Smoothness with respect to noise parameters for parabolic/hyperbolic Anderson model with regular or rough noise in space

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Abstract

In this work, we study the continuity in law of the solutions of two linear multiplicative SPDEs, namely, the parabolic Anderson model (PAM) and the hyperbolic Anderson model (HAM). The forcing term under investigation is examined in two cases: (i) the regular noise, with the spatial covariance given by the Riesz kernel of order $\alpha \in (0, d)$ in spatial dimension $d \geq 1$; (ii) the rough noise, which is a fractional noise in space with Hurst index $H < 1/2$ in spatial dimension $d = 1$. In both cases, the noise is assumed to be colored in time. The similar problem for the case of the white noise in time was considered in [13, 23].

For the initial condition, we consider two scenarios: (a) constant initial condition; (b) initial condition given by a signed Borel measure on $\mathbb{R}^d$. In the case of constant initial condition, we prove that the solution is continuous in law in the space $C([0, T] \times \mathbb{R}^d)$ of continuous functions, with respect to the spatial parameter (α or H) of the noise. In the case of general initial condition (given by a measure), the weak convergence of the solution with respect to spatial parameter of the noise is obtained in the space $C([t_0, T] \times \mathbb{R}^d)$. The solution is understood in Skorohod sense, using Malliavin Calculus, a stochastic calculus of variations theory that proves beneficial in exploring various aspects in the theory of SPDEs.

The results corresponding to cases (a) and (b) above are contained in the recent article [7], and respectively, in the preprint [30].

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Contents

Chapter 1

Introduction

For centuries, partial differential equations have served as a fundamental tool for describing various mathematical and physical phenomena, including heat conduction, fluid dynamics, and diffusion. Examples of basic partial differential equations include the wave equation, heat or diffusion equation, and Poisson's equation. These equations often encounter external influences, which are typically of a random nature. Consequently, incorporating this randomness leads to the formulation of stochastic partial differential equations (SPDEs), offering more robust models for studying these natural processes. SPDEs commonly emerge in applications across diverse fields such as physics, engineering, and finance.

There are three different approaches to study SPDEs in the mathematical literature, namely, the random field approach (see [42]), the infinite-dimensional approach (see [21]) and the analytical approach (see [32]). It turns out that each of these approaches has been fruitful in its own way; however, these three approaches are not equivalent in the sense that they cannot be applied to exactly the same family of SPDEs. In this work, we will focus on the random field approach, following more recent works for example [5, 19]. A gentle introduction to SPDEs using the random field approach is available in [2], providing an overview of fundamental concepts while aiming to engage readers and encourage further exploration of this topic. For a detailed review of SPDE theory, we refer to [20]. Over the past three decades, numerous authors have extensively researched this topic: see [1, 10, 11, 19, 40] for the existence and uniqueness of the solutions to certain equations and noises, [6, 14, 16] for intermittency, [27, 41] for large deviation principles and [22] for small ball probabilities.

When using the random field approach, the general form of a stochastic PDE is:

$$\mathcal{L}u(t,x) = \sigma\big(u(t,x)\big)\dot{W}(t,x) + b\big(u(t,x)\big), t > 0; x \in \mathbb{R}^d \tag{1.0.1}$$

with some initial conditions, where $\mathcal{L}$ is a partial differential operator with constant coefficients on $\mathbb{R}_+ \times \mathbb{R}^d$. It is known from the classical PDE theory that $\mathcal{L}$ has a fundamental solution G (which may be a distribution), i.e. $\mathcal{L}G = 0$. We let $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x)$ be a formal notation for the space-time derivative of the noise W, whose precise definition will be given below. The function b is referred to as the **drift** coefficient, while σ is called the **diffusion** coefficient. Both are subject to certain regularity conditions.

In this thesis, we will consider only Examples 1.0.1 and 1.0.2 below. For these examples, the fundamental solution associated to each operator $\mathcal{L}$, denoted by $G_t(\cdot)$, is a non-negative integrable function on $\mathbb{R}^d$. Let $\triangle = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ be the Laplacian operator. Our main focus will be on two examples:

Example 1.0.1. *The heat operator:* $\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\triangle$, *in any dimension* $d \geq 1$. *In this case,*

$$G_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\Big(- \frac{|x|^2}{2t}\Big), \tag{1.0.2}$$

where $|\cdot|$ *is the Euclidean norm on* $\mathbb{R}^d$.

Example 1.0.2. *The wave operator:* $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \triangle$, *in dimension* $d \leq 2$. *In this case,*

$$\begin{aligned} G_t(x) &= \frac{1}{2} 1_{\{|x|<t\}}, \ \textit{if } d = 1 \\ G_t(x) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \ \textit{if } d = 2. \end{aligned} \tag{1.0.3}$$

The Fourier transform of a function $\varphi \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x) \mathrm{d}x, \ \text{for all } \xi \in \mathbb{R}^d.$$

Here $\xi \cdot x = \sum_{i=1}^d \xi_i x_i$ is the inner product in $\mathbb{R}^d$. Note that in the case of heat operator:

$$\mathcal{F}G_t(\cdot)(\xi) = \exp\Big(- \frac{t|\xi|^2}{2}\Big), \ \text{for all } \xi \in \mathbb{R}, \tag{1.0.4}$$

and in the case of wave operator:

$$\mathcal{F}G_t(\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \ \text{for all } \xi \in \mathbb{R}. \tag{1.0.5}$$

In this work, we consider equation (1.0.1) (with drift term $b \equiv 0$) driven by a linear multiplicative Gaussian noise, i.e. $\sigma(u) \equiv u$. More precisely, we study the equation

$$\mathcal{L}u(t,x) = u(t,x)\dot{W}(t,x), t > 0; x \in \mathbb{R}^d, \tag{1.0.6}$$

where $\mathcal{L}$ is the heat or wave operator. If $\mathcal{L}$ is the heat operator, such equation is known in literature as the *parabolic Anderson model* (PAM), and by analogy, if the heat operator is replaced by the wave operator, it is known as the *hyperbolic Anderson model* (HAM).

The problem studied in this thesis is related to the continuity in law of the solution to equation (1.0.6) with respect to some parameters of the noise. We note that the same problem for the linear equation $\mathcal{L}u(t,x) = \dot{W}(t,x)$ (driven by the same type of noise) was studied in the author's Master's thesis [29].

It is very important to understand what kind of noise is associated to equation (1.0.6). In order to do this, we need to introduce some definitions.

Recall that a *fractional Brownian motion* (fBm) is a zero-mean Gaussian process $(B_t^{(H)})_{t\in\mathbb{R}_+}$ with covariance

$$E[B_t^{(H)}B_s^{(H)}] = R_H(t,s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \text{ for all } s,t \in \mathbb{R}_+.$$

The parameter $H \in (0,1)$ is called the *Hurst index*. If $H = 1/2$, then $R_H(t,s) = t \wedge s$ and the fBm coincides with the Brownian motion. If $H > 1/2$, the covariance of the fBm can be expressed as follows:

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v,$$

where

$$\alpha_H = H(2H-1). \tag{1.0.7}$$

For any $H \in (0,1)$, this covariance admits the spectral representation:

$$R_H(t,s) = c_H \int_{\mathbb{R}} \mathcal{F}1_{[0,t]}(\xi)\overline{\mathcal{F}1_{[0,s]}(\xi)}|\xi|^{1-2H}\mathrm{d}\xi,$$

where $\mathcal{F}1_{[0,t]}(\xi)$ is the Fourier transform of the indicator function $1_{[0,t]}$, given by:

$$\mathcal{F}1_{[0,t]}(\xi) = \int_0^t e^{-i\xi s}ds = \frac{1-e^{-i\xi t}}{i\xi} \quad \text{for all } \xi \in \mathbb{R},$$

and

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{1.0.8}$$

Throughout the thesis, we will use these definitions of the constants α_H and c_H.

A *fractional Brownian sheet* (fBs) with Hurst indices $H_0, H \in (0,1)$ is a zero-mean Gaussian process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ with covariance

$$E[W(t,x)W(s,y)] = R_{H_0}(t,s)R_H(x,y).$$

Coming back to (1.0.6), we assume that $\dot{W}$ is a zero-mean Gaussian process, whose covariance is given informally by:

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \gamma_0(t-s)\gamma(x-y).$$

The smoother γ, the more regular the noise and the solutions are. We can make the definition more precise. Let $W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a zero-mean Gaussian process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with covariance:

$$\begin{aligned}\mathbb{E}[W(\varphi)W(\psi)] &= \int_{\mathbb{R}_+^2}\int_{(\mathbb{R}^d)^2} \gamma_0(t-s)\gamma(x-y)\varphi(t,x)\psi(s,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} \gamma_0(t-s)\mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}\mu(\mathrm{d}\xi)\mathrm{d}t\mathrm{d}s =: \langle \varphi, \psi\rangle_{\mathcal{H}},\end{aligned}$$

for any $\varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$. Here $\mathcal{D}$ is the space of all infinitely differentiable functions whose compact support is contained in $\mathbb{R}_+ \times \mathbb{R}^d$ and $\mathcal{H}$ be a Hilbert space, obtained by completing $\mathcal{D}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

In this thesis, we will focus on two types of noise processes:

(I) the regular noise, with the spatial covariance given by the Riesz kernel of order $\alpha \in (0, d)$ in spatial dimension $d \geq 1$.

(II) the rough noise, which is a fractional noise in space with Hurst index $H < 1/2$ and $d = 1$.

More precisely, the spectral measure of the noise is given by:

$$\mu(d\xi) = \begin{cases} |\xi|^{-\alpha} d\xi & \text{with } \alpha \in (0,d) \text{ and } d \geq 1 \ (\textit{Case I: the regular case}) \\ c_H |\xi|^{1-2H} d\xi & \text{with } H \in (0,1/2) \text{ and } d = 1 \ (\textit{Case II: the rough case}) \end{cases}$$

In Case II, we assume that the constant c_H is given by (1.0.8). In both cases, the noise is assumed to be colored in time. The term "colored noise" refers to a type of stochastic process where the noise is not white, meaning it has some form of correlation in time, space, or both. The difference between these two noises is that: the spatial covariance of the regular noise is captured by the Riesz kernel, whereas for the rough noise, the covariance structure does not take the simple form of a Riesz kernel, the only way to express the covariance structure is using the Fourier transform. To emphasize the dependence on the parameter α or H, we denote the noise by W^α and W^H respectively. More details about the noise can be found in Section 2.1 and Section 4.1. See [3, 9, 28] for a sample of relevant references about the study of the regular noise and [4, 6] regarding the work in the case of the rough noise.

Just like ordinary differential equations (ODEs) and deterministic partial differential equations (PDEs), stochastic PDEs require initial conditions, which are also crucial in defining solutions, to determine the starting point of the stochastic process. We equip equation (1.0.6) with two different initial conditions: the constant initial condition and the general initial condition. More precisely,

$$u(0,\cdot) = u_0(\cdot) = \begin{cases} 1 & \text{type (a)} \\ \text{a signed Borel measure} & \text{type (b)} \end{cases}$$

Initial conditions of type (b) are called *rough*. So far, only the PAM with rough initial condition has been studied in the literature; see [3, 4, 14, 30].

We continue now with the definition of the solution. We say that a process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a *(mild) solution* to equation (1.0.6), if for any $t > 0$, $x \in \mathbb{R}^d$ with probability 1,

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)u(s,y)W(\delta s, \delta y), \tag{1.0.9}$$

where $G = G^h$ for the heat equation, respectively $G = G^w$ for the wave equation, is the fundamental solution of the heat and wave operator. The solution is understood in the *Skorohod* sense. This means that the stochastic integral in (1.0.9) is the divergence operator from Malliavin calculus. We refer the reader to [38] for mare details regarding Malliavin calculus. See also Appendix A.5.

Recalling Cases I and II mentioned above, we denote by u^α or u^H the solution to equation (1.0.6) to emphasize the dependence on the parameter α and H respectively. We use the same notation for both heat and wave equations, but we will specify each time when we need to consider separately each equation.

In the case of the heat equation, w is the solution of the deterministic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2} \triangle u(t,x), \quad t > 0, x \in \mathbb{R}^d \\ u(0,\cdot) = u_0, \end{cases}$$

and is given by $w(t,x) = (G_t * u_0)(x)$. Note that if $u_0 \equiv 1$, then $w \equiv 1$.

In the case of the wave equation, w is the solution of the deterministic equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \triangle u(t,x), \quad t > 0, x \in \mathbb{R}^d \ (d \leq 2) \\ u(0,\cdot) = u_0 \\ \dfrac{\partial u}{\partial t}(0,\cdot) = v_0 \end{cases}$$

and is given by $w(t,x) = (G_t * v_0)(x) + \frac{\partial u}{\partial t}(G_t * u_0)(x)$. In the present work, we consider only the case $u_0 \equiv 1$ and $v_0 \equiv 0$, so $w \equiv 1$.

The main results in this thesis give the continuity in law, with respect to the noise parameter α or H, of the solutions to SPDE (1.0.6) driven by the regular noise (Case I) or the rough noise (Case II), in the setting of the constant initial condition (type (a)) or the rough initial condition (type(b)). That is, we aim to show that

$$\begin{cases} u^{\alpha_n} \xrightarrow{d} u^{\alpha^*}, \text{whenever } \alpha_n \to \alpha^* & \text{Case I: the regular noise} \\ u^{H_n} \xrightarrow{d} u^{H^*}, \text{whenever } H_n \to H^* & \text{Case II: the rough noise} \end{cases}$$

In the case of the white noise in time (i.e. when $H_0 = 1/2$), the same problem of continuity in law of the solution was studied in [23, 24] for the linear equation, respectively the equation with multiplicative noise. In the case of the linear equation, the methods of [23] were extended to the colored noise in time (case $H_0 > 1/2$) in the author's Master thesis [29]. The same problem of the weak convergence of solutions has been explored in various related contexts; see [34, 35, 36, 37].

The proof of the main results will follow from the classical method of convergence of finite dimensional distributions (f.d.d) and tightness. For tightness, we will build upon the existing literature related to moment estimates for the increments of the solution, and show that these estimates hold uniformly in α or H, which allow us to apply Kolmogorov's theorem. More precisely, for any $(t,x) \in [0,T] \times \mathbb{R}^d$ (for any $(t',x') \in [t_0,T] \times \mathbb{R}^d$ when rough initial condition imposed), we will show that for any $p \geq 2$,

$$\begin{cases} \sup_{\alpha\in[a,b]} \mathbb{E}\big|u^\alpha(t',x') - u^\alpha(t,x)\big|^p \leq C\big(|t'-t| + |x'-x|\big)^{p\varepsilon/2} & \text{for heat equation,} \\ \sup_{\alpha\in[a,b]} \mathbb{E}\big|u^\alpha(t',x') - u^\alpha(t,x)\big|^p \leq C\big(|t'-t| + |x'-x|\big)^{p\varepsilon} & \text{for wave equation,} \end{cases}$$

while in case of the rough noise, we will prove that

$$\begin{cases} \sup_{H\in[a,b]} \mathbb{E}\big|u^H(t',x') - u^H(t,x)\big|^p \leq C\big(|t'-t| + |x'-x|\big)^{p\varepsilon/2} & \text{for heat equation,} \\ \sup_{H\in[a,b]} \mathbb{E}\big|u^H(t',x') - u^H(t,x)\big|^p \leq C\big(|t'-t| + |x'-x|\big)^{p\varepsilon} & \text{for wave equation.} \end{cases}$$

The range of the exponent $\varepsilon > 0$ appearing on the right-hand side of the above inequalities will be specified in each corresponding section.

For the finite dimensional distribution convergence, the main problem comes from the fact that, unlike the case of the linear equation, the solution is not a Gaussian process, and so identifying its covariance structure is not enough to characterize its law. We will prove that for any $k \geq 1$ and $(t_1, x_1), \dots, (t_k, x_k) \in [0, T] \times \mathbb{R}^d$, in the case of regular noise,

$$\big(u^{\alpha_n}(t_1, x_1), \dots, u^{\alpha_n}(t_k, x_k)\big) \xrightarrow{\mathrm{d}} \big(u^{\alpha^*}(t_1, x_1), \dots, u^{\alpha^*}(t_k, x_k)\big), \text{ as } n \to \infty,$$

and in the case of rough noise,

$$\big(u^{H_n}(t_1, x_1), \dots, u^{H_n}(t_k, x_k)\big) \xrightarrow{\mathrm{d}} \big(u^{H^*}(t_1, x_1), \dots, u^{H^*}(t_k, x_k)\big), \text{ as } n \to \infty.$$

It is worth mentioning that for establishing the f.d.d. convergence, the noise W^α, respectively W^H, does not have to be defined on the same probability space for all values α (or H). However, instead of f.d.d, we will prove the convergence in $L^2(\Omega)$, and for this, we need that all processes $(W^\alpha)_{\alpha \in (0,d)}$ (respectively, $(W^H)_{H \in (1/2,1)}$) be defined on the same probability space. To achieve this, we will use the following spectral representation of the noise:

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t, x) W^\alpha(\mathrm{d}t, \mathrm{d}x) = \int_{\mathbb{R} \times \mathbb{R}^d} \mathcal{F}\varphi(\tau, \xi) \sqrt{g_0(\tau)} |\xi|^{-\alpha/2} \widehat{W}(\mathrm{d}\tau, \mathrm{d}\xi)$$

and

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t, x) W^H(\mathrm{d}t, \mathrm{d}x) = \sqrt{c_H} \int_{\mathbb{R} \times \mathbb{R}^d} \mathcal{F}\varphi(\tau, \xi) \sqrt{g_0(\tau)} |\xi|^{\frac{1}{2} - H} \widehat{W}(\mathrm{d}\tau, \mathrm{d}\xi)$$

where g_0 is the density of the temporal spectral measure μ_0, $\mathcal{F}\varphi(\tau, \xi)$ is the Fourier transform of φ in both variables (t, x) and $\widehat{W}$ is a $\mathbb{C}$-valued Gaussian random measure. This construction of the noise is critical for proving the f.d.d. convergence (See Sections 2.4 and 4.3).

The following table summarizes the results presented in the thesis:

	Constant initial condition	**General initial condition**
Regular noise $\mu(\mathrm{d}\xi) = \lvert\xi\rvert^{-\alpha} d\xi$, $\alpha \in (0, d)$	Chapter 2 (Section 2 of [7])	Chapter 3 (Section 2 of [30])
Rough noise $\mu(\mathrm{d}\xi) = c_H \lvert\xi\rvert^{1-2H} d\xi$, $H \in (0, \frac{1}{2})$	Chapter 4 (Section 3 of [7])	Chapter 5 (Section 3 of [30])

We provide a brief overview of the steps required to demonstrate the main results, taking into account various noises and initial conditions. We first consider the space-time homogeneous noise W^α (the regular noise) to be the forcing term. In Chapter 2, we deal with PAM and HAM, equipped with constant initial condition. We introduce the noise in Section 2.1, which is specified by a general temporal covariance function γ_0 and a spatial covariance function γ given by the Riesz kernel with index α. In

Section 2.2, we prove the existence of the solution. In addition, we show that the p-th moment of the solution is bounded by an exponential function of t. We consider first the general temporal covariance function γ_0 and then we include one particular case when the temporal covariance of the noise is given by $\gamma_0(t) = \alpha_{H_0}|t|^{2H_0-2}$ with $H_0 \in (1/2, 1)$. The estimates for the moments of the increments of the solution are developed in Section 2.3 and are used to establish tightness. In Section 2.4, we prove the f.d.d. convergence, followed by the main result, which state that $u^{\alpha_n} \xrightarrow{d} u^{\alpha^*}$ in $C([0,T] \times \mathbb{R}^d)$.

In Chapter 3, we study PAM equipped with rough initial condition. The noise is the same as in Section 2.1, and has a general temporal covariance function γ_0. The existence of the solution and the fact that p-th moment of the solution is bounded by an exponential function of t are obtained in Section 3.1. We give some upper bounds for the moments of the increments of the solution in Section 3.2. The proofs of the f.d.d. convergence and the continuity in the law of the solution with respect to the noise parameter α is addressed in Section 3.3.

In Chapter 4, we focus on the rough noise W^H. This noise, which is introduced in Section 4.1, behaves in time like fBm with index $H_0 > 1/2$ and in space like the fBm with index $H < 1/2$. We treat the heat and wave equations separately. The existence of the solution is obtained under the condition $H_0 + H > 3/4$ for PAM, respectively $H > 1/4$ for HAM. (For HAM, we believe that this condition is not optimal since it does not involve the Hurst index H_0.) In Section 4.2, we obtain some estimates for the moments of the increments of the solution. The f.d.d. convergence and the continuity in law of the solution are proved in Section 4.3.

In Chapter 5, we return to PAM with rough initial condition, this time in the case when the noise is rough in space. The existence of the solution is obtained in Section 5.1 under the condition $H_0 + H > 3/4$. Tightness is established through the uniform moment estimates provided in Section 5.2, while Section 5.3 demonstrates the f.d.d. convergence and the continuity in law of the solution.

The Appendix covers some auxiliary results which are used in the thesis.

We specify the notation used in this thesis. We let $\mathcal{B}(\mathbb{R}^d)$ be the class of Borel sets of $\mathbb{R}^d$ and $\mathcal{B}_b(\mathbb{R}^d)$ be the class of bounded Borel sets of $\mathbb{R}^d$. We let $\mathcal{D}(\mathbb{R}^d)$ be the space of all infinitely differentiable functions whose support is compact and contained in $\mathbb{R}^d$ and $\mathcal{D}'(\mathbb{R}^d)$ be the space of distributions, i.e. the space of linear continuous functionals on $\mathcal{D}(\mathbb{R}^d)$. We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of rapidly decreasing functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, i.e. infinitely differentiable functions such that $\sup_{x\in\mathbb{R}^d} \left|x^\alpha D^{|\beta|} f(x)\right| < \infty$, for all $\alpha_i, \beta_i \in \mathbb{N}_0$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ are multi-indices and we define $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $D^{|\beta|} f(x) = \frac{\partial^{|\beta|} f(x)}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}$. We let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions, i.e. the space of linear continuous functionals on $\mathcal{S}(\mathbb{R}^d)$. Throughout this thesis, we denote $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n)$, $\mathbf{t} = (t_1, \cdots, t_n)$,

$\mathbf{s} = (s_1, \cdots, s_n)$ and we let $T_n(t) = \{(t_1, \ldots, t_n); 0 < t_1 < \ldots < t_n < t\}$. We use $\xi \cdot \mathbf{x} = \sum_{j=1}^{d} \xi_j x_j$ to denote the inner product in $\mathbb{R}^d$.

Chapter 2

PAM/HAM with regular noise and constant initial condition

In this chapter, we consider the Parabolic Anderson Model (PAM):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x \in \mathbb{R}^d \ (d \geq 1) \\ u(0,x) = 1, \quad x \in \mathbb{R}^d \end{cases} \tag{2.0.1}$$

and the Hyperbolic Anderson Model (HAM):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x \in \mathbb{R}^d \ (d \leq 2) \\ u(0,x) = 1, \quad x \in \mathbb{R}^d \\ \dfrac{\partial u}{\partial t}(0,x) = 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{2.0.2}$$

We assume that $\dot{W}$ is a space-time homogeneous Gaussian noise, which is correlated in both time and space with arbitrary temporal covariance kernel γ_0 and spatial covariance γ given by the Riesz kernel:

$$\gamma(x) = C_{d,\alpha}|x|^{-(d-\alpha)}, \text{ for } \alpha \in (0,d), \tag{2.0.3}$$

where

$$C_{d,\alpha} = \pi^{d/2} 2^{\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}. \tag{2.0.4}$$

For kernel γ_0, we need to introduce the following assumption, and we will assume it holds throughout the thesis.

Assumption A. *We assume that*

$$\mu_0(\mathrm{d}\tau) = g_0(\tau)\mathrm{d}\tau,$$

where g_0 is a non-negative function.

The precise definition of the noise and some of its properties are given in Section 2.1 below. The stochastic integral used for defining the solution is the Skorohod integral from using Malliavin calculus. For this reason, we say that this is a **Skorohod** solution. In Section 2.2, we give the definition of the solution and we review the classical results about the existence and uniqueness of the solution. More precisely, we show that for any $\alpha \in (\max(d-2,0),d)$, equations (2.0.1) and (2.0.2) have unique solutions.

The goal of this chapter is to study the continuity in law of the solution with respect to the parameter α of the noise (for fixed temporal covariance kernel γ_0), in the space of continuous functions on $C([0,T]\times\mathbb{R}^d)$. For this, we will apply the classical method of convergence of finite dimensional distributions (f.d.d) and tightness. For tightness, we will build upon the existing literature related to moment estimates for the increments of the solution, and show that these estimates hold uniformly in α. This will be achieved in Section 2.3. In particular, this will show that the solution has a modification with continuous sample paths in space and time. We will work with this modification. Finally, the f.d.d. convergence is shown in Section 2.4.

The main result for this chapter is the following theorem.

Theorem 2.0.1. *For any $\alpha \in (\max(d-2,0),d)$, let u^α be the solution of equation* (2.0.1) *or* (2.0.2). *Fix $\alpha^* \in (\max(d-2,0),d)$ and let $(\alpha_n)_{n\geq 1}$ be a sequence in $(\max(d-2,0),d)$ such that*

$$\lim_{n\to\infty}\alpha_n = \alpha^*.$$

Then for any $T>0$,

$$u^{\alpha_n} \xrightarrow{d} u^{\alpha^*} \text{ in } C([0,T]\times\mathbb{R}^d).$$

The main results of Chapters 2 and 4 (Theorems 2.0.1 and 4.0.1) have been published in the recent article [7].

2.1 The noise

In this section, we introduce the random noise perturbing equations (2.0.1) and (2.0.2). The noise has been considered by many authors [3, 8, 9]. In particular, the case of the white noise in time was introduced by Dalang in the seminal articles

[18] and [19]. The covariance structure of the noise W is specified by two locally integrable non-negative definite functions $\gamma_0 : \mathbb{R}_+ \to [0, \infty]$ in time and $\gamma : \mathbb{R}^d \to [0, \infty]$ in space. By non-negative definite, we mean that for any test function $\phi \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} (\phi * \widetilde{\phi})(t)\gamma_0(t)dt \geq 0$$

where $\widetilde{\phi}(t) = \phi(-t)$ denotes the reflection of ϕ. Moreover, by the Bochner-Schwartz theorem, γ is the Fourier transform (in $\mathcal{S}'(\mathbb{R}^d)$) of the tempered measure μ, written as $\gamma = \mathcal{F}\mu$, i.e.

$$\int_{\mathbb{R}^d} \varphi(x)\gamma(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(\mathrm{d}\xi), \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.1.1}$$

Similarly, the temporal covariance kernel γ_0 is the Fourier transform of a tempered measure μ_0 on $\mathbb{R}_+$, i.e. $\gamma_0 = \mathcal{F}\mu_0$ in $\mathcal{S}'(\mathbb{R})$. Note that for any functions $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \gamma_0(t-s)\phi_1(t)\phi_2(s)\mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}} \mathcal{F}\phi_1(\tau)\overline{\mathcal{F}\phi_2(\tau)}\mu_0(\mathrm{d}\tau).$$

More generally,

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \prod_{j=1}^{n} \gamma_0(t_j - s_j)\phi_1(t_1, \dots, t_n)\phi_2(s_1, \dots, s_n)\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}$$
$$= \int_{\mathbb{R}^n} \mathcal{F}\phi_1(\tau_1, \dots, \tau_n)\overline{\mathcal{F}\phi_2(\tau_1, \dots, \tau_n)}\mu_0(\mathrm{d}\boldsymbol{\tau}), \tag{2.1.2}$$

where $\mathbf{t} = (t_1, \dots, t_n)$, $\mathbf{s} = (s_1, \dots, s_n)$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$.

We assume that $W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is a zero-mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance:

$$\begin{aligned}
\mathbb{E}[W(\varphi)W(\psi)] &= \int_{\mathbb{R}_+^2} \int_{(\mathbb{R}^d)^2} \gamma_0(t-s)\gamma(x-y)\varphi(t,x)\psi(s,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \gamma_0(t-s)\mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}\mu(\mathrm{d}\xi)\mathrm{d}t\mathrm{d}s \\
&= \int_{\mathbb{R}\times\mathbb{R}^d} \mathcal{F}\varphi(\tau,\xi)\overline{\mathcal{F}\psi(\tau,\xi)}\mu_0(\mathrm{d}\tau)\mu(\mathrm{d}\xi) =: \langle \varphi, \psi \rangle_{\mathcal{H}},
\end{aligned}$$

for any $\varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$, where we denote by

$$\mathcal{F}\varphi(\tau,\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{-i\tau t}e^{-i\xi\cdot x}\varphi(t,x)\mathrm{d}x\mathrm{d}t,$$

the Fourier transform of function φ in both time and space variables. Some references use the terminology "colored in time" and "homogeneous in space" for the noise W. Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Throughout this chapter, we consider the noise W with general temporal covariance function γ_0 satisfying Assumption A and spatial covariance function is given by the Riesz kernel (2.0.3). In this case, it can be proved that (see [39]):

$$\mu(\mathrm{d}\xi) = |\xi|^{-\alpha}\mathrm{d}\xi.$$

To emphasize the dependence on the parameter α, we denote by $W^\alpha = \{W^\alpha(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$, a centered Gaussian process with covariance structure $\mathbb{E}[W^\alpha(\varphi)W^\alpha(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}_\alpha}$, where

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}_\alpha} &= C_{d,\alpha} \int_{\mathbb{R}_+^2} \int_{(\mathbb{R}^d)^2} \gamma_0(t-s)|x-y|^{-(d-\alpha)}\varphi(t,x)\psi(s,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \gamma_0(t-s)\mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|\xi|^{-\alpha}\mathrm{d}\xi\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\tau,\xi)\overline{\mathcal{F}\psi(\tau,\xi)}g_0(\tau)|\xi|^{-\alpha}\mathrm{d}\xi\mathrm{d}\tau, \end{aligned} \tag{2.1.3}$$

for any $\varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$.

One particular example that we are interested in is when the temporal covariance function $\gamma_0(t) = \alpha_{H_0}|t|^{2H_0-2}$ with $H_0 \in (1/2, 1)$. In this case, we say that the noise W^α is fractional in time with index $H_0 > 1/2$ and its covariance is given by:

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}_\alpha} &= \alpha_{H_0} C_{d,\alpha} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x)\psi(s,y)|t-s|^{2H_0-2}|x-y|^{-(d-\alpha)}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s \\ &= \alpha_{H_0} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|t-s|^{2H_0-2}|\xi|^{-\alpha}\mathrm{d}\xi\mathrm{d}t\mathrm{d}s, \end{aligned} \tag{2.1.4}$$

where $\alpha_{H_0} = H_0(2H_0 - 1)$. The motivation for this terminology comes from the fact that the fractional Brownian motion (fBm) with index $H \in (1/2, 1)$ is a zero-mean Gaussian process $(B_t^{(H)})_{t \geq 0}$ with covariance:

$$\mathbb{E}[B_t^{(H)} B_s^{(H)}] = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v.$$

In particular, if $d = 1$ and $\alpha = 2 - 2H$ for $H \in (1/2, 1)$, then the noise is also fractional in space.

2.2 Existence of solution

In this section, we consider equations (2.0.1) and (2.0.2) driven by Gaussian noise with covariance (2.1.3). The goal of this section is to prove that the solutions to (2.0.1) and (2.0.2) exist and to show that the p-th moment of these solutions are bounded by an exponential function of t. At the end of this section, we consider separately the case of the Gaussian noise with covariance (2.1.4), and we obtain more precise bounds for the moments of the solution, using an exponential function of t^ρ, for some $\rho > 0$. The results from this section are taken from [8, 9]. We include the details for the sake of completeness, but also because some of the intermediate steps will be needed in the subsequent sections.

Before introducing the definition of the solution, we need to recall some basic facts from Malliavin calculus. See Appendix A.5. We refer the reader to [38] for more details.

It is known that any square-integrable random variable X which is measurable with respect to the σ-field generated by W^α has the *Wiener-chaos expansion*:

$$X = \sum_{n \geq 0} I_n^\alpha(f_n)$$

for some $f_n \in \mathcal{H}_\alpha^{\otimes n}$, where $\mathcal{H}_\alpha^{\otimes n}$ is the n-th tensor product of $\mathcal{H}_\alpha$ and I_n^α is the multiple integral of order n with respect to W^α. Here $I_0^\alpha : \mathbb{R} \to \mathbb{R}$ is the identity map and $f_0 = \mathbb{E}(X)$. The terms in this series are orthogonal in $L^2(\Omega)$. If $\{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a random field such that each variable $X(t,x)$ has the chaos expansion:

$$X(t,x) = \sum_{n \geq 0} I_n^\alpha(f_n(\cdot, t, x))$$

then under suitable conditions, the Skorohod integral of X with respect to W^α is defined by:

$$\delta(X) := \int_0^\infty \int_{\mathbb{R}^d} X(t,x) W(\delta s, \delta y) = \sum_{n \geq 0} I_{n+1}^\alpha(f_n),$$

provided that the series on the right-hand side converges in $L^2(\Omega)$. This integral is used in the following definition.

Definition 2.2.1. *We say that a process $u^\alpha = \{u^\alpha(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a solution (in Skorohod sense) to equation* (2.0.1)*, respectively equation* (2.0.2)*, if for any $t > 0$, $x \in \mathbb{R}^d$ with probability 1,*

$$u^\alpha(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) u(s,y) W^\alpha(\delta s, \delta y), \tag{2.2.1}$$

where $G = G^h$ for equation (2.0.1), respectively $G = G^w$ for equation (2.0.2), is the fundamental solution of the heat and wave operator. The stochastic integral in (2.2.1) is interpreted in the Skorohod sense.

It was proved in [26] that if it exists, then the solution is unique and has the series expansion:

$$\begin{aligned} u^\alpha(t,x) &= 1 + \sum_{n\geq 1} \int_0^t \int_{\mathbb{R}^d} \int_0^{t_n} \int_{\mathbb{R}^d} \cdots \int_0^{t_2} \int_{\mathbb{R}^d} G_{t-t_n}(x-x_n) \cdots G_{t_2-t_1}(x_2-x_1) \\ &\qquad\qquad W^\alpha(\mathrm{d}t_1, \mathrm{d}x_1) \cdots W^\alpha(\mathrm{d}t_n, \mathrm{d}x_n) \\ &= 1 + \sum_{n\geq 1} I_n^\alpha(f_n(\cdot,t,x)) = \sum_{n\geq 0} I_n^\alpha(f_n(\cdot,t,x)), \end{aligned} \tag{2.2.2}$$

where the kernel $f_n(\cdot,t,x) \in \mathcal{H}_\alpha^{\otimes n}$ is given by

$$f_n(t_1,x_1,\cdots,t_n,x_n,t,x) = G_{t-t_n}(x-x_n) \cdots G_{t_2-t_1}(x_2-x_1) 1_{\{0<t_1<\cdots<t_n<t\}}, \tag{2.2.3}$$

and $I_n^\alpha(f_n(\cdot,t,x))$ is the multiple Wiener integral of order n with respect to W^α. We let $f_0(\cdot,t,x) = 1$. Observe that the second term of this series is

$$I_1^\alpha(f_1(\cdot,t,x)) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) W^\alpha(\mathrm{d}s, \mathrm{d}y),$$

which is the solution of the linear equation $\mathcal{L}u^\alpha = \dot{W}^\alpha$, where $\mathcal{L}$ is the heat or wave operator. This equation has been studied in [10, 11].

The solution exists if and only if the series $\sum_{n\geq 0} I_n^\alpha(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$, i.e.

$$\sum_{n\geq 0} \mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2 < \infty.$$

In this case,

$$\mathbb{E}|u^\alpha(t,x)|^2 = 1 + \sum_{n\geq 1} n! \|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} = \sum_{n\geq 0} n! \|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}},$$

where $\tilde{f}$ denotes the symmetrization of function f, given by

$$\tilde{f}(t_1,x_1,\ldots,t_n,x_n) = \frac{1}{n!} \sum_{\rho\in S_n} f(t_{\rho(1)}, x_{\rho(1)}, \ldots, t_{\rho(n}, x_{\rho(n)}),$$

the norm is given by

$$\|f\|^2_{\mathcal{H}_\alpha^{\otimes n}} = \int_{[0,t]^n} \int_{[0,t]^n} \prod_{i=1}^n \gamma_0(t_i - s_i) \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n \gamma(x_i - y_i)$$

$$f_n(t_1,x_1,\cdots,t_n,x_n,t,x)f_n(s_1,y_1,\cdots,s_n,y_n,t,x)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}$$

and S_n is the set of all permutations of $\{1,2,\ldots,n\}$.

After this brief introduction, we include below some preliminary results about the Fourier transform of the kernel $f_n(\cdot,t,x)$ that are needed in the sequel.

Lemma 2.2.2. *For any $0<t_1<\ldots<t_n<t$, the function*

$$(x_1,\ldots,x_n)\mapsto f_n(t_1,x_1,\ldots,t_n,x_n,t,x)$$

has the following Fourier transform:

$$\begin{aligned}\mathcal{F}f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)\\ = e^{-i(\xi_1+\cdots+\xi_n)\cdot x}\overline{\mathcal{F}G_{t_2-t_1}(\xi_1)}\cdots\overline{\mathcal{F}G_{t-t_n}(\xi_1+\cdots+\xi_n)}.\end{aligned} \tag{2.2.4}$$

Similarly to (2.1.2), we have: for any $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}\int_{(\mathbb{R}^d)^n}\int_{(\mathbb{R}^d)^n}\prod_{j=1}^n\gamma(x_i-y_i)\varphi(x_1,\cdots,x_n)\psi(y_1,\cdots,y_n)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\ =\int_{(\mathbb{R}^d)^n}\mathcal{F}\varphi(\xi_1,\cdots,\xi_n)\overline{\mathcal{F}\psi(\xi_1,\cdots,\xi_n)}\mu(\mathrm{d}\xi_1)\cdots\mu(\mathrm{d}\xi_n),\end{aligned} \tag{2.2.5}$$

where $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{y}=(y_1,\ldots,y_n)$. In our arguments below, we will use the following inequality whose proof is based on the Cauchy Schwarz inequality. This inequality has the advantage that it is valid for a general function γ_0, but the disadvantage that does not provide optimal bounds. In the particular case when $\gamma_0(t)=\alpha_{H_0}|t|^{2H_0-2}$, we will use a different inequality (Lemma 2.2.5 below) which provides sharper results.

Lemma 2.2.3 (Lemma 4.3 of [9])**.** *For any $n\geq 1$ and for any function $h:[0,t]^n\to\mathbb{R}$ which is either non-negative or integrable,*

$$\int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n\gamma_0(t_j-s_j)h(t_1,\ldots,t_n)\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}\leq\Gamma_{0,t}^n\int_{[0,t]^n}|h(t_1,\ldots,t_n)|\mathrm{d}\mathbf{t},$$

where $\Gamma_{0,t}=\int_{-t}^t\gamma_0(s)\mathrm{d}s=2\int_0^t\gamma_0(s)\mathrm{d}s$.

We are now in the position to present the first existence result. We note that the exponent of t in relation (2.2.6) is the same as in the case of the white noise in time (i.e. when $\gamma_0=\delta_0$). This is not optimal, since this exponent should contain some information about the temporal covariance of the noise.

Theorem 2.2.4. *Let W^α be the Gaussian noise with covariance* (2.1.3) *for some arbitrary function $g_0(\tau)$. Then for any $\alpha \in (\max\{d-2,0\}, d)$, equations* (2.0.1) *and* (2.0.2) *have unique solutions. Moreover, for any $p \geq 2$,*

$$\mathbb{E}|u^\alpha(t,x)|^p \leq C^{(1)} \exp\left(C^{(2)}\, p^k\, t\right), \tag{2.2.6}$$

where $C^{(1)} > 0$ and $C^{(2)} > 0$ are some constants depending on d and α, and

$$k = \begin{cases} k^h = \frac{4-(d-\alpha)}{2-(d-\alpha)} & \text{for heat equation,} \\ k^w = \frac{4-(d-\alpha)}{3-(d-\alpha)} & \text{for wave equation.} \end{cases} \tag{2.2.7}$$

Proof: We start by proving the existence of solution. Using (2.2.5), we have

$$\begin{aligned} &\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} \\ &= \int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)\int_{(\mathbb{R}^d)^n}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n \gamma(x_i-y_i)\tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t,x) \\ &\qquad \tilde{f}_n(s_1,y_1,\cdots,s_n,y_n,t,x)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &= \int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)A_n(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \end{aligned} \tag{2.2.8}$$

where

$$A_n(\mathbf{t},\mathbf{s}) = \int_{(\mathbb{R}^d)^n}\mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\cdots,\xi_n) \overline{\mathcal{F}\tilde{f}_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)}\mu(\mathrm{d}\xi_1)\cdots\mu(\mathrm{d}\xi_n).$$

Using Cauchy-Schwarz inequality, the fact that $ab \leq \frac{1}{2}(a^2+b^2)$ and Lemma 2.2.3, we have

$$\begin{aligned} \|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} &\leq \int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)A_n^{1/2}(\mathbf{t},\mathbf{t})A_n^{1/2}(\mathbf{s},\mathbf{s})\,\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq \frac{1}{2}\int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)\Big(A_n(\mathbf{t},\mathbf{t})+A_n(\mathbf{s},\mathbf{s})\Big)\,\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &= \int_{[0,t]^n}\Big(\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)A_n(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{s}\Big)\mathrm{d}\mathbf{t} \\ &\leq \Gamma_{0,t}^n\int_{[0,t]^n}A_n(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} = \Gamma_{0,t}^n\Big(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}A_n(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\Big) \end{aligned} \tag{2.2.9}$$

where S_n is the set of all permutation of $\{1, \cdots, n\}$ and in the last equation, we decomposed the set $[0,t]^n$ into $n!$ disjoint regions of the form $t_{\rho(1)} < \cdots < t_{\rho(n)}$ with $\rho \in S_n$. We study $A_n(\mathbf{t},\mathbf{t})$ first. Note that

$$A_n(\mathbf{t},\mathbf{t}) = \int_{(\mathbb{R}^d)^n} \left|\mathcal{F}\tilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n)\right|^2 \mu(\mathrm{d}\xi_1) \cdots \mu(\mathrm{d}\xi_n). \quad (2.2.10)$$

Fix $(t_1, \dots, t_n) \in [0,t]^n$. Let $\rho \in S_n$ be such that $0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t$, by Lemma 2.2.2, we have

$$\begin{aligned}
&\mathcal{F}\tilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) \\
&= \int_{(\mathbb{R}^d)^n} e^{-i(\xi_1 \cdot x_1 + \cdots + \xi_n \cdot x_n)} \tilde{f}_n(t_1, x_1, \cdots, t_n, x_n, t, x) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \frac{1}{n!} \int_{(\mathbb{R}^d)^n} e^{-i(\xi_1 \cdot x_1 + \cdots + \xi_n \cdot x_n)} f_n(t_{\rho(1)}, x_{\rho(1)}, \cdots, t_{\rho(n)}, x_{\rho(n)}, t, x) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \frac{1}{n!} \mathcal{F} f_n(t_{\rho(1)}, \cdot, \dots, t_{\rho(n)}, \cdot, t, x)(\xi_{\rho(1)}, \cdots, \xi_{\rho(n)}) \\
&= \frac{1}{n!} e^{-i(\xi_{\rho(1)} + \cdots + \xi_{\rho(n)}) \cdot \mathbf{x}} \prod_{j=1}^n \mathcal{F} G_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}) \qquad (2.2.11)
\end{aligned}$$

and therefore,

$$\left|\mathcal{F}\tilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n)\right|^2 = \frac{1}{(n!)^2} \prod_{j=1}^n \left|\mathcal{F} G_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)})\right|^2. \quad (2.2.12)$$

Coming back to relation (2.2.10) and using equation (2.2.12) and the change of variables $\xi'_j = \xi_{\rho(j)}$ for all $j = 1, \dots, n$, we obtain

$$\begin{aligned}
A_n(\mathbf{t},\mathbf{t}) &= \frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n \left|\mathcal{F} G_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)})\right|^2 \mu(\mathrm{d}\xi_{\rho(1)}) \cdots \mu(\mathrm{d}\xi_{\rho(n)}) \\
&= \frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n \left|\mathcal{F} G_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi'_1 + \cdots + \xi'_j)\right|^2 \mu(\mathrm{d}\xi'_1) \cdots \mu(\mathrm{d}\xi'_n) \\
&= \frac{1}{(n!)^2} \int_{\mathbb{R}^d} \left|\mathcal{F} G_{t_{\rho(2)} - t_{\rho(1)}}(\xi_1)\right|^2 \bigg(\int_{\mathbb{R}^d} \left|\mathcal{F} G_{t_{\rho(3)} - t_{\rho(2)}}(\xi_1 + \xi_2)\right|^2 \\
&\quad \cdots \bigg(\int_{\mathbb{R}^d} \left|\mathcal{F} G_{t - t_{\rho(n)}}(\xi_1 + \cdots + \xi_{n-1} + \xi_n)\right|^2 \mu(\mathrm{d}\xi_n) \bigg) \cdots \mu(\mathrm{d}\xi_2) \bigg) \mu(\mathrm{d}\xi_1) \\
&\le \frac{1}{(n!)^2} K_{d,\alpha} (t - t_{\rho(n)})^{r\alpha} \int_{\mathbb{R}^d} \left|\mathcal{F} G_{t_{\rho(2)} - t_{\rho(1)}}(\xi_1)\right|^2 \bigg(\int_{\mathbb{R}^d} \left|\mathcal{F} G_{t_{\rho(3)} - t_{\rho(2)}}(\xi_1 + \xi_2)\right|^2
\end{aligned}$$

$$\cdots\Big(\int_{\mathbb{R}^d}\Big|\mathcal{F}G_{t_{\rho(n)}-t_{\rho(n-1)}}(\xi_1+\cdots+\xi_{n-2}+\xi_{n-1})\Big|^2\mu(\mathrm{d}\xi_{n-1})\Big)\cdots\mu(\mathrm{d}\xi_2)\Big)\mu(\mathrm{d}\xi_1)$$
$$\le\cdots$$
$$\le\frac{1}{(n!)^2}K_{d,\alpha}^n(t-t_{\rho(n)})^{r_\alpha}(t_{\rho(n)}-t_{\rho(n-1)})^{r_\alpha}\cdots(t_{\rho(2)}-t_{\rho(1)})^{r_\alpha}, \tag{2.2.13}$$

where for the first inequality, we used Lemma A.2.1, and the constant r_α is given by

$$r_\alpha=\begin{cases}-(d-\alpha)/2 & \text{for heat equation,}\\ 2-(d-\alpha) & \text{for wave equation.}\end{cases}$$

We return to relation (2.2.9). Using the change of variables $t'_j=t_{\rho(j)}$ for all $j=1,\dots,n$, we get

$$\begin{aligned}\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}}&\le\Gamma_{0,t}^n\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\frac{1}{(n!)^2}K_{d,\alpha}^n\prod_{j=1}^n(t_{\rho(j+1)}-t_{\rho(j)})^{r_\alpha}\mathrm{d}\mathbf{t}\\&=\Gamma_{0,t}^n\frac{1}{(n!)^2}K_{d,\alpha}^n\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\prod_{j=1}^n(t_{\rho(j+1)}-t_{\rho(j)})^{r_\alpha}\mathrm{d}\mathbf{t}\\&=\frac{(\Gamma_{0,t}K_{d,\alpha})^n}{(n!)^2}n!\int_{0<t'_1<\cdots<t'_n<t}\prod_{j=1}^n(t'_{j+1}-t'_j)^{r_\alpha}\mathrm{d}\mathbf{t}\\&=\frac{(\Gamma_{0,t}K_{d,\alpha})^n}{n!}\frac{\big(\Gamma(r_\alpha+1)\big)^n}{\Gamma\big(n(r_\alpha+1)+1\big)}t^{n(r_\alpha+1)},\end{aligned}\tag{2.2.14}$$

where in the last equation, we used Lemma A.3.1. By Lemma A.1.3 and relation (2.2.14), there exists a constant $C_1>0$ depending on d and α such that

$$\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2=n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}}\le C_1^n\,t^{n(r_\alpha+1)}\frac{1}{(n!)^{r_\alpha+1}}. \tag{2.2.15}$$

This means that

$$\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2\le\begin{cases}C_1^n\,t^{n(2-(d-\alpha)/2)}\frac{1}{(n!)^{2-(d-\alpha)/2}}, & \text{for heat equation,}\\ C_1^n\,t^{n(3-(d-\alpha))}\frac{1}{(n!)^{3-(d-\alpha)}}, & \text{for wave equation.}\end{cases}$$

By Lemma A.1.6, we infer that

$$\mathbb{E}|u^\alpha(t,x)|^2=\sum_{n\ge0}\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2\le\sum_{n\ge0}\frac{C_1^nt^{n(r_\alpha+1)}}{(n!)^{r_\alpha+1}}\le C_2\,e^{C_3t},$$

where $C_2 > 0$, $C_3 > 0$ are some constants depending on d and α. This proves the existence of solution. For uniqueness, we refer reader to page 302-305 of [26] for the parabolic Anderson model, and Proposition 3.2 of [6] for the hyperbolic Anderson model.

When $p \geq 2$, using relation (2.2.15) and Minkowski inequality, we obtain

$$\begin{aligned}
\|u^\alpha(t,x)\|_p &= \Big\| \sum_{n\geq 0} I_n^\alpha(f_n(\cdot,t,x)) \Big\|_p \leq \sum_{n\geq 0} \big\| I_n^\alpha(f_n(\cdot,t,x)) \big\|_p \\
&\leq \sum_{n\geq 0} (p-1)^{n/2} \big\| I_n^\alpha(f_n(\cdot,t,x)) \big\|_2 \leq \sum_{n\geq 0} (p-1)^{n/2} \Big(C_1^n\, t^{n(r_\alpha+1)} \frac{1}{(n!)^{r_\alpha+1}} \Big)^{1/2} \\
&= \sum_{n\geq 0} \frac{\Big((p-1)^{1/2} C_1^{1/2}\, t^{(r_\alpha+1)/2} \Big)^n}{(n!)^{(r_\alpha+1)/2}} \leq C_4 \exp\Big(C_5 (p-1)^{\frac{1}{r_\alpha+1}}\, C_1^{\frac{1}{r_\alpha+1}}\, t \Big),
\end{aligned}$$

where we used Lemma A.5.3 for the second inequality and Lemma A.1.6 for the last inequality. Therefore,

$$\mathbb{E}|u^\alpha(t,x)|^p \leq C^{(1)} \exp\Big(C^{(2)}\, p^{\frac{r_\alpha+2}{r_\alpha+1}} t \Big),$$

where $C^{(1)} > 0$ and $C^{(2)} > 0$ are some constants depending on d and α.

∎

Now we consider one particular case when the temporal covariance of the noise is given by $\gamma_0(t) = \alpha_{H_0}|t|^{2H_0-2}$ with $H_0 \in (1/2, 1)$. We will use the following result, whose proof is based on Hölder inequality and Littlewood-Hardy inequality. We refer the reader to [33].

Lemma 2.2.5. *For any function $\varphi \in L^{1/H_0}(\mathbb{R}_+)$ with $H_0 \in (1/2, 1)$,*

$$\begin{aligned}
\alpha_{H_0}^n \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \prod_{i=1}^n |t_i - s_i|^{2H_0-2} \varphi(t_1,\cdots,t_n)\varphi(s_1,\cdots,s_n) \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\
\leq b_{H_0}^n \Big(\int_{\mathbb{R}_+^n} \big|\varphi(t_1,\cdots,t_n)\big|^{\frac{1}{H_0}} \mathrm{d}\mathbf{t} \Big)^{2H_0}
\end{aligned}$$

where $b_{H_0} > 0$ is a constant depending on H_0.

We include the theorem below for the sake of completeness, although it is not needed for the continuity in law of the solution.

Theorem 2.2.6. *Let W^α be the Gaussian noise with covariance* (2.1.4) *for some $H_0 \in (1/2, 1)$. Then for any $\alpha \in (\max(d-2, 0), d)$, the solution to equation* (2.0.1) *or* (2.0.2) *satisfies: for any $p \geq 2$,*

$$\mathbb{E}|u^\alpha(t,x)|^p \leq C^{(1)} \exp\left(C^{(2)}\, p^k\, t^\rho\right), \tag{2.2.16}$$

where $C^{(1)} > 0$ and $C^{(2)} > 0$ are some constants depending on H_0, d and α. The power k is given by (2.2.7) *and*

$$\rho = \begin{cases} \rho^h = \frac{4H_0-(d-\alpha)}{2-(d-\alpha)} & \text{for heat equation,} \\ \rho^w = \frac{2H_0+2-(d-\alpha)}{3-(d-\alpha)} & \text{for wave equation.} \end{cases} \tag{2.2.17}$$

Proof: We proceed similarly to the proof of the Theorem 2.2.4. In this case, relation (2.2.8) still holds. Using Cauchy-Schwarz inequality, the fact that $ab \leq \frac{1}{2}(a^2+b^2)$ and Lemma 2.2.5, it follows that

$$\begin{aligned} \|\tilde f_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} &\leq \int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i-s_i) A_n^{1/2}(\mathbf{t},\mathbf{t}) A_n^{1/2}(\mathbf{s},\mathbf{s})\,\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq b_{H_0}^n\Big(\int_{[0,t]^n}\Big|\big(A_n(\mathbf{t},\mathbf{t})\big)^{1/2}\Big|^{\frac{1}{H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} = b_{H_0}^n\Big(\int_{[0,t]^n}\big|A_n(\mathbf{t},\mathbf{t})\big|^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} \\ &= b_{H_0}^n\Big(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\big|A_n(\mathbf{t},\mathbf{t})\big|^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0}. \end{aligned} \tag{2.2.18}$$

We study $A_n(\mathbf{t},\mathbf{t})$ first. Note that relation (2.2.13) still holds. Returning to relation (2.2.18) and using the change of variables $t'_j = t_{\rho(j)}$ for all $j = 1, \ldots, n$, we get

$$\begin{aligned} \|\tilde f_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} &\leq b_{H_0}^n\Big(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\Big(\frac{1}{(n!)^2}K_{d,\alpha}^n\prod_{j=1}^n(t_{\rho(j+1)}-t_{\rho(j)})^{r_\alpha}\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} \\ &= b_{H_0}^n\frac{1}{(n!)^2}K_{d,\alpha}^n\Big(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\prod_{j=1}^n(t_{\rho(j+1)}-t_{\rho(j)})^{\frac{r_\alpha}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} \\ &= \frac{(b_{H_0}K_{d,\alpha})^n}{(n!)^2}\Big(n!\int_{0<t'_1<\cdots<t'_n<t}\prod_{j=1}^n(t'_{j+1}-t'_j)^{\frac{r_\alpha}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} \\ &= (n!)^{2H_0-2}(b_{H_0}K_{d,\alpha})^n\left(\frac{\big(\Gamma(\frac{r_\alpha}{2H_0}+1)\big)^n}{\Gamma\big(n(\frac{r_\alpha}{2H_0}+1)+1\big)}\right)^{2H_0} t^{n(r_\alpha+2H_0)}, \end{aligned} \tag{2.2.19}$$

where for the third equation, we used Lemma A.3.1. By Lemma A.1.3 and relation (2.2.19), there exist some constants $C_1 > 0$ and $C_2 > 0$ depending on H_0, d and α

such that

$$\begin{aligned}
&\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2\\
&= n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} \le (n!)^{2H_0-1}(b_{H_0}K_{d,\alpha})^n t^{n(r_\alpha+2H_0)}\left(\frac{\big(\Gamma(\frac{r_\alpha}{2H_0}+1)\big)^n}{\Gamma\big(n(\frac{r_\alpha}{2H_0}+1)+1\big)}\right)^{2H_0}\\
&= C_1^n\, t^{n(r_\alpha+2H_0)}(n!)^{2H_0-1}\left(\frac{1}{\Gamma\big(n(\frac{r_\alpha}{2H_0}+1)+1\big)}\right)^{2H_0} \le C_2^n\, t^{n(r_\alpha+2H_0)}\frac{1}{(n!)^{r_\alpha+1}}. \quad (2.2.20)
\end{aligned}$$

This means that

$$\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2 \le \begin{cases} C_2^n\, t^{n(\frac{4H_0+\alpha-d}{2})}\frac{1}{(n!)^{\frac{2+\alpha-d}{2}}}, & \text{for heat equation,}\\ C_2^n\, t^{n(2H_0+2+\alpha-d)}\frac{1}{(n!)^{3+\alpha-d}}, & \text{for wave equation.}\end{cases}$$

By Lemma A.1.6, we infer that

$$\mathbb{E}|u^\alpha(t,x)|^2 = \sum_{n\ge 0}\mathbb{E}\big|I_n^\alpha(f_n(\cdot,t,x))\big|^2 \le \sum_{n\ge 0}\frac{C_2^n t^{n(r_\alpha+2H_0)}}{(n!)^{r_\alpha+1}} \le C_3\exp\Big(C_4\, t^{\frac{r_\alpha+2H_0}{r_\alpha+1}}\Big),$$

where $C_3>0$ and $C_4>0$ are some constants depending on H_0, d and α. Note that the exponent of t is exactly the value ρ given by (2.2.17).

When $p\ge 2$, as in the proof of Theorem 2.2.4, we have

$$\begin{aligned}
\|u^\alpha(t,x)\|_p &\le \sum_{n\ge 0}(p-1)^{n/2}\big\|I_n^\alpha(f_n(\cdot,t,x))\big\|_2 \le \sum_{n\ge 0}(p-1)^{n/2}\left(\frac{C_2^n t^{n(r_\alpha+2H_0)}}{(n!)^{r_\alpha+1}}\right)^{1/2}\\
&\le C_5\exp\Big(C_6(p-1)^{\frac{1}{r_\alpha+1}}C_2^{\frac{1}{r_\alpha+1}}\, t^{\frac{r_\alpha+2H_0}{r_\alpha+1}}\Big),
\end{aligned}$$

where $C_5>0$ and $C_6>0$ are some constants depending on H_0, d and α and we used Lemma A.1.6 in the last inequality. This proves (2.2.16). ∎

2.3 Uniform moment estimates

Within this section, we present estimates for the moments of the increments of the solution to equation (2.0.1), respectively equation (2.0.2), driven by the noise W^α introduced in Section 2.1. These findings play an important role in establishing the weak convergence of the solution. We proceed as in the proof of Theorem 3.2 of [8] for heat equation, respectively in the proof of Theorem 8.3 of [9] for wave equation. We revisit the proofs of these results because we want to obtain bounds for the constants which are uniform for $\alpha\in[a,b]$.

Theorem 2.3.1. *Let u_h^α the solution to the heat equation (2.0.1) and u_w^α be the solution to the wave equation (2.0.2), with noise W^α as in Section 2.1. Let $[a,b]$ be a compact set such that*

$$\max\{0, d-2\} < a < b < d.$$

(a) (Heat equation) For any $p \geq 2$, $T > 0$ and $\beta \in (\frac{d-a}{2}, 1)$, there exist positive constants C_1^h and C_2^h (depending on β, a, b, p, d and T) such that for any $t', t \in [0,T]$ and $x', x \in \mathbb{R}^d$, we have

$$\sup_{\alpha \in [a,b]} \mathbb{E}|u_h^\alpha(t', x) - u_h^\alpha(t,x)|^p \leq C_1^h |t'-t|^{\frac{p(1-\beta)}{2}}, \tag{2.3.1}$$

and

$$\sup_{\alpha \in [a,b]} \mathbb{E}|u_h^\alpha(t, x') - u_h^\alpha(t,x)|^p \leq C_2^h |x'-x|^{p(1-\beta)}. \tag{2.3.2}$$

(b) (Wave equation) Let $K = [-M, M] \subset \mathbb{R}^d$ be an arbitrary subset, for some $M > 0$. For any $p \geq 2$, $T > 0$ and $\beta \in (\frac{d-a}{2}, 1)$, there exist positive constants C_1^w (depending on β, a, b, p, d and T) and C_2^w (depending on β, a, b, p, d, M and T) such that for any $t', t \in [0,T]$ and for any $x \in \mathbb{R}^d$, we have

$$\sup_{\alpha \in [a,b]} \mathbb{E}|u_w^\alpha(t', x) - u_w^\alpha(t,x)|^p \leq C_1^w |t'-t|^{p(1-\beta)}, \tag{2.3.3}$$

and for any $t \in [0,T]$ and for any $x', x \in K$,

$$\sup_{\alpha \in [a,b]} \mathbb{E}|u_w^\alpha(t, x') - u_w^\alpha(t,x)|^p \leq C_2^w |x'-x|^{p(1-\beta)}. \tag{2.3.4}$$

The following two results are needed for the proof of Theorem 2.3.1(a).

Lemma 2.3.2 (Lemma 2.1 of [8]). *Let G^h be the fundamental solution for heat equation. For any $t > 0$,*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G_t^h(\xi+\eta)\right|^2 \mu(\mathrm{d}\xi) = \int_{\mathbb{R}^d} \left|\mathcal{F}G_t^h(\xi)\right|^2 \mu(\mathrm{d}\xi) := k(t). \tag{2.3.5}$$

In particular, if the spectral measure μ is given by $\mu(\mathrm{d}\xi) = |\xi|^{-\alpha}\mathrm{d}\xi$ with $\alpha \in (0,d)$, then

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G_t^h(\xi+\eta)\right|^2 |\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} \left|\mathcal{F}G_t^h(\xi)\right|^2 |\xi|^{-\alpha}\mathrm{d}\xi := k_\alpha(t). \tag{2.3.6}$$

Proposition 2.3.3 (Proposition 3.1 of [8]). *If μ satisfies*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^\beta \mu(\mathrm{d}\xi) < \infty, \quad \text{for any } \beta \in (0,1), \tag{2.3.7}$$

then for any $t > 0$, $h > 0$ *and* $z \in \mathbb{R}^d$,

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}G^h_{t+h}(\xi+\eta)-\mathcal{F}G^h_t(\xi+\eta)\right|^2\mu(\mathrm{d}\xi) < C_\theta h^\theta t^{-\theta} k\Big(\frac{t}{2}\Big)$$

and

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|e^{-i(\xi+\eta)\cdot z}-1\right|^2\left|\mathcal{F}G^h_t(\xi+\eta)\right|^2\mu(\mathrm{d}\xi) < C_\theta |z|^{2\theta} t^{-\theta} k\Big(\frac{t}{2}\Big),$$

where $\theta = 1-\beta$, $C_\theta > 0$ *is a constant depending on* θ, *and recall that* $k(t)$ *is given by* (2.3.5). *In particular, if the spectral measure* μ *is given by* $\mu(\mathrm{d}\xi) = |\xi|^{-\alpha}\mathrm{d}\xi$ *with* $\alpha \in (0, d)$, *then for any* $\beta \in (\frac{d-\alpha}{2}, 1)$,

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}G^h_{t+h}(\xi+\eta)-\mathcal{F}G^h_t(\xi+\eta)\right|^2|\xi|^{-\alpha}\mathrm{d}\xi < C_\theta h^\theta t^{-\theta} k_\alpha\Big(\frac{t}{2}\Big) \tag{2.3.8}$$

and

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|e^{-i(\xi+\eta)\cdot z}-1\right|^2\left|\mathcal{F}G^h_t(\xi+\eta)\right|^2|\xi|^{-\alpha}\mathrm{d}\xi < C_\theta |z|^{2\theta} t^{-\theta} k_\alpha\Big(\frac{t}{2}\Big), \tag{2.3.9}$$

where $k_\alpha(t)$ *is given by* (2.3.6).

The next result is used for the proof of Theorem 2.3.1(b). We include its proof since we need uniform bounds for all $\alpha \in [a, b]$.

Lemma 2.3.4 (Proposition 7.4 of [17]). *Let* $\max\{d-2, 0\} < a < b < d$ *and* $\beta \in \big(\frac{d-a}{2}, 1\big)$ *be fixed. Then*

(*i*) *for any* $T > 0$, *there exists a constant* $C^{(1)} > 0$ *depending on* d, a, b, β *such that*

$$\sup_{\alpha\in[a,b]}\sup_{t\in[0,T]}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}G^w_{t+h}(\xi+\eta)-\mathcal{F}G^w_t(\xi+\eta)\right|^2|\xi|^{-\alpha}\mathrm{d}\xi \le C^{(1)}\,|h|^{2-2\beta};$$

(*ii*) *for any* $T > 0$, *there exists a constant* $C^{(2)} > 0$ *depending on* T, a, b, d, β *such that for any* $t \in [0, T]$

$$\sup_{\alpha\in[a,b]}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}G^w_t(\xi+\eta)\right|^2|\xi|^{-\alpha}\mathrm{d}\xi \le C^{(2)}\,t^{2-2\beta};$$

(*iii*) *for any* $T > 0$ *and for any compact set* $K = [-M, M] \subset \mathbb{R}^d$, *for some* $M > 0$, *there exists a constant* $C^{(3)} > 0$ *depending on* T, M, d, a, b *and* β *such that for any* $z \in K$,

$$\sup_{\alpha\in[a,b]}\sup_{t\in[0,T]}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\mathcal{F}G^w_t(\xi+\eta)\right|^2\left|1-e^{-i(\xi+\eta)\cdot z}\right|^2|\xi|^{-\alpha}\mathrm{d}\xi \le C^{(3)}|z|^{2-2\beta}.$$

Proof: Recall that the Fourier transform of the fundamental solution of the wave equation is

$$\mathcal{F}G_t^w(\xi) = \frac{\sin(t|\xi|)}{|\xi|},$$

in any dimension $d \geq 1$. We frequently use the following fact that for any $|x| > 1$,

$$\frac{1}{1+x^2} < \frac{1}{x^2} < \frac{2}{1+x^2}. \tag{2.3.10}$$

We start with part (i). Note that

$$\int_{\mathbb{R}^d} \left|\mathcal{F}G_{t+h}^w(\xi+\eta) - \mathcal{F}G_t^w(\xi+\eta)\right|^2 |\xi|^{-\alpha} \mathrm{d}\xi = I_1^{(1)}(\eta) + I_1^{(2)}(\eta), \tag{2.3.11}$$

where

$$I_1^{(1)}(\eta) := \int_{|\xi+\eta|<1} \frac{\left|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\right|^2}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi,$$
$$I_1^{(2)}(\eta) := \int_{|\xi+\eta|\geq 1} \frac{\left|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\right|^2}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi.$$

First, we consider $I_1^{(1)}(\eta)$. Using the trigonometric identity

$$\sin(a) - \sin(b) = 2\sin\big(\frac{a-b}{2}\big)\cos\big(\frac{a+b}{2}\big)$$

and the fact that $|\sin(x)| \leq x$ for all $x \geq 0$, we see that

$$\begin{aligned} &\left|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\right|^2 \\ &\qquad = 4\sin^2\left(\frac{h|\xi+\eta|}{2}\right)\cos^2\left(\frac{(2t+h)|\xi+\eta|}{2}\right) \leq h^2|\xi+\eta|^2. \end{aligned}$$

Therefore, using properties of the domain of integration of integral $I_1^{(1)}(\eta)$, we obtain that for any $h \in [0,1]$,

$$\begin{aligned} I_1^{(1)}(\eta) &\leq h^2 \int_{|\xi+\eta|<1} |\xi|^{-\alpha}\mathrm{d}\xi = h^2 \int_{|\xi+\eta|<1} \frac{1+|\xi+\eta|^2}{1+|\xi+\eta|^2}|\xi|^{-\alpha}\mathrm{d}\xi \\ &\leq h^2 \int_{|\xi+\eta|<1} \frac{2}{1+|\xi+\eta|^2}|\xi|^{-\alpha}\mathrm{d}\xi \leq 2h^{2(1-\beta)} \int_{|\xi+\eta|<1} \frac{1}{1+|\xi+\eta|^2}|\xi|^{-\alpha}\mathrm{d}\xi \\ &\leq 2h^{2(1-\beta)} \int_{|\xi+\eta|<1} \left(\frac{1}{1+|\xi+\eta|^2}\right)^\beta |\xi|^{-\alpha}\mathrm{d}\xi \leq 2h^{2(1-\beta)} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi+\eta|^2}\right)^\beta |\xi|^{-\alpha}\mathrm{d}\xi. \end{aligned} \tag{2.3.12}$$

Next, we study $I_1^{(2)}(\eta)$. Bounding the trigonometric functions by 1, using again the elementary properties of trigonometric functions and the fact that $|\sin(x)| \leq x$ for all $x \geq 0$, it follows that

$$\begin{aligned}
I_1^{(2)}(\eta) &= \int_{|\xi+\eta|\geq 1} \left(\frac{\big|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\big|}{|\xi+\eta|} \right)^{2\beta} \\
&\qquad\qquad \left(\frac{\big|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\big|}{|\xi+\eta|} \right)^{2(1-\beta)} |\xi|^{-\alpha} \mathrm{d}\xi \\
&= \int_{|\xi+\eta|\geq 1} \Big(\frac{\big|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\big|}{|\xi+\eta|} \Big)^{2\beta} \\
&\qquad\qquad \Big(\frac{2\Big|\sin\Big(\frac{h|\xi+\eta|}{2}\Big)\Big|\Big|\cos\Big(\frac{(2t+h)|\xi+\eta|}{2}\Big)\Big|}{|\xi+\eta|} \Big)^{2(1-\beta)} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq h^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \left(\frac{\big|\sin\big((t+h)|\xi+\eta|\big) - \sin\big(t|\xi+\eta|\big)\big|}{|\xi+\eta|} \right)^{2\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq h^{2(1-\beta)} 2^{2\beta} \int_{|\xi+\eta|\geq 1} \Big(\frac{1}{|\xi+\eta|^2} \Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq h^{2(1-\beta)} 4^{\beta} \int_{|\xi+\eta|\geq 1} \Big(\frac{2}{1+|\xi+\eta|^2} \Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq h^{2(1-\beta)} 8 \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2} \Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi, \qquad (2.3.13)
\end{aligned}$$

where we used relation (2.3.10) in the third last inequality. We combine relations (2.3.12) and (2.3.13). Taking the supremum over $\eta \in \mathbb{R}^d$ on both sides of (2.3.11) and using Lemma A.2.3, we obtain:

$$\begin{aligned}
&\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}G^w_{t+h}(\xi+\eta) - \mathcal{F}G^w_t(\xi+\eta)\big|^2 |\xi|^{-\alpha} \mathrm{d}\xi \\
&\quad \leq 10h^{2(1-\beta)} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2} \Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi = 10h^{2(1-\beta)} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi.
\end{aligned}$$

By Lemma A.2.2, the last integral can be bounded by a constant C depending on d, a, b, β, uniformly in $\alpha \in [a, b]$. Note that this integral is finite because $\beta \in (\frac{d-a}{2}, 1)$ which implies that $\beta \in (\frac{d-\alpha}{2}, 1)$ for all $\alpha \in [a, b]$.

