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**DYNAMIC MODELS OF SUSPENSION BRIDGES  
AND THEIR STABILITIES**

by

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A thesis submitted to the  
School of Graduate Studies and Research  
in partial fulfillment of the requirements for the degree of

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# **Dedication**

**In loving memory of my father  
Mohamed Harbi Ben Ounis (1906-1992)**

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# Abstract

Suspension bridges have a history of large-scale oscillations caused by wind, earthquake or traffic forces which may lead to structural failure. As a result of these oscillations and the probable resonance effects, the cables start to loosen and tighten producing a nonlinear effect. This nonlinear effect is very complex to model and only limited research has been conducted in this area. Therefore, there is a need to give a clear mathematical argument as to why suspension bridges oscillate and find the effect of nonlinearity behaviour of their cables. In order to clarify the oscillations and nonlinearity effect behaviour, this thesis presents a four dynamic Partial Differential Equations (PDE) models of suspension bridges. These models are generalized cases of those proposed by Lazer and McKenna. Further, an analytical study of the stability properties of these models under different types of dynamic loading is performed. Furthermore, for each loading situation, the results are illustrated by numerical simulation with physical interpretation.

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# Chapter 1

## Introduction

### 1.1 General

Suspension bridges have a history of large-scale oscillations that may lead to a catastrophic failure under high and even moderate forces such as wind, earthquake or traffic. Many people have seen the dramatic large-scale oscillations, followed by the collapse of the Tacoma Narrows suspension bridge. In the report on the destruction of the Tacoma Narrows bridge, eye-witnesses describe vertical and torsional motion of the bridge. As a result of these oscillations, the cables started to loosen and tighten, producing a nonlinear effect. Some other bridges have also exhibited vibration problems. The Bronx-Whitestone bridge, on which a traveller might often get seasick due to the large-scale motions, or the Golden Gate Bridge, which has exhibited travelling waves, had exhibited oscillatory behaviour due to the action of wind.

The cause and nature of the vertical and torsional oscillations can be studied by considering a simple model of a suspension bridge. Researchers such as Lazer and McKenna [10] have found that the bridge behaves like a particle of mass one at the end of a spring with spring constant,  $k$ , which is subject to a forcing term of frequency  $\mu/2\pi$ . If  $\mu$  is very close to the square root of  $k$ , the frequency just happened to be at a value very close to a resonant frequency of the bridge. i.e.. large oscillations occur.

Thus, even though the magnitude of the forcing term was small, the phenomenon of linear resonance was enough to explain the large oscillations and eventual collapse of the bridge.

In addition, it should be emphasized that because of the observed torsional motion, some of the cables were alternately loosening and tightening. This is the nonlinear effect which requires further study. Moreover, the restoring force due to a cable is such that it strongly resists expansion, but does not resist compression. Therefore, fundamental nonlinearity is very distinctive and should be considered in any dynamic analysis.

From an overview of what was said thus far, there is a need to give a clear and simple mathematical argument as to why suspension bridges oscillate and find the effect of nonlinearity behaviour of their cables.

## **1.2 Literature Review**

The problem of vibration of suspension bridges have been studied since 1920. Many researchers proposed different solutions and the following paragraphs describe some of the research related to the problem treated in this thesis.

The federal works agency report [1] by Amann, Von Karman and Woodruff, on the collapse of the Tacoma Narrows bridge in the State of Washington on November 7, 1940, as a result of wind action, created a widespread demand from both the general public and the engineering profession that steps be taken to inaugurate a comprehensive investigation of the problem of dynamic oscillation in suspension bridges, with a view to understand the causes of such destructive oscillations and develop design techniques to prevent their recurrence in future. The Advisory Board on the investigation of

suspension bridges undertook the following objectives:

- To determine the causes and the phenomenon of vibration in suspension bridges.
- To correlate and develop a rational theory explanatory of the phenomenon of vibration.
- To create new methods of design practice, applicable to future bridges.

Following the tragedy of the Tacoma Narrows bridge, a number of researchers, including Bleich, McCullough, Rosecrans, and Vincent [2], Wiles [3], Selberg [4], Abdel-Ghaffar [5] and Lazer and McKenna [10] started seriously studying the mathematical theory of vibration of suspension bridges.

Abdel-Ghaffar [5] gave a new methodology of free vibration analysis of suspension bridge with horizontal decks, utilizing a continuum approach to include the coupling between the vertical and torsional vibration and the effects of cross-sectional distortion. Variational principles were used to obtain the coupled equations of motion in their most general and nonlinear form.

Kawada and Hirai [6] examined a new approach through mathematical analysis and wind tunnel test to suspension bridge rehabilitation. They studied the relationship between the suspended mass and the limited oscillations of the long span suspension bridges. On the basis of their work, they put together the following conclusions:

- Suspension bridges demand the right amount of mass, otherwise, the result of oscillations will be disastrous and will charge an expensive compensation.
- If suspension bridges, with stream-lined stiffening girder, had any troubles, their cause must be mainly due to the lack of mass.

In [7], McKenna and Walter investigated nonlinear oscillations in a suspension

bridge, where the model was suggested by Lazer and McKenna [8]. As a model of suspension bridge they consider a one-dimensional beam of length  $L$  suspended by cables. When the cables are stretched, there is a restoring force which is assumed to be proportional to the amount of the stretching (Hooke's law). But when the beam moves in the opposite direction, then there is no restoring force exerted on it.

Oscillation induced fatigue of the structural members is a major factor limiting the useful life of a suspension bridge. Some relevant studies regarding oscillatory solutions were carried out by [9].

In reference [10], Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridges. They presented a one-dimensional mathematical model for the bridge, that takes account of the fact that the coupling provided by the stays (ties) connecting the suspension (main) cable to the deck of the road bed is fundamentally nonlinear.

McKenna and Walter [11] considered the existence of travelling wave solutions to a nonlinear beam equation which was proposed as a model for a suspension bridge in two previous articles [7] and [8].

In reference [12], a theory has been advanced which describes the one-dimensional vertical oscillation in suspension bridge, and which shows the behaviour of the following items:

- Suspension bridges are prone to large-scale oscillation;
- The occurrence of this large-scale oscillation is dependent on initial conditions;
- Large vertical oscillations can rapidly change, virtually instantaneously to torsional two-dimensional oscillations;

- The torsional oscillation was not of a single nodal type, but changed from no-noded to one-noded and back, both types of motion having the same period but with the no-noded motion having somewhat larger amplitude;
- Sometimes the torsional oscillation would be bigger at one end, indicating some asymmetry in the motion.

Jacover and McKenna [13] gave some explanation of why a nonlinearly suspended beam, subject to periodic forcing, could exhibit large-scale oscillations of a torsional type or, under the same periodic forcing term, could exhibit small linear torsional oscillations about equilibrium. The authors came up with the following conclusions:

- The long-term response to any given forcing term is highly sensitive to the initial conditions.
- In the presence of vertical forcing terms, the multiple-solution behaviour seems to be purely vertical.

In addition to the vibration and oscillation of suspension bridges which is typically a structural domain problem, instability and vibration problems are also encountered in space structures such as communication satellites, space shuttle and space station which are equipped with large antennas mounted on long flexible masts that could be modelled as beams [14-17]. These problems can be solved in similar fashion as the suspension bridge.

### **1.3 Thesis Objectives and Contributions**

This thesis presents a refinement of the model proposed by Laser and McKenna as described in [10]. The explicit objectives of this study can be outlined as follows:

- A presentation of the Partial Differential Equations (PDE) version of the Laser-McKenna suspension bridge model.

- A presentation of four suspension bridge dynamic models generalizing those proposed by Lazer and McKenna, including their stability under different types of dynamic loadings.
- An illustration of the numerical results from the models simulating the physical interpretation.

## 1.4 Outline of the thesis

The thesis is organized as follow:

- Chapter 1 contains the motivation to the problem of modelling and stabilization of the suspension bridge and a brief review of previous studies in this area.
- Chapters 2, presents a methodology for developing a rigorous dynamic models of suspension bridge. The complete dynamics of the suspension bridge in terms of coupled hyperbolic system is developed following four different approaches.
- An algorithm for numerical integration of the hyperbolic dynamics, and illustrative numerical results are also presented in Chapter 3 .
- Concluding remarks and suggestions for further research are presented in Chapter 4 .

# Chapter 2

## Dynamic Models of Suspension Bridge

### 2.1 General

This chapter shows the development of some models describing vibration in suspension bridges. Four models are presented. The suspension bridge model which is divided into a linear and a nonlinear model, a general nonlinear model without damping, and a general nonlinear model with damping. The stability of each system is also examined. For clarity of the method, a simple suspension bridge configuration is considered in Figure 2.1.

### 2.2 Some Relevant Function Spaces

Before going into the details of the modelling, it is worthwhile establishing some relevant function spaces. These functions spaces will be used for the development of the models and the analysis of their solutions.

Let  $\Sigma \subset R^n$  be an open bounded set with smooth boundary  $\partial\Sigma$  and let  $L_2(\Sigma)$  denote the space of equivalence classes of Lebesgue square integrable functions with the standard norm topology. Let  $H^m(\Sigma) \equiv H^m$ ,  $m \in N$ , denote the standard Sobolev

space with the usual norm topology and  $H_0^m(\Sigma) \equiv H_0^m \subset H^m$  denote the completion in the topology of  $H^m$  of  $C^\infty$  functions on  $\Sigma$  with compact support. From classical results on Sobolev spaces it is well known that the elements of  $H_0^m$  are those of  $H^m$  which, along with their conormal derivatives up to order  $m - 1$ , vanish on the boundary  $\partial\Sigma$ .

## 2.3 Suspension Bridge Model

A simplified model of a suspension bridge such as the one shown in Figure 2.1, is given by a coupled system of partial differential equations, taken from Lazer and McKenna [10] of the form:

$$\begin{cases} m_b(\partial^2/\partial t^2)z + \alpha(\partial^4/\partial x^4)z - F_0(y - z) = m_b g + f_1, & x \in (0, \ell) \equiv \Sigma, \quad t \geq 0; \\ m_c(\partial^2/\partial t^2)y - \beta(\partial^2/\partial x^2)y + F_0(y - z) = m_c g + f_2, & x \in (0, \ell) \equiv \Sigma, \quad t \geq 0 \end{cases} \quad (2.1)$$

The first equation describes the vibration of the road bed in the vertical plan and the second equation describes that of the main cable from which the road bed is suspended by tie cables ( stays).  $(\partial^k/\partial x^k)$  denotes the spatial derivative of order  $k$ . Here  $m_b, m_c$  are the masses per unit length of the road bed and the cable, respectively;  $\alpha$  and  $\beta$  are the flexural rigidity of the structure and coefficient of tensile strength of the cable, respectively. The function  $F_0$  represents the restraining force experienced both by the road bed and the suspension cable as transmitted through the tie lines (stays) thereby producing the coupling between the two. The functions  $f_1$  and  $f_2$  represent external as well as nonconservative forces generally time dependent.

Let  $z_s, y_s$  represent the static displacements (equilibrium positions) which are the solutions of the system of equations given below:

$$\begin{cases} \alpha(\partial^4/\partial x^4)z_s - F_0(y_s - z_s) = m_b g, & x \in (0, \ell); \\ -\beta(\partial^2/\partial x^2)y_s + F_0(y_s - z_s) = m_c g, & x \in (0, \ell) \end{cases} \quad (2.2)$$

Subtracting equation (2.2) from (2.1) we obtain the following system of equations

$$\begin{cases} m_b(\partial^2/\partial t^2)\bar{z} + \alpha(\partial^4/\partial x^4)\bar{z} - F(\bar{y} - \bar{z}) = f_1, & x \in \Sigma \equiv (0, \ell), \quad t \geq 0; \\ m_c(\partial^2/\partial t^2)\bar{y} - \beta(\partial^2/\partial x^2)\bar{y} + F(\bar{y} - \bar{z}) = f_2, & x \in (0, \ell), \quad t \geq 0 \end{cases} \quad (2.3)$$

where  $\bar{z} \equiv z - z_s$ ,  $\bar{y} \equiv y - y_s$  and the function  $F$  is given by

$$F(\zeta) \equiv F_0(\zeta + z_s - y_s) - F_0(z_s - y_s)$$

Note that  $F(0) = 0$  and throughout the rest of the thesis we assume that the displacements, again denoted by  $z$ ,  $y$  instead of  $\bar{z}$ ,  $\bar{y}$ , are measured relative to the static positions. The system of equations (2.3) is used as the general model.

### 2.3.1 Linear Model

A linear model is obtained by supporting the road bed with ties (stays) connected to two symmetrically placed main (suspension) cables one above and one below the road bed. In the absence of external forces, the dynamics of a linear suspension bridge around the equilibrium position can be described by a system of coupled partial differential equations as given below:

$$\begin{cases} m_b(\partial^2/\partial t^2)z + \alpha(\partial^4/\partial x^4)z - k(y - z) = 0, & x \in (0, \ell), \quad t \geq 0; \\ m_c(\partial^2/\partial t^2)y - \beta(\partial^2/\partial x^2)y + k(y - z) = 0, & x \in (0, \ell), \quad t \geq 0 \end{cases} \quad (2.4)$$

The quantity  $k$  denotes the stiffness coefficient of the stays (ties) connecting the road bed to the suspension cable. Assuming that the beam is clamped at both ends, the boundary conditions are given by

$$\begin{cases} z(t, 0) = z(t, \ell) = 0, & (\partial/\partial x)z(t, 0) = (\partial/\partial x)z(t, \ell) = 0 \\ y(t, 0) = y(t, \ell) = 0 \end{cases} \quad (2.5)$$

In case the beam is hinged at both ends the boundary conditions are given by

$$\begin{cases} z(t, 0) = z(t, \ell) = 0, & (\partial^2/\partial x^2)z(t, 0) = (\partial^2/\partial x^2)z(t, \ell) = 0 \\ y(t, 0) = y(t, \ell) = 0 \end{cases} \quad (2.6)$$

Other combinations, such as hinged on one side and clamped on the other, are also used. The initial conditions are given by

$$\begin{cases} z(0, x) = z_1(x), & (\partial/\partial t)z(0, x) = z_2(x), & x \in (0, \ell) \\ y(0, x) = y_1(x), & (\partial/\partial t)y(0, x) = y_2(x), & x \in (0, \ell) \end{cases} \quad (2.7)$$

Where  $z_1$ ,  $z_2$ ,  $y_1$  and  $y_2$  are suitable real valued functions defined on  $\Sigma = (0, \ell)$ . Using Fadeo-Galerkin method one can establish the existence and uniqueness of solutions of the system (2.4) to (2.7), (see Abstract Model in appendix A ). Given that  $z_1 \in H_0^2$ ,  $z_2 \in L_2(\Sigma)$ ,  $y_1 \in H_0^1$  and  $y_2 \in L_2(\Sigma)$ , the systems (2.4) to (2.7) have unique solutions  $\{z, y\} \in L_\infty(I, H_0^2 \times H_0^1)$  and  $\{\frac{\partial z}{\partial t}, \frac{\partial y}{\partial t}\} \in L_\infty(I, L_2(\Sigma) \times L_2(\Sigma))$ .

