ON THE MODELLING AND THE OPTIMAL CONTROL OF SYSTEMS WITH FEEL

by

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Submitted to the Department of Electrical Engineering
in partial fulfilment of the requirements for
the degree of Doctor of Philosophy

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1971

ABSTRACT

State space models of a'One Dimensional Feel system' and a 'Multi Dimensional Feel system' are obtained, and a detailed numerical analysis is carried out for the one dimensional case. A feel system always functions in conjunction with a manual control system. Hence a description for the human operator in the combined man-machine system is also obtained.

The optimisation problem considered in the design and operation of the feel systems is equivalent to that of tracking, and is solved by first converting it into an output regulator problem. A basis has been established for comparing the force reflecting characteristics of the various feel systems.

Effects of variations of the feel matrix, the weighting matrices and the intensity of the disturbances, and the signal transmission time delay on the performance of the system are also investigated.

ACKNOWLEDGEMENT

The author is very grateful to his supervisor, Professor G.S. Glinski, for having suggested the area of this dissertation and for his constant encouragement and advice during the investigation.

No thanks can express his gratitude to Dr. N.U. Ahmed for the kind words, stimulating discussions and critical comments without which this thesis would not have been possible.

Thanks are also due to Mme L. LeBlanc for her excellent typing.

The author also wishes to gratefully acknowledge the financial assistance received from the authorities of the Canadian Commonwealth Scholarship and Fellowship plan and the Indian Institute of Technology, New Delhi, India.

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CHAPTER I

INTRODUCTION

1.1 General

By definition, the conventional servomechanism is primarily a position transmitter; the output position is said to reflect the position of the input. Certain types especially the fast-response group, whose output speed is proportional to input speed, can also transmit velocity information, and this group is broad enough to satisfy most of the needs. However, a new era of servo usage has developed in which not only the position and the velocity but also the force between two mechanically isolated points is to be transmitted. The servo system for this application must be bilateral (see Fig 1.2) in contrast to the more common unilateral system (see Fig 1.1), permitting the transmission of force from the operator, to the load or 'slave', and simultaneously from the load back to the operator or 'master'. A servomechanism with these capabilities is said to be 'force reflecting'.

Force reflecting servomechanisms were originally developed along with and for use in the electrically connected master slave manipulators at the Argonne National Laboratory, U.S.A. A detailed discussion about their principle of operation and design can be found in [1] , [5] . Since the output power at the slave end can be made considerably larger than the input power at the master end, these servos, of late, found potential applications in the development of 'HumanExoskeletons' or 'Man Amplifiers' and a number of other cybernetic devices[8] . The principle of force reflection can also be found in the power steering of automobiles, ships and aircrafts. A discussion as to how the aerodynamic force feedbacks (or control feel) can be taken into consideration in the design of aircraft control systems is given in [6], [7] .

Systems containing force reflecting servomechanism as subsystems are said to be systems with a sense of feel at the input or simply 'feel'

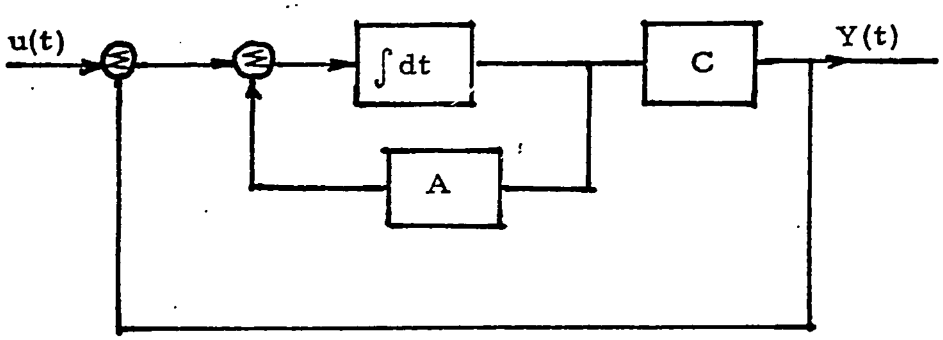

Fig 1.1 A Unilateral Servo Mechanism

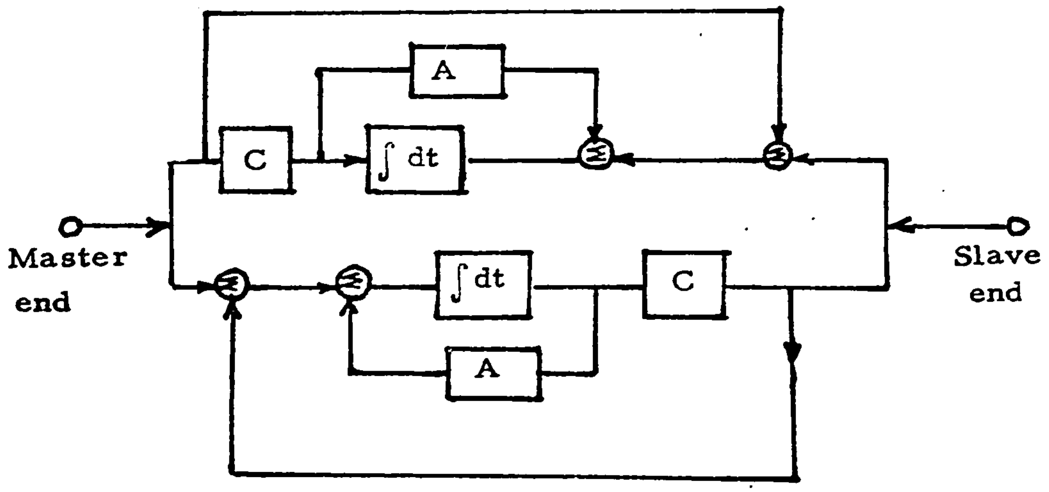

Fig 1.2 A Bilateral Servomechanism

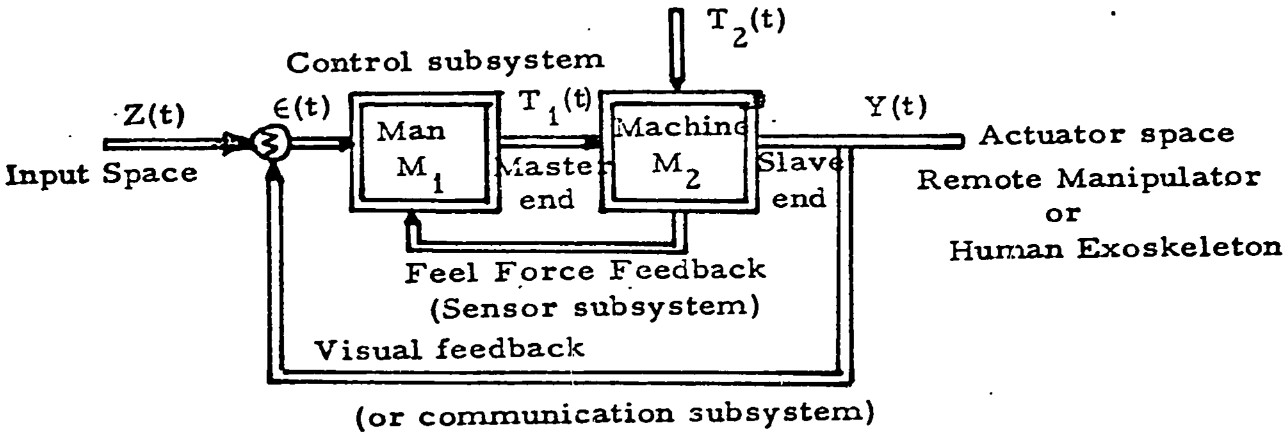

Fig 1.3 A Block Diagram representation of the combined Man-Machine System.

systems (eg. Remote Manipulators, Human Exoskeletons etc.) In this thesis, the term 'One Dimensional Feel System'[1] is used to refer to the force reflecting servomechanism. The term 'Multi Dimensional Feel System'[2] is used to refer to the feel system constructed by interconnecting several of the one dimensional feel systems. The feel systems always function in conjunction with a manual control system. They help the operator to project his inherent dexterity not only across the distance but through the physical barriers as well. A block diagram representation of the man machine system formed by combining the human operator and the multi-dimensional feel system is shown in Fig 1.3. The most generalised term used to refer to these man machine systems is 'Teleoperator'[8]. A generalised schematic of a teleoperator is given in Fig 1.4.

1.2 A Review of the Past Work.

Several attempts have been made in the past to obtain a mathematical description of a force reflecting servomechanism (or a one dimensional feel system), and to study its performance. For instance, the research sponsored by the Argonne National Laboratory, had resulted in the works of Chalmers, Arzbaecher, Schimidt and Thompson. A brief review of these is the following.

Chalmers, in [10], applies certain features of electrical network theory to force reflecting servomechanisms, by using electro-mechanical anologies. He showed that under certain conditions, this servo, can be represented by a passive two port mechanical (analogous electrical) network as shown in Fig 1.5, and therefore stable for all passive load conditions. Burnett[15], has also written a brief review of this network representation technique.

Arzbaecher, in the first part of his work[12], investigated the effects of positive and negative feedbacks on the behaviour of servomechanisms. In a major section devoted to the force reflecting servomechanisms he showed the possibilities of improving the sense of the feel by using force feedbacks, which effectively reduce the inertia and the friction of the system. He also

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*) A better terminology would have been 'single output' and 'multiple output' Feel systems.

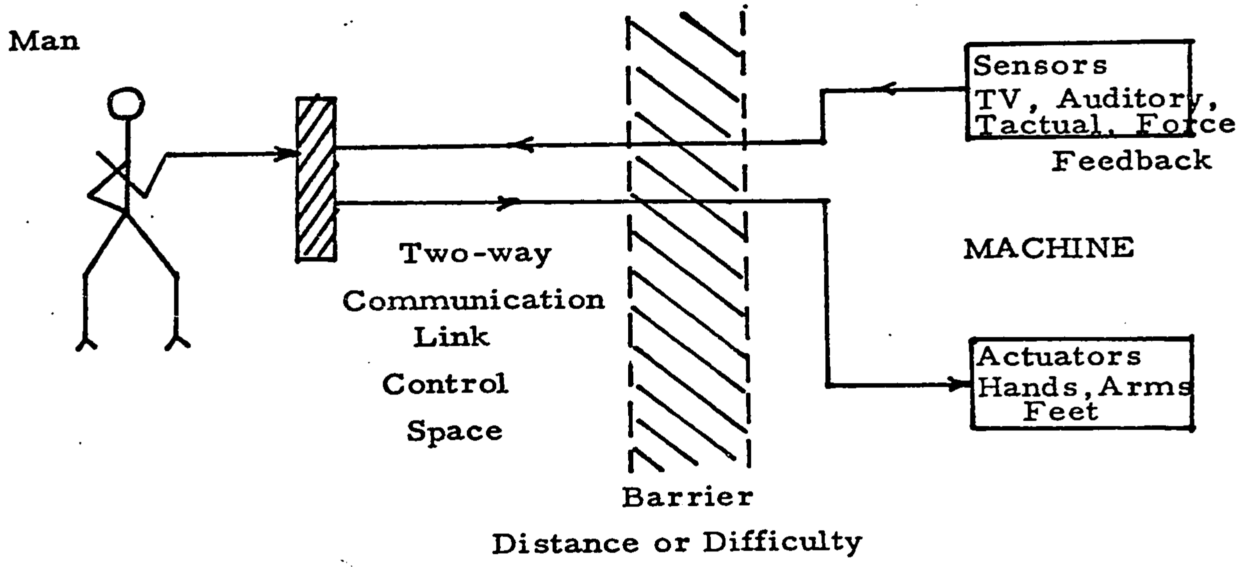
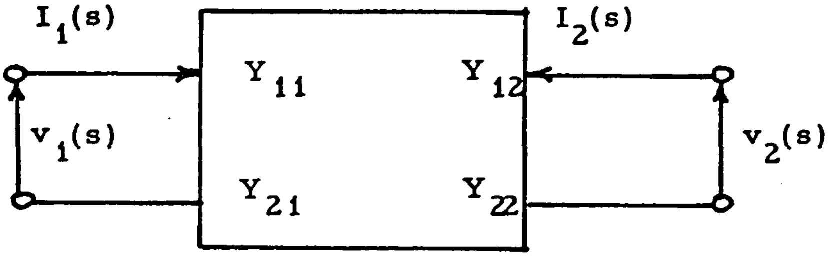
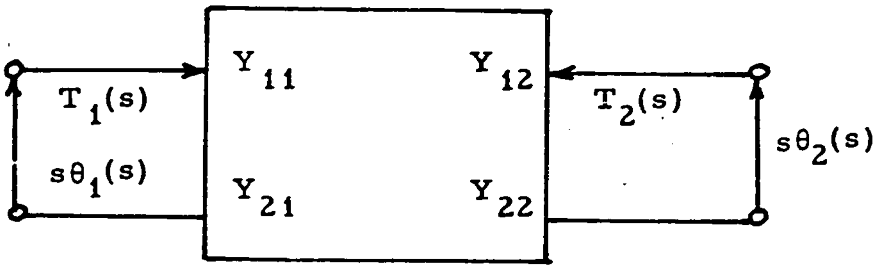


Fig 1.4 A Generalised schematic of a Tele-operator, incorporating dexterous actuators in the actuator space. The barrier between the control and operator space may be distance, a hostile environment or sheer magnitude (weight etc) of the task to be done.



A. Electrical Network.



B. Mechanical Network.

Fig1.5. A Network representation of the Force Reflecting Servomechanism.

developed certain stability criteria for these servos based on network theory stability concepts. Some preliminary work on inertia and friction reduction was done by Schimidt[14].

Thompson, in his work[13], investigated, the effects of the signal transmission time delay that results when the master and the slave ends are separated by long distances, on the stability of this servo. Remote manipulation with signal transmission time delay was first extensively studied by Sheridan and Ferrel[16]. Their research has been mainly concerned with the effects of this delay on the ability of the operator to perform certain manipulative tasks. The manipulator used in their experiment was a servo driven master slave positional system, with two translational motions and one grasping motion, and there was no force feedback. But, when the system was slightly modified, incorporating tactile force feedback in the grasping motion, instabilities occurred for certain values of the force feedback gains. This suggested that the signal transmission time delay may have a significant effect on the stability of the system. In [13] , Thompson, proved analytically as well as experimentally through extensive analog simulations, that for a given set of the gains of the transmission channels there exists a specific time delay and hence a distance of separation of the master and the slave ends, for the servo to be stable. The analytical study is based on a stability criterion for the time lag systems due to Pontryagin with suitable modifications.

Thus the long distance of operation of the force reflecting servomechanisms is restricted. In situations, wherein, it is necessary to have a distance of separation greater than that is possible with this arrangement, the problem can be overcome by eliminating the force feedbacks and replacing either completely or partially the human operator by a digital computer. Automatic manipulation was first attempted by Ernst[36]. He developed a computer controlled mechanical hand with a sense of touch that could 'feel' about to find blocks, stack them, and put them in a box, coping with a variety of disturbances and uncertainties. The M.A.C [37] project at M.I.T. has recently extended the work of Ernst to include visual inputs and to develop a hand eye system capable of manipulating objects. An example of a computer-manipulator system, which has limited intelligence and is intended to assist rather than replace an operator is 'The Supervisory Controlled Remote Manipulator'. For this system, Whitney[18] developed a state space model manipulative tasks. He showed that tasks such as 'positioning the jaw' or 'moving an object through a crowded environment' may be expressed in terms of discrete states. A state is defined to be the configuration of the task site. Position control of computer controlled manipulators was investigated by Krafchik[19] and Pieper[20]. Krafchik obtained an expression for the end point state vector in terms of the machine angles. He then developed an iterative procedure, using Newton-Raphson technique, to obtain a control that will drive the manipulator from its initial configuration to the desired end state. Pieper, developed a mathematical model which allows for a systematic description of the new and existing manipulators. Given the initial hand position and orientation in terms of the machine angles, he developed numerical procedures using the Newton-Raphson and the velocity methods, for determining a set of machine angles that will place the hand at the desired position and orientation.

Force reflecting servomechanisms have been considered as sub-systems of man-machine systems in [17]. Optimal control of feel systems is considered in [17] and [34]. Using some of the known results of optimal control theory a control is obtained in [17], which will force the response of the feel system, to follow a desired path. In [34], a control is obtained, which when applied to a feel system, will transfer an object optimally from one point in the actuator space to another. Criteria have been established in [17], [34] for comparing the force reflecting characteristics of the various feel systems. A procedure is outlined in [17], for designing a feel system which not only has a good force reflecting characteristic but also is compatible with the assumed characteristics of the human operator.

Finally, the changes in the gains of the signal transmission channels and the weighting matrices, affect the performance of the optimal feel systems obtained in [17] , [34] . The change in the system performance can be measured either as a change in the performance functional or as a change in the feel constant. In [35] , partial derivative techniques are used to obtain the sensitivities of the output positions the performance indices and the feel constants of the optimal feel systems obtained in [17]and[34] to variations in the gains of the transmission channels and the weighting matrices. Methods are also suggested in [35] for making use of the sensitivity functions thus obtained, in the design of feel systems.

The above review, by no means, is claimed to be complete. But it is hoped that it presents a good cross section of the available results in the fields of feel systems and remote manipulation. A summary of the objective and the contributions of the research described in this dissertation is given in the following section.

1.3 Objective and Contributions of the Dissertation.

In chapter I, a brief review is given of the previous works in the fields of 'feel' systems and remote manipulation.

In chapter II, the differential equations that describe the dynamic behaviour of a one dimensional feel system are given and a state space model is obtained. The state equations are solved on a digital computer using Fourth order Runga-Kutta numerical integration procedure, and the behaviour of a one dimensional feel system is studied for different feel matrices. The system responses for certain typical feel matrices are found to be basically different from those of a conventional servomechanism. The concept of a multi-dimensional feel system is introduced, and assuming that the cross coupling coefficients are known, a simple state space model is obtained for the multi-dimensional feel system. A multi-dimensional feel system can be combined with the human operator to form a multi-dimensional man machine system (Fig 1.3). Again, assuming that the cross coupling coefficients are known, a state space model is obtained for the human operator in a multiple axis situation.

In chapter III, the optimisation problem connected with the design and operation of the feel system (block M_2, Fig 1.3) is outlined. The optimisation problem is equivalent to that of tracking. Some of the existing techniques of optimal control theory are applied with suitable modifications to obtain a control that will cause the system response to follow a specified trajectory. The given tracking problem is converted into an output regulator problem by augmneting the state space of the original system and by a proper selection of an output vector. The transient solutions of the resulting matrix Riccati equation are utilised for determining the force reflecting characteristic of the system (block M_2, Fig 1.3). The steady state solutions are utilised for obtaining a description of the human operator (block M_1, Fig 1.3). Based on the numerical studies made on a one dimensional feel system, a design procedure is outlined for a multi-dimensional feel system.

Another interesting problem considered in chapter III, is that of transferring the output state (position) Y(t) of the system (block M_2, Fig 1.3), from a given initial state to the desired final state optimally. This required the solution of a two point boundary value problem. The two point boundary value problem is first converted into an initial value problem through the method of Riccati transformation and then solved.

The characteristics of an optimal feel system depend upon the choice of the feel matrix, the weighting matrices and the intensity of the disturbance on the system. Variations in these, affect the system performance. A knowledge of the sensitivity functions helps the designer to improve the quality of the design of the feel system.

In chapter IV, sensitivities of the output positions, performance indices and feel constants of the optimal feel systems obtained in chapter III, to variations in the feel matrix and the weighting matrices, are obtained.

Methods are outlined for determining a suitable combination of a feel matrix and the weighting matrices, using the sensitivity functions.

When the master and the slave ends are separated by long distances, there exists certain signal transmission time delay and is to be taken into account.

In chapter V, state space models are obtained for a one dimensional feel system, and a multi-dimensional feel system having signal transmission time delay. The differential difference equations for the one dimensional case are solved on a digital computer and the system performance is studied (i) for different feel matrices and (ii) for different delays. Methods are suggested for solving the associated optimal control problem.

In general, it is to be stated, that the main object of this dissertation is to apply existing theoretical results to Systems with Feel rather than to obtain new theoretical results.

CHAPTER II

STATE SPACE MODELS OF ONE DIMENSIONAL AND MULTI DIMENSIONAL FEEL SYSTEMS AND THE HUMAN OPERATOR

In this chapter, state space models have been obtained for a one dimensional feel system, a multi dimensional feel system and the human operator. An analysis is carried out for the one dimensional feel system, in order to study its behaviour.

2.1. One Dimensional Feel System

2.1.1 System Description

In this section, a brief description is given of the arrangements of the elements of a one dimensional feel system. The physical principles involved behind the operation and design of the various feel systems are well described and can be found in [3], [6], [8] .

In general, there are two basic ways in which the elements of a one dimensional feel system can be arranged. They are shown in Figs. 2.1 and 2.2. Although the system is completely symmetrical as shown and can operate equally well in either direction, the left hand side is usually referred to as the input'or master' end, and the right hand side as the output or 'slave' end. The arrangement shown in Fig 2.1 contains only one amplifier for mixing the rate and position error signals and for driving the master and the slave end actuators. The output power at the slave end can be made considerably larger than the input power at the master end by choosing a proper output-input force ratio - (a situation very suitable, for describing the principle of operation of G.E.C's 'Exoskeleton' or ' 'Man amplifier'). The arrangement shown in Fig 2.2 is the same as the one used in [13], except that there is no delay in the signal transmission channels. It contains two amplifiers one for driving the master end actuator and the other for driving the slave end actuator.

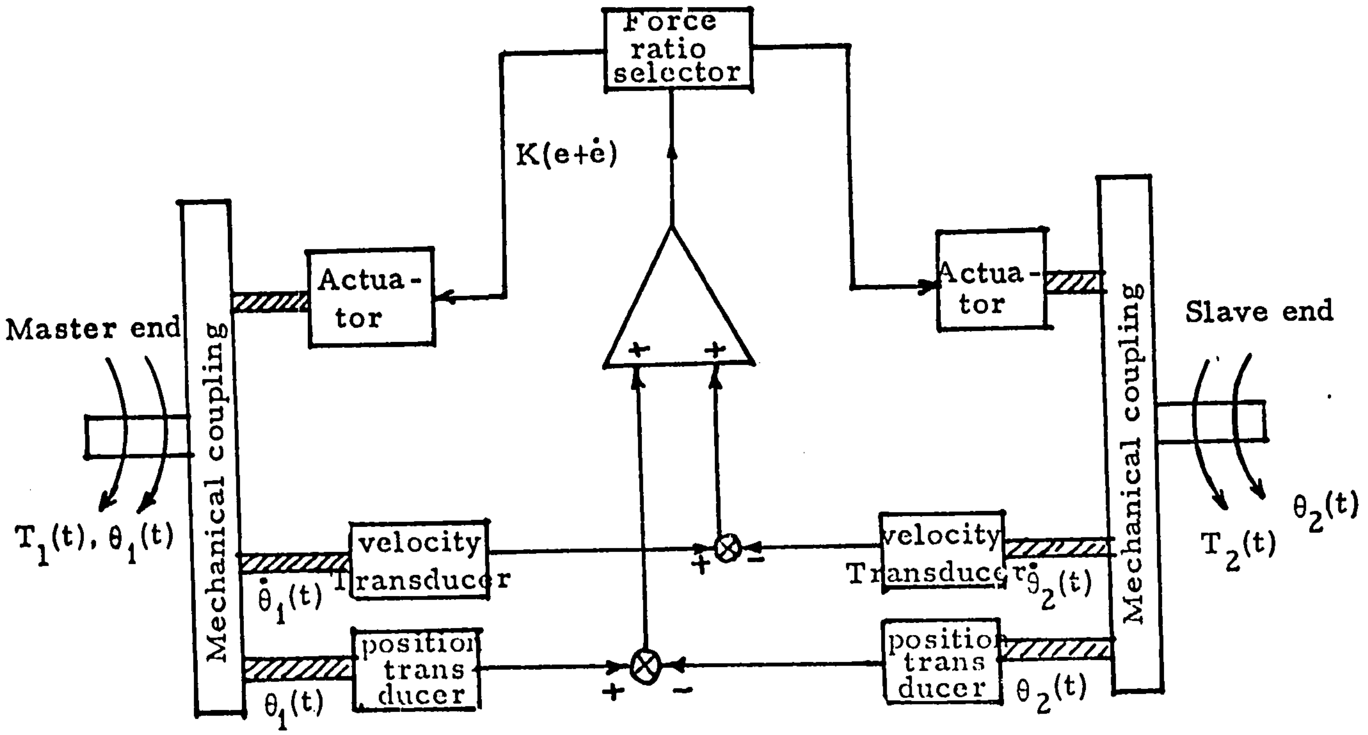


Fig 2.1 An Arrangement of the elements of a one dimensional feel system (using one amplifier).

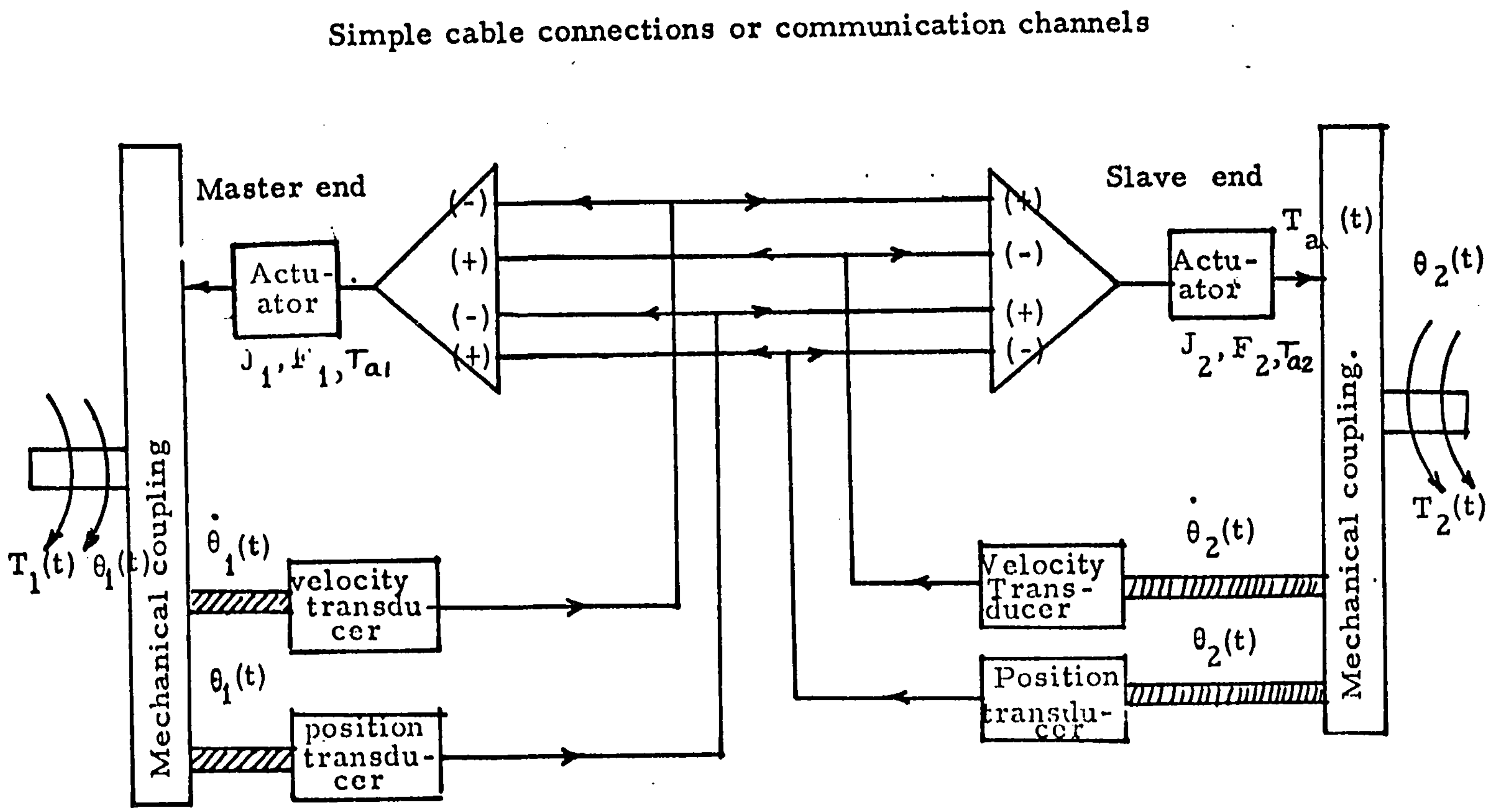


Fig 2.2. An Arrangement of the elements of a one dimensional feel system (using two Amplifiers).

Thefour signal channels transfer the signals between the two separated parts of the system. The power at the slave end can be made larger than that at the master end by suitably adjusting the gains of the signal channels. Both the arrangements, (of Figs 2.1 and 2.2), have two mechanical assemblies, each containing an actuator (can be either electric or hydraulic), a positional transducer, a velocity transducer, and an output shaft, connected together, with a mechanical coupling. The arrangement of Fig 2.2 contains that of Fig 2.1 as a special case, when each of the gains of the signal transmission channels are equal to a single pair of constants. A one dimensional feel system can also be made up of other arrangements of elements[3] or can have more elements such as force or torque transducers[12]. The arrangement shown in Fig 2.2 is considered as the representation of a one dimensional feel system in all the studies made in this thesis.

This system is essentially a combination of two positional servomechanisms, both operating to bring the input and output shafts into positional correspondance. If there is a positional difference, the same positional error signal appears at the input of each amplifier and the actuators tend to rotate in opposite directions, with equal torques in order to correct this error. If this action is prevented by externally applied torques T_1 and T_2, they must also be equal and opposite in order to balance the actuator torques, thus giving torque reflection (or force reflection in a linear motion system). Effectively the input and the output shafts appear to be coupled together by a torsional spring. In addition, the system also tends to produce velocity correspondance between the two shafts by using the velocity error signal. Accurate torque reflection, and position or velocity correspondance are achieved only under steady state conditions of zero or constant velocity because of the time and torque necessary to accelerate the system.

A mathematical model will now be obtained for the one dimensional feel system.

2.1.2 System Equations.

In most applications the components connected to the mechanical coupling shown in Fig 2.2, would rotate at different speeds. In the equations below however, all the quantities are referred to the output shaft, and the gear ratios are included in the gain constants. Therefore in the equations, it appears that each component rotates at the same speed. Also, the subscripts 1 and 2 refer to the input or master (left side) and output or slave (right side) of the system respectively.

The differential equations describing the dynamic performance of a one dimensional feel system are,

$$T_1(t) + T_{a1}(t) = J_1 \frac{d^2 \theta_1(t)}{dt^2} + F_1 \frac{d\theta_1(t)}{dt}$$

$$T_2(t) + T_{a2}(t) = J_2 \frac{d^2 \theta_2(t)}{dt^2} + F_2 \frac{d\theta_2(t)}{dt} \qquad (2.1.1)$$

where

T is an externally applied torque,

T_a is the torque developed by the actuator,

J is the total moment of inertia of the actuator and its gearing,

F is the coefficient of friction of the actuator and its gearing,

θ is the positional output.

The torques developed by the actuators (Fig 2.3 pp.12) depnd upon their positions and velocities and the gains of the signal transmission channels.

$$T_{a1}(t) = -K_{11}\theta_1(t) + K_{12}\theta_2(t) - D_{11}\frac{d\theta_1(t)}{dt} + D_{12}\frac{d\theta_2(t)}{dt}$$

$$T_{a2}(t) = K_{21}\theta_1(t) - K_{22}\theta_2(t) + D_{21}\frac{d\theta_1(t)}{dt} - D_{22}\frac{d\theta_2(t)}{dt} \qquad (2.1.2)$$

where

K_{ij} is the overall torque per unit of positional displacement constant that includes the transducer constant, amplifier gain, gearing ratio and actuator torque constant, and

D_{ij} is the overall torque per unit of velocity constant similar to K.

The first subscript indicates the destination and the second the source of the signal.

2.1.3 State Space Representation.

Let the positions and velocities at the master and the slave ends be chosen as the state variables, so that,

$$x_1(t) = \theta_1(t), \quad x_2(t) = \dot{\theta}_1(t), \quad x_3(t) = \theta_2(t), \text{ and } x_4(t) = \dot{\theta}_2(t) \tag{2.1.3}$$

A state space representation of (2.1.1) is then given by,

$$\dot{X}(t) = A\,X(t) + BT_1(t) + DT_2(t) \tag{2.1.4}$$

$$\text{where, } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_{11}/J_1 & -(D_{11}+F_1)/J_1 & K_{12}/J_1 & D_{12}/J_1 \\ 0 & 0 & 0 & 1 \\ K_{21}/J_2 & D_{21}/J_2 & -K_{22}/J_2 & -\dfrac{(D_{22}+F_2)}{J_2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 0 \end{bmatrix}, \text{and } D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J_2 \end{bmatrix}$$

The only output is the position of the slave so that

$$\theta_2(t) = y(t) = x_3(t) = Cx(t) \quad \text{where } C = [0 \ 0 \ 1 \ 0]$$

The matrix A in (2.1.4) can be written as the sum of two matrices A' and F, i.e A = A' + F

$$\text{where, } F \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ -K_{11}/J_1 & -D_{11}/J_1 & K_{12}/J_1 & D_{12}/J_1 \\ 0 & 0 & 0 & 0 \\ K_{21}/J_2 & D_{21}/J_2 & -K_{22}/J_2 & -D_{22}/J_2 \end{bmatrix} \tag{2.1.5}$$

The matrix F is referred to as 'Feel matrix' in this thesis, since the elements of F determine the force reflecting characteristic of the feel system, and

$$A' \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -F_{1/J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -F_{2/J_2} \end{bmatrix} \qquad (2.1.6)$$

The matrix A' is referred to as'actuator matrix'. The elements of A' are the coefficients in the differential equations (2.1.1) which are dependant on the system actuators.

This system is called a one dimensional feel system here, because there is only one position output. Several of these one dimensional feel systems can then be interconnected to form a multidimensional feel system.

A linear system representation of a one dimensional feel system is given in Fig 2.3. The torques developed by the actuators at the master and slave ends considered as state variable feedbacks. The elements that comprise the feel matrix are the gains of the signal transmission channels. The study of the effects of the changes in these gains on the performance of the system is equivalent to studying the changes in the system performance due to variations in the feel matrix F. This is considered in detail, in chapter IV.

In general $T_2(t)$ (of 2.1.4) can be any time function. It could even be a probabilistic disturbance. However, if $T_2(t)$ is simply a passive load on the system, an assumption later found to be very useful, in order to reduce the computational task, it can be replaced by a linear combination of the state variables in (2.1.4).

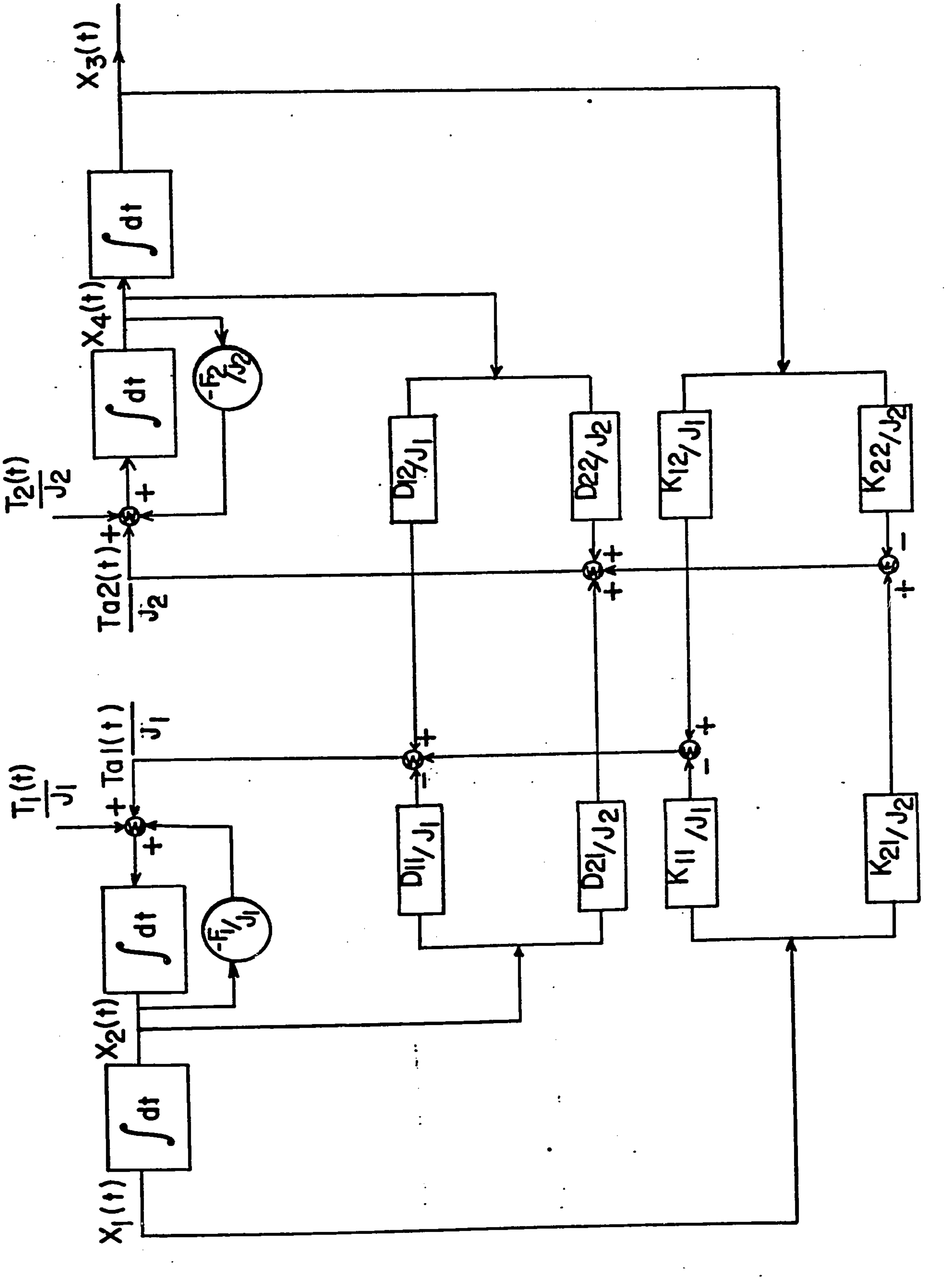


Fig. 2.3 A linear system representation of a one dimensional feel system.

Let the moment of inertia, coefficient of friction and the spring constants of the load be denoted by J_L, F_L and K_L respectively. Then,

$$-T_2(t) = J_L \, d^2 \theta_2(t)/dt^2 + F_L \, d\,\theta_2(t)/dt + K_L \, \theta_2(t)$$

$$= (\alpha/_{1+\alpha}) K_{21} x_1(t) + (\alpha/_{1+\alpha}) D_{21} x_2(t) + \left(\frac{\alpha-\beta}{1+\alpha}\right) K_{22} x_3(t) + \left[\frac{(\alpha-r)D_{22}+\alpha F_2}{(1+\alpha)}\right] x_4(t)$$

$$= S\,X(t) \qquad (2.1.7)$$

$$S = \begin{bmatrix} \left(\frac{\alpha}{1+\alpha}\right) K_{21} & \left(\frac{\alpha}{1+\alpha}\right) D_{21} & \left(\frac{\alpha-\beta}{1+\alpha}\right) K_{22} & \frac{((\alpha-\gamma)D_{22}+\alpha F_2)}{(1+\alpha)} \end{bmatrix} \qquad (2.1.8)$$

The loading factors α, β, and r are defined as follows[13]

$$\alpha = J_L/J_2 , \quad \beta = K_L/K_{22} \quad \text{and} \quad r = F_L/D_{22} \qquad (2.1.9)$$

Substituting (2.1.7) for $T_2(t)$ in (2.1.4), the description of a one dimensional feel system can be reduced to the form

$$\dot{X}(t) = A_1 X(t) + BT_1(t) \qquad (2.1.10)$$

and $\quad Y(t) = C\,X(t)$

where $A_1 = (A' + F - DS) \qquad (2.1.11)$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_{11}}{J_1} & -\dfrac{D_{11}+F_1}{J_1} & \dfrac{K_{12}}{J_1} & \dfrac{D_{12}}{J_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{K_{21}}{J_2(1+\alpha)} & \dfrac{D_{21}}{J_2(1+\alpha)} & -\dfrac{K_{22}(1+\beta)}{J_2(1+\alpha)} & -\dfrac{[D_{22}(1+r)+F_2]}{J_2(1+\alpha)} \end{bmatrix} \quad (2.1.12)$$

When the master and the slave ends are separated by long distances, there exists certain signal transmission delay. It is found [13] to have a significant effect on the stability of the system. Feel systems with signal transmission time delay are considered in chapter V.

Before proceeding to obtain a mathematical model for a Multi-dimensional feel system, let us examine some of the characteristics of a one dimensional feel system.