Now we prove part (ii). For any $t \in [0, T]$, we have

$$\int_{\mathbb{R}^d} \big|\mathcal{F}G^w_t(\xi+\eta)\big|^2 |\xi|^{-\alpha} \mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi = I_2^{(1)}(\eta) + I_2^{(2)}(\eta), \quad (2.3.14)$$

where

$$I_2^{(1)}(\eta) := \int_{|\xi+\eta|<1} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi \quad \text{and} \quad I_2^{(2)}(\eta) := \int_{|\xi+\eta|\geq 1} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi.$$

For the term $I_2^{(1)}(\eta)$, using the fact that $|\sin(x)| \leq x$ for all $x \geq 0$, we get

$$\begin{aligned}
I_2^{(1)}(\eta) &\leq t^2 \int_{|\xi+\eta|<1} |\xi|^{-\alpha} \mathrm{d}\xi \leq t^2 \int_{|\xi+\eta|<1} \frac{2}{1+|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq 2t^2 \int_{|\xi+\eta|<1} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq 2t^2 \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq 2T^{2\beta} t^{2(1-\beta)} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi. \qquad (2.3.15)
\end{aligned}$$

We next treat $I_2^{(2)}(\eta)$. Using the fact that $|\sin(x)| \leq x$ for all $x \geq 0$ and relation (2.3.10), we get

$$\begin{aligned}
I_2^{(2)}(\eta) &= \int_{|\xi+\eta|\geq 1} \left(\frac{\big|\sin(t|\xi+\eta|)\big|}{|\xi+\eta|}\right)^{2(1-\beta)} \left(\frac{\big|\sin(t|\xi+\eta|)\big|}{|\xi+\eta|}\right)^{2\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq t^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \left(\frac{\big|\sin(t|\xi+\eta|)\big|^2}{|\xi+\eta|^2}\right)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq t^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \Big(\frac{2}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq t^{2(1-\beta)} 2 \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi. \qquad (2.3.16)
\end{aligned}$$

We combine relations (2.3.15) and (2.3.16). Taking the supremum over $\eta \in \mathbb{R}^d$ on both sides of (2.3.14), there exist a constant C depending on T and β such that

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}G_t^w(\xi+\eta)\big|^2 |\xi|^{-\alpha} \mathrm{d}\xi \leq C^{(2)} t^{2(1-\beta)} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi.$$

The conclusion follows as for part (i) above, using Lemma A.2.3 and Lemma A.2.2.

For part (iii), let $K = [-M, M]$ be a compact set in $\mathbb{R}^d$ for some $M > 0$. Taking any $z \in K$ and for any $t \in [0, T]$, we have

$$\int_{\mathbb{R}^d} \big|\mathcal{F}G_t^w(\xi+\eta)\big|^2 \big|1 - e^{-i(\xi+\eta)\cdot z}\big|^2 |\xi|^{-\alpha} \mathrm{d}\xi = I_3^{(1)}(\eta) + I_3^{(2)}(\eta), \qquad (2.3.17)$$

where

$$\begin{aligned}
I_3^{(1)}(\eta) &:= \int_{|\xi+\eta|<1} \big|e^{-i(\xi+\eta)\cdot z} - 1\big|^2 \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi, \\
I_3^{(2)}(\eta) &:= \int_{|\xi+\eta|\geq 1} \big|e^{-i(\xi+\eta)\cdot z} - 1\big|^2 \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi.
\end{aligned}$$

We study $I_3^{(1)}(\eta)$ first. Using facts that $|\sin(x)| \leq x$ for all $x \geq 0$, $\big|1 - e^{ix}\big| \leq |x|$, we have:

$I_3^{(1)}(\eta)$

$$
\begin{aligned}
&\leq t^2 \int_{|\xi+\eta|<1} \left|e^{-i(\xi+\eta)\cdot z} - 1\right|^2 |\xi|^{-\alpha} \mathrm{d}\xi \leq T^2 \int_{|\xi+\eta|<1} \left|e^{-i(\xi+\eta)\cdot z} - 1\right|^2 \frac{2}{1+|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq T^2 \int_{|\xi+\eta|<1} |z|^2 |\xi+\eta|^2 \Big(\frac{2}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq 2T^2 |z|^2 \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq 2T^2 (2M)^{2\beta} |z|^{2(1-\beta)} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi. \qquad (2.3.18)
\end{aligned}
$$

Next, we consider $I_3^{(2)}(\eta)$. Using inequality $\left|1 - e^{ix}\right| \leq |x|$ and relation (2.3.10), we obtain

$$
\begin{aligned}
I_3^{(2)}(\eta) &= \int_{|\xi+\eta|\geq 1} \left|e^{-i(\xi+\eta)\cdot z} - 1\right|^{2\beta} \left|e^{-i(\xi+\eta)\cdot z} - 1\right|^{2(1-\beta)} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq \int_{|\xi+\eta|\geq 1} 2^{2\beta} \left|e^{-i(\xi+\eta)\cdot z} - 1\right|^{2(1-\beta)} \frac{1}{|\xi+\eta|^2} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq 2^{2\beta} |z|^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \frac{1}{|\xi+\eta|^{2\beta}} |\xi|^{-\alpha} \mathrm{d}\xi \leq 2^{2\beta} |z|^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \Big(\frac{2}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \\
&\leq 8|z|^{2(1-\beta)} \int_{|\xi+\eta|\geq 1} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq 8|z|^{2(1-\beta)} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi. \qquad (2.3.19)
\end{aligned}
$$

We combine relations (2.3.18) and (2.3.19). Taking the supremum over $\eta \in \mathbb{R}^d$ on both sides of (2.3.17), we obtain:

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G_t^w(\xi+\eta)\right|^2 \left|1 - e^{-i(\xi+\eta)\cdot z}\right|^2 |\xi|^{-\alpha} \mathrm{d}\xi \\
&\qquad \leq \big(2T^2(2M)^{2\beta} + 8\big) |z|^{2(1-\beta)} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi+\eta|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi.
\end{aligned}
$$

The conclusion follows by applying Lemma A.2.3 and Lemma A.2.2, as above. ∎

Proof of Theorem 2.3.1:

Before we begin the proof, we made some comments about the choice of parameters. We fix an interval $[a, b] \subset \big(\max\{d-2, 0\}, d\big)$. We fix $\beta \in \big(\frac{d-a}{2}, 1\big)$. We denote

$$\theta = 1 - \beta.$$

Note that for any $\alpha \in [a, b]$, β lies in the interval $\big(\frac{d-\alpha}{2}, 1\big)$, which means that: $\beta \in \Big(\frac{d-a}{2}, 1\Big) \subset \Big(\frac{d-\alpha}{2}, 1\Big)$,

$$K_{d,\beta,\alpha} := \int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi < \infty, \qquad (2.3.20)$$

and relations (2.3.8) and (2.3.9) holds.

We consider separately the heat and wave equations. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: we consider the heat equation} \begin{cases} \text{Step 1.a The time increments} \\ \text{Step 1.b The space increments} \end{cases} \\ \text{Step 2: we consider the wave equation} \begin{cases} \text{Step 2.a The time increments} \\ \text{Step 2.b The space increments} \end{cases} \end{cases}$$

Step 1: Heat equation

Step 1.a We start with the time increments. Let $t, t' \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. Assume that $h = t' - t > 0$ (The case $h < 0$ is similar). By Minkowski's inequality and Lemma A.5.3,

$$\begin{aligned} \|u^\alpha(t+h,x) - u^\alpha(t,x)\|_p &\le \sum_{n\ge 1}(p-1)^{n/2}\Big\| I_n^\alpha\big(f_n(\cdot,t+h,x) - f_n(\cdot,t,x)\big)\Big\|_2 \\ &\le \sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{2}{n!}\big(A_n^{(\alpha)}(t,h) + B_n^{(\alpha)}(t,h)\big)\Big)^{1/2}, \quad (2.3.21) \end{aligned}$$

where

$$A_n^{(\alpha)}(t,h) := (n!)^2\|\tilde{f}_n(\cdot,t+h,x)1_{[0,t]^n} - \tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}}, \qquad (2.3.22)$$

and

$$B_n^{(\alpha)}(t,h) := (n!)^2\|\tilde{f}_n(\cdot,t+h,x)1_{[0,t+h]^n\setminus[0,t]^n}\|^2_{\mathcal{H}_\alpha^{\otimes n}}. \qquad (2.3.23)$$

We study $A_n^{(\alpha)}(t,h)$ first. By (2.2.5),

$$\begin{aligned} A_n^{(\alpha)}(t,h) = (n!)^2 &\int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n \gamma_0(t_i - s_i)\int_{(\mathbb{R}^d)^n}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n\gamma(x_i - y_i) \\ &\Big(\tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t+h,x) - \tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t,x)\Big) \\ &\Big(\tilde{f}_n(s_1,y_1,\cdots,s_n,y_n,t+h,x) - \tilde{f}_n(s_1,y_1,\cdots,s_n,y_n,t,x)\Big)d\mathbf{x}d\mathbf{y}d\mathbf{t}d\mathbf{s} \\ = &\int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^n\gamma_0(t_i - s_i)\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s} \qquad (2.3.24) \end{aligned}$$

where

$$\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{s}) = \int_{(\mathbb{R}^d)^n}\mathcal{F}\big[g^{(n)}_{\mathbf{t}}(\cdot,t+h,x) - g^{(n)}_{\mathbf{t}}(\cdot,t,x)\big](\xi_1,\ldots,\xi_n)$$

$$\overline{\mathcal{F}\big[g_{\mathbf{s}}^{(n)}(\cdot,t+h,x)-g_{\mathbf{s}}^{(n)}(\cdot,t,x)\big](\xi_1,\dots,\xi_n)}\prod_{j=1}^{n}|\xi_j|^{-\alpha}\mathrm{d}\xi_j$$

and

$$g_{\mathbf{t}}^{(n)}(\cdot,t,x):=n!\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,x).$$

Using the Cauchy-Schwarz inequality and the fact that $ab\le(a^2+b^2)/2$, we have

$$\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{s})\le\Big(\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\psi_{t,h,n}^{(\alpha)}(\mathbf{s},\mathbf{s})\Big)^{1/2}\le\frac{1}{2}\Big(\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})+\psi_{t,h,n}^{(\alpha)}(\mathbf{s},\mathbf{s})\Big).$$

By Lemma 2.2.3, it follows that

$$A_n^{(\alpha)}(t,h)\le\int_{[0,t]^n}\int_{[0,t]^n}\prod_{i=1}^{n}\gamma_0(t_i-s_i)\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}\le\Gamma_{0,t}^{n}\int_{[0,t]^n}\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}. \tag{2.3.25}$$

We fix $\mathbf{t}=(t_1,\dots,t_n)\in[0,t]^n$ and we let $\rho\in S_n$ such that $t_{\rho(1)}<\dots<t_{\rho(n)}$. We define $u_j=t_{\rho(j+1)}-t_{\rho(j)}$ for $j=1,\dots,n$ and $t_{\rho(n+1)}=t$. Using relation (2.2.12), we obtain:

$$\begin{aligned}
\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})&=\int_{(\mathbb{R}^d)^n}\big|\mathcal{F}G_{u_1}^{h}(\xi_{\rho(1)})\big|^2\cdots\big|\mathcal{F}G_{u_{n-1}}^{h}(\xi_{\rho(1)}+\dots+\xi_{\rho(n-1)})\big|^2\\
&\qquad\big|\mathcal{F}[G_{u_n+h}^{h}-G_{u_n}^{h}](\xi_{\rho(1)}+\dots+\xi_{\rho(n)})\big|^2\prod_{j=1}^{n}|\xi_j|^{-\alpha}\mathrm{d}\xi_j\\
&=\int_{(\mathbb{R}^d)^n}\big|\mathcal{F}G_{u_1}^{h}(\xi_1')\big|^2\cdots\big|\mathcal{F}G_{u_{n-1}}^{h}(\xi_1'+\dots+\xi_{n-1}')\big|^2\\
&\qquad\big|\mathcal{F}[G_{u_n+h}^{h}-G_{u_n}^{h}](\xi_1'+\dots+\xi_n')\big|^2\prod_{j=1}^{n}|\xi_j'|^{-\alpha}\mathrm{d}\xi_j'\\
&=\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_1}^{h}(\xi_1)\big|^2|\xi_1|^{-\alpha}\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_2}^{h}(\xi_1+\xi_2)\big|^2|\xi_2|^{-\alpha}\dots\\
&\qquad\Big(\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_n+h}^{h}(\xi_1+\dots+\xi_n)-\mathcal{F}G_{u_n}^{h}(\xi_1+\dots+\xi_n)\big|^2|\xi_n|^{-\alpha}\mathrm{d}\xi_n\Big)\dots\mathrm{d}\xi_2\mathrm{d}\xi_1,
\end{aligned}$$

where we used the change of variables $\xi_j'=\xi_{\rho(j)}$ for all $j=1,\dots,n$. By Lemma 2.3.2 and Proposition 2.3.3, we get

$$\begin{aligned}
\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})&\le\prod_{j=1}^{n-1}\Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_j}^{h}(\xi_j+\eta)\big|^2|\xi_j|^{-\alpha}\mathrm{d}\xi_j\Big)\\
&\quad\times\Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_n+h}^{h}(\xi_n+\eta)-\mathcal{F}G_{u_n}^{h}(\xi_n+\eta)\big|^2|\xi_n|^{-\alpha}\mathrm{d}\xi_n\Big)
\end{aligned} \tag{2.3.26}$$

$$\leq C_\theta h^\theta \prod_{j=1}^{n-1} k_\alpha(u_j) u_n^{-\theta} k_\alpha(u_n/2), \tag{2.3.27}$$

where $k_\alpha(t)$ is given by (2.3.6). We denote

$$\rho_t^{(\alpha)} = \int_0^t s^{-\theta} k_\alpha(s) \mathrm{d}s.$$

Let $h_0^{(\alpha)}(t) = 1$ and for any $n \geq 1$,

$$h_n^{(\alpha)}(t) = \int_{0<t_1<\cdots<t_n<t} k_\alpha(t_2 - t_1) \dots k_\alpha(t_n - t_{n-1}) k_\alpha(t - t_n) \mathrm{d}t_1 \dots \mathrm{d}t_n.$$

By Lemma 2.6 of [16], $h_n^{(\alpha)}$ is non-decreasing.

Recall that $\mathcal{F}G_t^h(\xi) = \exp(-\frac{t|\xi|^2}{2})$. Using the polar coordinate $\xi = rz$ and the change of variable $x = r^2$, we have

$$\begin{aligned} k_\alpha(t) &= \int_{\mathbb{R}^d} e^{-t|\xi|^2} |\xi|^{-\alpha} \mathrm{d}\xi = \int_0^\infty \int_{S_1(0)} e^{-tr^2} r^{-\alpha} r^{d-1} \mathrm{d}z \mathrm{d}r \\ &= \Big(\int_{S_1(0)} \mathrm{d}z \Big) \int_0^\infty e^{-tr^2} r^{-\alpha+d-1} \mathrm{d}r = c_d \int_0^\infty e^{-tx} x^{\frac{-\alpha+d-1}{2}} \frac{1}{2} x^{-1/2} \mathrm{d}x \\ &= \frac{c_d}{2} \int_0^\infty e^{-tx} x^{\frac{-\alpha+d}{2}-1} \mathrm{d}x = \frac{c_d}{2} \frac{\Gamma\big(\frac{d-\alpha}{2}\big)}{t^{\frac{d-\alpha}{2}}}, \end{aligned} \tag{2.3.28}$$

where c_d is the area of the unit sphere $S_1(0) = \{z \in \mathbb{R}^d; |z| = 1\}$ in $\mathbb{R}^d$. By Lemma A.3.1 and relation (2.3.28), we obtain:

$$\begin{aligned} h_n^{(\alpha)}(t) &= \int_{T_n(t)} \prod_{j=1}^n k_\alpha(t_{j+1} - t_j) \mathrm{d}\mathbf{t} = \Big(\Big(\frac{c_d}{2}\Big) \Gamma\Big(\frac{d-\alpha}{2}\Big) \Big)^n \int_{T_n(t)} \prod_{j=1}^n \big(t_{j+1} - t_j\big)^{-\frac{d-\alpha}{2}} \mathrm{d}\mathbf{t} \\ &= \Big(\Big(\frac{c_d}{2}\Big) \Gamma\Big(\frac{d-\alpha}{2}\Big) \Big)^n \frac{\Gamma\Big(1 - \frac{d-\alpha}{2}\Big)^n}{\Gamma\Big(n\big(1 - \frac{d-\alpha}{2}\big) + 1\Big)} t^{n\left(1-\frac{d-\alpha}{2}\right)}, \end{aligned} \tag{2.3.29}$$

and

$$\begin{aligned} \rho_t^{(\alpha)} &= \int_0^t s^{-\theta} k_\alpha(s) \mathrm{d}s = \frac{c_d}{2} \Gamma\Big(\frac{d-\alpha}{2}\Big) \int_0^t s^{-\theta} s^{\frac{\alpha-d}{2}} \mathrm{d}s \\ &= \frac{c_d}{2} \Gamma\Big(\frac{d-\alpha}{2}\Big) \frac{2}{\alpha - d - 2\theta + 2} t^{\frac{\alpha-d-2\theta+2}{2}}. \end{aligned} \tag{2.3.30}$$

Note that for the last integral, we used the fact that $\frac{\alpha-d}{2} - \theta + 1 = \frac{\alpha-d}{2} + \beta > 0$, i.e. $\beta > \frac{\alpha-d}{2}$. This holds for all $\alpha \in [a,b]$, since $\beta > \frac{d-a}{2}$.

Putting together relations (2.3.25) and (2.3.27), we obtain

$$
\begin{aligned}
A_n^{(\alpha)}(t,h) &\le \Gamma_{0,t}^n \int_{[0,t]^n} C_\theta h^\theta \prod_{j=1}^{n-1} k_\alpha(u_j) u_n^{-\theta} k_\alpha(u_n/2) \mathrm{d}\mathbf{t} \\
&= C_\theta h^\theta \Gamma_{0,t}^n n! \int_{0<t_1<\cdots<t_n<t} \prod_{j=1}^{n-1} k_\alpha(t_{j+1}-t_j)(t-t_n)^{-\theta} k_\alpha\Big(\frac{t-t_n}{2}\Big) \mathrm{d}\mathbf{t} \\
&= C_\theta h^\theta \Gamma_{0,t}^n n! \int_0^t h_{n-1}^{(\alpha)}(t_n)(t-t_n)^{-\theta} k_\alpha\Big(\frac{t-t_n}{2}\Big) \mathrm{d}t_n \\
&\le C_\theta h^\theta \Gamma_{0,t}^n n! h_{n-1}^{(\alpha)}(t) \int_0^t (t-t_n)^{-\theta} k_\alpha\Big(\frac{t-t_n}{2}\Big) \mathrm{d}t_n = C_\theta h^\theta \Gamma_{0,t}^n n! h_{n-1}^{(\alpha)}(t) \rho_{t/2}^{(\alpha)}, \quad (2.3.31)
\end{aligned}
$$

where for the last line we used the fact that h_{n-1} is non-decreasing.

We would like to obtain upper bounds for $h_n^{(\alpha)}(t)$ and $\rho_t^{(\alpha)}$, for any $\alpha \in [a,b]$. Note that for any $\alpha \in [a,b]$, $\frac{d-\alpha}{2} \in \big[\frac{d-b}{2}, \frac{d-a}{2}\big] \subset (0,1)$. We used the fact that the function Γ is decreasing on $(0,1)$. Using relation (2.3.29), we have

$$
\begin{aligned}
h_n^{(\alpha)}(t) &\le \bigg(\frac{c_d}{2}\Gamma\Big(\frac{d-b}{2}\Big)\Gamma\Big(1-\frac{d-a}{2}\Big)\bigg)^n (t\vee 1)^{n(1-\frac{d-b}{2})} \frac{1}{\Gamma\Big(n\big(1-\frac{d-a}{2}\big)+1\Big)} \\
&\le C_1^n (t\vee 1)^{n(1-\frac{d-b}{2})} \frac{1}{(n!)^{1-\frac{d-a}{2}}}, \quad (2.3.32)
\end{aligned}
$$

where we used Lemma A.1.3 in the last inequality and $C_1 > 0$ is a constant depending only on d, a and b. Moreover, by (2.3.29)

$$
\rho_t^{(\alpha)} \le \frac{c_d}{2}\Gamma\Big(\frac{d-b}{2}\Big)\frac{2}{a-d-2\theta+2}\left(t\vee 1\right)^{\frac{b-d-2\theta+2}{2}} = C_2 \left(t\vee 1\right)^{\frac{b-d-2\theta+2}{2}}, \quad (2.3.33)
$$

where $C_2 > 0$ is a constant depending only on d, θ, a and b.

By relations (2.3.31), (2.3.32) and (2.3.33), we conclude that

$$
\begin{aligned}
&\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^{(\alpha)}(t,h)\Big)^{1/2} \le \sum_{n\ge 1}(p-1)^{n/2}\bigg(\frac{1}{n!}C_\theta h^\theta \Gamma_{0,t}^n n! h_{n-1}^{(\alpha)}(t)\rho_{t/2}^{(\alpha)}\bigg)^{1/2} \\
&= C_\theta^{1/2} h^{\theta/2}\Big(\rho_{t/2}^{(\alpha)}\Big)^{1/2} \sum_{n\ge 1}(p-1)^{n/2}\Gamma_{0,t}^{n/2}\Big(h_{n-1}^{(\alpha)}(t)\Big)^{1/2} \\
&\le C_\theta^{1/2} h^{\theta/2}\Big(C_2\left(T\vee 1\right)^{\frac{b-d-2\theta+2}{2}}\Big)^{1/2} \\
&\qquad \sum_{n\ge 1}(p-1)^{n/2}\Gamma_{0,T}^{n/2}\bigg(C_1^{n-1}\left(T\vee 1\right)^{(n-1)(1-\frac{d-b}{2})}\frac{1}{\big((n-1)!\big)^{1-\frac{d-a}{2}}}\bigg)^{1/2}
\end{aligned}
$$

$$= C_\theta^{1/2} h^{\theta/2}\Big(C_2\left(T\vee 1\right)^{\frac{b-d-2\theta+2}{2}}(p-1)\Gamma_{0,T}\Big)^{1/2} \sum_{n\geq 1}\frac{\left(\sqrt{(p-1)\Gamma_{0,T}C_1\big(T\vee 1\big)^{1-\frac{d-b}{2}}}\;\right)^{n-1}}{\big((n-1)!\big)^{(1-\frac{d-a}{2})/2}} := C_3\, h^{\theta/2}, \tag{2.3.34}$$

where $C_3 > 0$ is a constant depending on θ, a, b, p, d and T. We used the fact that functions $\Gamma_{0,t}$ is non-decreasing in t in the second inequality, and we applied Lemma A.1.6 in the last equation.

As for the term $B_n^{(\alpha)}(t,h)$, let $D_{t,h} = [0,t+h]^n \backslash [0,t]^n$, we note that

$$\begin{aligned} B_n^{(\alpha)}(t,h) &= (n!)^2 \int_{[0,t+h]^n}\int_{[0,t+h]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)1_{D_{t,h}}(\mathbf{t})1_{D_{t,h}}(\mathbf{s})\int_{(\mathbb{R}^d)^n}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n\gamma(x_i-y_i) \\ &\qquad \tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t+h,x)\tilde{f}_n(s_1,x_1,\cdots,s_n,x_n,t+h,x)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &= \int_{[0,t+h]^n}\int_{[0,t+h]^n}\prod_{i=1}^n \gamma_0(t_i-s_i)1_{D_{t,h}}(\mathbf{t})1_{D_{t,h}}(\mathbf{s})\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \end{aligned} \tag{2.3.35}$$

where

$$\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{s}) = \int_{(\mathbb{R}^d)^n}\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t+h,x)(\xi_1,\ldots,\xi_n)\overline{\mathcal{F}g_{\mathbf{s}}^{(n)}(\cdot,t+h,x)(\xi_1,\ldots,\xi_n)}\prod_{j=1}^n|\xi_j|^{-\alpha}\mathrm{d}\xi_j.$$

By Cauchy-Schwarz inequality and the fact that $ab \leq (a^2+b^2)/2$, we have

$$\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{s}) \leq \Big(\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\gamma_{t,h,n}^{(\alpha)}(\mathbf{s},\mathbf{s})\Big)^{1/2} \leq \frac{1}{2}\Big(\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})+\gamma_{t,h,n}^{(\alpha)}(\mathbf{s},\mathbf{s})\Big).$$

Using the symmetry of the function γ and Lemma 2.2.3, it follows that

$$\begin{aligned} B_n^{(\alpha)}(t,h) &\leq \int_{[0,t+h]^n}\int_{[0,t+h]^n}\prod_{i=1}^n\gamma_0(t_i-s_i)1_{D_{t,h}}(\mathbf{t})1_{D_{t,h}}(\mathbf{s})\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq \Gamma_{0,t+h}^n\int_{[0,t+h]^n}\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})\,\mathrm{d}\mathbf{t}. \end{aligned} \tag{2.3.36}$$

Applying the change of variables $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, $\xi_j' = \xi_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$, we have

$$\begin{aligned} \gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t}) = \int_{(\mathbb{R}^d)^n}&\left|\mathcal{F}G_{u_1}^h(\xi_{\rho(1)})\right|^2\cdots\left|\mathcal{F}G_{u_{n-1}}^h(\xi_{\rho(1)}+\ldots+\xi_{\rho(n-1)})\right|^2 \\ &\left|\mathcal{F}G_{u_n+h}^h(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})\right|^2\prod_{j=1}^n|\xi_j|^{-\alpha}\mathrm{d}\xi_j \end{aligned}$$

$$
\begin{aligned}
&= \int_{(\mathbb{R}^d)^n} \left|\mathcal{F}G^h_{u_1}(\xi_1')\right|^2 \cdots \left|\mathcal{F}G^h_{u_{n-1}}(\xi_1' + \ldots + \xi_{n-1}')\right|^2 \\
&\qquad\qquad \left|\mathcal{F}G^h_{u_n+h}(\xi_1' + \ldots + \xi_n')\right|^2 \prod_{j=1}^n |\xi_j'|^{-\alpha} \mathrm{d}\xi_j' \\
&= \int_{\mathbb{R}^d} \left|\mathcal{F}G^h_{u_1}(\xi_1)\right|^2 |\xi_1|^{-\alpha} \int_{\mathbb{R}^d} \left|\mathcal{F}G^h_{u_2}(\xi_1+\xi_2)\right|^2 |\xi_2|^{-\alpha} \ldots \\
&\qquad \left(\int_{\mathbb{R}^d} \left|\mathcal{F}G^h_{u_n+h}(\xi_1 + \ldots + \xi_n)\right|^2 |\xi_n|^{-\alpha} \mathrm{d}\xi_n \right) \ldots \mathrm{d}\xi_2 \mathrm{d}\xi_1 \\
&\le \prod_{j=1}^{n-1} \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G^h_{u_j}(\xi_j + \eta)\right|^2 |\xi_j|^{-\alpha} \mathrm{d}\xi_j \right) \\
&\qquad\qquad \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G^h_{u_n+h}(\xi_n + \eta)\right|^2 |\xi_n|^{-\alpha} \mathrm{d}\xi_n \right) \\
&= \prod_{j=1}^{n-1} k_\alpha(u_j) k_\alpha(u_n + h), \qquad (2.3.37)
\end{aligned}
$$

where we used Lemma 2.3.2 for the last equality and $k_\alpha(t)$ is given by (2.3.6). Note that $\Gamma_{0,t+h} < \Gamma_{0,T}$ since $t + h = t' \le T$. Observe that if $(t_1, \ldots, t_n) \in D_{t,h}$, then there exists at least one index i with $t_i > t$. So

$$
D_{t,h} = \bigcup_{\rho \in S_n} \{(t_1, \ldots, t_n); 0 \le t_{\rho(1)} \le \ldots \le t_{\rho(n)}, t < t_{\rho(n)} \le t + h\}.
$$

By relations (2.3.36) and (2.3.37), it follows that

$$
\begin{aligned}
B_n^{(\alpha)}(t,h) &\le \Gamma^n_{0,t+h} \int_{[0,t+h]^n} \prod_{j=1}^{n-1} k_\alpha(u_j) k_\alpha(u_n+h) 1_{D_{t,h}}(\mathbf{t})\, \mathrm{d}\mathbf{t} \\
&\le \Gamma^n_{0,T} \sum_{\rho \in S_n} \int_t^{t+h} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}} \prod_{j=1}^{n-1} k_\alpha(t_{\rho(j+1)} - t_{\rho(j)}) k_\alpha(t - t_{\rho(n)} + h) \mathrm{d}\mathbf{t} \\
&= \Gamma^n_{0,T} n! \int_t^{t+h} \int_{0<t_1<\ldots<t_n} \prod_{j=1}^{n-1} k_\alpha(t_{j+1} - t_j) k_\alpha(t - t_n + h) \mathrm{d}\mathbf{t} \\
&= \Gamma^n_{0,T} n! \int_t^{t+h} h^{(\alpha)}_{n-1}(t_n) k_\alpha(t - t_n + h) \mathrm{d}t_n \\
&\le \Gamma^n_{0,T} n! h^{(\alpha)}_{n-1}(t+h) \int_0^h k_\alpha(s) \mathrm{d}s \le \Gamma^n_{0,T} n! h^{(\alpha)}_{n-1}(t+h) h^\theta \rho^{(\alpha)}_h \\
&\le \Gamma^n_{0,T} n! h^{(\alpha)}_{n-1}(t+h) h^\theta \rho^{(\alpha)}_T, \qquad (2.3.38)
\end{aligned}
$$

since $\Gamma_{0,t}$ is non-decreasing in t. Hence, using relations (2.3.38), (2.3.32) and (2.3.33), we obtain

$$\begin{aligned}
&\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^{(\alpha)}(t,h)\Big)^{1/2} \leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}\Gamma_{0,T}^n n! h_{n-1}^{(\alpha)}(t+h)h^\theta \rho_T^{(\alpha)}\Big)^{1/2} \\
&= h^{\theta/2}\Big(\rho_T^{(\alpha)}\Big)^{1/2}\sum_{n\geq 1}(p-1)^{n/2}\Gamma_{0,T}^{n/2}\Big(h_{n-1}^{(\alpha)}(t+h)\Big)^{1/2} \\
&\leq h^{\theta/2}\Big(C_2\,(T\vee 1)^{\frac{b-d-2\theta+2}{2}}\Big)^{1/2} \\
&\qquad \sum_{n\geq 1}(p-1)^{n/2}\Gamma_{0,T}^{n/2}\Bigg(C_1^{n-1}(T\vee 1)^{(n-1)(1-\frac{d-b}{2})}\frac{1}{\big((n-1)!\big)^{1-\frac{d-a}{2}}}\Bigg)^{1/2} \\
&= h^{\theta/2}\Big(C_2\,(T\vee 1)^{\frac{b-d-2\theta+2}{2}}(p-1)\Gamma_{0,T}\Big)^{1/2}\sum_{n\geq 1}\frac{\Big(\sqrt{(p-1)\Gamma_{0,T}C_1(T\vee 1)^{1-\frac{d-b}{2}}}\Big)^{n-1}}{\big((n-1)!\big)^{(1-\frac{d-a}{2})/2}} \\
&:= C_4\,h^{\theta/2}, \qquad (2.3.39)
\end{aligned}$$

where $C_4 > 0$ is a constant depending on θ, a, b, p, d and T, and we used Lemma A.1.6 in the last equality.

Back to relation (2.3.21), combining relations (2.3.34) and (2.3.39), it follows that

$$\begin{aligned}
\|u^\alpha(t+h,x)-u^\alpha(t,x)\|_p &\leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{2}{n!}\big(A_n^{(\alpha)}(t,h)+B_n^{(\alpha)}(t,h)\big)\Big)^{1/2} \\
&\leq \sqrt{2}\Bigg(\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^{(\alpha)}(t,h)\Big)^{1/2}+\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^{(\alpha)}(t,h)\Big)^{1/2}\Bigg) \\
&\leq \sqrt{2}\big(C_3+C_4\big)h^{(1-\beta)/2} \qquad (2.3.40)
\end{aligned}$$

and therefore, taking supremum over $\alpha \in [a,b]$ on both sides of relation (2.3.40), we have relation (2.3.1).

Step 1.b Now we treat the spatial increments for the heat equation. For any $x, x' \in \mathbb{R}^d$, we let $z = x' - x$. By Minkowski's inequality and Lemma A.5.3,

$$\begin{aligned}
\|u^\alpha(t,x+z)-u^\alpha(t,x)\|_p &\leq \sum_{n\geq 1}(p-1)^{n/2}\big\|I_n^\alpha\big(f_n(\cdot,t,x+z)-f_n(\cdot,t,x)\big)\big\|_2 \\
&\leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}C_n^{(\alpha)}(t,z)\Big)^{1/2} \qquad (2.3.41)
\end{aligned}$$

where

$$C_n^{(\alpha)}(t,z) := (n!)^2 \|\tilde{f}_n(\cdot,t,x+z) - \tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} = \int_{[0,t]^{2n}} \prod_{i=1}^n \gamma_0(t_i - s_i)\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \tag{2.3.42}$$

and

$$\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{s}) = \int_{(\mathbb{R}^d)^n} \mathcal{F}\big[g^{(n)}_{\mathbf{t}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{t}}(\cdot,t,x)\big](\xi_1,\dots,\xi_n) \overline{\mathcal{F}\big[g^{(n)}_{\mathbf{s}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{s}}(\cdot,t,x)\big](\xi_1,\dots,\xi_n)} \prod_{j=1}^n |\xi_j|^{-\alpha}\mathrm{d}\xi_j. \tag{2.3.43}$$

Again, by Cauchy-Schwarz inequality and the fact that $ab \le (a^2+b^2)/2$, we have

$$\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{s}) \le \Big(\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\Psi^{(\alpha)}_{t,z,n}(\mathbf{s},\mathbf{s})\Big)^{1/2} \le \frac{1}{2}\Big(\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t}) + \Psi^{(\alpha)}_{t,z,n}(\mathbf{s},\mathbf{s})\Big).$$

Using Lemma 2.2.3, it follows that

$$C_n^{(\alpha)}(t,z) \le \int_{[0,t]^n}\int_{[0,t]^n} \prod_{i=1}^n \gamma_0(t_i-s_i)\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \le \Gamma^n_{0,t}\int_{[0,t]^n} \Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t})\,\mathrm{d}\mathbf{t} \tag{2.3.44}$$

where

$$\begin{aligned}\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t}) &= \int_{(\mathbb{R}^d)^n} \big|\mathcal{F}G^h_{u_1}(\xi_1)\big|^2 \cdots \big|\mathcal{F}G^h_{u_{n-1}}(\xi_1+\ldots+\xi_{n-1})\big|^2 \\ &\qquad \times \big|\mathcal{F}G^h_{u_n}(\xi_1+\ldots+\xi_n)\big|^2 \big|1-e^{-i(\xi_1+\ldots+\xi_n)\cdot z}\big|^2 \prod_{j=1}^n |\xi_j|^{-\alpha}\mathrm{d}\xi_j \\ &\le \prod_{j=1}^{n-1}\Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\mathcal{F}G^h_{u_j}(\xi_j+\eta)\big|^2|\xi_j|^{-\alpha}\mathrm{d}\xi_j\Big) \\ &\qquad \times \Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\mathcal{F}G^h_{u_n}(\xi_n+\eta)\big|^2\big|1-e^{-i(\xi_n+\eta)\cdot z}\big|^2|\xi_n|^{-\alpha}\mathrm{d}\xi_n\Big) \end{aligned} \tag{2.3.45}$$

$$\le C_\theta |z|^{2\theta}\prod_{j=1}^{n-1} k_\alpha(u_j)u_n^{-\theta}k_\alpha(u_n/2), \tag{2.3.46}$$

where we used Lemma 2.3.2 and Proposition 2.3.3 in the last inequality. Note that (2.3.46) is similar to (2.3.27). Therefore, the same argument as for (2.3.34) allows us to conclude that:

$$\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}C_n^{(\alpha)}(t,z)\Big)^{1/2} \le C_5\,|z|^\theta, \tag{2.3.47}$$

where $C_5 > 0$ is a constant depending on θ, a, b, p, d and T.

Returning to relation (2.3.41), we used relations (2.3.47), then

$$\|u^\alpha(t+h,x) - u^\alpha(t,x)\|_p \leq \sum_{n\geq 1} (p-1)^{n/2} \Big(\frac{1}{n!} C_n^{(\alpha)}(t,z)\Big)^{1/2} \leq C_5 \, |z|^{1-\beta} \tag{2.3.48}$$

and therefore, taking supremum over $\alpha \in [a,b]$ on both sides of relation (2.3.48), we have relation (2.3.2).

Step 2: Wave equation

Step 2.a We examine the time increments for the solution of the wave equation. The proof is similar to that of the heat equation. Assume that $h = t' - t > 0$ (The case $h < 0$ is similar). We fix $\mathbf{t} = (t_1, \ldots, t_n) \in [0,t]^n$ and we let $\rho \in S_n$ such that $t_{\rho(1)} < \ldots < t_{\rho(n)}$. We define $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \ldots, n$ and $t_{\rho(n+1)} = t$. Note that relation (2.3.26) still hold. To estimate $\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t})$, we use Lemma 2.3.4. It follows that

$$\begin{aligned}
\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t}) &\leq \prod_{j=1}^{n-1} \Big(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}G^w_{u_j}(\xi_j+\eta)\big|^2 |\xi_j|^{-\alpha} \mathrm{d}\xi_j \Big) \\
&\qquad \times \Big(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}G^w_{u_n+h}(\xi_n+\eta) - \mathcal{F}G^w_{u_n}(\xi_n+\eta)\big|^2 |\xi_n|^{-\alpha} \mathrm{d}\xi_n \Big) \\
&\leq \prod_{j=1}^{n-1} \Big(C_{T,d,a,b,\beta} u_j^{2(1-\beta)} \Big) \times \Big(C_{d,a,b,\beta} h^{2(1-\beta)} \Big) \leq C_1^n (u_1 \ldots, u_{n-1} h)^{2(1-\beta)},
\end{aligned} \tag{2.3.49}$$

where $C_1 > 0$ is a constant depending on T, d, a, b and β. Using relations (2.3.25), (2.3.49) and Lemma A.3.1 with $\beta_j = 2-2\beta$ for all $j = 1, \ldots, n$, there exists a constant $C_2 > 0$ depending on T, d, a, b and β such that

$$\begin{aligned}
A_n^{(\alpha)}(t,h) &\leq \Gamma_{0,t}^n \Big(\sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t} \psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t}) \mathrm{d}\mathbf{t} \Big) \\
&\leq h^{2-2\beta} \Gamma_{0,t}^n \, C_1^n \, n! \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} (t_{j+1}-t_j)^{2-2\beta} \mathrm{d}t_1 \ldots \mathrm{d}t_n \\
&= h^{2-2\beta} \Gamma_{0,t}^n \, C_1^n \, n! \int_0^t \Big(\int_{0<t_1<\ldots<t_n} \prod_{j=1}^{n-1} (t_{j+1}-t_j)^{2-2\beta} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \Big) \mathrm{d}t_n \\
&= h^{2-2\beta} \Gamma_{0,t}^n \, C_1^n \, n! \frac{\big(\Gamma(3-2\beta)\big)^{n-1}}{\Gamma\big((n-1)(3-2\beta)+1\big)} \int_0^t t_n^{(n-1)(3-2\beta)} \mathrm{d}t_n \\
&\leq h^{2-2\beta} \Gamma_{0,t}^n \, C_2^n \, n! \frac{1}{((n-1)!)^{3-2\beta}} t^{(n-1)(3-2\beta)+1},
\end{aligned} \tag{2.3.50}$$

where we used the change of variables $t_j = t_{\rho(j)}$ for all $j = 1, \dots, n$ in the second inequality and Lemma A.1.3 in the last inequality. Hence, by (2.3.50), there exists a constant $C_3 > 0$ depending on p, β, a, b, d and T such that

$$\begin{aligned}
&\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^{(\alpha)}(t,h)\Big)^{1/2} \\
&\leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}h^{2-2\beta}\Gamma_{0,t}^n\, C_2^n\, n!\frac{1}{((n-1)!)^{3-2\beta}}t^{(n-1)(3-2\beta)+1}\Big)^{1/2} \\
&\leq h^{1-\beta}\sum_{n\geq 1}(p-1)^{n/2}\left(\Gamma_{0,T}^n\, C_2^n\,\frac{1}{((n-1)!)^{3-2\beta}}(T\vee 1)^{(n-1)(3-2\beta)+1}\right)^{1/2} \\
&= h^{1-\beta}\Big((p-1)\Gamma_{0,T}C_2(T\vee 1)\Big)^{1/2}\sum_{n\geq 1}\frac{\left(\sqrt{(p-1)\Gamma_{0,T}C_2\big(T\vee 1\big)^{3-2\beta}}\,\right)^{n-1}}{\big((n-1)!\big)^{(3-2\beta)/2}} \\
&:= C_3\, h^{1-\beta},
\end{aligned} \tag{2.3.51}$$

where we used the fact that $\Gamma_{0,t}$ is non-decreasing in t and we applied Lemma A.1.6 in the last equation.

For the term $B_n^{(\alpha)}(t,h)$, note that the inequality in relation (2.3.37) still holds. By Lemma 2.3.4, there exists a constant C_4 depending on T, d, a, b and β such that

$$\begin{aligned}
\gamma_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t}) &\leq \prod_{j=1}^{n-1}\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_j}^w(\xi_j+\eta)\big|^2|\xi_j|^{-\alpha}\mathrm{d}\xi_j\right) \\
&\qquad\qquad\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G_{u_n+h}^w(\xi_n+\eta)\big|^2|\xi_n|^{-\alpha}\mathrm{d}\xi_n\right) \\
&\leq \prod_{j=1}^{n-1}\Big(C_{T,d,a,b,\beta}\, u_j^{2-2\beta}\Big)\times\Big(C_{T,d,a,b,\beta}\,(u_n+h)^{2-2\beta}\Big) \\
&= C_4^n\big(u_1\dots u_{n-1}(u_n+h)\big)^{2-2\beta},
\end{aligned} \tag{2.3.52}$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, for all $j = 1, \dots, n$ and $0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t = t_{\rho(n+1)}$. By relation (2.3.36) and Lemma A.3.1 with $\beta_j = 2-2\beta$ for all $j = 1, \dots, n$, we obtain

$$\begin{aligned}
&B_n^{(\alpha)}(t,h) \leq \Gamma_{0,t+h}^n C_4^n\int_{[0,t+h]^n}\big[u_1\dots u_{n-1}(u_n+h)\big]^{2-2\beta}1_{D_{t,h}}(\mathbf{t})\,\mathrm{d}\mathbf{t} \\
&\leq \Gamma_{0,t+h}^n C_4^n\sum_{\rho\in S_n}\int_t^{t+h}\int_{0<t_{\rho(1)}<\dots<t_{\rho(n)}}\prod_{j=1}^{n-1}(t_{\rho(j+1)}-t_{\rho(j)})^{2-2\beta}(t+h-t_{\rho(n)})^{2-2\beta}\mathrm{d}\mathbf{t} \\
&= \Gamma_{0,t+h}^n\, C_4^n\, n!\int_t^{t+h}\Big(\int_{0<t_1<\dots<t_n}\prod_{j=1}^{n-1}(t_{j+1}-t_j)^{2-2\beta}\mathrm{d}t_1\dots\mathrm{d}t_{n-1}\Big)(t+h-t_n)^{2-2\beta}\mathrm{d}t_n
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma_{0,t+h}^n\, C_4^n\, n! \frac{\big(\Gamma(3-2\beta)\big)^{n-1}}{\Gamma\big((n-1)(3-2\beta)+1\big)} \int_t^{t+h} t_n^{(n-1)(3-2\beta)}(t+h-t_n)^{2-2\beta}\mathrm{d}t_n \\
&= \Gamma_{0,t+h}^n\, C_4^n\, n! \frac{\big(\Gamma(3-2\beta)\big)^{n-1}}{\Gamma\Big((n-1)(3-2\beta)+1\Big)} \int_0^{h} (t+h-u)^{(n-1)(3-2\beta)}(u)^{2-2\beta}\mathrm{d}u \\
&\le \Gamma_{0,T}^n\, C_4^n\, n! \frac{\big(\Gamma(3-2\beta)\big)^{n-1}}{\Gamma\Big((n-1)(3-2\beta)+1\Big)} (T\vee 1)^{(n-1)(3-2\beta)} \frac{1}{3-2\beta} h^{3-2\beta} \\
&\le \Gamma_{0,T}^n\, C_5^n\, n! \frac{1}{((n-1)!)^{3-2\beta}} (T\vee 1)^{(n-1)(3-2\beta)} h^{2-2\beta}, \qquad (2.3.53)
\end{aligned}
$$

where we used the fact that function $\Gamma_{0,t}$ is non-decreasing in t and for the last inequality, we used Lemma A.1.3. Hence, using relation (2.3.53), we obtain

$$
\begin{aligned}
&\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^{(\alpha)}(t,h)\Big)^{1/2} \\
&\le \sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}\Gamma_{0,T}^n\, C_5^n\, n! \frac{1}{((n-1)!)^{3-2\beta}} (T\vee 1)^{(n-1)(3-2\beta)} h^{2-2\beta}\Big)^{1/2} \\
&= h^{1-\beta}\sum_{n\ge 1}(p-1)^{n/2}\Big(\Gamma_{0,T}^n\, C_5^n\, \frac{1}{((n-1)!)^{3-2\beta}} (T\vee 1)^{(n-1)(3-2\beta)}\Big)^{1/2} \\
&= h^{1-\beta}\Big((p-1)\Gamma_{0,T}\, C_5\Big)^{1/2} \sum_{n\ge 1} \frac{\Big(\sqrt{(p-1)\Gamma_{0,T}\, C_5\, \big(T\vee 1\big)^{(3-2\beta)}}\Big)^{n-1}}{\big((n-1)!\big)^{(3-2\beta)/2}} := C_6\, h^{1-\beta}, \qquad (2.3.54)
\end{aligned}
$$

where $C_6 > 0$ is a constant depending on T, d, a, b and β and we applied Lemma A.1.6 in the last equation.

Back to relation (2.3.21), combining relations (2.3.51) and (2.3.54), it follows that

$$
\begin{aligned}
\|u^\alpha(t+h,x)-u^\alpha(t,x)\|_p &\le \sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{2}{n!}\big(A_n^{(\alpha)}(t,h)+B_n^{(\alpha)}(t,h)\big)\Big)^{1/2} \\
&\le \sqrt{2}\bigg(\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^{(\alpha)}(t,h)\Big)^{1/2} + \sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^{(\alpha)}(t,h)\Big)^{1/2}\bigg) \\
&\le \sqrt{2}\big(C_3+C_6\big)h^{1-\beta} \qquad (2.3.55)
\end{aligned}
$$

and therefore, taking supremum over $\alpha \in [a,b]$ on both sides of relation (2.3.55), we have relation (2.3.3).

Step 2.b We examine the spatial increments for the solution of the wave equation. We denote by $K = [-M, M] \subset \mathbb{R}^d$ be an arbitrary subset, for some $M > 0$. For any $x, x' \in K$, we let $z = x' - x$. Note that inequalities (2.3.44) and (2.3.45) still hold. To estimate $\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t})$, we use Lemma 2.3.4. Then

$$\begin{aligned}\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t}) &\le \prod_{j=1}^{n-1}\Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G^w_{u_j}(\xi_j+\eta)\big|^2|\xi_j|^{-\alpha}\mathrm{d}\xi_j\Big)\\ &\qquad\times\Big(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathcal{F}G^w_{u_n}(\xi_n+\eta)\big|^2\big|1-e^{-i(\xi_n+\eta)\cdot z}\big|^2|\xi_n|^{-\alpha}\mathrm{d}\xi_n\Big)\\ &\le \prod_{j=1}^{n-1}\Big(C_{T,d,a,b,\beta}\,u_j^{2-2\beta}\Big)\times\Big(C_{T,M,d,a,b,\beta}|z|^{2-2\beta}\Big) \le C^n|z|^{2-2\beta}(u_1\dots u_{n-1})^{2-2\beta},\end{aligned} \tag{2.3.56}$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t_{\rho(n+1)} = t$ and C is a constant depending on T, M, d, a, b and β. Note that (2.3.56) is similar to (2.3.49). Therefore, the same argument as for (2.3.51) allows us to conclude that:

$$\sum_{n\ge1}(p-1)^{n/2}\Big(\frac{1}{n!}C^{(\alpha)}_n(t,z)\Big)^{1/2} \le C_7|z|^{1-\beta}, \tag{2.3.57}$$

where $C_7 > 0$ is a constant depending on p, T, M, d, a, b and β.

Returning to relation (2.3.41), we use relation (2.3.57). It follows that

$$\|u^\alpha(t+h,x) - u^\alpha(t,x)\|_p \le \sum_{n\ge1}(p-1)^{n/2}\Big(\frac{1}{n!}C^{(\alpha)}_n(t,z)\Big)^{1/2} \le C_7\,|z|^{1-\beta} \tag{2.3.58}$$

and therefore, taking supremum over $\alpha \in [a, b]$ on both sides of relation (2.3.58), we have relation (2.3.4). ∎

2.4 Continuity in law of the solution with respect to the noise parameter α

In this section, we consider equations (2.0.1) and (2.0.2) driven by the noise W^α introduced in Section 2.1. We prove that the solution of either one of these equations is continuous in law in the space of continuous functions $C([0,T]\times\mathbb{R}^d)$, with respect to the noise parameter α.

We first give the simultaneous construction of all noise processes $(W^\alpha)_{\alpha\in(0,d)}$ on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ using their spectral representations based on a complex-valued Gaussian measure. With this, the family of processes $(W^\alpha)_{\alpha\in(0,d)}$ are defined in a single probability space, and we can therefore apply Lemma 2.4.3 below to prove the finite dimensional distribution of u^{α_n} converge to those of u^{α^*}, when $n\to\infty$.

Definition 2.4.1. *Let $W_1=\{W_1(A);A\in\mathcal{B}_b(\mathbb{R}\times\mathbb{R}^d)\}$ and $W_2=\{W_2(A);A\in\mathcal{B}_b(\mathbb{R}\times\mathbb{R}^d)\}$ be two independent space-time Gaussian white noise processes defined on the same probability space $(\Omega,\mathcal{F},P)$ with covariance:*

$$\mathbb{E}[W_j(A)W_j(B)]=\frac{1}{2}\big|A\cap B\big|, j=1,2.$$

We say that

$$\widehat{W}(A):=W_1(A)+iW_2(A)$$

is a $\mathbb{C}$-valued Gaussian random measure, with control measure given by Lebesgue measure.

For any $\alpha\in(0,d)$ and for any function $\varphi\in\mathcal{S}(\mathbb{R}^{d+1})$, we define

$$\begin{aligned}\widehat{W}^\alpha(\varphi)&=\int_{\mathbb{R}^{d+1}}\varphi(\tau,\xi)\widehat{W}^\alpha(\mathrm{d}\tau,\mathrm{d}\xi)\\&:=\int_{\mathbb{R}^{d+1}}\varphi(\tau,\xi)\sqrt{g_0(\tau)}|\xi|^{-\alpha/2}\widehat{W}(\mathrm{d}\tau,\mathrm{d}\xi)=\widehat{W}\big(\varphi(\tau,\xi)\sqrt{g_0(\tau)}|\xi|^{-\alpha/2}\big).\end{aligned}\tag{2.4.1}$$

Formally, we can say

$$\widehat{W}^\alpha(\mathrm{d}\tau,\mathrm{d}\xi)=\sqrt{g_0(\tau)}|\xi|^{-\alpha/2}\widehat{W}(\mathrm{d}\tau,\mathrm{d}\xi).$$

We define

$$W^\alpha(\varphi)=\widehat{W}^\alpha(\mathcal{F}\varphi).\tag{2.4.2}$$

Then we have

$$\begin{aligned}\mathbb{E}[W^\alpha(\varphi)\overline{W^\alpha(\psi)}]&=\mathbb{E}[\widehat{W}^\alpha(\mathcal{F}\varphi)\overline{\widehat{W}^\alpha(\mathcal{F}\psi)}]\\&=\mathbb{E}\Big[\widehat{W}\big(\mathcal{F}\varphi(\tau,\xi)\sqrt{g_0(\tau)}|\xi|^{-\frac{\alpha}{2}}\big)\overline{\widehat{W}\big(\mathcal{F}\psi(\tau,\xi)\sqrt{g_0(\tau)}|\xi|^{-\frac{\alpha}{2}}\big)}\Big]\\&=\int_{\mathbb{R}^{d+1}}\mathcal{F}\varphi(\tau,\xi)\sqrt{g_0(\tau)}|\xi|^{-\frac{\alpha}{2}}\overline{\mathcal{F}\psi(\tau,\xi)}\sqrt{g_0(\tau)}|\xi|^{-\frac{\alpha}{2}}\mathrm{d}\tau\mathrm{d}\xi\\&=\int_{\mathbb{R}^{d+1}}\mathcal{F}\varphi(\tau,\xi)\overline{\mathcal{F}\psi(\tau,\xi)}\mu_0(\mathrm{d}\tau)|\xi|^{-\alpha}\mathrm{d}\xi.\end{aligned}$$

This means that the Gaussian process $W^\alpha=\{W^\alpha(\varphi);\varphi\in\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^d)\}$ has the desired covariance structure (2.1.3). More over, all processes $(W^\alpha)_{\alpha\in(0,d)}$ are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout this section, without any loss of generality, we assume the noise W^α perturbing equations (2.0.1) and (2.0.2) is given by (2.4.2).

Note that relation (2.4.2) can be written as

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}^d}\varphi(t,x)W^\alpha(\mathrm{d}t,\mathrm{d}x) &= \int_{\mathbb{R}\times\mathbb{R}^d}\mathcal{F}\varphi(\tau,\xi)\widehat{W}^\alpha(\mathrm{d}\tau,\mathrm{d}\xi)\\ &= \int_{\mathbb{R}\times\mathbb{R}^d}\mathcal{F}_t\big[\mathcal{F}_x\varphi(t,\cdot)(\xi)\big](\tau)\widehat{W}^\alpha(\mathrm{d}\tau,\mathrm{d}\xi),\end{aligned} \tag{2.4.3}$$

where $\mathcal{F}\varphi(\tau,\xi)$ is the Fourier transform of φ in both variables (t,x):

$$\mathcal{F}\varphi(\tau,\xi) = \int_{\mathbb{R}_+\times\mathbb{R}^d} e^{-i\tau t}e^{-i\xi\cdot x}\varphi(t,x)\mathrm{d}t\mathrm{d}x.$$

and the Fourier transforms appearing in the second equality are defined as follows:

$$\phi_\xi(t) := \mathcal{F}_x\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(t,x)\mathrm{d}x, \quad \text{for all } t>0$$

and $\mathcal{F}_t\big[\mathcal{F}_x\varphi(t,\cdot)(\xi)\big](\tau)$ is the Fourier transform of ϕ_ξ given by

$$\mathcal{F}\phi_\xi(\tau) = \int_{\mathbb{R}^d} e^{-i\tau t}\phi_\xi(t)\mathrm{d}t.$$

We have

$$\mathbb{E}\Big|\int_{\mathbb{R}\times\mathbb{R}^d} h(\tau,\xi)\widehat{W}(\mathrm{d}\tau,\mathrm{d}\xi)\Big|^2 = \int_{\mathbb{R}\times\mathbb{R}^d}\big|h(\tau,\xi)\big|^2\mathrm{d}\tau\mathrm{d}\xi.$$

A similar property holds for multiple integrals with respect to $\widehat{W}$, namely,

$$\begin{aligned}\mathbb{E}\Big|\int_{(\mathbb{R}\times\mathbb{R}^d)^n} & h(\tau_1,\xi_1,\dots,\tau_n,\xi_n)\widehat{W}(\mathrm{d}\tau_1,\mathrm{d}\xi_1)\dots\widehat{W}(\mathrm{d}\tau_n,\mathrm{d}\xi_n)\Big|^2\\ &= n!\int_{(\mathbb{R}\times\mathbb{R}^d)^n}\big|\tilde{h}(\tau_1,\xi_1,\dots,\tau_n,\xi_n)\big|^2\mathrm{d}\tau_1\mathrm{d}\xi_1\dots\mathrm{d}\tau_n\mathrm{d}\xi_n,\end{aligned} \tag{2.4.4}$$

where $\tilde{h}$ is the symmetrization of h.

Recall that the solution has the Wiener chaos expansion (2.2.2). By definition, $u^\alpha(t,x)$ is the $L^2(\Omega)$-limit of the sequence $\{u^\alpha_m(t,x)\}_{m\geq 1}$ defined by

$$u^\alpha_m(t,x) = \sum_{n=0}^m I^\alpha_n(f_n(\cdot,t,x)). \tag{2.4.5}$$

This means that

$$\mathbb{E}\big|u_m^\alpha(t,x)-u^\alpha(t,x)\big|^2 \to 0, \quad \text{as } m\to\infty \tag{2.4.6}$$

for any $\alpha \in (\max\{d-2,0\},d)$ fixed.

Proposition 2.4.2 below gives an extension of relation (2.4.3) to multiple Wiener integrals with respect to W^α. The proof is omitted.

Proposition 2.4.2. *Let $n \geq 1$, $f \in \mathcal{H}_\alpha^{\otimes n}$ such that $f(t_1,\cdot,\ldots,t_n,\cdot)$ is a function in $L^1(\mathbb{R}^d)$ for any $t_1,\ldots,t_n > 0$, then*

$$\begin{aligned}
&\int_{(\mathbb{R}_+\times\mathbb{R}^d)^n} f(t_1,x_1,\ldots,t_n,x_n)W^\alpha(\mathrm{d}t_1,\mathrm{d}x_1)\ldots W^\alpha(\mathrm{d}t_n,\mathrm{d}x_n)\\
&=\int_{(\mathbb{R}\times\mathbb{R}^d)^n} \mathcal{F}_t\big[\mathcal{F}_x f(t_1,\cdot,\ldots,t_n.\cdot)(\xi_1,\ldots,\xi_n)\big](\tau_1,\ldots,\tau_n)\widehat{W}^\alpha(\mathrm{d}\tau_1,\mathrm{d}\xi_1)\ldots\widehat{W}^\alpha(\mathrm{d}\tau_n,\mathrm{d}\xi_n)\\
&=\int_{(\mathbb{R}\times\mathbb{R}^d)^n} \mathcal{F}_t\phi_{\xi_1,\ldots,\xi_n}(\tau_1,\ldots,\tau_n)\\
&\qquad\qquad \prod_{j=1}^n\sqrt{g_0(\tau_j)}\prod_{j=1}^n|\xi_j|^{-\alpha/2}\widehat{W}(\mathrm{d}\tau_1,\mathrm{d}\xi_1)\ldots\widehat{W}(\mathrm{d}\tau_n,\mathrm{d}\xi_n),
\end{aligned}$$

where $\phi_{\xi_1,\ldots,\xi_n} = \mathcal{F}_x f(t_1,\cdot,\ldots,t_n.\cdot)(\xi_1,\ldots,\xi_n)$ is the Fourier transform of $f(t_1,\cdot,\ldots,t_n,\cdot)$ given by

$$\mathcal{F}_x f(t_1,\cdot,\ldots,t_n,\cdot)(\xi_1,\ldots,\xi_n) = \int_{(\mathbb{R}^d)^n} e^{-i\sum_{j=1}^n \xi_j\cdot x_j} f(t_1,x_1,\ldots,t_n,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n$$

and $\mathcal{F}_t\big[\mathcal{F}_x f(t_1,\cdot,\ldots,t_n.\cdot)(\xi_1,\ldots,\xi_n)\big](\tau_1,\ldots,\tau_n)$ is the Fourier transform of $\phi_{\xi_1,\ldots,\xi_n}$ given by

$$\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1,\ldots,\tau_n) = \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n \tau_j t_j}\phi_{\xi_1,\ldots,\xi_n}(t_1,\ldots,t_n)\mathrm{d}t_1\ldots\mathrm{d}t_n.$$

Note that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(t)\phi(s)\gamma_0(t-s)\mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}}\big|\mathcal{F}\phi(\tau)\big|^2 g_0(\tau)\mathrm{d}\tau.$$

This relationship can be extended to higher dimensions, as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(t_1,\ldots,t_n)\phi(s_1,\ldots,s_n)\prod_{j=1}^n\gamma_0(t_j-s_j)\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}\\
&\qquad= \int_{\mathbb{R}^n}\big|\mathcal{F}\phi(\tau_1,\ldots,\tau_n)\big|^2\prod_{j=1}^n g_0(\tau_j)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n.
\end{aligned}\tag{2.4.7}$$

The following result will be used in the proof of Theorem 2.0.1 below.

Lemma 2.4.3. *Let $X, (X_n)_n$ be random vectors in $\mathbb{R}^m$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $X = (X^1, \ldots, X^m)$ and $X_n = (X_n^1, \ldots, X_n^m)$. If*

$$\mathbb{E}\left|X_n^i - X^i\right|^2 \to 0, \quad \text{as } n \to \infty,$$

for all $i = 1, \ldots, m$, then

$$(X_n^1, \ldots, X_n^m) \xrightarrow{d} (X^1, \ldots, X^m), \text{ as } n \to \infty.$$

We will also use the following result from classical theory of metric spaces.

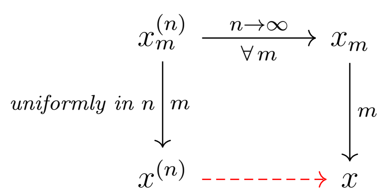
Lemma 2.4.4. *Let (E, d) be a metric space, $(x^{(n)})_m$, x_m, $x^{(n)}$ and $x \in E$ such that*

$$\begin{cases} (a) \quad d(x_m^{(n)}, x_m) \to 0, \text{ as } n \to \infty, \text{ for all } m \geq 1 \text{ fixed }, \\ (b) \quad d(x_m, x) \to 0, \text{ as } m \to \infty, \\ (c) \quad \sup_{n\geq 1} d(x_m^{(n)}, x^{(n)}) \to 0, \text{ as } m \to \infty. \end{cases}$$

Then

$$d(x^{(n)}, x) \to 0, \text{ as } n \to \infty.$$

This lemma can be illustrated by the following diagram:



In addition, for the proof of Theorem 2.0.1, we will use following result.

Lemma 2.4.5. *Under the hypothese of Theorem 2.0.1, for all $k \geq 1$ fixed, we have*

$$\mathbb{E}\left|I_k^{\alpha_n}(f_k(\cdot, t, x)) - I_k^{\alpha^*}(f_k(\cdot, t, x))\right|^2 \to 0, \text{ as } n \to \infty. \tag{2.4.8}$$

Proof: Note that by Proposition 2.4.2, we have

$$\begin{aligned} &I_k^{\alpha_n}(f_k(\cdot, t, x)) - I_k^{\alpha^*}(f_k(\cdot, t, x)) \\ &\quad = \int_{(\mathbb{R}_+\times\mathbb{R}^d)^k} f_k(t_1, x_1, \ldots, t_k, x_k, t, x) W^{\alpha_n}(\mathrm{d}t_1, \mathrm{d}t_k) \cdots W^{\alpha_n}(\mathrm{d}t_1, \mathrm{d}t_k) \\ &\qquad\qquad - \int_{(\mathbb{R}_+\times\mathbb{R}^d)^k} f_k(t_1, x_1, \ldots, t_k, x_k, t, x) W^{\alpha^*}(\mathrm{d}t_1, \mathrm{d}t_k) \cdots W^{\alpha^*}(\mathrm{d}t_1, \mathrm{d}t_k) \end{aligned}$$

$$= \int_{(\mathbb{R}\times\mathbb{R}^d)^k} H_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k)\widehat{W}(\mathrm{d}\tau_1,\mathrm{d}\xi_1)\cdots\widehat{W}(\mathrm{d}\tau_k,\mathrm{d}\xi_k), \tag{2.4.9}$$

where

$$H_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k) = \mathcal{F}_t\big[\mathcal{F}_x f_k(t_1,\cdot,\dots,t_k,\cdot,t,x)(\xi_1,\dots,\xi_k)\big](\tau_1,\dots,\tau_k) \prod_{j=1}^k \sqrt{g_0(\tau_j)}\Big(\prod_{j=1}^k|\xi_j|^{-\alpha_n/2} - \prod_{j=1}^k|\xi_j|^{-\alpha^*/2}\Big). \tag{2.4.10}$$

We denote by $\widetilde{H}_k$ the symmetrization of H_k. Then, by equation (2.4.4), we have

$$\begin{aligned} Q_n &= \mathbb{E}\big|I_k^{\alpha_n}(f_k(\cdot,t,x)) - I_k^{\alpha^*}(f_k(\cdot,t,x))\big|^2 \\ &= \mathbb{E}\Big|\int_{(\mathbb{R}_+\times\mathbb{R}^d)^k} H_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k)\widehat{W}(\mathrm{d}\tau_1,\mathrm{d}\xi_1)\cdots\widehat{W}(\mathrm{d}\tau_k,\mathrm{d}\xi_k)\Big|^2 \\ &= k!\int_{(\mathbb{R}\times\mathbb{R}^d)^k}\big|\widetilde{H}_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k)\big|^2\mathrm{d}\tau_1\mathrm{d}\xi_1\cdots\mathrm{d}\tau_k\mathrm{d}\xi_k = k!\big\|\widetilde{H}_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k)\big\|^2_{L^2_{\mathbb{C}}((\mathbb{R}\times\mathbb{R}^d)^k)} \end{aligned} \tag{2.4.11}$$

where $L^2_{\mathbb{C}}(\mathbb{R}\times\mathbb{R}^d)$ is the space of complex-valued functions φ on $\mathbb{R}\times\mathbb{R}^d$ such that $|\varphi|^2$ is integrable with respect to the Lebesgue measure. We will use the fact that $\|\widetilde{f}\|_{L^2(\mathbb{R}\times\mathbb{R}^d)^n} \le \|f\|_{L^2(\mathbb{R}\times\mathbb{R}^d)^n}$. Applying (2.4.7) to the function $\phi(t_1,\dots,t_k) = \mathcal{F}_x f_k(t_1,\cdot,\dots,t_k,\cdot,t,x)(\xi_1,\dots,\xi_k)$ which vanishes if $(t_1,\dots,t_k)\notin T_k(t)$, we obtain

$$\begin{aligned} Q_n &\le k!\big\|H_k(\tau_1,\xi_1,\dots,\tau_k,\xi_k)\big\|^2_{L^2_{\mathbb{C}}((\mathbb{R}\times\mathbb{R}^d)^k)} \\ &= k!\int_{\mathbb{R}^k}\int_{(\mathbb{R}^d)^k}\Big|\mathcal{F}_t\big[\mathcal{F}_x f_k(t_1,\cdot,\dots,t_k,\cdot,t,x)(\xi_1,\dots,\xi_k)\big](\tau_1,\dots,\tau_k)\Big|^2 \\ &\qquad \prod_{j=1}^k g_0(\tau_j)\Big|\prod_{j=1}^k|\xi_j|^{-\alpha_n/2} - \prod_{j=1}^k|\xi_j|^{-\alpha^*/2}\Big|^2\mathrm{d}\xi_1\dots\mathrm{d}\xi_k\mathrm{d}\tau_1\dots\mathrm{d}\tau_k \\ &= k!\int_{(\mathbb{R}^d)^k}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_k\Big|\prod_{j=1}^k|\xi_j|^{-\alpha_n/2} - \prod_{j=1}^k|\xi_j|^{-\alpha^*/2}\Big|^2 \\ &\qquad \Big(\int_{T_k(t)}\int_{T_k(t)}\prod_{j=1}^k\gamma_0(t_j-s_j)\mathcal{F}_x f_k(t_1,\cdot,\dots,t_k,\cdot,t,x)(\xi_1,\dots,\xi_k) \\ &\qquad \overline{\mathcal{F}_x f_k(s_1,\cdot,\dots,s_k,\cdot,t,x)(\xi_1,\dots,\xi_k)}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}\Big) \\ &= k!\int_{T_k(t)}\int_{T_k(t)}\prod_{j=1}^k\gamma_0(t_j-s_j)A_k^{(n)}(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}, \end{aligned} \tag{2.4.12}$$

where

$$A_k^{(n)}(\mathbf{t},\mathbf{s}) = \int_{(\mathbb{R}^d)^k} \mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\overline{\mathcal{F}_x f_k(s_1,\cdot,\ldots,s_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)} \Big|\prod_{j=1}^k |\xi_j|^{-\alpha_n/2} - \prod_{j=1}^k |\xi_j|^{-\alpha^*/2}\Big|^2 \mathrm{d}\xi_1\cdots\mathrm{d}\xi_k.$$

Using Cauchy-Schwartz inequality, the fact that $ab \le \frac{1}{2}(a^2+b^2)$ and Lemma 2.2.3, we have

$$\begin{aligned} Q_n &\le k! \int_{T_k(t)}\int_{T_k(t)} \prod_{j=1}^k \gamma_0(t_j-s_j)\big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2}\big(A_k^{(n)}(\mathbf{s},\mathbf{s})\big)^{1/2}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\le k! \int_{T_k(t)}\int_{T_k(t)} \prod_{j=1}^k \gamma_0(t_j-s_j)\frac{1}{2}\big(A_k^{(n)}(\mathbf{t},\mathbf{t}) + A_k^{(n)}(\mathbf{s},\mathbf{s})\big)\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &= k! \int_{T_k(t)}\int_{T_k(t)} \prod_{j=1}^k \gamma_0(t_j-s_j) A_k^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \le k!\ \Gamma_{0,t}^k \int_{T_k(t)} A_k^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} \\ &= k!\ \Gamma_{0,t}^k \int_{T_k(t)}\int_{(\mathbb{R}^d)^k} \Big|\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\Big|^2 \\ &\qquad \Big|\prod_{j=1}^k |\xi_j|^{-\alpha_n/2} - \prod_{j=1}^k |\xi_j|^{-\alpha^*/2}\Big|^2 \mathrm{d}\xi_1\cdots\mathrm{d}\xi_k\mathrm{d}\mathbf{t}. \end{aligned} \tag{2.4.13}$$

We want to show that the integral in relation (2.4.13) converges to 0 when $n \to \infty$. For this, we apply the Dominated Convergence Theorem. Note that the integrand converges pointwisely to 0 on $T_k(t) \times (\mathbb{R}^d)^k$, as $n \to \infty$. We now proceed to bound the integrand in relation (2.4.13) by an integrable function. First, we note that this integrand can be bounded by

$$\Big(\prod_{j=1}^k |\xi_j|^{-\alpha_n} + \prod_{j=1}^k |\xi_j|^{-\alpha^*}\Big)\Big|\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\Big|^2.$$

The two resulting integrals in the above are of the same type, the only difference being that the first one depends on n, whereas the second one does not. We therefore only consider the term of the integrand function that depends on n; the integrability of the other term will follow by a similar argument.