### 2.3.2 Stability of the Linear Model

In this subsection, we study the problem of stability of the suspension bridge described by the coupled linear system of hyperbolic partial differential equations (2.4) along with the boundary conditions (2.5) to (2.7).

The systems (2.4) to (2.7) is conservative. Indeed the total energy (Lyapunov function) is given by

$$\begin{aligned} E(t) \equiv & (1/2) \int_0^\ell m_b \left| \frac{\partial z}{\partial t} \right|^2 dx + (1/2) \int_0^\ell m_c \left| \frac{\partial y}{\partial t} \right|^2 dx \\ & + (1/2) \int_0^\ell \alpha \left| \frac{\partial^2 z}{\partial x^2} \right|^2 dx + (1/2) \int_0^\ell \beta \left| \frac{\partial y}{\partial x} \right|^2 dx \\ & + (1/2) \int_0^\ell k |(y - z)|^2 dx \end{aligned} \quad (2.8)$$

Differentiating  $E(t)$  with respect to time

$$\begin{aligned}
(d/dt)E(t) \equiv & (1/2) \int_0^\ell \left\{ 2m_b \left( \frac{\partial z}{\partial t} \frac{\partial^2 z}{\partial t^2} \right) dx + 2m_c \left( \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \right) + 2\alpha \left( \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial^2}{\partial x^2} \left( \frac{\partial z}{\partial t} \right) \right) \right) \right\} dx \\
& + (1/2) \int_0^\ell \left\{ 2\beta \left( \frac{\partial y}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial t} \right) \right) \right) + 2k(y-z) \left( \frac{\partial y}{\partial t} - \frac{\partial z}{\partial t} \right) \right\} dx
\end{aligned}$$

After integration by parts using initial and boundary conditions (2.5) or (2.6) , we obtain:

$$(d/dt)E = \int_0^\ell \left\{ \left( m_b \frac{\partial^2 z}{\partial t^2} + \alpha \frac{\partial^4 z}{\partial x^4} - k(y-z) \right) \frac{\partial z}{\partial t} + \left( m_c \frac{\partial^2 y}{\partial t^2} + \beta \frac{\partial^2 y}{\partial x^2} + k(y-z) \right) \frac{\partial y}{\partial t} \right\} dx$$

Since  $\{z, y\}$  is the solution of the systems (2.4) to (2.7), it follows from (2.4) and the above expression that  $(d/dt)E = 0$  and hence

$$E(t) = E(0), \quad \text{for all } t \geq 0 \tag{2.9}$$

This shows that the systems (2.4) to (2.7) is conservative and hence stable in the Lyapunov sense. However the system is not asymptotically stable with respect to the rest state though this is what is desirable for engineering structures.

**Remark 2.1**

The Lyapunov function defined in equation (2.8) actually represents the total energy of the linear system. We note that the first and the second terms in (2.8) respectively

represent the kinetic energy of the roadway and the main cables respectively; the third and fourth terms respectively give the potential energy of the roadbed and the main cables respectively. The last integral in (2.8) representing the elastic-potential energy in the stays cables (ties).

### 2.3.3 Nonlinear Model

If one of the set of tie cables (stays) above or below the road bed is removed the linear system of the subsection (2.3.1) turns into a nonlinear one and can be described as follows:

$$\begin{cases} m_b(\partial^2/\partial t^2)z + \alpha(\partial^4/\partial x^4)z - k\Psi(y - z) = 0, & x \in \Sigma \equiv (0, \ell), t \geq 0; \\ m_c(\partial^2/\partial t^2)y - \beta(\partial^2/\partial x^2)y + k\Psi(y - z) = 0, & x \in (0, \ell), t \geq 0 \end{cases} \quad (2.10)$$

where

$$\Psi(\xi) = \begin{cases} \xi & \text{if } \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.11)$$

This is subject to the same set of boundary conditions given by (2.5) and (2.6) and initial conditions shown in (2.7) .

### 2.3.4 Stability of the Nonlinear Model

Again the total system energy (Lyapunov Function) is given by:

$$E(t) \equiv (1/2) \int_0^\ell \left\{ m_b \left| \frac{\partial z}{\partial t} \right|^2 + m_c \left| \frac{\partial y}{\partial t} \right|^2 + \alpha \left| \frac{\partial^2 z}{\partial x^2} \right|^2 + \beta \left| \frac{\partial y}{\partial x} \right|^2 + k |\Psi(y - z)|^2 \right\} dx \quad (2.12)$$

In what follows, we compute the derivative of the Lyapunov function  $E(t)$ . For simplicity we denote by:

$$\Gamma(t) = \int_0^\ell k (\Psi(y - z))^2 dx$$

With the help of Heaviside function, it is not difficult to verify that

$$\begin{aligned} (d/dt)\Gamma(t) &= \int_0^\ell \frac{\partial}{\partial t} k (\Psi(y - z))^2 dx \\ &= 2k \int_0^\ell \Psi(y - z) \frac{\partial}{\partial t} \Psi(y - z) dx \\ &= 2k \int_0^\ell \Psi(y - z) \Psi'(y - z) \left( \frac{\partial y}{\partial t} - \frac{\partial z}{\partial t} \right) dx \\ &= 2k \int_0^\ell \Psi(y - z) \text{sign}(y - z) \left( \frac{\partial y}{\partial t} - \frac{\partial z}{\partial t} \right) dx \\ &= 2k \int_0^\ell \Psi(y - z) \left( \frac{\partial y}{\partial t} - \frac{\partial z}{\partial t} \right) dx. \end{aligned}$$

then

$$\begin{aligned} (d/dt)E(t) &= (1/2) \int_0^\ell \left\{ 2m_b \left( \frac{\partial z}{\partial t} \frac{\partial^2 z}{\partial t^2} \right) + 2m_c \left( \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \right) + 2\alpha \left( \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial^2}{\partial x^2} \left( \frac{\partial z}{\partial t} \right) \right) \right) \right\} dx \\ &\quad + (1/2) \int_0^\ell \left\{ 2\beta \left( \frac{\partial y}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial t} \right) \right) \right) \right\} dx + (1/2)(d/dt)\Gamma(t) \end{aligned}$$

After integration by parts using initial given by (2.7) and boundary conditions given by (2.5) and (2.6) we obtain:

$$(d/dt)E(t) = \int_0^\ell \left\{ \left( m_b \frac{\partial^2 z}{\partial t^2} + \alpha \frac{\partial^4 z}{\partial x^4} - k\Psi(y-z) \right) \frac{\partial z}{\partial t} + \left( m_c \frac{\partial^2 y}{\partial t^2} + \beta \frac{\partial^2 y}{\partial x^2} + k\Psi(y-z) \right) \frac{\partial y}{\partial t} \right\} dx$$

Thus, it follows from equation (2.10) and the above expression that

$$\dot{E}(t) = 0 \tag{2.9B}$$

and hence again the nonlinear system is also conservative.

$$(E(t) = E(0))$$

**Remark 2.2**

If we define by :

$$B_+ \equiv ((t, x), t \geq 0, x \in (0, \ell) \text{ for which } y > z)$$

and

$$B_- \equiv ((t, x), t \geq 0, x \in (0, \ell) \text{ for which } y \leq z)$$

we remark that in both cases the system of equations (2.10) and (2.11) is conservative.

## 2.4 General Nonlinear Model

In general the function  $F$  of model (2.3) can be taken as any function with its graph lying in the first and third quadrant of the plane  $R^2$ . However from physical point of view it makes sense only if  $F$  is a nondecreasing function of its argument. In any case let us consider the corresponding homogeneous system:

$$\begin{cases} m_b(\partial^2/\partial t^2)z + \alpha(\partial^4/\partial x^4)z - F(y - z) = 0, & x \in \Sigma \equiv (0, \ell), \quad t \geq 0; \\ m_c(\partial^2/\partial t^2)y - \beta(\partial^2/\partial x^2)y + F(y - z) = 0, & x \in (0, \ell), \quad t \geq 0 \end{cases} \quad (2.13)$$

This is subject to the same set of boundary and initial conditions as in (2.5/2.6) and (2.7) respectively.

### 2.4.1 Stability of the General Nonlinear Model

For simplicity we denote by:

$$G(\zeta) \equiv \int_0^\zeta F(\xi) d\xi \quad (2.14)$$

The total system energy is given by:

$$E(t) \equiv (1/2) \int_0^\ell \left\{ m_b \left| \frac{\partial z}{\partial t} \right|^2 + m_c \left| \frac{\partial y}{\partial t} \right|^2 + \alpha \left| \frac{\partial^2 z}{\partial x^2} \right|^2 + \beta \left| \frac{\partial y}{\partial x} \right|^2 + 2G(y - z) \right\} dx \quad (2.15)$$

Again it is easy to verify that

$$\dot{E}(t) = 0 \quad (2.9C)$$

and hence the nonlinear system (2.13) is also conservative.

In view of the above results we observe that in the absence of external forces a suspension bridge, linear or nonlinear, is conservative.

**Remark 2.3**

Interchange of energy between the main cables and the roadbed as transmitted through the tie lines (stays) may take place leading to changes in the structure velocity trajectories and in the amplitude of vibration of the roadway.

## 2.5 Aerodynamic Damping

In all the models given above aero-dynamic damping has been neglected. The addition of damping represents more the practical case of suspension bridge. Considering the model (2.3) and including aerodynamic damping, we can write

$$\begin{cases} m_b(\partial^2/\partial t^2)z + \alpha(\partial^4/\partial x^4)z - F(y - z) + f_1((\partial/\partial t)z) = 0, & x \in \Sigma \equiv (0, \ell), \quad t \geq 0; \\ m_c(\partial^2/\partial t^2)y - \beta(\partial^2/\partial x^2)y + F(y - z) + f_2((\partial/\partial t)y) = 0, & x \in (0, \ell), \quad t \geq 0 \end{cases} \quad (2.16)$$

This is subject to the same set of boundary and initial conditions as in (2.5/2.6) and (2.7) respectively. Using the energy function (2.15) and carrying out the differentiation one can verify that

$$(d/dt)E(t) = - \int_0^\ell \left\{ f_1 \left( \frac{\partial z}{\partial t} \right) \frac{\partial z}{\partial t} + f_2 \left( \frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} \right\} dx \quad (2.17)$$

It follows from this expression that if  $f_i(\xi)\xi \geq 0$  then  $\dot{E} \leq 0$ . Here we need a result similar that of LaSalle invariance to conclude that the system is asymptotically stable given that  $f_i(\zeta)\zeta > 0$ , for  $\zeta > 0$ . We give here an elementary proof of this. For this purpose we can choose the energy functional for the Lyapunov function. We introduce  $\mathcal{W} \equiv H_0^2 \times L_2(\Sigma) \times H_0^1 \times L_2(\Sigma)$  as the energy space and define

$$V(z_1, z_2, y_1, y_2) \equiv (1/2) \int_0^\ell \left\{ m_b |z_2|^2 + m_c |y_2|^2 + \alpha \left| \frac{\partial^2 z_1}{\partial x^2} \right|^2 + \beta \left| \frac{\partial y_1}{\partial x} \right|^2 + 2G(y_1 - z_1) \right\} dx \quad (2.18)$$

as the Lyapunov function on  $\mathcal{W}$ .

### 2.5.1 Stability of the Damped System

To prove the stability of the studied system we use the following theorem:

#### Theorem 2.1

Consider the system (2.16) and suppose the following assumptions hold:

(A1):  $f_i(0) = 0$ ,  $f_i(\zeta)\zeta > 0$  for  $\zeta \neq 0$ ,

(A2):  $F(\xi)\xi \geq 0$  for  $\xi \in R$ .

Then the system is asymptotically stable with respect to the rest state(zero state).

