Harmonic Analysis of Steady-state Conductive Heat Transfer in L-shaped Plates

by

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Abstract

The thesis presents a method of evaluating temperature distribution in a certain class of plates in terms of Fourier series. Detailed procedure developed for L-shaped plates is equally applicable to any other plate which can be divided into rectangular regions. It is shown that, within the computational limitations of any given hardware, the results obtained by the proposed technique are vastly superior to those obtained from the finite difference method. An interesting application of symmetry is also shown as a corollary.

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I. Introduction

Applications of harmonic analysis (Fourier series) to conductive heat transfer in rectangular plates are the subject of elementary heat transfer textbooks. Recently, A. Feingold [1] provided an exact solution for a particular class of such plates, namely those with linear temperature distribution along the boundaries.

Unfortunately, it is not possible to apply Fourier's ideas to L-shaped plates by a simple superposition of boundary conditions as it is done in rectangular plates, and only very exceptional boundary conditions would enable us to obtain an exact solution by Feingold's method (an example of this is discussed in Chapter VIII).

Numerical methods, such as for instance the finite difference method, are of course available and their utility has been greatly enhanced by the advent of digital computers. What is less generally known, is the degree to which the necessarily limited capacity of ordinarily available computing facilities affects the inaccuracy of the results. In fact, one of the achievements of the present work is to demonstrate the danger of excessive confidence in the realistically obtainable accuracy of the most commonly employed numerical method.

The central idea of this thesis is an extension of Fourier analysis as applied to rectangular plates to fields which can be divided into rectangular regions, an L-shaped plate being used as a convenient example.

The plate having thus been divided, the temperature distribution along the division lines is expressed in terms of Fourier series with undetermined coefficients. The temperature field in each region is then formulated as a function of these coefficients, and a series expression is written for the temperature gradient at each division line in the direction perpendicular to that line. The gradients in two adjoining regions are then matched at any desired number of equidistant points along their common boundary, leading to an equal number of equations in which the previously mentioned coefficients are the unknowns. Writing as many equations as we choose to have coefficients, we produce a linear system which is solved in the usual manner. Once the coefficients have been obtained, the temperatures can be calculated at any desired number of points throughout the entire plate irrespective of the choice of locations at which the gradients were matched.

The accuracy of this method is, of course, also subjected to the limitations of the available computing device and the crucial question now is whether, within that necessarily limited precision, the resulting approximation is better than the best that can be obtained from the same computer by using, say, the finite difference method.

As will be shown, the answer to this question appears to be clear, at least in those two cases for which complete sets of calculations have been prepared; the convergence of the method of undetermined coefficients is far superior to that of the finite difference method.

II. The Method of Undetermined Coefficients

Consider an L-shaped plate with adiabatic faces and with prescribed temperature distribution along its edges. It is assumed that the thermal conductivity is constant. The plate is subdivided as shown in Fig.1. The unknown temperature functions F(x) and G(y) along the separation lines as well as the given temperature functions along the boundaries are indicated in Fig.2.

Temperature T_A at a generic point of area A can be estimated as the sum of four temperatures obtained by the superposition of the four cases shown in Fig.3. Thus,

$$T_A = T_{A,1} + T_{A,2} + T_{A,3} + T_{A,4} \tag{1}$$

the first three temperature distributions on the right-hand side of Eq.(1) are easily obtainable from the boundary conditions, while $T_{A,4}$ contains the undetermined coefficients of the assumed series

$$F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{w_1} \tag{2}$$

$$T_{A,4} = \frac{2}{w_1} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi(h-y)}{w_1}}{\sinh \frac{m\pi h_2}{w_1}} \sin \frac{m\pi x}{w_1} \int_0^{w_1} F(x) \sin \frac{m\pi x}{w_1} dx \tag{3}$$

The development of Eq.(3) is shown in Appendix A.1.

Combining Eqs.(2) and (3) and rearranging, we get

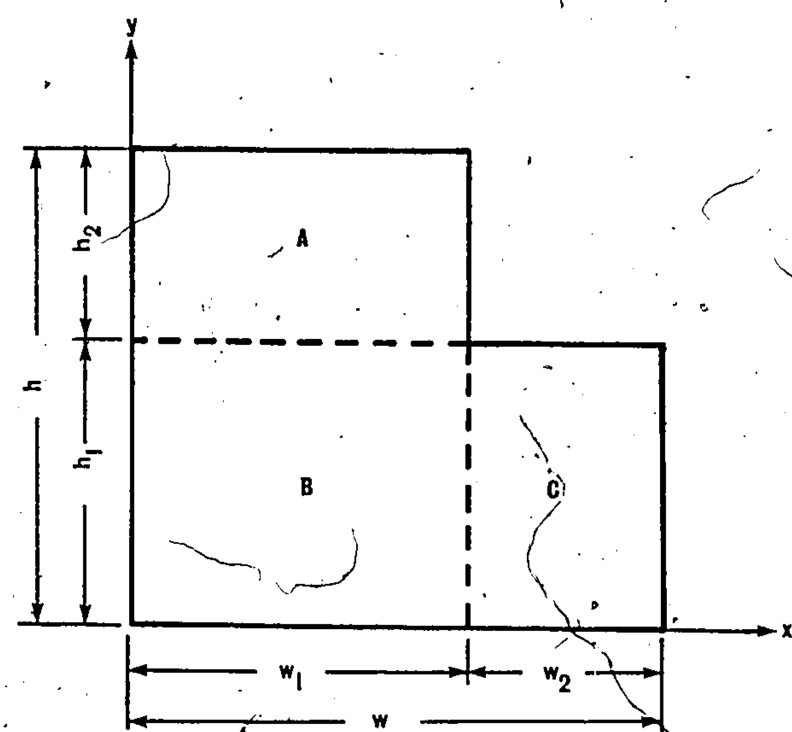


Fig. 1

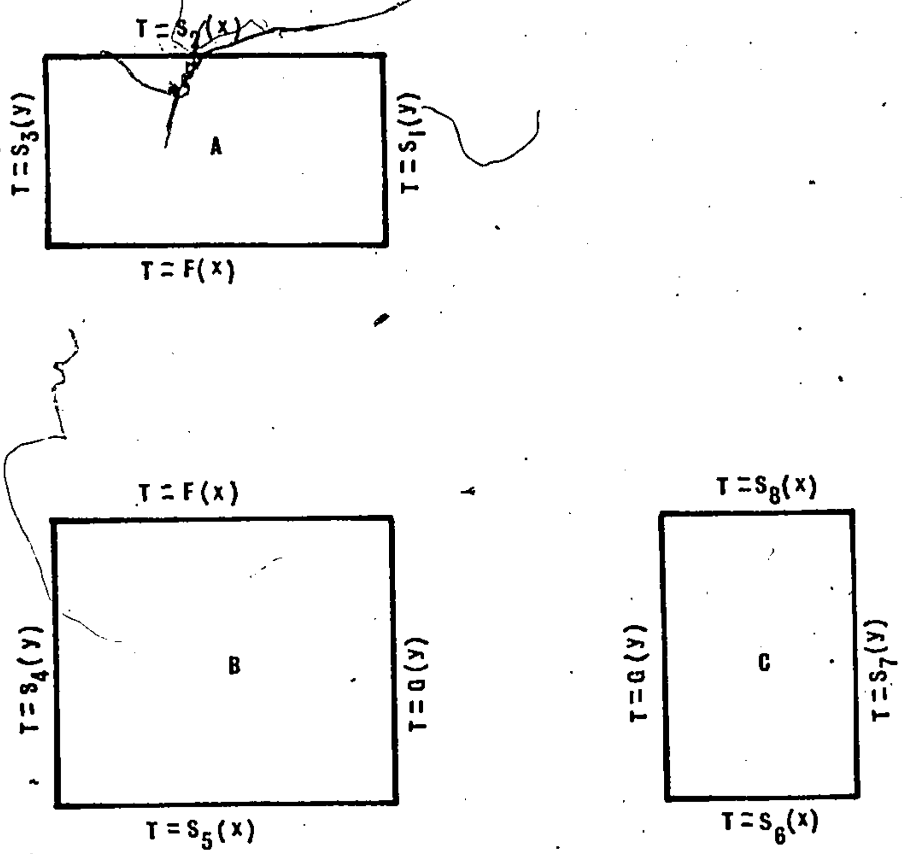


Fig. 2

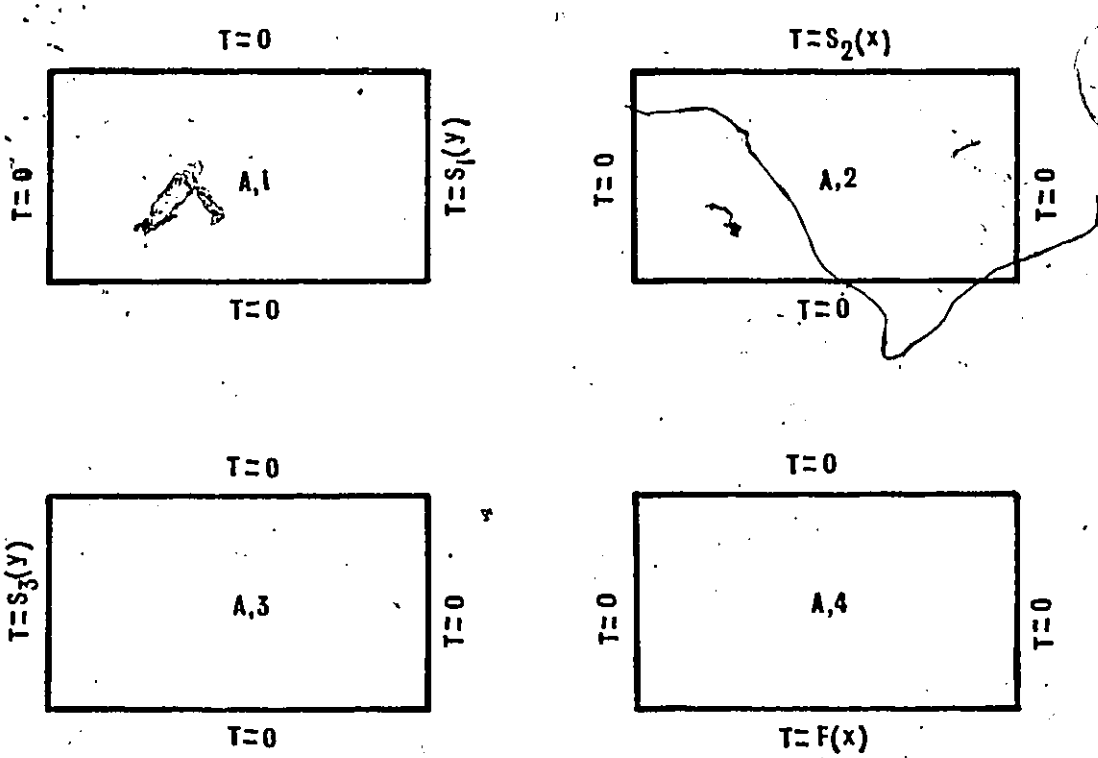


Fig. 3

$$T_{A,4} = \frac{2}{w_1} \sum_{m=1}^{\infty} A_n \frac{\sinh \frac{m\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{m\pi x}{w_1} \sum_{n=1}^{\infty} \int_0^{w_1} \sin \frac{n\pi x}{w_1}$$

$$\sin \frac{m\pi x}{w_1}\, dx \qquad (4)$$

The definite integral in Eq. (4) has the value of zero when $m \neq n$ and the value of $w_1/2$ when $m = n$, as is demonstrated in Appendix A.3.

