

THE ENVELOPING VON NEUMANN ALGEBRA
OF A C^* -ALGEBRA AND SOME OF ITS APPLICATIONS

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INTRODUCTION

A von Neumann algebra is an involutive subalgebra of $L(H)$, the algebra of all continuous linear operators on a Hilbert space, which is equal to its bicommutant. It is also called a ring of operators or a W^* -algebra. S. Sakai [11] has shown that a C^* -algebra is a von Neumann algebra if and only if it is the conjugate space of a Banach space (cf. § 2.3).

Knowing the fact that the bidual of a C^* -algebra A can be identified with the enveloping von Neumann algebra of A (cf. § 3.1), we are able to solve a number of problems in C^* -algebras by using von Neumann algebra methods. We do this in Chapters III and IV.

The contents of this thesis may briefly be described as follows. In Chapter I, we assemble together some basic definitions and results in C^* -algebra which are used throughout the work. In Chapter II, we define the four basic topologies on $L(H)$ and establish some useful properties of von Neumann algebras. We also discuss other definitions of a von Neumann algebra and give the proofs of their equivalence. The final section of this chapter (§ 2.5) is devoted to a characterization of factors of type I.

Chapter III is devoted to the discussion of the enveloping von Neumann algebra of a C^* -algebra. We show that the bidual

of a C*-algebra is isometrically isomorphic to its enveloping von Neumann algebra. We use the enveloping von Neumann algebra to obtain a characterization of the equivalence of irreducible representations of a C*-algebra defined by pure states. In fact, if π_f , π_g are two irreducible representations of a C*-algebra A corresponding to the pure states f and g on A , then π_f and π_g are equivalent if and only if the extensions \tilde{f} , \tilde{g} of f and g , respectively, to the enveloping von Neumann algebra of A have equivalent supports. In Chapter IV, we discuss the polar decomposition and the enveloping polar decomposition of a continuous linear functional on a C*-algebra. We give several applications of these decompositions; in particular, we show that if g is a hermitian continuous linear functional on a C*-algebra A , then there exists a unique couple (f, f') of positive functionals on A such that $g = f - f'$ and $\|g\| = \|f\| + \|f'\|$.

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Chapter I

C*-algebra

§ 1. Notation and terminology

Definition. Let A be an algebra over the field \mathbb{C} of complex numbers. A mapping $x \rightarrow x^*$ of A into A is called an involution if it satisfies the following conditions :

- (i) $(x^*)^* = x$;
- (ii) $(x + y)^* = x^* + y^*$;
- (iii) $(\lambda x)^* = \bar{\lambda} x^*$;
- (iv) $(xy)^* = y^* x^*$,

for all x, y in A and for all λ in \mathbb{C} . An algebra A with an involution is called an involution algebra.

Remark : Property (i) implies that an involution on A is necessarily a bijection.

An element x in an involutive algebra A is called hermitian if $x = x^*$. The set of all hermitian elements of A is denoted by A_h . Every element of A has a unique representation in the form

$$x = x_1 + ix_2 ,$$

where $x_1, x_2 \in A_h$. In fact, we have

$$x_1 = \frac{1}{2}(x + x^*) \quad \text{and} \quad x_2 = -\frac{1}{2}i(x - x^*) .$$

If a hermitian element x is of the form $x = yy^*$, for some y .

in A , then x is called a positive element of A and write $x \geq 0$. The set of all positive elements of A is denoted by A^+ .

Definition. A normed algebra with an involution such that $\|x^*\| = \|x\|$ for all x in A is called an involution normed algebra.

If A is complete, then A is called an involution Banach algebra. An involution Banach algebra A such that $\|x\|^2 = \|x^*x\|$ for all x in A is called a C^* -algebra.

A linear functional f defined on an involutive algebra A is called a positive functional if $f(x^*x) \geq 0$ for all x in A . If f is a linear functional defined on A , the functional f^* on A defined by $f^*(x) = \overline{f(x^*)}$ for all x in A is also linear and f^* is called the adjoint linear functional of f . It is easy to see that f^* has the following properties :

- (i) $f^{**} = f$;
- (ii) $(f + g)^* = f^* + g^*$;
- (iii) $(\lambda f)^* = \overline{\lambda} f^*$,

for all linear functionals f, g on A and for all λ in \mathbb{C} . If $f = f^*$, then f is called hermitian (or self-adjoint). Every linear functional f on A has a unique representation in the form

$$f = f_1 + if_2 ,$$

where f_1, f_2 are hermitian linear functionals. In fact, we have

$$f_1 = \frac{1}{2}(f + f^*) \quad \text{and} \quad f_2 = -\frac{1}{2}i(f - f^*) .$$

Let f and g be two linear functionals defined on an involutive algebra A . If $f - g$ is positive, we say that f

majorizes g and write $f \geq g$. Let f be a non-zero positive continuous functional on A . If every positive continuous functional g on A such that $g \leq f$ is of the form λf ($0 \leq \lambda \leq 1$), then f is called a pure functional. Let f be a non-zero continuous linear functional on an involutive normed algebra A . If $\|f\| = 1$, then f is called a state of A .

From now on, let H denote a (complex) Hilbert space and $L(H)$ the Banach algebra of all continuous linear operators on H , normed with the usual operator bound.

Definition. Let A and B be two involutive algebras. Then a homomorphism ϕ of A into B is called a *-homomorphism if $\phi(x^*) = (\phi(x))^*$ for all x in A . Similarly, an isomorphism with this property is called a *-isomorphism.

Definition. Let A be an involutive algebra. Then a *-homomorphism of A into $L(H)$ is called a representation of A in H .

Let π be a representation of an involutive algebra A in a Hilbert space H and let ξ be an element in H . The closure of $\pi(A)\xi$ is a closed vector subspace of H invariant with respect to $\pi(A)$. If this closed subspace is H , then ξ is called a topologically cyclic vector for π .

Let A be an involutive algebra and f a positive functional on A . Then, for all x and y in A , we have $f(y^*x) = \overline{f(x^*y)}$. In fact,

$$(1) \quad 0 \leq f((\alpha x + \beta y)^*(\alpha x + \beta y))$$

$$= |\alpha|^2 f(x^*x) + \bar{\beta}\alpha f(y^*x) + \beta\bar{\alpha} f(x^*y) + |\beta|^2 f(y^*y)$$

for all x, y in A and α, β in \mathbb{C} . Since $f(x^*x) \geq 0$ and $f(y^*y) \geq 0$ are real, $\bar{\beta}\alpha f(y^*x) + \beta\bar{\alpha} f(x^*y)$ is real for all α, β . Take $\alpha = \beta = 1$, then $f(y^*x) + f(x^*y)$ is real and so $\text{Im}(f(y^*x)) = -\text{Im}(f(x^*y))$. Take $\alpha = 1, \beta = i$, then $-if(y^*x) + if(x^*y)$ is real and so $\text{Re}(f(y^*x)) = \text{Re}(f(x^*y))$. Therefore $f(y^*x) = \overline{f(x^*y)}$. Moreover, the following Cauchy - Schwartz inequality holds :

$$|f(y^*x)|^2 \leq f(x^*x)f(y^*y)$$

for all x and y in A . In fact, in the inequality (1), let α be any real number and $\beta = f(y^*x)$, then we have

$$\begin{aligned} & \alpha^2 f(x^*x) + \alpha |f(y^*x)|^2 + \alpha f(y^*x)f(x^*y) + |f(y^*x)|^2 f(y^*y) \\ &= \alpha^2 f(x^*x) + 2\alpha |f(y^*x)|^2 + |f(y^*x)|^2 f(y^*y) \end{aligned}$$

Since this holds for all real α , the discriminant of the quadratic is negative or zero. Therefore the inequality follows. It is clear that if A has an identity, then

$$f(x^*) = \overline{f(x)} \quad \text{and} \quad |f(x)|^2 \leq f(1)f(x^*x)$$

holds for all x in A .

A set I is said to be directed upward or an increasing directed set if I is partially ordered and if for every α, β in I there is a γ in I such that $\alpha \leq \gamma, \beta \leq \gamma$. Let $(e_\alpha)_{\alpha \in I}$ be a family of elements in a normed algebra A , indexed by an increasing directed set I . The family $(e_\alpha)_{\alpha \in I}$ is called an approximate identity of A provided :

$$(i) \quad \|e_\alpha\| \leq 1 \quad \text{for each } \alpha \text{ in } I ;$$

$$(ii) \quad \lim_{\alpha} \|e_\alpha x - x\| = 0 \quad \text{and} \quad \lim_{\alpha} \|x e_\alpha - x\| = 0$$

for each x in A .

Let A be an algebra and let $\tilde{A} = A \times \mathbb{C}$ be the cartesian product of A and \mathbb{C} . Define in \tilde{A} the operations of addition, multiplication and multiplication by scalars as follows :

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta) ;$$

$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta) ;$$

$$\beta(x, \alpha) = (\beta x, \beta\alpha) ,$$

for all x, y in A and α, β in \mathbb{C} . Then with these operations, \tilde{A} becomes an associative algebra. Moreover, if A is an involutive normed algebra then \tilde{A} is an involutive normed algebra with the involution given by $(x, \alpha)^* = (x^*, \bar{\alpha})$ and the norm, for example, $\|(x, \alpha)\| = \|x\| + |\alpha|$. (\tilde{A} is also an involutive normed algebra under the norm $\|(x, \alpha)\| = (\|x\|^2 + |\alpha|^2)^{\frac{1}{2}}$ or $\|(x, \alpha)\| = \max(\|x\|, |\alpha|)$). It is clear that if A is complete then \tilde{A} is complete and that A can be considered as a maximal closed two sided ideal of \tilde{A} . \tilde{A} is called the algebra obtained from A by adjunction of an identity. If A is a C^* -algebra, then there exists a unique norm under which \tilde{A} is a C^* -algebra. In fact, if A has an identity element, then \tilde{A} is a C^* -algebra under the norm $\|(x, \alpha)\| = \max(\|x\|, |\alpha|)$. If A has no identity element, we consider the left regular representation $x \rightarrow L_x$ of \tilde{A} . Then under the norm $\|x\| = \|L_x\|$, \tilde{A} is a C^* -algebra. (cf. 1.3.3, 1.3.7 and 1.3.8 in [3])

If f is a positive continuous linear functional on an involutive normed algebra A , define \tilde{f} on \tilde{A} by the relation $\tilde{f}((x, \alpha)) = \alpha \|f\| + f(x)$. Then \tilde{f} is a positive continuous linear functional. \tilde{f} is called the canonical extension of f to \tilde{A} . (cf. 2.1.5 in [3])

Let A be an algebra over the complex field \mathbb{C} . We define in A the circle operation by the relation

$$x \circ y = x + y - xy.$$

An element of A which has a (left, right) inverse relative to the circle operation is said to be (left, right) quasi-regular. An element which is not (left, right) quasi-regular is called (left, right) quasi-singular.

Let A be a complex normed algebra with an identity. An element x in A is said to be left (right) regular provided there exists y in A such that $yx = 1$ ($xy = 1$). An element which is both left and right regular is called regular. If an element is not (left, right) regular then it is called (left, right) singular. When A has no identity, then every element of A is regarded as singular.

Let A be a complex algebra and x be any element of A . Then the spectrum of x in A is the set $Sp_A(x)$ of all complex numbers α such that $\alpha^{-1}x$ is quasi-singular, plus zero if x is singular. $\rho(x) = \sup \{ |\alpha| : \alpha \in Sp_A(x) \}$ is called the spectral radius of x . $\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}$ (cf. p30 in [9]).

$Sp_{\tilde{A}}(x)$ denotes the spectrum of x in \tilde{A} , where x is consi-

dered as an element of \tilde{A} . We have $\overline{Sp_A^!(x)} = Sp_A^!(x^*)$ and $0 \in Sp_A^!(x)$ for every x in A . It is clear that $Sp_A^!(x) = \{0\} \cup Sp_A(x)$. We observe that a hermitian element x in a C^* -algebra A is positive if and only if $Sp_A^!(x) \geq 0$ (cf. 1.1.6. and 1.6.1. in [3]).

Definition. A non-zero homomorphism of a commutative algebra A into its field is called a character of A .

Let A be a commutative complex Banach algebra. Each character of A is of norm ≤ 1 (cf. Corollary (3.1.7) in [9]). The set of all characters of A , with the weak*-topology, is a locally compact Hausdorff space and is called the carrier space of A , denoted by Φ_A . For each x in A , the function $\chi \longrightarrow \chi(x)$ on Φ_A is called the Gelfand transformation of x and is denoted by \mathcal{G}_x . If A is a commutative C^* -algebra then A is isometrically *-isomorphic to the algebra $C_0(\Phi_A)$ of all continuous complex-valued functions on Φ_A which vanish at infinity. (cf. Theorem (4.2.2) in [9])

Let $\{H_i : i \in I\}$ be a family of Hilbert spaces, and let H be the family of all functions (ξ_i) on I which satisfy the following conditions :

- (i) $\xi_i \in H_i$ for all i in I ;
- (ii) (ξ_i) contains at most a denumerable number of elements which are different from zero ;
- (iii) the series $\sum_i \|\xi_i\|^2$ converges .

It is easy to see that H is closed under the operations of addition, multiplication by scalars and the inner product given by the following relations :

$$(\xi_i) + (\eta_i) = (\xi_i + \eta_i) ;$$

$$\alpha(\xi_i) = (\alpha\xi_i) ;$$

$$((\xi_i), (\eta_i)) = \sum_i (\xi_i, \eta_i) ,$$

where $(\xi_i), (\eta_i) \in H$ and $\alpha \in \mathbb{C}$. In fact, with these operations H is a Hilbert space and is called the direct (Hilbert) sum of H_i , denoted by $\bigoplus_{i \in I} H_i$.

Let M denote the vector space of all bounded sequences of complex numbers $s = (s_1, s_2, \dots)$. Then M is a Banach space under the norm $\|s\| = \text{lub}_i |s_i|$. A Banach limit is any continuous linear functional defined on M such that

- (i) $f(s) = f(\mathcal{U}s)$ where $\mathcal{U}s = (s_2, s_3, \dots)$;
- (ii) $f(s) \geq 0$ if $s_i \geq 0$ for all i ;
- (iii) $f(e) = 1$ where $e = (1, 1, \dots)$.

We write $\text{LIM}_{n \rightarrow \infty} s_n = f(s)$. If s is a convergent sequence, then $\text{LIM}_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_n$. (cf. pp. 64-65 First course in functional analysis by C. Goffman and G. Pedrick and p.73 in [4]).

§ 2. Some useful results on C*-algebras .

Theorem (1.2.1). Let A be an involutive Banach algebra, B a C*-algebra and ϕ a *-homomorphism of A into B . Then

$$\|\phi(x)\| \leq \|x\| \quad \text{for all } x \text{ in } A .$$

Proof : [3] For each hermitian element y in B , $\|y^2\| = \|y^*y\| = \|y\|^2$. By induction, we have $\|y^{2^n}\|^{2^{-n}} = \|y\|$. Hence

$$(1) \quad \rho(y) = \|y\| ,$$

where $\rho(y)$ is the spectral radius of y . We have

$\text{Sp}_B^1(\varphi(x)) \subset \text{Sp}_A^1(x)$. In fact, if there exists $0 \neq \lambda \in \text{Sp}_B^1(\varphi(x))$ such that $\lambda \notin \text{Sp}_A^1(x)$, then $\lambda^{-1}x$ is quasi-regular. Hence $\lambda^{-1}\varphi(x)$ is quasi-regular and $\lambda \notin \text{Sp}_B^1(\varphi(x))$; a contradiction. Since $\rho(\varphi(x)) = \sup\{|\xi| : \xi \in \text{Sp}_B^1(\varphi(x))\} \leq \sup\{|\xi| : \xi \in \text{Sp}_A^1(x)\} = \rho(x)$, we have $\rho(\varphi(x)) \leq \rho(x) \leq \|x\|$ for every x in A . Hence, by (1) and the fact that x^*x is hermitian, it follows that

$$\begin{aligned} \|\varphi(x)\|^2 &= \|\varphi(x^*)\varphi(x)\| = \|\varphi(x^*x)\| = \rho(\varphi(x^*x)) \\ &\leq \|x^*x\| \leq \|x^*\| \|x\| = \|x\|^2 \end{aligned}$$

and this completes the proof.