#### Proof:

By virtue of equation (2.17) and the assumption (A1), it is clear that equation (\*) below, which is the derivative of equation (2.18), is less than zero.

$$\dot{E}(t) \equiv (d/dt)V \left( z, \frac{\partial z}{\partial t}, y, \frac{\partial y}{\partial t} \right) < 0 \quad (*)$$

whenever

$$\begin{cases} (\partial/\partial t)z \neq 0 \\ \text{or} \\ (\partial/\partial t)y \neq 0 \end{cases}$$

Hence  $E(t), t \geq 0$ , is a nonnegative monotone nonincreasing function of  $t$  and has a limit, say  $\lim_{t \rightarrow \infty} E(t) = r_0$ . If  $r_0 \neq 0$ , then along any solution trajectory of (2.16) starting from any point on the boundary of the energy ellipsoid,  $E_{r_0} \equiv \{(z_1, z_2, y_1, y_2) \in \mathcal{W} : V \leq r_0\}$ ,  $\dot{E} < 0$  and the system continues to dissipate energy and move towards the origin. We show that  $\dot{E}$  can vanish on an interval say,  $J \equiv [t_0, t_0 + \tau]$ , and the energy decay stalled, only if the rest state has been reached. We prove this by establishing a contradiction. Suppose that  $\dot{E}$  vanishes on the interval  $J$ , without having reached the rest state. Then, by virtue of assumption (A1), it follows from (2.17) that  $\{(\partial/\partial t)z \equiv 0, (\partial/\partial t)y \equiv 0\}$  on  $J$  and hence  $\{z, y\}$  must satisfy the equation

$$\begin{cases} a^2(\partial^4/\partial x^4)z - F(y - z) = 0, & x \in \Sigma = (0, \ell) \\ -b^2(\partial^2/\partial x^2)y + F(y - z) = 0, & x \in \Sigma = (0, \ell) \end{cases} \quad (2.19)$$

and either of the boundary conditions (2.5) or (2.6). Let  $\{z^0, y^0\}$  be a non-trivial solution of (2.19). Scalar multiplying the first equation by  $z = z^0$  and the second by  $y = y^0$  and integrating by parts using the boundary conditions and then adding the resulting expressions we arrive at

$$a^2 \left\| \frac{\partial^2 z^0}{\partial x^2} \right\|^2 + b^2 \left\| \frac{\partial y^0}{\partial x} \right\|^2 + (F(y^0 - z^0), y^0, z^0)_{L_2(\Sigma)} = 0. \quad (2.20)$$

Since by assumption (A2),  $F(\xi)\xi \geq 0$ ,  $\xi \in R$ , we have

$$(F(y^0 - z^0), y^0 - z^0)_{L_2(\Sigma)} = \int_0^\ell F(y^0(\xi) - z^0(\xi))(y^0(\xi) - z^0(\xi))d\xi \geq 0.$$

Hence

$$a^2 \left\| \frac{\partial^2 z^0}{\partial x^2} \right\|^2 + b^2 \left\| \frac{\partial y^0}{\partial x} \right\|^2 \leq 0. \quad (2.21)$$

This implies that

$$(\partial^2/\partial x^2)z^0 \equiv 0, \quad (\partial/\partial x)y^0 \equiv 0 \text{ on } \Sigma,$$

and hence it follows from the boundary conditions that  $z^0(\xi) \equiv 0$ ,  $y^0(\xi) \equiv 0$  on  $\Sigma = (0, \ell)$ . This contradiction proves the result. ■

In the presence of both viscous and structural damping  $f_1$  is a function of both  $(\partial/\partial t)z$  and  $(\partial^4/\partial x^4)((\partial/\partial t)z)$ . Assuming linearity  $f_1$  may be given by

$$f_1 \left( (\partial/\partial t)z, (\partial^4/\partial x^4)((\partial/\partial t)z) \right) = \gamma_{11}(\partial/\partial t)z + \gamma_{12}(\partial^4/\partial x^4)((\partial/\partial t)z).$$

For the suspension cable the structural damping is negligible. Assuming linear viscous damping  $f_2$  is given by

$$f_2((\partial/\partial t)y) = \gamma_{21}(\partial/\partial t)y.$$

Clearly from physical consideration  $\gamma_{11}, \gamma_{12}, \gamma_{21} \geq 0$ . Substituting in equation (2.16), it follows from (2.17) that  $\dot{E}(t) \leq 0$  and the system is asymptotically stable with respect to the origin.

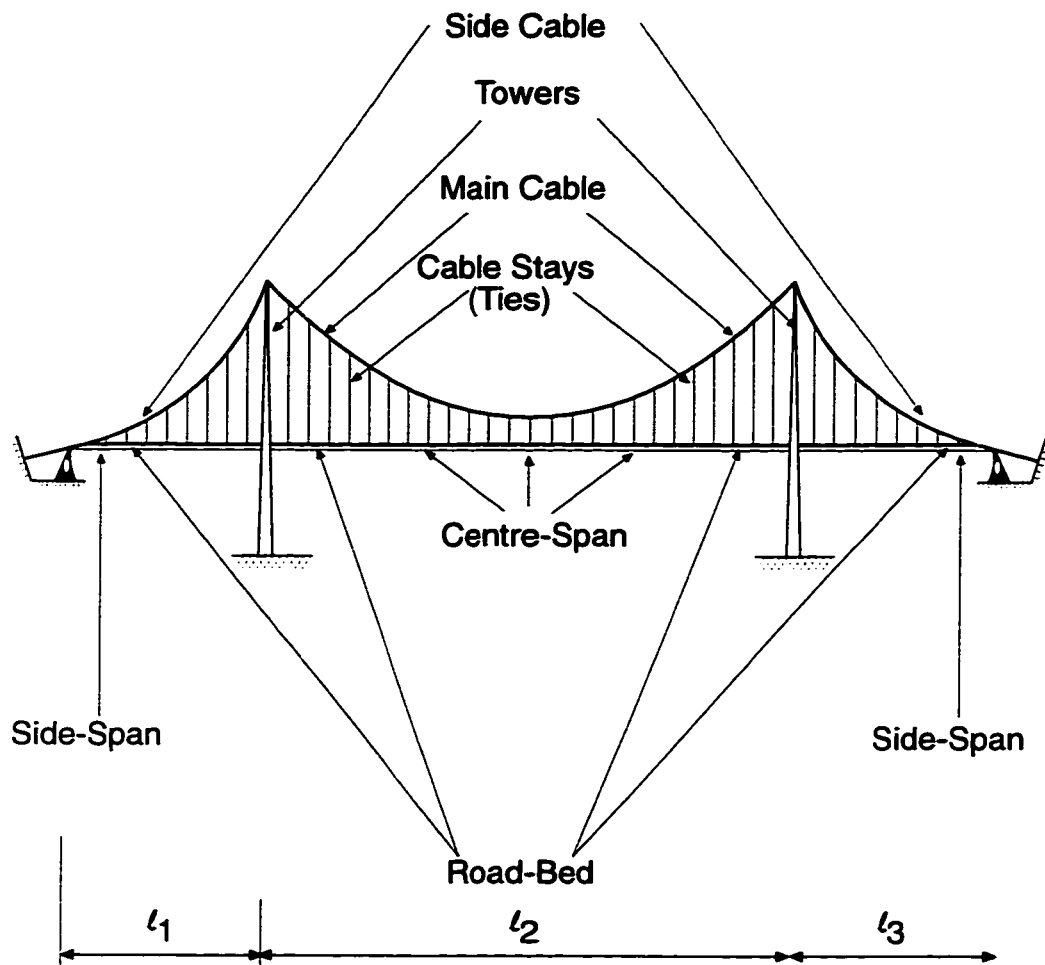


Figure 2.1: General Definition Diagram of Suspension Bridge

# Chapter 3

## Numerical Results and Discussions

### 3.1 General

This chapter presents some simulation results and discussions from a direct application of the developed models in chapter 2. The simulations are related to the typical suspension bridge shown in Figure 2.1.

For application, we solve a problem by finite-difference method utilizing central-difference approximations [20] and [21]. The size of time steps and space steps that are used, has been found to be sufficiently small to give highly accurate and stable solutions ( $\frac{\Delta t}{\Delta x} = 2.10^{-3}$ ). For the purpose of simulation, we assume that the roadbed and the cables are uniform. The following parameters are applied to the different models:

$$m_b = 10, \quad m_c = 1, \quad k = 100, \quad \alpha = 10, \quad \beta = 1$$

and

$$v = 30km/h, \quad g = 10m/s, \quad m_0 = 1, \quad \gamma = 0.01, \quad \delta = 0.01$$

In addition, the following boundary conditions are considered:

$$\begin{cases} z(t, 0) = z(t, \ell) = 0, & (\partial/\partial x)z(t, 0) = (\partial/\partial x)z(t, \ell) = 0 \\ y(t, 0) = y(t, \ell) = 0 \end{cases} \quad (3.1)$$

Finally, the initial conditions considered in the simulation process, can be summarised as follows:

$$\begin{cases} z(0, x) = z_1(x), & (\partial/\partial t)z(0, x) = 0, & x \in (0, \ell) \\ y(0, x) = y_1(x), & (\partial/\partial t)y(0, x) = 0, & x \in (0, \ell) \end{cases} \quad (3.2)$$

where

$$\begin{cases} z_1(x) = \varepsilon[(\cosh(\lambda) - \cos(\lambda))(\sinh(\lambda x) - \sin(\lambda x))] \\ -[(\sinh(\lambda) - \sin(\lambda))(\cosh(\lambda x) - \cos(\lambda x))] \\ y_1(x) = -x(1 - x) \end{cases} \quad (3.3)$$

with

$\varepsilon = -0.00012$  and  $\lambda$  satisfies

$$\cosh(\lambda)\cos(\lambda) = 1, \quad \lambda \neq 0$$

In total, five cases are considered for application purposes namely linear and nonlinear cases, a combined linear/nonlinear case, aerodynamic damping, and moving load. In each case, the total energy of the system and the vertical mid-span displacement of the cable and the roadbed are plotted against the time. In some cases, (linear and nonlinear), the velocity is also plotted against the time. The cases are presented in the following sections:

## 3.2 Linear Case

According to equation (2.9) a linear suspension bridge in the absence of damping is conservative. This is illustrated by the total energy plot as shown in Figure 3.1 (a) which illustrates a constant energy for the whole period of time. Figures 3.1 (b), and Figure 3.2 show the corresponding displacements and their rates at the mid point of the span. The Figures illustrate that the frequency of oscillation of the structure is roughly twice that of the cable.

## 3.3 Nonlinear Case

The nonlinear system is also subject to the same initial data as in the linear case. We have seen in subsection (2.3.2), equation (2.9B), that the nonlinear bridge is also conservative in the absence of damping. This is illustrated by numerical simulation result as shown in Figure 3.3 (a). In this Figure, there is a distortion at  $t = 0$  caused by the nonlinearity. The energy level for the complete period of time is lower because of the cut off due to nonlinearity [see equations 2.10 and 2.11]. Figure 3.3 (b) and Figure 3.4 show the displacements and their rates again at the mid point of the span. The cable is moving freely and is not interacting (no coupling) with the roadbed. This is because the Lebesgue measure of the set  $\Sigma_0 \equiv \{x \in (0, \ell) : y(0, x) \geq z(0, x)\}$  is less than  $\ell$ .

## 3.4 Linear/nonlinear

In this case we assume that the system is linear up to time  $t_s = 4$  after which the stays (tie cables) below the road bed breaks down (due possibly to seismic or other activities). Again Figure 3.5 (a) shows the conservation of energy with some wiggles due possibly to computational error caused by abrupt change. This constancy of energy level even after switching to nonlinear system is due to the fact that at the precise moment of

switching the set

$$\Sigma_s \equiv \{x \in (0, \ell) : y(t_s, x) - z(t_s, x) \geq 0\}$$

has (full) Lebesgue measure  $\lambda(\Sigma_s) = \ell$ , equal to the length of the span. Figure 3.5 (b) shows the displacement versus time. In this Figure, there is a sudden increase of the amplitude of oscillation of the upper cable and decrease of frequency. The same experiment was performed with different initial conditions as shown in Figure 3.6. We observe a break in the energy level at  $t_s = 4$ . In this case  $\lambda(\Sigma_s) < \ell$ . Thus there is a drop in the total energy level as a consequence of breakdown of tie cables. But note the consequent violent oscillation of the upper cables in Figure 3.6 (b).

### 3.5 Aerodynamic damping

The fourth simulation case is a numerical experiment with system (2.16) for two different cases:

- (i)  $f_1(\zeta) = \gamma \zeta$ , (damped roadbed)  $\gamma > 0$ , and  $f_2 = 0$ . (cable with no damping)
- (ii)  $f_1(\zeta) = \gamma \zeta$ ,  $\gamma > 0$ , and  $f_2(\zeta) = \delta \zeta$ , (damped cable)  $\delta > 0$ .

The results are shown in Figures 3.7 and 3.8. In Figure 3.7 (a) and 3.8 (a), there is a gradual decay in the total energy. After a certain time, the system is asymptotically stable with respect to the rest state as expected (see Theorem 2.1). The decay in the second case is reached faster than in the first case. The displacements at mid-span plotted against the time are shown in Figures 3.7 (b) and 3.8 (b) for the two cases, respectively. The oscillations of both the roadbed and the cable die out as time increases.

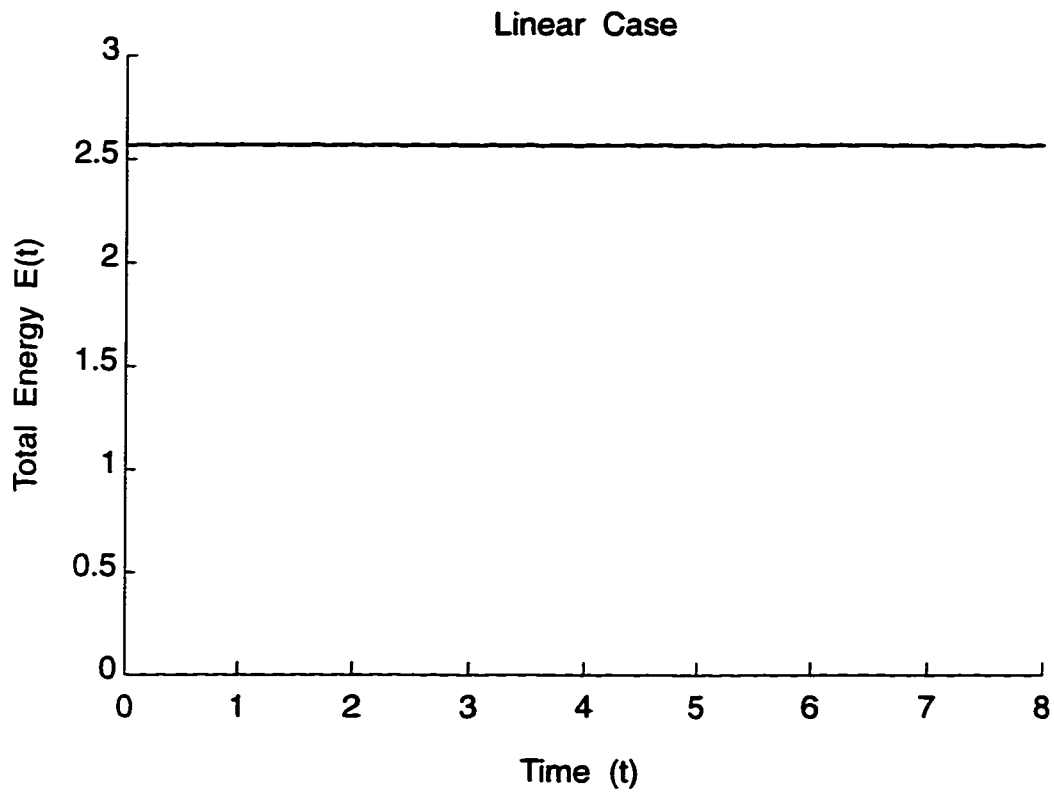
## 3.6 Moving Load

System (2.3) with **(1)**:  $F(\zeta) = k \zeta$  (linear case) and **(2)**:  $F(\zeta) = k\Psi(\zeta)$  (nonlinear case)  $f_2 = 0$  and  $f_1 \neq 0$ , were subjected to a moving load ( for example a moving truck) represented by

