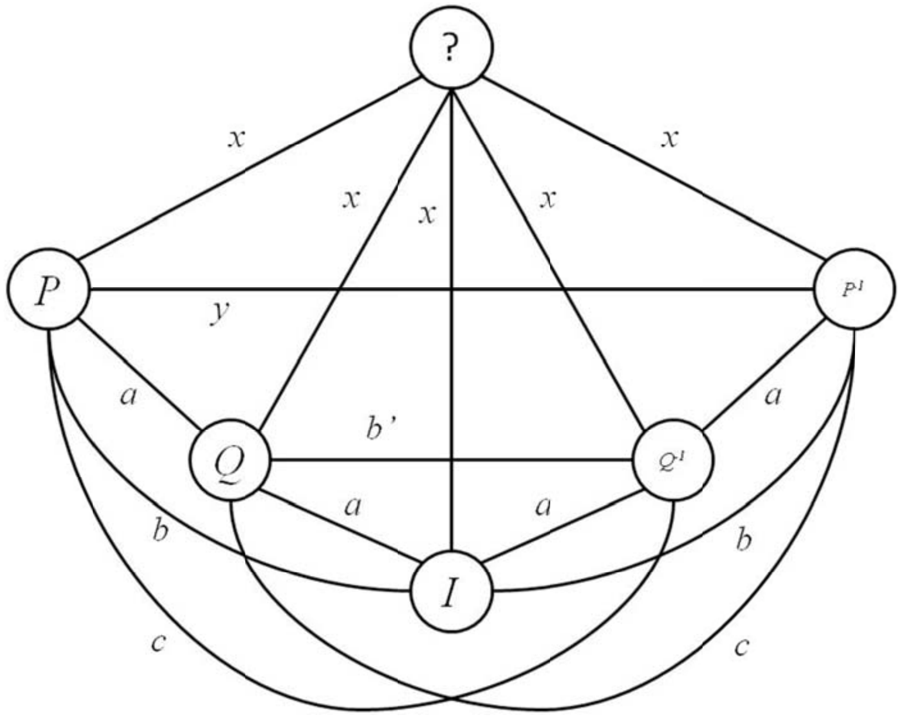
$$c \leq a + b, \tag{20}$$

$$c \leq a + b' \tag{21}$$

It can be shown that any combination of values (a, b, b', c, x, y) that satisfies relations (14) to (21) corresponds to a metric. Indeed, the non-negativity property is a direct consequence of condition C3. The symmetry property is obviously satisfied by definition of distance values. The inequalities (15) to (21) follow directly from the triangular inequality property requirement.

To obtain a particular metric, one has to select one possible combination of values (a, b, b', c, x, y), that is to say, select a representative point within the polyhedron defined by inequalities (14) to (21). This can be achieved in different ways.

Figure 1. Graphic representation of conditions C1 to C7



For instance, one could look for the centroid of the polyhedron. In this paper we propose a simplified approach based on the work of Jabeur *et al.* (2004). These authors have adopted similar conditions to the ones listed above except from those referring to the weak preference relation. Thus, without loss of consistency, we retain the numerical values they found: $D(P,?) = D(P^{-1}, ?) = D(I, ?) = x = 4/3$ and $D(P, P^{-1}) = y = 5/3$. Using these numerical values and the law of cosines from the two-dimensional representation of figure 2, we obtain the following system of equations:

$$\left(\frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 - 2\left(\frac{4}{3}\right)^2 \cos\gamma = 2\left(\frac{4}{3}\right)^2 (1-\cos\gamma) \tag{22}$$

$$D(Q, P)^2 = D(Q, I)^2 = 2\left(\frac{4}{3}\right)^2\left(1-\cos\frac{1}{4}\gamma\right) \tag{23}$$

$$D(I, P)^2 = D(Q, Q^{-1})^2 = 2\left(\frac{4}{3}\right)^2\left(1-\cos\frac{1}{2}\gamma\right) \tag{24}$$

$$D(Q, P^{-1})^2 = D(Q^{-1},P)^2 = 2\left(\frac{4}{3}\right)^2\left(1-\cos\frac{3}{4}\gamma\right) \tag{25}$$

By solving this system we obtain these values: $\gamma = 77.3644°$, $D(Q, P) = D(Q, I) = 0.448$, $D(I, P) = D(Q, Q^{-1}) = 0.8832$, $D(Q, P^{-1}) = D(Q^{-1}, P) = 1.2933$. Table 1 summarizes the values assigned to D.