2.1.4 An Analysis of a One Dimensional Feel System.

2.1.4.1 System with only Positional Feedback.

Consider the one dimensional feel system (2.1.10) with (D_{ij} i, j = 1,2) = 0, and $\alpha = \beta = r = 0$ in A_1 (2.1.12). Let the values of the gains of the positional feedback channels (K_{ij}, i, j = 1,2), the moments of inertia (J_i, i = 1,2), and the coefficients of friction (F_i, i = 1,2) of the master and the slave end actuator shafts be taken as unity. The arrangement of the elements for such a system is similar to that of Fig 2.2, except that it does not contain any velocity transducers.

The system matrix A_1 (2.1.12) is,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \qquad (2.1.13)$$

The feel matrix F^{1}(2.1.5) of the system is,

$$F^{1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \qquad (2.1.14)$$

The characteristic equation of the system is,

$$g(\lambda) = \lambda(\lambda+1)(\lambda^{2}+\lambda+2) \qquad (2.1.15)$$

The system matrix A_1 has eigenvalues, $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3, \lambda_4 = (-1 \pm \sqrt{7})/2$.

The transition matrix $\phi(t,t_o)$ of the system (2.1.10) is obtained as

$$\phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \qquad (2.1.16)$$

where,

$$\phi_{11}(t) = \begin{bmatrix} \frac{1}{2}\left[1+\frac{2\sqrt{2}}{\sqrt{7}}\, e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] & \frac{1}{2}\left[1-e^{-t}+\frac{2}{\sqrt{7}}\, e^{-t/2} \sin\frac{\sqrt{7}t}{2}\right] \\ \frac{-2}{\sqrt{7}}\, e^{-t/2} \sin\frac{\sqrt{7}t}{2} & \frac{1}{2}\left[e^{-t}-\frac{2\sqrt{2}}{\sqrt{7}}\, e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] \end{bmatrix}$$

$$\phi_{12}(t) = \begin{bmatrix} \frac{1}{2}\left[1-\frac{2\sqrt{2}}{\sqrt{7}}\, e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] & \frac{1}{2}\left[1-e^{-t}-\frac{2}{\sqrt{7}}\, e^{-t/2} \sin\frac{\sqrt{7}t}{2}\right] \\ \frac{2}{\sqrt{7}}\, e^{-t/2} \sin\frac{\sqrt{7}t}{2} & \frac{1}{2}\left[e^{-t}+\frac{2\sqrt{2}}{\sqrt{7}}\, e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] \end{bmatrix}.$$

$$\phi_{21}(t)=\begin{bmatrix} \frac{1}{2}\left[1-\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] & \frac{1}{2}\left[1-e^{-t}-\frac{2}{\sqrt{7}}\,e^{-t/2}\sin\frac{\sqrt{7}t}{2}\right] \\ \frac{2}{\sqrt{7}}\,e^{-t/2}\sin\frac{\sqrt{7}t}{2} & \frac{1}{2}\left[e^{-t}+\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right. \end{bmatrix}$$

$$\phi_{22}(t)=\begin{bmatrix} \frac{1}{2}\left[1+\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] & \frac{1}{2}\left[1-e^{-t}+\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\frac{\sqrt{7}t}{2}\right] \\ \frac{-2}{\sqrt{7}}\,e^{-t/2}\sin\frac{\sqrt{7}t}{2} & \frac{1}{2}\left[e^{-t}-\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] \end{bmatrix}$$

Let us examine the following cases.

(A) An Initial Displacement At The Master End.

Assume that no external torque is acting on the system (i.e $T_1 = 0$). Let the shaft at the master end be given an initial displacement of one radian, and be released (i.e $\theta_1(0) = 1$). It is required to find the responses $\theta_1(t)$ and $\theta_2(t)$.

Since $x(t) = \phi(t)\,x(0)$ (2.1.17)

and $x^T(0) = (1\ 0\ 0\ 0)$

$$x_1(t) = \theta_1(t) = \frac{1}{2}\left[1+\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] \tag{2.1.18}$$

$$x_3(t) = \theta_2(t) = \frac{1}{2}\left[1-\frac{2\sqrt{2}}{\sqrt{7}}\,e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}+\beta\right)\right] \tag{2.1.19}$$

and $\theta_2(t) \longrightarrow \frac{1}{2}$ as $t \longrightarrow \infty$.

Plots of (2.1.18) and (2.1.19) are given in Fig 2.4

It can be seen, that, the output position $\theta_2(t)$ resembles the step response of an ordinary servomechanism. Normally one would not expect this. In the case of an ordinary servomechanism, when an initial displacement is given at the input end, the output position would have settled down to zero after the initial transients had died down.

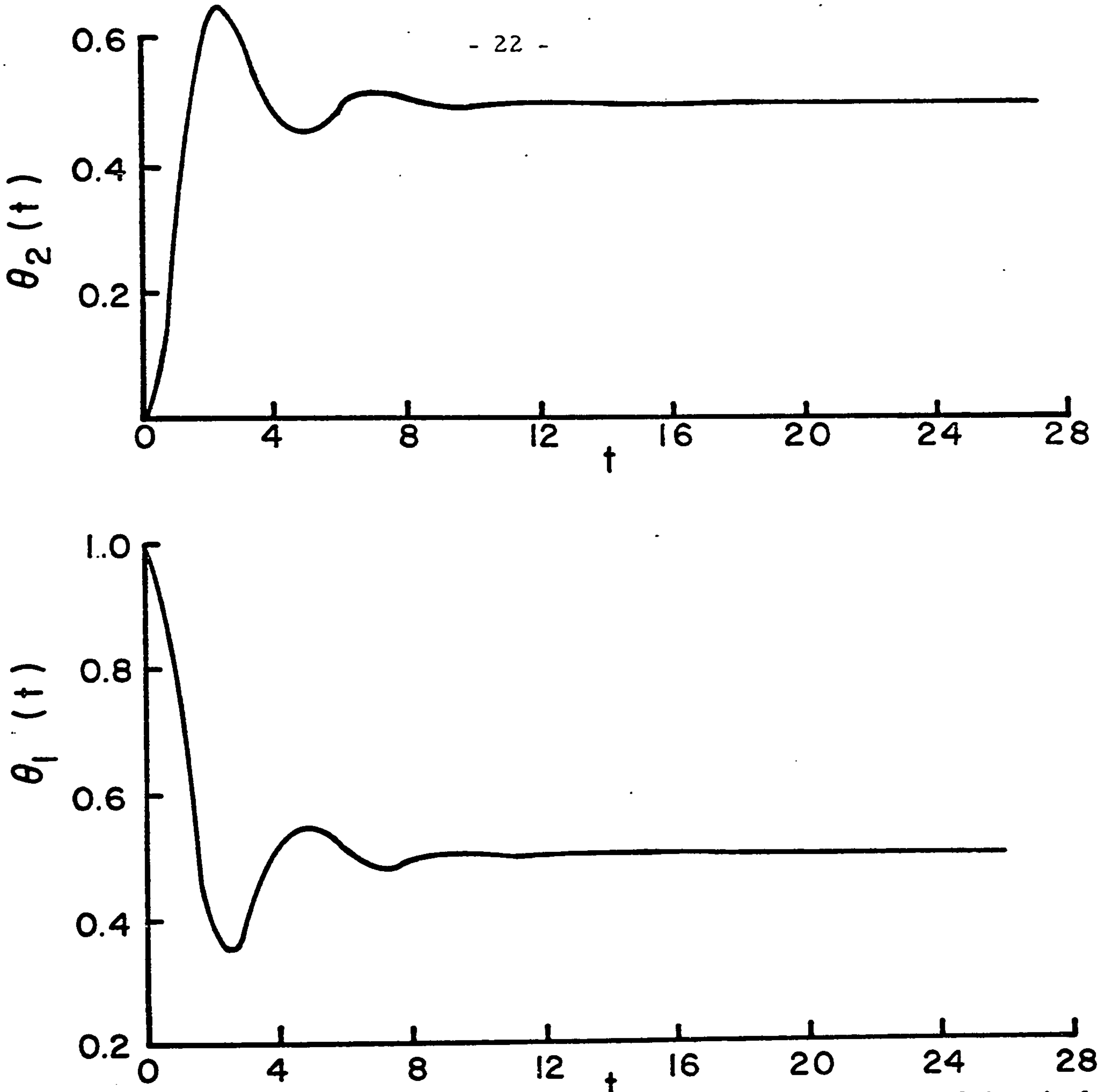


Fig. 2.4 System responses for a unit displacement of the shaft at the master end (system with Feel matrix. F^1).

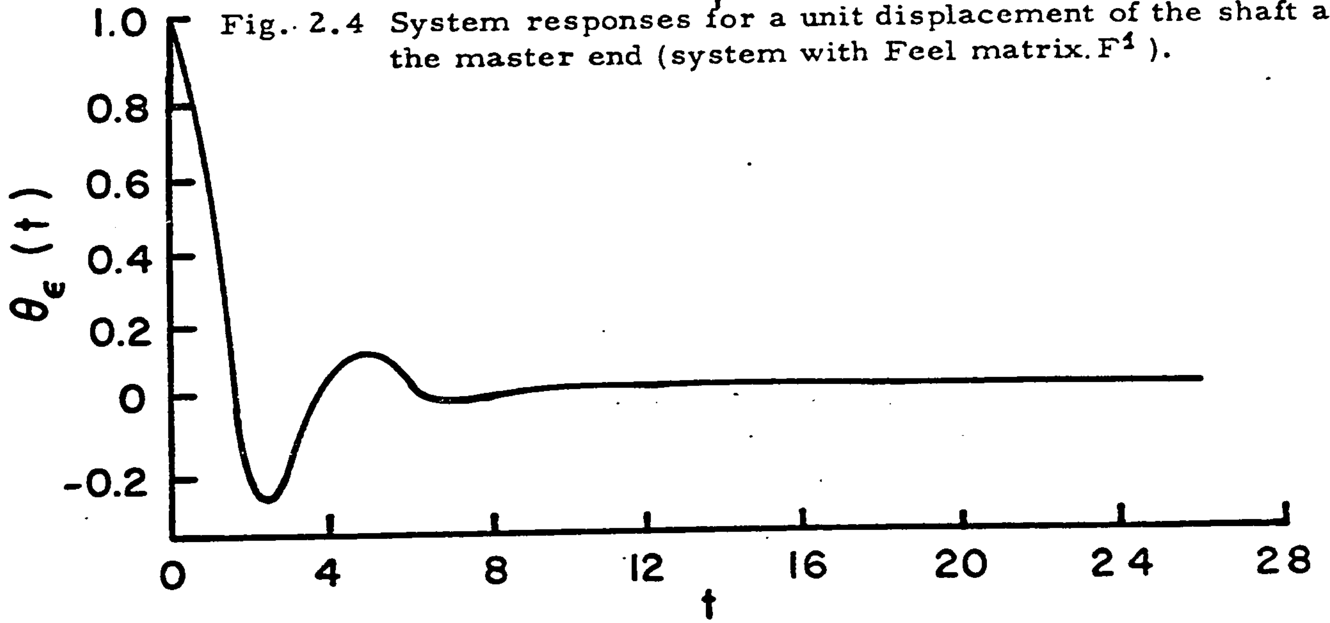


Fig. 2.5 Position error of the system for a unit displacement of shaft at the master end (system with Feel matrix F^1).

If, now, the ends are reversed, and the slave end is given an initial displacement of one radian (i.e $\theta_2(0) = 1$), the expressions for $\theta_1(t)$ and $\theta_2(t)$ are ;

$$\theta_1(t) = \frac{1}{2}\left[1 - \frac{2\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2} + \beta\right)\right] \qquad (2.1.20)$$

and $$\theta_2(t) = \frac{1}{2}\left[1 + \frac{2\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2} + \beta\right)\right] \qquad (2.1.21)$$

also $\theta_1(t) \longrightarrow \frac{1}{2}$ as $t \longrightarrow \infty$

The expression (2.1.20) for $\theta_1(t)$ is the same as (2.1.19) for $\theta_2(t)$.

Thus the system is truly bilateral i.e one can give an initial displacement at the master end and obtain certain motion at the slave end or vice versa. Also there is no special rest position for the system, such as $\theta_1 = 0 = \theta_2$, since with a proper initial displacement of one shaft, the other can be kept at any desired position.

The expression for the positional error $\theta_\epsilon(t)$ is given by

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = \frac{2\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2} + \beta\right) \qquad (2.1.22)$$

$\theta_\epsilon(t) \longrightarrow 0$ as $t \longrightarrow \infty$

A plot of $\theta_\epsilon(t)$ is given in Fig 2.5.

(B) Impulse Response.

The impulse response of the system H(t) is obtained using the relation, $H(t) = \phi(t) B$ (2.1.23)

where $B^T = (1 \; 0 \; 0 \; 0)$

The expressions for $\theta_1(t)$ and $\theta_2(t)$ are

$$\theta_1(t) = \frac{1}{2}\left[1 - e^{-t} + \frac{2}{\sqrt{7}} e^{-t/2} \sin\frac{\sqrt{7}t}{2}\right] \qquad (2.1.24)$$

$$\theta_2(t) = \frac{1}{2}\left[1 - e^{-t} - \frac{2}{\sqrt{7}} e^{-t/2} \sin\frac{\sqrt{7}t}{2}\right] \qquad (2.1.25)$$

The positional error $\theta_\epsilon(t)$ is given by

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = \frac{2}{\sqrt{7}} e^{-t/2} \sin\frac{\sqrt{7}t}{2} \qquad (2.1.26)$$

Plots of (2.1.24), (2.1.25) and (2.1.26) are given in Figs 2.6 and 2.7 respectively.

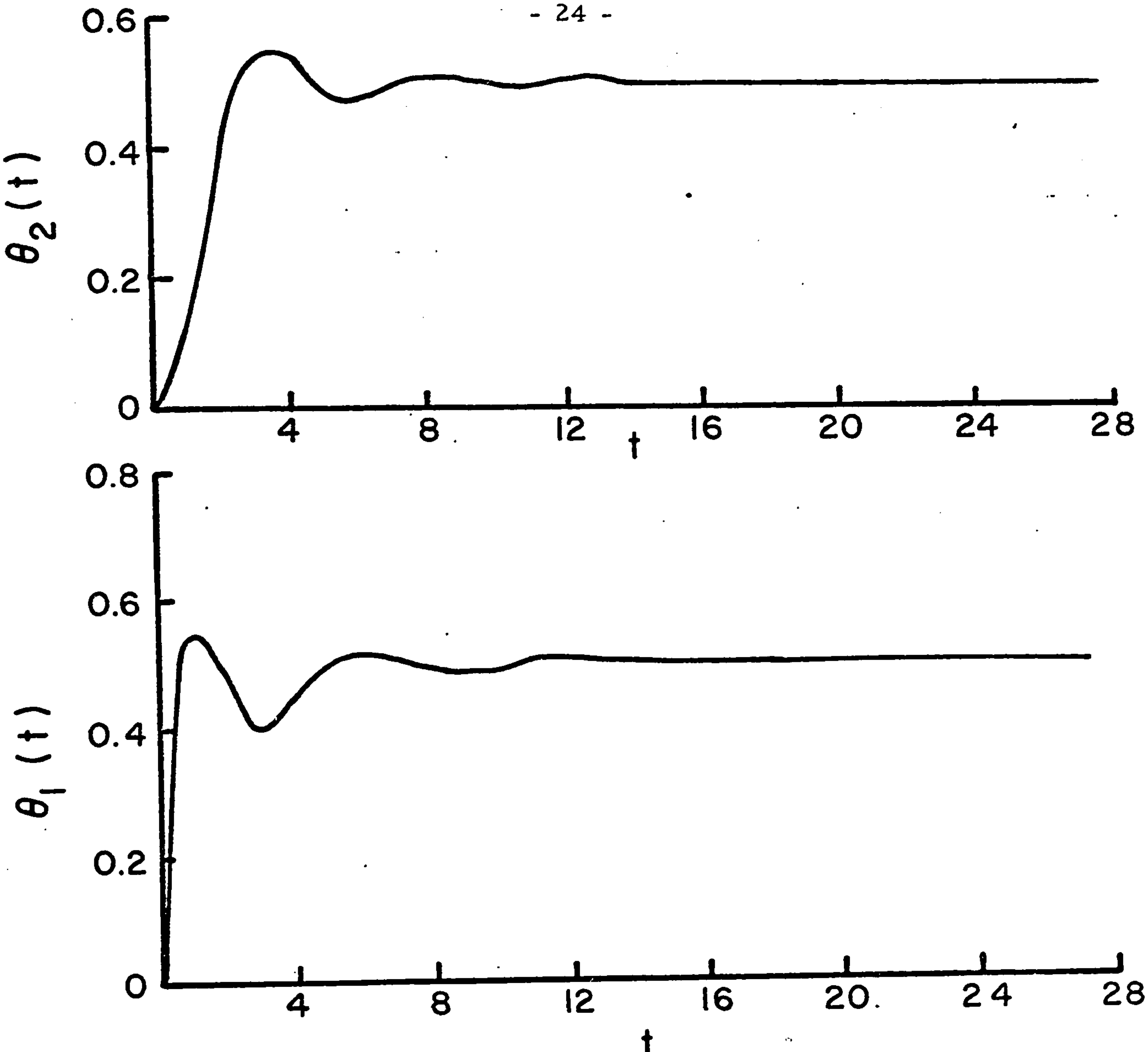


Fig. 2.6 System responses for a unit impulse torque applied at the master end

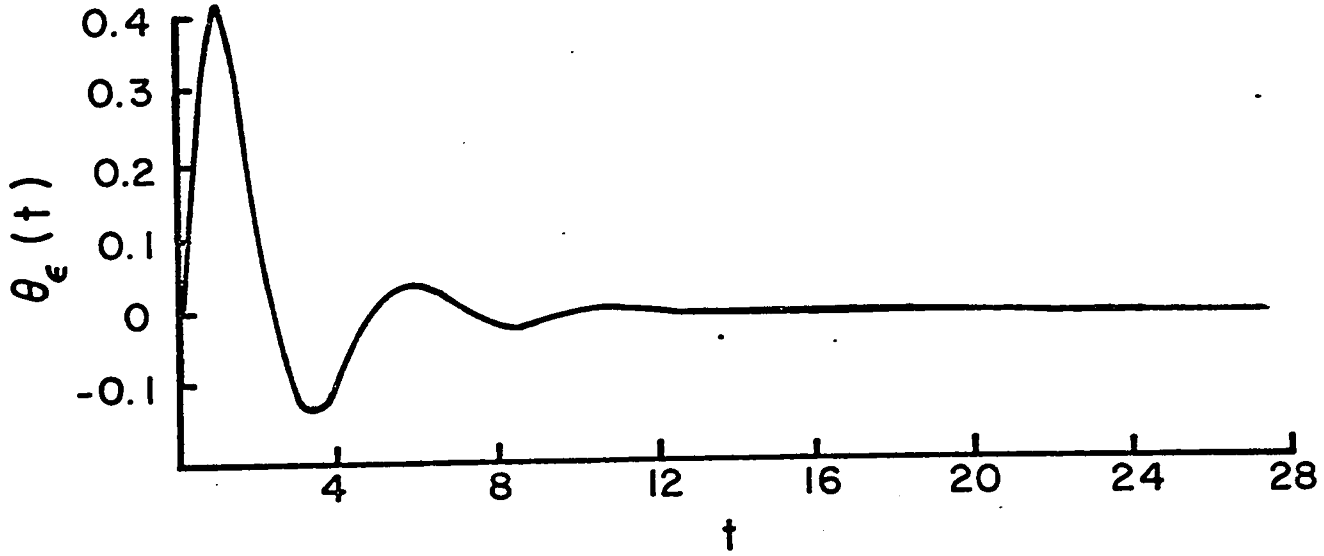


Fig. 2.7 Positional error of the system for a unit impulse torque applied at the master end.

Thus the impulse response of the system also resembles the response of an ordinary servomechanism to a step input. Also, if instead of applying a unit impulse at the master end, an impulse is given at the slave end then the expressions for $\theta_1(t)$ and $\theta_2(t)$ will be identical to (2.1.25) and (2.1.26) thus demonstrating again the bilateral property of the system.

(C) Unit Step Response.

The unit step response of the system, S(t), is obtained from the impulse response using the fact that,

$$S(t) = \int_o^t H(t-\tau)\ T(\tau)\,d\tau \qquad (2.1.27)$$

$$= \int_o^t H(t-\tau)\,d\tau\ .$$

The expressions for the resulting responses are given by

$$\theta_1(t) = \frac{1}{2}\,[t-(1-e^{-t}) + \frac{1}{2} - \frac{1}{2}\,e^{-t/_2}\cos\frac{\sqrt{7}t}{2} - \frac{1}{2\sqrt{7}}\,e^{-t/_2}\sin\frac{\sqrt{7}t}{2} \qquad (2.1.28)$$

$$\theta_2(t) = \frac{1}{2}\,[t-(1-e^{-t}) - \frac{1}{2} + \frac{1}{2}\,e^{-t/_2}\cos\frac{\sqrt{7}t}{2} + \frac{1}{2\sqrt{7}}\,e^{-t/_2}\sin\frac{\sqrt{7}t}{2} \qquad (2.1.29)$$

The positional error $\theta_\epsilon(t)$ is given by,

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = \frac{1}{2}\,[1 - \frac{2\sqrt{2}}{\sqrt{7}}e^{-t/_2}\sin\left(\frac{\sqrt{7}t}{2} + \beta\right)] \qquad (2.1.30)$$

Plots of (2.1.28), (2.1.29) and (2.1.30) are given in Figs 2.8 and 2.9 respectively.

It is observed that the system response resembles the response of an ordinary servomechanism for a ramp input. In fact $\theta_2(t)$ is proportional to t, for very large values of t. The error between the input and output positions, attains a constant value after the initial transients have died down. Thus the system response is unbounded, when a unit step torque is applied

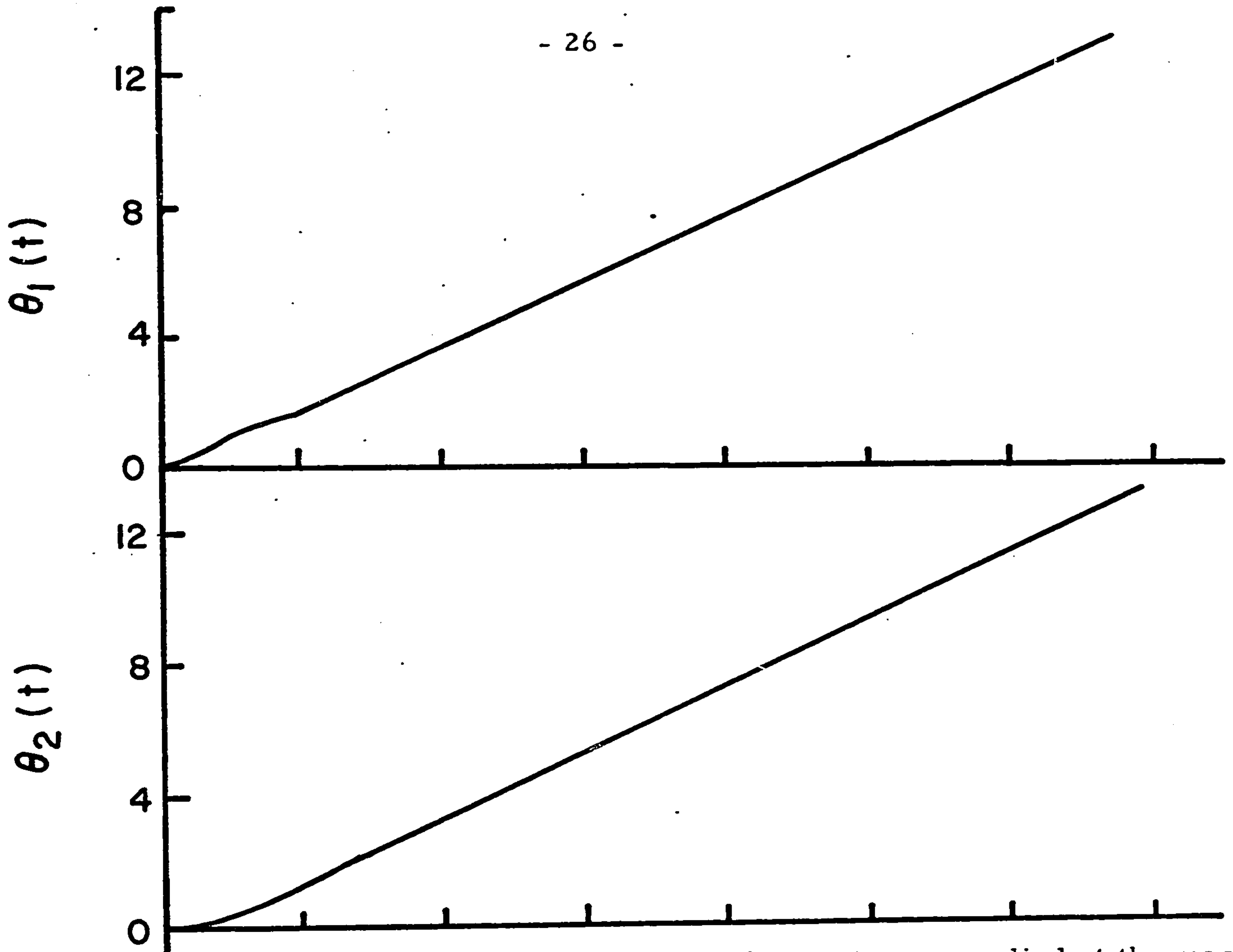


Fig. 2.8 System responses for a unit step torque applied at the master end (system with feel matrix F')

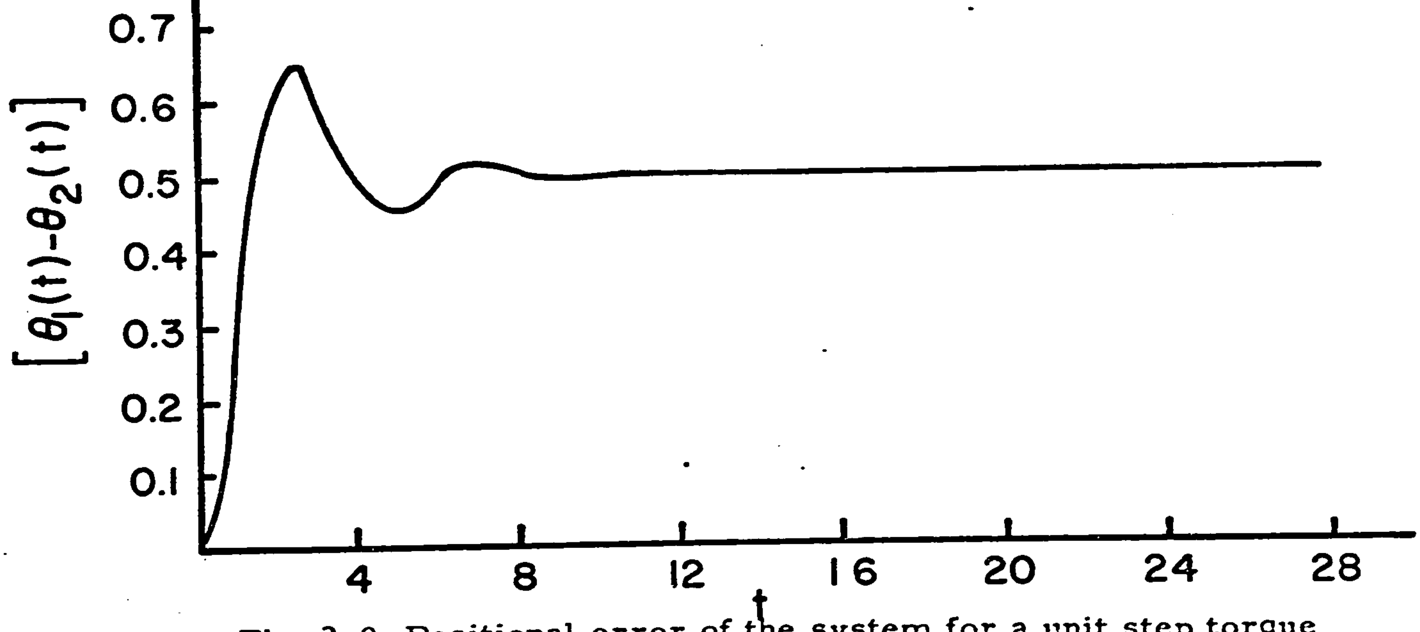


Fig. 2.9 Positional error of the system for a unit step torque applied at the master end (system with feel matrix F').

at the master end. The same is true when a unit step torque is applied at the slave end instead of at the master end.

(D) A Comparison of the Stability of the System with that of a Single Positional Servomechanism.

Some one dimensional feel systems consist of two positional servomechanisms connected back to back. The system response due to an initial condition $\theta_1(0)$ or $\theta_2(0)$ resembles the step response of a single positional servomechanism. It will be interesting then to compare the stability of the system with an initial condition, with that of a single positional servomechanism with a step position input.

Let the magnitude of the input to the single positional servomechanism be equal to half the magnitude of the initial condition on the system. The parameters (of 2.110) for the single positional servo are

$$A = \begin{bmatrix} 0 & 1 \\ -K & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ K \end{bmatrix} , \quad C = [\,0 \quad 1\,] \tag{2.1.31}$$

where K is the forward gain of the system

With $K = 1$, the expression for $\theta_2(t)$ is given by,

$$\theta_2(t) = \frac{1}{2}\left[1 - \frac{2e^{-t/2}}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2} + \alpha \right)\right] \tag{2.1.32}$$

If the value of the gain K is increased to 2, then $\theta_2(t)$ becomes,

$$\theta_2(t) = \frac{1}{2}\left[1 - \frac{2\sqrt{2}}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}t}{2} + \beta \right)\right] \tag{2.1.33}$$

The expression (2.1.33) is the same as (2.1.19)

Thus with a forward gain K, equal to twice the value used for obtaining the response of the one dimensional feel system (section 2.1.4.1, case A), and with an input whose magnitude is half that of the initial condition, the same response is obtained. Thus, a single positional servomechanism is stable for larger values of the gain K, than a bilateral servo constructed with two positional servomechanisms in its arms.

2.1.4.2 System With Derivative And Position Feedback.

Let the elements of the system be arranged as shown in Fig 2.1. Let the nominal values of the gains K_{ij}, D_{ij}, $i,j = 1,2$, the moments of inertia, J_i, $i = 1,2$, and the coefficients of friction F_i, $i = 1,2$ of the master and slave end actuators be taken as unity. The values of the loading factors α, β and r are assumed to be zero.

The system matrix A_1 (2.1.12) is,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \end{bmatrix} \tag{2.1.34}$$

The corresponding feel matrix F(2.1.5) is,

$$F^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix} \tag{2.1.35}$$

The system characteristic equation is

$$g(\lambda) = \lambda(\lambda + 2)(\lambda + 1)^2 \tag{2.1.36}$$

The eigenvalues of the system matrix A_1 are $\lambda_1 = 0$, $\lambda_2 = -2$, and $\lambda_3, \lambda_4 = -1$.

Consider again cases (A) to (C) examined in section 2.1.4.1.

(A) Initial Displacement at the Master End.

Assume that no external torque is acting on the system (i.e $T_1 = 0$). Proceeding as in section 2.1.4.1, the expressions for $\theta_1(t)$, $\theta_2(t)$, and $[\theta_1(t) - \theta_2(t)]$ can be obtained as

$$\theta_1(t) = \frac{1}{2}[1 + 2e^{-t} - e^{-2t}] \qquad (2.1.37)$$

$$\theta_2(t) = \frac{1}{2}[1 + e^{-2t} - 2e^{-t}] \qquad (2.1.38)$$

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = 2e^{-t} - e^{-2t} \qquad (2.1.39)$$

Plots of (2.1.37), (2.1.38) and (2.1.39) are given in figs 2.10 and 2.11 respectively. As in the case of a system with no derivative feedback, the system response (2.1.38) is similar to that of an ordinary servomechanism, with a step input. The only difference between the responses (2.1.19) and (2.1.38) is that the response (2.1.38) is highly damped as compared to (2.1.19). The steady state value of the output position remained the same (viz. $\frac{1}{2}$). The bilateral nature of the system can again be observed by giving an initial displacement at the slave end (i.e $\theta_2(0) = 1$), and noting that the resultant $\theta_1(t)$ is exactly the same as 2.1.38.

(B) Impulse response.

Proceeding as in section 2.1.4.1, the expressions for $\theta_1(t)$, $\theta_2(t)$ $[\theta_1(t) - \theta_2(t)]$ are obtained as,

$$\theta_1(t) = 1/2\ (1 - e^{-2t}) \qquad (2.1.40)$$

$$\theta_2(t) = 1/2\ (1 - 2e^{-t} + e^{-2t}) \qquad (2.1.41)$$

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = [e^{-t} - e^{-2t}] \qquad (2.1.42)$$

Plots of (2.1.40), (2.1.41), and (2.1.42) appear in figs 2.12 and 2.13 respectively.

It is observed that the output response (2.1.41) is similar to the response with no derivative feedback (2.1.25), except that the response (2.1.41) is highly damped.

(C) Step Response.

Again, proceeding as in section 2.1.4.1, the expressions for $\theta_1(t)$, $\theta_2(t)$ and $[\theta_1(t) - \theta_2(t)]$ are obtained as,

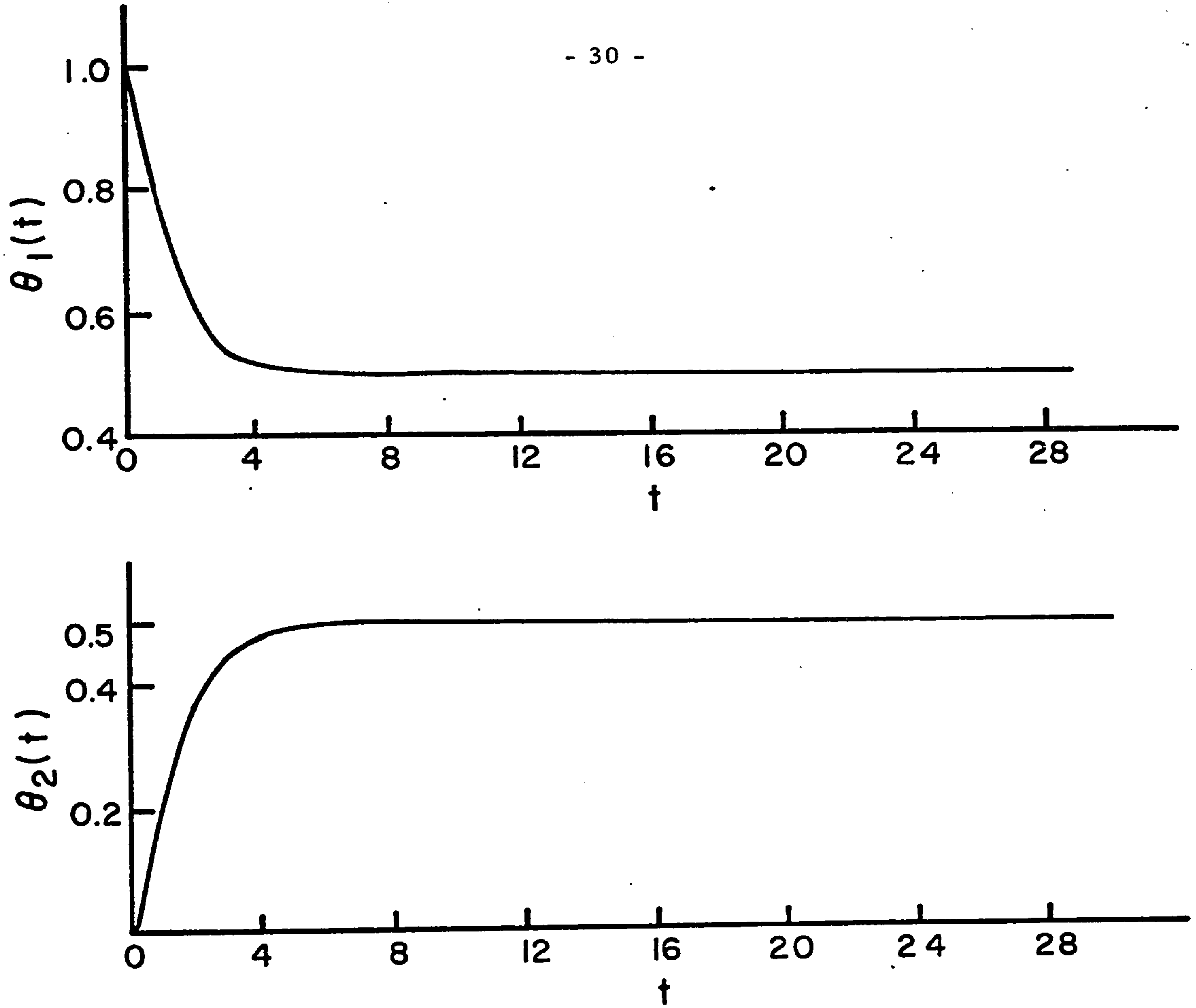


Fig 2.10 System responses for a unit displacement of shaft at the master end (system with Feel matrix F^2) .

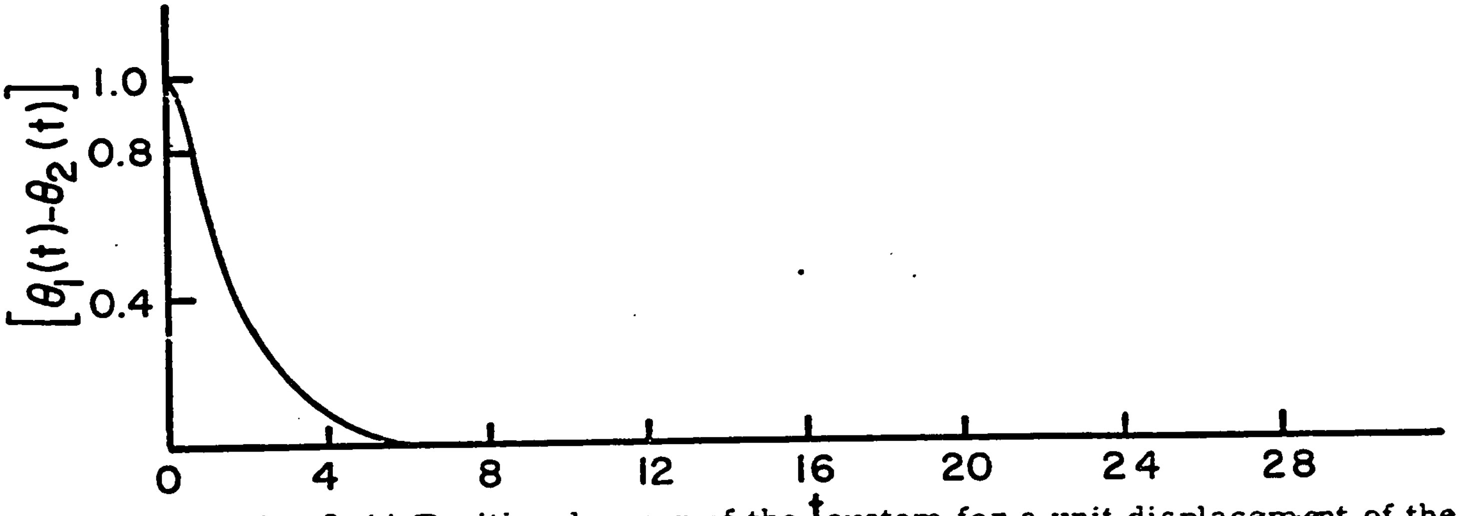


Fig. 2.11 Positional error of the system for a unit displacement of the shaft at the master end (system with Feel matrix F^2).