Theorem (1.2.2). Let A be a C^* -algebra, B an involutive normed algebra and φ an injective $*$ -homomorphism of A into B . Then

$$\|\varphi(x)\| \geq \|x\| \text{ for all } x \text{ in } A.$$

Proof : [3] Let $x \in A$. First, let us assume that x is a positive element. Replacing φ by its restriction to the sub- C^* -algebra of A generated by x , we may assume that A is commutative. Replacing B first by $\varphi(A)$, then by its completion, we may assume that B is commutative and complete. Replacing A by \tilde{A} , B by \tilde{B} (where \tilde{A} and \tilde{B} are the algebras obtained from A and B by adjunction of an identity), we may assume that A and B have identity elements. Briefly, we may assume that A and B are commutative, complete and with identity.

Let A and T be the carrier spaces of A and B respectively, then, by Corollary (3.1.11) in [9], S and T are compact Hausdorff spaces. For each f in T , $f \circ \varphi$ is a character of A , i.e., $f \circ \varphi \in S$ which we denote by $\varphi'(f)$. For a fixed x in A , $\varphi'(f)(x) = f(\varphi(x))$ is a continuous function of f . Given $f_0 \in T$, let

$$M = \{ g \in S : |(\varphi'(f_0) - g)(x)| < \varepsilon, \varepsilon > 0 \}$$

then M is an open neighborhood of $\varphi'(f_0)$ in S . But $\varphi'(f)(x)$ is a continuous function of f , hence

$$N = \{ f \in T : |(\varphi'(f_0) - \varphi'(f))(x)| < \varepsilon, \varepsilon > 0 \}$$

is an open neighborhood of f_0 in T . We have $\varphi'(N) \subset M$.

Since x and 1 are the generators of A , it follows that φ' is continuous (cf. pp. 122-123 in [9]). Since T is compact,

then $\varphi'(T)$ is a compact subset of A . If $\varphi'(T) \neq S$, then

there exist two continuous functions g and g' on S such

that $g = 1$ on $\varphi'(T)$ and $g' \neq 0$, $gg' = 0$. In fact, let

$U \subset S$ be an open set containing $\varphi'(T)$. By Urysohn Lemma,

there exists a continuous function g on S such that

$g|_{\varphi'(T)} = 1$ and $g|_{S-U} = 0$. Similarly there exists a continu-

ous function g' on S such that $g' \neq 0$ and $g'|_{\varphi'(T)} = 0$.

By Theorem (4.2.2) in [9], g and g' correspond to two elements

y and y' in A such that $yy' = 0$, $y \neq 0$ and $f(\varphi(y)) = 1$

for all f in T . Thus, by Corollary (3.1.9) in [9], $\varphi(x)$ is

regular. Now $xy = 0$ implies that $\varphi(x)\varphi(y) = \varphi(xy) = 0$.

Since $\varphi(x)$ is regular, it follows that $\varphi(y) = 0$. But this

is a contradiction because φ is an injection and $y \neq 0$.

Therefore $\varphi'(T) = S$ and

$$\|x\| = \sup_{h \in S} |h(x)| = \sup_{f \in T} |\varphi'(f)(x)| = \sup_{f \in T} |f(\varphi(x))| \leq \|\varphi(x)\|.$$

Now if x is any element of A , then we apply the argument

above to the element x^*x which is positive and obtain

$$\|x\|^2 = \|x^*x\| \leq \|\varphi(x^*x)\| = \|\varphi(x)^*\varphi(x)\| \leq \|\varphi(x)\|^2,$$

which completes the proof.

Remark: Let A be a C^* -algebra and let π be a representation

of A in a Hilbert space H . From Theorem (1.2.1) it follows

that $\|\pi(x)\| \leq \|x\|$ for all x in A . If π is a faithful representation, then from Theorems (1.2.1) and (1.2.2), it follows that $\|\pi(x)\| = \|x\|$ for all x in A .

Let A be a C^* -algebra, $(H_i)_{i \in I}$ a family of Hilbert spaces and $H = \bigoplus_{i \in I} H_i$. For each i in the index set I , let π_i be a representation of A in H_i . Then $\|\pi_i(x)\| \leq \|x\|$ for all x in A and all i in I . Let $(\xi_i) \in H$ and $x \in A$. Then $(\pi_i(x)\xi_i) \in H$; in fact

$$\begin{aligned} \|(\pi_i(x)\xi_i)\|^2 &= ((\pi_i(x)\xi_i), (\pi_i(x)\xi_i)) \\ (*) \quad &= \sum_i (\pi_i(x)\xi_i, \pi_i(x)\xi_i) = \sum_i \|\pi_i(x)\xi_i\|^2 \\ &\leq \sum_i \|\pi_i(x)\|^2 \|\xi_i\|^2 \leq \|x\|^2 \|(\xi_i)\|^2. \end{aligned}$$

Let $\pi(x)$ be the operator on H defined by

$$\pi(x)(\xi_i) = (\pi_i(x)\xi_i).$$

Then clearly $\pi(x)$ is a linear operator on H for each x in A and, by inequality (*), $\pi(x)$ is also bounded. Since each π_i is a representation of A in H_i , it is easily seen that π is a representation of A in H .

Theorem (1.2.3). Let A be a C^* -algebra with identity and f a continuous linear functional on A . Then f is positive if and only if $\|f\| = f(1)$.

Proof: ([3], [9]) Suppose f is positive. Since, for every x in A_h with $\|x\| \leq 1$, by 1.2.3. in [3], $1 - x$ has the form

y^*y , it follows that $f(1-x) \geq 0$, i.e., $f(x) \leq f(1)$.

If $x' \in A$ and $\|x'\| \leq 1$, then $\|x'^*x'\| \leq 1$ and , by

2.1.5. in [3] ,

$$|f(x')|^2 \leq f(1)f(x'^*x') \leq f(1)^2 .$$

Hence $\|f\| \leq f(1)$. But $\|f\| \geq f(1)$. Therefore $\|f\| = f(1)$.

Conversely, suppose $\|f\| = f(1)$. We may assume that

$\|f\| = 1$. Write $f = f_1 + if_2$ where f_1 and f_2 are hermi-

tian functionals. Since $f(1)$, $f_1(1)$ and $f_2(1)$ are real, it

follows that $f_2(1) = 0$. we show now that $f_2 = 0$. Let x be

an arbitrary hermitian element of A and set $u = \lambda - ix$, where

λ is an arbitrary real number. Then

$$\begin{aligned} (1) \quad \|u\|^2 &= \|u^*u\| = \|(\lambda - ix)^*(\lambda - ix)\| = \|(\lambda + ix)(\lambda - ix)\| \\ &= \|\lambda^2 + x^2\| \leq \lambda^2 + \|x\|^2 . \end{aligned}$$

Also

$$\begin{aligned} (2) \quad |f(u)|^2 &= |(f_1 + if_2)(\lambda - ix)|^2 = |\lambda - if_1(x) + f_2(x)|^2 \\ &= (\lambda + f_2(x))^2 + f_1^2(x) = \lambda^2 + 2\lambda f_2(x) + f_2^2(x) + f_1^2(x) . \end{aligned}$$

Therefore, solving for λ^2 in (2) and substituting in (1), we

obtain

$$\begin{aligned} \|u\|^2 &\leq |f(u)|^2 - 2\lambda f_2(x) - f_2^2(x) - f_1^2(x) + \|x\|^2 \\ &\leq |f(u)|^2 - 2\lambda f_2(x) + \|x\|^2 \leq \|u\|^2 - 2\lambda f_2(x) + \|x\|^2 . \end{aligned}$$

Hence $2\lambda f_2(x) \leq \|x\|^2$. Since this inequality holds for all

hermitian x and real λ , we have $f_2 = 0$. Hence f is a

hermitian linear functional. Now suppose there exists a hermitian

element x such that $x \geq 0$ while $f(x) < 0$. We can assume

$0 \leq x \leq 1$, so that , by spectral mapping theorem ,

$0 \leq 1 - x \leq 1$. Since $1 - x$ is hermitian, then

$$1 = f(1) = f(1 - x) + f(x) < f(1 - x) \leq \|1 - x\| \leq 1 .$$

But this is a contradiction and so f is a positive functional. This completes the proof.

Let A be an involutive Banach algebra and A_h be the set of all hermitian elements of A . Then, by Lemma 1 in [11], A_h is a real Banach space. Let f be a hermitian continuous linear functional on A . Then the mapping $f \rightarrow g = f|_{A_h}$ is a continuous isomorphism between the set of all hermitian continuous linear functionals on A and all real-valued continuous linear functionals on A_h . In fact, we have $\|f\| = \|g\|$. (cf. 1.2.6. in [3])

Theorem (1.2.4). Let A be a C^* -algebra and g a hermitian continuous linear functional on A . Then there exist two positive functionals f and f' on A such that $g = f - f'$ and $\|g\| = \|f\| + \|f'\|$.

Proof: ([3], [6]) Let B be the set of all positive functionals on A with norm ≤ 1 . Then, by 2.5.5. in [3], B is $\sigma(A^*, A)$ -compact (i.e., B is weak*-compact). Since $\sigma(A_h^*, A_h)$ is the topology on A_h^* induced by the topology $\sigma(A^*, A)$ on A^* , by the remark above, B can be considered as a $\sigma(A_h^*, A_h)$ -compact convex subset of the dual space A_h^* of A_h . The polar B^0 of B in A_h is equal to the closed unit ball S of A_h . In fact, let $C(B)$ be the set of all continuous real-valued functions on B . For each x in A_h , define a continuous real-valued function F_x on B by $F_x(f) = f(x)$ for all f in B . Then, by 2.6.3. in [3], the mapping $\phi : x \rightarrow F_x$ is an isome-

trix isomorphism of A_h onto $C(H)$. Hence S is isometrically isomorphic to the set

$$\begin{aligned} \{F_x \in C(B) : \|F_x\| \leq 1\} &= \{F_x \in C(B) : \sup_{f \in B} |F_x(f)| \leq 1\} \\ &= \{F_x \in C(B) : \sup_{f \in B} |f(x)| \leq 1\}. \end{aligned}$$

Now, by definition, $B^0 = \{x \in A_h : \sup_{f \in B} |f(x)| \leq 1\}$. Therefore $B^0 = S$. The closed unit ball S' of A_h^* is equal to the set $\{f' \in A_h^* : \sup_{x \in S} |f'(x)| \leq 1\}$, which is the polar

of S in A_h^* , i.e., $S' = S^0$. Thus S' is the bipolar of B in A_h^* . By Corollary 1, p.36 in [10], S' is the $\mathcal{U}(A_h^*, A_h)$ -closed absolutely convex envelope of B . Since B is (A^*, A) -closed and convex, then, by Theorem 3.4-F in [16], S' is the absolutely convex envelope of B . Therefore, by Theorem 3.4-E in [16], each element of S' is of the form $\alpha u - \beta v$ with u, v in B and α, β positive scalars, $\alpha + \beta \leq 1$. Let us assume that $\|g\| = 1$. Then the restriction g_h of g to A_h belongs to S' . Thus $g_h = f - f'$ with $f = \alpha u$ and $f' = \beta v$. Hence $g = f - f'$ where $f \geq 0$, $f' \geq 0$. We also have

$$1 = \|g\| \leq \|f\| + \|f'\| \leq \alpha + \beta \leq 1.$$

Hence $\|g\| = \|f\| + \|f'\|$. This completes the proof.

Let A be an involutive Banach algebra with an approximate identity and f a positive continuous functional on A . We would like to recall briefly the construction of a representation of A corresponding to f . Let \tilde{A} be the involutive Banach algebra obtained from A by adjunction of an identity, \tilde{f} the

canonical extension of f to \tilde{A} and $N = \{x \in \tilde{A} : \tilde{f}(x^*x) = 0\}$. Define an inner product on \tilde{A}/N by $(\tilde{x}, \tilde{y}) = \tilde{f}(y^*x)$ where $x, y \in A$ and \tilde{x}, \tilde{y} represent the equivalence classes of x and y in \tilde{A}/N . Then $H_f^1 = \tilde{A}/N$ becomes a (Hausdroff) prehilbert space. Let H_f be the Hilbert space completion of H_f^1 . For each x in \tilde{A} , let $\pi(x)$ be the operator in H_f^1 obtained by passing to the quotient of the left multiplication by x in \tilde{A} . Let ξ_f be the canonical image of 1 in H_f^1 . Then we have the following results:

(i) Every $\pi(x)$ can be uniquely extended to a continuous linear operator $\pi(x)$ on H_f ;

(ii) The mapping $x \rightarrow \pi(x)$ ($x \in A$) is a representation of A in H_f ;

(iii) ξ_f is topologically cyclic for $\pi(A)$;

(iv) $f(x) = (\pi(x)\xi_f, \xi_f)$ for all x in A .

H_f , π and ξ_f are called the space, representation and vector defined by f , respectively. (cf. 2.4.4. in [3])

Chapter II

von Neumann algebra

§ 1. Topologies on $L(H)$.

Definition. The topologies on $L(H)$ defined by the following semi-norms are called the strong, weak, ultrastrong and ultra-weak topologies respectively :

$$(i) \quad T \rightarrow \|T\xi\| \quad (T \in L(H), \xi \in H) \quad (\text{strong})$$

$$(ii) \quad T \rightarrow |(T\xi, \eta)| \quad (T \in L(H), \xi, \eta \in H) \quad (\text{weak})$$

$$(iii) \quad T \rightarrow \left[\sum_{i=1}^{\infty} \|T\xi_i\|^2 \right]^{\frac{1}{2}} \quad (T \in L(H), \xi_i \in H \text{ and}$$

$$\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty) \quad (\text{ultrastrong})$$

$$(iv) \quad T \rightarrow \left| \sum_{i=1}^{\infty} (T\xi_i, \eta_i) \right| \quad (T \in L(H), \xi_i, \eta_i \in H$$

$$\text{and } \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty) \quad (\text{ultraweak})$$

These topologies are related as follows :

$$\begin{array}{ccc} \text{norm topology} & > & \text{ultrastrong topology} & > & \text{strong topology} \\ & & \vee & & \vee \\ & & \text{ultraweak topology} & > & \text{weak topology} \end{array}$$

(" > " means stronger than) (for the proof of most of these relations , see M. A. Naimark , Normed Rings , Noordhoff, Groningen (1959) pp. 441 - 444)

Theorem (2.1.1). Strong (resp. weak) and ultrastrong (resp. ultraweak) topologies coincide on each bounded subset of $L(H)$.

Proof: [2] Without any loss of generality, we may assume that $A = \{T : \|T\| < 1\}$. Let N be an ultrastrong neighborhood of 0 in A . We may assume that

$$N = \left\{ T \in A : \left(\sum_{i=1}^{\infty} \|T\xi_i\|^2 \right)^{\frac{1}{2}} < 1, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \xi_i \in H \right\}.$$

Since $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, there exists a positive integer n

such that $\sum_{i=n}^{\infty} \|\xi_i\|^2 < \frac{1}{2}$. Let $N' = \left\{ T \in A : \sum_{i=1}^{n-1} \|T\xi_i\|^2 < \frac{1}{2} \right\}$.

It is clear that N' is a strong neighborhood of 0 in A . For each T in N' , we have

$$\begin{aligned} \sum_{i=1}^{\infty} \|T\xi_i\|^2 &= \sum_{i=1}^{n-1} \|T\xi_i\|^2 + \sum_{i=n}^{\infty} \|T\xi_i\|^2 \\ &< \frac{1}{2} + \|T\| \sum_{i=n}^{\infty} \|\xi_i\|^2 < \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Thus $T \in N$, $N' \subset N$ and N is a strong neighborhood of 0 . Hence strong topology coincides with ultrastrong topology on A .

Let M' be an ultraweak neighborhood of 0 in A . We may assume that

$$M' = \left\{ T \in A : \left| \sum_{i=1}^{\infty} (T\xi_i, \eta_i) \right| < 1, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty \right\}.$$

Since $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$, then $\sum_{i=1}^{\infty} \|\xi_i\| \|\eta_i\| < \infty$.

Hence there exists a positive integer n such that

$$\sum_{i=n}^{\infty} \|\xi_i\| \|\eta_i\| < \frac{1}{2}. \text{ Let } M = \left\{ T \in A : \left| \sum_{i=1}^{n-1} (T\xi_i, \eta_i) \right| < \frac{1}{2} \right\}.$$

Then M is a weak neighborhood of 0 in A . For each T in M , we have

$$\begin{aligned} \left| \sum_{i=1}^{\infty} (T\xi_i, \eta_i) \right| &\leq \left| \sum_{i=1}^{n-1} (T\xi_i, \eta_i) \right| + \left| \sum_{i=n}^{\infty} (T\xi_i, \eta_i) \right| \\ &\leq \frac{1}{2} + \|T\| \sum_{i=n}^{\infty} \|\xi_i\| \|\eta_i\| \leq \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Thus $M \subset M'$. Hence ultraweak topology coincides with weak topology on A . This completes the proof of the theorem.

