

# **Infinite Choice Transferable Utility Games and an Application to Wage Distribution in Firms**

**by Bruce Calvert**

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**Supervisor: Professor Roland Pongou**

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## Abstract

*Much work has been done on extending the Shapley value to finite multi-choice transferable utility games. However, little of this work has dealt with infinite choice games. This paper extends the Shapley value to infinite choice games and shows that this Shapley value satisfies six simple axioms: weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance. In addition, this paper gives a similar axiomatization for extending the Shapley value to finite multi-choice games. Justification for using this axiomatization for wage distribution in firm problems is given and compared to the justification for the Aumann-Shapley value. Taking advantage of the fact that total-factor productivity is positively correlated with firm size, this paper uses 2012 Current Population Survey data to test if workers are being paid their expected Shapley value in earnings. Union membership is found to have a positive effect on bargaining strength that is statistically significant at the 95% confidence level.*

## 1. Introduction

Income inequality and income distribution have become issues of increasing concern over the past few decades. Income inequality, as measured by the Gini coefficient, has risen in a large majority of OECD countries in the three decades prior to 2011 (OECD 2011), the income share received by the top 1% of income earners has more than doubled from 1976 to 2014 (Piketty and Saez 2003) and real median household income in the US has slowly declined since its peak in 1999 (United States Bureau of the Census 2016). There have been many explanations for the observed increase in income inequality over the past few decades. Increasing income inequality has been linked to declining unionization (Card et al. 2003), declining real values of minimum wages (Autor et al. 2010), decreasing tax levels and tax progressivity (Wagstaff et al. 1999), globalization (Alderson and Nielson 2002), and skill biased technological change (Vivarelli 2012). This increasing income inequality may cause one to ask the question 'is the current income distribution fair?'

There may be reasons why people are not being paid a wage equal to their marginal product of labour. Firstly, the decisions of a firm are often made by a CEO or other executives who have interests that do not align with the best interests of the firm. In particular, these individuals have incentives to increase their own pay even at the expense of the firm. A study by Cooper et al. (2014) found that CEO pay is negatively related to future stock returns, suggesting that CEOs overpay themselves. Secondly, low income individuals tend to be more risk averse, which may reduce their negotiating power and thus their relative income. This is consistent with economic theory, which tends to find that "an agent's degree of risk aversion... reduces the agent's bargaining power" (Li et al. 2015). In addition, it has been found that sense of power leads to increased risk taking (Anderson and Galinsky 2006), which can result in high income individuals taking more risks leading to even higher income. Finally, some individuals may have more 'friends' within a firm, who they can collude with in order to increase their negotiating power and thus their relative wages.

Whatever the reason for the departure between marginal product of labour and observed wages, if such a departure exists it may be empirically observable. However, an individual's marginal product of labour within a firm is something that can be difficult to evaluate. An individual within a firm has interactions with many other individuals and they may help the firm be productive not just through the work they do on their own, but also in how they interact with others. This problem of interdependence means that some concept similar to marginal product of labour needs to be used in order to test for such a departure. This is where the concept of an allocation procedure comes in, a procedure by which one can take information about how a worker's effort (where effort is something empirically observable such as hours worked) influences a firm's output and use that information to determine how much each worker should be paid. One approach to determine what allocation procedure should be used is to justify some basic axioms that one thinks a reasonable allocation procedure should have until there are enough axioms to define an allocation procedure.

Two popular axioms for justifying an allocation procedure are the symmetry axiom and the efficiency axiom. In the symmetry axiom, two workers that are equally productive and contribute equally to the firm should get the same pay. This axiom is often justified on grounds of egalitarianism in that people should get the equal pay for equal work. This paper uses a weak symmetry axiom for multi-choice games which is weaker than the symmetry axiom used in other papers. As for the efficiency axiom, this is satisfied if the sum of the wages to the workers (shareholders can be thought of as workers in that they contribute to the firm's output) equals the total profits of the firm. If total profits are less than the sum of wages then the firm would go bankrupt, so such an arrangement is unfeasible. Alternatively, if the sum of wages is less than the total profits then it would be possible to make 1 worker better off without making any workers worse off, so such an allocation is undesirable. Thus, efficiency can be justified on the grounds that an allocation that is both feasible and maximizes the wellbeing of workers must necessarily be efficient.

Another commonly used axiom is the null worker axiom. In this axiom, if a worker's effort does not affect the output (such a worker is called a null worker) then that individual's allocation is zero. This axiom can be justified on the basis that one should not get paid if they do not work. Alternatively, the null worker axiom can be justified by appealing to practicality. If this axiom does not hold then one could take a random firm in the US, add a random person in Mongolia to the firm and argue that this person should get some share of the profits; one can continue this process until every single person on the planet is added to the firm, in which case the allocation problem becomes impractically large and difficult to solve. A third way to justify this axiom is by noting that there is zero incentive for the non-null workers to cooperate and share firm output with null workers; the non-null workers can simply work together, share profits and exclude null workers. Thus the null worker axiom can be thought of as a

feasibility constraint. A similar axiom called no pay for no work, where workers that do not work get zero pay, can be justified on similar grounds.

The additivity axiom is also a popular axiom. In the additivity axiom, if one can represent a firm as a sum of two firms then the allocation and individual should receive under the first firm should equal the sum the individual receives under the latter two firms. There are some cases where additivity can make intuitive sense, such as if one takes a firm where workers work 5 days a week and treats it as 5 separate firms for each day of the week; in this case the behaviour of the workers and the output they produce has not changed so the allocation of output should not change, thus additivity should be satisfied. Alternatively, the additivity axiom can be justified on grounds of simplicity in that linear or additive allocation procedures tend to be simpler and therefore easier to calculate and use in practice. It has been shown that symmetry, efficiency, additivity and the null worker axiom uniquely characterize an allocation procedure known as the Shapley value (Shapley 1953) for situations where the level of effort is binary for all workers (i.e. either the workers work or do not work).

In realistic scenarios, workers in a firm have more options than work or not work. A worker can choose to work hard, sleep on the job, or trick a co-worker into doing their work for them. There may be many ways in which individuals can vary their level of effort and affect firm output. Because of this, it may be more realistic to treat each worker's effort as multi-choice. However, for such multi-choice problems, symmetry, efficiency, additivity and the null worker axiom are not sufficient to determine an allocation procedure. Furthermore, it may be difficult to measure worker effort as firms do not have perfect information on the behaviour of their workers and there may be different ways to measure effort: such as hours worked, number of words typed, or number of assignments completed. Because of the desire for an additional axiom to determine an allocation procedure and because of the problem of measuring effort, it may be desirable to require that the allocation procedure is somewhat independent of the way in which effort is measured. However, if an axiom is introduced to make the allocation procedure completely independent of effort then all workers will end up with the same pay regardless of effort; which is likely undesirable as it eliminates the incentive to work. Instead, an axiom known as strong scale invariance can be introduced, where the allocation procedure is invariant under analytic, invertible and strictly increasing transformations of the way in which an individual's effort is measured. It is shown in this paper that weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance uniquely characterize an allocation procedure similar to the Shapley value for infinite choice transferable utility games.

For multi-choice allocation problems, other allocation procedures have been used. In particular, a very popular allocation procedure known as the Aumann-Shapley value was developed by Aumann and Shapley (1974). It has been demonstrated that the axioms of efficiency, additivity, proportionality and scale invariance uniquely characterize the Aumann-Shapley value when the number of choices are

infinite (Monderer and Neyman 1988). Here, scale invariance is an axiom where the allocation procedure is invariant upon changing the definition of effort by multiplying the effort of all workers by a positive real number, such as measuring effort by minutes worked instead of hours worked. As for proportionality, it means that there is no incentive to trick the firm either by a worker pretending to be two workers or by two workers pretending to be a single worker; proportionality can be justified on the grounds that it is unfeasible to avoid it as avoiding it allows for an incentive for workers to cheat. The Aumann-Shapley value has been given alternative axiomatizations (Young 1985b) and has been applied and axiomatized for situations where workers only have finite choices (Sprumont 2004; Albizuri et al. 2014).

The Aumann-Shapley value was originally created to deal with cost allocation problems, problems where one has to distribute the costs of production to multiple buyers, such as multiple NATO countries purchasing fighter jets from a single buyer. For cost allocation problems, money (which is analogous to effort in the case of wage distribution problems) is easily transferable between buyers and thus the proportionality axiom is easy to justify as it is easy for buyers to cheat the system by transferring money to other buyers. However, in the case of wage distribution in firms, it is difficult for workers to transfer effort to other workers as they often have different skills and the firm usually knows who is working and who is not working, thus the proportionality axiom is difficult to justify. Alternatively, for cost allocation problems, the ratio of costs paid by different buyers can usually be determined objectively (such as if money is used), thus strong scale invariance might be difficult to justify. In addition, the buyers may not necessarily be single humans, thus symmetry may be difficult to justify on grounds of egalitarianism. As a result, the Aumann-Shapley value is better suited for cost allocation problems while the Shapley value defined in this paper is better suited for wage distribution in firm problems.

Therefore, if one accepts that weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance are reasonable axioms for wage distribution in firm problems then the Shapley value is justified for wage distribution in firm problems and as a result can be used to test if there is a departure between observed wages and what the workers should be paid. In reality, society rarely knows the production function of a firm. However, given enough empirical data, one can estimate a model of the production function by looking at how total firm output varies depending on its input. By using this estimated production function it is possible to estimate the departure between observed wages and the Shapley value.

Neoclassical economic models often assume that individuals are paid their marginal product of labour and that an individual's marginal product of labour does not depend much on the actions of others within a firm. Generally, these assumptions are not true. It is well known that firm size is positively correlated with total-factor productivity and, as a result, an additional worker in a firm may

have an external positive impact on the productivity of others within the firm. This external effect means that the Shapley value for such firms is non-trivial, which allows one to test for a possible divergence between the expected Shapley value of a worker in a firm and the observed wage. This paper uses Current Population Survey (CPS) data to see if workers are paid their expected Shapley value in earnings; the usage of CPS data gives one an economy wide perspective. Differences between observed earnings and the expected Shapley values may be due to differences in bargaining strength. Union membership is found to have a positive effect on bargaining strength that is statistically significant at the 95% confidence level.