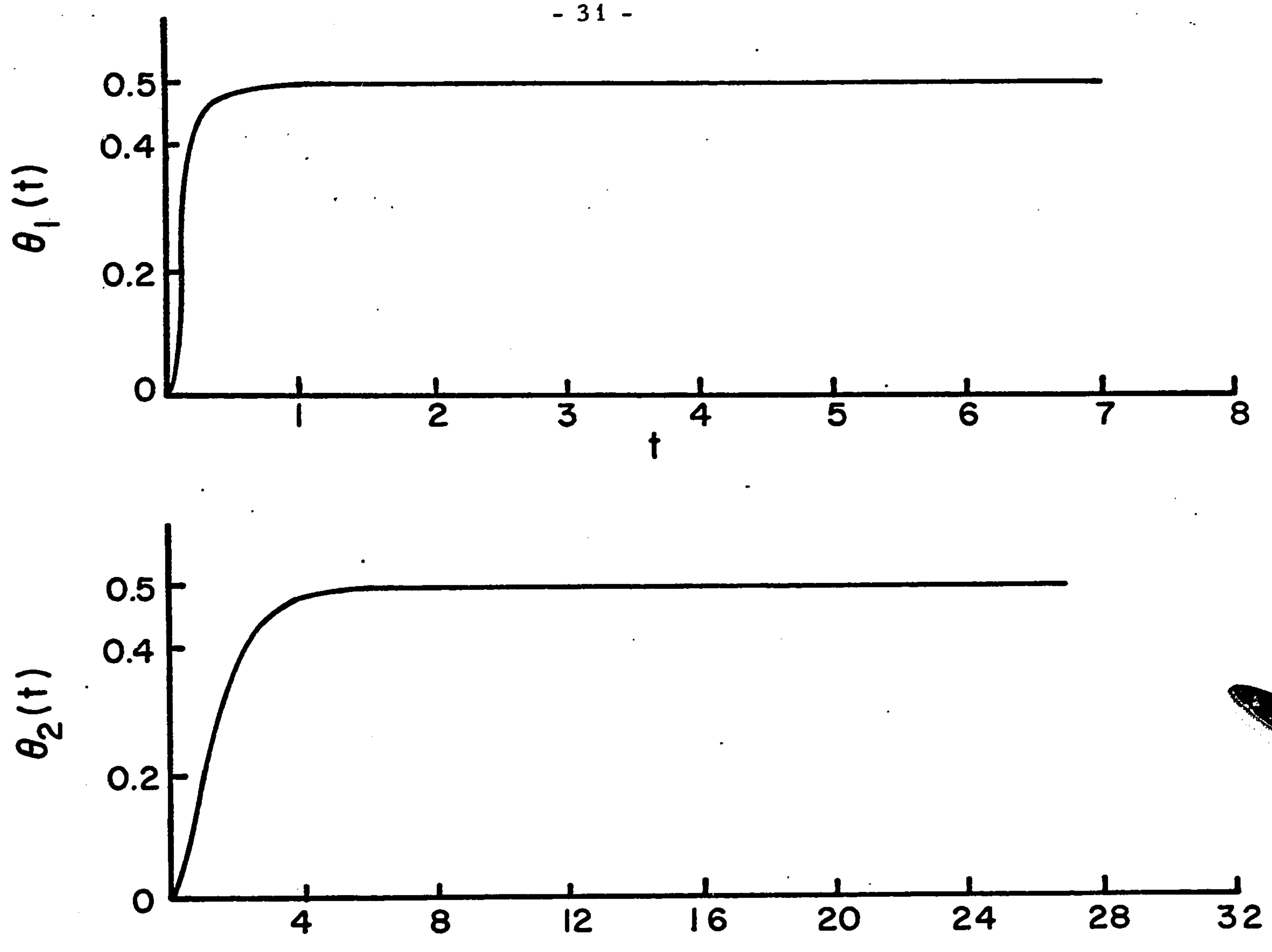


Fig. 2.12 System responses for a unit impulse torque at the master end (system with Feel matrix F^2).

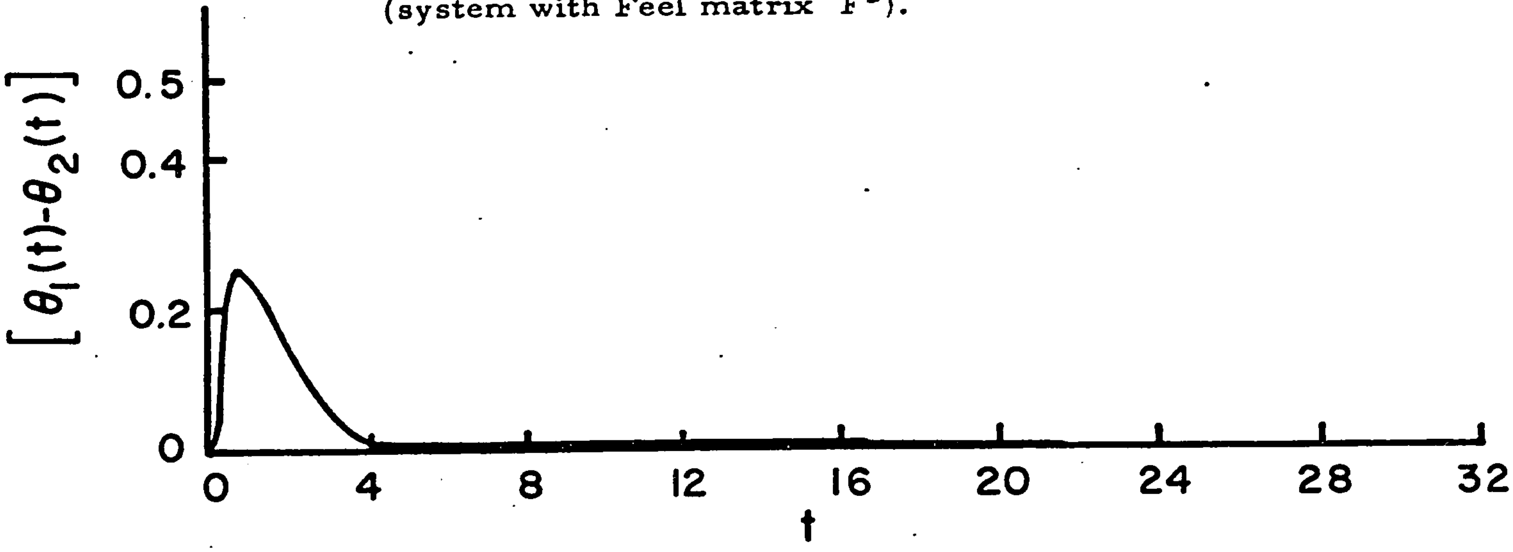


Fig. 2.13 Positional error of the system with a unit impulse torque applied at the master end (system with Feel matrix F^2).

$$\theta_1(t) = \frac{1}{2}\left(t - \frac{1}{2} + \frac{1}{2}e^{-2t}\right) \qquad (2.1.43)$$

$$\theta_2(t) = \frac{1}{2}\left(t - \frac{3}{2} + 2e^{-t} - \frac{1}{2}e^{-2t}\right) \qquad (2.1.44)$$

$$\theta_\epsilon(t) = [\theta_1(t) - \theta_2(t)] = \frac{1}{2}(1 + e^{-2t} - 2e^{-t}) \qquad (2.1.45)$$

Plots of (2.1.43), (2.1.44), and (2.1.45) are shown in Figs 2.14 and 2.15 respectively.

It is found that the step response of the system is unbounded, and that the positional error attains a constant value after the initial transients had died down. Thus the introduction of the derivative feedback has not produced any changes in the three basic responses of the system. The only difference is that, for the system with derivative feedback, the responses are highly damped, as is expected.

The difference in the behaviours of the systems (2.1.14), (2.1.35), and an ordinary servomechanism can be attributed to the fact that the system matrices (2.1.13), (2.1.34) possess an eigenvalue equal to zero. This makes the systems marginally stable. The problem will not arise in the case of the system shown in Fig 2.2, if the values of the gains K_{ij}, D_{ij}, $i, j = 1,2$ are made unequal. The system matrix will not have an eigenvalue equal to zero, and the behaviour of the system is identical to that of an ordinary servomechanism. The same is true, when a passive load is present on the systems (2.1.14) and (2.1.35). A one dimensional feel system whose behaviour is identical to that of an ordinary servomechanism, is considered in section 2.1.5.

Another interesting extreme case is the one when $F_1 = 0 = F_2$. The characteristic equation of the system (K_{ij}, D_{ij}, $i, j = 1,2 = 1, J_i = 1, i = 1,2$ and $\alpha = \beta = r = 0$) is,

$$g(\lambda) = \lambda^2(\lambda^2 + 2\lambda + 2) \qquad (2.1.46)$$

Thus the system matrix has two eigen values equal to zero. This system, when excited by an impulse torque, and after the initial transients die down, continues to move with a constant velocity. During this motion

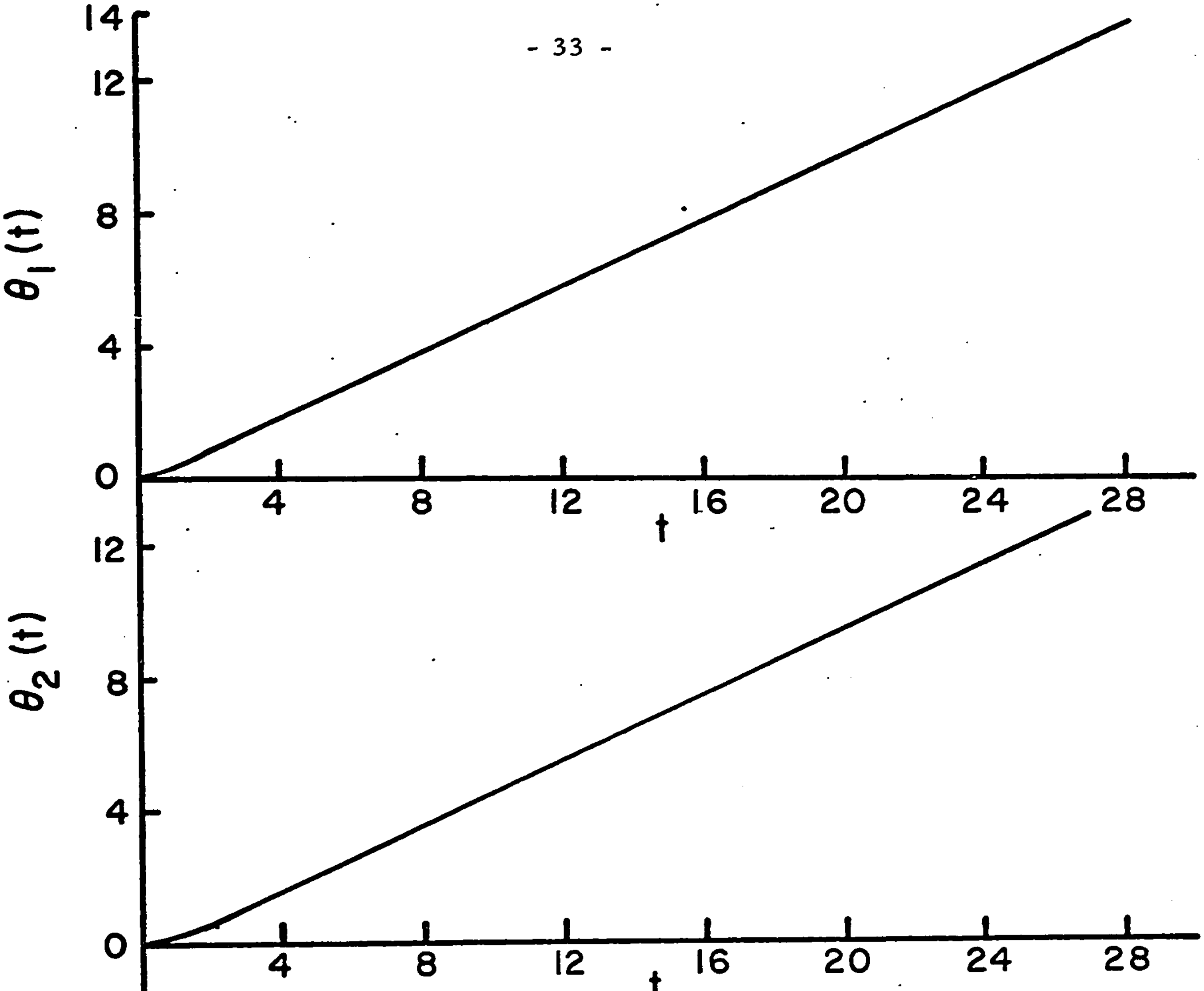


Fig. 2.14 System responses for a unit step torque applied at the master end (system with Feel matrix F^2).

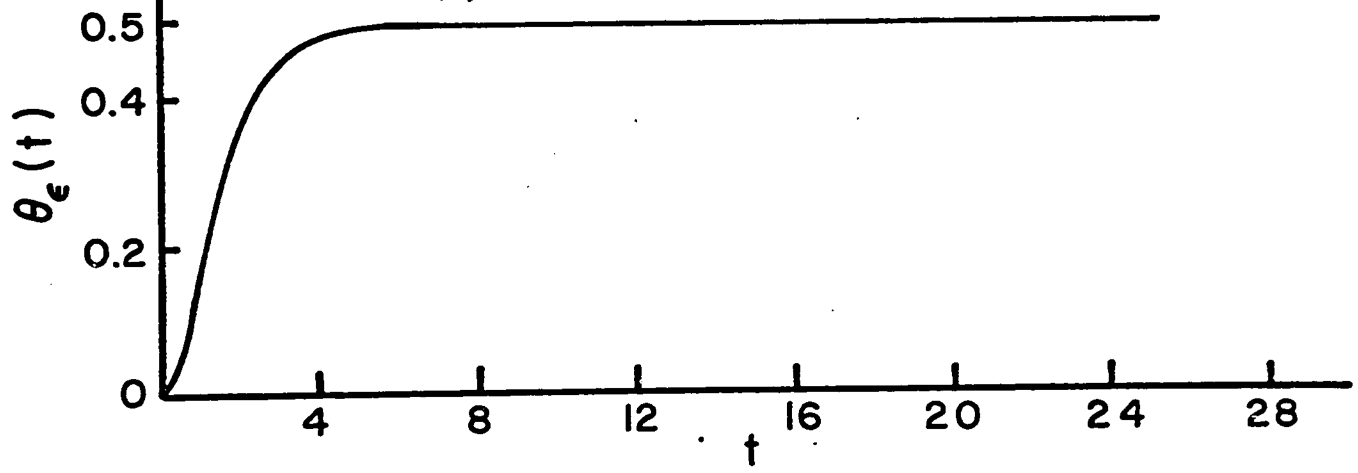


Fig. 2.15 Positional error of the system for a system with step torque applied at the master end (system with Feel matrix F^2).

the system does not dissipate any energy.

It may be stated that the system matrix A_1 of a one dimensional feel system may possess either one or two eigenvalues equal to zero, and the system still be stable. Hence, while designing the system the values of the relative gains of the signal channels must be chosen in such a way that the eigenvalues of A_1 are non-positive. The system responses described in section 2.1.4, and plotted in Figs 2.4 to 2.15 are obtained by solving the system equations (2.1.10) on an I.B.M. 360 digital computer using Fourth order Rungakutta numberical integration procedure. When the system matrix A_1 is chosen arbitrarily, it is not possible to obtain the system responses through the analysis given section 2.1.4 . They must be obtained through the numerical solution of (2.1.4).

2.1.5 A Discussion of Force Reflection or Feel.

The force reflected at the master end or the feel force felt by the operator is the most important quantity in the study of feel systems. It represents to a certain scale, the actual forces acting at the slave end of the system.

A mathematical expression (2.1.2) for the force reflected at the master end is given by,

$$T_{a1}(t) = -K_{11}\,\theta_1(t) + K_{12}\,\theta_2(t) - D_{11}\,\dot{\theta}_1(t) + D_{12}\,\dot{\theta}_2(t) \qquad (2.1.2')$$

In general, the force reflected is a function of the gains K_{ij} and D_{ij} $(i,j = 1,2)$ of the transmission channels. The ratio of output load forces $(-T_2(t))$ to the force reflected $T_{a1}(t)$ is usually, referred to as 'output input force ratio'. For a system, whose elements are arranged as shown in Fig 2.1, the desired force ratio can be obtained through a voltage devider called the 'force ratio selector'. For a system whose elements are arranged as shown in Fig 2.2, the required force ratio can be obtained by a proper selection of the gains (K_{ij} and D_{ij}, $i,j = 1,2$). For illustration, three different combinations of the gains K_{ij} and D_{ij} are chosen, and the forces reflected at the master end (2.1.2), and the output load forces

$-T_2(t)$ (2.1.7) are computed. The values of the gains and the loading factors α, β, r selected are such that the system matrix A_1(2.1.12) remains invariant.

The system matrix A_1 is chosen as,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1.22 & 0.66 & 0.55 \\ 0 & 0 & 0 & 1 \\ 0.6 & 0.5 & -1.2 & -1.4 \end{bmatrix} \qquad (2.1.47)$$

The corresponding feel matrix F^3 is,

$$F^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 0.66 & 0.55 \\ 0 & 0 & 0 & 0 \\ 0.6 & 0.5 & -1 & -1 \end{bmatrix} \qquad (2.1.48)$$

$(J_i = 1,\ i = 1,2,\ F_i = 0.22,\ i = 1,2)$

The system equations (2.1.10) with A_1 given by (2.1.47)are solved to obtain $x_1(t)$ ($= \theta_1(t)$), and $x_3(t)$ ($= \theta_2(t)$). Plots of $\theta_1(t)$ and $\theta_2(t)$ are given in Fig 2.16. Plots of force reflected at the master end, and the output load forces, corresponding to the three combinations of the gains K_{ij}, D_{ij} as shown in Table 2.1 below, are given in Fig. 2.17

No.	K_{11}	K_{22}	K_{21}	K_{12}	D_{11}	D_{22}	D_{21}	D_{12}
1.	1	1	0.66	0.66	1	1	0.55	0.55
2.	1	0.8	0.72	0.66	1	0.82	0.60	0.55
3.	1	0.4	0.84	0.66	1	0.46	0.70	0.55

Table 2.1

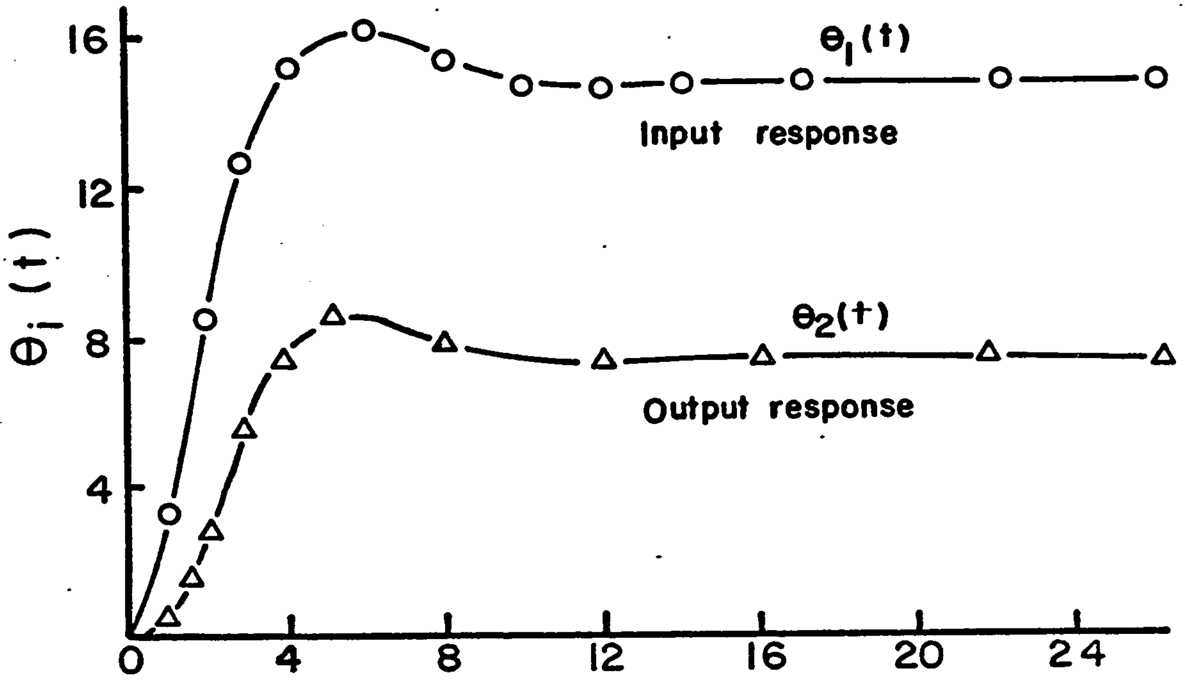


Fig. 2.16 Input and output responses for the system with Feel matrix F^3 (the gains K_{ij} and D_{ij} are unequal).

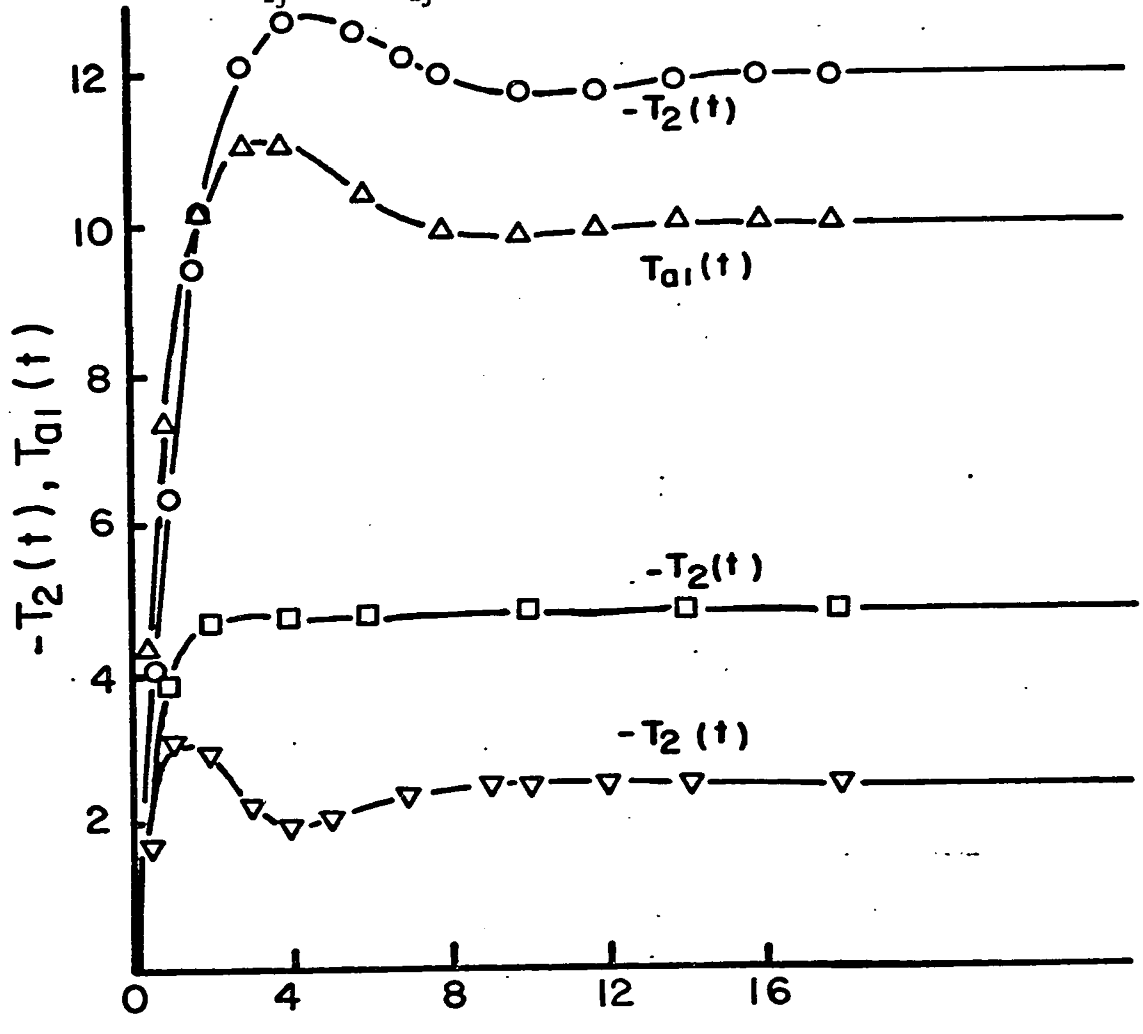


Fig. 2.17 Force reflected at the master end, and the output load forces for the system feel matrix F^3.

It is observed from an inspection of Fig 2. 17 that the magnitude of the output forces can be considerably increased by suitably adjusting the gains, K_{ij}, D_{ij}, $i,j = 1,2$. A better comparison is obtained by determining the steady state values of the output - input force ratios. In the steady state situation (s.s.) $T_{a1}(t)$ is given by

$$T_{a1}(t)\Big|_{s.s.} = -T_1(t)\Big|_{s.s.} = -K_{11}\,\theta_1(t)\Big|_{s.s.} + K_{12}\,\theta_2(t)\Big|_{s.s.} \qquad (2.1.49)$$

and $-T_2(t)$ is given by,

$$-T_2(t)\Big|_{s.s.} = -T_{a2}(t)\Big|_{s.s.} = -K_{22}(1+\beta)\,\theta_2(t)\Big|_{s.s.} + K_{21}\theta_1(t)\Big|_{s.s.} \qquad (2.1.50)$$

The steady-state output-input force ratios for the three combinations chosen are,

$$\frac{-T_2(t)}{T_{a1}(t)}\Bigg|_{s.s.} = 0.238,\ 0.48,\ 1.2 \qquad (2.1.51)$$

Thus it may be concluded that by a proper selection of the parameters (the transmission gains K_{ij} and D_{ij}, $i,j = 1,2$, and the loading factors α, β, r) the overall steady state gain of the system (Fig 2.2, equation 2.1.10) can be increased. (about 6 times for the example considered).

2.2. Multi Dimensional Feel System.

As mentioned in section 2.1.3, several of the one dimensional feel systems can be interconnected in a suitable manner to form a large class of multi-dimensional feel systems. The output positions of the various one dimensional feel systems have to be property co-ordinated, so that the multi-dimensional feel system will be able to position an object in the desired manner. For example, it is known that a remote manipulator has at least seven degrees of freedom, and there is a force reflecting servomechanism in each degree of freedom.

Definition

'n' Dimensional Feel System.

A feel system constructed by interconnecting n, one dimensional feel systems is called an 'n' dimensional feel system.

Definition

'n' Dimensional Man Machine System.

A system formed by combining an 'n' dimensional feel system with the human operator is called an 'n' dimensional man machine system.

Fig 2.18 shows a block diagram representation of a one dimensional man machine system, while Fig 2.19 shows a block diagram representation of a two dimensional man-machine system.

In the formulation of a mathematical model for a multi-dimensional feel system, the most important factor to be taken into consideration is the interaction between the inputs and the outputs of the various axes. This has been taken into account by assuming [23] that the equivalent error in torque in any axis, is a linear combination of the errors in torque in all the axes. To start with a two dimensional situation is considered, and then generalized.

2.2.1. Two Dimensional Feel System.

2.2.1.1 System Equations.

A and B represent the two axes of a two dimensional feel system (Fig 2.19). $\{X_{Ai}, X_{Bi}, i = 1,2,3,4\}$ are the state variables, Y_A and Y_B are the output positions of axes A and B, and $Y = \begin{bmatrix} Y_A \\ Y_B \end{bmatrix}$ is the resulting output position vector. T_{A1}, T_{B1} are the input torque to the two dimensional feel system, T_{Aa1}, T_{Ba1} and T_{Aa2}, T_{Ba2} are the torques developed by the actuators at the master and slave ends respectively. T_{A2} and T_{B2} are the load torques; K_{ij}, D_{ij}, $i,j = 1,2$ are the gains of the transmission channels and J_i, F_i, $i = 1,2$ are the moments of inertia and coefficients of friction of the actuator shafts (Fig 2.18).

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1) A better terminology would have been 'n-axis man-machine system'.

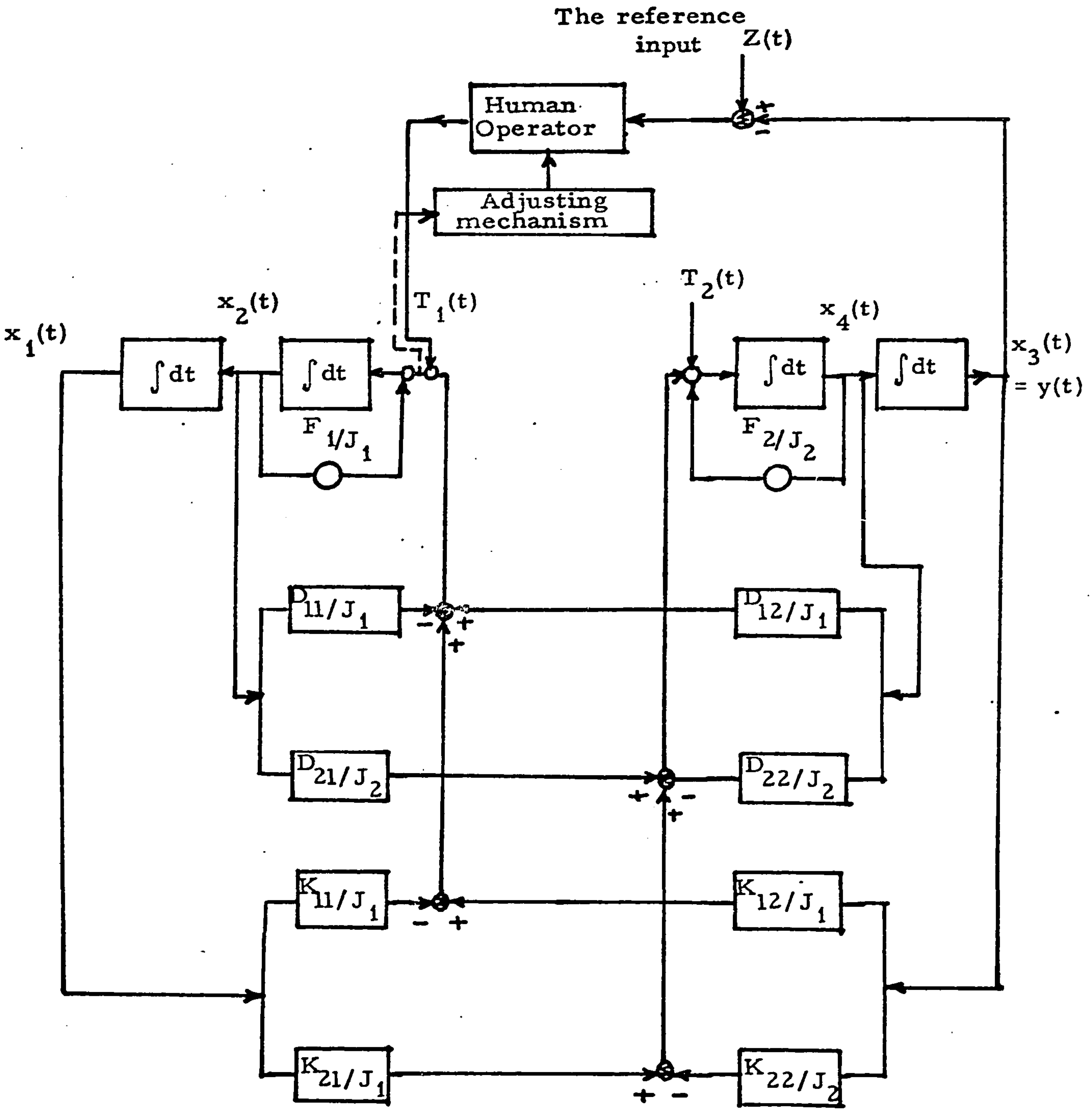


Fig 2. 18 A Block Diagram representation of a one dimensional Man-machine system.

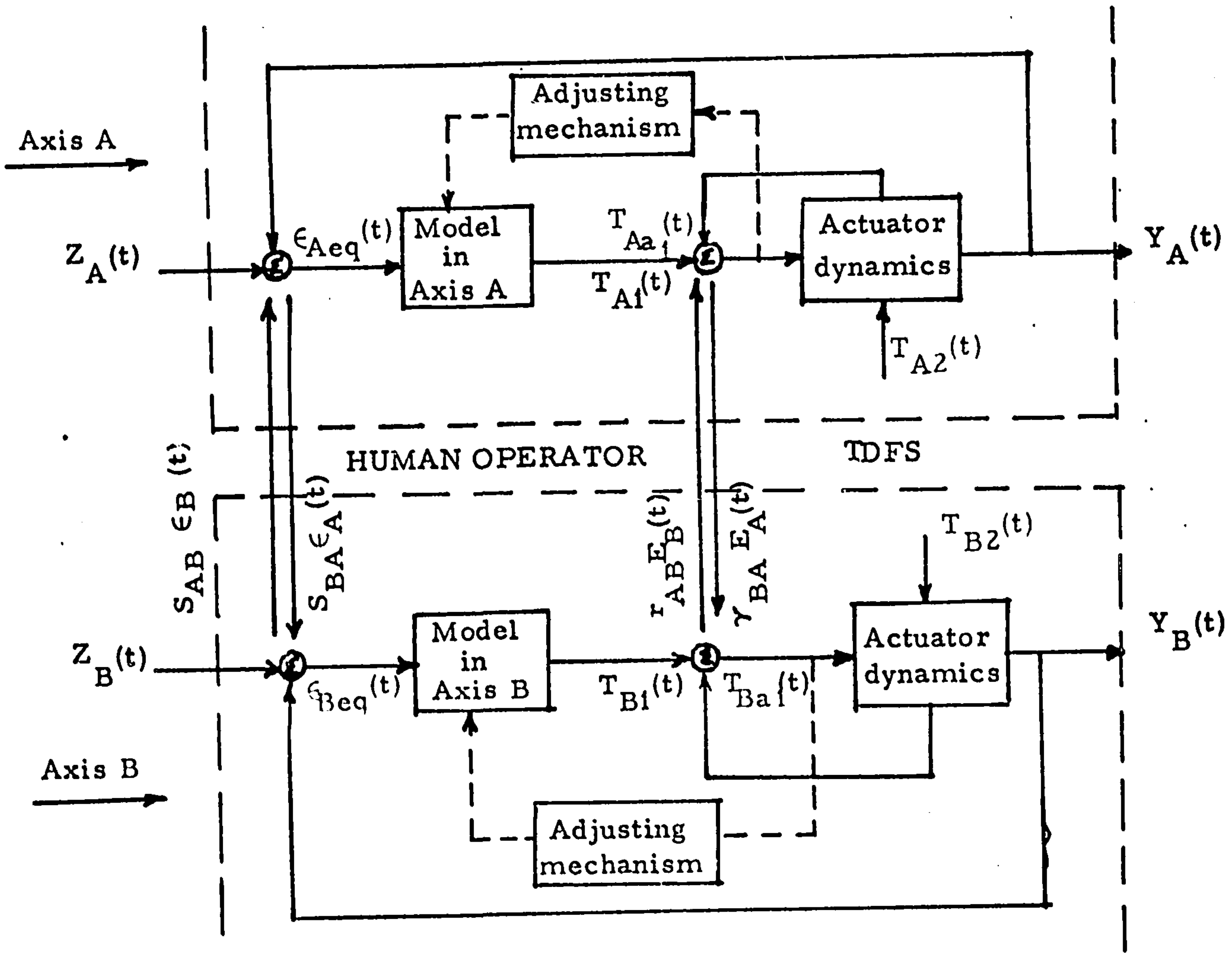


Fig 2.19 A Block Diagram Representation of the Two Dimensional Man-Machine System.

The cross coupling effect is taken into account by defining suitable coupling coefficients r_{AB}, and r_{BA}. (The first subscript refers to the axis in which the motion is being studied and second the axis whose input interacts with the input of the other axis).

The expressions for the equivalent errors in the two axes are

$$E_{Aeq}(t) = E_A(t) + r_{AB} E_B(t) , \qquad E_{Beq}(t) = E_B(t) + r_{BA} E_A(t) \tag{2.2.1}$$

where

$$E_A(t) = T_{A1}(t) + T_{Aa1}(t)$$

$$E_B(t) = T_{B1}(t) + T_{Ba1}(t)$$

The differential equations that describe the dynamics of a two dimensional feel system are,

$$[T_{A1}(t) + T_{Aa1}(t)] + r_{AB}[T_{B1}(t) + T_{Ba1}(t)] = J_1\ddot{\theta}_{A1}(t) + F_1\dot{\theta}_{A1}(t)$$

$$[T_{A2}(t) + T_{Aa2}(t)] = J_2\ddot{\theta}_{A2}(t) + F_2\dot{\theta}_{A2}(t)$$

$$[T_{B1}(t) + T_{Ba1}(t)] + r_{BA}[T_{A1}(t) + T_{Aa1}(t)] = J_1\ddot{\theta}_{B1}(t) + F_1\dot{\theta}_{B1}(t)$$

$$[T_{B2}(t) + T_{Ba2}(t)] = J_2\ddot{\theta}_{B2}(t) + F_2\dot{\theta}_{B2}(t)$$

where

$$T_{Aa1}(t) = -K_{A11}\theta_{A1}(t) + K_{A12}\theta_{A2}(t) - D_{A11}\frac{d\theta_{A1}}{dt}(t) + D_{A12}\frac{d\theta_{A2}}{dt}(t)$$

$$T_{Aa2}(t) = K_{A21}\theta_{A1}(t) - K_{A22}\theta_{A2}(t) + D_{A21}\frac{d\theta_{A1}}{dt}(t) - D_{A22}\frac{d\theta_{A2}}{dt}(t)$$

$$T_{Ba1}(t) = -K_{B11}\theta_{B1}(t) + K_{B12}\theta_{B2}(t) - D_{B11}\frac{d\theta_{B1}}{dt}(t) + D_{B12}\frac{d\theta_{B2}}{dt}(t)$$

$$T_{Ba2}(t) = K_{B21}\theta_{B1}(t) - K_{B22}\theta_{B2}(t) + D_{B21}\frac{d\theta_{B1}}{dt}(t) - D_{B22}\frac{d\theta_{B2}}{dt}(t) \tag{2.2.2}$$

and θ_{Ai} and θ_{Bi}, $i = 1,2$ are the output positions of the actuators at the master and slave end in the axes A and B respectively.

2.2.1.2 State Space Representation.

Let the positions and velocities of the actuators at the master and slave ends be chosen as the state variables so that

$$\left.\begin{array}{lcl} \theta_{A1} & = & x_{A1} \\ \dot{\theta}_{A1} & = & x_{A2} \\ \theta_{A2} & = & x_{A3} \\ \dot{\theta}_{A2} & = & x_{A4} \end{array}\right\} \triangleq \bar{X}_A, \qquad \left.\begin{array}{lcl} \theta_{B1} & = & x_{B1} \\ \dot{\theta}_{B1} & = & x_{B2} \\ \theta_{B2} & = & x_{B3} \\ \dot{\theta}_{B2} & = & x_{B4} \end{array}\right\} \triangleq \bar{X}_B .$$

A state space representation of equations(2.2.2) is given by

$$\dot{X} = (A' + F) X + BT_1 + DT_2 \qquad (2.2.3)$$

$$Y = CX$$

where

$$A' = \left[\begin{array}{c:c} A_o' & 0 \\ \hdashline 0 & A_o' \end{array}\right], \quad F = \left[\begin{array}{c:c} B_3F_A & S_1BEF_B \\ \hdashline S_2BEF_A & B_3F_B \end{array}\right], B = \left[\begin{array}{c} B_1 \\ \hdashline B_2 \end{array}\right], \quad D = \left[\begin{array}{c} D_1 \\ \hdashline D_2 \end{array}\right].$$

$$X = \begin{bmatrix} \bar{X}_A \\ \bar{X}_B \end{bmatrix}, Y = \begin{bmatrix} Y_A \\ Y_B \end{bmatrix}, C = \left[\begin{array}{cccc:cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$A_o' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F_1}{J_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{F_2}{J_2} \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{J_1} & \frac{r_{AB}}{J_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ \frac{r_{BA}}{J_1} & \frac{1}{J_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{J_2} & 0 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_2} \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$S_1 = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$S_2 = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$F_A = \begin{bmatrix} -K_{A11} & -D_{A11} & K_{A12} & D_{A12} \\ K_{A21} & D_{A21} & -K_{A22} & -D_{A22} \end{bmatrix}$$

$$F_B = \begin{bmatrix} -K_{B11} & -D_{B11} & K_{B12} & D_{B12} \\ K_{B21} & D_{B21} & -K_{B22} & -D_{B22} \end{bmatrix}$$

2.2.2 n-dimensional Feel System.

The mathematical model of an n dimensional feel system is obtained by generalizing (2.2.3) (corresponding to n = 2).

It is given by,

$$\dot{X}(t) = [A'(t) + F(t)]\; X(t) + B(t)\, T_1(t) + D(t)\, T_2(t) \qquad (2.2.4)$$

$$Y(t) = C(t)\, X(t)$$

where

$X(t)$ is a $n \times 1$ state vector

$A'(t)$ is $4n \times 4n$ actuator matrix

$F(t)$ is $4n \times 4n$ Feel matrix

$T_1(t)$ is a $n \times 1$ input torque vector (force in the case of a linear motion system)

$T_2(t)$ is a $n \times 1$ disturbance torque (force) vector

$Y(t)$ is a $n \times 1$ output position vector

$T_1(t)$, the input torque to the n dimensional feel system is the output torque of the human operator. $T_2(t)$ could be either an active disturbance (deterministic or probabilistic) or a passive load on the system. As mentioned in section 2.1.3, when $T_2(t)$ is a passive load on the system, it can be expressed as a linear combination of the state variables (2.1.7) and can be eliminated from the system equations (2.2.4). The term $F(t)\, X(t)$ in (2.2.4) represents the torques (forces) reflected at the master and slave ends. The feel matrix $F(t)$ is the most important system parameter, and may be selected in such a way that the system performance is optimised in some sense. The problem of optimal selection of the feel matrix F is considered in detail in chapter III.