§ 2. Properties of von Neumann algebras and their preduals

Let M be a subset of $L(H)$. We call the commutant M' of M the set of all elements of $L(H)$ which commute with the elements of M . $(M')' = M''$ is called the bicommutant of M . It is clear that $1 \in M'$ and M' is a subalgebra of $L(H)$. If $M \subset N$, then $M' \supset N'$. Since $M \subset M''$, then $M' \supset (M'')' = M''''$, $M' \subset (M'')'' = M''''$ and $M' = M''''$.

Thus we have

$$M' = M'''' = M^{(5)} = \dots\dots\dots$$

$$M \subset M'' = M^{(4)} = \dots\dots\dots$$

Theorem (2.2.1). M' is weakly closed in $L(H)$.

Proof. Let $\{T_\alpha\}$ be a net in M' which converges to some T_0 in $L(H)$ in the weak topology, i.e., $(T_\alpha \xi, \eta) \rightarrow (T_0 \xi, \eta)$ for all ξ, η in H . Hence $(T_\alpha T \xi, \eta) \rightarrow (T_0 T \xi, \eta)$ and $(T_\alpha \xi, T^* \eta) \rightarrow (T_0 \xi, T^* \eta)$ for all T in $L(H)$. Thus $(T T_\alpha \xi, \eta) \rightarrow (T T_0 \xi, \eta)$. Since $T T_\alpha = T_\alpha T$ for all T in M , $(T_0 T \xi, \eta) = (T T_0 \xi, \eta)$ and $T_0 T = T T_0$, i.e., $T_0 \in M'$.

It is clear that $L(H)$ with the involution $T \rightarrow T^*$ (where T^* is the adjoint operator of T) is a C^* -algebra.

If $M \subset L(H)$ is invariant under the involution, i.e., $M^* \subset M$, then M' is an involutive subalgebra of $L(H)$; in fact if $T_0 \in M'$, then $T_0^* T = (T^* T_0)^* = (T_0 T^*)^* = T T_0^*$ for all T in M , and $T_0^* \in M'$. If $M^* \subset M$, M'' is an involutive subalgebra of $L(H)$.

Definition. An involutive subalgebra A of $L(H)$ is called a von Neumann algebra if $A = A''$.

Remarks :

- (i) A von Neumann algebra A is weakly closed, hence (norm) closed. Thus A is a C^* -algebra. It is clear that $L(H)$ is a von Neumann algebra.
- (ii) If M is a subset of $L(H)$ such that $M^* \subset M$, then M' and M'' are von Neumann algebras because $M' = M''''$ and $M'' = M^{(4)}$. If A is a von Neumann algebra containing M , then $M'' \subset A'' = A$. Hence M'' is the smallest von Neumann algebra containing M .

If M is any subset of $L(H)$ and $N = M \cup M^*$, then it is clear that any von Neumann algebra which contains M also contains N and conversely. Thus there exists a smallest von Neumann algebra containing M . By Remark (ii), this von Neumann algebra is N'' . N'' is called the von Neumann algebra generated by M .

Notation :

- (i) For all ξ, η in H and T in $L(H)$, let $\omega_{\xi, \eta}(T) = (T\xi, \eta)$ and $\omega_{\xi, \xi} = \omega_{\xi}$. It is clear that $\omega_{\xi, \eta}$ is weakly continuous on

$L(H)$ and ω_ξ is positive.

(ii) Let X^* denote the dual (conjugate) space of a locally convex topological space X .

(iii) Let L_w denote the set of all weakly continuous linear functionals on $L(H)$ and L_* denote the (norm) closure of L_w in L^* .

(iv) For each couple of vectors ξ, η in H , let $\{\xi, \eta\}$ denote a linear mapping of rank 1 on H into H such that

$$\{\xi, \eta\} \gamma = (\gamma, \xi) \eta \quad \text{for all } \gamma \text{ in } H. \text{ Since}$$

$$\|\{\xi, \eta\} \gamma\| \leq \|\xi\| \|\eta\| \|\gamma\|$$

then $\{\xi, \eta\}$ is continuous and hence it is completely continuous.

Theorem (2.2.2). Let f be a linear functional on $L(H)$.

(i) The following conditions are equivalent :

$$(ia) \quad f = \sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i} \quad , \text{ with } \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty ;$$

(ib) f is ultraweakly continuous ;

(ic) f is ultrastrongly continuous.

(ii) If f is ultrastrongly continuous, then f belong to L_* .

Proof: [2] (i). (ia) \Rightarrow (ib) \Rightarrow (ic) is clear. We have only to prove (ic) \Rightarrow (ia). Suppose that f is ultrastrongly continuous. Since $L(H)$ with the ultrastrong topology is a locally convex space, then, by Theorem 3.8-C in [16], there exists a semi-norm P on $L(H)$ such that $|f(T)| \leq P(T)$ ($T \in L(H)$).

In other words, this means that there exist ξ_i in H ($i = 1, 2, \dots$) and $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$ such that $|f(T)| \leq \left[\sum_{i=1}^{\infty} \|T \xi_i\|^2 \right]^{\frac{1}{2}}$

for all T in $L(H)$. Let $\tilde{H} = \bigoplus_{i=1}^{\infty} H_i$ where $H_i = H$. For

each T in $L(H)$, we define an operator \tilde{T} on \tilde{H} by $\tilde{T}(\tilde{Y}) = (TY_i)$ for all $\tilde{Y} = (Y_i)$ in \tilde{H} . It is clear that the set $M = \{ \tilde{T}\tilde{\xi} : T \in L(H) \}$ is a subspace of \tilde{H} . Define a linear functional ϕ on M by $\phi(\tilde{T}\tilde{\xi}) = f(T)$. Then, as

$$|\phi(\tilde{T}\tilde{\xi})| = |f(T)| \leq \left[\sum_{i=1}^{\infty} \|TY_i\|^2 \right]^{1/2} = \|(T\xi_i)\| = \|\tilde{T}\tilde{\xi}\|$$

ϕ is continuous on M . By Hahn - Banach Theorem, ϕ can be extended to a continuous linear functional ϕ' on \tilde{H} . By Riesz representation theorem for linear functionals on a Hilbert space, there exists a $\tilde{\eta} = (\eta_i) \in \tilde{H}$ such that $\phi'(\tilde{Y}) = (\tilde{Y}, \tilde{\eta})$ for all \tilde{Y} in \tilde{H} . In particular,

$$f(T) = \phi(\tilde{T}\tilde{\xi}) = (\tilde{T}\tilde{\xi}, \tilde{\eta}) = \sum_{i=1}^{\infty} (T\xi_i, \eta_i) .$$

Hence

$$f = \sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i} .$$

(ii). Let $f = \sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i}$ and $f_n = \sum_{i=1}^n \omega_{\xi_i, \eta_i}$. Then $f_n \in L_{\infty}$

and

$$\|f - f_n\| = \sup_{\|T\| \leq 1} |f(T) - f_n(T)| = \sup_{\|T\| \leq 1} \left| \sum_{i=n+1}^{\infty} (T\xi_i, \eta_i) \right|$$

$$\leq \sum_{i=n+1}^{\infty} \|\xi_i\| \|\eta_i\| .$$

Since $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$, then

$$\sum_{i=n+1}^{\infty} \|\xi_i\| \|\eta_i\| \rightarrow 0 \text{ as } n \rightarrow \infty . \text{ Therefore } \|f - f_n\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence f belongs to L_* .

Lemma (2.2.3). Let $g = \sum_{j=1}^p \{\xi_j, \eta_j\}$ and $g' = \sum_{k=1}^q \{\xi'_k, \eta'_k\}$

be two continuous linear mappings of finite rank on H into H .

If $g = g'$, then

$$\sum_{j=1}^p \omega_{\xi_j, \eta_j} = \sum_{k=1}^q \omega_{\xi'_k, \eta'_k} .$$

Proof : [2] For each couple of vectors γ, γ' in H , we have

$$\begin{aligned} \sum_{j=1}^p (\{\gamma, \gamma'\} \xi_j, \eta_j) &= \sum_{j=1}^p (\xi_j, \gamma) (\gamma', \eta_j) \\ &= (\gamma', \sum_{j=1}^p (\gamma, \xi_j) \eta_j) \\ &= (\gamma', g(\gamma)) = (\gamma', g'(\gamma)) \\ &= (\gamma', \sum_{k=1}^q (\gamma, \xi'_k) \eta'_k) \\ &= \sum_{k=1}^q (\{\gamma, \gamma'\} \xi'_k, \eta'_k) . \end{aligned}$$

Let F be the set of all operators of finite rank in H . Then F is weakly dense in $L(H)$. (Since $F'' = L(H)$ and since $\{T\xi : T \in F, \xi \in H\}$ is dense in H , by Corollary 1 of Theorem 2, p.44 in [2], F is weakly dense in $L(H)$.) By the weak continuity and linearity, we have

$$\sum_{j=1}^p (T\xi_j, \eta_j) = \sum_{k=1}^q (T\xi'_k, \eta'_k)$$

for all T in $L(H)$, i.e.,

$$\sum_{j=1}^p \omega_{\xi_j, \eta_j} = \sum_{k=1}^q \omega_{\xi'_k, \eta'_k} .$$

Lemma (2.2.4) Let $f \in L_\omega$. Then there exists two orthogonal systems $(e_1, \dots, e_n), (e'_1, \dots, e'_n)$ and positive numbers λ_i ($i = 1, \dots, n$) such that

$$\|f\| = \sum_{i=1}^n \lambda_i \quad \text{and} \quad f = \sum_{i=1}^n \lambda_i \omega_{e_i, e_i'} :$$

Proof: [2] By (Proposition 11, Chap. IV, § 2, in [1]),

$$f = \sum_{j=1}^p \omega_{\xi_j, \eta_j} .$$

The mapping $g = \sum_{j=1}^p \{\xi_j, \eta_j\}$ is continuous and of finite

rank, hence it is completely continuous (cf. p.12 in [13]). By

Theorem 7 in [13] p.18, there exist two orthonormal systems

$(e_i)_{1 \leq i \leq n}$ and $(e_i')_{1 \leq i \leq n}$ and positive scalars λ_i such that

$$g = \sum_{i=1}^n \lambda_i \{e_i, e_i'\} = \sum_{i=1}^n \{\lambda_i e_i, e_i'\} . \text{ By Lemma (2.2.3),}$$

$$f = \sum_{i=1}^n \omega_{\lambda_i e_i, e_i'} = \sum_{i=1}^n \lambda_i \omega_{e_i, e_i'} . \text{ Thus}$$

$$|f(T)| \leq \sum_{i=1}^n \lambda_i \|T\| \|e_i\| \|e_i'\| = \|T\| \sum_{i=1}^n \lambda_i .$$

Hence $\|f\| \leq \sum_{i=1}^n \lambda_i$. Set $T' = \sum_{i=1}^n \{e_i, e_i'\}$ then

$\|T'\| \leq 1$. In fact, let $(e_\alpha)_{\alpha \in \Delta}$ be an orthonormal basis of H , which contains $(e_i)_{1 \leq i \leq n}$. For each ξ in H , write

$$\xi = \sum_{k=1}^{\infty} (\xi, e_k) e_k . \text{ Then}$$

$$T'(\xi) = \sum_{i=1}^n \sum_{k=1}^{\infty} (\xi, e_k) \{e_i, e_i'\} e_k = \sum_{i=1}^n (\xi, e_i) e_i' ,$$

$$\|T'(\xi)\|^2 = \left(\sum_{i=1}^n (\xi, e_i) e_i', \sum_{i=1}^n (\xi, e_i) e_i' \right)$$

$$= \sum_{i=1}^n |(\xi, e_i)|^2 \leq \sum_{i=1}^{\infty} |(\xi, e_i)|^2 = \|\xi\|^2 .$$

Therefore we have

$$\begin{aligned} f(T) &= \sum_{i=1}^n \lambda_i (T e_i, e_i) = \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n \{e_j, e_j\} e_i, e_i \right) \\ &= \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n (e_i, e_j) e_j, e_i \right) = \sum_{i=1}^n \lambda_i . \end{aligned}$$

Thus $\|f\| \geq \sum_{i=1}^n \lambda_i$. This completes the proof.

Theorem (2.2.5). If f belongs to L_* , then f is ultraweakly continuous.

Proof: [2] Since $f \in L_*$, we may write $f = \sum_{k=1}^{\infty} f_k$ with

$f_k \in L_{\omega}$ and $\|f_k\| \leq 2^{-k}$. By Lemma (2.2.4),

$$f_k = \sum_{i=1}^{n_k} \lambda_i^k \omega_{e_i^k, e_i^k} \quad \text{with } \|e_i^k\| = \|e_i^k\| = 1, \lambda_i^k \geq 0 \text{ and}$$

$$\sum_{i=1}^{n_k} \lambda_i^k = \|f_k\| \leq 2^{-k} . \text{ Hence}$$

$$\begin{aligned} f &= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \omega_{(a_i^k)^{\frac{1}{2}} e_i^k, (a_i^k)^{\frac{1}{2}} e_i^k} \right) \\ \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \|(a_i^k)^{\frac{1}{2}} e_i^k\|^2 \right) &\leq \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \lambda_i^k \right) \\ &\leq \sum_{k=1}^{\infty} 2^{-k} < \infty . \end{aligned}$$

Similarly, $\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \|(a_i^k)^{\frac{1}{2}} e_i^k\|^2 \right) < \infty$. Hence f is ul-

traweakly continuous.

Corollary (2.2.6). L_* coincides with the set of all ultraweakly continuous linear functionals on $L(H)$.

Proof: This is immediate from Theorems (2.2.2) and (2.2.5).

Theorem (2.2.7). $L(H)$ is the dual space of L_* (considered as a Banach space).

Proof: [2] Let E be the dual space of L_* . For each T in $L(H)$, define \hat{T} in E by $\hat{T}(f) = f(T)$ for all f in L_* .

Let θ be a linear mapping on $L(H)$ into E such that

$$\theta(T) = \hat{T}. \quad \theta \text{ is continuous on } L(H) \text{ with topology } \sigma(L(H), L_*)$$

into E with topology $\sigma(E, L_*)$. In fact, for each T_0 in

$$L(H), \text{ the set } N = \{ G \in E : |\hat{T}_0(f) - G(f)| < \varepsilon, f \in L_*, \varepsilon > 0 \}$$

is a neighborhood of $\theta(T_0) = \hat{T}_0$ in $\sigma(E, L_*)$. Let

$$N' = \{ T \in L(H) : |f(T_0) - f(T)| < \varepsilon, f \in L_* \}. \text{ Then } N' \text{ is}$$

a neighborhood of T_0 in $\sigma(L(H), L_*)$. Since $\hat{T}(f) = f(T)$,

$\theta(N') \subset N$ and so θ is continuous. By Corollary (2.2.6),

$\sigma(L(H), L_*)$ is just the ultraweak topology on $L(H)$. If

$\theta(T) = \hat{T} = 0$ for some T in $L(H)$, then $f(T) = 0$ for all f

in L_* . Since $\sigma(L(H), L_*)$ is a locally convex space, $T = 0$.

Therefore θ is one-one. Let B_1 and E_1 be the closed unit

balls of $L(H)$ and E , respectively. By (Corollary 3 of Theorem

1, Chap. IV, § 2 in [1]), B_1 is $\sigma(L(H), L_*)$ -compact. Since

$$\theta \text{ is continuous and } \|\theta(T)\| = \|\hat{T}\| = \sup_{\|f\| \leq 1} |\hat{T}(f)| = \sup_{\|f\| \leq 1} |f(T)| \leq$$

$$\leq \|T\|, \quad \theta(B_1) \text{ is } \sigma(E, L_*)\text{-compact and contained in } E_1.$$

$\theta(B_1)$ is dense in E_1 . In fact, if it were not, there would

exist $e_0 \in E_1$ and $\varepsilon_0 > 0$ such that $|e_0(f) - \hat{T}(f)| > \varepsilon_0$

for all f in L_* and T in B_1 . Choose f_0 in L_* such

that $\|f_0\| = 1$. Since B_1 is ultraweakly compact and f_0 is

ultraweakly continuous, there exists $T_0 \in B_1$ such that

$f_0(T_0) = 1$. We may assume $e(f_0) - f_0(T_0) > \epsilon_0$ (otherwise, consider $-f_0$). Hence $e(f_0) > \epsilon_0 + f_0(T_0) > 1$. But $|e_0(f_0)| \leq \|e_0\| \|f_0\| \leq 1$; a contradiction. Thus $\theta(B_1) = E_1$ because $\theta(B_1)$ is closed. Moreover, θ is isometric; for if it were not, there would exist a y in B_1 such that $\|\theta(y)\| < \|y\|$. Hence $\theta(y/\|\theta(y)\|) \in E_1$ and $y/\|\theta(y)\| \notin A_1$. Therefore θ is not one-to-one; a contradiction. Thus $\theta(L(H)) = E$, i.e., $L(H)$ is isometrically isomorphic to E . This completes the proof.