Hence,

$$T_{A,4} = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \qquad (5)$$

The substitution of Eq. (5) into Eq. (1) yields

$$T_A = T_{A,1} + T_{A,2} + T_{A,3} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \qquad (6)$$

and the temperature gradient in the y direction is

$$\frac{\partial T_A}{\partial y} = \frac{\partial T_{A,1}}{\partial y} + \frac{\partial T_{A,2}}{\partial y} + \frac{\partial T_{A,3}}{\partial y}$$

$$- \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n\, n \frac{\cosh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \qquad (7)$$

Approaching the line separating A from B this temperature gradient becomes

$$\left(\frac{\partial T_A}{\partial y}\right)_{y=h_1} = \left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{A,2}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{A,3}}{\partial y}\right)_{y=h_1}$$

$$- \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \, n \coth \frac{n\pi h_2}{w_1} \sin \frac{n\pi x}{w_1} \tag{8}$$

With the notation shown in Fig.4, we can write in the same manner

$$T_C = T_{C,1} + T_{C,2} + T_{C,3} + T_{C,4} \tag{9}$$

Assuming the temperature function G(y) is in the form

$$G(y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{h_1} \tag{10}$$

where C_n are the undetermined coefficients, just as A_n were in Eq.(2),

$$T_{C,4} = \frac{2}{h_1} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi(w-x)}{h_1}}{\sinh \frac{m\pi w_2}{h_1}} \sin \frac{m\pi y}{h_1} \int_0^{h_1} G(y) \sin \frac{m\pi y}{h_1} \, dy \tag{11}$$

Clearly, applying the same reasoning which got us from Eq.(3) to Eq.(5), we can now write

$$T_{C,4} = \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi(w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \tag{12}$$

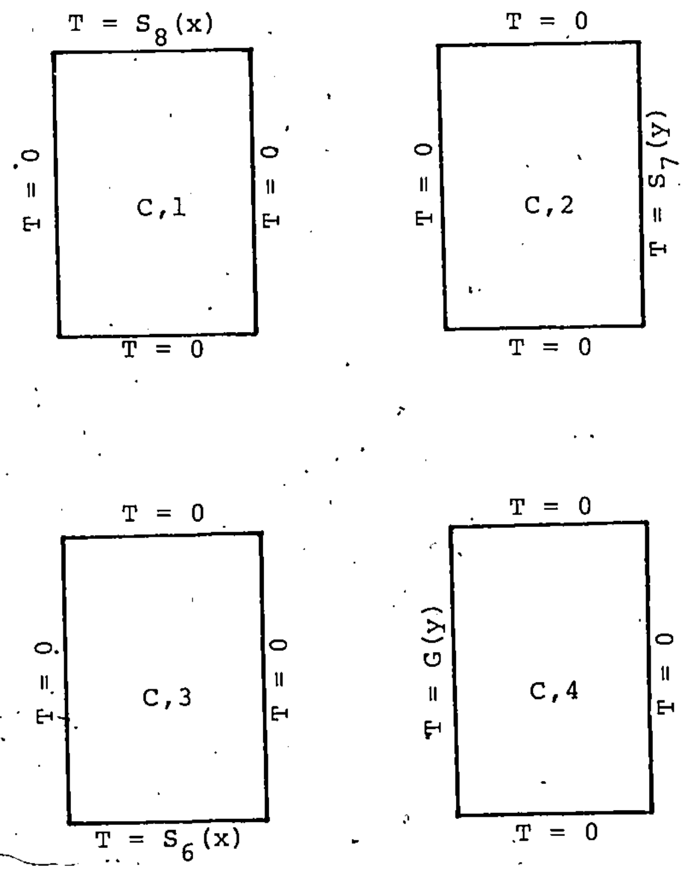


Fig. 4

Combining Eqs.(9) and (12), we get

$$T_C = T_{C,1} + T_{C,2} + T_{C,3} + \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi(w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \tag{13}$$

and the temperature gradient in x direction is

$$\frac{\partial T_C}{\partial x} = \frac{\partial T_{C,1}}{\partial x} + \frac{\partial T_{C,2}}{\partial x} + \frac{\partial T_{C,3}}{\partial x} - \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \, n \frac{\cosh \frac{n\pi(w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1}$$

Approaching the line separating C from B, this temperature gradient becomes

$$\left(\frac{\partial T_C}{\partial x}\right)_{x=w_1} = \left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{C,2}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{C,3}}{\partial x}\right)_{x=w_1} - \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \, n \coth \frac{n\pi w_2}{h_1} \sin \frac{n\pi y}{h_1} \tag{14}$$

For area B, following again a similar reasoning, we obtain with the notation of Fig.5

$$T_B = T_{B,1} + T_{B,2} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1} + \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \tag{15}$$

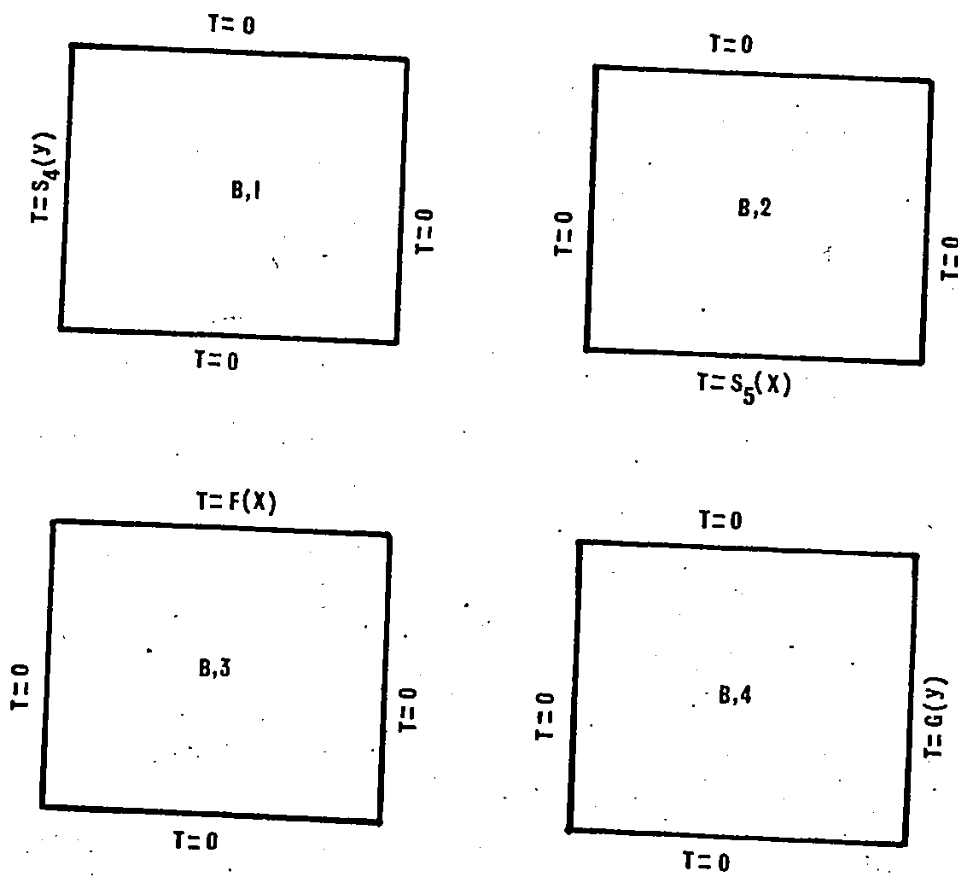


Fig. 5

We shall now develop the expressions for the temperature gradients corresponding to those in Eqs.(8), and (14) but approaching the separation lines from the side of region B.

$$\frac{\partial T_B}{\partial y} = \frac{\partial T_{B,1}}{\partial y} + \frac{\partial T_{B,2}}{\partial y} + \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \; n \; \frac{\cosh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \; n \; \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \cos \frac{n\pi y}{h_1}$$

Along the line $y = h_1$, knowing that $\cos n\pi = (-1)^n$, the temperature gradient will be

$$\left(\frac{\partial T_B}{\partial y}\right)_{y=h_1} = \left(\frac{\partial T_{B,1}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{B,2}}{\partial y}\right)_{y=h_1} + \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \; n \coth \frac{n\pi h_1}{w_1}$$

$$\sin \frac{n\pi x}{w_1} + \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n \; n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \tag{16}$$

The temperature gradient in x-direction is

$$\frac{\partial T_B}{\partial x} = \frac{\partial T_{B,1}}{\partial x} + \frac{\partial T_{B,2}}{\partial x} + \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \; n \; \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \cos \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \; n \; \frac{\cosh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1}$$

Approaching the line $x = w_1$ this temperature gradient becomes

$$\left(\frac{\partial T_B}{\partial x}\right)_{x=w_1} = \left(\frac{\partial T_{B,1}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{B,2}}{\partial x}\right)_{x=w_1} = \frac{\pi}{h_1}\sum_{n=1}^{\infty} C_n\, n \coth\frac{n\pi w_1}{h_1} \sin\frac{n\pi y}{h_1} + \frac{\pi}{w_1}\sum_{n=1}^{\infty} A_n \frac{(-1)^n\, n \sinh\frac{n\pi y}{w_1}}{\sinh\frac{n\pi h_1}{w_1}} \quad (17)$$

Clearly, for continuity we must have

$$\left(\frac{\partial T_A}{\partial y}\right)_{y=h_1} = \left(\frac{\partial T_B}{\partial y}\right)_{y=h_1} \quad (18)$$

and

$$\left(\frac{\partial T_C}{\partial x}\right)_{x=w_1} = \left(\frac{\partial T_B}{\partial x}\right)_{x=w_1} \quad (19)$$

Combining Eqs.(8), (16) and (18), and rearranging, we get

$$\frac{\pi}{w_1}\sum_{n=1}^{\infty} A_n\, n \left(\coth\frac{n\pi h_1}{w_1} + \coth\frac{n\pi h_2}{w_1}\right) \sin\frac{n\pi x}{w_1} + \frac{\pi}{h_1}\sum_{n=1}^{\infty} C_n \frac{(-1)^n\, n \sinh\frac{n\pi x}{h_1}}{\sinh\frac{n\pi w_1}{h_1}} = R(x) \quad (20)$$

where

$$R(x) = \left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{A,2}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{A,3}}{\partial y}\right)_{y=h_1} - \left(\frac{\partial T_{B,1}}{\partial y}\right)_{y=h_1} - \left(\frac{\partial T_{B,2}}{\partial y}\right)_{y=h_1}$$

Similarly, combining Eqs.(14), (17) and (19), we obtain

$$\frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \, n \left(\coth \frac{n\pi w_1}{h_1} + \coth \frac{n\pi w_2}{h_1}\right) \sin \frac{n\pi y}{h_1} + \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \frac{(-1)^n \, n \sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} = R(y) \qquad (21)$$

where

$$R(y) = \left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{C,2}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{C,3}}{\partial x}\right)_{x=w_1} - \left(\frac{\partial T_{B,1}}{\partial x}\right)_{x=w_1} - \left(\frac{\partial T_{B,2}}{\partial x}\right)_{x=w_1}$$

Eq.(20) must be valid at any point along the line $y = h_1$ and, therefore, it may be written as many times as we choose for different values of x between zero and w_1. Similarly, Eq.(21) can be written for any number of values of y between zero and h_1. If we desire to truncate each series after the first p terms, we shall have to determine p coefficients A_n and p coefficients C_n, for a total of 2p coefficients.

Clearly, if we wrote 2p equations (arbitrarily distributed between those of type (20) and those of type (21)), we would have a linear system of 2p equations with 2p unknowns which can readily be solved.

Once the coefficients have been determined in this manner, Eqs.(6), (13) and (15) can be employed to calculate the temperature in regions A, C and B, respectively. Such temperature can be calculated at any point, independently, of the choice of values of x and y used in Eqs.(20) and (21) for the purpose of determining the coefficients A_n and C_n.

III. Accuracy

Like any solution based on harmonic analysis, our method will increase its accuracy with the increase of number of terms in the truncated series. The capacity of computational facilities provides here the limit as it would in any numerical method. An idea of the rapidity of convergence can be obtained by comparing several sets of solutions with the number of equations increasing in arithmetical progression, taking first 2p = k, then 2p = k + 2m, then 2p = k + 4m, etc. The difference between the temperatures calculated from two consecutive sets must diminish with the increased number of equations, otherwise the method does not converge. When that difference is deemed to be sufficiently small, we may assume to be close to the "true" value.

In order to get a realistic appreciation of the practical worth of our method, we shall apply it in the following chapters to a variety of boundary conditions.

IV. Sinusoidal Temperature Distributions along Some Segments of the Boundary

Consider the boundary conditions shown in Fig.6 where we have sinusoidal temperature distributions on two segments of the boundary while the remaining segments are maintained at zero. Clearly, we need only to find R(x) and R(y) in Eqs.(20) and (21), respectively.

Adopting the nomenclature of Figs.2, 3, 4 and 5, we have in this case

$$T_{A,2} = T_{A,3} = T_{B,1} = T_{B,2} = T_{C,2} = T_{C,3} = 0 \tag{22}$$

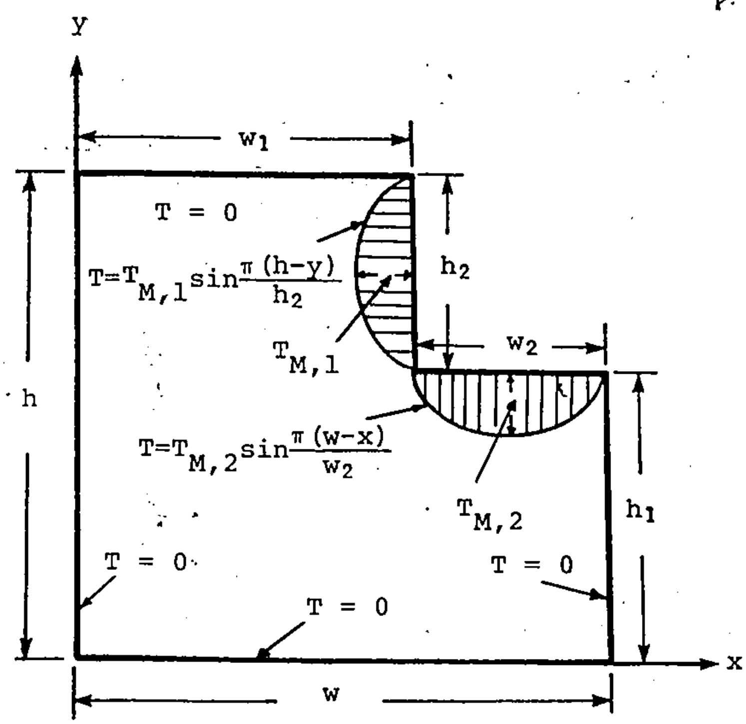
and, therefore, all the derivatives of these temperatures are also zero. This reduces R(x) and R(y) to the following simple expressions

$$R(x) = \left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} \quad \text{and} \quad R(y) = \left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} \tag{23}$$

From Eq.(119) in Appendix A.2, taking into account the transformation of the coordinate system, we have

$$T_{A,1} = -\frac{2}{h_2} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi x}{h_2}}{\sinh \frac{m\pi w_1}{h_2}} \sin \frac{m\pi(h-y)}{h_2} \int_{h}^{h_1} S_1(y) \sin \frac{m\pi(h-y)}{h_2}\, dy \tag{24}$$

but in this case



y
w1
T = 0
T=TM,1sinπ(h-y)/h2
h2
TM,1
w2
h
T=TM,2sinπ(w-x)/w2
TM,2
h1
T = 0
T = 0
T = 0
x
w

Fig. 6

$$S_1(y) = T_{M,1} \sin \frac{\pi(h-y)}{h_2} \tag{25}$$

Combining Eqs. (24) and (25), we get

$$T_{A,1} = -\frac{2\, T_{M,1}}{h_2} \sum_{m=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \sin \frac{m\pi(h-y)}{h_2} \int_h^{h_1} \sin \frac{\pi(h-y)}{h_2} \cdot \sin \frac{m\pi(h-y)}{h_2}\, dy \tag{26}$$

The definite integral in Eq. (26) has the value of zero when $m \neq 1$ and the value of $-h_2/2$ when $m = 1$, as is demonstrated in Appendix A.4 (compare with a similar procedure adopted with respect to Eq. (4)).