2.3 A Mathematical Model For The Human Operator.

2.3.1 Introduction.

A multidimensional feel system always functions in conjunction with a manual control (sub) system, which is the most important subsystem in the combined man machine system. Hence, it is also necessary to obtain a description for the human operator. In this section, a mathematical model is obtained for the human operator appearing as a subsystem in an 'n' dimensional man machine system (Fig 2.18, for n = 2). This model, aims to describe the control behaviour of a human. The (feel) system being controlled has 'n' degrees of freedom (or n axes), and there are two inputs to the operator viz. (i) Visual input and (ii) feel force (sensory information).

In the past several attempts have been made to obtain a model for the human operator, when there is only visual input, and when the controlled element has one degree of freedom. They include linear, [27] quasilinear, [28] and sampled[24] data models. Only recently, optimal control theory has been made use of in [25], to obtain a description of the human operator. There exists some literature [26], [29], [17] regarding modelling of a human operator, controlling a system with multiple degrees of freedom. But practically none exists for multiple inputs. The difficulty of multiple inputs is overcome in this work by assuming that the feel force felt by the operator is not an additional input but a feedback to the human, that suitably modifies the visual response.

The cross coupling that exists between the inputs to and outputs of the human operator in the various axes, is taken into consideration in the formulation of an 'n' axis model for the human operator. The operator's inability to distinguish or separate 'dot motions' (refer to [29])in one axis from the other is described by the term 'input cross coupling' or 'perceptual cross coupling'. The inability of an operator to perform a control manoeuvre purely in one axis without producing some effect in the other axes is described by the term 'output cross-coupling' or 'motor cross coupling'. Of these, the motor cross coupling has already been taken into account in the formulation of a mathematical model for an 'n' dimensional feel system. In this section perceptual cross coupling is considered. This is done by assuming that the equivalent error in position in any axis is given by a linear combination of the errors in position in all the axes.

2.3. 2 Single Axis Model.

The single axis model assumed, in the formulation of an 'n' axis model for the human operator is due to McRuer and Krendel[27], and is given in the form of a transfer function by

$$G(s) = \frac{T_{1(s)}}{\epsilon(s)} = \frac{K \exp(-\tau_d s)(1 + s\tau_2)}{(1 + s\tau_1)(1 + s\tau_n)} \qquad (2.3.1)$$

where $T_1(s)$, $\epsilon(s)$ are the Laplace transforms of the output torque $T_1(t)$ of the human operator, and the error in position $\epsilon(t)$ respectively

It contains,

(1) a term $\exp(-\tau_d s)$ due to the reaction time of the central nervous system.

(2) One lag term with time constant τ_n, attributed to the neuromuscular impedance.

(3) One lead-lag combination, with time constants τ_1 and τ_2.

(4) The gain K, adjustable by the operator to maintain good low frequency performance and yet keep the system stable at high frequencies.

The differential equation corresponding to (2.3.1) is given by

$$\ddot{T}_1(t) + a_1 \dot{T}_1(t) + a_2 T_1(t) = a_3 \dot{\epsilon}(t-\tau_d) + a_4 \epsilon(t-\tau_d) \qquad (2.3.2)$$

where $a_1 = \left(\frac{1}{\tau_1} + \frac{1}{\tau_n}\right)$, $a_2 = \left(\frac{1}{\tau_1 \tau_n}\right)$, $a_3 = \left(\frac{K\tau_2}{\tau_1 \tau_n}\right)$, $a_4 = \left(\frac{K}{\tau_1 \tau_n}\right)$

2.3.3. Two-Axis Model.

2.3.3.1 The Differential equations

A and B represent the two axes of a two dimensional man-machine system (Fig 2.16). $Z_A(t)$, $Z_B(t)$, are the reference inputs to the axis A and B and $Y_A(t)$, $Y_B(t)$ are the corresponding output positions. S_{AB} and S_{BA} are the perceptual cross coupling coefficients.

The expressions for the equivalent errors in position in axes A and B are,

$$\epsilon_{Aeq}(t) = \epsilon_A(t) + S_{AB}\epsilon_B(t)$$
$$\epsilon_{Beq}(t) = \epsilon_B(t) + S_{BA}\epsilon_A(t) \qquad (2.3.3)$$

where $\epsilon_A(t) = Z_A(t) - Y_A(t)$

and $\epsilon_B(t) = Z_B(t) - Y_B(t)$

The differential equations that describe the operators dynamics are,

$$\ddot{T}_{A1}(t) + \alpha_1 \dot{T}_{A1}(t) + \alpha_2 T_{A1}(t) = \alpha_3(\dot{\epsilon}_A(t-\tau_d) + S_{AB}\dot{\epsilon}_B(t-\tau_d)) + \alpha_4(\epsilon_A(t-\tau_d) + S_{AB}\epsilon_B(t-\tau_d))$$

$$\ddot{T}_{B1}(t) + \alpha_1 \dot{T}_{B1}(t) + \alpha_2 T_{B1}(t) = \alpha_3(\dot{\epsilon}_B(t-\tau_d) + S_{BA}\dot{\epsilon}_A(t-\tau_d)) + \alpha_4(\epsilon_B(t-\tau_d) + S_{BA}\epsilon_A(t-\tau_d)) \qquad (2.3.4)$$

2.3.3.2 State-Space Representation.

In order to obtain a state space representation of (2.3.4), the following are chosen as the state variables.

$x_1 = T_{A1}$, $x_2 = \dot{T}_{A1}$, $x_3 = T_{B1}$ and $x_4 = \dot{T}_{B1}$.

The state space representation of (2.3.4) is given by

$$\dot{X}_m(t) = PX_a(t) + R_1 U(t-\tau_d) + R_2 \dot{U}(t-\tau_d) \qquad (2.3.5)$$

and $Y(t) = C_1 X_m(t)$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha_2 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 & -\alpha_1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 \\ \alpha_4 & \alpha_4 S_{AB} \\ 0 & 0 \\ \alpha_4 S_{BA} & \alpha_4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ \alpha_3 & \alpha_3 S_{AB} \\ 0 & 0 \\ \alpha_3 S_{BA} & \alpha_3 \end{bmatrix}$$

$$U(t) = \begin{bmatrix} \epsilon_A(t) \\ \epsilon_B(t) \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Y(t) = \begin{bmatrix} T_{A1}(t) \\ T_{B1}(t) \end{bmatrix}$$

2.3.4 n-axis Model.

An n axis model for the human operator is obtained by generalising (2.3.4) (corresponding to n = 2), and obtaining the corresponding state space representation.

It is given by,

$$\dot{X}_m(t) = P\, X_a(t) + R_1\, U(t-\tau_d) + R_2\, \dot{U}(t-\tau_d) \qquad (2.3.6)$$

and $Y(t) = C_1\, X_m(t)$

where X_m is a $2n \times 1$ state vector

P is a $2n \times 2n$ matrix

R_1 is a $2n \times n$ matrix

R_2 is a $2n \times n$ matrix

U is a $n \times 1$ input positional error vector

Y is a $n \times 1$ output torque vector

τ_d is the reaction time delay

2.3.5 Discussions.

It might appear, that, the interaction between the output torques (T_{A1}, T_{B1}) of the human operator in axes A and B, has not been taken into consideration in obtaining (2.3.5) and hence (2.3.6). It is actually contained in the terms $\epsilon_A(t)$ and $\epsilon_B(t)$, since these errors are functions of both T_{A1} and T_{B1}. It can be observed that (2.3.5) represents a two axis model for the human operator for only visual input. It does not take into account the effect of the feel force feedback. This is true for the n-axis model (2.3.6) also. It is still acceptable, since it is stated earlier that the feel force will not be considered as an additional input, but as a feedback to the human. The effect of the feel force feedback is to suitably modify the magnitudes of the coefficients α_1, α_2, α_3 and α_4 (adjusting mechanism of Fig 2.18) and hence the response otherwise obtained through the visual input.

The reaction time τ_d is usually very small, and also consideration of τ_d does not in any way affect the modelling methodology for the human operator. Thus τ_d is taken as zero in all the further studies made for the human operator. The problem of determination of the coefficients a_1, a_2, a_3 and a_4 using optimal control theory is considered in detail in chapter III.

2.4 A Mathematical Model for A Multi-Dimensional Man-Machine System.

Combining the 'n' dimensional feel system (2.2.4') and the human operator (2.3.6), a mathematical model for the 'n' dimensional man-machine system can be obtained.

The system of differential equations that describe an 'n' dimensional man-machine system are:

$$\begin{aligned} \dot{X}(t) &= (A' + F)\,X(t) + BT_1(t) + DT_2(t) \\ \dot{X}_m(t) &= PX_m(t) + R_1\,U(t) + R_2\,\dot{U}(t) \\ T_1(t) &= C_1\,X_m(t) \\ Y(t) &= C\,X(t) \\ U(t) &= Z(t) - Y(t) \end{aligned} \tag{2.4.1}$$

This completes the mathematical modelling of an 'n' dimensional feel system, the human operator and an 'n' dimensional man-machine system. Using (2.4.1), and the existing techniques of optimal control theory, a control is obtained in chapter 3, that will force the system response to follow a desired path. The problem of transferring an object optimally from one point in the actuator space to another with a feel system is also considered. A procedure is described for designing an 'n' dimensional feel system compatible with the assumed characteristics of the human operator (determination of F, P, R_1 and R_2).

CHAPTER III

OPTIMAL CONTROL OF SYSTEMS WITH FEEL

3.1 Introduction

It is shown in the previous chapter, that an 'n' dimensional feel system, can be represented by,

$$\dot{X}(t) = [A'(t) + F(t)]\, X(t) + B(t)\, T_1(t) + D(t)\, T_2(t) \qquad (2.2.4')$$

$$Y(t) = C\,X(t)$$

In this chapter, optimal control of feel systems described by (2.2.4') is considered.

The problem is to transfer an object from one point in the actuator space (Fig 1.3 pp 2) to another optimally, using the feel system (2.2.4'). The desired path, i.e the path along which it is ideal to transfer the object, is assumed to be known in advance. This assumption is justified, because of the following reason.

While experimenting with actual three degree of freedom remote manipulators, it was observed by G. Wolfman, A. Nicholson and H. Growth[30] that a human operator, who is the controller in the combined man-machine system, with a free choice, will tend to apply such a control as to minimize the difference between the straight line joining the initial and the final positions and the actual output position. Stated in more general terms, it simply means, that given all the constraints in the actuator space, the human operator, will first determine (or picture mentally) the optimal path of transfer $Z(t)$, and then apply such a control as to make the system response $Y(t)$, follow $Z(t)$ as closely as possible.

Two types of optimisation problems have been considered. They are: (i) the optimal tracking problem and (ii) the optimal transfer problem. For the tracking problem, a control is obtained, which will force the response of the system, to follow a specified path. For the optimal transfer problem, a control is obtained which, when applied to the feel system, will transfer an object from one point in the actuator space to another, and will minimize a given quadratic cost functional.

Factors that influence the design of a feel system are also investigated. A major problem associated with the design of a feel system is the choice of a proper feel matrix, and hence, criteria have been established based on which such a choice can be made. The feel matrix selected should be such that it optimises the system performance in some sense (subject to the constraint that the response obtained for the human operator model matches with the actual human response.)

Assuming that the optimal control of (2.2.4') is the optimal output of the human operator, for the tracking problem, the parameters P, R_1 and R_2 of the state space model (2.3.6) for the human operator have also been obtained.

Let us consider the optimal tracking problem first.

3.2 Statement of the Optimal Tracking Problem.

An object is initially at the position $Y(o) = Y_o$. It is to be transferred from $Y(o) = Y_o$ to the given final position $Y(T) = Y_f$ at $t = T$. $Z(t)$ is the desired path from Y_o to Y_f. The objective, is to obtain a control $T_1(t)$, which will force the response of the system (2.2.4') $Y(t)$, to follow $Z(t)$, as closely as possible, and which will minimize the quadratic cost functional,

$$J(T_1) = \frac{1}{2} < e(T), N\, e(T) > + \frac{1}{2} \int_0^T \{ < e(t), Q(t) e(t) > + < T_1(t), R T_1(t) > \} \, dt \qquad (3.2.1)$$

where $e(t) = (Z(t) - Y(t))$,

and N, Q and R are $n \times n$ weighting matrices.

The most important remark to be made about (3.2.1), is, regarding the nature of $Q(t)$. $Q(t)$, in the study of manual control systems, is referred to as 'the matrix of non-uniform importance'. It simply means that the weight attached to the error, changes with time. This time varying nature of Q(t) is very important. Since the actuator space in which the human operator positions the object is known in advance, the operator can change the importance he attaches to the error at any particular time, like for instance, he may penalise very heavily even for small errors, while carrying the object through a crowded environment, while paying no attention at all to the error at other times. Another important fact is that, we usually assume in most of the optimal control problems, that the non-diagonal elements of the weighting matrix Q(t) are zero. But actually they represent the cross-coupling effects of the errors in position in the various axes. Since, cross-coupling effects have been taken into consideration in obtaining (2.2.4'), one should also penalize for excessive cross-coupling, and as such, the non-diagonal elements should not be taken as zero. Also Q(t) should be positive definitive.

R(t) is a positive definite, time varying matrix. Ideally the non-diagonal elements should not be zero, since they represent the penalties imposed for excessive input cross-coupling.

The matrix N, penalises for the terminal error e(T), and is a constant positive semi-definite matrix.

3.3 Derivation of the Optimal Control Law.

The Hamiltonian H for the system (2.2.4') and the cost $J(T_1)$ (3.2.1) is given by,

$$H = L(X,t) + < p, f(X,t) > \tag{3.3.1}$$

$$= \frac{1}{2} < [Z(t) - C(t)\, X(t)],\ Q(t)[Z(t)-C(t)X(t)]> + \frac{1}{2} < T_1(t),\ R(t)\, T_1(t)>$$
$$+ < A(t)\, X(t),\ p(t)> + <B(t)\, T_1(t),\ p(t)> + <D(t)\, T_2(t), p(t)> \tag{3.3.2}$$

where, p(t), called, the co-state vector, is the solution of the vector differential equation,

$$\dot{p}(t) = - \frac{\partial H}{\partial T_1}(t) = 0 \tag{3.3.3}$$

which implies that

$$\left(\frac{\partial H}{\partial T_1}\right)(t) = R(t)\,T_1(t) + B^T(t)\,p(t) = 0 \qquad (3.3.4)$$

Hence, $T_1(t) = -\,R^{-1}(t)\,B^T(t)\,p(t)$ (3.3.5)

The assumption that R(t) is positive definite for all $t \in [0,T]$ guarantees the existence of $R^{-1}(t)$ for all $t \in [0,T]$.

The necessary condition $\left(\frac{\partial H}{\partial T_1}\right)(t) = 0$ yields only an extremum of H, with respect to T_1. In order for the extremum of H to be a minimum, with respect to T_1, the matrix $\left(\frac{\partial^2 H}{\partial T_1^2}\right)(t)$ must be positive definite.

From (3.3.4) we find that $\left(\frac{\partial^2 H}{\partial T_1^2}\right)(t) = R(t)$ (3.3.6)

Since R(t) is assumed to be positive definite, it follows that the control $T_1(t)$ (3.3.5) does indeed minimize H and hence is optimal.

The condition (3.3.2) yields

$$\dot{p}(t) = -\,C^T(t)\,Q(t)\,C(t)\,X(t) - A^T(t)\,p(t) + C^T(t)\,Q(t)\,Z(t) \qquad (3.3.7)$$

From (2.2.4!) and (3.3.4), we obtain the relation

$$\dot{X}(t) = A(t)\,X(t) - B(t)\,R^{-1}(t)\,B^T(t)\,p(t) + D(t)\,T_2(t) \qquad (3.3.8)$$

Combining (3.3.7) and (3.3.8) we obtain the following canonical equations.

$$\begin{bmatrix} \dot{X}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -C^T(t)Q(t)C(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} X(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} 0 \\ C^T(t)Q(t)Z(t) \end{bmatrix} + \begin{bmatrix} D(t)T_2(t) \\ 0 \end{bmatrix} \qquad (3.3.9)$$

This is a system of 8n linear time varying differential equations. The vectors $C^T(t)\,Q(t)\,Z(t)$ and $D(t)\,T_2(t)$ act as the forcing functions. At $t = 0$ 4n boundary conditions are provided by the initial state $X(0) = X_o$.

At $t = T$, 4n boundary conditions are provided by the transversality condition (see [40], p 302).

$$\begin{aligned} p(T) &= \frac{\partial}{\partial X(T)}\left[\tfrac{1}{2} < e(T),\, N e(T) > \right] \\ &= C^T(T)\,N\,C(T)\,X(T) - C^T(T)\,N\,Z(T) \qquad (3.3.10) \end{aligned}$$

Let $\phi(t,t_o)$ be the 8n x 8n fundamental matrix for the system (3.3.9)

Then,

$$\begin{bmatrix} X(T) \\ p(T) \end{bmatrix} = \phi(T,t) \begin{bmatrix} X(t) \\ p(t) \end{bmatrix} + \int_t^T \phi^{-1}(\tau;t) \begin{bmatrix} 0 \\ C^T(\tau)Q(\tau)Z(\tau) \end{bmatrix} d\tau + \int_t^T \phi^{-1}(\tau,t) \begin{bmatrix} D(\tau)T_2(\tau) \\ 0 \end{bmatrix} d\tau \tag{3.3.11}$$

Equation (3.3.11) is rewritten as

$$\begin{bmatrix} X(T) \\ p(T) \end{bmatrix} = \begin{bmatrix} \phi_{11}(T;t) & \phi_{12}(T;t) \\ \phi_{21}(T;t) & \phi_{22}(T;t) \end{bmatrix} \begin{bmatrix} X(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{3.3.12}$$

where

$$u_1(t) = \int_t^T \phi^{-1}(\tau;t)D(\tau)\,T_2(\tau)\,d\tau$$

$$u_2(t) = \int_t^T \phi^{-1}(\tau;t)\,C^T(\tau)\,Q(\tau)\,Z(\tau)\,d\tau$$

and $\phi_{ij}(T;t)$ i,j = 1,2 are 4n x 4n submatrices formed by partioning $\phi(T;t)$.

From (3.3.12) it follows that

$$\begin{aligned} X(T) &= \phi_{11}(T;t)\,X(t) + \phi_{12}(T;t)\,p(t) + r_1(t) \\ p(T) &= \phi_{21}(T;t)\,X(t) + \phi_{22}(T;t)\,p(t) + r_2(t) \end{aligned} \tag{3.3.13}$$

where $r_1(t) = \phi_{11}(T;t)\,u_1(t) + \phi_{12}(T;t)\,u_2(t)$

and $r_2(t) = \phi_{21}(T;t)\,u_1(t) + \phi_{22}(T;t)\,u_2(t)$

Assuming that the necessary inverse exists, it can be shown using (3.3.10) and (3.3.13) that,

$$p(t) = K(t)\,X(t) - h(t) \tag{3.3.14}$$

where

$$K(t) = [\phi_{22}(T;t)-C^T(T)NC(T)\phi_{12}(T;t)]^{-1}\,[C^T(T)NC(T)\phi_{11}(T;t)-\phi_{21}(T;t)]$$

and $h(t) = [\phi_{22}(T;t)-C^T(T)NC(T)\phi_{12}(T;t)]^{-1}\,[r_2(t)-C^T(T)NZ(T)-C^T(T)NC(T)r_1(t)]$

It may be noticed that for the optimal output regulator problem[40], the term h(t) does not appear in the expression (3.3.14) for the co-state p(t).

Differentiating (3.3.14) with respect to t

$$\dot{p}(t) = K(t)\,\dot{X}(t) + \dot{K}(t)\,X(t) - \dot{h}(t) \quad . \tag{3.3.15}$$

Eliminating $\dot{X}(t)$ from (3.3.15) using (3.3.8), we obtain,

$$\dot{p}(t) = K(t)\,[A(t)\,X(t) - B(t)R^{-1}(t)B^{T}(t)p(t) + D(t)T_2(t)] + \dot{K}(t)X(t) - \dot{h}(t) \tag{3.3.16}$$

A comparison (3.3.7) and (3.3.16) suggest that the following relations must be satisfied

$$\dot{K}(t) = -A^{T}(t)\,K(t) - K(t)A(t) + K(t)\,B(t)\,R^{-1}(t)\,B^{T}(t)\,K(t) - C^{T}(t)\,Q(t)C(t) \tag{3.3.17}$$

and
$$\dot{h}(t) = -[A(t) - B(t)R^{-1}(t)\,B^{T}(t)\,K(t)]^{T}h(t) + K(t)\,D(t)T_2(t) - C^{T}(t)Q(t)Z(t) \tag{3.3.18}$$

The boundary conditions for (3.3.17) and (3.3.18) may be obtained as follows:

From (3.3.14) at $t = T$, $p(T) = K(T)\,X(T) - h(T)$ (3.3.19)

From (3.3.10) $p(T) = C^{T}(T)NC(T) - C^{T}(t)\,N\,Z(T)$ (3.3.20)

Since (3.3.19) and (3.3.20) must hold for all X(T) we conclude that,

$$K(T) = C^{T}(T)N\,C(T) \quad . \tag{3.3.21}$$

$$h(T) = C^{T}(T)N\,Z(T) \tag{3.3.22}$$

Solving equations (3.3.17) and (3.3.18) with the boundary conditions (3.3.21) and (3.3.22), and substituting K(t) and h(t) in (3.3.14), p(t) can be obtained. Then using (3.3.5), the optimal control $T_1(t)$ can be determined.

The optimal trajectory is obtained by solving the following differential equation, obtained from (3.3.8) and (3.3.14).

$$\dot{X}(t) = [A(t) - B(t)R^{-1}(t)\,B^{T}(t)\;K(t)]X(t) + B(t)R^{-1}(t)B^{T}(t)h(t) + D(t)T_2(t)$$
$$X(o) = X_o \quad . \tag{3.3.23}$$

It will be also interesting to obtain an expression for the minimal cost $J^{*}[X(t), t]$ for all X(t), using Hamilton Jacobi equation.

The Hamilton Jacobi equation for the system (2.2.4') and the cost functional $J(T_1)$(3.2.1) is

$$\frac{\partial J^*}{\partial t} + \min_{T_1(t)} \{ \frac{1}{2} < e(t), Q(t)\, e(t)> + \frac{1}{2} < T_1(t), R(t) T_1(t)> + < A(t)\, X(t), \frac{\partial J^*}{\partial X}(t) >$$

$$+ < B(t) T_1(t)\, \frac{\partial J^*}{\partial X}(t) > + < D(t) T_2(t), \frac{\partial J^*}{\partial X}(t) > \} = 0 \qquad (3.3.24)$$

where $e(t) = [Z(t) - C(t)\, X(t)]$

The control that minimises the expression in the braces is given by,

$$T_1(t) = -R^{-1}(t)\, B^T(t)\, \frac{\partial J^*}{\partial X}(t) \qquad (3.3.25)$$

Substituting equation (3.3.25) in equation (3.3.24)

$$\frac{\partial J^*}{\partial t} + \frac{1}{2} < [Z(t) - C(t)\, X(t)], Q(t) [Z(t) - C(t)\, X(t)] > + < A(t)\, X(t), \frac{\partial J^*}{\partial X}(t)>$$

$$+ \frac{1}{2} < R^{-1}(t)\, B^T(t)\, \frac{\partial J^*}{\partial X}(t), \frac{\partial J^*}{\partial X}(t)> + < D(t)\, T_2(t), \frac{\partial J^*}{\partial X}(t)> \qquad (3.3.26)$$

$$= 0 .$$

The equation for the minimal cost appears to be of the form

$$J^* [X(t), t] = \frac{1}{2} < X(t), K(t) X(t)> - <h(t), X(t)> + \varphi(t) \qquad (3.3.27)$$

where $\varphi(t)$ satisfies the differential equation

$$\dot{\varphi}(t) = -\frac{1}{2}[< Z(t), Q(t)\, Z(t)> - <h(t), B(t) R^{-1}(t) B^T(t) h(t)> + <h(t), D(t) T_2(t) >] \qquad (3.3.28)$$

Differentiating (3.3.27) ω.r.t. t and X(t), yields

$$\frac{\partial J^*}{\partial t} = \frac{1}{2} < X(t), \dot{K}(t)\, X(t)> - <\dot{h}(t), X(t)> + \dot{\varphi}(t) \qquad (3.3.29)$$

$$\frac{\partial J^*}{\partial X}(t) = K(t)\, X(t) - h(t) \qquad (3.3.30)$$

Substituting (3.3.29) and (3.3.30) in (3.3.26), we obtain

$$\frac{1}{2} < X(t), \dot{K}(t)X(t)> - < \dot{h}(t), X(t)> + \dot{\varphi}(t) + <A(t)X(t), \frac{\partial J^*}{\partial X}(t) > + < D(t)T_2(t), \frac{\partial J}{\partial X}(t) >$$

$$+ \frac{1}{2} < [Z(t) - C(t)X(t)], Q(t) [Z(t) - C(t)X(t)] > + \frac{1}{2} < R^{-1}(t)B^T(t)\frac{\partial J^*}{\partial X}(t), \frac{\partial J^*}{\partial X}(t)> = 0$$

or

$$\frac{1}{2} < X(t), [\dot{K}(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)K(t) + C^T(t)Q(t)C(t)] X(t) >$$

$$- <(\dot{h}(t) + A^T(t)h(t) - K(t) B(t)R^{-1}(t) B^T(t)h(t) + C^T(t) Q(t)Z(t) - K(t)D(t)T_2(t)), X(t)> =0$$

$$= 0 \qquad (3.3.31)$$

By vitue of equations (3.3.17) and (3.3.18), (3.3.31) is true for all X(t). Hence the equation (3.3.27) assumed for the minimum cost is true.

3.4 Discussions.

Let us now examine the differential equations (3.3.17), (3.3.18) and (3.3.23) obtained in section 3.3. It can be seen that the matrix Riccati equation (3.3.17) and the boundary condition (3.3.19) are independent of the desired path Z(t), and the desired output force - $T_2(t)$. This means that the gain matrix K(t) is completely specified, once the system (2.2.4') and the cost functional (3.2.1), and the terminal time T are specified.

The dynamical behaviours of both the systems (3.3.18) and (3.3.23) are also independent of Z(t) and - $T_2(t)$. Z(t) and $-T_2(t)$ can be considered as the forcing functions to the dynamical system that generates h(t).

Let $\psi(t, t_o)$ be the fundamental matrix of the system (3.3.18). We can then conclude immediately that

$$C^T(T)NZ(T) = h(T) = \psi(T;t)[h(t) - \int_t^T \psi^{-1}(\tau;t)C^T(\tau)Q(\tau)Z(\tau)d\tau - $$

$$- \int_t^T \psi^{-1}(\tau;t)K(\tau)D(\tau)d\tau] \qquad (3.4.1)$$

or for all $t \in [0, T]$ that

$$h(t) = \psi^{-1}(T;t)h(\tau) + \int_t^T \psi^{-1}(\tau;t)C^T(\tau)Q(\tau)Z(\tau)d\tau + \int_t^T \psi^{-1}(\tau;t)K(\tau)D(\tau)T_2(\tau)d\tau$$

$$(3.4.2)$$

The implication of (3.4.2) is that, in order to evaluate $h(t)$, $t \in [0,T]$ we must know $Z(\tau)$ and $-T_2(\tau)$ for all $\tau \in [t,T]$. In other words, in order to evaluate the present values of $h(t)$, we must know all the future values of the desired path $Z(t)$, and the output force $-T_2(t)$.

Since the optimal control is given by,

$$T_1(t) = -R^{-1}(t)\, B^T(t)\, [K(t)\, X(t) - h(t)] \,, \tag{3.4.3}$$

We further conclude that the present value of the optimal control $T_1(t)$, depends upon the future values of $Z(t)$ and $-T_2(t)$. But, it is not possible to have such information in many practical problems. If the future values of $Z(t)$ and $-T_2(t)$ are not available, then the control (3.4.3) cannot be realized. For, the particular tracking problem under consideration, however, $Z(t)$ and $-T_2(t)$ are assumed to be known for all $t \in [0,T]$.

Let us examine the computational aspect of determining $K(t)$ and $h(t)$. Since the matrix $K(t)$, is independent of $Z(t)$ and $-T_2(t)$, it can be precomputed. Knowledge of $Z(t)$ and $-T_2(t)$ will enable us to compute $h(t)$ backward in time for all $t \in [0,T]$. Then the solution $h(t)$ of (3.3.18) can be stored, and can be made use of in the forward integration of (3.3.23). Alternatively, the initial value $h(o)$ can be computed using the formula,

$$h(o) = \psi^{-1}(T;0)C^T(T)N\, Z(T) + \int_o^T \psi^{-1}(\tau,0)C^T(\tau)Q(\tau)\, Z(\tau)d\tau + \int_o^T \psi^{-1}(\tau,0)K(\tau)T_2(\tau)d\tau \tag{3.4.4}$$

We can then use the computed value of $h(o)$ as the initial condition for the system (3.3.16) and generate the solution $h(t)$ forward in time. Fig 3.1 illustrates the dynamical system which generates $h(t)$, once $h(o)$ has been precomputed. Fig 3.2 shows the structure of the overall system for the tracking problem.

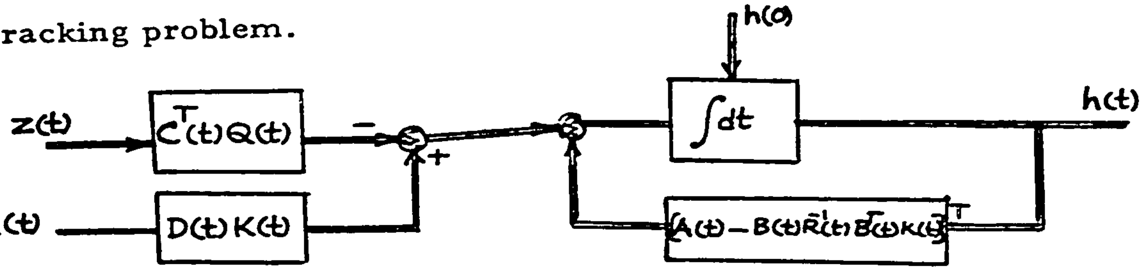


Fig 3.1 Simulation of the system (3.3.18) for generating $h(t)$. It is assumed that $h(o)$ and $K(t)$ have been precomputed.

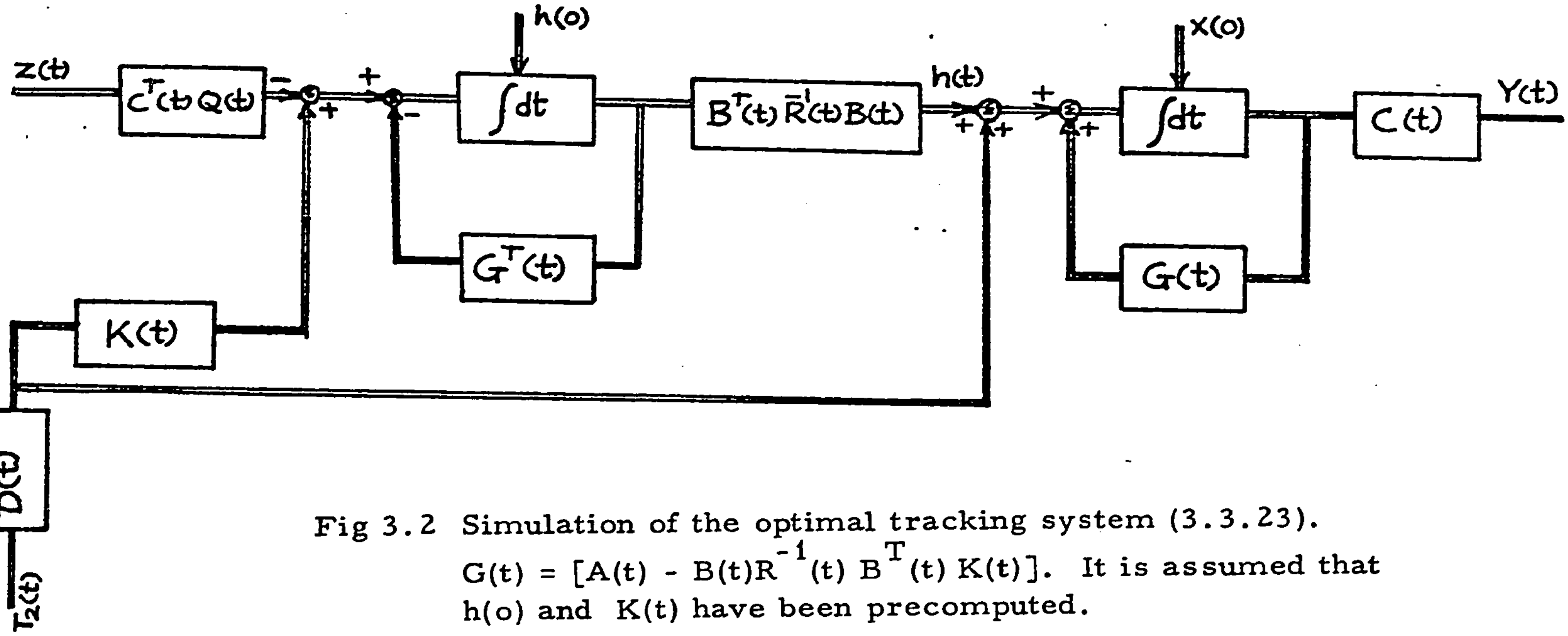


Fig 3.2 Simulation of the optimal tracking system (3.3.23). $G(t) = [A(t) - B(t)R^{-1}(t)\ B^T(t)\ K(t)]$. It is assumed that h(o) and K(t) have been precomputed.

Though it looks very simple from the above discussion, it will be extremely difficult to obtain a real time solution to the optimal tracking problem. This is because of the computations required for generating h(t). Hence, an alternative formulation of the problem is required.

The given tracking problem, hence, is converted into an output regulator problem and then solved. This is described in the following section.

3.5 Conversion of the Tracking Problem into an Output Regulator Problem.

The following assumptions are made:

(i) Z(t) is the solution of a given linear homogeneous system described by the differential equation,

$$\dot{Z}(t) = E\ Z(t) \qquad (3.5.1)$$

(ii) $-T_2(t)$, the required output force of the system, (2.2.4') is also the solution of a linear homogeneous system described by the differential equation

$$\dot{T}_2(t) = M\ T_2(t) \qquad (3.5.2)$$

Also, it will be explained later, that in order to obtain a description for the human operator, a steady state solution to the optimal tracking problem is required. It is claimed by Athans and Falb ([40]; pp 801) that no steady state solution exists for the tracking problem considered in section 3.3, except an approximate solution, as the terminal time $T \to \infty$. However, there exists a solution for the output regulator problem at $T = \infty$. ([40] pp 785).

Hence, with the assumptions (3.5.1), (3.5.2) and by a judicious selection of an output vector, the optimal tracking problem is converted into an output regulator problem.

The solution to the modified tracking problem is obtained by solving the following system of differential equations.

$$\begin{aligned} \dot{Z}(t) &= E\,Z(t) \\ \dot{T}_2(t) &= M\,T_2(t) \\ \dot{X}(t) &= (A'(t) + F(t))\,X(t) + B(t)T_1(t) + D(t)\,T_2(t) \qquad (3.5.3) \end{aligned}$$

A new system is formed by augmenting the state space of the original system (2.2.4') and having $Z(t)$, $T_2(t)$, and $X(t)$ as the state variables.

It is described by,

$$\dot{X}_a(t) = A_a(t)\,X_a(t) + B_a(t)\,T_1(t) \qquad (3.5.4)$$

and $\quad Y_a(t) = C_a\,X_a(t)$

where $X_a(t) = \begin{bmatrix} Z(t) \\ T_2(t) \\ X(t) \end{bmatrix}$, $A_a(t) = \begin{bmatrix} E & 0 & 0 \\ 0 & M & 0 \\ 0 & D(t) & [A'(t)+F(t)] \end{bmatrix}$, $B_a(t) = \begin{bmatrix} 0 \\ 0 \\ B(t) \end{bmatrix}$

C_a is chosen in such a manner as to make the output vector of the augmented system $Y_a(t)$, $= e(t) = Z(t) - Y(t)$, where $Y(t)$ is the output position of the original system.

The problem is to find a control $T_1(t)$, that will minimise the cost functional,

$$J(T_1) = \frac{1}{2} < e(T), N e(T) > + \frac{1}{2}\int_0^T [< e(t), Q(t)e(t) > + < T_1(t), R(t)T_1(t) >]\,dt \qquad (3.5.5)$$

The Hamiltonian H for the system (3.5.4) and the cost functional (3.5.5) is

$$H = \frac{1}{2} < C_a X_a(t), Q(t)\ C_a X_a(t)> + \frac{1}{2} < T_1(t), R(t)\,T_1(t)> + < A_a X_a(t), p(t)> +$$

$$+ \ < B_a(t) T_1(t), p(t) > \qquad (3.5.6)$$

Again as in section in 3.3, the control $T_1(t)$,is assumed to be unconstrained, then we must have along the optimal trajectory,

$$\left(\frac{\partial H}{\partial T_1}\right)(t) = 0$$

which imples that $\left(\frac{\partial H}{\partial T_1}\right)(t) = R(t)\ T_1(t) + B_a^T(t)\ p(t) = 0$

and $T_1(t) = -R^{-1}(t)\ B_a^T(t)\ p(t)$ (3.5.7)

For the ouptut regulator problem, it is known[40] that the co-state p(t) is a linear function of the state $X_a(t)$, and is given by,

$$p(t) = K(t)\ X_a(t) \qquad (3.5.8)$$

using (3.5.7) and (3.5.8) the optimal control can be obtained as

$$T_1(t) = -R^{-1}\ B_a^T(t)\ K(t)\ X_a(t) \qquad (3.5.9)$$

Differentiating the Hamiltonian H(3.5.6) ω.r.t. $X_a(t)$ yields,

$$\dot{p}(t) = -\left(\frac{\partial H}{\partial X_a}\right)(t) = C_a^T\ Q(t) C_a + A_a^T(t)\ K(t)\ X_a(t) \qquad (3.5.10)$$

Also, differentiating (3.5.8) we obtain,

$$\dot{p}(t) = K(t)\ \dot{X}_a(t) + \dot{K}(t)\ X_a(t)$$

$$= \dot{K}(t)\ X_a(t) + K(t)\,[A_a(t)\ X_a(t) + B_a(t)\ T_1(t)] \qquad (3.5.11)$$

From a comparison of equation (3.5.10) and (3.5.11) for $\dot{p}(t)$, we obtain

$$\dot{K}(t) = -K(t)\ A_a(t) - A_a^T(t)\ K(t) + K(t)\ B_a(t) R^{-1}(t)\ B_a^T(t)\ K(t) - C_a^T\ Q(t) C_a \qquad (3.5.12)$$

The boundary condition for (3.5.12) may be obtained as follows.

From the transversality condition [40, pp 302] ,

$$p(T) = \frac{\partial}{\partial X_a(T)}\left[\frac{1}{2} < C_a\ X_a(T),\ N C_a X_a(T) > \right] \qquad (3.5.13)$$

$$= C_a^T\ N C_a\ X_a(T) \qquad (3.5.14)$$

Also from equation (3.5.8), we have at $t = T$

$$p(T) = K(T)\, X_a(T) \tag{3.5.15}$$

Since (3.5.13) and (3.5.14) must hold for all $X_a(T)$, we conclude that

$$K(T) = C_a^T\, N\, C_a \tag{3.5.16}$$

Using (3.5.9) and (3.5.4) we can obtain an equation for the optimal trajectory of the augmented system

$$\dot{X}_a(t) = [A_a(t) - L(t)]\, X_a(t) \tag{3.5.17}$$

where $L(t) = B_a(t)\, R^{-1}(t)\, B_a^T(t)\, K(t)$

$L(t)$ is partitioned as shown below

$$L(t) \triangleq \begin{bmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t) \\ L_{21}(t) & L_{22}(t) & L_{23}(t) \\ L_{31}(t) & L_{32}(t) & L_{33}(t) \end{bmatrix} \tag{3.5.18}$$

$$Y_a(t) = C_a X_a(t) \quad \text{where} \quad C_a = [\, C_1 \,\vdots\, 0 \,\vdots\, C_2 \,] \tag{3.5.19}$$

The optimal trajectory for the original system is the solution of the differential equation.

$$\dot{X}(t) = (A'(t) + F(t) - L_{33}(t))X(t) + (D(t) - L_{32}(t))\, T_2(t) - L_{31}(t)\, Z(t)$$

with the initial condition $X(o) = X_o$ (3.5.20)

The optimal output position is given by $Y(t) = C_1 Z(t) + C_2 X(t)$ (3.5.21)

The structure of the optimal system (3.5.20) is shown in Fig 3.3 below:

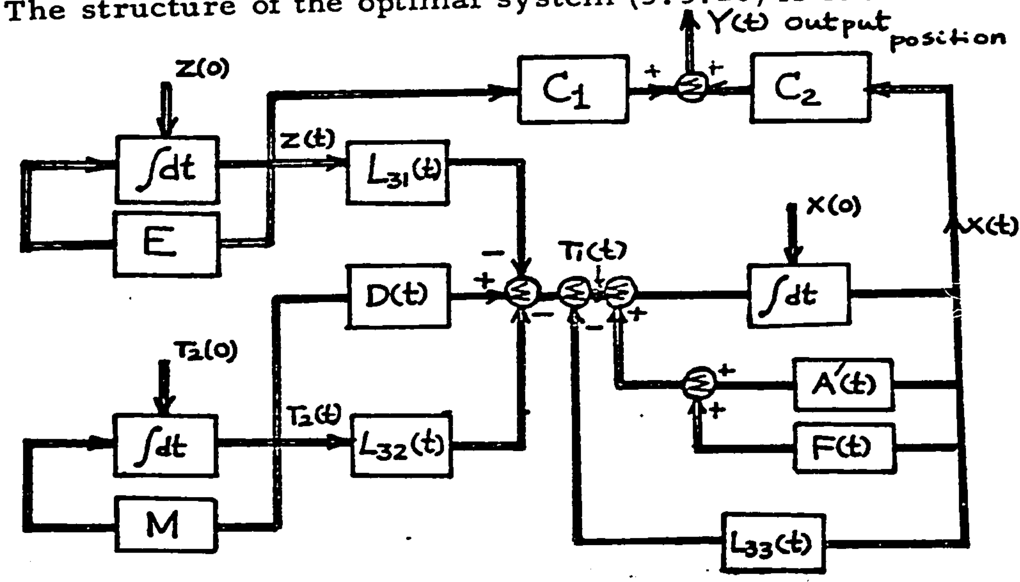


Fig 3.3. Structure of the optimal feedback system

3.6 Solution of the Output Regulator Problem.

Equation (3.5.20) can be used for investigating the performance of an optimal 'n' dimensional feel system. However, in order to reduce the computational task, numerical solution is attempted only for the one dimensional case. $T_2(t)$ is assumed to be a passive load on the system, and is replaced by a linear combination of the state variables (2.1.7) in (2.1.4). Also, the resulting system (2.1.10) is assumed to be time invariant.