Theorem (2.2.8). Let A be a von Neumann algebra and A_ω (resp. A_*) be the set of all weakly (resp. ultraweakly) continuous linear functionals on A . Then A_* is the (norm) closure of A_ω in A^* and A is the dual space of A_* .

Proof: [2] By Theorem (2.2.2) (resp. Lemma (2.2.4)), each element of A_* (resp. A_ω) is of the form $\sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i}$ with

$$\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty \quad (\text{resp. } \sum_{i=1}^n \omega_{\xi_i, \eta_i}).$$

Hence A_* is the (norm) closure of A_ω . By Corollary (2.2.6), $\mathcal{U}(L(H), L_*)$ is just the ultraweak topology on $L(H)$ and therefore A is $\mathcal{U}(L(H), L_*)$ -closed. By Theorem (2.2.7), $L(H)$ is the dual space of L_* . Hence, by Theorem 4.62-A in [16], A is just the vector subspace of $L(H)$ orthogonal to the set $A^\perp = \{f \in L_* : f(A) = 0\}$, i.e., $A = (A^\perp)^\perp$. The mapping $T \rightarrow \bar{T}$, where $T \in (L_*)^* = L(H)$ and \bar{T} is defined by $\bar{T}(f + A^\perp) = T(f)$ ($f \in L_*$), is an isometric isomorphism of $(A^\perp)^\perp = A$ onto $(L_*/A^\perp)^*$; we write $A \cong (L_*/A^\perp)^*$ (cf. p.72 in [4]). Thus $A^* \cong (L_*/A^\perp)^{**} \supset L_*/A^\perp$; that is to say

L_*/A^\perp is a subspace of A^* . Since every $f \in L_*/A^\perp \subset A^*$ is ultraweakly continuous on A and $f(A) \neq 0$, then $f \in A_*$. On the other hand, for each g in A_* , g can be extended to an ultraweakly continuous linear functional on $L(H)$ (cf. Theorem (4.3.3)), $g \in L_*/A^\perp$. Hence $L_*/A^\perp = A_*$. This completes the proof.

Definition. (with the notation of Theorem (2.2.8)) A_* is called the predual of A .

Remark : By Theorem (2.2.2), A_* is also the set of all ultrastongly continuous linear functionals on A .

§ 3. Alternative definitions of a von Neumann algebra.

There are several ways to define a von Neumann algebra. In this section, we shall discuss some alternative definitions of a von Neumann algebra and give the proofs of their equivalence.

Theorem (2.3.1). Let A be an involutive subalgebra of $L(H)$. The weak, strong, ultraweak and ultrastrong closures of A coincide. Let B be this closure and K be the closed vector subspace of H generated by $T\xi$ ($T \in A, \xi \in H$). If $K = H$, then $B = A''$.

Proof. The first part of the theorem follows immediately from Theorem 2, Chap. I, § 3, in [2], By Theorem (2.2.1), A'' is weakly closed. Since A'' contains A , then A'' also contains B . Because $K = H$, by Lemma 6, in [2], p.43, we have $A'' \subset B$. Hence $A'' = B$.

Corollary (2.3.2). Let A be a weakly closed involutive subalgebra of $L(H)$. Then A is a von Neumann algebra if and only if $1 \in A$.

Proof: If A is a von Neumann algebra, then $1 \in A'' = A$. Conversely if $1 \in A$, then $K = H$. By Theorem (2.3.1), A is a von Neumann algebra.

Corollary (2.3.3). Let $1 \in A$. Then A is a von Neumann algebra if and only if A is (ultraweakly, ultrastrongly, weakly, strongly) closed.

Proof: This is clear.

Thus we can give another definition of a von Neumann algebra.

Definition . An involutive subalgebra A of $L(H)$ is called a von Neumann algebra provided that $1 \in A$ (or $K = H$) and A is (ultrastrongly, strongly, ultraweakly, weakly) closed.

In what follows, we assume that A is a C^* -algebra which is the dual space of some Banach space B and A_1 is the closed unit ball of A . It is well-known that A_1 is $\mathcal{V}(A, B)$ -compact.

Lemma (2.3.4). A has an identity element.

Proof: Since A_1 is convex and $\mathcal{V}(A, B)$ -compact, by Krein - Milmann Theorem, A_1 contains some extremal point. Hence, by Theorem 2.1 in [12], A has an identity.

Theorem (2.3.5). A is a von Neumann algebra.

Proof: [11] Let P be the set of all positive $\mathcal{V}(A, B)$ -contin-

uous linear functionals on A . For each f in P , let π_f , ξ_f and H_f be the representation, vector and Hilbert space defined by f , H the direct sum of H_f and π the direct sum of π_f ($f \in P$). If $\pi(x) = 0$ for some x in A , then, by Lemma 2, in [11], $x = 0$. Thus π is one-to-one. By Theorems (1.2.1) and (1.2.2), π is isometric. Therefore $\pi(A)$ is isometrically isomorphic to A . By Lemma (2.3.4), $1 \in \pi(A)$. Hence, in order to prove that A is a von Neumann algebra, we have only to prove $\pi(A)$ is ultraweakly closed. Since $\pi(A_1 \cap A_h)$ is strongly closed (cf. p. 771, lines 20-21 in [12]), by Theorem 1 (iv), Chap. I § 3 in [2], $\pi(A_h)$ is ultraweakly closed. Since $A = A_h + iA_h$ (i.e., every $x \in A$ is of the form $x = x_1 + ix_2$ with $x_1, x_2 \in A_h$), $\pi(A) = \pi(A_h) + i\pi(A_h)$. For each z in $\overline{\pi(A)}$, the ultraweak closure of $\pi(A)$, let $\{z_\alpha\}$ be a net in $\pi(A)$ such that $z_\alpha \rightarrow z$ in the ultraweak topology. Since the mapping $y \rightarrow y^*$ is ultraweakly continuous on $L(H)$ (cf. p.36, in [2]), $z_\alpha^* \rightarrow z^*$ in the ultraweak topology. Write

$$z = z_1 + iz_2 = \frac{1}{2}(z + z^*) + i\frac{1}{2i}(z - z^*) ,$$

$$z_\alpha^* = z_{\alpha 1} + iz_{\alpha 2} = \frac{1}{2}(z_\alpha + z_\alpha^*) + i\frac{1}{2i}(z_\alpha - z_\alpha^*) .$$

Since $z_{\alpha 1}, z_{\alpha 2} \in \pi(A_h)$ and $z_{\alpha 1} \rightarrow z_1, z_{\alpha 2} \rightarrow z_2$ ultraweakly and, since $\pi(A_h)$ is ultraweakly closed, z_1 and z_2 belong to $\pi(A_h)$. Therefore $z \in \pi(A)$. This completes the proof.

Remark : It follows from Theorems (2.2.8) and (2.3.5) that a C^* -algebra is a von Neumann algebra if and only if it is a dual space of some Banach space. Therefore we can give another definition of a von Neumann algebra due to S. Sakai [11].

Definition. A C^* -algebra is called a von Neumann algebra if it is a dual space of some Banach space.

§ 4. The support of a normal positive functional

In what follows, let P_X denote the (orthogonal) projection on H such that $P_X(H) = X$.

Let T belong to $L(H)$. It is clear that the set $\{\xi \in H : T\xi = 0\}$ is a closed vector subspace of H . Let X be its orthogonal complement. Then $E = P_X$ is called the support of T . E is the smallest projection in $L(H)$ satisfying the equality $TE_1 = T$, where E_1 is a projection in $L(H)$. Let M be the closure of $T(H)$ and let $F = P_M$, then P_M is the support of T^* (cf. Appendix III in [2]).

Let $U \in L(H)$ and E be the support of U . U is called partially isometric if U is isometric on $X = E(H)$. $U(H) = U(X)$ is a closed vector subspace of H . Let $M = U(X)$ and $F = P_M$. We say that E (resp. X) is the initial projection (resp. initial subspace) of U and F (resp. M) the final projection (resp. final subspace) of U . U^* is also partially isometric, having initial projection F and final projection E . We have $U^*U = E$ and $UU^* = F$ (cf. Appendix III in [2]).

Let $T \in L(H)$, E the support of T , F the support of T^* , $X = E(H)$, $M = F(H)$ and $|T| = (T^*T)^{\frac{1}{2}}$. Then $|T|$ has support E and the closure of $|T|(H)$ is equal to X . The linear mapping $|T|\xi \rightarrow T\xi$ ($\xi \in H$) is isometric from $|T|(H)$ onto $T(H)$ and therefore can be extended to an isometrically continuous linear mapping V of X onto M . Let U be a partially isometric operator on H with support E such

that U coincides with V on X . Then $T = U|T|$. This equality is called the polar decomposition of T . This decomposition is unique, The equality $T^* = U^*(U|T|U^*)$ is the polar decomposition of T^* . We also have $|T^*| = U|T|U^*$ and $|T| = U^*|T^*|U$. (cf. Appendix III in [2]).

Lemma (2.4.1). If M is an ultraweakly closed subset of $L(H)$, then $M^* = \{T^* : T \in M\}$ is also ultraweakly closed.

Proof: Let $\overline{M^*}$ be the ultraweak closure of M^* . For each x in $\overline{M^*}$, let $\{x_\alpha^*\}$ be a net in M^* which converges ultraweakly to x . Since the mapping $y \rightarrow y^*$ is ultraweakly continuous on $L(H)$, then $x_\alpha = x_\alpha^{**} \rightarrow x^*$ ultraweakly. Hence $x^* \in M$ and $x^{**} = x \in M^*$.

Lemma (2.4.2). Let A be a von Neumann algebra and M an ultraweakly closed left ideal of A . Then there exists a unique projection E in A such that $M = AE$. If M is a two sided ideal, then E belongs to the center of A .

Proof: [2] Let $N = M \cap M^*$. Then N is an involutive subalgebra and by Lemma (2.4.1), N is ultraweakly closed. Let E be the greatest projection in N (the existence of E is given by Theorem 2, p.43 in [2]). Since $N \subset M$, $AE \subset M$. Now, for each T in M let $T = U|T|$ be its polar decomposition. Since $U^*T = U^*U|T| = |T|$, $|T| \in M$ and, since $|T| = |T|^* \in M^*$, $|T| \in N$. But, by Theorem 2 in [2], p. 43, we have $|T|E = |T|$. Therefore $TE = U|T|E = U|T| = T$ and so $T \in AE$. Hence $M = AE$. If F is a projection of A such that $M = AF$, then $E = EF$, so that $E \leq F$. Similarly, we have $F \leq E$. Hence

$F = E$. If M is a two-sided ideal, then $ET \in M$ for all T in A . Let G be a unitary element of A (i.e., $GG^* = G^*G = 1$). Since $EG \in M$, then $(EG)E = EG$. Thus $(EG)EG^* = EGG^* = E$. Then $E = E^* = GEG^*E = GEG^*$ because $GEG^* \in M$. Therefore $EG = GEG^*G = GE$. By Proposition 3 in [2], p.4, E commutes with every element of A and hence belongs to the center of A .

Definition. A positive linear functional f defined on a von Neumann algebra A is called normal if it belongs to the predual of A .

Theorem (2.4.3). Let A be a von Neumann algebra and f a normal positive linear functional on A . Then, among the projections G in A such that $f(G) = 0$, there exists a projection F greater than all the others. We also have that $f(TF) = f(FT) = 0$ for all T in A .

Proof: [2] Set $M = \{T \in A : f(T^*T) = 0\}$. Then M is a left ideal, for if $T' \in A$, $T \in M$, by 2.1.5. in [3],

$$0 \leq f((T'T)^*(T'T)) = f(T^*(T'^*T')T) \leq \|T'^*T'\|f(T^*T) = 0.$$

By Cauchy-Schwartz inequality, we have

$$M = \left\{ T \in A : f(S^*T) = 0, \text{ for all } S \text{ in } A \right\}.$$

Since the mapping $T \rightarrow S^*T$ and f are ultraweakly continuous,

$\{T \in A : f(S^*T) = 0\}$ is ultraweakly closed for all S^* in A

and since $M = \bigcap_{S \in A} \{T \in A : f(S^*T) = 0\}$, M is ultraweakly

closed. By Lemma (2.4.2), there exists a unique projection $F \in M$

such that $M = \{T \in A : T = TF\}$. Thus $f(F) = f(F^*F) = 0$.

If G is a projection in A such that $f(G) = 0$, then

$f(G) = f(G*G) = 0$. Hence $G \in M$ and $G = GF$; so that $F \geq G$.

Finally, for each T in A , we have

$$|f(TF)|^2 = |f((T^*) * F)|^2 \leq f(TT^*)f(F*F) = 0.$$

Similarly, $|f(FT)|^2 \leq f(T^*T)f(F) = 0$.

Definition. (with the above notation) The projection $E = 1 - F$ is called the support of f .

§ 5. A characterization of factors of type I

Definition. A von Neumann algebra is called a factor if its center contains only scalar multiples of the identity operator.

Definition. A von Neumann algebra is called of type I if it is isometrically isomorphic to a von Neumann algebra whose commutant is abelian.

Definition. Let A, B be two von Neumann algebras and ϕ a homomorphism of A into B . ϕ is called normal if for each increasingly directed subset $F \subset A^+$ with least upper bound $x \in A^+$, then $\phi(x)$ is the least upper bound of $\phi(F)$.

Definition. Let A be an involutive algebra, H a Hilbert space and π a representation of A in H . π is called topologically irreducible if the only closed subspaces invariant with respect to $\pi(A)$ are (0) and H .

Let A be a von Neumann algebra. Then the center Z of A is equal to $A \cap A'$. Let $T \in A$, E the support of T and $B = \{G \in Z : G \text{ is a projection, } TG = T\}$. By definition

of E , $G \geq E$ ($G \in B$). Therefore, by Lemma (2.5.4), there exists the smallest projection F in Z such that $F \geq E$. Since $T^*G = (GT)^* = (TG)^* = T^*$ for all G in B , F is also the smallest projection in Z which is greater than the support of T^* . As $TG = TE = T$, we have $TGF = TEF$, i.e., $TF = TE = T$. Since $1 - F$ is the largest projection in Z such that $T(1 - F) = 0$, it follows that if P is any projection in Z such that $TP = 0$, then P is orthogonal to F . F is called the central support of T .

Lemma(2.5.1). Let A be an involutive subalgebra of $L(H)$ and X a closed vector subspace of H . Then X is invariant with respect to A (i.e., $T(X) \subset X$ for all T in A) if and only if the projection $P = P_X \in A'$.

Proof. [2] If $P \in A'$, then, for each T in A and x in X , we have $Tx = TPx = PTx$. Therefore $Tx \in X$ ($x \in X, T \in A$). Conversely, if X is invariant with respect to A , then, for each T in A , we have $TP(H) \subset T(X) \subset X$, so that $PTP(\xi) = TP(\xi)$ for all ξ in H . Thus

$$PT = (T^*P)^* = (PT^*P)^* = PTP = TP$$

and so $P \in A'$.

Lemma (2.5.2). Let R be a family of positive continuous functionals defined on a C^* -algebra A , $\pi = \bigoplus_{f \in R} \pi_f$ and

$H = \bigoplus_{f \in R} H_f$. Then π is non-degenerate, i.e., $\overline{\pi(A)H} = H$

(norm closure).

Proof : Let ξ_f be the vector defined by f ($f \in R$). It is well-known that $\pi_f(A)\xi_f = H_f$. Considering ξ_f as a vector in H , we have $\overline{\pi(A)H} \supset \pi(A)\xi_f = H_f$ ($f \in R$). Hence $H \supset \overline{\pi(A)H} \supset \bigoplus_{f \in R} H_f = H$. This completes the proof.

Corollary (2.5.3). The weak closure of $\pi(A)$ is a von Neumann algebra.

Proof : This is immediate from Theorem (2.3.1) and Lemma (2.5.2).

