

# De Bruijn Graphs and Lamplighter Groups

by

Shathaa Alharthy

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Department of Mathematics and Statistics  
Faculty of Science  
University of Ottawa

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## Abstract

De Bruijn graphs were originally introduced for finding a superstring representation for all fixed length words of a given finite alphabet. Later they found numerous applications, for instance, in *DNA* sequencing. Here we study a relationship between de Bruijn graphs and the family of lamplighter groups (a particular class of wreath products). We show how de Bruijn graphs and their generalizations can be presented as Cayley and Schreier graphs of lamplighter groups.

Keywords: De Bruijn graph, languages, group, wreath product, lamplighter.

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## Dedication

It is my genuine gratefulness and warmest regard that I dedicate this work to my family members for the generous support they provided me throughout my entire life. First of all, this would mean nothing if I could never thank my father Hussien who is not with me but I know that you are feeling me. I can never forget your words, and you have taught me to never concede defeat. Also, I dedicate to especially mom Hamad, who have always loved me and whose good examples have taught me to work hard for achievement. In addition, a special and profound thank to my brother Fahad and my sisters Abeer, Aziza, Joharh and Haifa who offered invaluable support and humour over the years.

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# Chapter 1

## Introduction

The general theory of graphs is a field of intensive research. During the last 50 years with the development of technologies and computer science, efforts have been made to provide mathematical frameworks to better understand graphs properties and their relations to other branches of mathematics. This thesis explores a link between graphs as mathematical objects and groups as frameworks. Group theory concepts have proved to be very useful to study and understand symmetries. This thesis connects three main ideas: Graphs, Languages and Groups with the focus on De Bruin graphs in particular and organized as follows.

Chapter 1 will outline the foundation of graph theory. The concepts and terminologies and general properties will be presented. A special attention will be given to Eulerian and Hamiltonian graphs. As these properties are especially important for De Bruin graphs.

Chapter 2 introduces De Bruijn graphs and their generalizations. De Bruijn graphs are a family of directed graphs with nodes labeled by  $n$ -tuples over an alphabet set. The properties of De Bruijn graphs discussed in this chapter show that they are a good choice for constructing interconnection networks. We first examine the superstring problem which first appeared in the original paper of De Bruijn [8]. We then explore the properties of Hamiltonian and Eulerian cycles that occur on De Bruijn graphs. The superstring problem was originally defined as the question about the existence of a circular string that contains each binary substring of length  $k$  exactly once [8]. The problem can be eas-

ily solved for small values of  $k$ . For instance, when  $k$  equals 3, the list of all 3-mers is  $\{000, 001, 010, 011, 100, 101, 110, 111\}$ , and the superstring is  $s = 000111010$ . But as  $k$  increases, the problem becomes very difficult to solve. De Bruijn graphs are useful for solving this problem and its variants. We also present ways in which they can be applied both in theory and in practice. We discuss some theorems (specifically Eulerian and Hamiltonian properties) proven by De Bruijn in the study of the structure of De Bruijn Graphs. These results are useful for applications of De Bruijn graphs networks (structure with objects connected by communication paths) problems and genomic assembly [29]. Indeed, when examining a network, there are desirable properties such as existence of a eulerian path and shortest path which make De Bruijn graphs a good choice for investigation. De Bruijn graphs make excellent choices for a network's structure because they have a relatively large number of nodes and few connections at each node and, yet still maintain short paths between nodes [2]. Since De Bruijn graphs and languages are closely related, we present general concepts from the theory of languages. A language is a set of words over an alphabet. De Bruijn original problem was a search of the shortest binary string that contains exactly once each binary word of length  $K$ . Later the theory was extended to languages over different alphabet set such as the genetic alphabet hence, providing biological applications of graph theory including *DNA* sequencing and genomic assembly.

In chapter 3, we present a group interpretation for graphs in general and de Bruijn graphs in particular. The concept of groups is widely used in Mathematics. A group is an algebraic structure and is presented in various ways, making it easier and efficient for mathematicians to study and model real-life phenomena. As a results, groups are of first interest to mathematicians as to what algebraic structure they represent and what model can be derived from them[32]. The lamplighter group is derived from the idea of the lamplighters in the early times before the time of electric lamp posts, when streets in Europe were illuminated by oil lamps on top of lamp posts. Lamplighters were then the people whose job was to light the lamps along the street nightly. From the group point of view, the lamplighter stands in a space that extends infinitely in both directions with infinitely many lamp posts indexed by the set of natural integers [24]. The lamplighter can then move right or left from one lamp post to the next; light the lamp where he is

currently at or turn off a lamp that he has previously lit. The group consisting of all such actions of the lamplighter is the infinite lamplighter group. Instead of a lamplighter that extends indefinitely in both directions with infinite lamps, the finite lamplighter group is a system consisting of finitely many lamps arranged in a circle around a single pointer called the lighter, which can be turned clockwise to go to the next lamp, or light the current lamp it is currently pointed at[23]. Lamplighter groups are a special cases of a semi-group structure known as wreath product.

Finally, Chapter 4 we discuss we graphs associated with the Lamplighter groups and show how they are related with De Bruijn graphs.

# Chapter 2

## General theory of Graphs

In this chapter, we present the background from the general graph theory. In particular, we discuss such of concepts the graph theory as oriented graphs and non-oriented graphs, Isomorphism of graphs, Eulerian and Hamiltonian graphs.

The theory of graphs is an integral part of Mathematics. It was originated [33] from the study of famous Koenigsberg bridge problem by the Swiss mathematician Euler [14]. He solved this problem by introducing what is currently known as the notion of an Eulerian graph. A graph is called Eulerian if it is possible, starting from any vertex, to find a path which goes along each edge once and only once. Since then, graph theory has been widely applied to many fields including Biology, Engineering, Finance, Business [18] [15].

A graphs is essentially the same as a binary relation. It is a set of vertices or nodes connected with edges or arrows. Graphs are powerful mathematical tools used to model practical situations. Graphs are used to study molecules, data structures, data algorithms, to optimize distribution tours, to schedule problems etc [15].

In the first section of the paper, the general notion of a graph and its variants, including non-oriented graphs, trees, multi-graphs, and hypergraphs are presented. Two representations of graphs, namely adjacency matrix and list are introduced.

The other chapter will include families of graphs, De Bruijn graphs, lamplighter groups, and slider graphs.

## 2.1 Graphs

A *graph* ( $G$ ) is a mathematical structure made up of vertices ( $V$ ) nodes, or points which are connected by arrows or edges ( $E$ ). Below we shall specify this definition for several particular classes of graphs.

## 2.2 Oriented graphs

**Oriented graphs:** An oriented graph on a set  $V$  is a subset  $E$  of the product set  $V \times V$  where  $V$  is either finite or infinite. If  $V$  is finite, the graph is finite. In this case if  $E$  is also finite and it contains at most  $|V|^2$  edges.

The elements of  $E$  are called arrows. If  $(x, y)$  is in  $E$ , then  $x$  is called the origin of the arrow  $(x, y)$  and  $y$  is called the destination of this arrow. If  $x = y$  then the arrow  $(x, x)$  is called a loop. Two arrows are adjacent if they have at least one endpoint in common.

The order of vertices in  $E$  is important, For example, if  $(x, y)$  is an arrow then  $(y, x)$  may or may not be an arrow. In the first case, the arrow  $(x, y)$  is reversible.

Let  $a = (x, y) \in E$ . The  $a$  is an outward arrow to vertex  $x$  and inward arrow to vertex  $y$ . The set of out arrows of a vertex  $x$  is denoted by  $W^+(a)$  and the set of in arrows of a vertex  $x$ , is denoted by  $W^-(a)$ . We define the out-degree from a vertex  $v$  by the number of out arrows from that vertex. In contrast, the in-degree of a vertex  $v$  is the number of edges coming in that vertex. Figure 2.1 represents an example of an oriented graph [36].

### Example:

Consider the graph  $G = (V, E)$  where the set of vertices is  $V = \{a, b, c, d, e\}$  and the edges set  $E = \{(a, b), (a, e), (b, b), (b, c), (c, c), (c, d), (c, e), (d, a), (e, a), (e, d)\}$ . This is shown in Figure 2.1 below.

For example,  $W^+(a) = \{(a, b), (a, e)\}$ , and  $W^-(a) = \{(e, a), (d, a)\}$ . The degree of vertex  $a$ , is  $\deg(a) = 2 + 2 = 4$ .

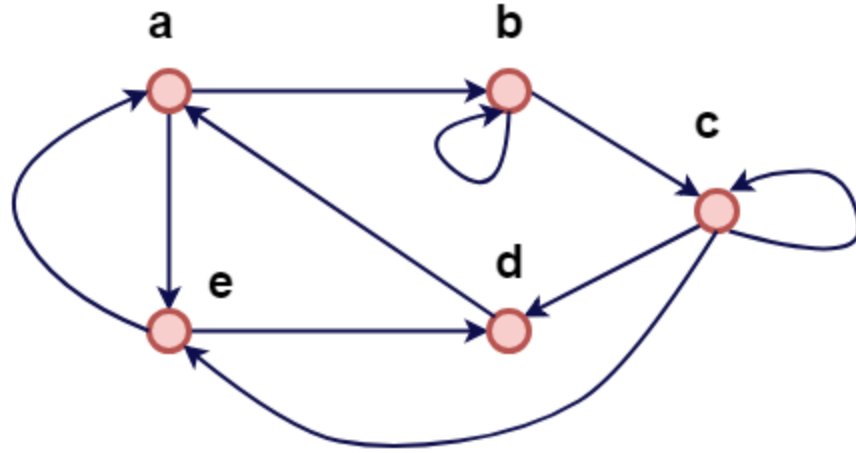


Figure 2.1: An example of an oriented graph

## 2.3 Multigraphs

**Multigraph:** A multi-set is a set within which an element can be repeated more than once. For example,  $\{1, 1, 2, 3\}$ ,  $\{1, 2, 3\}$  and  $\{1, 2, 2, 3\}$  are distinct multi-sets. To distinguish copies of an element  $x$  of a multi-set, it is convenient to index them. For example. Set the multi-set  $\{1_1, 1_2, 1_3, 2_1, 2_2\}$ . This way of proceeding will make it easy to define functions over a multi-set. A multigraph  $G = (V, E)$  is a graph for which the set  $E$  of arrows is a multi-set. In other words, there may be more than one arrow connecting two given vertices. An example of the representation of a multigraph is given in the figure 2.2.

Let  $p \geq 1$ . A  $p$ -graph is a multigraph  $G = (V, E)$  for which every edge of  $E$  is repeated at most  $p$  times.

## 2.4 Simple graphs

**Simple graphs:** A graph  $G = (V, E)$  is simple if it is not a multi-graph and if the set  $E$  is irreflexive, that is for  $v \in V$ .  $(v, v)$  is not in  $E$  i.e,  $G$  does not contain a loop. Figure 2.3 depicts an example of a simple graph [9].

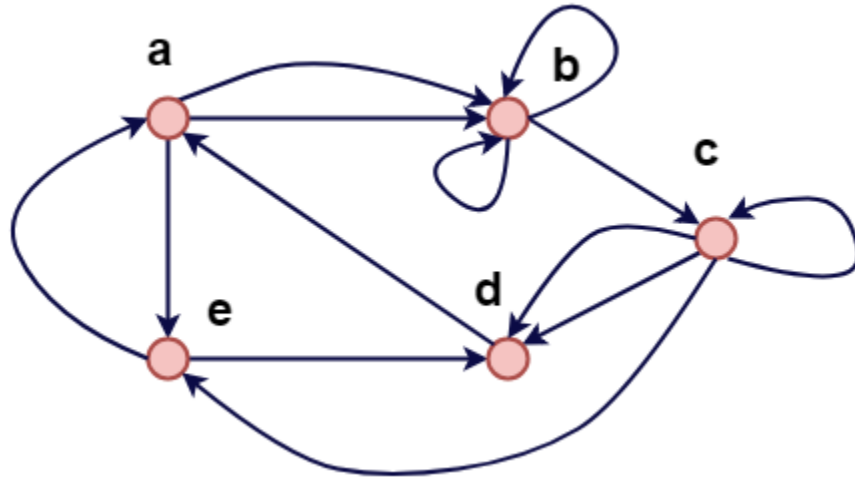


Figure 2.2: An example of a multigraph

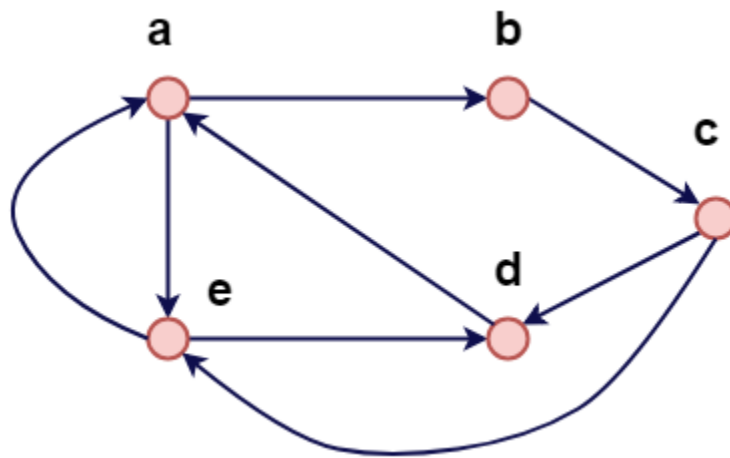


Figure 2.3: An example of a simple graph

## 2.5 Non-oriented graphs

Undirected or non-oriented graphs are a special case of graphs. It is non-oriented if each arrow of the graph is reversible. If  $G = (V, E)$  is a graph (or a multi-graph) and  $E$  a symmetrical relation on  $V$ , then  $G$  is an undirected graph or an undirected multi-graph. In other words,  $G$  is not directed if  $\forall x, y \in V : (x, y) \in E \Rightarrow (y, x) \in E$ . In this case, the sagittal representation of  $G$  is simplified by drawing a segment between  $x$  and  $y$ . The definitions previously encountered (Oriented graph to simple graph) are easily adapted to the case of undirected graphs. By **the Handshaking lemma** [3], if  $G = (V, E)$  is a non-oriented multigraph that can be described by the equation 2.1.

$$\sum_{v \in V} \deg v = 2|E| \quad (2.1)$$

## 2.6 Bipartite graphs

A graph  $G = (V, E)$  is bipartite if the set  $V$  can be partitioned into two sets  $X$  and  $Y$  such that  $E \subseteq X \times Y$ . If  $|X| = m$ ,  $|Y| = n$  and  $E = X \times Y$ , then it is complete bipartite graph, denoted by  $K_{mn}$ . Figure 2.4 depicts a bipartite graph [9].