The augmented system for this case is,

$$\dot{X}_a(t) = A_a X_a(t) + B_a T_1(t) \tag{3.6.1}$$

and $$Y_a(t) = C_a X_a(t)$$

where,

$$A_a = \begin{bmatrix} E & 0 \\ & \\ 0 & A_1 \end{bmatrix}, \; B_a \begin{bmatrix} 0 \\ \text{----} \\ B \end{bmatrix}, \text{ and } C_a = [C_1 \quad C_2] \tag{3.6.2}$$

The matrix E is assumed to be

$$E = \begin{bmatrix} 0 & 1 \\ & \\ -e_1 & -e_2 \end{bmatrix} \tag{3.6.3}$$

The matrix A_1 is given by (2.1.11), and is reproduced here, for easy reference.

$$A_1 = (A' + F + SD) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_{11}/J_1 & -(D_{11}+F_1/J_1) & K_{12}/J_1 & D_{12}/J_1 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_{21}}{J_2(1+\alpha)} & \dfrac{D_{21}}{J_2(1+\alpha)} & -\dfrac{K_{22}(1+\beta)}{J_2(1+\alpha)} & -\dfrac{[D_{22}(1+r)+F_2]}{J_2(1+\alpha)} \end{bmatrix} \tag{3.6.4}$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.6.5}$$

The weighting matrices are chosen to be of the form

$$R = r, \quad \text{and} \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \qquad (3.6.6)$$

The matrix L(t), for this case, is of the form

$$L(t) = \begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix}, \qquad (3.6.7)$$

and is given by

$$L(t) = B_a R^{-1} B_a^T K(t)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{k_{14}(t)}{J_1^2 r} & \frac{k_{24}(t)}{J_1^2 r} & \frac{k_{34}(t)}{J_1^2 r} & \frac{k_{44}(t)}{J_1^2 r} & \frac{k_{45}(t)}{J_1^2 r} & \frac{k_{46}(t)}{J_1^2 r} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.6.8)$$

The optimal trajectory of the original system (2.1.10), is the solution of the differential equation,

$$\dot{X}(t) = [A_1 - L_{22}(t)]\, X(t) - L_{21}(t)\, Z(t) \qquad (3.6.9)$$

with $X(o) = X_o$

To obtain the solution of (3.6.9), the gain matrix K(t), which is the solution of the matrix Riccati equation (3.5.12) is required.

This in turn requires the solution of the following 21 first order nonlinear differential equations, obtained by expanding (3.5.12).

$$\dot{k}_{11}(t) = 2k_{12}(t)\, e_1 + k^2_{14}(t)/r - q_1$$

$$\dot{k}_{12}(t) = - k_{11}(t) + k_{12}(t)\, e_2 + k_{22}(t)\, e_1 + k_{14}(t)\, k_{24}(t)/r$$

$$\dot{k}_{13}(t) = k_{14}(t)\, K_{11}/J_1 - k_{16}(t)\, K_{21}/J_2(1+a) + k_{23}(t)\, e_1 + k_{14}(t)\, k_{34}(t)/r$$

$$\dot{k}_{14}(t) = -k_{13}(t) + k_{14}(t)\,[D_{11}+F_1]/J_1 - \frac{k_{16}(t)D_{21}}{J_2(1+a)} + k_{24}(t)\, e_1 + k_{14}(t)\, k_{44}(t)/r$$

$$\dot{k}_{15}(t) = - k_{14}(t)K_{12}/J_1 + k_{16}(t)K_{22}(1+\beta)/J_2(1+a) + k_{25}(t)e_1 + k_{14}(t)k_{45}(t)/r + q_1$$

$$\dot{k}_{16}(t) = -k_{14}(t)D_{12}/J_1 - k_{15}(t) + k_{16}(t)\,[D_{22}(1+r)+F_2]/J_2(1+a) + k_{26}(t)e_1 \frac{k_{14}(t)k_{46}(t)}{r}$$

$$\dot{k}_{22}(t) = - 2k_{12}(t) + 2k_{22}(t)\, e_2 + k^2_{24}(t)/r - q_2$$

$$\dot{k}_{23}(t) = - k_{24}(t)K_{11}/J_1 - k_{26}(t)\, K_{21}/J_2(1+a) - k_{13}(t) + k_{23}(t)e_2 + k_{24}(t)k_{34}(t)/r$$

$$\dot{k}_{24}(t) = - k_{23}(t) + k_{24}(t)(D_{11}+F_1)/J_1 - k_{26}(t)D_{21}/J_2(1+a) - k_{14}(t) + k_{24}(t)e_2 + k_{24}(t)k_{44}(t)/r$$

$$\dot{k}_{25}(t) = - k_{24}(t)K_{12}/J_1 + k_{26}(t)K_{22}(1+\beta)/J_2(1+a) - k_{15}(t) + k_{25}(t)e_2 + \frac{k_{24}(t)k_{45}(t)}{r}.$$

$$\dot{k}_{26}(t) = - k_{24}(t)D_{12}/J_1 + k_{26}(t)[D_{22}(1+r) + F_2]/J_2(1+a) - k_{16}(t) + k_{26}(t)e_2 + q_2 + \\ + k_{24}(t)k_{46}(t)/r$$

$$\dot{k}_{33}(t) = 2k_{34}(t)K_{11}/J_1 - 2\, k_{36}(t)K_{21}/J_2(1+a) + k^2_{34}(t)/r$$

$$\dot{k}_{34}(t) = - k_{33}(t) + k_{34}(t)(D_{11}+F_1)/J_1 - k_{36}(t)D_{21}/J_2(1+a) + k_{44}(t)K_{11}/J_1 \\ - k_{46}(t)K_{21}/J_2(1+a) + k_{34}(t)\, k_{44}(t)/r$$

$$\dot{k}_{35}(t) = - k_{34}(t)\, K_{12}/J_1 + \frac{k_{36}(t)K_{22}(1+\beta)}{J_2(1+a)} + \frac{k_{45}(t)K_{11}}{J_1} - \frac{k_{56}(t)K_{21}}{J_2(1+a)} + \frac{k_{34}(t)k_{45}(t)}{r}$$

$$\dot{k}_{36}(t) = - k_{34}(t)D_{12}/J_1 - k_{35}(t) + k_{36}(t)[D_{22}(1+r)+F_2]/J_2(1+a) + k_{34}(t)k_{46}(t)/r \\ + k_{46}(t)K_{11}/J_1 - k_{66}(t)\, K_{21}/J_2(1+a)$$

$$\begin{aligned}
\dot{k}_{44}(t) &= -2k_{34}(t) + 2k_{44}(t)(D_{11}+F_1)/J_1 - 2k_{46}(t)D_{21}/J_2(1+\alpha) + k_{44}^2(t)/r \\
\dot{k}_{45}(t) &= -k_{44}(t)K_{12}/J_1 - k_{45}(t) + k_{46}(t)[D_{22}(1+r) + F_2]/J_2(1+\alpha) - k_{36}(t) \\
&\quad + k_{46}(t)(D_{11}+F_1)/J_1 - k_{46}(t)D_{21}/J_2(1+\alpha) + k_{44}(t)k_{46}(t)/r \\
\dot{k}_{46}(t) &= -k_{44}(t)\,D_{12}/J_1 - k_{45}(t) + k_{46}(t)\,[D_{22}(1+r)+F_2]/J_2(1+\alpha) - k_{36}(t) - k_{46}(t)\frac{(D_{11}+F_1)}{J_1} \\
&\quad + k_{44}(t)\,k_{46}(t)/r + k_{66}(t)\,D_{21}/J_2(1+\alpha) \\
\dot{k}_{55}(t) &= -2k_{45}(t)K_{12}/J_1 + 2k_{55}(t)\,\frac{K_{22}(1+\beta)}{J_2(1+\alpha)} + k_{45}^2(t)/r - q_1 \\
\dot{k}_{56}(t) &= -k_{45}(t)D_{12}/J_1 - k_{55}(t) + k_{56}(t)[D_{22}(1+r)+F_2]/J_2(1+\alpha) - k_{46}(t)\,K_{12}/J_1 \\
&\quad + k_{45}(t)k_{46}(t)/r + k_{66}(t)K_{22}(1+\beta)/J_2(1+\alpha) \\
\dot{k}_{66}(t) &= -2k_{46}(t)D_{12}/J_1 - 2k_{56}(t) + 2k_{66}(t)[D_{22}(1+r)+F_2]/J_2(1+\alpha) + k_{66}^2(t)/r - q_2
\end{aligned} \tag{3.6.10}$$

In general, if K is a n x n matrix, there will be $n(n+1)/2$ first order nonlinear differential equations to be solved. Since K is symmetric, there will be $(n^2-n)/2$ non-diagonal elements, and n diagonal elements, and hence the total number of elements is $n(n+1)/2$. .

The solution of the matrix Riccati equation (3.5.12) is obtained through the numerical integration of $\dot{K}(t)$, using the following approximation,

$$\dot{K}(t) \simeq \frac{K(t+\Delta) - K(t)}{\Delta} \tag{3.6.11}$$

The Riccati equation is solved backwards in time, using a small negative step of Δ.

The boundary condition (3.5.15) used for solving (3.5.12) is

$$K(T) = C_a^T \, N \, C_a = 0 \tag{3.6.12}$$

For higher order systems, the number of equations to be integrated is large, and takes a significant amount of computer time. Special computational procedures such as 'Automatic synthesis matrix iterative procedure'[31] or Negative exponential solution [32] have to be made use of in such cases. The Fourth order Runge Kutta numerical integration procedure with a negative increment of time can also be used for obtaining the solution of (3.5.12).

After computing the gain matrix K(t), its values are stored, and the system equations (3.6.4) are integrated forward to obtain the optimal state trajectories, and hence the optimal output positions. A family of solutions is obtained (i) for different feel matrices (ii) for different weighting factors and (iii) for different loads. A discussion of the numerical results is given in section 3.7. The flow chart for computing the gain matrix K(t) and the optimal output positions is shown in Fig 3.4. The flow chart is drawn using conventional techniques of Fortran IV (51).

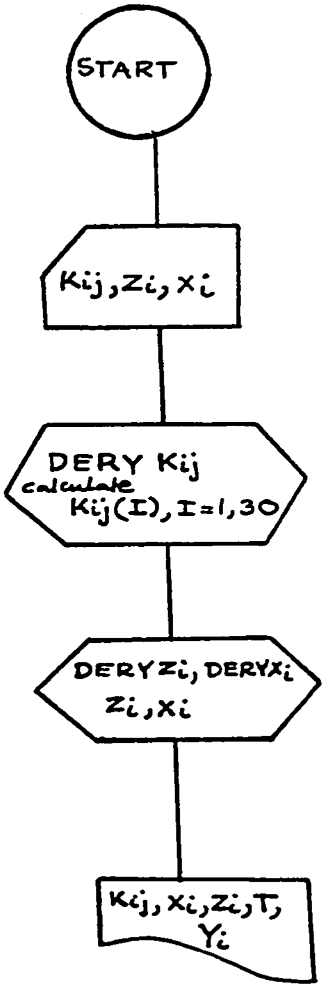


Fig 3.4. Flow chart for computing the gain matrix K(t) and optimal position Y(t).

3.7 Discussion of Numerical Results.

While obtaining the numerical solutions of (3.6.9), the following initial condition has been assumed for the augmented state vector,

$$X_a^T(o) = [5 \quad 0 \quad 5 \quad 0 \quad 5 \quad 0] \qquad (3.7.1)$$

The matrix E, which defines the reference path, Z(t), is chosen as,

$$E = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.7 \end{bmatrix} \qquad (3.7.2)$$

Fig 3.5 shows the system behaviour for different feel matrices of (2.1.5). Figs 3.6 and 3.7, illustrate the system performance for different Q matrices (of 3.5.5), and for different loads (of 2.1.7) respectively. The results of Fig 3.5 are very useful since they establish a basis for comparing the force reflecting characteristics of the various feel systems.

The feel matrix (2.1.5) will be modified when there is a load on the system, and is given by

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -K_{11}/J_1 & -D_{11}/J_1 & K_{12}/J_1 & D_{12}/J_1 \\ 0 & 0 & 0 & 0 \\ K_{21}/J_2(1+\alpha) & D_{21}/J_2(1+\alpha) & -\dfrac{K_{22}(1+\beta)}{J_2(1+\alpha)} & -\dfrac{D_{22}(1+r)}{J_2(1+\alpha)} \end{bmatrix} \qquad (3.7.3)$$

The typical feel matrices (3.7.3) chosen are:

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0.9 & -1.2 & -1.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0 & -1.2 & -0.2 \end{bmatrix}$$

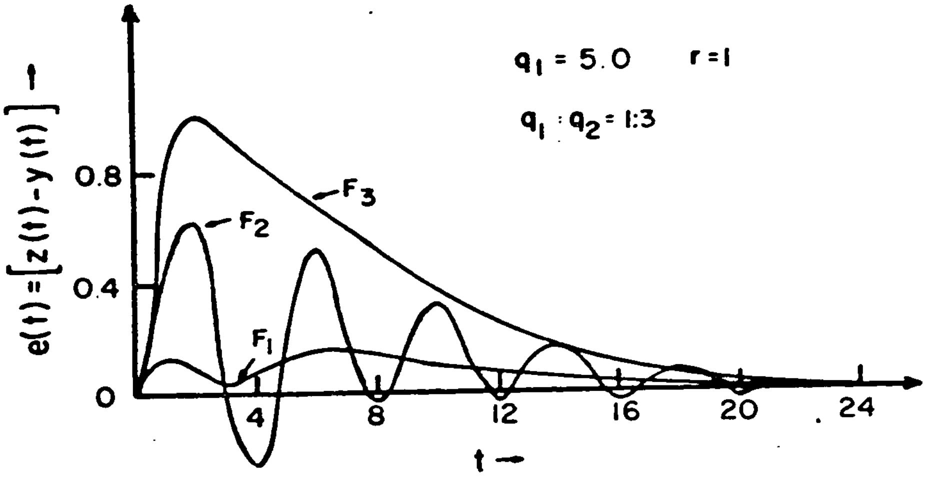


Fig 3.5 System behaviour for different feel matrices.

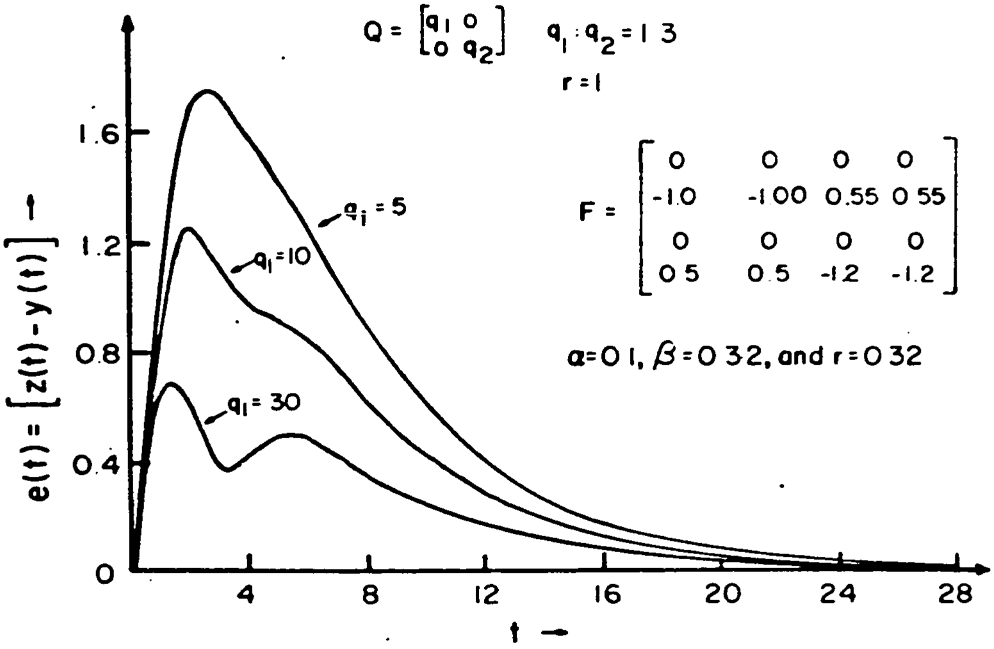


Fig 3.6 System behaviour for different Q matrices.

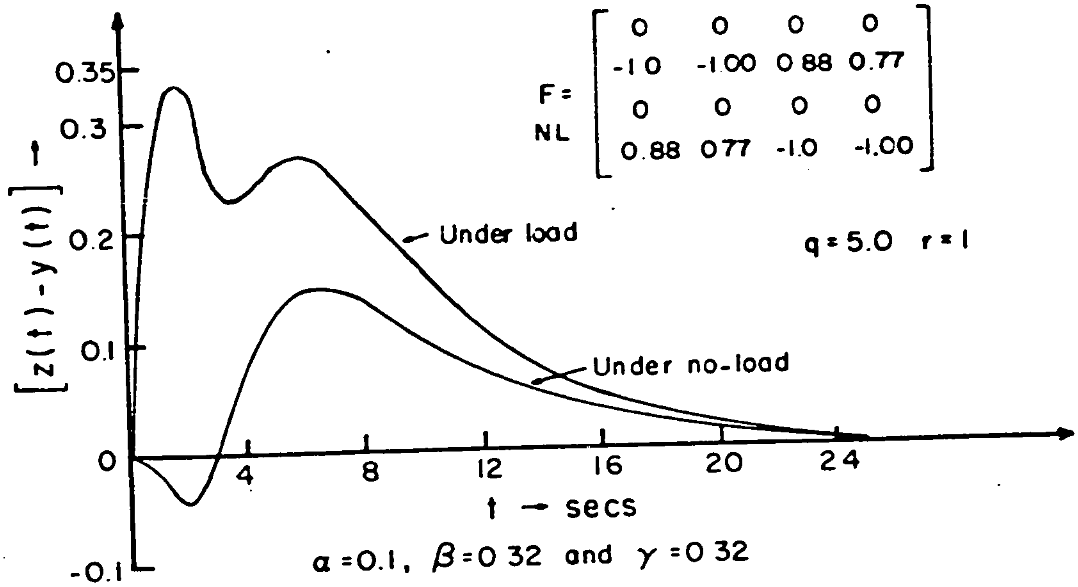


Fig 3.7 System behaviour for two loads.

$$F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.0 & -1.0 & 0.66 & 0.55 \\ 0 & 0 & 0 & 0 \\ 0.6 & 0.5 & -1.2 & -1.2 \end{bmatrix} \qquad (3.7.4)$$

Through an observation of the optimal output positions Y(t) for different feel matrices it is found that the better the quality of force reflection of the feel system, the smaller will be the deviation, e(t), of Y(t) from Z(t). Thus, of the three feel matrices (3.7.4) considered, the system with the feel matrix F_1 has the best force reflecting characteristic, where as the system with the feel matrix F_3 has the worst.

The difference between the matrices F_1 and F_2 is that in F_2, the values of the gains D_{ij}, i, j = 1, 2, are equal to zero, i.e the system with the feel matrix F_2, has only position feedback, and no information is available about the rate of change of position. Hence, for this system, the errors in tracking are much larger than in the case of the system with the feel matrix F_1. However, it does not mean, that by simply providing a system with rate feedback, a better feel can be obtained. For instance, the system with the feel matrix F_3, which has both rate and position feedbacks, has a poorer feel compared to the system with the feel matrix F_2. Thus a good feel also depends upon the relative values of the gains K_{ij} and D_{ij}. Similar conclusions are also reached, through actual experimentation with force reflecting servomechanisms (one dimensional feel system).

It is observed in [12], that by using positive tachometric and acceleration feedbacks around each end of a force reflective servomechanism, the effective inertia and friction of the system actuators can be reduced by the same amount, and this improves the no load feel of the servo. This result is verified here, using the criterion stated above. The changes in F_i, J_i, i = 1, 2, are taken into account, by a suitable modification of the feel matrix F (2.1.5), and the corresponding results are plotted in Fig 3.8.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$q_1 : q_2 = 1 : 3, \quad q_1 = 5.0$

$r = 1$

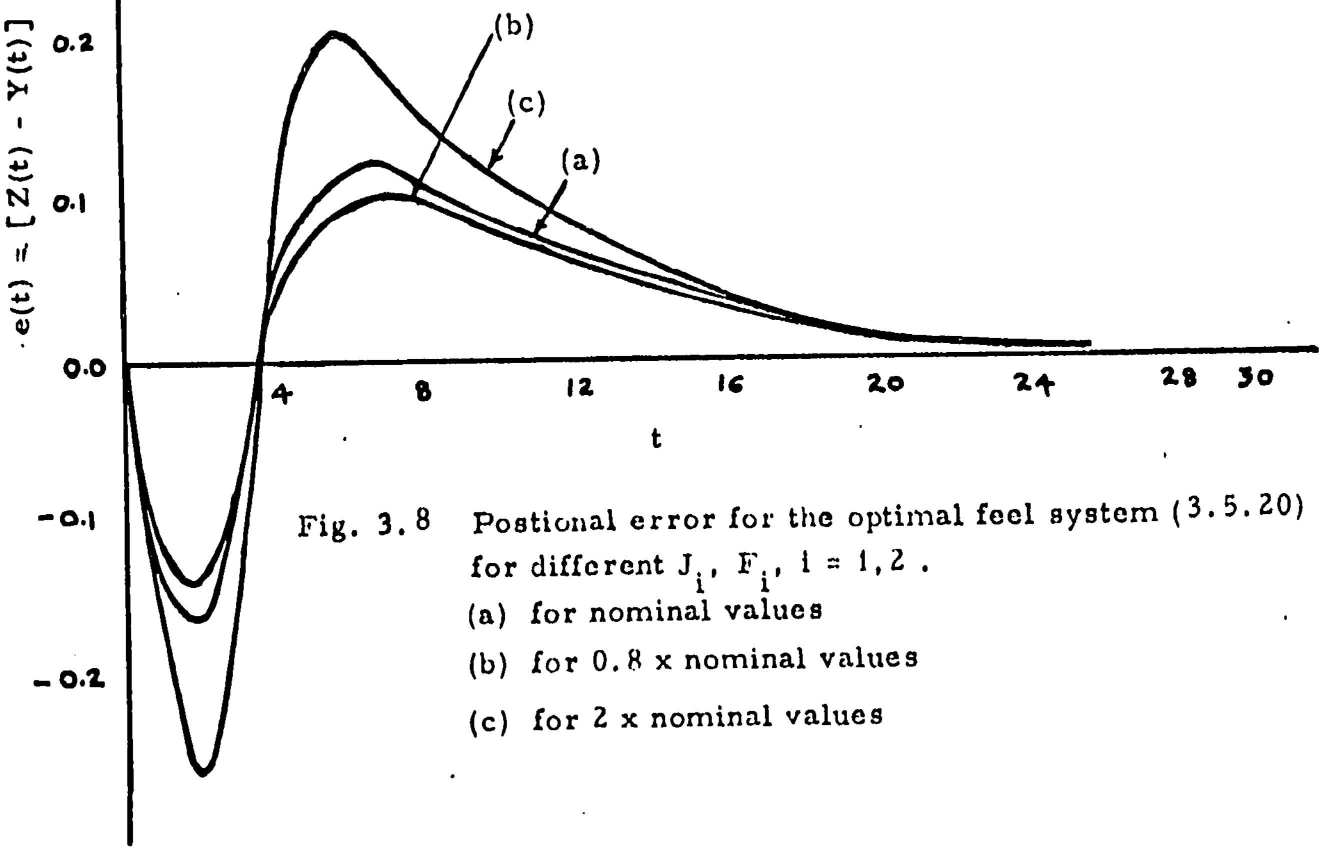


Fig. 3.8 Postional error for the optimal feel system (3.5.20) for different J_i, F_i, $i = 1,2$.

(a) for nominal values

(b) for 0.8 x nominal values

(c) for 2 x nominal values

It can be seen from Fig 3.8 that the errors in the follow up are much smaller with lower values of J_i, F_i i = 1, 2 of (2. 1. 4). Thus the no-load feel of the system is better with lower values of J_i and F_i.

A measure of the quality of force reflection of a one dimensional feel system can be obtained, by defining a feel constant,

$$S^1 = \int_o^T |e(t)| \, dt \qquad (3.7.5)$$

where $e(t) = (Z(t) - y(t))$

For the two dimensional feel system, a feel vector can be defined as

$$S^2 = \begin{bmatrix} S_A \\ S_B \end{bmatrix} = \begin{bmatrix} \int_o^T |e_A(t)| \, dt \\ \int_o^T |e_B(t)| \, dt \end{bmatrix} \qquad (3.7.6)$$

where $e_A(t) = [Z_A(t) - Y_A(t)]$

and $e_B(t) = [Z_B(t) - Y_B(t)]$

$$\|S\| = \left(\sum_{i=A,B} S_i^2 \right)^{\frac{1}{2}} \qquad (3.7.7)$$

Hence the smaller the feel constant of the system, the better is the quality of force reflection.

From Fig 3.5, it is observed that the heavier the penalty, the better is the follow up.

Fig 3. 9 shows the optimal output positions for different feel matrices. Of the three matrices (3. 7. 4) selected, best tracking is obtained for the system with the feel matrix F_1 when $q_1 = 7$, and $q_1 : q_2 = 1 : 3$.

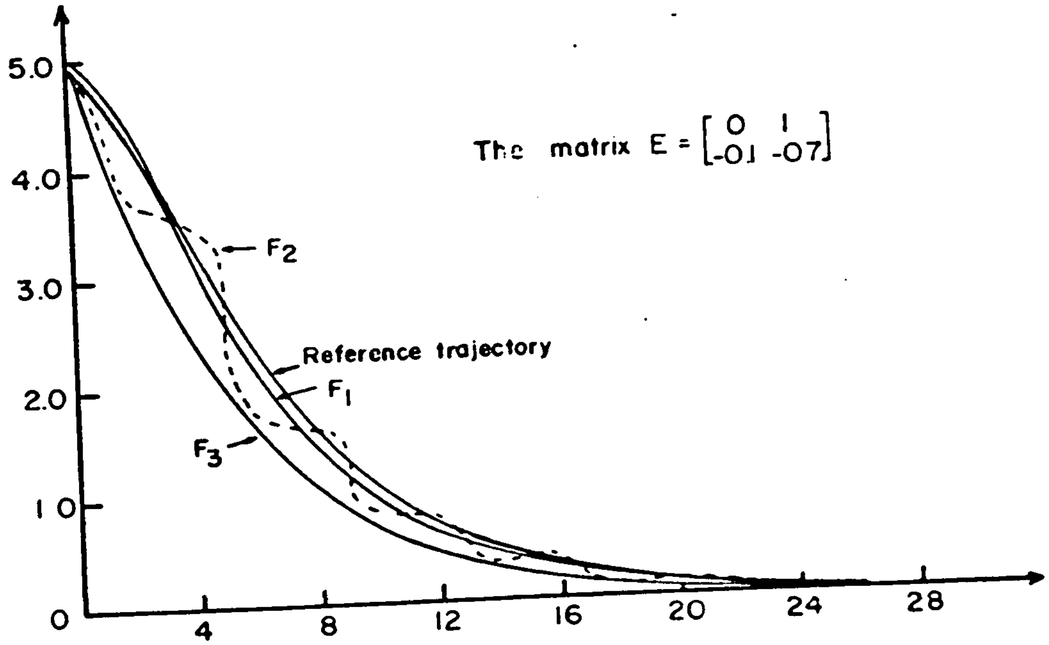


Fig 3. 9 Optimal output positions for different feel matrices.

3.8 Human Operator Model (as an optimal controller)

In section 2.3, a mathematical model (2.3.6) has been obtained for the human operator in terms of the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of the differential equation (2.3.2), perceptual cross coupling coefficients S_{AB} and S_{BA} and the reaction time delay τ_d. The perceptual cross coupling coefficients are assumed to be given, and hence the description of the human operator is obtained once the coefficients $\alpha_1, \dots \alpha_4$ are determined. In this section, a procedure is described for the determination of the coefficients $\alpha_1 \dots \alpha_4$, using some of the known results of optimal control theory. The procedure is based on the assumption that a well trained and a well motivated human operator behaves in a nearly optimal manner, and that he applies such a control as to minimize a given quadratic cost functional. Under this assumption, the function of the human operator can be replaced by an optimal controller. The description of the human operator is then the same as that of the optimal controller. The optimal control of the system (2.2.4') is the same as the optimal output of the human operator (2.3.6).

3.8.1 One Axis Model.

The steady-state behaviour of the optimal system (3.6.9) is described by,

$$\dot{x}(t) = [A_1 - \hat{L}_{22}] X(t) - \hat{L}_{21} Z(t) \tag{3.8.1}$$

where

$$\hat{L} \triangleq \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\hat{k}_{14}}{r} & \frac{\hat{k}_{24}}{r} & \frac{\hat{k}_{34}}{r} & \frac{\hat{k}_{44}}{r} & \frac{\hat{k}_{54}}{r} & \frac{\hat{k}_{64}}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= B_a R^{-1} B_a^T \hat{K} \tag{3.8.2}$$

and $\hat{K}$ is the steady-state solution of the matrix Riccati equation (3.5.12)

i.e $\lim_{t \to \infty} K(t) = \hat{K}$ (3.8.3) .

For illustration the matrix A_1 (2.1.11) is selected as,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & -1 & -2 \end{bmatrix} \qquad (3.8.4)$$

The values of the loading factors α , β , r appearing in expression (2.1.11) are equal to zero. The weighting factor r(3.8.1) is taken as unity.

An expansion of (3.8.1) with A_1 given by (3.8.4) and L by (3.8.2) gives the following state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -(1+\hat{k}_{34})\, x_1(t) - (2+\hat{k}_{44})\, x_2(t) + (0.5-\hat{k}_{54})\, x_3(t) + (0.5-\hat{k}_{64}) x_4(t) \\ &\quad - \hat{k}_{14}\, z_1(t) - \hat{k}_{24}\, z_2(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= 0.5\, x_1(t) + 0.5\, x_2(t) - x_3(t) - 2\, x_4(t) \end{aligned} \qquad (3.8.5)$$

Let $x_3(s)$, $z_1(s)$ $E(s)$ and $T_1(s)$ be the Laplace transforms of the output position x_3, input reference position Z_1, the positional error $e = z_1 - x_3$, and the input torque T_1 respectively.

Using Laplace transform, it can be shown that

$$\frac{X_3(s)}{Z_1(s)} = \frac{(\hat{k}_{14} + s\,\hat{k}_{24})}{\{2(s+1)\,[s^2 + 2(s+\hat{k}_{44}) + (\hat{k}_{34}+0.75)] + (\hat{k}_{54} + s\,\hat{k}_{64})\}} \qquad (3.8.6)$$

The transfer function of the one dimensional feel system (3. 6. 1) is given by,

$$P_1(s) = \frac{X_3(s)}{T_1(s)} = \frac{1}{2(s+1)(s^2+2s+0.75)} \qquad (3.8.7)$$

Using (3. 8. 6) it can be shown that

$$\frac{X_3(s)}{E(s)} = \frac{X_3(s)}{[T_1(s)-X_3(s)]} = \frac{(\hat{k}_{14} + s\hat{k}_{24})}{\{2(s+1)[(s^2+2s+0.75)+(0.75+\hat{k}_{34})] + (\hat{k}_{54}-\hat{k}_{14})+s(\hat{k}_{64}-\hat{k}_{24})\}} \qquad (3.8.8)$$

Using (3. 8. 7) and (3. 8. 8) the transfer function of the optimal regulator (and hence the human operator) is obtained as

$$G(s) = \frac{T_1(s)}{E(s)} = \frac{2\hat{k}_{24}(s+\hat{k}_{14}/\hat{k}_{24})(s+1)(s^2+2s+0.75)}{\{2(s+1)[s^2+(2+\hat{k}_{44})s+(\hat{k}_{34}+0.75)] + \hat{k}_{54}-\hat{k}_{14})+s(\hat{k}_{46}-\hat{k}_{64})\}} \qquad (3.8.9)$$

In order to obtain G(s), the steady state gains $\hat{k}_{i4}$, $i = 1,6$ are required, which in turn require the steady state solution K of the matrix Riccati equation (3. 5. 12). The terminal time T, and the terminal condition K(T) (3. 6. 12) are considered as the starting time and initial condition respectively of equation (3. 5. 12). As $T \to \infty$, the transient due to the initial condition K(T) does down, and the steady state internal.during which K(t) = K increases.

In this work, in order to obtain K, the matrix Riccati equation (3. 5. 12) is integrated backwards in time using the approximation (3. 6. 11) for $\dot{K}(t)$ and with the terminal condition (3. 6. 12) till each of the elements K_{ij}, $i, j, 1 = 6$ of K(t) reaches approximately constant value. A flow chart for computing the steady state gain matrix $\hat{K}$ is given in Fig 3. 10.

The steady state values of the optimal gains $\hat{k}_{i4}$, $i = 1,6$ for three Q matrices of (3. 5. 5) (r = 1), obtained from the steady state

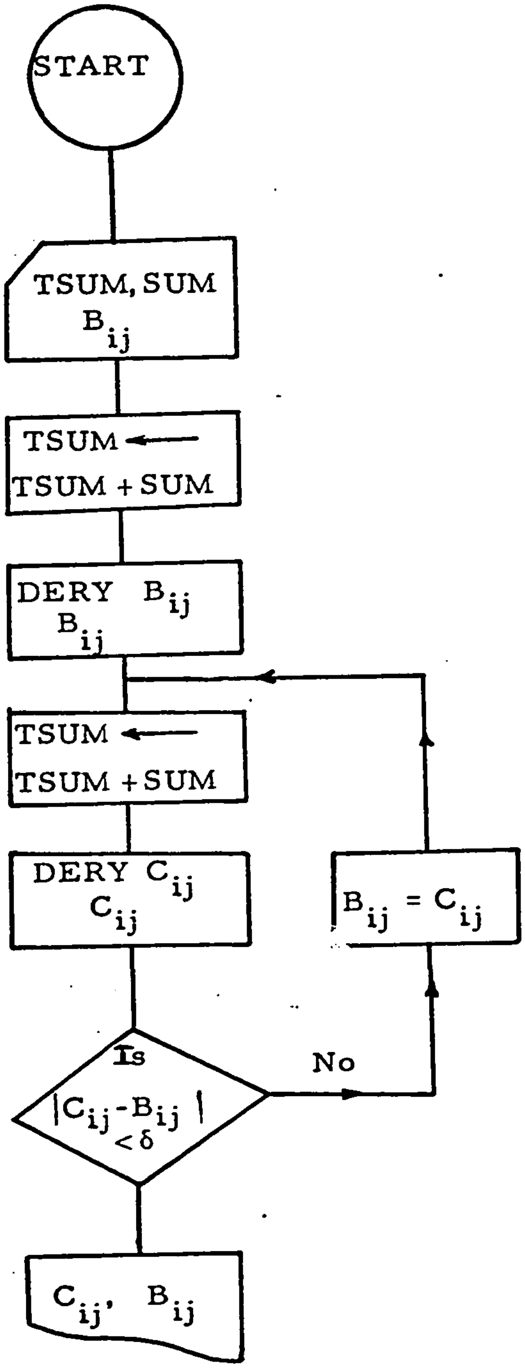


Fig 3.10 Flow chart for computing with steady state gain matrix K.

solution of the matrix Riccati equation (3.5.12) are given in Table 3.1.

Trial No.	q_1	q_2	r	$\hat{k}_{14}$	$\hat{k}_{24}$	$\hat{k}_{34}$	$\hat{k}_{44}$	$\hat{k}_{54}$	$\hat{k}_{64}$
1	1.0	1.5	1	0.4172	0.6445	0.0767	0.0767	0.1498	0.1593
2	5.0	15.0	1	1.7714	3.1580	0.3294	0.5861	0.5337	1.9150
3	10.0	30.0	1	2.5626	4.5437	0.3667	0.6298	0.7122	1.3185

Table 3.1.

Using the data of trial No. 2, and after proper simplifications, the transfer function of the optimal regulator (or human operator) (3.8.9) is obtained as

$$G(s) = \frac{3.17(s+0.555)\,(s^2+2s+0.75)}{(s+2.4)(s+0.19)} \tag{3.8.10}$$

(3.8.10) can be written in the form,

$$G(s) = \frac{K(1+s\tau_1)\,(s^2+2s+0.75)}{(1+s\tau_2)\,(1+s\tau_n)} \tag{3.8.11}$$

where K = 3.5, τ_2 = 1.8, τ_1 = 0.42 and τ_n = 5.3.

It can be observed that the difference between the one axis model (2.3.1) assumed in section 2.3, for the human operator is the presence of the term $(s^2+2s+0.75) = (s+0.5)(s+1.5)$ in the numerator or(two zeros at - 0.5 and -1.5). The inference is that the expression (3.8.11) is obtained, with a deterministic input, where as the expression (2.3.1) is for random inputs. So for the specific problem under consideration the operator was able to learn the task more effectively, and was able to cancel two poles of the system transfer function $P_1(s)$ viz - 0.5, -1.5 (see 3.8.7). Perhaps this would be the difference between the operator's model for deterministic and random inputs. Also the reaction time delay τ_d is usually very small.

Assuming that the term $(s^2+2s+0.75)$ is used to cancel two of the poles of (3.8.7), it is dropped from (3.8.11). The modified transfer function G'(s) for the optimal regulator (or human operator) is then given by

$$G'(s) = \frac{3.5\,(1 + 1.8s)}{(1+0.42s)\,(1+5.3s)} \qquad (3.8.12)$$

The modified system transfer function, then, is

$$P_1'(s) = \frac{1}{2(s+1)} \qquad (3.8.13)$$

It can be seen that $G(s) = G'(s)\,P_1'(s)$.
By converting the frequency domain expression (3.8.12) into time domain the following differential equation is obtained.

$$\ddot{T}_1(t) + a_1\,\dot{T}_1(t) + a_2\,T_1(t) = a_3\,\dot{\epsilon}(t) + a_4\,\epsilon(t) \qquad (3.8.14)$$

where $\epsilon(t) = [z(t) - y(t)]$

where $a_1 \simeq 5.7$, $a_2 \simeq 0.46$, $a_3 \simeq 2.9$ and $a_4 \simeq 1.6$

Equation (3.8.14) (which is similar to 2.3.2 with $\tau_d = 0$), describes the behaviour of a human operator controlling a one dimensional feel system (3.8.7).

3.8.2. Two Axis Model.

It is shown in section (2.3) that the two axis model of a human operator is given by (2.3.5). Since the coupling coefficients S_{AB}, and S_{BA} are assumed to be known, the only unknowns are the matrices P, R_1, and R_2. They are obtained as,

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.46 & -5.7 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.46 & -5.7 \end{bmatrix},\ R_1 = \begin{bmatrix} 0 & 0 \\ 1.6 & 1.6S_{AB} \\ 0 & 0 \\ 1.6S_{BA} & 1.6 \end{bmatrix},\ \text{and } R_2 = \begin{bmatrix} 0 & 0 \\ 2.9 & 2.9S_{AB} \\ 0 & 0 \\ 2.9S_{BA} & 2.9 \end{bmatrix} \qquad (3.8.12)$$

In a similar manner, the human operator model can be obtained for any n.