This paper is organized as follows. Section 2 gives a literature review of the literature relevant to extending the Shapley value to multi-choice games. Section 3 provides basic definitions and extends the Shapley value to infinite choice games using six basic axioms: weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance. Section 4 provides definitions and an axiomatization for extending the Shapley value to finite multi-choice games using similar axioms. Section 5 gives a simple example of the Shapley value and compares it to the Aumann-Shapley value. Section 6 explains the data used. Section 7 describes the methodology and gives the results of this methodology. Section 8 discusses the results and concludes.

## **2. Literature Review**

The Shapley value was first proposed by Lloyd Shapley (1953) as a solution to the problem of allocating coalition winnings for cooperative transferable utility games. Shapley came up with his solution by first considering the set of allocation procedures that use all the possible outcomes of a game to assign a real number to each player within the winning coalition; this real number represents the value of that player playing the game given the winning coalition. The Shapley value is the unique allocation procedure that satisfies 4 basic axioms: symmetry, additivity, efficiency and the null worker axiom. Shapley notes that the Shapley value is the expected earnings of a player if player order in the winning coalition is randomized and then each player in the winning coalition is given their marginal contribution to coalition value of the coalition that consists of all players that come before that player in the ordering. Shapley's work is very important as it is the basis for the Shapley value and shows that axiomatization can be used to find solutions to complicated games.

However, the axiomatization proposed by Shapley is not the only axiomatization of the Shapley value. Young (1985a) showed that the Shapley value is the unique allocation procedure that satisfies symmetry, efficiency and strong monotonicity. Strong monotonicity is the property where if a player's marginal contribution to each coalition stays the same or increases then the player's allocation of

earnings should not decrease. Young's result is very important as it arguably gives a more economic and more intuitive axiomatic basis for the Shapley value by replacing the null worker axiom and the additivity axiom with strong monotonicity; there are many economic applications where it is easy to justify strong monotonicity but harder to justify additivity and the null worker axiom. Hart and Mas-Colell (1989) found another interesting axiomatization of the Shapley value using symmetry, efficiency, consistency and transferable utility invariance. Alternative axiomatizations such as those by Hart and Mas-Colell or Young help strengthen the justification for using the Shapley value in real world applications.

An important contribution to the literature on Shapley values was made by Roth (1977). Roth considered the von Neumann-Morgenstern utility gain that an individual gains out of playing a position in a cooperative transferable utility game. Roth showed that this utility gain is proportional to the Shapley value if and only if the individual is neutral with respect to ordinary and strategic risk. In this context, ordinary risk is the risk associated with uncertainty about which position a player will play in a game and strategic risk is the uncertainty about which coalition will end up being formed as the winning coalition. Roth's result is important as it suggests that, under some conditions, the Shapley value corresponds to the von Neumann-Morgenstern utility gain. In the real world, risk neutrality in individuals is likely violated as shown by various literature reviews of measures of risk aversion in humans such as by Outreville (2014). However, if the magnitudes of the potential earnings due to the coalition game are relatively small compared to the individuals' total incomes then the individuals are approximately risk neutral for the coalition game, so the Shapley value corresponds roughly to the von Neumann-Morgenstern utility gain.

Aumann and Shapley (1974) contributed significantly to extending the Shapley value to infinite choice games by considering non-atomic games. Aumann and Shapley considered the cost-sharing problem where finite individuals may each demand an arbitrary quantity of goods; the total cost of the goods is then shared among the individuals. Aumann and Shapley treated each unit of good demanded by each of the individuals as a separate individual and considered the sum of the Shapley values of the goods demanded by each individual. Aumann and Shapley took the limit as the number of individuals approaches infinity to obtain the Aumann-Shapley value, which corresponds to individuals paying the integral of their marginal costs from the origin (zero demand for all players) to the total demand of all players. The Aumann-Shapley result is axiomatized using efficiency, proportionality, additivity and scale invariance (Monderer and Neyman 1988). The Aumann-Shapley result has been given an alternative axiomatization that uses strong monotonicity instead of additivity (Young 1985b) and has been axiomatized and extended to discrete cases (Sprumont 2004; Albizuri et al. 2014). The Aumann-Shapley result is important as it suggests that an axiom similar scale invariance such as strong scale invariance may be a useful when trying to generalize the Shapley value to infinite choice games.

Attempts have been made to generalize the Shapley value to multi-choice games. Hsiao and Raghavan (1993) extended the Shapley value to multi-choice cooperative transferable utility games using four basic axioms. The Hsiao and Raghavan extension assigns a vector rather than a single value to each worker and requires predetermined weights assigned to each of the possible options that workers can take when forming coalitions. The Hsiao and Raghavan result is limited in applicability, although Hsiao and Raghavan point out that one may be able to justify a weight vector by looking at market conditions. Freixas and Zwicker (2003) generalize the concept of cooperative transferable utility games to finite multi-choice games with a finite number of outputs; Freixas (2005) extends the Shapley-Shubik index (the Shapley-Shubik index is an index that represents the share of voting power of individuals and was developed by Shapley and Shubik (1954)) for these games and axiomatizes the index when the number of outputs is two using four basic axioms. Like Hsiao and Raghavan, the results of Freixas and Zwicker require worker effort to be ordered and have pre-assigned weights.

There have been some recent attempts to generalize and axiomatize the Shapley value to finite choice games. Adam (2014) generalized the Shapley value to finite multi-choice games using four basic axioms: efficiency, symmetry, additivity and the null player axiom. The symmetry axiom used by Adam is much stronger than the weak symmetry axiom used in this paper so is called the strong symmetry axiom for convenience. Ravindra (2015) found a similar axiomatization using efficiency, strong symmetry and strong monotonicity. The Shapley value has even been axiomatized using variants of efficiency, strong symmetry and strong monotonicity for cases where the production environment is uncertain (Pongou and Tondji 2018). The result of Pongou and Tondji is particularly important as it explains how one can combine their *a priori* Shapley value with Bayesian updating to obtain a Bayesian Shapley value, which can have useful practical applications such as by an employer in a firm trying to allocate wages. A recent paper by Aguiar, Pongou and Tondji (2018) shows that the distance between the Shapley value and the observed wage distribution can be decomposed into violations of the axioms of the Shapley value, which suggests that it may be of interest to try to understand differences between observations and the Shapley value.

While attempts have been made to generalize the Shapley value to multi-choice games, very little of the literature deals with infinite-choice games. Furthermore, for the axiomatizations that do exist for multi-choice games, they usually use a strong symmetry axiom that may be harder to justify than the weak symmetry axiom or they are similar to the Aumann-Shapley value so are poorly applicable to wage distribution in firm problems. This paper generalizes the Shapley value to infinite-choice games by using six basic axioms: efficiency, weak symmetry, additivity, the null worker axiom, no pay for no work and strong scale invariance. This paper also gives a similar axiomatization for finite multi-choice games using similar axioms.

### 3. Axiomatization for Infinite Choice Games

**Definition 1:** Let  $N = \{1, \dots, n\}$  be the set of  $n \in \mathbb{Z}_+$  workers of a cooperative transferable utility game. A production function  $f$  is an analytic function that maps  $\mathbb{R}_+^n \rightarrow \mathbb{R}$  such that  $f((0, \dots, 0)) = 0$ .

**Definition 2:** Let  $F$  be the set of production functions. An allocation procedure  $\psi(f, X)$  is a function that maps  $(F, \mathbb{R}_+^n) \rightarrow \mathbb{R}^n$ , where  $X \in \mathbb{R}_+^n$  is called an effort set.

**Definition 3:** Given  $m \in \mathbb{Z}_+$ ,  $X \in \mathbb{R}_+^m$  and  $Z \subseteq \{1, \dots, m\}$ , define  $X_Z \in \mathbb{R}_+^m$  by

$$(X_Z)_j = \begin{cases} X_j & \text{if } j \in Z \\ 0 & \text{else} \end{cases} \quad \forall j \in \{1, \dots, m\}.$$

**Definition 4:** The Shapley value  $\varphi$  is the allocation procedure defined by

$$\varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)).$$

**Definition 5:** Define  $\tau_{i,j}$  as the permutation operator that permutes the  $i$ th and  $j$ th coordinates of an  $n$  dimensional space.

**Definition 6:** An allocation procedure  $\psi$  satisfies weak symmetry if and only if

$$f(X) = f(\tau_{i,j}(X)) \quad \forall X \in \mathbb{R}_+^n \Rightarrow \psi_i(f, X) = \psi_j(f, \tau_{i,j}(X)) \quad \forall X \in \mathbb{R}_+^n \quad \forall i, j \in N \quad \forall f \in F.$$

**Definition 7:** An allocation procedure  $\psi$  satisfies efficiency if and only if

$$f(X) = \sum_{i=1}^n \psi_i(f, X) \quad \forall X \in \mathbb{R}_+^n \quad \forall f \in F.$$

**Definition 8:** An allocation procedure  $\psi$  satisfies additivity if and only if

$$\psi(f + g, X) = \psi(f, X) + \psi(g, X) \quad \forall X \in \mathbb{R}_+^n \quad \forall f, g \in F.$$

**Definition 9:** A worker  $k$  is said to be a null worker of  $f$  if and only if  $f(X) = f(X_{N \setminus \{k\}}) \quad \forall X \in \mathbb{R}_+^n$ .

**Definition 10:** An allocation procedure  $\psi$  satisfies the null worker axiom if and only if  $i$  is a null worker of  $f \Rightarrow \psi_i(f, X) = 0 \quad \forall X \in \mathbb{R}_+^n$ .

**Definition 11:** An allocation procedure  $\psi$  satisfies no pay for no work if and only if  $X_i = 0 \Rightarrow$

$$\psi_i(f, X) = 0 \quad \forall X \in \mathbb{R}_+^n \quad \forall f \in F.$$

**Definition 12:** A strong scale transformation is a function  $\mu$  that maps  $\mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  such that  $\mu(X) = (\mu_1(X_1), \dots, \mu_n(X_n)) \quad \forall X \in \mathbb{R}_+^n$ , where  $\mu_i$  is an analytic, invertible and strictly increasing function that maps  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$  with inverse  $\mu_i^{-1} \quad \forall i \in N$ . Note that by this definition,  $\mu$  has an inverse  $\mu^{-1}$  defined as  $\mu^{-1}(X) = (\mu_1^{-1}(X_1), \dots, \mu_n^{-1}(X_n)) \quad \forall X \in \mathbb{R}_+^n$ .