Figure 2. Two-dimensional representation of conditions C1 to C7

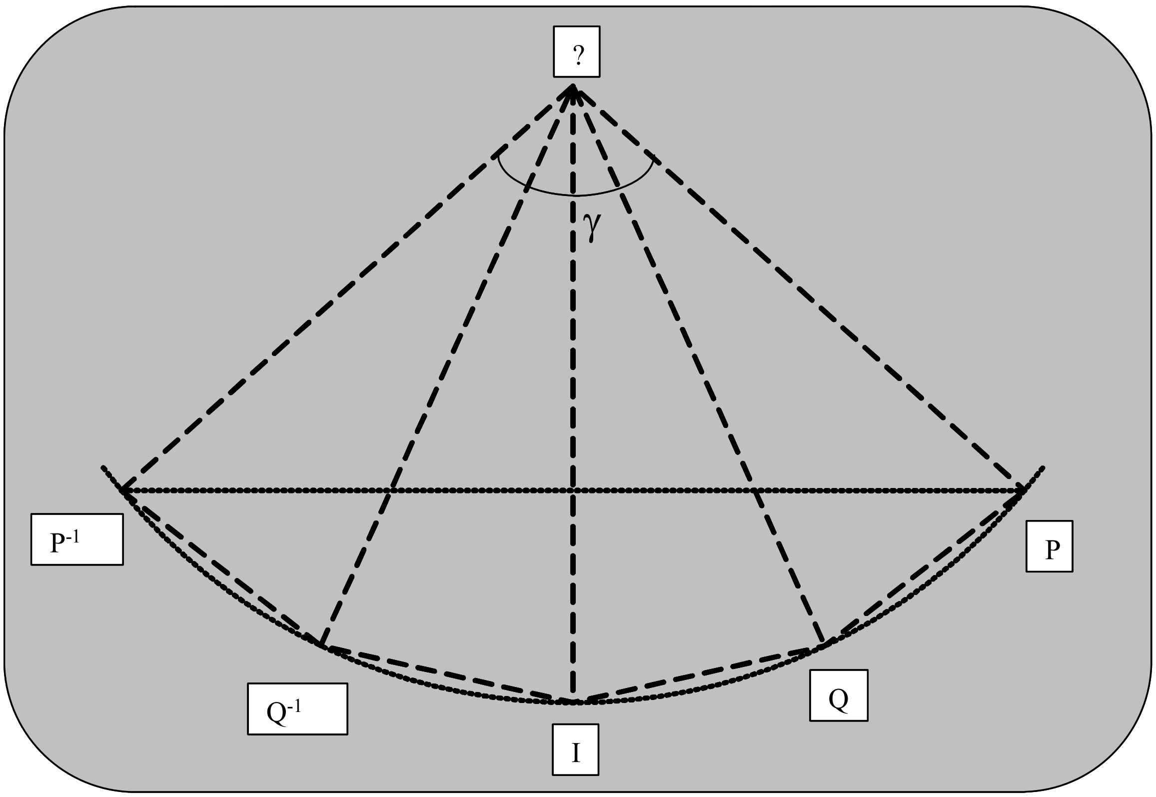


Table 1. Numerical values assigned to the distance measure *D*

	$a_i\ I\ a_k$	$a_i\ Q\ a_k$	$a_i\ P\ a_k$	$a_i\ ?\ a_k$	$a_i\,Q^{-1}\ a_k$	$a_i\,P^{-1}\ a_k$
$a_i\ I\ a_k$	0	0.448	0.8832	1.3333	0.448	0.8832
$a_i\ Q\ a_k$	0.448	0	0.448	1.3333	0.8832	1.2933
$a_i\ P\ a_k$	0.8832	0.448	0	1.3333	1.2933	1.6667
$a_i\ ?\ a_k$	1.3333	1.3333	1.3333	0	1.3333	1.3333
$a_i\,Q^{-1}\ a_k$	0.448	0.8832	1.2933	1.3333	0	0.448
$a_i\,P^{-1}\ a_k$	0.8832	1.2933	1.6667	1.3333	0.448	0

3. Multiple Criteria Aggregation Procedure for Mixed Evaluations Including the Weak preference Relation

A multiple criteria aggregation procedure (MCAP) for mixed evaluations was designed to address the issue of different types of information imperfections (uncertainty, imprecision, …) that can be present in the same decision situation (Ben Amor *et al*., 2007). It leads to non-valued global (multicriterion) preference relations where the p.r.s. (preference relational system) consists of the strict preference, the indifference and the incomparability relations. The procedure is here extended to include the weak preference relation in its p.r.s. Enriching the preference relational system will give more freedom to the decision-maker and allow more nuance in the preference modeling. This MCAP is based on pairwise comparisons and proceeds through two main steps. In the first step, local (unicriterion) preference relations are constructed between all the pairs of alternatives. This construction is based on an extension of the stochastic dominance concepts to the mixed evaluations context (distributional dominance). These concepts will here be adapted to accommodate the weak preference relation (section 3.1). Local preference relations are aggregated in the second step into global (multicriterion) preference relations. The aggregation step uses an algorithm developed in a group decision context and adapted to multicriteria aggregation (Jabeur and Martel, 2007a). This algorithm will here be adapted using the new distance measure D to handle the weak preference relation (section 3.3). The resulting global relations can then be exploited according to the decisional problematic at hand (choice, ranking or sorting). Exploitation step is discussed in section 3.4 using adapted procedures from the work of Jabeur and Martel (2007a).

3.1. General modeling framework and notation

Our procedure is based on the (A, X, E) model for discrete multiple criteria where $A = \{a_1,...,a_i,...,a_m\}$ is a set of potential alternatives, $X = \{X_1,...,X_j,...,X_n\}$ is a finite set of attributes (or criteria) and E is a performance table or an evaluation matrix:

$$
E = \begin{array}{c} \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_m \end{array}
\begin{array}{c} \begin{array}{ccccc} X_1 & \cdots & X_j & \cdots & X_n \end{array} \\
\begin{bmatrix} e_{11} & \cdots & e_{1j} & \cdots & e_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{i1} & \cdots & e_{ij} & \cdots & e_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{m1} & \cdots & e_{mj} & \cdots & e_{mn} \end{bmatrix} \end{array} \tag{26}
$$

Several methods associate to each attribute/criterion X_j a coefficient of relative importance (or weight) w_j, where $\sum_{j=1}^{n} w_j = 1$.