In general, a family of optimal controllers is obtained for different combinations of the feel matrix and the weighting matrices Q and R. The controller whose response matches with the response of a human operator obtained through actual experimentation with a human operator and a feel system represents his correct description. Also the same procedure is used when $T_2(t)$ is a passive load on the system. The model (2. 3. 5) obtained with a certain load on the system will be different from (3. 8. 12). This may, perhaps, illustrate the difference in the human operator description for two types of feel. Thus, it is easy to see that the human operator description (matrices P, R_1, and R_2 of (2. 3.5)) also depends on the feel force feedback. Since the main interest here is the modelling methodology, no attempt has been made to obtain a specific description.

3. 8. 3 Remark.

Another interesting application of optimal control theory in the modelling of the human operator is to consider him not only as an optimal controller as in this section, but also as an Information processor. The model due to Baron and Kleinman[26] contains the elements for the operators instrument monitoring, data reconstruction and control behaviour as well as means for representing his inherent limitations. The operator's control behaviour is assumed to be that of an ideal feedback controller, and the human data reconstruction was chosen so as to obtain a best estimate of the state of the controlled plant based on information obtained by sampling the various instruments. The structure of the model proposed in [26] is shown in Fig 3. 11 below.

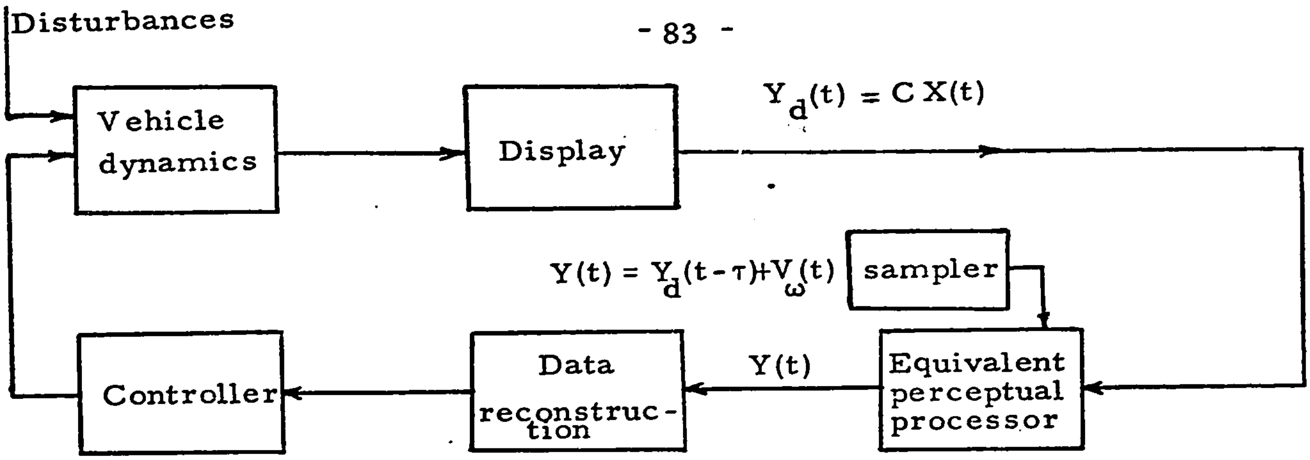


Fig 3.13. The structure of the human operator model proposed in [26]

Based on the numerical studies made in sections (3.7) and (3.8) for the one dimensional case, an algorithm for designing a multi-dimensional feel system is described in the next section.

3.9 An Algorithm for the Design of a Multi-Dimensional Feel System.

An algorithm for the design of a multi-dimensional feel system is described below.

(1) Select arbitrarily any feel matrix F, and a set of weighting matrices Q and R. For the combination chosen, compute the feel constant (3.7.4), using transient solution of the matrix Riccati equation (3.5.12)

(2) The human operator model depends upon the steady state solution of the matrix Riccati equation K. For the combination of the feel matrix and the weighting matrices chosen, obtain the operators model (3.8.12)

(3) Actually a family of optimal controllers or operator models will be obtained. Out of these, the controller whose response matches with the response of the human operator obtained through actual experimentation with the feel system and the human operator represents his correct description.

(4) Of the possible such models, choose the one with the minimum feel constant. The feel system thus obtained will not only have a good force reflecting characteristic but also will be compatible with the assumed

characteristics of the human operator.

3.10. A Two Point Boundary Value Problem Arising In the Theory of Feel Systems (or the Optimal Transfer Problem) :

Another interesting problem that arises in the theory of Feel systems is that of obtaining a control which when applied to an 'n' dimensional feel system, will transfer an object from one point in the actuator space (state) to another, optimally. It requires the solution of a two point boundary value problem. This optimal transfer problem differs from the optimal tracking problem discussed in section 3.5 in the sense that the trajectory along which this transfer takes place need not be restricted to follow a specified path.

Consider the 'n' dimensional feel system (2.2.4′)

$$\dot{X}(t) = [A'(t) + F(t)]\, X(t) + B(t)\, T_1(t) + D(t)\, T_2(t)$$

$$Y(t) = C(t)\, X(t) \qquad (3.10.1)$$

Let $T_2(t)$ be a passive load on the system (3.10.1). Then, as stated in section 2.3, $T_2(t)$ can be expressed as a linear combination of the state variables $T_2(t) = S\, X(t)$ (3.10.2)

Substituting (3.10.2) for $T_2(t)$ in (3.10.1), we obtained, the system description in the form:

$$\dot{X}(t) = A_1(t)\, X(t) + B(t)\, T_1(t)$$

and $$Y(t) = C(t)\, X(t) \qquad (3.10.3)$$

where $$A_1(t) = [A'(t) + F(t) - D(t)\, S]$$

The optimal transfer problem may now be stated as follows:

3.10.1 Statement of the Problem.

An object is initially at the point (state) $X(t_o) = X_o$, at $t = t_o$, in the actuator space. It is to be transferred to the point (state) $X(t_f) = X_f$ at $t = t_f$, in the actuator space using an 'n' dimensional feel system described by (3.10.3). It is required to obtain a control $T_1(t)$, which when applied to the feel system (3.10.3) will accomplish this, and will minimize the following quadratic cost functional

$$J(T_1) = \frac{1}{2} \int_{t_o}^{t_f} \{< X(t), Q(t), X(t)> + < T_1(t), R(t) T_1(t)> \} dt \quad (3.10.4)$$

where Q(t) and R(t) respectively are 4n x 4n, and n x n positive definite weighting matrices. It is assumed that both Q(t) and R(t) are diagonal matrices.

3.10.2 Solution of the Problem.

The above problem can be expressed as a two point boundary value problem, using the Minimum Principle of Pontryagin[40].

The Hamiltonian H, for the system (3.10.3), and the cost functional (3.10.4) is

$$H = \frac{1}{2} < X(t), Q(t)X(t)> + \frac{1}{2} < T_1(t), R(t) T_1(t)> + < A_1(t) X(t), p(t)> + < B(t) T_1(t), p(t)> \quad (3.10.5)$$

The condition $(\partial H / \partial T_1)(t) = 0$ yields the equation

$$(\partial H / \partial T_1)(t) = 0 = R(t) T_1(t) + B^T p(t)$$

The optimal control $T_1(t) = -R^{-1}(t) B^T(t) p(t)$ (3.10.6)

Since R is positive definite, the control $T_1(t)$ minimizes H.

The costate vector p(t) is the solution of

$$\dot{p}(t) = - (\partial H / \partial X)(t) = -Q(t) X(t) - A_1^T(t) p(t) \quad (3.10.7)$$

(3.10.3), (3.10.6), and (3.10.7) can be combined into the form .

$$\begin{bmatrix} \dot{X}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & -B(t) R^{-1}(t) B^T(t) \\ -Q(t) & -A_1^T(t) \end{bmatrix} \begin{bmatrix} X(t) \\ p(t) \end{bmatrix} \quad (3.10.8)$$

It is given that $X(t_o) = X_o$, and $X(t_f) = X_f$ (3.10.9)

The solution of (3.10.8), determines the optimal control $T_1(t)$ and hence the optimal state trajectory X(t).

Several methods exist for the solution of the two point boundary value problem (3.10.8), a comprehensive summary of which is given in [38]. Two simple methods for the solution of the two point boundary value problem are given in the following:

3.10.2.1 The Method of Complimentary Functions

Let the transition matrix of the system (3.10.8) be denoted by $\phi(t,t_o)$. The solution of (3.10.8) is given by,

$$\begin{bmatrix} X(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t,t_o) & \phi_{12}(t,t_o) \\ \phi_{21}(t,t_o) & \phi_{22}(t,t_o) \end{bmatrix} \begin{bmatrix} X(t_o) \\ p(t_o) \end{bmatrix} \tag{3.10.10}$$

where $\phi_{ij}(t,t_o)$ are 4n x 4n submatrices formed by partioning $\phi(t,t_o)$

From (3.10.10), we may write at $t = t_f$

$$X(t_f) = \phi_{11}(t_f,t_o)X(t_o) + \phi_{12}(t_f,t_o)p(t_o) \tag{3.10.11}$$

$$p(t_f) = \phi_{21}(t_f,t_o)\,X(t_o) + \phi_{22}(t_f,t_o)p(t_o) \tag{3.10.12}$$

Assuming that the necessary inverse exists, one can obtain from (3.10.11) that

$$p(t_o) = \phi_{12}^{-1}(t_f,t_o)\,[X(t_f) - \phi_{11}(t_f,t_o)\,X(t_o)] \tag{3.10.13}$$

Since $X(t_f)$ and $X(t_o)$ are given, $p(t_o)$ can be computed.
The optimal control $T_1(t)$ is determined using the relation (3.10.6)

3.10.2.2. Method of Riccati Transformation.

The Riccati transformation used is

$$X(t) = S(t)\,p(t) + g(t) \qquad t_o \le t \le t_f \tag{3.10.14}$$

where S(t) is a 4n x 4n matrix and g(t) is a 4n vector.

Differentiating (3.10.14) ω.r.t. t, we obtain

$$\dot{X}(t) = \dot{S}(t)\,p(t) + S(t)\,\dot{p}(t) + \dot{g}(t) \tag{3.10.15}$$

Eliminating $\dot{X}(t)$ and $\dot{p}(t)$ from (3.10.15) using the relations (3.10.16) and (3.10.17), given below and obtaining from (3.10.8) and (3.10.14),

$$\dot{X}(t) = A_1(t)\,[S(t)\,p(t) + g(t)] - B(t)\,R^{-1}(t)\,B^T(t) \tag{3.10.16}$$

and $$\dot{p}(t) = Q(t)\,[S(t)\,p(t) + g(t)] - A_1^T(t)\,p(t), \tag{3.10.17}$$

we obtain,

$$[A_1(t)S(t)-\dot{S}(t)-B(t)R^{-1}(t)B^T(t)+S(t)Q(t)S(t)+S(t)A_1^T(t)]p(t)+[A_1(t)g(t)-\dot{g}(t)+S(t)Q(t)]=0 \qquad (3.10.18)$$

Assuming (3.10.18) to hold for arbitrary p(t), it follows that the coefficient of p(t), should vanish. This yields the following differential equation for S(t) and g(t).

$$\dot{S}(t) = A_1(t)\,S(t) + S(t)A_1^T(t) + S(t)\,Q(t)S(t)-B(t)R^{-1}(t)B^T(t) \qquad (3.10.19)$$

$$\dot{g}(t) = (A_1(t) + S(t)\,Q(t))\,g(t) \qquad (3.10.20)$$

Since (3.10.14) is also true at $t = t_f$ and $X(t_f) = X_f$ is known, it is evident that $S(t_f) = 0$ and $g(t_f) = X_f$. (3.10.21)

$p(t_o)$ is obtained by integrating (3.10.19) and (3.10.20) backwards using terminal conditions (3.10.21), and the relation

$$p(t_o) = S^{-1}(t_o)\,[X(t_o) - g(t_o)] \qquad (3.10.22)$$

Integrating equation (3.10.17), forwards with the initial condition $p(t_o)$ determined from (3.10.22), p(t) can be obtained. Then, using (3.10.14), the optimal state trajectory X(t) can be computed.

The optimal control is again given by (3.10.6)

3.10.2.3 Numerical Results

In order to reduce the computational task, the numerical solution is attempted only for the one dimensional case (n = 1). The solution of the resulting two point boundary value problem (3.10.8) is obtained using the method of Riccati Transformation. The system (3.10.3) is assumed to be time invariant.

The solution of (3.10.17), (3.10.19), and (3.10.20) is based on the following approximation:

$$\frac{d}{dt}\,f(t) \simeq \left[\frac{f(t+\Delta)-f(t)}{\Delta}\right] \qquad (3.10.23)$$

A flow chart for the computer solution of the problem is given in Fig 3.12.

In order to illustrate the actual procedure the following example is considered.

The parameters A_1, B, and C of the system (3.10.3) are chosen as

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1.22 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0.9 & 0.9 & -1.2 & -1.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad C = I \tag{3.10.24}$$

The initial and the final states of the system are assumed to be $X^T(t_o) = [1.0\ 0.5\ 1.0\ 0.5]$ and $X^T(t_f) = [0.6 0.4 0.6 0.4]$

$t_o = 0$ and $t_f = 5$

The weighting matrices Q and R are selected as

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \quad \text{and} \quad R = r$$

where $q_1 = q_2 = 1$, $q_3 = q_4 = 2$, and $r = 1$ (3.10.25)

3.10.2.3.1 Determination of $S(t_o)$

In order to determine $S(t_o)$, the following first order nonlinear differential equations, obtained by expanding (3.10.19), have to be solved backward in time using the terminal condition $S_{ij}(5) = 0$, i, j = 1,4

$$\dot{s}_{11}(t) = s_{12}(t)+s_{21}(t)+q_2 s_{12}(t)s_{21}(t)+q_1 s_{11}^2(t)+q_3 s_{13}(t)s_{31}(t)+q_4 s_{14}(t)s_{41}(t)$$

$$\dot{s}_{12}(t) = s_{22}(t)-K_{11}/J_1 s_{11}(t)-(D_{11}+F_1)/J_1 s_{12}(t)+K_{12}/J_1 s_{13}(t)+D_{12}/J_1 s_{14}(t)$$
$$+ q_1 s_{11}(t)s_{12}(t)+q_2 s_{12}(t)s_{23}(t)+q_3 s_{13}(t)s_{32}(t)+q_4 s_{14}(t)\, s_{42}(t)$$

$$\dot{s}_{13}(t) = s_{23}(t)+s_{14}(t)+q_1 s_{11}(t)s_{13}(t)+q_2 s_{12}(t)s_{23}(t)+q_3 s_{13}(t)\, s_{33}(t)$$
$$+ q_4 s_{14}(t)\, s_{43}(t)$$

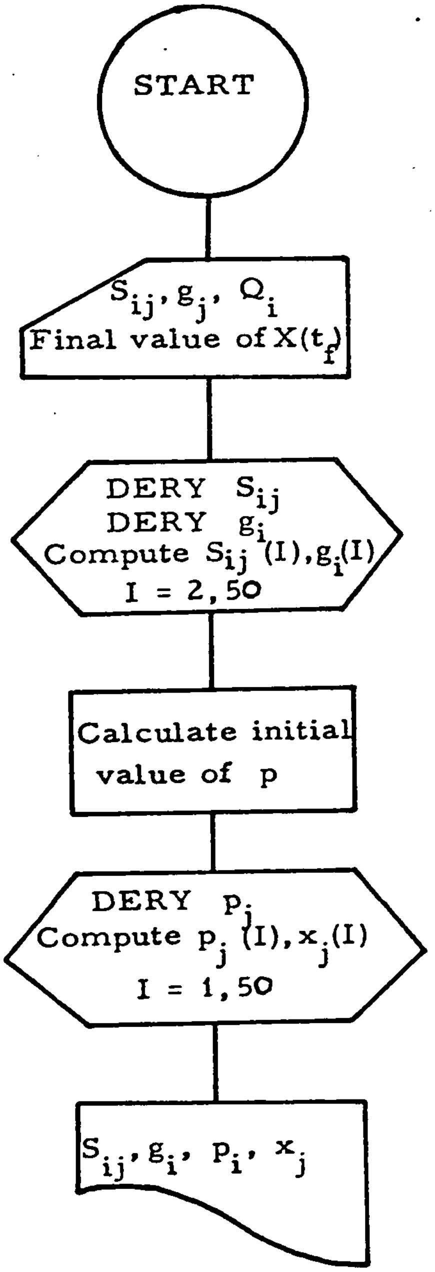


Fig 3.12 Flow chart for the computer solution of the two point boundary value problem (or optimal transfer problem).

$$\dot{s}_{14}(t)= s_{24}(t) +K_{21}/J_2(1+a)\, s_{11}(t)+D_{21}/J_2(1+a)\, s_{12}(t)- \frac{K_{22}(1+\beta)}{J_2(1+a)}\, s_{13}(t)$$

$$- \frac{[D_{22}(1+r)+F_2]}{J_2(1+a)}\, s_{14}(t)+q_1 s_{11}(t)s_{14}(t)+q_2 s_{12}(t)s_{24}(t)+q_3 s_{13}(t)s_{34}(t)$$

$$+ q_4 s_{14}(t)\, s_{44}(t)$$

$$\dot{s}_{21}(t) = - K_{11}/J_1\, s_{11}(t)-(D_{11}+F_1)J_1\, s_{21}(t)+K_{12}/J_1\, s_{31}(t)+D_{12}/J_1\, s_{41}(t)+s_{22}(t)$$

$$+ q_1\, s_{11}(t)s_{21}(t)+q_2 s_{21}(t)s_{22}(t)+ q_3 s_{23}(t)s_{31}(t)+q_4 s_{24}(t)\, s_{41}(t)$$

$$\dot{s}_{22}(t) = - K_{11}/J_1\, s_{12}(t)-2(D_{11}+F_1)/J_1\, s_{22}(t)+K_{12}/J_1\, s_{32}(t)+D_{12}/J_1\, s_{42}(t)- \frac{K_{11}}{J_1}\, s_{21}(t)$$

$$+ K_{12}/J_1\, s_{23}(t)+D_{12}/J_1\, s_{24}(t)+q_1 s_{21}(t)s_{12}(t)+q_2 s_{22}(t)+q_3 s_{23}(t)s_{24}(t)+q_4 s_{24}(t)s_{42}(t) - 1$$

$$\dot{s}_{23}(t) = -K_{11}/J_1\, s_{13}(t)-(D_{11}+F_1)/J_1\, s_{23}(t)+K_{12}/J_1\, s_{33}(t)+D_{12}/J_1\, s_{43}(t)+s_{24}(t)$$

$$+ q_1 s_{21}(t)s_{13}(t)+q_2 s_{22}(t)s_{23}(t)+q_3 s_{23}(t)s_{33}(t)+q_4 s_{24}(t)s_{43}(t)$$

$$\dot{s}_{24}(t) = -K_{11}/J_1\, s_{14}(t)-(D_{11}+F_1)/J_1\, s_{24}(t)+K_{12}/J_1\, s_{34}(t)+D_{12}/J_1\, s_{44}(t)+ \frac{K_{21}}{J_2(1+a)}\, s_{21}(t)$$

$$+ \frac{D_{21}}{J_2(1+a)}\, s_{22}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\, s_{23}(t) - \frac{[D_{22}(1+r)+F_2]}{J_2(1+a)}\, s_{24}(t)+q_1 s_{21}(t)s_{14}(t)$$

$$+ q_2 s_{22}(t)\, s_{24}(t)+q_3 s_{23}(t)s_{34}(t)+q_4 s_{24}(t)\, s_{44}(t)\ .$$

$$\dot{s}_{31}(t) = s_{41}(t)+s_{32}(t)+q_1 s_{31}(t)s_{11}(t)+q_2 s_{32}(t)s_{21}(t)+q_3 s_{33}(t)s_{31}(t) + q_4 s_{34}(t)s_{41}(t)\ .$$

$$\dot{s}_{32}(t) = s_{42}(t) - \frac{K_{11}}{J_1}\, s_{31}(t) - \frac{(D_{11}+F_1)}{J_1}\, s_{32}(t)+ \frac{K_{12}}{J_1}\, s_{33}(t)+ \frac{D_{12}}{J_1}\, s_{34}(t)$$

$$+ q_1 s_{31}(t)s_{12}(t)+q_2 s_{32}(t)s_{22}(t)+q_3 s_{33}(t)s_{32}(t)+q_4 s_{34}(t)s_{42}(t)\ .$$

$$\dot{s}_{33}(t) = s_{43}(t)+s_{34}(t)+q_1 s_{31}(t)s_{13}(t)+q_2 s_{32}(t)s_{23}(t)+q_3 s^2_{33}(t)+q_4 s_{34}(t)s_{43}(t)\ .$$

$$\dot{s}_{34}(t) = s_{44}(t)+ \frac{K_{21}}{J_2(1+a)}\ s_{31}(t)+ \frac{D_{21}}{J_2(1+a)}\ s_{32}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\ s_{33}(t)$$

$$- [\frac{D_{22}(1+r)+F_2}{J_2(1+a)}]\ s_{34}(t)+q_1 s_{31}(t) s_{14}(t)+q_2 s_{32}(t) s_{24}(t)+q_3 s_{33}(t) s_{34}(t)+q_4 s_{34}(t) s_{44}(t)\ .$$

$$\dot{s}_{41}(t) = \frac{K_{21}}{J_2(1+a)}\ s_{11}(t)+ \frac{D_{21}}{J_2(1+a)}\ s_{21}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\ s_{31}(t)- [\frac{D_{22}(1+r)+F_2}{J_2(1+a)}] s_{41}(t)+s_{42}(t)$$

$$+ q_1 s_{41}(t) s_{11}(t)+q_2 s_{42}(t) s_{21}(t)+q_3 s_{43}(t) s_{31}(t)+q_4\ s_{44}(t)\ s_{41}(t)\ .$$

$$\dot{s}_{42}(t) = \frac{K_{21}}{J_2(1+a)}\ s_{12}(t) + \frac{D_{21}}{J_2(1+a)}\ s_{22}(t) - \frac{K_{22(1+\beta)}}{J_2(1+a)}\ s_{32}(t) - \frac{(D_{22}(1+r)+F_2)}{J_2(1+a)}\ s_{42}(t)$$

$$- \frac{K_{11}}{J_1}\ s_{41}(t) - \frac{(D_{11}+F_1)}{J_1}\ s_{42}(t)+\frac{K_{12}}{J_1} s_{43}(t)+ \frac{D_{12}}{J_1}\ s_{44}(t)+q_1\ s_{41}(t) s_{12}(t)$$

$$+\ q_2 s_{42}(t) s_{22}(t)+q_3 s_{43}(t) s_{32}(t)+ q_4 s_{44}(t)\ s_{42}(t).$$

$$\dot{s}_{43}(t) = \frac{K_{21}}{J_2(1+a)}\ s_{13}(t)+ \frac{D_{21}}{J_2(1+a)}\ s_{23}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\ s_{33}(t) - \frac{(D_{22}(1+r)+F_2)}{J_2(1+a)}\ s_{43}(t)$$

$$+\ s_{44}(t) + q_1 s_{41}(t) s_{13}(t)+q_2 s_{42}(t) s_{23}(t)+q_3 s_{43}(t) s_{33}(t)+q_4 s_{44}(t) s_{43}(t)$$

$$\dot{s}_{44}(t) = \frac{K_{21}}{J_2(1+a)}\ s_{14}(t)+ \frac{D_{21}}{J_2(1+a)}\ s_{24}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\ s_{34}(t) - 2\ \frac{(D_{22}(1+r)+F_2)}{J_2(1+a)}\ s_{44}(t)$$

$$+\ \frac{K_{21}}{J_2\ (1+a)}\ s_{41}(t) + \frac{D_{21}}{J_2(1+a)}\ s_{42}(t) - \frac{K_{22}(1+\beta)}{J_2(1+a)}\ s_{43}(t)+q_1 s_{41}(t) s_{14}(t)+q_2 s_{42}(t) s_{24}(t)$$

$$+ q_3\ s_{43}(t)\ s_{34}(t) + q_4 s_{44}^2(t)\ . \qquad (3.10.26)$$

For the system (3. 10. 24) $S(t_o)$ is obtained as

$$S(t_o) = \begin{bmatrix} 0.4223 & -0.2092 & -0.1437 & -0.1586 \\ -0.2092 & 1.0530 & 0.0136 & -0.4645 \\ -0.1437 & 0.0136 & 0.2374 & -0.3411 \\ -0.1587 & -0.4645 & -0.3411 & 0.7636 \end{bmatrix} \qquad (3.10.27)$$

Incidentally, through numerical results, it was found that $S(t)$ is Symmetric
(For Proof See P. 762 [40])

3. 10. 2. 3. 2. Determination of $g(t_o)$.

In order to detrmine $g(t_o)$, the following first order nonlinear differential equations, obtained by expanding (3. 10. 20), are integrated backwards with the terminal condition $g(t_f) = X(t_f)$.

$$\dot{g}_1(t) = q_1 s_{11}(t) g_1(t) + (1 + q_2 s_{12}(t)) g_2(t) + q_3 s_{13}(t) g_3(t) + q_4 s_{14}(t) g_4(t)$$

$$\begin{aligned}\dot{g}_2(t) = {} & (q_1 s_{21}(t) - K_{11/J_1})\, g_1(t) + (q_2 s_{22}(t) - [\frac{D_{11}+F_1}{J_1}])\, g_2(t) + (q_3 s_{23}(t) + \frac{K_{12}}{J_1})\, g_3(t) \\ & + (q_4 s_{24}(t) + D_{12/J_1})\, g_4(t)\end{aligned}$$

$$\dot{g}_3(t) = q_1 s_{31}(t) g_1(t) + q_2 s_{32}(t) g_2(t) + q_3 s_{33}(t) g_3(t) + (1 + q_4 s_{34}) g_4(t)$$

$$\begin{aligned}\dot{g}_4(t) = {} & [q_1 s_{41}(t) + K_{21/J_2(1+\alpha)}]\, g_1(t) + (q_2 s_{42}(t) + D_{21/J_2(1+\alpha)})\, g_2(t) \\ & + [q_3 s_{43}(t) - \frac{K_{22}(1+\beta)}{J_2(1+\alpha)}]\, g_3(t) + \{q_4 s_{44}(t) - \frac{[D_{22}(1+r) + F_2]}{J_2(1+\alpha)}\}\, g_4(t)\end{aligned} \tag{3. 10. 28}$$

For the system (3. 10.24) $g(t_o)$ is obtained as

$$g(t_o) = \begin{bmatrix} -3.554 \\ 0.8367 \\ -5.56 \\ -3.28 \end{bmatrix} \tag{3. 10. 29}$$

3. 10. 2. 3. 3. Determination of $p(t_o)$.

Using the relation (3. 10. 22) and (3. 10. 27) and (3. 10. 29) $p(t_o)$ is computed.

$$p(t_o) = \begin{bmatrix} -36.77 \\ -25.78 \\ -54.80 \\ -42.80 \end{bmatrix} \tag{3. 10. 30}$$

3.10.2.3.4. Determination of p(t).

Using the value of $p(t_o)$ (3.10.30) obtained as above as the initial condition, the following equations may be integrated forwards to obtain p(t)

$$\dot{p}_1(t) = -q_1 s_{11}(t)p_1(t)+(1-q_1 s_{12}(t))p_2(t)- q_1 s_{13}(t)p_3(t) - (\frac{K_{21}}{J_2(1+a)}+q_1 s_{14}(t))p_4(t) -q_1\ g_1(t).$$

$$\dot{p}_2(t) = -(1+q_2 s_{21}(t))p_1(t) +\left(\frac{(D_{11}+F_1)}{J_1} -q_2 s_{22}(t)\right) p_2(t)-q_2 s_{23}(t)p_3(t)-q_2 g_2(t) -\left(\frac{D_{21}}{J_2(1+a)} +q_2 s_{24}(t) \right) p_4(t)$$

$$\dot{p}_3(t) = - q_3 s_{31}(t)p_1(t)- (q_3 s_{32}(t)+\frac{K_{12}}{J_1}) p_2(t)-q_3 s_{33}(t)p_3(t)- q_3 g_3(t) +\left(\frac{K_{22}(1+a)}{J_2(1+a)} - q_3 s_{43}(t) \right) p_4(t)$$

$$\dot{p}_4(t) = -q_4 s_{41}(t)p_1(t) - (q_4 s_{42}(t) + \frac{D_{12}}{J_1}) p_2(t) - (1+q_4 s_{43}(t))\ p_3(t)-q_4 g_4(t) +\left(\frac{(D_{22}(1+r)+F_2)}{J_2(1+a)} - q_4 s_{44}(t)\right) p_4(t) \qquad (3.10.31).$$

3.10.2.3.5. Determination of the optimal control $T_1(t)$, the optimal trajectory X(t), and the optimal output position Y(t).

The optimal control $T_1(t) = -p_2(t)$ (3.10.32).

The optimal trajectory $X^T(t) = [x_1(t)\ \ x_2(t)\ x_3(t)\ x_4(t)]$ is computed using the following relations obtained by expanding (3.10.14).

$$x_1(t) = s_{11}(t)p_1(t)+s_{12}(t)\ p_2(t)+ s_{13}(t)p_3(t) + s_{14}(t)p_4(t)$$

$$x_2(t) = s_{21}(t)p_1(t)+s_{22}(t)p_2(t) + s_{23}(t)p_3(t) + s_{24}(t)p_4(t)$$

$$x_3(t) = s_{31}(t)p_1(t) + s_{32}(t)p_2(t) +s_{33}(t)p_3(t)+ s_{34}(t)p_4(t)$$

$$x_4(t) = s_{41}(t)p_1(t) + s_{42}(t)p_2(t) + s_{43}(t)p_3(t) + s_{44}(t)p_4(t) \qquad (3.10.33)$$

The optimal output position is given by $y(t) = x_3(t)$ (3. 10. 34)
y(t), for the system (3. 10. 24) considered as an example, is shown in Fig 3. 13 (graph F_1)

3. 10. 2. 3.6 . A Comparison of the Force Reflecting Characteristics of Various Feel Systems.

In order to obtain a comparison of the relative force reflecting characteristics of the various feel systems, equations (3. 10. 3) are solved for the one dimensional case for three different feel matrices using the parameters (3. 10. 25). Fig 3. 13 shows the optimal output positions Y(t), for three typical feel matrices F_1, F_2 and F_3 selected as shown below (The system considered in the example above has the feel matrix F_1).

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0.9 & -1.2 & -1.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0 & -1.2 & -0.2 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 0.66 & 0.55 \\ 0 & 0 & 0 & 0 \\ 0.6 & 0.5 & -1.2 & -1.2 \end{bmatrix} \qquad (3.\,10.\,35)$$

It is to be noted that the three feel matrices F_1, F_2, and F_3(3. 10. 35) selected are the same as the feel matrices (3. 7. 1) considered in section 3. 7. This is done in order to obtain a comparison of the results obtained in this section and in section 3. 7. From an examination of the optimal output positions, it appears that they tend to oscillate around the straight line joining the initial and the final positions. $(y(t_o)) = 1.0$ and $y(t_f) = 0.6$ for the example). Let this straight line be denoted by m(t). The control strategy seems to be, that the human operator, will try to transfer the object from the initial position $y(t_o)$ to the final position $y(t_f)$ in such a manner as to minimize the difference between y(t) and m(t). While experimenting with actual three degree of freedom remote manipulators the same strategy was noted by G. Wolfman, A. Nicholson and H. Growth, and they described it in [30].

Let the difference in position between y(t) and m(t) be denoted by e(t) i.e $e(t) = [m(t) - y(t)]$. The inference drawn from Fig 3.14 is that the better the feel matrix, the smaller will be the values of e(t). Of the three feel matrices (3.10.38) considered, the system with the feel matrix F_1 has the best force reflecting characteristic, whereas the system with the feel matrix F_3 has the worst. The characteristisation of the feel matrices obtained here agrees with the one obtained in section 3.7. The discussion about the nature of the feel matrices F_1, F_2, F_3 given in section 3.7, applies to this section as well. Fig 3.14 show the optimal output positions for different Q matrices.

As in section 3.7, a measure of the quality of force reflection of the one dimensional feel system can be obtained by a defining a feel constant, S

$$S = \int_{t_o}^{t_f} |e(t)|\, dt \qquad (3.10.36)$$

For the 'n' dimensional feel system, the feel constant, can be obtained from the relation

$$\| S \| = \left(\sum_{i=1}^{n} S_i^2 \right)^{\frac{1}{2}} \qquad (3.10.37)$$

In a similar manner, the effect of reducing the moments of inertia and the coefficients of friction of the system actuators, on the no-load feel of the system can also be investigated.

This completes the study of optimal control of systems with feel. Numerical solutions have been obtained in the above for the one dimensional case for both the optimal tracking and the optimal transfer problems. Variations in the gains of the signal transmission channels, the weighting matrices and the intensity of the disturbances, affect the performance of the optimal feel systems obtained in this chapter. This problem is investigated, through sensitivity analyses, in the next chapter.

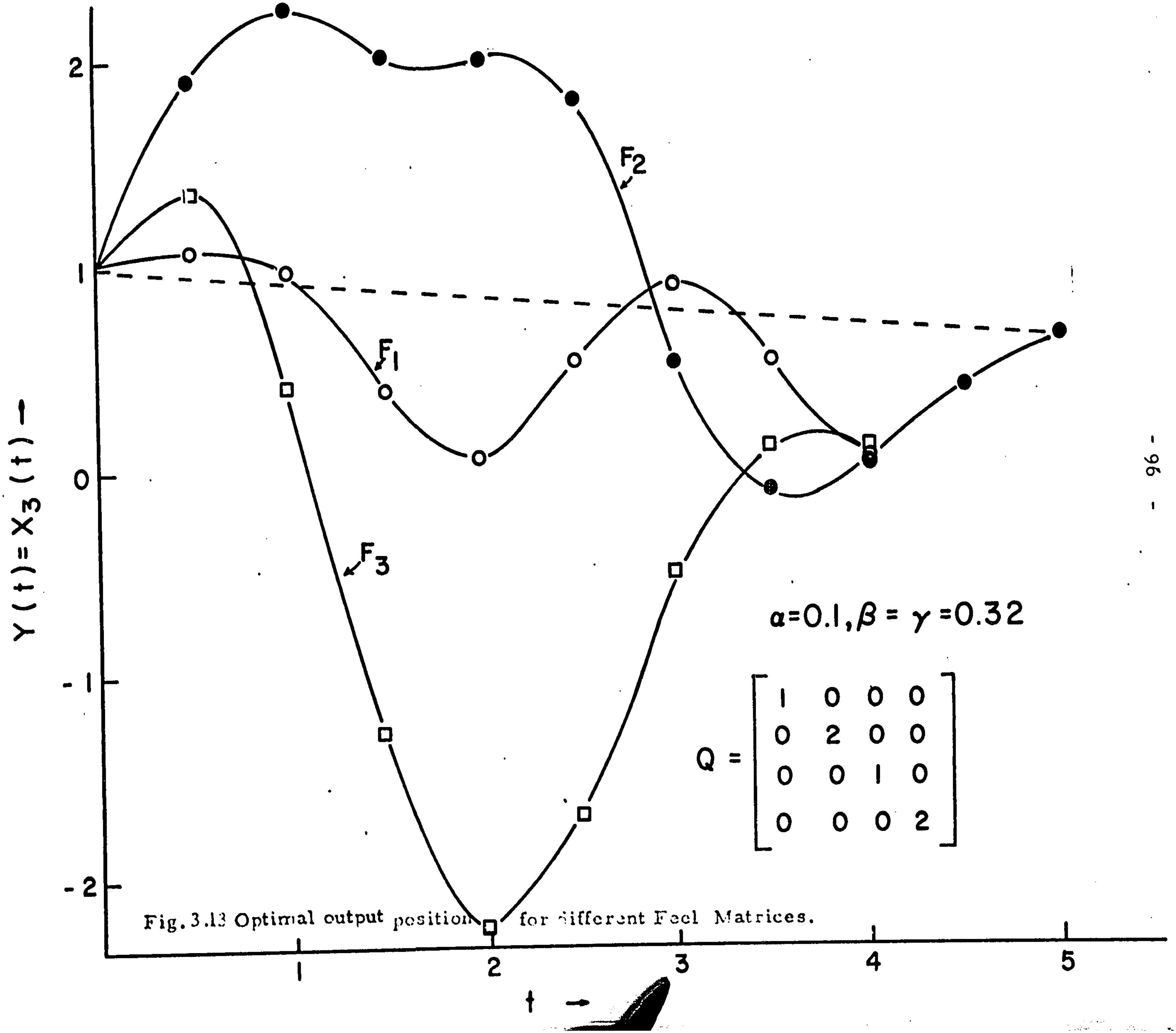


Fig. 3.13 Optimal output position for different Feel Matrices.

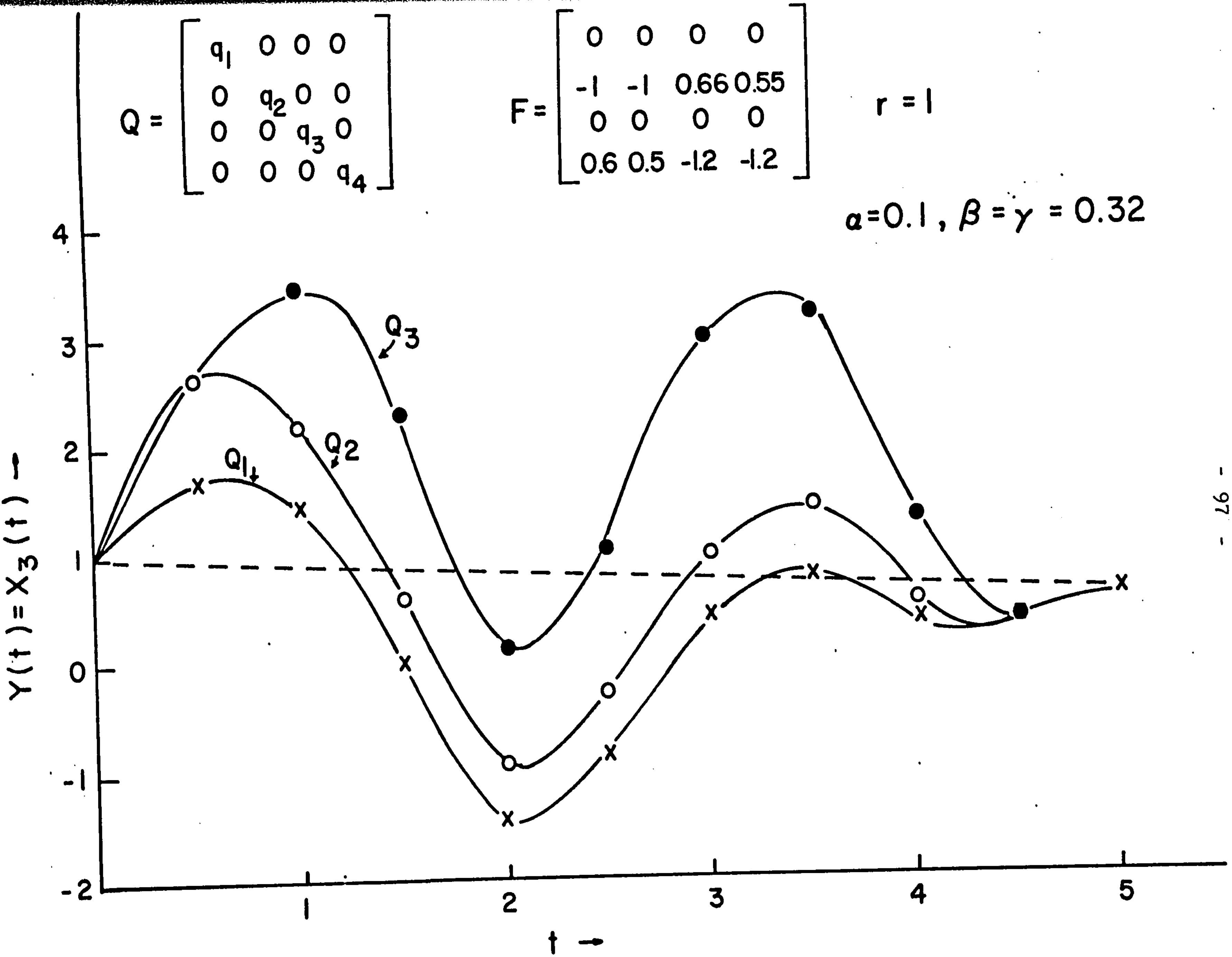


Fig 3. 14 Optimal output positions for different Q matrices.