Recall that $\max\{d-2,0\} < \alpha^* < d$. Fix numbers a and b such that $\max\{d-2,0\} < a < \alpha^* < b < d$. Since $\alpha_n \to \alpha^*$, there exists $N \in \mathbb{N}$ such that

$$a \le \alpha_n \le b, \text{ for all } n \ge N.$$

For any $\alpha_n \in [a,b] \subset \big(\max\{d-2,0\}, d\big)$, we know that

$$|\xi_j|^{-\alpha_n} \leq |\xi_j|^{-b} 1_{\{|\xi_j|\leq 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}. \tag{2.4.14}$$

This allows us to bound our integrand, provided that we can show that

$$I := \int_{T_k(t)} \int_{(\mathbb{R}^d)^k} \Big|\mathcal{F}_x f_k(t_1,\cdot,\dots,t_k,\cdot,t,x)(\xi_1,\dots,\xi_k)\Big|^2 \prod_{j=1}^{k} \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\leq 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \mathrm{d}\boldsymbol{\xi} \mathrm{d}\mathbf{t} < \infty. \tag{2.4.15}$$

Note that

$$I = \int_{T_k(t)} \left(\int_{(\mathbb{R}^d)^k} \prod_{j=1}^{k} \big|\mathcal{F}_x G_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\leq 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \mathrm{d}\boldsymbol{\xi} \right) \mathrm{d}\mathbf{t}. \tag{2.4.16}$$

We estimate separately $\mathrm{d}\boldsymbol{\xi}$ integral above. We need to consider separately the heat equation and wave equation. For the heat equation, using (2.4.16) and the fact that

$$\big|\mathcal{F}_x G_t^h(\xi)\big| = \Big|e^{-\frac{t|\xi|^2}{2}}\Big| \leq 1, \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^d,$$

for any $j = 1,\dots,k$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\mathcal{F}_x G^h_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\leq 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \mathrm{d}\xi_j \\
&\quad = \int_{|\xi_j|\leq 1} \big|\mathcal{F}_x G^h_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 |\xi_j|^{-b} \mathrm{d}\xi_j \\
&\qquad\qquad + \int_{|\xi_j|>1} \big|\mathcal{F}_x G^h_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 |\xi_j|^{-a} \mathrm{d}\xi_j \\
&\quad \leq \int_{|\xi_j|\leq 1} |\xi_j|^{-b} \mathrm{d}\xi_j + \int_{\mathbb{R}^d} \big|\mathcal{F}_x G^h_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 |\xi_j|^{-a} \mathrm{d}\xi_j \\
&\quad \leq \frac{c_d}{d-b} + K_{d,a}\big(t_{j+1}-t_j\big)^{-\frac{d-a}{2}},
\end{aligned} \tag{2.4.17}$$

where we used Lemma A.2.1 in last inequality. Let $C_{a,b,d} = \frac{c_d}{d-b} + K_{d,a}$. Coming back to relation (2.4.16), by (2.4.17), we have

$$I \leq C^k_{a,b,d} \int_{T_k(t)} \prod_{j=1}^{k} \Big(1 + \big(t_{j+1}-t_j\big)^{-\frac{d-a}{2}}\Big) \mathrm{d}\mathbf{t} < \infty, \tag{2.4.18}$$

due to Lemma A.3.1.

Next we study the wave equation. Using (2.4.16) and the fact that

$$\big|\mathcal{F}_x G_t^w(\xi)\big| = \Big|\frac{\sin(t|\xi|)}{|\xi|}\Big| \le t, \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^d,$$

for any $j = 1, \dots, k$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\mathcal{F}_x G^w_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 \big(|\xi_j|^{-b}1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a}1_{\{|\xi_j|>1\}}\big)\mathrm{d}\xi_j \\
&= \int_{|\xi_j|\le 1} \big|\mathcal{F}_x G^w_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 |\xi_j|^{-b}\mathrm{d}\xi_j + \int_{|\xi_j|>1} \big|\mathcal{F}_x G^w_{t_{j+1}-t_j}(\xi_1+\cdots+\xi_j)\big|^2 |\xi_j|^{-a}\mathrm{d}\xi_j \\
&\le t^2 \int_{|\xi_j|\le 1} |\xi_j|^{-b}\mathrm{d}\xi_j + \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}_x G^w_{t_{j+1}-t_j}(\xi_j+\eta)\big|^2 |\xi_j|^{-a}\mathrm{d}\xi_j \\
&\le t^2 \frac{c_d}{d-b} + K_{d,a}(t-t_k)^{2-(d-a)} \le t^2\frac{c_d}{d-b} + K_{d,a}\, t^{2-(d-a)}, \qquad (2.4.19)
\end{aligned}$$

where for the second inequality, we used Lemma A.2.1. Coming back to relation (2.4.16), by (2.4.19), we have

$$I \le \left(t^2\frac{c_d}{d-b} + K_{d,a}\, t^{2-(d-a)}\right)^k \int_{T_k(t)} \mathrm{d}\mathbf{t} < \infty.$$

This finishes the proof of (2.4.15) and concludes the justification of the application of the Dominated Convergence Theorem. ∎

Proof of Theorem 2.0.1: From Theorem 3.2 of [8] (for PAM) and Theorem 8.3 of [9] (for HAM), we know that the process u^α has a continuous modification (see relations (2.3.1), (2.3.2), (2.3.3) and (2.3.4) above). We work with this modification, which we denote also by u^α. We need to prove that the finite dimensional distribution convergence and the sequence of probability measures induced by $(u^{\alpha_n})_{n\ge 1}$ is tight in the space of $C([0,T]\times\mathbb{R}^d)$. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: Finite dimensional distribution convergence} \\ \text{Step 2: Tightness} \end{cases}$$

Step 1: Finite dimensional distribution convergence

We have to prove that for any $k \ge 1$ and $(t_1,x_1),\dots,(t_k,x_k) \in [0,T]\times\mathbb{R}^d$,

$$\big(u^{\alpha_n}(t_1,x_1),\dots,u^{\alpha_n}(t_k,x_k)\big) \xrightarrow{\mathrm{d}} \big(u^{\alpha^*}(t_1,x_1),\dots,u^{\alpha^*}(t_k,x_k)\big), \text{ as } n\to\infty.$$

For this, by Lemma 2.4.3, it will be enough to prove that for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\mathbb{E}\left|u^{\alpha_n}(t,x) - u^{\alpha^*}(t,x)\right|^2 \to 0, \text{ as } n \to \infty. \tag{2.4.20}$$

To prove relation (2.4.20), we apply Lemma 2.4.4 with $E = L^2(\Omega)$, equipped with the norm $\|\cdot\|_{L^2(\Omega)}$. Recall the diagram of Lemma 2.4.4,

$$\begin{array}{ccc} u_m^{\alpha_n} & \xrightarrow[\forall m]{n\to\infty} & u_m^{\alpha^*} \\ \text{uniformly in } n \Big\downarrow m & & \Big\downarrow m \\ u^{\alpha_n} & \dashrightarrow & u^{\alpha^*} \end{array}$$

Therefore it will suffice to show that

$$\begin{cases} (a) \quad \mathbb{E}\left|u_m^{\alpha_n}(t,x) - u_m^{\alpha^*}(t,x)\right|^2 \to 0, \text{ as } n \to \infty, \text{ for all } m \text{ fixed}, \\ (b) \quad \mathbb{E}\left|u_m^{\alpha^*}(t,x) - u^{\alpha^*}(t,x)\right|^2 \to 0, \text{ as } m \to \infty, \\ (c) \quad \sup_{n\geq 1} \mathbb{E}\left|u_m^{\alpha_n}(t,x) - u^{\alpha_n}(t,x)\right|^2 \to 0, \text{ as } m \to \infty. \end{cases}$$

Note that part (b) is automatically satisfied by relation (2.4.6), so we only need to prove (a) and (c).

For part (a), by Lemma 2.4.5, we have

$$\mathbb{E}\left|I_k^{\alpha_n}(f_k(\cdot,t,x)) - I_k^{\alpha^*}(f_k(\cdot,t,x))\right|^2 \to 0, \text{ as } n \to \infty, \tag{2.4.21}$$

Hence for all m fixed,

$$\begin{aligned} \mathbb{E}\left|u_m^{\alpha_n}(t,x) - u_m^{\alpha^*}(t,x)\right|^2 &= \mathbb{E}\left|\sum_{k=1}^m I_k^{\alpha_n}(f_k(\cdot,t,x)) - I_k^{\alpha^*}(f_k(\cdot,t,x))\right|^2 \\ &\leq m \sum_{k=1}^m \mathbb{E}\left|I_k^{\alpha_n}(f_k(\cdot,t,x)) - I_k^{\alpha^*}(f_k(\cdot,t,x))\right|^2 \to 0. \end{aligned}$$

For part (c), it is enough to show that for all compact set $[a,b] \subset (\max\{d-2,0\},d)$,

$$\sup_{\alpha\in[a,b]} \mathbb{E}\left|u_m^{\alpha}(t,x) - u^{\alpha}(t,x)\right|^2 \to 0, \tag{2.4.22}$$

as $m \to \infty$. Note that

$$u^{\alpha}(t,x) - u_m^{\alpha}(t,x) = \sum_{k\geq m+1} I_k^{\alpha}(f_k(\cdot,t,x)).$$

By the orthogonality of the Wiener chaos space and relation (2.2.14), we have

$$\begin{aligned}\mathbb{E}\Big|u^\alpha(t,x)-u_m^\alpha(t,x)\Big|^2 &= \sum_{k\geq m+1}\mathbb{E}\Big|I_k^\alpha(f_k(\cdot,t,x))\Big|^2\\ &\leq \sum_{k\geq m+1}\Gamma_{0,t}^k K_{d,\alpha}^k\frac{\big(\Gamma(r_\alpha+1)\big)^k}{\Gamma\big(k(r_\alpha+1)+1\big)}t^{k(r_\alpha+1)},\end{aligned} \tag{2.4.23}$$

where we recall that

$$r_\alpha=\begin{cases}-(d-\alpha)/2 & \text{for heat equation,}\\ 2-(d-\alpha) & \text{for wave equation.}\end{cases}$$

From the proof of Lemma A.2.1, we know that $K_{d,\alpha}\leq c_d\big(\frac{1}{d-\alpha}+\frac{1}{2-(d-\alpha)}\big)$. Therefore, for any $\alpha\in[a,b]$,

$$K_{d,\alpha}\leq c_d\Big(\frac{1}{d-b}+\frac{1}{2-(d-a)}\Big):=K_d. \tag{2.4.24}$$

We consider separately the heat equation and wave equation. For heat equation, note that

$$1-\frac{d-a}{2}<1+r_\alpha=1-\frac{d-\alpha}{2}<1-\frac{d-b}{2}.$$

It is known that there exists $x_0\in(1,2)$ such that gamma function $\Gamma(x)$ is decreasing on $(0,x_0)$ and increasing on (x_0,∞). We pick $m_0\geq 1$ such that $m_0\Big(1-\frac{d-a}{2}\Big)>x_0$. For any $m\geq m_0$ and for any $k\geq m$, we have $k\Big(1-\frac{d-\alpha}{2}\Big)\geq k\Big(1-\frac{d-a}{2}\Big)>x_0$, which implies that for any $\alpha\in[a,b]$,

$$\Gamma\Big(k\big(1-\frac{d-a}{2}\big)+1\Big)\leq\Gamma\Big(k\big(1-\frac{d-\alpha}{2}\big)+1\Big),$$

and

$$\frac{1}{\Gamma\big(k(r_\alpha+1)+1\big)}=\frac{1}{\Gamma\Big(k\big(1-\frac{d-\alpha}{2}\big)+1\Big)}\leq\frac{1}{\Gamma\Big(k\big(1-\frac{d-a}{2}\big)+1\Big)}. \tag{2.4.25}$$

Moreover, since $x_0>1>1-\frac{d-\alpha}{2}\geq 1-\frac{d-a}{2}$, we have

$$\Gamma(r_\alpha+1)=\Gamma\Big(1-\frac{d-\alpha}{2}\Big)\leq\Gamma\Big(1-\frac{d-a}{2}\Big)$$

and

$$\big(\Gamma(r_\alpha+1)\big)^k\leq\Big(\Gamma\big(1-\frac{d-a}{2}\big)\Big)^k. \tag{2.4.26}$$

Finally, for any $\alpha \in [a,b] \subset (\max\{d-2,0\},d)$, it follows that

$$t^{k(r_\alpha+1)} = \begin{cases} t^{k\left(1-\frac{d-\alpha}{2}\right)} \leq 1 & \text{if } t<1 \\ t^{k\left(1-\frac{d-\alpha}{2}\right)} \leq t^{k\left(1-\frac{d-b}{2}\right)} & \text{if } t \geq 1 \end{cases}$$

Therefore, we get

$$t^{k(r_\alpha+1)} \leq (t\vee 1)^{k\left(1-\frac{d-b}{2}\right)}. \tag{2.4.27}$$

Returning to relation (2.4.23), using relations (2.4.24), (2.4.25), (2.4.26) and (2.4.27), we obtain

$$\sup_{\alpha\in[a,b]} \sum_{k\geq m+1} \mathbb{E}\big|I_k^\alpha(f_k(\cdot,t,x))\big|^2 \leq \sum_{k\geq m+1} \Gamma_{0,t}^k K_d^k \frac{\left(\Gamma(1-\frac{d-a}{2})\right)^k}{\Gamma\left(k(1-\frac{d-a}{2})+1\right)}(t\vee 1)^{k(1-\frac{d-b}{2})} \to 0$$

as $m\to\infty$ since

$$\sum_{k\geq 1} \Gamma_{0,t}^k K_d^k \frac{\left(\Gamma(1-\frac{d-a}{2})\right)^k}{\Gamma\left(k(1-\frac{d-a}{2})+1\right)}(t\vee 1)^{k(1-\frac{d-b}{2})} < \infty$$

due to the fact that $1-\frac{d-a}{2}>0$.

Now we study the wave equation. Note that inequality (2.4.24) still holds. Using the same approach as in the heat case above, we pick $m_0\geq 1$ such that $m_0\big(3-(d-a)\big) > x_0$. For any $m\geq m_0$ and for any $k\geq m$, we have $k\big(3-(d-\alpha)\big) \geq k\big(3-(d-a)\big) > x_0$, which implies that for any $\alpha\in[a,b]$,

$$\Gamma\Big(k\big(3-(d-a)\big)+1\Big) \leq \Gamma\Big(k\big(3-(d-\alpha)\big)+1\Big),$$

and

$$\frac{1}{\Gamma\Big(k\big(3-(d-\alpha)\big)+1\Big)} \leq \frac{1}{\Gamma\Big(k\big(3-(d-a)\big)+1\Big)}. \tag{2.4.28}$$

Moreover, for any $\alpha \in [a,b]\subset(\max\{d-2,0\},d)$ with $d\leq 2$, we have

$$1 < 3-(d-a) \leq 1+r_\alpha = 3-(d-\alpha) \leq 3-(d-b) < 3.$$

Hence

$$\big(\Gamma(r_\alpha+1)\big)^k = \big(\Gamma(3-(d-\alpha))\big)^k \leq (\Gamma(3))^k. \tag{2.4.29}$$

Finally, for any $\alpha \in [a,b] \subset (\max\{d-2,0\},d)$, it follows that

$$t^{k(r_\alpha+1)} = \begin{cases} t^{k(3-(d-\alpha))} \leq 1 & \text{if } t<1 \\ t^{k(3-(d-\alpha))} \leq t^{k(3-(d-b))} & \text{if } t \geq 1 \end{cases}$$

Therefore, we get

$$t^{k(r_\alpha+1)} \leq (t \vee 1)^{k(3-(d-b))}. \tag{2.4.30}$$

Returning to relation (2.4.23), using relations (2.4.24), (2.4.28), (2.4.29) and (2.4.30), we get

$$\sup_{\alpha\in[a,b]} \sum_{k\geq m+1} \mathbb{E}\big|I_k^\alpha(f_k(\cdot,t,x))\big|^2 \leq \sum_{k\geq m+1} \Gamma_{0,t}^k K_d^k \frac{(\Gamma(3))^k}{\Gamma\Big(k\big(3-(d-a)\big)+1\Big)} (t\vee 1)^{k(3-(d-b))} \to 0$$

as $m \to \infty$ since

$$\sum_{k\geq 1} \Gamma_{0,t}^k K_d^k \frac{(\Gamma(3))^k}{\Gamma\Big(k\big(3-(d-a)\big)+1\Big)} (t\vee 1)^{k(3-(d-b))} < \infty$$

due to the fact that $3-(d-a) > 0$. This completes the proof.

Step 2: Tightness

Note that condition (i) of Theorem A.4.2 holds automatically since $u^{\alpha_n}(0,0) = 1$ for all $n \geq 1$. It remains to prove condition (ii) of Theorem A.4.2.

We choose a compact set $[a,b]$ such that

$$\max\{d-2, 0\} < a < \alpha^* < b < d.$$

Since $\alpha_n \to \alpha^*$, there exists $N \in \mathbb{N}$ such that $\alpha_n \in [a,b]$ for all $n \geq N$. Fix $\beta \in \big(\frac{d-a}{2}, 1\big)$. Let $t, t' \in [0,T]$ and $x, x' \in K = [-M, M] \subset \mathbb{R}^d$ be arbitrary. For the heat equation, by relations (2.3.1) and (2.3.2) in Theorem 2.3.1, we have:

$$\begin{aligned}
\sup_{n\geq N} \mathbb{E}\big|u^{\alpha_n}(t',x') - u^{\alpha_n}(t,x)\big|^p &\leq \sup_{\alpha\in[a,b]} \mathbb{E}\big|u^{\alpha}(t',x') - u^{\alpha^*}(t,x)\big|^p \\
&\leq 2^{p-1}\Big\{\sup_{\alpha\in[a,b]} \mathbb{E}\big|u^\alpha(t',x) - u^\alpha(t,x)\big|^p + \sup_{\alpha\in[a,b]} \mathbb{E}\big|u^\alpha(t,x') - u^\alpha(t,x)\big|^p\Big\} \\
&\leq 2^{p-1}\Big(C_1^h|t'-t|^{p(1-\beta)/2} + C_2^h|x'-x|^{p(1-\beta)}\Big) \\
&\leq 2^{p-1}\Big(C_1^h|t'-t|^{p(1-\beta)/2} + C_2^h(2M\vee 1)^{p(1-\beta)/2}|x'-x|^{p(1-\beta)/2}\Big) \\
&\leq C^h\big(|t'-t| + |x'-x|\big)^{p(1-\beta)/2}
\end{aligned}$$

where $C^h := 2^p \max\{C_1^h, C_2^h(2M\vee 1)^{p(1-\beta)/2}\}$. Condition (ii) of Theorem A.4.2 follows since $\frac{p(1-\beta)}{2} > 2$, if we choose $p > \frac{4}{1-\beta}$.

For the wave equation, by relations (2.3.3) and (2.3.4) in Theorem 2.3.1, we have:

$$\sup_{n\geq N} \mathbb{E}\big|u_w^{\alpha_n}(t',x') - u_w^{\alpha_n}(t,x)\big|^p \leq \sup_{\alpha\in[a,b]} \mathbb{E}\big|u_w^{\alpha}(t',x') - u_w^{\alpha^*}(t,x)\big|^p$$

$$\leq 2^{p-1}\left\{ \sup_{\alpha\in[a,b]} \mathbb{E}\left|u^{\alpha}(t',x) - u^{\alpha}(t,x)\right|^{p} + \sup_{\alpha\in[a,b]} \mathbb{E}\left|u^{\alpha}(t,x') - u^{\alpha}(t,x)\right|^{p} \right\}$$

$$\leq 2^{p-1}\left(C_1^w |t'-t|^{p(1-\beta)} + C_2^w |x'-x|^{p(1-\beta)} \right) \leq C^w \big(|t'-t| + |x'-x|\big)^{p(1-\beta)}$$

where $C^w := 2^p \max\{C_1^w, C_2^w\}$. Condition (ii) of Theorem A.4.2 follows since $p(1-\beta) > 2$, if we choose $p > \frac{2}{1-\beta}$.

∎

Chapter 3

PAM with regular noise and general initial condition

In this chapter, we consider the following *Parabolic Anderson Model*:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}^\alpha(t,x), \quad t>0, x\in\mathbb{R}^d \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{3.0.1}$$

with initial condition given by a non-negative Borel measure u_0 on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} e^{-a|x|^2} u_0(\mathrm{d}x) < \infty, \text{ for all } a > 0. \tag{3.0.2}$$

Initial conditions of this type are sometimes called *rough*. In this chapter, the noise W^α which is randomly perturbing equation (3.0.1) is exactly the same as in Section 2.1. More precisely, its covariance is given by relation (2.1.3). Throughout this chapter, G denotes the fundamental solution of the heat equation. Let $w(t,x)$ be the solution of the deterministic heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), \; t>0, x\in\mathbb{R}^d$$

with the same initial condition as (3.0.1), is that

$$w(t,x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(\mathrm{d}y). \tag{3.0.3}$$

Note that (3.0.2) is the necessary and sufficient condition for $w(t,x)$ to be well-defined. Therefore (3.0.2) is the most general condition for the Parabolic Anderson Model. This includes the case $u_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$ for a function u_0 on $\mathbb{R}^d$. The case $u_0 = 1$ was considered in Chapter 2.

The objective of this chapter is to show the continuity in the law of the solution with respect to the noise parameter α, in the space of continuous functions on $C([t_0, T] \times \mathbb{R}^d)$. To accomplish this, we will prove finite-dimensional distributions (f.d.d) convergence and tightness: F.d.d convergence is proved in Section 3.3, while tightness will follow from the uniform moment estimates developed in Section 3.2.

The main result for this chapter is the following theorem, which is an extension of Theorem 2.0.1 to general initial conditions.

Theorem 3.0.1. *For any $\alpha \in (\max(d-2, 0), d)$, let u^α be the solution of equation (3.0.1). Fix $\alpha^* \in (\max(d-2, 0), d)$ and let $(\alpha_n)_{n\geq 1}$ be a sequence in $(\max(d-2, 0), d)$ such that*

$$\lim_{n\to\infty} \alpha_n = \alpha^*.$$

Then for any $T > 0$ and for any $0 < t_0 < T$,

$$u^{\alpha_n} \xrightarrow{d} u^{\alpha^*} \text{ in } C([t_0, T] \times \mathbb{R}^d).$$

We note that in Theorem 3.0.1, the convergence is only on compact sets of the form $[t_0, T] \times K$ with $0 < t_0 < T$. This limitation is due to the uniform moment estimates that we obtained in Theorem 3.2.1 below, which is needed for the proof of tightness.

The main results of Chapters 4 and 5 (Theorems 4.0.1 and 5.0.1) have been studied in the preprint [30].

3.1 Existence of solution

In this section, we consider equation (3.0.1) driven by Gaussian noise with covariance (2.1.3). We prove that the solution to (3.0.1) exists and show that its moments are bounded by an exponential function of t. This was proved in [3]. We provide the details for the sake of completeness, and also because certain intermediate steps will be required in the following sections.

For the definition below, we recall that δ is the skorohod integral from Malliavin calculus; see Appendix A.5.

Definition 3.1.1. *Let $\mathcal{F}_t$ be the filtration generated by the noise W^α whose covariance structure is given by (2.1.3). We say that a process $u^\alpha = \{u^\alpha(t, x); t \geq 0, x \in \mathbb{R}^d\}$ is a* **(mild) solution** *of equation (3.0.1) if for any $t > 0$ and $x \in \mathbb{R}^d$, $u^\alpha(t, x)$ is $\mathcal{F}_t$-measurable, $\mathbb{E}|u^\alpha(t, x)|^2 < \infty$ and the following integral equation holds:*

$$u^\alpha(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y) u(s, y) W^\alpha(\delta s, \delta y), \tag{3.1.1}$$

i.e. $v^{(t,x)} \in Dom(\delta)$ *and* $u(t,x) = w(t,x) + \delta(v^{(t,x)})$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *where*

$$v^{(t,x)}(s,y) = 1_{[0,t]}(s) G_{t-s}(x-y) u(s,y), \quad s \geq 0, y \in \mathbb{R}^d. \tag{3.1.2}$$

The solution has the series expansion:

$$\begin{aligned} u^\alpha(t,x) &= w(t,x) + \sum_{n\geq 1} \int_0^t \int_{\mathbb{R}^d} \int_0^{t_n} \int_{\mathbb{R}^d} \cdots \int_0^{t_2} \int_{\mathbb{R}^d} G_{t-t_n}(x-x_n) \cdots G_{t_2-t_1}(x_2-x_1) \\ &\qquad\qquad w(t_1,x_1) W^\alpha(\mathrm{d}t_1, \mathrm{d}x_1) \cdots W^\alpha(\mathrm{d}t_n, \mathrm{d}x_n) \\ &= w(t,x) + \sum_{n\geq 1} I_n^\alpha(f_n(\cdot,t,x)) = \sum_{n\geq 0} I_n^\alpha(f_n(\cdot,t,x)), \end{aligned} \tag{3.1.3}$$

where $f_0(\cdot,t,x) = w(t,x)$ and for $n \geq 1$, the kernel $f_n(\cdot,t,x)$ is given by:

$$\begin{aligned} & f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \\ &\quad = G_{t-t_n}(x-x_n) \cdots G_{t_2-t_1}(x_2-x_1) w(t_1,x_1) 1_{\{0<t_1<\cdots<t_n<t\}}. \end{aligned} \tag{3.1.4}$$

We recall that I_n^α is the multiple Wiener integral of order n with respect to W^α.

The kernel f_n given in (3.1.4) has a different expression compared to the one given in (2.2.3), taking in account the rough initial condition. Therefore, we will have a different estimate for the Fourier transform of this kernel f_n.

We recall the following result.

Theorem 3.1.2 (Theorem 2.3 of [3])**.** *Suppose that* $f_n(\cdot,t,x) \in \mathcal{H}_\alpha^{\otimes n}$ *for any* $t > 0$, $x \in \mathbb{R}^d$ *and* $n \geq 1$. *The equation* (3.0.1) *has a unique solution if and only if for any* $t > 0$ *and* $x \in \mathbb{R}^d$, *the series* $\sum_{n\geq 0} I_n^\alpha(f_n(\cdot,t,x))$ *converges in* $L^2(\Omega)$, *i.e.*

$$I_{t,x} := \sum_{n\geq 0} n! \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} < \infty. \tag{3.1.5}$$

In this case, $\mathbb{E}|u^\alpha(t,x)|^2 = I_{t,x}$.

The goal of this section is to prove the following result.

Theorem 3.1.3. *For any* $\alpha \in (\max\{0, d-2\}, d)$, *for any non-negative and non-negative definite kernel* γ_0 *satisfying* (2.1.3) *and for any initial measure* u_0 *satisfying* (3.0.2), *equation* (3.0.1) *has a unique solution. Moreover, for any* $p \geq 2$,

$$\mathbb{E}|u^\alpha(t,x)|^p \leq C^{(1)} w^p(t,x) \exp\left(C^{(2)}\, p^{\frac{4-(d-\alpha)}{2-(d-\alpha)}} \Gamma_{0,t}^{\frac{2}{2-(d-\alpha)}}\; t \right) \tag{3.1.6}$$

where $\Gamma_{0,t} = 2\int_0^t \gamma_0(s)\mathrm{d}s$ *and* $C^{(1)} > 0$ *and* $C^{(2)} > 0$ *are some constants depending on* d *and* α.

We postpone the proof of Theorem 3.1.3 for the end of this section. We start with some preliminary results. In order to show the kernel $f_n(\cdot,t,x)\in\mathcal{H}_\alpha^{\otimes n}$, we need an alternative expression of this kernel, which is obtained as follows. Suppose that $0<t_1<\ldots<t_n<t$. Using the definition of $w(t,x)$ in (3.0.3), we see that

$$\begin{aligned} &f_n(t_1,x_1,\cdots,t_n,x_n,t,x)\\ &\quad=\int_{\mathbb{R}^d} G_{t-t_n}(x-x_n)\cdots G_{t_2-t_1}(x_2-x_1)G_{t_1}(x_1-x_0)u_0(\mathrm{d}x_0). \end{aligned} \tag{3.1.7}$$

To study the Fourier transform on kernel $f_n(\cdot,t,x)$, we need the following lemma.

Lemma 3.1.4 (Lemma 2.3.7 of [14]). *For $t,s>0$ and $x,y\in\mathbb{R}^d$,*

$$G_t(x)G_s(y)=G_{\frac{ts}{t+s}}\Big(\frac{sx+ty}{t+s}\Big)G_{t+s}(x-y).$$

Proof: Note that

$$G_t(x)G_s(y)=\frac{1}{(2\pi t)^{d/2}\,(2\pi s)^{d/2}}\exp\Big(-\frac{|x|^2}{2t}-\frac{|y|^2}{2s}\Big).$$

and

$$G_{\frac{ts}{t+s}}\Big(\frac{sx+ty}{t+s}\Big)G_{t+s}(x-y)=\frac{1}{\big(2\pi\big)^d\big(ts\big)^{d/2}}\exp\Bigg(-\frac{\big|\frac{sx+ty}{t+s}\big|^2}{2\big(\frac{ts}{t+s}\big)}-\frac{|x-y|^2}{2(t+s)}\Bigg).$$

It remains to show that

$$\frac{|x|^2}{t}+\frac{|y|^2}{s}=\frac{\big|\frac{sx+ty}{t+s}\big|^2}{\frac{ts}{t+s}}+\frac{|x-y|^2}{t+s},.$$

and this can be proved by direct calculation. ∎

By Lemma 3.1.4, we have

$$\begin{aligned} G_{t_2-t_1}(x_2-x_1)G_{t_1}(x_1-x_0)&=G_{t_2-t_1}(x_2-x_1)G_{t_1}(x_0-x_1)\\ &=G_{\frac{t_1(t_2-t_1)}{t_2}}\Big(\frac{t_1(x_2-x_1)+(t_2-t_1)(x_0-x_1)}{t_2}\Big)\times G_{t_2}(x_2-x_0)\\ &=G_{\left(1-\frac{t_1}{t_2}\right)t_1}\Big(\frac{t_2-t_1}{t_2}x_0+\frac{t_1}{t_2}x_2-\frac{(t_2-t_1)+t_1}{t_2}x_1\Big)\times G_{t_2}(x_2-x_0)\\ &=G_{\left(1-\frac{t_1}{t_2}\right)t_1}\bigg(\Big(1-\frac{t_1}{t_2}\Big)x_0+\frac{t_1}{t_2}x_2-x_1\bigg)\times G_{t_2}(x_2-x_0). \end{aligned} \tag{3.1.8}$$

Similarly, we obtain:

$$G_{t_3-t_2}(x_3-x_2)G_{t_2}(x_2-x_0)=G_{\left(1-\frac{t_2}{t_3}\right)t_2}\left(\left(1-\frac{t_2}{t_3}\right)x_0+\frac{t_2}{t_3}x_3-x_2\right)\times G_{t_3}(x_3-x_0).$$

We continue in this manner. In the last step, we get:

$$G_{t-t_n}(x-x_n)G_{t_n}(x_n-x_0)=G_{\left(1-\frac{t_n}{t}\right)t_n}\left(\left(1-\frac{t_n}{t}\right)x_0+\frac{t_n}{t}x-x_n\right)\times G_t(x-x_0).$$

Therefore, using the notation $t_{n+1}=t$ and $x_{n+1}=x$, we can rewrite the kernel $f_n(\cdot,t,x)$ as follows:

$$\begin{aligned}&f_n(t_1,x_1,\cdots,t_n,x_n,t,x)\\&\quad=\int_{\mathbb{R}^d}\mu_0(\mathrm{d}x_0)\,G_t(x-x_0)\prod_{j=1}^{n}G_{\left(1-\frac{t_j}{t_{j+1}}\right)t_j}\left(\left(1-\frac{t_j}{t_{j+1}}\right)x_0+\frac{t_j}{t_{j+1}}x_{j+1}-x_j\right).\end{aligned}$$

Note that it can be proved that $f_n(t_1,x_1,\cdots,t_n,x_n,t,x)\in L^1(\mathbb{R}^{nd})$, using the fact $\int_{\mathbb{R}^d}G_t(x-y)\mathrm{d}y=1$, for any $x\in\mathbb{R}^d$. The following result gives the form of the Fourier transform of the kernel $f_n(\cdot,t,x)$.

Lemma 3.1.5 (Lemma 2.5 of [3])**.** *For any $0<t_1<\ldots<t_n<t=t_{n+1}$ and for any $\xi_1,\ldots,\xi_n\in\mathbb{R}^d$, we have*

$$\begin{aligned}&\mathcal{F}f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)\\&\quad=\prod_{k=1}^{n}\exp\Big\{-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big\}\exp\Big\{-\frac{i}{t}\Big(\sum_{j=1}^{n}t_j\xi_j\Big)\cdot x\Big\}\\&\qquad\times\int_{\mathbb{R}^d}\exp\Big\{-i\Big[\sum_{j=1}^{n}\Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big\}G_t(x-x_0)u_0(\mathrm{d}x_0).\end{aligned}$$

Proof: Note that for any function $\varphi\in L^1(\mathbb{R}^d)$, $\mathcal{F}\varphi(x-\cdot)(\xi)=\mathrm{e}^{-i\xi\cdot x}\overline{\mathcal{F}\varphi(\xi)}$. Hence

$$\mathcal{F}G_t(x-\cdot)(\xi)=\int_{\mathbb{R}^d}\mathrm{e}^{-i\xi\cdot y}G_t(x-y)\mathrm{d}y=\mathrm{e}^{-i\xi\cdot x}\overline{\mathcal{F}G_t(\xi)}.\tag{3.1.9}$$

We start with the $\mathrm{d}x_1$ integral. By equation (3.1.8) and (3.1.9), we have

$$\begin{aligned}&\int_{\mathbb{R}^d}e^{-i\xi_1\cdot x_1}G_{\left(1-\frac{t_1}{t_2}\right)t_1}\left(\left(1-\frac{t_1}{t_2}\right)x_0+\frac{t_1}{t_2}x_2-x_1\right)\mathrm{d}x_1\\&\qquad=\exp\Big\{-i\xi_1\cdot\Big[\Big(1-\frac{t_1}{t_2}\Big)x_0+\frac{t_1}{t_2}x_2\Big]\Big\}\overline{\mathcal{F}G_{\left(1-\frac{t_1}{t_2}\right)t_1}(\xi_1)}.\end{aligned}$$

Next we calculate the $\mathrm{d}x_2$ integral. Using again (3.1.9):

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{-i\xi_2\cdot x_2}\exp\Big\{-i\xi_1\cdot\Big(\frac{t_1}{t_2}x_2\Big)\Big\}G_{\left(1-\frac{t_2}{t_3}\right)t_2}\Big((1-\frac{t_2}{t_3})x_0+\frac{t_2}{t_3}x_3-x_2\Big)\mathrm{d}x_2\\&=\int_{\mathbb{R}^d}\exp\Big\{-i\Big(\xi_2+\frac{t_1}{t_2}\xi_1\Big)\cdot x_2\Big\}G_{\left(1-\frac{t_2}{t_3}\right)t_2}\Big((1-\frac{t_2}{t_3})x_0+\frac{t_2}{t_3}x_3-x_2\Big)\mathrm{d}x_2\\&=\exp\Big\{-i\Big(\xi_2+\frac{t_1}{t_2}\xi_1\Big)\cdot\Big[\Big(1-\frac{t_2}{t_3}\Big)x_0+\frac{t_2}{t_3}x_3\Big]\Big\}\overline{\mathcal{F}G_{\left(1-\frac{t_2}{t_3}\right)t_2}\Big(\xi_2+\frac{t_1}{t_2}\xi_1\Big)}.\end{aligned}$$

We continue in this manner. In the last step, we find the following $\mathrm{d}x_n$ integral:

$$\begin{aligned}&\int_{\mathbb{R}^d}\exp\Big\{-i\Big(\xi_n+\frac{\sum_{j=1}^{n-1}t_j\xi_j}{t_n}\Big)\cdot x_n\Big\}G_{\left(1-\frac{t_n}{t}\right)t_n}\Big((1-\frac{t_n}{t})x_0+\frac{t_n}{t}x-x_n\Big)\mathrm{d}x_n\\&=\exp\Big\{-i\Big(\xi_n+\frac{\sum_{j=1}^{n-1}t_j\xi_j}{t_n}\Big)\cdot\Big[\Big(1-\frac{t_n}{t}\Big)x_0+\frac{t_n}{t}x\Big]\Big\}\overline{\mathcal{F}G_{\left(1-\frac{t_n}{t}\right)t_n}\Big(\xi_n+\frac{\sum_{j=1}^{n-1}t_j\xi_j}{t_n}\Big)}.\end{aligned}$$

Combining all these calculations above, it follows that

$$\begin{aligned}&\mathcal{F}f_n(t_1,\cdot,\dots,t_n,\cdot,t,x)(\xi_1,\dots,\xi_n)\\&=\prod_{k=1}^n\overline{\mathcal{F}G_{\left(1-\frac{t_k}{t_{k+1}}\right)t_k}\Big(\frac{\sum_{j=1}^k t_j\xi_j}{t_k}\Big)}\exp\Big\{-i\frac{(\sum_{j=1}^n t_j\xi_j)\cdot x}{t}\Big\}\int_{\mathbb{R}^d}G_t(x-x_0)\\&\qquad\times\exp\Big\{-i\Big[\sum_{j=1}^{n-1}\Big(1-\frac{t_j}{t_n}\Big)\xi_j+\Big(1-\frac{t_n}{t}\Big)\Big(\xi_n+\frac{\sum_{j=1}^{n-1}t_j\xi_j}{t_n}\Big)\Big]\cdot x_0\Big\}u_0(\mathrm{d}x_0)\\&=\prod_{k=1}^n\overline{\mathcal{F}G_{\left(1-\frac{t_k}{t_{k+1}}\right)t_k}\Big(\frac{\sum_{j=1}^k t_j\xi_j}{t_k}\Big)}\exp\Big\{-i\frac{(\sum_{j=1}^n t_j\xi_j)\cdot x}{t}\Big\}\int_{\mathbb{R}^d}G_t(x-x_0)\\&\qquad\times\exp\Big\{-i\Big[\sum_{j=1}^{n}\Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big\}u_0(\mathrm{d}x_0),\end{aligned}$$

where for the last line, we used

$$\sum_{j=1}^{n-1}\Big(1-\frac{t_j}{t_n}\Big)\xi_j+\Big(1-\frac{t_n}{t}\Big)\Big(\xi_n+\frac{\sum_{j=1}^{n-1}t_j\xi_j}{t_n}\Big)=\sum_{j=1}^n\Big(1-\frac{t_j}{t}\Big)\xi_j.$$

Using (1.0.4), we have

$$\mathcal{F}G_{\left(1-\frac{t_k}{t_{k+1}}\right)t_k}\Big(\frac{\sum_{j=1}^k t_j\xi_j}{t_k}\Big)=\exp\Big\{-\frac{1}{2}\Big(1-\frac{t_k}{t_{k+1}}\Big)t_k\Big|\frac{\sum_{j=1}^k t_j\xi_j}{t_k}\Big|^2\Big\}.$$

This complete the proof. ∎

For the proof of Theorem 3.1.3, we need to show (3.1.5). Recall that the norm of $f \in \mathcal{H}_\alpha^{\otimes n}$ is given by:

$$\|f\|^2_{\mathcal{H}_\alpha^{\otimes n}} = \int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n \gamma_0(t_j - s_j)\,\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{s}\int_{\mathbb{R}^{nd}}\mu(\mathrm{d}\xi_1)\dots\mu(\mathrm{d}\xi_n)$$
$$\mathcal{F}f(t_1,\cdot,\dots,t_n,\cdot,t,x)(\xi_1,\dots,\xi_n)\overline{\mathcal{F}f(s_1,\cdot,\dots,s_n,\cdot,t,x)(\xi_1,\dots,\xi_n)}. \quad (3.1.10)$$

We denote

$$\mathbb{E}|I_n^\alpha(f_n(\cdot,t,x))|^2 = n!\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_\alpha^{\otimes n}} =: \frac{1}{n!}k_n(t,x).$$

With this notation, condition (3.1.5) becomes:

$$\sum_{n\geq 0}\frac{1}{n!}k_n(t,x) < \infty. \quad (3.1.11)$$

Using the definition of the norm in $\mathcal{H}_\alpha^{\otimes n}$, we see that

$$k_n(t,x) = \int_{[0,t]^{2n}}\prod_{j=1}^n \gamma_0(t_j - s_j)\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s},$$

where

$$\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{nd}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n \gamma(x_j - y_j) g_{\mathbf{t},t,x}^{(n)}(x_1,\dots,x_n) g_{\mathbf{s},t,x}^{(n)}(x_1,\dots,x_n)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$
$$= \int_{\mathbb{R}^{nd}}\mathcal{F}g_{\mathbf{t},t,x}^{(n)}(\xi_1,\dots,\xi_n)\overline{\mathcal{F}g_{\mathbf{s},t,x}^{(n)}(\xi_1,\dots,\xi_n)}\mu(\mathrm{d}\xi_1)\dots\mu(\mathrm{d}\xi_n) \quad (3.1.12)$$

and

$$g_{\mathbf{t},t,x}^{(n)}(x_1,\dots,x_n) = n!\widetilde{f}_n(t_1,x_1,\dots,t_n,x_n,t,x)$$
$$= \sum_{\rho\in S_n} G_{t-t_{\rho(n)}}(x - x_{\rho(n)})\dots G_{t_{\rho(2)}-t_{\rho(1)}}(x_{\rho(2)} - x_{\rho(1)})$$
$$\times\, w\big(t_{\rho(1)}, x_{\rho(1)}\big)1_{\{0<t_{\rho(1)}<\dots<t_{\rho(n)}<t\}}.$$

To estimate $k_n(t,x)$, using Cauchy-Schwarz inequality and the fact $ab \leq \frac{1}{2}(a^2 + b^2)$, we have

$$\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s}) \leq \big(\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2}\big(\psi_{t,x}^{(n)}(\mathbf{s},\mathbf{s})\big)^{1/2} \leq \frac{1}{2}\Big(\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t}) + \psi_{t,x}^{(n)}(\mathbf{s},\mathbf{s})\Big). \quad (3.1.13)$$

Hence, using relation (3.1.13) and applying Lemma 2.2.3 to the function $h(\mathbf{t}) = \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})$, we have

$$k_n(t,x) \leq \int_{[0,t]^{2n}} \prod_{j=1}^{n} \gamma_0(t_j - s_j)\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \leq \Gamma_{0,t}^n \int_{[0,t]^n} \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}$$
$$= \Gamma_{0,t}^n \sum_{\rho \in S_n} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t} \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}. \tag{3.1.14}$$

The next result gives an estimate for $\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})$.

Lemma 3.1.6 (Lemma 3.2 of [3])**.** *If* $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t =: t_{\rho(n+1)}$, *then*

$$\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t}) \leq w^2(t,x) \int_{\mathbb{R}^{nd}} \exp\Big\{ -\sum_{k=1}^{n} \Big(\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \Big| \sum_{j=1}^{k} t_{\rho(j)}\xi_j \Big|^2 \Big) \Big\} \mu(\mathrm{d}\xi_1)\ldots\mu(\mathrm{d}\xi_n).$$

Proof: Note that $n!\mathcal{F}\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)$ has the same expression as $\mathcal{F}f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)$ in which we replace (t_j,ξ_j) by $(t_{\rho(j)},\xi_{\rho(j)})$. This expression is given by Lemma 3.1.5. By (3.1.12), we have

$$\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t}) = \int_{\mathbb{R}^{nd}} \Big| \mathcal{F}g_{\mathbf{t},t,x}^{(n)}(\xi_1,\ldots,\xi_n) \Big|^2 \mu(\mathrm{d}\xi_1)\ldots\mu(\mathrm{d}\xi_n)$$
$$= (n!)^2 \int_{\mathbb{R}^{nd}} \Big| \mathcal{F}\widetilde{f}_n(t_1,x_1,\ldots,t_n,x_n,t,x)(\xi_1,\ldots,\xi_n) \Big|^2 \mu(\mathrm{d}\xi_1)\ldots\mu(\mathrm{d}\xi_n)$$
$$= \int_{\mathbb{R}^{nd}} \Big| \prod_{k=1}^{n} \exp\Big\{ -\frac{1}{2}\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \Big| \sum_{j=1}^{k} t_{\rho(j)}\xi_{\rho(j)} \Big|^2 \Big\} \Big|^2 \Big| \exp\Big\{ -\frac{i}{t}\Big(\sum_{j=1}^{n} t_{\rho(j)}\xi_{\rho(j)} \Big) \cdot x \Big\} \Big|^2$$
$$\times \Big| \int_{\mathbb{R}^d} \exp\Big\{ -i\Big[\sum_{j=1}^{n} \Big(1 - \frac{t_{\rho(j)}}{t}\Big)\xi_{\rho(j)} \Big] \cdot x_0 \Big\} G_t(x-x_0)u_0(\mathrm{d}x_0) \Big|^2 \mu(\mathrm{d}\xi_{\rho(1)})\ldots\mu(\mathrm{d}\xi_{\rho(n)})$$
$$\leq \int_{\mathbb{R}^{nd}} \exp\Big\{ -\sum_{k=1}^{n} \Big(\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \Big| \sum_{j=1}^{k} t_{\rho(j)}\xi_{\rho(j)} \Big|^2 \Big) \Big\}$$
$$\times \Big(\int_{\mathbb{R}^d} |G_t(x-x_0)| u_0(\mathrm{d}x_0) \Big)^2 \mu(\mathrm{d}\xi_{\rho(1)})\ldots\mu(\mathrm{d}\xi_{\rho(n)})$$
$$= w^2(t,x) \int_{\mathbb{R}^{nd}} \exp\Big\{ -\sum_{k=1}^{n} \Big(\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \Big| \sum_{j=1}^{k} t_{\rho(j)}\xi_{\rho(j)} \Big|^2 \Big) \Big\} \mu(\mathrm{d}\xi_{\rho(1)})\ldots\mu(\mathrm{d}\xi_{\rho(n)}),$$

where we used the trivial inequality $\Big| \int \ldots u_0(\mathrm{d}x_0) \Big|^2 \leq \Big(\int |\ldots| u_0(\mathrm{d}x_0) \Big)^2$. ∎

We return to (3.1.14). Using the change of variables $t'_j = t_{\rho(j)}$, $\xi'_j = \xi_{\rho(j)}$ for all $j = 1, \dots, n$ (we denote $t'_{n+1} = t$) and Lemma 3.1.6, we see that

$$\frac{1}{n!}k_n(t,x) \leq \frac{1}{n!}w^2(t,x)\Gamma_{0,t}^n \sum_{\rho \in S_n} \int_{0<t_{\rho(1)}<\dots<t_{\rho(n)}<t}$$

$$\int_{\mathbb{R}^{nd}} \exp\Big\{ -\sum_{k=1}^{n} \Big(\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)}t_{\rho(k)}} \Big| \sum_{j=1}^{k} t_{\rho(j)}\xi_{\rho(j)} \Big|^2 \Big) \Big\} \mu(\mathrm{d}\xi_{\rho(1)}) \dots \mu(\mathrm{d}\xi_{\rho(n)}) \mathrm{d}\mathbf{t}$$

$$= \Gamma_{0,t}^n w^2(t,x) \int_{0<t_1<\dots<t_n<t} I_t^{(n)}(t_1,\dots,t_n)\mathrm{d}\mathbf{t}, \tag{3.1.15}$$

where

$$I_t^{(n)}(t_1,\dots,t_n) = \int_{\mathbb{R}^{nd}} \exp\Big\{ -\sum_{k=1}^{n} \Big(\frac{t_{k+1} - t_k}{t_{k+1}t_k} \Big| \sum_{j=1}^{k} t_j\xi_j \Big|^2 \Big) \Big\} \mu(\mathrm{d}\xi_1) \dots \mu(\mathrm{d}\xi_n). \tag{3.1.16}$$

We now use the following maximum principle to continue studying $\alpha_n(t,x)$.

Lemma 3.1.7 (Lemma 3.4 of [3]). *Let μ be a tempered measure on $\mathbb{R}^d$ such that its Fourier transform in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ is a locally integrable function γ, i.e. (2.1.1) holds. Assume that γ is non-negative. Then for any $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\psi * \widetilde{\psi}$ is non-negative, where $\widetilde{\psi}(x) = \psi(-x)$ for all $x \in \mathbb{R}^d$, we have:*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}\psi(\xi+\eta)\big|^2 \mu(\mathrm{d}\xi) = \int_{\mathbb{R}^d} \big|\mathcal{F}\psi(\xi)\big|^2 \mu(\mathrm{d}\xi). \tag{3.1.17}$$

In particular, for any $a > 0$ and $t > 0$,

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-a|t\xi+\eta|^2} \mu(\mathrm{d}\xi) = \int_{\mathbb{R}^d} e^{-a|t\xi|^2} \mu(\mathrm{d}\xi). \tag{3.1.18}$$

Now we introduce some new notations. Following [16], define

$$k_\alpha(t) := C_{d,\alpha} \int_{\mathbb{R}^d} |z|^{-(d-\alpha)} G_t(z)\mathrm{d}z,$$

where $C_{d,\alpha}$ is given by relation (2.0.4). By (1.0.4) and (2.1.1), we see that

$$k_\alpha(t) = \int_{\mathbb{R}^d} \exp\Big(-\frac{t|\xi|^2}{2} \Big) |\xi|^{-\alpha}\mathrm{d}\xi. \tag{3.1.19}$$

Using the change of variable $\xi' = \sqrt{t}\xi$, then $\mathrm{d}\xi = t^{-d/2}\mathrm{d}\xi'$, it follows that

$$k_\alpha(t) = \int_{\mathbb{R}^d} e^{-t|\xi|^2/2}|\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} e^{-|\xi'|/2}t^{\alpha/2}|\xi'|^{-\alpha}t^{-d/2}\mathrm{d}\xi' = t^{-(d-\alpha)/2}C^{(1)}_{d,\alpha}, \quad (3.1.20)$$

where

$$C^{(1)}_{d,\alpha} := \int_{\mathbb{R}^d} e^{-|\xi|^2/2}|\xi|^{-\alpha}\mathrm{d}\xi = \frac{c_d}{2}\Gamma\Big(\frac{d-\alpha}{2}\Big). \quad (3.1.21)$$

See (2.3.28) with $t = 1$.

Lemma 3.1.8 (Lemma 3.5 of [3]). *For any $0 < t_1 < \ldots < t_n < t := t_{n+1}$, we have that*

$$I^{(n)}_t(t_1,\ldots,t_n) \le \prod_{j=1}^n k_\alpha\Big(\frac{2(t_{j+1}-t_j)t_j}{t_{j+1}}\Big).$$

Proof: Let $a_i = \frac{t_{i+1}-t_i}{t_{i+1}t_i}$, for all $i = 1,\ldots,n$. Then

$$I^{(n)}_t(t_1,\ldots,t_n) = \int_{\mathbb{R}^{nd}} \exp\Big\{-\sum_{k=1}^n a_k\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big\}\mu(\mathrm{d}\xi_1)\ldots\mu(\mathrm{d}\xi_n)$$
$$= \Big(\int_{\mathbb{R}^d} e^{-a_1|t_1\xi_1|^2}\mu(\mathrm{d}\xi_1)\Big) \times \Big(\int_{\mathbb{R}^d} e^{-a_2|t_1\xi_1+t_2\xi_2|^2}\mu(\mathrm{d}\xi_2)\Big) \times \ldots \times \Big(\int_{\mathbb{R}^d} e^{-a_n|t_1\xi_1+\ldots+t_n\xi_n|^2}\mu(\mathrm{d}\xi_n)\Big).$$

For the the last integral, we note that for any $\xi_1,\ldots,\xi_{n-1} \in \mathbb{R}^d$, by Lemma 3.1.7,

$$\int_{\mathbb{R}^d} e^{-a_n|t_1\xi_1+\ldots+t_n\xi_n|^2}\mu(\mathrm{d}\xi_n) \le \int_{\mathbb{R}^d} e^{-a_n|t_n\xi_n|^2}\mu(\mathrm{d}\xi_n) = k_\alpha\Big(\frac{2(t_{n+1}-t_n)t_n}{t_{n+1}}\Big).$$

The other integrals are estimated similarly. ∎

Coming back to (3.1.15) and using Lemma 3.1.8, we obtain:

$$\begin{aligned}\frac{1}{n!}k_n(t,x) &\le \Gamma^n_{0,t}w^2(t,x)\int_{T_n(t)} I^{(n)}_t(t_1,\ldots,t_n)\mathrm{d}\mathbf{t}\\ &\le \Gamma^n_{0,t}w^2(t,x)\int_{T_n(t)}\prod_{j=1}^n k_\alpha\Big(\frac{2(t_{j+1}-t_j)t_j}{t_{j+1}}\Big)\mathrm{d}\mathbf{t}\\ &= \Gamma^n_{0,t}w^2(t,x)\int_{T_n(t)} C^{(1)}_{d,\alpha}\Big(\frac{2(t_2-t_1)t_1}{t_2}\Big)^{-(d-\alpha)/2}\ldots C^{(1)}_{d,\alpha}\Big(\frac{2(t-t_n)t_n}{t}\Big)^{-(d-\alpha)/2}\mathrm{d}\mathbf{t}\\ &= \Gamma^n_{0,t}w^2(t,x)\big(C^{(1)}_{d,\alpha}\big)^n 2^{-(d-\alpha)n/2}t^{(d-\alpha)/2}\int_{T_n(t)}\big[t_1(t_2-t_1)\ldots(t-t_n)\big]^{-(d-\alpha)/2}\mathrm{d}\mathbf{t},\end{aligned} \quad (3.1.22)$$

where for the second last line, we used (3.1.20).

Lemma 3.1.9 (Lemma 4.1 of [3]). *For any $h > -1$, we have*

$$\int_{0<t_1<\ldots<t_n<t} \big[t_1(t_2-t_1)\ldots(t-t_n)\big]^h \mathrm{d}t_1\ldots\mathrm{d}t_n = \frac{\Gamma(h+1)^{n+1}}{\Gamma\Big((n+1)(h+1)\Big)} t^{n(h+1)+h}.$$

We are ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3: By Theorem 3.1.2, to prove the existence of solution, we have to show that relation (3.1.11) holds. Using (3.1.22) and Lemma 3.1.9, we have:

$$\frac{1}{n!}k_n(t,x) \le \Gamma_{0,t}^n w^2(t,x)\big(C_{d,\alpha}^{(2)}\big)^n \frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^{n+1}}{\Gamma\big((n+1)(1-\frac{d-\alpha}{2})\big)} t^{n(1-\frac{d-\alpha}{2})}, \tag{3.1.23}$$

where $C_{d,\alpha}^{(2)} = C_{d,\alpha}^{(1)} 2^{-(d-\alpha)/2}$. Note that to apply Lemma 3.1.9, we need to impose condition that $\alpha > d-2$. By Lemma A.1.4, there exists constants $C_{d,\alpha}^{(3)} > 0$ and $C_{d,\alpha}^{(4)} > 0$ such that

$$\begin{aligned}\frac{1}{n!}k_n(t,x) &\le \Gamma_{0,t}^n w^2(t,x)\big(C_{d,\alpha}^{(2)}\big)^n \frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^{n+1}}{\Gamma\big(n(1-\frac{d-\alpha}{2})+1-\frac{d-\alpha}{2}\big)} t^{n(1-\frac{d-\alpha}{2})}\\ &= \Gamma_{0,t}^n w^2(t,x)\big(C_{d,\alpha}^{(3)}\big)^n \frac{1}{\Gamma\big(n(1-\frac{d-\alpha}{2})+1-\frac{d-\alpha}{2}\big)} t^{n(1-\frac{d-\alpha}{2})}\\ &\le \Gamma_{0,t}^n w^2(t,x)\big(C_{d,\alpha}^{(4)}\big)^n t^{n(1-\frac{d-\alpha}{2})} \frac{1}{(n!)^{1-\frac{d-\alpha}{2}}}.\end{aligned} \tag{3.1.24}$$

With relation (3.1.24) and Lemma A.1.6, the upper bound for the second moment of solution u^α can be found as follows:

$$\mathbb{E}|u^\alpha(t,x)|^2 = \sum_{n\ge 0} \mathbb{E}|I_n^\alpha(f_n(\cdot,t,x))|^2 \le \Gamma_{0,t}^n w^2(t,x)\big(C_{d,\alpha}^{(4)}\big)^n \sum_{n\ge 0} \frac{t^{n(1-\frac{d-\alpha}{2})}}{(n!)^{1-\frac{d-\alpha}{2}}} < C_1 e^{C_2 t},$$

where C_1, C_2 are some positive constants. This proves the existence of solution.

Next, we prove (3.1.6). Using Minkowski inequality, hypercontractivity and relation (3.1.24), we obtain

$$\|u^\alpha(t,x)\|_p = \Big\|\sum_{n\ge 0} I_n^\alpha(f_n(\cdot,t,x))\Big\|_p \le \sum_{n\ge 0}\Big\|I_n^\alpha(f_n(\cdot,t,x))\Big\|_p$$

$$\le \sum_{n\ge 0}(p-1)^{n/2}\Big\|I_n^\alpha(f_n(\cdot,t,x))\Big\|_2 \le w(t,x)\sum_{n\ge 0}(p-1)^{n/2}\left(\Gamma_{0,t}^n\big(C_{d,\alpha}^{(4)}\big)^n t^{n(1-\frac{d-\alpha}{2})}\frac{1}{(n!)^{1-\frac{d-\alpha}{2}}}\right)^{1/2}$$

$$= w(t,x)\sum_{n\geq 0}\frac{\left(\sqrt{(p-1)\Gamma_{0,t}C^{(4)}_{d,\alpha}\,t^{(1-\frac{d-\alpha}{2})}}\right)^n}{(n!)^{(1-\frac{d-\alpha}{2})/2}}$$

$$\leq C_3 w(t,x)\exp\left(C_4(p-1)^{\frac{2}{2-(d-\alpha)}}\,\Gamma_{0,t}^{\frac{2}{2-(d-\alpha)}}\,\left(C^{(4)}_{d,\alpha}\right)^{\frac{2}{2-(d-\alpha)}}\,t\right),$$

where we used Lemma A.1.6 for the last inequality. This proves (3.1.6).

Remark 3.1.10. If $\gamma_0 = \alpha_{H_0}|t|^{2H_0-2}$ for some $H_0 \in (0,1/2)$, then $\Gamma_{0,t} = 2\int_0^t H_0(2H_0-1)s^{2H_0-2} = 2H_0 t^{2H_0-1}$, and relation (3.1.6) become

$$\mathbb{E}|u^\alpha(t,x)|^p \leq C^{(3)} w^p(t,x)\exp\left(C^{(4)}\,p^{\frac{4-(d-\alpha)}{2-(d-\alpha)}}\,t^{\frac{4H_0-(d-\alpha)}{2-(d-\alpha)}}\right),$$

which coincides with (2.2.16) in Theorem 2.2.6.

3.2 Uniform moment estimates

In this section, we include some estimates for the moments of the increments of the solution to equation (3.0.1) with the noise W^α introduced in Section 2.1. The result below is essentially given by Theorem 1.3 of [3], and will be used in the proof of tightness in Section 3.3. We include the details of the proof since we need the explicit form of the constants, and also because we correct a mistake from the proof of Theorem 1.3 of [3].

Theorem 3.2.1. *Let u^α be the solution to equation* (3.0.1) *with noise W^α as in Section 3.1. Let K be a compact set contained in $\mathbb{R}^d$ and $[a,b]$ be a compact set such that*

$$\max\{d-2,0\} < a < b < d.$$

We fix $T > t_0 > 0$. For any $p \geq 2$, $d_0 \in (0, 1-\frac{d-a}{2})$ and $\beta \in (\frac{d-a}{2}+d_0, 1)$, there exist positive constants C_1 and C_2 (depending on d, t_0, T, p, a, b, d_0, and β) such that for any $t', t \in [t_0, T]$ and $x', x \in K$, we have

$$\sup_{\alpha\in[a,b]}\mathbb{E}|u^\alpha(t',x)-u^\alpha(t,x)|^p \leq C_1|t'-t|^{\frac{p(1-\beta)}{2}} \tag{3.2.1}$$

and

$$\sup_{\alpha\in[a,b]}\mathbb{E}|u^\alpha(t,x')-u^\alpha(t,x)|^p \leq C_2|x'-x|^{p(1-\beta)}. \tag{3.2.2}$$

Proof: We proceed as **Step 1** in the proof of Theorem 2.3.1. A road map of this proof is the following:

$$\begin{cases}\text{Step 1: The time increments}\\ \text{Step 2: The space increments}\end{cases}$$

Step 1: We start with the time increments. Fix $0 < t_0 < T$. Let $t, t' \in [t_0, T]$ and $x \in K$ be arbitrary. Assume that $h = t' - t > 0$ (The case $h < 0$ is similar). We use (2.3.21).

We study $A_n^{(\alpha)}(t,h)$ first. By (2.3.25), we know that

$$A_n^{(\alpha)}(t,h) \leq \Gamma_{0,t}^n \int_{[0,t]^n} \psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} = \Gamma_{0,t}^n \sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t} \psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}. \quad (3.2.3)$$

We estimate $\psi_{t,h,n}^{(\alpha)}(\mathbf{t},\mathbf{t})$. Assume for simplicity that $0 < t_1 < \ldots < t_n < t$. Using Lemma 3.1.5, we obtain the following estimates (which is a correction of the one given in the proof of Lemma B.3 of [3]):

$$\begin{aligned}
&\Big|\mathcal{F}\big(g_{\mathbf{t}}^{(n)}(\cdot, t+h, x) - g_{\mathbf{t}}^{(n)}(\cdot, t, x)\big)(\xi_1, \ldots, \xi_n)\Big| \\
&\qquad = \prod_{k=1}^{n-1} \exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)(I_1 - I_2) \\
&\qquad \leq \prod_{k=1}^{n-1} \exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big) \cdot \Big(I_1^{(1)}\big|I_1^{(2)} - I_2^{(2)}\big| + \big|I_2^{(2)}\big|\big|I_1^{(1)} - I_2^{(1)}\big|\Big) \\
&\qquad =: K_1 + K_2
\end{aligned} \quad (3.2.4)$$

where $I_1 = I_1^{(1)} \times I_1^{(2)}$ and $I_2 = I_2^{(1)} \times I_2^{(2)}$, with

$$\begin{aligned}
I_1^{(1)} &= \exp\Big(-\frac{1}{2}\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \\
I_1^{(2)} &= \int_{\mathbb{R}^d} \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot x\Big) \\
&\qquad\qquad \exp\Big(-i\Big[\sum_{j=1}^n\Big(1-\frac{t_j}{t+h}\Big)\xi_j\Big]\cdot x_0\Big)\, G_{t+h}(x-x_0)u_0(\mathrm{d}x_0) \\
&= \int_{\mathbb{R}^d} \exp\Big(-i\Big(\sum_{j=1}^n \xi_j\Big)\cdot x_0\Big) \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x-x_0)\Big)\, G_{t+h}(x-x_0)u_0(\mathrm{d}x_0) \\
I_2^{(1)} &= \exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \\
I_2^{(2)} &= \int_{\mathbb{R}^d} \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot x\Big) \exp\Big(-i\Big[\sum_{j=1}^n\Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big)\, G_t(x-x_0)u_0(\mathrm{d}x_0)
\end{aligned}$$

$$= \int_{\mathbb{R}^d} \exp\Big(-i\Big(\sum_{j=1}^n \xi_j\Big)\cdot x_0\Big) \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) G_t(x-x_0)u_0(\mathrm{d}x_0).$$

We first consider K_1. Note that

$$\begin{aligned}\Big|I_1^{(2)} - I_2^{(2)}\Big| &= \Big| \int_{\mathbb{R}^d} \exp\Big(-i\Big(\sum_{j=1}^n \xi_j\Big)\cdot x_0\Big)\Big\{ \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) G_{t+h}(x-x_0) \\ &\qquad - \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) G_t(x-x_0)\Big\} u_0(\mathrm{d}x_0)\Big| \\ &\le \int_{\mathbb{R}^d} \Big| \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) G_{t+h}(x-x_0) \\ &\qquad - \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) G_t(x-x_0)\Big| u_0(\mathrm{d}x_0) \le T_1 + T_2 \qquad (3.2.5)\end{aligned}$$

where

$$\begin{aligned}T_1 &= \int_{\mathbb{R}^d} \Big| \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big)\Big| \Big| G_{t+h}(x-x_0) - G_t(x-x_0)\Big| u_0(\mathrm{d}x_0) \\ &\le \int_{\mathbb{R}^d} \Big| G_{t+h}(x-x_0) - G_t(x-x_0)\Big| u_0(\mathrm{d}x_0)\end{aligned}$$

and

$$\begin{aligned}T_2 = \int_{\mathbb{R}^d} \Big| &\exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big) \\ &- \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot (x-x_0)\Big)\Big| G_t(x-x_0)u_0(\mathrm{d}x_0).\end{aligned}$$

We study T_1 first. We use the inequality $1-e^{-x} \le x^\theta$ for any $x>0$ and $\theta \in [0,1]$. Here θ is a constant which will be chosen later. Then, for any $t \in [t_0, T]$, we have

$$\begin{aligned}T_1 &\le \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} \Big| e^{-\frac{|x-x_0|^2}{2(t+h)}} - e^{-\frac{|x-x_0|^2}{2t}}\Big| u_0(\mathrm{d}x_0) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-x_0|^2}{2(t+h)}} \Big| 1 - e^{-\frac{h|x-x_0|^2}{2t(t+h)}}\Big| u_0(\mathrm{d}x_0) \\ &\le \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-x_0|^2}{2(t+h)}} \Big(\frac{h|x-x_0|^2}{2t(t+h)}\Big)^\theta u_0(\mathrm{d}x_0) \\ &\le h^\theta \frac{1}{(2\pi t)^{d/2}} \frac{1}{(2t^2)^\theta} \int_{\mathbb{R}^d} e^{-\frac{|x-x_0|^2}{2T}} |x-x_0|^{2\theta} u_0(\mathrm{d}x_0) \le h^\theta C_{t_0,d,\theta,T}. \qquad (3.2.6)\end{aligned}$$

Note that the last integral is finite due to condition (3.0.2), since

$$\int_{|x-x_0|\le 1} \exp\Big(-\frac{|x-x_0|^2}{2T}\Big)|x-x_0|^\theta u_0(\mathrm{d}x_0) \le \int_{|x-x_0|\le 1} \exp\Big(-\frac{|x-x_0|^2}{2T}\Big)u_0(\mathrm{d}x_0) < \infty$$

and

$$\begin{aligned}&\int_{|x-x_0|>1} \exp\Big(-\frac{|x-x_0|^2}{2T}\Big)|x-x_0|^\theta u_0(\mathrm{d}x_0)\\ &\qquad\le \int_{|x-x_0|>1} \exp\Big(-\frac{|x-x_0|^2}{2T}\Big)\exp\big(c|x-x_0|^\theta\big)u_0(\mathrm{d}x_0) < \infty,\end{aligned}$$

using the fact that $x^\theta \le e^{cx}$ for any $x>0$ and $c>0$.