In our context, it should be observed that the information imperfection would notably appear in the evaluations e_{ij}. The modeling framework adopted for the MCAP for mixed evaluations accepts, in uncertain decision-making situations, stochastic, possibilistic or "evidential" evaluations. It also includes fuzzy or ordinal evaluations. It is based on the fact that the theory of evidence (Shafer, 1976) offers a general modeling framework where possibilities and probabilities can be seen as special cases.

Let us consider for each attribute X_j, the set of H possible nature states $\Omega^j = \{\omega_1^j, ..., \omega_h^j, ..., \omega_H^j\}$. For each alternative a_i, evaluation e_{ij} according to attribute X_j depends on the occurrence of a particular nature state. e_{ij}^h is then the evaluation of alternative a_i when ω_h^j occurs with h = 1,2,…, H. The performance matrix for attribute X_j is then defined according to table 2.

Table 2. Performance matrix for attribute X_j

Alternatives	
a_1 $\vdots$ a_i $\vdots$ a_m	

A priori information available about the nature states (probability functions, belief masses, possibility measures, …) is integrated as in table 3. This table is based on the theory of evidence (Shafer, 1976) where a priori information is represented by belief masses associated to the so called focal elements. Focal elements are subsets $B_{h'}^{j} \subseteq \Omega^{j}$ $(h' = 1,...,2^{H})$ of the nature states set $\Omega^{j} = \{\omega_1^j,...,\omega_h^j,...,\omega_H^j\}$, that influence the evaluations of the alternatives on a given attribute X_j. Evaluation e_{ij} of alternative a_i on attribute X_j, is then represented by a subset of values $C_{ij}^{h'} \subseteq \mathrm{e}_{ij} = \{e_{ij}^1,...,e_{ij}^h,...,e_{ij}^H\}$ which depends on the subsets of nature states (focal elements) $B_{h'}^{j} \subseteq \Omega^{j}$, $(h' = 1,...,2^{H})$.

Table 3. Performance matrix for attribute X_j integrating a priori information

Alternatives	Any focal elements
a_1 $\vdots$ a_i $\vdots$ a_m	
A priori belief masses	

In this modeling framework, one can consider that a possibilistic or a probabilistic modeling is a special case when a priori information is represented by possibility or probability distributions respectively. In fact, for a possibilistic attribute where a priori information is characterized by possibility distributions π_{ij}, belief masses are associated with focal elements that are embedded $B_1^j \subseteq ... \subseteq B_{h'}^j \subseteq ... \subseteq B_{H'}^j$. Possibility measures $\Pi(B_{h'}^j)$ correspond to the plausibilities of the embedded focal elements (Ben Amor *et al*., 2007). For stochastic attributes, the focal elements $B_{h'}^j$ are reduced to singletons $\{\omega_h^j\}$, $(h = 1,2,...,H$. Associated belief masses correspond to probability measures $P(\omega_j^h)$. In this case, evaluations are characterized by random variables X_{ij} with a priori probability distributions f_{ij} .

3.2. Local Preference Relations

The first phase of our procedure consists of building local preference relations between two alternatives. Let H_j be the local relation between two alternatives a_i and a_k on a criterion X_j. We have: $a_i\ H_j\ a_k$ where $H_j = \{P, P^{-1}, Q, Q^{-1}, I, ?\}$ with I the indifference relation, P the strict preference relation, P^{-1} the inverse strict preference relation, Q the weak preference relation, Q^{-1} the inverse weak preference relation and ? the incomparability relation.

In order to build these relations, the approach based on the extension of the stochastic dominance concept was privileged as in Ben Amor *et al*. (2007). This approach is consistent with a decision-maker whose attitude towards risk follows DARA (Decreasing Absolute Risk Aversion) utility functions (Martel and Zaras, 1995). In the case where the evaluation e_{ij} is a random variable, the results of stochastic dominance can be directly applied in order to establish preference relations (indifference, strict preference, inverse strict preference and incomparability).

The stochastic dominance concepts can be extended to ambiguous probabilities (Langewish and Choobineh, 1996). Similarly we can process fuzzy, possibilistic or evidential criteria if we apply some known transformations to the functions characterizing the evaluation of an alternative (fuzzy membership functions, possibility distributions, belief masses, …) to give them properties similar to those of a probability distribution (Ben Amor *et al*., 2007). In such cases, the stochastic dominance principle is extended to a more general dominance structure that will be called "distributional dominance".

A limit of the existing MCAP for mixed evaluations is that it includes a single level of preference (strict preference). In this paper we propose to distinguish between strict and weak preference relations. For this purpose, the distributional dominance results are adapted in the following way:

The indifference relation is defined as in Ben Amor *et al*. (2007):

$$e_{ij} I_j e_{kj} \Leftrightarrow |H_1(x)| = |F_{ij}(x) - F_{kj}(x)| \le s_j \,,\ \forall x \in \left[x_{j*}, x_j^*\right] \qquad (27)$$

where x_{j^*} and x_j^* are respectively the inferior and superior limits of the evaluation scale of attribute/criterion X_j, $s_j \geq 0$ is a predetermined threshold, F_{ij} and F_{kj} are the cumulative probability distributions for the evaluations of alternatives a_i and a_k on attribute X_j and x is a modality of this attribute.