Lemma (2.5.4). Let A be a von Neumann algebra, Z its center and P a projection in A . Then there exists the smallest projection P' in Z such that $P' \geq P$.

Proof : Let $\{P_\alpha\}_{\alpha \in \Delta}$ be the family of all (orthogonal) projection in Z which are greater than P . Since $1 \in Z$ and $1 \geq P$, $\{P_\alpha\}_{\alpha \in \Delta}$ is not empty. Let K_α be the range of P_α . Then K_α is closed for each $\alpha \in \Delta$. Let $P' = P \bigwedge_{\alpha \in \Delta} K_\alpha$. Then P' is an orthogonal projection. Since $P_\alpha \in Z$, $TK_\alpha \subset K_\alpha$ for every $T \in A$ and every $\alpha \in \Delta$. Therefore $TP' = P'T$ for every $T \in A$ and so $P' \in A'$. Similarly $TK_\alpha \subset K_\alpha$ for every $T \in A'$ and $\alpha \in \Delta$. Hence $P' \in A'' = A$ and so $P' \in Z$. It is clear that $P' \leq P_\alpha$ for every $\alpha \in \Delta$ and by definition of P' , $P' \geq P$.

Theorem (2.5.5). In order that a factor A , considered as a Banach space, be the bidual space of a C^* -algebra C , it is necessary and sufficient that A be of type I.

Proof : [14] Let A be a factor of type I. Then A is $*$ -isomorphic to $L(H)$ for some suitable Hilbert space H (cf. P.121 in [2]). Let C be the set of all completely continuous opera-

tors on H . Then C is a C^* -algebra (cf. Lemma 4 and Theorem 5 in [13] p.13) and, by Theorem 3 in [13] p.48, its bidual C^{**} can be identified with $L(H)$. This proves the sufficiency of the condition.

Conversely let us suppose that A is the bidual of a C^* -algebra C . Let S be the set of all pure states on C , and for each f in S , let H_f and π_f be the space and representation defined by f . Let $\pi = \bigoplus_{f \in S} \pi_f$, $\tilde{H} = \bigoplus_{f \in S} H_f$ and let M be the weak closure of $\pi(C)$ in $L(\tilde{H})$. By Corollary (2.5.3), M is a von Neumann algebra. Let $P_f = P_{H_f}$, for $f \in S$. Identifying $P_f \tilde{H}$ with H_f , we see that H_f is invariant with respect to $\pi(C)$ and therefore with respect to M . Thus, by Lemma (2.5.1), $P_f \in M'$ and so $P_f M P_f = M P_f$. By Proposition 1 in [2] p.18, $M P_f$ is a von Neumann algebra for each $f \in S$. Clearly $\pi(C) P_f \subset M P_f$. Since $M P_f$ can be considered as a von Neumann algebra of operators in H_f and since $\pi(C) P_f$ can be identified with $\pi_f(C)$, we can write $M P_f \supset \pi_f(C)$. Since f is pure, by 2.5.4. in [3], π_f is irreducible and so, by 2.3.1. in [3], $(\pi_f(C))'$ consists of the scalar multiples of the identity operator. Therefore $(\pi_f(C))'' = L(H_f)$. Since π_f is irreducible, by Theorem (2.3.1), $\overline{\pi_f(C)} = \pi_f(C)''$, where $\overline{\pi_f(C)}$ denotes the weak closure of $\pi_f(C)$ in $L(H_f)$. But $M P_f$ is weakly closed, hence $M P_f \supset \overline{\pi_f(C)}$ and so $M P_f = L(H_f)$. Therefore $M P_f$ is of type I. Now, by Lemma (2.5.4), there exists the smallest projection P_f' in the center of M such that $P_f \leq P_f'$. Then $M P_f'$ is $*$ -isomorphic to $M P_f$. In fact, it is clear that P_f' is the central support of P_f . By

Proposition 1 in [2] p.18, MP_f' and MP_f are von Neumann algebras, $(M'P_f')' = (MP_f')'$ and $(M'P_f)' = (MP_f)'$. It is clear that P_f' is the identity of $M'P_f'$ and $P_f = P_f P_f' \in M'P_f'$. Hence, by Proposition 2 in [2], p.19, $(M'P_f')' = MP_f'$ is *-isomorphic to $(M'P_f P_f')' = MP_f$. Let f and g be any two elements of S and let $G = P_g'(H)$. Since $P_f P_f' = P_g' P_g$, either $H_f \subseteq G$ or $H_f \supseteq G$. Now, for each x in A and $\eta \in G$, we have

$$\pi_f(x) = \pi_f(x) P_f'(\eta) = P_g' \pi_f(x)(\eta) \in G.$$

Therefore G is invariant with respect to $\pi_f(x)$. Since π_f is irreducible, $H_f \not\subseteq G$. Hence $H_f \subset G$ holds for all f in S and so $\bigoplus_{f \in S} H_f \subset G \subset \tilde{H}$. Thus $G = \tilde{H}$ and so $P_g' = 1_{\tilde{H}}$. Since MP_g is *-isomorphic to $MP_g' = M$ and MP_g is of type I, it follows that M is of type I.

Let T be the set of all states on C and $\pi' = \bigoplus_{g \in T} \pi_g$. By Theorem 14 in [7], A is isomorphic or anti-isomorphic to the weak closure W of $\pi'(C)$ in $L(H_{\pi'})$. Since the type of factor is invariant for anti-isomorphism, we may assume that A is isomorphic to W . By Theorem 2 in [14], there exists a normal homomorphism ϕ of A onto M . A does not contain non-trivial ultraweakly closed two-sided ideals. (In fact, if N is a non-zero ultraweakly closed two-sided ideal in A , then, by Lemma(2.4.2), there exists a projection E in the center of A such that $N = EA$. Since A is a factor, E is necessarily equal to 1. Hence $1 \in N$ and $N = A$). As ϕ is ultraweakly continuous, its kernel $K = (0)$ (cf, Appendix A27 in [3]) and so ϕ is one-to-one. Hence A , being *-isomorphic to M , is a factor of type I.

Chapter III

The enveloping von Neumann algebra of a C*-algebra

§ 3.1. The enveloping von Neumann algebra and some of its properties

Definition. Let Q be the family of all positive continuous functionals on a C*-algebra A . The representation $\pi = \bigoplus_{f \in Q} \pi_f$ is called the universal representation of A .

Remark :

- (i) By Lemma (2.5.2), π is nondegenerate.
- (ii) By Theorems (1.2.1) and (1.2.2), π is isometric.

Lemma (3.1.1). If A is a von Neumann algebra, then each element of the predual P of A is a linear combination of normal positive functionals on A .

Proof: [2] Let $f \in P$. By Theorem (2.2.2), each element of P is of the form $\sum_{j=1}^{\infty} \omega_{\xi_j, \eta_j}$, where $\xi_j, \eta_j \in H$ and $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$ and $\sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty$. Now, for all T in $L(H)$,

we have

$$\begin{aligned} 4(T\xi_j, \eta_j) &= (T(\xi_j + \eta_j), \xi_j + \eta_j) - (T(\xi_j - \eta_j), \xi_j - \eta_j) \\ &\quad + i(T(\xi_j + i\eta_j), \xi_j + i\eta_j) \\ &\quad - i(T(\xi_j - i\eta_j), \xi_j - i\eta_j) \end{aligned}$$

Thus,

$$f = \frac{1}{4} \left(\sum_{j=1}^{\infty} \omega_{\xi_j + \eta_j} - \sum_{j=1}^{\infty} \omega_{\xi_j - \eta_j} + i \sum_{j=1}^{\infty} \omega_{\xi_j + i\eta_j} - i \sum_{j=1}^{\infty} \omega_{\xi_j - i\eta_j} \right)$$

$$- i \sum_{j=1}^{\infty} \omega_{\xi_j - i\eta_j} .$$

As each functional having the form ω_{ξ} is positive, the lemma follows.

Theorem (3.1.2). Let A be a C^* -algebra, π its universal representation and B the weak closure of $\pi(A)$ in $L(H_{\pi})$.

Then

(i) Every normal positive functional on B is of the form ω_{ξ} ($\xi \in H_{\pi}$). Every ultraweakly continuous linear functional on B is weakly continuous.

(ii) If $f \in A^*$, then there exists a unique weakly continuous linear functional \tilde{f} on B such that $\tilde{f}(\pi(x)) = f(x)$ for all x in A .

(iii) The mapping $f \rightarrow \tilde{f}$ is an isometric isomorphism of A^* onto the predual of B which transforms the set of all positive functionals on A into the set of all normal positive functionals on B ; we also have $(f^*)^{\sim} = (\tilde{f})^*$ for all f in A^* .

(iv) For each y in B , let \hat{y} be the linear functional $f \rightarrow \tilde{f}(y)$ on A^* . Then the mapping ψ defined by $\psi(y) = \hat{y}$ is an isometric isomorphism of B onto A^{**} whose composition with π (i.e., $\psi \circ \pi$) is the canonical injection of A into A^{**} .

(v) This isomorphism ψ is bicontinuous for the weak (operator) topology on B and the $\sigma(A^{**}, A^*)$ -topology on A^{**} .

Remark : By Corollary (2.5.3), B is a von Neumann algebra.

Proof of Theorem (3.1.2) : [3] For each f' in $(\pi(A))^*$, we define the functional f on A by

$$(1) \quad f(x) = f'(\pi(x)) \quad x \in A .$$

Then f is linear and continuous because f' is linear and

$$|f(x)| = |f'(\pi(x))| \leq \|f'\| \|\pi(x)\| = \|f'\| \|x\| .$$

Thus $\|f\| \leq \|f'\|$. Since $|f'(\pi(x))| = |f(x)| \leq \|f\| \|x\| = \|f\| \|\pi(x)\|$

then $\|f'\| \leq \|f\|$ and hence $\|f'\| = \|f\|$. Also, for each f in

A^* , (1) defines a linear functional f' in $(\pi(A))^*$ such

that $\|f'\| = \|f\|$. Therefore the mapping ϕ defined by

$$\phi(f) = f' \text{ is an isometric isomorphism of } A^* \text{ onto } (\pi(A))^* .$$

Since $\pi(x) \geq 0$ if and only if $x \geq 0$, it follows that f

is positive if and only if f' is positive.

(i) Let \tilde{f} be a normal positive functional on B . Then $f' = \tilde{f}|_{\pi(A)}$ is positive on $\pi(A)$. Hence $f = \phi^{-1}(f')$ is positive on A . Let π_f and ξ_f be the representation and vector defined by f , then we have $f(x) = (\pi_f(x)\xi_f, \xi_f)$ for all x in A . Since $\pi = \bigoplus_{f \in Q} \pi_f$, then $f(x) = (\pi(x)\xi, \xi)$

(where $\xi = \xi_f$ is considered as a vector in H_π). Thus $f'(\pi(x)) = f(x) = (\pi(x)\xi, \xi)$ for all x in A . Since, by Theorem (2.3.1), $\pi(A)$ is ultraweakly dense in B , then, by the ultraweak continuity, $\tilde{f}(y) = (y\xi, \xi)$ for all y in B , i.e., $\tilde{f} = \omega_\xi$. Hence \tilde{f} is weakly continuous. Therefore it follows from Lemma (3.1.1) that each ultraweakly continuous linear functional on B is weakly continuous. This proves (i).

(ii) If h' is a positive functional on $\pi(A)$, then $h = \phi^{-1}(h')$ is positive on A and so $h \in Q$ (where Q is the set of all positive linear functionals on A). Therefore $h'(\pi(x)) = h(x) = (\pi(x)\xi_h, \xi_h)$. Hence h' is weakly conti-

nuous on $\mathcal{K}(A)$. Since each g in $\mathcal{K}(A)^*$ is of the form $g = g_1 + ig_2$, where g_1 and g_2 are hermitian, by Theorem (1.2.4), g is a linear combination of positive functionals on $\mathcal{K}(A)$ and therefore weakly continuous. It follows that if $f \in A^*$, then $f' = \Phi(f)$ is weakly continuous on $\mathcal{K}(A)$. Hence f' can be extended to a unique weakly continuous linear functional \tilde{f} on B . Moreover, we have $\tilde{f}(\mathcal{K}(x)) = f'(\mathcal{K}(x)) = f(x)$ for all x in A . This proves (ii).

(iii). If $f' \in (\mathcal{K}(A))^*$, then f is weakly continuous and hence is ultraweakly continuous. Let \tilde{f} be the unique extension of f' to B by weak continuity. By 12.1.1. in [3], the mapping $f' \longrightarrow \tilde{f}$ is an isometric isomorphism of $(\mathcal{K}(A))^*$ onto the predual of B . Since the mapping $f \longrightarrow f'$ is an isomorphism of A^* onto $(\mathcal{K}(A))^*$, $f \longrightarrow \tilde{f}$ is an isometric isomorphism of A^* onto the predual of B . If $f \geq 0$, then, by the same argument as in (ii), $f' = \Phi(f) = \omega_{\xi_f}$ on $\mathcal{K}(A)$. Hence, by the weak continuity, $\tilde{f} = \omega_{\xi_f}$ on B . Therefore \tilde{f} is positive and normal. If $f \in A^*$, then $f^* \in A^*$. Since $f'^* = f'^*$, $f'^* \in (\mathcal{K}(A))^*$. We have

$$(f'^*)(\mathcal{K}(x)) = f^*(x) = \overline{f(x^*)} = \overline{f^*(\mathcal{K}(x)^*)} = (f^*)^*(\mathcal{K}(x))$$

for all x in A . Hence, by the weak continuity, $(f'^*)^{\sim}(y) = (f^*)^{\sim}(y)$ for all y in B . This proves (iii).

(iv). Since $(\mathcal{K}(A))^*$ is isometrically isomorphic to A^* , by 12.1.1. in [3], we get (iv).

(v). From (i) we know that the predual of B is also the set of all weakly continuous linear functionals on B . By (iii), every weakly continuous linear functional on B has the form \tilde{f}

with $f \in A^*$. Now a sub-base of 0-neighborhoods N for the weak topology in B is the family of sets of the form

$$\{ y \in B : |f(y)| < \varepsilon, \varepsilon > 0, f \in A^* \}$$

and a sub-base of 0-neighborhoods N' for the topology $\mathcal{T}(A^{**}, A^*)$ in A^{**} is the family of sets of the form

$$\{ \hat{y} \in A^{**} : |\hat{y}(f)| < \varepsilon, \varepsilon > 0, f \in A^* \}.$$

Since the isomorphism $y \rightarrow \hat{y}$ transforms the elements of N into the elements of N' and its inverse transforms the elements of N' into the elements of N , it follows that $y \rightarrow \hat{y}$ is bicontinuous.

Definition. (with above notation) B is called the enveloping von Neumann algebra of A and π is called the canonical homomorphism of A into B .

Corollary (3.1.3). The bidual A^{**} of a C^* -algebra A is isometrically isomorphic to the enveloping von Neumann algebra of A .

Proof: This is immediate from Theorem (3.1.3).

§ 3.2. Equivalence of representations

Definition. Let $x \rightarrow \pi_1(x)$ and $x \rightarrow \pi_2(x)$ be two representations of an involutive algebra A in two Hilbert spaces H_1 and H_2 respectively. The two representations are said to be equivalent if there exists an isometric isomorphism U of H_2 onto H_1 such that $U\pi_2(x) = \pi_1(x)U$ for all x in A . In this case, we write $\pi_1 \cong \pi_2$.

Remark : It is easily seen that the relation $\pi_1 \cong \pi_2$ is an equivalence relation.

Definition. Let A be a von Neumann algebra, E and F be two projections in A . We say that E and F are equivalent (relative to A) and write $E \sim F$ if there exists an element U in A such that $U^*U = E$, $UU^* = F$. We write $E \prec F$ or $F \succ E$ if there exists a projection P in A such that $P \sim E$ and $P \leq F$.

Remarks :

(i) U is isometric on $E(H)$, for if $x \in E(H)$, then

$$(x, x) = (Ex, x) = (U^*Ux, x) = (Ux, Ux) .$$

(ii) It is clear that the relation $E \sim F$ is an equivalence relation and that the relation $E \prec F$ is reflexive and transitive (cf. p.225 in [2]).