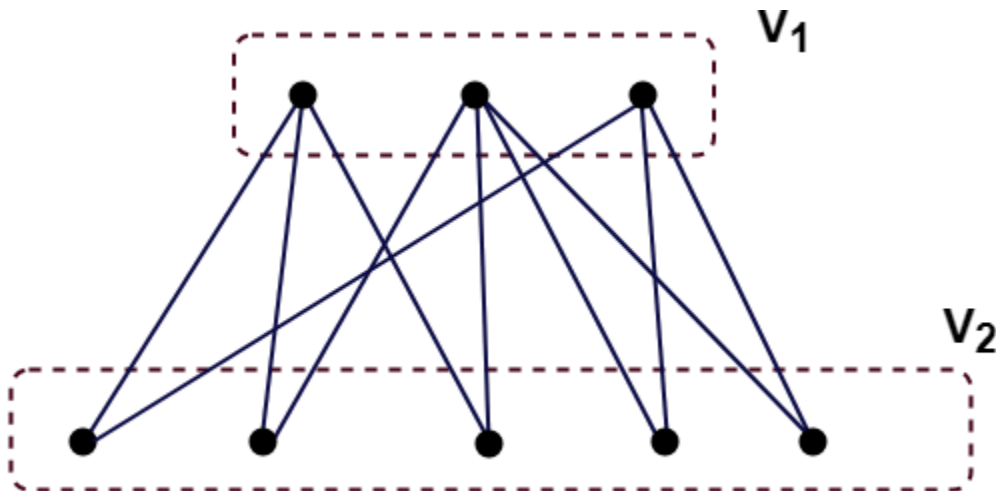


Figure 2.4: An example of a bipartite graph

This notion can be generalized by defining  $n$ -party graphs, for  $n \geq 2$ . To do this,  $V$  must be partitioned in  $n$  subsets  $V_1, \dots, V_n$  such that  $E \subseteq \bigcup V_i \times V_j$  for distinct indices  $i$  and  $j$ .

## 2.7 Graphs and Eulerian paths

A path (or a circuit) of a multi-graph  $G$  is Eulerian if it passes once and only once by each arrow or arc of  $G$ . Such a path (or circuit) may of course pass more than once by the same vertex. In other words, a Eulerian path (or Eulerian circuit) is a trail (or a closed trail) passing through each arrow or arc of  $G$  [34]. A Eulerian multigraph is a graph that possesses a Eulerian circuit.

### 2.7.1 Example:

The graph  $G$  has an Eulerian paths which is "dcbaeb" and the augmented graph  $G'$  (with an added dashed edge) has an Eulerian circuit "eabcdbe". Figure 2.5 shows the graph  $G$  has Eulerian paths and Eulerian circuit

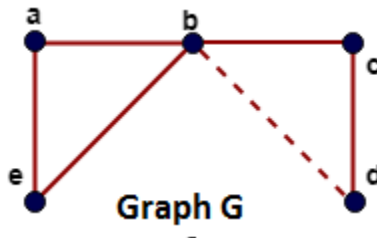


Figure 2.5: Graph G

## 2.8 Connected graphs

A graph is called connected if there is a path from any vertex to any other vertex. Thus, there is a path between every pair of vertices. In a connected graph, there are no unreachable vertices. A graph  $G$  is called disconnected if there exist two vertices such that no path

has those vertices as endpoints. For example, a graph without arrow with two or more vertices is disconnected.

### Example

A graph without edges with two or more vertices is disconnected. Figure 2.6 shows an example of a disconnected graph. Connectivity is an important property of graph theory. In an undirected graph, two vertices  $u$  and  $v$  are called connected if the graph contains a

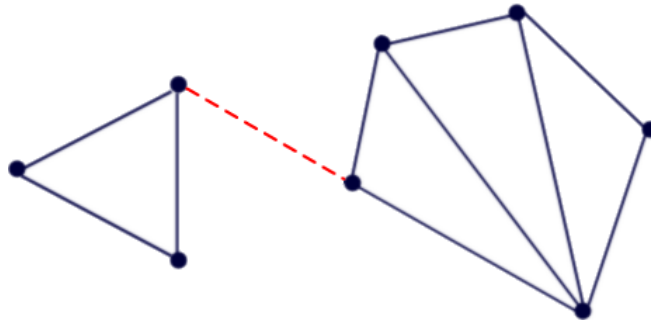


Figure 2.6: A connected graph becomes disconnected when dashed edge is removed

path from  $u$  to  $v$ . Otherwise, they are called disconnected. If the vertices are connected by a path of length 1 (single arrow), the vertices are called adjacent. A directed graph is called weakly connected if replacing all of its arrows with edges produces a connected graph [14].

**Theorem 1.** *A connected non-oriented finite multi-graph  $G = (V, E)$  possesses an Euler circuit if and only if the degree of each vertex is even. An undirected multi-graph has a eulerian path joining two vertices  $X$  and  $Y$  if and only if  $X$  and  $Y$  are the only vertices of odd degree. Moreover, a finitely oriented and connected multi-graph  $G = (V, E)$  has an Euler circuit if and only if the incoming half-degree each vertex is equal to its outgoing semi-degree [14].*

**Lemma 1.** *Given a finite and weakly connected graph  $G$ , the graph  $G$  is Eulerian if and only if, for all vertices  $x$ , the number of in-degrees equals the number of out-degrees [31].*

## 2.9 Trees

A graph  $G$  is a *tree* if it has no loops and does not contain any cycles. Also, The vertices of degree 1 in a tree are its called *leaves*. A *forest* is a graph with each connected component a tree. Figure 2.7 shows a tree and a forest of 2 trees.

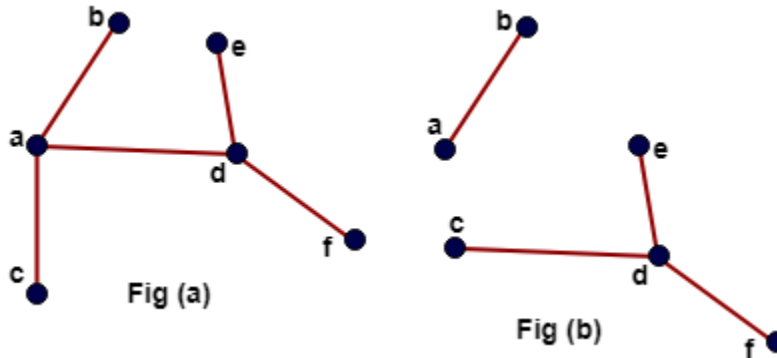


Figure 2.7: A tree and a forest

## 2.10 Hamiltonian graphs and paths

In the previous section, graph  $G$  with an Eulerian path passing once and only once by each arrow of  $G$  was determined. A path (or circuit) passing once and only once by each vertex of  $G$  is called Hamiltonian. A Hamiltonian graph is a graph that contains a Hamiltonian path or circuit [30].

## 2.11 Isomorphic graphs

Let  $G_i = (V_i, E_i), i = 1, 2, \dots$ , be two graphs. A map  $f: V_1 \rightarrow V_2$  is a homomorphism if  $(x, y) \in E_1 \Rightarrow (f(x), f(y)) \in E_2$  [35]. Example of graph homomorphisms are shows in Figure 2.8 below.

A graph  $G$  is called isomorphic to  $H$  if there is a bijection between their sets of vertices which preserves edges or arrows.

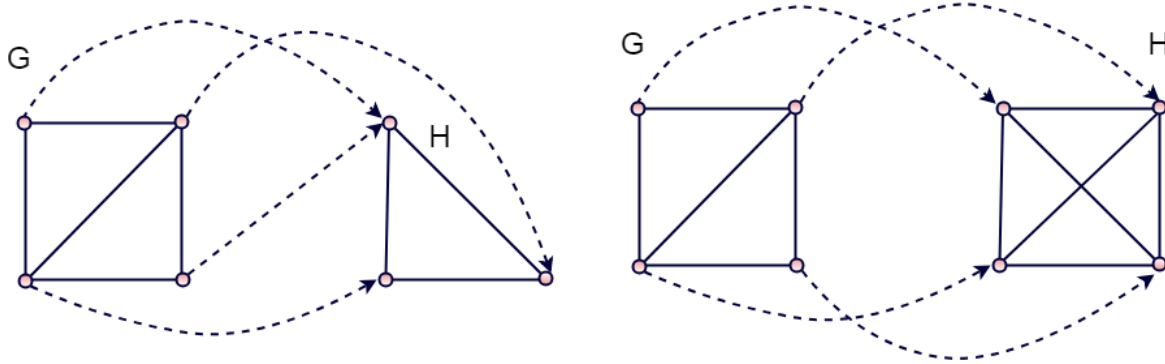


Figure 2.8: Homomorphic graphs

Figure 2.9 below represent isomorphic graphs. With

$$\varphi: a \rightarrow 4, b \rightarrow 5, c \rightarrow 6, d \rightarrow 1, e \rightarrow 2, f \rightarrow 1.$$

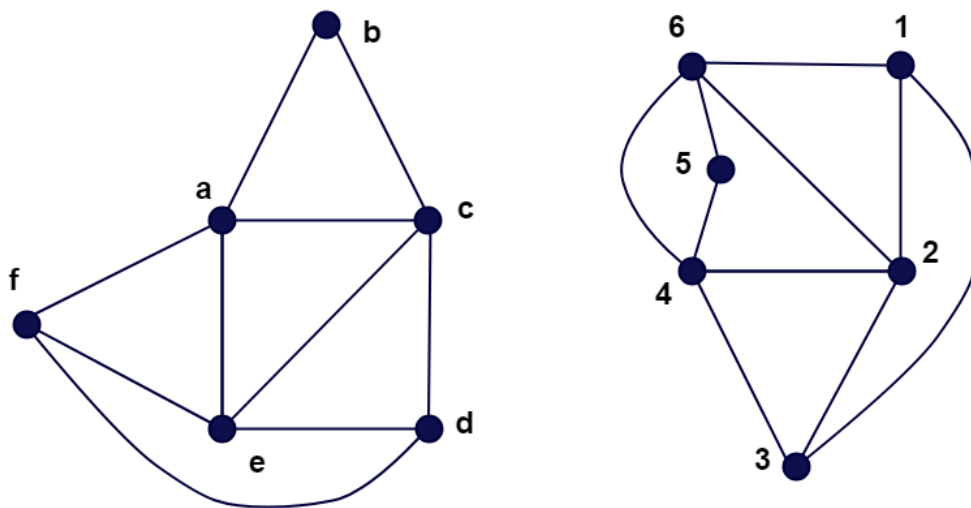


Figure 2.9: Isomorphic graphs

An automorphism of  $G$  is an isomorphism of  $G$  into  $G$ . The set of automorphism of  $G$  provided with the law of composition of applications forms a group, noted  $\text{Aut}(G)$ . That is a subgroup of the symmetric group  $S_n$  of the permutations of  $n$  elements. A graph for which  $\text{Aut}(G)$  is reduced to the identity is called asymmetric.

## 2.12 Matrix representation of a graph

Let  $G = (V, E)$  be an undirected multi-graph whose vertices are given by set  $V = \{v_1, \dots, v_n\}$ . The adjacency matrix of  $G$  is the matrix  $A(G)$  whose element  $ij$ -th is equal to the number of edges  $(v_i, v_j)$  present in  $E$ , with  $1 \leq i, j \leq n$ . It is therefore a symmetric matrix with integer.