Hence,

$$T_{A,1} = T_{M,1} \frac{\sinh \frac{\pi x}{h_2}}{\sinh \frac{\pi w_1}{h_2}} \sin \frac{\pi(h-y)}{h_2} \tag{27}$$

The temperature gradient in y direction is

$$\frac{\partial T_{A,1}}{\partial y} = -\frac{\pi\, T_{M,1} \sinh \frac{\pi x}{h_2}}{h_2 \sinh \frac{\pi w_1}{h_2}} \cos \frac{\pi(h-y)}{h_2}$$

Along the line $y = h_1$, knowing that $h_2 = h - h_1$ and $\cos \pi = -1$, the gradient will be

$$\left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} = \frac{\pi \; T_{M,1} \sinh \frac{\pi x}{h_2}}{h_2 \sinh \frac{\pi w_1}{h_2}} \tag{28}$$

From Eqs.(6), (22) and (27), we get

$$T_A = T_{M,1} \frac{\sinh \frac{\pi x}{h_2}}{\sinh \frac{\pi w_1}{h_2}} \sin \frac{\pi (h-y)}{h_2}$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi (h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \tag{29}$$

Using Eqs.(23) and (28), Eq.(20) becomes now

$$\frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \; n(\coth \frac{n\pi h_1}{w_1} + \coth \frac{n\pi h_2}{w_1}) \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} = \frac{\pi \; T_{M,1} \sinh \frac{\pi x}{h_2}}{h_2 \sinh \frac{\pi w_1}{h_2}} \tag{20a}$$

Following the same reasoning based on

$$S_8(x) = T_{M,2} \sin \frac{\pi (w-x)}{w_2} \tag{30}$$

we obtain

$$T_{C,1} = T_{M,2} \frac{\sinh \frac{\pi y}{w_2}}{\sinh \frac{\pi h_1}{w_2}} \sin \frac{\pi (w-x)}{w_2} \tag{31}$$

The temperature gradient in x direction approaching the line $x = w_1$ is now

$$\left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} = \frac{\pi \; T_{M,2} \sinh \frac{\pi y}{w_2}}{w_2 \sinh \frac{\pi h_1}{w_2}} \tag{32}$$

and Eqs. (13), (22) and (31) yield

$$T_C = T_{M,2} \frac{\sinh \frac{\pi y}{w_2}}{\sinh \frac{\pi h_1}{w_2}} \sin \frac{\pi (w-x)}{w_2}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi (w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \tag{33}$$

Bearing in mind Eqs. (23) and (32), we can rewrite Eq. (21) in this form:

$$\frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \, n \left(\coth \frac{n\pi w_1}{h_1} + \coth \frac{n\pi w_2}{h_1}\right) \sin \frac{n\pi y}{h_1}$$

$$+ \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \frac{(-1)^n \, n \sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} = \frac{\pi \; T_{M,2} \sinh \frac{\pi y}{w_2}}{w_2 \sinh \frac{\pi h_1}{w_2}} \tag{21a}$$

As for T_B, we observe that according to Eq.(22), $T_{B,1}$ and $T_{B,2}$ appearing in Eq.(15) are both zero. Thus

$$T_B = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (34)$$

We are now ready to determine the values of coefficients A_n and C_n by means of the solution of a number of simultaneous linear equations which can be written by repeating Eqs.(20a) and (21a) for different values of n as many times as we please (see Chapter II). These values of A_n and C_n are now inserted into the series in Eqs.(29), (33) and (34) to calculate T_A, T_C and T_B, respectively.

Evidently, in numerical calculations we must assign a number representing so many units of length to each segment of plate boundary. We have chosen integral dimensions for the sake of simplicity, taking however care that the plate will not be symmetrical or exhibit any particularly simple length ratio, in order to preserve generality. We have used the same dimensions for all our examples. The dimensions and the particular values of $T_{M,1}$ and $T_{M,2}$ used in our calculations are shown in Fig.7. The computation has been carried out using built-in subroutine called GELG [2].

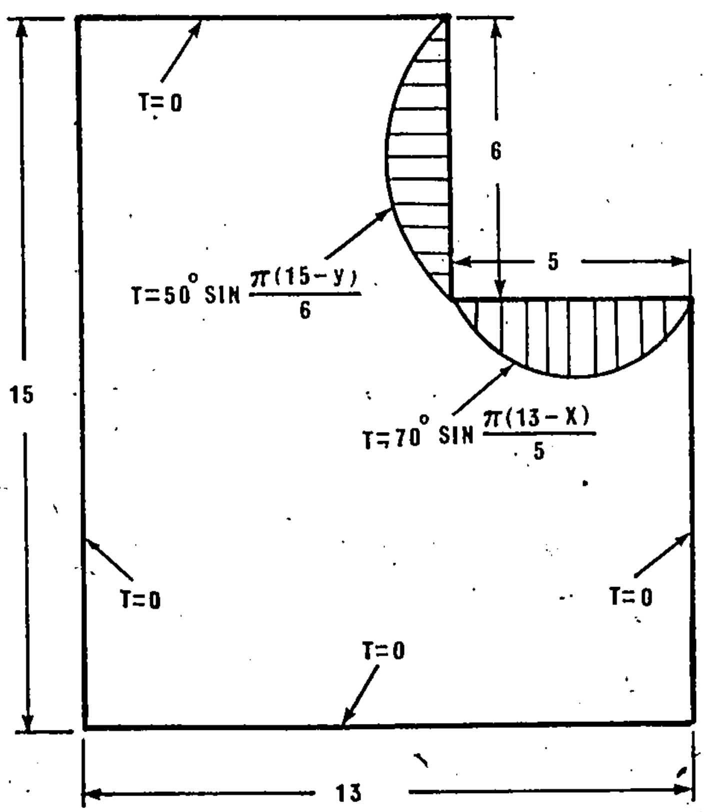

T=0
6
5
$T=50^\circ \sin \frac{\pi(15-y)}{6}$
15
$T=70^\circ \sin \frac{\pi(13-X)}{5}$
T=0
T=0
T=0
13

Fig. 7

We had to determine the maximum number p of coefficients in each series which our computer could process. This turned out to be 37, meaning that we had a total of 74 equations containing 37 A_n and 37 C_n coefficients. As can be seen later, we have also performed calculations with several smaller numbers of coefficients in order to test the rapidity of convergence of our method.

Table 1 contains the values of all the coefficients when their total number was $2 \times 37 = 74$.

Figure 8 depicts the family of isothermal curves for the boundary conditions and dimensions shown in Fig.7. In this case, as also in those considered further below, the isotherms were plotted on the basis of temperatures computed at intervals of 0.2 units of length throughout the entire plate.

Table 1. The Coefficients Obtained Using Eqs.(20a) and (21a) for the Boundary Conditions Shown in Fig.7

n	A_n	C_n
1	1.09672×10	1.36763×10
2	-4.21351×10^{0}	-5.61987×10^{0}
3	2.22875×10^{0}	2.94885×10^{0}
4	-1.39208×10^{0}	-1.81427×10^{0}
5	9.57355×10^{-1}	1.23118×10^{0}
6	-7.00520×10^{-1}	-8.91254×10^{-1}
7	5.35106×10^{-1}	6.75039×10^{-1}
8	-4.21808×10^{-1}	-5.28527×10^{-1}
9	3.40519×10^{-1}	4.24366×10^{-1}
10	-2.80041×10^{-1}	-3.47487×10^{-1}
11	2.33742×10^{-1}	2.89018×10^{-1}
12	-1.97450×10^{-1}	-2.43441×10^{-1}
13	1.68423×10^{-1}	2.07167×10^{-1}
14	-1.44816×10^{-1}	-1.77791×10^{-1}
15	1.25351×10^{-1}	1.53645×10^{-1}
16	-1.09083×10^{-1}	-1.33534×10^{-1}
17	9.53453×10^{-2}	1.16591×10^{-1}
18	-8.36253×10^{-2}	-1.02172×10^{-1}
19	7.35423×10^{-2}	8.97892×10^{-2}
20	-6.48031×10^{-2}	-7.90665×10^{-2}
21	5.71605×10^{-2}	6.97103×10^{-2}
22	-5.04423×10^{-2}	-6.14939×10^{-2}
23	4.45013×10^{-2}	5.42315×10^{-2}
24	-3.92127×10^{-2}	-4.77738×10^{-2}
25	3.44794×10^{-2}	4.20002×10^{-2}
26	-3.02266×10^{-2}	-3.68107×10^{-2}
27	2.63824×10^{-2}	3.21253×10^{-2}
28	-2.28954×10^{-2}	-2.78766×10^{-2}
29	1.97159×10^{-2}	2.40016×10^{-2}
30	-1.68057×10^{-2}	-2.04601×10^{-2}
31	1.41352×10^{-2}	1.72048×10^{-2}
32	-1.16663×10^{-2}	-1.42001×10^{-2}
33	9.37796×10^{-3}	1.14161×10^{-2}
34	-7.25334×10^{-3}	-8.82819×10^{-3}
35	5.26839×10^{-3}	6.41138×10^{-3}
36	-3.40663×10^{-3}	-4.14799×10^{-3}
37	1.65559×10^{-3}	2.01527×10^{-3}

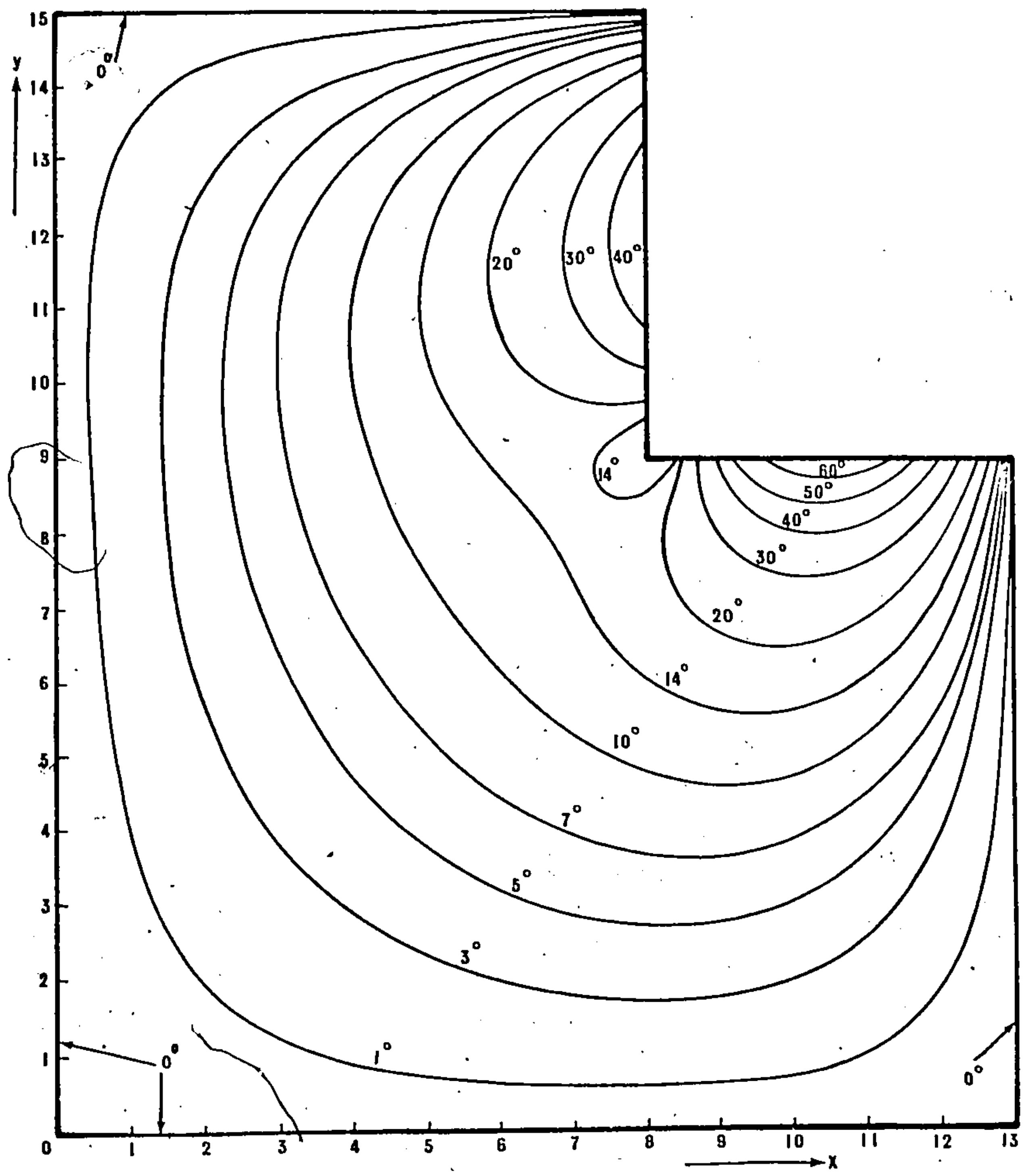


Fig. 8 The Family of Isothermal Curves for the Boundary Conditions and Dimensions Shown in Fig.7.