In what follows, let A be a C^* -algebra, B the enveloping von Neumann algebra of A , f and g two pure states on A , π_f and π_g the (irreducible) representations of A defined by f and g respectively; \tilde{f} and \tilde{g} the normal positive linear functionals on B induced by f and g (cf. Theorem (3.1.3)), $e_{\tilde{f}}$ and $e_{\tilde{g}}$ the supports of \tilde{f} and \tilde{g} , and $\tilde{\pi}_{\tilde{f}}$ and $\tilde{\pi}_{\tilde{g}}$ the (irreducible) representations of B defined by \tilde{f} and \tilde{g} ; $\tilde{\pi}_f$ and $\tilde{\pi}_g$ the representations of B in H_f and H_g , respectively, such that $\tilde{\pi}_f(\pi(x)) = \pi_f(x)$ and $\tilde{\pi}_g(\pi(x)) = \pi_g(x)$ for all x in A , where π is the universal representation of A . By 12.15 in [3], $\tilde{\pi}_f$ and $\tilde{\pi}_g$ are normal

representations of B .

Lemma (3.2.1). $\pi_f \cong \tilde{\pi}_f$.

Proof : Since $\pi_f(A)\xi_f = H_f$, $\tilde{\pi}_f(B)\xi_f = H_f$. For each x in A , we have (with the notation as in Theorem (3.1.2))

$$f'(\pi(x)) = f(x) = (\pi_f(x)\xi_f, \xi_f) = (\tilde{\pi}_f(\pi(x))\xi_f, \xi_f).$$

Since $\pi(A)$ is ultraweakly dense in B and \tilde{f} is ultraweakly continuous, then, by the ultraweak continuity, we have

$$\tilde{f}(y) = (\tilde{\pi}_f(y)\xi_f, \xi_f) \text{ for all } y \text{ in } B. \text{ But } \tilde{f}(y) = (\tilde{\pi}_f(y)\xi_f, \xi_f)$$

for all y in B . Hence, by Theorem (4.4.3) in [9], $\pi_f \cong \pi_g$.

Theorem (3.2.2). $\pi_f \cong \pi_g$ if and only if $\pi_f \cong \pi_g$.

Proof : if $\pi_f \cong \pi_g$, then, by Lemma (3.2.1), $\tilde{\pi}_f \cong \tilde{\pi}_g$.

Therefore there exists an isometric isomorphism U from H_f onto H_g such that $U\tilde{\pi}_f(y) = \tilde{\pi}_g(y)U$ for all y in B .

Therefore $U\pi_f(x) = \pi_g(x)U$ for all x in A . Hence $\pi_f \cong \pi_g$.

Conversely, let us suppose $\pi_f \cong \pi_g$. Then there exists an isometric isomorphism V from H_f onto H_g such that

$$V\pi_f(x) = \pi_g(x)V \text{ for all } x \text{ in } A. \text{ When } A \text{ is regarded as}$$

a subalgebra of B , by means of the universal representation

of A , A is ultraweakly dense in B . Since $\tilde{\pi}_f(\pi(x)) = \pi_f(x)$

and $\tilde{\pi}_g(\pi(x)) = \pi_g(x)$ for every x in A and since $\tilde{\pi}_f$

and $\tilde{\pi}_g$ are ultraweakly continuous (cf. Appendix A27 in [3]),

it follows that $V\tilde{\pi}_f(y) = \tilde{\pi}_g(y)V$ for all y in B . Hence

$$\tilde{\pi}_f \cong \tilde{\pi}_g \text{ and so, by Lemma (3.2.1), } \pi_f \cong \pi_g.$$

Theorem (3.2.3). \tilde{f} is a pure state.

Proof : By Theorem (3.1.2), $\|\tilde{f}\| = \|f\| = 1$. By 2.5.4. in [3] , we have only to prove that $\pi_{\tilde{f}}$ is irreducible. Since f is pure, π_f is an irreducible representation of A . Therefore $\tilde{\pi}_f$ is an irreducible representation of $\pi(A)$ on H_f . As $B \supset \pi(A)$, $\tilde{\pi}_f$ is an irreducible representation of B on H_f . But by Lemma (3.2.1), $\tilde{\pi}_f \cong \pi_{\tilde{f}}$. Hence $\pi_{\tilde{f}}$ is irreducible , and the proof is complete.

Lemma (3.2.4). $e_{\tilde{f}}$ is a minimal projection in B .

Proof : Let $N = \{x \in B : \tilde{f}(x^*x) = 0\}$. Then N is an ultraweakly closed left ideal and so, by Lemma (2.4.2), there exists a unique projection F in B such that $N = BF$. By definition we have $e_{\tilde{f}} = 1 - F$. It is evident that $BF \oplus Be_{\tilde{f}} = B$ and $BF \cap Be_{\tilde{f}} = (0)$. Since \tilde{f} is a pure state, by 2.9.5. in [3] , N is a maximal modular ideal of B . Therefore $e_{\tilde{f}}$ is a minimal projection in B .

Since $e_{\tilde{f}}$ is a minimal projection in B , for x, y in B , we have $e_{\tilde{f}}xy^*xe_{\tilde{f}} = \lambda e_{\tilde{f}}$, where λ is a scalar . Let $\langle xe_{\tilde{f}}, ye_{\tilde{f}} \rangle = \lambda$. Then $\langle xe_{\tilde{f}}, ye_{\tilde{f}} \rangle$ is an inner product on $Be_{\tilde{f}}$ which makes $Be_{\tilde{f}}$ into a Hilbert space (cf. p.160, Annihilator algebras by E.F. Bonsall and A. W. Goldie, Proc. London Math. Soc. (3) 4, (1954)). We shall show later on that the given norm in $Be_{\tilde{f}}$ is identical with the norm defined by this inner product (cf. proof of Theorem (3.2.7)).

Lemma (3.2.5). Let π_1 be the left regular representation of B in $Be_{\tilde{f}}$ (i.e., $\pi_1(x)(ye_{\tilde{f}}) = xye_{\tilde{f}}$ for all x, y in B). Then $\pi_1 \cong \pi_{\tilde{f}}$.

Proof : Since \tilde{f} is a pure state, the mapping U defined by $U(ye_{\tilde{f}}) = \pi_{\tilde{f}}(ye_{\tilde{f}})\xi_{\tilde{f}}$, for all y in B , is an isometric isomorphism from $Be_{\tilde{f}}$ onto $H_{\tilde{f}}$ (cf. Theorem 2 in [15]). Moreover, for all x, y in B , we have

$$\begin{aligned} U\pi_1(x)(ye_{\tilde{f}}) &= U(xye_{\tilde{f}}) = \pi_{\tilde{f}}(xye_{\tilde{f}})\xi_{\tilde{f}} = \pi_{\tilde{f}}(x)\pi_{\tilde{f}}(ye_{\tilde{f}})\xi_{\tilde{f}} \\ &= \pi_{\tilde{f}}(x)U(ye_{\tilde{f}}) \end{aligned}$$

Therefore $U\pi_1(x) = \pi_{\tilde{f}}(x)U$ and $\pi_1 \cong \pi_{\tilde{f}}$.

Lemma (3.2.6). Let $E_{\tilde{f}}$ and $E_{\tilde{g}}$ be the central supports of $e_{\tilde{f}}$ and $e_{\tilde{g}}$ in B . Then $E_{\tilde{f}}$ and $E_{\tilde{g}}$ are minimal projections in the center Z of B . Moreover, if $E_{\tilde{f}} = E_{\tilde{g}}$, then $e_{\tilde{f}} \sim e_{\tilde{g}}$ in B .

Proof : Let h be a projection in Z such that $h < E_{\tilde{f}}$. Since h and $e_{\tilde{f}}$ commute, $he_{\tilde{f}}$ is a projection element of B . Then $he_{\tilde{f}} < E_{\tilde{f}}e_{\tilde{f}} = e_{\tilde{f}}$. As $e_{\tilde{f}}$ is a minimal projection in B , $he_{\tilde{f}} = 0$. Thus $(1-h)e_{\tilde{f}} = e_{\tilde{f}}$ and so $1-h \geq E_{\tilde{f}} > h$. Hence $h = 0$. Therefore $E_{\tilde{f}}$ is a minimal projection in Z . Suppose that $E_{\tilde{f}} = E_{\tilde{g}}$. Then there exist two non-zero equivalent projections G_1 and G_2 in B such that $G_1 < e_{\tilde{f}}$ and $G_2 < e_{\tilde{g}}$ (cf. Lemma 1 in [2] p.227). Therefore, by definition, there exists two non-zero projections P_1 and P_2 in B such that $G_1 \sim P_1$ and $P_1 \leq e_{\tilde{f}}$; $G_2 \sim P_2$ and $P_2 \leq e_{\tilde{g}}$. But $e_{\tilde{f}}$ and $e_{\tilde{g}}$ are minimal projections. Hence $P_1 = e_{\tilde{f}}$ and $P_2 = e_{\tilde{g}}$. Consequently $e_{\tilde{f}} \sim G_1 \sim G_2 \sim e_{\tilde{g}}$.

Theorem (3.2.7). $\pi_{\tilde{f}} \cong \pi_{\tilde{g}}$ if and only if $e_{\tilde{f}} \sim e_{\tilde{g}}$.

Proof : [17] If $\pi_{\tilde{f}} \cong \pi_{\tilde{g}}$, then, by Lemma (2.3.1), $\pi_{\tilde{f}} \cong \pi_{\tilde{g}}$.

Since $\pi_{\tilde{f}}$ (resp $\pi_{\tilde{g}}$) is equivalent to the left regular representation π_1 of B in $Be_{\tilde{f}}$ (resp. π_2 of B in $Be_{\tilde{g}}$), we have $\pi_1 \cong \pi_2$. Therefore there exists an isometric isomorphism U from $Be_{\tilde{f}}$ onto $Be_{\tilde{g}}$ such that $\pi_2(x) = U\pi_1(x)U^{-1}$ for all x in B . Then for all x, y in B , we have

$$U\pi_1(x)U^{-1}(ye_{\tilde{g}}) = \pi_2(x)(ye_{\tilde{g}}) = xye_{\tilde{g}}.$$

Let $E_{\tilde{f}}$ and $E_{\tilde{g}}$ be the central supports of $e_{\tilde{f}}$ and $e_{\tilde{g}}$ respectively. By Lemma (3.2.6), both $E_{\tilde{f}}$ and $E_{\tilde{g}}$ are minimal projections in the center Z of B and therefore either $E_{\tilde{f}}E_{\tilde{g}} = 0$ or $E_{\tilde{f}} = E_{\tilde{g}}$ holds. If $E_{\tilde{f}}E_{\tilde{g}} = 0$, then, for all x in B , we have

$$(1) \quad U\pi_1(E_{\tilde{f}})U^{-1}(xe_{\tilde{g}}) = \pi_2(E_{\tilde{f}})(xe_{\tilde{g}}) = x(E_{\tilde{f}}E_{\tilde{g}})e_{\tilde{g}} = 0.$$

On the other hand, for all y in B , we have

$$(2) \quad \pi_1(E_{\tilde{f}})(ye_{\tilde{f}}) = E_{\tilde{f}}ye_{\tilde{f}} = yE_{\tilde{f}}e_{\tilde{f}} = ye_{\tilde{f}}.$$

Since U is one-to-one and onto, for each y in B , there exists x in B such that $ye_{\tilde{f}} = U^{-1}(xe_{\tilde{g}})$. Thus, by (2), we have $\pi_1(E_{\tilde{f}})U^{-1}(xe_{\tilde{g}}) = ye_{\tilde{f}}$. Therefore (1) implies that $U(ye_{\tilde{f}}) = 0$ for all y in B ; a contradiction. Hence $E_{\tilde{f}} = E_{\tilde{g}}$ and therefore, by Lemma (3.2.1), $e_{\tilde{f}} \sim e_{\tilde{g}}$.

Conversely, let us suppose that $e_{\tilde{f}} \sim e_{\tilde{g}}$. By definition, there exists a partially isometric operator v in B such that $v^*v = e_{\tilde{f}}$ and $vv^* = e_{\tilde{g}}$. Define a mapping U on $Be_{\tilde{f}}$ to $Be_{\tilde{g}}$ by $U(ye_{\tilde{f}}) = yv^*e_{\tilde{g}}$. For each y in B , we have $ye_{\tilde{g}} = ye_{\tilde{g}}e_{\tilde{g}} = yvv^*e_{\tilde{g}} = U(yve_{\tilde{f}})$ and hence U is an onto mapping. Moreover, for each y in B , we have

$$\begin{aligned}
(3) \quad \|U(ye_{\tilde{F}})\|^2 &= \|yv^*e_{\tilde{g}}\|^2 = \|yv^*vv^*\|^2 = \|ye_{\tilde{F}}v^*\|^2 \\
&= \| (ye_{\tilde{F}}v^*)^*(ye_{\tilde{F}}v^*) \| = \|ve_{\tilde{F}}y^*ye_{\tilde{F}}v^*\| \\
&= \|e_{\tilde{F}}y^*ye_{\tilde{F}}v^*\| = \| (ve_{\tilde{F}}y^*ye_{\tilde{F}})^* \| \\
&= \|ve_{\tilde{F}}y^*ye_{\tilde{F}}\| = \|e_{\tilde{F}}y^*ye_{\tilde{F}}\| = \|ye_{\tilde{F}}\|^2 .
\end{aligned}$$

Let $|\cdot|$ be the norm in $Be_{\tilde{F}}$ defined by the inner product $\langle xe_{\tilde{F}}, ye_{\tilde{F}} \rangle = e_{\tilde{F}}y^*xe_{\tilde{F}}$. Then $|ye_{\tilde{F}}| = \|ye_{\tilde{F}}\|$. In fact, by definition

$$|ye_{\tilde{F}}|^2 = \langle ye_{\tilde{F}}, ye_{\tilde{F}} \rangle = \lambda \quad \text{and} \quad \lambda e_{\tilde{F}} = e_{\tilde{F}}y^*ye_{\tilde{F}} .$$

Then

$$\|ye_{\tilde{F}}\|^2 = \|e_{\tilde{F}}y^*ye_{\tilde{F}}\| = \|\lambda e_{\tilde{F}}\| = \lambda \|e_{\tilde{F}}\| = \lambda .$$

Hence $|ye_{\tilde{F}}| = \|ye_{\tilde{F}}\|$. Therefore (3) shows that U is isometric on $Be_{\tilde{F}}$ onto $Be_{\tilde{g}}$. Finally, for all x and y in B , we have

$$\begin{aligned}
U\pi_1(x)U^{-1}(ye_{\tilde{F}}) &= U\pi_1(x)U^{-1}(ye_{\tilde{F}}^2) = U\pi_1(x)U^{-1}(yv^*e_{\tilde{g}}) \\
&= U\pi_1(x)(yve_{\tilde{F}}) = U(xyve_{\tilde{F}}) = xyvv^*e_{\tilde{g}} \\
&= xye_{\tilde{g}} = \pi_2(x)(ye_{\tilde{g}}) .
\end{aligned}$$

Hence $\pi_1 \cong \pi_2$. Therefore $\pi_{\tilde{F}} \cong \pi_{\tilde{g}}$ and $\pi_{\tilde{F}} \cong \pi_{\tilde{g}}$. This completes the proof.

Chapter IV

The polar decompositions of the elements of the predual of a von Neumann algebra

§ 1. The existence theorem of the polar decompositions

Let A be a C^* -algebra, A^* its dual, f an element in A^* and x an element in A . Define a functional $x.f$ (resp. $f.x$) on A by $(x.f)(y) = f(xy)$ (resp. $f.x(y) = f(yx)$) for all y in A . Then it is clear that $x.f$ and $f.x$ are linear and

$$x.f = \sup_{\|y\| \leq 1} |f(xy)| \leq \|f\| \|x\|, \quad \|f.x\| \leq \|f\| \|x\|.$$

Thus $x.f$ and $f.x$ belong to A^* . For each y in A , we have

$$(x.f)^*(y) = \overline{(x.f)(y^*)} = \overline{f(xy^*)} = f^*(yx^*) = f^*.x^*(y).$$

Hence $(x.f)^* = f^*.x^*$. Let A be a von Neumann algebra and f ultraweakly continuous. Since the mappings $y \rightarrow xy$ and $y \rightarrow yx$ are ultraweakly continuous on A , $x.f$ and $f.x$ are also ultraweakly continuous.

Theorem (4.1.1). Let A be a von Neumann algebra, f an element of the predual of A . Then there exists a pair (p, u) with the following properties :

- (i) p is a normal positive functional on A and $\|p\| = \|f\|$;
- (ii) u is a partially isometric element in A whose final projection is equal to the support of p ;
- (iii) $f = u.p$, $p = u^*.p$.