It can be noted that the elements of the adjacency matrix of a simple graph belong to the binary set  $\{0, 1\}$ . Figure 2.10 is an example of a graph with the adjacency matrix.

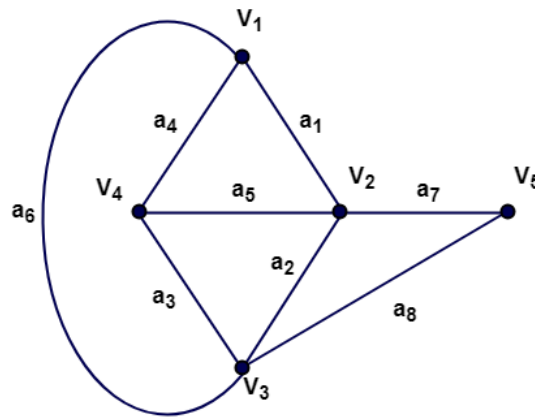


Figure 2.10: An example of a hypergraph

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

# Chapter 3

## Theory of De Bruijn Graphs and Generalization

This chapter presents the theory underlying the construction and properties of de Bruijn graphs and its applications. A generalisation of de Bruijn graphs to non oriented networks will follow along with the connection with slider graphs and lamplighter groups that we will present in the next chapter.

The *De Bruijn graphs* were defined in 1946 by *Nicolaas De Bruijn*, a Dutch professor of Mathematics at Eindhoven University. De Bruijn's graphs make it possible to represent the overlap of all words of a given length by a directed graph. Research has been carried out on the properties of Bruijn graphs, particularly in mathematics and biology. In this chapter, the De Bruijn graphs will be introduced and their properties will be defined. Also, different variants will be exposed that have been developed in the literature.

The history of De Bruijn graphs were described by Nicolaas De Bruijn in "A combinatorial problem" [8]. In his article, the author provides the solution to a problem called "Superstring" which was originally (De Bruijn, 1946) defined as a circular string that contains each binary substring of length  $k$  (also known as  $k$ -mer) exactly once.

What is the shortest possible superstring that contains all the possibilities of length  $k$  from a given alphabet? In the binary alphabet, the only symbols are 1 and 0. The set

of words of length  $k$  is simply the set of binary strings of length  $k$ . In other words, for a length  $k$  equal to 3, the superstring problem is to find the shortest word containing the set of binary string of length 3  $\{000, 001, 010, 011, 100, 101, 110, 111\}$ . The total number of words are  $d^n$  where  $(d = 2, k = 3)$ , then we have  $2^3 = 8$ . In this case, the solution is 0001110100. In order to obtain the mathematical solution to this problem, De Bruijn is inspired by the solution of  $L$ . Euler of the famous problem of the seven bridges in the Prussian town of Konigsberg as depicted in Figure 3.1 .De Brijn graph uses the notions of an Eulerian graph (see chapter 1)

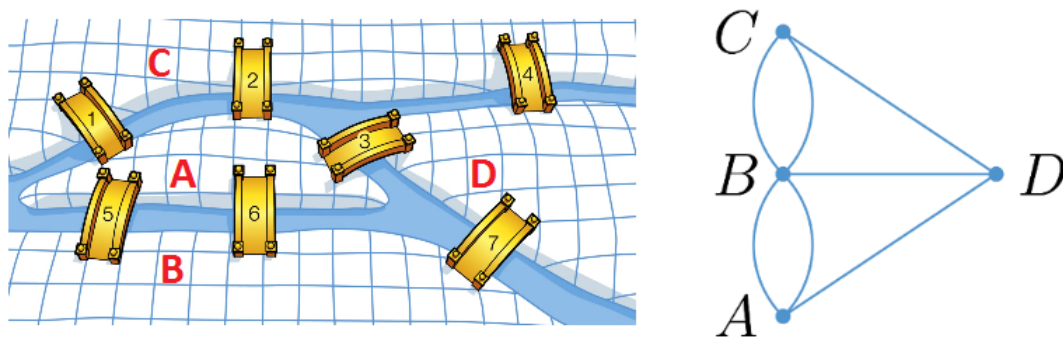


Figure 3.1: Problem of the bridges of the city Konigsberg [22]

De Bruijn solution is to build a graph where each node corresponds to a word of length  $(k - 1)$  of the alphabet. One node is connected to another if its suffix corresponds to the prefix of the node to which he is connected. For example, in Figure 3.2 below, an edge starts from node 000 to node 001 because the suffix one is the prefix of the other. In addition, each edge corresponds to a word of length  $k$  whose suffix is the starting node and the prefix is the arrival node.

### 3.0.1 Example

Considering for  $(k = 4, d = 2)$ , which  $2^4 = 16$  a Eulerian cycle of the graph  
 (0000, 0001, 0011, 0110, 1100, 1001, 0010, 0101, 1011, 0111, 1111, 1110, 1101, 1010, 0100, 1000)  
 . There are 16 vertices since  $2^k = 16$ . The shortest cyclic Superstring corresponds to the word formed by the first character of each edge of the cycle is (0000110010111101).

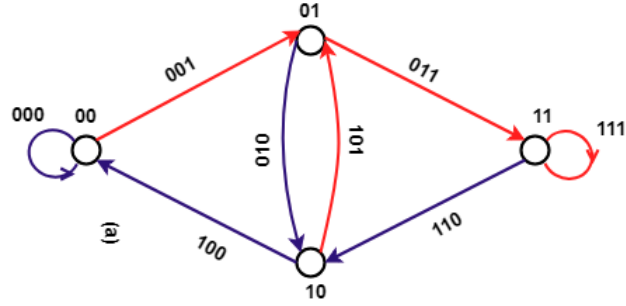


Figure 3.2: De Bruijn graph for a  $k = 3$ .

We notice that each vertex of the graph has even degree according to Eulerian theorem on chapter (1). The superstring for a length of word is 0000110010111101000. In Figure below shows the  $B_2^4$  [7, 28].

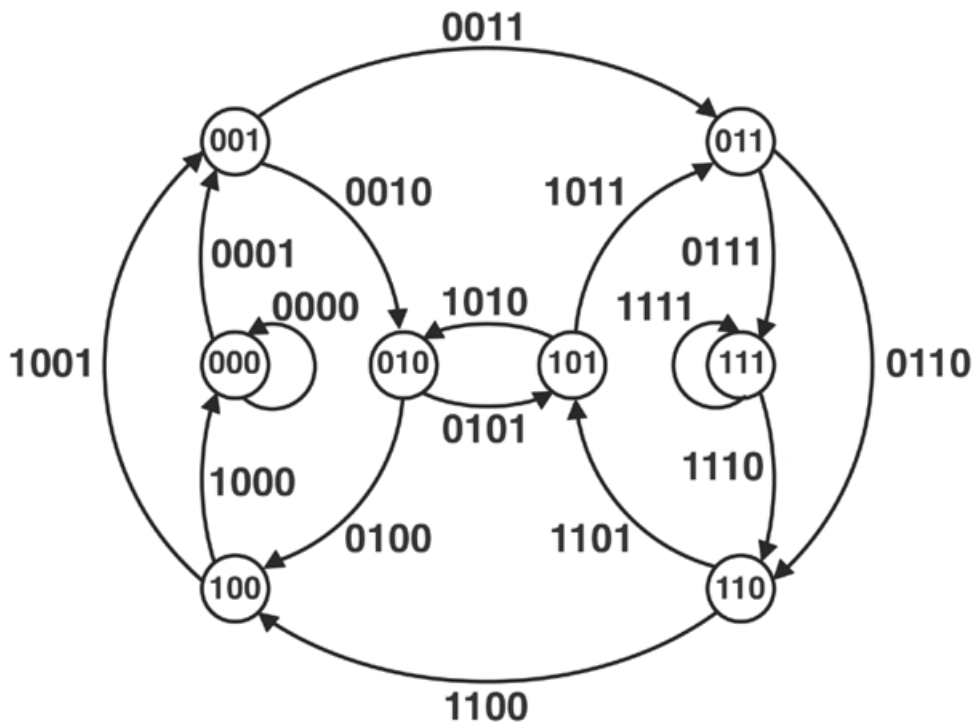


Figure 3.3: De Bruijn graph for a  $k = 4$ .

### 3.0.2 Definition

Given a character alphabet  $A$  with  $d$  symbols, *the De Bruijn graph* denoted by  $B_d^n$ , is defined as the oriented (directed) graph with  $d^n$  vertices, where each vertex is labeled by an  $n$ -tuple in the set  $A^n$ . If we let  $a_i$  denote a character of the alphabet set, then each edge is defined as the following ordered pair  $((a_1, a_2, a_3, \dots, a_n), (a_2, a_3, a_4, \dots, a_{n+1}))$ . Thus, if  $x$  and  $y$  are words of length  $n$ , there is a directed edge from  $x$  to  $y$  if and only if there characters symbols  $a, b$  such that  $xb = ay$ .

In this term, a De Bruijn graph is an oriented graph of order  $d = 2^n$  constructed from the SuperString ( $S$ ) of which the vertices are the factors of length  $d$  of ( $S$ ), and the edges  $a_i$  are the factors of length  $d + 1$  of ( $S$ ) between two factors of length  $d$  of  $a_i$ . Of course, this definition is extremely difficult to use for implementation purposes because it requires the knowledge of SuperString [34].

## 3.1 Undirected De Bruijn graphs:

A similar structure of Bruijn graph can also be defined for undirected graphs from the properties of the directed De Bruijn graph. We denote by  $UB_d^n$ , the undirected De Bruijn graph with the following properties[21][14]:

- If there is a loop in the graph, then remove it.
- If  $(u, v)$  is an arrow in  $UB_d^n$ , then  $(u, v)$  is an edge in  $UB_d^n$ .
- If  $(u, v)$  and  $(v, u)$  are both arrows in  $UB_d^n$ , then  $(u, v)$  is a single edge in  $UB_d^n$ .

**The undirected De Bruijn graph** is therefore obtained by transforming it in bidirectional graph by removing the direction of the arrows of the graph.

**Theorem 2.** *For  $d \geq 2$  and  $n \geq 1$ , the de Bruijn undirected graph  $UB_d^n$  has the following properties:*

- 1-  $UB_d^n$  has the maximum degree  $2d$  and the minimum degree  $2d - 2$ .

2-  $UB_d^n$  has diameter  $n$ .

3-  $UB_d^n$  has connectivity  $2d - 2$ .

Proposition (1) is clearly true. By the definition of  $UB_d^n$  the diameter of  $UB_d^n$  is less than that of  $B_d^n$  which is  $n$ . Moreover, since the distance between the vertices  $00 \dots 00$  and  $11 \dots 11$  is  $n$ , the proposition (2) is true. The proof of the assertion (3) is the interested reader is referred to Esfahanian and Hakimi [12]see , who proves this.

### Example

The simple De Bruijn graph with  $n = 2$  and  $d = 2$  is defined over the binary alphabet set  $\{0, 1\}$ . Denoted by  $B_2^2$ , it is an example of De Bruijn graph with a word  $x$  of length 16 over the binary set is  $x = 0000101111010011$  In the Figure 3.4 below shows (a) is The De Bruijn graph with alphabet  $\{0, 1\}$  whose vertices are 2-tuples. (b) is The undirected De Bruijn graph  $B_2^2$ .

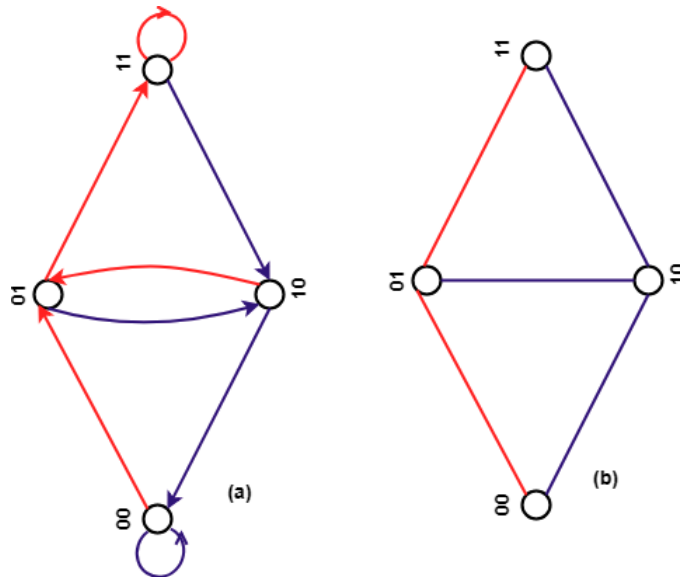


Figure 3.4: (a)Directed and (b)undirected  $B_2^2$

## Example

There are more examples of De Bruijn graph for  $d = 2$  and  $n \in \{1, 2, 3\}$  where one group of edges are colored in blue and labeled by 0, another group colored in red and labeled by 1, as depicted in Figure 3.5 [19].