V. Linear Temperature Distribution along Segments of the Boundary

The example treated in the previous chapter was particularly simple because of the sinusoidal distribution along two segments of the boundary. In any other case, the formulation of temperatures $T_{A,1}$ and $T_{C,1}$ is a little more complicated. The following example will illustrate the procedure.

Consider the boundary conditions shown in Fig.9. Equations (22) and (23) are still valid. It will be convenient to express the temperature functions along segments 1 and 8 of the boundary in terms of coordinates x_1 and y_1 and transform the coordinate system to x and y at the end. Thus,

$$S_1(y_1) = \begin{cases} \dfrac{2\, T_{M,1}\, y_1}{h_2} & \text{when } 0 < y_1 < \dfrac{h_2}{2} \\[2ex] \dfrac{2\, T_{M,1}\,(h_2 - y_1)}{h_2} & \text{when } \dfrac{h_2}{2} < y_1 < h_2 \end{cases} \tag{35}$$

$$S_8(x_1) = \begin{cases} \dfrac{2\, T_{M,2}\, x_1}{w_2} & \text{when } 0 < x_1 < \dfrac{w_2}{2} \\[2ex] \dfrac{2\, T_{M,2}\,(w_2 - x_1)}{w_2} & \text{when } \dfrac{w_2}{2} < x_1 < w_2 \end{cases} \tag{36}$$

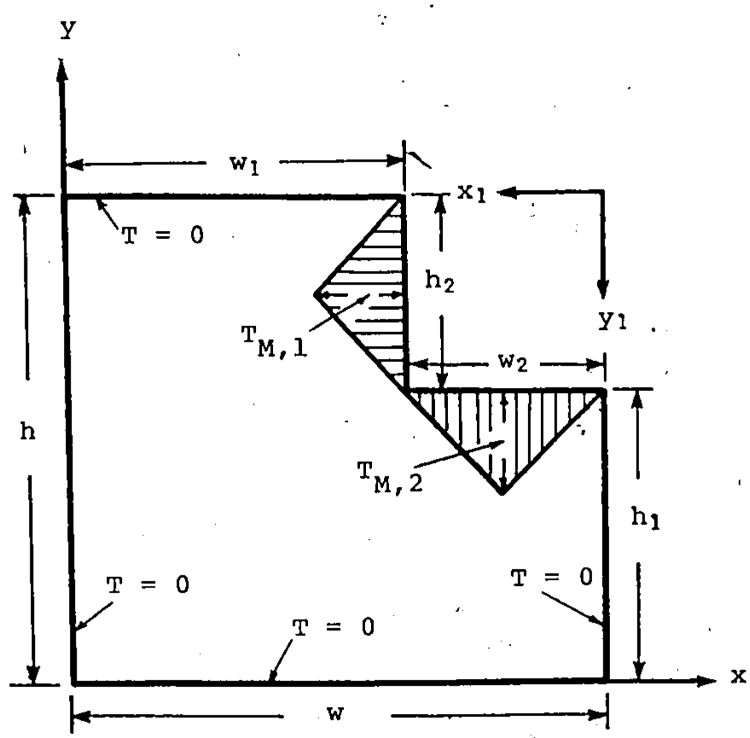


Fig. 9

Before going any further, we must expand Eqs.(35) and (36) into Fourier series.

For Eq.(35), we shall write

$$S_1(y_1) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi y_1}{h_2} \tag{37}$$

where

$$B_n = \frac{2}{h_2}\int_0^{h_2} S_1(y_1) \sin \frac{n\pi y_1}{h_2}\, dy_1$$

or, substituting for $S_1(y_1)$ the appropriate expression given in Eq.(35),

$$B_n = \frac{2}{h_2}\int_0^{\frac{h_2}{2}} \frac{2\, T_{M,1}\, y_1}{h_2} \sin \frac{n\pi y_1}{h_2}\, dy_1 + \frac{2}{h_2}\int_{\frac{h_2}{2}}^{h_2} \frac{2\, T_{M,1}(h_2 - y_1)}{h_2} \sin \frac{n\pi y_1}{h_2}\, dy_1 \tag{38}$$

and finally

$$B_n = \frac{4\ T_{M,1}}{h_2}\int_0^{\frac{h_2}{2}} y_1 \sin\frac{n\pi y_1}{h_2}\,dy_1 + \frac{4\ T_{M,1}}{h_2}\int_{\frac{h_2}{2}}^{h_2} \sin\frac{n\pi y_1}{h_2}\,dy_1$$

$$- \frac{4\ T_{M,1}}{h_2^2}\int_{\frac{h_2}{2}}^{h_2} y_1 \sin\frac{n\pi y_1}{h_2}\,dy_1 \qquad (39)$$

The integration will be performed with the aid of standard integral tables [5]. The general expression

$$\int y_1 \sin a y_1 dy_1 = \frac{1}{a^2}\sin a y_1 - \frac{y_1}{a}\cos a y_1 \qquad (40)$$

will give us

$$\int_0^{\frac{h_2}{2}} y_1 \sin\frac{n\pi y_1}{h_2}\,dy_1 = \left[\frac{h_2^2}{n^2\pi^2}\sin\frac{n\pi y_1}{h_2} - \frac{h_2 y_1}{n\pi}\cos\frac{n\pi y_1}{h_2}\right]_0^{\frac{h_2}{2}}$$

$$= \frac{h_2^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{h_2^2}{2n\pi}\cos\frac{n\pi}{2} \qquad (41)$$

and

$$\int_{\frac{h_2}{2}}^{h_2} y_1 \sin\frac{n\pi y_1}{h_2}\,dy_1 = \left[\frac{h_2^2}{n^2\pi^2}\sin\frac{n\pi y_1}{h_2} - \frac{h_2 y_1}{n\pi}\cos\frac{n\pi y_1}{h_2}\right]_{\frac{h_2}{2}}^{h_2}$$

$$= -\frac{h_2^2}{n\pi}\cos n\pi - \frac{h_2^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{h_2^2}{2n\pi}\cos\frac{n\pi}{2} \qquad (42)$$

The middle integral in Eq.(39) is simply

$$\int_{\frac{h_2}{2}}^{h_2} \sin \frac{n\pi y_1}{h_2}\, dy_1 = \left[- \frac{h_2}{n\pi} \cos \frac{n\pi y_1}{h_2} \right]_{\frac{h_2}{2}}^{h_2}$$

$$= - \frac{h_2}{n\pi} \cos n\pi + \frac{h_2}{n\pi} \cos \frac{n\pi}{2} \tag{43}$$

After the substitution of Eqs.(41), (42) and (43), Eq.(39) becomes

$$B_n = \frac{8\ T_{M,1}}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

$$\text{But} \quad \sin \frac{n\pi}{2} = \begin{cases} 0 & \text{when n is even} \\ (-1)^{\frac{n-1}{2}} & \text{when n is odd} \end{cases}$$

thus

$$B_n = \frac{8\ T_{M,1} (-1)^{\frac{n-1}{2}}}{n^2 \pi^2} \qquad n = 1,\ 3,\ 5,\ \ldots\ldots \tag{44}$$

and, with this, Eq.(37) reads

$$S_1(y_1) = \frac{8\ T_{M,1}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi y_1}{h_2} \tag{45}$$

To return to the original coordinate system, we substitute $y_1 = h - y$

$$S_1(y) = \frac{8\,T_{M,1}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi(h-y)}{h_2} \qquad (46)$$

Similarly, Fourier expansion for the temperature function $S_8(x_1)$ is

$$S_8(x_1) = \frac{8\,T_{M,2}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi x_1}{w_2}$$

and, with the substitution of $x_1 = w - x$,

$$S_8(x) = \frac{8\,T_{M,2}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi(w-x)}{w_2} \qquad (47)$$

Now we are ready to evaluate the temperature distributions $T_{A,1}$ and $T_{C,1}$.

Recall Eq.(24):

$$T_{A,1} = -\frac{2}{h_2} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi x}{h_2}}{\sinh \frac{m\pi w_1}{h_2}} \sin \frac{m\pi(h-y)}{h_2} \int_{h}^{h_1} S_1(y) \sin \frac{m\pi(h-y)}{h_2}\, dy$$

Substituting Eq.(46) into Eq.(24) and bearing in mind that

$$\int_{h}^{h_1} \sin \frac{m\pi(h-y)}{h_2} \sin \frac{n\pi(h-y)}{h_2}\, dy = \begin{cases} 0 & \text{when } m \neq n \\ -\frac{h_2}{2} & \text{when } m = n \end{cases}$$

(see Appendix A.4), we get

$$T_{A,1} = \frac{8\ T_{M,1}}{\pi^2} \sum_{n=1,3,5...}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi x}{h_2}}{n^2 \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi(h-y)}{h_2} \tag{48}$$

Consequently,

$$\left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} = \frac{8\ T_{M,1}}{\pi\ h_2} \sum_{n=1,3,5...}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \tag{49}$$

Equation (6), which will be used to calculate the temperatures in region A, is transformed by Eqs.(22) and (48) into

$$T_A = \frac{8\ T_{M,1}}{\pi^2} \sum_{n=1,3,5...}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi x}{h_2}}{n^2 \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi(h-y)}{h_2}$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \tag{50}$$

In a manner analogous to the procedure developed above in relation to $T_{A,1}$, we can obtain

$$T_{C,1} = \frac{8\ T_{M,2}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi y}{w_2}}{n^2 \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi (w-x)}{w_2} \tag{51}$$

$$\left(\frac{\partial T_{C,I}}{\partial x}\right)_{x=w_1} = \frac{8\ T_{M,2}}{\pi w_2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi y}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \tag{52}$$

and

$$T_C = \frac{8\ T_{M,2}}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi y}{w_2}}{n^2 \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi (w-x)}{w_2}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi (w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \tag{53}$$

The temperatures in region B are calculated from Eq. (15) which, taking into account Eq. (22), reads

$$T_B = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (54)$$

We must now return to Eqs. (20) and (21) from which we shall be calculating the values of undetermined coefficients A_n and C_n. In each case, all the derivatives ~~on~~ the right-hand side, except the ones calculated in Eqs. (49) and (52), are zero. Hence, Eqs. (20) and (21) become respectively

$$\frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \, n \left(\coth \frac{n\pi h_1}{w_1} + \coth \frac{n\pi h_2}{w_1}\right) \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n \, n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} ,$$

$$= \frac{8\, T_{M,1}}{\pi h_2} \sum_{n=1,3,5...}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \qquad (20b)$$

and

$$\frac{\pi}{h_1}\sum_{n=1}^{\infty} C_n \; n\left(\coth \frac{n\pi w_1}{h_1} + \coth \frac{n\pi w_2}{h_1}\right) \sin \frac{n\pi y}{h_1}$$

$$+ \frac{\pi}{w_1}\sum_{n=1}^{\infty} A_n \frac{(-1)^n \; n \sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}},$$

$$= \frac{8\, T_{M,1}}{\pi w_2} \sum_{n=1,3,5...}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sinh \frac{n\pi y}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \qquad (21b)$$

Figure 10 shows the dimensions and temperatures used in the numerical calculation which again was performed for several values of p as in Chapter IV. The maximum value of p was again 37 and the coefficients calculated with this value of p are reported in Table 2. The temperature map appears in Fig.11.