For the term T_2, using the fact that $\left|e^{-ia}-e^{-ib}\right| \le |e^{-ia}||1-e^{-i(b-a)}| \le |b-a|^\theta$, for all $\theta\in[0,1]$, we have:

$$\begin{aligned}T_2 &\le \int_{\mathbb{R}^d}\left|\frac{\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x-x_0)}{t} - \frac{\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x-x_0)}{t+h}\right|^\theta G_t(x-x_0)u_0(\mathrm{d}x_0)\\ &= \int_{\mathbb{R}^d}\left|\frac{h\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x-x_0)}{t(t+h)}\right|^\theta G_t(x-x_0)u_0(\mathrm{d}x_0)\\ &\le h^\theta\frac{1}{t^{2\theta}}\Big|\sum_{j=1}^n t_j\xi_j\Big|^\theta\int_{\mathbb{R}^d}|x-x_0|^\theta G_t(x-x_0)u_0(\mathrm{d}x_0) \le h^\theta C_{t_0,\theta}\Big|\sum_{j=1}^n t_j\xi_j\Big|^\theta. \qquad (3.2.7)\end{aligned}$$

Combining relations (3.2.5), (3.2.6) and (3.2.7), we infer that there exist a constant $C^{(1)}_{t_0,d,\theta,T}$ such that

$$\begin{aligned}K_1 &\le \prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\times\exp\Big(-\frac{1}{2}\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\\ &\qquad\times\Big(h^\theta C_{t_0,d,\theta,T} + h^\theta C_{t_0,\theta}\Big|\sum_{j=1}^n t_j\xi_j\Big|^\theta\Big)\\ &\le h^\theta C^{(1)}_{t_0,d,\theta,T}\prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\\ &\qquad\times\exp\Big(-\frac{1}{2}\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\times\Big(1+\Big|\sum_{j=1}^n t_j\xi_j\Big|^\theta\Big)\\ &\le h^\theta C^{(1)}_{t_0,d,\theta,T}\prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\end{aligned}$$

$$\times \exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^2\Big) \times \Big(1+\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^\theta\Big)$$

where for the last inequality, we used the fact that

$$\frac{t+h-t_n}{t_n(t+h)} = \frac{1}{t_n} - \frac{1}{t+h} > \frac{1}{t_n} - \frac{1}{t} = \frac{t-t_n}{t_n t}.$$

We need the following inequality: for any $A > 0$ and $\theta > 0$, there exists a constant $C_\theta > 0$ depending on θ such that

$$e^{-Ax^2} x^{2\theta} \le C_\theta A^{-\theta} e^{-\frac{A}{2}x^2}, \text{ for all } x \ge 0. \tag{3.2.8}$$

(See relation (5.2) in [3]). Using inequality (3.2.8) with $A = \frac{t-t_n}{t_n t}$ and $x = \Big|\sum_{j=1}^n t_j\xi_j\Big|$, we obtain:

$$\exp\Big(-\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^2\Big)\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^{2\theta} \le C_\theta\Big(\frac{t-t_n}{t_n t}\Big)^{-\theta} \exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^2\Big). \tag{3.2.9}$$

Note that for any $t \in [t_0, T]$, we have

$$\Big(\frac{t-t_n}{t_n t}\Big)^{-\theta} \le T^{2\theta}(t-t_n)^{-\theta}. \tag{3.2.10}$$

Hence, denoting $t_{n+1} = t$, by relations (3.2.9) and (3.2.10), we have

$$\begin{aligned}
K_1^2 \le{}& h^{2\theta}\big(C^{(1)}_{t_0,d,\theta,T}\big)^2\Big\{\prod_{k=1}^{n}\exp\Big(-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^{k} t_j\xi_j\Big|^2\Big) \\
&+ \prod_{k=1}^{n-1}\exp\Big(-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^{k} t_j\xi_j\Big|^2\Big) \times \exp\Big(-\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^2\Big) \times \Big|\sum_{j=1}^{n} t_j\xi_j\Big|^{2\theta}\Big\} \\
\le{}& h^{2\theta}\big(C^{(1)}_{t_0,d,\theta,T}\big)^2\Big\{\prod_{k=1}^{n}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^{k} t_j\xi_j\Big|^2\Big) \\
&+ \prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^{k} t_j\xi_j\Big|^2\Big) \times C_\theta\Big(\frac{t-t_n}{t_n t}\Big)^{-\theta}\exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^{n} t_j\xi_j\Big|^2\Big)\Big\} \\
\le{}& h^{2\theta} C^{(2)}_{t_0,d,\theta,T}\prod_{k=1}^{n}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^{k} t_j\xi_j\Big|^2\Big)\big[1+(t-t_n)^{-\theta}\big].
\end{aligned} \tag{3.2.11}$$

We consider next the term K_2. Note that using the fact that $(1-e^{-x})^2 \le 1-e^{-x} \le x^\theta$ for all $x>0$ and $\theta \in [0,1]$, for any $t \in [t_0, T]$, we get:

$$
\begin{aligned}
\Big|I_1^{(1)} - I_2^{(1)}\Big|^2 &= \Big|\exp\Big(-\frac{1}{2}\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) - \exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\Big|^2 \\
&= \exp\Big(-\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\Big|1-\exp\Big(-\frac{1}{2}\frac{h}{t(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\Big)\Big|^2 \\
&\le \exp\Big(-\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\Big(\frac{1}{2}\frac{h}{t(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)^\theta \\
&\le h^\theta C_{t_0,\theta}\exp\Big(-\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big)\Big|\sum_{j=1}^n t_j\xi_j\Big|^{2\theta} \\
&\le h^\theta C_{t_0,\theta,T}\Big(\frac{t-t_n}{t_n t}\Big)^{-\theta}\exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big), \qquad (3.2.12)
\end{aligned}
$$

where we used relation (3.2.9) for the last inequality. Moreover, we see that

$$
\begin{aligned}
|I_2^{(2)}| &\le \int_{\mathbb{R}^d}\Big|\exp\Big(-i\Big(\sum_{j=1}^n \xi_j\Big)\cdot x_0\Big)\exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x-x_0)\Big)\Big|G_t(x-x_0)u_0(\mathrm{d}x_0) \\
&\le w(t,x). \qquad (3.2.13)
\end{aligned}
$$

Therefore, by (3.2.10), (3.2.12) and (3.2.13), we obtain:

$$
\begin{aligned}
K_2^2 \le \prod_{k=1}^{n-1}&\exp\Big(-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)w^2(t,x)h^\theta\Big(\frac{t-t_n}{t_n t}\Big)^{-\theta} \\
&\qquad C_{t_0,\theta,T}\exp\Big(-\frac{1}{2}\frac{t-t_n}{t_n t}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \\
\le h^\theta C^{(3)}_{t_0,d,\theta,T}&(t-t_n)^{-\theta}\prod_{k=1}^n \exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big), \qquad (3.2.14)
\end{aligned}
$$

where for the last inequality, we use the fact that $w(t,x)$ is uniformly bounded by a constant $C_{t_0,d}$ in $[t_0,T]\times K$ (by Lemma B.2 of [3] which says that $w(t,x)$ is continuous on $(0,\infty)\times\mathbb{R}^d$). Note that K_1^2 is bounded by a factor of $h^{2\theta}$, while K_2^2 is bounded by a factor of h^θ, and h^θ dominates.

We continue to estimate $\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t})$. Using (3.2.11), (3.2.14) and recalling the definition of $k_\alpha(t)$ in (3.1.19), we have

$$
\psi^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t}) = \int_{(\mathbb{R}^d)^n}\Big|\mathcal{F}\Big(g^{(n)}_{\mathbf{t}}(\cdot,t+h,x) - g^{(n)}_{\mathbf{t}}(\cdot,t,x)\Big)(\xi_1,\dots,\xi_n)\Big|^2\prod_{k=1}^n|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}
$$

$$
\begin{aligned}
&\leq \int_{(\mathbb{R}^d)^n} 2(K_1^2+K_2^2)\prod_{k=1}^n |\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}\\
&= h^\theta 2\Big(C^{(2)}_{t_0,d,\theta,T}+C^{(2)}_{t_0,d,\theta,T}(t-t_n)^{-\theta}+C^{(3)}_{t_0,d,\theta,T}(t-t_n)^{-\theta}\Big)\\
&\qquad\qquad \int_{(\mathbb{R}^d)^n}\prod_{k=1}^n \exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\prod_{k=1}^n|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}\\
&\leq h^\theta C^{(4)}_{t_0,d,\theta,T}(t-t_n)^{-\theta}\int_{(\mathbb{R}^d)^n}\exp\Big(-\sum_{k=1}^n\frac{\frac{t_{k+1}}{2}-\frac{t_k}{2}}{\frac{t_{k+1}}{2}\frac{t_k}{2}}\Big|\sum_{j=1}^k\frac{t_j}{2}\xi_j\Big|^2\Big)\prod_{k=1}^n|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi} \quad (3.2.15)\\
&\leq h^\theta C^{(4)}_{t_0,d,\theta,T}(t-t_n)^{-\theta}\prod_{j=1}^n k_\alpha\Big(\frac{2(\frac{t_{j+1}}{2}-\frac{t_j}{2})(\frac{t_j}{2})}{\frac{t_{j+1}}{2}}\Big)\\
&= h^\theta C^{(4)}_{t_0,d,\theta,T}(t-t_n)^{-\theta}\big(C^{(1)}_{d,\alpha}\big)^n t^{(d-\alpha)/2}\big[t_1(t_2-t_1)\dots(t-t_n)\big]^{-(d-\alpha)/2}, \qquad (3.2.16)
\end{aligned}
$$

where for the last inequality, we used Lemma 3.1.8 and for the last equation, we used (3.1.22). A similar estimate holds in the case $t_{\rho(1)}<\ldots<t_{\rho(n)}$, replacing t_j by $t_{\rho(j)}$. Note that $C^{(1)}_{d,\alpha}$ (defined in (3.1.21)) can be bounded by some constant. To see this, we use the inequality $e^{-x}\leq\frac{1}{1+x}$ for all $x\geq 0$. Similarly to the proof of Lemma A.2.1, for any $\alpha\in[a,b]$, we know

$$
C^{(1)}_{d,\alpha}=\int_{\mathbb{R}^d}e^{-|\xi|^2/2}|\xi|^{-\alpha}\mathrm{d}\xi\leq\int_{\mathbb{R}^d}\frac{2}{2+|\xi|^2}|\xi|^{-\alpha}\mathrm{d}\xi\leq c_d\Big(\frac{1}{d-\alpha}+\frac{2}{2-(d-\alpha)}\Big)\leq C_{d,a,b}. \qquad (3.2.17)
$$

We return to relation (3.2.3). Let $t'_j=t_{\rho(j)}$ for all $j=1,\ldots,n$ and $t=t_{\rho(n)+1}$. By relations (3.2.16) and (3.2.17),

$$
\begin{aligned}
A_n^{(\alpha)}(t,h)&\leq h^\theta\Gamma_{0,t}^n\, n!\, C^{(4)}_{t_0,d,\theta,T}C^n_{d,a,b}t^{(d-\alpha)/2}\\
&\qquad\int_{0<t_1<\ldots<t_n<t}\big[t_1(t_2-t_1)\dots(t_n-t_{n-1})\big]^{-\frac{d-\alpha}{2}}(t-t_n)^{-\frac{d-\alpha}{2}-\theta}\mathrm{d}\mathbf{t}\\
&= h^\theta\Gamma_{0,t}^n\, n!\, C^{(4)}_{t_0,d,\theta,T}C^n_{d,a,b}t^{\frac{d-\alpha}{2}}\frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^n}{\Gamma\big(n(1-\frac{d-\alpha}{2})\big)}\int_0^t t_n^{(n-1)(1-\frac{d-\alpha}{2})-\frac{d-\alpha}{2}}(t-t_n)^{-\frac{d-\alpha}{2}-\theta}\mathrm{d}t_n\\
&\leq h^\theta\Gamma_{0,t}^n\, n!\, C^{(4)}_{t_0,d,\theta,T}C^n_{d,a,b}\frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^n}{\Gamma\big(n(1-\frac{d-\alpha}{2})\big)}t^{\frac{d-\alpha}{2}}t^{(n-1)(1-\frac{d-\alpha}{2})}\int_0^t t_n^{-\frac{d-\alpha}{2}}(t-t_n)^{-\frac{d-\alpha}{2}-\theta}\mathrm{d}t_n\\
&= h^\theta\Gamma_{0,t}^n\, n!\, C^{(4)}_{t_0,d,\theta,T}C^n_{d,a,b}\frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^n}{\Gamma\big(n(1-\frac{d-\alpha}{2})\big)}t^{n\big(1-\frac{d-\alpha}{2}\big)-\theta}\frac{\Gamma(1-\frac{d-\alpha}{2})\Gamma(1-\frac{d-\alpha}{2}-\theta)}{\Gamma(2-(d-\alpha)-\theta)}
\end{aligned}
\qquad (3.2.18)
$$

where we used Lemma 3.1.9 in the first equality. For the last equality we used the identity: for any $a > -1$, $b > -1$

$$\int_0^t s^a(1-s)^b \mathrm{d}s = t^{a+b+1}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}.$$

For this calculation, we need $\theta < 1 - \frac{d-\alpha}{2}$. But we also need to bound $\Gamma(1 - \frac{d-\alpha}{2} - \theta)$ for all $\alpha \in [a, b]$. Since $1 - \frac{d-\alpha}{2} > 1 - \frac{d-a}{2} > \theta$, $1 - \frac{d-\alpha}{2} - \theta$ can in principle be very close to 0 where the Γ function explodes. Therefore, we have to choose first a constant $d_0 \in (0, 1 - \frac{d-a}{2})$, and then choose

$$0 < \theta < 1 - \frac{d-a}{2} - d_0. \tag{3.2.19}$$

With this choice, $1 - \frac{d-\alpha}{2} - \theta > d_0$ for all $\alpha \in [a, b]$, and since Γ is decreasing on $[0, 1]$, $\Gamma\Big(1 - \frac{d-\alpha}{2} - \theta\Big) < \Gamma(d_0)$, for all $\alpha \in [a, b]$. This means that we can take

$$\theta = 1 - \beta,$$

where β is arbitrary such that

$$\frac{d-a}{2} + d_0 < \beta < 1$$

and $d_0 \in (0, 1 - \frac{d-a}{2})$ is arbitrary. This specifies our choice of θ.

Note that for any $\alpha \in [a, b] \subset (\max\{d-2, 0\}, d)$,

$$1 - \frac{d-a}{2} < 1 - \frac{d-\alpha}{2} < 1 - \frac{d-b}{2}$$

and since Γ is decreasing on $(0, 1)$, we have $\Gamma\Big(1 - \frac{d-\alpha}{2}\Big) < \Gamma\Big(1 - \frac{d-a}{2}\Big)$. For the denominator of (3.2.18),

$$1 - \frac{d-a}{2} - d_0 \le 1 - \frac{d-\alpha}{2} - d_0 \le 2 - (d-\alpha) - \theta \le 2 - (d-b)$$

and $\Gamma(2 - (d-\alpha) - \theta) \ge \min\Big\{\Gamma\big(1 - \frac{d-a}{2} - d_0\big), \Gamma\big(2 - (d-b)\big)\Big\} =: c$. By Lemma A.1.4, we have

$$\frac{1}{\Gamma\Big(n\big(1 - \frac{d-\alpha}{2}\big)\Big)} \le \frac{1}{\Gamma\Big(n\big(1 - \frac{d-a}{2}\big)\Big)} \le \big(c^{(1)}\big)^n \frac{1}{(n!)^{1-\frac{d-a}{2}}}, \tag{3.2.20}$$

$$\Gamma\Big(1 - \frac{d-\alpha}{2}\Big)^n \le \Gamma\Big(1 - \frac{d-a}{2}\Big)^n, \tag{3.2.21}$$

and

$$t^{n\left(1-\frac{d-\alpha}{2}\right)-\theta} \leq (T\vee 1)^{n\left(1-\frac{d-b}{2}\right)-\theta}. \tag{3.2.22}$$

We come back to (3.2.18). Putting together relations (3.2.20), (3.2.21) and (3.2.22), we have

$$A_n^{(\alpha)}(t,h) \leq h^{\theta}\Gamma_{0,t}^n\, n!\, C^{(5)}_{t_0,d,\theta,T} C^n_{d,a,b}\big(c^{(1)}\big)^n \frac{\Gamma\big(1-\frac{d-a}{2}\big)^n}{(n!)^{1-\frac{d-a}{2}}}(T\vee 1)^{n\left(1-\frac{d-b}{2}\right)-\theta}. \tag{3.2.23}$$

Hence

$$\sup_{\alpha\in[a,b]}\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^{(\alpha)}(t,h)\Big)^{1/2} \leq C^{(1)}_{d,t_0,T,p,a,b,\beta}h^{\theta/2}, \tag{3.2.24}$$

where $C^{(1)}_{d,t_0,T,p,a,b,\beta}$ is a constant depending on d, t_0, T, p, a, b and β.

As for the term $B_n^{(\alpha)}(t,h)$, by (2.3.36)

$$\begin{aligned} B_n^{(\alpha)}(t,h) &\leq \Gamma^n_{0,t+h}\int_{[0,t+h]^n}\gamma^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})\,\mathrm{d}\mathbf{t} \\ &= \Gamma^n_{0,t+h}\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)},\, t<t_{\rho(n)}<t+h}\gamma^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}. \end{aligned} \tag{3.2.25}$$

We estimate $\gamma^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t})$. Assume for simplicity that $0<t_1<\ldots<t_n$, and $t<t_n<t+h$. By Lemma 3.1.5, we have

$$\begin{aligned} &\big|\mathcal{F}g^{(n)}_{\mathbf{t}}(\cdot,t+h,x)(\xi_1,\ldots,\xi_n)\big|^2 \\ &= \Big|\prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\exp\Big(-\frac{1}{2}\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \\ &\quad \exp\Big(-\frac{i}{t+h}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot x\Big)\int_{\mathbb{R}^d}\exp\Big(-i\Big[\sum_{j=1}^n\Big(1-\frac{t_j}{t+h}\Big)\xi_j\Big]\cdot x_0\Big)\,G_{t+h}(x-x_0)u_0(\mathrm{d}x_0)\Big|^2 \\ &\leq w^2(t+h,x)\prod_{k=1}^{n-1}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)\exp\Big(-\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \end{aligned}$$

and hence for any $(t,x)\in[t_0,T]\times K$,

$$\gamma^{(\alpha)}_{t,h,n}(\mathbf{t},\mathbf{t}) \leq C_{t_0,d}\int_{(\mathbb{R}^d)^n}\prod_{k=1}^{n-1}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)$$

$$\exp\Big(-\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^{n}t_j\xi_j\Big|^2\Big)\prod_{j=1}^{n}|\xi_j|^{-\alpha}\mathrm{d}\xi_j. \quad (3.2.26)$$

A similar estimate holds if $t_{\rho(1)} < \ldots < t_{\rho(n)} < t+h$. By relations (3.2.25) and (3.2.26), it follows that

$$\begin{aligned}
&B_n^{(\alpha)}(t,h)\\
&\le \Gamma_{0,t+h}^n C_{t_0,d}\sum_{\rho\in S_n}\int_t^{t+h}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}}\Big(\int_{(\mathbb{R}^d)^n}\prod_{k=1}^{n-1}\exp\Big(-\frac{t_{\rho(k+1)}-t_{\rho(k)}}{t_{\rho(k)}t_{\rho(k+1)}}\Big|\sum_{j=1}^{k}t_{\rho(j)}\xi_j\Big|^2\Big)\\
&\qquad\exp\Big(-\frac{t+h-t_{\rho(n)}}{t_{\rho(n)}(t+h)}\Big|\sum_{j=1}^{n}t_j\xi_j\Big|^2\Big)\prod_{j=1}^{n}|\xi_j|^{-\alpha}\mathrm{d}\xi_j\Big)\mathrm{d}\mathbf{t}\\
&= \Gamma_{0,t+h}^n C_{t_0,d}\, n!\int_t^{t+h}\int_{0<t_1<\ldots<t_n}\Big(\int_{(\mathbb{R}^d)^n}\prod_{k=1}^{n-1}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\\
&\qquad\exp\Big(-\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^{n}t_j\xi_j\Big|^2\Big)\prod_{j=1}^{n}|\xi_j|^{-\alpha}\mathrm{d}\xi_j\Big)\mathrm{d}\mathbf{t}\\
&= \Gamma_{0,t+h}^n C_{t_0,d}\, n!\int_t^{t+h}\int_{0<t_1<\ldots<t_n} I_{t+h}^{(n)}(t_1,\ldots,t_n)\,\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}\mathrm{d}t_n, \qquad (3.2.27)
\end{aligned}$$

where we recall the definition of $I_{t+h}^{(n)}(t_1,\ldots,t_n)$ is given in (3.1.16). Using Lemma 3.1.8 and relation (3.1.22), we have

$$\begin{aligned}
I_{t+h}^{(n)}(t_1,\ldots,t_n) &\le \prod_{j=1}^{n-1}k_\alpha\Big(\frac{2(t_{j+1}-t_j)t_j}{t_{j+1}}\Big)\times k_\alpha\Big(\frac{2(t+h-t_n)t_n}{t+h}\Big)\\
&= \big(C_{d,\alpha}^{(1)}\big)^n 2^{-(d-\alpha)n/2}\big[t_1(t_2-t_1)\ldots(t_n-t_{n-1})\big]^{-(d-\alpha)/2}\Big(\frac{t+h-t_n}{t+h}\Big)^{-(d-\alpha)/2}\\
&\le C_{d,a,b}^n\big[t_1(t_2-t_1)\ldots(t_n-t_{n-1})\big]^{-(d-\alpha)/2}\Big(\frac{t+h-t_n}{t+h}\Big)^{-(d-\alpha)/2}. \qquad (3.2.28)
\end{aligned}$$

Using Lemma 3.1.9, relations (3.2.20) and (3.2.21), we have

$$\begin{aligned}
&\int_{0<t_1<\ldots<t_n}\big[t_1(t_2-t_1)\ldots(t_n-t_{n-1})\big]^{-(d-\alpha)/2}\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}\\
&\quad=\frac{\Gamma\Big(1-\frac{d-\alpha}{2}\Big)^n}{\Gamma\Big(n\big(1-\frac{d-\alpha}{2}\big)\Big)}t_n^{(n-1)\left(1-\frac{d-\alpha}{2}\right)-\frac{d-\alpha}{2}} \le \big(c^{(1)}\big)^n\frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}}t_n^{(n-1)\left(1-\frac{d-\alpha}{2}\right)-\frac{d-\alpha}{2}}.
\end{aligned}$$

(3.2.29)

We return to relation (3.2.27). Using the change of variable $s = t + h - t_n$, by (3.2.28) and (3.2.29), we have

$$
\begin{aligned}
&B_n^{(\alpha)}(t,h)\\
&\le \Gamma_{0,t+h}^n C_{t_0,d}\, n! C_{d,a,b}^n \big(c^{(1)}\big)^n \frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}} \int_t^{t+h} t_n^{(n-1)\left(1-\frac{d-\alpha}{2}\right)-\frac{d-\alpha}{2}} \Big(\frac{t+h-t_n}{t+h}\Big)^{-(d-\alpha)/2} \mathrm{d}t_n\\
&= \Gamma_{0,t+h}^n C_{t_0,d}\, n! C_{d,a,b}^n \big(c^{(1)}\big)^n \frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}} \Big(\frac{1}{t+h}\Big)^{-(d-\alpha)/2} \int_0^h (t+h-s)^{n\left(1-\frac{d-\alpha}{2}\right)-1} s^{-(d-\alpha)/2} \mathrm{d}s\\
&\le \Gamma_{0,t+h}^n C_{t_0,d}\, n! C_{d,a,b}^n \big(c^{(1)}\big)^n \frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}} \Big(\frac{1}{t+h}\Big)^{-(d-\alpha)/2} T^{n\left(1-\frac{d-\alpha}{2}\right)-1} \int_0^h s^{-(d-\alpha)/2} \mathrm{d}s\\
&\le \Gamma_{0,t+h}^n C_{t_0,d}\, n! C_{d,a,b}^n \big(c^{(1)}\big)^n \frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}} (t+h)^{(d-\alpha)/2} T^{n\left(1-\frac{d-\alpha}{2}\right)-1} \frac{2}{2-(d-\alpha)} h^{1-(d-\alpha)/2}\\
&\le \Gamma_{0,t+h}^n C_{t_0,d}\, n! C_{d,a,b}^n \big(c^{(1)}\big)^n \frac{\Gamma\Big(1-\frac{d-a}{2}\Big)^n}{(n!)^{1-\frac{d-a}{2}}} T^{(n-1)\left(1-\frac{d-b}{2}\right)} \frac{2}{2-(d-a)} h^{\theta}, \qquad (3.2.30)
\end{aligned}
$$

for any $h \in (0,1)$, $\theta \in \big(0, 1-\frac{d-a}{2}\big)$ and $\alpha \in [a,b]$. Therefore,

$$
\sup_{\alpha\in[a,b]} \sum_{n\ge 1} (p-1)^{n/2} \Big(\frac{1}{n!} B_n^{(\alpha)}(t,h)\Big)^{1/2} \le C_{d,t_0,T,p,a,b}^{(2)} h^{\theta/2}, \qquad (3.2.31)
$$

where $C_{d,t_0,T,p,a,b}^{(2)}$ is a constant depending on d, t_0, T, p, a and b.

Combining relations (3.2.24) and (3.2.31), it follows that relation (3.2.1) holds.

Step 2: Now we treat the spatial increments. For any $x, x' \in K$, we let $z = x' - x$. We use (2.3.41). By (2.3.44),

$$
C_n^{(\alpha)}(t,z) \le \Gamma_{0,t}^n \int_{[0,t]^n} \Psi_{t,z,n}^{(\alpha)}(\mathbf{t},\mathbf{t})\, \mathrm{d}\mathbf{t}. \qquad (3.2.32)
$$

We estimate $\Psi_{t,z,n}^{(\alpha)}(\mathbf{t},\mathbf{t})$. Assume for simplicity that $0 < t_1 < \ldots < t_n < t_{n+1} = t$. Using Lemma 3.1.5, we obtain

$$
\begin{aligned}
&\big(\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x+z) - \mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)\big)(\xi_1,\ldots,\xi_n)\\
&\qquad = \prod_{k=1}^n \exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big)(J_1+J_2) \qquad (3.2.33)
\end{aligned}
$$

where

$$\begin{aligned}
J_1 &:= \Big(\exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot(x+z)\Big) - \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot x\Big)\Big) \\
&\qquad \int_{\mathbb{R}^d} \exp\Big(-i\Big[\sum_{j=1}^n \Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big)\, G_t(x+z-x_0)u_0(\mathrm{d}x_0) \\
J_2 &:= \exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot x\Big) \\
&\qquad \int_{\mathbb{R}^d} \exp\Big(-i\Big[\sum_{j=1}^n \Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big)\Big(G_t(x+z-x_0)-G_t(x-x_0)\Big)u_0(\mathrm{d}x_0).
\end{aligned}$$

We study J_1 first. We use the fact that $|1-e^{-ix}|^2 \le x^{2\theta}$, for any $\theta \in (0,1)$. Hence, we get

$$\begin{aligned}
J_1^2 &\le \Big|\exp\Big(-\frac{i}{t}\Big(\sum_{j=1}^n t_j\xi_j\Big)\cdot z\Big) - 1\Big|^2 \Big(\int_{\mathbb{R}^d} G_t(x+z-x_0)u_0(\mathrm{d}x_0)\Big)^2 \\
&\le \Big|\Big(\frac{\sum_{j=1}^n t_j\xi_j}{t}\Big)\cdot z\Big|^{2\theta} w^2(t,x+z) \le C^{(5)}_{t_0,d,\theta,T}\Big|\sum_{j=1}^n t_j\xi_j\Big|^{2\theta}|z|^{2\theta}, \qquad (3.2.34)
\end{aligned}$$

using the fact that $w(t,x)$ is uniformly bounded on compact sets.

Next, we consider J_2. We will use the following inequality (given by Lemma 4.1 of [15]): for any $\alpha \in (0,1]$,

$$\big|G_t(x) - G_t(y)\big| \le \frac{C_\alpha}{t^{\alpha/2}}\big[G_{2t}(x) + G_{2t}(y)\big]|x-y|^\alpha,$$

for any $t > 0$ and $x, y \in \mathbb{R}^d$, where $C_\alpha > 0$ is a constant depending on α. Using above inequality, we have

$$\begin{aligned}
J_2^2 &\le \Big|\int_{\mathbb{R}^d} \exp\Big(-i\Big[\sum_{j=1}^n \Big(1-\frac{t_j}{t}\Big)\xi_j\Big]\cdot x_0\Big)\Big(G_t(x+z-x_0)-G_t(x-x_0)\Big)u_0(\mathrm{d}x_0)\Big|^2 \\
&\le \Big(\int_{\mathbb{R}^d}\Big|G_t(x+z-x_0)-G_t(x-x_0)\Big|u_0(\mathrm{d}x_0)\Big)^2 \\
&\le \Big(\int_{\mathbb{R}^d}\frac{C_\theta}{t^{\theta/2}}\big[G_{2t}(x+z-x_0)+G_{2t}(x-x_0)\big]|z|^\theta u_0(\mathrm{d}x_0)\Big)^2 \\
&\le |z|^{2\theta}\frac{C_\theta}{t_0^\theta}\Big(w(2t,x+z)+w(2t,x)\Big)^2 \le C^{(6)}_{t_0,d,\theta,T}|z|^{2\theta}, \qquad (3.2.35)
\end{aligned}$$

where we use the fact that $w(t,x)$ is uniformly bounded in $[t_0,T]\times K$.

We come back to relation (3.2.33). Combining relations (3.2.34) and (3.2.35), it follows that there exists a constant $C^{(7)}_{t_0,d,\theta,T}>0$ depending on t_0, d, θ and T such that

$$\begin{aligned}&\Big|\big(\mathcal{F}g^{(n)}_{\mathbf{t}}(\cdot,t,x+z)-\mathcal{F}g^{(n)}_{\mathbf{t}}(\cdot,t,x)\big)(\xi_1,\dots,\xi_n)\Big|^2\\&\le 2\prod_{k=1}^{n}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\big(J_1^2+J_2^2\big)\le |z|^{2\theta}C^{(7)}_{t_0,d,\theta,T}(F_1+F_2),\end{aligned}\tag{3.2.36}$$

where

$$F_1:=\prod_{k=1}^{n}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\Big|\sum_{j=1}^{n}t_j\xi_j\Big|^{2\theta},$$

and

$$F_2:=\prod_{k=1}^{n}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big).$$

We only need to study F_1. Using relations (3.2.9) and (3.2.10), we obtain

$$\begin{aligned}F_1&\le\prod_{k=1}^{n-1}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\\&\qquad\times C_\theta\Big(\frac{t-t_n}{t_nt}\Big)^{-\theta}\exp\Big(-\frac{1}{2}\frac{t-t_n}{t_nt}\Big|\sum_{j=1}^{n}t_j\xi_j\Big|^2\Big)\\&\le\prod_{k=1}^{n}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)C_\theta T^{2\theta}(t-t_n)^{-\theta}.\end{aligned}\tag{3.2.37}$$

By (3.2.36) and (3.2.37), we get:

$$\begin{aligned}\Psi^{(\alpha)}_{t,z,n}(\mathbf{t},\mathbf{t})&=\int_{(\mathbb{R}^d)^n}\Big|\mathcal{F}\Big(g^{(n)}_{\mathbf{t}}(\cdot,t,x+z)-g^{(n)}_{\mathbf{t}}(\cdot,t,x)\Big)(\xi_1,\dots,\xi_n)\Big|^2\prod_{k=1}^{n}|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}\\&\le|z|^{2\theta}C^{(7)}_{t_0,d,\theta,T}\Big(C_\theta T^{2\theta}(t-t_n)^{-\theta}\int_{(\mathbb{R}^d)^n}\prod_{k=1}^{n}\exp\Big(-\frac{1}{2}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\prod_{k=1}^{n}|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}\\&\qquad+\int_{(\mathbb{R}^d)^n}\prod_{k=1}^{n}\exp\Big(-\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{j=1}^{k}t_j\xi_j\Big|^2\Big)\prod_{k=1}^{n}|\xi_k|^{-\alpha}\mathrm{d}\boldsymbol{\xi}\Big)\end{aligned}\tag{3.2.38}$$

$$
\begin{aligned}
&\leq |z|^{2\theta} C^{(8)}_{t_0,d,\theta,T}\Bigg((t-t_n)^{-\theta}\prod_{j=1}^{n} k_\alpha\Big(\frac{2(\frac{t_{j+1}}{2}-\frac{t_j}{2})(\frac{t_j}{2})}{\frac{t_{j+1}}{2}}\Big)+\prod_{j=1}^{n} k_\alpha\Big(\frac{2(t_{j+1}-t_j)t_j}{t_{j+1}}\Big)\Bigg)\\
&= |z|^{2\theta} C^{(9)}_{t_0,d,\theta,T}\big(C^{(1)}_{d,\alpha}\big)^n t^{(d-\alpha)/2}\\
&\quad\Bigg(\big[t_1(t_2-t_1)\dots(t-t_n)\big]^{-(d-\alpha)/2}(t-t_n)^{-\theta}+\big[t_1(t_2-t_1)\dots(t-t_n)\big]^{-(d-\alpha)/2}\Bigg),
\end{aligned}
$$

where for the last inequality we used Lemma 3.1.8 and for the last equality we used (3.1.22). Similarly to relation (3.2.23), we obtain:

$$
\begin{aligned}
C_n^{(\alpha)}(t,z) \leq{}& |z|^{2\theta}\Gamma^n_{0,t} n! C^{(9)}_{t_0,d,\varepsilon,T}\big(C^{(1)}_{d,\alpha}\big)^n\big(c^{(1)}\big)^n\\
&\Bigg(\frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^n}{(n!)^{1-\frac{d-\alpha}{2}}}(T\vee 1)^{n(1-\frac{d-\alpha}{2})-\theta}+\frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^{n+1}}{\big((n+1)!\big)^{1-\frac{d-\alpha}{2}}}(T\vee 1)^{n(1-\frac{d-\alpha}{2})}\Bigg)
\end{aligned}
\tag{3.2.39}
$$

and therefore,

$$
\sup_{\alpha\in[a,b]}\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}C_n^{(\alpha)}(t,z)\Big)^{1/2} \leq C^{(3)}_{d,t_0,T,p,a,b,\beta}|z|^{\theta}, \tag{3.2.40}
$$

where $C^{(3)}_{d,t_0,T,p,a,b,\beta}$ is a constant depending on d, t_0, T, p, a, b and β.

We come back to (2.3.41). Combining relations (3.2.39) and (3.2.40), it follows that relation (3.2.2) holds.

∎

3.3 Continuity in law of the solution with respect to the noise parameter α

In this section, we consider equation (3.0.1) driven by the noise W^α introduced in Section 2.1, which is specified by a general temporal covariance function γ_0 and a spatial covariance function γ given by the Riesz kernel of index α. We prove that the solution of this equation is continuous in law in the space of continuous functions $C([t_0,T]\times\mathbb{R}^d)$, with respect to the noise parameter α.

As in the proof of Theorem 2.0.1, we apply Lemma 2.4.3 to prove the convergence of the finite dimensional distribution of u^{α_n} to u^{α^*}, when $n\to\infty$. For this, we need the whole family of processes $\{W^\alpha;\alpha\in(0,d)\}$ to be defined in a single probability space. We use the same construction of this family of noise processes, as in Section 2.4. We will use relations (2.4.3), (2.4.4), (2.4.7) and Proposition 2.4.2.

Recall that the solution has the Wiener chaos expansion (3.1.3). By definition, $u^\alpha(t,x)$ is the $L^2(\Omega)$-limit of the sequence $\{u_m^\alpha(t,x)\}_{m\geq 1}$ defined by

$$u_m^\alpha(t,x)=\sum_{n=0}^{m} I_n^\alpha(f_n(\cdot,t,x)). \tag{3.3.1}$$

This means that

$$\mathbb{E}\Big|u_m^\alpha(t,x)-u^\alpha(t,x)\Big|^2\to 0,\quad \text{as } m\to\infty \tag{3.3.2}$$

for any $\alpha\in(\max\{0,d-2\},d)$ fixed.

The proof of Theorem 3.0.1 is based on the following result, where proof is similar to the proof of Lemma 2.4.5.

Lemma 3.3.1. *Under the hypothesis of Theorem 3.0.1, for all $k\geq 1$ fixed, we have*

$$\mathbb{E}\Big|I_k^{\alpha_n}(f_k(\cdot,t,x))-I_k^{\alpha^*}(f_k(\cdot,t,x))\Big|^2\to 0,\ \textit{as } n\to\infty. \tag{3.3.3}$$

Proof: Recall that by relation (2.4.13), we have

$$\begin{aligned} Q_n &:= \mathbb{E}\Big|I_k^{\alpha_n}(f_k(\cdot,t,x))-I_k^{\alpha^*}(f_k(\cdot,t,x))\Big|^2 \\ &\leq k!\ \Gamma_{0,t}^k\int_{T_k(t)}\int_{(\mathbb{R}^d)^k}\Big|\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\Big|^2 \\ &\qquad\qquad \Big|\prod_{j=1}^{k}|\xi_j|^{-\alpha_n/2}-\prod_{j=1}^{k}|\xi_j|^{-\alpha^*/2}\Big|^2 \mathrm{d}\xi_1\cdots \mathrm{d}\xi_k \mathrm{d}\mathbf{t}. \end{aligned} \tag{3.3.4}$$

The difference is that here we use a different expression for $\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)$ than in Chapter 2 (see Lemma 3.1.5).

As in the proof of Lemma 2.4.5, we show that the integral in (3.3.4) converges to 0 when $n\to\infty$, by applying the Dominated Convergence Theorem. We need to show that the functions:

$$g_n(\mathbf{t},\boldsymbol{\xi}):=\Big|\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\Big|^2\prod_{j=1}^{k}|\xi_j|^{-\alpha_n}$$

can be bounded by a function $g(\mathbf{t},\boldsymbol{\xi})$ which is integrable over $T_k(t)\times(\mathbb{R}^d)^k$

Recall that $\max\{0,d-2\}<\alpha^*<d$. Fix numbers a and b such that $\max\{0,d-2\}<a<\alpha^*<b<d$. Since $\alpha_n\to\alpha^*$, there exists $N\in\mathbb{N}$ such that

$$a\leq\alpha_n\leq b,\ \text{for all } n\geq N.$$

For any $\alpha_n \in [a,b] \subset \big(\max\{0, d-2\}, d\big)$, we know that

$$|\xi_j|^{-\alpha_n} \le |\xi_j|^{-b} 1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}. \tag{3.3.5}$$

Hence a natural candidate for g is

$$\begin{aligned} g(\mathbf{t}, \boldsymbol{\xi}) &= \Big|\mathcal{F}_x f_k(t_1, \cdot, \dots, t_k, \cdot, t, x)(\xi_1, \dots, \xi_k)\Big|^2 \prod_{j=1}^k \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \\ &\le w^2(t,x) \prod_{j=1}^k \exp\Big\{ -\frac{t_{j+1}-t_j}{t_j t_{j+1}} \Big|\sum_{i=1}^j t_i \xi_i\Big|^2\Big\} \\ &\qquad \prod_{j=1}^k \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big), \end{aligned} \tag{3.3.6}$$

where for the inequality we used Lemma 3.1.5. So we need to check that $g(\mathbf{t}, \boldsymbol{\xi})$ is integrable, i.e. $\int_{T_k(t)} \int_{(\mathbb{R}^d)^k} g(\mathbf{t}, \boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} \mathrm{d}\mathbf{t} < \infty$. By (3.3.6), it is enough to prove that

$$\begin{aligned} I := w^2(t,x) \int_{T_k(t)} \prod_{j=1}^k \Bigg(\int_{\mathbb{R}^d} \exp\Big\{ -\frac{t_{j+1}-t_j}{t_j t_{j+1}} \Big|\sum_{i=1}^j t_i \xi_i\Big|^2\Big\} \\ \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \mathrm{d}\xi_j \Bigg) \mathrm{d}\mathbf{t} < \infty. \end{aligned} \tag{3.3.7}$$

We prove (3.3.7). To simplify writing, we let $m_j = \frac{t_{j+1}-t_j}{t_j t_{j+1}}$ and $p_j = \frac{(t_{j+1}-t_j)t_j}{t_{j+1}}$, for all $j = 1, \dots, k$. Notice that by Lemma 3.1.7, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{-m_j \left|\sum_{i=1}^j t_i \xi_i\right|^2} \Big(|\xi_j|^{-b} 1_{\{|\xi_j|\le 1\}} + |\xi_j|^{-a} 1_{\{|\xi_j|>1\}}\Big) \mathrm{d}\xi_j \\ &\quad = \int_{|\xi_j|\le 1} e^{-m_j \left|\sum_{i=1}^j t_i \xi_i\right|^2} |\xi_j|^{-b} \mathrm{d}\xi_j + \int_{|\xi_j|>1} e^{-m_j \left|\sum_{i=1}^j t_i \xi_i\right|^2} |\xi_j|^{-a} \mathrm{d}\xi_j \\ &\quad \le \int_{|\xi_j|\le 1} |\xi_j|^{-b} \mathrm{d}\xi_j + \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-m_j \left|t_j \xi_j + \eta\right|^2} |\xi_j|^{-a} \mathrm{d}\xi_j \\ &\quad = \frac{c_d}{d-b} + \int_{\mathbb{R}^d} e^{-m_j |t_j \xi_j|^2} |\xi_j|^{-a} \mathrm{d}\xi_j = \frac{c_d}{d-b} + \int_{\mathbb{R}^d} e^{-p_j \xi_j^2} |\xi_j|^{-a} \mathrm{d}\xi_j, \end{aligned} \tag{3.3.8}$$

where c_d is the area of the unit sphere $S_1(0) = \{z \in \mathbb{R}^d; |z| = 1\}$ in $\mathbb{R}^d$. We use the change of variables $\xi_j' = \sqrt{p_j} \xi_j$, then $\mathrm{d}\xi_j = p_j^{-d/2} \mathrm{d}\xi_j'$, for all $j = 1, \dots, k$. It follows that

$$\int_{\mathbb{R}^d} e^{-p_j \xi_j^2} |\xi_j|^{-a} \mathrm{d}\xi_j = \int_{\mathbb{R}^d} e^{-|\xi_j'|^2} \Big|\frac{\xi_j'}{\sqrt{p_j}}\Big|^{-a} p_j^{-d/2} \mathrm{d}\xi_j' = p_j^{-\frac{d-a}{2}} \int_{\mathbb{R}^d} e^{-|\xi_j|^2} |\xi_j|^{-a} \mathrm{d}\xi_j$$

$$\leq p_j^{-\frac{d-a}{2}} K_{d,a}, \tag{3.3.9}$$

where $K_{d,a} = \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} |\xi|^{-a} \mathrm{d}\xi < \infty$ for any $a \in (d-2, d)$.

We return to relation (3.3.7). Let $C_{a,b,d} = \frac{c_d}{d-b} + K_{d,a}$ be a constant depending on a, b and d. Using (3.3.8) and (3.3.9), we obtain:

$$\begin{aligned} I &\leq w^2(t,x) \int_{T_k(t)} \prod_{j=1}^{k} \Big(\frac{c_d}{d-b} + p_j^{-\frac{d-a}{2}} K_{d,a} \Big) \mathrm{d}\mathbf{t} \\ &\leq w^2(t,x) C_{a,b,d}^k \int_{T_k(t)} \prod_{j=1}^{k} \Big(1 + \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \Big) \mathrm{d}\mathbf{t}. \end{aligned} \tag{3.3.10}$$

To evaluate the integral, note that for any $0 < t_1 < \ldots < t_k < t_{k+1} = t$, we have

$$\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \leq t_{j+1} - t_j \leq t.$$

Therefore

$$\Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \geq t^{-\frac{d-a}{2}},$$

and multiplying by $t^{\frac{d-a}{2}}$ on both side, we obtain:

$$\Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} t^{\frac{d-a}{2}} \geq 1$$

Then, replacing 1 in the integrand in (3.3.10), we get:

$$1 + \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \leq \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \Big(1 + t^{\frac{d-a}{2}} \Big).$$

Hence, there exist a constant $C_T > 0$ such that

$$\begin{aligned} &\prod_{j=1}^{k} \Big(1 + \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \Big) \\ &\qquad \leq \Big(1 + t^{\frac{d-a}{2}} \Big)^k \prod_{j=1}^{k} \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \leq C_T^k \prod_{j=1}^{k} \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}}. \end{aligned}$$

Returning to (3.3.10), it follows that

$$I \leq w^2(t,x) C_{a,b,d,T}^k \int_{T_k(t)} \prod_{j=1}^{k} \Big(\frac{(t_{j+1}-t_j)t_j}{t_{j+1}} \Big)^{-\frac{d-a}{2}} \mathrm{d}\mathbf{t}$$

$$= w^2(t,x) C^k_{a,b,d,T} \frac{\Gamma\left(\frac{2-(d-a)}{2}\right)^{n+1}}{\Gamma\left((n+1)\left(\frac{2-(d-a)}{2}\right)\right)} t^{n\left(\frac{2-(d-a)}{2}\right)-\frac{d-a}{2}} < \infty.$$

where we used Lemma 3.1.9 in the equality and this finishes the justification of the application of the Dominated Convergence Theorem. ∎

Proof of Theorem 3.0.1: From Theorem 1.3 of [3], we know that the process u^α has a continuous modification. We work with this modification, which we denote also by u^α. We need to prove the finite dimensional distribution convergence and the fact that the sequence of probability measures induced by $(u^{\alpha_n})_{n\geq 1}$ is tight in the space of $C([t_0,T]\times\mathbb{R}^d)$. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: Finite dimensional distribution convergence} \\ \text{Step 2: Tightness} \end{cases}$$

Step 1: Finite dimensional distribution convergence

In this step, we have to prove that for any $k \geq 1$ and $(t_1,x_1),\dots,(t_k,x_k) \in [t_0,T]\times\mathbb{R}^d$,

$$\left(u^{\alpha_n}(t_1,x_1),\dots,u^{\alpha_n}(t_k,x_k)\right) \xrightarrow{\text{d}} \left(u^{\alpha^*}(t_1,x_1),\dots,u^{\alpha^*}(t_k,x_k)\right), \text{ as } n\to\infty.$$

For this, by Lemma 2.4.3, it will be enough to prove that for all $(t,x)\in[t_0,T]\times\mathbb{R}^d$

$$\mathbb{E}\left|u^{\alpha_n}(t,x) - u^{\alpha^*}(t,x)\right|^2 \to 0, \text{ as } n\to\infty. \tag{3.3.11}$$

To prove relation (3.3.11), we use the same argument as in the proof of Theorem 2.0.1. For part (c), using Lemma 3.3.1, it remains to show that for all compact set $[a,b]\subset(\max\{0,d-2\},d)$,

$$\sup_{\alpha\in[a,b]} \mathbb{E}\left|u^\alpha_m(t,x) - u^\alpha(t,x)\right|^2 \to 0, \text{ as } m\to\infty. \tag{3.3.12}$$

Recall that $u^\alpha_m(t,x) = \sum_{k=0}^m I^\alpha_k(f_k(\cdot,t,x))$ and $u^\alpha(t,x) = \sum_{k\geq 0} I^\alpha_k(f_k(\cdot,t,x))$. Hence

$$u^\alpha(t,x) - u^\alpha_m(t,x) = \sum_{k\geq m+1} I^\alpha_k(f_k(\cdot,t,x)).$$

By the orthogonality of the Wiener chaos space and relation (3.1.23), we have

$$\mathbb{E}\left|u^\alpha(t,x) - u^\alpha_m(t,x)\right|^2 = \sum_{k\geq m+1} \mathbb{E}\left|I^\alpha_k(f_k(\cdot,t,x))\right|^2$$

$$\leq \sum_{k\geq m+1} C_{t_0,d}\Gamma_{0,t}^k \big(C_{d,\alpha}^{(2)}\big)^k \frac{\Gamma\big(1-\frac{d-\alpha}{2}\big)^{k+1}}{\Gamma\big((k+1)(1-\frac{d-\alpha}{2})\big)} t^{k(1-\frac{d-\alpha}{2})}, \tag{3.3.13}$$

where $C_{d,\alpha}^{(2)} = C_{d,\alpha}^{(1)} 2^{-(d-\alpha)/2}$ and $C_{d,\alpha}^{(1)} := \int_{\mathbb{R}^d} e^{-|\xi|^2/2}|\xi|^{-\alpha}\mathrm{d}\xi$. We will bound $C_{d,\alpha}^{(1)}$ using (3.2.17).

We return to relation (3.3.12). Using the same arguments as for (2.4.25), (2.4.26) and (2.4.27), we see that

$$\begin{aligned}
&\sup_{\alpha\in[a,b]} \sum_{k\geq m+1} \mathbb{E}\big|I_k^{\alpha}(f_k(\cdot,t,x))\big|^2 \\
&\quad \leq \sum_{k\geq m+1} C_{t_0,d}\Gamma_{0,T}^k C_{d,a,b}^k \frac{\big(\Gamma(1-\frac{d-a}{2})\big)^{k+1}}{\Gamma\Big((k+1)\big(1-\frac{d-a}{2}\big)\Big)} (t\vee 1)^{k(1-\frac{d-b}{2})} \to 0
\end{aligned}$$

as $m\to\infty$ since

$$\sum_{k\geq 1} C_{t_0,d}\Gamma_{0,t}^k C_{d,a,b}^k \frac{\big(\Gamma(1-\frac{d-a}{2})\big)^{k+1}}{\Gamma\Big((k+1)\big(1-\frac{d-a}{2}\big)\Big)} (t\vee 1)^{k(1-\frac{d-b}{2})} < \infty,$$

due to the fact that $1-\frac{d-a}{2} > 0$.

Step 2: Tightness

This follows exactly as in the proof of Theorem 2.0.1, using the uniform bounds for the moments of the increments of solution given by Theorem 3.2.1.

∎

Chapter 4

PAM/HAM with rough noise and constant initial condition

In this chapter, we consider the following Parabolic Anderson Model (PAM):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}^H(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x)=1, \quad x\in\mathbb{R} \end{cases} \tag{4.0.1}$$

and the Hyperbolic Anderson Model (HAM):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}^H(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x)=1, \quad x\in\mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x)=0, \quad x\in\mathbb{R}, \end{cases} \tag{4.0.2}$$

with *rough* noise W^H. More precisely, W^H is a Gaussian noise which is fractional in time with index $H_0\in(1/2,1)$ and fractional in space with index $H\in(0,1/2)$. We first study the existence and the moment estimates of solution to (4.0.1), respectively (4.0.2). The existence of solution is obtained under the condition $H_0+H>3/4$ for equation (4.0.1), respectively $H>1/4$ for equation (4.0.2). The objective of this chapter is to show the continuity in the law of the solution with respect to the noise parameter H, in the space of continuous functions on $C([0,T]\times\mathbb{R})$, where the temporal parameter H_0 is fixed. The results presented in this chapter are contained in Section 3 of [7].

Theorem 4.0.1 below is the main result for this chapter. In this theorem, we use subindices h and w to indicate the corresponding solutions for the heat equation, respectively the wave equation. We drop these indices in the proofs to simplify the writing, whenever the argument is valid for both equations.

Theorem 4.0.1. *Let $H_0 \in (1/2, 1)$ be fixed. For any $H \in (0, 1/2)$, let u^H be the solutions of equations* (4.0.1) *or* (4.0.2)*, according to Definition 4.1.1. Let $(H_n)_{n\geq 1}$ be an arbitrary sequence in $(0, 1/2)$ such that*

$$\lim_{n\to\infty} H_n \to H^*.$$

(a) In the case of the heat equation (4.0.1)*, if*

$$H_0 + H^* > 3/4,$$

then

$$u_h^{H_n} \xrightarrow{d} u_h^{H^*} \text{ in } C([0,T]\times\mathbb{R}).$$

(b) In the case of the wave equation (4.0.2)*, if*

$$H^* > 1/4,$$

then

$$u_w^{H_n} \xrightarrow{d} u_w^{H^*} \text{ in } C([0,T]\times\mathbb{R}).$$

4.1 The noise and the existence of solution

In this section, we introduce the noise and prove the existence of solution to equation. The random term W^H is a Gaussian noise which behaves in time like fBm with index $H_0 > 1/2$ and in space like the fBm with index $H < 1/2$. More precisely, it is given by a family of centered Gaussian random variables $W^H = \{W^H(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following covariance structure:

$$\begin{aligned} E[W^H(\varphi)W^H(\psi)] &= \alpha_{H_0} c_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|t-s|^{2H_0-2}|\xi|^{1-2H}\mathrm{d}\xi\mathrm{d}t\mathrm{d}s \\ &=: \langle \varphi, \psi\rangle_{\mathcal{H}_H}, \end{aligned} \tag{4.1.1}$$

for any $\varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$, where $\mathcal{H}_H$ is the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_H}$. Here $\alpha_{H_0} = H_0(2H_0 - 1)$ and c_H is given in relation (1.0.8). Throughout this chapter, H_0 is fixed and we are interested in the convergence with respect to H. For this reason, we emphasize the dependence on H using the notation $\mathcal{H}_H$.

Definition 4.1.1. *We say that a process $u^H = \{u^H(t,x); t \geq 0, x \in \mathbb{R}\}$ is a solution (in Skorohod sense) to equation* (4.0.1)*, respectively equation* (4.0.2)*, if for any $t > 0$, $x \in \mathbb{R}$ with probability 1,*

$$u^H(t,x) = 1 + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u(s,y)W^H(\delta s, \delta y), \tag{4.1.2}$$

where G is the fundamental solution of the heat operator, respectively wave operator.

Note that as in Chapter 2 and 3, the stochastic integral in (4.1.2) is interpreted in the Skorohod sense. i.e. it is given by the divergence operator from Malliavin calculus.

The solution has the series expansion:

$$\begin{aligned} u^H(t,x) &= 1 + \sum_{n\geq 1} \int_0^t \int_{\mathbb{R}} \int_0^{t_n} \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} G_{t-t_n}(x-x_n) \cdots G_{t_2-t_1}(x_2-x_1) \\ &\qquad W^H(\mathrm{d}t_1, \mathrm{d}x_1) \cdots W^H(\mathrm{d}t_n, \mathrm{d}x_n) \\ &= 1 + \sum_{n\geq 1} I_n^H(f_n(\cdot,t,x)) = \sum_{n\geq 0} I_n^H(f_n(\cdot,t,x)), \end{aligned} \tag{4.1.3}$$

where the kernel $f_n(\cdot,t,x) \in \mathcal{H}_H^{\otimes n}$ is given by

$$f_n(t_1,x_1,\cdots,t_n,x_n,t,x) = G_{t-t_n}(x-x_n)\cdots G_{t_2-t_1}(x_2-x_1)1_{\{0<t_1<\cdots<t_n<t\}}, \tag{4.1.4}$$

and $I^H : \mathcal{H}_H^{\otimes n} \to \mathcal{H}_H$ is the multiple Wiener integral of order n with respect to W^H. We let $f_0(\cdot,t,x) = 1$. Observe that the second term of this series is

$$I_1^H(f_1(\cdot,t,x)) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(\mathrm{d}s, \mathrm{d}y),$$

which is the solution of the linear equation $\mathcal{L}u^H = \dot{W}^H$, where $\mathcal{L}$ is the heat or wave operator.

The solution exists if and only if the series $\sum_{n\geq 0} I_n^H(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$, i.e.

$$\sum_{n\geq 0} \mathbb{E}\big|I_n^H(f_n(\cdot,t,x))\big|^2 < \infty$$

and we have

$$\mathbb{E}|u^H(t,x)|^2 = 1 + \sum_{n\geq 1} n! \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}} = \sum_{n\geq 0} n! \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}}.$$

For the proof of existence of the solution, we need Lemma 4.1.2 below, which can be derived from Lemma 3.1 of [5].

Lemma 4.1.2. *In the case of the heat equation, the integral* $\int_{\mathbb{R}} |\mathcal{F}G_t^h(\xi)|^2 |\xi|^\alpha \mathrm{d}\xi$ *is finite if and only if* $\alpha > -1$ *and in this case,*

$$\int_{\mathbb{R}} |\mathcal{F}G_t^h(\xi)|^2 |\xi|^\alpha \mathrm{d}\xi = \Gamma\Big(\frac{1+\alpha}{2}\Big)\ t^{-(1+\alpha)/2}.$$

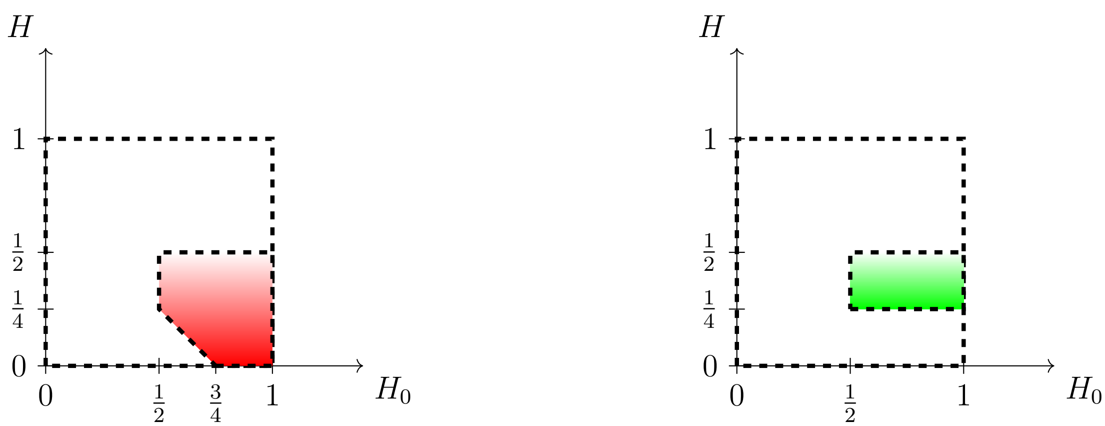
In the case of the wave equation, the integral $\int_{\mathbb{R}} |\mathcal{F}G_t^w(\xi)|^2|\xi|^\alpha \mathrm{d}\xi$ *is finite if and only if* $\alpha \in (-1,1)$*, and in this case, we have*

$$\int_{\mathbb{R}} |\mathcal{F}G_t^w(\xi)|^2|\xi|^\alpha \mathrm{d}\xi = 2^{1-\alpha}\widetilde{C}_\alpha \, t^{1-\alpha},$$

where $\widetilde{C}_\alpha$ *is given by*

$$\widetilde{C}_\alpha = \begin{cases} (1-\alpha)^{-1}\Gamma(\alpha)\sin(\pi\alpha/2) & \text{if } \alpha \in (0,1), \\ \alpha^{-1}(1-\alpha)^{-1}\Gamma(1+\alpha)\sin(\pi\alpha/2) & \text{if } \alpha \in (-1,0), \\ \pi/2 & \text{if } \alpha = 0. \end{cases}$$

We now give the main results of this section, regarding the existence of the unique solution. Below we include the picture showing the range of the parameters for equation 4.0.1 (PAM) and equation 4.0.2 (HAM).



(a) PAM (b) HAM

Figure 4.1: The shaded area on the left represents the region $\{H_0 \in (1/2,1), H \in (0,1/2)$ and $H + H_0 > 3/4\}$ for PAM; The shaded area on the right represents the region $\{H_0 \in (1/2,1), H \in (0,1/2)$ and $H > 1/4\}$ for HAM.

We use the following identity:

Lemma 4.1.3. *For any* $n \geq 2$ *and* $x_1, \dots, x_n \in \mathbb{R}_+$*, it holds that*

$$x_1 \prod_{k=2}^{n} (x_k + x_{k-1}) = \sum_{\mathbf{a} \in A_n} \prod_{j=1}^{n} x_j^{a_j},$$

where A_n *is a set of indices* $\mathbf{a} = (a_1, \dots, a_n)$ *such that card*$(A_n) = 2^{n-1}$ *and*

$$a_1 \in \{1,2\}, a_n \in \{0,1\}, a_2, \dots, a_{n-1} \in \big\{0,1,2\big\},$$

$$\sum_{j=1}^{i} a_j \in \{i, i+1\} \text{ for } i = 1, \dots, n-1, |\mathbf{a}| = \sum_{j=1}^{n} a_j = n,$$
$$a_i + a_{i+1} \in \{1, 2, 3\} \text{ for } i = 2, \dots, n-2,$$
$$a_1 + a_2 \in \{2, 3\} \text{ and } a_{n-1} + a_n \in \{1, 2\}.$$

Theorem 4.1.4. *(a) For any $H_0 \in (1/2, 1)$ and $H \in (0, 1/2)$ such that*

$$H_0 + H > 3/4,$$

equation (4.0.1) has a unique solution u^H. Moreover, for any $p \geq 2$,

$$\mathbb{E}|u^H(t,x)|^p \leq (C_1^h)^p \exp\left(C_2^h \; p^{k^h} t^{\rho^h}\right),$$

where

$$k^h = \frac{H+1}{H}, \; \rho^h = \frac{2H_0 + H - 1}{H}$$

and C_1^h, C_2^h are some positive constants depending on H_0 and H.

(b) For any $H_0 \in (1/2, 1)$ and $H \in (1/4, 1/2)$, equation (4.0.2) has a unique solution u^H. Moreover, for any $p \geq 2$,

$$\mathbb{E}|u^H(t,x)|^p \leq (C_1^w)^p \exp\left(C_2^w \; p^{k^w} t^{\rho^w}\right),$$

where

$$k^w = \frac{2H+2}{2H+1}, \; \rho^w = \frac{2H_0 + 2H}{2H+1}$$

and C_1^w, C_2^w are some positive constants depending on H_0 and H.

Proof: **Step 1**: We start with a general argument which is valid for both heat and wave equations. We proceed similarly to the proof of Theorem 2.2.6 (or Theorem 2.2.4). In this case,

$$\|\tilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}_H^{\otimes n}} = \int_{[0,t]^n} \int_{[0,t]^n} \alpha_{H_0}^n \prod_{j=1}^{n} |t_j - s_j|^{2H_0 - 2} A_n(\mathbf{t}, \mathbf{s}) \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s},$$

where

$$A_n(\mathbf{t}, \mathbf{s}) = c_H^n \int_{\mathbb{R}^n} \mathcal{F}\tilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) \overline{\mathcal{F}\tilde{f}_n(s_1, \cdot, \dots, s_n, \cdot, t, x)(\xi_1, \cdots, \xi_n)} \prod_{j=1}^{n} |\xi_j|^{1-2H} \; \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.$$

Using Lemma 2.2.5 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}} &\le \int_{[0,t]^n}\int_{[0,t]^n}\Big(\alpha_{H_0}^n\prod_{j=1}^n|t_j-s_j|^{2H_0-2}\Big)A_n^{1/2}(\mathbf{t},\mathbf{t})A_n^{1/2}(\mathbf{s},\mathbf{s})\,\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}\\ &\le b_{H_0}^n\Big(\int_{[0,t]^n}\Big|\big(A_n(\mathbf{t},\mathbf{t})\big)^{1/2}\Big|^{\frac{1}{H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} = b_{H_0}^n\Big(\int_{[0,t]^n}\big|A_n(\mathbf{t},\mathbf{t})\big|^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0}\\ &= b_{H_0}^n\Big(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\big|A_n(\mathbf{t},\mathbf{t})\big|^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\Big)^{2H_0} \qquad (4.1.5)\end{aligned}$$

where S_n is the set of all permutations of $\{1,\cdots,n\}$. We study $A_n(\mathbf{t},\mathbf{t})$ first. Note that

$$A_n(\mathbf{t},\mathbf{t}) = c_H^n\int_{\mathbb{R}^n}\big|\mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)\big|^2\prod_{j=1}^n|\xi_j|^{1-2H}\ \mathrm{d}\xi_1\cdots\mathrm{d}\xi_n. \qquad (4.1.6)$$

Fix $(t_1,\ldots,t_n)\in[0,t]^n$. Let $\rho\in S_n$ be such that $0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t$. By (2.2.12) and the change of variables $\xi'_j=\xi_{\rho(j)}$ for all $j=1,\ldots,n$, we have

$$\begin{aligned}A_n(\mathbf{t},\mathbf{t}) &= \frac{1}{(n!)^2}c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^n\Big|\mathcal{F}G_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\cdots+\xi_{\rho(j)})\Big|^2\prod_{j=1}^n|\xi_{\rho(j)}|^{1-2H}\ \mathrm{d}\xi_{\rho(1)}\cdots\mathrm{d}\xi_{\rho(n)}\\ &= \frac{1}{(n!)^2}c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^n\Big|\mathcal{F}G_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi'_1+\cdots+\xi'_j)\Big|^2\prod_{j=1}^n|\xi'_j|^{1-2H}\ \mathrm{d}\xi'_1\cdots\mathrm{d}\xi'_n.\end{aligned}$$

Unlike the proof of Theorem 2.2.4, we can not apply Lemma A.2.1 here to estimate $A_n(\mathbf{t},\mathbf{t})$ since $H<1/2$. Instead of this, we use the change of variables $\eta_j=\xi'_1+\cdots+\xi'_j$ for all $j=1,\ldots,n$, we obtain

$$A_n(\mathbf{t},\mathbf{t}) = \frac{1}{(n!)^2}c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^n\Big|\mathcal{F}G_{t_{\rho(j+1)}-t_{\rho(j)}}(\eta_j)\Big|^2|\eta_1|^{1-2H}\prod_{j=2}^n|\eta_j-\eta_{j-1}|^{1-2H}\mathrm{d}\eta_1\cdots\mathrm{d}\eta_n. \qquad (4.1.7)$$

Using the fact that $(a+b)^p\le a^p+b^p$, for any $p\in(0,1)$ and $a,b>0$, we have

$$|\eta_j-\eta_{j-1}|^{1-2H}\le(|\eta_j|+|\eta_{j-1}|)^{1-2H}\le|\eta_j|^{1-2H}+|\eta_{j-1}|^{1-2H}. \qquad (4.1.8)$$

Taking the product for $j=2,\ldots,n$, we get

$$\prod_{j=2}^n|\eta_j-\eta_{j-1}|^{1-2H}\le\prod_{j=2}^n\big(|\eta_j|^{1-2H}+|\eta_{j-1}|^{1-2H}\big). \qquad (4.1.9)$$

By Lemma 4.1.3,

$$|\eta_1|^{1-2H}\prod_{j=2}^{n}\Big(|\eta_j|^{1-2H}+|\eta_{j-1}|^{1-2H}\Big)=\sum_{\alpha\in D_n}|\eta_1|^{\alpha_1}|\eta_2|^{\alpha_2}\cdots|\eta_n|^{\alpha_n}=\sum_{\alpha\in D_n}\prod_{j=1}^{n}|\eta_j|^{\alpha_j}, \tag{4.1.10}$$

where D_n is the set of multi-indices $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)$ with $\alpha_j=(1-2H)a_j$ for $j=1,\dots,n$ and $\mathbf{a}=(a_1,\dots,a_n)\in A_n$. Notice that we have the following properties:

$$\alpha_1\in\big\{1-2H,2(1-2H)\big\},\alpha_n\in\big\{0,1-2H\big\},\alpha_2,\dots,\alpha_{n-1}\in\big\{0,1-2H,2(1-2H)\big\}$$

and

$$|\boldsymbol{\alpha}|=\sum_{j=1}^{n}\alpha_j=n(1-2H).$$

Hence,

$$|\eta_1|^{1-2H}\prod_{j=2}^{n}|\eta_j-\eta_{j-1}|^{1-2H}\leq\sum_{\alpha\in D_n}|\eta_1|^{\alpha_1}|\eta_2|^{\alpha_2}\cdots|\eta_n|^{\alpha_n}. \tag{4.1.11}$$

The complete description of the set D_n is omitted here. Coming back to relation (4.1.7), by inequality (4.1.11), we have

$$A_n(\mathbf{t},\mathbf{t})\leq\frac{1}{(n!)^2}c_H^n\sum_{\alpha\in D_n}\prod_{j=1}^{n}\Big(\int_{\mathbb{R}}\big|\mathcal{F}G_{t_{\rho(j+1)}-t_{\rho(j)}}(\eta_j)\big|^2|\eta_j|^{\alpha_j}\mathrm{d}\eta_j\Big). \tag{4.1.12}$$

At this point, we need to consider separately the heat and wave equations.