Proof : [3] We may assume that $\|f\| = 1$. Let $B = \{x \in A : \|x\| \leq 1\}$

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and $B' = \{x \in B : f(x) = 1\}$. By (Corollary 3 to Theorem 1, Chapter IV, § 2 in [1]), B is weakly compact and hence ultraweakly compact. Since f is ultraweakly continuous, there exists an x in B such that $|f(x)| = 1$. By multiplying by a scalar, we may assume that $f(x) = 1$, so that $x \in B'$ and $B' \neq \emptyset$. Since B' is ultraweakly closed and contained in B , B' is ultraweakly compact. It is clear that B' is also convex. Therefore, by Krein - Milmann Theorem, B' has an extremal point v . It is easily seen that v is also an extremal point of B . In fact, if $v = \lambda s + (1 - \lambda)t$ where $0 < \lambda < 1$ and $s, t \in B$, then

$$1 = f(v) = \lambda f(s) + (1 - \lambda)f(t).$$

Since s, t are in B and $\|f\| = 1$, $|f(s)| \leq 1$ and $|f(t)| \leq 1$. Write $f(s) = a + ib$ and $f(t) = c + id$, where a, b, c and d are real numbers. Since $1 = \lambda f(s) + (1 - \lambda)f(t)$, $b = d = 0$. Hence $1 = \lambda a + (1 - \lambda)c$ and so $a = c = 1$. Thus $f(s) = f(t) = 1$, i.e., s and t are in B' . As v is an extremal point in B' , $v = s = t$, so that v is also an extremal point of B . By 12.2.2. in [3], v^*v is a projection, Hence $\|v\| = \|v^*v\|^{1/2} = 1$. For each ξ in $v^*v(H)$, we have

$$\|\xi\|^2 = (\xi, \xi) = (v^*v\xi, \xi) = (v\xi, v\xi) = \|v\xi\|^2.$$

Therefore v is partially isometric. Let $p = v.f$. Clearly p is ultraweakly continuous and $\|p\| \leq \|v\|\|f\| = 1$. Since $p(1) = f(v) = 1$, $\|p\| = 1$, and so, by Theorem (1.2.3), p is positive. Hence p is normal. Because v^*v is the initial projection of v , $vv^*v = v$. Thus $p(v^*v) = f(vv^*v) = f(v) = 1$, and $p(1 - v^*v) = 0$. Let e be the support of p . Then,

$1 - v^*v \leq 1 - e$, so that $e \leq v^*v$. Put $u = ev^*$, then it is clear that u is partially isometric. Now, since $uu^* = ev^*ve = e^2 = e$, u has final projection e . For each x in A , we have

$$(u^*.f)(x) = f(u^*x) = f(vex) = v.f(ex) = p(ex) = p(x).$$

Hence $p = u^*.f$. Let $u^*u = e'$. Then

$$\|e'.f\| \leq \|u^*u\|\|f\| = \|uu^*\|\|f\| = \|e\|\|f\| = 1.$$

On the other hand, since

$$\|u^*\| \leq \|v\|\|e\| = 1$$

and

$$(e'.f)(u^*) = f(u^*uu^*) = f(u^*e) = f(u^*) = f(ve) = p(e) = 1.$$

We have $\|e'.f\| \geq 1$. Thus $\|e'.f\| = 1 = \|f\|$. By 12.2.3. in [3], $e'.f = f$. For all x in A , we have

$$\begin{aligned} (u.p)(x) &= p(ux) = f(vux) = f(vev^*x) = f(u^*ux) \\ &= f(e'.x) = (e'.f)(x) = f(x). \end{aligned}$$

Thus $f = u.p$ and the proof is complete.

Remark: If f is a normal functional, the support E of f is the same as that of p , and $E = uu^*$.

Definition. (with the notation of Theorem (4.1.1)) p is called the absolute value of f and is denoted by $|f|$. The equality $f = u.|f|$ is called the polar decomposition of f .

Remark: By 12.2.4. in [3], this decomposition is unique.

Definition. Let A be a C^* -algebra, B its enveloping von Neumann algebra, f a continuous linear functional on A and f^2

the unique extension of f to B . The polar decomposition $\tilde{f} = v \cdot |\tilde{f}|$ is called the enveloping polar decomposition of f .

Lemma (4.1.2). Let A be a C^* -algebra with identity, f a continuous linear functional on A , p a (continuous) positive functional on A and v an element of A such that $\|v\| \leq 1$, $v \cdot p = f$ and $\|p\| = \|f\|$. If u is an element of A such that $\|u\| \leq 1$, $u \cdot f \geq 0$ and $\|u \cdot f\| = \|f\|$, then $u \cdot f = p$.

Proof: [3] We may assume that $\|p\| = \|f\| = 1$. Let ξ_p and π_p be the vector and representation defined by p , then

$$\begin{aligned} (\xi_p, \pi_p(u^*v^*)\xi_p) &= (\pi_p(vu)\xi_p, \xi_p) = p(vu) = f(u) \\ &= (u \cdot f)(1) = \|u \cdot f\| = \|f\| = 1. \end{aligned}$$

Since

$$(\xi_p, \xi_p) = (\pi_p(1)\xi_p, \xi_p) = p(1) = \|p\| = 1,$$

we obtain

$$\begin{aligned} (\xi_p, \xi_p) &= (\xi_p, \pi_p(u^*v^*)\xi_p) \leq \|\xi_p\| \|\pi_p(u^*v^*)\xi_p\| \\ &\leq \|\xi_p\|^2 \|u^*v^*\| \leq \|\xi_p\|^2 \|u\| \|v\| \leq \|\xi_p\|^2. \end{aligned}$$

Hence $\pi_p(u^*v^*)\xi_p = \xi_p$. Furthermore, we have

$$\begin{aligned} (u \cdot f)(x) &= f(ux) = p(vux) = (\pi_p(vux)\xi_p, \xi_p) \\ &= (\pi_p(x)\xi_p, \pi_p(u^*v^*)\xi_p) = (\pi_p(x)\xi_p, \xi_p) = p(x) \end{aligned}$$

for all x in A . Hence $u \cdot f = p$.

Theorem (4.1.3). Let A be a von Neumann algebra and f an element of the predual of A . Then the absolute values of the polar decomposition and the enveloping polar decomposition of f coincide, i.e., $|\tilde{f}| = |\tilde{f}|$, where $|\tilde{f}|$ is the unique exten-

sion of $|f|$ to B .

Proof : [3] Let $f = u \cdot |f|$ be the polar decomposition of f , B the enveloping von Neumann algebra of A and π the universal representation of A . For convenience, we write $|f| = p$. By Theorem (3.1.2), there exist two weakly continuous linear functionals \tilde{f} and \tilde{p} on B such that $\tilde{f}(\pi(x)) = f(x)$ and $\tilde{p}(\pi(x)) = p(x)$ for all x in A . By Theorem (4.1.1), we have $\|\tilde{p}\| = \|p\| = \|f\| = \|\tilde{f}\|$ and $\tilde{f}(\pi(u^*x)) = f(u^*x) = u^* \cdot f(x) = p(x)$ for all x in A . Since $\pi(A)$ is ultraweakly dense in B , by the ultraweak continuity, we have $\tilde{f}((\pi(u^*))y) = \tilde{p}(y)$ for all y in B , i.e., $\pi(u^*) \cdot \tilde{f} = \tilde{p}$. Hence $\|\pi(u^*) \cdot \tilde{f}\| = \|\tilde{p}\| = \|\tilde{f}\|$. We also have $\|\pi(u^*)\| = \|u\| \leq 1$. Let $\tilde{f} = v \cdot |\tilde{f}|$ be the enveloping polar decomposition of \tilde{f} . By Lemma (4.1.2), we have $\pi(u^*) \cdot \tilde{f} = |\tilde{f}|$. Hence $\tilde{p} = |\tilde{f}|$. This completes the proof.

Remark : The polar decomposition of f , in general, is distinct from its enveloping polar decomposition.

Example : [3] Let H be a Hilbert space with a countable orthonormal basis (e_1, e_2, \dots) . For each e_n , $\mathbb{C}e_n$ is a one dimensional closed subspace of H . Let $P_n = P_{\mathbb{C}e_n}$ for $n = 1, 2, \dots$. Then it is clear that $P_n \in L(H)$ and $P_n P_m = 0$ if $n \neq m$. Hence $\sum_{n=1}^{\infty} P_n$ is a projection. Let A be the von Neumann algebra generated by $\{P_n\}$. Since $P_n P_m = P_m P_n$ ($n, m = 1, 2, \dots$), the set $\{P_n\}$ is contained in the center of A . We show now that each element T of A is of the form $\sum_{n=1}^{\infty} \lambda_n P_n$ with

the sequence of scalars $\{\lambda_n\}_{n=1}^{\infty}$ bounded. Let $T \in A$. Since T commutes with P_n , $Te_n = \lambda_n e_n$ and so $Te_n = \lambda_n P_n e_n$ ($n = 1, 2, \dots$). Since for each ξ in H , $\xi = \sum_{n=1}^{\infty} \beta_n e_n$, we have

$$T\xi = \sum_{n=1}^{\infty} \beta_n Te_n = \sum_{n=1}^{\infty} \beta_n \lambda_n P_n e_n = \sum_{n=1}^{\infty} \lambda_n P_n (\xi).$$

Thus each $T \in A$ is of the form $\sum_{n=1}^{\infty} \lambda_n P_n$ and, since $\|T\| \geq \|Te_n\| = \|\lambda_n\|$ ($n = 1, 2, \dots$), $\{\lambda_n\}$ is bounded.

Let f be the linear functional on A defined by

$$f(T) = f\left(\sum_{n=1}^{\infty} \lambda_n P_n\right) = \sum_{n=1}^{\infty} n^{-2} \lambda_n,$$

where $T = \sum_{n=1}^{\infty} \lambda_n P_n$. Since $\sup |\lambda_n| < \infty$, $\sum_{n=1}^{\infty} n^{-2} \lambda_n < \infty$.

Thus f is well-defined. Moreover, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \omega_{\frac{1}{n}} e_n (T) &= \sum_{n=1}^{\infty} n^{-2} (Te_n, e_n) = \sum_{n=1}^{\infty} n^{-2} (\lambda_n P_n e_n, e_n) \\ &= \sum_{n=1}^{\infty} n^{-2} \lambda_n = f(T). \end{aligned}$$

It follows that f is positive and ultraweakly continuous, hence normal. Let E be the support of f and $F = 1 - E$. Then, by Theorem (2.4.2), $f(TF) = f(FT) = 0$ for all T in A . Since F is of the form $F = \sum_{n=1}^{\infty} \alpha_n P_n$, for each P_j , we have

$$0 = f(FP_j) = f\left(\sum_{n=1}^{\infty} \alpha_n P_n P_j\right) = f(\alpha_j P_j) = \alpha_j f(P_j) = j^{-2} \alpha_j.$$

Hence $\alpha_j = 0$ and so $F = 0$. Thus $E = 1$.

Let g be a continuous linear functional on A defined by

$$g(T) = g\left(\sum_{n=1}^{\infty} \lambda_n P_n\right) = \text{LIM}_{n \rightarrow \infty} \lambda_n,$$

where $\text{LIM}_{n \rightarrow \infty} \lambda_n$ denotes the Banach limit of the sequence $\{\lambda_n\}$.

Then

$$g(T^*T) = g\left(\sum_{n=1}^{\infty} |\lambda_n|^2 P_n\right) = \lim_{n \rightarrow \infty} |\lambda_n|^2 \geq 0.$$

Hence g is positive. Let B be the enveloping von Neumann algebra of A and let \tilde{f} and \tilde{g} be the unique extensions of f and g to B . Then \tilde{f} and \tilde{g} are normal functionals on B (cf. Theorem (3.1.2)). Let $Q_n = \sum_{i=0}^{\infty} P_{n+i} \in A \subset B$. As $P_{n+i} P_{n+j} = 0$ for $i \neq j$, Q_n is a projection. Since $\{Q_n\}$ is a decreasing sequence of projections, it converges to a projection Q in the strong topology. But, by Lemma 1 in [11], B^+ is strongly closed. Hence, since $Q_n \in B^+$, $Q \in B^+$. Since Q is the greatest lower bound of the sequence $\{Q_n\}$ and since \tilde{g} is normal and $\tilde{g}(Q_n) = 1$ for all n , we have $\tilde{g}(Q) = 1$ (cf. Appendix A 25, in [3]). Thus $Q \neq 0$. On the other hand, $\tilde{f}(Q_n) = \sum_{i=0}^{\infty} (n+i)^{-2}$. Since $\sum_{i=0}^{\infty} (n+i)^{-2}$ is convergent, $\tilde{f}(Q_n) \rightarrow 0$ as $n \rightarrow \infty$. As \tilde{f} is normal on B , $\tilde{f}(Q) = 0$. Let \tilde{E} be the support of \tilde{f} and $\tilde{F} = 1 - \tilde{E}$. Then, by Theorem (2.4.2), $\tilde{F} \geq Q$, so that $\tilde{F} \neq 0$ and consequently $\tilde{E} = 1 - \tilde{F} \neq 1$. Thus the support of f relative to A is 1, and relative to B , is not equal to 1. This shows that the polar decomposition of f is different from its enveloping polar decomposition. If $f = u.p$ and $\tilde{f} = v.\tilde{p}$, then $uu^* = E = 1$ and $vv^* = \tilde{E} \neq 1$.

§ 2. Some applications of the polar decompositions

Theorem (4.2.1). Let A be a von Neumann algebra, f an ultraweakly continuous linear functional on A and p a (continuous) positive functional on A . Then $p = |f|$ if and only if

$\|p\| = \|f\|$ and $|f(x)|^2 \leq \|f\|p(x^*x)$ for all x in A .

Proof : [5] Let $f = u|f|$ be the polar decomposition of f .

If $p = |f|$, then we have $\|p\| = \|f\|$ and

$$\begin{aligned} |f(x)|^2 &= |u.p(x)|^2 = |p(ux)|^2 \leq p(u^*u)p(x^*x) \\ &\leq \|p\|p(x^*x) = \|f\|p(x^*x) \end{aligned}$$

Conversely, let us suppose that $\|p\| = \|f\|$ and $|f(x)|^2 \leq \|f\|p(x^*x)$.

Let H_p , π_p and ξ_p be the space, representation and vector defined by p , Then, for all x in A , we have

$$|f(x)|^2 \leq \|f\|p(x^*x) = \|f\|(\pi_p(x^*x)\xi_p, \xi_p) = \|f\|\|\pi_p(x)\xi_p\|^2$$

$$|f(x)|^2 = |u^*.f(x)|^2 = |f(u^*x)|^2 \leq \|f\|\|\pi_p(u^*x)\xi_p\|^2$$

Since uu^* is a projection, $(1 - uu^*)$ is also a projection.

Now, as $x^*x - x^*uu^*x = x^*(1 - uu^*)x = ((1 - uu^*)x)^*(1 - uu^*)x$, we have $p(x^*x - x^*uu^*x) \geq 0$, i.e., $p(x^*x) \geq p(x^*uu^*x)$. Hence

$$\|\pi_p(u^*x)\xi_p\| \leq \|\pi_p(x)\xi_p\|. \text{ Therefore, } |f(x)|^2 \leq \|f\|\|\pi_p(x)\xi_p\|^2.$$

Since $\pi_p(A)\xi_p = H_p$, we can define a linear functional ϕ on H_p by $\phi(\pi_p(x)\xi_p) = \|f\|^{-\frac{1}{2}}|f(x)|$. Then, as

$$|\phi(\pi_p(x)\xi_p)| \leq \|\pi_p(x)\xi_p\|,$$

ϕ is continuous. Therefore there exists a ξ' in H such that

$$\phi(\pi_p(x)\xi_p) = (\pi_p(x)\xi_p, \xi')$$
 for all x in A . Hence

$$|f(x)| = (\pi_p(x)\xi_p, \|f\|^{\frac{1}{2}}\xi') = (\pi_p(x)\xi_p, \xi)$$

where $\xi = \|f\|^{\frac{1}{2}}\xi'$. Choose y in A such that $\pi_p(y)\xi_p = \xi$.

Then

$$(\xi, \xi)^2 = |f(y)|^2 \leq \|f\|\|\pi_p(y)\xi_p\|^2 = \|f\|\|\xi\|^2.$$

Hence $\|\xi\| \leq \|f\|^{\frac{1}{2}}$. We also have $\|p\| = p(1) = (\pi_p(1)\xi_p, \xi_p) =$

$= \|\xi_p\|^2$. Therefore

$$\begin{aligned} (\xi_p, \xi) &= (\pi_p(1)\xi_p, \xi) = |f|(1) = \|f\| = \|p\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}} \\ &\geq \|\xi_p\|\|\xi\| \end{aligned}$$

But $(\xi_p, \xi) \leq \|\xi_p\|\|\xi\|$. Hence $(\xi_p, \xi) = \|\xi_p\|\|\xi\|$ and $(\xi_p, \xi) = \|f\| = \|p\| = \|\xi_p\|^2$. Therefore $\xi = \xi_p$ and $|f|(x) = (\pi_p(x)\xi_p, \xi_p) = p(x)$ for all x in A . This completes the proof.