**Definition 13:** An allocation procedure  $\psi$  satisfies strong scale invariance if and only if

$$\mu \text{ is a strong scale transformation} \Rightarrow \psi(f, X) = \psi(f \circ \mu^{-1}, \mu(X)) \quad \forall X \in \mathbb{R}_+^n \quad \forall f \in F.$$

**Remark:** It may help to recall the definition of an analytic function. There are many equivalent definitions of an analytic function; one definition is that an analytic function is a function that can be expressed as a Taylor series around any point in its domain. While all analytic functions are  $C^\infty$ , not all

$C^\infty$  functions are analytic such as the function  $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$ . However, all complex analytic functions are  $C^\infty$ , although this paper deals with real functions. Definitions and properties of analytic functions are available in various undergraduate mathematics textbooks such as in Kreyszig (2006).

The requirement that  $f$  be analytic is added to make the proof of Theorem 1 feasible. More general proofs may exist. In practice, this restriction is not severe since non-analytic functions can be well approximated by analytic functions (such as by using a fourier series), so the result of this proof can be considered to justify the use of the Shapley value  $\varphi$  for a much more general class of functions in the context of wage distribution in firm problems. This includes the production function used later in this paper in the empirical analysis, which is given a slightly discontinuous form when new workers are added to the firm for the sake of convenience and simplicity of calculations.

**Theorem 1:** Let  $f$  be a production function.  $\psi$  is an allocation procedure for  $f$  that satisfies weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance if and only if  $\psi$  is the Shapley value  $\varphi$ .

**Proof:**

Sufficiency:

Suppose that worker  $i$  is a null worker of  $f$ . Then  $f(X_{Z \cup \{i\}}) = f(X_Z) \forall Z \subseteq N \setminus \{i\} \forall X \in \mathbb{R}_+^n$ . Thus  $\varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) = 0$ .

Therefore, the Shapley value satisfies the null worker axiom.

Suppose that  $X_i = 0$ . Then  $f(X_{Z \cup \{i\}}) = f(X_Z) \forall Z \subseteq N \setminus \{i\} \forall X \in \mathbb{R}_+^n$ . Thus  $\varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) = 0$ .

Therefore, the Shapley value satisfies no pay for no work.

$$\begin{aligned}
\varphi(f + g, X) &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( (f + g)(X_{Z \cup \{i\}}) - (f + g)(X_Z) \right) \\
&= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(X_{Z \cup \{i\}}) - f(X_Z) \right) + \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( g(X_{Z \cup \{i\}}) - g(X_Z) \right) \\
&= \varphi(f, X) + \varphi(g, X)
\end{aligned}$$

Therefore, the Shapley value satisfies additivity.

$$\begin{aligned}
\text{Suppose that } f(X) &= f(\tau_{i,j}(X)) \quad \forall X \in \mathbb{R}_+^n. \text{ Then } \varphi_i(f, X) = \\
\sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} &\left( f(X_{Z \cup \{i\}}) - f(X_Z) \right) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(\tau_{i,j}(X_{Z \cup \{i\}})) - f(\tau_{i,j}(X_Z)) \right) \\
&= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(\tau_{i,j}(X)_{\tau_{i,j}(Z \cup \{i\})}) - f(\tau_{i,j}(X)_{\tau_{i,j}(Z)}) \right) \\
&= \sum_{Z \subseteq N \setminus \{j\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(X_{Z \cup \{i\}}) - f(X_Z) \right) = \varphi_j(f, X) \quad \forall X \in \mathbb{R}_+^n
\end{aligned}$$

Therefore, the Shapley value satisfies weak symmetry.

Let  $\omega(N)$  be the set of random permutations of  $N$ . Given  $Q \in \omega(N)$  let  $\aleph(Q, j) = \{Q_k \text{ such that } k < l\}$ , where  $l$  is the smallest integer such that  $Q_l = j$ . Then

$$\begin{aligned}
\sum_{i=1}^n \varphi_i(f, X) &= \sum_{i=1}^n \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(X_{Z \cup \{i\}}) - f(X_Z) \right) \\
&= \sum_{i=1}^n \sum_{Q \in \omega(N \setminus \{i\})} \frac{1}{n!} \left( f(X_{\aleph(Q, i) \cup \{i\}}) - f(X_{\aleph(Q, i)}) \right) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} \left( \left( f(X_{\aleph(Q, Q_1) \cup \{Q_1\}}) - f(X_{\aleph(Q, Q_1)}) \right) + \dots + \left( f(X_{\aleph(Q, Q_n) \cup \{Q_n\}}) - f(X_{\aleph(Q, Q_n)}) \right) \right) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} \left( f(X_{\aleph(Q, Q_n) \cup \{Q_n\}}) - f(\rho(X_{\aleph(Q, Q_1)})) \right) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} \left( f(X_N) - f(X_\emptyset) \right) = \sum_{Q \in \omega(N)} \frac{1}{n!} f(X) = f(X)
\end{aligned}$$

Therefore, the Shapley value satisfies efficiency.

Suppose that  $\mu$  is a strong scale transformation of  $\mathbb{R}_+^n$ .

$$\begin{aligned}
\varphi(f \circ \mu^{-1}, \mu(X)) &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( (f \circ \mu^{-1})(\mu(X)_{Z \cup \{i\}}) - (f \circ \mu^{-1})(\mu(X)_Z) \right) \\
&= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(\mu^{-1}(\mu(X))_{Z \cup \{i\}}) - f(\mu^{-1}(\mu(X))_Z) \right) \\
&= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(X_{Z \cup \{i\}}) - f(X_Z) \right) = \varphi(f \circ \mu^{-1}, \mu(X))
\end{aligned}$$

Therefore, the Shapley value satisfies strong scale invariance.

Therefore,  $\psi$  is the Shapley value  $\Rightarrow \psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance.

**Necessity:**

Suppose that  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance.

Because  $f$  is analytic it can be expressed as an infinite sum of polynomials.

Let  $f(Y) = \sum_{a_1, \dots, a_n \in \mathbb{N}} C_{a_1, \dots, a_n} \prod_{j \in N} Y_j^{a_j} \forall Y \in \mathbb{R}_+^n$ , where  $C_{a_1, \dots, a_n}$  are real constants and  $C_{0, \dots, 0} = 0$  since  $f((0, \dots, 0)) = 0$ .

Consider  $g$  such that  $g(Y) = C_{a_1, \dots, a_n} \prod_{j \in N} Y_j^{a_j} \forall Y \in \mathbb{R}_+^n$  for some  $a_1, \dots, a_n \in \mathbb{N}$ , where  $\exists l \in \{1, \dots, n\}$  such that  $a_l \neq 0$ , and let  $X \in \mathbb{R}_+^n$ . Note that  $g$  satisfies the definition of a production function.

Let  $S$  be the set of workers such that  $j \in S$  implies that  $a_j > 0$  and  $X_j > 0$ .

For a given  $j \in S$  let  $\mu_j: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\mu_j(Y_j) = \left(\frac{Y_j}{X_j}\right)^{a_j} \forall Y_j \in \mathbb{R}_+$ .

Since  $a_j > 0$ ,  $X_j > 0$  and  $\mu_j$  is the well known power function,  $\mu_j$  is an analytic function. Furthermore,  $\mu_j$  is invertible and strictly increasing, with inverse  $\mu_j^{-1}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\mu_j^{-1}(Y_j) = X_j Y_j^{\frac{1}{a_j}} \forall Y_j \in \mathbb{R}_+$ .

For  $j \in N \setminus S$  let  $\mu_j: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\mu_j$  is the identity function. The identity function is analytic, invertible and strictly increasing, with itself as an inverse.

Define  $\mu: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  by  $\mu(Y) = (\mu_1(Y_1), \dots, \mu_n(Y_n)) \forall Y \in \mathbb{R}_+^n$ .  $\mu$  satisfies the definition of a strong scale transformation and has inverse  $\mu^{-1}(Y) = (\mu_1^{-1}(Y_1), \dots, \mu_n^{-1}(Y_n)) \forall Y \in \mathbb{R}_+^n$ .

$$(g \circ \mu^{-1})(Y) = C_{a_1, \dots, a_n} \prod_{j \in S} X_j^{a_j} Y_j \prod_{j \in N \setminus S} Y_j^{a_j} \quad \forall Y \in \mathbb{R}_+^n.$$

Let  $i, k \in S$ .  $(g \circ \mu^{-1})(\tau_{i,k}(Y)) = C_{a_1, \dots, a_n} \prod_{j \in S} X_j^{a_j} Y_j \prod_{j \in N \setminus S} Y_j^{a_j} = (g \circ \mu^{-1})(Y) \quad \forall Y \in \mathbb{R}_+^n$ . Thus

$$\psi_i(g \circ \mu^{-1}, \mu(X)) = \psi_k(g \circ \mu^{-1}, \tau_{i,k}(\mu(X))) \text{ by the weak symmetry axiom. But } \mu_j(X_j) = \begin{cases} 1 & \text{if } j \in S \\ X_j & \text{else} \end{cases}. \text{ Thus } \mu(X) = \tau_{i,k}(\mu(X)). \text{ So } \psi_i(g \circ \mu^{-1}, \mu(X)) = \psi_k(g \circ \mu^{-1}, \mu(X)).$$

By strong scale invariance, if  $i, k \in S$  then  $\psi_i(g, X) = \psi_i(g \circ \mu^{-1}, \mu(X)) = \psi_k(g \circ \mu^{-1}, \mu(X)) = \psi_k(g, X)$ .

By efficiency,  $g(X) = \sum_{j \in N} \psi_j(g, X) = \sum_{j \in S} \psi_j(g, X) + \sum_{j \in N \setminus S} \psi_j(g, X)$ .

If  $a_i = 0$  then  $i$  is a null worker of  $g$ , thus  $\psi_i(g, X) = 0 = \varphi_i(g, X)$  by the null worker axiom.

If  $X_i = 0$  then  $\psi_i(g, X) = 0 = \varphi_i(g, X)$  by no pay for no work.

$$\text{Thus } g(X) = \sum_{j \in S} \psi_j(g, X) + \sum_{j \in N \setminus S} \psi_j(g, X) = \sum_{j \in S} \psi_j(g, X) = |S| \psi_i(g, X) \quad \forall i \in S. \\ \Rightarrow \psi_i(g, X) = \frac{g(X)}{|S|} \quad \forall i \in S.$$

But as  $\varphi$  also satisfies weak symmetry, efficiency, the null worker axiom, no pay for no work and strong scale invariance,  $\varphi_i(g, X) = \frac{g(X)}{|S|} \quad \forall i \in S$ .