CHAPTER IV

SENSITIVITY ANALYSIS OF OPTIMAL FEEL SYSTEMS

The performance of the optimal feel systems obtained in chapter III is affected due to variations in the feel matrix, the weighting matrices selected, and the intensity of the disturbances on the system. The change in the system performance can be measured either as a change in the performance index or as a change in the feel constant. In this chapter, using partial derivative techniques, sensitivities have been obtained for the optimal tracking system and the optimal transfer system, due to changes in the feel matrix and the weighting matrices. An application of these sensitivity functions to the design of feel systems has also been considered.

4. 1 Sensitivity Equations for a General Linear System.

Consider the linear system, described by the differential equation

$$\frac{\partial X}{\partial t}(t, a_1 \ldots a_m) = A(a_1 \ldots a_m)\, X(t, a_1 \ldots a_m) + B(a_1 \ldots a_m) u(t)$$

$$Y(t, a_1 \ldots a_m) = C(a_1 \ldots a_m)\, X(t, a_1 \ldots a_m) + D(a_1 \ldots a_m)\, u(t) \qquad (4.1.1)$$

The system parameters chosen are $(a_1 \ldots a_m)$. The possible dependance of the output signals on the parameters is shown by explicitly indicating these as the arguments of the time functions. The matrices A, B, C, and D are time invariant, but they depend on the parameters $a_1 \ldots a_m$. Let it be assumed that the partial derivatives of the matrices with respect to the parameters exist, and that they are continuous functions with respect to these parameters. Then the signals are continuous and differentiable functions of the parameters. Let it be assumed that all the second order partial derivatives are continuous.

Taking the partial derivatives of the vector equation (4. 1. 1) ω. r. t. a_i, and noting that $\partial^2 X(t)/\partial t.\, \partial a_i = \partial^2 X(t)/\partial a_i \partial t$, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial X(t)}{\partial a_i}\right) = A\left(\frac{\partial X(t)}{\partial a_i}\right) + \left(\frac{\partial A}{\partial a_i}\right) X(t) + \left(\frac{\partial B}{\partial a_i}\right) u(t) \tag{4. 1. 2}$$

and $$\left(\frac{\partial Y(t)}{\partial a_i}\right) = C\left(\frac{\partial X(t)}{\partial a_i}\right) + \left(\frac{\partial C}{\partial a_i}\right) X(t) + \left(\frac{\partial D}{\partial a_i}\right) u(t)$$

In equation (4. 1. 2), the partial derivatives are evaluated ω. r. t. a_i for fixed a_i. Thus (4. 1. 2) is an ordinary differential equation for the time functions $\frac{\partial X}{\partial a_i}(t)$ and $\frac{\partial Y}{\partial a_i}(t)$ respectively. The solution of (4. 1. 2) yields the sensitivity functions $\partial X/\partial a_i(t)$ and $\partial Y/\partial a_i$

4. 2 Sensitivity Analysis of the Optimal Tracking System.

It is shown in chapter III, that the optimal output position for the tracking system is given by,

$$Y(t) = C X_a(t) \tag{4. 2. 1}$$

where the augmented state vector $X_a(t)$, is the solution of the vector differential equation

$$\dot{X}_a(t) = [A_a - B_a R^{-1} B_a^T K(t)]\; X_a(t) .$$

with $$X_a(o) = X_a^o . \tag{4. 2. 2}$$

The augmented system matrix A_a can be written as the sum of two matrices A'_a and F_a where,

$$A'_a = \begin{bmatrix} E & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A' \end{bmatrix} \text{ and } F_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix} \tag{4. 2. 3}$$

The optimal control $T_1(t)$ is given by, $T_1(t) = -R^{-1} B_a^T K(t) X_a(t)$ (4. 2. 4)

where K(t) is the solution of the matrix Riccati equation,

$$\dot{K}(t) = -K(t)\, A_a - A_a^T K(t) + K(t)\, B_a R^{-1} B_a^T K(t) - C_a^T Q C_a \qquad (4.2.5)$$

with $K(T) = C_a^T N C_a$

The performance index $J(T_1)$ minimised is

$$J(T_1) = \frac{1}{2} < X_a(T), C_a^T N C_a X_a(T) > + \frac{1}{2} \int_0^T \{ < X_a(t), C_a^T Q C_a X_a(t) > + < T_1(t), R\, T_1(t) > \} \, dt \qquad (4.2.6)$$

where N, Q, R are n x n weighting matrices. Further they are assumed to be diagonal.

Let us now proceed to obtain the sensitivites of the optimal output position due to changes in F, N, Q, and R.

4.2.1 Effect of Variations of the Feel Matrix on the Output Position of the System.

Let the nominal values of $X_a(t)$, $K(t)$, A_a and A_1 be $x_a^o(t)$, $K^o(t)$, A_a^o and A_1^o respectively.

Differentiation of (4.2.2), with respect to f_{ij}, an element of the feel matrix F, yields the following vector differential equation

$$\frac{\partial}{\partial f_{ij}} (\dot{X}_a(t)) = \frac{d}{dt} \left[\frac{\partial X_a(t)}{\partial f_{ij}} \right] = [A_a^o - B_a R^{-1} B_a^T K^o(t)] \frac{\partial X_a(t)}{\partial f_{ij}} + \left[\frac{\partial F_a}{\partial f_{ij}} - B_a R^{-1} B_a^T \frac{\partial K(t)}{\partial f_{ij}} \right] X_a^o(t) \qquad (4.2.7)$$

The initial condtion for (4.2.7), $\left. \frac{\partial X_a(t)}{\partial f_{ij}} \right|_{t=o} = 0$, since the initial state X_a^o is independent of the feel matrix.

The solution of (4.2.7) requires the sensitivity of the feedback gain matrix $K(t)$ with respect to changes in f_{ij}. The equation which determines this sensitivity can be obtained by differentiating (4.2.5) with respect to f_{ij}. That is $\partial K(t)/\partial f_{ij}$ is the solution of the linear matrix differential equation

$$\frac{d}{dt}\left[\frac{\partial K(t)}{\partial f_{ij}}\right] = -\left(\frac{\partial K(t)}{\partial f_{ij}}\right) A_1^o(t) - A_1^{oT}(t)\left(\frac{\partial K(t)}{\partial f_{ij}}\right) + P(t) \qquad (4.2.8)$$

where $A_1^o(t) = \left[A_a^o - B_a R^{-1} B_a^T K^o(t)\right]$

and $P(t) = K^o(t)\,\frac{\partial F_a}{\partial f_{ij}} + \left(\frac{\partial F_a}{\partial f_{ij}}\right)^T K^o(t)$

The solution of (4.2.8) may be obtained as follows:

Let $\partial K(t)/\partial f_{ij}$ be denoted by K_{ij}^1

Let $V(t)$ and $W(t)$ denote the solutions of

$$\dot{V}(t) = V(t)\,A_1^{o\,T}(t) \qquad V(o) = I \qquad (4.2.9)$$

$$\dot{W}(t) = A_1^o(t)\,W(t) \qquad W(o) = I \qquad (4.2.10)$$

Then $\frac{d}{dt}\left[V K_{ij} W\right] = V \dot{K}_{ij}^1 W + V K_{ij}^1 \dot{W} + \dot{V} K_{ij}^1 W$

$$= V \dot{K}_{ij}^1 W + V K_{ij}^1 A_1^o W + V A_1^{oT} K_{ij}^1 W \qquad (4.2.11)$$

Multiplying (4.2.8) on the left by V and on the right by W we have

$$V \dot{K}_{ij}^1 W = -V K_{ij}^1 A_1^o W - V A_1^{oT} K_{ij}^1 W + V P W \qquad (4.2.12)$$

Using (4.2.11) and (4.2.12) it can be shown that

$$\frac{d}{dt}(V K_{ij}^1 W) = V P W \qquad (4.2.13)$$

Integrating (4.2.13) we have

$$(V K_{ij}^1 W)(t) = C + \int_o^t V(\tau)\,P(\tau)\,W(\tau)\,d\tau \qquad (4.2.14)$$

Assuming that V(t) and W(t) for $t \geq o$ are non-singular, it follows from (4.2.13) that

$$K^1_{ij}(t) = V^{-1}(t)\, C W^{-1}(t) + \int_o^t V^{-1}(t) V(\tau)\, p(\tau)\, W(\tau)\, W^{-1}(t)\, d\tau \qquad (4.2.15)$$

Further $K(T) = N$ is a given fixed matrix

$$K^1_{ij}(T) = \partial K(t)/_{\partial f_{ij}} \Big|_{t=T} = \lceil 0 \rceil \qquad (4.2.16)$$

Further the solution of (4.2.9) is given by

$$V(t) = e^{\int_o^t A_1^{oT}(\sigma) d\sigma} \qquad (4.2.17)$$

and that of (4.2.10) is given by

$$W(t) = e^{\int_o^t A_1^{o}(\sigma) d\sigma} \qquad (4.2.18)$$

Therefore substituing V(t) and W(t) obtained from (4.2.17) and (4.2.18) in (4.2.15), the sensitivity of the gain matrix K(t) to changes in the elements of the feel matrix F is obtained as $K^1_{ij}(t)$ for $t \geq o$ and $i, j = 1, 2, \ldots 4n$. The sensitivity of Y(t) due to variations of the feel matrix may now be obtained by solving (4.2.7) using K^1_{ij} of (4.2.15) and using the relation

$$\partial Y(t)/_{\partial f_{ij}} = C({}^{\partial X_a(t)/\partial f_{ij}}) \qquad (4.2.19)$$

4.2.2. Effect of Variations of Q on Y(t).

Differentiation of (4.2.2) with respect to q_i, an element of the weighting matrix Q, yields the vector differential equation,

$$\frac{d}{dt} \left[\frac{\partial X_a(t)}{\partial q_i} \right] = A_1^o(t) \frac{\partial X_a(t)}{\partial q_i} - B_a R^{-1} B_a^T \frac{\partial K(t)}{\partial q_i} X_a^o(t) \qquad (4.2.20)$$

The initial condition for (4.2.20) is given by $\dfrac{\partial X_a(t)}{\partial q_i} \Big|_{t=o} = 0$

The solution of (4.2.20) requires the sensitivity of K(t) to changes in q_i denoted by $\partial K(t)/\partial q_i$. The equation which determines this sensitivity is obtained by differentiating (4.2.5) with respect to q_i; i.e $\partial K(t)/_{\partial q_i}$ is the solution of the linear matrix differential equation,

$$\frac{d}{dt}\left[\frac{\partial K(t)}{\partial q_i}\right] = -\left[\frac{\partial K(t)}{\partial q_i}\right] A_1^o(t) - A_1^T(t)\left[\frac{\partial K(t)}{\partial q_i}\right] - C_a^T C_a \qquad (4.2.21)$$

The boundary condition for (4.2.21) is $\partial K(t)/_{\partial q_i}\Big|_{t=T} = \partial N/_{\partial q_i} = [\,0\,]$

The sensitivity of Y(t) due to variations in the weighting matrix Q may now be obtained by solving (4.2.20) using the solution of (4.2.21), and using the relation,

$$\partial Y(t)/\partial q_i = C\,(\,\partial X_a(t)/\partial q_i\,) \qquad (4.2.22)$$

4.2.3 Effect of Variations in N on Y(t).

Differentiation of (4.2.5) with respect to n_i, and element of the weighting matrix N, yields the following vector differential equation,

$$\frac{d}{dt}\left[\frac{\partial X_a(t)}{\partial n_i}\right] = A_1^o(t)\,\frac{\partial X_a(t)}{\partial n_i} - B_a R^{-1} B_a^T\,\partial K(t)/\partial n_i\; X_a^o(t) \qquad (4.2.23)$$

The initial condition for (4.2.23) is $\frac{\partial X_a(t)}{\partial n_i}\Big|_{t=o} = 0$

The solution of (4.2.23) requires the sentivity of K(t) with respect to changes in n_i, denoted by $\partial K/_{\partial n_i}(t)$. The equation which determines this sensitivity can be obtained by differentiating (4.2.5) with respect to n_i. That is, $\partial K(t)/_{\partial n_i}$ is the solution of the matrix differential equation

$$\frac{d}{dt}\left[\frac{\partial K(t)}{\partial n_i}\right] = -\left[\frac{\partial K(t)}{\partial n_i}\right] A_1^o(t) - A_1^{o\,T}(t)\left[\frac{\partial K(t)}{\partial n_i}\right] \qquad (4.2.24)$$

The boundary condition for (4.2.24) if

$$\partial K(t)/\partial n_i\Big|_{t=T} = \frac{\partial N}{\partial n_i} = \text{diag}\;(\delta_{i1} \cdots \delta_{in})$$

Then, the sensitivity of Y(t) due to variations in the weighting matrix N, may be obtained by solving (4.2.23) using the solution of (4.2.24), and using the relation,

$$\partial Y(t)/_{\partial n_i} = C\left[\frac{\partial X_a(t)}{\partial n_i}\right] \qquad (4.2.25)$$

4.2.4 Effect of Variations in R on Y(t).

Differentiation of (4.2.5) with respect to r_i, an element of the weighting matrix R, yields the following differential equation,

$$\frac{d}{dt}\left[\frac{\partial X_a(t)}{\partial r_i}\right] = A_1^o(t)\,\frac{\partial X_a(t)}{\partial r_i} - \left[B_a R_i B_a^T K^o(t) + B_a R^{-1} B_a^T \frac{\partial K(t)}{\partial r_i}\right] X_a^o(t) \tag{4.2.26}$$

where $R_i = \partial R^{-1}/\partial r_i = \partial/\partial r_i \left[\mathrm{diag}\left(1/r_1, 1/r_2, \ldots\, 1/r_n\right)\right]$

$$= -\,\mathrm{diag}\left(\delta_{i1}/r_1^2, \delta_{i2}/r_2^2, \ldots\, \delta_{in}/r_n^2\right) \tag{4.2.27}$$

The initial condition for (4.2.26) is $\partial X_a/\partial r_i \big|_{t=o} = 0$

The solution of (4.2.26) requires the sensitivity of K(t) due to changes in r_i, denoted by $\partial K/\partial r_i(t)$. The equation which determines this sensitivity is obtained by differentiating (4.2.5) ω.r.t. r_i.

i.e $\partial K/\partial r_i(t)$ is the solution of the linear matrix differential equation,

$$\frac{d}{dt}\left[\frac{\partial K}{\partial r_i}(t)\right] = -\left(\frac{\partial K}{\partial r_i}(t)\right) A_1^o(t) - A_1^{oT}(t)\left(\frac{\partial K}{\partial r_i}(t)\right) + K^o(t) B_a R_i B_a^T K^o(t) \tag{4.2.28}$$

The boundary condition for (4.2.28) is $\frac{\partial K}{\partial r_i}(t)\Big|_{t=T} = \frac{\partial N}{\partial r_i} = [0]$

The sensitivity of Y(t) due to changes in r_i, may be obtained by solving (4.2.26) using the solution of (4.2.28) and using the relation

$$\frac{\partial Y}{\partial r_i}(t) = C\left(\frac{\partial X_a}{\partial r_i}\right)(t) \tag{4.2.29}$$

The change in the system performance as a result of changes in the feel matrix and the weighting matrices may be measured either as a change in the performance index or as a change in the feel constant. This is illustrated for variations in the feel matrix.

4.2.5 Effect of Variations in F on $J(T_1)$.

It is known that the minimum value of the performance index $J(T_1)$ (4.2.6) is given by,

$$J^*(T_1) = \frac{1}{2} < X_a(o), K(o) X_a(o) > \qquad (4.2.30)$$

differentiating (4.2.30) ω.r.t. f_{ij}, we obtain

$$\partial J^*(T_1)/_{\partial f_{ij}} = < X_a(t), K(t) \frac{\partial X_a(t)}{\partial f_{ij}} > + \frac{1}{2} < X_a(t), \frac{\partial K}{\partial f_{ij}}(t) X_a(t) > \qquad (4.2.31)$$

substituting the solutions (4.2.7) and (4.2.8) in (4.2.31) $\partial J^* (T_1)/_{\partial f_{ij}}$ can be determined

4.2.6 Effect of Variations in F on the Feel Constant.

4.2.6.1 One Dimensional Feel System.

As mentioned in chapter III, the quality of force reflection of a one dimensional feel system, can be obtained by computing the feel constant S,

$$S^1 = \int_o^T |e(t)| \, dt, \quad \text{where } e(t) = z(t) - y(t) \qquad (4.2.31)$$

(The superscript denotes the dimension of the feel system)

$$= \int_o^T | C_a X_a(t) | \, dt \qquad (4.2.32)$$

differentiating (4.2.32) ω.r.t. f_{ij} we have,

$$\frac{\partial S^1}{\partial f_{ij}} = \int_o^T \text{sign} (C_a X_a^o) C_a \frac{\partial X_a}{\partial f_{ij}} (t) \, dt. \qquad (4.2.33)$$

For the one dimensional feel system, a sensitivity index may be defined as

$$S_1 = (\sum_{i,j} (\frac{\partial S^1}{\partial f_{ij}})^2)^{1/2} \qquad (4.2.34)$$

(The subscript denotes the dimension of the feel system).

4.2.6.2 'n' Dimensional Feel System.

For the 'n' dimensional feel system, the sensitivity of the feel vector due to variations in f_{ij}, is given by

$$S^n = \begin{bmatrix} \partial S^1/\partial f_{ij} \\ \vdots \\ \partial S^k/\partial f_{ij} \\ \vdots \\ \partial S^n/\partial f_{ij} \end{bmatrix}, \quad \text{where} \quad S^k = \int_{t_o}^{t_f} | Z^k(t) - y^k(t) | \, dt \qquad (4.2.35)$$

The sensitivity index for the 'n' dimensional feel system may be defined as

$$S_n = \Big(\sum_k \sum_{i,j} \big(\partial S^k/\partial f_{ij} \big)^2 \Big)^{1/2} \qquad (4.2.36)$$

In general, the feel matrix and the weighting matrices chosen should be such that S_n is as small as possible.

4.3 Sensitivity Analysis of the Optimal Transfer System.

For the optimal transfer system, the output position is given by,

$$Y(t) = C_1 X(t) \qquad (4.3.1)$$

where $X(t) = S(t)\, p(t) + g(t) \qquad t_o \le t \le t_f \qquad (4.3.2)$

S(t), g(t), and p(t) in (4.3.2) are the solutions of the differential equations

$$\dot{S}(t) = A_1 S(t) + S(t) A_1^T + S(t)\, Q\, S(t) - B R^{-1} B^T \qquad (4.3.3)$$

with $S(t_f) = [0]$,

$$\dot{g}(t) = (A_1 + S(t)\, Q)\, g(t) \qquad (4.3.4)$$

with $g(t_f) = X_f$.

and $\dot{p}(t) = -Q\,[S(t)\, p(t) + g(t)] - A_1^T p(t) \qquad (4.3.5)$

with $p(t_o) = S^{-1}(t_o)\,[X(t_o) - g(t_o)] \qquad (4.3.5a)$

The optimal control $T_1(t) = -R^{-1} B^T p(t) \qquad (4.3.6)$

The cost functional $J(T_1)$ minimised is

$$J(T_1) = \frac{1}{2} \int_{t_o}^{t_f} \{< X(t), Q\, X(t) > + < T_1(t), R\, T_1(t) > \}\, dt \tag{4.3.7}$$

Proceeding again as in section 4.2, the sensitivities of the optimal output positions due to changes in the feel matrix and the weighting matrices can be obtained. The procedure will be illustrated only for variations in the feel matrix

4.3.1 Effect of Variations in the Feel Matrix on the output position of the System.

Let the nominal values of S, p, g, A_1 and A_3 be S^o, p^o, g^o, A_1^o and A_3^o respectively. The sensitivity of the optimal state trajectory due to variations in f_{ij}, may be obtained by differentiating (4.3.2) ω.r.t. f_{ij}.

i.e $$\frac{\partial X}{\partial f_{ij}}(t) = S^o(t)\frac{\partial p}{\partial f_{ij}}(t) + \frac{\partial S}{\partial f_{ij}}(t)\, p^o(t) + \frac{\partial g}{\partial f_{ij}}(t) \tag{4.3.8}$$

Thus, it requires the knowledge of the sensitivities of $S(t)$, $g(t)$, and $p(t)$ due to changes in f_{ij}. They may be obtained by differentiating (4.3.3), (4.3.4) and (4.3.5) ω.r.t. f_{ij}.

Thus, $\frac{\partial S}{\partial f_{ij}}(t)$ is the solution of the linear matrix differential equation.

$$\frac{d}{dt}\left[\frac{\partial S}{\partial f_{ij}}(t)\right] = A_3^o(t)\frac{\partial S}{\partial f_{ij}}(t) + \frac{\partial S}{\partial f_{ij}}(t)\, A_3^{oT}(t) + L_1(t) \tag{4.3.9}$$

where $A_3^o(t) = [A_1^o + S^o(t)\, Q]$

and $L_1(t) = \frac{\partial F}{\partial f_{ij}} S^o(t) + \left(\frac{\partial F}{\partial f_{ij}} S^o(t)\right)^T$

The boundary condition for (4.3.9) is $\frac{\partial S}{\partial f_{ij}}(t_f) = [0]$

$\frac{\partial g}{\partial f_{ij}}(t)$ is the solution of the linear vector differential equation

$$\frac{d}{dt}\left[\frac{\partial g}{\partial f_{ij}}(t)\right] = A_3^o(t)\frac{\partial g}{\partial f_{ij}}(t) + \frac{\partial A_3}{\partial f_{ij}}(t)\, g^o(t) \tag{4.3.10}$$

The boundary condition for (4. 3. 10) is given by $\frac{\partial g}{\partial f_{ij}}(t_f) = 0$ and $\frac{\partial p}{\partial f_{ij}}(t)$ is the solution of the linear vector differential equation

$$\frac{d}{dt}\left[\frac{\partial p}{\partial f_{ij}}(t)\right] = -A_3^{oT}(t)\,\frac{\partial p}{\partial f_{ij}}(t) - L_2(t) \qquad (4.3.11)$$

where $L_2(t) = \left(\frac{\partial A_3}{\partial f_{ij}}\right)^T p^o(t) + Q\,\frac{\partial g}{\partial f_{ij}}(t)$

The boundary condition for (4. 3. 11) may be obtained by differentiating (4. 3. 5a) ω. r. t. f_{ij}, so that

$$\frac{\partial p}{\partial f_{ij}}(t_o) = -\left(S^{-1}(t_o)\right)\left[\frac{\partial S}{\partial f_{ij}}(t)\,p^o(t) + \frac{\partial g}{\partial f_{ij}}(t)\right] \qquad (4.3.11a)$$

The sensitivity of the optimal output position due to changes in f_{ij} can then be obtained from the relation

$$Y(t) = C_1\,\frac{\partial X}{\partial f_{ij}}(t) \qquad (4.3.12)$$

The change in the system performance can be measured, as before, either as a change in the performance index or as a change in the feel constant. The procedure will be illustrated for variations in the feel matrix.

4. 3. 2 Effect of Changes in the Feel Matrix on $\overset{*}{J}(x(t),t)$ and S.

It can be shown that the minimal value of the cost (4. 3. 7) is given by

$$J^*(x(t),t) = \frac{1}{2} < X(t),\ S^{-1}(t)\,X(t) > \qquad (4.3.13)$$

where $S^{-1}(t)$ satisfies the matrix Riccati equation

$$\frac{d}{dt}\left(S^{-1}(t)\right) = -S^{-1}(t)\,A_1 - A_1^T\,S^{-1}(t) + S^{-1}(t)\,B\,R^{-1}\,B^T S^{-1}(t) - Q\,.$$

This equation may be obtained from (4. 3. 3) by premultiplying and post multiplying both sides by S.

Differentiating (4. 3. 13) ω. r. t. f_{ij}, we obtain

$$\partial J^*(T_1)/\partial f_{ij} = < X^o(t),\ S^{o^{-1}}(t)\ \partial X/\partial f_{ij}(t) > + \frac{1}{2} < X^o(t), S^{o^{-1}}(t)\ \partial S/\partial f_{ij}(t) S^{o^{-1}}(t) X^o(t) > \qquad (4.3.14)$$

Since $\frac{\partial X}{\partial f_{ij}}$ (t), and $\frac{\partial S}{\partial f_{ij}}$ (t) can be obtained from (4. 3. 8) and (4. 3. 9) $\partial J^*(x(t),t)/\partial f_{ij}$ can be computed.

The sensitivity of the feel constant due to changes in f_{ij} may be obtained, proceeding as in section 4. 2. 6. For instance, for the one dimensional feel system, it is given by

$$\partial S^1/\partial f_{ij} = \int_{t_o}^{t_f} \text{sign } e^o(t) \ \partial e/\partial f_{ij} (t) \ dt$$

where $e(t) = [m(t) - y(t)]$, and $m(t)$ is the straight line joining the initial and the final positions.

4. 3. 3. Remark.

In the above analyses, $T_2(t)$ has been assumed to be a passive load on the system. It will be interesting to carry out the same, considering $T_2(t)$ as a noise. A similar problem has been treated in [41], where the effect of the noise intensity on the performance index for plants described by stochastic differential equations has been investigated.

4. 4 Application of Perturbation Techniques.

Partial derivative techniques have been used in the above to obtain the sensitivity of the optimal state trajectory to changes in the system parameters for the optimal feel systems obtained in chapter III.

In simpler cases, perturbation techniques can also be used [42] to find the same. This method is illustrated in this section.

Let the given linear system be described by the differential equation,

$$\dot{X} = AX \tag{4.4.1}$$

Let the system matrix A be perturbed by an amount ϵ D, where D is a matrix of the same dimension as A and ϵ is a scalar quantity.

$$\text{Then } \dot{X}_1 = (A + \epsilon D) X_1 \tag{4.4.2}$$

using the perturbation method, the solution of (4. 4. 2) may be obtained as

$$X_1(t) = X(t) + \epsilon \int_o^t \phi(t-\tau)\, D\, x(\tau)\, d\tau + 0(\epsilon) \qquad (4.4.3)$$

Since $X(t) = \phi(t)\, X(o)$ and $X(\tau) = \phi(\tau)\, X(o)$

we have $$X_1(t) = \phi(t)\, X(o) + \epsilon \int_o^t \phi(t-\tau)\, D\, \phi(\tau)\, X(o)\, d\tau \qquad (4.4.4)$$

Therefore $$[X_1(t) - X(t)] = \Delta X(t) = \epsilon \int_o^t \phi(t-\tau)\, D\, \phi(\tau)\, X(o)\, d\tau \qquad (4.4.5)$$

Let us consider an example

Let a linear system be described by an 'n' th order differential equation

$$a_n \frac{d^n x}{dx^n} + a_{n-1} \frac{d^{n-1} x}{dx^{n-1}} + \ldots\ldots + a_1 x = 0 \qquad (4.4.6)$$

A state space representation of (4.8.6) is given by,

$$\dot{X} = A X \quad (4.4.7) \qquad \text{where } A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ -a_1 & -a_2 & -a_3 & \ldots & -a_n \end{bmatrix}$$

Let the initial condition for (4.4.7) be $X^T(o) = [1, 0, \ldots 0]$

Let each of the coefficients $a_1 \ldots a_n$ be changed by the same percentage amount. Then the change in the system matrix is ϵA, and $D = A$.

The change in the state vector $\Delta X(t)$ (4.4.5) is given by

$$\Delta X(t) = \epsilon \int_o^t \phi(t-\tau)\, A\, \phi(\tau)\, X(o)\, d\tau$$

$$= \epsilon \int_o^t \phi(t-\tau) \begin{bmatrix} \phi_{21}(\tau) \\ \vdots \\ \phi_{(n-1)1}(\tau) \\ \psi(\tau) \end{bmatrix} d\tau \qquad (4.4.8)$$

where $\psi(\tau) = -a_1 \phi_{11}(\tau) - a_2 \phi_{21}(\tau) - \ldots\ldots - a_n \phi_{n1}(\tau)$

4.5 Use of Sensitivity Functions in the Design of Feel Systems.

It is stated in chapter III, that the design of a good feel system depends upon a proper selection of a feel matrix and the weighting matrices, N, Q, and R. Either the heuristic procedure or the two level design technique described in [39] can be employed for obtaining such a combination using the sensitivity functions. The procedure will be illustrated for the optimal tracking problem.

4.5.1. The Heuristic Procedure.

The method consists of the following steps:

(i) choose an initial set of a feel matrix, and the weighting matrices. Compute the optimal control, and the optimal trajectory for the assumed feel matrix and the weighting matrices.

(ii) Determine whether the present combination yields a control which will produce an acceptable response. If the response is acceptable, then the problem is solved. Otherwise proceed to step (iii)

(iii) Determine the sensitivity functions $\partial X/\partial f_{ij}$, $\partial X/\partial q_i$, $\partial X/\partial n_i$ and $\partial X/\partial r_i$ from the information obtained either in section 4.2. Let the present values of the feel matrix and weighting matrices be denoted as follows.

$$f^o = \begin{bmatrix} f_{11}^o \\ f_{12}^o \\ \vdots \\ f_{4n4n} \end{bmatrix}, \quad q^o = \begin{bmatrix} q_1^o \\ q_2^o \\ \vdots \\ q_n^o \end{bmatrix}, \quad n^o = \begin{bmatrix} n_1^o \\ n_2^o \\ \vdots \\ n_n^o \end{bmatrix}, \text{ and } r^o = \begin{bmatrix} r_1^o \\ r_1^o \\ \vdots \\ r_n^o \end{bmatrix} \qquad (4.5.1)$$

Expanding $X(t,f,q,n,r)$ in a Taylor's series about the nominal point (f^o, q^o, n^o, r^o) gives

$$X(t,f,q,n,r) = X(t\ \ f^o, q^o, n^o, r^o) + \frac{\partial X}{\partial f}\Big|_o (f-f^o) + \frac{\partial X}{\partial f}\Big|_o (q-q^o) + \frac{\partial X}{\partial n}\Big|_o (n-n^o)$$
$$+ \frac{\partial X}{\partial r}\Big|_o (r-r^o) + \text{higher order terms} \qquad (4.5.2)$$

where $\frac{\partial X}{\partial f}\Big|_o \frac{\partial X}{\partial q}\Big|_o, \frac{\partial X}{\partial n}\Big|_o$ and $\frac{\partial X}{\partial r}\Big|_o$ are Jacobian matrices evaluated at (f^o, q^o, n^o, r^o)

Examine the reasons for the 'unacceptability' of the system response. For instance, for the longitudinal vehicle control problem considered in [39], it is shown that for the initial choice of the weighting matrices, the vehicle, in order to achieve the desired final position, must attain a maximum acceleration of about 24 ft/sec^2. This is too great on acceleration to expect from the vehicle, and also it would not present a comfortable ride. Thus the acceleration profile of the vehicle is not acceptable. Actually, using this procedure, a set of weighting matrices have been obtained for which the vehicle will be able to attain the desired position with a peak acceleration of about 2.9 ft/sec^2. This is acceptable.

For the present problem, the acceptability of the system response may be based on the facts, that the value of the feel constant should be within a certain range, and that the accelerations and the velocities to be attained by the system while positioning an object at a desired location should not exceed the system capabilities. Also, the model obtained for the human operator in the combined man-machine system for the given combination should match with the model obtained through actual experimentation (see [17]).

Using the sensitivity functions obtained in section 4.3, and the relation (4.5.2), small changes in the feel matrix and the weighting matrices can be determined, which will improve the system response.

Steps (i) through (iii) are repeated iteratively until a suitable combination of a feel matrix, the weighting matrices and control are determined. Alternatively the sensitivity functions $\partial X/\partial f \ldots \partial X/\partial r$ could perhaps be used with numerical techniques in a two level design approach.

4.5.2 The Two Level Design Technique.

This method requires the definition of a second performance functional J_2, in addition to (4.2.6), as a measure of the unacceptability of the system response. It may be defined as

$$J_2 = \int_0^T L(X_c)\, dt \qquad (4.5.3)$$

where X_c is a subset of the state variables, for which, for a given combination of a feel matrix and the weighting matrices, the system response is unacceptable; even though the state trajectory is optimal in the sense that J is minimised. In general there may be difficulties in choosing an appropriate performance functional J_2. For the particular problem, the feel constant may be chosen as J_2.

Obviously J_2 is also a function of the feel matrix and the weighting matrices. Then the two level design procedure would involve using an iterative numerical procedure tochoose the feel matrix and the weighting matrices so that,

$$J_2^* = \min_{f,q,n,r} J_2(X_c) \qquad (4.5.4)$$

The control $T_1(t)$ will be chosen so as to minimise J for the given combination

If a gradient type method is used in the minimisation indicated in (4.5.4), then the gradient components $\partial J_2/\partial u$ can be obtained using the relation.

$$\partial J_2/\partial u = \int_0^T \text{grad } L(X_c) \frac{\partial X}{\partial u}\, dt \qquad (4.5.5)$$

where $u \in \{f, q, n, r\}$ and $\partial X/\partial u$ are the sensitivity functions obtained in section 4.2.

It is believed that a feel system designed using the sensitivity functions will be superior in performance to a feel system obtained through the procedure described in [17].

This completes the study of systems with feel having no signal transmission time delay. However, when the master and the slave ends are separated by long distances, there exists certain time delay in the signal transmission channels and is to be taken into account. Mathematical modelling, and optimal control of feel systems with delay is considered in the next chapter.

CHAPTER V

FEEL SYSTEMS WITH SIGNAL TRANSMISSION TIME DELAY

In the previous chapters we have considered feel systems that do not have any delay in the signal transmission channels. However, when the master and the slave ends of an 'n' dimensional feel system are separated by long distances, there exists certain signal transmission time delay and is to be taken into account. The presence of the delay adversely effects the stability of the feel system and the gains of the transmission channels K_{ij} and $D_{ij}(i_{jj} = 1, 2)$ must be so chosen as to make the system stable. For a given set of the gains, there exists a specific time delay and hence a distance of separation of the master and the slave ends for the system to be stable. The delay also modifies the force reflecting characteristic of the system. In this chapter, state space models are obtained for a one dimensional feel system and an 'n' dimensional feel system having signal transmission time delay. The performance of a one dimensional feel system is studied (i) for different feel matrices and (ii) for different delays. Finally, the optimal control of a feel system with delay is considered.

5. 1. One Dimensional Feel System.

5. 1. 1. System Description.

Let the signal transmission time delay between the master and the slave ends be denoted by τ. The differential equations that describe the dynamic performance of a one dimensional feel system are:

$$\begin{aligned} T_1(t) + T_{a1}(t) &= J_1 \ddot{\theta}_1(t) + F_1 \dot{\theta}_1(t) \\ T_2(t) + T_{a2}(t) &= J_2 \ddot{\theta}_2(t) + F_2 \dot{\theta}_2(t) \end{aligned} \qquad (5.1.1)$$

where the torques developed by the actuators $T_{a1}(t)$ and $T_{a2}(t)$ are given by

$$\begin{aligned} T_{a1}(t) &= -K_{11}\theta_1(t) + K_{12}\theta_2(t) - D_{11}\dot{\theta}_1(t) + D_{12}\dot{\theta}_2(t-\tau) \\ T_{a2}(t) &= K_{21}\theta_1(t-\tau) - K_{22}\theta_2(t) + D_{21}\dot{\theta}_1(t-\tau) - D_{22}\dot{\theta}_2(t) \end{aligned} \qquad (5.1.2)$$

The only comment to be made about the performance of the system (5.1.1) is that, if $\tau = 0$, and if each of the gains K_{ij} and $D_{ij}(i,j = 1,2)$ is equal to a single pair of constants K and D respectively, then, the motor torques would dependon the differences in position and velocity between the input and output shafts. The actuators are connected in such a way that they tend to drive each side of the system in the direction as to reduce these errors to zero. When τ is not equal to zero, and K_{ij}'s and D_{ij}'s are unequal in (5.1.1), correct error signals are not available instantaneously at either end. As mentioned already in chapter II, the effects on the stability of the system due to the usage of error signals that are modified by τ, and unequal gains of the transmission channels, is one of the major considerations of Thompsons'[13] work.

The initial conditions for the system (5.1.1) are chosen as

$$\left.\begin{aligned} \theta_1(t) &= a \\ \theta_2(t) &= 0 = \dot{\theta}_1(t) = \dot{\theta}_2(t) \end{aligned}\right\} \quad \text{for } -\tau \le t \le 0 \qquad (5.1.3)$$

5.1.2 State Space Representation.

Choosing the positions and velocities at the master and the slave ends as the state variables, a state space representation of (5.1.1) is obtained as

$$\begin{aligned} \dot{X}(t) &= (A + F_1)\, X(t) + F_2\, X(t-\tau) + BT_1(t) + DT_2(t) \quad t \ge 0 \\ Y(t) &= C\, X(t) \end{aligned} \qquad (5.1.4)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -F_1/J_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -F_2/J_2 \end{bmatrix} F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -K_{11}/J_1 & -D_{11}/J_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -K_{22}/J_2 & -D_{22}/J_2 \end{bmatrix} F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K_{12}/J_1 & D_{12}/J_1 \\ 0 & 0 & 0 & 0 \\ K_{21}/J_2 & D_{21}/J_2 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \end{bmatrix}, \quad C = [\,0 \quad 0 \quad 1 \quad 0\,] \text{ and } X^T(t) = [x_1(t)\, x_2(t)\, x_3(t)\, x_4(t)]$$

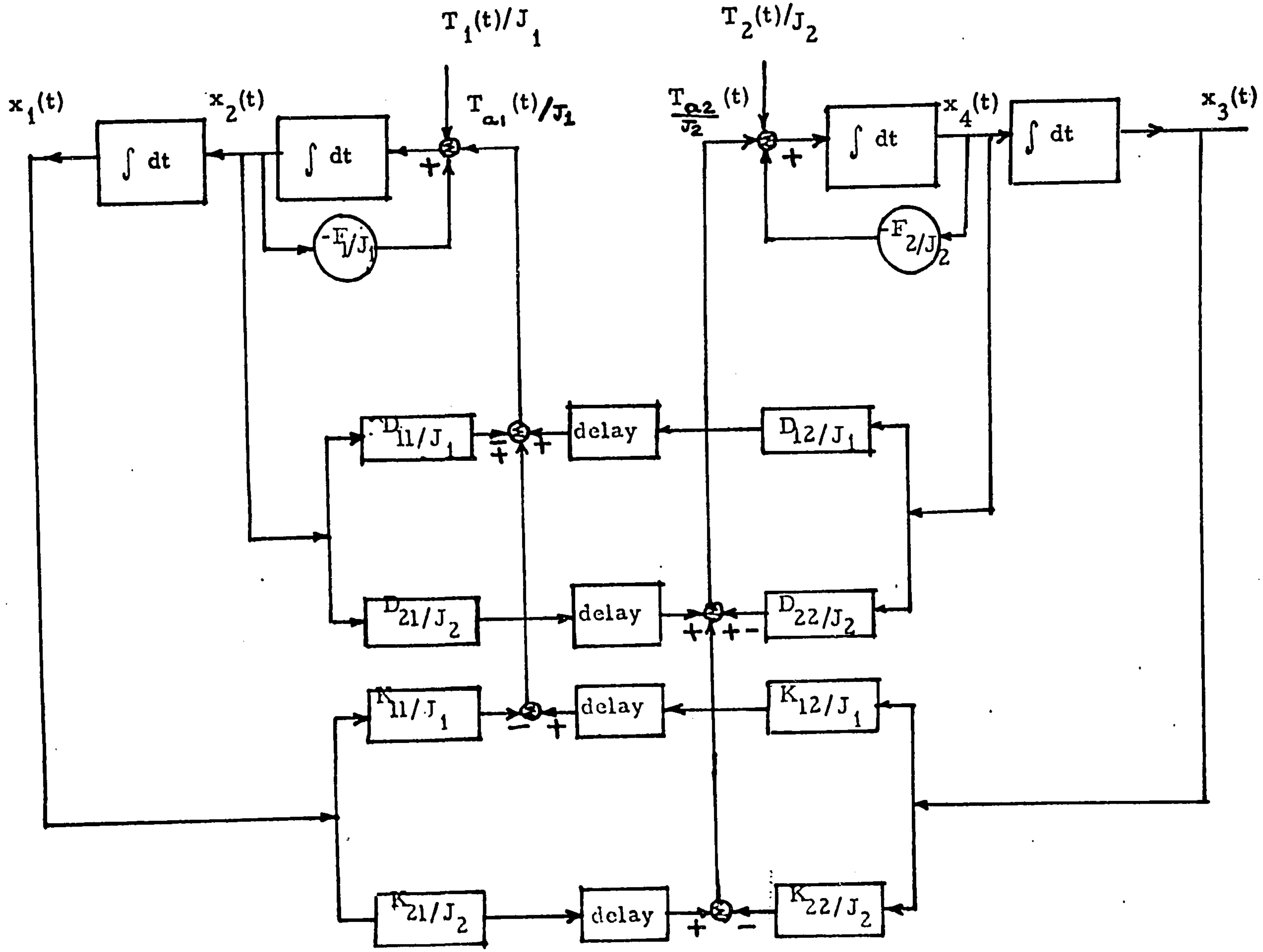


Fig 5.1. A Linear System representation of a one dimensional feel system with signal transmission time delay.
]

The initial function for the system (5.1.4) is $X(t) = G(t) = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $-\tau \leq t \leq 0$, (5.1.5)

A linear system representation of the one dimensional feel system (5.1.4) is shown in Fig 5.1.