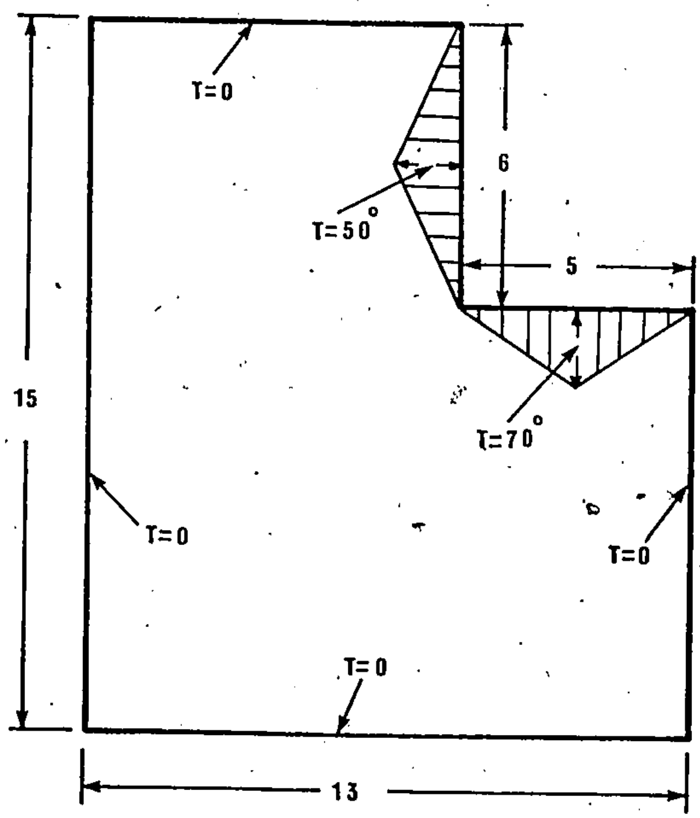


Fig. 10

Table 2. The Coefficients Obtained Using Eqs.(20b) and (21b) for the Boundary Conditions Shown in Fig.10

n	A_n	C_n
1	8.88972×10^{0}	1.10856×10
2	-3.41536×10^{0}	-4.55530×10^{0}
3	1.80655×10^{0}	2.39025×10^{0}
4	-1.12836×10^{0}	-1.47059×10^{0}
5	7.75993×10^{-1}	9.97960×10^{-1}
6	-5.67816×10^{-1}	-7.22425×10^{-1}
7	4.33744×10^{-1}	5.47167×10^{-1}
8	-3.41911×10^{-1}	-4.28409×10^{-1}
9	2.76016×10^{-1}	3.43978×10^{-1}
10	-2.26991×10^{-1}	-2.81664×10^{-1}
11	1.89460×10^{-1}	2.34271×10^{-1}
12	-1.60045×10^{-1}	-1.97326×10^{-1}
13	1.36522×10^{-1}	1.67924×10^{-1}
14	-1.17387×10^{-1}	-1.44112×10^{-1}
15	1.01609×10^{-1}	1.24540×10^{-1}
16	-8.84196×10^{-2}	-1.08239×10^{-1}
17	7.72821×10^{-2}	9.45054×10^{-2}
18	-6.77813×10^{-2}	-8.28171×10^{-2}
19	5.96120×10^{-2}	7.27810×10^{-2}
20	-5.25220×10^{-2}	-6.40897×10^{-2}
21	4.63351×10^{-2}	5.65055×10^{-2}
22	-4.08867×10^{-2}	-4.98457×10^{-2}
23	3.60698×10^{-2}	4.39586×10^{-2}
24	-3.17821×10^{-2}	-3.87249×10^{-2}
25	2.79470×10^{-2}	3.40450×10^{-2}
26	-2.45012×10^{-2}	-2.98384×10^{-2}
27	2.13870×10^{-2}	2.60402×10^{-2}
28	-1.85592×10^{-2}	-2.25958×10^{-2}
29	1.59805×10^{-2}	1.94554×10^{-2}
30	-1.36207×10^{-2}	-1.65846×10^{-2}
31	1.14567×10^{-2}	1.39455×10^{-2}
32	-9.45694×10^{-3}	-1.15100×10^{-2}
33	7.60346×10^{-3}	9.25383×10^{-3}
34	-5.88063×10^{-3}	-7.15591×10^{-3}
35	4.27058×10^{-3}	5.19721×10^{-3}
36	-2.76026×10^{-3}	-3.36221×10^{-3}
37	1.34078×10^{-3}	1.63349×10^{-3}

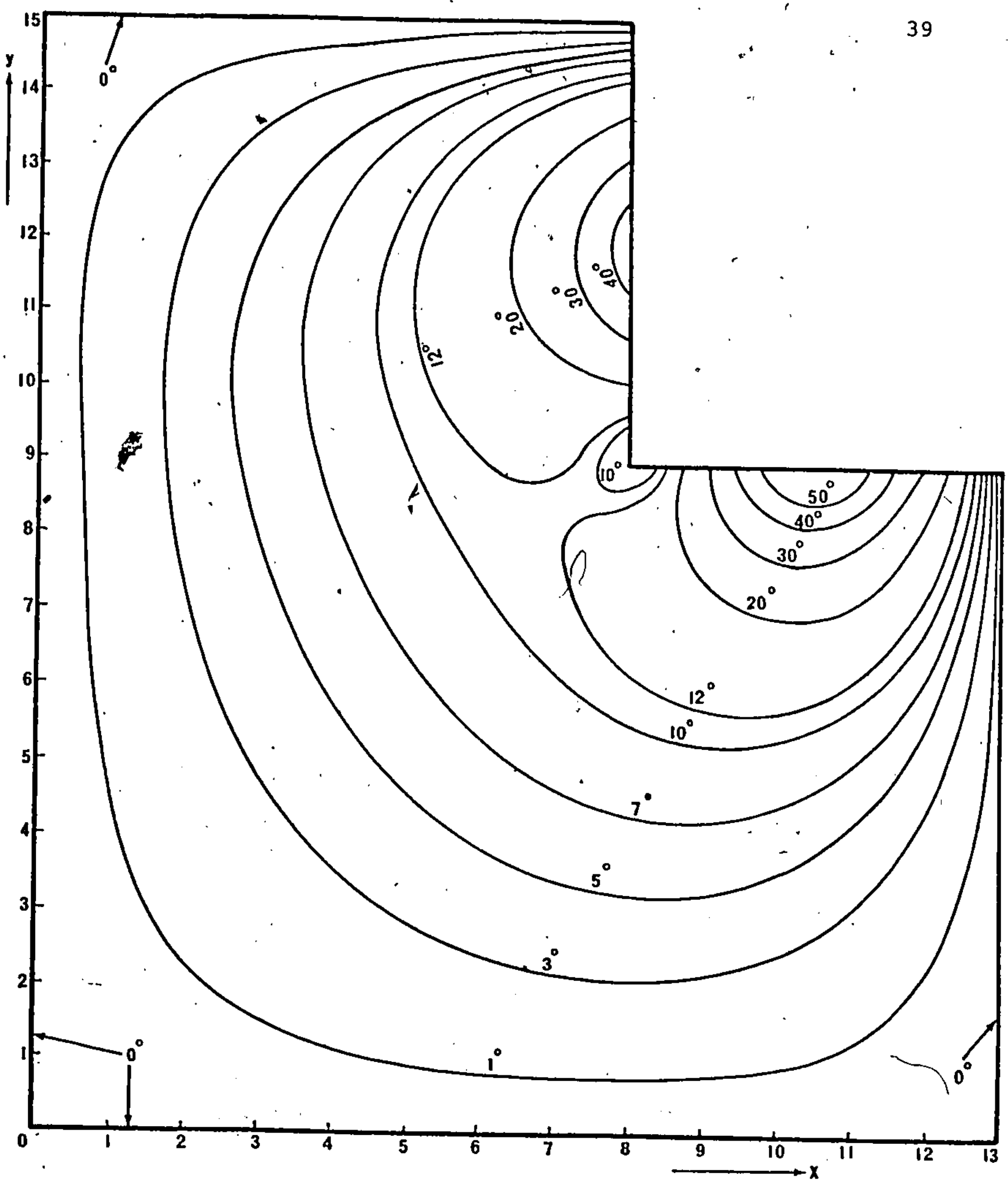


Fig. 11 The Temperature Map for the Boundary Conditions and Dimensions Shown in Fig.10.

VI. Comparison between the Method of Undetermined Coefficients and Finite Difference Method

In the finite difference method applied to a two-dimensional field, the entire field is covered with a grid. A set of difference equations involving the magnitudes of the required variables at grid points can be written so as to turn the problem into the solution of a linear system.

The closer the mesh, the more accurate will be the calculated values. The true degree of accuracy is, of course, not known since no exact solution is generally available. In exceptional cases where an exact closed-form solution does exist, there is no point in looking for an approximate one.

Nevertheless, it is of practical importance to have some idea as to the degree of convergence of any numerical method we may be employing. In the case of finite difference method, the best one can do is to determine the variations of the results with the increase of the number of points until the computing capacity has been exhausted, and, if this variation is not too great, one assumes that one is fairly close to an "exact" temperature distribution.

For a square mesh the algorithm for the finite difference method corresponding to Laplace's equation is [3]

$$4\,T_{m,n} = T_{m,n-1} + T_{m,n+1} + T_{m-1,n} + T_{m+1,n} \tag{55}$$

with the meaning of the symbols shown in Fig.12.

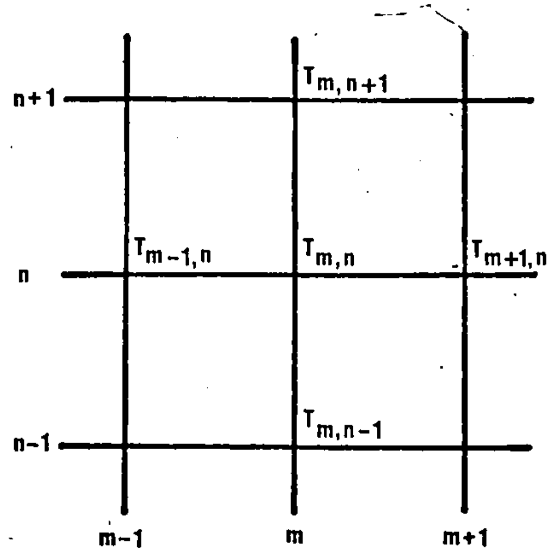


Fig. 12

We have considered the results obtained for the two cases illustrated in Chapters IV and V which fit the boundary conditions indicated in Figs.7 and 10, respectively. It was thought that the comparison will be clearer, if rather simple boundary conditions were used. The two examples employ respectively, non-linear and linear temperature distributions along the segments of the boundary with non-zero temperatures, in order to see whether linearity affects the outcome of our comparisons.

For the finite difference method, two intervals were employed - a unit of length and one half of that unit - involving 138 and 605 equations respectively.

Even with 605 x 605 matrix special measures were required to accommodate the problem to the available storage space. Subroutine GELB [2] allowed us to cope with this matrix size, but a still smaller mesh interval could not be used even with this additional software. It may be worthwhile noting that with the interval of 0.2 the number of equations would rise to 3986.

Tables 3 and 4 allow to compare the results of temperature calculations with various degrees of approximation. Table 3 corresponds to Fig.7 and Table 4 to Fig.10. For each point four values were obtained using the method of undetermined coefficients with 2x22, 2x27, 2x32 and 2x37 coefficients respectively.

Table 3. Comparison of the Result of Temperature Calculations for the Case Shown in Fig.7

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series, p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
1	1	0.25330	0.25493	0.25600	0.25675	0.22565	0.24633
1	2	0.50767	0.51096	0.51312	0.51463	0.45201	0.49364
1	3	0.76327	0.76825	0.77152	0.77381	0.67917	0.74206
1	4	1.01867	1.02535	1.02974	1.03281	0.90611	0.99029
1	5	1.27047	1.27877	1.28422	1.28804	1.13061	1.23521
1	6	1.51295	1.52263	1.52899	1.53345	1.34933	1.47176
1	7	1.73726	1.74787	1.75484	1.75974	1.55738	1.69208
1	8	1.92892	1.93981	1.94697	1.95201	1.74601	1.88312
1	9	2.06304	2.07287	2.08088	2.08659	1.89683	2.02213
1	10	2.10670	2.11593	2.12201	2.12629	1.97436	2.07322
1	11	2.01389	2.02145	2.02643	2.02992	1.92619	1.99121
1	12	1.74890	1.75455	1.75827	1.76088	1.70236	1.73641
1	13	1.30027	1.30397	1.30641	1.30813	1.28203	1.29503
1	14	0.69526	0.69709	0.69829	0.69913	0.69076	0.69377
2	1	0.50538	0.50862	0.51074	0.51223	0.45057	0.49159
2	2	1.01382	1.02036	1.02465	1.02766	0.90323	0.98597
2	3	1.52638	1.53633	1.54286	1.54744	1.35855	1.48410
2	4	2.04059	2.05402	2.06284	2.06902	1.81467	1.98366
2	5	2.54978	2.56660	2.57765	2.58539	2.26704	2.47852
2	6	3.04321	3.06302	3.07603	3.08515	2.70934	2.95915
2	7	3.50556	3.52745	3.54184	3.55194	3.13418	3.41226
2	8	3.91226	3.93484	3.94969	3.96012	3.52983	3.81656
2	9	4.22087	4.24037	4.25381	4.26378	3.86695	4.13087
2	10	4.34986	4.36882	4.38133	4.39010	4.07444	4.28017
2	11	4.20431	4.21965	4.22975	4.23685	4.02802	4.15858
2	12	3.68830	3.69960	3.70705	3.71229	3.60122	3.66474
2	13	2.76384	2.77117	2.77600	2.77939	2.73500	2.75528
2	14	1.48508	1.48866	1.49102	1.49267	1.48101	1.48337

Table 3 (Continued)

x	y	Method of undetermined coefficients — Number of undetermined coefficients in each series: p = 22	p = 27	p = 32	p = 37	Finite difference method — Square mesh of unit-length side	Square mesh of one half unit-length side
3	1	0.75406	0.75882	0.76194	0.76413	0.67342	0.73382
3	2	1.51524	1.52491	1.53126	1.53570	1.35176	1.47419
3	3	2.28713	2.30196	2.31169	2.31850	2.03716	2.22428
3	4	3.06686	3.08710	3.10038	3.10969	2.72698	2.98137
3	5	3.84418	3.86990	3.88679	3.89863	3.41352	3.73582
3	6	4.60325	4.63406	4.65429	4.66848	4.08682	4.47332
3	7	5.32622	5.36083	5.38358	5.39953	4.74020	5.17899
3	8	5.99013	6.02617	6.04988	6.06653	5.37215	5.83561
3	9	6.54078	6.58052	6.59969	6.61757	5.96671	6.40030
3	10	6.87023	6.90000	6.91960	6.93338	6.42837	6.75837
3	11	6.77490	6.79838	6.81386	6.82472	6.51024	6.70586
3	12	6.05259	6.06944	6.08055	6.08836	5.93953	6.02157
3	13	4.59847	4.60915	4.61619	4.62114	4.57573	4.59090
3	14	2.49161	2.49675	2.50014	2.50252	2.49831	2.49239
4	1	0.99498	1.00112	1.00515	1.00797	0.89134	0.96909
4	2	2.00485	2.01739	2.02561	2.03137	1.79323	1.95193
4	3	3.03870	3.05812	3.07086	3.07978	2.71136	2.95672
4	4	4.09456	4.12148	4.13914	4.15152	3.64256	3.98123
4	5	5.15660	5.19157	5.21452	5.23060	4.57322	5.01017
4	6	6.19916	6.24219	6.27045	6.29025	5.48424	6.01956
4	7	7.20201	7.25176	7.28445	7.30737	6.36758	6.99236
4	8	8.16616	8.21899	8.25374	8.27811	7.25186	7.93752
4	9	9.08654	9.13198	9.16302	9.18816	8.19937	8.86371
4	10	9.80861	9.85088	9.87873	9.89828	9.16211	9.64427
4	11	9.99024	10.02211	10.04311	10.05787	9.64506	9.89933
4	12	9.18182	9.20366	9.21805	9.22816	9.07092	9.15061
4	13	7.11963	7.13297	7.14177	7.14794	7.13005	7.12024
4	14	3.90362	3.90990	3.91404	3.91695	3.93650	3.91100