Thus  $\psi_i(g, X) = \varphi_i(g, X) \quad \forall i \in N \quad \forall X \in \mathbb{R}_+^n$  as the choice of  $X$  is arbitrary.

$$\text{Using additivity one obtains } \psi_i(f, X) = \psi_i(\sum_{a_1, \dots, a_n \in N} g_{a_1, \dots, a_n}, X) \\ = \sum_{a_1, \dots, a_n \in N} \psi_i(g_{a_1, \dots, a_n}, X) = \sum_{a_1, \dots, a_n \in N} \varphi_i(g_{a_1, \dots, a_n}, X) = \varphi_i(\sum_{a_1, \dots, a_n \in N} g_{a_1, \dots, a_n}, X) \\ = \varphi_i(f, X) \quad \forall i \in N \quad \forall X \in \mathbb{R}_+^n$$

Therefore,  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null player axiom and strong scale invariance  $\Rightarrow \psi$  is the Shapley value.

Therefore,  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null player axiom and strong scale invariance if and only if  $\psi$  is the Shapley value.

#### 4. Axiomatization for Finite Choice Games

Various concepts defined in section 3 are redefined here for the cases of finite choice games.

**Definition 14:** Let  $N = \{1, \dots, n\}$  be the set of  $n$  players of a cooperative transferable utility game and let  $E \in \mathbb{Z}_+$  be the number of effort levels that each player can choose from. A production function  $f$  is a function that maps  $\mathbb{Z}_E^n \rightarrow \mathbb{R}$  such that  $f(0, \dots, 0) = 0$ .

**Definition 15:** Let  $F$  be the set of production functions. An allocation procedure  $\psi(f, X)$  is a function that maps  $(F, \mathbb{R}_+^n) \rightarrow \mathbb{R}^n$ , where  $X \in \mathbb{Z}_E^n$  is called an effort set.

**Definition 16:** Given  $m \in \mathbb{Z}_+$ ,  $X \in \mathbb{Z}_E^n$  and  $Z \subseteq \{1, \dots, m\}$ , define  $X_Z \in \mathbb{Z}_E^m$  by

$$(X_Z)_j = \begin{cases} X_j & \text{if } j \in Z \\ 0 & \text{else} \end{cases} \quad \forall j \in \{1, \dots, m\}.$$

**Definition 17:** The Shapley value  $\varphi$  is the allocation procedure defined by

$$\varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)).$$

**Definition 18:** An allocation procedure  $\psi$  satisfies weak symmetry if and only if

$$f(X) = f(\tau_{i,j}(X)) \quad \forall X \in \mathbb{Z}_E^n \Rightarrow \psi_i(f, X) = \psi_j(f, X) \quad \forall X \in \mathbb{Z}_E^n \quad \forall i, j \in N \quad \forall f \in F.$$

**Definition 19:** An allocation procedure  $\psi$  satisfies efficiency if and only if

$$f(X) = \sum_{i=1}^n \psi_i(f, X) \quad \forall X \in \mathbb{Z}_E^n \quad \forall f \in F.$$

**Definition 20:** An allocation procedure  $\psi$  satisfies additivity if and only if

$$\psi(f + g, X) = \psi(f, X) + \psi(g, X) \quad \forall X \in \mathbb{Z}_E^n \quad \forall f, g \in F.$$

**Definition 21:** A worker  $k$  is said to be a null worker of  $f$  if and only if  $f(X) = f(X_{N \setminus \{k\}}) \quad \forall X \in \mathbb{Z}_E^n$ .

**Definition 22:** An allocation procedure  $\psi$  satisfies the null worker axiom if and only if

$$i \text{ is a null worker of } f \Rightarrow \psi_i(f, X) = 0 \quad \forall X \in \mathbb{Z}_E^n.$$

**Definition 23:** An allocation procedure  $\psi$  satisfies invariance under re-ordering of non-zero effort levels if and only if it satisfies the following property:

If  $f$  and  $g$  are two production functions where  $\exists i \in N$  and  $\exists j, k \in \mathbb{Z}_E \setminus \{0\}$  such that

$$f(X) = g(X_1, \dots, X_{i-1}, \beth_{j,k}(X_i), X_{i+1}, \dots, X_n) \quad \forall X \in \mathbb{Z}_E^n, \text{ where } \beth_{j,k}: E \rightarrow E \text{ such that } \beth_{j,k}(l) =$$

$$\begin{cases} k & \text{if } l = j \\ j & \text{if } l = k, \text{ then } \psi(f, X) = \psi(g, (X_1, \dots, X_{i-1}, \beth_{j,k}(X_i), X_{i+1}, \dots, X_n)) \quad \forall X \in \mathbb{Z}_E^n. \\ l & \text{else} \end{cases}$$

**Remark:** Invariance under re-ordering of non-zero effort levels resembles strong scale invariance in the finite context. No rescaling operation exists for finite sets of effort so an alternative axiom is needed for finite choice games. Invariance under re-ordering of non-zero effort levels allows one to swap the order of any two non-zero effort levels for a player and preserve the allocation procedure. The proof for the Shapley value in the finite choice context is similar to the proof in the infinite choice context, although slightly different due to the different axiomatization.

**Theorem 2:**  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom and invariance under re-ordering of non-zero effort levels iff  $\psi$  is the Shapley value  $\varphi$ .

**Proof:**

Sufficiency:

Suppose that worker  $i$  is a null worker of  $f$ . Then  $f(X_{Z \cup \{i\}}) = f(X_Z) \forall Z \subseteq N \setminus \{i\} \forall X \in \mathbb{Z}_E^n$ . Thus

$$\varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) = 0.$$

Therefore, the Shapley value satisfies the null worker axiom.

$$\begin{aligned} \varphi(f + g, X) &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} ((f + g)(X_{Z \cup \{i\}}) - (f + g)(X_Z)) \\ &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) + \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (g(X_{Z \cup \{i\}}) - g(X_Z)) \\ &= \varphi(f, X) + \varphi(g, X) \end{aligned}$$

Therefore, the Shapley value satisfies additivity.

Suppose that  $f(X) = f(\tau_{i,j}(X)) \forall X \in \mathbb{Z}_E^n$ . Then  $\varphi_i(f, X) =$

$$\begin{aligned} \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(\tau_{i,j}(X_{Z \cup \{i\}})) - f(\tau_{i,j}(X_Z))) \\ &= \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(\tau_{i,j}(X)_{\tau_{i,j}(Z \cup \{i\})}) - f(\tau_{i,j}(X)_{\tau_{i,j}(Z)})) \\ &= \sum_{Z \subseteq N \setminus \{j\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) = \varphi_j(f, X) \forall X \in \mathbb{Z}_E^n \end{aligned}$$

Therefore, the Shapley value satisfies weak symmetry.

Let  $\omega(N)$  be the set of random permutations of  $N$ . Given  $Q \in \omega(N)$  let  $\kappa(Q, j) = \{Q_k \text{ such that } k < j\}$ , where  $l$  is the smallest integer such that  $Q_l = j$ . Then

$$\begin{aligned}
\sum_{i=1}^n \varphi_i(f, X) &= \sum_{i=1}^n \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) \\
&= \sum_{i=1}^n \sum_{Q \in \omega(N \setminus \{i\})} \frac{1}{n!} (f(X_{\kappa(Q, i) \cup \{i\}}) - f(X_{\kappa(Q, i)})) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} \left( (f(X_{\kappa(Q, Q_1) \cup \{Q_1\}}) - f(X_{\kappa(Q, Q_1)})) + \dots + (f(X_{\kappa(Q, Q_n) \cup \{Q_n\}}) - f(X_{\kappa(Q, Q_n)})) \right) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} (f(X_{\kappa(Q, Q_n) \cup \{Q_n\}}) - f(X_{\kappa(Q, Q_n)})) \\
&= \sum_{Q \in \omega(N)} \frac{1}{n!} (f(X_N) - f(X_\emptyset)) = \sum_{Q \in \omega(N)} \frac{1}{n!} f(X) = f(X)
\end{aligned}$$

Therefore, the Shapley value satisfies efficiency.

As the definition of the Shapley value  $\left( \varphi_i(f, X) = \sum_{Z \subseteq N \setminus \{i\}} \frac{|Z|!(n-|Z|-1)!}{n!} (f(X_{Z \cup \{i\}}) - f(X_Z)) \right)$  depends only on production levels where the effort of each worker is either zero or at the current level of effort, the Shapley value depends only on 1 non-zero effort level for each player. As a result, swapping two non-zero effort levels will not change the Shapley value, so the Shapley value satisfies invariance under re-ordering of non-zero effort levels.

Therefore,  $\psi$  is the Shapley value  $\Rightarrow \psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom and invariance under re-ordering of non-zero effort levels.

#### Necessity:

Suppose  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom and invariance under re-ordering of non-zero effort levels.

The production function  $f$  can be expressed as  $f = \sum_{Y \in \mathbb{Z}_E^n} C_Y \tilde{f}_Y$ , where  $C_Y \in \mathbb{R} \forall Y \in \mathbb{Z}_E^n$ ,  $C_{0, \dots, 0} = 0$  since  $f((0, \dots, 0)) = 0$ , and  $\tilde{f}_Y(X) = \begin{cases} 1 & \text{if } X_i = Y_i \forall i \in \{j \in \{1, \dots, n\} | Y_j \neq 0\} \\ 0 & \text{else} \end{cases}$ .

Consider  $\tilde{f}_Y$  such that  $Y \neq (0, \dots, 0)$ . Note that  $\tilde{f}_Y$  satisfies the definition of a production function.