Step 2: In this case, we consider the heat equation. Using relation (4.1.12) and Lemma 4.1.2, we get:

$$\begin{aligned}A_n(\mathbf{t},\mathbf{t})&\leq\frac{1}{(n!)^2}c_H^n\sum_{\alpha\in D_n}\prod_{j=1}^{n}\Gamma\Big(\frac{1+\alpha_j}{2}\Big)\big(t_{\rho(j+1)}-t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{2}}\\&\leq\frac{1}{(n!)^2}C_{H,1}^n\sum_{\alpha\in D_n}\prod_{j=1}^{n}\big(t_{\rho(j+1)}-t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{2}},\end{aligned} \tag{4.1.13}$$

where

$$C_{H,1}=c_H\max\Big\{\Gamma\Big(\frac{1}{2}\Big),\Gamma\Big(1-H\Big),\Gamma\Big(\frac{3-4H}{2}\Big)\Big\}. \tag{4.1.14}$$

In order to apply Lemma 4.1.2, we need $\alpha_j>-1$ for $j=1,\dots,n$. This is clearly true, in fact $\alpha_j\geq 0$ for all $j=1,\dots,n$. Coming back to relation (4.1.5), using the change of variables $t_{\rho(j)} = t'_j$ for $i = j, \dots, n$ and the fact that $\Big(\sum_{i=1}^N a_i\Big)^b \leq \sum_{i=1}^N a_i^b$ for any $0 < b < 1$, we obtain

$$\begin{aligned}
&\mathbb{E}\big|I_n^H(f_n(\cdot,t,x))\big|^2 = n!\big\|\widetilde{f}_n(\cdot,t,x)\big\|^2_{\mathcal{H}_H^{\otimes n}} \\
&\leq n!\frac{1}{(n!)^2} b_{H_0}^n C_{H,1}^n \Bigg[n! \int_{T_n(t)} \Big(\sum_{\alpha \in D_n} \prod_{j=1}^n (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{2}} \Big)^{1/2H_0} \mathrm{d}\mathbf{t} \Bigg]^{2H_0} \\
&= (n!)^{2H_0-1} C_{H,1}^n b_{H_0}^n \Bigg[\int_{T_n(t)} \Big(\sum_{\alpha \in D_n} \prod_{j=1}^n (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{2}} \Big)^{1/2H_0} \mathrm{d}\mathbf{t} \Bigg]^{2H_0} \\
&\leq (n!)^{2H_0-1} C_{H,1}^n b_{H_0}^n \Bigg[\sum_{\alpha \in D_n} \int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}} \mathrm{d}\mathbf{t} \Bigg]^{2H_0}.
\end{aligned}$$

For each fixed $\boldsymbol{\alpha} \in D_n$, we apply Lemma A.3.1 with $\beta_j = -\frac{1+\alpha_j}{4H_0}$, for all $j = 1, \dots, n$. Hence

$$|\beta| = \sum_{j=1}^n \beta_j = -\frac{n + n(1-2H)}{4H_0} = \frac{Hn - n}{2H_0}.$$

To apply Lemma A.3.1, we need $\beta_j > -1$ for all $j = 1, \dots, n$. When $\alpha_j = 2(1-2H)$, this leads to the restriction $H_0 + H > 3/4$. We obtain:

$$\begin{aligned}
\int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}} \mathrm{d}\mathbf{t} &= \prod_{j=1}^n \Gamma\Big(-\frac{1+\alpha_j}{4H_0} + 1 \Big) \frac{t^{\frac{Hn-n}{2H_0}+n}}{\Gamma\big(\frac{Hn-n}{2H_0} + n + 1\big)} \\
&\leq C_{H,2}^n \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{\Gamma\big(n\frac{2H_0+H-1}{2H_0} + 1\big)}, \qquad (4.1.15)
\end{aligned}$$

where

$$C_{H,2} = \max\Bigg\{ \Gamma\Big(-\frac{1}{4H_0} + 1 \Big), \Gamma\Big(-\frac{1-H}{2H_0} + 1 \Big), \Gamma\Big(-\frac{3-4H}{4H_0} + 1 \Big) \Bigg\}. \qquad (4.1.16)$$

Since the estimate in relation (4.1.15) does not depend on $\alpha \in D_n$ and $\mathrm{card}(D_n) = 2^{n-1}$, we have

$$\begin{aligned}
\mathbb{E}\big|I_n^H(f_n(\cdot,t,x))\big|^2 &\leq (n!)^{2H_0-1} C_{H,1}^n b_{H_0}^n \Bigg[2^{n-1} C_{H,2}^n \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{\Gamma(n\frac{2H_0+H-1}{2H_0} + 1)} \Bigg]^{2H_0} \\
&\leq C_{H_0,H,1}^n (n!)^{2H_0-1} \frac{t^{n(2H_0+H-1)}}{\big(\Gamma(n\frac{2H_0+H-1}{2H_0} + 1)\big)^{2H_0}}, \qquad (4.1.17)
\end{aligned}$$

where $C_{H_0,H,1} = C_{H,1} b_{H_0} (2C_{H,2})^{2H_0}$. By Lemma A.1.3, we have

$$\mathbb{E}\big|I_n^H(f_n(\cdot,t,x))\big|^2 \leq C_{H_0,H,2}^n (n!)^{2H_0-1} \frac{t^{n(2H_0+H-1)}}{(n!)^{\frac{2H_0+H-1}{2H_0}\cdot 2H_0}} = \frac{(C_{H_0,H,2} t^{2H_0+H-1})^n}{(n!)^H}, \tag{4.1.18}$$

where $C_{H_0,H,2}$ is a constant depending on H_0 and H.

For any $p \geq 2$, using relation (4.1.18), Minkowski inequality and Lemma A.5.3 (hypercontractivity), we obtain

$$\begin{aligned}
\|u^H(t,x)\|_p &= \Big\|\sum_{n\geq 0} I_n^H(f_n(\cdot,t,x))\Big\|_p \leq \sum_{n\geq 0}\big\|I_n^H(f_n(\cdot,t,x))\big\|_p \\
&\leq \sum_{n\geq 0}(p-1)^{n/2}\big\|I_n^H(f_n(\cdot,t,x))\big\|_2 \leq \sum_{n\geq 0}(p-1)^{n/2}\left(\frac{(C_{H_0,H,2}t^{2H_0+H-1})^n}{(n!)^H}\right)^{1/2} \\
&\leq \sum_{n\geq 0}\frac{\left(\sqrt{p\, C_{H_0,H,2}\ t^{2H_0+H-1}}\right)^n}{(n!)^{H/2}} \leq C_1^h \exp\left(C_2^h\ p^{\frac{1}{H}}\ t^{\frac{2H_0+H-1}{H}}\right),
\end{aligned}$$

where we used Lemma A.1.6 in the last inequality, and hence,

$$\mathbb{E}|u^H(t,x)|^p = \|u^H(t,x)\|_p^p \leq (C_1^h)^p \exp\left(C_2^h\ p^{\frac{H+1}{H}}\ t^{\frac{2H_0+H-1}{H}}\right),$$

where C_1^h and C_2^h are some constants depending on H_0 and H.

Step 3: In this case, we consider the wave equation. By relation (4.1.12) and Lemma 4.1.2, we get

$$\begin{aligned}
A_n(\mathbf{t},\mathbf{t}) &\leq \frac{1}{(n!)^2} c_H^n \sum_{\alpha\in D_n}\prod_{j=1}^n 2^{1-\alpha_j}\widetilde{C}_{\alpha_j}\ (t_{\rho(j+1)} - t_{\rho(j)})^{1-\alpha_j} \\
&\leq \frac{1}{(n!)^2} C_{H,3}^n \sum_{\alpha\in D_n}\prod_{j=1}^n \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{1-\alpha_j},
\end{aligned} \tag{4.1.19}$$

where

$$2^{1-\alpha_j}\widetilde{C}_{\alpha_j} = \begin{cases} \pi & \text{if } \alpha_j = 0, \\ 2^{2H}\frac{\Gamma(1-2H)\sin\left(\pi(1-2H)/2\right)}{2H} & \text{if } \alpha_j = 1-2H, \\ 2^{4H-1}\frac{\Gamma(2-4H)\sin\left(\pi(1-2H)\right)}{4H-1} & \text{if } \alpha_j = 2(1-2H) \end{cases}$$

and $C_{H,3}$ is given by

$$C_{H,3} = c_H \max\left\{\pi, \frac{\Gamma\big(1-2H\big)}{H}, \frac{2\Gamma\big(2-4H\big)}{4H-1}\right\}.$$

To apply Lemma 4.1.2, we need $\alpha_j \in (-1, 1)$ for $j = 1, \ldots, n$. When $\alpha_j = 2(1-2H)$, the condition $\alpha_j < 1$ imposes the restriction $H > 1/4$. Coming back to relation (4.1.5), using the change of variables $t_{\rho(j)} = t'_j$ for $j = 1, \ldots, n$ and the fact that $\left(\sum_{i=1}^N a_i\right)^b \leq \sum_{i=1}^N a_i^b$ for any $0 < b < 1$, we obtain

$$
\begin{aligned}
\mathbb{E}\left|I_n^H(f_n(\cdot,t,x))\right|^2 &= n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}} \\
&\leq n!\frac{1}{(n!)^2} b_{H_0}^n C_{H,3}^n \left[n! \int_{T_n(t)} \Big(\sum_{\alpha\in D_n}\prod_{j=1}^n (t_{j+1}-t_j)^{1-\alpha_j}\Big)^{1/2H_0} \mathrm{d}\mathbf{t}\right]^{2H_0} \\
&= (n!)^{2H_0-1} C_{H,3}^n b_{H_0}^n \left[\int_{T_n(t)} \Big(\sum_{\alpha\in D_n}\prod_{j=1}^n (t_{j+1}-t_j)^{1-\alpha_j}\Big)^{1/2H_0} \mathrm{d}\mathbf{t}\right]^{2H_0} \\
&\leq (n!)^{2H_0-1} C_{H,3}^n b_{H_0}^n \left[\sum_{\alpha\in D_n}\int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{\frac{1-\alpha_j}{2H_0}} \mathrm{d}\mathbf{t}\right]^{2H_0}.
\end{aligned}
$$

For each fixed $\boldsymbol{\alpha} \in D_n$, we apply Lemma A.3.1 with

$$
\beta_j = \frac{1-\alpha_j}{2H_0}, \text{ for all } j = 1, \ldots, n.
$$

Hence

$$
|\beta| = \sum_{j=1}^n \beta_j = \frac{n-|\alpha|}{2H_0} = \frac{Hn}{H_0}.
$$

To apply Lemma A.3.1, we need $\beta_j > -1$ for all $j = 1, \ldots, n$, which imposes no restriction on H. We obtain:

$$
\int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{\frac{1-\alpha_j}{2H_0}} \mathrm{d}\mathbf{t} = \frac{\prod_{j=1}^n \Gamma(\frac{1-\alpha_j}{2H_0}+1)}{\Gamma(\frac{Hn}{H_0}+n+1)} t^{\frac{Hn}{H_0}+n} \leq C_{H,4}^n \frac{t^{n\frac{H_0+H}{H_0}}}{\Gamma(n\frac{H_0+H}{H_0}+1)}, \quad (4.1.20)
$$

where

$$
C_{H,4} = \max\left\{\Gamma\Big(\frac{1}{2H_0}+1\Big), \Gamma\Big(\frac{H}{H_0}+1\Big), \Gamma\Big(\frac{4H-1}{2H_0}+1\Big)\right\}.
$$

Since estimate of relation (4.1.20) does not depend on $\boldsymbol{\alpha} \in D_n$ and $\mathrm{Card}(D_n) = 2^{n-1}$, we have

$$
\begin{aligned}
\mathbb{E}\left|I_n^H(f_n(\cdot,t,x))\right|^2 &\leq (n!)^{2H_0-1} C_{H,3}^n b_{H_0}^n \left[2^{n-1} C_{H,4}^n \frac{t^{n\frac{H_0+H}{H_0}}}{\Gamma(n\frac{H_0+H}{H_0}+1)}\right]^{2H_0} \\
&\leq C_{H_0,H,3}^n (n!)^{2H_0-1} \frac{t^{n(2H_0+2H)}}{\Big(\Gamma(n\frac{H_0+H}{H_0}+1)\Big)^{2H_0}}, \qquad (4.1.21)
\end{aligned}
$$

where $C_{H_0,H,3} = C_{H,3} b_{H_0} (2C_{H,4})^{2H_0}$. By Lemma A.1.3, we obtain:

$$\mathbb{E}\big|I_n^H(f_n(\cdot,t,x))\big|^2 \leq C_{H_0,H,4}^n (n!)^{2H_0-1} \frac{t^{n(2H_0+2H)}}{(n!)^{\frac{H_0+H}{H_0}\cdot 2H_0}} = \frac{\big(C_{H_0,H,4}\ t^{(2H_0+2H)}\big)^n}{(n!)^{2H+1}}, \quad (4.1.22)$$

where $C_{H_0,H,4}$ is a constant depending on H_0 and H.

For any $p \geq 2$, using relation (4.1.22), Minkowski inequality and Lemma A.5.3 (hypercontractivity), we obtain

$$\begin{aligned}\|u^H(t,x)\|_p &= \Big\|\sum_{n\geq 0} I_n^H(f_n(\cdot,t,x))\Big\|_p \leq \sum_{n\geq 0} \big\|I_n^H(f_n(\cdot,t,x))\big\|_p \\ &\leq \sum_{n\geq 0}(p-1)^{n/2}\big\|I_n^H(f_n(\cdot,t,x))\big\|_2 \leq \sum_{n\geq 0}(p-1)^{n/2}\left(\frac{\big(C_{H_0,H,4}\ t^{(2H_0+2H)}\big)^n}{(n!)^{2H+1}}\right)^{1/2} \\ &\leq \sum_{n\geq 0}\frac{\left(\sqrt{p\ C_{H_0,H,4}\ t^{2H_0+2H}}\right)^n}{(n!)^{(2H+1)/2}} \leq C_1^w \exp\left(C_2^w\ p^{\frac{1}{2H+1}}\ t^{\frac{2H_0+2H}{2H+1}}\right)\end{aligned}$$

where we used Lemma A.1.6 in the last inequality, and therefore,

$$\mathbb{E}|u^H(t,x)|^p = \|u^H(t,x)\|_p^p \leq (C_1^w)^p \exp\left(C_2^w\ p^{\frac{2H+2}{2H+1}}\ t^{\frac{2H_0+2H}{2H+1}}\right),$$

where C_1^w and C_2^w are some constants depending on H_0 and H. ∎

4.2 Uniform moment estimates

In this section, we include some estimates for the moments of the increments of the solution to equation (4.0.1) and (4.0.2) with the noise introduced in Section 4.1. The results in this section are used in the proof of tightness in Section 4.3. Throughout Theorem 4.2.1 below, we use subindices h and w to indicate that corresponding solution for heat equation, respectively wave equation.

Theorem 4.2.1. *Let u_h^H and u_w^H be the solution to equation* (4.0.1)*, respectively equation* (4.0.2) *with noise W^H as in Section 4.1.*

(a) *Let $[a,b]$ be a compact set such that*

$$\max\left\{0, \frac{3-4H_0}{4}\right\} < a < b < \frac{1}{2}.$$

For any $p \geq 2$ and $T > 0$, there exists positive constants C_1^h and C_2^h such that for any $t', t \in [0,T]$ and for any $x', x \in \mathbb{R}$, we have

$$\sup_{H\in[a,b]} \mathbb{E}|u_h^H(t',x) - u_h^H(t,x)|^p \leq C_1^h |t'-t|^{p\varepsilon} \tag{4.2.1}$$

and

$$\sup_{H\in[a,b]} \mathbb{E}|u_h^H(t,x') - u_h^H(t,x)|^p \leq C_2^h |x'-x|^{p\delta} \tag{4.2.2}$$

for any

$$0 < \varepsilon < \frac{2H_0(1-c_0)+a-1}{2} \quad \text{and} \quad 0 < \delta < 2H_0(1-c_0)+a-1 \tag{4.2.3}$$

where $c_0 \in \Big(0, \frac{2H_0+a-1}{2H_0}\Big)$.

(b) *Let $[a,b]$ be a compact set such that*

$$\frac{1}{4} < a < b < \frac{1}{2}.$$

For any $p \geq 2$ and $T > 0$, there exists a positive constant C_1^w such that for any $t', t \in [0,T]$ and for any $x \in \mathbb{R}$, we have

$$\sup_{H\in[a,b]} \mathbb{E}|u_w^H(t',x) - u_w^H(t,x)|^p \leq C_1^w |t'-t|^{p\delta} \tag{4.2.4}$$

for any

$$0 < \delta < a.$$

For any $p \geq 2$ and $T > 0$, there exists a positive constant C_2^w such that for any $t \in [0,T]$ and for any $x', x \in \mathbb{R}$,

$$\sup_{H\in[a,b]} \mathbb{E}|u_w^H(t,x') - u_w^H(t,x)|^p \leq C_2^w |x'-x|^{p\delta} \tag{4.2.5}$$

for any

$$\frac{c_1}{2} < \delta < a \tag{4.2.6}$$

where $c_1 \in \big(0, 2a\big)$.

Proof: We treat separately the heat and wave equations. The difference between the proofs for the heat equation and the wave equation is that in the case of the wave equation, instead of the Littlewood-Hardy inequality (given by Lemma 2.2.5), we use the Cauchy-Schwartz inequality and Lemma 2.2.3. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: we consider the heat equation} \begin{cases} \text{Step 1.a The time increments.} \\ \text{Step 1.b The space increments.} \end{cases} \\ \text{Step 2: we consider the wave equation} \begin{cases} \text{Step 2.a The time increments.} \\ \text{Step 2.b The space increments.} \end{cases} \end{cases}$$

Step 1: Heat equation

Step 1.a We start with the time increments. Let $t,t' \in [0,T]$ and $x \in \mathbb{R}$ be arbitrary. Assume that $h = t' - t > 0$ (The case $h < 0$ is similar). Similarly to (2.3.21),

$$\|u^H(t+h,x) - u^H(t,x)\|_p \leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{2}{n!}\big(A_n^H(t,h) + B_n^H(t,h)\big)\Big)^{1/2}, \quad (4.2.7)$$

where

$$A_n^H(t,h) := \|n!\tilde{f}_n(\cdot,t+h,x)1_{[0,t]^n} - n!\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}}, \quad (4.2.8)$$

and

$$B_n^H(t,h) := \|n!\tilde{f}_n(\cdot,t+h,x)1_{[0,t+h]^n\setminus[0,t]^n}\|^2_{\mathcal{H}_H^{\otimes n}}. \quad (4.2.9)$$

We study $A_n^H(t,h)$ first. Let $\mathcal{P}_H$ be the completion of $\mathcal{D}(\mathbb{R})$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{P}_H}$ where

$$\langle\varphi,\psi\rangle_{\mathcal{P}_H} = c_H\int_{\mathbb{R}}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}|\xi|^{1-2H}\mathrm{d}\xi$$

for any $\varphi,\psi \in \mathcal{D}(\mathbb{R})$. We have

$$A_n^H(t,h) = \alpha_{H_0}^n\int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n|t_j - s_j|^{2H_0-2}\psi_{t,h,n}^H(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \quad (4.2.10)$$

where

$$\begin{aligned}\psi_{t,h,n}^H(\mathbf{t},\mathbf{s}) &= c_H^n\int_{\mathbb{R}^n}\Big(\mathcal{F}\big[g_{\mathbf{t}}^{(n)}(\cdot,t+h,x) - g_{\mathbf{t}}^{(n)}(\cdot,t,x)\big](\xi_1,\dots,\xi_n)\Big) \\ &\qquad \Big(\overline{\mathcal{F}\big[g_{\mathbf{s}}^{(n)}(\cdot,t+h,x) - g_{\mathbf{s}}^{(n)}(\cdot,t,x)\big](\xi_1,\dots,\xi_n)}\Big)\prod_{j=1}^n|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi} \\ &= \Big\langle g_{\mathbf{t}}^{(n)}(\cdot,t+h,x) - g_{\mathbf{t}}^{(n)}(\cdot,t,x), g_{\mathbf{s}}^{(n)}(\cdot,t+h,x) - g_{\mathbf{s}}^{(n)}(\cdot,t,x)\Big\rangle_{\mathcal{P}_H^{\otimes n}}\end{aligned} \quad (4.2.11)$$

and $g_{\mathbf{t}}^{(n)}(\cdot,t,x) := n!\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,x)$.

We use Cauchy-Schwartz inequality followed by an application of Lemma 2.2.5 to the function $\varphi(\mathbf{t}) = \big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{1/2}$. We obtain:

$$\begin{aligned} A_n^H(t,h) &\le \alpha_{H_0}^n \int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n |t_j-s_j|^{2H_0-2}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\psi^H_{t,h,n}(\mathbf{s},\mathbf{s})\Big)^{1/2}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\le b_{H_0}^n\bigg(\int_{[0,t]^n}\big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\bigg)^{2H_0}. \end{aligned} \tag{4.2.12}$$

Fix $\mathbf{t}=(t_1,\dots,t_n)\in[0,t]^n$ and pick ρ such that $t_{\rho(1)}<\dots<t_{\rho(n)}<t$. Recall that, by Lemma 2.2.2

$$\begin{aligned} &\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)(\xi_1,\dots,\xi_n) \\ &\quad = e^{-i(\xi_1x_1+\dots+\xi_nx_n)}\prod_{j=1}^{n-1}\mathcal{F}G^h_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(j)})\mathcal{F}G^h_{t-t_{\rho(n)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(n)}). \end{aligned} \tag{4.2.13}$$

Therefore,

$$\begin{aligned} &\mathcal{F}\big[g_{\mathbf{t}}^{(n)}(\cdot,t+h,x)-g_{\mathbf{t}}^{(n)}(\cdot,t,x)\big](\xi_1,\dots,\xi_n) \\ &\qquad = e^{-i(\xi_1x_1+\dots+\xi_nx_n)}\prod_{j=1}^{n-1}\mathcal{F}G^h_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(j)}) \\ &\qquad\qquad \Big(\mathcal{F}G^h_{t+h-t_{\rho(n)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(n)})-\mathcal{F}G^h_{t-t_{\rho(n)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(n)})\Big). \end{aligned} \tag{4.2.14}$$

We denote $u_j = t_{\rho(j+1)}-t_{\rho(j)}$, for $j=1,\dots,n-1$ and $u_n = t-t_{\rho(n)}$. Using the change of variables $\xi'_j=\xi_{\rho(j)}$ for $j=1,\dots,n$, we obtain:

$$\begin{aligned} \psi^H_{t,h,n}(\mathbf{t},\mathbf{t}) &= c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\Big|\mathcal{F}G^h_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(j)})\Big|^2 \\ &\qquad \Big|\mathcal{F}G^h_{t+h-t_{\rho(n)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(n)})-\mathcal{F}G^h_{t-t_{\rho(n)}}(\xi_{\rho(1)}+\dots+\xi_{\rho(n)})\Big|^2\prod_{j=1}^n|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi} \\ &= c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\Big|\mathcal{F}G^h_{u_j}(\xi'_1+\dots+\xi'_j)\Big|^2 \\ &\qquad \Big|\mathcal{F}G^h_{u_n+h}(\xi'_1+\dots+\xi'_n)-\mathcal{F}G^h_{u_n}(\xi'_1+\dots+\xi'_n)\Big|^2\prod_{j=1}^n|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi} \end{aligned}$$

$$
\begin{aligned}
&= c_H^n \int_{\mathbb{R}^n} \prod_{j=1}^{n-1} \left|\mathcal{F}G_{u_j}^h(\eta_j)\right|^2 \left|\mathcal{F}G_{u_n+h}^h(\eta_n) - \mathcal{F}G_{u_n}^h(\eta_n)\right|^2 \prod_{j=1}^{n} |\eta_j - \eta_{j-1}|^{1-2H} \mathrm{d}\boldsymbol{\eta} \\
&\le c_H^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \Big(\int_{\mathbb{R}} \left|\mathcal{F}G_{u_j}^h(\eta_j)\right|^2 |\eta_j|^{\alpha_j} \mathrm{d}\eta_j \Big) \Big(\int_{\mathbb{R}} \left|\mathcal{F}G_{u_n+h}^h(\eta_n) - \mathcal{F}G_{u_n}^h(\eta_n)\right|^2 |\eta_n|^{\alpha_n} \mathrm{d}\eta_n \Big)
\end{aligned} \tag{4.2.15}
$$

where for the last equality, we use the change of variables $\eta_j = \xi_1 + \ldots + \xi_j$ for $j = 1, \ldots, n$ with $\eta_0 = 0$ and for the inequality above, we used relation (4.1.11). Note that

$$
1 - e^{-x} \le x^{\varepsilon}, \text{ for all } x > 0 \text{ and } \varepsilon \in [0,1], \tag{4.2.16}
$$

To prove (4.2.16), note that if $x \ge 1$, $1 - e^{-x} \le 1 \le x^{\varepsilon}$ for any $\varepsilon \ge 0$, and if $x < 1$, then $1 - e^{-x} \le x \le x^{\varepsilon}$ for any $\varepsilon \in [0,1]$. Using (4.2.16), we get:

$$
\begin{aligned}
\left|\mathcal{F}G_{u_n+h}^h(\eta_n) - \mathcal{F}G_{u_n}^h(\eta_n)\right|^2 &= \left| e^{-\frac{(u_n+h)|\eta_n|^2}{2}} - e^{-\frac{u_n|\eta_n|^2}{2}} \right|^2 = \left| e^{-\frac{u_n|\eta_n|^2}{2}} (1 - e^{-\frac{h|\eta_n|^2}{2}}) \right|^2 \\
&\le e^{-u_n|\eta_n|^2} \Big(\frac{h|\eta_n|^2}{2} \Big)^{2\varepsilon} \le h^{2\varepsilon} e^{-u_n|\eta_n|^2} |\eta_n|^{4\varepsilon}.
\end{aligned} \tag{4.2.17}
$$

We return to relation (4.2.15). By (4.2.17), we obtain:

$$
\begin{aligned}
\psi_{t,h,n}^H(\mathbf{t},\mathbf{t}) &\le h^{2\varepsilon} c_H^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \Big(\int_{\mathbb{R}} e^{-u_j|\eta_j|^2} |\eta_j|^{\alpha_j} \mathrm{d}\eta_j \Big) \Big(\int_{\mathbb{R}} e^{-u_n|\eta_n|^2} |\eta_n|^{\alpha_n + 4\varepsilon} \mathrm{d}\eta_n \Big) \\
&= h^{2\varepsilon} c_H^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \Gamma\Big(\frac{1+\alpha_j}{2} \Big) u_j^{-\frac{1+\alpha_j}{2}} \cdot \Gamma\Big(\frac{1+\alpha_n+4\varepsilon}{2} \Big) u_n^{-\frac{1+\alpha_n+4\varepsilon}{2}},
\end{aligned}
$$

where in the last equation, we used Lemma 4.1.2. For the application of Lemma 4.1.2, we need $\alpha_j > -1$ for $j = 1, \ldots, n-1$ and $\alpha_n + 4\varepsilon > -1$. This is clearly satisfied since $\alpha_j \ge 0$ for all $j = 1, \ldots, n$.

By relation (4.1.13), we see that

$$
c_H^n \prod_{j=1}^{n-1} \Gamma\Big(\frac{1+\alpha_j}{2} \Big) \le c_H C_{H,1}^{n-1}, \tag{4.2.18}
$$

where

$$
\begin{aligned}
C_{H,1} &= c_H \max\Big\{ \Gamma\Big(\frac{1}{2}\Big), \Gamma\Big(1-H\Big), \Gamma\Big(\frac{3-4H}{2}\Big) \Big\} \\
&= c_H \Gamma\Big(\frac{1}{2}\Big) \le \frac{1}{\pi}\sqrt{\pi} = \frac{1}{\sqrt{\pi}} < 1
\end{aligned} \tag{4.2.19}
$$

since $c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \leq \frac{1}{\pi}$. Recall that $\alpha_n \in \{0, 1-2H\}$ and $\varepsilon \in [0,1]$. We have

$$\frac{1}{2} < \frac{1+4\varepsilon}{2} \leq \frac{1+\alpha_n+4\varepsilon}{2} \leq \frac{1+1-2H+4}{2} = 3-H < 3$$

and

$$\Gamma\Big(\frac{1+\alpha_n+4\varepsilon}{2}\Big) < \Gamma\big(3\big) = 2.$$

Hence

$$\psi^H_{t,h,n}(\mathbf{t},\mathbf{t}) \leq 2h^{2\varepsilon} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{2}} \big(t - t_{\rho(n)}\big)^{-\frac{1+\alpha_n+4\varepsilon}{2}}. \tag{4.2.20}$$

Taking power $\frac{1}{2H_0}$ on both sides of relation (4.2.20) above, we obtain:

$$\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \leq \big(2h^{2\varepsilon}\big)^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{4H_0}} \big(t - t_{\rho(n)}\big)^{-\frac{1+\alpha_n+4\varepsilon}{4H_0}}. \tag{4.2.21}$$

We now integrate $\big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}$ over $[0,t]^n$. Using relation (4.2.21) and the change of variables $t'_j = t_{\rho(j)}$ for all $j = 1,\ldots,n$, we get:

$$\begin{aligned}
\int_{[0,t]^n} \Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} &= \sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t} \Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \\
&\leq \big(2h^{2\varepsilon}\big)^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \sum_{\rho\in S_n} \int_{0<t'_1<\ldots<t'_n<t} \prod_{j=1}^{n-1} \big(t'_{j+1} - t'_j\big)^{-\frac{1+\alpha_j}{4H_0}} \big(t - t'_n\big)^{-\frac{1+\alpha_n+4\varepsilon}{4H_0}} \mathrm{d}\mathbf{t} \\
&= \big(2h^{2\varepsilon}\big)^{\frac{1}{2H_0}} n! \sum_{\boldsymbol{\alpha}\in D_n} \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} \big(t_{j+1} - t_j\big)^{-\frac{1+\alpha_j}{4H_0}} \big(t - t_n\big)^{-\frac{1+\alpha_n+4\varepsilon}{4H_0}} \mathrm{d}\mathbf{t}, \quad (4.2.22)
\end{aligned}$$

where S_n is the set of all permutation of $\{1,\ldots,n\}$ and in the first equation, we decompose the set $[0,t]^n$ into $n!$ disjoint regions of the form $t_{\rho(1)} < \cdots < t_{\rho(n)}$ with $\rho \in S_n$.

Now we want to evaluate the integral in relation (4.2.22). We apply Lemma A.3.1 with $\beta_j = -\frac{1+\alpha_j}{4H_0}$ for all $j = 1,\ldots,n-1$ and $\beta_n = -\frac{1+\alpha_n+4\varepsilon}{4H_0}$. Note that

$$\begin{aligned}
|\beta| = \sum_{j=1}^{n} \beta_j &= -\sum_{j=1}^{n-1} \frac{1+\alpha_j}{4H_0} - \frac{1+\alpha_n+4\varepsilon}{4H_0} = -\frac{1}{4H_0}\Big[\sum_{j=1}^{n}(1+\alpha_j) + 4\varepsilon\Big] \\
&= -\frac{n+n(1-2H)+4\varepsilon}{4H_0} = -\frac{1-H}{2H_0}\cdot n - \frac{\varepsilon}{H_0}
\end{aligned}$$

and

$$|\beta| + n = \left(1 - \frac{1-H}{2H_0}\right) \cdot n - \frac{\varepsilon}{H_0} = \frac{2H_0 + H - 1}{2H_0} \cdot n - \frac{\varepsilon}{H_0}.$$

To apply Lemma A.3.1, we need $\beta_j > -1$ for all $j = 1, \ldots, n$, which means $\frac{1+\alpha_j}{4H_0} < 1$ for $j = 1, \ldots, n-1$ and $\frac{1+\alpha_n+4\varepsilon}{4H_0} < 1$. When $\alpha_j = 2(1-2H)$, we use the condition $4H_0 + 4H > 3$. When $\alpha_n = 1 - 2H$, we use the fact that

$$\varepsilon < \frac{2H_0 + H - 1}{2}, \tag{4.2.23}$$

which is a consequence of condition (4.2.3). Therefore,

$$\begin{aligned} &\int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} \left(t_{j+1} - t_j\right)^{-\frac{1+\alpha_j}{4H_0}} \left(t - t_n\right)^{-\frac{1+\alpha_n+4\varepsilon}{4H_0}} \mathrm{d}\mathbf{t} \\ &\quad = \frac{\prod_{j=1}^{n} \Gamma(\beta_j + 1)\, t^{|\beta|+n}}{\Gamma(|\beta| + n + 1)} = \frac{\prod_{j=1}^{n-1} \Gamma\left(1 - \frac{1+\alpha_j}{4H_0}\right) \Gamma\left(1 - \frac{1+\alpha_n+4\varepsilon}{4H_0}\right) t^{\frac{2H_0+H-1}{2H_0} \cdot n - \frac{\varepsilon}{H_0}}}{\Gamma\left(\frac{2H_0+H-1}{2H_0} \cdot n - \frac{\varepsilon}{H_0} + 1\right)}. \end{aligned} \tag{4.2.24}$$

Recall that $\alpha_1 \in \{1-2H, 2(1-2H)\}$, $\alpha_n \in \{0, 1-2H\}$ and $\alpha_j \in \{0, 1-2H, 2(1-2H)\}$ for $j = 2, \ldots, n-1$. Therefore, $0 \le \alpha_j \le 2(1-2H)$ for all $j = 1, \ldots, n$ which implies that for any $H \in [a, b]$

$$0 < 1 - \frac{3-4a}{4H_0} < 1 - \frac{3-4H}{4H_0} \le 1 - \frac{1+\alpha_j}{4H_0} \le 1 - \frac{1}{4H_0} < \frac{3}{4}.$$

Because the Gamma function Γ is decreasing on $(0, x_0)$ and increasing on (x_0, ∞) with $x_0 \approx 1.4$, we know that

$$\prod_{j=1}^{n-1} \Gamma\left(1 - \frac{1+\alpha_j}{4H_0}\right) \le \left(\Gamma\left(1 - \frac{3-4a}{4H_0}\right)\right)^{n-1}. \tag{4.2.25}$$

To find a bound for $\Gamma\left(1 - \frac{1+\alpha_n+4\varepsilon}{4H_0}\right)$, we need to study the range of possible values for $1 - \frac{1+\alpha_n+4\varepsilon}{4H_0}$. Since $\alpha_n \in \{0, 1-2H\}$, $H \in [a, b]$ and $\varepsilon \in [0, 1]$, we have

$$1 - \frac{1-a+2\varepsilon}{2H_0} \le 1 - \frac{1-H+2\varepsilon}{2H_0} \le 1 - \frac{1+\alpha_n+4\varepsilon}{4H_0} \le 1 - \frac{1}{4H_0}.$$

If we simply choose $\varepsilon \in [0, 1]$ such that $1 - \frac{1-a+2\varepsilon}{2H_0} > 0$ (i.e. $\varepsilon < \frac{2H_0+a-1}{2}$), then the lower bound $1 - \frac{1-a+2\varepsilon}{2H_0}$ can be in principle, be very close to 0: as ε approaches $\frac{2H_0+a-1}{2}$, the lower bound $1 - \frac{1-a+2\varepsilon}{2H_0}$ approaches 0. This is a problem since $\lim_{x \to 0^+} \Gamma(x) = \infty$, and hence, we are not able to bound the term $\Gamma\Big(1-\frac{1+\alpha_n+4\varepsilon}{4H_0}\Big)$. To avoid this problem, we choose an arbitrary value $c_0>0$ such that

$$c_0<1-\frac{1-a+2\varepsilon}{2H_0}.$$

Note that this is equivalent to $2\varepsilon<2H_0(1-c_0)-(1-a)$. Since $\varepsilon>0$, we must choose $c_0>0$ such that $2H_0(1-c_0)-(1-a)>0$, i.e.

$$c_0<\frac{2H_0+a-1}{2H_0}.$$

With this choice of c_0 and ε, we have

$$\Gamma\Big(1-\frac{1+\alpha_n+4\varepsilon}{4H_0}\Big)\le\Gamma(c_0). \tag{4.2.26}$$

Hence, there exist a constant $c^{(1)}_{H_0,a}>0$ depending on H_0 and a such that

$$\prod_{j=1}^{n-1}\Gamma\Big(1-\frac{1+\alpha_j}{4H_0}\Big)\Gamma\Big(1-\frac{1+\alpha_n+4\varepsilon}{4H_0}\Big)\le\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0). \tag{4.2.27}$$

Next, recall that relation (4.2.3) implies (4.2.23). Hence, for any $H\in[a,b]$, we have

$$\frac{(n-1)(2H_0+H-1)}{2H_0}<\frac{n(2H_0+H-1)}{2H_0}-\frac{\varepsilon}{H_0}<\frac{n(2H_0+H-1)}{2H_0}<\frac{n(2H_0+b-1)}{2H_0}.$$

It follows that

$$t^{\frac{2H_0+H-1}{2H_0}\cdot n-\frac{\varepsilon}{H_0}}\le\begin{cases}1 & \text{if } t<1,\\ t^{\frac{n(2H_0+b-1)}{2H_0}} & \text{if } t\ge1.\end{cases}$$

Therefore, we get:

$$t^{\frac{2H_0+H-1}{2H_0}\cdot n-\frac{\varepsilon}{H_0}}\le\big(t\vee1\big)^{\frac{n(2H_0+b-1)}{2H_0}}. \tag{4.2.28}$$

Finally, we pick $m_0\ge1$ such that $(m_0-1)\cdot\frac{2H_0+a-1}{2H_0}>x_0$. For any $n\ge m_0$, we have

$$\frac{n(2H_0+H-1)}{2H_0}-\frac{\varepsilon}{H_0}>\frac{(n-1)(2H_0+a-1)}{2H_0}>x_0,$$

which implies that for any $H\in[a,b]$,

$$\Gamma\Big(\frac{n(2H_0+H-1)}{2H_0}-\frac{\varepsilon}{H_0}+1\Big)>\Gamma\Big(\frac{(n-1)(2H_0+a-1)}{2H_0}+1\Big).$$

By Lemma A.1.3, we know that there exists a positive constant $c^{(2)}_{H_0,a}$ depending on H_0 and a such that

$$\Gamma\Big(\frac{2H_0+H-1}{2H_0}\cdot n-\frac{\varepsilon}{H_0}+1\Big)>\Gamma\Big(\frac{(n-1)(2H_0+a-1)}{2H_0}+1\Big)$$
$$\geq (c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}. \tag{4.2.29}$$

We come back to (4.2.24). Using relations (4.2.27), (4.2.28) and (4.2.29), we obtain:

$$\int_{0<t_1<\ldots<t_n<t}\prod_{j=1}^{n-1}\big(t_{j+1}-t_j\big)^{-\frac{1+\alpha_j}{4H_0}}\big(t-t_n\big)^{-\frac{1+\alpha_n+4\varepsilon}{4H_0}}\mathrm{d}\mathbf{t}$$
$$\leq\frac{\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}}. \tag{4.2.30}$$

Returning to equality (4.2.22) and using relation (4.2.30) and the fact that $\operatorname{card}(D_n)=2^{n-1}$, we obtain:

$$\int_{[0,t]^n}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\leq 2h^{\frac{\varepsilon}{H_0}}n!2^{n-1}\frac{\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}}$$
$$\leq 2^{2n}h^{\frac{\varepsilon}{H_0}}\frac{\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}} \tag{4.2.31}$$

where for the last inequality, we used fact that $n!\leq 2^n(n-1)!$.

We take power $2H_0$ in the above estimate. Then relation (4.2.12) becomes:

$$A^H_n(t,h)\leq b^n_{H_0}\Bigg(2^{2n}h^{\frac{\varepsilon}{H_0}}\frac{\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}}\Bigg)^{2H_0}$$
$$=b^n_{H_0}h^{2\varepsilon}4^{2nH_0}\frac{\Big(\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\Big)^{2H_0}\big(t\vee 1\big)^{n(2H_0+b-1)}}{\big(c^{(2)}_{H_0,a}\big)^{(n-1)2H_0}\big[(n-1)!\big]^{a-1}}. \tag{4.2.32}$$

Therefore, by relation (4.2.32), we conclude that

$$\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A^H_n(t,h)\Big)^{1/2}\leq C^{(1)}_{p,a,b,c_0,H_0,T}\,h^{\varepsilon}, \tag{4.2.33}$$

where $C^{(1)}_{p,a,b,c_0,H_0,T} > 0$ is a constant depending on p, a, b, c_0, H_0, and T.

As for the term $B^H_n(t,h)$, let $D_{t,h} = [0,t+h]^n \backslash [0,t]^n$, we note that

$$\begin{aligned} & B^H_n(t,h) \\ & = \alpha^n_{H_0} \int_{[0,t+h]^n} \int_{[0,t+h]^n} \prod_{j=1}^n |t_j - s_j|^{2H_0-2} 1_{D_{t,h}}(\mathbf{t}) 1_{D_{t,h}}(\mathbf{s})\, \gamma^H_{t,h,n}(\mathbf{t},\mathbf{s}) \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \end{aligned} \tag{4.2.34}$$

where

$$\begin{aligned} \gamma^H_{t,h,n}(\mathbf{t},\mathbf{s}) &= c^n_H \int_{\mathbb{R}^n} \mathcal{F} g^{(n)}_{\mathbf{t}}(\cdot, t+h, x) \overline{\mathcal{F} g^{(n)}_{\mathbf{s}}(\cdot, t+h, x)} \prod_{j=1}^n |\xi_j|^{1-2H} \mathrm{d}\boldsymbol{\xi} \\ &= \left\langle g^{(n)}_{\mathbf{t}}(\cdot, t+h, x), g^{(n)}_{\mathbf{s}}(\cdot, t+h, x) \right\rangle_{\mathcal{P}^{\otimes n}_H} \end{aligned} \tag{4.2.35}$$

and $g^{(n)}_{\mathbf{t}}(\cdot, t+h, x) := n! \tilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, t+h, x)$.

We use Cauchy-Schwartz inequality followed by an application of Lemma 2.2.5 to the function $\varphi(\mathbf{t}) = \left(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\right)^{1/2} 1_{D_{t,h}}(\mathbf{t})$, we have

$$\begin{aligned} B^H_n(t,h) \le \alpha^n_{H_0} & \int_{[0,t+h]^n} \int_{[0,t+h]^n} \prod_{j=1}^n |t_j - s_j|^{2H_0-2} 1_{D_{t,h}}(\mathbf{t}) 1_{D_{t,h}}(\mathbf{s}) \\ & \qquad \left(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\right)^{1/2} \left(\gamma^H_{t,h,n}(\mathbf{s},\mathbf{s})\right)^{1/2} \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ \le b^n_{H_0} & \left(\int_{[0,t+h]^n} \left(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t}) \mathrm{d}\mathbf{t} \right)^{2H_0}. \end{aligned} \tag{4.2.36}$$

Fix $\mathbf{t} = (t_1, \ldots, t_n) \in [0, t+h]^n$ and pick ρ such that $t_{\rho(1)} < \ldots < t_{\rho(n)} < t+h$ and $t < t_{\rho(n)} < t+h$, we get

$$\begin{aligned} & \mathcal{F} g^{(n)}_{\mathbf{t}}(\cdot, t+h, x)(\xi_1, \ldots, x_n) \\ & = e^{-i(\xi_1 x_1 + \cdots + \xi_n x_n)} \prod_{j=1}^{n-1} \mathcal{F} G^h_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}) \mathcal{F} G^h_{t+h-t_{\rho(n)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(n)}) \end{aligned}$$

where we used Lemma 2.2.2. We denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, for $j = 1, \ldots, n-1$ and $u_n = t - t_{\rho(n)}$. Using the change of variables $\xi'_j = \xi_{\rho(j)}$ for $j = 1, \ldots, n$ and $\eta_j = \xi'_1 + \ldots + \xi'_j$ for $j = 1, \ldots, n$ with $\eta_0 = 0$, we obtain:

$$\begin{aligned} & \gamma^H_{t,h,n}(\mathbf{t},\mathbf{t}) \\ & = c^n_H \int_{\mathbb{R}^n} \prod_{j=1}^{n-1} \left| \mathcal{F} G^h_{t_{\rho(j+1)} - t_{\rho(j)}}(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}) \right|^2 \end{aligned}$$

$$\Big|\mathcal{F}G^h_{t+h-t_{\rho(n)}}(\xi_{\rho(1)}+\cdots+\xi_{\rho(n)})\Big|^2\prod_{j=1}^n|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi}$$

$$
\begin{aligned}
&= c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\Big|\mathcal{F}G^h_{u_j}(\xi'_1+\cdots+\xi'_j)\Big|^2\Big|\mathcal{F}G^h_{u_n+h}(\xi'_1+\cdots+\xi'_n)\Big|^2\prod_{j=1}^n|\xi'_j|^{1-2H}\mathrm{d}\boldsymbol{\xi}\\
&= c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\big|\mathcal{F}G^h_{u_j}(\eta_j)\big|^2\big|\mathcal{F}G^h_{u_n+h}(\eta_n)\big|^2\prod_{j=1}^n|\eta_j-\eta_{j-1}|^{1-2H}\mathrm{d}\boldsymbol{\eta}\\
&\le c_H^n\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\big|\mathcal{F}G^h_{u_j}(\eta_j)\big|^2\big|\mathcal{F}G^h_{u_n+h}(\eta_n)\big|^2\Big(\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^n|\eta_j|^{\alpha_j}\Big)\mathrm{d}\boldsymbol{\eta}\\
&= c_H^n\sum_{\boldsymbol{\alpha}\in D_n}\Big(\prod_{j=1}^{n-1}\int_{\mathbb{R}}\big|\mathcal{F}G^h_{u_j}(\eta_j)\big|^2|\eta_j|^{\alpha_j}\mathrm{d}\eta_j\Big)\Big(\int_{\mathbb{R}}\big|\mathcal{F}G^h_{u_n+h}(\eta_n)\big|^2|\eta_n|^{\alpha_n}\mathrm{d}\eta_n\Big) \qquad (4.2.37)\\
&= c_H^n\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}\Gamma\Big(\frac{1+\alpha_j}{2}\Big)u_j^{-\frac{1+\alpha_j}{2}}\cdot\Gamma\Big(\frac{1+\alpha_n}{2}\Big)(u_n+h)^{-\frac{1+\alpha_n}{2}}, \qquad (4.2.38)
\end{aligned}
$$

where for the inequality, we used relation relation (4.1.11) and in the last equation we used Lemma 4.1.2. For the application of Lemma 4.1.2, we need $\alpha_j > -1$ for $j = 1, \ldots, n$. This is clearly satisfied since $\alpha_j \ge 0$ for all $j = 1, \ldots, n$. By relations (4.2.18), we get:

$$c_H^n\prod_{j=1}^n\Gamma\Big(\frac{1+\alpha_j}{2}\Big)\le C_{H,1}^n\le\Big(\frac{1}{\sqrt{\pi}}\Big)^n<1. \qquad (4.2.39)$$

Combining relations (4.2.38) and (4.2.39), we obtain:

$$\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\le\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}u_j^{-\frac{1+\alpha_j}{2}}\cdot(u_n+h)^{-\frac{1+\alpha_n}{2}}. \qquad (4.2.40)$$

Hence, taking power $\frac{1}{2H_0}$ on both sides of relation (4.2.40), we obtain:

$$\Big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\le\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}u_j^{-\frac{1+\alpha_j}{4H_0}}(u_n+h)^{-\frac{1+\alpha_n}{4H_0}}. \qquad (4.2.41)$$

We now integrate $\big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}1_{D_{t,h}}(\mathbf{t})$ over $[0,t+h]^n$. Using relation (4.2.41) and the change of variables $t'_j = t_{\rho(j)}$ for all $j = 1, \ldots, n$, we get:

$$\int_{[0,t+h]^n}\Big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}$$

$$
\begin{aligned}
&= \sum_{\rho\in S_n}\int_t^{t+h}\left(\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}}\left(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}}\,\mathrm{d}t_{\rho(1)}\ldots\mathrm{d}t_{\rho(n-1)}\right)\mathrm{d}t_{\rho(n)}\\
&\le \sum_{\rho\in S_n}\int_t^{t+h}\Bigg(\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}}\\
&\qquad \sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}(t_{\rho(j+1)}-t_{\rho(j)})^{-\frac{1+\alpha_j}{4H_0}}(t+h-t_{\rho(n)})^{-\frac{1+\alpha_n}{4H_0}}\mathrm{d}t_{\rho(1)}\ldots\mathrm{d}t_{\rho(n-1)}\Bigg)\mathrm{d}t_{\rho(n)}\\
&\le n!\sum_{\boldsymbol{\alpha}\in D_n}\int_t^{t+h}J^h(t_n)\cdot(t+h-t_n)^{-\frac{1+\alpha_n}{4H_0}}\mathrm{d}t_n \qquad (4.2.42)
\end{aligned}
$$

where

$$
\begin{aligned}
J^h(t_n) &= \int_{0<t_1<\ldots<t_{n-1}<t_n}\prod_{j=1}^{n-1}(t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}}\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}\\
&= \frac{\prod_{j=1}^{n-1}\Gamma\big(\beta_j+1\big)\ t_n^{\sum_{j=1}^{n-1}\beta_j+(n-1)}}{\Gamma\Big(\sum_{j=1}^{n-1}\beta_j+(n-1)+1\Big)}, \qquad (4.2.43)
\end{aligned}
$$

where the second equality follows by Lemma A.3.1 with $\beta_j = -\frac{1+\alpha_j}{4H_0}$ for all $j = 1,\ldots,n-1$. To evaluate $J^h(t_n)$, notice that similarly to relation (4.1.15), we have

$$
\prod_{j=1}^{n-1}\Gamma\big(\beta_j+1\big)=\prod_{j=1}^{n-1}\Gamma\Big(1-\frac{1+\alpha_j}{4H_0}\Big)\le C_{H,2}^{n-1} \qquad (4.2.44)
$$

where $C_{H,2}$ is given in (4.1.16). We now show how to find an upper bound for $C_{H,2}$, uniformly in $H\in[a,b]$. Note that for any $H_0\in(1/2,1)$, $-\frac{1}{4H_0}+1\in(1/2,3/4)$, we have

$$
\Gamma\Big(-\frac{1}{4H_0}+1\Big)\le\Gamma\Big(\frac{1}{2}\Big),
$$

and $-\frac{1-H}{2H_0}+1\in\big(1-\frac{1-a}{2H_0},1-\frac{1-b}{2H_0}\big)\subset(0,3/4)$, we get

$$
\Gamma\Big(-\frac{1-H}{2H_0}+1\Big)\le\Gamma\Big(\frac{2H_0+a-1}{2H_0}\Big).
$$

Using the condition $a>3/4-H_0$, we see that

$$
-\frac{3-4H}{4H_0}+1\in\big(1-\frac{3-4a}{4H_0},1-\frac{3-4b}{4H_0}\big)\subset(0,3/4),
$$

then

$$
\Gamma\Big(-\frac{3-4H}{4H_0}+1\Big)\le\Gamma\Big(\frac{4H_0+4a-3}{4H_0}\Big).
$$

Thus,

$$C_{H,2} \leq \max\left\{\Gamma\Big(\frac{1}{2}\Big), \Gamma\Big(\frac{2H_0+a-1}{2H_0}\Big), \Gamma\Big(\frac{4H_0+4a-3}{4H_0}\Big)\right\} := c_{a,2}. \tag{4.2.45}$$

Therefore, with relation (4.2.44), we have

$$\prod_{j=1}^{n-1} \Gamma\big(\beta_j+1\big) \leq C_{H,2}^{n-1} \leq c_{a,2}^{n-1}. \tag{4.2.46}$$

Moreover, for any $H \in [a,b]$, we have

$$\frac{(n-1)(4H_0+2a-2)-(1-2a)}{4H_0} \leq \sum_{j=1}^{n-1} \beta_j + (n-1) \leq \frac{(n-1)(2H_0+b-1)}{2H_0}. \tag{4.2.47}$$

We prove relation (4.2.47) below. Note that

$$\sum_{j=1}^{n-1} \beta_j + (n-1) = \frac{-\sum_{j=1}^{n-1}(1+\alpha_j)+4H_0(n-1)}{4H_0} = \frac{-(n-1)-\sum_{j=1}^{n-1}\alpha_j+4H_0(n-1)}{4H_0}$$
$$= \frac{(n-1)(4H_0-1)-\sum_{j=1}^{n-1}\alpha_j}{4H_0}.$$

We know that

$$\sum_{j=1}^{n-1}\alpha_j = \sum_{j=1}^{n}\alpha_j - \alpha_n = n(1-2H)-\alpha_n.$$

Therefore, since $\alpha_n \in \{0, 1-2H\}$ and $H \in [a,b]$, we have

$$(n-1)(1-2b) \leq (n-1)(1-2H) \leq \sum_{j=1}^{n-1}\alpha_j \leq n(1-2H) \leq n(1-2a)$$

and

$$-\frac{n(1-2a)}{4H_0} \leq -\frac{\sum_{j=1}^{n-1}\alpha_j}{4H_0} \leq -\frac{(n-1)(1-2b)}{4H_0}.$$

This leads to

$$\frac{(n-1)(4H_0+2a-2)-(1-2a)}{4H_0} \leq \sum_{j=1}^{n-1} \beta_j + (n-1) \leq \frac{(n-1)(2H_0+b-1)}{2H_0}.$$

We now return to the main estimate. We pick $m_0 \geq 1$ such that

$$\frac{(m_0-1)(4H_0+2a-2)-(1-2a)}{4H_0} > x_0,$$

where $x_0 \in (1,2)$ is such that Gamma function $\Gamma(\cdot)$ is increasing on (x_0,∞). By relation (4.2.47), for any $n \geq m_0$, we have

$$\sum_{j=1}^{n-1} \beta_j + (n-1) \geq \frac{(n-1)(4H_0+2a-2)-(1-2a)}{4H_0} > x_0,$$

which implies that for any $n \geq m_0$, there exists a positive constant $c^{(3)}_{H_0,a}$ depending on H_0 and a such that

$$\begin{aligned}\Gamma\Big(\sum_{j=1}^{n-1}\beta_j + (n-1)+1\Big) &> \Gamma\Big(\frac{(n-1)(4H_0+2a-2)-(1-2a)}{4H_0}+1\Big)\\ &\geq (c^{(3)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}, \end{aligned} \tag{4.2.48}$$

where for the last inequality, we used Lemma A.1.4. By relation (4.2.47), it follows that

$$t_n^{\sum_{j=1}^{n-1}\beta_j+(n-1)} \leq \begin{cases} 1 & \text{if } t_n < 1, \\ t_n^{\frac{(n-1)(2H_0+b-1)}{2H_0}} & \text{if } t_n \geq 1 \end{cases} \leq (T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}}. \tag{4.2.49}$$

Now, returning to (4.2.43) and using relations (4.2.46), (4.2.48) and (4.2.49), we get:

$$J^h(t_n) \leq \frac{c_{a,2}^{n-1}}{(c^{(3)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}}\,(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}}. \tag{4.2.50}$$

We continue to integrate $\big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})$ over $[0,t+h]^n$. Using relations (4.2.42), (4.2.50) and the change of variable $s = t+h-t_n$, we have

$$\begin{aligned}&\int_{[0,t+h]^n}\Big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}\\ &\quad\leq n!\frac{c_{a,2}^{n-1}}{(c^{(3)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}}\sum_{\boldsymbol{\alpha}\in D_n}\int_t^{t+h}(t+h-t_n)^{-\frac{1+\alpha_n}{4H_0}}\mathrm{d}t_n\\ &\quad= n!\frac{c_{a,2}^{n-1}}{(c^{(3)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}}\sum_{\boldsymbol{\alpha}\in D_n}\int_0^{h} s^{-\frac{1+\alpha_n}{4H_0}}\mathrm{d}s\end{aligned}$$

$$= n! \frac{c_{a,2}^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \frac{1}{1-\frac{1+\alpha_n}{4H_0}} h^{1-\frac{1+\alpha_n}{4H_0}}. \tag{4.2.51}$$

Note that, since $\alpha_n \in \{0, 1-2H\}$ and $H \in [a,b]$, we have

$$\frac{1}{1-\frac{1+\alpha_n}{4H_0}} < \frac{2H_0}{2H_0+H-1} < \frac{2H_0}{2H_0+a-1} =: c_{H_0,a}^{(4)}.$$

Recall that relation (4.2.3) implies that relation (4.2.23) holds. Hence, we have

$$0 < \frac{\varepsilon}{H_0} < \frac{2H_0+a-1}{2H_0} \le 1 - \frac{1+\alpha_n}{4H_0} = \frac{4H_0-1-\alpha_n}{4H_0} \le 1-\frac{1}{4H_0} \le \frac{3}{4}$$

which implies for any $h\in(0,1)$,

$$h^{1-\frac{1+\alpha_n}{4H_0}} \le h^{\frac{\varepsilon}{H_0}}.$$

Therefore, using the fact that $n! \le 2^n (n-1)!$ and relation (4.2.51), we obtain:

$$\begin{aligned}
&\int_{[0,t+h]^n} \Big(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t} \\
&\le n! \frac{c_{a,2}^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{4H_0+2a-2}{4H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} c_{H_0,a}^{(4)} h^{\frac{\varepsilon}{H_0}} \\
&\le h^{\frac{\varepsilon}{H_0}} 2^n \frac{c_{a,2}^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} 2^{n-1} c_{H_0,a}^{(4)}.
\end{aligned}$$

We take power $2H_0$ in the above estimate, then relation (4.2.36) becomes:

$$\begin{aligned}
B_n^H(t,h) &\le b_{H_0}^n \Bigg(h^{\frac{\varepsilon}{H_0}} 2^n \frac{c_{a,2}^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}}(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} 2^{n-1} c_{H_0,a}^{(4)}\Bigg)^{2H_0} \\
&= h^{2\varepsilon} b_{H_0}^n 2^{n2H_0} \frac{c_{a,2}^{(n-1)2H_0}(c_{H_0,a}^{(4)})^{2H_0}}{(c_{H_0,a}^{(3)})^{(n-1)2H_0}\big[(n-1)!\big]^{a-1}}(T\vee 1)^{(n-1)(2H_0+b-1)} 2^{(n-1)2H_0}.
\end{aligned} \tag{4.2.52}$$

Therefore, by relation (4.2.52), we conclude that

$$\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^H(t,h)\Big)^{1/2} \le C_{p,a,b,H_0,T}^{(2)}\, h^{\varepsilon}, \tag{4.2.53}$$

where $C^{(2)}_{p,a,b,H_0,T} > 0$ is a constant depending on p, a, b, c_0, H_0, and T.

Returning to (4.2.7) and using (4.2.33) and (4.2.53), it follows that

$$\|u^H(t+h,x) - u^H(t,x)\|_p \leq \sqrt{2}\Big(C^{(1)}_{p,a,b,H_0,T} + C^{(2)}_{p,a,b,H_0,T}\Big)h^{\varepsilon}.$$

Relation (4.2.1) follows by taking power p and then taking supremum over $H \in [a,b]$.

Step 1.b We examine the spatial increments for the solution of the heat equation. For any $x, x' \in \mathbb{R}$, we let $z = x' - x$. By relation (2.3.41),

$$\|u^H(t,x+z) - u^H(t,x)\|_p \leq \sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}C^H_n(t,z)\Big)^{1/2}, \tag{4.2.54}$$

where

$$\begin{aligned} C^H_n(t,z) &:= \|n!\tilde{f}_n(\cdot,t,x+z) - n!\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}} \\ &= \alpha^n_{H_0}\int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n |t_j - s_j|^{2H_0-2}\Psi^H_{t,z,n}(\mathbf{t},\mathbf{s})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}, \end{aligned} \tag{4.2.55}$$

and we denote by

$$\begin{aligned} \Psi^H_{t,z,n}(\mathbf{t},\mathbf{s}) &= c^n_H \int_{\mathbb{R}^n} \mathcal{F}\big[g^{(n)}_{\mathbf{t}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{t}}(\cdot,t,x)\big](\xi_1,\ldots,\xi_n) \\ &\qquad \overline{\mathcal{F}\big[g^{(n)}_{\mathbf{s}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{s}}(\cdot,t,x)\big](\xi_1,\ldots,\xi_n)}\prod_{j=1}^n|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi} \\ &= \Big\langle g^{(n)}_{\mathbf{t}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{t}}(\cdot,t,x), g^{(n)}_{\mathbf{s}}(\cdot,t,x+z) - g^{(n)}_{\mathbf{s}}(\cdot,t,x)\Big\rangle_{\mathcal{P}_H^{\otimes n}} \end{aligned} \tag{4.2.56}$$

and $g^{(n)}_{\mathbf{t}}(\cdot,t,x) := n!\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)$.

We use Cauchy-Schwartz inequality followed by an application of Lemma 2.2.5 to the function $\varphi(\mathbf{t}) = \big(\Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\big)^{1/2}$. We obtain:

$$\begin{aligned} C^H_n(t,z) &\leq \alpha^n_{H_0}\int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n|t_j-s_j|^{2H_0-2}\Big(\Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\Psi^H_{t,z,n}(\mathbf{s},\mathbf{s})\Big)^{1/2}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq b^n_{H_0}\bigg(\int_{[0,t]^n}\Big(\Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\bigg)^{2H_0}. \end{aligned} \tag{4.2.57}$$

Fix $\mathbf{t} = (t_1, \ldots, t_n) \in [0,t]^n$ and pick ρ such that $t_{\rho(1)} < \ldots < t_{\rho(n)} < t_{\rho(n+1)} = t$. By (4.2.13),

$$\mathcal{F}\big[g_{\mathbf{t}}^{(n)}(\cdot,t,x+z) - g_{\mathbf{t}}^{(n)}(\cdot,t,x)\big](\xi_1,\ldots,\xi_n)$$
$$= e^{-i(\xi_1 x_1+\cdots+\xi_n x_n)}\big(e^{-i(\xi_1+\ldots+\xi_n)z} - 1\big)\prod_{j=1}^{n}\mathcal{F}G^h_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\cdots+\xi_{\rho(j)}). \tag{4.2.58}$$

We denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, for $j = 1,\ldots,n$. Using the change of variables $\xi'_j = \xi_{\rho(j)}$ for $j = 1,\ldots,n$, we obtain:

$$\Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})$$
$$= c_H^n \int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big|\mathcal{F}G^h_{t_{\rho(j+1)}-t_{\rho(j)}}(\xi_{\rho(1)}+\cdots+\xi_{\rho(j)})\Big|^2\Big|1-e^{-i(\xi_{\rho(1)}+\cdots+\xi_{\rho(n)})z}\Big|^2\prod_{j=1}^{n}|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi}$$
$$= c_H^n \int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big|\mathcal{F}G^h_{u_j}(\xi'_1+\cdots+\xi'_j)\Big|^2\Big|1-e^{-i(\xi'_1+\cdots+\xi'_n)z}\Big|^2\prod_{j=1}^{n}|\xi'_j|^{1-2H}\mathrm{d}\boldsymbol{\xi}$$
$$\le |z|^{2\delta}c_H^n \int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big|\mathcal{F}G^h_{u_j}(\xi_1+\cdots+\xi_j)\Big|^2\Big|\xi_1+\ldots+\xi_n\Big|^{2\delta}\prod_{j=1}^{n}|\xi_j|^{1-2H}\mathrm{d}\boldsymbol{\xi}$$
$$= |z|^{2\delta}c_H^n \int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big|\mathcal{F}G^h_{u_j}(\eta_j)\Big|^2|\eta_n|^{2\delta}\prod_{j=1}^{n}|\eta_j-\eta_{j-1}|^{1-2H}\mathrm{d}\boldsymbol{\eta}$$
$$\le |z|^{2\delta}c_H^n \int_{\mathbb{R}^n}\prod_{j=1}^{n}\Big|\mathcal{F}G^h_{u_j}(\eta_j)\Big|^2|\eta_n|^{2\delta}\Big(\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n}|\eta_j|^{\alpha_j}\Big)\mathrm{d}\boldsymbol{\eta} \tag{4.2.59}$$

where for the last equality, we use the change of variables $\eta_j = \xi_1 + \ldots + \xi_j$ for $j = 1,\ldots,n$ with $\eta_0 = 0$ and for the inequality above, we used relation (4.1.11). Note that for the first inequality above, we used:

$$|1-e^{-ix}|^2 = 2\big(1-\cos(x)\big) \le x^{2\delta}, \text{ for all } x > 0 \text{ and } \delta \in [0,1]. \tag{4.2.60}$$

To prove (4.2.60), note that if $x \ge 1$, $1-\cos(x) \le 1 \le x^{2\delta}$ for any $\delta \ge 0$, and if $x < 1$, then $1-\cos(x) \le \frac{1}{2}x^2 \le \frac{1}{2}x^{2\delta}$ for any $\delta \in [0,1]$. (Note that (4.2.60) is the analogue of (4.2.16) for complex exponents.)

We return to relation (4.2.59). By (4.2.60), we obtain:

$$\Psi^H_{t,z,n}(\mathbf{t},\mathbf{t}) \le |z|^{2\delta}c_H^n\sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}\Big(\int_{\mathbb{R}}e^{-u_j|\eta_j|^2}|\eta_j|^{\alpha_j}\mathrm{d}\eta_j\Big)\Big(\int_{\mathbb{R}}e^{-u_n|\eta_n|^2}|\eta_n|^{\alpha_n+2\delta}\mathrm{d}\eta_n\Big)$$

$$= |z|^{2\delta} c_H^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \Gamma\Big(\frac{1+\alpha_j}{2}\Big) u_j^{-\frac{1+\alpha_j}{2}} \cdot \Gamma\Big(\frac{1+\alpha_n+2\delta}{2}\Big) u_n^{-\frac{1+\alpha_n+2\delta}{2}}, \tag{4.2.61}$$

where in the last equation, we used Lemma 4.1.2. For the application of Lemma 4.1.2, we need $\alpha_j > -1$ for $j = 1, \dots, n-1$ and $\alpha_n + 2\delta > -1$. They are satisfied since $\alpha_j \geq 0$ for all $j = 1, \dots, n$ and $\delta \in [0, 1]$.