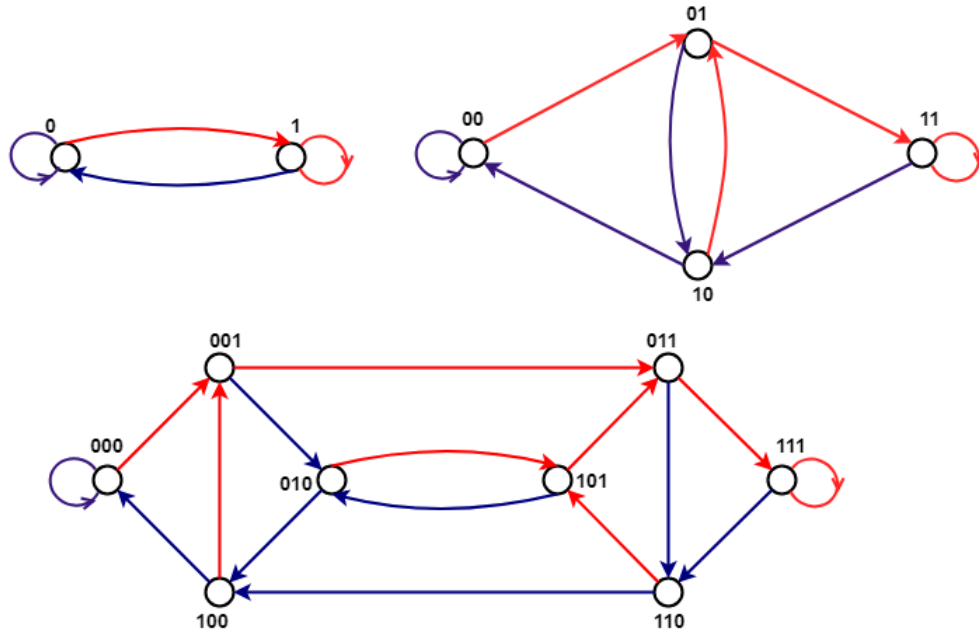


Figure 3.5: De Bruijn graphs  $B_1^2, B_2^2, B_3^2$

## 3.2 Properties of De Bruijn graphs

### 3.2.1 Doubling of a De Bruijn graph

A De Bruijn graphs can be expanded from a graph of words length  $n$  to words of length  $n + 1$  with the following procedure we using the algorithm described below [3][21]:

- 1- The vertices of De Bruijn graph  $B_d^{n+1}$  are the edge of  $B_d^n$ .

For instance, for  $n = 3$  if we have two adjacent vertices  $(a_1a_2a_3)$  and  $(a_2a_3a_4)$  create an edge in the De Bruijn graph  $B_d^3$  and corresponds to the single vertex  $(a_1a_2a_3a_4)$

in  $B_d^4$ . To be more specific in the De Bruijn graph  $B_3^4$ , we have  $(201) \xrightarrow{(2011)} (011)$  in  $B_3^4$ . In the doubling of a De Bruijn graph from  $B_3^3$  to  $B_3^4$ , the edge  $(201)(011)$  in  $B_3^3$  corresponds to a vertex in  $B_3^4$  which is  $(2011)$

- 2- To create the edge of the De Bruijn graph  $B_d^{n+1}$ , draw an arrow in  $B_d^{n+1}$  from each adjacent point of arrow in  $B_d^n$ .

For Instance in  $B_3^3$ , suppose we have  $(122) \xrightarrow{(1222)} (222) \xrightarrow{(2220)} (220)$ . Then, the doubling De Bruijn graph will be  $B_3^4$   $(1222) \rightarrow (2220)$  [19].

See Figure 3.6 to see an example of the doubling process from  $B_2^2$  to  $B_2^3$

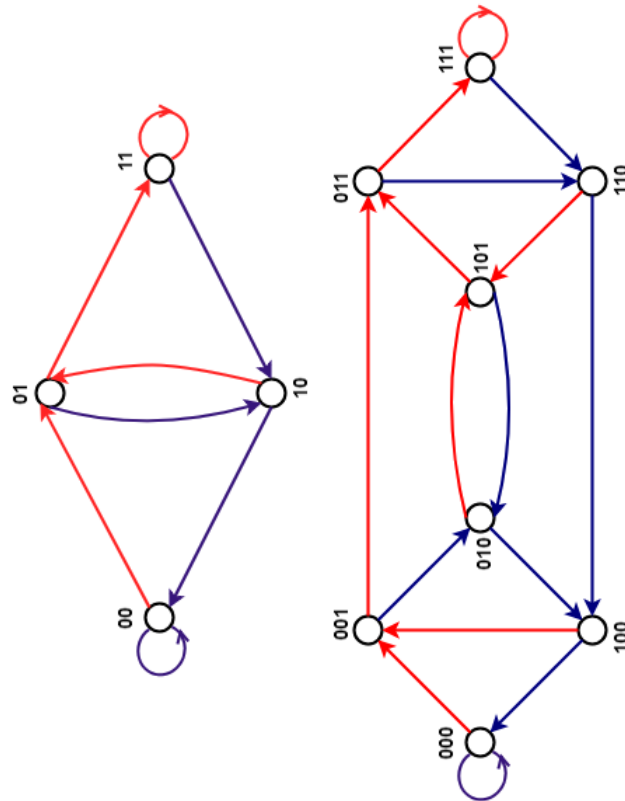


Figure 3.6: Doubling process from  $B_2^2, B_2^3$

Remark: The numbers of vertices in the doubling De Bruijn graph is  $d^{n+1}$  because the number of the edges in  $B_d^n$  is  $d^{n+1}$  similarly the numbers of arrows in  $B_d^{n+1}$  is  $d^{n+1}$ .

### 3.2.2 The degrees and distances of the vertices of De Bruijn graphs

Another interesting property of the Bruijn graphs that carries many applications especially in the communication networks in the property related to the degrees and distances of the vertices. For a given De Bruijn graph ( $B_d^n$ ), we know the total number of vertices is  $d^n$  and this can be extremely large depending on the size  $n$  of the words. However, there is a small in degree and out degree for each vertex which means very few connections between vertices. Moreover, the distance between any two nodes is relatively small since it does not exceed  $n$ . Finally, the number of paths between the vertices is large; in practice this has significant values since it allows a sender to transmit simultaneously many packages of information to a receiver without interference [21].

### 3.2.3 The order of De Bruijn graph

A De Bruijn graph is of the order  $k = 2^n$  with  $n$  integer, which represents the level of the graph. A level 3 De Bruijn graph is therefore a graph possessing 8 vertices. The degree of a  $n$ -level Bruijn graph equals its order multiplied by 2, thus  $2^{(n+1)}$  since from each node leaves two edges.

The maximum path length equals the level of the graph De Bruijn and is therefore a function of the logarithm in base 2 of the order of the De Bruijn graph:  $P_{\max} = n = \log_2(d)$ . The average path length of a De Bruijn is more difficult to define, but bounds and estimates can be found. The average path is estimated by  $\log_2(n - 1)$ .

#### Example

For a level 3 De Bruijn graph, starting from 101, the propagation in the network proceeds as in Figure 3.7. We can note that after three jumps, all the nodes have been reached. However, several nodes have been affected differently at different propagation stages like the 011 nodes. Consequently, there is an increase in the value of the average path with respect to its real value.

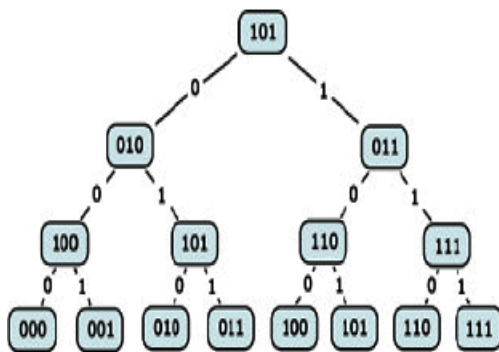


Figure 3.7: Tree structure of De Bruijn graph showing the path starting with 101

### 3.3 Alphabet and words

An alphabet  $A$  is a finite set of elements, usually referred to as symbols or letters. For instance, the English alphabet contains 26 symbols, the binary alphabet has only two symbols,  $\{0, 1\}$ . The words of an alphabet  $A$  are sequences  $u = u_1u_2u_3\dots$  of symbols of the set  $A$ . Those words can be finite or infinite in length. For example, the word  $u = 0101101$  of the alphabet  $A = \{0, 1\}$  is finite whereas the word  $v = 000111111\dots$  is infinite. We call the power set  $A_n = \{u_1u_2u_3\dots u_n \text{ with each } u_i \in A\}$ , the set of words of fixed length  $n$ . When  $n = 0$ , the  $n^{\text{th}}$  power set is the empty word denoted by the zero power of  $A$ ,  $A_0 = \{\lambda\}$ , which consists of one single word, the empty word  $\lambda$ . The empty word  $\lambda$  has no characters or symbols, its a blank space.

We also define  $A^*$  the set of all finite words of any given length, and by  $A^+$  the set of all nonempty finite words of any given length. Finally, to process infinitely long sequences of symbols of an alphabet, we introduce the omega power set  $A_\omega = \{u = u_1u_2u_3/u_i \in A\}$  the set of infinite words.

The length of a word  $u \in A_n$  is denoted by  $|u| = n$  and  $|u| = \infty$  for words of the omega power set  $A_\omega$ .

A word  $v$  from the set  $A^*$  is a subword of a given word  $u$ , if  $v$  can be written as  $v = u_iu_{i+1}\dots u_{j-1}$  for some integers  $i$  and  $j$  such that  $0 \leq i \leq j \leq |u|$ . We define the subset  $D$  of  $A^+$ ,  $D \subseteq A^+$ , as the set of forbidden words [18].

### 3.4 Subshifts

The subshift set is a subset of the omega power space defined by the finite forbidden words. More precisely, it is defined as  $\Sigma_D = \{u \in A_\omega \text{ for each subword } v \text{ of } u, v \notin D\}$ .

For an alphabet  $A$  and a set  $D$  of forbidden words a nonempty set  $\Sigma$  is called a subshift, if the equality  $\Sigma = \Sigma_D$  holds for some set  $D$  in the power set  $A$ . When  $D$  is a finite set, the subshift is of finite type (*SFT*). The order of a subshift (*SFT*) is the smallest integer  $p \geq 2$  such that there is a subset  $D$  in the power set  $A_p$  with  $\Sigma = \Sigma_D$ . Consider the binary alphabet  $A = \{0, 1\}$ . The subshifts  $SFT \Sigma\{00, 111\} = \Sigma\{000, 001, 111\}$  has order 3. Examples of *SFT* of order 2 in the binary alphabet are:  $\Sigma\{00, 11\} = \{(01)^\omega, (10)^\omega\}$  which is a finite subshift, and  $\Sigma\{11\} = \{0, 10^\omega\}$ , which is uncountable. Some subshifts are of infinite type, such as the soliton subshift of words which contain at most one occurrence of the binary symbol 1. The forbidden set for the soliton is:  $D = \{10^n 1 / n \geq 0\}$  [18].

#### Example

We now consider the subshift  $\Sigma$ , subset of the language  $A^{\mathbb{Z}}$  and the language  $L$  over  $\Sigma$ . In addition, we impose the restriction that consecutive symbols of a given word ( $w$ ) are distinct. Such a language is also referred to as language of irreducible words of the free product of multiples copies of the group  $Z_2$  over the alphabet  $A$ . This subshift is often called the Kautz subshift.

#### Example

The language of infinite words over the binary alphabet that forbid the word 11 is a subshift of finite type. On the contrary, the set of infinite words, that have an even number of 1's between 2 zeros is a subshift but not of finite type.

## 3.5 Language

A language is a set of words that is finite strings of symbols taken from the alphabet over which the language is defined. The language of a subshift  $\Sigma$  of the power set  $A^\omega$  is the set of finite words defined as:  $L(\Sigma) = \{u \in A^\omega \text{ there is at least } x \text{ in } \Sigma \text{ such that } u \text{ is a subword of } x\}$ . This language represents words which occur as subwords of infinite words.

Consider  $w$  as a finite word from alphabet  $A$  which belongs to language  $L$  where  $L = L(w)$ . If  $w$  is a word in *DNA* then its associated graphs will be Bruijn graphs which are represented as  $p(l) = p(w)$ .

Given a language alphabet set  $A$  and an integer  $n$ , *circular slider graphs* are general subgraphs of the De Bruijn graph  $B_n^A$  that are parametrized by subsets  $E$  of the language  $A_{n+1}$ . The vertices or nodes of the subgraphs are formed by the set of all initial and final segments of length  $n$  from  $E$ . Moreover, an element  $w$  of  $E$  creates an arrow between an initial and a final segment. A *factorial language* is a language for which if a word ( $w$ ) is in the language, then any truncation of the word ( $w$ ) is also in the language. In general, the subgraphs of the De Bruijn graphs are generated by the language of words of length  $n + 1$  from a factorial language  $L$ , that is  $L_{n+1} = A_{n+1}L$ . A graph that possesses those properties is called a factorial slider graph and its set of vertices is  $L_n$ . If for a given word  $w$  in  $L$ , there are symbols  $a, b$  such that  $(awb)$  is also in  $L$ , the language set  $L$  is said to be *prolongable or extendable*. For extendable languages  $L$ , there exists a corresponding language of subwords of a certain subshift. The following provides particular examples of graphs over language and subshifts [18].