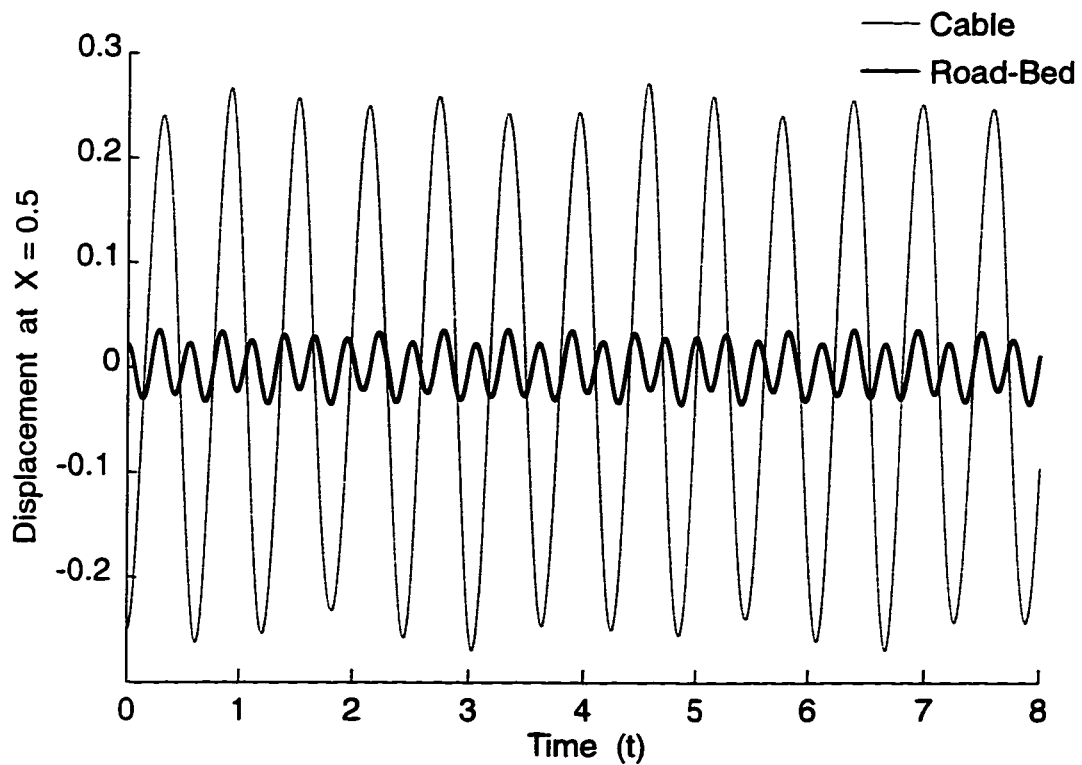
$$f_1(t, x) \equiv -m_0 g \mathcal{C}_{e(t)(x)}$$

$$e(t) \equiv (vt, vt + L) \cap (0, \ell)$$

where  $m_0$  is the mass of the vehicle per unit length,  $L$  is its length, and  $v$  is its velocity and  $\mathcal{C}_e$  stands for the characteristic function of the set  $e$ . The velocity  $v$  was chosen so that the vehicle passes the bridge in 4 units of time. Figure 3.9 (a) shows the variation of energy level with time and Figure 3.9 (b) shows the corresponding response. Note that the system is left with some residual energy after the passage of the vehicle (Figure 3.9 (a)). This is due to the absence of friction or damping. For the nonlinear system similar results are shown in Figure 3.10.

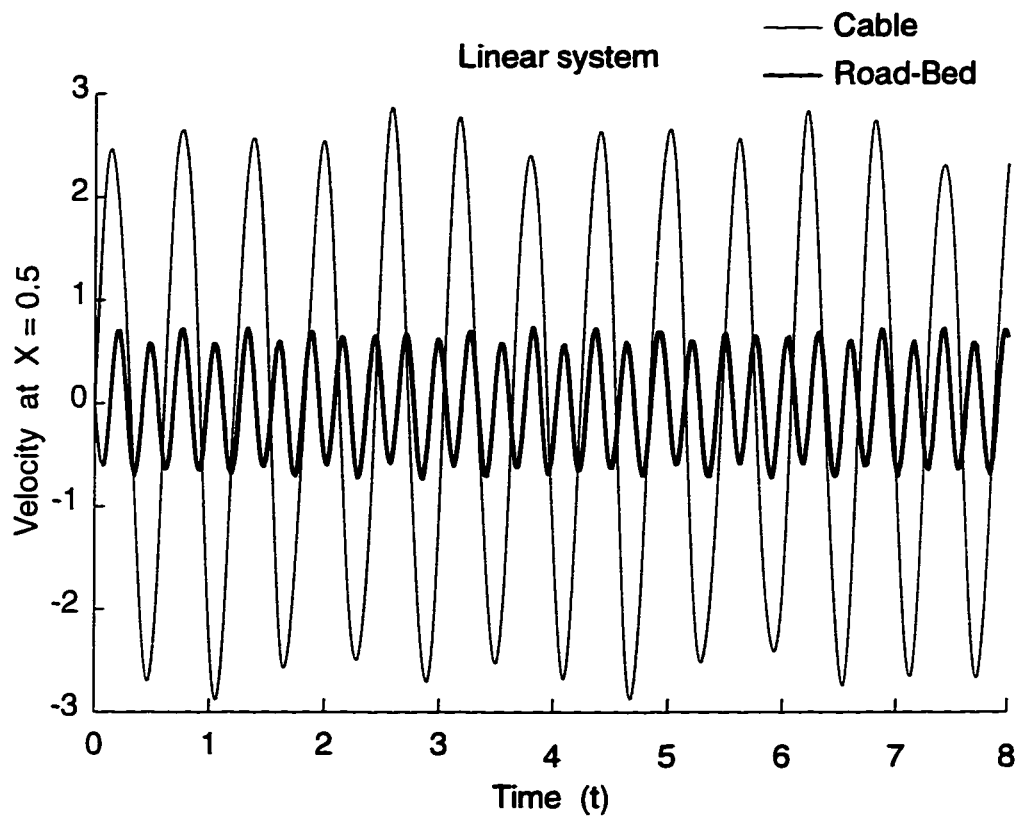


(a) Total Energy vs Time



(b) Displacement at Mid-Span vs Time

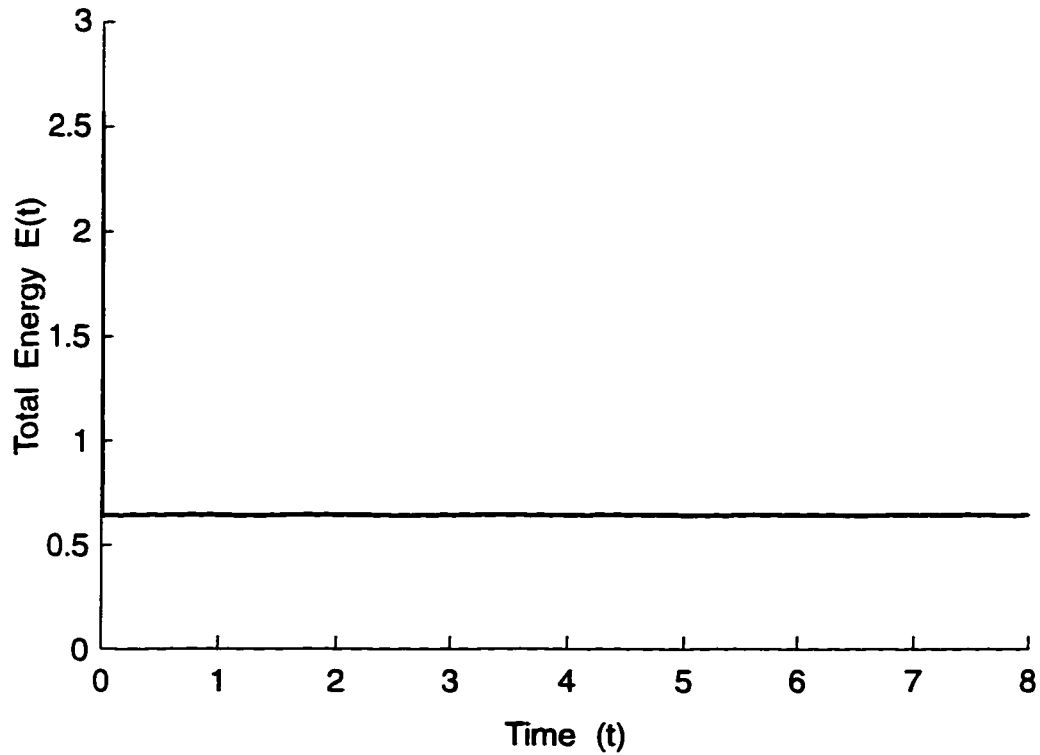
Figure 3.1: Linear System (Total Energy and Displacement)



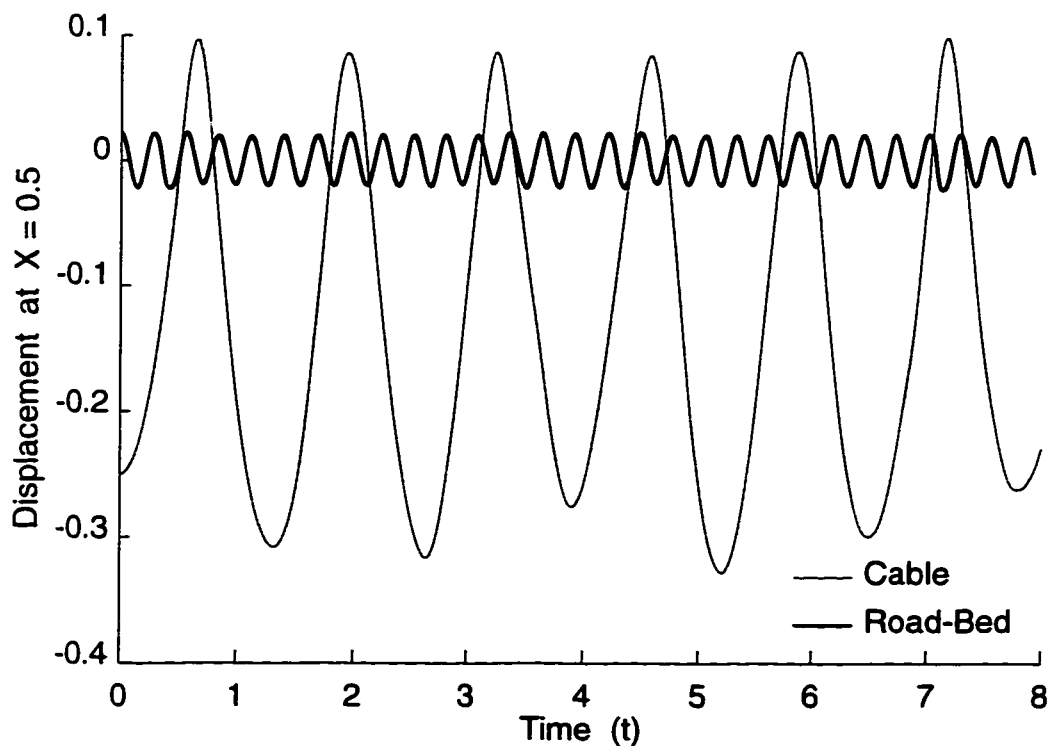
Velocity at Mid-Span vs Time

Figure 3.2: Velocity (for Linear System)