Let  $\mu$  be a transformation of the definition of effort defined by

$$\mu_i(X) = \begin{cases} \sum_{Y_i, E-1} (X_i) & \text{if } i \in S \\ X_i & \text{else} \end{cases}, \text{ where } S = \{j \in \{1, \dots, n\} | Y_j \neq 0\}. \text{ As } \psi \text{ satisfies invariance under re-ordering of non-zero effort levels, } \psi(\tilde{f}_Y, X) = \psi(\tilde{f}_Y \circ \mu^{-1}, \mu(X)) \quad \forall X \in \mathbb{Z}_E^n.$$

$$\text{Note that } (\tilde{f}_Y \circ \mu^{-1})(X) = \begin{cases} 1 & \text{if } X_j = E - 1 \quad \forall j \in S \\ 0 & \text{else} \end{cases}.$$

Thus if  $i, k \in S$  then  $\psi_i(\tilde{f}_Y \circ \mu^{-1}, X) = \psi_k(\tilde{f}_Y \circ \mu^{-1}, X)$  as  $\psi$  satisfies weak symmetry.

$$\Rightarrow \psi_i(\tilde{f}_Y, X) = \psi_i(\tilde{f}_Y \circ \mu^{-1}, \mu(X)) = \psi_k(\tilde{f}_Y \circ \mu^{-1}, \mu(X)) = \psi_k(\tilde{f}_Y, X).$$

Suppose that  $i \notin S$ . Then  $\tilde{f}_Y(X) = \tilde{f}_Y(X_{N \setminus \{i\}}) \quad \forall X \in \mathbb{Z}_E^n$

$$\Rightarrow \psi_i(\tilde{f}_Y, X) = 0 = \varphi_i(\tilde{f}_Y, X) \text{ by the null worker axiom.}$$

Suppose that  $i \in S$ . As  $\psi$  satisfies the efficiency axiom,

$$\tilde{f}_Y(X) = \sum_{j=1}^n \psi_j(\tilde{f}_Y, X) = \sum_{j \notin S} \psi_j(\tilde{f}_Y, X) + \sum_{j \in S} \psi_j(\tilde{f}_Y, X)$$

$$= 0 + |S| \cdot \psi_i(\tilde{f}_Y, X)$$

$$\Rightarrow \psi_i(\tilde{f}_Y, X) = \tilde{f}_Y(X) / |S|.$$

However, if  $i \in S$  then, as  $\varphi$  also satisfies weak symmetry, efficiency, the null worker axiom and invariance under re-ordering of non-zero effort levels,  $\varphi_i(\tilde{f}_Y, X) = \tilde{f}_Y(X) / |S|$ .

$$\text{Thus } \varphi_i(\tilde{f}_Y, X) = \tilde{f}_Y(X) / |S| = \psi_i(\tilde{f}_Y, X).$$

$$\Rightarrow \varphi(\tilde{f}_Y, X) = \psi(\tilde{f}_Y, X).$$

As  $\psi$  satisfies additivity, one obtains

$$\psi(f, X) = \psi\left(\sum_{Y \in \mathbb{Z}_E^n} C_Y \tilde{f}_Y, X\right) = \sum_{Y \in \mathbb{Z}_E^n} C_Y \psi(\tilde{f}_Y, X) = \sum_{Y \in \mathbb{Z}_E^n} C_Y \varphi(\tilde{f}_Y, X)$$

As  $\varphi$  also satisfies additivity, one obtains

$$\psi(f, X) = \sum_{Y \in \mathbb{Z}_E^n} C_Y \varphi(\tilde{f}_Y, X) = \varphi\left(\sum_{Y \in \mathbb{Z}_E^n} C_Y \tilde{f}_Y, X\right) = \varphi(f, X)$$

Therefore,  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom and invariance under re-ordering of non-zero effort levels  $\Rightarrow \psi$  is the Shapley value.

Therefore,  $\psi$  is an allocation procedure that satisfies weak symmetry, efficiency, additivity, the null worker axiom and invariance under re-ordering of non-zero effort levels if and only if  $\psi$  is the Shapley value.

## 5. Example of Shapley Value

To illustrate how to use the Shapley value and to demonstrate how the Shapley value defined for this paper differs from the Aumann-Shapley value, it can help to have a simple example. Suppose that there are two workers in a firm such that the firm's production function is  $f(X_1, X_2) = X_1^{1/3} X_2^{2/3}$ , where  $X_1$  and  $X_2$  correspond to the hours worked by worker 1 and worker 2 respectively. Suppose that both workers work exactly 1 hour.

Under the Shapley value defined in this paper, worker 1 is allocated

$$\begin{aligned}\varphi_1(f, (1,1)) &= \sum_{Z \subseteq \{1,2\} \setminus \{1\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f((1,1)_{Z \cup \{1\}}) - f((1,1)_Z) \right) \\ &= \frac{f(1,1) - f(0,1) + f(1,0) - f(0,0)}{2} = \frac{1 - 0 + 0 - 0}{2} = \frac{1}{2}.\end{aligned}$$

Similarly, under the Shapley value, worker 2 is allocated

$$\begin{aligned}\varphi_2(f, (1,1)) &= \sum_{Z \subseteq \{1,2\} \setminus \{2\}} \frac{|Z|!(n-|Z|-1)!}{n!} \left( f(\rho((1,1)_{Z \cup \{2\}})) - f((1,1)_Z) \right) \\ &= \frac{f(1,1) - f(1,0) + f(0,1) - f(0,0)}{2} = \frac{1 - 0 + 0 - 0}{2} = \frac{1}{2}.\end{aligned}$$

For a production function  $f$  that maps  $\mathbb{R}_+^n \rightarrow \mathbb{R}$  the Aumann-Shapley value  $\phi$  of worker  $i$  is defined as  $\phi_i(f, X) = X_i \int_0^1 f_i(tX) dt$ , where  $f_i(X) = \frac{\partial f(X)}{\partial X_i}$ .

For this example, worker 1's Aumann-Shapley value can be calculated as follows:

$$\begin{aligned}f_1(X) &= \frac{\partial f(X)}{\partial X_1} = \frac{\partial X_1^{-2/3} X_2^{2/3}}{\partial X_1} = \frac{2}{3} X_1^{-5/3} X_2^{2/3} \Rightarrow \phi_1(f, (1,1)) = 1 \cdot \int_0^1 f_1(t \cdot (1,1)) dt = 1 \cdot \int_0^1 f_1((t, t)) dt \\ &= 1 \cdot \int_0^1 \frac{2}{3} t^{-5/3} t^{2/3} dt = \int_0^1 \frac{2}{3} t^{-1} dt = \frac{2}{3}.\end{aligned}$$

Similarly, worker 2's Aumann-Shapley value can be calculated as follows:

$$\begin{aligned}f_2(X) &= \frac{\partial f(X)}{\partial X_2} = \frac{\partial X_1^{1/3} X_2^{-1/3}}{\partial X_2} = \frac{1}{3} X_1^{1/3} X_2^{-4/3} \Rightarrow \phi_2(f, (1,1)) = 1 \cdot \int_0^1 f_2(t \cdot (1,1)) dt = 1 \cdot \int_0^1 f_2((t, t)) dt \\ &= 1 \cdot \int_0^1 \frac{1}{3} t^{1/3} t^{-4/3} dt = \int_0^1 \frac{1}{3} t^{-1} dt = \frac{1}{3}.\end{aligned}$$

In this case, the Shapley value and the Aumann-Shapley value give slightly different results. The more traditional Shapley value axiomatized in this paper tends to give slightly more egalitarian allocations of firm output since it satisfies the symmetry axiom, whereas the Aumann-Shapley value does not satisfy the symmetry axiom.

## 6. Data

Data for earnings weight, age, educational attainment, industry, class of worker, weeks worked per year, usual hours worked per week, hourly wage, union membership, and firm size of the March 2012 current population survey (CPS) were obtained from the Integrated Public Use Microdata Series website (Flood et al. 2015). March 2012 data was used because it is relatively recent, contains data on job tenure (although this job tenure data was ultimately not used in this paper), and came up early in a search. The Integrated Public Use Microdata Series website provided a STATA do file to extract the data.

Once in STATA, observations that did not have a valid hourly wage or a valid firm size were removed, since they did not have sufficient information to perform the analysis with. Observations that did not obtain their wages from a private firm were excluded since governments don't necessarily have the same resource constraints as private firms and because self-employed or unpaid family-workers are unlikely to have reliable hourly wages and hours worked. This leaves 6678 total observations.

Years of schooling is estimated using educational attainment data. Table 1 indicates how many years of schooling are assumed for a given level of educational attainment. Years of experience by individual is estimated from the formula  $\text{experience} = \text{age} - 5 - \text{schooling}$ . Hours worked per year is calculated as the product of numbers of weeks worked times hours worked per week, and wage income is calculated as the product of hours worked per year times hourly wage.

**Table 1: Education Category vs Assumed Years of Schooling**

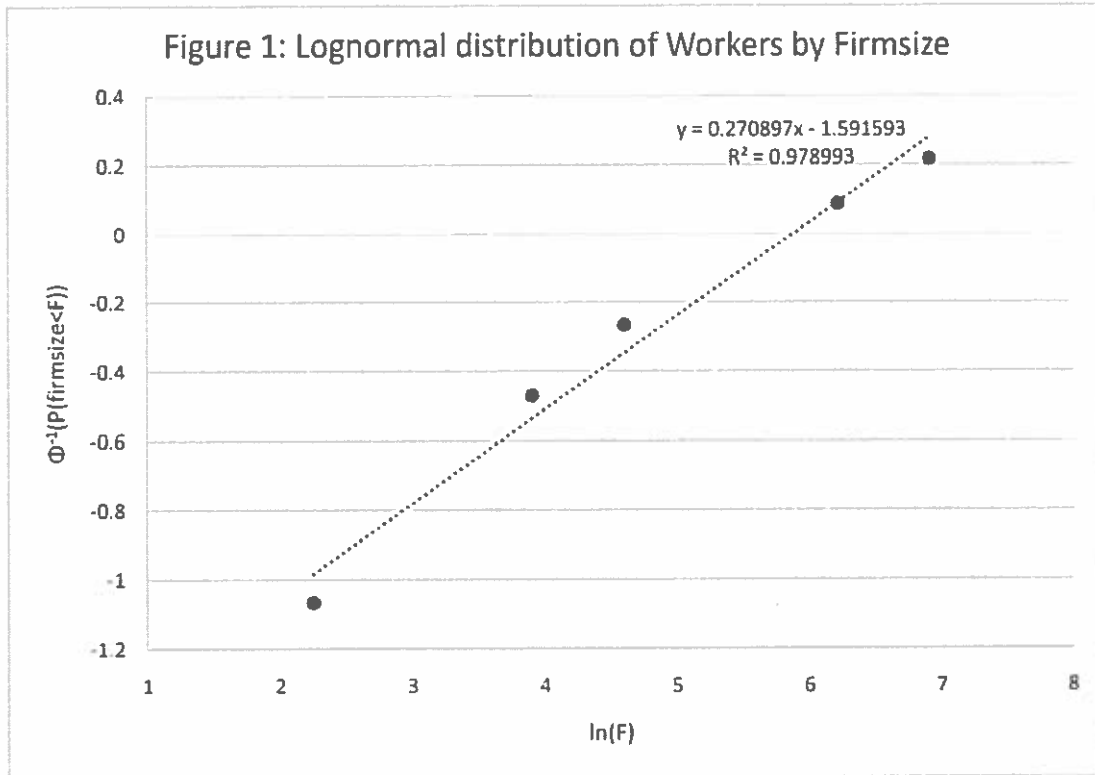
Education Category	Assumed Years of Schooling
None, preschool or kindergarten	0
Grades 1, 2, 3, 4	2.5
Grades 5 or 6	5.5
Grade 7	7
Grade 8	8
Grade 9	9
Grade 10	10
Grade 11	11
Grade 12	12
High school diploma or equivalent	12
Some college	13
Associate degree	14
Bachelor's degree	16
Master's degree	17
Professional school degree	18
Doctorate degree	20

## 7. Method and Results

Firm size is given in 6 categories: 1-9 workers, 10-49 workers, 50-99 workers, 100-499 workers, 500-999 workers and 1000+ workers. In order to estimate the effect of firm size on total-factor productivity, this paper assumes that there is only 1 representative firm, with a representative number of workers, for each firm size category. But without a distribution of workers by firm size, it is difficult to determine a representative number of workers by firm size category. Since firm size must be positive and the density of workers by firm size is likely smaller for small or large firm sizes compared to medium firm sizes, it might be reasonable to assume that the distribution of workers by firm size is roughly a lognormal distribution with mean  $\mu$  and standard deviation  $\sigma$ .