Theorem (4.2.2). Let f and g be two elements of the predual of a von Neumann algebra A and let $f = u \cdot |f|$, $g = v \cdot |g|$ and $f + g = w \cdot |f + g|$ be the polar decompositions of f , g and $f + g$, respectively. Then, for all x in A , we have

$$|f + g|(x)|^2 \leq (\|f\| + \|g\|)(|f|(x^*x) + |g|(x^*x))$$

Proof : [5] By Theorem (4.1.1), $|f + g| = w^* \cdot (f + g)$. Therefore

$$\begin{aligned} |f + g|(x)|^2 &= |w^* \cdot (f + g)(x)|^2 = |(f + g)(w^*x)|^2 \\ &= |f|(uw^*x) + |g|(vw^*x)|^2 \\ &\leq (|f|(uw^*x) + |g|(vw^*x))|^2 \\ &\leq (|f|(x^*x)^{\frac{1}{2}}|f|(uw^*wu)^{\frac{1}{2}} + |g|(x^*x)^{\frac{1}{2}}|g|(vw^*wv^*)^{\frac{1}{2}})^2 \\ &\leq (\|f\|^{\frac{1}{2}}|f|(x^*x)^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}|g|(x^*x)^{\frac{1}{2}})^2 \\ &= \|f\||f|(x^*x) + \|g\||g|(x^*x) + \\ &\quad + 2\|f\|^{\frac{1}{2}}\|g\|^{\frac{1}{2}}|f|(x^*x)^{\frac{1}{2}}|g|(x^*x)^{\frac{1}{2}} \\ &\leq \|f\||f|(x^*x) + \|g\||g|(x^*x) + \|f\||g|(x^*x) \\ &\quad + \|g\||f|(x^*x) \\ &= (\|f\| + \|g\|)(|f|(x^*x) + |g|(x^*x)) \end{aligned}$$

Theorem (4.2.3). Let A be a C^* -algebra with identity and $\{f_\alpha\}$ a net in A^* converging to an $f \in A^*$ in the weak* topology. If $\|f_\alpha\|$ also converges to $\|f\|$, then the net $|f_\alpha|$ converges to $|f|$ in the weak* topology.

Proof: [5] (Identify f, f_α and A with $\tilde{f}, \tilde{f}_\alpha$ and $\mathcal{K}(A)$ as in Theorem (3.1.2)). Let $f = u \cdot |f|$ and $f_\alpha = u_\alpha \cdot |f_\alpha|$ be the enveloping polar decompositions of f and f_α respectively. Since $\| |f| \| = \|f\|$ and $\| |f_\alpha| \| = \|f_\alpha\|$, we have that $\| |f_\alpha| \| \rightarrow \| |f| \|$ and so we may assume that the net $\{|f_\alpha|\}$ is bounded. As bounded closed subsets in A^* are weakly* compact, then, by Theorem 2, Chap. V in [8], $\{|f_\alpha|\}$ has at least one cluster point. Therefore we have only to show that $|f|$ is the only cluster point of $\{|f_\alpha|\}$. Let $|f_\beta|$ be a subnet weakly* converging to a functional p , then $|f_\beta|(x^*x) \rightarrow p(x^*x)$ ($x \in A$) and so p is positive. By Theorem (4.2.1), $|f_\beta(x)|^2 \leq \| |f_\beta| \| |f_\beta|(x^*x)$ for all x in A . Therefore $|f(x)|^2 \leq \|f\| p(x^*x)$ for all x in A . By Corollary (3.1.3), A^{**} is the enveloping von Neumann algebra of A , so that A is ultraweakly dense in A^{**} . When regarded as elements of the predual of A^{**} , f and p are ultraweakly continuous, therefore $|f(y)|^2 \leq \|f\| p(y^*y)$ for all y in A^{**} . As $\| |f_\beta| \| \rightarrow \|f\|$ and $\| |f_\beta| \| = |f_\beta|(1) \rightarrow p(1) = \|p\|$, $\|f\| = \|p\|$. Thus, by Theorem (4.2.1), $p = |f|$. This completes the proof.

Definition. Let A be a von Neumann algebra, f and f' two normal positive functionals on A and e and e' their supports respectively. Then f and f' are called disjoint if $ee' = 0$.

Theorem (4.2.3). Let A be a C^* -algebra with identity and $\{f_\alpha\}$ a net in A^* converging to an $f \in A^*$ in the weak* topology. If $\|f_\alpha\|$ also converges to $\|f\|$, then the net $|f_\alpha|$ converges to $|f|$ in the weak* topology.

Proof: [5] (Identify f, f_α and A with $\tilde{f}, \tilde{f}_\alpha$ and $\pi(A)$ as in Theorem (3.1.2)). Let $f = u \cdot |f|$ and $f_\alpha = u_\alpha \cdot |f_\alpha|$ be the enveloping polar decompositions of f and f_α respectively. Since $\| |f| \| = \|f\|$ and $\| |f_\alpha| \| = \|f_\alpha\|$, we have that $\| |f_\alpha| \| \rightarrow \| |f| \|$ and so we may assume that the net $\{|f_\alpha|\}$ is bounded. As bounded closed subsets in A^* are weakly* compact, then, by Theorem 2, Chap. V in [8], $\{|f_\alpha|\}$ has at least one cluster point. Therefore we have only to show that $|f|$ is the only cluster point of $\{|f_\alpha|\}$. Let $|f_\beta|$ be a subnet weakly* converging to a functional p , then $|f_\beta|(x^*x) \rightarrow p(x^*x)$ ($x \in A$) and so p is positive. By Theorem (4.2.1), $|f_\beta(x)|^2 \leq \| |f_\beta| \| |f_\beta|(x^*x)$ for all x in A . Therefore $|f(x)|^2 \leq \|f\| p(x^*x)$ for all x in A . By Corollary (3.1.3), A^{**} is the enveloping von Neumann algebra of A , so that A is ultraweakly dense in A^{**} . When regarded as elements of the predual of A^{**} , f and p are ultraweakly continuous, therefore $|f(y)|^2 \leq \|f\| p(y^*y)$ for all y in A^{**} . As $\| |f_\beta| \| \rightarrow \| |f| \|$ and $\| |f_\beta| \| = |f_\beta|(1) \rightarrow p(1) = \|p\|$, $\| |f| \| = \|p\|$. Thus, by Theorem (4.2.1), $p = |f|$. This completes the proof.

Definition. Let A be a von Neumann algebra, f and f' two normal positive functionals on A and e and e' their supports respectively. Then f and f' are called disjoint if $ee' = 0$.

Theorem (4.2.4). Let A be a von Neumann algebra and g an ultraweakly continuous hermitian linear functional on A . Then there exists a unique pair (f, f') of disjoint normal positive functionals on A such that $g = f - f'$.

Proof : [3] Let A_* be the predual of A . By Theorem (1.2.4), there exists two positive functionals f and f' in A^* such that $g = f - f'$ and $\|g\| = \|f\| + \|f'\|$. Let B be the enveloping von Neumann algebra of A and let \tilde{f} , \tilde{f}' and \tilde{g} be the unique extensions of f , f' and g to B . Then, by Theorem (3.1.2), f and f' are normal and $\|\tilde{g}\| = \|g\|$, $\|\tilde{f}'\| = \|f'\|$, $\|\tilde{f}\| = \|f\|$. Therefore $\|\tilde{g}\| = \|\tilde{f}\| + \|\tilde{f}'\|$ and so, by 12.3.1. (ii), in [3], $|\tilde{g}| = \tilde{f} + \tilde{f}'$. But, by Theorem (4.1.3), $|\tilde{g}| = \tilde{|g|}$. Hence $|g| = f + f'$. Since $|g|$ and g are in A_* and since $f = \frac{1}{2}(|g| + g)$ and $f' = \frac{1}{2}(|g| - g)$, f and f' are in A_* and hence normal. Therefore, by 12.3.1. (i) in [3], f and f' are disjoint. Suppose that we also have $g = h - h'$, where h and h' are disjoint normal positive functionals on A . Then $f + f' = |g| = h + h'$ and $f - f' = g = h - h'$ and so we get $f = h$, $f' = h'$. This completes the proof.

Corollary (4.2.5). Let A be a C^* -algebra and g a hermitian continuous linear functional on A . Then there exists a unique couple (f, f') of positive functionals on A such that $g = f - f'$ and $\|g\| = \|f\| + \|f'\|$.

Proof : [3] The existence of such a decomposition is given by Theorem (1.2.4). The uniqueness follows from Theorem (4.2.4) and Theorem (3.1.2) (iii), which allows us to identify isomet-

rically g, f and f' with elements in the predual of the enveloping von Neumann algebra of A .

§ 3. Extensions of functionals

Let H be a Hilbert space. In what follows C_H is the one-dimensional von Neumann algebra. $C_H = \{\lambda \cdot 1_H : \lambda \text{ is scalar}\}$.

Let H_1 and H_2 be two (complex) Hilbert spaces and let H_0 be their algebraic tensor product, which is a (complex) vector space. Then there exists on H_0 a unique prehilbert space structure defined by the inner product

$$(\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2) = (\xi_1, \eta_1)(\xi_2, \eta_2),$$

where $\xi_i, \eta_i \in H_i$ ($i = 1, 2$). H_0 is also a Hausdorff space. The Hilbert space completion of H_0 is called the tensor product Hilbert space of H_1 and H_2 and is denoted by $H_1 \otimes H_2$.

Let $(e_i)_{i \in I}$ be an orthonormal basis of H_2 . Then each ξ in H can be expressed uniquely in the form $\xi = \sum_{i \in I} \xi_1^i \otimes e_i$, where $(\xi_1^i)_{i \in I}$ is a family of elements in H_1 , containing at most a countable number of elements which are different from zero and such that $\sum_{i \in I} \|\xi_1^i\|^2 < \infty$. We also have

$$\|\xi\|^2 = \sum_{i \in I} \|\xi_1^i\|^2 \quad (\text{cf. pp. 22 - 24 in [2]}).$$

Let A_1 and A_2 be two von Neumann algebras on two Hilbert spaces H_1 and H_2 respectively. Let $H = H_1 \otimes H_2$.

Then the operators of the form

$$x_1 \otimes x_2 + y_1 \otimes y_2 + \dots + z_1 \otimes z_2,$$

where $x_i, y_i, \dots, z_i \in A_i$ ($i = 1, 2$), form an involutive subalgebra A_0 of $L(H)$, containing the identity operator 1_H of $L(H)$. The von Neumann algebra generated by A_0 is called the tensor product von Neumann algebra of A_1 and A_2 and is denoted by $A_1 \otimes A_2$. If $A_2 = C_{H_2}$, then the mapping $x_1 \rightarrow x_1 \otimes 1_{H_2}$ ($x_1 \in A_1$) is an *-isomorphism of A_1 onto $A_1 \otimes C_{H_2}$ (cf. pp.24 - 26 in [2]).

Lemma(4.3.1). Let A be a von Neumann algebra. If $\omega_{\xi, \eta}$ is positive on A , then there exists a γ in H such that $\omega_{\xi, \eta} = \omega_\gamma$.

Proof : [2] For each T in A^+ , we have

$$\begin{aligned} 4\omega_{\xi, \eta}(T) &= 2(T\xi, \eta) + 2(\xi, T\eta) = 2(T\xi, \eta) + 2(T\eta, \xi) \\ &= (T(\xi + \eta), (\xi + \eta)) - (T(\xi - \eta), (\xi - \eta)) \\ &\leq (T(\xi + \eta), \xi + \eta) = \omega_{\xi + \eta}(T) \end{aligned}$$

Thus $4\omega_{\xi, \eta}$ is majorized by $\omega_{\xi + \eta}$, and so, by Lemma 1, in [2], p.50, there exists a T' in A^+ such that

$$4\omega_{\xi, \eta} = \omega_{T'(\xi + \eta)}, \text{ i.e., } \omega_{\xi, \eta} = \omega_{\frac{1}{2}T'(\xi + \eta)}. \text{ This completes the proof.}$$

Lemma (4.3.2). Let $f = \sum_{n=1}^n \omega_{\xi_i, \eta_i}$ (resp. $f = \sum_{n=1}^{\infty} \omega_{\xi_i, \eta_i}$)

with $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$) be a linear

functional on a von Neumann algebra A . Let K be a finite dimensional (resp. infinite dimensional with a countable basis) Hilbert space and ϕ the mapping $x \rightarrow x \otimes 1_K$ from A onto $A \otimes C_K$. Then there exist ξ and η in $H \otimes K$ such that

$f(x) = (\varphi(x)\xi, \eta)$. Moreover, if $\xi_i = \eta_i$ for all i , then we can put $\xi = \eta$.

Proof : [2] Let us suppose, for example, that $f = \sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i}$

with $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$. Let $(e_i)_{i=1}^{\infty}$

be an orthogonal basis of K , and let

$$\xi = \sum_{i=1}^{\infty} \xi_i \otimes e_i, \quad \eta = \sum_{i=1}^{\infty} \eta_i \otimes e_i$$

Then $\xi, \eta \in H \otimes K$. We have

$$\begin{aligned} (\varphi(x)\xi, \eta) &= (x \otimes 1_K(\xi), \eta) \\ &= \left(\sum_{i=1}^{\infty} (x \otimes 1_K)(\xi_i \otimes e_i), \sum_{i=1}^{\infty} \eta_i \otimes e_i \right) \\ &= \left(\sum_{i=1}^{\infty} x\xi_i \otimes e_i, \sum_{i=1}^{\infty} \eta_i \otimes e_i \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n x\xi_i \otimes e_i, \sum_{i=1}^n \eta_i \otimes e_i \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (x\xi_i \otimes e_i, \eta_i \otimes e_i) \\ &= \sum_{i=1}^{\infty} (x\xi_i, \eta_i)(e_i, e_i) \\ &= \sum_{i=1}^{\infty} (x\xi_i, \eta_i) = f(x). \end{aligned}$$

If $\xi_i = \eta_i$, then $\xi = \eta$. This completes the proof.

Theorem (4.3.3). Let f be an ultraweakly continuous linear functional defined on a von Neumann algebra A . Then there exists an ultraweakly continuous linear functional g on $L(H)$, extending f (i.e., $g|_A = f$) and such that $\|g\| = \|f\|$.

Proof : ([3], [12]) (we use the notation of Lemma (4.3.2))

Since f is ultraweakly continuous, $f = \sum_{i=1}^{\infty} \omega_{\xi_i, \eta_i}$ with

$$\xi_i, \eta_i \in H \quad \text{and} \quad \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty.$$

By Lemma (4.3.2), we have $f(x) = (\varphi(x)\xi, \eta)$ with ξ, η in $H \otimes K$. Suppose that $f \geq 0$. Then, by Lemma (4.3.1), there exists γ in $H \otimes K$ such that $f(x) = (\varphi(x)\gamma, \gamma)$, where

$$\gamma = \sum_{i=1}^{\infty} \gamma_i \otimes e_i, \quad \text{with } \gamma_i \in H \quad \text{and} \quad \sum_{i=1}^{\infty} \|\gamma_i\|^2 < \infty.$$

Then

$$f(x) = (\varphi(x)\gamma, \gamma) = \sum_{i=1}^{\infty} (x\gamma_i, \gamma_i) \quad (x \in A).$$

Define $g(x) = \sum_{i=1}^{\infty} (x\gamma_i, \gamma_i)$ ($x \in L(H)$). Then g is a normal positive functional on $L(H)$ and so, by Theorem (1.2.3), $\|g\| = g(1) = f(1) = \|f\|$. If f is not a positive functional, let $f = u \cdot |f|$ be the polar decomposition of f . By Theorem (4.1.1), $|f|$ is a normal positive functional on A . By the above argument, $|f|$ can be extended to a normal positive functional h on $L(H)$. Let $g = u \cdot h$. Then g is an ultraweakly continuous linear functional on $L(H)$ and is an extension of f . We also have

$$\|g\| = \|u \cdot h\| \leq \|h\| = h(1) = |f|(1) = \|f\| = \|f\|.$$

But g extends f , hence $\|g\| = \|f\|$. This completes the proof.

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