Nonlinear Case

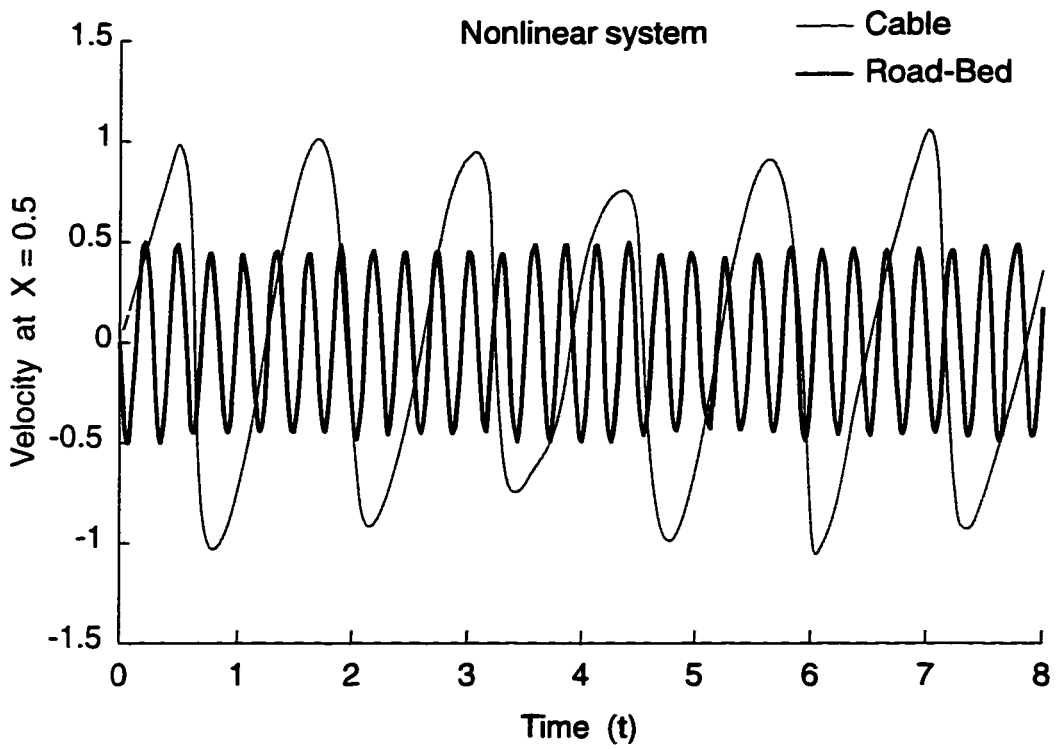


(a) Total Energy vs Time



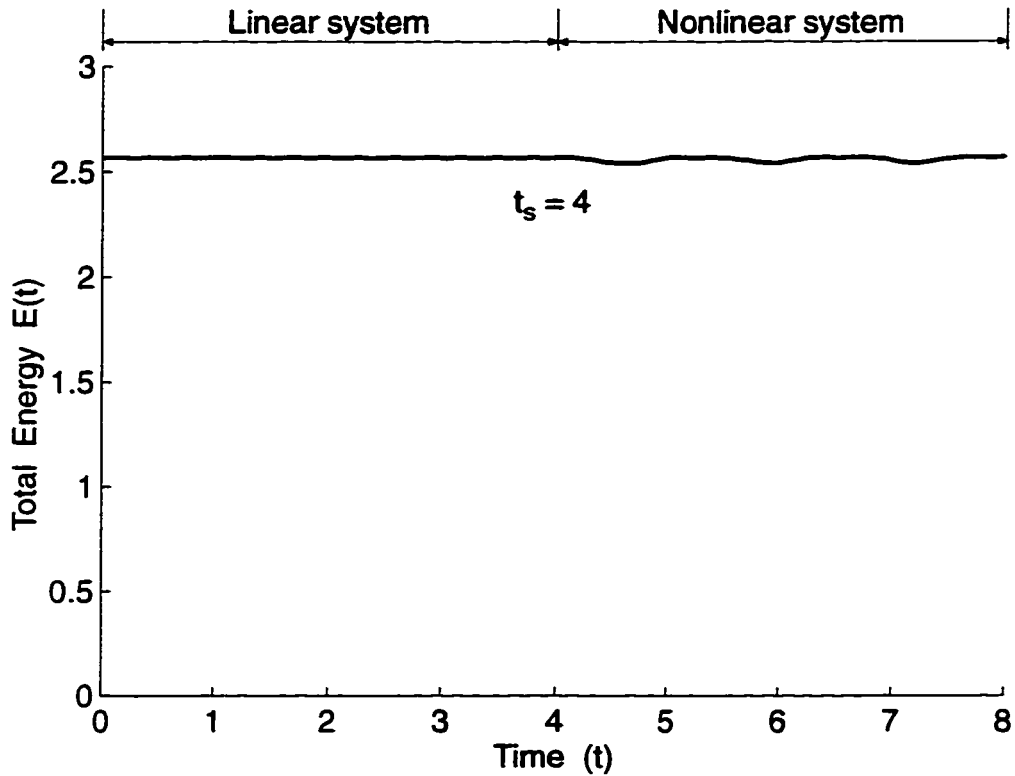
(b) Displacement at Mid-Span vs Time

Figure 3.3: Nonlinear System (Total Energy and Displacement)

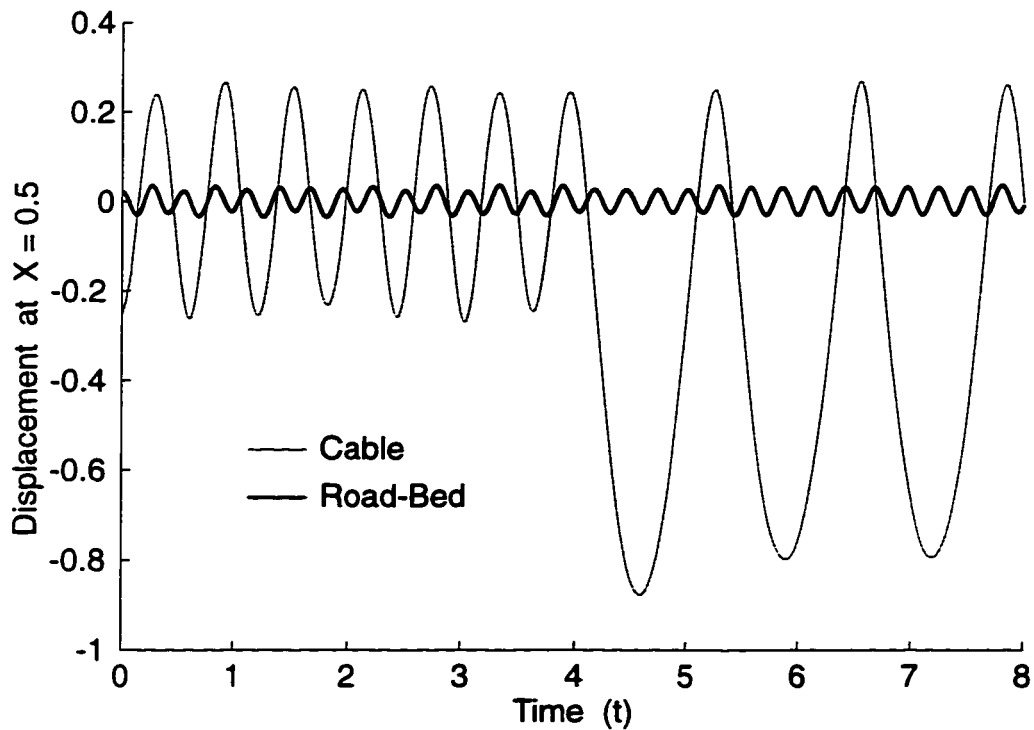


Velocity at Mid-Span vs Time

Figure 3.4: Velocity (for Nonlinear System)

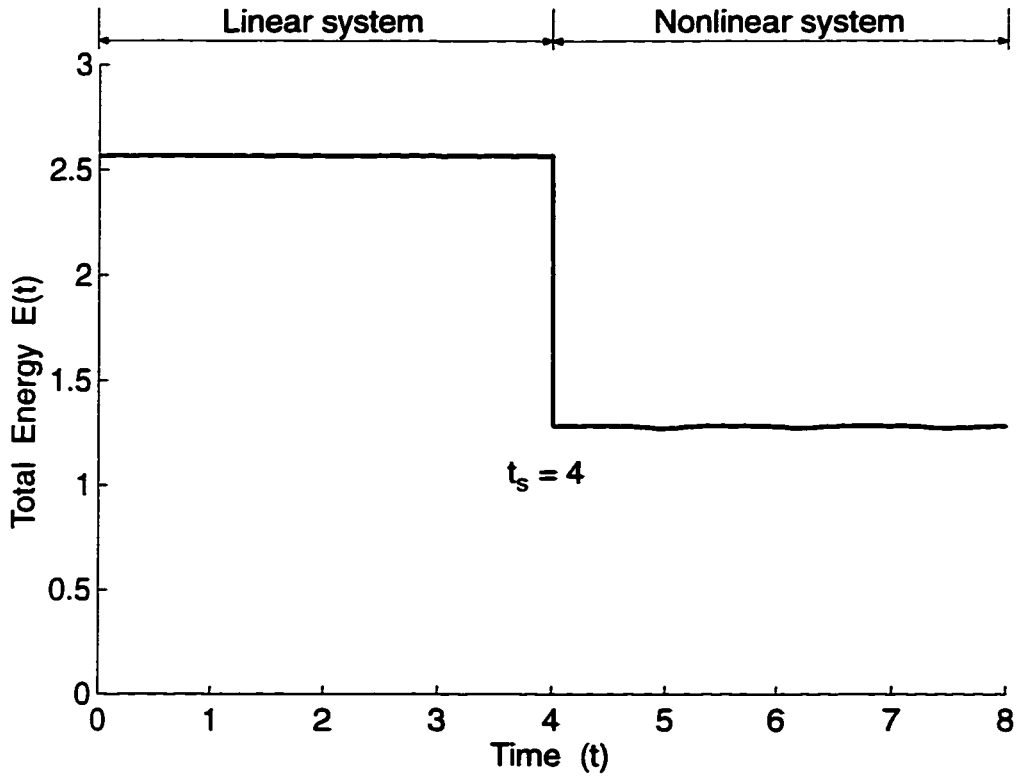


(a) Total Energy vs Time

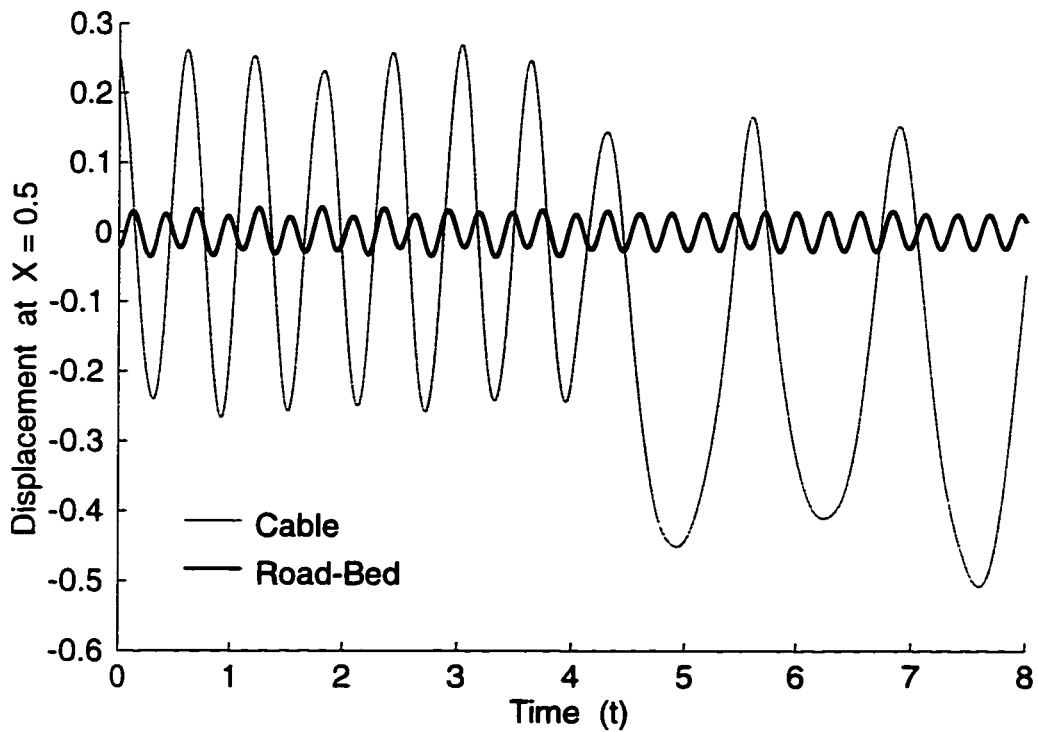


(b) Displacement at Mid-Span vs Time

Figure 3.5: Linear/Nonlinear System (Total Energy and Displacement)



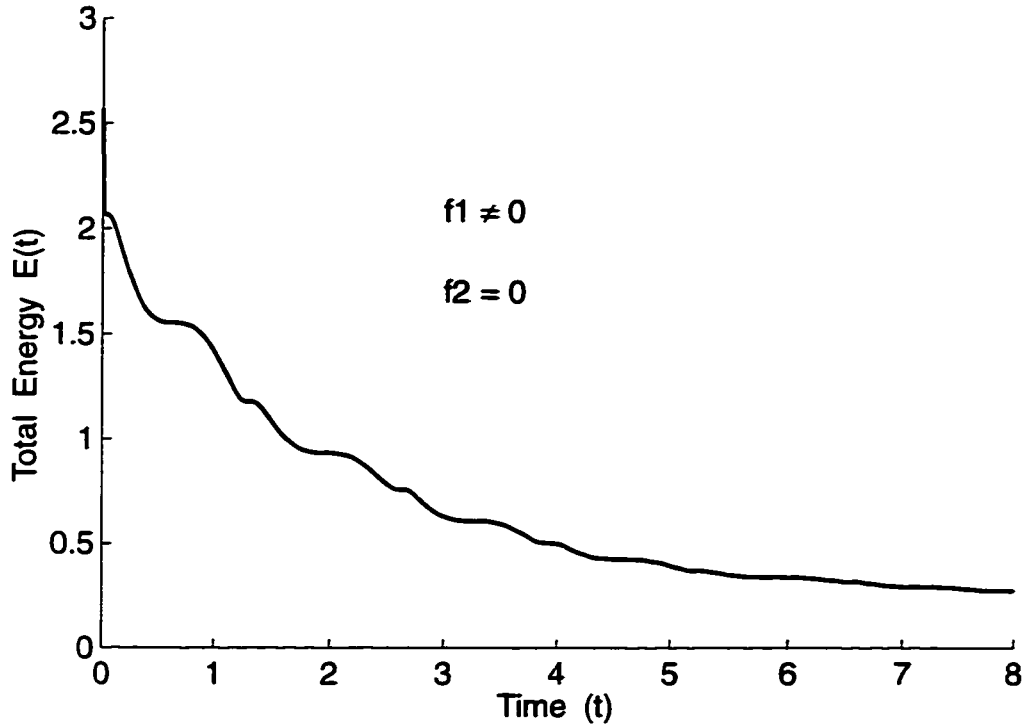
(a) Total Energy vs Time



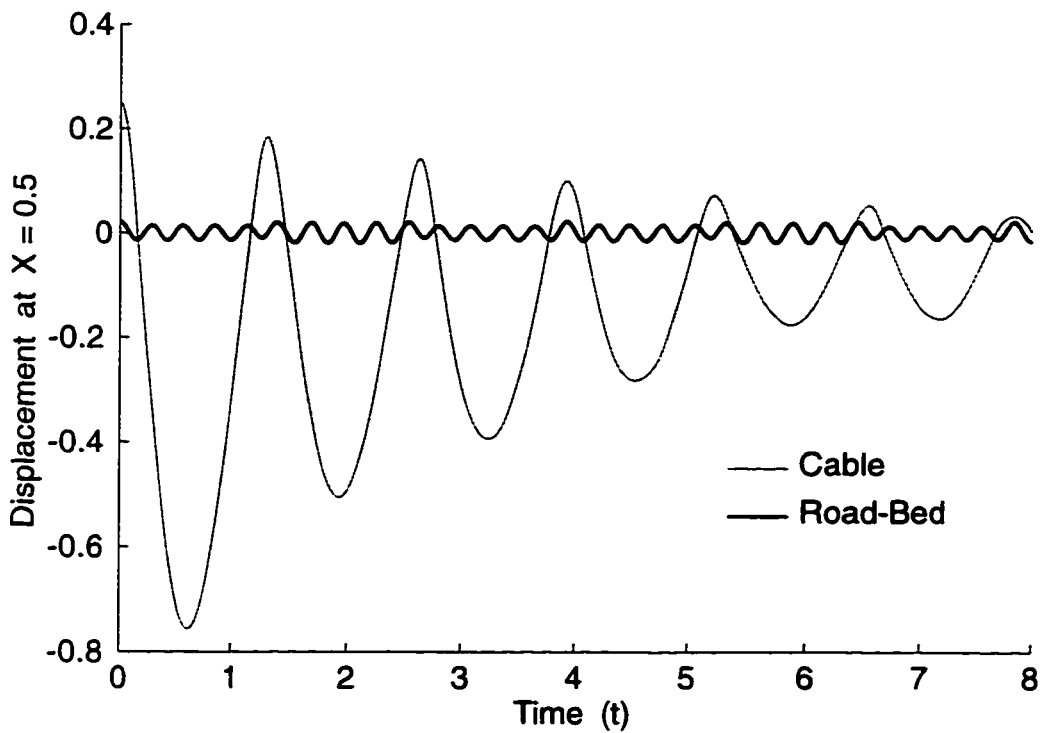
(b) Displacement at Mid-Span vs Time

Figure 3.6: Linear/Nonlinear System (Total Energy and Displacement)