Similarly to relation (4.2.39), we see that

$$c_H^n \prod_{j=1}^{n-1} \Gamma\Big(\frac{1+\alpha_j}{2}\Big) \leq c_H C_{H,1}^{n-1} \leq \frac{1}{\pi}\big(\frac{1}{\sqrt{\pi}}\big)^{n-1}.$$

Since $\alpha_n \in \{0, 1-2H\}$ and $\delta \in [0,1]$,

$$\frac{1}{2} < \frac{1+2\delta}{2} \leq \frac{1+\alpha_n+2\delta}{2} \leq \frac{1+1-2H+2\delta}{2} < 2 - H < 2$$

we have

$$\Gamma\Big(\frac{1+\alpha_n+2\delta}{2}\Big) < \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}.$$

Hence

$$\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t}) \leq |z|^{2\delta} \Big(\frac{1}{\sqrt{\pi}}\Big)^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{2}} \big(t - t_{\rho(n)}\big)^{-\frac{1+\alpha_n+2\delta}{2}}$$
$$\leq |z|^{2\delta} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{2}} \big(t - t_{\rho(n)}\big)^{-\frac{1+\alpha_n+2\delta}{2}}. \tag{4.2.62}$$

Taking power $\frac{1}{2H_0}$ on both sides of (4.2.62) above, we obtain:

$$\Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \leq \big(|z|^{2\delta}\big)^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{-\frac{1+\alpha_j}{4H_0}} \big(t - t_{\rho(n)}\big)^{-\frac{1+\alpha_n+2\delta}{4H_0}}. \tag{4.2.63}$$

We now integrate $\big(\psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}$ over $[0,t]^n$. Using relation (4.2.63) and the change of variables $t_j' = t_{\rho(j)}$ for all $j = 1, \dots, n$, we get:

$$\int_{[0,t]^n} \Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} d\mathbf{t}$$
$$= \sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\dots<t_{\rho(n)}<t} \Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} d\mathbf{t}$$

$$\leq \left(|z|^{2\delta}\right)^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \sum_{\rho\in S_n} \int_{0<t'_1<\ldots<t'_n<t} \prod_{j=1}^{n-1} \left(t'_{j+1}-t'_j\right)^{-\frac{1+\alpha_j}{4H_0}} \left(t-t'_n\right)^{-\frac{1+\alpha_n+2\delta}{4H_0}} d\mathbf{t}$$

$$= |z|^{\frac{\delta}{H_0}} n! \sum_{\boldsymbol{\alpha}\in D_n} \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} \left(t_{j+1}-t_j\right)^{-\frac{1+\alpha_j}{4H_0}} \left(t-t_n\right)^{-\frac{1+\alpha_n+2\delta}{4H_0}} d\mathbf{t}. \quad (4.2.64)$$

Notice that the sum appearing in the previous estimate is the same as (4.2.22) in which we replace ε by $\delta/2$ in the exponent of $(t-t_n)$. Using the same argument as the one leading to relation (4.2.26) (and replacing ε by $\delta/2$), we obtain that for any $c_0 \in \left(0, 1-\frac{1-a}{2H_0}\right)$ and for any δ such that

$$0 < \frac{\delta}{2} < \frac{2H_0(1-c_0)+a-1}{2},$$

we have

$$\Gamma\Big(1-\frac{1+\alpha_n+2\delta}{4H_0}\Big) \leq \Gamma(c_0).$$

Hence, relation (4.2.30) in which we replace ε by $\delta/2$ becomes:

$$\int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} \left(t_{j+1}-t_j\right)^{-\frac{1+\alpha_j}{4H_0}} \left(t-t_n\right)^{-\frac{1+\alpha_n+2\delta}{4H_0}} d\mathbf{t}$$
$$\leq \frac{\left(c^{(5)}_{H_0,a}\right)^{n-1}\Gamma(c_0)\left(t\vee 1\right)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(6)}_{H_0,a})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}}. \quad (4.2.65)$$

where $c^{(5)}_{H_0,a}$ and $c^{(6)}_{H_0,a}$ are some positive constants depending on H_0 and a.

Exactly as in the case of relation (4.2.33), we obtain:

$$\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}C_n^H(t,z)\Big)^{1/2} \leq C^{(3)}_{p,a,b,H_0,T}\,|z|^{\delta}, \quad (4.2.66)$$

where $C^{(3)}_{p,a,b,H_0,T} > 0$ is a constant depending on p, a, b, H_0, and T. Therefore, taking power p and then taking supremum over $H \in [a,b]$ on both sides of relation (4.2.66), we have relation (4.2.2), using (4.2.54).

Step 2: Wave equation

Step 2.a We examine the time increments for the solution of the wave equation. Assume that $h = t'-t > 0$, and the case $h < 0$ is similar. Note that relation (4.2.7) still holds, where $A_n^H(t,h)$ and $B_n^H(t,h)$ are given by (4.2.10), respectively (4.2.34).

We study $A_n^H(t,h)$ first. Using Cauchy-Schwarz inequality and the fact that $ab \le (a^2+b^2)/2$, we have

$$\psi^H_{t,h,n}(\mathbf{t},\mathbf{s}) \le \Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\psi^H_{t,h,n}(\mathbf{s},\mathbf{s})\Big)^{1/2} \le \frac{1}{2}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})+\psi^H_{t,h,n}(\mathbf{s},\mathbf{s})\Big).$$

By Lemma 2.2.3, it follows that

$$\begin{aligned} A_n^H(t,h) &\le \int_{[0,t]^n}\int_{[0,t]^n} \alpha^n_{H_0}\prod_{j=1}^n |t_j-s_j|^{2H_0-2}\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\le \Gamma^n_{0,t}\int_{[0,t]^n}\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}, \end{aligned} \tag{4.2.67}$$

where $\Gamma_{0,t} = 2\int_0^t \alpha_{H_0}|t-s|^{2H_0-2}\mathrm{d}s = 2H_0t^{2H_0-1}$. We denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, for $j = 1,\dots,n-1$ and $u_n = t - t_{\rho(n)}$. Note that $\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})$ can be estimated by (4.2.15), in which G^h is replaced by G^w.

To study the second integral in (4.2.15), we need to consider α_n separately. Recall that $\alpha_n \in \{0, 1-2H\}$ and $h \in [0,1]$. To investigate the behaviour of various integral including the Fourier transform of G^w_t, we use arguments similar to those given in the proof of Proposition 7.4 of [17] (see also Lemma 2.3.4).

Case 1: If $\alpha_n = 0$, using Plancherel theorem, we have:

$$\begin{aligned} &\int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2\mathrm{d}\eta_n = 2\pi\int_{\mathbb{R}}\big|G^w_{u_n+h}(x)-G^w_{u_n}(x)\big|^2\mathrm{d}x \\ &\quad = 2\pi\int_{\mathbb{R}}\big(1_{\{|x|<u_n+h\}} - 1_{\{|x|<u_n\}}\big)^2\mathrm{d}x = 2\pi\int_{\mathbb{R}} 1_{\big\{[-u_n-h,-u_n]\cup[u_n,u_n+h]\big\}}\mathrm{d}x = 4\pi h. \end{aligned}$$

Case 2: If $\alpha_n = 1-2H$, we split $\mathbb{R}$ into $\{|\eta_n| \le 1\}$ and $\{|\eta_n| > 1\}$. Then

$$\int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2|\eta_n|^{1-2H}\mathrm{d}\eta_n = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_{|\eta_n|\le 1}\frac{\big|\sin\big((u_n+h)|\eta_n|\big)-\sin(u_n|\eta_n|)\big|^2}{|\eta_n|^2}|\eta_n|^{1-2H}\mathrm{d}\eta_n \\ I_2 &= \int_{|\eta_n|>1}\frac{\big|\sin\big((u_n+h)|\eta_n|\big)-\sin(u_n|\eta_n|)\big|^2}{|\eta_n|^2}|\eta_n|^{1-2H}\mathrm{d}\eta_n. \end{aligned}$$

We study I_1 first. Using the trigonometric identity

$$\sin(a)-\sin(b) = 2\sin\big(\frac{a-b}{2}\big)\cos\big(\frac{a+b}{2}\big)$$

and the fact that $\sin(x) \le x$ for all $x > 0$, we see that

$$\big|\sin\big((u_n+h)|\eta_n|\big) - \sin(u_n|\eta_n|)\big|^2 = 4\sin^2\Big(\frac{h|\eta_n|}{2}\Big)\cos^2\Big(\frac{(2u+h)|\eta_n|}{2}\Big)$$
$$\le 4\Big(\frac{h|\eta_n|}{2}\Big)^2\cos^2\Big(\frac{(2u+h)|\eta_n|}{2}\Big) \le h^2|\eta_n|^2.$$

Therefore

$$I_1 \le h^2 \int_{|\eta_n|\le 1} |\eta_n|^{1-2H} \mathrm{d}\eta_n = \frac{1}{1-H} h^2. \tag{4.2.68}$$

Next, we consider I_2. Writing $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$, we have

$$\Big|\frac{\sin\big((u_n+h)|\eta_n|\big) - \sin(u_n|\eta_n|)}{|\eta_n|}\Big|^2 = \frac{\big|e^{iu_n|\eta_n|}(e^{ih|\eta_n|}-1) - e^{-iu_n|\eta_n|}(e^{-ih|\eta_n|}-1)\big|^2}{4|\eta_n|^2}$$
$$\le \frac{\big|e^{iu_n|\eta_n|}\big|^2\big|e^{ih|\eta_n|}-1\big|^2 + \big|e^{-iu_n|\eta_n|}\big|^2\big|e^{-ih|\eta_n|}-1\big|^2}{2|\eta_n|^2} \le h^{2\delta}|\eta_n|^{2\delta-2},$$

where we used relation (4.2.60) in the last inequality. Hence

$$I_2 \le h^{2\delta} \int_{|\eta_n|>1} |\eta_n|^{2\delta-2H-1} \mathrm{d}\eta_n = \frac{1}{H-\delta}\, h^{2\delta}, \tag{4.2.69}$$

which is valid if we assume that $0 < \delta < H$. To ensure that this condition holds uniformly for all $H \in [a,b]$, we need to choose:

$$0 < \delta < a.$$

Combining relations (4.2.68) and (4.2.69), we obtain for any $H \in [a,b]$:

$$\int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2|\eta_n|^{1-2H}\mathrm{d}\eta_n \le \frac{1}{1-H}h^2 + \frac{1}{H-\delta}h^{2\delta}$$
$$\le \max\Big\{\frac{1}{1-H}, \frac{1}{H-\delta}\Big\}h^{2\delta} \le \max\Big\{\frac{1}{1-b}, \frac{1}{a-\delta}\Big\}h^{2\delta}.$$

Therefore, for any $\alpha_n \in \{0, 1-2H\}$ and $0 < \delta < a$, it follows that

$$\int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2|\eta_n|^{\alpha_n}\mathrm{d}\eta_n$$
$$\le \max\Big\{\int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2\mathrm{d}\eta_n, \int_{\mathbb{R}} \big|\mathcal{F}G^w_{u_n+h}(\eta_n) - \mathcal{F}G^w_{u_n}(\eta_n)\big|^2|\eta_n|^{1-2H}\mathrm{d}\eta_n\Big\}$$
$$\le \max\Big\{4\pi h, \max\Big\{\frac{1}{1-b}, \frac{1}{a-\delta}\Big\}h^{2\delta}\Big\} < c_{a,b,\delta}\, h^{2\delta}, \tag{4.2.70}$$

where $c_{a,b,\delta}$ is a constant depending on a, b and δ.

We return to relation (4.2.15) (with G^h replaced by G^w). Using Lemma 4.1.2 and relation (4.2.70), we obtain:

$$\psi^H_{t,h,n}(\mathbf{t},\mathbf{t}) \leq h^{2\delta}\, c_{a,b,\delta}\, c_H^n \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \Big(2^{1-\alpha_j} \widetilde{C}_{\alpha_j} u_j^{1-\alpha_j}\Big), \tag{4.2.71}$$

where we recall that

$$2^{1-\alpha_j}\widetilde{C}_{\alpha_j} = \begin{cases} \pi & \text{if } \alpha_j = 0, \\ 2^{2H} \frac{\Gamma(1-2H)\sin\big(\pi(1-2H)/2\big)}{2H} & \text{if } \alpha_j = 1-2H, \\ 2^{4H-1}\frac{\Gamma(2-4H)\sin\big(\pi(1-2H)\big)}{4H-1} & \text{if } \alpha_j = 2(1-2H). \end{cases}$$

For the application of Lemma 4.1.2, we need $-1 < \alpha_j < 1$ for $j = 1, \ldots, n-1$. This is clearly satisfied since $0 \leq \alpha_j \leq 2(1-2H) < 1$ for $j = 1, \ldots, n-1$, due to the condition $H > 1/4$.

Note that

$$c_H^n \prod_{j=1}^{n-1} 2^{1-\alpha_j}\widetilde{C}_{\alpha_j} \leq c_H C_{H,3}^{n-1} \tag{4.2.72}$$

where $C_{H,3}$ is given by

$$C_{H,3} = c_H \max\left\{\pi, \frac{\Gamma\big(1-2H\big)}{H}, \frac{2\Gamma\big(2-4H\big)}{4H-1}\right\}.$$

We now show how to find an upper bound for $C_{H,3}$, uniformly in $H \in [a,b]$. For any $H \in [a,b] \subset (1/4, 1/2)$, we have

$$\frac{\Gamma\big(1-2H\big)}{H} \leq \frac{\Gamma\big(1-2b\big)}{a} \quad \text{and} \quad \frac{2\Gamma\big(2-4H\big)}{4H-1} \leq \frac{2\Gamma\big(2-4b\big)}{4a-1}.$$

Hence

$$C_{H,3} \leq \frac{1}{\pi}\max\left\{\pi, \frac{\Gamma\big(1-2b\big)}{a}, \frac{2\Gamma\big(2-4b\big)}{4a-1}\right\} =: c_{a,b,3}. \tag{4.2.73}$$

We come back to relation (4.2.71). Using relations (4.2.72), (4.2.73) and the fact that $c_H \leq \frac{1}{\pi} < 1$, we have:

$$\psi^H_{t,h,n}(\mathbf{t},\mathbf{t}) \leq h^{2\delta}\, c_{a,b,\delta}\, c_H C_{H,3}^{n-1} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} u_j^{1-\alpha_j}$$

$$\leq h^{2\delta}\, c_{a,b,\delta} c_{a,b,3}^{n-1} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \left(t_{\rho(j+1)} - t_{\rho(j)}\right)^{1-\alpha_j}. \tag{4.2.74}$$

Note that (4.2.74) is the analogue of (4.2.20) (that we obtained for the heat equation).

We now integrate $\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})$ over $[0,t]^n$. Using (4.2.74) and the change of variables $t'_j = t_{\rho(j)}$ for all $j = 1,\dots,n$, we get:

$$\begin{aligned}
\int_{[0,t]^n} \psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\,\mathrm{d}\mathbf{t} &= \sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t} \psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\,\mathrm{d}\mathbf{t} \\
&\leq h^{2\delta}\, c_{a,b,\delta}\, c_{a,b,3}^{n-1} \sum_{\boldsymbol{\alpha}\in D_n} \sum_{\rho\in S_n} \int_{0<t'_1<\ldots<t'_n<t} \prod_{j=1}^{n-1} \left(t'_{j+1} - t'_j\right)^{1-\alpha_j} \mathrm{d}t'_1 \ldots \mathrm{d}t'_{n-1}\mathrm{d}t'_n \\
&= h^{2\delta}\, c_{a,b,\delta}\, c_{a,b,3}^{n-1}\, n! \sum_{\boldsymbol{\alpha}\in D_n} \int_{0<t_n<t} J^w(t_n)\mathrm{d}t_n,
\end{aligned} \tag{4.2.75}$$

where in the first equation, we decomposed the set $[0,t]^n$ into $n!$ disjoint regions of the form $t_{\rho(1)} < \cdots < t_{\rho(n)}$ with $\rho \in S_n$ and we used notation:

$$J^w(t_n) = \int_{0<t_1<\ldots<t_{n-1}<t_n} \prod_{j=1}^{n-1} \left(t_{j+1} - t_j\right)^{1-\alpha_j} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1}.$$

We give the explicit form of $J^w(t_n)$. For this, we apply Lemma A.3.1 with $\beta_j = 1 - \alpha_j$ for all $j = 1,\dots,n-1$. Note that

$$\begin{aligned}
\sum_{j=1}^{n-1} \beta_j = \sum_{j=1}^{n-1}(1-\alpha_j) = (n-1) - \sum_{j=1}^{n-1} \alpha_j = (n-1) - \Big(\sum_{j=1}^{n} \alpha_j - \alpha_n\Big) \\
= (n-1) - \Big(n(1-2H) - \alpha_n\Big) = 2Hn - 1 + \alpha_n
\end{aligned}$$

and

$$\sum_{j=1}^{n-1} \beta_j + (n-1) = 2Hn - 1 + \alpha_n + (n-1) = n(2H+1) + (\alpha_n - 2).$$

To apply Lemma A.3.1, wee need $\beta_j > -1$ for all $j = 1,\dots,n-1$, which means $\alpha_j < 2$ for all $j = 1,\dots,n-1$. This is clearly satisfied since $H < 1/2$. Therefore,

$$J^w(t_n) = \int_{0<t_1<\ldots<t_{n-1}<t_n} \prod_{j=1}^{n-1} (t_{j+1} - t_j)^{1-\alpha_j} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1}$$

$$= \frac{\prod_{j=1}^{n-1} \Gamma\big(\beta_j+1\big)\ t_n^{\sum_{j=1}^{n-1}\beta_j+(n-1)}}{\Gamma\Big(\sum_{j=1}^{n-1}\beta_j+(n-1)+1\Big)} = \frac{\prod_{j=1}^{n-1} \Gamma\big(2-\alpha_j\big)\ t_n^{n(2H+1)+(\alpha_n-2)}}{\Gamma\Big(n(2H+1)+(\alpha_n-1)\Big)}. \quad (4.2.76)$$

Hence

$$\int_{0<t_n<t} J^w(t_n)\,\mathrm{d}t_n = \frac{\prod_{j=1}^{n-1} \Gamma\big(2-\alpha_j\big)\, t^{n(2H+1)+(\alpha_n-1)}}{\Gamma\Big(n(2H+1)+\alpha_n\Big)}. \quad (4.2.77)$$

Recall that $\alpha_1 \in \big\{1-2H, 2(1-2H)\big\}$, $\alpha_j \in \big\{0, 1-2H, 2(1-2H)\big\}$ for $j = 2, \ldots, n-1$ and $\alpha_n \in \big\{0, 1-2H\big\}$ (see Lemma 4.1.3 with $\alpha_j = (1-2H)a_j$). Therefore, $0 \le \alpha_j \le 2(1-2H)$ for all $j = 1, \ldots, n-1$ which implies that for any $H \in [a, b]$

$$1 < 4a \le 4H \le 2 - \alpha_j \le 2.$$

Because the Gamma function Γ is decreasing on $(0, x_0)$ and increasing on (x_0, ∞) with $x_0 \approx 1.4$, we know that

$$\prod_{j=1}^{n-1} \Gamma\big(2-\alpha_j\big) \le 1. \quad (4.2.78)$$

Moreover, we have

$$n(2a+1)-1 \le n(2H+1)-1 \le n(2H+1)+(\alpha_n-1) \le n(2H+1)-2H \le n(2b+1)-2a.$$

It follows that

$$t^{n(2H+1)+(\alpha_n-1)} \le \begin{cases} 1 & \text{if } t < 1, \\ t^{n(2b+1)-2a} & \text{if } t \ge 1. \end{cases}$$

Therefore, we get:

$$t^{n(2H+1)+(\alpha_n-1)} \le \big(t \vee 1\big)^{n(2b+1)-2a}. \quad (4.2.79)$$

Finally, we pick $m_0 \ge 1$ such that $m_0(2a+1)-1 > x_0$. For any $n \ge m_0$, we have

$$n(2H+1)+(\alpha_n-1) > n(2a+1)-1 > x_0,$$

which implies that for any $H \in [a, b]$,

$$\Gamma\Big(n(2H+1)+(\alpha_n-1)\Big) > \Gamma\Big(n(2a+1)-1\Big).$$

By Lemma A.1.3, we know that there exists a positive constant $c_a^{(7)}$ depending on a such that

$$\Gamma\Big(n(2H+1)+\alpha_n\Big) > \Gamma\Big(n(2a+1)\Big) \ge (c_a^{(7)})^n \big[n!\big]^{2a+1}. \quad (4.2.80)$$

We come back to (4.2.77). Using relations (4.2.78), (4.2.79) and (4.2.80), we obtain:

$$\int_{0<t_n<t} J^w(t_n)\mathrm{d}t_n \leq \frac{\big(t\vee 1\big)^{n(2b+1)-2a}}{(c_a^{(7)})^n\big[n!\big]^{2a+1}}. \tag{4.2.81}$$

We return to equality (4.2.75). By relation (4.2.81) and the fact that $\mathrm{card}(D_n) = 2^{n-1}$, then

$$\int_{[0,t]^n} \psi_{t,h,n}^H(\mathbf{t},\mathbf{t})\,\mathrm{d}\mathbf{t} \leq h^{2\delta}\, c_{a,b,\delta}\, c_{a,b,3}^{n-1}\, n!\, 2^{n-1} \frac{\big(t\vee 1\big)^{n(2b+1)-2a}}{(c_a^{(7)})^n\big[n!\big]^{2a+1}} \tag{4.2.82}$$

Therefore, by relations (4.2.67) and (4.2.82), we conclude that

$$\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A_n^H(t,h)\Big)^{1/2} \leq := C_{p,a,b,\delta,T}^{(4)}\, h^{\delta}, \tag{4.2.83}$$

where $C_{p,a,b,\delta,T}^{(4)} > 0$ is a constant depending on p, a, b, δ, and T.

We now study $B_n^H(t,h)$. Recall that $B_n^H(t,h)$ is given by (4.2.34). Using Cauchy-Schwarz inequality and the fact that $ab \leq (a^2+b^2)/2$ again, we have

$$\gamma_{t,h,n}^H(\mathbf{t},\mathbf{s}) \leq \Big(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\gamma_{t,h,n}^H(\mathbf{s},\mathbf{s})\Big)^{1/2} \leq \frac{1}{2}\Big(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t}) + \gamma_{t,h,n}^H(\mathbf{s},\mathbf{s})\Big).$$

By Lemma 2.2.3, it follows that

$$\begin{aligned} B_n^H(t,h) &\leq \int_{[0,t+h]^n}\int_{[0,t+h]^n} \alpha_{H_0}^n \prod_{j=1}^n |t_j-s_j|^{2H_0-2}\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq \Gamma_{0,t}^n \int_{[0,t+h]^n} \gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}. \end{aligned} \tag{4.2.84}$$

Note that relation (4.2.37) still holds, in which G^h is replaced by G^w. Using Lemma 4.1.2, we obtain:

$$\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t}) \leq c_H^n \sum_{\boldsymbol{\alpha}\in D_n}\Big(\prod_{j=1}^{n-1} 2^{1-\alpha_j}\widetilde{C}_{\alpha_j} u_j^{1-\alpha_j}\Big)\Big(2^{1-\alpha_n}\widetilde{C}_{\alpha_n}(u_n+h)^{1-\alpha_n}\Big). \tag{4.2.85}$$

For the application of Lemma 4.1.2, we need $-1<\alpha_j<1$ for $j=1,\dots,n$. This is clearly satisfied since $0\leq \alpha_j \leq 2(1-2H)\leq 1$ for all $j=1,\dots,n$, due to the fact that $H\geq \frac{1}{4}$.

Note that

$$c_H^n \prod_{j=1}^n 2^{1-\alpha_j}\widetilde{C}_{\alpha_j} \leq C_{H,3}^n \leq c_{a,b,3}^n.$$

where $C_{H,3}$ and $c_{a,b,3}$ is given by (4.2.72), respectively (4.2.73). Then, we obtain the following inequality, which is the analogue of relation (4.2.40) (that we obtained for the heat equation):

$$\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t}) \leq c^n_{a,b,3} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} u_j^{1-\alpha_j} \cdot (u_n+h)^{1-\alpha_n}. \tag{4.2.86}$$

We now integrate $\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})$ over $[0,t+h]^n$. Using relation (4.2.86) and the change of variables $t'_j = t_{\rho(j)}$ for all $j=1,\dots,n$, we get:

$$\begin{aligned}
&\int_{[0,t+h]^n} \gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t} \\
&\quad = \sum_{\rho\in S_n} \int_t^{t+h} \left(\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}} \gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\, \mathrm{d}t_{\rho(1)}\ldots \mathrm{d}t_{\rho(n-1)} \right) \mathrm{d}t_{\rho(n)} \\
&\quad \leq c^n_{a,b,3} \sum_{\rho\in S_n} \int_t^{t+h} \Bigg(\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}} \\
&\qquad\qquad \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} (t_{\rho(j+1)}-t_{\rho(j)})^{1-\alpha_j} (t+h-t_{\rho(n)})^{1-\alpha_n} \mathrm{d}t_{\rho(1)}\ldots \mathrm{d}t_{\rho(n-1)} \Bigg) \mathrm{d}t_{\rho(n)} \\
&\quad \leq c^n_{a,b,3}\, n! \sum_{\boldsymbol{\alpha}\in D_n} \int_t^{t+h} J^w(t_n)\cdot (t+h-t_n)^{1-\alpha_n} \mathrm{d}t_n
\end{aligned} \tag{4.2.87}$$

where $J^w(t_n)$ is given by (4.2.76). We would like to find an upper bound for $J^w(t_n)$ uniformly in $H\in[a,b]$. Recalling that $\alpha_n \in \{0, 1-2H\}$, we have

$$n(2H+1)+(\alpha_n-2) \leq (n-1)(2H+1) \leq (n-1)(2b+1).$$

It follows that

$$t_n^{n(2H+1)+(\alpha_n-2)} \leq \begin{cases} 1 & \text{if } t_n<1 \\ t_n^{(n-1)(2b+1)} & \text{if } t_n \geq 1 \end{cases} \leq (T\vee 1)^{(n-1)(2b+1)}. \tag{4.2.88}$$

Moreover, we have

$$n(2H+1)+(\alpha_n-1) \geq n(2H+1)-1 \geq n(2a+1)-1.$$

We pick $m_0 \geq 1$ such that $m_0(2a+1)-1 > x_0$. For any $n \geq m_0$, we have

$$n(2H+1)+(\alpha_n-1) \geq (n-1)(2a+1)+2a > x_0,$$

which implies that for any $H \in [a,b]$,

$$\Gamma\Big(n(2H+1)+(\alpha_n-1)\Big) > \Gamma\Big((n-1)(2a+1)+2a\Big) \geq (c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a+1}, \tag{4.2.89}$$

where we applied Lemma A.1.4 for the last inequality and $c_a^{(8)}$ is a constant depending on a. Using relations (4.2.78), (4.2.88), (4.2.89), we know that

$$J^w(t_n) \leq \frac{(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a+1}}, \tag{4.2.90}$$

which is the analogue of relation (4.2.50) that we obtained for the heat equation.

We return to (4.2.87). By (4.2.90) and the change of variable $s = t+h-t_n$, we have

$$\begin{aligned}
&\int_{[0,t+h]^n} \gamma^H_{t,h,n}(\mathbf{t},\mathbf{t}) 1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t} \\
&\leq c^n_{a,b,3}\, n! \frac{(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a+1}} \sum_{\boldsymbol{\alpha}\in D_n} \int_t^{t+h} (t+h-t_n)^{1-\alpha_n}\mathrm{d}t_n \\
&\leq c^n_{a,b,3}\, 2^{n-1}(n-1)!\, \frac{(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a+1}} \sum_{\boldsymbol{\alpha}\in D_n} \int_0^h s^{1-\alpha_n}\mathrm{d}s \\
&= c^n_{a,b,3}\, \frac{2^n(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a}} \sum_{\boldsymbol{\alpha}\in D_n} \frac{1}{2-\alpha_n} h^{2-\alpha_n},
\end{aligned} \tag{4.2.91}$$

where we used the fact that $n! \leq 2^{n-1}(n-1)!$ in the second inequality. Note that, since $\alpha_n \in \{0, 1-2H\}$ and $H \in [a,b]$, we have

$$2a+1 \leq 2H+1 \leq 2-\alpha_n \leq 2,$$

which implies

$$\frac{1}{2} \leq \frac{1}{2-\alpha_n} \leq \frac{1}{2a+1} < 2$$

and for any $h \in (0,1)$ and $\delta \in (0,a)$,

$$h^{2-\alpha_n} \leq h^{2a+1} \leq h^{2\delta+1} \leq h^{2\delta}.$$

Therefore, using the fact that $\mathrm{Card}(D_n) = 2^{n-1}$ and relation (4.2.91), we obtain:

$$\int_{[0,t+h]^n} \gamma^H_{t,h,n}(\mathbf{t},\mathbf{t}) 1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t} \leq c^n_{a,b,3}\, \frac{2^n(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a}} \sum_{\boldsymbol{\alpha}\in D_n} 2h^{2\delta}$$

$$\leq h^{2\delta}4^n c_{a,b,3}^n \frac{(T\vee 1)^{(n-1)(2b+1)}}{(c_a^{(8)})^{n-1}\big[(n-1)!\big]^{2a}}. \tag{4.2.92}$$

Hence, by relations (4.2.84) and (4.2.92), we conclude that

$$\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}B_n^H(t,h)\Big)^{1/2} \leq C_{p,a,b,T}^{(5)}\, h^{\delta}, \tag{4.2.93}$$

where $C_{p,a,b,T}^{(5)} > 0$ is a constant depending on p, a, b and T.

Relation (4.2.4) follows by combining relations relation (4.2.7), (4.2.83) and (4.2.93).

Step 2.b We now examine the space increments for the solution of the wave equation. Note that relation (4.2.54) still holds where $C_n^H(t,z)$ is given by (4.2.55). Using Cauchy-Schwarz inequality and the fact that $ab \leq (a^2+b^2)/2$, we have

$$\Psi_{t,z,n}^H(\mathbf{t},\mathbf{s}) \leq \Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\Big)^{1/2}\Big(\Psi_{t,z,n}^H(\mathbf{s},\mathbf{s})\Big)^{1/2} \leq \frac{1}{2}\Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t}) + \Psi_{t,z,n}^H(\mathbf{s},\mathbf{s})\Big).$$

By Lemma 2.2.3, it follows that

$$\begin{aligned} C_n^H(t,z) &\leq \int_{[0,t]^n}\int_{[0,t]^n} \alpha_{H_0}^n \prod_{j=1}^n |t_j - s_j|^{2H_0-2}\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \\ &\leq \Gamma_{0,t}^n \int_{[0,t]^n} \Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}. \end{aligned} \tag{4.2.94}$$

Note that (4.2.59) still holds, with G^h replaced by G^w. Hence, for any $\delta \in [0,1]$,

$$\begin{aligned} &\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t}) \\ &\leq |z|^{2\delta}c_H^n \sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}\Big(\int_{\mathbb{R}} \frac{|\sin(u_j|\eta_j|)|^2}{|\eta_j|^2}|\eta_j|^{\alpha_j}\mathrm{d}\eta_j\Big)\Big(\int_{\mathbb{R}} \frac{|\sin(u_n|\eta_n|)|^2}{|\eta_n|^2}|\eta_n|^{\alpha_n+2\delta}\mathrm{d}\eta_n\Big) \\ &= |z|^{2\delta}c_H^n \sum_{\boldsymbol{\alpha}\in D_n}\prod_{j=1}^{n-1}\Big(2^{1-\alpha_j}\widetilde{C}_{\alpha_j}u_j^{1-\alpha_j}\Big)\cdot\Big(2^{1-\alpha_n-2\delta}\widetilde{C}_{\alpha_n+2\delta}\, u_n^{1-\alpha_n-2\delta}\Big), \end{aligned} \tag{4.2.95}$$

where in the last equation, we used Lemma 4.1.2. Note that (4.2.95) is the analogue of relation (4.2.61) that we have obtained for the heat equation. For the application of Lemma 4.1.2, we need $\alpha_j \in (-1,1)$ for $j=1,\dots,n-1$ and $\alpha_n + 2\delta \in (-1,1)$. The first condition is satisfied since $0 \leq \alpha_j \leq 2(1-2H) \leq 1$ for all $j=1,\dots,n-1$ due to our condition $H > 1/4$. However, the condition $\alpha_n + 2\delta < 1$ introduces the restriction $0 < \delta < H$ in the case when $\alpha_n = 1-2H$. To ensure that this holds for any $H \in [a,b]$, we need to take

$$0 < \delta < a.$$

Similarly to relation (4.2.72), we see that

$$c_H^n \prod_{j=1}^{n-1} 2^{1-\alpha_j} \widetilde{C}_{\alpha_j} \le c_H C_{H,3}^{n-1} \le \frac{1}{\pi} c_{a,b,3}^{n-1} < c_{a,b,3}^{n-1}.$$

Since $\alpha_n \in \{0, 1-2H\}$ and $\delta \in (0, H)$,

$$1 = 2^0 < 2^{1-\alpha_n - 2\delta} < 2^1 = 2$$

and $\alpha_n + 2\delta \in (0,1)$, we have

$$\widetilde{C}_{\alpha_n+2\delta} = \frac{1}{1-\alpha_n-2\delta} \Gamma\big(\alpha_n + 2\delta\big) \sin\Big(\frac{\pi(\alpha_n+2\delta)}{2}\Big).$$

To bound $\Gamma\big(\alpha_n+2\delta\big)$, we need to study the range of possible values $\alpha_n + 2\delta$. If $\alpha_n = 1-2H$, then $\alpha_n + 2\delta \ge 1 - 2H \ge 1-2b$ and $\Gamma(\alpha_n+2\delta) \le \Gamma(1-2b)$. The problem is when $\alpha_n = 0$. If we simply choose $\delta \in (0,H)$, then $\alpha_n + 2\delta = 2\delta$ can be in principle, be very close to 0. This is a problem since $\lim\limits_{x\to 0^+} \Gamma(x) = \infty$, and hence, we are not able to bound the term $\Gamma\big(\alpha_n+2\delta\big)$. To avoid this problem, we choose an arbitrary value $c_1 > 0$ such that

$$c_1 < 2\delta.$$

With this choice of c_1 and δ, we have

$$\Gamma\big(\alpha_n+2\delta\big) \le \max\big\{\Gamma(c_1), \Gamma(1-2b)\big\} =: k$$

and

$$\frac{1}{1-\alpha_n-2\delta} \le \frac{1}{2H-c_1} \le \frac{1}{2a-c_1}$$

Therefore,

$$\widetilde{C}_{\alpha_n+2\delta} \le \frac{k}{2a-c_1}.$$

Hence, relation (4.2.95) becomes:

$$\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t}) \le |z|^{2\delta}\, c_{a,b,3}^{n-1}\, \frac{k}{2a-c_1} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{j=1}^{n-1} \big(t_{\rho(j+1)} - t_{\rho(j)}\big)^{1-\alpha_j} \big(t - t_{\rho(n)}\big)^{1-\alpha_n-2\delta}. \tag{4.2.96}$$

We now integrate $\psi^H_{t,z,n}(\mathbf{t},\mathbf{t})$ over $[0,t]^n$. Using relation (4.2.96) and the change of variables $t'_j = t_{\rho(j)}$ for all $j=1,\dots,n$, we get:

$$\begin{aligned}
\int_{[0,t]^n} \Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} &= \sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\dots<t_{\rho(n)}<t} \Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} \\
&\le |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1} \\
&\qquad \sum_{\boldsymbol{\alpha}\in D_n}\sum_{\rho\in S_n}\int_{0<t'_1<\dots<t'_n<t}\prod_{j=1}^{n-1}\big(t'_{j+1}-t'_j\big)^{1-\alpha_j}\big(t-t'_n\big)^{1-\alpha_n-2\delta}\mathrm{d}\mathbf{t} \\
&= |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1}\, n!\sum_{\boldsymbol{\alpha}\in D_n}\int_{0<t_1<\dots<t_n<t}\prod_{j=1}^{n-1}\big(t_{j+1}-t_j\big)^{1-\alpha_j}\big(t-t_n\big)^{1-\alpha_n-2\delta}\mathrm{d}\mathbf{t} \\
&= |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1}\, n!\sum_{\boldsymbol{\alpha}\in D_n}\int_{0<t_n<t} J^w(t_n)\big(t-t_n\big)^{1-\alpha_n-2\delta}\mathrm{d}t_n, \qquad (4.2.97)
\end{aligned}$$

where $J^w(t_n)$ is given in (4.2.76).

Using relation (4.2.90) and the fact that $\mathrm{Card}(D_n) = 2^{n-1}$, we have:

$$\begin{aligned}
&\int_{[0,t]^n} \Psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t} \\
&\le |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1}\, n!\frac{(T\vee 1)^{(n-1)(2b+1)}}{(c^{(8)}_a)^{n-1}\big[(n-1)!\big]^{2a+1}}\sum_{\boldsymbol{\alpha}\in D_n}\int_{0<t_n<t}\big(t-t_n\big)^{1-\alpha_n-2\delta}\mathrm{d}t_n \\
&\le |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1}\, 2^n(n-1)!\,\frac{(T\vee 1)^{(n-1)(2b+1)}}{(c^{(8)}_a)^{n-1}\big[(n-1)!\big]^{2a+1}}\sum_{\boldsymbol{\alpha}\in D_n}\frac{1}{2-\alpha_n-2\delta}t^{2-\alpha_n-2\delta} \\
&\le |z|^{2\delta}\, c^{n-1}_{a,b,3}\,\frac{k}{2a-c_1}\,\frac{2^n(T\vee 1)^{(n-1)(2b+1)}}{(c^{(8)}_a)^{n-1}\big[(n-1)!\big]^{2a}}2^{n-1}\frac{1}{2a-c_1}(T\vee 1)^2
\end{aligned}$$

where in the last inequality, we used the fact that for any $H\in[a,b]$ and $\delta<c_1$,

$$\frac{1}{2-\alpha_n-2\delta} < \frac{1}{1-\alpha_n-2\delta} \le \frac{1}{2H-c_1} \le \frac{1}{2a-c_1}$$

and

$$t^{2-\alpha_n-2\delta} \le \begin{cases} 1 & \text{if } t<1 \\ t^2 & \text{if } t\ge 1\end{cases} \le (T\vee 1)^2.$$

Exactly as in the case of relation (4.2.83), we obtain:

$$\sum_{n\ge 1}(p-1)^{n/2}\Big(\frac{1}{n!}C^H_n(t,z)\Big)^{1/2} \le C^{(6)}_{p,a,b,k,c_1,T}\,|z|^{\delta},$$

where $C^{(6)}_{p,a,b,k,c_1,T} > 0$ is a constant depending on p, a, b, k, c_1, and T. Relation (4.2.5) follows.

∎

4.3 Continuity in law of the solution with respect to the noise parameter H

In this section, we consider equations (4.0.1) and (4.0.2) driven by the noise W^H introduced in Section 4.1, which is fractional in time with index $H_0 \in (1/2, 1)$ and fractional in space with index $H \in (0, 1/2)$. We prove that the solution of either one of these equations is continuous in law in the space of continuous functions $C([0,T]\times\mathbb{R})$, with respect to the spatial Hurst index H.

As in the proof of Theorem 2.0.1, we apply Lemma 2.4.3 to prove the convergence of the finite dimensional distribution of u^{H_n} to u^{H^*}, when $n \to \infty$. For this, we need the whole family of processes $\{W^H; H \in (0, 1/2)\}$ to be defined on the same probability space. To do this, we give the simultaneous construction of all noise processes $\{W^H; H \in (0, 1/2)\}$ on the same probability space, which has a spectral representation in terms of the complex-valued Gaussian measure $\widehat{W}$ given by definition 2.4.1.

Before we begin, note that for any function $\phi \in \mathcal{S}(\mathbb{R})$,

$$\alpha_{H_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(t)\phi(s)|t-s|^{2H_0-2}\mathrm{d}t\mathrm{d}s = c_{H_0} \int_{\mathbb{R}} \left|\mathcal{F}\phi(\tau)\right|^2 |\tau|^{1-2H_0}\mathrm{d}\tau,$$

where $\alpha_{H_0} = H_0(2H_0-1)$ and $c_{H_0} = \frac{\Gamma(2H_0+1)\sin(\pi H_0)}{2\pi}$. This relationship can be extended to higher dimensions, as follows:

$$\alpha^n_{H_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(t_1,\ldots,t_n)\phi(s_1,\ldots,s_n) \prod_{j=1}^n |t_j - s_j|^{2H_0-2}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}$$
$$= c^n_{H_0} \int_{\mathbb{R}^n} \left|\mathcal{F}\phi(\tau_1,\ldots,\tau_n)\right|^2 \prod_{j=1}^n |\tau_j|^{1-2H_0}\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n. \tag{4.3.1}$$

Similarly to (2.4.1) with $g_0(\tau)$ replaced by $c_{H_0}|\tau|^{1-2H_0}$ and $|\xi|^{-\alpha}$ replaced by $c_H|\xi|^{1-2H}$, for any $H \in (0, 1/2)$ and for any function $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we set

$$\widehat{W}^H(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(\tau,\xi)\widehat{W}^H(\mathrm{d}\tau,\mathrm{d}\xi)$$

$$
\begin{aligned}
&:= \sqrt{c_{H_0}}\sqrt{c_H} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(\tau, \xi) |\tau|^{\frac{1}{2}-H_0} |\xi|^{\frac{1}{2}-H} \widehat{W}(\mathrm{d}\tau, \mathrm{d}\xi) \\
&= \widehat{W}\Big(\sqrt{c_{H_0}}\sqrt{c_H} \varphi(\tau, \xi) |\tau|^{\frac{1}{2}-H_0} |\xi|^{\frac{1}{2}-H} \Big).
\end{aligned}
$$

Formally, we can say

$$
\widehat{W}^H(\mathrm{d}\tau, \mathrm{d}\xi) = \sqrt{c_{H_0}}\sqrt{c_H} |\tau|^{\frac{1}{2}-H_0} |\xi|^{\frac{1}{2}-H} \widehat{W}(\mathrm{d}\tau, \mathrm{d}\xi).
$$

Similarly to (2.4.2), we define

$$
W^H(\varphi) = \widehat{W}^H(\mathcal{F}\varphi). \tag{4.3.2}
$$

Then we have

$$
\begin{aligned}
\mathbb{E}\Big[W^H(\varphi) \overline{W^H(\psi)} \Big] &= \mathbb{E}\Big[\widehat{W}^H(\mathcal{F}\varphi) \overline{\widehat{W}^H(\mathcal{F}\psi)} \ \Big] \\
&= c_{H_0} c_H \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}\varphi(\tau, \xi) \overline{\mathcal{F}\psi(\tau, \xi)} |\tau|^{1-2H_0} |\xi|^{1-2H} \mathrm{d}\tau \mathrm{d}\xi. \qquad (4.3.3)
\end{aligned}
$$

That means that we can construct isonormal Gaussian processes $W^H = \{W^H(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ which have exactly the covariance structure (4.1.1) and are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout this section, without any loss of generality, we assume the noise perturbing equations (2.0.1) and (2.0.2) is determined by the Gaussian process W^H.

Note that relation (4.3.2) can be written as

$$
\begin{aligned}
\int_{\mathbb{R}_+ \times \mathbb{R}} \varphi(t, x) W^H(\mathrm{d}t, \mathrm{d}x) &= \int_{\mathbb{R}^2} \mathcal{F}\varphi(\tau, \xi) \widehat{W}^H(\mathrm{d}\tau, \mathrm{d}\xi) \\
&= \int_{\mathbb{R}^2} \mathcal{F}_t \big[\mathcal{F}_x \varphi(t, \cdot)(\xi) \big](\tau) \widehat{W}^H(\mathrm{d}\tau, \mathrm{d}\xi). \qquad (4.3.4)
\end{aligned}
$$

Recall that the second moment of the multiple integral with respect to $\widehat{W}$ is given by relation (2.4.4) when $d = 1$.

In Section 4.1, we proved the solution u^H has the Wiener chaos expansion (4.1.3). By definition, $u^H(t, x)$ is the $L^2(\Omega)$-limit of the sequence $\{u^H_m(t, x)\}_{m \geq 1}$ defined by

$$
u^H_m(t, x) = \sum_{n=0}^{m} I^H_n(f_n(\cdot, t, x)). \tag{4.3.5}
$$

This means that

$$
\mathbb{E}\big| u^H_m(t, x) - u^H(t, x) \big|^2 \to 0, \quad \text{as } m \to \infty \tag{4.3.6}
$$

for any $H \in (0, 1/2)$ fixed.

The proof of Theorem 4.0.1 is based on the following result.

Lemma 4.3.1. *Under the hypothesis of Theorem 4.0.1, for all $k \geq 1$ fixed, we have*

$$\mathbb{E}\Big|I_k^{H_n}(f_k(\cdot,t,x)) - I_k^H(f_k(\cdot,t,x))\Big|^2 \to 0, \ \text{as } n \to \infty. \tag{4.3.7}$$

Proof: We denote $\mathbf{t} = (t_1, \ldots, t_k)$, $\mathbf{s} = (t_1, \ldots, t_k)$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_j)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_j)$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_j)$. Similarly to (2.4.9), we have

$$I_k^{H_n}(f_k(\cdot,t,x)) - I_k^H(f_k(\cdot,t,x)) = \int_{\mathbb{R}^{2k}} H_k(\tau_1, \xi_1, \ldots, \tau_k, \xi_k)\widehat{W}(\mathrm{d}\tau_1, \mathrm{d}\xi_1)\cdots\widehat{W}(\mathrm{d}\tau_k, \mathrm{d}\xi_k),$$

where

$$\begin{aligned}H_k(\tau_1, \xi_1, \ldots, \tau_k, \xi_k) &= \mathcal{F}_t\big[\mathcal{F}_x f_k(t_1, \cdot, \ldots, t_k, \cdot, t, x)(\xi_1, \ldots, \xi_k)\big](\tau_1, \ldots, \tau_k) \\ &\qquad (c_{H_0})^{k/2}\prod_{j=1}^k |\tau_j|^{\frac{1}{2}-H_0}\Big(c_{H_n}^{k/2}\prod_{j=1}^k |\xi_j|^{\frac{1}{2}-H_n} - c_H^{k/2}\prod_{j=1}^k |\xi_j|^{\frac{1}{2}-H}\Big).\end{aligned}$$

Similarly to relation (2.4.12), we have

$$\begin{aligned}Q_n &:= \mathbb{E}\big|I_k^{H_n}(f_k(\cdot,t,x)) - I_k^H(f_k(\cdot,t,x))\big|^2 \\ &\leq k!\int_{T_k(t)}\int_{T_k(t)}\prod_{j=1}^k \alpha_{H_0}|t_j - s_j|^{2H_0-2}A_k^{(n)}(\mathbf{t},\mathbf{s})\,\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s},\end{aligned}$$

where

$$\begin{aligned}A_k^{(n)}(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^k} &\mathcal{F}_x f_k(t_1, \cdot, \ldots, t_k, \cdot, t, x)(\xi_1, \ldots, \xi_k)\overline{\mathcal{F}_x f_k(s_1, \cdot, \ldots, s_k, \cdot, t, x)(\xi_1, \ldots, \xi_k)} \\ &\left|c_{H_n}^{k/2}\prod_{j=1}^k \big|\xi_j\big|^{\frac{1}{2}-H_n} - c_H^{k/2}\prod_{j=1}^k \big|\xi_j\big|^{\frac{1}{2}-H}\right|^2 \mathrm{d}\boldsymbol{\xi}.\end{aligned} \tag{4.3.8}$$

Using Cauchy-Schwartz inequality, we obtain

$$Q_n \leq k!\int_{T_k(t)}\int_{T_k(t)}\prod_{j=1}^k \alpha_{H_0}|t_j - s_j|^{2H_0-2}\big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2}\big(A_k^{(n)}(\mathbf{s},\mathbf{s})\big)^{1/2}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}. \tag{4.3.9}$$

Recall that we require $H > 3/4 - H_0$ for heat equation, and $H > 1/4$ for wave equation. For this reason, we consider separately the heat and wave equation. Since the condition $H > 1/4$ does not depend on H_0, the argument for the wave equation can be reduced to the case of the white noise in time, treated in [23]. For heat equation, we need to derive a new argument.

Step 1: We consider the wave equation.

In this case, using relation (4.3.9), inequality $ab \leq \frac{1}{2}(a^2+b^2)$ and Lemma 2.2.3 with $\gamma_0(t) = \alpha_{H_0}|t|^{2H_0-2}$, we have

$$Q_n \leq k!\,\Gamma_{0,t}^k \int_{T_k(t)} A_k^{(n)}(\mathbf{t},\mathbf{t})\mathrm{d}\mathbf{t}$$
$$= k!\,\Gamma_{0,t}^k \int_{T_k(t)} \int_{\mathbb{R}^k} \left|\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k)\right|^2$$
$$\left|c_{H_n}^{k/2}\prod_{j=1}^k \left|\xi_j\right|^{\frac{1}{2}-H_n} - c_H^{k/2}\prod_{j=1}^k \left|\xi_j\right|^{\frac{1}{2}-H}\right|^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\mathbf{t}$$

where $\Gamma_{0,t} = 2\alpha_{H_0}\int_0^t |t-s|^{2H_0-2}\mathrm{d}s = 2H_0 t^{2H_0-1}$. Recall that if $0 < t_1 < \ldots < t_k < t$, then

$$\mathcal{F}_x f_k(t_1,\cdot,\ldots,t_k,\cdot,t,x)(\xi_1,\ldots,\xi_k) = e^{-i(\xi_1+\ldots+\xi_k)x}\prod_{j=1}^k \mathcal{F}G_{t_{j+1}-t_j}(\xi_1+\ldots+\xi_j).$$

Using the change of variables $\eta_j = \xi_1+\ldots+\xi_j$ for all $j = 1,\ldots,k$ with $\eta_0 = 0$, we obtain:

$$Q_n \leq k!\,\Gamma_{0,t}^k \int_{T_k(t)} \int_{\mathbb{R}^k} \prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\xi_1+\ldots+\xi_j)\right|^2$$
$$\left|c_{H_n}^{k/2}\prod_{j=1}^k \left|\xi_j\right|^{\frac{1}{2}-H_n} - c_H^{k/2}\prod_{j=1}^k \left|\xi_j\right|^{\frac{1}{2}-H}\right|^2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\mathbf{t}$$
$$= k!\,\Gamma_{0,t}^k \int_{T_k(t)} \int_{\mathbb{R}^k} \prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j)\right|^2$$
$$\left|c_{H_n}^{k/2}\prod_{j=1}^k \left|\eta_j-\eta_{j-1}\right|^{\frac{1}{2}-H_n} - c_H^{k/2}\prod_{j=1}^k \left|\eta_j-\eta_{j-1}\right|^{\frac{1}{2}-H}\right|^2 \mathrm{d}\boldsymbol{\eta}\mathrm{d}\mathbf{t}. \quad (4.3.10)$$

We want to show that the integral in relation (4.3.10) converges to 0 when $n \to \infty$. This was shown in the proof of Theorem 4.1 of [23]. We include this argument for the sake of completeness.

We apply the Dominated Convergence Theorem. Note that the integrand of (4.3.10) converges pointwisely to 0 on $T_k(t) \times \mathbb{R}^k$, as $n \to \infty$ since c_H is a continuous function of H (see relation (1.0.8)). We now proceed to bound this integrand by an integrable function. First, we note that this integrand can be bounded by

$$\left(c_{H_n}^k \prod_{j=1}^k \left|\eta_j-\eta_{j-1}\right|^{1-2H_n} + c_H^k \prod_{j=1}^k \left|\eta_j-\eta_{j-1}\right|^{1-2H}\right)\prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j)\right|^2. \quad (4.3.11)$$

The two resulting integrals in the above are of the same type, and the only difference is that the first one depends on n, whereas the second one does not. We therefore only consider the term of the integrand function that depends on n. We need to show that

$$F_n(\mathbf{t},\boldsymbol{\eta}) := c_{H_n}^k \prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j)\right|^2 \prod_{j=1}^k \left|\eta_j - \eta_{j-1}\right|^{1-2H_n} \tag{4.3.12}$$

is bounded by a function $F(\mathbf{t},\boldsymbol{\eta})$ which is integrable on $T_k(t)\times\mathbb{R}^k$.

Fix number a and b such that $1/4 < a < H < b < 1/2$. Since $H_n \to H$, there exists $N\in\mathbb{N}$ such that

$$a \le H_n \le b, \text{ for all } n \ge N.$$

Since all H_n are included in a compact set $[a,b]$ and the constant c_H is defined by (1.0.8), we see that c_{H_n} is bounded by a constant $c>0$. Hence, using relations (4.1.9) and (4.1.10), we have

$$\begin{aligned} F_n(\mathbf{t},\boldsymbol{\eta}) &\le c^k \prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j)\right|^2 \prod_{j=1}^k \left|\eta_j - \eta_{j-1}\right|^{1-2H_n} \\ &\le c^k \sum_{\mathbf{a}\in A_k} \prod_{j=1}^k \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j)\right|^2 \prod_{j=1}^k |\eta_j|^{(1-2H_n)a_j} \end{aligned} \tag{4.3.13}$$

where A_k is a set of indices $\mathbf{a}=(a_1,\dots,a_k)$ such that $\text{card}(A_k)=2^{k-1}$,

$$a_1\in\{1,2\},\quad a_n\in\{0,1\},\quad a_2,\dots,a_{k-1}\in\big\{0,1,2\big\}$$

and

$$|\mathbf{a}| = \sum_{j=1}^k a_j = k.$$

We define functions $f_0, f_1, f_2 : \mathbb{R}_+\to\mathbb{R}_+$ as follows: $f_0(r)=1$,

$$f_1(r) = \begin{cases} r^{1-2a} & \text{if } r\ge 1,\\ 1 & \text{if } r<1,\end{cases} \qquad \text{and} \qquad f_2(r) = \begin{cases} r^{2(1-2a)} & \text{if } r\ge 1,\\ 1 & \text{if } r<1.\end{cases}$$

We have the following estimates:

$$|\eta_j|^{(1-2H_n)a_j} = \begin{cases} 1 = f_0(|\eta_j|) & \text{if } a_j = 0, \\ |\eta_j|^{1-2H_n} \leq f_1(|\eta_j|) = \begin{cases} |\eta_j|^{1-2a} & \text{if } |\eta_j| \geq 1, \\ 1 & \text{if } |\eta_j| < 1, \end{cases} & \text{if } a_j = 1, \\ |\eta_j|^{(1-2H_n)2} \leq f_2(|\eta_j|) = \begin{cases} |\eta_j|^{2(1-2a)} & \text{if } |\eta_j| \geq 1, \\ 1 & \text{if } |\eta_j| < 1, \end{cases} & \text{if } a_j = 2. \end{cases} \tag{4.3.14}$$

So, from relation (4.3.13) and the above estimates (4.3.14), we infer that

$$F_n(\mathbf{t}, \boldsymbol{\eta}) \leq c^k \sum_{\mathbf{a} \in A_k} \prod_{j=1}^k \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j) \right|^2 \prod_{j=1}^k f_{a_j}(|\eta_j|) =: F(\mathbf{t}, \boldsymbol{\eta}). \tag{4.3.15}$$

It remains to prove $F(\mathbf{t}, \boldsymbol{\eta})$ is integrable over $T_k(t) \times \mathbb{R}^k$. We first show that $F(\mathbf{t}, \cdot)$ is integrable on $\mathbb{R}^k$. Notice that

$$\int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta} = c^k \sum_{\mathbf{a} \in A_k} \prod_{j=1}^k I(a_j)$$

where

$$I(a_j) := \int_{\mathbb{R}} \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta_j) \right|^2 f_{a_j}(|\eta_j|) \mathrm{d}\eta_j.$$

For fixed $\mathbf{a} \in A_k$, we evaluate separately the $\mathrm{d}\eta_j$ integral above and we drop subindex j from η_j to simplify writing. Note that

• If $a_j = 0$, using Lemma 4.1.2 with $\alpha = 0$, we have

$$I(a_j) = I(0) = \int_{\mathbb{R}} \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta) \right|^2 \mathrm{d}\eta = \pi(t_{j+1} - t_j).$$

• If $a_j = 1$, using Lemma 4.1.2 with $\alpha = 1 - 2a \in (0, 1/2)$ and estimate of $f_1(|\eta_j|)$ in (4.3.14), we have

$$\begin{aligned} I(a_j) = I(1) &= \int_{\mathbb{R}} \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta) \right|^2 f_1(|\eta|) \mathrm{d}\eta \\ &= \int_{|\eta| \leq 1} \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta) \right|^2 \mathrm{d}\eta + \int_{|\eta| > 1} \left| \mathcal{F}G^w_{t_{j+1}-t_j}(\eta) \right|^2 |\eta|^{1-2a} \mathrm{d}\eta \end{aligned}$$

$$\begin{aligned}&\leq \int_{\mathbb{R}} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 \mathrm{d}\eta + \int_{\mathbb{R}} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 |\eta|^{1-2a} \mathrm{d}\eta\\ &= \pi(t_{j+1}-t_j) + 2^{2a}\widetilde{C}_{1-2a}(t_{j+1}-t_j)^{2a}.\end{aligned}$$

• If $a_j = 2$, using again Lemma 4.1.2 with $\alpha = 2(1-2a) \in (0,1)$ and estimate of $f_2(|\eta_j|)$ in (4.3.14), we have

$$\begin{aligned}I(a_j) = I(2) &= \int_{\mathbb{R}} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 f_2(|\eta|)\mathrm{d}\eta\\ &= \int_{|\eta|\leq 1} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 \mathrm{d}\eta + \int_{|\eta|>1} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 |\eta|^{2(1-2a)} \mathrm{d}\eta\\ &\leq \int_{\mathbb{R}} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 \mathrm{d}\eta + \int_{\mathbb{R}} \left|\mathcal{F}G^w_{t_{j+1}-t_j}(\eta)\right|^2 |\eta|^{2(1-2a)} \mathrm{d}\eta\\ &= \pi(t_{j+1}-t_j) + 2^{4a-1}\widetilde{C}_{2(1-2a)}(t_{j+1}-t_j)^{4a-1}.\end{aligned}$$

Therefore, there exist a constant $c_a > 0$ depending on a such that

$$I(a_j) \leq c_a \Big\{(t_{j+1}-t_j) + (t_{j+1}-t_j)^{2a} + (t_{j+1}-t_j)^{4a-1}\Big\},$$

for all $j = 1, \ldots, k$ and for any $\mathbf{a} \in A_k$. We use the fact that

$$x^b < t^{b-a}x^a, \text{ for any } 0 < x < t \text{ and } a < b. \tag{4.3.16}$$

We have

$$t_{j+1}-t_j < t^{1-2a}(t_{j+1}-t_j)^{2a} \text{ and } (t_{j+1}-t_j)^{2a} < t^{1-2a}(t_{j+1}-t_j)^{4a-1}.$$

Hence

$$\begin{aligned}(t_{j+1}-t_j) + (t_{j+1}-t_j)^{2a} + (t_{j+1}-t_j)^{4a-1} &< \big(t^{2(1-2a)} + t^{1-2a} + 1\big)(t_{j+1}-t_j)^{4a-1}\\ &:= c_{t,a}(t_{j+1}-t_j)^{4a-1}.\end{aligned}$$

Therefore, for any $\mathbf{a} \in A_k$,

$$I(a_j) \leq c_a c_{t,a}(t_{j+1}-t_j)^{4a-1}, \text{ for any } j = 1, \ldots, k.$$

Since $\mathrm{card}(A_k) = 2^{k-1}$, we obtain:

$$\int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \leq 2^{k-1} c_a^k c_{t,a}^k \prod_{j=1}^k (t_{j+1}-t_j)^{4a-1}. \tag{4.3.17}$$

This proves the integrability of $F(\mathbf{t}, \cdot)$ on $\mathbb{R}^k$. Moreover, by relation (4.3.17), we get

$$\int_{T_k(t)} \int_{\mathbb{R}^k} \prod_{j=1}^k F(\mathbf{t}, \boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\mathbf{t} \leq \int_{T_k(t)} 2^{k-1} c_a^k c_{t,a}^k \prod_{j=1}^k (t_{j+1}-t_j)^{4a-1} \mathrm{d}\mathbf{t}$$

$$= 2^{k-1} c_a^k c_{t,a}^k \int_{T_k(t)} \prod_{j=1}^{k} (t_{j+1}-t_j)^{4a-1} \mathrm{d}\mathbf{t} = 2^{k-1} c_a^k c_{t,a}^k \frac{\big(\Gamma(4a)\big)^k t^{4ka}}{\Gamma(4ka+1)} < \infty,$$

where we used Lemma A.3.1 for the last equality. Note that to apply Lemma A.3.1, we need $4a-1 > -1$, i.e. $a > 0$. This is clearly satisfied. This finishes the justification of the application of the Dominated Convergence Theorem in the case of the wave equation.

Step 2: We consider the heat equation.

Recall that we want to prove that

$$Q_n = \mathbb{E}\Big| I_k^{H_n}(f_k(\cdot, t, x)) - I_k^H(f_k(\cdot, t, x))\Big|^2 \to 0, \quad \text{as } n \to \infty.$$

We will use estimate (4.3.9) that we obtained above for Q_n. Instead of using Cauchy-Schwartz inequality (as in the case of wave equation), we will use Lemma 2.2.5. We obtain:

$$Q_n \le k!\, b_{H_0}^k \Big(\int_{T_k(t)} \big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2H_0} \mathrm{d}\mathbf{t}\Big)^{2H_0}, \tag{4.3.18}$$

where we recall that the integrand $A_k^{(n)}(\mathbf{t},\mathbf{t})$ is given in relation (4.3.8) by an integral over $\mathbb{R}^k$. Using the change of variables $\eta_j = \xi_j + \ldots + \xi_j$ for all $j = 1, \ldots, k$ with $\eta_0 = 0$, we have

$$\begin{aligned} &A_k^{(n)}(\mathbf{t},\mathbf{t}) \\ &= \int_{\mathbb{R}^k} \prod_{j=1}^{k} \Big| \mathcal{F} G^h_{t_{j+1}-t_j}(\xi_1 + \ldots + \xi_j)\Big|^2 \Big| c_{H_n}^{k/2} \prod_{j=1}^{k} \big|\xi_j\big|^{\frac12 - H_n} - c_H^{k/2} \prod_{j=1}^{k} \big|\xi_j\big|^{\frac12 - H}\Big|^2 \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^k} \prod_{j=1}^{k} \Big| \mathcal{F} G^h_{t_{j+1}-t_j}(\eta_j)\Big|^2 \Big| c_{H_n}^{k/2} \prod_{j=1}^{k} \big|\eta_j - \eta_{j-1}\big|^{\frac12 - H_n} - c_H^{k/2} \prod_{j=1}^{k} \big|\eta_j - \eta_{j-1}\big|^{\frac12 - H}\Big|^2 \mathrm{d}\boldsymbol{\eta}. \end{aligned} \tag{4.3.19}$$

Hence, it suffices to show that

$$\int_{T_k(t)} \big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2H_0} \mathrm{d}\mathbf{t} \to 0, \quad \text{as } n \to \infty.$$

For this, we will apply the Dominated Convergence Theorem. We will have to prove that for any $\mathbf{t} \in T_k(t)$,

$$A_k^{(n)}(\mathbf{t},\mathbf{t}) \to 0, \quad \text{as } n \to \infty \tag{4.3.20}$$

and there exists a function $h_k(\mathbf{t})$ such that

$$\big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2H_0} \le h_k(\mathbf{t}) \;\text{ for all } n \ge 1, \mathbf{t} \in T_k(t) \;\text{ and }\; \int_{T_k(t)} h_k(\mathbf{t}) \mathrm{d}\mathbf{t} < \infty. \tag{4.3.21}$$

We first prove (4.3.20). Let $\mathbf{t} \in T_k(t)$ be fixed. We denote $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ the integrand in (4.3.19). $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ converges to 0 as $n \to \infty$, as we noticed before. So (4.3.20) will follow by Dominated Convergence Theorem. To justify the application of this theorem, we need to bound $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ by an integrable function, i.e. we have to show that there exists a function B such that

$$B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le B(\mathbf{t},\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbb{R}^k, n \ge 1 \text{ and } \int_{\mathbb{R}^k} B(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

Note that

$$B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le 2\big(B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) + B_2(\mathbf{t},\boldsymbol{\eta})\big)$$

where

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) := \prod_{j=1}^{k} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 c^k_{H_n} \prod_{j=1}^{k} \left|\eta_j - \eta_{j-1}\right|^{1-2H_n} \tag{4.3.22}$$

$$B_2(\mathbf{t},\boldsymbol{\eta}) := \prod_{j=1}^{k} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 c^k_{H} \prod_{j=1}^{k} \left|\eta_j - \eta_{j-1}\right|^{1-2H}. \tag{4.3.23}$$

The two terms $B_1^{(n)}(\mathbf{t},\boldsymbol{\eta})$ and $B_2(\mathbf{t},\boldsymbol{\eta})$ are of the same form and the second one does not depend on n. We will prove that there exists a function F such that for all $n \ge 1$, $\mathbf{t} \in T_k(t)$ and $\boldsymbol{\eta} \in \mathbb{R}^k$,

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le F(\mathbf{t},\boldsymbol{\eta}) \text{ and } \int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

Recall that we impose the condition $H_0 + H > 3/4$. Hence $H > \ell := \max\{3/4 - H_0, 0\}$. Fix number a and b such that $\ell < a < H < b < 1/2$. Since $H_n \to H$, there exists $N \in \mathbb{N}$ such that

$$\ell < a \le H_n \le b < 1/2, \text{ for all } n \ge N.$$

Since all H_n are included in a compact set $[a,b]$ and the constant c_H is defined by (1.0.8), we see that c_{H_n} is bounded by a constant $c > 0$. Invoking again relations (4.1.9) and (4.1.10), we obtain

$$\begin{aligned} B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) &\le c^k \sum_{\mathbf{a}\in A_k} \prod_{j=1}^{k} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \prod_{j=1}^{k} \left|\eta_j\right|^{(1-2H_n)a_j} \\ &\le c^k \sum_{\mathbf{a}\in A_k} \prod_{j=1}^{k} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \prod_{j=1}^{k} f_{a_j}(|\eta_j|) := F(\mathbf{t},\boldsymbol{\eta}) \end{aligned}$$

where for the last inequality, we used estimates in (4.3.14). By the same argument $B_2(\mathbf{t},\boldsymbol{\eta}) \le F(\mathbf{t},\boldsymbol{\eta})$. Hence $B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le 2F(\mathbf{t},\boldsymbol{\eta})$ and so

$$A_k^{(n)}(\mathbf{t},\mathbf{t}) = \int_{\mathbb{R}^k} B^{(n)}(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \le 4\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}. \tag{4.3.24}$$

It remains to prove that $F(\mathbf{t},\cdot)$ is integrable on $\mathbb{R}^k$, i.e.

$$\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

We also give an estimate for this integral, which will be needed for the proof of relation (4.3.21) above. We have

$$\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} = c^k \sum_{\mathbf{a}\in A_k}\prod_{j=1}^{k} I(a_j)$$

where

$$I(a_j) := \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 f_{a_j}(|\eta_j|)\mathrm{d}\eta_j. \tag{4.3.25}$$

For fixed $\mathbf{a} \in A_k$, we evaluate separately the $\mathrm{d}\eta_j$ integral above. Notice that by Lemma 4.1.2,

- If $a_j = 0$, we have

$$\begin{aligned} I(a_j) = I(0) &= \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \mathrm{d}\eta_j = \int_{\mathbb{R}} e^{-(t_{j+1}-t_j)|\eta_j|^2}\mathrm{d}\eta_j \\ &= \sqrt{\pi}(t_{j+1}-t_j)^{-\frac{1}{2}}. \end{aligned}$$

- If $a_j = 1$, using the definition of $f_1(|\eta_j|)$ in (4.3.14), we have

$$\begin{aligned} I(a_j) = I(1) &= \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 f_1(|\eta_j|)\mathrm{d}\eta_j \\ &= \int_{|\eta|\le 1} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \mathrm{d}\eta_j + \int_{|\eta|>1} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 |\eta_j|^{1-2a}\mathrm{d}\eta_j \\ &\le \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \mathrm{d}\eta_j + \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 |\eta_j|^{1-2a}\mathrm{d}\eta_j \\ &= \sqrt{\pi}(t_{j+1}-t_j)^{-\frac{1}{2}} + \Gamma(1-a)(t_{j+1}-t_j)^{-(1-a)}. \end{aligned}$$

- If $a_j = 2$, using the definition of $f_2(|\eta_j|)$ in (4.3.14), we have

$$I(a_j) = I(2) = \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 f_2(|\eta_j|)\mathrm{d}\eta_j$$

$$
\begin{aligned}
&= \int_{|\eta|\le 1} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \mathrm{d}\eta_j + \int_{|\eta|>1} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 |\eta_j|^{2(1-2a)} \mathrm{d}\eta_j \\
&\le \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 \mathrm{d}\eta_j + \int_{\mathbb{R}} \left|\mathcal{F}G^h_{t_{j+1}-t_j}(\eta_j)\right|^2 |\eta_j|^{2(1-2a)} \mathrm{d}\eta_j \\
&= \sqrt{\pi}(t_{j+1}-t_j)^{-\frac{1}{2}} + \Gamma\Big(\frac{3-4a}{2}\Big)(t_{j+1}-t_j)^{-\frac{3-4a}{2}}.
\end{aligned}
$$

Therefore, there exist a constant $c_a > 0$ depending on a such that

$$
I(a_j) \le c_a \Big\{ (t_{j+1}-t_j)^{-\frac{1}{2}} + (t_{j+1}-t_j)^{-(1-a)} + (t_{j+1}-t_j)^{-\frac{3-4a}{2}} \Big\}, \tag{4.3.26}
$$

for all $j = 1, \dots, k$ and for any $\mathbf{a} \in A_k$. Using relation (4.3.16), we get

$$
(t_{j+1}-t_j)^{-\frac{1}{2}} < t^{\frac{1}{2}-a}(t_{j+1}-t_j)^{-(1-a)} \text{ and } (t_{j+1}-t_j)^{-(1-a)} < t^{\frac{1}{2}-a}(t_{j+1}-t_j)^{-\frac{3-4a}{2}}.
$$

Hence, there exist a constant $c_{t,a} > 0$ depending on t and a such that

$$
\begin{aligned}
(t_{j+1}-t_j)^{-\frac{1}{2}} + (t_{j+1}-t_j)^{-(1-a)} + (t_{j+1}-t_j)^{-\frac{3-4a}{2}} &< \big(t^{1-2a} + t^{\frac{1}{2}-a} + 1\big)(t_{j+1}-t_j)^{-\frac{3-4a}{2}} \\
&:= c_{t,a}(t_{j+1}-t_j)^{-\frac{3-4a}{2}}. \qquad (4.3.27)
\end{aligned}
$$

Therefore, for any $\mathbf{a} \in A_k$,

$$
I(a_j) \le c_a c_{t,a} (t_{j+1}-t_j)^{-\frac{3-4a}{2}}, \text{ for any } j = 1, \dots, k.
$$

Since $\mathrm{card}(A_k) = 2^{k-1}$, we obtain:

$$
\int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta} \le 2^{k-1} c^k c_a^k c_{t,a}^k \prod_{j=1}^k (t_{j+1}-t_j)^{-\frac{3-4a}{2}}. \tag{4.3.28}
$$

This proves the integrability of $F(\mathbf{t}, \cdot)$ and so relation (4.3.20) follows by the Dominated Convergence Theorem.