5.1.3 An Analysis of the One Dimensional Feel System.

In this section, the responses of (5.1.4) are obtained for an initial displacement of the shafts at the master and the slave ends (i) for different feel matrices $F = (F_1, F_2)$ and (ii) for different delays τ. Computational results obtained for a feel system with derivative feedback for a unit torque input at the master end are also given. Finally a comparison of the responses obtained here is made, with similar responses obtained in chapter II, for a system with no signal transmission time delay.

The responses of (5.1.4) are computed using the relation

$$X(t) = K(t)G(o) + \int_{-\tau}^{o} K(t-\theta-\tau)F_2 G(\theta)d\theta + \int_{o}^{t} K(t-\theta)[BT_1(\theta)+DT_2(\theta))\,d\theta \qquad (5.1.6)$$

where $K(t)$, the fundamental matrix of (5.1.4) satisfies the following matrix differential difference equation,

$$\dot{K}(t) = (A+F_1)K(t) + F_2 K(t-\tau) \qquad (5.1.7)$$

$$K(t) = 0 \text{ for } t < 0$$

$$= I \text{ for } t = 0$$

5.1.3.1. Evaluation of the Fundamental Matrix K(t).

In order to obtain $K(t)$, the following first order differential difference equations, obtained by expanding (5.1.7) have to be solved.

$$\dot{k}_{11}(t) = k_{21}(t)$$

$$\dot{k}_{12}(t) = k_{22}(t)$$

$$\dot{k}_{13}(t) = k_{23}(t)$$

$$\dot{k}_{14}(t) = k_{24}(t)$$

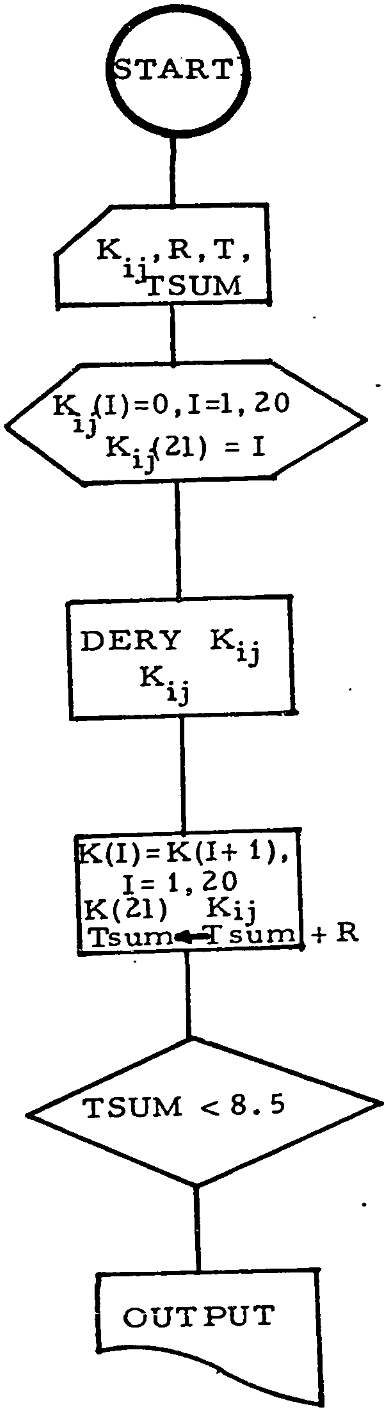


Fig 5.2 Flow chart for computing the fundamental matrix K(t)

$$\dot{k}_{21}(t) = -K_{11}/J_1\, k_{11}(t) - (D_{11}+F_1)/J_1\, k_{21}(t) + K_{12}/J_1\, k_{31}(t-\tau) + D_{12}/J_1\, k_{41}(t-\tau)$$

$$\dot{k}_{22}(t) = -K_{11}/J_1\, k_{12}(t) - (D_{11}+F_1)/J_1\, k_{22}(t) + K_{12}/J_1\, k_{32}(t-\tau) + D_{12}/J_1\, k_{42}(t-\tau)$$

$$\dot{k}_{23}(t) = -K_{11}/J_1\, k_{13}(t) - (D_{11}+F_1)/J_1\, k_{23}(t) + K_{12}/J_1\, k_{33}(t-\tau) + D_{12}/J_1\, k_{43}(t-\tau)$$

$$\dot{k}_{24}(t) = -K_{11}/J_1\, k_{14}(t) - (D_{11}+F_1)/J_1\, k_{24}(t) + K_{12}/J_1\, k_{34}(t-\tau) + D_{12}/J_1\, k_{44}(t-\tau)$$

$$\dot{k}_{31}(t) = k_{41}(t)$$

$$\dot{k}_{32}(t) = k_{42}(t)$$

$$\dot{k}_{33}(t) = k_{43}(t)$$

$$\dot{k}_{34}(t) = k_{44}(t)$$

$$\dot{k}_{41}(t) = -K_{22}/J_2\, k_{31}(t) - (D_{22}+F_2)/J_2\, k_{41}(t) + K_{21}/J_2\, k_{11}(t-\tau) + D_{21}/J_2\, k_{21}(t-\tau)$$

$$\dot{k}_{42}(t) = -K_{22}/J_2\, k_{32}(t) - (D_{22}+F_2)/J_2\, k_{42}(t) + K_{21}/J_2\, k_{12}(t-\tau) + D_{21}/J_2\, k_{22}(t-\tau)$$

$$\dot{k}_{43}(t) = -K_{22}/J_2\, k_{33}(t) - (D_{22}+F_2)/J_2\, k_{43}(t) + K_{21}/J_2\, k_{13}(t-\tau) + D_{21}/J_2\, k_{23}(t-\tau)$$

$$\dot{k}_{44}(t) = -K_{22}/J_2\, k_{34}(t) - (D_{22}+F_2)/J_2\, k_{44}(t) + K_{21}/J_2\, k_{14}(t-\tau) + D_{21}/J_2\, k_{24}(t-\tau) \tag{5.1.8}$$

5.1.3.1.1. Digital Computer Solution.

A digital computer programme has been written for solving (5.1.8) a flow chart of which appears in Fig 5.2. For simpler cases the "method of steps" [48] can be used to determine K(t). An example is given below.

5.1.3.1.2 The "Method of Steps".

Let a linear time lag system be represented by

$$\dot{X}(t) = A\,X(t) + F_2\,X(t-\tau) + B\,T_1(t) \qquad t \geq 0 \tag{5.1.9}$$

and $X(t) = G(t), \; -\tau \leq t \leq 0$

where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The fundamental matrix K(t) for (5. 1. 9) satisfies the matrix differential difference equation,

$$\dot{K}(t) = A\,K(t) + F_2\,K(t-\tau) \qquad (5.1.10)$$

$$K(t) = 0 \quad \text{for} \quad t < 0,$$

$$= I \quad \text{for} \quad t = 0.$$

Expanding (5. 1. 10) we have,

$$\dot{k}_{11}(t) = k_{21}(t-\tau) \qquad (5.1.11)$$

$$\dot{k}_{12}(t) = k_{22}(t-\tau) \qquad (5.1.12)$$

$$\dot{k}_{21}(t) = k_{11}(t) \qquad (5.1.13)$$

$$\dot{k}_{22}(t) = k_{12}(t) \qquad (5.1.14)$$

Let us first determine $k_{11}(t)$.

From equation (5. 1. 11) and (5. 1. 13) we have

$$\ddot{k}_{11}(t) = k_{11}(t-\tau) \qquad (5.1.15)$$

For the first step i. e $t \in [0, \tau)$

$$\ddot{k}_{11}(t) = k_{11}(t-\tau) = 0 \qquad (5.1.16)$$

Integrating (5. 1. 16) we have

$$\dot{k}_{11}(t) = C_1 .$$

Since $\dot{k}_{11}(o) = 0$, $C_1 = 0$ and hence $\dot{k}_{11}(t) = 0$ (5. 1. 17)

Integrating (5. 1. 17) we have

$$k_{11}(t) = C_2$$

Since $k_{11}(o) = 1$, $C_2 = 1$, and we have $k_{11}(t) = 1$ for $t \in [0, \tau)$

For the second step i. e $t \in [\tau, 2\tau)$

$$\ddot{k}_{11}(t) = k_{11}(t-\tau) = 1 \qquad (5.1.18)$$

Integrating (5. 1. 18) we have

$$\dot{k}_{11}(t) = t + C_3$$

But since $\dot{k}_{11}(\tau) = 0$, $C_3 = -\tau$, and hence

$$\dot{k}_{11}(t) = (t-\tau) \qquad (5.1.19)$$

Integrating (5.1.19), we have

$$k_{11}(t) = \frac{(t-\tau)^2}{2} + C_4$$

Since the solution of $k_{11}(t)$ is continuous we have $k_{11}(\tau) = 1$, and hence $C_4 = 1$

Thus $$\dot{k}_{11}(t) = 1 + \frac{(t-\tau)^2}{2} \quad \text{for } t \in [\tau, 2\tau) \qquad (5.1.20)$$

Generalising (5.1.20) we obtain, for any t

$$k_{11}(t) = \sum_{n=o}^{\lceil t/\tau \rceil} \frac{(t-n\tau)^{2n}}{(2n)!} \qquad (5.1.21)$$

$k_{22}(t)$ can be obtained by solving

$$\ddot{k}_{22}(t) = k_{22}(t-\tau) \qquad (5.1.22)$$

Equation (5.1.22) has the same form and initial condition as (5.1.15).

Hence $k_{22}(t) = k_{11}(t)$

$k_{21}(t)$ can be obtained by solving

$$\ddot{k}_{21}(t) = k_{21}(t-\tau) \text{ with } \dot{k}_{21}(o) = 1 \text{ and } k_{21}(o) = 0 \qquad (5.1.23)$$

It can be shown that $$k_{21}(t) = \sum_{n=o}^{\lceil t/\tau \rceil} \frac{(t-n\tau)^{2n+1}}{(2n+1)!} \qquad (5.1.24)$$

$$\ddot{k}_{12}(t) = k_{12}(t-\tau) \text{ with } \dot{k}_{12}(o) = 0 \text{ and } k_{12}(o) = 0 \qquad (5.1.25)$$

It can be shown that $$k_{12}(t) = \sum_{n=1}^{\lceil t/\tau \rceil} \frac{(t-n\tau)^{2n-1}}{(2n-1)!}$$

5.1.3.2 Computation of the System Responses.

For the one dimensional feel system (5.1.4), the positions of the shafts at the master ($\theta_m(t)$) and the slave ends ($\theta_s(t)$) are then obtained using the relations ($a = 1$).

$$\theta_m(t) = x_1(t) = k_{11}(t) + \int_{-\tau}^{o} k_{14}(t-\theta-\tau)\,\frac{K_{21}}{J_2}\,d\theta + \int_o^t k_{12}(t-\theta)\,\frac{T_1(\theta)}{J_1}\,d\theta + \int_o^t k_{14}(t-\theta)\,\frac{T_2(\theta)}{J_2} \qquad (5.1.26)$$

$$\theta_s(t) = x_3(t) = k_{31}(t) + \int_{-\tau}^{o} k_{34}(t-\theta-\tau)\frac{K_{21}}{J_2}\, d\theta + \int_{o}^{t} k_{32}(t-\theta)\,\frac{T_1(\theta)}{J_1}\, d\theta \; + \int_{o}^{t}\frac{k_{34}(t-\theta)T_2(\theta)}{J_2}\, d\theta \qquad (5.1.27)$$

5.1.3.3. A Discussion of The Numerical Results.

Figs 5.3 to 5.6 show the output positions of the shafts at the master and the slave ends for an initial displacement of 1 rad. at the master end for a signal transmission time delay of 0.5 secs, for three typical feel matrices F^1, F^2 and F^3. Figs 5.3 and 5.4 also, show, the positions of the shafts of the system with feel matrix F^1 for delays of 0, 0.2 sec, and 0.5 secs. Fig 5.5 also shows the position of the slave end of the system with feel matrix F^2 for a unit step torque input at the master end. It was found that the basic nature of the responses obtained here is the same as in the case of a system with no delay (chapter II). For instance, when each of the gains K_{ij} and D_{ij} (i_{jj} = 1, 2) is equal to unity, with an initial displacement of the shaft at the master end, the system behaves as if a unit step torque is applied at the master end. Also the steady state position attained by the slave end is equal to half the magnitude of the initial displacement at the master end. The responses obtained when derivative feedback is present, (as in the case of a system with feel matrix F^2) are well damped, compared to the case when there is only position feedback. With increases in time delay, the system responses tend to be more and more oscillatory. However, when the gains K_{ij}'s and D_{ij}'s are unequal, (as in the case of a system with feel matrix F^3 the system responses obtained are similar to that of an ordinary servomechanism.

Let us now examine the bilateral nature of the system. Let each of the gains K_{ij} and D_{ij} be equal to a single pair of constants K and D respectively. Let a unit displacement be given to the shaft at the master end and be released. Let the resulting positions at the master and the slave ends due to such a

$$F_1' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \text{ and } F_2' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

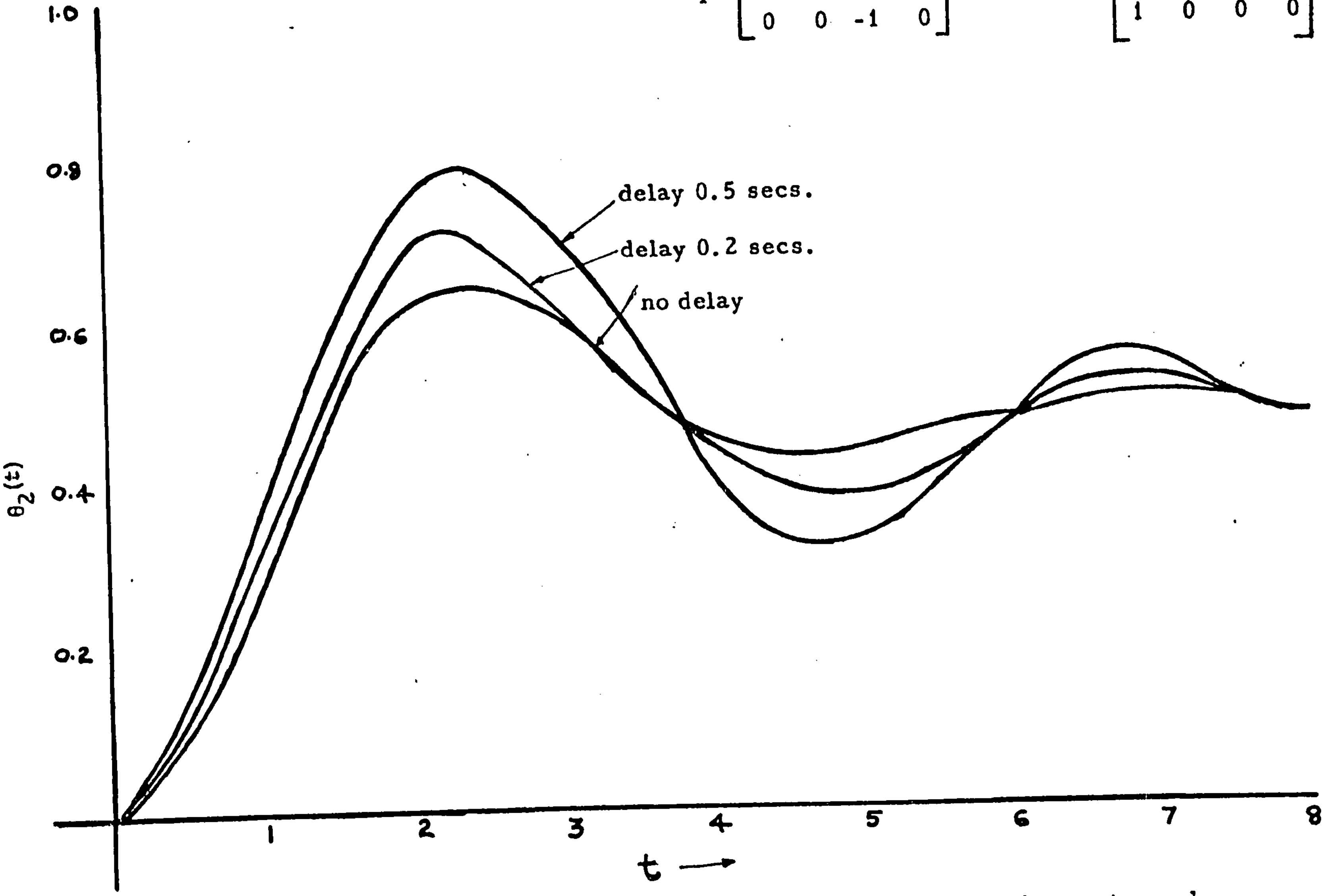


Fig 5.3 Position of the slave end for a unit displacement of the shaft at the master end (system with Feel matrix pair (F_1', F_2'))

$$F'_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \text{ and } F'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

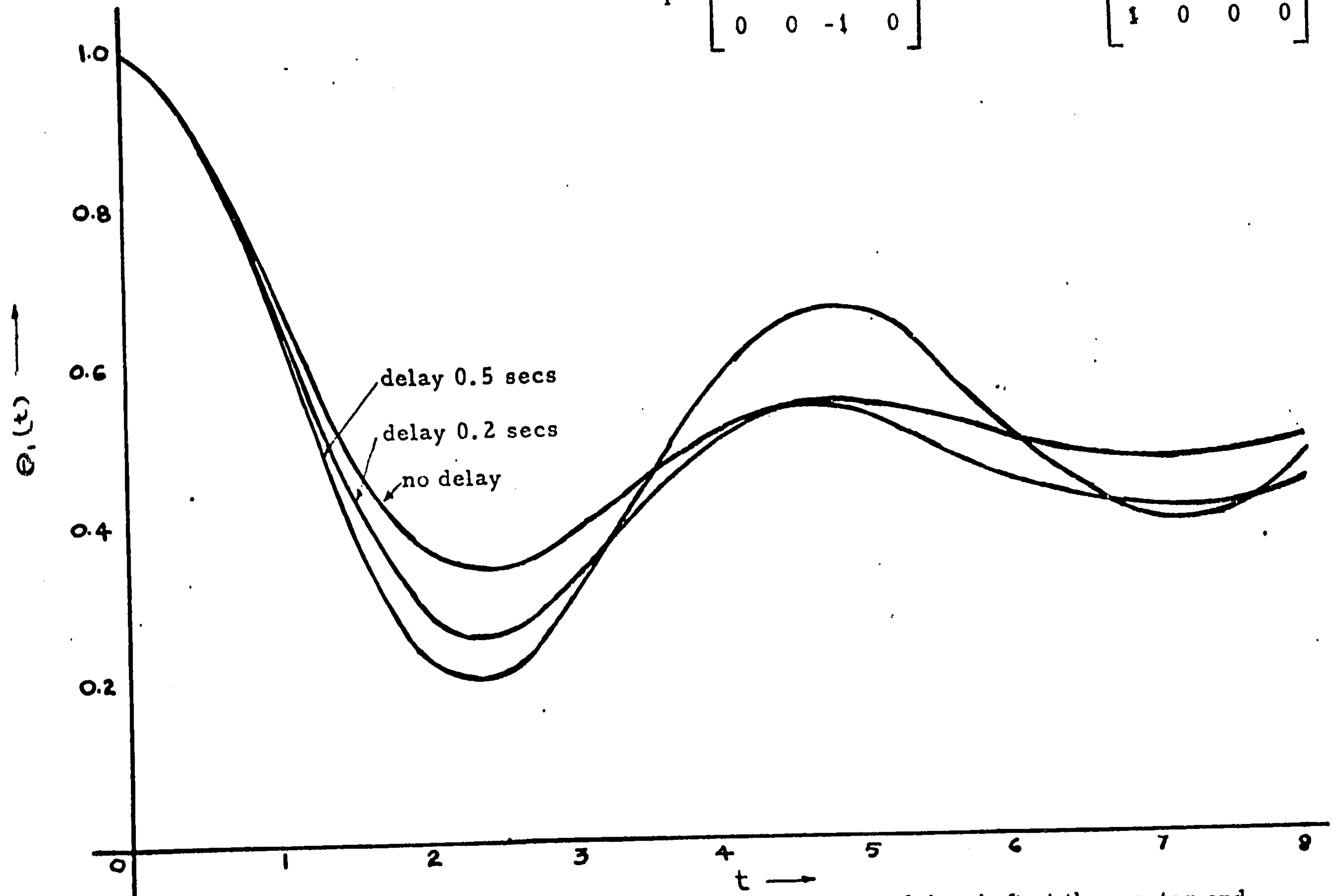


Fig. 5.4 Position of the master end for a unit displacement of the shaft at the master end

[system with Feel matrix pair (F'_1, F'_2)]

displacement at the master end be denoted by $\theta_{m,m}(t)$ and $\theta_{s,m}(t)$ respectively. Then,

$$\theta_{m,m}(t) = x_1(t) = k_{11}(t) + \int_{-\tau}^{0} k_{14}(t-\theta-\tau)\,\frac{K_{21}}{J_2}\,d\theta \tag{5.1.28}$$

$$\theta_{s,m}(t) = x_3(t) = k_{31}(t) + \int_{-\tau}^{0} k_{34}(t-\theta-\tau)\,\frac{K_{21}}{J_2}\,d\theta \tag{5.1.29}$$

$(a = 1)$

If now a unit displacement is given to the shaft at the slave end and released, the corresponding responses denoted by $\theta_{m,s}(t)$ and $\theta_{s,s}(t)$ respectively are given by

$$\theta_{m,s}(t) = x_1(t) = k_{13}(t) + \int_{-\tau}^{0} k_{12}(t-\theta-\tau)\,\frac{K_{12}}{J_1}\,d\theta \tag{5.1.30}$$

$$\theta_{s,s}(t) = x_3(t) = k_{33}(t) + \int_{-\tau}^{0} k_{32}(t-\theta-\tau)\,\frac{K_{12}}{J_1}\,d\theta \tag{5.1.31}$$

Even when there is a delay in the signal transmission, numerical results reveal that,

$$\begin{aligned} k_{13}(t) &= k_{31}(t) & k_{11}(t) &= k_{33}(t) \\ k_{12}(t) &= k_{34}(t) & k_{14}(t) &= k_{32}(t) \end{aligned} \tag{5.1.32}$$

Since $K_{12} = K = K_{21}$ and $J_1 = J = J_2$, we have

$$\theta_{s,m}(t) = \theta_{m,s}(t)$$

and

$$\theta_{m,m}(t) = \theta_{s,s}(t) \tag{5.1.33}$$

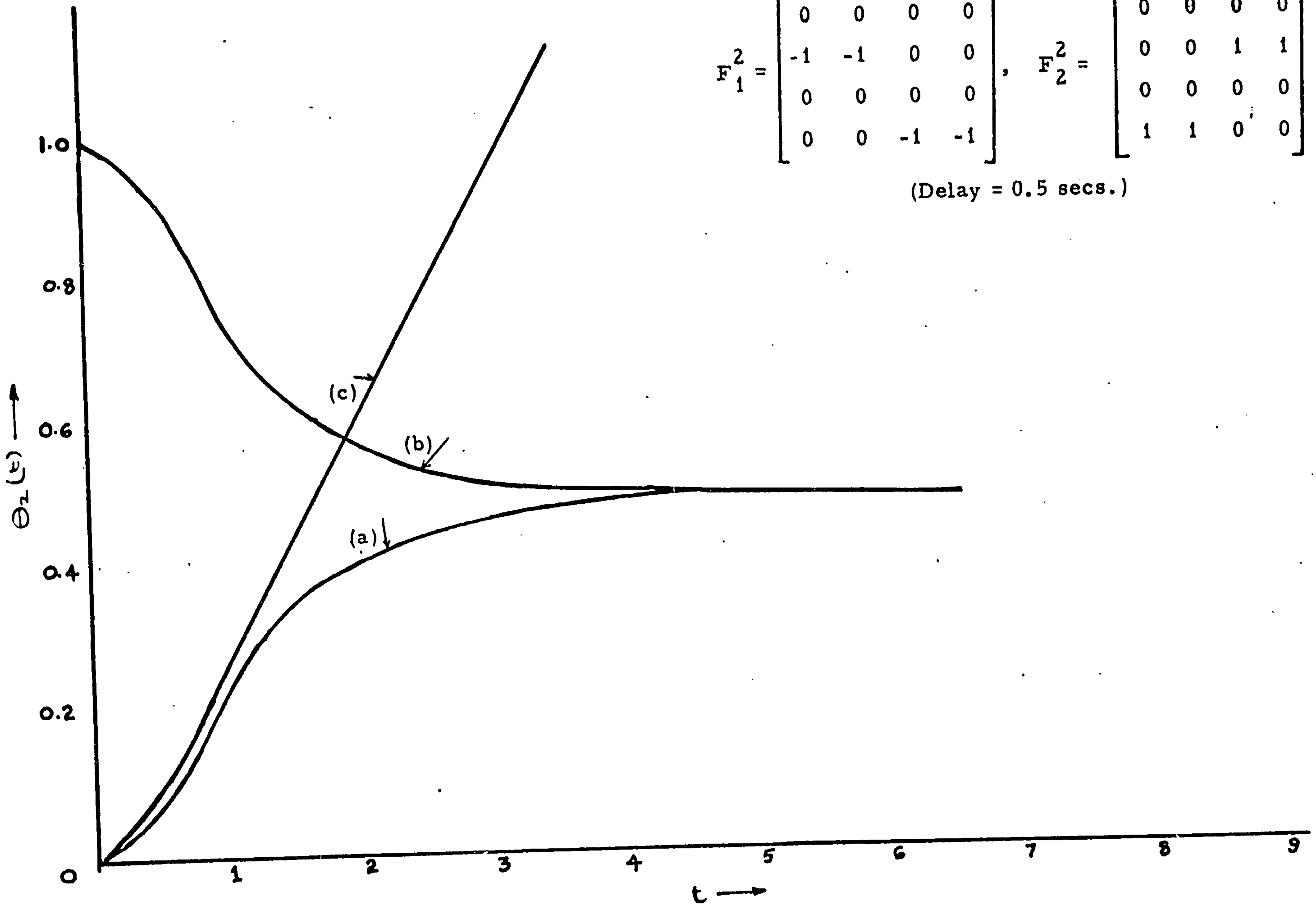
Thus the system (5.1.4) is bilateral.

This bilateral property of the system can also shown to be true, by considering the case when there is an external torque acting on the system. For, if a unit torque is applied at the master end, the resulting position at the slave end $(\theta_{s,m}(t))$ is given by

$$\theta_{s,m}(t) = k_{31}(t) + \int_{-\tau}^{0} k_{34}(t-\theta-\tau)\,\frac{K_{21}}{J_2}\,d\theta + \int_{0}^{t} k_{32}(t-\theta)\,\frac{T_1(\theta)}{J_1}\,d\theta \tag{5.1.34}$$

Now, if a unit torque is given at the slave end, the resulting position at the master end $(\theta_{m,s}(t))$ is given by

$$\theta_{m,s}(t) = k_{13}(t) + \int_{-\tau}^{0} k_{12}(t-\theta-\tau)\,\frac{K_{12}}{J_1}\,d\theta + \int_{0}^{t} k_{14}(t-\theta)\,\frac{T_2(\theta)}{J_2}\,d\theta \tag{5.1.35}$$



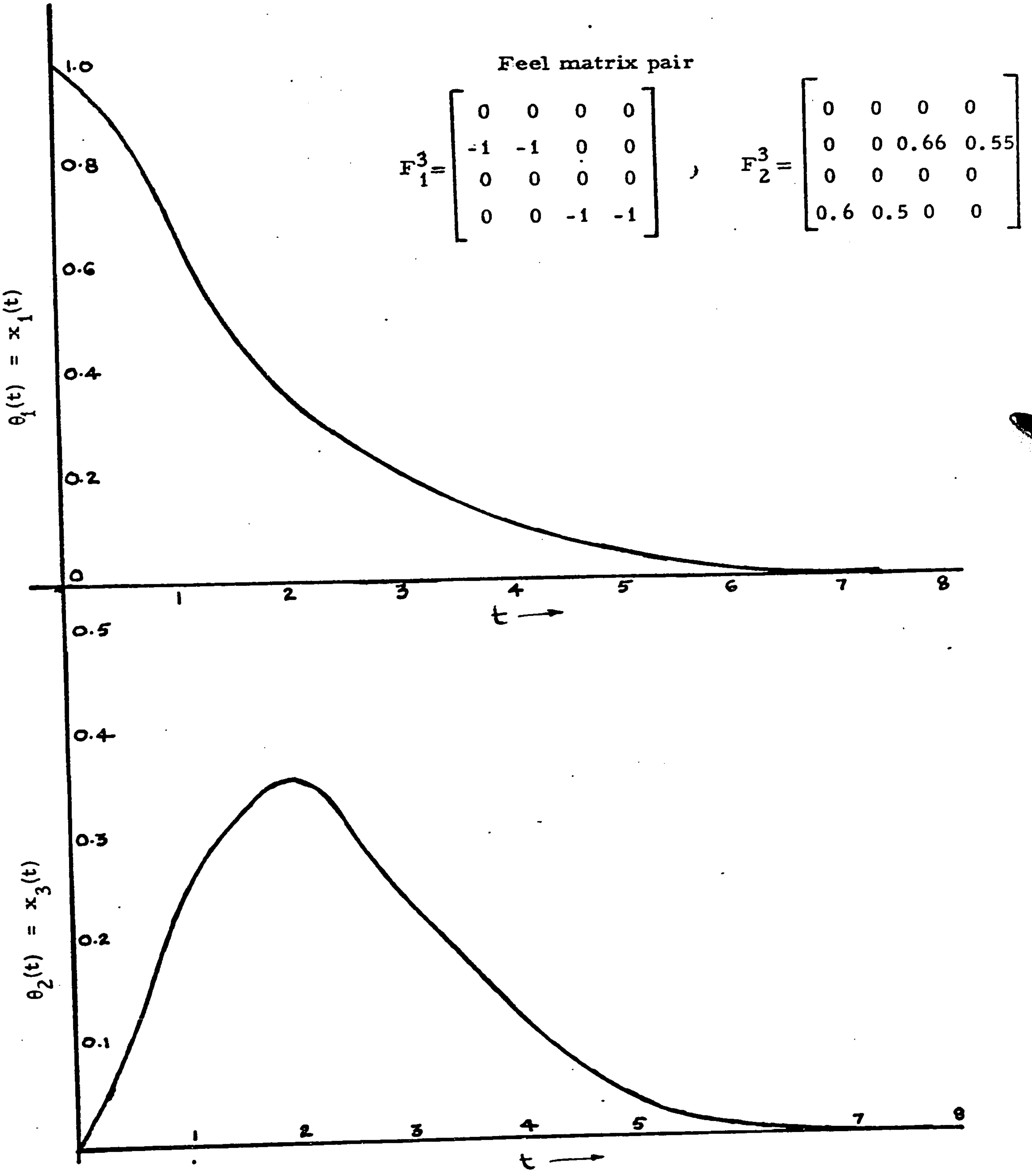
$$F_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad F_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(Delay = 0.5 secs.)

Fig. 5.5 (a) Position of the slave for a unit displacement of the shaft at the master end

(b) Position of the master for a unit displacement of the shaft at the master end

(c) Position of the slave for a unit step torque applied at the master end [(All for the system with the Feel matrix pair F_1^2, F_2^2)].



Feel matrix pair

$$F_1^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad F_2^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.66 & 0.55 \\ 0 & 0 & 0 & 0 \\ 0.6 & 0.5 & 0 & 0 \end{bmatrix}$$

Fig. 5.6 System responses for a unit displacement of the shaft at the master end (System with Feel matrix pair (F_1^3, F_2^3) for delay of 0.5 secs)

If $T_1(t) = T_2(t)$, then it is easy to see that $\dot{\theta}_{s,m}(t) = \theta_{m,s}(t)$ (5.1.36)

However, if the gains K_{ij} and D_{ij} are unequal or when there is a passive load on the system then, the system can no longer be bilateral.

From Fig 5.5 it can be seen that the system response with derivative feedback (the system with feel matrix F^2) for a unit torque at the master end is unbounded. However, for the system with feel matrix F^3, the response will resemble that of an ordinary servomechanism and is bounded. Thus, as mentioned already, the stability of the system (5.1.4) depends for a given time delay, on the magnitudes of the gains K_{ij} and D_{ij}. In order that the system be stable, the gains should be restricted, for a given time delay, to a certain region in the gain parameter space.

5.2. 'n' Dimensional Feel System.

Again from differential equations similar to (2.2.2)(corresponding to $n = 2$ and $\tau = 0$), the state space description of an 'n' dimensional feel system with signal transmission time delay, can be obtained in the form,,

$$\begin{aligned} \dot{X}(t) &= (A + F_1)\,X(t) + F_2\,X(t-\tau) + BT_1(t) + DT_2(t), && (5.2.1) \\ Y(t) &= C\,X(t), && t \geq 0 \\ X(t) &= G(t), && -\tau \leq t \leq 0 . \end{aligned}$$

where

$X(t)$	is	$4n \times 1$	state vector
$A(t)$	is	$4n \times 4n$	actuator matrix
F_1	is	$4n \times 4n$	feel matrix
F_2	is	$4n \times 4n$	feel matrix
$T_1(t)$	is	$n \times 1$	input torque vector
$T_2(t)$	is	$n \times 1$	output torque vector
τ	is		signal transmission delay
$Y(t)$	is	$n \times 1$	output position vector.

As before, when $T_2(t)$ is a passive load on the system, it can be expressed as a linear combination of the state variables i.e $-T_2(t) = S\,X(t)$ and can be eliminated from (5.2.1)

5.3 Optimal Control of the System.

Some progress has been made in recent years, in obtaining solutions to the optimal control problems associated with systems described by differential difference equations. Kharatishvili [43] has approached this problem by extending Pontryagin's Maximum Principle. The actual solution involves the solution of a two point boundary value problem in which advances and delays are present. Time optimal control of systems with delay has been investigated by Oguztoreli[52]. For a time invariant system with infinite upper limit, Krasovski[45] has developed the form of a controller and a performance measure. However, the solution to his equations for the controller functional is very difficult, and requires various approximations to obtain a practical numerical solution. Aggarwal[46] used the results of Krasovski to develop a set of partial differential equations (or generalized Riccati equations) which may be solved for the optimal feedback controller. Mueller[47] obtained a numerical open loop solution of the optimal regulator problem with a finite optimization interval.

The optimisation problem that is connected with the operation of (5.2.1) is similar to the problem considered in section 3.1 and may be solved using the methods suggested by Mueller[47] and Aggarwal[46].

The optimal output positions for the system (5.2.1) may be obtained (i) for different feel matrices, (ii) for different weighting matrices and (iii) for different delays. Again, from an observation of the output positions, a criterion may be established for comparing the force reflecting characteristics of the various feel systems with signal transmission time delay.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

6.1 Conclusions

A state space model has been obtained for a one dimensional feel system. Through an anlysis of the system, it is found, that its behaviour is different from that of an ordinary servomechanism, when each of the gains of the signal transmission channels K_{ij}, D_{ij}, $i,j = 1,2$, is equal to a single pair of constants K and D respectively. For instance, when an initial displacement is given to the shaft either at the master and or at the slave end, the system behaves as if it is a positional servomechanism with a step torque input. The system response is found to be unbounded, when a step torque is applied at the input. This is also true in the case of a system which has only one amplifier for mixing the rate and the position error signals (Fig 2.1). The reason for the difference is that the system matrices possess either one or two eignevalues equal to zero. However, if the gains K_{ij} and D_{ij} are unequal, then the behaviour is identical to that of an ordinary servomechanism. Also, the stability of the system is found to be dependent on the relative values of the gains the signal transmission channels.

Several of the one dimensional feel systems are interconnected to form a multi dimensional feel system. This generalised term is used here instead of such specific terms as 'Remote Manipulator' or 'Human Exoskeleton'. A multi-dimensional man machine system is formed by combining a multi-dimensional feel system and the human operator. Because of the presence of the operator two types of cross couplings are present: (i) Input or perceptual cross coupling and (ii) output or motor cross coupling. Assuming that the coupling coefficients are given, simple state space models are obtained for a multi-dimensional feel system and the human operator.

The optimisation problem connected with the design and operation of a feel system is found to be equivalent to that of tracking. Using some of the known results of optimal control theory, a control has been obtained, that will force the response of the system, to follow a specified path. Optimal output positions are computed (i) for different feel matrices (ii) for different weighting matrices and (iii) for different loads. The results of (i) are the most important, since they establish a basis for comparing the force reflecting characteristics of the various feel systems. A measure of the 'feel' of the system can be obtained through the computation of a feel constant, which is the integral of the absolute value of the difference between the path specified and the actual output position. The smaller the value of the feel constant, the better is the feel.

Under the assumption that a well trained and a well motivated human operator behaves in a near optimal manner, and that he applies such a control as to minimize a given quadratic cost functional, the function of the human operator can be replaced by an optimal controller. In order to obtain the description of the optimal controller and hence that of the operator, the steady solution of the matrix Riccati equation for the system is required. Actually a family of optimal controllers will be obtained for different combinations of the feel matrix and the weighting matrices. The controller whose response matches with that of a human obtained through actual experimentation represents 'his' correct description. The model obtained for the human operator is found to be dependent on the feel force feedback.

Based on the numerical studies made for a one dimensional feel system, a design procedure is outlined for a multi-dimensional feel system.

The problem of transferring an object, optimally, using a feel system, from one point in the actuator space to another is also considered. Some of the known results of optimal control theory are used to obtain a control which will accomplish this, minimizing a given quadratic cost functional. This required the solution of a two point boundary value problem.

The trajectories along which the transfer takes place depend on the feel matrix and weighting matrices. It is found that they tend to oscillate around the straight line (m (t)) joining the initial and the final positions. The control strategy seems to be that the human operator will try to transfer the object from its initial position to the final position in such a manner, as to minimize the difference,e(t),between the straight line m(t), and the actual trajectory. It is found that the better the quality of force reflection of the system the smaller will be e(t). The characterisation of the feel matrices obtained for this optimal transfer problem agrees with the one that is obtained for the tracking problem. A measure of the feel can be obtained by integrating the absolute value of e(t).

Variations in the feel matrix, the weighting matrices and the intensity of the disturbances affect the performance of a feel system. The change in the system performance can be measured either as a change in the performance functional or as a change in the feel constant. Sensitivities of the output positions, the performance indices and the feel constants of the optimal feel systems obtained in chapter III, to changes in the feel matrix and weighting matrices are obtained using partial derivative techniques. The sensitivity functions thus determined can be utilised to determine a suitable combination of a feel matrix and the weighting matrices for a feel system.

When the master and the slave ends of a feel system are separated by long distances there exist certain signal transmission time delay and is to be taken into account. An anlysis of a one dimensional feel system with delay, revealed that the basic nature of the system responses remained the same as in the case of a system with no delay. With increases in delay, the system responses tend to be more oscillatory. For a given time delay, the gains of the signal transmission channels should be restricted to a certain region in the gain parameter space in order that the system may be stable. Methods are suggested for solving the associated optimal control problem.

6.2 Suggestions for Future Research.

Optimal control of systems with feel has been the main subject in this investigation. In order to reduce the computational task, numerical solutions have been obtained only for the one dimensional case (i) for the tracking problem and (ii) for the optimal transfer problem. Hence, assuming suitable values for the coupling coefficients, numerical solutions may be obtained for feel systems having more than one degree of freedom. The computational task will be quite complex, since for each degree of freedom, the order of the system increases by four. Also, in the case of the tracking problem, special computational procedures such as 'Automatic synthesis matrix iterative procedure', 'Negative exponential solution' have to be used to solve higher order matrix, Riccati equations. Although the system is considered as linear, some of the components such as the system actuators are truly nonlinear, and it would be interesting to study how the nonlinearity affects the system performance.

Another interesting area for further investigation is the optimal control of feel systems with signal transmission time delay. A numerical solution may be obtained for the associated optimal control problem, and the effect of the signal delay on the quality of force reflection of a feel system with time delay. As in the case of a system with no delay, a criterion may be established for comparing the force reflecting characteristics of various feel systems with time delay.

But the most important of all is the following. In this thesis, it is assumed that the optimal control of the system, is the optimal output of the human, and accordingly a description is obtained for the human operator in terms of the coefficients of the differential equation(2.3.6). Since no limitations are placed on the human capabilities (i.e the optimal control of the system is unconstrained), it may turn out at times that the model obtained for the human operator may be unrealistic. In order to take into account, the physiological and psychological limitations of the human operator in performing a particular task, the magnitudes of the coefficients of the differential equation (2. 3. 6) may be made to lie within certain limits. These limits can be established through actual experimentation. The problem to be solved, then, is to obtain a set of coefficients of the differential equation whose value lie within these limits and for which the system performance is optimal in some sense.

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