Otherwise, i.e. for the cases where $|H_1(x)| > s_j$, ($F_{ij} \neq F_{kj}$), preference or incomparability will occur. We propose to define strict preference, weak preference and incomparability as follows:

$$
\begin{aligned}
&e_{ij} P_j e_{kj} \Leftrightarrow e_{ij}\ FDD_j\ e_{kj} \\
&e_{ij} Q_j e_{kj} \Leftrightarrow e_{ij}\ SDD_j \text{ or } TDD_j\ e_{kj} \text{ and } e_{ij} \text{ non } FDD_j\ e_{kj} \qquad (28) \\
&e_{ij}\ ?_j\ e_{kj} \Leftrightarrow e_{ij} \text{ non} DD_j\ e_{kj}
\end{aligned}
$$

where *FDD*, *SDD* and *TDD* respectively stand for first, second and third order distributional dominance and *DD* means that one of the three dominance types is verified.

Let us note that in the case of a deterministic criterion the distribution functions are step functions and so only two results are possible: either indifference or strict preference. This is a limit of the current procedure that does not include indifference or preference thresholds as in the context of punctual evaluations (pseudo-criterion notion). Let us note also that in the case of punctual evaluations, locally there is no incomparability. It is one of the characteristics of a criterion as defined by Roy (1985).

3.3. Aggregating Local Preference Relations

Once the local preference relations are established, the second stage of the procedure aims at aggregating n binary relations ($a_i\ H_j\ a_k$) in order to obtain a global binary relation ($a_i\ H\ a_k$) by considering the relative importance of the criteria.

To achieve this, the algorithm (AL3) developed by Jabeur and Martel (2007a) in a group decision-making context is adapted to handle the weak preference relation. AL3 starts by defining, for each pair of alternatives (a_i, a_k), an index $\Phi^H(a_i, a_k)$ where $H \in \{P, P^{-1}, I, ?\}$. This index measures the divergence between the global relation H and the local binary relations relating this pair of alternatives on each criterion. This divergence is quantified by using the distance measure Δ between two binary relations *H* relating two alternatives (Jabeur *et al.* 2004) when $H \in \{P, P^{-1}, I, ?\}$.

In order to include the weak preference and the inverse weak preference relations one can use the extended distance measure *D*. The following adjustments to the divergence index calculations need to be made. The index $\Phi^H(a_i, a_k)$ is then determined as follows:

$$\Phi^H(a_i, a_k) = \sum_{j=1}^{n} w_j D(H, R_j(a_i, a_k))$$

$$\text{where } R_j(a_i, a_k) = \begin{cases} P & if \quad a_i P_j a_k \\ Q & if \quad a_i Q_j a_k \\ I & if \quad a_i I_j a_k \\ ? & if \quad a_i \, ?_j \, a_k \\ Q^{-1} & if \quad a_k Q_j a_i \\ P^{-1} & if \quad a_k P_j a_i \end{cases}, \qquad (29)$$

w_j is the normalized coefficient of the relative importance of criterion X_j, $R_j(a_i, a_k)$ represents the local binary relation comparing a_i to a_k on criterion X_j, *D* is the extended distance measure between the binary relations relating the pairs of alternatives and $H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}$ is the global relation. In the literature, various methods are proposed to establish the relative importance of the criteria w_j. Regarding this issue, one can refer to Mousseau (1995) and Roy and Mousseau (1996).

Once the divergence indexes have been computed, we identify, for each pair of alternatives (a_i, a_k), the set of the global relations $\mathbf{H^*}$ that minimize the indexes of divergence $\Phi^H(a_i, a_k)$, that is:

$$\mathbf{H^*} = \left\{ H^* / \Phi^{H^*} = \underset{H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}}{Min} \Phi^H(a_i, a_k) \right\} \qquad (30)$$

Therefore, if the set $\mathbf{H^*}$ is reduced to a singleton, i.e. there is only one global relation H^* that minimizes the indexes of divergence $\Phi^H(a_i, a_k)$, then H^* will be retained for the pair of alternatives (a_i, a_k) globally. In the opposite case, i.e. more that one distinct relation minimizes the indexes of divergence $\Phi^H(a_i, a_k)$; priority rules have to be applied between the binary relations $\{P, P^{-1}, Q, Q^{-1}, I, ?\}$. We extend the two priority rules proposed by Jabeur *et al.* (2004) to include weak preference. Jabeur *et al.* (2004) state that the preference relation priority is higher than that of the indifference relation and that the indifference relation priority is higher than that of the incomparability relation. Likewise, we can consider that the preference relation has higher priority than the indifference relation, which has higher priority than the weak preference relation. The latter stands as we consider the weak preference relation as a state of hesitation between strict preference and indifference. Obviously, other priority rules can be reasonably adopted.