Now we prove (4.3.21). By relations (4.3.24) and (4.3.28) , for any $n \ge 1$ and $\mathbf{t} \in T_k(t)$, we have

$$
\begin{aligned}
\Big(A_k^{(n)}(\mathbf{t}, \mathbf{t})\Big)^{1/(2H_0)} &\le \Big(4 \int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta}\Big)^{1/(2H_0)} \\
&\le 2^{(k+1)/(2H_0)} \Big(c^k c_a^k c_{t,a}^k \prod_{j=1}^k (t_{j+1}-t_j)^{-\frac{3-4a}{2}}\Big)^{1/(2H_0)} =: h_k(\mathbf{t})
\end{aligned}
$$

Note that $h_k(\mathbf{t})$ is integrable over the symplex $T_k(t)$ by Lemma A.3.1 since $\frac{3-4a}{4H_0} < 1$ which is equivalent to $a > 3/4 - H_0$. This finishes the justification of the application of the Dominated Convergence Theorem in the case of the heat equation.

∎

Proof of Theorem 4.0.1: By Theorem 4.2.1, the process u^H has a continuous modification. We work with this modification, which we denote also by u^H. We need to prove the finite dimensional distribution convergence and the fact that the sequence of probability measures induced by $(u^{H_n})_{n\geq 1}$ is tight in the space of $C([0,T]\times\mathbb{R})$. More precisely, we will apply Theorem A.4.3. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: Finite dimensional distribution convergence} \begin{cases} \text{Step 1.a the heat equation} \\ \text{Step 1.b the wave equation} \end{cases} \\ \text{Step 2: Tightness} \end{cases}$$

Step 1: Finite dimensional distribution convergence

We have to prove that for any $k\geq 1$ and $(t_1,x_1),\dots,(t_k,x_k)\in[0,T]\times\mathbb{R}$,

$$\big(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\big)\overset{\mathrm{d}}{\to}\big(u^{H^*}(t_1,x_1),\dots,u^{H^*}(t_k,x_k)\big),\ \text{as } n\to\infty.$$

For this, by Lemma 2.4.3, it will be enough to prove that for all $(t,x)\in[0,T]\times\mathbb{R}$

$$\mathbb{E}\big|u^{H_n}(t,x)-u^{H^*}(t,x)\big|^2\to 0,\ \text{as } n\to\infty. \tag{4.3.29}$$

To prove relation (4.3.29), we use the same argument as in the proof of Theorem 2.0.1, by replacing α_n and α^* by H_n and H^*, respectively. More precisely, we need to prove that

$$\begin{cases} (a) \quad \mathbb{E}\big|u_m^{H_n}(t,x)-u_m^{H}(t,x)\big|^2\to 0,\ \text{as } n\to\infty,\ \text{for all } m \text{ fixed}, \\ (b) \quad \mathbb{E}\big|u_m^{H}(t,x)-u_m^{H}(t,x)\big|^2\to 0,\ \text{as } m\to\infty, \\ (c) \quad \sup_{n\geq 1}\mathbb{E}\big|u_m^{H_n}(t,x)-u^{H_n}(t,x)\big|^2\to 0,\ \text{as } m\to\infty. \end{cases}$$

Note that part (b) is automatically satisfied by relation (4.3.6), while part (a) follows by Lemma 4.3.1.

For part (c), it is enough to show that for all compact set $[a,b]$,

$$\sup_{H\in[a,b]}\mathbb{E}\big|u_m^H(t,x)-u^H(t,x)\big|^2\to 0,\ \text{as } m\to\infty. \tag{4.3.30}$$

Recall that $u_m^H(t,x)=\sum_{k=0}^m I_k^H(f_k(\cdot,t,x))$ and $u^H(t,x)=\sum_{k\geq 0} I_k^H(f_k(\cdot,t,x))$. Hence

$$u^H(t,x)-u_m^H(t,x)=\sum_{k\geq m+1} I_k^H(f_k(\cdot,t,x)).$$

We consider separately the heat and wave equation.

Step 1.a: We study the heat equation

We consider the compact set $[a,b]$ such that $\ell < a < b < 1/2$, where $\ell = \max\{3/4 - H_0, 0\}$. By the orthogonality of the Wiener chaos space and relation (4.1.17), we have

$$\begin{aligned}\mathbb{E}\Big|u^H(t,x) - u^H_m(t,x)\Big|^2 &= \sum_{k\geq m+1} \mathbb{E}\Big|I^H_k(f_k(\cdot,t,x))\Big|^2 \\ &\leq \sum_{k\geq m+1} C^k_{H_0,H,1}(k!)^{2H_0-1}\frac{t^{k(2H_0+H-1)}}{\Big(\Gamma\Big(k\frac{2H_0+H-1}{2H_0}+1\Big)\Big)^{2H_0}} \end{aligned} \tag{4.3.31}$$

where $C_{H_0,H,1} = C_{H,1}b_{H_0}(2C_{H,2})^{2H_0}$. Recall that

$$C_{H,1} = c_H \max\Big\{\Gamma\Big(\frac{1}{2}\Big), \Gamma\Big(1-H\Big), \Gamma\Big(\frac{3-4H}{2}\Big)\Big\}$$

and

$$C_{H,2} = \max\Big\{\Gamma\Big(-\frac{1}{4H_0}+1\Big), \Gamma\Big(-\frac{1-H}{2H_0}+1\Big), \Gamma\Big(-\frac{3-4H}{4H_0}+1\Big)\Big\}.$$

Note that $C_{H,1}$ and $C_{H,2}$ have been studied in Section 4.2 (see relations (4.2.19) and (4.2.45)). Therefore, we have

$$\sup_{H\in[a,b]} C_{H_0,H,1} \leq c_{a,b,1}b_{H_0}(2c_{a,2})^{2H_0} =: C_{H_0,a,b,1}. \tag{4.3.32}$$

Moreover, note that

$$\frac{2H_0+a-1}{2H_0} < \frac{2H_0+H-1}{2H_0} < \frac{2H_0+b-1}{2H_0}.$$

Recall that $x_0 \in (1,2)$ is the point such that the Gamma function $\Gamma(x)$ is decreasing on $(0,x_0)$ and increasing on (x_0,∞). We pick $m_0 \geq 1$ such that $m_0\Big(\frac{2H_0+a-1}{2H_0}\Big) > x_0$. For any $k \geq m \geq m_0$, we have $k\Big(\frac{2H_0+H-1}{2H_0}\Big) \geq k\Big(\frac{2H_0+a-1}{2H_0}\Big) > x_0$, which implies that for any $H \in [a,b]$,

$$\Gamma\Big(k\cdot\frac{2H_0+a-1}{2H_0}+1\Big) \leq \Gamma\Big(k\cdot\frac{2H_0+H-1}{2H_0}+1\Big),$$

and

$$\left(\frac{1}{\Gamma\big(k\cdot\frac{2H_0+H-1}{2H_0}+1\big)}\right)^{2H_0} \leq \left(\frac{1}{\Gamma\big(k\cdot\frac{2H_0+a-1}{2H_0}+1\big)}\right)^{2H_0}. \tag{4.3.33}$$

Next, for any $H \in [a, b]$ we have

$$0 < 2H_0 + a - 1 < 2H_0 + H - 1 < 2H_0 + b - 1,$$

it follows that

$$t^{k(2H_0+H-1)} \leq \begin{cases} 1 & \text{if } t < 1, \\ t^{k(2H_0+b-1)} & \text{if } t \geq 1. \end{cases}$$

Therefore, we get

$$t^{k(2H_0+H-1)} \leq (t \vee 1)^{k(2H_0+b-1)}. \tag{4.3.34}$$

Finally, by Lemma A.1.3, there exists a constant $c^{(1)}_{H_0,a} > 0$ depending on H_0 and a such that

$$\Gamma\big(k \cdot \frac{2H_0 + a - 1}{2H_0} + 1\big) \geq \big(c^{(1)}_{H_0,a}\big)^k (k!)^{\frac{2H_0+a-1}{2H_0}}. \tag{4.3.35}$$

Returning to relation (4.3.31), we use relations (4.3.32), (4.3.33), (4.3.34) and (4.3.35), we obtain

$$\sup_{H\in[a,b]} \sum_{k\geq m+1} \mathbb{E}\big|I_k^H(f_k(\cdot, t, x))\big|^2 \leq \sum_{k\geq m+1} C^k_{H_0,a,b,2} \frac{(t \vee 1)^{k(2H_0+b-1)}}{(k!)^a} \to 0$$

as $m \to \infty$ since

$$\sum_{k\geq 1} \frac{\big(C_{H_0,a,b,2}(t \vee 1)^{(2H_0+b-1)}\big)^k}{(k!)^a} < \infty,$$

due to Lemma A.1.6.

Step 1.b: We consider the wave equation

In this case $H \in (1/4, 1/2)$, we have to show that for any compact set $[a, b]$ such that $1/4 < a < b < 1/2$, relation (4.3.30) holds. By the orthogonality of the Wiener chaos space and relation (4.1.21), we have

$$\begin{aligned} \mathbb{E}\Big|u^H(t, x) - u^H_m(t, x)\Big|^2 &= \sum_{k\geq m+1} \mathbb{E}\big|I_k^H(f_k(\cdot, t, x))\big|^2 \\ &\leq \sum_{k\geq m+1} C^k_{H_0,H,3} (k!)^{2H_0-1} \frac{t^{k(2H_0+2H)}}{\Big(\Gamma(k\frac{H_0+H}{H_0} + 1)\Big)^{2H_0}}, \end{aligned} \tag{4.3.36}$$

where $C_{H_0,H,3} = C_{H,3} b_{H_0} (2C_{H,4})^{2H_0}$. Recall that

$$C_{H,3} = c_H \max\left\{\pi, \frac{\Gamma\big(1 - 2H\big)}{H}, \frac{2\Gamma\big(2 - 4H\big)}{4H - 1}\right\}$$

and

$$C_{H,4} = \max\left\{\Gamma\Big(\frac{1}{2H_0}+1\Big), \Gamma\Big(\frac{H}{H_0}+1\Big), \Gamma\Big(\frac{4H-1}{2H_0}+1\Big)\right\}.$$

Note that $C_{H,3}$ has been studied in Section 4.2 (see relation (4.2.73)). We now consider $C_{H,4}$. For any $H_0 \in (1/2,1)$, we have $x_0 < \frac{3}{2} < \frac{1}{2H_0}+1 < 2$ and hence

$$\Gamma\Big(\frac{1}{2H_0}+1\Big) < \Gamma(2) = 1.$$

Moreover, we see that $\frac{5}{4} < \frac{a}{H_0}+1 < \frac{H}{H_0}+1 < \frac{b}{H_0}+1 < 2$. Then

$$\Gamma\Big(\frac{H}{H_0}+1\Big) \le \max\left\{\Gamma\Big(\frac{a}{H_0}+1\Big), \Gamma\Big(\frac{b}{H_0}+1\Big)\right\}.$$

Next, we have $1 < \frac{4a-1}{2H_0}+1 < \frac{4H-1}{2H_0}+1 < \frac{4b-1}{2H_0}+1 < 2$. Then

$$\Gamma\Big(\frac{4H-1}{2H_0}+1\Big) \le \max\left\{\Gamma\Big(\frac{4a-1}{2H_0}+1\Big), \Gamma\Big(\frac{4b-1}{2H_0}+1\Big)\right\}.$$

Hence,

$$C_{H,4} \le \max\left\{1, \Gamma\Big(\frac{a}{H_0}+1\Big), \Gamma\Big(\frac{b}{H_0}+1\Big), \Gamma\Big(\frac{4a-1}{2H_0}+1\Big), \Gamma\Big(\frac{4b-1}{2H_0}+1\Big)\right\} := c_{a,b,4}.$$

Therefore

$$\sup_{H\in[a,b]} C_{H_0,H,3} \le c_3 b_{H_0} (2c_4)^{2H_0} =: C_{H_0,a,b,3}. \tag{4.3.37}$$

Moreover, note that

$$\frac{H_0+a}{H_0} < \frac{H_0+H}{H_0} < \frac{H_0+b}{H_0}.$$

Using the same approach as in the heat case above, we pick $m_0 \ge 1$ such that $m_0 \cdot \frac{H_0+a}{H_0} > x_0$. For any $k \ge m \ge m_0$, we have $k \cdot \frac{H_0+H}{H_0} \ge k \cdot \frac{H_0+a}{H_0} > x_0$, which implies that for any $H \in [a,b]$,

$$\Gamma\Big(k\cdot\frac{H_0+a}{H_0}+1\Big) \le \Gamma\Big(k\cdot\frac{H_0+H}{H_0}+1\Big),$$

and

$$\left(\frac{1}{\Gamma\big(k\cdot\frac{H_0+H}{H_0}+1\big)}\right)^{2H_0} \le \left(\frac{1}{\Gamma\big(k\cdot\frac{H_0+a}{H_0}+1\big)}\right)^{2H_0}. \tag{4.3.38}$$

Next, for any $H \in [a,b] \subset (1/4, 1/2)$, it follows that

$$t^{k(2H_0+2H)} \le \begin{cases} 1 & \text{if } t < 1, \\ t^{k(2H_0+2b)} & \text{if } t \ge 1. \end{cases}$$

Therefore, we get

$$t^{k(2H_0+2H)} \le (t \vee 1)^{k(2H_0+2b)}. \tag{4.3.39}$$

Finally, by Lemma A.1.3, there exists a constant $c^{(2)}_{H_0,a} > 0$ depending on H_0 and a such that

$$\Gamma\big(k \cdot \frac{H_0+a}{H_0} + 1\big) \ge \big(c^{(2)}_{H_0,a}\big)^k (k!)^{\frac{H_0+a}{H_0}}. \tag{4.3.40}$$

Returning to relation (4.3.36), we use relations (4.3.37), (4.3.38), (4.3.39) and (4.3.40), we have

$$\sup_{H\in[a,b]} \sum_{k\ge m+1} \mathbb{E}\big|I^H_k(f_k(\cdot,t,x))\big|^2 \le \sum_{k\ge m+1} C^k_{H_0,a,b,4} \frac{(t\vee 1)^{k(2H_0+2b)}}{(k!)^{1+2a}} \to 0,$$

as $m \to \infty$ since

$$\sum_{k\ge 1} \frac{\big(C_{H_0,a,b,4}(t\vee 1)^{(2H_0+2b)}\big)^k}{(k!)^{1+2a}} < \infty,$$

due to Lemma A.1.6.

Step 2: Tightness

This follows exactly as in the proof of Theorem 2.0.1, using the uniform bounds for the moments of the increments of solution given by Theorem 4.2.1.

∎

Chapter 5

PAM with rough noise and general initial condition

In this chapter, we consider the following Parabolic Anderson Model:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}^H(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x) = u_0, \quad x\in\mathbb{R} \end{cases} \tag{5.0.1}$$

with rough initial condition given by a non-negative Borel measure μ_0 on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{-a|x|^2} u_0(\mathrm{d}x) < \infty, \text{ for all } a>0. \tag{5.0.2}$$

The results presented in this chapter are contained in Section 3 of the preprint [30].

The noise $\dot{W}^H$ is assumed to be a zero-mean Gaussian that is fractional in time with parameter $H_0 \in (1/2, 1)$ and fractional in space with parameter $H \in (0, 1/2)$. Throughout this chapter, G denotes the fundamental solution of the heat equation. Let $w(t,x)$ be the solution of the deterministic heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x),\ t>0, x\in\mathbb{R}$$

with the same initial condition as (5.0.1), i.e.

$$w(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(\mathrm{d}y). \tag{5.0.3}$$

Note that (5.0.2) is the necessary and sufficient condition for $w(t,x)$ to be well-defined. Therefore (5.0.2) is the most general condition for the Parabolic Anderson Model. This includes the case $u_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$ for a function u_0 on $\mathbb{R}$. The case $u_0 = 1$ was considered in Chapter 4.

The goal of this chapter is to study the continuity in law of the solution with respect to the parameter H in the space of continuous functions on $C([t_0,T]\times\mathbb{R})$, when the temporal parameter H_0 is fixed. The proof uses the classical method of convergence of finite dimensional distributions (f.d.d) and tightness. Section 5.3 establishes the convergence of finite-dimensional distributions, while tightness will be established from the uniform moment estimates presented in Section 5.2.

The following theorem is the main result for this chapter. This result is an extension of Theorem 4.0.1 (a) (the case of the heat equation with constant initial condition) to general initial conditions.

Theorem 5.0.1. *We fix $0<t_0<T$ and $H_0\in(1/2,1)$. Let u^H be the solution of equation* (5.0.1)*. Define $\ell=\max(3/4-H_0,0)$. Let $(H_n)_{n\geq 1}$ be an arbitrary sequence in $(0,1/2)$. If $H_n\to H^*\in(\ell,1/2)$, then*

$$u^{H_n}\xrightarrow{d}u^{H^*}\ \text{in } C([t_0,T]\times\mathbb{R}).$$

Note that the convergence is only on compact sets of the form $[t_0,T]\times K$ with $0<t_0<T$. This limitation is due to the uniform moment estimates that we obtain in Theorem 5.2.1 below, which is needed for the proof of tightness. The techniques that are used for the proof of Theorem 5.0.1 are specific to the heat equation (e.g. we use the semigroup property of the heat kernel) and cannot be extended to the wave equation. Continuity in law for the solution of the Hyperbolic Anderson model (HAM) with noise W^H as above and general initial condition remains an open problem.

5.1 The existence of solution

In this section we give the definition of the solution to (5.0.1), driven by the rough noise introduced in Section 4.1 and we show the existence of unique solution. This was proved in [4] under the condition $H_0+H>3/4$. We include this argument, because for our problem we need the explicit form of the constants.

Definition 5.1.1. *Let $\mathcal{F}_t$ be the filtration generated by the noise W^H whose covariance structure is given by* (4.1.1)*. We say that a process $u^H=\{u^H(t,x);t\geq 0,x\in\mathbb{R}\}$ is a* **(mild) solution** *of equation* (5.0.1) *if for any $t>0$ and $x\in\mathbb{R}$, $u^H(t,x)$ is $\mathcal{F}_t$-measurable, $\mathbb{E}|u^H(t,x)|^2<\infty$ and the following integral equation holds:*

$$u^H(t,x)=w(t,x)+\int_0^t\int_{\mathbb{R}^d}G_{t-s}(x-y)u(s,y)W^H(\delta s,\delta y), \tag{5.1.1}$$

i.e. $v^{(t,x)}\in Dom(\delta)$ and $u(t,x)=w(t,x)+\delta(v^{(t,x)})$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, where

$$v^{(t,x)}(s,y)=1_{[0,t]}(s)G_{t-s}(x-y)u(s,y),\quad s\geq 0,y\in\mathbb{R}. \tag{5.1.2}$$

The solution has the series expansion:

$$u^H(t,x) = w(t,x) + \sum_{n\geq 1} I_n^H(f_n(\cdot,t,x)) = \sum_{n\geq 0} I_n^H(f_n(\cdot,t,x)), \tag{5.1.3}$$

where the kernel $f_n(\cdot,t,x) \in \mathcal{H}_H^{\otimes n}$ is given by

$$\begin{aligned} &f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \\ &\quad = G_{t-t_n}(x-x_n)\cdots G_{t_2-t_1}(x_2-x_1)w(t_1,x_1)1_{\{0<t_1<\cdots<t_n<t\}}, \end{aligned} \tag{5.1.4}$$

and I_n^H is the multiple Wiener integral of order n with respect to W^H. We let $I_0^H(x) = x$ and $f_0(\cdot,t,x) = w(t,x)$ by convention. Similarly to the case of the regular noise, the kernel f_n given in (5.1.4) has another expression, compared to the case in (4.1.4).

The goal of this section is to prove the following result.

Theorem 5.1.2 (Theorem 1.1 of [4]). *For any $H_0 \in (1/2,1)$ and $H \in (0,1/2)$ such that*

$$H_0 + H > 3/4,$$

for any non-negative and non-negative definite kernel γ_0 satisfying (2.1.3) *and for any initial measure u_0 satisfying* (5.0.2)*, equation* (5.0.1) *has a unique solution. Moreover, for any $p \geq 2$ and for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\mathbb{E}|u^H(t,x)|^p \leq C_1^p w^p(t,x) \exp\left(C_2\, p^{\frac{H+1}{H}}\, t^{\frac{2H_0+H-1}{H}}\right) \tag{5.1.5}$$

where $C_1 > 0$ and $C_2 > 0$ are some constants depending on H_0 and H.

Proof: We follow a similar approach as outlined in the proof of Theorem 1.1 in [4], which involves five steps:

Step 1: In this step, we obtain a preliminary bound for $\mathbb{E}|I_n^H(f_n(\cdot,t,x))|^2$. Note that

$$\mathbb{E}|I_n^H(f_n(\cdot,t,x))|^2 = n!\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}_H^{\otimes n}} =: \frac{1}{n!}k_n(t,x)$$

where

$$k_n(t,x) = \alpha_{H_0}^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H_0-2} \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s}) \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s},$$

and $\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s})$ is defined as the same as in (3.1.12) with $\mu(\mathrm{d}\xi_j) = c_H|\xi_j|^{1-2H}\mathrm{d}\xi_j$, for all $j = 1,\ldots,n$. By applying relation (3.1.13) and Lemma 2.2.5 to the function $\varphi(\mathbf{t}) = \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})$, we obtain:

$$\mathbb{E}|I_n^\alpha(f_n(\cdot,t,x))|^2 \leq \frac{1}{n!} b_{H_0}^n \left(\sum_{\rho\in S_n} \int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t} \left(\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \right)^{2H_0}. \tag{5.1.6}$$

By Lemma 3.1.6 with $d=1$, for any $\rho \in S_n$ fixed, we obtain:

$$\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t} \left(\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \leq w^{1/H_0}(t,x) \int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t} \mathrm{d}\mathbf{t}$$
$$\times \left(c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{\rho(k+1)}-t_{\rho(k)}}{t_{\rho(k+1)}t_{\rho(k)}} \Big| \sum_{i=1}^{k} t_{\rho(i)}\xi_i \Big|^2 \right\} |\xi_k|^{1-2H} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \right)^{\frac{1}{2H_0}}.$$

Using the change of variables $t'_k = t_{\rho(k)}$ for $k=1,\ldots,n$ and $t'_{n+1} = t_{\rho(n+1)}$, we notice that the integral on the right-hand side above is not affected by the choice of ρ. Hence, for any ρ, we get:

$$\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t} \left(\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \leq w^{1/H_0}(t,x) \int_{0<t_1<\ldots<t_n<t} I_t^{(n)}(t_1,\ldots,t_n)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \tag{5.1.7}$$

where

$$I_t^{(n)}(t_1,\ldots,t_n) = c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{k+1}-t_k}{t_{k+1}t_k} \Big| \sum_{i=1}^{k} t_i\xi_i \Big|^2 \right\} |\xi_k|^{1-2H} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \tag{5.1.8}$$

and $t_{n+1} = t$.

Step 2: In this step, we provide an estimate for $I_t^{(n)}$. Using the change of variables $z_i = t_i\xi_i$ for $i=1,\ldots,k$, followed by $\eta_k = \sum_{i=1}^{k} z_i$ for $k=1,\ldots,n$, we have:

$$\begin{aligned} I_t^{(n)}(t_1,\ldots,t_n) &= c_H^n \Big(\prod_{k=1}^{n} t_k\Big)^{2H-2} \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{k+1}-t_k}{t_{k+1}t_k} \Big| \sum_{i=1}^{k} z_i \Big|^2 \right\} |z_k|^{1-2H} \mathrm{d}\mathbf{z} \\ &= c_H^n \Big(\prod_{k=1}^{n} t_k\Big)^{2H-2} \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{k+1}-t_k}{t_{k+1}t_k} |\eta_k|^2 \right\} |\eta_1|^{1-2H} \prod_{k=2}^{n} |\eta_k - \eta_{k-1}|^{1-2H} \mathrm{d}\boldsymbol{\eta} \\ &\leq c_H^n \Big(\prod_{k=1}^{n} t_k\Big)^{2H-2} \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{k+1}-t_k}{t_{k+1}t_k} |\eta_k|^2 \right\} |\eta_1|^{1-2H} \prod_{k=2}^{n} \left(|\eta_k|^{1-2H} + |\eta_{k-1}|^{1-2H}\right) \mathrm{d}\boldsymbol{\eta}. \end{aligned} \tag{5.1.9}$$

By relation (4.1.11),

$$I_t^{(n)}(t_1,\ldots,t_n) \leq c_H^n \Big(\prod_{k=1}^{n} t_k\Big)^{2H-2} \sum_{\alpha \in D_n} \prod_{k=1}^{n} \left\{ \int_{\mathbb{R}} \exp\left(-\frac{t_{k+1}-t_k}{t_{k+1}t_k} |\eta_k|^2 \right) |\eta_k|^{\alpha_k} \mathrm{d}\eta_k \right\}, \tag{5.1.10}$$

where the set D_n is described after relation (4.1.10). Using Lemma 4.1.2, we have:

$$I_t^{(n)}(t_1,\ldots,t_n) \leq c_H^n \Big(\prod_{k=1}^{n} t_k\Big)^{2H-2} \sum_{\alpha \in D_n} \prod_{k=1}^{n} \Gamma\Big(\frac{1+\alpha_k}{2}\Big) \Big(\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big)^{-\frac{1+\alpha_k}{2}}$$

$$\leq C_{H,1}^{n} \sum_{\alpha \in D_n} t^{\frac{1+\alpha_n}{2}} t_1^{\frac{4H-3+\alpha_1}{2}} \Big(\prod_{k=2}^{n} t_k^{\frac{4H-2+\alpha_{k-1}+\alpha_k}{2}} \Big) \prod_{k=1}^{n} (t_{k+1}-t_k)^{-\frac{1+\alpha_k}{2}} \tag{5.1.11}$$

where for the second inequality we used the constant $C_{H,1}$ defined by (4.1.14).

Step 3: In this step, we continue with the estimation of $\mathbb{E}|I_n^H(f_n(\cdot,t,x)))|^2$ using Step 2. Taking power $1/2H_0$ on both sides of (5.1.11) and using (5.1.6) and (5.1.7), we obtain:

$$\begin{aligned} &\mathbb{E}\big|I_n^H(f_n(\cdot,t,x)))\big|^2 \\ &\leq b_{H_0}^2 w^2(t,x) C_{H,1}^n (n!)^{2H_0-1} \Big(\sum_{\alpha \in D_n} t^{\frac{1+\alpha_n}{4H_0}} \int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} (t_{k+1}-t_k)^{\widetilde{\beta}_k} \mathrm{d}\mathbf{t} \Big)^{2H_0} \end{aligned} \tag{5.1.12}$$

where

$$\widetilde{\alpha}_k = \begin{cases} \frac{4H-3+\alpha_1}{4H_0}, & k=1 \\ \frac{4H-2+\alpha_{k-1}+\alpha_k}{4H_0}, & k=2,\ldots,n \end{cases} \tag{5.1.13}$$

and

$$\widetilde{\beta}_k = -\frac{1+\alpha_k}{4H_0},\ k=1,\ldots,n. \tag{5.1.14}$$

In order to apply Lemma A.3.2 to evaluate the integral in (5.1.12), we need to verify that the hypotheses of this lemma are satisfies. This verification has been done in pages 66-67 of the Master's thesis [31]. We omit the details. By applying Lemma A.3.2, we obtain:

$$\int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\tilde{\alpha}_k} (t_{k+1}-t_k)^{\tilde{\beta}_k} \mathrm{d}\mathbf{t} = \frac{\Gamma(\widetilde{\alpha}_1+1) \prod_{k=1}^{n} \Gamma(\widetilde{\beta}_k+1)}{\Gamma\big(|\widetilde{\alpha}|+|\widetilde{\beta}|+n+1\big)} \gamma_n t^{|\widetilde{\alpha}|+|\widetilde{\beta}|+n} \tag{5.1.15}$$

with

$$\gamma_n = \gamma_n(\alpha_1,\ldots,\alpha_n) := \prod_{k=1}^{n-1} \frac{\Gamma\big(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1+\widetilde{\alpha}_{k+1}\big)}{\Gamma\big(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1\big)}. \tag{5.1.16}$$

It is possible to prove that

$$\gamma_n \leq 1.$$

This proof require significant effort and is based on some combinatorial arguments. We refer the reader to Step 4 of the proof of Theorem 1.1 in [4].

Step 4: In this step, we give an estimate for the integral appearing in (5.1.15). Note that

$$
\begin{aligned}
&|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1 = \left(\widetilde{\alpha}_1 + \widetilde{\beta}_1\right) + \sum_{k=2}^{n} \left(\widetilde{\alpha}_k + \widetilde{\beta}_k\right) + n + 1 \\
&= \frac{H-1}{H_0} + \sum_{k=2}^{n} \frac{4H - 3 + \alpha_{k-1}}{4H_0} + n + 1 = \frac{H-1}{H_0} + (n-1)\frac{4H-3}{4H_0} + \frac{|\alpha| - \alpha_n}{4H_0} + n + 1 \\
&= n\frac{2H_0 + H - 1}{2H_0} - \frac{1+\alpha_n}{4H_0} + 1 \qquad (5.1.17) \\
&\geq n\frac{2H_0 + H - 1}{2H_0} - \frac{1-H}{2H_0} + 1. \qquad (5.1.18)
\end{aligned}
$$

For n large enough, by Lemma A.1.4, there exists a constant $C^{(1)}_{H_0,H}$ such that

$$\Gamma\Big(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1\Big) \geq \left(C^{(1)}_{H_0,H}\right)^n (n!)^{\frac{2H_0+H-1}{2H_0}}. \qquad (5.1.19)$$

Moreover,

$$\Gamma(\widetilde{\alpha}_1 + 1) \leq \max\left\{\Gamma\Big(\frac{2H_0 + H - 1}{2H_0}\Big), \Gamma\Big(1 - \frac{1}{4H_0}\Big)\right\} =: C^{(2)}_{H_0,H} \qquad (5.1.20)$$

and

$$\prod_{k=1}^{n} \Gamma(\widetilde{\beta}_k + 1) = \prod_{k=1}^{n} \Gamma\Big(-\frac{1+\alpha_k}{4H_0} + 1\Big) \leq C^n_{H,2} \qquad (5.1.21)$$

(the inequality holds by (4.2.44)) where $C_{H,2}$ is given in (4.1.16). Hence,

$$
\begin{aligned}
t^{\frac{1+\alpha_n}{4H_0}} \int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\tilde{\alpha}_k} (t_{k+1} - t_k)^{\tilde{\beta}_k} \mathrm{d}\mathbf{t} \\
\leq \left(C^{(1)}_{H_0,H}\right)^{-n} C^{(2)}_{H_0,H} C^n_{H,2} (n!)^{-\frac{2H_0+H-1}{2H_0}} t^{n\frac{2H_0+H-1}{2H_0}}. \qquad (5.1.22)
\end{aligned}
$$

Step 5: In this step, we will continue the previous calculations to arrive at conclusion (5.1.5). We come back to (5.1.12). Since relation (5.1.22) does not depend on $\alpha \in D_n$ and $\mathrm{card}(D_n) = 2^{n-1}$, by (5.1.22), we have

$$
\begin{aligned}
&\mathbb{E}\left|I^H_n(f_n(\cdot, t, x)))\right|^2 \\
&\leq b^2_{H_0} w^2(t,x) C^n_{H,1} (n!)^{2H_0 - 1} \left(2^{n-1} \left(C^{(1)}_{H_0,H}\right)^{-n} C^{(2)}_{H,H_0} C^n_{H,2} (n!)^{-\frac{2H_0+H-1}{2H_0}} t^{n\frac{2H_0+H-1}{2H_0}}\right)^{2H_0} \\
&= \frac{w^2(t,x)\left(C_{H_0,H,1} t^{2H_0+H-1}\right)^n}{(n!)^H} \qquad (5.1.23)
\end{aligned}
$$

where $C_{H_0,H,1}$ is a constant depending on H_0 and H.

For any $p \geq 2$, using relation (5.1.23), Minkowski inequality and Lemma A.5.3 (hypercontractivity), we obtain

$$\begin{aligned}
&\|u^H(t,x)\|_p \\
&\quad \leq \sum_{n\geq 0}(p-1)^{n/2}\big\|I_n^H(f_n(\cdot,t,x))\big\|_2 \leq \sum_{n\geq 0}(p-1)^{n/2}\left(\frac{w^2(t,x)\big(C_{H_0,H,1}t^{2H_0+H-1}\big)^n}{(n!)^H}\right)^{1/2} \\
&\quad \leq w(t,x)\sum_{n\geq 0}\frac{\left(\sqrt{p\ C_{H_0,H,1}\ t^{2H_0+H}-1}\ \right)^n}{(n!)^{H/2}} \leq C_1 w(t,x)\exp\left(C_2\ p^{\frac{1}{H}}\ t^{\frac{2H_0+H-1}{H}}\right),
\end{aligned}$$

where we used Lemma A.1.6 in the last inequality. Taking power p, we obtain:

$$\mathbb{E}|u^H(t,x)|^p = \|u^H(t,x)\|_p^p \leq (C_1)^p w^p(t,x)\exp\left(C_2\ p^{\frac{H+1}{H}}\ t^{\frac{2H_0+H-1}{H}}\right),$$

where C_1 and C_2 are some constants depending on H_0 and H. ∎

Remark 5.1.3. The bound in Theorem 5.1.2 depends on x. But this result is not used for the continuity in law. We included it only for the sake of completeness. All that is needed is that $\mathbb{E}|u^H(t,x)|^p$ is finite. We showed a little bit more: we showed a bound which includes an exponential function of x. If u_0 is a function (i.e. the measure u_0 has a density function, denoted also u_0) and this function is bounded, then $w(t,x)$ is bounded, and the result in Theorem 5.1.2 matches the known results from the literature for PAM with constant initial condition.

Remark 5.1.4. Proving a result similar to Theorem 5.1.2 for the hyperbolic Anderson model is an open problem.

Remark 5.1.5. Note that the exponents of p and t in (5.1.5) are the same as in the case of the Parametric Anderson Model with the same type of noise W^H as above and constant initial condition; see Theorem 4.1.4 (a). Note that (5.1.5) can be written as:

$$\overline{\lim_{t\to\infty}}\, t^{-\frac{2H_0+H-1}{H}}\log\mathbb{E}\big|u(t,x)\big|^p \leq p\,\overline{\lim_{t\to\infty}}\, t^{-\frac{2H_0+H-1}{H}}\log w(t,x) + C_2 p^{\frac{H+1}{H}}.$$

We conjecture that the previous inequality is actually an equality, and we can replace the two $\overline{\lim}$ by lim.

5.2 Uniform moment estimates

In this section, we provide some estimates for the moments of the increments of the solution to equation (5.0.1) driven by the noise introduced in Section 4.1. The results in this section are used for the proof of tightness.

Theorem 5.2.1. *Let u^H be the solution to equation* (5.0.1) *with noise W^H as in Section 4.1. Let K be a compact set contained in $\mathbb{R}$ and $[a,b]$ be a compact set such that*

$$\max\left\{0, \frac{3-4H_0}{4}\right\} < a < b < \frac{1}{2}. \tag{5.2.1}$$

We fix $T > t_0 > 0$. For any $p \geq 2$, there exists positive constants C_1 and C_2 such that for any $t', t \in [t_0, T]$ and for any $x', x \in K$, we have

$$\sup_{H\in[a,b]} \mathbb{E}|u^H(t',x) - u^H(t,x)|^p \leq C_1|t'-t|^{p\theta/2} \tag{5.2.2}$$

and

$$\sup_{H\in[a,b]} \mathbb{E}|u^H(t,x') - u^H(t,x)|^p \leq C_2|x'-x|^{p\theta} \tag{5.2.3}$$

for any

$$0 < \theta < 2H_0(1-c_0) + a - 1 \tag{5.2.4}$$

where $c_0 \in \left(0, \frac{2H_0+a-1}{2H_0}\right)$.

Proof: A road map of this proof is the following: $\begin{cases} \text{Step 1: the time increments,} \\ \text{Step 2: the space increments.} \end{cases}$

Step 1: We start with the time increments. Let $t, t' \in [t_0, T]$ and $x \in K$ be arbitrary. Assume that $h = t' - t > 0$ (the case $h < 0$ is similar). Similarly to (4.2.7), we have:

$$\|u^H(t+h,x) - u^H(t,x)\|_p \leq \sum_{n\geq 1} (p-1)^{n/2} \left(\frac{2}{n!}\left(A_n^H(t,h) + B_n^H(t,h)\right)\right)^{1/2}, \tag{5.2.5}$$

where $A_n^H(t,h)$ and $B_n^H(t,h)$ are given by (4.2.8), respectively (4.2.9).

We study $A_n^H(t,h)$ first. Using the same approach as in (4.2.12), we know that

$$A_n^H(t,h) \leq b_{H_0}^n \left(\int_{[0,t]^n} \left(\psi_{t,h,n}^H(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t}\right)^{2H_0}, \tag{5.2.6}$$

where $\psi_{t,h,n}^H(\mathbf{t},\mathbf{t})$ is defined in (4.2.11). Note that the Fourier transform of the kernel $f_n(\cdot,t,x)$ is given by Lemma 3.1.5 for any $d \geq 1$. Here, we use this formula for $d = 1$.

We will use techniques which are similar to those of Chapter 3 (the proof of Theorem 3.2.1), where we studied the Parabolic Anderson Model with regular noise in space W^α (colored in time) and general initial condition. The difference is that here we consider the rough noise in space W^H, instead of W^α. Basically, we replace the Riesz kernel $|\xi|^{-\alpha}$ by the fractional kernel $c_H|\xi|^{1-2H}$.

To estimate $\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})$, we use estimates (3.2.4), (3.2.11) and (3.2.14), for the increments of the Fourier transform appearing in (4.2.11). We assume for simplicity that $t_1 < \ldots < t_n$. Similarly to (3.2.15), we obtain that for any $\theta \in [0,1]$, there exists $C^{(1)}_{t_0,\theta,T}$ such that:

$$\begin{aligned}
&\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\\
&\leq h^\theta C^{(1)}_{t_0,\theta,T}\big(t-t_n\big)^{-\theta} c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^n \exp\Big(-\frac{\frac{t_{k+1}}{2}-\frac{t_k}{2}}{\frac{t_k}{2}\frac{t_{k+1}}{2}}\Big|\sum_{j=1}^k \frac{t_j}{2}\xi_j\Big|^2\Big)|\xi_k|^{1-2H}\mathrm{d}\boldsymbol{\xi}\\
&= h^\theta C^{(1)}_{t_0,\theta,T}\big(t-t_n\big)^{-\theta} I^{(n)}_{t/2}(t_1/2,\ldots,t_n/2) \qquad (5.2.7)
\end{aligned}$$

where $I^{(n)}_t$ is given by (5.1.8). Using (5.1.11) for estimating $I^{(n)}_t$, we obtain:

$$\begin{aligned}
&\big(t-t_n\big)^{-\theta} I^{(n)}_{t/2}(t_1/2,\ldots,t_n/2)\\
&\leq C^n_{H,1} 2^n \sum_{\alpha\in D_n} t^{\frac{1+\alpha_n}{2}} t_1^{\frac{4H-3+\alpha_1}{2}} \Big(\prod_{k=2}^n t_k^{\frac{4H-2+\alpha_{k-1}+\alpha_k}{2}}\Big)\Big(\prod_{k=1}^{n-1}\big(t_{k+1}-t_k\big)^{-\frac{1+\alpha_k}{2}}\Big)(t-t_n)^{-\frac{1+\alpha_n}{2}-\theta}
\end{aligned} \qquad (5.2.8)$$

where $C_{H,1}$ is defined in (4.1.14). Taking power $\frac{1}{2H_0}$ on both sides of (5.2.7) above, we obtain:

$$\Big(\psi^H_{t,z,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \leq \Big(h^\theta C^{(1)}_{t_0,\theta,T} C^n_{H,1} 2^n\Big)^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{k=1}^n t_k^{\widetilde{\alpha}_k} \prod_{k=1}^{n-1}\big(t_{k+1}-t_k\big)^{\widetilde{\beta}_k}\big(t-t_n\big)^{\widetilde{\beta}_{n,\theta}} \qquad (5.2.9)$$

where $\widetilde{\alpha}_k$ are given by (5.1.13) for $k=1,\ldots,n$, $\widetilde{\beta}_k$ is given by (5.1.14) for $k=1,\ldots,n-1$ and $\widetilde{\beta}_{n,\theta} = -\frac{1+\alpha_n+2\theta}{4H_0}$. A similar estimate holds for arbitrary $(t_1,\ldots,t_n) \in [0,t]^n$ with $t_{\rho(1)} < \ldots < t_{\rho(n)}$ for same permutation $\rho \in S_n$, where S_n is the set of all permutation of $\{1,\ldots,n\}$.

We now integrate $\big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}$ over $[0,t]^n$, we get:

$$\int_{[0,t]^n}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t} = \sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}$$

$$\leq \left(h^{\theta} C^{(1)}_{t_0,\theta,T} C^n_{H,1} 2^n\right)^{\frac{1}{2H_0}} n! \sum_{\boldsymbol{\alpha}\in D_n} \int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} \prod_{k=1}^{n-1} \left(t_{k+1}-t_k\right)^{\widetilde{\beta}_k} \left(t_{n+1}-t_n\right)^{\widetilde{\beta}_{n,\theta}} \mathrm{d}\mathbf{t}, \tag{5.2.10}$$

where in the first equation, we decompose the set $[0,t]^n$ into $n!$ disjoint regions of the form $t_{\rho(1)} < \cdots < t_{\rho(n)}$ with $\rho \in S_n$.

Now we want to evaluate the integral in relation (5.2.10). To apply Lemma A.3.2, we need $\widetilde{\alpha}_k > -1$, $\widetilde{\beta}_k > -1$ for all $k = 1, \ldots, n-1$ and $\widetilde{\beta}_{n,\theta} > -1$. We have already mentioned that $\widetilde{\alpha}_1 > -1, \ldots, \widetilde{\alpha}_n > -1$, $\widetilde{\beta}_1 > -1, \ldots, \widetilde{\beta}_{n-1} > -1$; see the statement after (5.1.14). So it remains to check that $\widetilde{\beta}_{n,\theta} > -1$. Recall that $\alpha_n \in \{0, 1-2H\}$. When $\alpha_n = 0$, we need $-\frac{1+2\theta}{4H_0} > -1$, i.e. $\theta < 2H_0 - 1/2$. When $\alpha_n = 1-2H$, we need $-\frac{2+2H+2\theta}{4H_0} > -1$, i.e. $\theta < 2H_0 + H - 1$. Note that $2H_0 - 1/2 > 2H_0 + H - 1$ since $H < 1/2$. Hence, we need

$$0 < \theta < 2H_0 + H - 1. \tag{5.2.11}$$

Since we require (5.2.11) to hold for all $H \in [a,b]$, we need to impose the condition

$$0 < \theta < 2H_0 + a - 1. \tag{5.2.12}$$

Moreover, condition (A.3.2) (which does not involve $\widetilde{\beta}_{n,\theta}$) has been verified in **Step 3** of the proof of Theorem 5.1.2. Then by Lemma A.3.2, we obtain:

$$\begin{aligned} &\int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} \prod_{k=1}^{n-1} \left(t_{k+1}-t_k\right)^{\widetilde{\beta}_k} \left(t_{n+1}-t_n\right)^{\widetilde{\beta}_{n,\theta}} \mathrm{d}\mathbf{t} \\ &\qquad = \frac{\Gamma(\widetilde{\alpha}_1+1)\prod_{k=1}^{n-1}\Gamma(\widetilde{\beta}_k+1)\Gamma(\widetilde{\beta}_{n,\theta}+1)}{\Gamma\big(|\widetilde{\alpha}|+|\widetilde{\beta}|+n+1\big)} \gamma_n t^{|\widetilde{\alpha}|+|\widetilde{\beta}|+n} \end{aligned} \tag{5.2.13}$$

with

$$\gamma_n = \gamma_n(\alpha_1, \ldots, \alpha_n) = \prod_{k=1}^{n-1} \frac{\Gamma\big(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1}\big)}{\Gamma\big(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1\big)}.$$

Note that γ_n does not depend on $\widetilde{\beta}_{n,\theta}$, and therefore has the same expression as in (5.1.16). Hence

$$\gamma_n \leq 1.$$

Recall that $\alpha_1 \in \big\{1-2H, 2(1-2H)\big\}$, $\alpha_n \in \big\{0, 1-2H\big\}$ and $\alpha_i \in \big\{0, 1-2H, 2(1-2H)\big\}$ for $i = 2, \ldots, n-1$. Therefore, $0 \leq \alpha_i \leq 2(1-2H)$ for all $i = 1, \ldots, n$ which implies that for any $H \in [a,b]$,

$$\frac{2H_0 + a - 1}{2H_0} \leq \frac{2H_0 + H - 1}{2H_0} \leq 1 + \frac{4H - 3 + \alpha_1}{4H_0} \leq 1 - \frac{1}{4H_0} < \frac{3}{4}.$$

Hence,

$$\Gamma(\widetilde{\alpha}_1+1) \leq \Gamma\Big(\frac{2H_0+a-1}{2H_0}\Big) =: c_{H_0,a,1}. \tag{5.2.14}$$

Now we want to bound

$$\prod_{k=1}^{n-1} \Gamma(\widetilde{\beta}_k+1) \times \Gamma(\widetilde{\beta}_{n,\theta}+1).$$

Using relation (4.2.25) and property of the Gamma function, for any $H \in [a,b]$, we have:

$$\prod_{k=1}^{n-1} \Gamma(\widetilde{\beta}_k+1) = \prod_{k=1}^{n-1} \Gamma\Big(1-\frac{1+\alpha_k}{4H_0}\Big) \leq \Big(\Gamma\big(1-\frac{3-4a}{4H_0}\big)\Big)^{n-1} =: \big(c^{(1)}_{H_0,a}\big)^{n-1}. \tag{5.2.15}$$

To bound $\Gamma(1+\widetilde{\beta}_{n,\theta})$, we need to study the range of possible values for $1-\frac{1+\alpha_n+2\theta}{4H_0}$. Since $\alpha_n \in \{0, 1-2H\}$, $H \in [a,b]$ and $\theta \in [0,1]$, we have

$$1-\frac{1-a+\theta}{2H_0} \leq 1-\frac{1-H+\theta}{2H_0} \leq 1-\frac{1+\alpha_n+2\theta}{4H_0} \leq 1-\frac{1}{4H_0}.$$

If we simply choose θ such that $\theta < 2H_0+a-1$, then the lower bound $1-\frac{1-a+\theta}{2H_0}$ can be in principle, be very close to 0: as θ approaches $2H_0+a-1$, the lower bound $1-\frac{1-a+\theta}{2H_0}$ approaches 0. This is a problem since $\lim\limits_{x\to 0^+}\Gamma(x)=\infty$, and hence, we are not able to find a lower bound for the term $\Gamma\Big(1-\frac{1+\alpha_n+2\theta}{4H_0}\Big)$. To avoid this problem, we choose an arbitrary value $c_0>0$ such that

$$c_0 < 1-\frac{1-a+\theta}{2H_0}.$$

Note that this is equivalent to $\theta < 2H_0(1-c_0)-(1-a)$. Since $\theta>0$, we must choose $c_0>0$ such that $2H_0(1-c_0)-(1-a)>0$, i.e.

$$c_0 < \frac{2H_0+a-1}{2H_0}.$$

With this choice of c_0 and θ, we have

$$\Gamma\Big(1-\frac{1+\alpha_n+2\theta}{4H_0}\Big) \leq \Gamma(c_0).$$

Hence, there exist a constant $c^{(1)}_{H_0,a}>0$ depending on H_0 and a such that

$$\prod_{k=1}^{n-1} \Gamma(\widetilde{\beta}_k+1) \times \Gamma(\widetilde{\beta}_{n,\theta}+1) = \prod_{k=1}^{n-1} \Gamma\Big(1-\frac{1+\alpha_k}{4H_0}\Big)\Gamma\Big(1-\frac{1+\alpha_n+2\theta}{4H_0}\Big) \leq \big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0). \tag{5.2.16}$$

Note that by (5.1.18), we know that

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + n = \frac{n(2H_0 + H - 1)}{2H_0} - \frac{1 + \alpha_n}{4H_0} - \frac{\theta}{2H_0}.$$

For any $H \in [a, b]$, the upper bound of $|\widetilde{\alpha}| + |\widetilde{\beta}| + n$ is

$$\begin{aligned} |\widetilde{\alpha}| + |\widetilde{\beta}| + n &\leq \frac{n(2H_0 + H - 1)}{2H_0} - \frac{1 + \alpha_n}{4H_0} \leq \frac{n(2H_0 + H - 1)}{2H_0} - \frac{1}{4H_0} \\ &\leq \frac{n(2H_0 + b - 1)}{2H_0} - \frac{1}{4H_0} \leq \frac{n(2H_0 + b - 1)}{2H_0} \end{aligned}$$

and the lower bound of $|\widetilde{\alpha}| + |\widetilde{\beta}| + n$ is

$$\begin{aligned} |\widetilde{\alpha}| + |\widetilde{\beta}| + n &\geq \frac{(n-1)(2H_0 + H - 1)}{2H_0} - \frac{1 + \alpha_n}{4H_0} \geq \frac{(n-1)(2H_0 + H - 1)}{2H_0} - \frac{1 - H}{2H_0} \\ &\geq \frac{(n-1)(2H_0 + a - 1)}{2H_0} - \frac{1 - a}{2H_0}, \end{aligned}$$

where in the first inequality, we used (5.2.11), which is a consequence of (5.2.4).

It follows that

$$t^{|\widetilde{\alpha}| + |\widetilde{\beta}| + n} \leq \begin{cases} 1 & \text{if } t < 1, \\ t^{\frac{n(2H_0 + b - 1)}{2H_0}} & \text{if } t \geq 1. \end{cases}$$

Therefore, we get:

$$t^{|\widetilde{\alpha}| + |\widetilde{\beta}| + n} \leq \big(t \vee 1\big)^{\frac{n(2H_0 + b - 1)}{2H_0}}. \tag{5.2.17}$$

Finally, we pick $m_0 \geq 1$ such that $\frac{(m_0 - 1)(2H_0 + a - 1)}{2H_0} - \frac{1 - a}{2H_0} > x_0$, where $x_0 \in (1, 2)$ is such that Gamma function $\Gamma(\cdot)$ is increasing on (x_0, ∞). For any $n \geq m_0$, we have

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + n > \frac{(n-1)(2H_0 + a - 1)}{2H_0} - \frac{1 - a}{2H_0} > x_0,$$

which implies that for any $H \in [a, b]$,

$$\Gamma\Big(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1\Big) > \Gamma\Big(\frac{(n-1)(2H_0 + a - 1)}{2H_0} - \frac{1 - a}{2H_0} + 1\Big).$$

By Lemma A.1.4, we know that there exists a positive constant $c_{H_0,a}^{(2)}$ depending on H_0 and a such that

$$\Gamma\Big(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1\Big) \geq (c_{H_0,a}^{(2)})^{n-1} \big[(n-1)!\big]^{\frac{2H_0 + a - 1}{2H_0}}. \tag{5.2.18}$$

We come back to (5.2.13). Using relations (5.2.14), (5.2.16), (5.2.17) and (5.2.18), we obtain:

$$\int_{0<t_1<\ldots<t_n<t}\prod_{k=1}^{n}t_k^{\widetilde{\alpha}_k}\prod_{k=1}^{n-1}\big(t_{k+1}-t_k\big)^{\widetilde{\beta}_k}\big(t-t_n\big)^{\widetilde{\beta}_{n,\theta}}\mathrm{d}\mathbf{t}$$
$$\leq \frac{c_{H_0,a,1}\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}}. \tag{5.2.19}$$

Returning to equality (5.2.10) and using relation (5.2.19) and the fact that $\mathrm{card}(D_n)=2^{n-1}$, we obtain:

$$\int_{[0,t]^n}\Big(\psi^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}}\mathrm{d}\mathbf{t}\leq h^{\frac{\theta}{2H_0}}(C^{(1)}_{t_0,\theta,T}2^n)^{\frac{1}{2H_0}}n!2^{n-1}\frac{c_{H_0,a,1}\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}}$$
$$\leq h^{\frac{\theta}{2H_0}}(C^{(1)}_{t_0,\theta,T}2^n)^{\frac{1}{2H_0}}2^{2n-1}\frac{c_{H_0,a,1}\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}}, \tag{5.2.20}$$

where for the first inequality, we used the fact that $C_{H,1}<1$ due to (4.2.19) and for the last inequality, we used fact that $n!\leq 2^n(n-1)!$.

We take power $2H_0$ in the above estimate. Then relation (5.2.6) becomes:

$$A^H_n(t,h)\leq b^n_{H_0}\left(h^{\frac{\theta}{2H_0}}(C^{(1)}_{t_0,\theta,T}2^n)^{\frac{1}{2H_0}}2^{2n-1}\frac{c_{H_0,a,1}\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\big(t\vee 1\big)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c^{(2)}_{H_0,a})^{n-1}\big[(n-1)!\big]^{\frac{a-1}{2H_0}}}\right)^{2H_0}$$
$$=b^n_{H_0}h^{\theta}2^{n(1+4H_0)-2H_0}C^{(1)}_{t_0,\theta,T}\frac{\Big(c_{H_0,a,1}\big(c^{(1)}_{H_0,a}\big)^{n-1}\Gamma(c_0)\Big)^{2H_0}\big(t\vee 1\big)^{n(2H_0+b-1)}}{\big(c^{(2)}_{H_0,a}\big)^{(n-1)2H_0}\big[(n-1)!\big]^{a-1}}. \tag{5.2.21}$$

Therefore, by relation (5.2.21), we conclude that

$$\sum_{n\geq 1}(p-1)^{n/2}\Big(\frac{1}{n!}A^H_n(t,h)\Big)^{1/2}\leq C^{(1)}_{p,a,b,\theta,c_0,H_0,T}\,h^{\theta/2}, \tag{5.2.22}$$

where $C^{(1)}_{p,a,b,\theta,c_0,H_0,T}>0$ is a constant depending on p, a, b, θ, c_0, H_0, and T.

As for the term $B^H_n(t,h)$, by (4.2.34) and (4.2.36), we have

$$B^H_n(t,h)\leq b^n_{H_0}\bigg(\int_{[0,t+h]^n}\big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}\bigg)^{2H_0}, \tag{5.2.23}$$

where $\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})$ is given in (4.2.35).

We estimate $\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})$. For this, we use the Fourier transform of the kernel $f_n(\cdot,t,x)$ given in Lemma 3.1.5 with $d=1$. Assume for simplicity that $0<t_1<\ldots<t_n$ and $t<t_n<t+h$. Similarly to (3.2.26), and using an argument similar to (5.1.9), followed by (5.1.11), we obtain:

$$\begin{aligned}\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t}) &\leq C_{t_0,d}\, c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^{n-1} \exp\Big(-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big|\sum_{j=1}^k t_j\xi_j\Big|^2\Big) \\ &\qquad \exp\Big(-\frac{t+h-t_n}{t_n(t+h)}\Big|\sum_{j=1}^n t_j\xi_j\Big|^2\Big) \prod_{k=1}^n |\xi_k|^{1-2H}\mathrm{d}\xi_k \\ &\leq C_{t_0,d}\, c_H^n \Big(\prod_{k=1}^n t_k\Big)^{2H-2} \sum_{\boldsymbol{\alpha}\in D_n} \Big(\prod_{k=1}^{n-1}\int_{\mathbb{R}} \exp\Big(-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\Big)|\eta_k|^{\alpha_k}\mathrm{d}\eta_k\Big) \\ &\qquad \Big(\int_{\mathbb{R}} \exp\Big(-\frac{t+h-t_n}{t_n(t+h)}\Big)|\eta_n|^{\alpha_n}\mathrm{d}\eta_n\Big) \\ &= C_{t_0,d}\, c_H^n \Big(\prod_{k=1}^n t_k\Big)^{2H-2} \sum_{\boldsymbol{\alpha}\in D_n} \prod_{k=1}^{n-1} \Gamma\Big(\frac{1+\alpha_k}{2}\Big)\Big(\frac{t_{k+1}-t_k}{t_{k+1}t_k}\Big)^{-\frac{1+\alpha_k}{2}} \\ &\qquad \times \Gamma\Big(\frac{1+\alpha_n}{2}\Big)\Big(\frac{t+h-t_n}{t_n(t+h)}\Big)^{-\frac{1+\alpha_n}{2}} \\ &\leq C_{t_0,d} \sum_{\boldsymbol{\alpha}\in D_n} t_1^{\frac{4H-3+\alpha_1}{2}} \Big(\prod_{k=2}^n t_k^{\frac{4H-2+\alpha_{k-1}+\alpha_k}{2}}\Big)(t+h)^{\frac{1+\alpha_n}{2}} \\ &\qquad \prod_{k=1}^{n-1}\big(t_{k+1}-t_k\big)^{-\frac{1+\alpha_k}{2}}\big(t+h-t_n\big)^{-\frac{1+\alpha_n}{2}} \end{aligned} \tag{5.2.24}$$

where for the last inequality, we used (4.2.39).

Hence, taking power $\frac{1}{2H_0}$ on both sides of relation (5.2.24), we obtain:

$$\Big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \leq C_{t_0,d}^{\frac{1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n}\prod_{k=1}^n t_k^{\widetilde{\alpha}_k}\big(t_{k+1}-t_k\big)^{\widetilde{\beta}_k}(t+h)^{\frac{1+\alpha_n}{4H_0}}, \tag{5.2.25}$$

where $\widetilde{\alpha}_k$ are given by (5.1.13) and $\widetilde{\beta}_k$ are given by (5.1.14) for all $k=1,\ldots,n$. A similar estimate holds for arbitrary $(t_1,\ldots,t_n)\in[0,t]^n$ with $t_{\rho(1)}<\ldots<t_{\rho(n)}$ for same permutation $\rho\in S_n$, where S_n is the set of all permutation of $\{1,\ldots,n\}$.

We now integrate $\big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})$ over $[0,t+h]^n$. Using relation (5.2.25), we get:

$$\int_{[0,t+h]^n} \Big(\gamma^H_{t,h,n}(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})\mathrm{d}\mathbf{t}$$

$$
\begin{aligned}
&= \sum_{\rho \in S_n} \int_t^{t+h} \left(\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}} \left(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}t_{\rho(1)} \ldots \mathrm{d}t_{\rho(n-1)} \right) \mathrm{d}t_{\rho(n)} \\
&\le C_{t_0,d}^{\frac{1}{2H_0}} n! \int_t^{t+h} \Bigg(\int_{0<t_1<\ldots<t_{n-1}<t_n} \\
&\qquad \sum_{\boldsymbol{\alpha} \in D_n} (t+h)^{\frac{1+\alpha_n}{4H_0}} \prod_{k=1}^{n-1} t_k^{\widetilde{\alpha}_k} (t_{k+1}-t_k)^{\widetilde{\beta}_k} (t+h-t_n)^{\widetilde{\beta}_n} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \Bigg) \mathrm{d}t_n \\
&\le C_{t_0,d}^{\frac{1}{2H_0}} n! \sum_{\boldsymbol{\alpha} \in D_n} (t+h)^{\frac{1+\alpha_n}{4H_0}} \int_t^{t+h} J(t_n) \cdot (t+h-t_n)^{\widetilde{\beta}_n} \mathrm{d}t_n
\end{aligned} \tag{5.2.26}
$$

where

$$
J(t_n) = \int_{0<t_1<\ldots<t_{n-1}<t_n} \prod_{k=1}^{n-1} t_k^{\widetilde{\alpha}_k} (t_{k+1}-t_k)^{\widetilde{\beta}_k} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1}.
$$

By Lemma A.3.2 (whose hypotheses are verified; see the statement after (5.1.14)), we have

$$
J(t_n) = \frac{\Gamma(\widetilde{\alpha}_1+1) \prod_{k=1}^{n-1} \Gamma(\widetilde{\beta}_k+1)}{\Gamma\big(\sum_{k=1}^{n-1} \widetilde{\alpha}_k + \sum_{k=1}^{n-1} \widetilde{\beta}_k + n\big)} \gamma_{n-1} t^{\sum_{k=1}^{n-1} \widetilde{\alpha}_k + \sum_{k=1}^{n-1} \widetilde{\beta}_k + n-1}, \tag{5.2.27}
$$

with

$$
\gamma_{n-1} = \gamma_n(\alpha_1, \ldots, \alpha_{n-1}) = \prod_{k=1}^{n-2} \frac{\Gamma\big(\sum_{i=1}^k (\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1}\big)}{\Gamma\big(\sum_{i=1}^k (\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1\big)}.
$$

What is new here is that we consider only the product of the first $n-1$ terms. Nevertheless, the same argument as in the the proof of Theorem 1.1 in [4] can be used to obtain that $\gamma_{n-1} \le 1$.

We now want to evaluate $J(t_n)$. Note that by (5.1.18), we see that

$$
\begin{aligned}
\sum_{k=1}^{n-1} \widetilde{\alpha}_k + \sum_{k=1}^{n-1} \widetilde{\beta}_k + n - 1 &= |\widetilde{\alpha}| + |\widetilde{\beta}| + n - \widetilde{\alpha}_n - \widetilde{\beta}_n - 1 \\
&= \frac{n(2H_0+H-1)}{2H_0} - \frac{4H-2+\alpha_{n-1}+\alpha_n}{4H_0} - 1.
\end{aligned}
$$

For any $H \in [a,b]$, the upper bound of $\sum_{k=1}^{n-1} \widetilde{\alpha}_k + \sum_{k=1}^{n-1} \widetilde{\beta}_k + n - 1$ is:

$$
\sum_{k=1}^{n-1} \widetilde{\alpha}_k + \sum_{k=1}^{n-1} \widetilde{\beta}_k + n - 1 \le \frac{n(2H_0+H-1)}{2H_0} - \frac{4H-2}{4H_0} - 1
$$

$$\leq \frac{n(2H_0+b-1)}{2H_0} - \frac{2H_0+2a-1}{2H_0}$$

and the lower bound of $\sum_{k=1}^{n-1}\widetilde{\alpha}_k + \sum_{k=1}^{n-1}\widetilde{\beta}_k + n - 1$ is:

$$\begin{aligned}\sum_{k=1}^{n-1}\widetilde{\alpha}_k + \sum_{k=1}^{n-1}\widetilde{\beta}_k + n - 1 &\geq \frac{n(2H_0+H-1)}{2H_0} - \frac{4H-2+2(1-2H)+(1-2H)}{4H_0} - 1\\ &= \frac{n(2H_0+H-1)}{2H_0} - \frac{2H_0+H-1}{2H_0} - \frac{3-4H}{4H_0}\\ &\geq \frac{(n-1)(2H_0+a-1)}{2H_0} - \frac{3-4a}{4H_0}. \qquad (5.2.28)\end{aligned}$$

By (5.2.28), it follows that

$$t^{\sum_{k=1}^{n-1}\widetilde{\alpha}_k+\sum_{k=1}^{n-1}\widetilde{\beta}_k+n-1} \leq \begin{cases} 1 & \text{if } t<1 \\ t^{\frac{n(2H_0+b-1)}{2H_0}-\frac{2H_0+2a-1}{2H_0}} & \text{if } t\geq 1 \end{cases} \leq (T\vee 1)^{\frac{n(2H_0+b-1)}{2H_0}-\frac{2H_0+2a-1}{2H_0}}. \qquad (5.2.29)$$

We pick $m_0 \geq 1$ such that

$$\frac{(m_0-1)(2H_0+a-1)}{2H_0} - \frac{3-4a}{4H_0} > x_0,$$

where $x_0 \in (1,2)$ is such that Gamma function $\Gamma(\cdot)$ is increasing on (x_0,∞). By relation (5.2.28), for any $n \geq m_0$, we have

$$\sum_{k=1}^{n-1}\widetilde{\alpha}_k + \sum_{k=1}^{n-1}\widetilde{\beta}_k + n - 1 \geq \frac{(n-1)(2H_0+a-1)}{2H_0} - \frac{3-4a}{4H_0} > x_0,$$

which implies that for any $n \geq m_0$, there exists a positive constant $c_{H_0,a}^{(3)}$ depending on H_0 and a such that

$$\begin{aligned}\Gamma\Big(\sum_{k=1}^{n-1}\widetilde{\alpha}_k + \sum_{k=1}^{n-1}\widetilde{\beta}_k + n\Big) &> \Gamma\Big(\frac{(n-1)(2H_0+a-1)}{2H_0} - \frac{3-4a}{4H_0} + 1\Big)\\ &\geq (c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}, \qquad (5.2.30)\end{aligned}$$

where for the last inequality, we used Lemma A.1.4.

We return to (5.2.27). Using relations (5.2.14), (5.2.15), (5.2.29) and (5.2.30), we get

$$J(t_n) \leq \frac{c_{H_0,a,1}\big(c_{H_0,a}^{(1)}\big)^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\big[(n-1)!\big]^{\frac{2H_0+a-1}{2H_0}}}\,(T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}}. \qquad (5.2.31)$$

We continue to integrate $\left(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t})$ over $[0,t+h]^n$. Using relations (5.2.26), (5.2.31) and the change of variable $s = t+h-t_n$, we have

$$\begin{aligned}
&\int_{[0,t+h]^n} \left(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t}) \mathrm{d}\mathbf{t} \\
&\quad \le C_{t_0,d}^{\frac{1}{2H_0}} n! \frac{c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}} (T\vee 1)^{\frac{n(2H_0+b-1)}{2H_0}-\frac{2H_0+2a-1}{2H_0}} \\
&\qquad\qquad \sum_{\boldsymbol{\alpha}\in D_n} (t+h)^{\frac{1+\alpha_n}{4H_0}} \int_t^{t+h} (t+h-t_n)^{-\frac{1+\alpha_n}{4H_0}} \mathrm{d}t_n \\
&\quad \le n! \frac{C_{t_0,d}^{\frac{1}{2H_0}} c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}}{(c_{H_0,a}^{(3)})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}} (T\vee 1)^{\frac{2-2H}{4H_0}+\frac{n(2H_0+b-1)}{2H_0}-\frac{2H_0+2a-1}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \int_0^h s^{-\frac{1+\alpha_n}{4H_0}} \mathrm{d}s \\
&\quad \le n! \frac{C_{t_0,d}^{\frac{1}{2H_0}} c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}(C_{T,a,b}\vee 1)}{(c_{H_0,a}^{(3)})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}} (T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} \frac{1}{1-\frac{1+\alpha_n}{4H_0}} h^{1-\frac{1+\alpha_n}{4H_0}}.
\end{aligned} \tag{5.2.32}$$

Note that, since $\alpha_n \in \{0, 1-2H\}$ and $H \in [a,b]$, we have

$$\frac{1}{1-\frac{1+\alpha_n}{4H_0}} < \frac{2H_0}{2H_0+H-1} < \frac{2H_0}{2H_0+a-1} =: c_{H_0,a}^{(4)}.$$

Recall that relation (5.2.4) implies that relation (5.2.11) holds. Hence, for any θ satisfying (5.2.12), we have

$$0 < \frac{\theta}{2H_0} < \frac{2H_0+a-1}{2H_0} \le 1 - \frac{1+\alpha_n}{4H_0} = \frac{4H_0-1-\alpha_n}{4H_0} \le 1 - \frac{1}{4H_0} \le \frac{3}{4}$$

which implies for any $h \in (0,1)$,

$$h^{1-\frac{1+\alpha_n}{4H_0}} \le h^{\frac{\theta}{2H_0}}.$$

Therefore, using the fact that $n! \le 2^n (n-1)!$ and relation (5.2.32), we obtain:

$$\begin{aligned}
&\int_{[0,t+h]^n} \left(\gamma_{t,h,n}^H(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} 1_{D_{t,h}}(\mathbf{t}) \mathrm{d}\mathbf{t} \\
&\quad \le n! \frac{C_{t_0,d}^{\frac{1}{2H_0}} c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}(C_{T,a,b}\vee 1)}{(c_{H_0,a}^{(3)})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}} (T\vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} c_{H_0,a}^{(4)} h^{\frac{\theta}{2H_0}}
\end{aligned}$$

$$\leq h^{\frac{\theta}{2H_0}} 2^n \frac{C_{t_0,d}^{\frac{1}{2H_0}} c_{H_0,a,1} \big(c_{H_0,a}^{(1)}\big)^{n-1} (C_{T,a,b} \vee 1)}{(c_{H_0,a}^{(3)})^{n-1} \big[(n-1)!\big]^{\frac{a-1}{2H_0}}} \, (T \vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} 2^{n-1} c_{H_0,a}^{(4)}.$$

We take power $2H_0$ in the above estimate, then relation (5.2.23) becomes:

$$B_n^H(t,h) \leq b_{H_0}^n \Bigg(h^{\frac{\theta}{2H_0}} 2^n \frac{C_{t_0,d}^{\frac{1}{2H_0}} c_{H_0,a,1} \big(c_{H_0,a}^{(1)}\big)^{n-1} (C_{T,a,b} \vee 1)}{(c_{H_0,a}^{(3)})^{n-1} \big[(n-1)!\big]^{\frac{a-1}{2H_0}}} \, (T \vee 1)^{\frac{(n-1)(2H_0+b-1)}{2H_0}} 2^{n-1} c_{H_0,a}^{(4)} \Bigg)^{2H_0}$$

$$= h^{\theta} b_{H_0}^n 2^{2H_0(2n-1)} \frac{C_{t_0,d} \big(c_{H_0,a}^{(1)}\big)^{(n-1)2H_0} \big(c_{H_0,a,1} c_{H_0,a}^{(4)}\big)^{2H_0} (C_{T,a,b} \vee 1)}{(c_{H_0,a}^{(3)})^{(n-1)2H_0} \big[(n-1)!\big]^{a-1}} (T \vee 1)^{(n-1)(2H_0+b-1)}.$$

Therefore, by above estimation, we conclude that

$$\sum_{n\geq 1} (p-1)^{n/2} \Big(\frac{1}{n!} B_n^H(t,h) \Big)^{1/2} \leq C_{p,a,b,H_0,T}^{(2)} \, h^{\theta/2}, \tag{5.2.33}$$

where $C_{p,a,b,H_0,T}^{(2)} > 0$ is a constant depending on p, a, b, c_0, H_0, and T.