### 3.5.1 Example

Consider two finite languages  $L$  and  $M$  defined by:  $L = \{0, 01, 001\}$ ,  $M = \{0, 01, 10\}$  We compute  $L^2$  as the concatenation of the words in  $L$ .

$$L^2 = \{00, 001, 0001, 010, 0101, 01001, 0010, 00101, 001001\}$$

The size of  $L^2$  is  $|L^2| = 9$ . Also, we have  $M^2 = \{00, 001, 010, 0101, 0110, 100, 1001, 1010\}$ , with size  $|M^2| = 8$ . So, we write that  $|L^2| > |M^2|$ , and  $|L^2| = |L|^2$  but  $|M^2| < |M|^2$ . The

fact that  $|L^2| = |L|^2$  is due to unique factorisation of the elements in the set  $L$  (which is not the case for  $M$  because  $(01)0 = 0(10)$ ) [4].

### 3.5.2 Example

We let  $w$  be a finite word of the alphabet set  $A$ . Then  $L_{(w)}$  denotes the factorial language over  $A$ . In application such as Bioinformatics, the word  $w$  represents a *DNA* string over the genetic alphabet set  $\{A, G, T, C\}$ . Each word  $w$  is a gene and codes for a physical trait of an organism. The corresponding De Bruijn graph  $L_{(w)}^n$  are composed of genes of length  $n$ .

#### Definition of closed language

A language is prefix-closed if it contain all prefixes of any of its elements. For example, the language  $\{0, 00, 001, 0001\}$  is prefix-closed. However, the language is suffix-closed if it contain all suffixe of any of its elements. For example, the language  $L = \{1, 11, 011, 0011\}$ . Another suffix-closed language is  $L = \{0^n1, n > 0\}$  which is  $= \{01, 001, 0001, \dots\}$ . Some languages are called factors if they contain all factors of any of their elements. The language  $L = \{0^n1, n > 0\}$  is not factorial [4].

#### Example

The set of word over the binary alphabet  $\{0, 1\}$  that contain an even number of 1's  $L = \{\epsilon, 0, 00, 11, 000, 011, \dots\}$ .

## 3.6 Generalizations Of De Bruijn Graphs

### 3.6.1 Infinite words

In combinatorial words, we work with an infinite words (has length  $L = \infty$ ) sequences from a finite (which is a word of a given length  $L$ , where  $L$  is integer) alphabet set. A De

Bruijn graph of order  $d$  of the word  $w$  is composed of vertices which are factors of length  $d$  of  $w$  (i.e. finite words of the form  $W_i \dots W_{i+d-1}$ ) and for which there exist a label edge  $\sigma$  between the vertices  $ax$  and  $x\sigma$  if and only if  $ax\sigma$  is a factor of word  $w$  of length  $d + 1$ . De Bruijn graph can be also be con started to represent infinite words. For example, Figure 3.8 represents the graph of De Bruijn of order 3 of the periodic word  $W = abababab \dots$  [11].



Figure 3.8: Generalized De Bruijn graph, 2 vertices.

These graphs are in direct relationship with certain combinatorial properties of the corresponding words. For example, an infinite word is periodic (i.e., there are finite words  $u$  and  $v$  such as  $w = uvuv \dots$ ) if and only if for any sufficiently larged, De Bruijn graph of order  $d$  contains a single cycle of which all vertices have an outgoing degree of 1 [10].

### 3.6.2 Generalization of De Bruijn graphs to any number of vertices

The original defintion of De Bruijn graphs is based on a fixed number of vertices represented by fininte strings of a given alphabet set. An extensions of De Bruijn graphs to any number of vertices was proposed independently By (Imase and Itoh, and Reddy, Radhan and Kuhl)[10]. One of the advantage of the De Bruijn graphs is the fact that they have short diameters and simple finding path strategy, which make them very suitable in real life application such as, communication networks and multiprocessor system. Unfortunately the De Bruijn graphs are limited with respect to the number of nods or vertices. There are exactly  $d^n$  vertices in the  $B_d^n$  De Bruijn graphs, and each vertex is represented by a string or vector of  $n$  symbole from the alphabet set. The edge (or arrows) of the De Brijn graphs are definid as follows:

$$i \longrightarrow di + r \pmod{d^n} \text{ For } 0 \leq i \leq d^n - 1, 0 \leq r \leq d - 1 \quad (3.1)$$

### Example:

Let  $(d = 2, n = 2)$ , then we have  $0 \leq i \leq 3, 0 \leq r \leq 1$ . In Figure below shows that  $B_2^2$  generalization to any number of vertices  $n$  is denoted by  $G_B(n, d)$  as follows:

$$i \longrightarrow d_{n-1}i + r \pmod{d^n} \text{ For } 0 \leq i \leq n - 1, 0 \leq r \leq d - 1 \quad (3.2)$$

It is known that the number of loops in the  $B_d^n$  De Bruijn graph is  $d$ . The following theorem gives the number of the loops in the generalized de Bruijn graph  $G_B(n, d)$

**Theorem 3.** *Let  $\gcd(n, d) = g$ , then the number of the loops in  $G_{B_d^n}$  is  $g \lceil \frac{d}{g} \rceil$  where  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$ . Moreover, the smallest number of loops in  $G_{B_d^n}$  is  $d$  when  $\gcd(n, d) = 1$ , the maximum number of loop in  $G_{B_d^n}$  is  $(2^{d^2})$  when  $\frac{d-1}{n}$*

[10].

### 3.6.3 Rauzy graphs

De Bruijn graphs can be generalized in many ways. One important class of graphs can be used as a natural extension of De Bruijn graphs. We consider the alphabet set  $\{0, 1, 2, 3, \dots, d - 1\}$ . Let  $F$  be the set of forbidden words defining a subshift. The construction of the Rauzy graph from the De Bruijn graph  $B_d^n$  is follows:

- Erase all edge and vertices in  $B_d^n$  containing subsequence in  $F$ . We note that the De Bruijn graph  $B_d^n$  is Rauzy graph for  $F = \{\} = \phi$

### Example

See Figure 3.9 shows Rauzy graphs for  $d = 2, N \in \{1, 2, 3\}$  and  $F = 11$  where one group of arrows are colored in blue and labeled by 0, another group colored in red and labeled by 1, as depicted in Figure Rauzy graphs are annuated with specific forbidden words any effect some of the structure of these words. They are important tools to deal with terms of infinite words. A right factor term  $x$  of an infinite sequence word  $w$  is such that if  $a \neq b$ , then  $xa$  and  $xb$  are again factor of the word  $w$  [19].

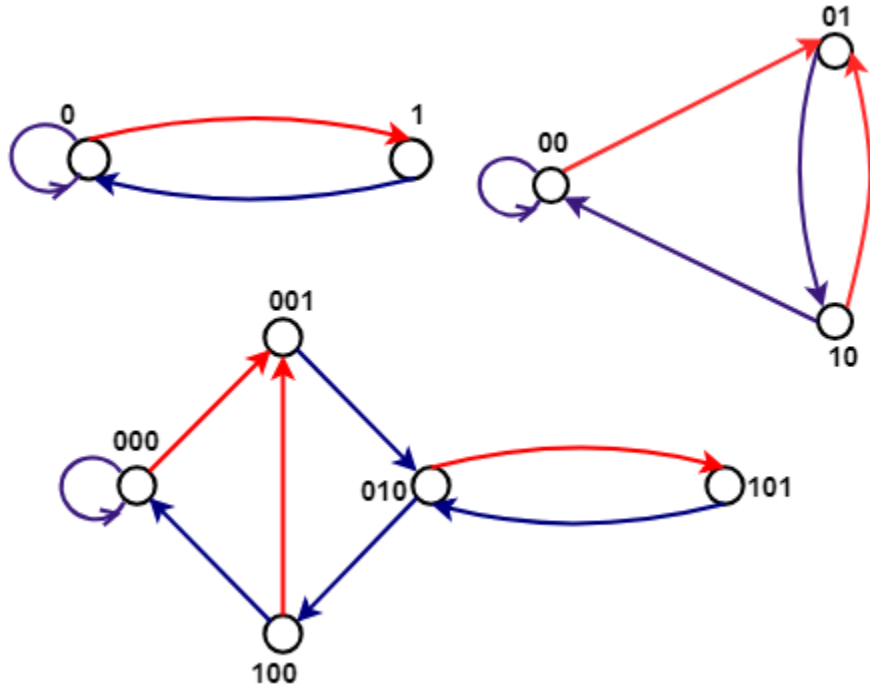


Figure 3.9: Rauzy digraphs for  $d = 2, N \in \{1, 2, 3\}$ .

## 3.7 Slider graphs

### 3.7.1 Introduction

De Bruijn graphs have been presented in the previous sections as directed graphs of words arrangements. They represent overlapping words between consecutive sub-words. Specifically, using the binary alphabet  $(0, 1)$  the De Bruijn graph of length 3 represents all  $2^3 = 8$  possible circular arrangements of 0 and 1, in which each 3-symbol word is shown exactly ones. Those 3-letter words are the vertices and the arrows are pairs of 3 words with length 2 (or  $n - 1$  in general). Graphically, the arrow  $(x, y)$  is defined in such a way that:

$$x = a_1 a_2 a_3 a_n \rightarrow y = a_2 a_3 a_4 a_{n+1}$$

Each De Bruijn transition is basically the conjunction of two operations: removing

the initial symbol  $a_1$  and adding a new alphabet symbol  $a_{n+1}$ . We now present another representation of De Bruijn graph based on circular words.

### 3.7.2 Spider slider graphs

We first recall that the tensor product of two digraphs  $M$  and  $N$  is a new digraph denoted by  $K = M \oplus N$  such that the vertex set of  $(K)$  is the direct product of the vertex sets of  $M$  and  $N$ . Also, the edge or arrow sets of  $(K)$  is the direct product of the edge sets of  $M$  and  $N$ . Spider-web graphs are defined as tensor product  $C \oplus B_m^n$ , where  $m$  is the alphabet set size  $n$  is the length of words of the De Bruijn graph, and  $C$  is the cyclic group determined by the generator 1, of the associated Cayley directed graph. The tensor product  $A \oplus L$  of the Cayley digraph  $A$  of a cyclic group  $A$  and a circular slider graph  $(L)$  is called a spider slider graph [17].

### 3.7.3 Circular words

De Bruijn graphs have been presented using linear words. But non linear words can also be used to represent De Bruijn graphs. In particular, circular words equipped with a pointer (which separates the initial and last symbols of the linear code word) can be used to represent vertices, which are  $n$ -symbols words.

To see how this is possible, first we observe that, given a vocabulary  $V$  of words of length  $n$ , there is a bijective (or one to one correspondence) relation between linear code words and circular code words. Suppose we are given a code word  $x = a_1a_2a_3a_n$ . Each word alphabet word  $a_j$  is indexed by integer  $j$ . The indices values  $j = 1, 2, 3, \dots, n$  can be replaced with new indices  $i = j(\text{ mod } n)$ , such that  $j$  varies from  $n$  to  $1(\text{ mod } n)$ . The De Bruijn transitions are then obtained according to the following procedure. Let  $S$  be the circular shift operator.  $(S_x)_i = (x)_{i+1(\text{ mod } n)}$ , Then the transition rule is:  $x \longrightarrow y \Leftrightarrow (y)_i = (S_x)_i$  And for any symbol letter  $a$  in the code word  $x$ , the symbol  $a$  in the transition word  $y$  is given by:  $(a_i) = a_{i+1(\text{ mod } n)}$ . The change of word structure from linear to circular is equivalent to a change of coordinate system. The symbols of each

word move with respect to a fixed pointer coordinate system to moving the pointer in the opposite direction with respect to fixed symbols in the symbols coordinate systems [17].

# Chapter 4

## Lamplighters groups over cyclic graphs

In this chapter we first introduce the basic concepts of group theory. Cayley and Schreier graphs of a group are then presented. We also introduce the general notion of wreath products as a framework for lamplighters which are special cases of wreath products. Finally, we conclude this chapter with general results regarding lamplighter over cyclic groups and the generators.

### 4.1 Group theory background

A group  $G$  denoted by  $\{G, *\}$  is a set of elements with a binary operation denoted by  $*$  (called the group operation), that associates to each ordered pair  $(a, b)$  of elements in  $G$  an element  $x = a * b$  in  $G$ , such that the following axioms are satisfied:

- It is **Closure**, If  $a$  and  $b$  belong to  $G$ .
- It is **associative**, if  $a * (b * c) = (a * b) * c$  for all  $a, b, c$  in  $G$ .
- It is **Identity element**, there is an element  $e$  in  $G$  such that  $a * e = e * a = a$  for all  $a$  in  $G$ .