Once the priority rules are applied, we analyze the cardinality of $\mathbf{H}^*$. If $\mathbf{H}^*$ contains a unique relation H^*, then H^* will be retained for the pair of alternatives (a_i, a_k) globally. In the opposite case, i.e. when $\mathbf{H}^*$ contains two relations of the same priority (necessarily the preference (P) and the inverse preference (P^{-1}) relations or the weak preference (Q) and the inverse weak preference (Q^{-1}) relations), we extend the intersection rule used by Roy (1978) to combine two complete orders, and therefore, the incomparability relation ($\mathbf{H}^* \equiv ?$) will be retained for the pair of alternatives (a_i, a_k) globally.

3.4. Exploitation of the global preference relational system

Once the global p.r.s. is determined, it can be exploited according to different decisional problematics (e.g. the choice problematic, the ranking problematic, or the sorting problematic). In the literature, several exploitation procedures can be found. One can quote the works of Roy (1968), Colson (2000) and Jabeur and Martel (2007a) for the choice problematic; the works of Roy (1978), Brans and Vincke (1985), Colson (2000) and Jabeur *et al.* (2010) for the ranking problematic; and the works of Yu (1992) and Jabeur and Martel (2007b) for the sorting problematic.

In Jabeur and Martel (2007a) the authors propose a detailed review of original procedures to exploit a p.r.s. according to three decisional problematics: the choice, ranking and sorting problematics. In the case of a choice problematic, for instance, they proposed a procedure based on the alternative performance concept. An extension to this procedure allowing to handle the weak preference relation will be illustrated here.

The main idea of this procedure is to build, without erasing any information contained in the prs-graph, a subset of the "best" alternatives, called Θ, by using their performance (this concept will be defined later). This procedure will be adapted as follows. First, we determine, for each alternative a_i, the following sets:

- Successors' set $\Psi^{++}(a_i)$ and predecessors' set $\Psi^{--}(a_i)$:

$$\Psi^{++}(a_i) = \{a_k / a_i P a_k, i \neq k\} \text{ and } \Psi^{--}(a_i) = \{a_k / a_k P a_i, i \neq k\}, \tag{31}$$

- Weak successors' set $\Psi^{+}(a_i)$ and weak predecessors' set $\Psi^{-}(a_i)$:

$$\Psi^{+}(a_i) = \{a_k / a_i Q a_k, i \neq k\} \text{ and } \Psi^{-}(a_i) = \{a_k / a_k Q a_i, i \neq k\} \tag{32}$$

- the set of all alternatives with which a_i is indifferent:

$$\Psi^{\approx}(a_i) = \{a_k / a_i \approx a_k, i \neq k\} \tag{33}$$

- The set of alternatives which are incomparable:

$$N^{isol} = \{a_i / \Psi^{++}(a_i) = \Psi^{--}(a_i) = \Psi^{+}(a_i) = \Psi^{-}(a_i) = \Psi^{\approx}(a_i) = \varnothing \ \}. \tag{34}$$

Second, we calculate for each alternative $a_i \in A \setminus N^{isol}$ the index $\Pi(a_i)$, which measures its performance in the prs-graph. Introducing the weak preference relation, this performance will be defined as follows:

$$\Pi(a_i) = card(\Psi^{++}(a_i)) - card(\Psi^{--}(a_i)) + 0.5(card(\Psi^{+}(a_i)) - card(\Psi^{-}(a_i))) \tag{35}$$

where 0.5 is a discounting factor to distinguish between weak and strict preference. The value 0.5 is purely arbitrary. It is main purpose is to reduce the impact of weak successors and weak predecessors compared to successors and predecessors respectively, in assessing the performance of an alternative.

Note that the index $\Pi(a_i)$ considers only the preference relations (strict and weak) to assess the performance of an alternative a_i. Thus, in order to take into account the effect of the indifference relations in the performance of an alternative a_i, Jabeur and Martel (2007a) add to $\Pi(a_i)$ the average of the performances of all the alternatives that are indifferent to a_i in the prs-graph. Let:

$$\begin{cases} \hat{\Pi}(a_i) = \Pi(a_i) \quad if \quad card(\Psi^{\approx}(a_i)) = 0 \\ \hat{\Pi}(a_i) = \Pi(a_i) + \dfrac{1}{card(\Psi^{\approx}(a_i))} \sum\limits_{a_k \in \Psi^{\approx}(a_i)} \Pi(a_k) \quad otherwise \end{cases} \tag{36}$$

It is noteworthy to mention that the additional $\frac{1}{card(\Psi^{\approx}(a_i))} \sum_{a_k \in \Psi^{\approx}(a_i)} \Pi(a_k)$ component allows to correct upward or downward the initial performance of the alternative a_i expressed by the index $\Pi(a_i)$. Finally, Jabeur and Martel (2007a) build the Θ set in the following manner: $\Theta = N^{isol} \cup \hat{N}$ where $\hat{N} = \{ a_i / \underset{i}{Max} \hat{\Pi}(a_i) \}$. In other words, Θ will contain the alternatives which have obtained the highest performance $\hat{\Pi}(*)$ and those which are incomparable to them. The last step of this exploitation procedure consists in "purifying" Θ. In this step, any alternative a_i such as $\exists a_k \in \Theta \ and \ a_k \in \Psi^{--}(a_i)$, has to be removed from the prs-graph engendered by Θ, i.e. the prs-subgraph composed of binary relations which link only the alternatives belonging to Θ. If the purified Θ is nonempty, then it will contain the "best" alternatives. In the opposite case, i.e. the purified Θ is empty, we say that before this operation, Θ contained the alternatives which obtained the best score.