Back to relation (5.2.5), combining relations (5.2.22) and (5.2.33), it follows that

$$\|u^H(t+h,x) - u^H(t,x)\|_p \leq \sqrt{2} \Big(C_{p,a,b,\theta,c_0,H_0,T}^{(1)} + C_{p,a,b,H_0,T}^{(2)} \Big) h^{\theta/2} \tag{5.2.34}$$

and therefore, taking power p and then taking supremum over $H \in [a,b]$ on both sides of relation (5.2.34), we have relation (5.2.2).

Step 2: We examine the spatial increments for the solution of the heat equation. For any $x, x' \in K$, we let $z = x' - x$. By (4.2.54), we have

$$\|u^H(t,x+z) - u^H(t,x)\|_p \leq \sum_{n\geq 1} (p-1)^{n/2} \Big(\frac{1}{n!} C_n^H(t,z) \Big)^{1/2}, \tag{5.2.35}$$

where $C_n^H(t,z)$ is given in (4.2.55). Recall that by (4.2.57), we have:

$$C_n^H(t,z) \leq b_{H_0}^n \bigg(\int_{[0,t]^n} \Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t}) \Big)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \bigg)^{2H_0}, \tag{5.2.36}$$

where $\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})$ is given by (4.2.56).

We estimate $\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})$. Assume for simplicity that $t_1 < \ldots < t_n < t_{n+1} = t$. By (3.2.36), there exists a constant $C_{t_0,\theta,T}^{(2)} > 0$ such that

$$\Big| \big(\mathcal{F} g_{\mathbf{t}}^{(n)}(\cdot,t,x+z) - \mathcal{F} g_{\mathbf{t}}^{(n)}(\cdot,t,x) \big)(\xi_1,\ldots,\xi_n) \Big|^2 \leq |z|^{2\theta} C_{t_0,\theta,T}^{(2)} \big(F_1 + F_2 \big)$$

where we recall that F_1 and F_2 are given by

$$\begin{aligned} F_1 &:= \prod_{k=1}^{n} \exp\Big(- \frac{t_{k+1} - t_k}{t_k t_{k+1}} \Big| \sum_{j=1}^{k} t_j \xi_j \Big|^2 \Big) \Big| \sum_{j=1}^{n} t_j \xi_j \Big|^{2\theta} \\ &\leq C_\theta T^\theta (t - t_n)^{-\theta} \prod_{k=1}^{n} \exp\Big(- \frac{1}{2} \frac{t_{k+1} - t_k}{t_k t_{k+1}} \Big| \sum_{i=1}^{k} t_i \xi_i \Big|^2 \Big), \end{aligned}$$

(where the inequality is due to (3.2.37)) and

$$F_2 := \prod_{k=1}^{n} \exp\Big(- \frac{t_{k+1} - t_k}{t_k t_{k+1}} \Big| \sum_{j=1}^{k} t_j \xi_j \Big|^2 \Big).$$

Hence, applying a similar approach as in (3.2.38), we have

$$\Psi^H_{t,z,n}(\mathbf{t}, \mathbf{t}) \leq |z|^{2\theta} C^{(3)}_{t_0,\theta,T} \big(I_1 + I_2 \big) \tag{5.2.37}$$

where

$$\begin{aligned} I_1 &:= C_\theta T^\theta (t - t_n)^{-\theta} c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\Big\{ - \frac{1}{2} \frac{t_{k+1} - t_k}{t_{k+1} t_k} \Big| \sum_{j=1}^{k} t_j \xi_j \Big|^2 \Big\} \big| \xi_k \big|^{1-2H} \mathrm{d}\boldsymbol{\xi} \\ &\leq C_\theta T^\theta (t - t_n)^{-\theta} c_H^n I^{(n)}_{t/2}(t_1/2, \ldots, t_n/2) \end{aligned}$$

and

$$I_2 := c_H^n \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\Big\{ - \frac{t_{k+1} - t_k}{t_{k+1} t_k} \Big| \sum_{j=1}^{k} t_j \xi_j \Big|^2 \Big\} \big| \xi_k \big|^{1-2H} \mathrm{d}\boldsymbol{\xi} = I^{(n)}_t(t_1, \ldots, t_n).$$

Here we used (5.2.7) for the estimate of I_1 and we recall that $I^{(n)}_t(t_1, \ldots, t_n)$ is given by (5.1.8). Recall that $c_H \leq \frac{1}{\sqrt{\pi}}$.

We now take power $1/(2H_0)$ on (5.2.37). We use the fact that for any $p \in (0, 1)$,

$$(a + b)^p \leq a^p + b^p, \text{ for all } a, b \in \mathbb{R}_+$$

we obtain:

$$\begin{aligned} \Big(\Psi^H_{t,z,n}(\mathbf{t}, \mathbf{t}) \Big)^{\frac{1}{2H_0}} &\leq |z|^{\frac{\theta}{H_0}} \big(C^{(3)}_{t_0,\theta,T} \big)^{\frac{1}{2H_0}} \\ &\Big(C_\theta^{\frac{1}{2H_0}} T^{\frac{\theta}{2H_0}} \Big((t - t_n)^{-\theta} I^{(n)}_{t/2}(t_1/2, \ldots, t_n/2) \Big)^{\frac{1}{2H_0}} + \Big(I^{(n)}_t(t_1, \ldots, t_n) \Big)^{\frac{1}{2H_0}} \Big). \end{aligned}$$

We estimate separately the two terms in the inequality above. We start with $C_\theta^{\frac{1}{2H_0}} T^{\frac{\theta}{2H_0}} \Big((t-t_n)^{-\theta} I_{t/2}^{(n)}(t_1/2,\dots,t_n/2)\Big)^{\frac{1}{2H_0}}$. Using (5.2.8), we have:

$$
\begin{aligned}
&C_\theta^{\frac{1}{2H_0}} T^{\frac{\theta}{2H_0}} \Big((t-t_n)^{-\theta} I_{t/2}^{(n)}(t_1/2,\dots,t_n/2)\Big)^{\frac{1}{2H_0}} \\
&\quad \le C_\theta^{\frac{1}{2H_0}} T^{\frac{\theta}{2H_0}} C_{H,1}^{\frac{n}{2H_0}} 2^{\frac{n}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} t^{\frac{1+\alpha_n}{4H_0}} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} \prod_{k=1}^{n-1} \big(t_{k+1}-t_k\big)^{\widetilde{\beta}_k} \big(t-t_n\big)^{\widetilde{\beta}_{n,\theta}}
\end{aligned}
$$

where $C_{H,1}$ is defined in (4.1.14) and $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\beta}_{n,\theta}$ are defined in (5.2.9).

For the second term, using (5.1.11) and taking power $\frac{1}{2H_0}$, we obtain:

$$
\Big(I_t^{(n)}(t_1,\dots,t_n)\Big)^{\frac{1}{2H_0}} \le C_{H,1}^{\frac{n}{2H_0}} \sum_{\alpha\in D_n} t^{\frac{1+\alpha_n}{4H_0}} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} (t_{k+1}-t_k)^{\widetilde{\beta}_k}
$$

where $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ are given in (5.1.14).

Putting these estimates for the two terms together, we obtain that for any $0 < t_1 < \ldots < t_n < t$, there exists a constant $C_{t_0,\theta,T}^{(4)} > 0$ such that

$$
\begin{aligned}
&\Big(\Psi_{t,z,n}^H(\mathbf{t},\mathbf{t})\Big)^{\frac{1}{2H_0}} \\
&\le |z|^{\frac{\theta}{H_0}} \big(C_{t_0,\theta,T}^{(4)}\big)^{\frac{1}{2H_0}} C_{H,1}^{\frac{n}{2H_0}} \Bigg(2^{\frac{n}{2H_0}} \sum_{\boldsymbol{\alpha}\in D_n} t^{\frac{1+\alpha_n}{4H_0}} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} \prod_{k=1}^{n-1} \big(t_{k+1}-t_k\big)^{\widetilde{\beta}_k} \big(t-t_n\big)^{\widetilde{\beta}_{n,\theta}} \\
&\qquad + \sum_{\boldsymbol{\alpha}\in D_n} t^{\frac{1+\alpha_n}{4H_0}} \prod_{k=1}^{n} t_k^{\widetilde{\alpha}_k} (t_{k+1}-t_k)^{\widetilde{\beta}_k} \Bigg). \qquad (5.2.38)
\end{aligned}
$$

A similar estimate holds for arbitrary $(t_1,\dots,t_n) \in [0,t]^n$ with $t_{\rho(1)} < \ldots < t_{\rho(n)}$ for same permutation $\rho \in S_n$, where S_n is the set of all permutation of $\{1,\dots,n\}$. Since the previous bound is the same for all $\rho \in S_n$, this introduce a factor $n!$. Recall that $C_{H,1} < 1$ (see (4.2.19)).

We now integrate $\big(\Psi_{t,h,n}^H(\mathbf{t},\mathbf{t})\big)^{\frac{1}{2H_0}}$ over $[0,t]^n$. For the first integral, we use (5.2.19), which gives a uniform bound for all $H \in [a,b]$. The second integral has been estimated in (5.1.22), but this bound was not uniform in H. Below we give a revised form of (5.1.22) which is valid for all $H \in [a,b]$. Recall that the integral appearing in (5.1.22) is given by (5.1.15). $\Gamma\Big(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1\Big)$ is bounded below by (5.2.18). Moreover, using (5.1.20), we have:

$$
\Gamma(\widetilde{\alpha}_1 + 1) \le C_{H_0,H}^{(2)} \le C_{H,2} \le c_{a,2},
$$

where $C_{H,2}$ and $c_{a,2}$ are given by (4.1.16), respectively (4.2.45). Using (5.1.21),

$$\prod_{k=1}^{n} \Gamma(\widetilde{\beta}_k + 1) \leq C_{H,2}^n \leq c_{a,2}^n.$$

By (5.1.17), the polynomial function of t in the right side of (5.1.15) is $t^{|\widetilde{\alpha}|+|\widetilde{\beta}|+n} = t^{n\frac{2H_0+H-1}{2H_0} - \frac{1+\alpha_n}{4H_0}}$ and the factor $t^{-\frac{1+\alpha_n}{4H_0}}$ cancels when we multiplying the integral by $t^{\frac{1+\alpha_n}{4H_0}}$, as required when integrating the right hand side of (5.2.38) with respect to $t_1, \ldots, t_n$. Finally,

$$t^{n\frac{2H_0+H-1}{2H_0}} \leq (t \vee 1)^{n\frac{2H_0+b-1}{2H_0}}.$$

Therefore, for any $H \in [a, b]$, we have:

$$t^{\frac{1+\alpha_n}{4H_0}} \int_{0<t_1<\ldots<t_n<t} \prod_{k=1}^{n} t_k^{\tilde{\alpha}_k} (t_{k+1} - t_k)^{\tilde{\beta}_k} \mathrm{d}\mathbf{t} \leq c_{a,2}^{n+1} \frac{(t \vee 1)^{n\frac{2H_0+b-1}{2H_0}}}{(c_{H_0,a}^{(2)})^{n-1} [(n-1)!]^{\frac{2H_0+a-1}{2H_0}}} \tag{5.2.39}$$

Using relations (5.2.10) and (5.2.39), we get:

$$\begin{aligned} &\int_{[0,t]^n} \left(\Psi_{t,z,n}^H(\mathbf{t}, \mathbf{t})\right)^{\frac{1}{2H_0}} \mathrm{d}\mathbf{t} \\ &\leq |z|^{\frac{\theta}{H_0}} \left(C_{t_0,\theta,T}^{(4)}\right)^{\frac{1}{2H_0}} 2^{\frac{n}{2H_0}} n! \frac{\left(c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}\Gamma(c_0) + c_{a,2}^{n+1}\right)(t \vee 1)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c_{H_0,a}^{(2)})^{n-1}\left[(n-1)!\right]^{\frac{2H_0+a-1}{2H_0}}} \\ &\leq |z|^{\frac{\theta}{H_0}} \left(C_{t_0,\theta,T}^{(4)}\right)^{\frac{1}{2H_0}} 2^{\frac{(1+2H_0)n}{2H_0}} \frac{\left(c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}\Gamma(c_0) + c_{a,2}^{n+1}\right)(t \vee 1)^{\frac{n(2H_0+b-1)}{2H_0}}}{(c_{H_0,a}^{(2)})^{n-1}\left[(n-1)!\right]^{\frac{a-1}{2H_0}}} \end{aligned}$$

where we used the fact that $n! \leq 2^n (n-1)!$.

Returning to (5.2.36), we obtain:

$$C_n^H(t, z) \leq |z|^{2\theta} b_{H_0}^n C_{t_0,\theta,T}^{(4)} 2^{(1+2H_0)n} \frac{\left(c_{H_0,a,1}\left(c_{H_0,a}^{(1)}\right)^{n-1}\Gamma(c_0) + c_{a,2}^{n+1}\right)^{2H_0} (T \vee 1)^{n(2H_0+b-1)}}{(c_{H_0,a}^{(2)})^{(n-1)2H_0}\left[(n-1)!\right]^{a-1}} \tag{5.2.40}$$

and therefore, for any $H \in [a, b]$

$$\sum_{n\geq 1} (p-1)^{n/2} \Big(\frac{1}{n!} C_n^H(t, z)\Big)^{1/2} \leq C_{t_0,T,p,a,b,c_0,\theta}^{(3)} |z|^{\theta},$$

where $C^{(3)}_{t_0,T,p,a,b,c_0,\theta}$ is a constant depending on t_0, T, p, a, b, c_0 and θ. This conclusion follows from (5.2.35).

∎

5.3 Continuity in law of the solution with respect to the noise parameter H

In this section, we consider equation (5.0.1) driven by the noise W^H introduced in Section 4.1, which is fractional in time with index $H_0 \in (1/2, 1)$ and rough in space with index $H \in (0, 1/2)$. Our goal is to show the weak convergence of the solution of this equation in the space of continuous functions $C([t_0, T] \times \mathbb{R})$, with respect to the noise parameter H. We use the same method as in Section 2.4.

As in Section 4.3, we construct a family of isonormal Gaussian process $\{W^H; H \in (0, 1/2)\}$ with covariance given by (4.1.1) which are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that relations (4.3.3) and (4.3.4) still hold in this case.

Recall that the solution has the Wiener chaos expansion (5.1.3). By definition, $u^H(t, x)$ is the $L^2(\Omega)$-limit of the sequence $\{u^H_m(t, x)\}_{m\geq 1}$ defined by

$$u^H_m(t, x) = \sum_{n=0}^{m} I^H_n(f_n(\cdot, t, x)). \tag{5.3.1}$$

This means that

$$\mathbb{E}\big|u^H_m(t, x) - u^H(t, x)\big|^2 \to 0, \quad \text{as } m \to \infty \tag{5.3.2}$$

for any $H \in (0, 1/2)$ fixed.

The proof of Theorem 5.0.1 is based on the following result.

Lemma 5.3.1. *Let $\ell = \max\{3/4 - H_0, 0\} < H^* < 1/2$ and $H_n \to H^*$, as $n \to \infty$. For all $k \geq 1$ and $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$\mathbb{E}\Big|I^{H_n}_k(f_k(\cdot, t, x)) - I^{H^*}_k(f_k(\cdot, t, x))\Big|^2 \to 0, \ \textit{as } n \to \infty. \tag{5.3.3}$$

Proof: We apply the same argument in the proof of Lemma 4.3.1. By relation (4.3.18), we have

$$Q_n = \mathbb{E}\big|I^{H_n}_k(f_k(\cdot, t, x)) - I^{H^*}_k(f_k(\cdot, t, x))\big|^2 \leq k!\, b^k_{H_0}\Big(\int_{T_k(t)} \big(A^{(n)}_k(\mathbf{t}, \mathbf{t})\big)^{1/2H_0} d\mathbf{t}\Big)^{2H_0},$$

where $A_k^{(n)}(\mathbf{t},\mathbf{t})$ is given by relation (4.3.8). What is new here, compared to Section 4.3, is the form of the Fourier transform of f_k in the space variables, which is now given by Lemma 3.1.5. Using the change of variables $z_i = t_i\xi_i$ for $i = 1, \dots, k$, followed by $\eta_k = \sum_{i=1}^k z_i$ for $k = 1, \dots, n$, we have

$$\begin{aligned}
A_k^{(n)}(\mathbf{t},\mathbf{t}) &\le w^2(t,x)\int_{\mathbb{R}^k}\prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_jt_{j+1}}\Big|\sum_{i=1}^j t_i\xi_i\Big|^2\Big) \\
&\qquad \Big|c_{H_n}^{k/2}\prod_{j=1}^k\big|\xi_j\big|^{\frac12-H_n} - c_{H^*}^{k/2}\prod_{j=1}^k\big|\xi_j\big|^{\frac12-H^*}\Big|^2 \mathrm{d}\boldsymbol{\xi} \\
&= w^2(t,x)\Big(\prod_{j=1}^k t_j^{-1}\Big)\int_{\mathbb{R}^k}\prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_jt_{j+1}}\Big|\sum_{i=1}^j z_i\Big|^2\Big) \\
&\qquad \Big|c_{H_n}^{k/2}\prod_{j=1}^k t_j^{H_n-\frac12}\prod_{j=1}^k\big|z_j\big|^{\frac12-H_n} - c_{H^*}^{k/2}\prod_{j=1}^k t_j^{H^*-\frac12}\prod_{j=1}^k\big|z_j\big|^{\frac12-H^*}\Big|^2 \mathrm{d}\boldsymbol{z} \\
&= w^2(t,x)\int_{\mathbb{R}^k}\prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_jt_{j+1}}\big|\eta_j\big|^2\Big) \\
&\qquad \Big|c_{H_n}^{k/2}\prod_{j=1}^k t_j^{H_n-1}\prod_{j=1}^k\big|\eta_j-\eta_{j-1}\big|^{\frac12-H_n} - c_{H^*}^{k/2}\prod_{j=1}^k t_j^{H^*-1}\prod_{j=1}^k\big|\eta_j-\eta_{j-1}\big|^{\frac12-H^*}\Big|^2 \mathrm{d}\boldsymbol{\eta}
\end{aligned} \tag{5.3.4}$$

where $\mathrm{d}\boldsymbol{z} = \mathrm{d}z_1\cdots\mathrm{d}z_k$, $\mathrm{d}\boldsymbol{\eta} = \mathrm{d}\eta_1\cdots\mathrm{d}\eta_k$ and $\eta_0 = 0$.

Hence, it suffices to show that

$$\int_{T_k(t)}\big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2H_0}\mathrm{d}\mathbf{t} \to 0, \quad \text{as } n\to\infty.$$

For this, we will use Dominated Convergence Theorem. Namely, we will prove that (4.3.20) and (4.3.21) hold for any $\mathbf{t}\in T_k(t)$, i.e.

$$A_k^{(n)}(\mathbf{t},\mathbf{t}) \to 0, \quad \text{as } n\to\infty \tag{5.3.5}$$

and there exists a function $h_k(\mathbf{t})$ such that

$$\big(A_k^{(n)}(\mathbf{t},\mathbf{t})\big)^{1/2H_0} \le h_k(\mathbf{t}) \ \text{ for all } n\ge 1, \mathbf{t}\in T_k(t) \ \text{ and } \int_{T_k(t)} h_k(\mathbf{t})\mathrm{d}\mathbf{t} < \infty. \tag{5.3.6}$$

We first prove (5.3.5). Let $\mathbf{t}\in T_n(t)$ be fixed. We denote $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ the integrand in (5.3.4). $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ converges to 0 as $n\to\infty$, since $H_n\to H^*$. So (5.3.5) will follow by Dominated Convergence Theorem. To justify the application of this theorem, we need to bound $B^{(n)}(\mathbf{t},\boldsymbol{\eta})$ by an integrable function, i.e. we have to show that there exists a function B such that

$$B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le B(\mathbf{t},\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbb{R}^k, n \ge 1 \text{ and } \int_{\mathbb{R}^k} B(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

Note that

$$B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le 2\big(B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) + B_2(\mathbf{t},\boldsymbol{\eta})\big)$$

where

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) := \prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_j t_{j+1}}\big|\eta_j\big|^2\Big) c_{H_n}^k \prod_{j=1}^k t_j^{2H_n-2} \prod_{j=1}^k \big|\eta_j-\eta_{j-1}\big|^{1-2H_n}$$

$$B_2(\mathbf{t},\boldsymbol{\eta}) := \prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_j t_{j+1}}\big|\eta_j\big|^2\Big) c_{H^*}^k \prod_{j=1}^k t_j^{2H^*-2} \prod_{j=1}^k \big|\eta_j-\eta_{j-1}\big|^{1-2H^*}.$$

(The terms $B_1^{(n)}$ and $B_2^{(n)}$ are different than (4.3.22) and (4.3.23).) The two terms $B_1^{(n)}(\mathbf{t},\boldsymbol{\eta})$ and $B_2(\mathbf{t},\boldsymbol{\eta})$ are of the same form and the second one does not depend on n. We will prove that there exists a function F such that for all $n \ge 1$, $\mathbf{t} \in T_k(t)$ and $\boldsymbol{\eta} \in \mathbb{R}^k$,

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le F(\mathbf{t},\boldsymbol{\eta}) \text{ and } \int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

Fix number a and b such that $\ell < a < H^* < b < 1/2$. Since $H_n \to H^*$, there exists $N \in \mathbb{N}$ such that

$$\ell < a \le H_n \le b < 1/2, \text{ for all } n \ge N.$$

Since all H_n are included in a compact set $[a,b]$ and the constant c_{H^*} defines a continuous function of H^*, we see that c_{H_n} is bounded by a constant $c > 0$. Invoking again relations (4.1.9) and (4.1.10), we obtain

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) \le c^k \sum_{\mathbf{a}\in A_k} \prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_j t_{j+1}}\big|\eta_j\big|^2\Big) \prod_{j=1}^k t_j^{2H_n-2} \prod_{j=1}^k \big|\eta_j\big|^{(1-2H_n)a_j},$$

Note that for any $H_n \in [a,b]$,

$$\prod_{j=1}^k t_j^{2H_n-2} = \Big(\prod_{j=1}^k t_j^{-1}\Big)^{2-2H_n} \le \prod_{j=1}^k \big(t_j^{-1} \vee 1\big)^{2-2H_n} \le \prod_{j=1}^k \big(t_j^{-1} \vee 1\big)^{2-2a}$$

$$= \prod_{j=1}^k \big(t_j \wedge 1\big)^{2a-2} \le \prod_{j=1}^k \big(t_j \vee 1\big)^{2a-2}.$$

Hence

$$B_1^{(n)}(\mathbf{t},\boldsymbol{\eta}) \leq c^k \Big(\prod_{j=1}^k (t_j \vee 1)^{2a-2}\Big) \sum_{\mathbf{a}\in A_k} \prod_{j=1}^k \exp\Big(-\frac{t_{j+1}-t_j}{t_j t_{j+1}}\Big|\eta_j\Big|^2\Big) \prod_{j=1}^k f_{a_j}(|\eta_j|) := F(\mathbf{t},\boldsymbol{\eta})$$

where for the last inequality, we used estimates in (4.3.14). By the same argument $B_2(\mathbf{t},\boldsymbol{\eta}) \leq F(\mathbf{t},\boldsymbol{\eta})$. Hence $B^{(n)}(\mathbf{t},\boldsymbol{\eta}) \leq 2F(\mathbf{t},\boldsymbol{\eta})$ and so

$$A_k^{(n)}(\mathbf{t},\mathbf{t}) = \int_{\mathbb{R}^k} B^{(n)}(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \leq 4\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}. \tag{5.3.7}$$

It remains to prove that $F(\mathbf{t},\cdot)$ is integrable on $\mathbb{R}^k$, i.e.

$$\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} < \infty.$$

We also give an estimate for this integral, which will be needed for the proof of relation (5.3.6) above. We have

$$\int_{\mathbb{R}^k} F(\mathbf{t},\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} = c^k \Big(\prod_{j=1}^k (t_j \vee 1)^{2a-2}\Big) \sum_{\mathbf{a}\in A_k} \prod_{j=1}^k I(a_j) \tag{5.3.8}$$

where

$$I(a_j) := \int_{\mathbb{R}} \exp\Big(-\frac{t_{j+1}-t_j}{t_j t_{j+1}}\Big|\eta_j\Big|^2\Big) f_{a_j}(|\eta_j|)\mathrm{d}\eta_j = \int_{\mathbb{R}} \big|\mathcal{F}G^h_{u_j}(\eta_j)\big|^2 f_{a_j}(|\eta_j|)\mathrm{d}\eta_j,$$

where $u_j = \frac{t_{j+1}-t_j}{t_j t_{j+1}}$, $j = 1,\ldots,k$. For fixed $\mathbf{a} \in A_k$, we evaluate separately the $\mathrm{d}\eta_j$ integral above. Using the same argument as for the study of $I(a_j)$ given by (4.3.25), with $t_{j+1}-t_j$ replaced by u_j, we obtain the following estimate which is the analogue of (4.3.26). More precisely, for all $j = 1,\ldots,k$ and for any $\mathbf{a} \in A_k$, we have

$$I(a_j) \leq c_a\Big\{(t_{j+1}-t_j)^{-\frac{1}{2}}(t_j t_{j+1})^{\frac{1}{2}} + (t_{j+1}-t_j)^{-(1-a)}(t_j t_{j+1})^{1-a} + (t_{j+1}-t_j)^{-\frac{3-4a}{2}}(t_j t_{j+1})^{\frac{3-4a}{2}}\Big\}.$$

We multiply this inequality by $(t_j \vee 1)^{2a-2}$. Note that

$$\begin{cases} (t_j \vee 1)^{2a-2}(t_j t_{j+1})^{\frac{1}{2}} \leq t_{j+1}^{\frac{1}{2}} \frac{1}{(t_j\vee 1)^{2-2a}}(t_j \vee 1)^{\frac{1}{2}} \leq t_{j+1}^{\frac{1}{2}}, & \text{since } \frac{1}{2} < 2-2a, \\ (t_j \vee 1)^{2a-2}(t_j t_{j+1})^{1-a} \leq t_{j+1}^{1-a} \frac{1}{(t_j\vee 1)^{1-a}}(t_j \vee 1)^{1-a} \leq t_{j+1}^{1-a}, & \text{since } 1-a < 2-2a, \\ (t_j \vee 1)^{2a-2}(t_j t_{j+1})^{\frac{3-4a}{2}} \leq t_{j+1}^{\frac{3-4a}{2}} \frac{1}{(t_j\vee 1)^{\frac{3-4a}{2}}}(t_j \vee 1)^{\frac{3-4a}{2}} \leq t_{j+1}^{\frac{3-4a}{2}}, & \text{since } \frac{3-4a}{2} < 2-2a. \end{cases}$$

Note that $\frac{1}{2} < 1-a < \frac{3-4a}{2}$. We have $t_{j+1}^{\frac{1}{2}}$, t_{j+1}^{1-a} and $t_{j+1}^{\frac{3-4a}{2}}$ are all dominated by $(T\vee 1)^{\frac{3-4a}{2}}$. Hence,

$$\begin{aligned}(t_j \vee 1)^{2a-2} I(a_j) &\leq c_a t_{j+1}^{\frac{3-4a}{2}} \left\{ (t_{j+1}-t_j)^{-\frac{1}{2}} + (t_{j+1}-t_j)^{-(1-a)} + (t_{j+1}-t_j)^{-\frac{3-4a}{2}} \right\} \\ &\leq c_a c_{t,a} (T\vee 1)^{\frac{3-4a}{2}} (t_{j+1}-t_j)^{-\frac{3-4a}{2}}\end{aligned}$$

where for the last inequality, we need (4.3.27).

Since $\mathrm{card}(A_k) = 2^{k-1}$, using (5.3.8), we obtain:

$$\int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta} \leq 2^{k-1} c^k c_a^k c_{t,a}^k (T\vee 1)^{k\cdot\frac{3-4a}{2}} \prod_{j=1}^{k} (t_{j+1}-t_j)^{-\frac{3-4a}{2}}. \tag{5.3.9}$$

This proves the integrability of $F(\mathbf{t}, \cdot)$ and so, relation (5.3.5) follows by the Dominated Convergence Theorem.

Now we prove (5.3.6). By relations (5.3.7) and (5.3.9) , for any $n \geq 1$ and $\mathbf{t} \in T_k(t)$, we have

$$\begin{aligned}\left(A_k^{(n)}(\mathbf{t},\mathbf{t})\right)^{\frac{1}{2H_0}} &\leq \left(4 \int_{\mathbb{R}^k} F(\mathbf{t}, \boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta}\right)^{\frac{1}{2H_0}} \\ &\leq \left(2^{k+1} c^k c_a^k c_{t,a}^k (T\vee 1)^{k\cdot\frac{3-4a}{2}} \prod_{j=1}^{k} (t_{j+1}-t_j)^{-\frac{3-4a}{2}}\right)^{\frac{1}{2H_0}} =: h_k(\mathbf{t}).\end{aligned}$$

Note that $h_k(\mathbf{t})$ is integrable over the symplex $T_k(t)$ by Lemma A.3.1 since $-\frac{3-4a}{4H_0} > -1$ which is equivalent to $a > \frac{3}{4} - H_0$.

∎

Proof of Theorem 5.0.1: From Theorem 1.3 of [4], we know that the process u^{H^*} has a continuous modification. We work with this modification, which we denote also by u^{H^*}. We need to prove the finite dimensional distribution convergence and the fact that the sequence of probability measures induced by $(u^{H_n})_{n\geq 1}$ is tight in the space of $C([t_0, T] \times \mathbb{R})$. A road map of this proof is the following:

$$\begin{cases} \text{Step 1: Finite dimensional distribution convergence} \\ \text{Step 2: Tightness} \end{cases}$$

Step 1: Finite dimensional distribution convergence

In this step, we have to prove that for any $k \geq 1$ and $(t_1, x_1), \ldots, (t_k, x_k) \in [0, T] \times \mathbb{R}$,

$$\big(u^{H_n}(t_1, x_1), \ldots, u^{H_n}(t_k, x_k)\big) \xrightarrow{\mathrm{d}} \big(u^{H^*}(t_1, x_1), \ldots, u^{H^*}(t_k, x_k)\big), \text{ as } n \to \infty.$$

For this, by Lemma 2.4.3, it will be enough to prove that for all $(t, x) \in [0, T] \times \mathbb{R}$

$$\mathbb{E}\Big|u^{H_n}(t, x) - u^{H^*}(t, x)\Big|^2 \to 0, \text{ as } n \to \infty. \tag{5.3.10}$$

To prove relation (5.3.10), we use the same argument as in the proof of Theorem 2.0.1. For part (c), using Lemma 5.3.1, it remains to show that for all compact set $[a, b]$,

$$\sup_{H \in [a,b]} \mathbb{E}\Big|u^H_m(t, x) - u^H(t, x)\Big|^2 \to 0, \text{ as } m \to \infty. \tag{5.3.11}$$

Recall that $u^H_m(t, x) = \sum_{k=0}^m I^H_k(f_k(\cdot, t, x))$ and $u^H(t, x) = \sum_{k \geq 0} I^H_k(f_k(\cdot, t, x))$. Hence

$$u^H(t, x) - u^H_m(t, x) = \sum_{k \geq m+1} I^H_k(f_k(\cdot, t, x)).$$

By the orthogonality of the Wiener chaos space and relations (5.1.12) and (5.1.15), we have

$$\begin{aligned}
\mathbb{E}\Big|u^H(t, x) - u^H_m(t, x)\Big|^2 &= \sum_{k \geq m+1} \mathbb{E}\big|I^H_k(f_k(\cdot, t, x))\big|^2 \\
&\leq \sum_{k \geq m+1} b^2_{H_0} w^2(t, x) C^k_{H,1} (k!)^{2H_0 - 1} \left(2^{k-1} \frac{\Gamma(1 + \widetilde{\alpha}_1) \prod_{j=1}^k \Gamma(1 + \widetilde{\beta}_j)}{\Gamma\big(|\widetilde{\alpha}| + |\widetilde{\beta}| + k + 1\big)} \gamma_k t^{|\widetilde{\alpha}| + |\widetilde{\beta}| + k + \frac{1+\alpha_k}{4H_0}}\right)^{2H_0},
\end{aligned} \tag{5.3.12}$$

where $\widetilde{\alpha}_j$, $\widetilde{\beta}_j$ are given in (5.1.13), respectively (5.1.14) with $|\widetilde{\alpha}| = \sum_{j=1}^k \widetilde{\alpha}_j$, $|\widetilde{\beta}| = \sum_{j=1}^k \widetilde{\beta}_j$, and γ_k is given by (5.1.16).

By (4.2.39), we know $C_{H,1} \leq 1$. Recall that in **Step 3** of the proof in Theorem 5.1.2, we saw that $\gamma_k \leq 1$. Note that by (5.1.18), for any $H \in [a, b]$, we have:

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + k + 1 > \frac{k(2H_0 + a - 1)}{2H_0} - \frac{1 - a}{4H_0} + 1$$

and

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + k + \frac{1 + \alpha_k}{4H_0} = \frac{k(2H_0 + H - 1)}{2H_0} < \frac{k(2H_0 + b - 1)}{2H_0}.$$

For k large enough, by Lemma A.1.4, there exists a constant $C^{(1)}_{H_0,a}$ such that

$$\Gamma\Big(|\widetilde{\alpha}| + |\widetilde{\beta}| + k + 1\Big) \geq \big(C^{(1)}_{H_0,a}\big)^k (k!)^{\frac{2H_0+a-1}{2H_0}}. \tag{5.3.13}$$

and

$$t^{2H_0\left(|\widetilde{\alpha}|+|\widetilde{\beta}|+k+\frac{1+\alpha_k}{4H_0}\right)} \leq \begin{cases} 1 & \text{if } t<1, \\ t^{k(2H_0+b-1)} & \text{if } t \geq 1. \end{cases}$$

Therefore, we get

$$t^{2H_0\left(|\widetilde{\alpha}|+|\widetilde{\beta}|+k+\frac{1+\alpha_k}{4H_0}\right)} \leq (t \vee 1)^{k(2H_0+b-1)}. \tag{5.3.14}$$

Finally, using relations (5.1.20) and (5.1.21), we have:

$$C^{(2)}_{H_0,a} \leq C_{H,2} \leq c_{a,2} \tag{5.3.15}$$

and

$$\prod_{j=1}^{k} \Gamma(\widetilde{\beta}_j + 1) \leq C^k_{H,2} \leq c^k_{a,2} \tag{5.3.16}$$

where $C_{H,2}$ is given in (4.1.16) and the inequalities (5.3.15) and (5.3.16) are due to (4.2.45).

We return to relation (5.3.12). Using relations (5.3.13), (5.3.14), (5.3.15) and (5.3.16), we obtain that there exists a constant $C_{H_0,a,b,1} > 0$ such that

$$\sup_{H\in[a,b]} \sum_{k\geq m+1} \mathbb{E}\big|I^H_k(f_k(\cdot,t,x))\big|^2 \leq \sum_{k\geq m+1} C^k_{H_0,a,b,1} \frac{(t\vee 1)^{k(2H_0+b-1)}}{(k!)^a}.$$

The last term clearly converges to 0 as $m \to \infty$.

Step 2: Tightness

This follows exactly as in the proof of Theorem 2.0.1, using the uniform bounds for the moments of the increments of solution given by Theorem 5.2.1.

∎

Appendix A

Auxiliary results

In this appendix chapter, we include various results which are used in this thesis.

A.1 Useful inequalities

In this section, we give some inequalities which play an important role in the thesis. We begin by recalling two integral inequalities: Minkowski's inequality and Hölder's inequality.

Lemma A.1.1 (Minkowski inequality for integrals, [39]). *For any measurable spaces* $(\mathcal{X}, \mathcal{E}, \mu)$ *and* $(\mathcal{Y}, \mathcal{F}, \nu)$ *and for any measurable function* $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$*, the following inequality holds for any* $p \geq 1$*:*

$$\Big\| \int_{\mathcal{X}} f(x, \cdot)\mu(\mathrm{d}x) \Big\|_{L^p(\mathcal{Y})} \leq \int_{\mathcal{X}} \|f(x, \cdot)\|_{L^p(\mathcal{Y})}\mu(\mathrm{d}x),$$

where $\|g\|_{L^P(\mathcal{Y})} = \Big(\int_{\mathcal{Y}} |g(y)|^p \nu(\mathrm{d}y) \Big)^{1/p}$.

Lemma A.1.2 (Hölder inequality for a finite measure space). *Let* $(E, \mathcal{E}, \mu)$ *be a finite measure space. If* $g : E \to \mathbb{R}$ *measurable, then for any* $p \geq 1$*,*

$$\left| \int_E g(x)\mu(\mathrm{d}x) \right|^p \leq [\mu(E)]^{p-1} \int_E |g(x)|^p \mu(\mathrm{d}x).$$

The following two results give some inequalities about the Gamma function.

Lemma A.1.3. *For any* $a > 0$*, there exist some constants* $C_1 > 0$ *and* $C_2 > 0$ *depending on* a *such that*

$$C_1 C_{a,1}^n (n!)^a \leq \Gamma(an + 1) \leq C_2 C_{a,2}^n (n!)^a,$$

where $C_{a,1} = \min\{a^a, a^a 2^{(1-a)/2}\}$ *and* $C_{a,2} = \max\{a^a, a^a 2^{(1-a)/2}\}$.

Proof: By Stirling formula, we know that

$$\Gamma(n+1) = n! \sim n^n e^{-n} (2\pi n)^{1/2}. \tag{A.1.1}$$

Therefore, taking any power $a > 0$ and multiplying $(2\pi n)^{-a/2}$ on both sides of relation (A.1.1), it follows that

$$(n^n e^{-n})^a \sim (n!)^a (2\pi n)^{-a/2}.$$

Hence,

$$\begin{aligned}\Gamma(an+1) &\sim (an)^{an} e^{-an} (2\pi an)^{1/2} = (n^n e^{-n})^a (2\pi an)^{1/2} a^{an} \\ &\sim (n!)^a (a)^{an+1/2} (2\pi n)^{(1-a)/2}.\end{aligned} \tag{A.1.2}$$

Using equation (A.1.2), we know that for any $a > 0$, there exists some constants $C_1 > 0$ and $C_2 > 0$ depending on a such that

$$C_1 K_n (n!)^a \le \Gamma(an+1) \le C_2 K_n (n!)^a, \tag{A.1.3}$$

where $K_n = a^{an} n^{(1-a)/2}$. To see this, we consider separately the case $a < 1$ and $a > 1$. When $a < 1$, $1 - a > 0$. For the upper bound, using the fact that $n \le 2^n$ for any $n \ge 1$, we have

$$n^{(1-a)/2} \le (2^n)^{(1-a)/2} = (2^{(1-a)/2})^n.$$

Therefore,

$$K_n \le a^{an} (2^{(1-a)/2})^n = (a^a 2^{(1-a)/2})^n. \tag{A.1.4}$$

For the lower bound, using the fact that $n^{(1-a)/2} \ge 1$ for any $a < 1$, we get

$$K_n \ge a^{an} = (a^a)^n. \tag{A.1.5}$$

We now consider the case when $a > 1$. For the upper bound, we have $n^{(1-a)/2} \le 1$ since $1 - a < 0$. Therefore,

$$K_n \le a^{an} = (a^a)^n. \tag{A.1.6}$$

For the lower bound, using the fact that $n^{(1-a)/2} \ge (2^n)^{(1-a)/2}$ for any $a > 1$, we obtain

$$K_n \ge a^{an} (2^{(1-a)/2})^n = (a^a 2^{(1-a)/2})^n. \tag{A.1.7}$$

Combining relations (A.1.4) and (A.1.6),

$$K_n \le \max\{(a^a 2^{(1-a)/2})^n, (a^a)^n\} := C_{a,2}^n$$

where $C_{a,2} = \max\{a^a, a^a 2^{(1-a)/2}\}$. Similarly, Combining relations (A.1.5) and (A.1.7), we have

$$K_n \ge \min\{(a^a 2^{(1-a)/2})^n, (a^a)^n\} := C_{a,1}^n$$

where $C_{a,1} = \min\{a^a, a^a 2^{(1-a)/2}\}$. ∎

Lemma A.1.4. *For any $a > 0$ and any $b \in \mathbb{R}$, there exist some positive constants c_b, C_b depending on b such that for any $n \geq 1$*

$$c_b^n (n!)^a \leq \Gamma(an + 1 + b) \leq C_b^n (n!)^a.$$

Proof: This follows from the relation

$$\lim_{n\to\infty} \frac{\Gamma(an+1+b)}{\Gamma(an+1)n^b} = 1$$

and Stirling's formula. See for instance relation (2.10) of [25]. ∎

The following result gives some estimates for the *Mittag-Leffler function* given by:

$$E_{a,b}(x) = \sum_{n\geq 0} \frac{x^n}{\Gamma(an+b)}, \; x > 0, a > 0, b > 0. \tag{A.1.8}$$

Lemma A.1.5. *For any $a > 0$, there exists some constants $C_a > 0$ and $C'_a > 0$ depending on a such that*

$$C'_a \exp\{x^{1/a}\} \leq \sum_{n\geq 0} \frac{x^n}{\Gamma(an+1)} \leq C_a \exp\{x^{1/a}\},$$

for all $x > 0$.

Proof: Using the fact that

$$\lim_{x\to\infty} \frac{E_{a,b}(x)}{x^{(1-b)/a} \exp\{x^{1/a}\}} = \frac{1}{a}.$$

When $b = 1$,

$$\lim_{x\to\infty} \frac{E_{a,1}(x)}{\exp\{x^{1/a}\}} = \frac{1}{a}.$$

Therefore, there exists a constant $C_a > 0$ depending on a such that

$$E_{a,1}(x) \leq C_a \exp\{x^{1/a}\}, \;\; \text{for all } x > 0. \tag{A.1.9}$$

Similarly, there exists a constant $C'_a > 0$ depending on a such that

$$E_{a,1}(x) \geq C'_a \exp\{x^{1/a}\}, \;\; \text{for all } x > 0. \tag{A.1.10}$$

∎

Combining Lemmas A.1.3 and A.1.5, we obtain the following result.

Lemma A.1.6. *Let $x > 0$ by arbitrary. For any $a > 0$, there exists some positive constants C_1, C_2, C_1' and C_2' depending on a such that*

$$C_1' \exp\{C_2' x^{1/a}\} \leq \sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_1 \exp\{C_2 x^{1/a}\}$$

Proof: By Lemma A.1.3, there exists some positive constants C_1^*, C_2^*, $C_{a,1}$ and $C_{a,2}$ depending on a, so that for all $n \geq 0$,

$$C_1^* C_{a,1}^n \leq \frac{\Gamma(an+1)}{(n!)^a} \leq C_2^* C_{a,2}^n.$$

Therefore,

$$C_1^* \frac{C_{a,1}^n}{\Gamma(an+1)} \leq \frac{1}{(n!)^a} \leq C_2^* \frac{C_{a,2}^n}{\Gamma(an+1)}. \tag{A.1.11}$$

Using the definition in (A.1.8) and taking the sum over $n \geq 0$ on relation (A.1.11), we obtain

$$C_1^* E_{a,1}(C_{a,1}x) \leq \sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_2^* E_{a,1}(C_{a,2}x). \tag{A.1.12}$$

By relation (A.1.9), there exists a constant $C_2 > 0$ depending on a such that

$$E_{a,1}(C_{a,2}x) \leq C_2 \exp\{(C_{a,2}x)^{1/a}\}.$$

Coming back to (A.1.12), it follows that

$$\sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_2^* C_2 \exp\{C_{a,2}^{1/a} x^{1/a}\}.$$

Similarly, by relation (A.1.10), there exists a constant $C_1 > 0$ depending on a such that

$$E_{a,1}(C_{a,1}x) \geq C_1 \exp\{(C_{a,1}x)^{1/a}\}.$$

Using relation (A.1.12), we obtain

$$\sum_{n\geq 0} \frac{x^n}{(n!)^a} \geq C_1^* C_1 \exp\{C_{a,1}^{1/a} x^{1/a}\}.$$

∎

Finally, we recall the Hardy-Littlewood-Sobolev Inequality.

Theorem A.1.7 (Theorem 1, page 119 of [39]). *For any $f \in L^1(\mathbb{R}_+)$ and $\alpha \in (0,1)$, we define the fractional integral of f of order α by:*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty |x-y|^{\alpha-1} f(y) \mathrm{d}y.$$

Then for any $\alpha \in (0,1)$ and $1 < p < q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \alpha$, we have

$$\|I^\alpha f\|_{L^q(0,\infty)} \leq A_{p,q} \|f\|_{L^p(0,\infty)} \tag{A.1.13}$$

where $A_{p,q} > 0$ is a constant depend on p and q.

The following result is very useful for estimating various integrals involving the fractional kernel $|t-s|^{2H-2}$ with $H \in (1/2, 1)$. It appeared for the first time in [33]. Its proof follows by applying Hölder's inequality with exponent $p = 1/H$, followed by Lemma A.1.7 with $\alpha = 2H - 1$.

Lemma A.1.8. *For any $H \in (\frac{1}{2}, 1)$ and for any function $\varphi \in L^{1/H}(\mathbb{R}_+)$,*

$$\alpha_H \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H-2} \, \mathrm{d}t\mathrm{d}s \leq b_H \Big(\int_0^\infty |\varphi(t)|^{1/H} \mathrm{d}t \Big)^{2H},$$

where $\alpha_H = H(2H-1)$ and $b_H > 0$ is a constant depend on H. More precisely, $b_H = \alpha_H \Gamma(2H-1) A_H$, where A_H is the constant appearing in inequality (A.1.13) with $\alpha = 2H-1$, $p = \frac{1}{H}$, and $q = \frac{1}{1-H}$.

A.2 Estimates for integrals involving the Riesz kernel

In this section, we present some estimates related to various integrals involving the Riesz kernel.

Lemma A.2.1. *For any $\alpha \in \big(\max\{d-2, 0\}, d\big)$, there exists a constant $K_{\alpha,d}$ depending on α and d such that*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathcal{F}G_t(\xi+\eta)\big|^2 |\xi|^{-\alpha} \mathrm{d}\xi \leq K_{d,\alpha} \, t^{r_\alpha},$$

where

$$K_{d,\alpha} = \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} \mathrm{d}\xi \leq c_d \Big(\frac{1}{d-\alpha} + \frac{1}{2-(d-\alpha)} \Big)$$

and

$$r_\alpha = \begin{cases} -(d-\alpha)/2 & \text{for heat equation,} \\ 2-(d-\alpha) & \text{for wave equation.} \end{cases}$$

Here we denote c_d the area of the unit sphere $S_1(0) = \{z \in \mathbb{R}^d; |z| = 1\}$ in $\mathbb{R}^d$.

Proof: We first consider the heat kernel. Recall that $\mathcal{F}G_t^h(\xi) = \exp\{-\frac{1}{2}t|\xi|^2\}$. We use the change of variable $\xi' = \sqrt{t}(\xi+\eta)$, then $\mathrm{d}\xi = t^{-d/2}\mathrm{d}\xi'$. It follows that

$$\int_{\mathbb{R}^d} \left|\mathcal{F}G_t^h(\xi+\eta)\right|^2 |\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} e^{-|\xi+\eta|^2 t}|\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} e^{-t\left|\frac{\xi'}{\sqrt{t}}\right|^2}\left|\frac{\xi'}{\sqrt{t}} - \eta\right|^{-\alpha} t^{-d/2}\mathrm{d}\xi'$$
$$= t^{-(d-\alpha)/2} I(\sqrt{t}\eta)$$

where for any $a \in \mathbb{R}^d$, we define

$$I(a) := \int_{\mathbb{R}^d} e^{-|\xi|^2}|\xi - a|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} e^{-|\xi+a|^2}|\xi|^{-\alpha}\mathrm{d}\xi.$$

We use the inequality $e^{-x} \le \frac{1}{1+x}$ for all $x \ge 0$. Hence,

$$I(a) \le \int_{\mathbb{R}^d} \frac{1}{1+|\xi+a|^2}|\xi|^{-\alpha}\mathrm{d}\xi \le \sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{1+|\xi+a|^2}|\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}|\xi|^{-\alpha}\mathrm{d}\xi,$$

where the equality follows by Lemma A.2.3 with $\beta = 1$. Note that for any $\alpha \in \big(\max\{d-2, 0\}, d\big)$,

$$K_{d,\alpha} = \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}|\xi|^{-\alpha}\mathrm{d}\xi < \infty.$$

To see this, we consider separately integrals on $\{|\xi| \le 1\}$ and $\{|\xi| > 1\}$.

Case 1: when $|\xi| \le 1$, we use the inequality $\frac{1}{1+|\xi|^2} \le 1$, and therefore,

$$\int_{|\xi|\le 1} \frac{1}{1+|\xi|^2}|\xi|^{-\alpha}\mathrm{d}\xi \le \int_{|\xi|\le 1} |\xi|^{-\alpha}\mathrm{d}\xi = c_d\int_0^1 r^{-\alpha}r^{d-1}\mathrm{d}r < \frac{c_d}{d-\alpha},$$

since $\alpha < d$.

Case 2: when $|\xi| > 1$, using the inequality $\frac{1}{1+|\xi|^2} \le \frac{1}{|\xi|^2}$, we get

$$\int_{|\xi|>1} \frac{1}{1+|\xi|^2}|\xi|^{-\alpha}\mathrm{d}\xi \le \int_{|\xi|>1} |\xi|^{-\alpha-2}\mathrm{d}\xi = c_d\int_1^\infty r^{-\alpha-2}r^{d-1}\mathrm{d}r < \frac{c_d}{\alpha-d+2},$$

since $\alpha > d-2$.

Now we study the wave kernel. Recall that $\mathcal{F}G^w(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}$. We use the change of variable $\xi' = t(\xi+\eta)$, then $\mathrm{d}\xi = t^{-d}\mathrm{d}\xi'$. We have

$$\int_{\mathbb{R}^d} \left|\mathcal{F}G_t^w(\xi+\eta)\right|^2 |\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2}|\xi|^{-\alpha}\mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{\sin^2(|\xi'|)}{|\xi'/t|^2}\left|\frac{\xi'}{t} - \eta\right|^{-\alpha} t^{-d}\mathrm{d}\xi'$$
$$= t^{2+\alpha-d}\int_{\mathbb{R}^d} \frac{\sin^2(|\xi|)}{|\xi|^2}|\xi - t\eta|^{-\alpha}\mathrm{d}\xi = t^{2+\alpha-d}J(t\eta)$$

where for any $a \in \mathbb{R}^d$, we define

$$J(a) = \int_{\mathbb{R}^d} \frac{\sin^2(|\xi|)}{|\xi|^2} |\xi - a|^{-\alpha} \mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{\sin^2(|\xi + a|)}{|\xi + a|^2} |\xi|^{-\alpha} \mathrm{d}\xi.$$

Using the inequality $\frac{\sin^2(x)}{x^2} \leq \frac{2}{1+x^2}$ for all $x > 0$, we get

$$\begin{aligned} J(a) &\leq 2 \int_{\mathbb{R}^d} \frac{1}{1 + |\xi + a|^2} |\xi|^{-\alpha} \mathrm{d}\xi \leq 2 \sup_{a \in \mathbb{R}^d} \frac{1}{1 + |\xi + a|^2} |\xi|^{-\alpha} \mathrm{d}\xi \\ &= 2 \int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} |\xi|^{-\alpha} \mathrm{d}\xi < \infty, \end{aligned}$$

for any $\alpha \in \big(\max\{d-2, 0\}, d\big)$, where for the equality, we used Lemma A.2.3 with $\beta = 1$. ∎

Lemma A.2.2. *Let $[a, b]$ be an arbitrary subset contained in $\big(\max\{d-2, 0\}, d\big)$. For any $\beta \in (\frac{d-a}{2}, 1)$ and for any $\alpha \in [a, b]$, we have*

$$K_{d,\beta,\alpha} := \int_{\mathbb{R}^d} \Big(\frac{1}{1 + |\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi < C_{d,a,b,\beta},$$

where $C_{d,a,b,\beta}$ is a positive constant depending on d, a, b and β.

Proof: We consider separately integrals on $\{|\xi| \leq 1\}$ and $\{|\xi| > 1\}$. Note that

$$K_{d,\beta,\alpha} = K^{(1)}_{d,\beta,\alpha} + K^{(2)}_{d,\beta,\alpha} \tag{A.2.1}$$

where

$$K^{(1)}_{d,\beta,\alpha} := \int_{|\xi| \leq 1} \Big(\frac{1}{1 + |\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \quad \text{and} \quad K^{(2)}_{d,\beta,\alpha} := \int_{|\xi| > 1} \Big(\frac{1}{1 + |\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi.$$

We first study the $K^{(1)}_{d,\beta,\alpha}$. If $|\xi| \leq 1$, then $\frac{1}{1+|\xi|^2} \leq 1$, and hence

$$K^{(1)}_{d,\beta,\alpha} \leq \int_{|\xi| \leq 1} |\xi|^{-\alpha} \mathrm{d}\xi = c_d \int_0^1 r^{-\alpha} r^{d-1} \mathrm{d}r = \frac{c_d}{d - \alpha} \leq \frac{c_d}{d - b},$$

since $\alpha < d$, where c_d is the area of the unit sphere $S_1(0) = \{z \in \mathbb{R}^d; |z| = 1\}$ in $\mathbb{R}^d$.

Next, we consider $K^{(2)}_{d,\beta,\alpha}$. If $|\xi| > 1$,then we use the fact that $\frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$, and hence

$$K^{(2)}_{d,\beta,\alpha} \leq \int_{|\xi| > 1} \Big(\frac{1}{|\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi = c_d \int_1^{\infty} r^{-\alpha - 2\beta} r^{d-1} \mathrm{d}r = \frac{c_d}{\alpha - d + 2\beta} \leq \frac{c_d}{a - d + 2\beta},$$

since $\alpha > d - 2\beta$. Therefore, there exists a constant $C_{d,a,b,\beta} > 0$ such that

$$\int_{\mathbb{R}^d} \Big(\frac{1}{1+|\xi|^2}\Big)^{\beta} |\xi|^{-\alpha} \mathrm{d}\xi \leq c_d \Big(\frac{1}{d-b} + \frac{1}{a-d+2\beta}\Big) < C_{d,a,b,\beta}.$$

∎

We recall also the following result, which has been used frequently in the thesis, and is valid for an arbitrary function f (not just for the Riesz kernel).

Lemma A.2.3 (Lemma 4.1 of [9]). *Let μ be a tempered measure on $\mathbb{R}^d$ whose Fourier transform in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ is a locally-integrable function $f : \mathbb{R}^d \to [0, \infty]$ such that $f(x) < \infty$ a.e. Then for any $\beta > 0$,*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi+\eta|^2}\right)^{\beta} \mu(\mathrm{d}\xi) = \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{\beta} \mu(\mathrm{d}\xi).$$

A.3 Beta-type integrals on the simplex

In this appendix, we give two results about some beta-type integrals on the simplex. Note that Lemma A.3.2 is an extension of Lemma A.3.1, which is used in Chapters 3 and 5 of the thesis, for the case of the rough initial condition.

Lemma A.3.1. *Let $T_n(t) = \{0 < t_1 < \ldots < t_n < t\}$ be the simplex. For any $\beta_1, \ldots, \beta_n > -1$,*

$$\int_{T_n(t)} \prod_{j=1}^{n} (t_{j+1} - t_j)^{\beta_j} \mathrm{d}t_1 \cdots \mathrm{d}t_n = \frac{\prod_{j=1}^{n} \Gamma(\beta_j + 1) t^{|\beta|+n}}{\Gamma(|\beta| + n + 1)} \tag{A.3.1}$$

where $|\beta| = \sum_{j=1}^{n} \beta_j$.

Lemma A.3.2 (Lemma A.1 in [4]). *Suppose that $\alpha_i > -1$, $\beta_i > -1$ for any $i = 1, \ldots, n$, and*

$$\sum_{i=1}^{k} (\alpha_i + \beta_i) + k + 1 + \alpha_{k+1} > 0 \ \text{ for all } k = 1, \ldots, n-1. \tag{A.3.2}$$

Then by setting $t_{n+1} = t$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $|\beta| = \sum_{i=1}^{n} \beta_i$, we have that

$$I_n(t, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) := \int_{0<t_1<\ldots<t_n<t} \prod_{i=1}^{n} t_i^{\alpha_i} (t_{i+1} - t_i)^{\beta_i} \mathrm{d}\mathbf{t}$$

$$= \frac{\Gamma(\alpha_1 + 1) \prod_{i=1}^{n} \Gamma(\beta_i + 1)}{\Gamma\big(|\alpha| + |\beta| + n + 1\big)} \prod_{k=1}^{n-1} \frac{\Gamma\big(\sum_{i=1}^{k} (\alpha_i + \beta_i) + k + 1 + \alpha_{k+1}\big)}{\Gamma\big(\sum_{i=1}^{k} (\alpha_i + \beta_i) + k + 1\big)} t^{|\alpha|+|\beta|+n}.$$

A.4 Convergence of probability measures

In this section we consider the convergence of probability measures on the space $C([0,1]\times\mathbb{R})$ of continuous functions $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, equipped with the sup norm topology.

Recall that a collection $X=\{X(t,x)\}_{(t,x)\in[0,1]\times\mathbb{R}}$ of random variables defined on the same probability space is called a **multi-parameter stochastic process** or **random field**.

A random element in $C([0,1]\times\mathbb{R})$ is a function $X:\Omega\to C([0,1]\times\mathbb{R})$ which is $\mathcal{F}/\mathcal{C}([0,1]\times\mathbb{R})$-measurable, where $\mathcal{C}([0,1]\times\mathbb{R})$ is the Borel σ-field on $C([0,1]\times\mathbb{R})$. It can be proved that $\mathcal{C}([0,1]\times\mathbb{R})$ coincides with the σ-field generated by the projection maps $\pi_{t,x}:C([0,1]\times\mathbb{R})\to\mathbb{R}$ given by

$$\pi_{t,x}(f)=f(t,x),\ \text{for all } t\in[0,1],x\in\mathbb{R}.$$

Hence any random field $X=\Big\{X(t,x)\Big\}_{t\in[0,1],\ x\in\mathbb{R}}$ with continuous paths is a random element in $C([0,1]\times\mathbb{R})$.

If $(X_n)_{n\geq1}$ and X are processes with sample paths in $C([0,1]\times\mathbb{R}^d)$, we say that $(X_n)_{n\geq1}$ *converges in distribution* to X (and we write $X_n\xrightarrow{d}X$) if $(\mu_{X_n})_{n\geq1}$ converges weakly to μ_X, where μ_{X_n},μ_X are the laws of X_n, respectively X on $C([0,1]\times\mathbb{R}^d)$.

We recall the following well-known result, which can be proved by classical methods (see Theorem 8.1 of [12] for the corresponding result for $C([0,1])$).

Theorem A.4.1. *Let $X_n=\{X_n(t,x)\}$ and $X=\{X(t,x)\}$ be random fields with sample paths in $C([0,1]\times\mathbb{R})$. If the following conditions hold:*

(i) $\Big(X_n(t_1,x_1),...,X_n(t_k,x_k)\Big)\xrightarrow{d}\Big(X(t_1,x_1),...,X(t_k,x_k)\Big)$ *for all* $t_1,...,t_k\in[0,1]$, $x_1,...,x_k\in\mathbb{R}$ *and* $k\geq1$,

(ii) $\big(\mu_{X_n}\big)_n$ *is tight, where μ_n is the law of X_n on $C([0,1]\times\mathbb{R})$,*

then $X_n\xrightarrow{d}X$ in $C([0,1]\times\mathbb{R})$.

The following theorem gives a useful criterion for proving tightness, and is borrowed from [43]. Note that the corresponding criterion for the tightness for probability measures on $C([0,1])$ is not explicitly stated in Billingsley's book [12].

Theorem A.4.2 (Proposition 2.3 of [43])**.** *Let $X_n=\{X_n(t,x)\}_{t\in[0,1],\ x\in\mathbb{R}},n\geq1$ be random fields with sample paths in $C([0,1]\times\mathbb{R})$. Suppose that the following two conditions hold:*

(i) $\sup_{n\geq1}\mathbb{E}|X_n(0,0)|^{p'}<\infty,$ *for some* $p'>0$,

(ii) for any compact set $J \subset \mathbb{R}$, there exist $p > 0, \delta > 2, C > 0$ and $N \in \mathbb{N}$ such that

$$\mathbb{E}\Big|X_n(t', x') - X_n(t, x)\Big|^p \leq C\Big(|t' - t| + |x' - x|\Big)^{\delta}, \tag{A.4.1}$$

for all $t', t \in [0, 1], x', x \in J$, and $n \geq N$,

then $(X_n)_n$ is tight in $C([0, 1] \times \mathbb{R})$.

Combining Theorems A.4.1 and A.4.2, we obtain the following result, which is used several times in the thesis.

Theorem A.4.3. *Let $X_n = \{X_n(t, x)\}$ and $X = \{X(t, x)\}$ be random fields with sample paths in $C([0, 1] \times \mathbb{R})$. If the following conditions hold:*

(i) $\Big(X_n(t_1, x_1), ..., X_n(t_k, x_k)\Big) \xrightarrow{d} \Big(X(t_1, x_1), ..., X(t_k, x_k)\Big)$ for all $t_1, ..., t_k \in [0, 1]$, $x_1, ..., x_k \in \mathbb{R}$ and $k \geq 1$,

(ii) $\sup_{n\geq 1} \mathbb{E}|X_n(0, 0)|^{p'} < \infty$, for some $p' > 0$,

(iii) for any compact set $J \subset \mathbb{R}$, there exist $p > 0, \delta > 2, C > 0$ and $N \in \mathbb{N}$ such that relation (A.4.1) *holds, for all $t', t \in [0, 1], x', x \in J$, and $n \geq N$,*

then $X_n \xrightarrow{d} X$ in $C([0, 1] \times \mathbb{R})$.

Remark A.4.4. Theorem A.4.3 remains valid (with obvious modification) for random fields with sample paths in $C([0, T] \times \mathbb{R})$ for any $T > 0$.

A.5 Basic results from Malliavin calculus

In this section, we recall some basic concepts and facts about Malliavin calculus, and we refer the readers to the monograph [38] for more details. Since the noise introduced in Section 2.1 is not a martingale in time, the stochastic integral with respect to W cannot be defined in the Itô sense. To define the concept of solution we use the divergence operator from Malliavin calculus (which coincides with the Skorohod integral, in the case of the Brownian motion).

Let $W = \{W(\varphi); \varphi \in \mathcal{H}\}$ be an isonormal Gaussian process, i.e. a zero-mean Gaussian process indexed by elements of a Hilbert space $\mathcal{H}$ such that

$$E[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

Every square-integrable random variable $F \in L^2(\Omega)$ which is measurable with respect to W has the Wiener chaos expansion:

$$F = \mathbb{E}[F] + \sum_{n\geq 1} I_n(f_n), \text{ for some } f_n \in \mathcal{H}^{\otimes n}, \tag{A.5.1}$$

where $\mathcal{H}^{\otimes n}$ is the n-th tensor product of $\mathcal{H}$ and I_n is the multiple Wiener integral with respect to W. Note that

$$\mathbb{E}[I_n(f)I_m(g)] = \begin{cases} n!\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{H}^{\otimes n}} & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

where $\tilde{f}$ is the symmetrization of f in all n variables:

$$\tilde{f}(x_1, \cdots, x_n) := \frac{1}{n!} \sum_{\rho \in S_n} f(x_{\rho(1)}, \cdots, x_{\rho(n)}),$$

and S_n is the set of all permutations of $\{1, 2, \cdots, n\}$. If F has the chaos expansion (A.5.1), then

$$E|F|^2 = \sum_{n\geq 0} E|I_n(f_n)|^2 = \sum_{n\geq 0} n!\|\tilde{f}_n\|^2_{\mathcal{H}^{\otimes n}}.$$

Let $\mathcal{S}$ be the class of "smooth" random variables, i.e variables of the form

$$F = f(W(\varphi_1), \cdots, W(\varphi_n)), \tag{A.5.2}$$

where $f \in C_b^\infty(\mathbb{R}^n), \varphi_i \in \mathcal{H}, n \geq 1$, and $C_b^\infty(\mathbb{R}^n)$ is the class of bounded C^∞-functions on $\mathbb{R}^n$, whose partial derivatives of all orders are bounded.

Definition A.5.1 (Malliavin derivative)**.** *The* Malliavin derivative *of a smooth random variable F of the form* (A.5.2) *is the $\mathcal{H}$-valued random variable given by:*

$$DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(W\big(\varphi_1), \ldots, W(\varphi_n)\big)\Big)\varphi_i.$$

We endow $\mathcal{S}$ with the norm

$$\|F\|_{\mathbb{D}^{1,2}} := (\mathbb{E}|F|^2)^{1/2} + (\mathbb{E}\|DF\|^2_{\mathcal{H}})^{1/2}.$$

The operator D can be extended to the space $\mathbb{D}^{1,2}$, the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$.

Definition A.5.2 (Divergence operator)**.** *The divergence operator δ is the adjoint of the operator D. The domain of δ, denoted by Dom(δ), is the set of $u \in L^2(\Omega, \mathcal{H})$ such that*

$$|\mathbb{E}\langle DF, u\rangle_{\mathcal{H}}| \leq c(\mathbb{E}|F|^2)^{1/2}, \quad \text{for all } F \in \mathbb{D}^{1,2}$$

where c is a constant depending on u.

If $u \in \mathrm{Dom}(\delta)$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$\mathbb{E}[F\delta(u)] = \mathbb{E}\langle DF, u\rangle_{\mathcal{H}}, \quad \text{for all } F \in \mathbb{D}^{1,2}.$$

In particular, $\mathbb{E}[\delta(u)] = 0$. If $u \in \mathrm{Dom}(\delta)$, we use the notation

$$\delta(u) = \int_{\mathbb{R}^d} u(x) W(\delta x),$$

and we say that $\delta(u)$ is the Skorohod integral of u with respect to W.

The following lemma is used in the proof of Theorems 2.2.4 and 2.2.6. We refer the reader to page 62 of [38] for more details.

Lemma A.5.3 (Hypercontractivity)**.** *Let $\mathcal{H}_n$ be the n-th Wiener chaos space. For any $F \in \mathcal{H}_n$,*

$$\|F\|_p \leq (p-1)^{n/2} \|F\|_2 \tag{A.5.3}$$

for any $p \geq 2$, where $\|F\|_p = \left(\mathbb{E}|F|^p\right)^{1/p}$ is the norm in $L^p(\Omega)$.

Proof: We use the fact that for any $1 < p < q < \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on any Wiener chaos $\mathcal{H}_n$. More precisely, let $t > 0$ such that $q = 1 + e^{2t}(p-1)$. Then for every $F \in \mathcal{H}_n$, we have

$$e^{-nt}\|F\|_q \leq \|F\|_p. \tag{A.5.4}$$

(see page 62 of [38]). Taking $t = \frac{1}{2}\ln\left(\frac{q-1}{p-1}\right)$, by relation (A.5.4), we have

$$\|F\|_q \leq e^{nt}\|F\|_p = \left(\frac{q-1}{p-1}\right)^{n/2}\|F\|_p. \tag{A.5.5}$$

In relation (A.5.5), if we replace the values (q,p) by $(p,2)$, we obtain (A.5.3). ∎

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