Table 3 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series				Finite difference method	
		p = 22	p = 27	p = 32	p = 37	Square mesh of unit-length side	Square mesh of one half unit-length side
5	1	1.22004	1.22730	1.23206	1.23539	1.09873	1.18993
5	2	2.46859	2.48349	2.49326	2.50011	2.21844	2.40642
5	3	3.76577	3.78912	3.80444	3.81516	3.37249	3.66786
5	4	5.11408	5.14706	5.16869	5.18384	4.55869	4.97555
5	5	6.48915	6.53321	6.56212	6.58237	5.75255	6.30501
5	6	7.83946	7.89589	7.93292	7.95886	6.90934	7.60643
5	7	9.11813	9.18663	9.23161	9.26313	7.99405	8.83581
5	8	10.36920	10.44499	10.49481	10.52973	9.06834	10.04263
5	9	11.75606	11.83081	11.88017	11.91473	10.41681	11.43519
5	10	13.27289	13.33007	13.36774	13.39418	12.37551	13.04146
5	11	14.22930	14.26883	14.29489	14.31319	13.83696	14.12450
5	12	13.63108	13.65614	13.67266	13.68427	13.56909	13.61238
5	13	10.85981	10.87432	10.88387	10.89059	10.93704	10.87783
5	14	6.04225	6.04887	6.05323	6.05630	6.11764	6.06065
6	1	1.41526	1.42324	1.42847	1.43214	1.28514	1.38324
6	2	2.88072	2.89721	2.90802	2.91559	2.60930	2.81388
6	3	4.43708	4.46318	4.48029	4.49227	4.00148	4.32966
6	4	6.10152	6.13904	6.16364	6.18087	5.46716	5.94473
6	5	7.84528	7.89695	7.93084	7.95456	6.96889	7.62786
6	6	9.56593	9.63552	9.68116	9.71312	8.40652	9.27683
6	7	11.09493	11.18634	11.24632	11.28833	9.63087	10.72799
6	8	12.40823	12.51921	12.59212	12.64322	10.61065	11.95267
6	9	14.20234	14.29562	14.36263	14.42078	12.02404	13.66525
6	10	17.22273	17.29672	17.3455	17.37978	16.08615	16.91928
6	11	20.10226	20.14528	20.17365	20.19359	19.75819	20.01033
6	12	20.38924	20.41336	20.42926	20.44044	20.43137	20.39890
6	13	16.76849	16.78149	16.79007	16.79608	16.93129	16.80925
6	14	9.47571	9.48143	9.48520	9.48785	9.59703	9.50616

Table 3 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series, p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
7	1	1.55883	1.56702	1.57239	1.57615	1.43254	1.52805
7	2	3.19831	3.21530	3.22642	3.23421	2.93212	3.13344
7	3	4.99294	5.01999	5.03772	5.05013	4.55696	4.88678
7	4	6.99694	7.03632	7.06212	7.08019	6.33956	6.83699
7	5	9.20923	9.26469	9.30103	9.32648	8.24933	8.97525
7	6	11.49250	11.57062	11.62182	11.65765	10.11699	11.15452
7	7	13.41382	13.52704	13.60122	13.65315	11.51227	12.93832
7	8	14.15045	14.32004	14.43125	14.50910	11.71933	13.53249
7	9	14.71900	14.97288	15.05238	15.13048	10.98253	13.70436
7	10	21.18454	21.26489	21.31798	21.35532	20.18674	20.91631
7	11	28.81377	28.84671	28.86845	28.88374	28.67831	28.78220
7	12	31.32690	31.34270	31.35312	31.36044	31.46733	31.36354
7	13	26.56477	26.57271	26.57793	26.58160	26.75975	26.61435
7	14	15.21159	15.21497	15.21720	15.21877	15.33919	15.24388
8	1	1.61761	1.62805	1.63497	1.63803	1.51290	1.59479
8	2	3.35922	3.37555	3.38597	3.39313	3.12967	3.30278
8	3	5.33504	5.35494	5.37416	5.38694	4.95472	5.24077
8	4	7.65999	7.70459	7.72426	7.74378	7.08479	7.52546
8	5	10.45075	10.50408	10.54075	10.56698	9.57189	10.24668
8	6	13.70478	13.77476	13.81739	13.84916	12.29986	13.36920
8	7	16.94099	17.03754	17.12845	17.16425	14.58178	16.39294
8	8	17.94174	18.24722	18.40274	18.44232	13.77188	16.91270
9	1	1.56785	1.57475	1.57926	1.58243	1.48939	1.54889
9	2	3.28189	3.29614	3.30547	3.31201	3.11892	3.24256
9	3	5.30858	5.33116	5.34594	5.35629	5.04747	5.24584
9	4	7.85690	7.88937	7.91064	7.92551	7.47290	7.76502
9	5	11.20106	11.24575	11.27499	11.29546	10.65357	11.07010
9	6	15.69270	15.75280	15.79212	15.81961	14.92878	15.50768
9	7	21.72191	21.80061	21.85204	21.88795	20.74312	21.47731
9	8	29.60129	29.68973	29.74731	29.78745	28.78642	29.40901

Table 3 (Continued)

x	y	Method of undetermined coefficients				Finite difference method	
		Number of undetermined coefficients in each series				Square mesh of unit-length side	Square mesh of one half unit-length side
		p = 22	p = 27	p = 32	p = 37		
10	1	1.37195	1.37747	1.38107	1.38360	1.32574	1.36068
10	2	2.90022	2.91155	2.91898	2.92417	2.80914	2.87803
10	3	4.77527	4.79302	4.80465	4.81278	4.64334	4.74328
10	4	7.26825	7.29325	7.30962	7.32106	7.10578	7.22912
10	5	10.80928	10.84230	10.86391	10.87902	10.64064	10.76919
10	6	16.13844	16.17940	16.20616	16.22487	16.01859	16.11177
10	7	24.64066	24.68584	24.71533	24.73593	24.67558	24.65761
10	8	39.21538	39.25069	39.27371	39.28976	39.48579	39.30042
11	1	1.02512	1.02893	1.03143	1.03318	1.00444	1.01996
11	2	2.18443	2.19223	2.19734	2.20091	2.14853	2.17548
11	3	3.64873	3.66081	3.66872	3.67425	3.61104	3.63938
11	4	5.67903	5.69566	5.70655	5.71416	5.66625	5.67615
11	5	8.72502	8.74613	8.75993	8.76958	8.78461	8.74105
11	6	13.62673	13.65109	13.66700	13.67813	13.82943	13.68093
11	7	22.04683	22.07068	22.08626	22.09712	22.45483	22.15674
11	8	37.38982	37.40570	37.41608	37.42331	37.90735	37.52882
12	1	0.54976	0.55170	0.55297	0.55387	0.54349	0.54813
12	2	1.17755	1.18151	1.18510	1.8592	1.16951	1.17542
12	3	1.98511	1.99119	1.99516	1.99794	1.98603	1.98518
12	4	3.13341	3.14165	3.14703	3.15080	3.16356	3.14095
12	5	4.90950	4.91963	4.92630	4.93094	5.00196	4.93326
12	6	7.86258	7.87379	7.88112	7.88624	8.05968	7.91403
12	7	13.08365	13.09396	13.10069	13.10540	13.40735	13.16873
12	8	22.75462	22.76105	22.76524	22.76817	23.11488	22.84911

Table 4. Comparison of the Result of Temperature Calculations for the Case Shown in Fig.10

x	y	Method of undetermined coefficients — Number of undetermined coefficients in each series: p = 22	p = 27	p = 32	p = 37	Finite difference method — Square mesh of unit-length side	Square mesh of one half unit-length side
1	1	0.20532	0.20664	0.20751	0.20812	0.17621	0.19194
1	2	0.41150	0.41417	0.41592	0.41714	0.35306	0.38452
1	3	0.61868	0.62272	0.62537	0.62723	0.53077	0.57780
1	4	0.82571	0.83112	0.83468	0.83717	0.70876	0.77082
1	5	1.02980	1.03653	1.04095	1.04404	0.88570	0.96143
1	6	1.22635	1.23420	1.23935	1.24297	1.05953	1.14625
1	7	1.40816	1.41677	1.42242	1.42639	1.22714	1.32002
1	8	1.56352	1.57235	1.57816	1.58224	1.38229	1.47337
1	9	1.67223	1.68019	1.68669	1.69131	1.51058	1.58877
1	10	1.70755	1.71504	1.71997	1.72343	1.58281	1.63708
1	11	1.63240	1.63852	1.64256	1.64539	1.55435	1.58021
1	12	1.41768	1.42226	1.42527	1.42739	1.38091	1.38376
1	13	1.05396	1.05696	1.05894	1.06033	1.04294	1.03493
1	14	0.56349	0.56497	0.56594	0.56662	0.56248	0.55525
2	1	0.40965	0.41227	0.41399	0.41519	0.35177	0.38317
2	2	0.82177	0.82708	0.83055	0.83299	0.70528	0.76823
2	3	1.23724	1.24531	1.25060	1.25431	1.06124	1.15573
2	4	1.65404	1.66493	1.67207	1.67708	1.41857	1.54390
2	5	2.06677	2.08041	2.08936	2.09564	1.77450	1.92846
2	6	2.46673	2.48279	2.49333	2.50073	2.12527	2.30311
2	7	2.84149	2.85924	2.87091	2.87909	2.46674	2.65945
2	8	3.17116	3.18946	3.20150	3.20995	2.79145	2.98315
2	9	3.42131	3.43713	3.44803	3.45609	3.07722	3.24339
2	10	3.52550	3.54088	3.55101	3.55812	3.26630	3.37971
2	11	3.40788	3.42032	3.42851	3.43426	3.25369	3.30283
2	12	2.98998	2.99915	3.00519	3.00943	2.92636	2.92444
2	13	2.24029	2.24623	2.25014	2.25289	2.22835	2.20515
2	14	1.20340	1.20630	1.20821	1.20955	1.20700	1.18874

Table 4 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series, p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
3	1	0.61121	0.61508	0.61761	0.61938	0.52557	0.57234
3	2	1.22821	1.23605	1.24119	1.24479	1.05506	1.14923
3	3	1.85387	1.86590	1.87378	1.87930	1.59034	1.73269
3	4	2.48590	2.50231	2.51308	2.52062	2.12980	2.32040
3	5	3.11596	3.13683	3.15052	3.16011	2.66845	2.90527
3	6	3.73125	3.75623	3.77263	3.78413	3.20033	3.47776
3	7	4.31727	4.34533	4.36377	4.37670	3.72312	4.02967
3	8	4.85541	4.88464	4.90385	4.91734	4.23952	4.55262
3	9	5.30175	5.33396	5.34950	5.36400	4.74057	5.01790
3	10	5.56704	5.59118	5.60707	5.61824	5.15144	5.33540
3	11	5.49152	5.51056	5.52310	5.53191	5.26775	5.33348
3	12	4.90779	4.92146	4.93046	4.93678	4.84250	4.81715
3	13	3.72737	3.73604	3.74174	3.74575	3.73709	3.68319
3	14	2.01788	2.02205	2.02479	2.02672	2.03719	2.00071
4	1	0.80650	0.81148	0.81474	0.81703	0.69546	0.75663
4	2	1.62507	1.63523	1.64190	1.64657	1.39906	1.52311
4	3	2.46307	2.47882	2.48914	2.49637	2.11525	2.30492
4	4	3.31892	3.34075	3.35506	3.36509	2.84185	3.00995
4	5	4.17977	4.20813	4.22673	4.23976	3.56916	3.89479
4	6	5.02484	5.05974	5.08263	5.09869	4.28448	4.67345
4	7	5.83772	5.87807	5.90455	5.92313	4.98587	5.42701
4	8	6.61923	6.66207	6.69023	6.70999	5.70293	6.17205
4	9	7.36527	7.40210	7.42725	7.44763	6.49411	6.92872
4	10	7.94216	7.97644	7.99901	8.01486	7.33115	7.60672
4	11	8.09773	8.12358	8.14060	8.15255	7.82339	7.89090
4	12	7.45096	7.46866	7.48032	7.48852	7.43880	7.35101
4	13	5.77091	5.78173	5.78886	5.79387	5.84031	5.73101
4	14	3.15577	3.16086	3.16422	3.16657	3.20467	3.14202