Suppose that the distribution of workers by firm size is roughly a lognormal distribution with mean  $\mu$  and standard deviation  $\sigma$ . Then the distribution of workers as a function of the natural logarithm of firm size is a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ . This implies that the probability that a worker's firm size is less than  $F$  is  $\Phi((\ln(F) - \mu)/\sigma)$ , where  $\Phi$  is the standard normal cumulative probability distribution function. This implies that  $\Phi^{-1}(P(\text{firm size} < F)) = (\ln(F) - \mu)/\sigma$ . Thus a plot of  $\Phi^{-1}(P(\text{firm size} < F))$  vs  $\ln(F)$  should produce a straight line.

Given the firm size categories for the CPS data,  $P(\text{firm size} < F)$  can be determined for  $F = 9.5, 49.5, 99.5, 499.5, \text{ and } 999.5$ . These values are determined using STATA (Appendix 2) and a plot of  $\Phi^{-1}(P(\text{firm size} < F))$  vs  $\ln(F)$  is created using excel (Figure 1). From the figure, it is clear that the relationship of  $\Phi^{-1}(P(\text{firm size} < F))$  vs  $\ln(F)$  is very close to linear, indicating that the assumption of a lognormal distribution is quite reasonable. For the line of best fit of Figure 1,  $\mu$  is estimated to be 3.691 and  $\sigma$  is estimated to be 5.875. Using these estimates, the worker-weighted expected firm size by firm size category is estimated numerically using Matlab (Appendix 3) to be 2.95, 25.30, 71.96, 249.52, 719.37, and 1478.67 for firm size categories 1-9 workers, 10-49 workers, 50-99 workers, 100-499 workers, 500-999 workers and 1000+ workers respectively. Each firm size category is represented by a single representative firm that has a number of workers equal to the expected firm size rounded to the nearest integer. The results of the STATA code (Appendix 1) are exported to a text file, which are then imported by Matlab (Appendix 4).



Suppose that the production of each worker per unit time is a Cobb-Douglas production function of physical capital and human capital, i.e.  $y_i = A_i k_i^\alpha h_i^\beta$ , where  $y_i$  is worker  $i$ 's wage,  $A_i$  is worker  $i$ 's total-factor productivity,  $k_i$  is worker  $i$ 's physical capital,  $h_i$  is worker  $i$ 's human capital and  $\alpha$  and  $\beta$  are constants. Next, suppose that there is perfect physical capital mobility (which might be a reasonable assumption given that all observations of the CPS data are in the USA). Then the level of physical capital of a worker depends on the worker's human capital and total-factor productivity, such that we can simplify the worker's production function per unit time to  $y_i = B_i h_i^\gamma$ , where  $B_i = A_i^{1/(1-\alpha)}$  represents the worker's factor productivity and  $\gamma = \beta/(1 - \alpha)$  is a constant.

Next, suppose that the output of firm  $f$  is proportional to the sum of the earnings of its workers  $Y_f$  (note that some of the firm's income will go towards paying for physical capital, however, under a Cobb-Douglas production function, physical capital's share of income is fixed), a worker's production is proportional to the amount of time they work and a worker's factor productivity depends only on the firm size such that  $B_i = KN_f^\delta$ , where  $K$  and  $\delta$  are positive real numbers and  $N_f$  is the number of workers in worker  $i$ 's firm  $f$ . Then the sum of the earnings of the workers of firm  $f$  is  $Y_f = KN_f^\delta \cdot \sum_{\text{worker } i \in \text{Firm } f} h_i^\gamma t_i$ , where  $t_i$  is the number of hours worked per year by individual  $i$ .

In addition, suppose that the worker's human capital depends only on schooling and experience and roughly follows the Mincer earnings function such that  $\ln(h_i) = C + \beta_1 S_i + \beta_2 X_i$ , where  $C$ ,  $\beta_1$  and  $\beta_2$  are constants,  $S_i$  is the years of schooling of person  $i$  and  $X_i$  is the years of experience of person  $i$ . Then we have  $Y_i = e^{\beta_0} N_i^{\delta} \cdot \sum_{\text{worker}_i \in \text{Firm}_f} e^{\beta_1 S_i + \beta_2 X_i} \bar{t}_i$ , where  $\beta_0$  is a constant. Note that the Mincer earnings function usually includes an experience squared term, however this was dropped as there are only 6 firm categories in the CPS data set, so the number of degrees of freedom to estimate this model is very low and overfitting is a concern.

$\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  need to be estimated in order to test if wage distribution within firms follows the Shapley value. Note that the total worker earnings equation from the last paragraph can be approximated as  $Y_i \approx e^{\beta_0} N_i^{\delta+1} e^{\beta_1 \bar{S}_f + \beta_2 \bar{X}_f} \bar{t}_f$ , where  $\bar{S}_f$  is the average years of schooling for workers in firm  $f$ ,  $\bar{X}_f$  is the average years of experience for workers in firm  $f$  and  $\bar{t}_f$  is the average number of hours worked per year by workers in firm  $f$ . The natural logarithm of this approximation is  $\ln(Y_i) \approx \beta_0 + (\delta+1)\ln(N_i) + \beta_1 \bar{S}_f + \beta_2 \bar{X}_f + \ln(\bar{t}_f)$ . Thus  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  can be estimated by performing a weighted ordinary least squares (OLS) regression of  $\ln(Y_i/\bar{t}_f) = \beta_0 + (\delta+1)\ln(N_i) + \beta_1 \bar{S}_f + \beta_2 \bar{X}_f + \varepsilon_i$ , where  $\varepsilon_i$  is the residual of the model.

As  $Y_i$  is the total earnings of workers within firm  $f$ ,  $\ln(Y_i) - \ln(N_i) = \ln(\bar{I}_f)$ , where  $\bar{I}_f$  is the average wage income of workers in firm  $i$ . Thus the regression equation becomes  $\ln(\bar{I}_f/\bar{t}_f) \approx \beta_0 + \delta \ln(N_i) + \beta_1 \bar{S}_f + \beta_2 \bar{X}_f + \varepsilon_i$ .  $\bar{t}_f$ ,  $\bar{I}_f$ ,  $\bar{S}_f$ , and  $\bar{X}_f$  are calculated for each firm by weighting each individual by their CPS earnings weight. The weight of each firm is assumed to be the sum of the CPS earnings weights of its workers. This weighted regression is estimated using Matlab (Appendix 4) and the results are indicated on Table 2. As expected  $\hat{\delta}, \hat{\beta}_1, \hat{\beta}_2 > 0$  and are statistically significant at the 95% confidence level, indicating that both schooling and experience have a positive effect on production, and firms with more workers generally have higher levels of factor productivity.

**Table 2:** Estimate of worker production function using simple OLS model (standard errors are given in parentheses, a \* indicates a value that is statistically significant at the 95% confidence level)

$\hat{\beta}_0$	$\hat{\delta}$	$\hat{\beta}_1$	$\hat{\beta}_2$
1.1481* (0.4186)	0.0213* (0.0036)	0.0946* (0.0292)	0.0148* (0.0048)

The results Table 2 only give estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  under the approximation  $N_i^{\delta} \sum_{\text{worker}_i \in \text{Firm}_f} e^{\beta_1 S_i + \beta_2 X_i} \bar{t}_i \approx N_i^{\delta+1} e^{\beta_1 \bar{S}_f + \beta_2 \bar{X}_f} \bar{t}_f$ . It is possible to estimate  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  without this approximation. One method to estimate this is the by using the Gauss-Newton algorithm. Under the Gauss-Newton algorithm, one starts with an initial guess for  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  (which in this case can be

the results of Table 2). One then computes the Jacobian matrix  $J$  of output with respect to the current estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ . One then modifies the initial guess by  $(J'J)^{-1}J'R$ , where  $R$  is the residual matrix (the difference between actual firm output and expected firm output under the model with the current estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ ) in order to get a more accurate guess. One can keep iterating until a desired level of convergence is reached.

For the model  $Y_f = e^{\beta_0 N_f} \delta \sum_{worker_i \in Firm_f} e^{\beta_1 S_i + \beta_2 X_i} t_i$ , the Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial Y_f}{\partial \delta} & \frac{\partial Y_f}{\partial \beta_0} & \frac{\partial Y_f}{\partial \beta_1} & \frac{\partial Y_f}{\partial \beta_2} \end{bmatrix}$$

$$= [\ln(N_f) Y_f, Y_f, e^{\beta_0 N_f} \delta \sum_{worker_i \in Firm_f} S_i e^{\beta_1 S_i + \beta_2 X_i} t_i, e^{\beta_0 N_f} \delta \sum_{worker_i \in Firm_f} X_i e^{\beta_1 S_i + \beta_2 X_i} t_i]$$

More accurate estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  are calculated using the Gauss-Newton algorithm and Matlab (Appendix 4). The method is iterated until the loss in unexplained variation between consecutive steps is less than 0.1%. The standard errors of the estimates are determined by using the approximation that the covariance matrix is  $(J'J)^{-1}$ . The results are given on Table 3. Again,  $\hat{\delta}, \hat{\beta}_1, \hat{\beta}_2 > 0$  and are statistically significant.