- **Inverse element**, for each element  $a$  in  $G$ , there is an element  $\alpha = a^{-1}$  in the set  $G$  such that  $a * \alpha = \alpha * a = e$ .

**Example:** The set  $S_N$  of permutations on the operation  $(*)$  which is  $\{1, 2, \dots, N\}$ . It is called the symmetric group  $S_N$ , and  $n$  be a fixed positive integer. The operation  $*$  of composition of permutations is a group with  $e = (1, 2, \dots, N)$ , which is a function  $* : G \times G \rightarrow G$ . For example, when  $N = 3$ , and let  $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ , and  $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  ( $\sigma_1 * \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ ). Also we have  $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ , and  $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ . Then we have, ( $\sigma_1 * \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ ).

A group that has a finite number of elements is referred to as a finite group, and the order of the group is equal to the number of elements in the group. Otherwise, the group is an infinite group. A group is called abelian or commutative if it satisfies the following additional condition:

- **Commutativity:**  $a * b = b * a$ , for all  $a, b$  in  $G$ .

The set of integers (positive, negative, and 0) with addition  $(+)$  is an abelian group. The set of real numbers without 0 multiplication  $(*)$  is also an abelian group. However, the set  $S_N$  of permutations is not an abelian group. When the group operation is addition  $(+)$ , the identity element is referred to as  $(0)$ . In that case, the inverse element of  $a$ , is denoted  $-a$ . And the subtraction operation is defined as:

$$a - b = a + (-b)$$

Exponentiation is defined as repeated application of the group operation represented as  $'*'$ . Hence, for any element  $a$  in the group  $G$ ,  $a^3 = a * a * a$ . Also we have:  $a^0 = e$ , where  $e$  is the identity element, and  $a^{-n} = \alpha^n$ , where  $\alpha$  is the inverse element for  $a$  [25].

### 4.1.1 Cyclic groups

**Cyclic group:** A group  $G$  is cyclic if every element  $a$ , of the group  $G$  is a power  $a^k$  (where  $k \in \mathbb{Z}$ ) of a fixed element  $a$  in  $G$ . The element  $a$  is called to generate the group  $G$ , or to be a generator of  $G$ , which can be described by the following equation:

$$G = \{a^k / k \in \mathbb{Z}\} = \langle a \rangle \quad (4.1)$$

A cyclic group is always abelian (commutative) and it can be an *infinite* cyclic group, which is isomorphic to  $\mathbb{Z}$  the set of integers, or *finite* cyclic group of order  $n$  which is isomorphic to the group  $\mathbb{Z}/n\mathbb{Z}$ , of the integers modulo  $n$  [27].

#### Example

A class example of an infinite cyclic group is the group of integers under  $(+)$ . The group generated by 1, which is  $\mathbb{Z} = \langle 1 \rangle$ . The group generated by 1 contain:

$$\langle 1 \rangle = \{\dots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

We note that  $|\mathbb{Z}| = \infty$ , so it is indeed possible for groups of infinite order to be cyclic.

Moreover, a class example of finite cyclic group is the group of integers mod  $n$  under  $(+)$ . The elements are:  $\{0, 1, 2, \dots, n - 1\}$ . Then we get  $\mathbb{Z} = \langle 1 \rangle$ .

## 4.2 Cayley and Schreier graphs

In general, given a generating set  $A$ , the Cayley graph of a group  $G$  with respect to  $A$  is defined as the graph whose vertices are elements of  $G$  and such that, for each element  $u$  in  $G$  and  $v$  in  $A$ , there is an arrow from  $u$  to  $uv$ , labelled  $v$  [5, 1].

In this term, the cayley graph of a group  $G$  is a geometric representation of the group with very few assumptions on  $G$ . More precisely, if  $S$  is a generating set of  $G$ , the Cayley graph of  $G$  with respect to  $S$  is the graph defined as follows: the vertices are the elements

of  $G$ . An arrow connects vertex  $g$  to vertex  $g'$  and is labelled with the generator element  $s$  ( $s$  belongs to  $S$ ) if and only if  $g * s = g'$ . The Cayley graph depends on the generating set.

The Cayley graph of a free group  $F_2 = \langle a, b \rangle$  is an infinite tree, for which each vertex has 4 arrows, one in each direction of the elements  $a, b, a^{-1}, b^{-1}$ . The equation presents the Cayley graph of two generators:

$$\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1 \rangle \tag{4.2}$$

shown in Figure 4.1 below, an example of the Cayley graph. Each red arrow corresponds to the generator  $a$ , and the blue facing upwards corresponds to the generator  $b$ .

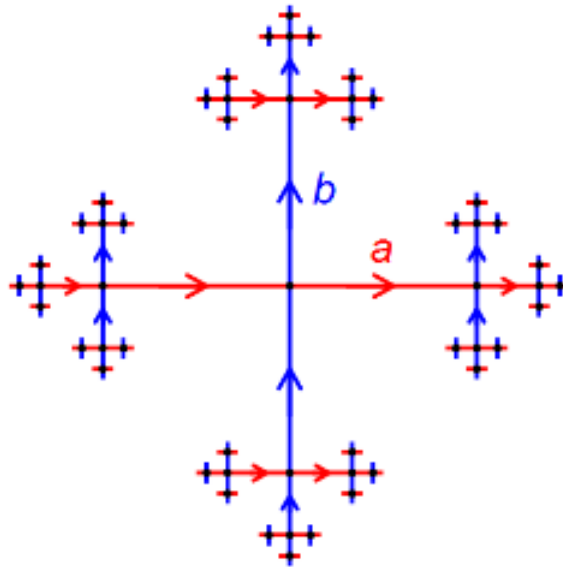


Figure 4.1: The Cayley graph of the group  $\mathbb{Z}^2$  with the generator  $(a, b)$  and  $(b, a)$ .

### Example

Cayley graph on  $\mathbb{Z}$  with one generators  $\{a\}$  is depicted in Figure 4.2

Let  $\{a^2, a^3\} \in \mathbb{Z}$ , Figure 4.3 below show the sample of Cayley graph on  $\mathbb{Z}$ , with two generators.



Figure 4.2: The Cayley graph with one generator



Figure 4.3: The Cayley graph with two generator

### 4.2.1 Generalizations of Cayley graphs

Consider a group  $G$  with a generating set  $S$ . Now we consider a subgroup  $H$  of  $G$ . Given  $G$  and a binary operation  $(*)$ , the subset  $H$  of  $G$  is a subgroup of  $G$  if  $H$  also forms a group under the operation  $(*)$ . It is usually denoted  $H \leq G$ . We also introduce the quotient space  $H \backslash G = \{Hg : g \in G\}$ .

### 4.2.2 Schreier graphs

The *Schreier graphs* of  $H$  with respect to the generating set  $S$  is obtained as follows: the vertices are the elements of  $G/H$  and an arrow connects  $Hg$  to  $Hgt$  and labelled with the generator  $s$  in  $S$  if and only if  $Hg * s = Hgt$ . These graphs are a generalization of Cayley graphs.

#### Properties of Schreier graphs

- Each Schreier graph has a root,  $H$  or the identity group in case of a Cayley graph.
- Schreier graphs are also regular. Indeed, from each vertex  $v$ , and a given generator  $s$ , there exists one incoming and outgoing arrow labelled with  $s$ . Thus, all vertices have the same degree, making the graph regular. If we count loops as two degrees, the total degrees in a finite Schreier graph is always even [5].

Figure 4.4 shows the Schreier graph of  $F_2 = \langle a, b \rangle$ .

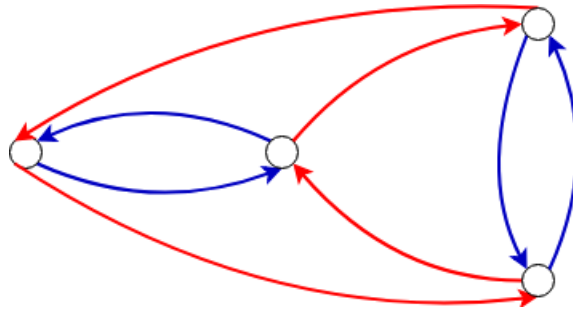


Figure 4.4: Schreier graph

## 4.3 Wreath products

### 4.3.1 Product or Direct product

Let  $(A, \bullet)$  and  $(B, \blacklozenge)$  be two groups. We define their cartesian product as the set  $G = A \times B = \{(a, b) \mid a \text{ is in } A, \text{ and } b \text{ is in } B\}$ . We define a new group operation  $(*)$  on  $G$  as follows:

- The elements of  $G$  are the elements of the cartesian product  $A \times B$ .
- For any two elements  $(a, b)$  and  $(x, y)$  in  $G$ , we have:  $(a, b) * (x, y) = (a \bullet x, b \blacklozenge y)$ .
- The identity element of  $G$  is  $(e_A, e_B)$ .
- The inverse of an element  $(a, b)$  of  $G$  is defined as  $(a, b)^{-1} = (a^{-1}, b^{-1})$ .

The new group  $(G, *)$  is called the product or direct product of the groups  $(A, \bullet)$  and  $(B, \blacklozenge)$ . This definition of a group product over two groups can be extended to a family of groups [26].

## 4.3.2 Semi direct product

### Internal semi direct product

Suppose  $G$  is a group with the group operation  $(*)$ , and let  $A, B$  denote two subgroups of  $G$ . Moreover, suppose  $A$  is a normal subgroup in  $G$ . The group  $(G, *)$  is an internal (or inner) semi direct product of the normal subgroup  $A$  by the subgroup  $B$  if and only if for every element  $x$  in  $G$ , there is a unique pair  $(a, b)$  in  $A \times B$ , such that  $x = a * b$ . In other words, each element  $x$  of  $G$  has unique decomposition  $a * b$ . The groups  $A$  and  $B$  are said to be complements in  $G$ . And if  $e$  is the identity element in  $G$ , one can write:  $A \cap B = \{e\}$ , with  $G = A * B$  [13].

### Example

Let  $G$  be a cyclic group of order 6 generated by an element  $x$ . So,  $G = \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5\}$ , where  $x^6 = e$ . The following two subgroups of  $G$  are respectively of order 2 and 3:

$$A = \langle x^3 \rangle = \{e, x^3\}, B = \langle x^2 \rangle = \{e, x^2, x^4\}.$$

Then,  $G$  is an inner semi direct product of its subgroups  $A$  and  $B$ .

**Remark:** Since  $A$  is normal in  $G$ , it follows that for every  $x$  in  $G$ ,  $x * A = A * x$  (the left coset generated by  $x$  equals the right coset generated by  $x$ ). For each element  $b$  of  $B$ , we know the map:

$f_b : A \rightarrow A$ , such that for every  $a$  in  $A$ ,  $f_b(a) = b * a * b^{-1}$  (action of  $b$  on  $a$ , conjugation by  $b$ ) is an automorphism of group, an element of  $Aut(A)$ . Moreover, the map of the action of  $B$  on  $A$  defined by  $f : B \rightarrow Aut(A)$ , such that for every  $b$  in  $B$ , corresponds the previous map  $f_b$  defines by  $f_b(a) = b * a * b^{-1}$  (action of  $b$  on  $a$ , conjugation by  $b$ ) is a homomorphism of groups.

Given two elements of  $G$ ,  $x = a_1 * b_1$ ,  $y = a_2 * b_2$ , we have  $x * y = a_1 * b_1 * a_2 * b_2 = a_1 * (b_1 * a_2 * (b_1)^{-1}) * b_1 * b_2$ , and the term  $a_1 * (b_1 * a_2 * (b_1)^{-1})$  is still an element of  $A$  because  $A$  is normal, and  $(b_1 * b_2)$  is in  $B$ . So,  $B$  acts by conjugation on  $A$  through the homomorphism map  $f$  describes above.

The internal semi product  $G$  can be thought as the product of  $A$  and  $B$  with respect to the homomorphism  $f$ . We write:  $G = A \ltimes_f B$ . It seems reasonable to define a semi direct product on two subgroups  $A$  and  $B$ , as long as the group  $B$  acts on  $A$  through some morphism of group  $f$ .