Table 4 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series, p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
5	1	0.98893	0.99482	0.99867	1.00137	0.85721	0.93056
5	2	2.00095	2.01304	2.02096	2.02651	1.73043	1.88064
5	3	3.05241	3.07135	3.08376	3.09245	2.62975	2.86316
5	4	4.14531	4.17205	4.18958	4.20186	3.55319	3.87730
5	5	5.25989	5.29563	5.31905	5.33546	4.48188	4.90206
5	6	6.35442	6.40018	6.43018	6.45121	5.38256	5.89805
5	7	7.39086	7.44641	7.48286	7.50841	6.23296	6.83532
5	8	8.40494	8.46640	8.50677	8.53509	7.09222	7.76719
5	9	9.52909	9.58970	9.62974	9.65774	8.20178	8.88558
5	10	10.71845	10.76482	10.79535	10.81679	9.85554	10.25450
5	11	11.53327	11.56533	11.58644	11.60128	11.25589	11.29604
5	12	11.09003	11.11035	11.12374	11.13315	11.24899	11.01658
5	13	8.80208	8.81384	8.82159	8.82703	8.98071	8.78748
5	14	4.85753	4.86290	4.86643	4.86891	4.94116	4.85248
6	1	1.14716	1.15364	1.15788	1.16084	1.00295	1.08415
6	2	2.33502	2.34840	2.35715	2.36328	2.03572	2.20405
6	3	3.59656	3.61772	3.63158	3.64129	3.12014	3.38734
6	4	4.94569	4.97612	4.99606	5.01002	4.25929	4.64199
6	5	6.35913	6.40104	6.42850	6.44772	5.42252	5.93867
6	6	7.75384	7.81027	7.84725	7.87315	6.53094	7.19162
6	7	8.99320	9.06732	9.11592	9.14997	7.47112	8.27135
6	8	10.05771	10.14771	10.20678	10.24821	8.23124	9.17058
6	9	11.51198	11.58762	11.64191	11.68906	9.36524	10.48970
6	10	13.77016	13.83016	13.86969	13.89748	12.63335	13.17132
6	11	16.28632	16.32121	16.34419	16.36037	16.09564	16.07001
6	12	16.73067	16.75023	16.76312	16.77219	17.32057	16.76416
6	13	13.58410	13.59464	13.60159	13.60647	13.89230	13.61550
6	14	7.49067	7.49531	7.49837	7.50052	7.57927	7.49090

Table 4 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
7	1	1.26353	1.27018	1.27453	1.27757	1.11886	1.20104
7	2	2.59245	2.60623	2.61524	2.62155	2.28935	2.46177
7	3	4.04711	4.06906	4.08342	4.09348	3.55578	3.83576
7	4	5.67149	5.70343	5.72434	5.73898	4.94130	5.35754
7	5	7.46471	7.50968	7.53913	7.55976	6.41800	7.01186
7	6	9.31545	9.37881	9.42029	9.44933	7.84757	8.66787
7	7	10.87281	10.96461	11.02473	11.06682	8.88937	9.96320
7	8	11.46990	11.60741	11.69753	11.76063	8.99635	10.28163
7	9	11.93074	12.13662	12.20103	12.26434	8.39461	10.26752
7	10	16.30118	16.36632	16.40933	16.43962	15.21685	15.73958
7	11	23.23586	23.26257	23.28020	23.29259	23.17278	23.10829
7	12	26.47403	26.48683	26.49527	26.50121	28.04549	26.84795
7	13	21.41290	21.41934	21.42357	21.42654	21.68866	21.45930
7	14	11.45977	11.46251	11.46432	11.46559	11.48365	11.45071
8	1	1.31117	1.31965	1.32525	1.32774	1.18315	1.25757
8	2	2.72287	2.73612	2.74456	2.75036	2.44705	2.60416
8	3	4.32441	4.34056	4.35613	4.36648	3.87235	4.13099
8	4	6.20895	6.24512	6.26105	6.27686	5.53212	5.92724
8	5	8.47103	8.51429	8.54401	8.56527	7.46061	8.05502
8	6	11.10868	11.16542	11.19997	11.22571	9.55199	10.46132
8	7	13.73190	13.81019	13.88397	13.91304	11.24232	12.68683
8	8	14.54310	14.79082	14.91686	14.94893	10.47020	12.75577
9	1	1.27085	1.27644	1.28010	1.28266	1.16669	1.22546
9	2	2.66018	2.67175	2.67931	2.68460	2.44334	2.56643
9	3	4.30289	4.32121	4.33319	4.34157	3.95444	4.15443
9	4	6.36806	6.39441	6.41164	6.42369	5.85417	6.15385
9	5	9.07604	9.11228	9.13598	9.15255	8.34033	8.77794
9	6	12.69903	12.74777	12.77962	12.80190	11.65747	12.28965
9	7	17.46899	17.53282	17.57449	17.60361	16.05771	16.92722
9	8	23.09538	23.16710	23.21376	23.24628	21.64214	22.55670

Table 4 (Continued)

x	y	Method of undetermined coefficients: Number of undetermined coefficients in each series, p = 22	p = 27	p = 32	p = 37	Finite difference method: Square mesh of unit-length side	Square mesh of one half unit-length side
10	1	1.11206	1.11654	1.11946	1.12151	1.04027	1.07981
10	2	2.35083	2.36003	2.36604	2.37025	2.20519	2.28588
10	3	3.87073	3.88513	3.89455	3.90114	3.64793	3.77290
10	4	5.89171	5.91199	5.92525	5.93452	5.58978	5.76312
10	5	8.76363	8.79042	8.80793	8.82016	8.38902	8.61406
10	6	13.09430	13.12750	13.14919	13.16435	12.67988	12.95642
10	7	20.05826	20.09489	20.11879	20.13548	19.68890	20.02341
10	8	32.33820	32.36684	32.38548	32.39850	32.04068	32.54510
11	1	0.83093	0.83402	0.83605	0.83747	0.78921	0.81133
11	2	1.77063	1.77696	1.78110	1.78399	1.68922	1.73253
11	3	2.95759	2.96739	2.97379	2.97828	2.84231	2.90432
11	4	4.60354	4.61703	4.62585	4.63202	4.46801	4.54356
11	5	7.07420	7.09132	7.10250	7.11032	6.94612	7.02692
11	6	11.05838	11.07813	11.09102	11.10004	10.98411	11.06508
11	7	17.95578	17.97513	17.98773	17.99655	17.97737	18.09378
11	8	30.85844	30.87134	30.87973	30.88559	30.83179	31.19127
12	1	0.44561	0.44719	0.44822	0.44894	0.42734	0.43665
12	2	0.95448	0.95769	0.95979	0.96126	0.92016	0.93762
12	3	1.60900	1.61393	1.61715	1.61940	1.56407	1.58713
12	4	2.53936	2.54604	2.55041	2.55346	2.49383	2.51877
12	5	3.97630	3.98454	3.98992	3.99368	3.94325	3.96848
12	6	6.35218	6.36127	6.36721	6.37136	6.33306	6.37236
12	7	10.46713	10.47549	10.48095	10.48476	10.40490	10.51612
12	8	17.54567	17.55089	17.55429	17.55666	17.30916	17.56833

As can be seen, the difference between consecutive values obtained by this method are rather small, and, more importantly, there is in every case a monotonic diminishing rate of increase indicating that we are quite close to an assymptotic "exact" value.

As opposed to this, the two values obtained from the finite difference method with 138 and 605 mesh points, respectively, differ considerably from each other.

Since the algorithm used in the finite difference method tends to give more correct results as the mesh size becomes smaller, we have calculated the temperatures at 3986 points (mesh size 0.2) with our method using 2x37 coefficients and subjected the temperatures thus obtained for sets of points such as the ones shown in Fig.13 to a test using Eq.(55). The results are extremely accurate, as can be judged from the samples reported in Tables 5 and 6 which refer to the boundary conditions shown in Figs.7 and 10, respectively.

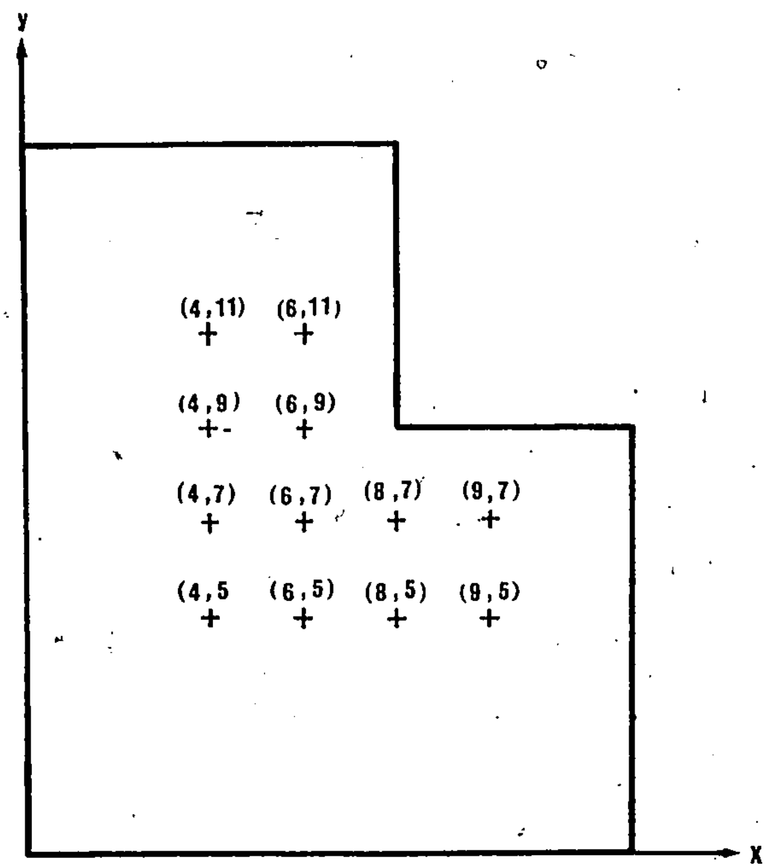


Fig. 13

Table 5. Application of Eq.(55) with Mesh Interval of 0.2 to Temperatures Calculated by the Method of Undetermined Coefficients with p = 37 for the Case Shown in Fig.7

x	y	$T_{x,y}$	$T_1 = T_{x-0.2,y}$	$T_2 = T_{x+0.2,y}$	$T_3 = T_{x,y-0.2}$	$T_4 = T_{x,y+0.2}$	$\frac{1}{4}(T_1+T_2+T_3+T_4)$
4	5	5.23060	4.96266	5.49933	5.01541	5.44500	5.23060
4	7	7.30737	6.92147	7.69514	7.10782	7.50504	7.30737
4	9	9.18816	8.65713	9.72644	9.01475	9.35439	9.18818
4	11	10.05787	9.34373	10.81220	10.09189	9.98363	10.05786
6	5	7.95456	7.67873	8.23069	7.59658	8.31222	7.95456
6	7	11.28833	10.87148	11.71652	10.99530	11.57022	11.28838
6	9	14.42078	13.97351	14.79169	13.97222	14.91818	14.41390
6	11	20.19359	18.84462	21.65002	19.76683	20.51321	20.19367
8	5	10.56698	10.34375	10.77205	9.96430	11.19188	10.567995
8	7	17.16425	16.33371	18.07004	16.56328	17.72873	17.17394
10	5	10.87902	11.08525	10.60329	10.05429	11.77339	10.87906
10	7	24.73593	24.51831	24.72377	22.67622	27.02628	24.73615

Table 6. Application of Eq.(55) with Mesh Interval of 0.2 to Temperatures Calculated by the Method of Undetermined Coefficients with p = 37 for the Case Shown in Fig.10

x	y	$T_{x,y}$	$T_1 = T_{x-0.2,y}$	$T_2 = T_{x+0.2,y}$	$T_3 = T_{x,y-0.2}$	$T_4 = T_{x,y+0.2}$	$\frac{1}{4}(T_1+T_2+T_3+T_4)$
4	5	4.23976	4.02258	4.45759	4.06534	4.41356	4.23977
4	7	5.92313	5.61033	6.23745	5.76138	6.08336	5.92313
4	9	7.44763	7.01717	7.88397	7.30707	7.5981	7.44701
4	11	8.15255	7.57372	8.76398	8.17753	8.09500	8.15256
6	5	6.44772	6.22414	6.67154	6.15755	6.73763	6.44772
6	7	9.14997	8.81209	9.49705	8.91245	9.37846	9.15001
6	9	11.68906	11.32652	11.98971	11.32546	12.03608	11.66944
6	11	16.36037	15.27024	17.53526	15.95349	16.68262	16.36040
8	5	8.56527	8.38431	8.73026	8.07675	9.07175	8.56577
8	7	13.91304	13.23974	14.59590	13.42583	14.37048	13.90799
10	5	8.82016	8.98618	8.59753	8.15105	9.54602	8.820195
10	7	20.13548	19.90721	20.16556	18.43919	22.03090	20.13572

VII. Convergence of the Method of Undetermined Coefficients

An examination of the results obtained by the method of undetermined coefficients which is reported in Tables 3 and 4 reveals that the temperature T(x,y) calculated at any given point increases with the increased number p of coefficients employed. The rate of increase, as we have pointed out, diminishes visibly, and if we were to plot a T-p curve, such a curve would be assymptotic to a horizontal line passing through the true value of T.

It may be interesting to fit curves of this kind to the tabulated data in order to obtain a feeling for what the assymptotic value in each case may be.

A hyperbola of the form

$$T = \frac{-a}{p + b} + c \qquad (56)$$

is probably the simplest curve which can be made to pass through three points while fitting the required condition

$$\left(\frac{dT}{dp}\right)_{p\to\infty} = 0 \qquad (57)$$

In fact, for $p\to\infty$ we have $T = c$.