### Damped System (Nonlinear Case)



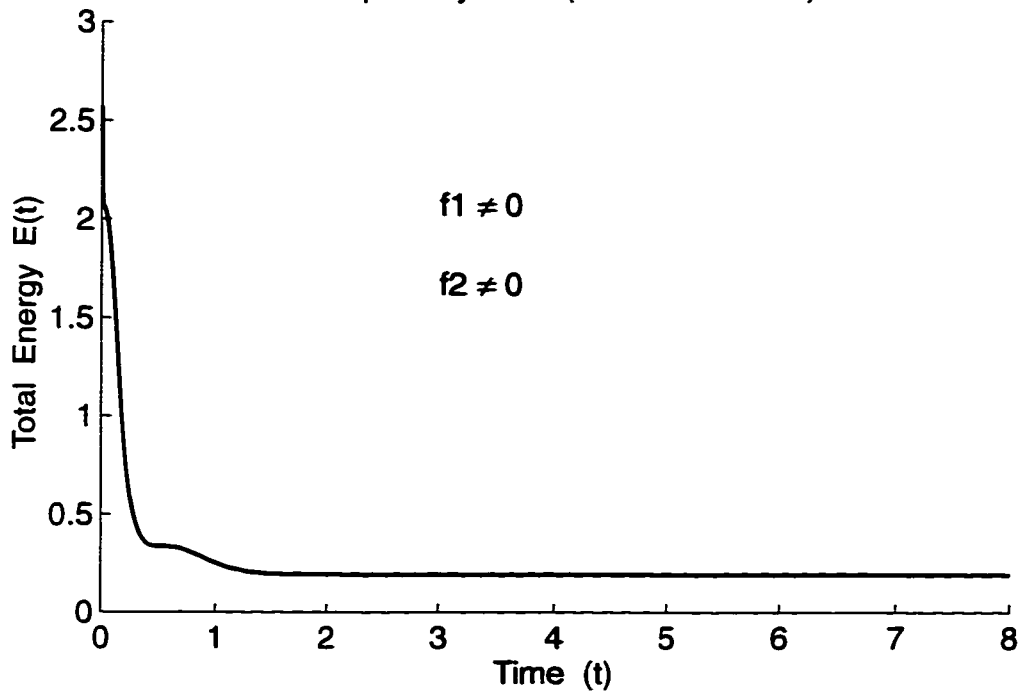
(a) Total Energy vs Time



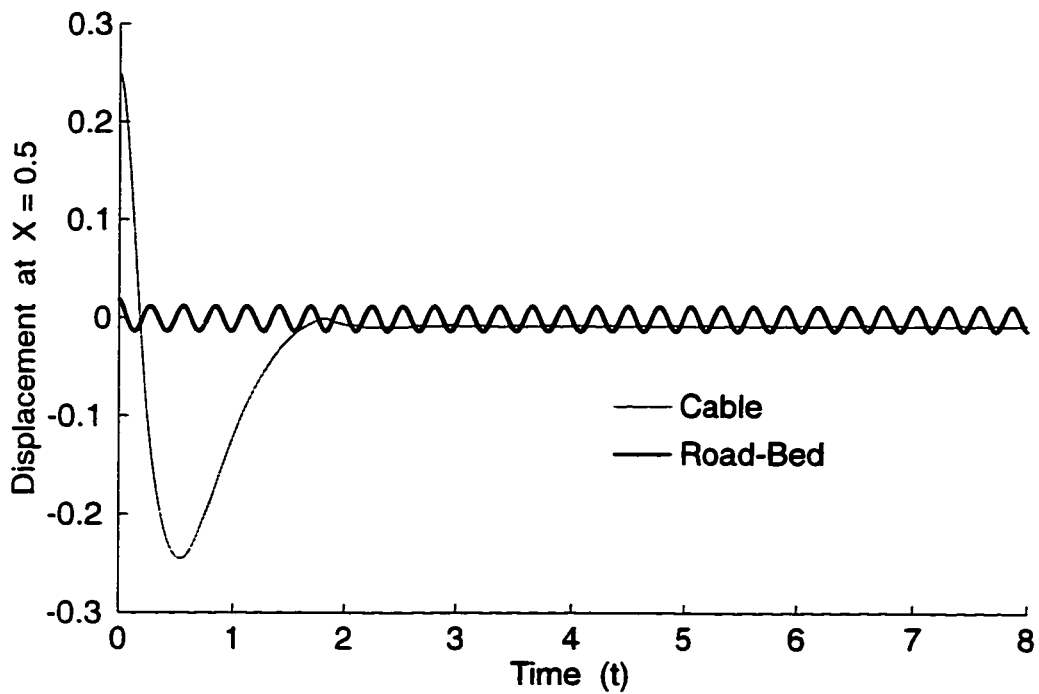
(b) Displacement at Mid-Span vs Time

Figure 3.7: Damped System (Total Energy and Displacement)

### Damped System (Nonlinear Case)

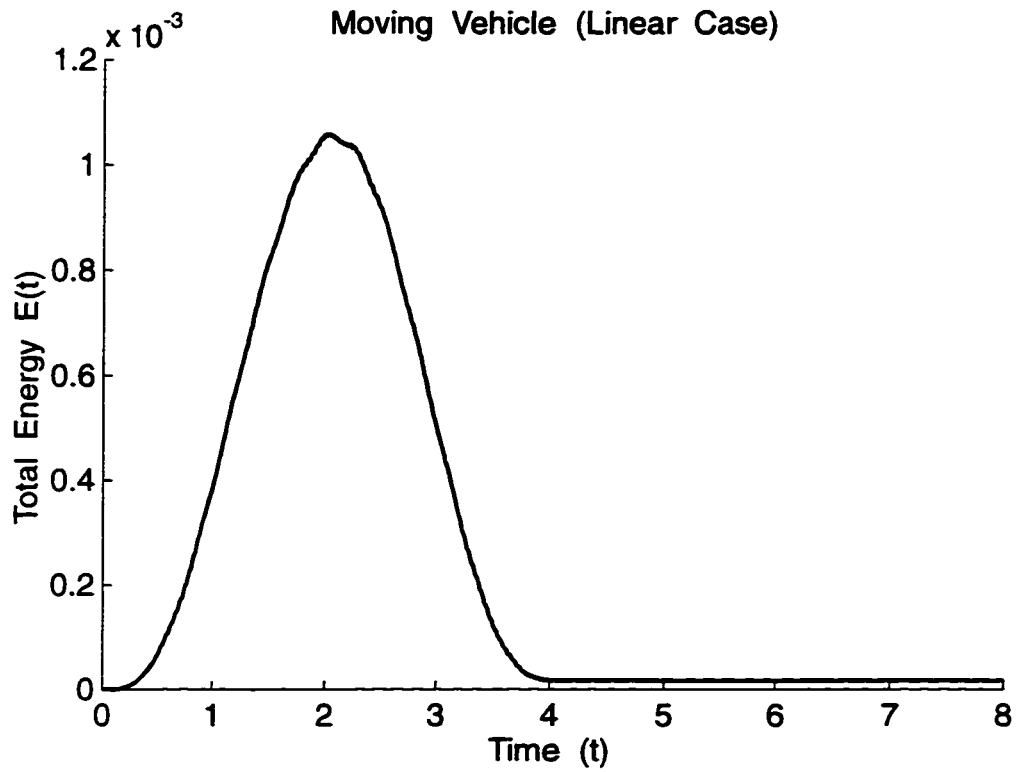


(a) Total Energy vs Time

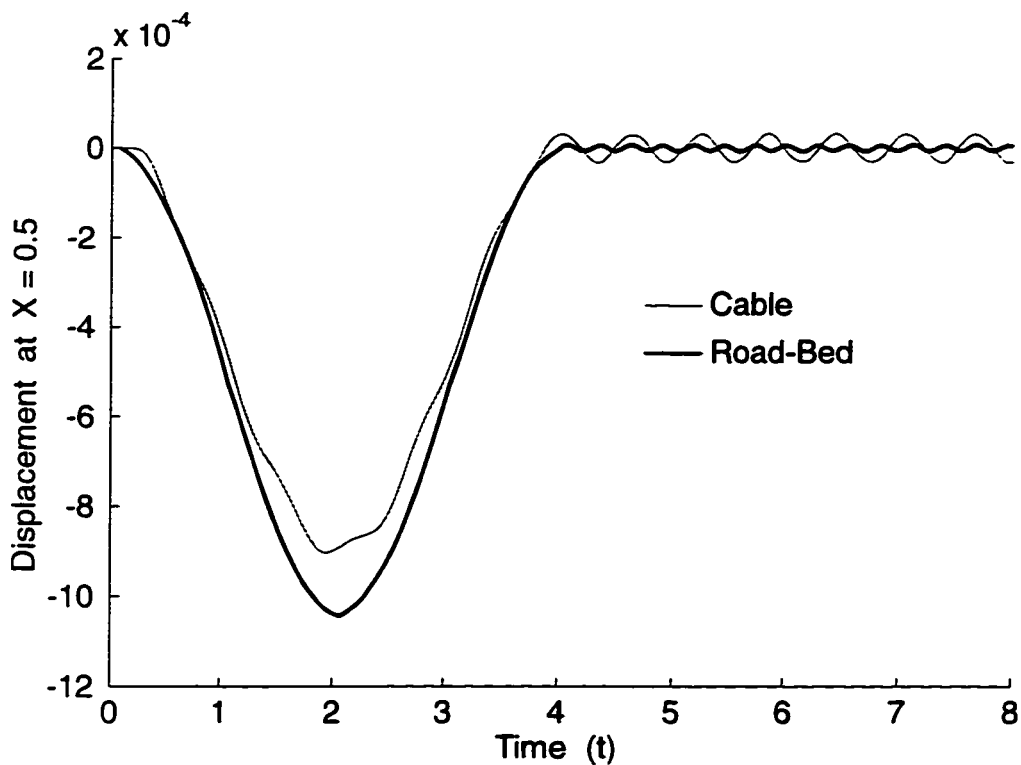


(b) Displacement at Mid-Span vs Time

Figure 3.8: Damped System (Total Energy and Displacement)

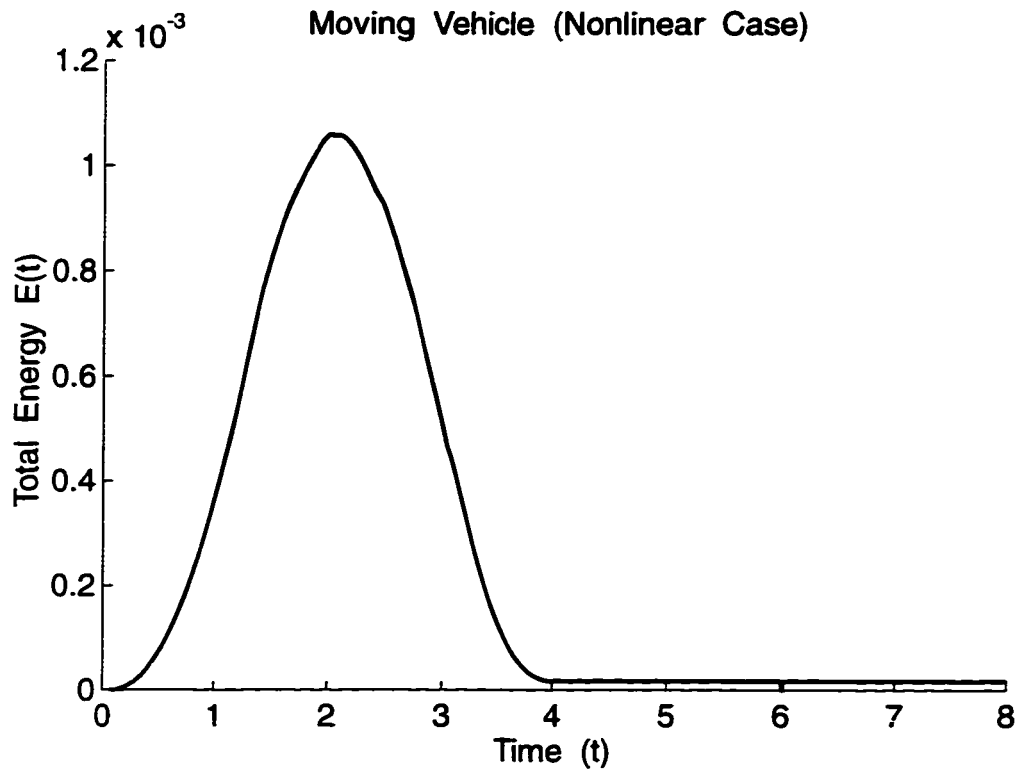


(a) Total Energy vs Time

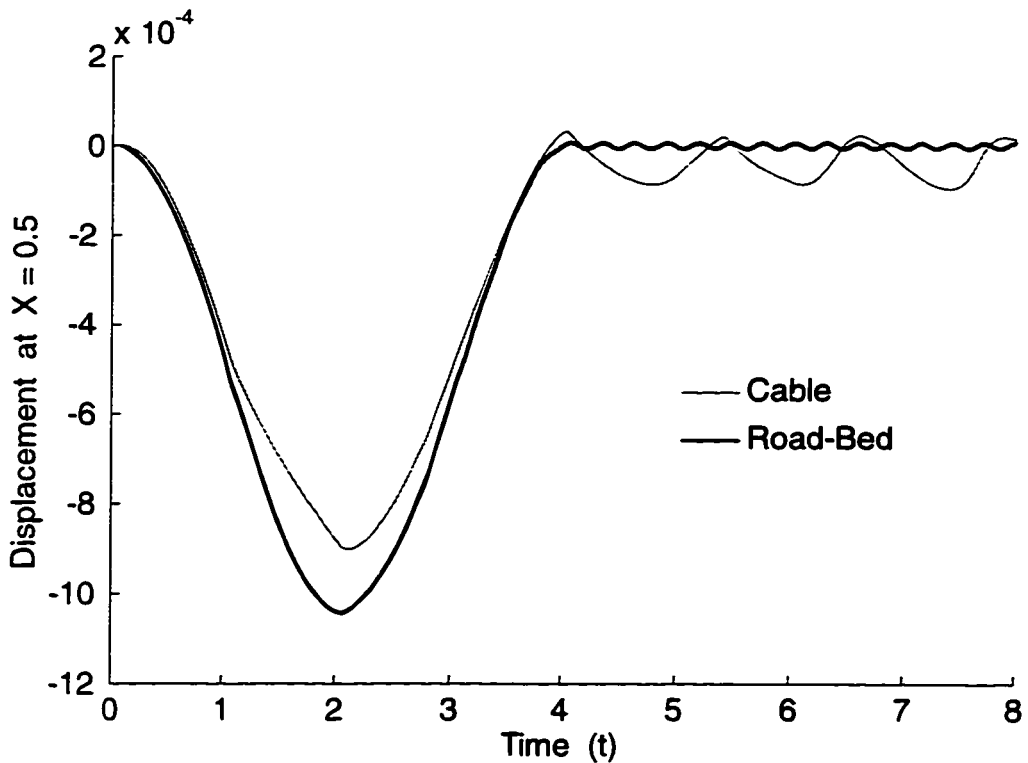


(b) Displacement at Mid-Span vs Time

Figure 3.9: Moving Vehicle (Total Energy and Displacement)