### 4.3.3 External semi direct product

The notion of external semi product generalizes the concept of direct product of groups. It uses the notions of an automorphism of a group and of the group of automorphisms. Let  $f : G_1 \rightarrow G_2$  be such that  $f(a * b) = f(a)f(b)$  is a group homomorphism. For example, the map from  $Z$  to  $Z_2$  that assign the value 1 to odd numbers and 0 to even numbers is a homomorphism. In this case, for instance,  $f(5) = 1, f(4) = 0, f(7) = 0$ . We have  $f(5 + 7) = f(12) = 0$ , but  $f(5) + f(7) = 1 + 1 = 0$ . So,  $f(5 + 7) = f(5) + f(7)$ . Here we use  $+$  to denote the group operations in  $Z$  and  $Z_2$ . When  $G_1 = G_2$ ,  $f$  is called a group automorphism. For example, for the group of integers  $Z$  the map  $f(z) = -z$  is an automorphism. Given two automorphisms  $f_1$  and  $f_2$  of a group  $G$ , the composition  $f_1 \circ f_2$  is again an automorphism. Thus the set of all automorphisms on  $G$  denoted by  $Aut(G)$  is itself a group. In the case of the group  $Z$  there is only one non-trivial automorphism, so that the group of automorphisms of  $Z$  is  $Z_2$ . Now, let  $(A, \bullet)$  and  $(B, \blacklozenge)$  be two groups (not even related). Let  $f$  be a homomorphism map from  $B$  to the group  $Aut(A)$  of all automorphisms of  $A$ . We define the external semi direct product  $G$  of  $A$  and  $B$  with respect to  $f$  as the Cartesian product set  $A \times B$  with the following properties:

- 1- The underlying set of  $G$  is the cartesian product  $A \times B$ .
- 2- The group operation is defined as follows:  $(a, b) * (x, y) = (a \bullet f(b)(x), b \blacklozenge y) = (a \bullet f_b(x), b \blacklozenge y)$ , where the element  $f(b)(x)$  is the image of  $x$  by the map  $f$  under the action of  $b$ .
- 3- The identity element is  $(e_B, e_A)$
- 4- The inverse of an element is  $(a, b)^{-1} = (f_b^{-1}(a^{-1}), b^{-1})$ .

We write  $G = A \ltimes_f B$ ; it is the smallest group generated by copies of  $A$  and  $B$ . The order of  $G$  is the product of the orders of  $A$  and  $B$ . A special case is obtained by choosing the homomorphism map  $f$  as the trivial map  $f_b(a) = a$ . In that case,  $(G, *)$  is simply the direct product of  $(A, \bullet)$  and  $(B, \blacklozenge)$ .

### Example

We let and the group operation  $(*)$  is the matrix multiplication. Let  $A$  be the subgroup of  $G$  that consists of diagonal matrices, and let  $B$  denote the subgroup of  $G$  that consists of diagonal matrices.  $B$  is clearly normal in  $G$ , this is because the two sets below coincide when  $a = 1$ .

Moreover, the intersection of  $A$  and  $B$  reduces to the identity element. We need to show that  $G = B * A$ , that is any element of  $G$  is uniquely decomposed as the product of one element in  $A$  and another element in  $B$ .

So,  $G = B \ltimes A$ . More abstractly, let's define a homomorphism map  $f$  as follows:  $f : A \rightarrow \text{Aut}(B)$ , such that  $f_a(b) = a * b$  (additive group of the group of automorphisms of  $B$ ).

Since  $B$  is isomorphic to  $R$  and  $A$  is isomorphic to  $R^*$ ,  $G$  is also isomorphic to the external semi product  $G = R \ltimes_f R^*$ . Another way to view  $G$  is as a group of affine transformations  $T(x) = ax + b$  where the group operation  $*$  is the composition of transformations. The affine transformation  $T$  does two things to  $x$ . First, it scales  $x$  by the non-zero number  $a$ , and second it translates by  $b$ . We define the two subgroups, the translation group  $B = \{x \rightarrow x + b\}$  and the scaling group  $A = \{x \rightarrow ax, a \text{ is non-zero}\}$ . Then,  $G$  is the semi product of  $B$  and  $A$ .

### 4.3.4 Wreath product

Consider two groups  $A$  and  $B$ , where  $A$  is called the active or base group and  $B$  is called the passive or states group. The wreath product of  $A$  and  $B$  can be constructed as follows. First, we form the direct sum of copies of the group  $B$ , one copy for each element  $a$  in  $A$ .

Using the pointwise multiplication of the direct sum of copies of  $B$  indexed by set  $A$ . We define  $F = (\oplus_{a \in A} B)$ , as the group of configurations that take values different from the identity of  $B$  at finitely many points.

We can now define the wreath product  $G$  denoted by  $G = A \wr B$ , as the semi-product of the groups  $A$  and  $F = (\oplus_{a \in A} B)$ . The action of the group  $A$  on the direct sum is the standard Cayley action of  $A$  on  $A$ . For example, given a vector  $f$  in group  $F$ , the element  $a$  in group  $A$  induces a permutation of the components of the vector  $f$ , by taking the component in position  $x$ , to the position  $a * x$ , for each  $x$  in the set  $A$ .

### Example

As a simple example for wreath product, the group  $\mathbb{Z}_2 \wr \mathbb{Z}_3$  which is the semi-direct product of  $(\mathbb{Z}_2)^3$  and  $\mathbb{Z}_3$ . To which we associate the automorphism  $\phi$  from  $\mathbb{Z}_3$  to  $Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$  is such that  $\phi(1)$  is the cyclic permutation of  $(a, b, c)$  into  $(b, c, a)$ . Hence, it can readily be verified that the composition  $[(0, 1, 1), 1] * [(1, 0, 0), 1] = [(0, 1, 0), 2]$  belongs to the group  $\mathbb{Z}_2 \wr \mathbb{Z}_3$ .

### From Wreath product to lamplighter

Wreath products provide a way of constructing interesting examples of groups [6]. As a convention, in  $G = A \wr B$ , the active group  $A$  is always on the left and the passive group on the right. Moreover, we assume the active group  $A$  is cyclic and commutative or abelian. When  $A$  is finite,  $A \wr B$  is a circular group sometimes referred to as the lamplighter group. When  $A$  is infinite, the group  $A \wr B$  is called linear group. The Lamplighter group is thus a special wreath product. It is the restricted wreath product  $\mathbb{Z}_2 \wr \mathbb{Z}$ . This group that we will denote by  $L_2$  is defined as the semi-direct product: of the infinite direct sum  $(\dots \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots)$  with  $\mathbb{Z}$ . For a given element  $(a, f)$  in the product group  $G$ , the component  $a$  is the lamplighter position, whereas the component  $f$  is the lamp configuration. It should be noted that each element of the infinite direct sum above has finitely many non-zero components.

## 4.4 Lamplighters over $\mathbb{Z}$

### 4.4.1 Definition

The lamplighters group over  $\mathbb{Z}$  is defined as the wreath product  $\mathbb{Z}_n \wr \mathbb{Z}$ . Each element of  $\mathbb{Z}$  gives the position of the lamplighter and the integer  $n$  denotes the possible states of the lamplighter. When  $n = 2$ , the lamplighter is a group defined by a 2-states automaton. This means the lamplighter can be in two states: ON/OFF.

### 4.4.2 Dynamic of the lamplighters

We consider a dynamical system with a static object and several types of modifications that can be dynamically performed upon this object. Suppose a road is lined with streetlamps in either direction, indexed by the integers. We call such a structure the lampstand. A finite number of lamps along the lampstand are illuminated while all others remain off, and a lamplighter stands at one lamp. We may dynamically modify this object by allowing the lamplighter to walk in either direction and change the state of the lamp at which is standing. This system can serve as a visual model for several groups. This group is referred to as the Lamplighter Group [24, 16].

#### Example

Let  $(L)$  denote the set of all possible configurations as described above. An arbitrary configuration in  $(L)$  could be represented symbolically, as follows. Let  $c_0 \in (L)$  be the configuration with lamps at indices  $-2$  and  $1$  illuminated, and the lamplighter positioned at lamp at index  $-1$ . The picture below shows  $c_0$ , where a closed (**dark**) circle represents an illuminated bulb, an open circle (**white**) represents an unilluminated one, and the position of the lamplighter is indicated by an arrow pointing at the index corresponding to the lamp. Let  $\varepsilon \in L$  be the configuration in which all lamps are unilluminated and the lamplighter stands at index  $0$ . We will refer to this configuration as the empty lampstand.

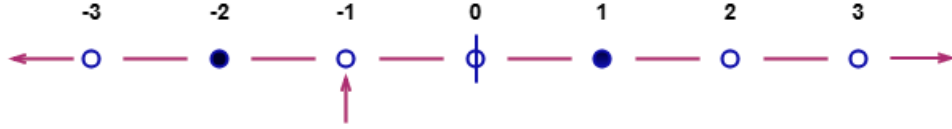


Figure 4.5: Pictorial of a lamplighter system

The dynamics of the system are meant to mimic the movements of the lamplighter about the lampstand and the changes it makes to the state of individual lamps. The following three tasks describe what the lamplighter may perform on any arbitrary configuration in  $(L)$ .

- Switch on/off the lamp at which the he currently stands.
- Move one lamp to his right.
- Move one lamp to his left.

We can notice that by performing tasks (2) or (3), a certain number of times,  $k \in \mathbb{Z}$  times, the lamplighter achieves the task of having moved  $k$  lamps to either the right or left respectively [24].

**Tasks and lamplighter dynamic:** Let  $T = \{\alpha, \tau, \tau'\}$  be a set where each element in  $T$  is a function from the lamp set  $(L)$  to itself such that, given  $x \in (L)$ , we define  $\alpha(x)$  is the configuration resulting from having performed task (1) on  $x$ . The configuration  $\tau(x)$  results from having performed task (2) on  $x$ . Finally,  $\tau'(x)$  is the configuration resulting from having performed task (3) on  $x$ .

As applying a function in  $T$  to a configuration in  $L$  is equivalent to performing its corresponding task on this configuration, we will use these terms interchangeably.

Revisiting our example configuration  $c_0$ , we indicate  $\alpha(c_0)$ ,  $\tau(c_0)$ , and  $\tau'(c_0)$  on Figure 4.6 below [24].

It is possible to obtain any configuration in  $(L)$  by applying a finite sequence of tasks to the  $\varepsilon$  configuration, the empty lampstand. For example, in configuration  $c_0$ , the con-

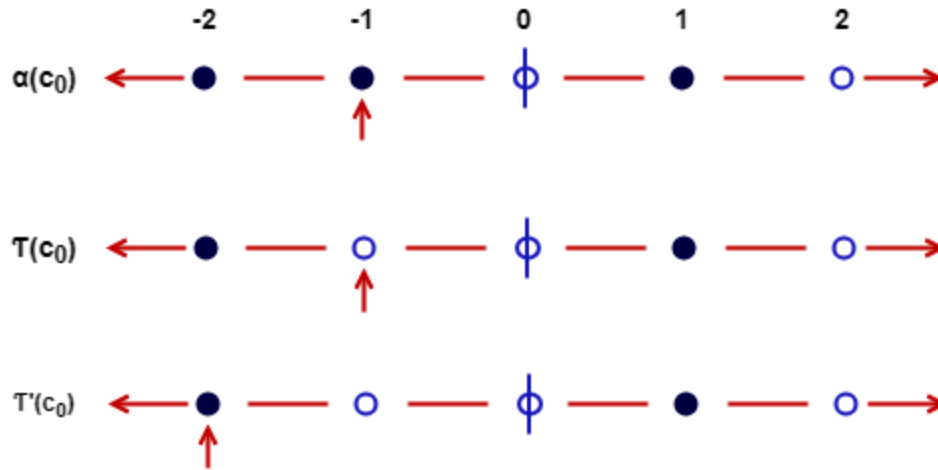


Figure 4.6: Applications of tasks on  $C_0$

figuration with lamps at indices  $-2$  and  $1$  illuminated and the lamplighter positioned at lamp at index  $-1$ . We can get to this configuration using the following sequence of tasks:

- Move the lamplighter two lamps to his left (to index  $-2$ ).
- Switch on/off the lamp at which the lamplighter stands (index  $-2$  on).
- Move the lamplighter three lamps to his right (to index  $1$ ).
- Switch on/off the lamp at which the lamplighter stands (index  $1$  on).
- Move the lamplighter two lamps to his left (to index  $-1$ ).

Such a sequence is not unique. We could also use this sequence of tasks to achieve the same end:

- Move the lamplighter one lamp to his right (to index  $1$ ).
- Switch on/off the lamp at which the lamplighter stands (index  $1$  on).
- Move the lamplighter three lamps to his left (to index  $-2$ ).
- Switch on/off the lamp at which the lamplighter stands (index  $-2$  on).
- Move the lamplighter one lamps to his right (to index  $-1$ ).

## 4.5 New Generators

### 4.5.1 Generators of the Lamplighters group

This part is concerned with the generators of the group  $G$ . A generating set  $S$  of the group  $G = A \wr B$  is a subset of  $G$  such that every element of  $G$  can be expressed as the combination (under the group operation) of finitely many elements of  $S$  and their inverses. So, if  $S$  is a subset of  $G$ , then we use the notation  $\langle S \rangle$ , to denote the subgroup generated by  $S$ . That is the smallest subgroup of  $G$  containing every element of  $S$ . When  $G = \langle S \rangle$ , then  $S$  generates  $G$ , and the elements in  $S$  are called generators or group generators. If  $S$  is the empty set, then  $\langle S \rangle$  is the trivial group  $\{e\}$ .