4. Numerical Example

Let us consider the following numerical example adapted from Ben Amor *et al*. (2007) to illustrate the use of the extended distance measure *D* and its application to the MCAP for mixed evaluations including the weak preference relation.

$A = \{a_1, a_2, a_3, a_4\}$ and $X = \{X_1, X_2, X_3, X_4\}$ where X_1 is a stochastic attribute, X_2 is an evidential attribute, X_3 is a possibilistic attribute and X_4 is a fuzzy attribute. For each attribute, the inputs consist of the performance matrix, a priori information, and the indifference threshold.

Table 4. Performance matrices for X_1, X_2, X_3 and X_4

a_i	□1					□2			□3			X_4	Associated fuzzy numbers
	ω_1^1	ω_2^1	ω_3^1	ω_4^1	ω_5^1	ω_1^2	ω_2^2	ω_3^2	ω_1^3	ω_2^3	ω_3^3		
a_1	3	1	2	6	4	90	90	110	35	70	45	Weak	(0, .1, 0, .2)
a_2	1	3	4	6	3	120	90	100	45	30	45	Fair	(.5, .5, .2, .2)
a_3	2	1	2	3	5	110	130	80	45	60	25	High	(.9, 1, .2, 0)
a_4	6	1	4	4	2	80	110	120	40	70	30	More or less weak	(.2, .2, .2, .2)

Table 4 gives the performance matrices for the four attributes. The evaluations according to X_4 are linguistic variables represented by trapezoidal fuzzy numbers (a, b, c, d) (see figure 3).

Table 5 gives the a priori information for attribute X_1, while table 6 gives the a priori information for X_2 and X_3. For attribute X_4, a priori information is implicitly contained in the performance matrix. We consider the same indifference threshold for all attributes, i.e. $s_1 = s_2 = s_3 = s_4 = s = 0.05$.

Figure 3. Membership function for fuzzy trapezoidal number (a, b, c, d)

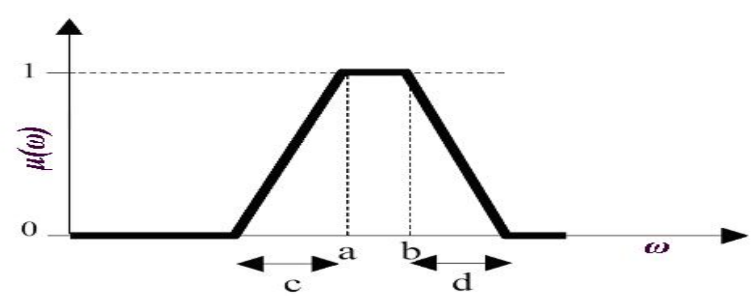


Table 5. A priori information for X_1

	ω_1^1	ω_2^1	ω_3^1	ω_4^1	ω_5^1
$P(\omega_h^1)$	0.28	0.23	0.10	0.28	0.11

Table 6. A priori information for X_2 and X_3

	Ø	$\{\omega_1^2\}$	$\{\omega_2^2\}$	$\{\omega_3^2\}$	$\{\omega_1^2, \omega_2^2\}$	$\{\omega_1^2, \omega_3^2\}$	$\{\omega_2^2, \omega_3^2\}$	$\{\omega_1^2, \omega_2^2, \omega_2^2\}$
$m(B_{h'}^2)$			0.1		0.6			0.3
$m(B_{h'}^3)$				0.2		0.2		0.6
$\Pi(B_{h'}^3)$	0	0.8	0.6	1.0	0.8	1	1	1

Local preference relations: Stochastic dominance results are directly applied in the case of X_1. The obtained results and the corresponding local preference relations are given in table 7.

After applying appropriate transformations (Ben Amor *et al.*, 2007), the dominance stochastic results for X_j $j = 2, 3, 4$, led to the local preference relations H_j listed in table 8.

Table 7. Stochastic dominance results and local preference relations for X_1

	Stochastic dominance results				H_1			
Alternatives	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4
a_1	*	*IND*	*FSD*	?	*	I	P	?
a_2	*IND*	*	?	-	I	*	?	Q^{-1}
a_3	-	?	*	-	P^{-1}	?	*	P^{-1}
a_4	?	*TSD*	*FSD*	*	?	Q	P	*

Table 8. Local preference relations H_j for attributes X_j, j = 2, 3, 4

Relations	H_2				H_3				H_4			
Alternatives	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4
a_1	*	P^{-1}	?	?	*	Q	P	P	*	P^{-1}	P^{-1}	P^{-1}
a_2	P	*	?	Q	Q^{-1}	*	Q	Q	P	*	P^{-1}	P
a_3	?	?	*	P	P^{-1}	Q^{-1}	*	Q^{-1}	P	P	*	P
a_4	?	Q^{-1}	P^{-1}	*	P^{-1}	Q^{-1}	Q	*	P	P^{-1}	P^{-1}	*

Aggregating local preference relations: This step consists in applying the adapted (AL3) algorithm. Thus, we start by computing the divergence index $\Phi^H(a_i, a_k)$ for each relation $H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}$ and for each alternatives pair. This operation is performed using tables 7 and 8, and the distance measure given in table 3. Considering the relative importance vector $\boldsymbol{W_j}$ = [.4, .2, .3, .1], the obtained divergence indexes are given in table 9. For instance, we have:

$\Phi^I(a_1, a_2) = 0.4D(I, I) + 0.2D(I, P^{-1}) + 0.3D(I, Q) + 0.1D(I, P^{-1}) = (0.4 \times 0) + (0.2 \times 0.8832) + (0.3 \times 0.448) + (0.1 \times 0.8832) = 0.399$.