**Table 3:** Estimate of worker production function using Gauss-Newton algorithm (standard errors are given in parentheses, a \* indicates a value that is statistically significant at the 95% confidence level)

$\hat{\beta}_0$	$\hat{\delta}$	$\hat{\beta}_1$	$\hat{\beta}_2$
1.3481* (0.1385)	0.0224* (0.0012)	0.0729* (0.0083)	0.0157* (0.0022)

Once estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  are calculated for the model of worker earnings, it is possible to test the hypothesis that individuals in firms are paid their Shapley value. Consider a firm  $f$  of  $N_f$  individuals. According to Theorem 1 of this paper, the Shapley value of an individual  $i$  in this firm is

$$\varphi_i = \sum_{Z \subseteq Firm_f \setminus \{i\}} \frac{|Z|! (N_f - |Z| - 1)!}{N_f!} [Y_f(Z \cup \{i\}) - Y_f(Z)]$$

where  $Y_f(Z)$  corresponds to the sum of worker earnings of firm  $f$  that would occur if firm  $f$  only employed the set of workers  $Z$ , assuming that hours and effort of workers in  $Z$  are the same as the amount of hours and effort that the workers currently put into firm  $f$ .

In the CPS data set, total worker earnings of a firm of size  $N_f$  does not exist, but one can infer total worker earnings of a representative firm of size  $N_f$ . If the number of individuals in society is very large then it may be a reasonable simplifying assumption to assume that the level of human capital and hours worked of a new employee are independent of the current employees already in the firm. In this case, the expected Shapley value of an individual  $i$  in a representative firm of size  $N_f$  is

$$E(\varphi_i) = \frac{1}{N_f} \sum_{j=0}^{N_f-1} E \left( [Y_f(Z \cup \{i\}) - Y_f(Z)] \mid Z \text{ is a random set of } j \text{ individuals that work for a firm of firm size } N_f \right)$$

$$\Rightarrow E(\varphi_i) = \frac{1}{N_f} \sum_{j=0}^{N_f-1} [e^{\beta_0(j+1)^\delta} (jE(e^{\beta_1 S + \beta_2 X_t}) + e^{\beta_1 S_i + \beta_2 X_i t_i}) - e^{\beta_0 j^{\delta+1}} E(e^{\beta_1 S + \beta_2 X_t})]$$

$$\Rightarrow E(\varphi_i) = \frac{e^{\beta_0}}{N_f} \sum_{j=0}^{N_f-1} [(j+1)^\delta e^{\beta_1 S_i + \beta_2 X_i t_i} + j((j+1)^\delta - j^\delta) E(e^{\beta_1 S + \beta_2 X_t})]$$

where  $E(e^{\beta_1 S + \beta_2 X_t})$  is the expected value of  $e^{\beta_1 S + \beta_2 X_t}$  for a worker in a representative firm of size  $N_f$ .

Using Matlab,  $E(e^{\beta_1 S + \beta_2 X_t})$  is calculated for each firm size category (using the earnings weights) and, using this value, the expected Shapley value for each of the 6678 individuals is calculated. In many cases, there is a divergence between the Shapley value of earnings predicted by the model and actual earnings. An explanation for this is that some individuals may have more bargaining power than others, so may be able to obtain a relatively higher wage. Perhaps an individual is in a union and is therefore able to take advantage of collective bargaining. Perhaps an individual has stayed with a company for a longer period of time and has therefore been able to gain more friends and connections. Alternatively, perhaps the firm is not accurately evaluating the value of a person's education.

One way to test if being in a union, having more experience or having a higher education has an effect on a person's bargaining power is to perform the following regression:

$$\ln(I_i/E(\varphi_i)) = \mu_0 + \mu_1 S_i + \mu_2 X_i + \mu_3 Union_i + \epsilon_i$$

The above equation is estimated in Matlab using a weighted OLS regression, where the weight of each individual is proportional to their earnings weight. The results are given in Table 4.

**Table 4:** Estimate of unionization, education and experience on bargaining strength using the best estimates of table 3 (standard errors are given in parentheses, a \* indicates a value that is statistically significant at the 95% confidence level)

$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$
-0.0898* (0.0342)	0.0004 (0.0024)	-0.0045* (0.0004)	0.2675* (0.0197)

From table 4, unionization has a positive and statistically significant effect on a person's bargaining strength, schooling has a negligible and a non-statistically significant impact on bargaining strength, and experience has a negative and statistically significant effect on a person's bargaining strength. Note that the human capital model for this section only includes linear terms for schooling and experience, whereas the Mincer earnings function usually includes an experience squared term which tends to be negative. Because of this and because of the fact that experience squared is correlated with experience, the result of a negative impact of experience on bargaining strength may be due to this specification error. Unfortunately, as there are only 6 firm size categories in the CPS data, making the model more complex than it currently is may result in overfitting of the model.

One issue with the estimates of table 4 is that they do not include the uncertainty due to the estimates of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ . Given that  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  are simultaneously estimated, the estimates are likely correlated, so it is difficult to analytically determine the total uncertainty of how unionization, education and experience affect bargaining strength. One relatively easy way to take into account this additional uncertainty is to perform Monte-Carlo simulations.

For the Monte-Carlo simulation to take into account the uncertainty in the estimates from Table 3, one starts by obtaining the Cholesky matrix of the covariance matrix used in the computation of Table 3. One then multiplies the Cholesky matrix by a vector of 4 independent random variables with standard normal distributions to obtain random values of  $\delta$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ . These values are then used to obtain estimates of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$ . The Cholesky matrix of the covariance matrix used to obtain the estimates of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  is then used to obtain random values of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  by multiplying the Cholesky matrix by a vector of 4 independent random variables with standard normal distributions. These random values of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  are a Monte-Carlo simulation. After performing 1000 Monte-Carlo simulations, the means and standard errors of the simulations of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  are determined. Table 5 gives the estimates of  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  that result from 1000 Monte-Carlo simulations.

**Table 5:** Estimate of unionization, education and experience on bargaining strength using 1000 Monte-Carlo simulations (standard errors are given in parentheses, a \* indicates a value that is statistically significant at the 95% confidence level)

$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$
-0.0902 (0.1406)	0.0005 (0.0082)	-0.0045* (0.0020)	0.2675* (0.0009)

The results of Table 5 are similar to the results of Table 4; unionization has a positive and statistically significant effect on a person’s bargaining strength, schooling has a negligible and a non-statistically significant impact on bargaining strength, and experience has a negative and statistically significant effect on a person’s bargaining strength. However, the significance of the experience term has been reduced considerably and is now only slightly more than twice the standard error in magnitude. Given that the model of human capital used likely has specification error that biases the estimate of the impact of experience on bargaining strength, it is not unreasonable to suggest that the true impact of experience on bargaining strength may not be negative and statistically significant.

Note that the results from Table 5 do not include the uncertainty due to the lognormal approximation of the distribution of workers by firm size as well as the rounding error that results from rounding the expected firm size by category to the nearest integer. It is difficult to take this uncertainty into account without a way to analytically extend the formula of the Shapley value to cases with a non-discrete number of players. However, given that the rounding error is relatively small (<1.7%) and that the lognormal fit as indicated by Figure 1 is relatively good, it is likely that this additional uncertainty is small compared to the uncertainty already taken into account in Table 5.

## 8. Discussion and Conclusion

The results of this paper indicate the importance of estimating non-linear models rather than their linear approximations and ensuring that all sources of error are properly propagated. The estimates of how workers affect firm production are up to 30% different when the Gauss-Newton algorithm is used to estimate the non-linear model (Table 3) than when the model is linearly approximated (Table 2). Furthermore, using Monte-Carlo simulations to propagate the error due to the uncertainty of how workers affect firm production increased the standard error of the estimate of the effect of experience on bargaining strength by a factor of 5 (Tables 4 and 5). Ultimately, the results of tables 4 and 5 suggest that unionization has a positive and statistically significant effect at the 95% confidence level on bargaining strength. An alternative explanation for this result is that unionized workers are more productive than non-unionized workers of comparable experience, education and firm size; although this seems unlikely. The results of Table 5 do not take into account the error associated with the lognormal approximation of the distribution of workers by firm size; but the linear

relationship of Figure 1 and the fact that the rounding error is less than 1.7% suggests that this is not a big issue.

A major limitation of this paper is that the CPS data only has 6 firm size categories and, as a result, the functional form of the production function is very simplistic in order to avoid overfitting. Specification error due to the simplistic functional form of the production function may have caused the observed statistically significant (at the 95% confidence level) and negative impact of experience on bargaining strength. This result of a negative impact of experience on bargaining strength is contrary to expectations since more experienced workers are more likely to have accumulated more 'friends' within a firm, which may allow them to enjoy greater bargaining power than less experienced workers. It is recommended that future studies try to take advantage of more firm size categories and more months of observation in order to allow for a better specification of the production function without running into overfitting problems, although using more months of observation may cause issues due to temporal correlation of observations. One should not read too much into the numeric results of this paper, rather the purpose of using CPS data to test if individuals are paid their expected Shapley value in earnings is to show how one can use the Shapley value defined in this paper to answer real world questions.

Taking advantage of the correlation between firm size and factor productivity allows this paper to test if individuals are being paid their expected Shapley value in earnings. Discrepancies between observed earnings and the expected Shapley value are found and may arise due to differences in bargaining strength. This paper finds that unionization has a statistically significant (at the 95% confidence level) and positive impact on bargaining strength. The definition of Shapley value used for this paper is an extension of the Shapley value to infinite choice games and is justified using six simple axioms: weak symmetry, efficiency, additivity, the null worker axiom, no pay for no work and strong scale invariance. This paper gives a similar axiomatization for finite multi-choice games. These axiomatizations help to further the literature on the Shapley value and on wage distribution in firm problems by extending the Shapley value to infinite choice games and by using the results to empirically test for discrepancies between observed wages and the Shapley value. Overall, this paper helps to improve understanding of wage distribution in firms, which may eventually help to explain the observed increase in wage inequality over the past few decades.