(a) Total Energy vs Time



(b) Displacement at Mid-Span vs Time

Figure 3.10: Moving Vehicle (Total Energy and Displacement)

# Chapter 4

## Summary, Conclusions and Further Research

### 4.1 Summary and Conclusions

This thesis presented an analysis of the suspension bridge model proposed by Lazer and McKenna using a simplified Partial Differential Equations (PDE) solution. Four models have been developed namely, a linear and nonlinear model, a general nonlinear model without damping, and a general nonlinear model with damping. The dynamic behaviour of the system under different situations and associated simulation results and physical interpretations have been studied. Five simulation cases were studied including, linear and nonlinear cases, combined linear/nonlinear case, aerodynamic damping, and moving load. From this study the following conclusions can be drawn:

- The frequency of oscillation of the structure is roughly twice that of the cable.
- The initial energy level in nonlinear case drops significantly due to stays cables failure.
- In linear/nonlinear case, the total energy in the system may drop or stay at the same level of initial energy depending on the initial conditions of the system.
- In case of aerodynamic damping the system is asymptotically stable with respect to the rest as expected.
- In the study of moving load case under linear and nonlinear conditions the system shows that is left with some residual energy after the passage of the vehicle.

- In addition, analysis of an abstract model and indication of its usefulness in determining the regularity of solutions and constructing stochastic models has also been presented.

## 4.2 Further Research

During the course of this study, it became evident that more research is needed to comprehend the behaviour and stability of suspension bridges subjected to vibration forces. The needed research includes, but is not limited to, the following:

- A more elaborate model such as the one suggested in reference [6] and its stochastic version to carry out similar analysis including simulation is still needed.
- A study on nonlinear torsional vibration of two-dimensional suspension bridge is essential. Future works have to focus on developing guidelines for introducing stability of suspension bridges.

# Appendix A

## ABSTRACT MODEL

This section considers the system (2.3) with either of the boundary conditions (2.5) or (2.6) and reformulate (2.3) as an ordinary differential equation on an appropriate Hilbert space. The abstract formulation has many advantages as we shall see later in the sequel. First of all we write equation (2.3) in its normalized form as follows

$$\begin{cases} (\partial^2/\partial t^2)z + a^2(\partial^4/\partial x^4)z = F_1(t, x, y, z), & x \in \Sigma, \quad t \geq 0, \\ (\partial^2/\partial t^2)y - b^2(\partial^2/\partial x^2)y = F_2(t, x, y, z), & x \in \Sigma, \quad t \geq 0 \end{cases} \quad (\text{A.1})$$

where

$$\begin{cases} a^2 \equiv (\alpha/m_b) \\ b^2 \equiv (\beta/m_c) \\ F_1(t, x, y, z) \equiv (1/m_b)(F(y - z) + f_1) \\ F_2(t, x, y, z) \equiv (1/m_c)(-F(y - z) + f_2) \end{cases} \quad (\text{A.2})$$

We introduce the Hilbert space  $H$  and the vector space  $V$  as follows:

$$H \equiv L_2(\Sigma) \times L_2(\Sigma), \quad V \equiv H_0^2 \times H_0^1 \quad (\text{A.3})$$

With the first space having equipped with the standard inner product. The second space is equipped with the scalar product and norms as follows:

$$\begin{cases} \langle \phi, \psi \rangle_V \equiv (D^2\phi_1, D^2\psi_1)_{L_2(\Sigma)} + (D\phi_2, D\psi_2)_{L_2(\Sigma)} \\ \|\phi\|_V \equiv \left( \|D^2\phi_1\|_{L_2(\Sigma)}^2 + \|D\phi_2\|_{L_2(\Sigma)}^2 \right)^{1/2} \end{cases} \quad (\text{A.4})$$

where  $D^k$  denotes the spatial derivative of order  $k$ . By virtue of Poincare inequality the norms

$$\|v\|_{H^m} \equiv \left( \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L_2(\Sigma)} \right)^{1/2} \quad \text{and} \quad \|v\|_{H_0^m} \equiv \left( \sum_{|\alpha|=m} \|D^\alpha v\|_{L_2(\Sigma)} \right)^{1/2}$$

are equivalent and hence the space  $V$  endowed with the scalar product and norm as defined by (A.4) is also a Hilbert space. Note that the embedding  $V \hookrightarrow H$  is continuous and dense and moreover it is actually compact. Letting  $V^*$  denote the topological dual of  $V$  and identifying  $H$  with its dual we have

$$V \hookrightarrow H \hookrightarrow V^*$$

with injections being compact. Note that  $V^* = H^{-2} \times H^{-1}$  where  $H^{-s}$ ,  $s > 0$ , denote the Sobolev spaces with negative exponents which are actually distributions. Now we introduce the bilinear form

$$c(\phi, \psi) \equiv a^2(D^2\phi_1, D^2\psi_1)_{L_2(D)} + b^2(D\phi_2, D\psi_2)_{L_2(D)} \quad (\text{A.5})$$

It is clear that the map

$$c : V \times V \longrightarrow \mathbb{R} \quad (\text{A.6})$$

is a continuous bilinear form. Indeed it is easy to verify that for  $d \equiv 2(a^2 + b^2)$

$$c(\phi, \psi) \leq d \|\phi\|_V \|\psi\|_V, \quad \text{for all } \phi, \psi \in V \quad (\text{A.7})$$

Thus there exists a linear operator  $A \in \mathcal{L}(V, V^*)$  such that

$$c(\phi, \psi) = \langle A\phi, \psi \rangle_{(V^*, V)} \quad \text{for all } \phi, \psi \in V \quad (\text{A.8})$$

The operator  $A$  is precisely the realization of the formal differential operator

$$A(D)\phi \equiv \begin{pmatrix} a^2 D^4 \phi_1 \\ -b^2 D^2 \phi_2 \end{pmatrix} \quad (\text{A.9})$$

with the boundary conditions (2.5/2.6). Similarly for  $d_0 \equiv \min\{a^2, b^2\}$  we have

$$c(\phi, \phi) \geq d_0 \|\phi\|_{V^2}, \quad \text{for all } \phi \in V.$$

Thus  $A$  is coercive. Define the operator

$$F(t, \phi) \equiv \begin{pmatrix} F_1(t, \cdot, \phi_1(t, \cdot), \phi_2(t, \cdot)) \\ F_2(t, \cdot, \phi_1(t, \cdot), \phi_2(t, \cdot)) \end{pmatrix} \quad (\text{A.10})$$

Now defining the state variable for each  $t \geq 0$ , as

$$\phi(t) \equiv \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \begin{pmatrix} z(t, \cdot) \\ y(t, \cdot) \end{pmatrix},$$

which are functions defined on  $\Sigma$ , we can reformulate the system (A.1) as an abstract second order differential equation on the Hilbert space  $H$  given by

$$(d^2/dt^2)\phi + A\phi = F(t, \phi), \quad t \geq 0, \quad \phi(0) \equiv \phi_{01}, \quad \dot{\phi}(0) = \phi_{02} \quad (\text{A.11})$$

where the initial state  $\{\phi_{01}, \phi_{02}\}$  is given by equation (2.7). With this preparation we can now prove a result on the existence, uniqueness and regularity of solutions of the Cauchy problem (A.11). For any Banach space  $X$  and  $1 \leq p \leq \infty$ , let  $L_p^{loc}(X) \equiv L_p^{loc}(R_0, X)$  denote the class of  $X$  valued functions on  $R_0 \equiv (0, \infty)$  having  $X$ -norms locally  $p$ -th power integrable.

**Definition A.1**

The Cauchy problem (A.11) is said to have a weak solution if there exists a  $\phi$  satisfying

(i):  $\phi \in L_2^{loc}(V)$   $\dot{\phi} \in L_2^{loc}(H)$ ,  $\ddot{\phi} \in L_2^{loc}(V^*)$ ,

(ii): for all  $\eta \in C_0^\infty(I)$ , and any  $v \in V$

$$- \int_I (\dot{\phi}(t), v) \dot{\eta}(t) dt + \int_I \langle \phi(t), A^* v \rangle \eta(t) dt = \int_I (F(t, \phi(t)), v)_H \eta(t) dt \quad (\text{A.12})$$

(iii):  $\phi(0) = \phi_{01}$ ,  $\dot{\phi}(0) = \phi_{02}$ .

First we prove an apriori bound. Let  $I \equiv [0, T]$ ,  $T < \infty$ . We introduce the following assumptions for  $F$  :

(F1):  $F : I \times H \rightarrow H$ , is Lebesgue measurable in  $t$  on  $I$  and Borel measurable in the second variable on  $H$ .

(F2): there exists a constant  $K \in R$ , such that

$$\| F(t, \zeta) \|_H^2 \leq K^2 (1 + \| \zeta \|_H^2), \zeta \in H.$$

**Lemma A.2**

Suppose the hypothesis (F1) and (F2) hold and let  $\phi_{01} \in V$  and  $\phi_{02} \in H$ . Then any solution  $\phi$  of (A.11), if one exists, satisfies the following conditions:

(i): there exist constants  $M, \omega > 0$  such that

$$\| \dot{\phi}(t) \|_H^2 + \| \phi(t) \|_V^2 \leq M e^{\omega t}, \quad t \geq 0 \quad (\text{A.13})$$

(ii):  $\phi \in L_\infty(I, V)$ ,  $\dot{\phi} \in L_\infty(I, H)$ , and  $\ddot{\phi} \in L_\infty(I, V^*)$ .

**Proof.**

Let  $\phi$  be any solution of the Cauchy problem (A.11). Using the  $V^* - V$  pairing and scalar multiplying the equation with  $\dot{\phi}$  on either side we obtain

$$\langle \ddot{\phi}, \dot{\phi} \rangle_{(V^*, V)} + c(\phi, \dot{\phi}) = \langle F(t, \phi), \dot{\phi} \rangle_{(V^*, V)} = (F(t, \phi), \dot{\phi})_H \quad (\text{A.14})$$

where  $c$  is the bilinear form as given by (A.5). It is not difficult to show that

$$\langle \ddot{\phi}(t), \dot{\phi}(t) \rangle = (1/2)d/dt \left( \|\dot{\phi}(t)\|_H^2 \right)$$

in the sense of distributions. Clearly the bilinear form  $c$  is symmetric and hence

$$2c(\phi(t), \dot{\phi}(t)) = (d/dt)c(\phi(t), \phi(t)).$$

Using these facts in (A.14) we obtain

$$(1/2)d/dt \left( \|\dot{\phi}(t)\|_{H^2}^2 + c(\phi(t), \phi(t)) \right) = (F(t, \phi(t)), \dot{\phi})_H, \quad t \geq 0.$$

Integrating this and recalling the norm topology of  $V$  and using Schwarz inequality and the embedding constant, say  $\delta > 0$ , for the injection  $V \hookrightarrow H$ , we arrive at

$$\begin{aligned} \|\dot{\phi}(t)\|_H^2 + d_0 \|\phi(t)\|_V^2 &\leq \|\phi_{02}\|_H^2 + d \|\phi_{01}\|_V^2 + \int_0^t \|\dot{\phi}(s)\|_H^2 ds \\ &+ K^2 \int_0^t (1 + \delta^2 \|\phi(s)\|_V^2) ds, \quad t \geq 0 \end{aligned} \quad (\text{A.15})$$

Defining

$$\left\{ \begin{array}{l} \bar{d}_0 = \min\{1, d_0\} \\ \bar{d} = \max\{1, d\} \\ \bar{\delta}^2 = \max\{1, \delta^2\} \\ d_1 \equiv (\bar{d}/\bar{d}_0) \\ d_2 \equiv (1 + (K\bar{\delta})^2)/\bar{d}_0 \end{array} \right.$$

one can rewrite equation (A.15) as

$$(1 + \|\dot{\phi}(t)\|_H^2 + \|\phi(t)\|_V^2) \leq d_1 (1 + \|\phi_{02}\|_H^2 + \|\phi_{01}\|_V^2) \tag{A.16}$$

$$+ d_2 \int_0^t (1 + \|\dot{\phi}(s)\|_H^2 + \|\phi(s)\|_V^2) ds, t \geq 0$$

Defining

$$\eta(t) \equiv 1 + \|\dot{\phi}(t)\|_H^2 + \|\phi(t)\|_V^2$$

it follows from (A.16) that

$$\eta(t) \leq d_1 \eta(0) + d_2 \int_0^t \eta(s) ds, t \geq 0.$$

Hence by Gronwall inequality we have

$$\eta(t) \leq d_1 \eta(0) e^{d_2 t} \equiv M e^{\omega t}, t \geq 0$$

which implies (A.13). Hence (i) follows. The first two inclusions of (ii) follow from (i). The last inclusion follows from (i) and the inequality

$$\|\bar{\phi}(t)\|_V^2 \leq c_0 + c_1 \|\phi(t)\|_V^2, \text{ for all } t \geq 0,$$

where the constants  $c_0 = c_0(\delta, K)$ ,  $c_1 = c_1(d, \delta, K)$  are dependent on the parameters as indicated. This completes the proof. ■

The following result is believed to have independent interest in other applications.