Table 9. Indexes of divergence

Alternative pairs	Φ^I	Φ^Q	Φ^P	$\Phi^?$	$\Phi^{Q^{-1}}$	$\Phi^{P^{-1}}$
(a_1, a_2)	**0.399**	0.567	0.988	1.333	0.579	0.741
(a_1, a_3)	0.973	0.71	**0.433**	1.067	1.217	1.433
(a_1, a_4)	1.153	1.064	0.967	**0.533**	1.233	1.3
(a_2, a_3)	1.023	0.929	1.101	**0.533**	1.11	1.188
(a_2, a_4)	0.491	**0.398**	0.741	1.333	0.571	0.992
(a_3, a_4)	0.753	0.917	1.055	1.333	**0.567**	0.634

For each alternatives pair, the set of global relations $\mathbf{H}^*$ minimizing the indexes of divergence is identified. In this case, the minimum is unique for each alternatives pair. Therefore, $\mathbf{H}^*$ is reduced to a singleton. The global preference relations are given in table 10. For instance, we have for a_1 and a_2:

$\underset{H}{Min}\,\Phi^H(a_1, a_2) = Min\,\{0.399, 0.567, 0.988, 1.333, 0.579, 0.741\} = 0.399$, thus $\mathbf{H}^* = \{I\}$.

Table 10. Global preference relations

Alternatives	a_1	a_2	a_3	a_4
a_1	$*$	I	P	$?$
a_2	I	$*$	$?$	Q
a_3	P^{-1}	$?$	$*$	Q^{-1}
a_4	$?$	Q^{-1}	Q	$*$

On the basis of table 10, we can draw the prs-graph associated to the global preference relations (see Figure 4).

In order to identify Θ, i.e. the "best" alternatives set, without erasing the slightest information contained in the prs-graph, the exploitation procedure based on the alternative performance determines, for each alternative a_i, the sets $\Psi^{++}(a_i)$, $\Psi^{+}(a_i)$, $\Psi^{--}(a_i)$ $\Psi^{-}(a_i)$, $\Psi^{\approx}(a_i)$ and the performance indexes $\Pi(a_i)$ and $\hat{\Pi}(a_i)$. All these results are presented in Table 11.

Figure 4. Global prs-graph

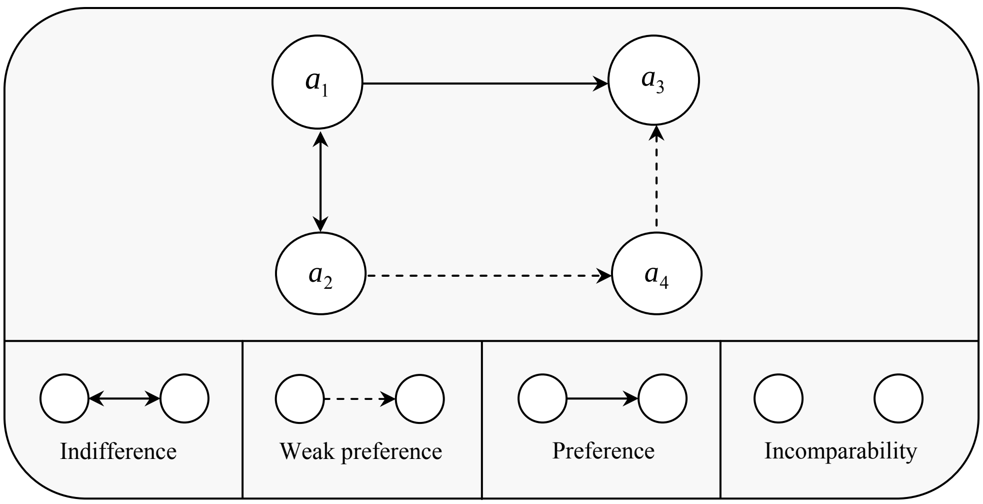


Table 11. The results of the exploitation procedure based on the alternatives performance

a_i	$\Psi^{++}(a_i)$	$\Psi^{+}(a_i)$	$\Psi^{--}(a_i)$	$\Psi^{-}(a_i)$	$\Psi^{I}(a_i)$	$\Pi(a_i)$	$\hat{\Pi}(a_i)$
a_1	$\{a_3\}$	---	---	---	$\{a_2\}$	1	1.5
a_2		$\{a_4\}$	---	---	$\{a_1\}$	0.5	1.5
a_3	---	---	$\{a_1\}$	$\{a_4\}$	---	-1.5	-1.5
a_4	---	$\{a_3\}$	---	$\{a_2\}$	---	0	0

According to table 11, alternative a_1 and a_2 have obtained the best performance, thus the “best” alternative set will contain a_1 and a_2, i.e. $\Theta = \{a_1, a_2\}$. Conducting the purification operation in this case didn’t affect Θ as a_1 and a_2 are indifferent.