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## Appendices

### Appendix 1 – STATA Code

```
cd "Z:\"  
  
do cps_00001.do  
  
keep if hourwage < 99.98  
  
keep if hourwage > 0  
  
keep if classwkr == 21  
  
keep if firmsize != 0  
  
generate schooling = 0  
  
replace schooling = 2.5 if educ == 10  
replace schooling = 5.5 if educ == 20  
replace schooling = 8 if educ == 30  
replace schooling = 9 if educ == 40  
replace schooling = 10 if educ == 50  
replace schooling = 11 if educ == 60  
replace schooling = 12 if educ == 71  
replace schooling = 12 if educ == 73  
replace schooling = 13 if educ == 81  
replace schooling = 14 if educ == 91  
replace schooling = 14 if educ == 92  
replace schooling = 16 if educ == 111  
replace schooling = 17 if educ == 123  
replace schooling = 18 if educ == 124  
replace schooling = 20 if educ == 125  
  
generate schooling2 = schooling^2
```

```

generate exp = age - 5 - schooling
generate exp2 = exp^2
generate hours = wkswork1*uhrswork
generate income = hours*hourwage
summarize i.firmsize [aweight = earnwt]
generate firmsize2 = 3
replace firmsize2 = 25 if firmsize == 4
replace firmsize2 = 72 if firmsize == 6
replace firmsize2 = 250 if firmsize == 7
replace firmsize2 = 719 if firmsize == 8
replace firmsize2 = 1479 if firmsize == 9
generate Union = 1 - (union == 1)
outsheet schooling exp hours income firmsize2 Union earnwt using Shapley.txt

```

**Appendix 2 – Weighted Distribution of Workers by Firm Size calculated by STATA**

Variable	Obs	Weight	Mean	Std. Dev.	Min	Max
-----+-----						
firmsize						
10 to 49	6663	62252120.8	.1763765	.3811688	0	1
50 to 99	6663	62252120.8	.0757627	.2646379	0	1
100 to 499	6663	62252120.8	.1396592	.3466591	0	1
500 to 999	6663	62252120.8	.050749	.2195011	0	1
-----+-----						
1000+	6663	62252120.8	.4138897	.4925662	0	1

### Appendix 3 – Matlab Code to calculate expected value of worker firm size by firm size category

```
N = [0.005:0.01:99999.5]; %range of values used for numeric approximation
PDF = lognpdf(N,5.87527,3.69144); %probability distribution
NPDF = N.*PDF; %probability distribution weighted by firm size
i = 1;
EN = zeros(6,1); %Expected firm size by firm size category
SumPDF = zeros(6,1);
SumNPDF = zeros(6,1);
while N(i) < 9.5
SumPDF(1) = SumPDF(1) + PDF(i);
SumNPDF(1) = SumNPDF(1) + NPDF(i);
i = i + 1;
end
while N(i) < 49.5
SumPDF(2) = SumPDF(2) + PDF(i);
SumNPDF(2) = SumNPDF(2) + NPDF(i);
i = i + 1;
end
while N(i) < 99.5
SumPDF(3) = SumPDF(3) + PDF(i);
SumNPDF(3) = SumNPDF(3) + NPDF(i);
i = i + 1;
end
while N(i) < 499.5
SumPDF(4) = SumPDF(4) + PDF(i);
SumNPDF(4) = SumNPDF(4) + NPDF(i);
i = i + 1;
end
while N(i) < 999.5
SumPDF(5) = SumPDF(5) + PDF(i);
SumNPDF(5) = SumNPDF(5) + NPDF(i);
i = i + 1;
end
while true
SumPDF(6) = SumPDF(6) + PDF(i);
SumNPDF(6) = SumNPDF(6) + NPDF(i);
if i == 9999950
break
end
i = i + 1;
end

EN = SumNPDF./SumPDF;
```

#### Appendix 4 – Matlab Code that performs Regressions and Monte Carlo Simulation

```
clear
S = dlmread('Z:\Shapley.txt','\t','A2..A6679'); %schooling
X = dlmread('Z:\Shapley.txt','\t','E2..B6679'); %experience
t = dlmread('Z:\Shapley.txt','\t','C2..C6679'); %hours worked
I = dlmread('Z:\Shapley.txt','\t','D2..D6679'); %income
N = dlmread('Z:\Shapley.txt','\t','E2..E6679'); %Firm Size
Union = dlmread('Z:\Shapley.txt','\t','F2..F6679'); %Union Membership
W = dlmread('Z:\Shapley.txt','\t','G2..G6679'); %Person weight

FirmW = zeros(6,1); %Total weight of Firms by firm size
FirmS = zeros(6,1); %Weighted average schooling by firm size
FirmX = zeros(6,1); %Weighted average experience by firm size
Firmt = zeros(6,1); %Weighted average hours worked by firm size
FirmI = zeros(6,1); %Weighted average income by firm size
FirmN = [3;25;72;250;719;1479]; %firm size categories
FirmY = zeros(6,1); %Total weighted output by firm size

for i = 1:6678
    for j = 1:6
        if N(i) == FirmN(j)
            FirmW(j) = FirmW(j) + W(i);
            FirmS(j) = FirmS(j) + S(i)*W(i);
            FirmX(j) = FirmX(j) + X(i)*W(i);
            Firmt(j) = Firmt(j) + t(i)*W(i);
            FirmI(j) = FirmI(j) + I(i)*W(i);
        end
    end
end

FirmS = FirmS./FirmW;
FirmX = FirmX./FirmW;
Firmt = Firmt./FirmW;
FirmY = FirmI;
FirmI = FirmI./FirmW;

%Linear regression to estimate production function
FlnIt = log(FirmI./Firmt);
Z = [ones(6,1), log(FirmN), FirmS, FirmX];
DFirmW = diag(FirmW);
Beta = inv(Z'*DFirmW*Z)*Z'*DFirmW*FlnIt; %estimates of production function
error = FlnIt - Z*Beta;
sigma = sqrt(error'*DFirmW*error/(6-4));
errorBeta = sigma*sqrt(diag(inv(Z'*DFirmW*Z))); %standard errors of estimates

%Stop code here to obtain linear estimates before starting Gauss-Newton
algorithm

%Gauss-Newton Algorithm to better estimate production function
J = zeros(6,4);
temp = 0;
for i = 1:6678
    for j = 1:6
        if N(i) == FirmN(j)
```

```

        temp = exp(Beta(1) + Beta(3)*S(i) +
Beta(4)*X(i))*t(i)*N(i)^Beta(2)*W(i);
        J(j,:) = J(j,:) + [1,log(N(i)),S(i),X(i)]*temp;
    end
end
end
error = FirmY - J(:,1);
sigma = sqrt(error'*DFirmW*error/(6-4));
errorBeta = sigma*sqrt(diag(inv(J'*DFirmW*J)));

iterations = 0;
while true
    iterations = iterations + 1;
    dBeta = inv(J'*DFirmW*J)*J'*DFirmW*error;
    Beta = Beta + dBeta;
    oldsigma = sigma;

    J = zeros(6,4);
    temp = 0;
    for i = 1:6678
        for j = 1:6
            if N(i) == FirmN(j)
                temp = exp(Beta(1) + Beta(3)*S(i) +
Beta(4)*X(i))*t(i)*N(i)^Beta(2)*W(i);
                J(j,:) = J(j,:) + [1,log(N(i)),S(i),X(i)]*temp;
            end
        end
    end
    error = FirmY - J(:,1);
    sigma = sqrt(error'*DFirmW*error/(6-4));
    errorBeta = sigma*sqrt(diag(inv(J'*DFirmW*J)));
    if log(oldsigma) - log(sigma) < 0.001
        break
    end
end
end

%Calculation of Shapley value for each individual in the survey
Expected = zeros(6,1);
for i = 1:6678
    for j = 1:6
        if N(i) == FirmN(j)
            Expected(j) = Expected(j) + exp(Beta(3)*S(i) +
Beta(4)*X(i))*t(i)*W(i);
        end
    end
end
Expected = Expected./FirmW;

EShapley = zeros(6678,1);
temp2 = 0;
for i = 1:6678
    for j = 1:6
        if N(i) == FirmN(j)
            temp = exp(Beta(3)*S(i) + Beta(4)*X(i))*t(i);
            for k = 0:(N(i)-1)

```

```

                temp2 = (k+1)^Beta(2)*temp + k*((k+1)^Beta(2) -
k^Beta(2))*Expected(j);
                EShapley(i) = EShapley(i) + exp(Beta(1))/N(i)*temp2;
            end
        end
    end
end

%Regression that tries to explain ratio of actual income to Shapley value
Z2 = [ones(6678,1), S, X, Union];
DW = diag(W);
lnI = log(I./EShapley);
Mu = inv(Z2'*DW*Z2)*Z2'*DW*lnI;
difference = lnI - Z2*Mu;
errorMu = sqrt(difference'*DW*difference/(6678-4)*diag(inv(Z2'*DW*Z2)));

%Monte Carlo Simulation
Q = 1000; %#of simulations
Mu_Simulated = zeros(4,Q);
BetaTest = zeros(4,1);
Cholesky = chol(error'*DFirmW*error/(6-4)*inv(J'*DFirmW*J),'lower');
Cholesky2 = zeros(4,4);
for l = 1:Q
    BetaTest = Beta + Cholesky*normrnd(0,1,[4,1]);

    %Calculation of Shapley value for for each randomized set of Beta
    Expected = zeros(6,1);
    for i = 1:6678
        for j = 1:6
            if N(i) == FirmN(j)
                Expected(j) = Expected(j) + exp(BetaTest(3)*S(i) +
BetaTest(4)*X(i))*t(i)*W(i);
            end
        end
    end
    Expected = Expected./FirmW;

    EShapley = zeros(6678,1);
    for i = 1:6678
        for j = 1:6
            if N(i) == FirmN(j)
                temp = exp(BetaTest(3)*S(i) + BetaTest(4)*X(i))*t(i);
                for k = 0:(N(i)-1)
                    temp2 = (k+1)^BetaTest(2)*temp + k*((k+1)^BetaTest(2) -
k^BetaTest(2))*Expected(j);
                    EShapley(i) = EShapley(i) + exp(BetaTest(1))/N(i)*temp2;
                end
            end
        end
    end
end

%Obtain random values of Mu
lnI = log(I./EShapley);
Mu = inv(Z2'*DW*Z2)*Z2'*DW*lnI;
difference = lnI - Z2*Mu;
Choleksy2 = chol(difference'*DW*difference/(6678-4)*inv(Z2'*DW*Z2));

```

```
    Mu_Simulated(:,1) = Mu + Cholesky2*normrnd(0,1,[4,1]);
    disp(int2str(1));
end
%Obtain means and standard errors of Monte Carlo results
MeanMu = mean(Mu_Simulated,2);
StdMu = std(Mu_Simulated,0,2);
```