**Lemma A.3**

Let  $X, Y$  be two arbitrary Banach spaces with the embedding  $X \hookrightarrow Y$  being compact. Let  $I = [0, T]$  be a finite interval,  $p \in (1, \infty)$  and  $B \subset L_\infty(I, X)$  a bounded set for which there exists a constant  $\bar{b} > 0$ , such that

$$\int_I \|\dot{f}\|_Y^p dt \leq \bar{b}, \text{ for all } f \in B.$$

Then  $B$  is a relatively compact subset of  $L_p(I, Y)$ .

**Proof.**

Clearly the set  $B$  considered as a subset of  $L_p(I, Y)$  is bounded. Since  $B$  is a bounded subset of  $L_\infty(I, X)$ , for almost all  $t \in I$ , the set

$$B(t) \equiv \{f(t), f \in B\} \subset X$$

is bounded and since the embedding  $X \hookrightarrow Y$  is compact,  $B(t)$  is a relatively compact subset of  $Y$ . For each  $f \in B$ ,  $\dot{f} \in L_p(I, Y)$  and we have

$$f(t+h) = f(t) + \int_t^{t+h} \dot{f}(s) ds, \quad \text{for almost all } t, t+h \in I.$$

For  $h \in \mathbb{R}$  define  $I(h) \equiv I \cap [0, T-h]$ . Then we have

$$\begin{aligned} \int_{I(h)} \|f(t+h) - f(t)\|_Y^p dt &= \int_{I(h)} \left\| \int_t^{t+h} \dot{f}(s) ds \right\|_Y^p dt \\ &\leq |h|^{p-1} T \left( \int_I \|\dot{f}(t)\|_Y^p dt \right) \\ &\leq |h|^{p-1} T \bar{b} \end{aligned} \quad (\text{A.17})$$

for all  $f \in B$ . Therefore

$$\lim_{h \rightarrow 0} \int_{I(h)} \|f(t+h) - f(t)\|_Y^p dt = 0$$

uniformly with respect to  $f \in B$ . Thus we have shown that the set  $B$  satisfies the following conditions :

- (i):  $B$  is a bounded subset of  $L_p(I, Y)$ ,
- (ii): the  $t$  sections  $B(t)$  of  $B$  are relatively compact subsets of  $Y$  and
- (iii):  $\lim_{h \rightarrow 0} \int_{I(h)} \|f(t+h) - f(t)\|_Y^p dt = 0$  uniformly with respect to  $f \in B$ .

Thus  $B$  is a relatively compact subset of  $L_p(I, Y)$ . This completes the proof.  $\blacksquare$

Now we are prepared to prove the existence of a solution of equation (A.11) in the sense of definition A.1.

**Theorem A.4**

Suppose the assumptions (F1) and (F2) hold and the map  $x \ni H \longrightarrow F(t, x) \in H$  is continuous. Then for each  $\phi_{01} \in V, \phi_{02} \in H$  the Cauchy problem (A.11) has at least one weak solution.

**Proof.**

Since the injection  $V \hookrightarrow H$  is compact there exists a complete system of functions  $\{\phi_i, i \in N\}$  which is orthogonal in  $V$  and  $V^*$  and orthonormal in  $H$ . We use Galerkin approach and define

$$\begin{cases} \xi_i^n \equiv (\phi_{01}, \phi_i), & i = 1, 2, 3 \dots n \in N \\ \dot{\xi}_i^n \equiv (\phi_{02}, \phi_i), & i = 1, 2, 3 \dots n \in N \end{cases} \quad (\text{A.18})$$

and expand  $\phi_{01}$  and  $\phi_{02}$  with respect to the system  $\{\phi_i, i \in N\}$  and note that

$$\phi_{01}^n \equiv \sum_{1 \leq i \leq n} \xi_i^n \phi_i \xrightarrow{s} \phi_{01} \quad \text{in } V \quad \phi_{02}^n \equiv \sum_{1 \leq i \leq n} \dot{\xi}_i^n \phi_i \xrightarrow{s} \phi_{02} \quad \text{in } H \quad (\text{A.19})$$

By projection to the  $n$ -dimensional subspace spanned by  $\{\phi_i, i = 1, 2, \dots, n\}$  we obtain the system of ordinary differential equations

$$\langle d^2/dt^2 \phi^n(t), \phi_j \rangle + \langle A\phi^n, \phi_j \rangle = (F(t, \phi^n(t)), \phi_j), \quad 1 \leq j \leq n \quad (\text{A.20})$$

with initial conditions given by the left hand members of (A.19). Since  $F$  is measurable in  $t$  and continuous in the second variable and satisfies the growth condition (F2), this system of equations has at least one solution  $\phi^n$  satisfying  $\phi^n \in AC(I, V), \dot{\phi}^n \in AC(I, V) \subset AC(I, H)$ , with  $\phi^n(0) = \phi_{01}^n$  and  $\dot{\phi}^n(0) = \phi_{02}^n$ . By virtue of the a priori bounds (see Lemma A.2) the sequence  $\{\phi^n, \dot{\phi}^n\}$  is contained in a bounded subset of  $L_\infty(I, V) \times L_\infty(I, H)$  and hence there exists a subsequence of this sequence (re-labeled as above) and a  $\phi \in L_\infty(I, V)$  with  $\dot{\phi} \in L_\infty(I, H)$  such that

$$\phi^n \xrightarrow{w^*} \phi \text{ in } L_\infty(I, V), \quad \dot{\phi}^n \xrightarrow{w^*} \dot{\phi} \text{ in } L_\infty(I, H) \quad (\text{A.21})$$

Multiplying equation (A.20) by  $\eta \in C_0^\infty(I)$  and integrating by parts over  $I$  we have

$$-\int_I (\dot{\phi}^n(t), \phi_j) \dot{\eta}(t) dt + \int_I \langle \phi^n(t), A^* \phi_j \rangle \eta(t) dt = \int_I (F(t, \phi^n(t)), \phi_j) \eta(t) dt \quad (\text{A.22})$$

Letting  $n \rightarrow \infty$  in the first two terms of (A.22) it follows from (A.21) that

$$\begin{cases} \lim_{n \rightarrow \infty} -\int_I (\dot{\phi}^n(t), \phi_j) \dot{\eta}(t) dt = -\int_I (\dot{\phi}(t), \phi_j) \dot{\eta}(t) dt \\ \lim_{n \rightarrow \infty} \int_I \langle \phi^n(t), A^* \phi_j \rangle \eta(t) dt = \int_I \langle \phi(t), A^* \phi_j \rangle \eta(t) dt \end{cases} \quad (\text{A.23})$$

It remains to verify that

$$\lim_{n \rightarrow \infty} \int_I (F(t, \phi^n(t)), \phi_j) \eta(t) dt = \int_I (F(t, \phi(t)), \phi_j) \eta(t) dt \quad (\text{A.24})$$

Here we use the embedding result (Lemma A.3). Take  $X = V, Y = H, p = 2$  and  $B = \{\phi^n, n = 1, 2, \dots\}$ . By virtue of Lemma A.2,  $B$  satisfies the conditions of Lemma A.3 and hence the family  $B = \{\phi^n\}$  is relatively compact in  $L_2(I, H)$ . Thus, through a subsequence if necessary,  $\phi^n$  converges to  $\phi$  strongly in  $L_2(I, H)$  and hence there exists a subsequence that converges almost every where to  $\phi$ . Using this fact and the assumptions on  $F$ , (A.24) follows from Lebesgue dominated convergence theorem. Since the set  $\{\phi_j\}$  is complete, we conclude that

$$-\int_I (\dot{\phi}(t), v) \dot{\eta}(t) dt + \int_I \langle \phi(t), A^* v \rangle \eta(t) dt = \int_I (F(t, \phi(t)), v)_H \eta(t) dt \quad (\text{A.25})$$

for any  $v \in V$ . Thus (ii) of Definition (A.1) is satisfied. Now integrating the first term of (A.25) by parts once we have

$$\int_I (\langle \bar{\phi}(t), v \rangle + \langle A\phi(t), v \rangle - (F(t, \phi(t)), v)_H) \eta(t) dt = 0 \quad (\text{A.26})$$

for all  $\eta \in C_0^\infty$ . Hence the equality

$$\bar{\phi}(t) + A\phi(t) = F(t, \phi(t)), t \in I \quad (\text{A.27})$$

holds in the sense of  $V^*$ -valued distributions. Thus (i) of Definition A.1 follows from Lemma A.2. To prove that  $\phi$  as constructed above also satisfies the given initial conditions, multiply (A.20) by  $\eta \in C^2(I)$  satisfying  $\eta(T) = 0, \dot{\eta}(T) = 0$ ; and scalar multiply (A.27) by  $\phi_j \eta(t)$  and integrate by parts. Subtracting one from the other and letting  $n \rightarrow \infty$  one arrives at

$$\lim_{n \rightarrow \infty} ((\dot{\phi}(0) - \phi_{02}^n, \phi_j) \eta(0) + (\phi_{01}^n - \phi(0), \phi_j) \dot{\eta}(0)) = 0 \quad (\text{A.28})$$

since  $\eta \in C^2$  (satisfying the end conditions) is arbitrary, and  $\{\phi_j\}$  is complete in the triple  $\{V, H, V^*\}$ , the conclusion follows from (A.19) and (A.28). This completes the proof of Theorem A.4. ■

Additional regularity property of the solution is given in the following corollary.

**Corollary A.5.**

The solutions of the Cauchy problem (A.11) satisfy the following regularity properties:

$$\phi \in C(I, V), \quad \dot{\phi} \in C(I, H)$$

**Proof.**

The proof is given by parabolic regularization using the result (ii) of Lemma A.2, and Lemma A.1 of [Lions-Magenes, p275]. ■

**Theorem A.6.**

If in addition to the assumptions of Theorem A.4, the operator  $F$  satisfies also the Lipschitz condition

$$\| F(t, \zeta) - F(t, \xi) \|_{H^2} \leq K^2 (\| \zeta - \xi \|_{H^2}), \zeta, \xi \in H,$$

then the system (A.11) has a unique weak solution and the solution is continuously dependent on the initial data.

In view of the above regularity result we can extend the result of Theorem A.4 to cover the system model

$$(d^2/dt^2)\phi + A\phi = F(t, \phi, \dot{\phi}), t \geq 0, \quad \phi(0) \equiv \phi_{01}, \quad \dot{\phi}(0) = \phi_{02} \quad (\text{A.29})$$

Defining  $E \equiv V \times H$  and  $\psi = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$  we can rewrite equation (A.29) as a first order evolution equation on the Hilbert space  $E$  as given below

$$\dot{\psi} + \mathcal{A}\psi = \tilde{F}(t, \psi), \quad t \geq 0, \quad \psi(0) = \psi_0 \equiv \begin{pmatrix} \phi(0) \\ \dot{\phi}(0) \end{pmatrix} \quad (\text{A.30})$$

where the operators  $\mathcal{A}$  and  $\tilde{F}$  are given by

$$\mathcal{A} \equiv \begin{pmatrix} 0 & -I_d \\ A & 0 \end{pmatrix}, \quad \tilde{F}(t, \psi) \equiv \begin{pmatrix} 0 \\ F(t, \phi, \dot{\phi}) \end{pmatrix}$$

with  $I_d$  being the identity operator in  $H$ .

**Theorem A.7**

Suppose  $\tilde{F}$  satisfy the following hypotheses

(G1):  $\| \tilde{F}(t, \zeta) \|_E \leq K(1 + \| \zeta \|_E)$

(G2): For every  $r > 0$  there exists a constant  $K_r \geq 0$ , such that

$$\| \tilde{F}(t, \zeta) - \tilde{F}(t, \xi) \|_E \leq K_r (\| \zeta - \xi \|_E),$$

for all  $\zeta, \xi \in B_r$ , where  $B_r \equiv \{\eta \in E : \|\eta\|_E \leq r\}$ .

Then for each  $\psi_0 \in E$  the Cauchy problem (A.30) has a unique mild solution  $\psi \in C(I, E)$ .

**Proof**

(Outline) We follow the procedure as in [19, Theorem 5.3.3]. In view of Corollary A.5, the operator  $-\mathcal{A}$  generates a  $C_0$ -semigroup,  $S(t), t \geq 0$ , on  $E$  and hence by the variation of constants formula one can rewrite the evolution equation (A.30) as an integral equation

$$\psi(t) = S(t)\psi_0 + \int_0^t S(t-s)\tilde{F}(s, \psi(s))ds, t \in I \tag{A.31}$$

Using the assumptions (G1) and (G2) and Banach fixed point theorem one can easily establish the existence and uniqueness of a solution  $\psi \in C(I_{\tau_1}, E)$  of the integral equation (A.31) over the interval say  $I_{\tau_1} \equiv [0, \tau_1]$  for sufficiently small but positive  $\tau_1$ . The hypothesis (G1) implies finiteness of  $\psi(\tau_1)$ . Starting with this as the initial state one solves the integral equation

$$\psi(t) = S(t - \tau_1)\psi(\tau_1) + \int_{\tau_1}^t S(t-s)\tilde{F}(s, \psi(s))ds, t \in I_{\tau_1, \tau_2} \equiv [\tau_1, \tau_2] \tag{A.32}$$

where again  $\tau_2$  is chosen suitably so that the integral operator is a contraction. Since  $I$  is compact, by repeating this and piecing together these uniquely defined local solutions on  $[0, \tau_1], [\tau_1, \tau_2], [\tau_2, \tau_3], \dots$  one obtains a unique solution for the original interval  $I$ . This is the mild solution of equation (A.30). This completes the proof. ■

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