5. Conclusion

In this paper we proposed a new distance measure between two preorders where the weak preference relation has been taken into account. The multicriteria aggregation procedure designed to handle mixed evaluations (stochastic, fuzzy, possibilistic or else) was adatped to lead to an enriched preference relational system including the weak preference relation. For this purpose, the new measure of distance was used to aggregate different unicriterion preorders into one multicriterion preorder. An adapted exploitation procedure was also proposed. For future work, the distance measure can be improved and the stochastic dominance can be extended to conclude about preferences for a decision-maker whose attitude towards risk follows INARA (INcreasing Absolute Risk Aversion) utility function (Zaras and Martel, 1994).

References

Ben Amor S, Jabeur K and Martel J-M. Multiple criteria aggregation procedure for mixed evaluations. *European Journal of Operational Research* 2007; 181(3); 1506-1515.

Ben Khélifa S, Martel, J-M. A Distance-Based Collective Weak Ordering. *Group Decision and Negotiation* 2001; 10(4); 317-329.

Brans J-P, Vincke Ph. A preference ranking organization method: The PROMETHEE method, *Management Science* 1985; 31; 647-656.

Bogart, KP. Preference Structures II: Distances Between Asymmetric Relations. *SIAM Journal On Applied Mathematics* 1975*;* 29(2), 254–262.

Colson G. The OR's prize winner and the software ARGOS: How a multijudge and multicriteria ranking GDSS helps a jury to attribute a scientific award. *Computers and Operations Research* 2000; 27; 741-755.

Cook, WD and Kress M. (1985). Ordinal Ranking with Intensity of Preference. *Management Science* 1985; 31(1); 26–32.

Cook, WD, Kress M and Seiford LM. An Axiomatic Approach To Distance on Partial Orderings. *R.A.I.R.O. Recherche Opérationnelle* 1986a; 20(2); 115-122.

Cook, WD, Kress M, and Seiford LM. Information And Preference in Partial Orders: A Bimatrix Representation. *Psychometrika* 1986b; 51(2); 197–207.

Cook, WD and Seiford LM. Priority Ranking And Consensus Formation. *Management Science* 1978; 24(16); 1721–1733.

Jabeur K, Martel J-M. A collective choice method based on individual preferences relational systems (p.r.s.). *European Journal of Operational Research* 2007a; 177; 1549-1565.

Jabeur K, Martel J-M. An Ordinal Sorting Method for Group Decision-Making. *European Journal of Operational Research* 2007b; 180(3); 1272-1289.

Jabeur, K, Martel J-M and Guitouni A. Deriving a minimum distance-based collective pre-order: A binary mathematical programming approach. *OR Spectrum*; 2010; 1-20.

Jabeur K, Martel J-M, Ben Khélifa S. A Distance-Based Collective Preorder Integrating the relative Importance of the Group's Members. *Group Decision and Negotiation* 2004; 13; 327-349.

Kemeny, JG and J. L. Snell JL. Preference Ranking: An Axiomatic Approach. *In Mathematical Models in The Social Sciences*; 1962; 9–23.

Langewish A, Choobineh F. Stochastic dominance tests for ranking alternatives under ambiguity. *European Journal of Operational Research* 1996; 95; 139-154.

Martel J-M, Zaras K. Stochastic dominance in multicriterion analysis under risk. *Theory and Decision* 1995; 39; 31-49.

Mousseau V. Eliciting information concerning the relative importance of criteria. In: Pardalos P, Siskos Y and Zopounidis C (Eds), *Advances in Multicriteria Analysis*, Kluwer Academic: Dordrecht; 1995. p. 17-43.

Roy B. Classement et choix en présence de points de vue multiples (la méthode ELECTRE). *R.I.R.O.* 1968; 2; 57-75.

Roy B. ELECTRE III: Un algorithme de classement fondé sur une représentation floue des préférences en présence de critères multiples. *Cahiers du Centre d'Étude de Recherche Opérationnelle* 1978; 20; 3-24.

Roy B. *Méthodologie Multicritère d'Aide à la Décision*. Economica : Paris; 1985.

Roy B and Mousseau V. A theoretical framework for analyzing the notion of relative importance of criteria. *Journal of Multi-Criteria Decision Analysis* 1996; 5; 145-159.

Roy B, Slowinski R. Criterion of distance between technical programming and socio-economic priority. *R.A.I.R.O. Recherche Opérationnelle* 1993; 27(1); 45-60.

Shafer G. *A Mathematical Theory of Evidence*. Princeton University Press: Princeton, NJ; 1976.

Yu W. Aide multicritère à la décision dans le cadre d'une problématique du tri : concepts, méthodes et applications. Ph.D. Thesis, UER Sciences de l'organisation, Université de Paris-Dauphine, 1992.

Zaras K, Martel J-M 1995. Multiattribute analysis based on stochastic dominance. In : Machina, Meunier B (Eds), *Models and Experiments on Risk and Rationality*, Kluwer Academic Publishers: Dordrecht; 1994. p. 225-248.