When  $S$  contains a single element,  $\langle S \rangle$  is usually written as  $\langle x \rangle$ . In this case,  $\langle x \rangle$  is the cyclic subgroup of the powers of  $x$ . For finite groups, it is also equivalent to saying that  $x$  has order  $|G|$ .

We define  $T^a[f(x)] = f(x - a)$ . So that the group multiplication in  $G$  is:

$$(a_1, f_1) * (a, f_2) = (a_1 + a, f_1 * T_1^a(f_2))$$

If  $e$  is the identity of the group  $B$ , and  $\{\}$  represents the empty configuration, that is  $\{\}(a) = e$ , for every element  $a$  in  $A$ , then the pair  $(0, \{\})$  is the identity element of  $G = A \wr B$  [17].

### 4.5.2 Example Standard generators

The group  $G$  possesses many standard generators, including  $(-1, \{\}), (+1, \{\})$  referred to as the walk generators.

### 4.5.3 Example switch generator of $G$

Another generator of  $G$  is defined as followed. Let  $K$  denote a symmetric generating set of group  $B$ . And let  $\delta(a, b, x)$  be a configuration such as:  $\delta(a, b, x) = b$ , if  $x = a$  and

$\delta(a, b, x) = e$ , if  $x$  is not equal  $a$ . Then  $(0, \delta(0, b))$ , for  $b$  in  $K$  is a generator of  $G$ . this generator is called switch generator of  $G$ .

Using the property of the group multiplication in  $G$  (above):

we have:

$$(a, f) * (\pm 1, \{\}) = (a \pm 1, f)$$

$$(a, f) * (0, \delta(0, b)) = (a, f * \delta(a, b))$$

The first identity implies that the action of the walk generator maintains the lamp configuration  $f$  static, as it moves the lamplighter along  $A$  one step to the left  $(-1)$  or to the right  $(+1)$ .

The second identity implies that the action of the switch generator maintains fixed or static the position  $a$  of the lamplighter in group  $A$ , but the state of the lamp at location  $a$ , changes as it moves from  $f(a)$  to  $f(a)*b$ .

#### 4.5.4 Example Walk switch generator of $G$

Yet another generator of  $G = A \wr B$  is as follows. For any  $b$  in  $B$ ,

- Let  $B^+ = \{(1, \delta(1, b))\} = \{(1, \{\}) * (0, \delta(0, b))\}$
- Let  $B^- = \{(-1, \delta(0, b))\} = \{(0, \delta(0, b)) * (-1, \{\})\}$
- Define set  $BB$ , as the union  $(B^+ \cup B^-)$ .

By the group multiplication in  $G$ , it follows that:

- $(a, f) * (1, \delta(1, b)) = (a + 1, f * \delta(a + 1, b))$

This identity implies that the lamplighter moves from location  $a$  to next  $a + 1$ , and the configuration values changes from  $f(a + 1)$  to  $f(a + 1)*b$ . we can think of the generators from  $B^+$  as walk-right switch, whereas the generators from  $B^-$  are walk-left switch.

To see this graphically, let's denote the lamplighter configuration by  $\Phi$ , and rather than pointing the lamplighter at the position  $(a)$ , we will rather use a sliding window of width 2 over the positions  $a$  and  $(a + 1)$  which distinguishes the edges between  $(a)$  and  $(a + 1)$ . Figure 4.7 represents Lamplighters window slider [20, 17].

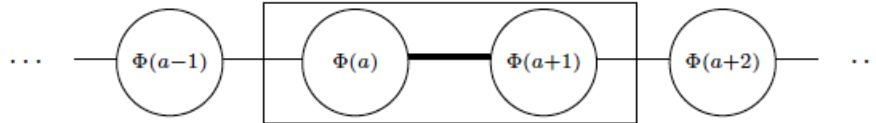


Figure 4.7: Lamplighters window slider

The multiplication by an element  $(1, \delta(1, b))$  is a walk-right switch generator which corresponds to shifting the slider one position to the right and multiply the state at the intersection of the old and new slider by  $b$ , as shown in Figure 4.8 .

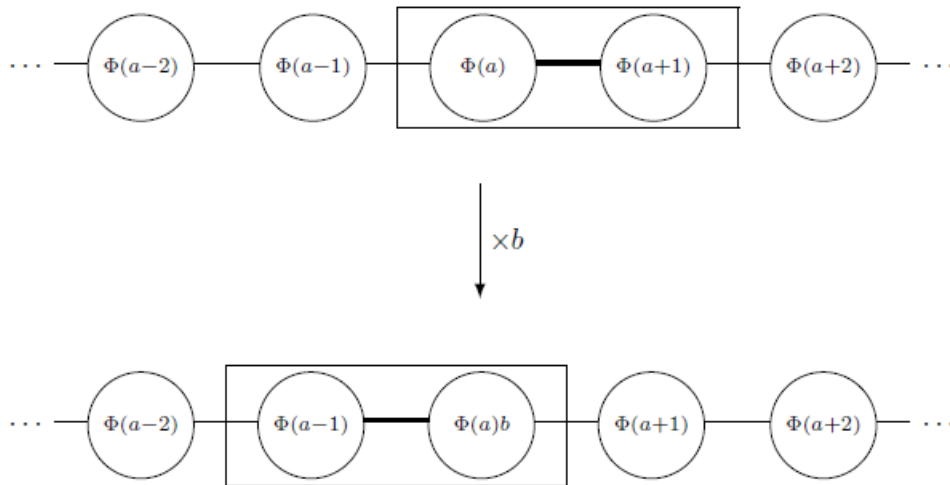


Figure 4.8: Walk-rightswitch generators.

Symmetrically, the multiplication by an element  $(-1, \delta(0, b))$  is a walk-left switch generator which corresponds to shifting the slider one position to the left and multiply the state at the intersection of the old and new slider by  $b$ , as shown in Figure 4.9 below.

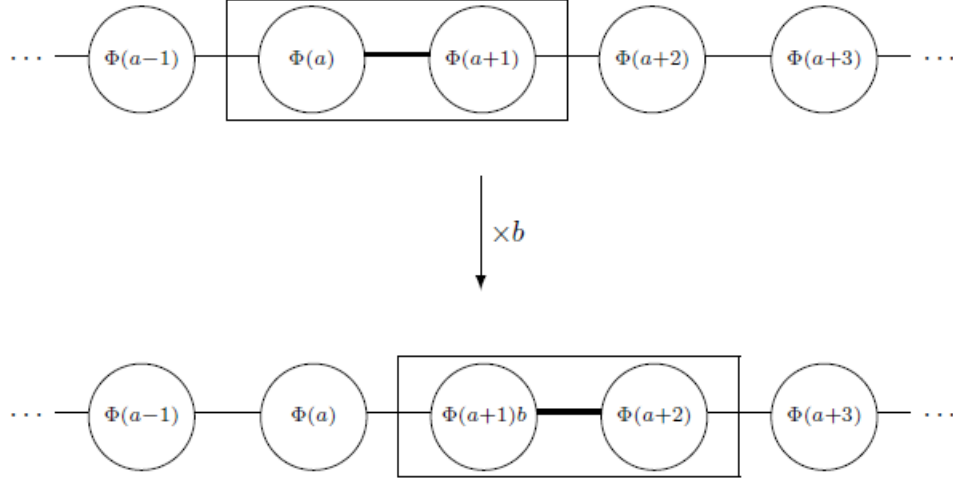


Figure 4.9: Walk-rightswitch generators.

#### 4.5.5 Generator of the lamplighter group $L_2$

**Lemma 2.** *The lamplighter group  $L_2$  is generated by two elements, one of order 2 and other of infinite order.*

The geometric representation of the  $G$  can shed light on the use of the term lamplighter. For example, let  $t = (\{\}, 1)$  and  $a = (\{0\}, 0)$  two elements of  $L_2$ . The action of  $\mathbb{Z}$  on  $A$  has the effect of adding 1 to each coordinate. Thus,  $t * a = (\{1\}, 1)$ . And in general,  $t^n * a = (\{n\}, n)$ , and also  $t^n * a * t^{-n} = (\{n\}, 0)$ . Thus, any arbitrary element of the group  $L_2$  can be written as the product:

$$((n_1, n_2, n_3, \dots, n_m), k) = (t^{n_1} * a * t^{-n_1}) * (t^{n_2} * a * t^{-n_2}) * (t^{n_m} * a * t^{-n_m}) * t^k$$

The set  $(a, t)$  is a generator of the lamplighter group  $L_2$ . To illustrate, consider the element  $z = t^3 * a * t^{-2} * a * t$ . Geometrically, the element  $z$  consists of the following algorithm:

- Move the lamplighter 3 units right
- Light lamp at  $x = 3$

- Move the lamplighter 2 units left
- Light lamp at  $x = 1$
- Move the lamplighter 1 unit right

Another equivalent algorithm for the same element is:

- Move the lamplighter 1 unit right
- Light lamp at  $x = 1$
- Move the lamplighter 2 units right
- Light lamp at  $x = 3$
- Move the lamplighter 1 unit left

The equivalence of the two algorithms indicate that the elements  $t^3 * a * t^{-2} * a * t$  and  $t * a * t^2 * a * t^{-1}$  of  $L_2$  are identical.

## 4.6 Lamplighters and slider graphs

### 4.6.1 Word length in lamplighters $L_2$

For  $z \in L_2$ , we need a formula that will allow us to find the length of the shortest word  $w$  in  $\{a, t, t^{-1}\}$  such that  $z = w$  when  $w$  is considered as a product. In addition, to specifying the length of such a word, the result of this section will yield the word itself. Consider the following example. Let  $g = (\{2, -1, 1, 3\}, 1)$  in the lamplighter  $L_2$ .

We have seen that it is possible to express this element as a word in  $\{a, t, t^{-1}\}$  by multiplying our generators by the identity as follows: for each index at which a bulb is illuminated in  $g$ , do the following procedure:

- Move the lamplighter to that index,

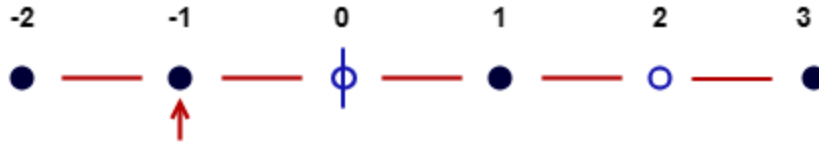


Figure 4.10: lamplighter

- Change the state of the bulb at that index,
- Return the lamplighter to the origin,
- Move the lamplighter to the index of his final destination.

We may express this process as the following word.

$$Z = t^{-1}(t^{-1}at^{-1})(t^{-3}at^3)(t^{-2}at^2)(t^{-1}at^1)$$

This word has length  $d = 19$ , and we may reduce it by combining the exponents of adjacent generators and obtain a word in which the lamplighter does not return to the origin in between stops at other indices:  $at^{-4}at^5at^3at$ .

This last word corresponds to stopping the lamplighter at indices 1,  $-2$ , 3, and  $-1$  (in that order) to illuminate each bulb, and then finally stopping at his final destination  $-1$ . Summing the exponents of the reduced form of this word, we see that this word has a length of 17.

## 4.6.2 Lamplighter over cyclic groups

### Slider graphs

The use of a slider or sliding window (instead of a pointer) can be mathematically convenient. More specifically, we consider a slider of width 2 that covers one alphabet symbol on each side of the pointer. Using this approach, the De Bruijn graph of word length  $n$  over the alphabet  $A$  and denoted  $B_n^A$  by is now represented by the complete circular slider graph of span  $n$  over set  $A$ .

This graph is an oriented graph with vertices and arrows with the following properties. The vertices are the circular words of length  $n$  (also called slider pointed) in the alphabet set  $A$ , and the arrows are the slider transitions. Each slider transition moves the slider one position clockwise and eventually change the letter at the intersection position of the old and new slider windows.

Slider graphs can then be generalized as a subgraph of the complete circular slider graph. Variants of the slider graphs family include periodic slider graphs and transversally Markov circular slider graphs. Periodic slider graphs are invariant with respect to the circular shift operator  $S$ , whereas transversally Markov circular slider graphs impose a condition on the change  $a_1 \rightarrow a_{n+1}$  that is the arrow  $(a_1, a_{n+1})$  must be an arrow of a directed graph over the alphabet set  $A$ .

**Theorem 4.** *Given a finite cyclic group set  $A$  and alphabet set  $B$ . We consider a generating set  $B'$ . For the group  $A \wr B$  defined as follows [16] [17]:*

$$B_+ = \{(1, \delta(1, b))\}_{b \in B}$$

$$B_- = \{(-1, \delta(0, b))\}_{b \in B}$$

$B' = B_+ \cup B_-$ . We also defined the group right action:  $b(a, \phi) = b(0, \phi)(a, \phi)$  Then, the Schreier digraph  $Sch(B^{|A|}, B_+)$  of the action with respect to the generating set  $B_+$  is isomorphic to the De Bruijn graph of span  $|A|$  over the alphabet set  $B$ .

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