We have calculated the values of a, b and c at each point shown in Fig.13 using the temperature values for p = 27, p = 32 and p = 37 in Tables 3 and 4. We have also checked whether the resulting hyperbolas would give a good approximation to the tabulated value of T for p = 22. This check gave surprisingly good results, thus confirming the reasonableness of our choice of curve form.

For example, for point (4,5) in Table 7, the hyperbola becomes

$$T = \frac{-3.05178}{p - 3.59389} + 5.32195$$

which for p = 22 gives T = 5.15615, agreeing within four significant figures with the tabulated value presented in Table 3, and for $p \to \infty$ gives T = 5.32195. If this last number indeed gives us a good idea of what the "true" value of T may be, then it would appear that our best approximation was within less than 2 percent off that true value. Similarly, convincing results were obtained for the other points; these results are presented in Tables 7 and 8.

Table 7. Convergence of the Method of Undetermined Coefficients for a Set of Points Shown in Fig.13 for the Boundary Conditions Presented in Fig.7

x	y	Temperature obtained using the method of undetermined coefficients				The constants obtained for the hyperbola in Eq.(56)			T for p=22 obtained from Eq.(56)
		p = 22	p = 27	p = 32	p = 37	$c=T_{p\to\infty}$	a	b	
4	5	5.15660	5.19157	5.21452	5.23060	5.32195	3.05178	-3.59389	5.15615
4	7	7.20201	7.25176	7.28445	7.30737	7.43783	4.36509	-3.54043	7.20136
4	9	9.08654	9.13198	9.16302	9.18816	9.42754	12.59403	15.61017	9.09269
4	11	9.99024	10.02211	10.04311	10.05787	10.14246	2.84665	-3.34615	9.98985
6	5	7.84528	7.89695	7.93084	7.95456	8.08893	4.47757	-3.67650	7.84456
6	7	11.09493	11.18634	11.24632	11.28833	11.52676	7.95831	-3.62215	11.09372
6	9	14.20234	14.29562	14.36263	14.42078	15.24223	62.12801	38.63205	14.21756
6	11	20.10226	20.14528	20.17365	20.19359	20.30786	3.84562	-3.34638	20.10170
8	5	10.45075	10.50408	10.54075	10.56698	10.72501	5.55084	-1.87548	10.44919
8	7	16.94099	17.03754	17.12845	17.16425	17.24656	1.35783	-20.50390	16.33898
10	5	10.80928	10.84230	10.86391	10.87902	10.96438	2.83789	-3.75385	10.80885
10	7	24.64066	24.68584	24.71533	24.73593	24.85200	3.85026	-3.82790	24.64012

Table 8. Convergence of the Method of Undetermined Coefficients for a Set of Points Shown in Fig.13 for the Boundary Conditions Presented in Fig.10

x	y	Temperature obtained using the method of undetermined coefficients				The constants obtained for the hyperbola in Eq.(56)			T for p=22 obtained from Eq.(56)
		p = 22	p = 27	p = 32	p = 37	$c=T_{p\to\infty}$	a	b	
4	5	4.17977	4.20813	4.22673	4.23976	4.31375	2.47085	-3.60682	4.17942
4	7	5.83772	5.87807	5.90455	5.92313	6.02911	3.55223	-3.48101	5.83729
4	9	7.36527	7.40210	7.42725	7.44763	7.64216	10.25659	15.72537	7.37028
4	11	8.09773	8.12358	8.14060	8.15255	8.22083	2.29224	-3.42998	8.09739
6	5	6.35913	6.40104	6.42850	6.44772	6.55660	3.62853	-3.67476	6.35860
6	7	8.99320	9.06732	9.11592	9.14997	9.34339	6.46056	-3.59794	8.99231
6	9	11.51198	11.58762	11.64191	11.68906	12.35893	50.93477	39.03641	11.52444
6	11	16.28632	16.32121	16.34419	16.36037	16.45355	3.14886	-3.20588	16.28600
8	5	8.47103	8.51429	8.54401	8.56527	8.69338	4.50061	-1.86998	8.46981
8	7	13.73190	13.81019	13.88397	13.91304	13.97991	1.10352	-20.49810	13.24517
10	5	8.76363	8.79042	8.80793	8.82016	8.88905	2.28447	-3.83712	8.76327
10	7	20.05826	20.09489	20.11879	20.13548	20.22944	3.11460	-3.85160	20.05782

VIII. A Case for Which a Closed-form Solution Exists

Feingold, [1] has shown that for a rectangular plate with temperature varying linearly along each segment of the boundary there exists a closed-form solution for the temperature field. He has shown, in fact, that the temperatures along any line x = constant or y = constant in such a plate also vary linearly.

This allows us to construct for an L-shaped plate a set of boundary conditions such that the entire temperature field be known exactly.

Consider the rectangular plate in Fig.14. According to Feingold, the temperature at a generic point (x,y) of that plate is

$$T(x,y) = T_{Max.} \frac{xy + (w-x)(h-y)}{wh}$$

If we now took the L-shaped plate used in our previous examples as being a part of a rectangular 13x15 plate, we could choose for convenience $T_{Max.} = 13(15) = 195$. The temperature at any point of the plate would then be

$$T(x,y) = xy + (13-x)(15-y) \tag{58}$$

and, in particular at the point N (Fig.15)

$$T_N = 102 \text{ deg.}$$

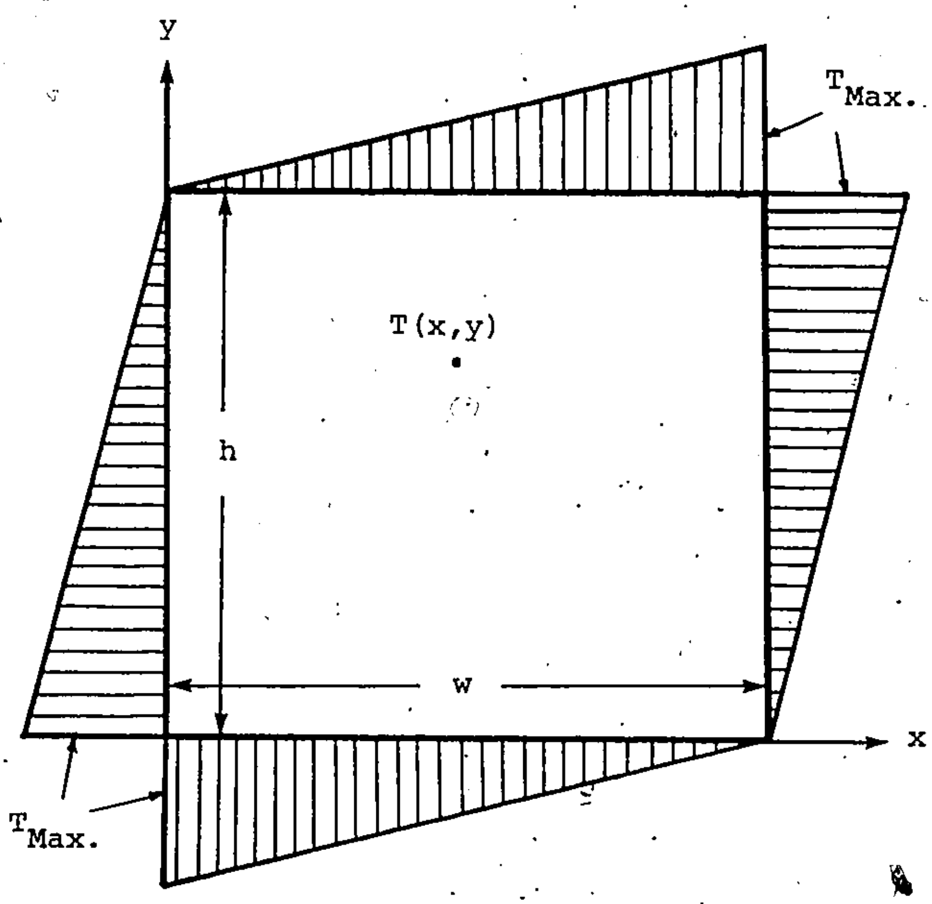


Fig. 14

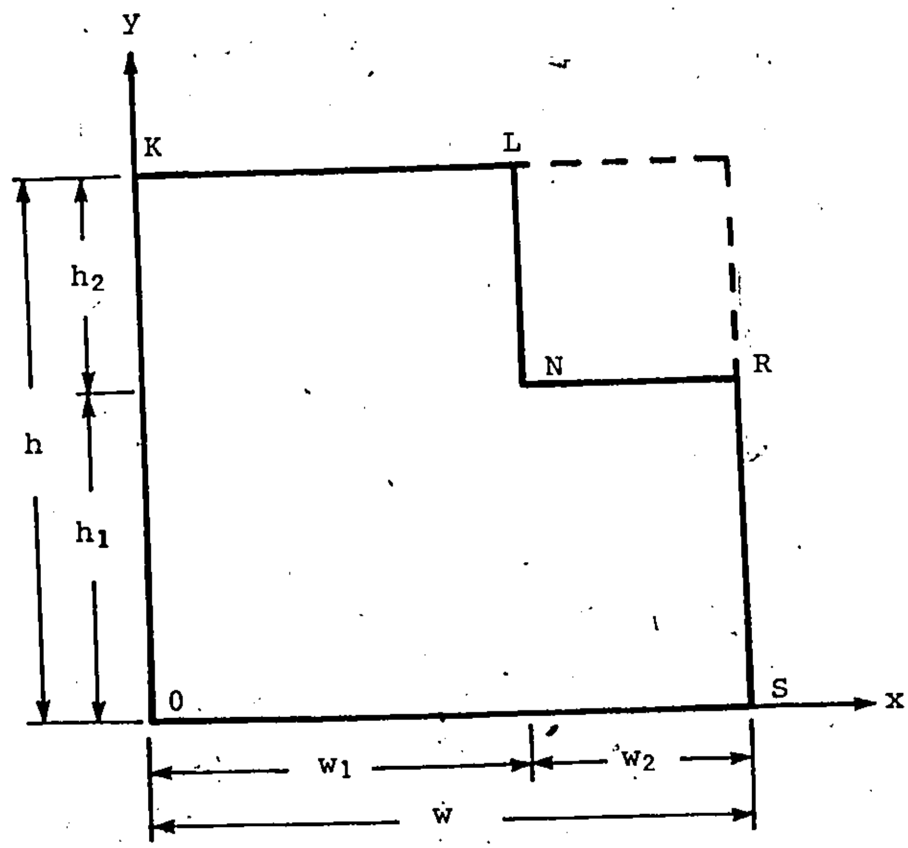


Fig. 15

On the other hand, from the original boundary conditions,

$$T_L = 195\,\frac{8}{13} = 120\ \text{deg.} \quad \text{and} \quad T_R = 195\,\frac{9}{15} = 117\ \text{deg.}$$

Bearing in mind that, as said before, the temperature variations along LN and NR are linear, we now have a set of boundary conditions for an L-shaped plate for which Eq.(58) gives the field temperatures exactly.

The object of this whole exercise is to calculate the same temperatures with the method of undetermined coefficients and see by how much will the results thus obtained differ from the exact ones.

Figure 16 represents our special boundary conditions. Figure 17 shows the three regions, A, B and C, with the appropriate temperature functions bearing the same nomenclature as in Chapter II.

The temperature in region A can be obtained by superposition of the five boundary conditions shown in Fig.18.

$$T_A = T_{A,1-1} + T_{A,1-2} + T_{A,2} + T_{A,3} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \qquad (59)$$

For $T_{A,1-1}$, we have the solution in integral form

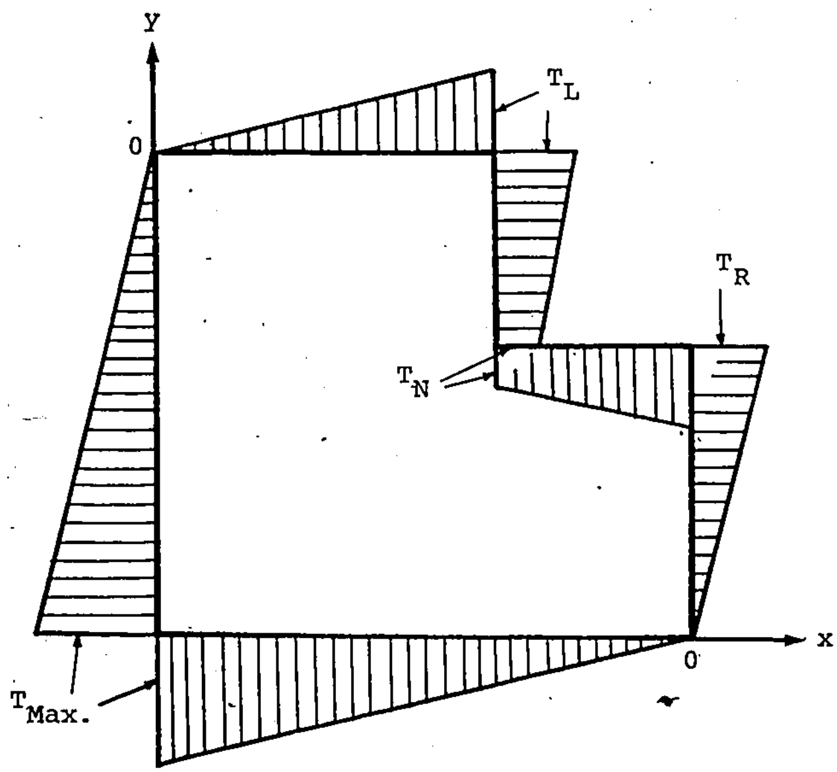
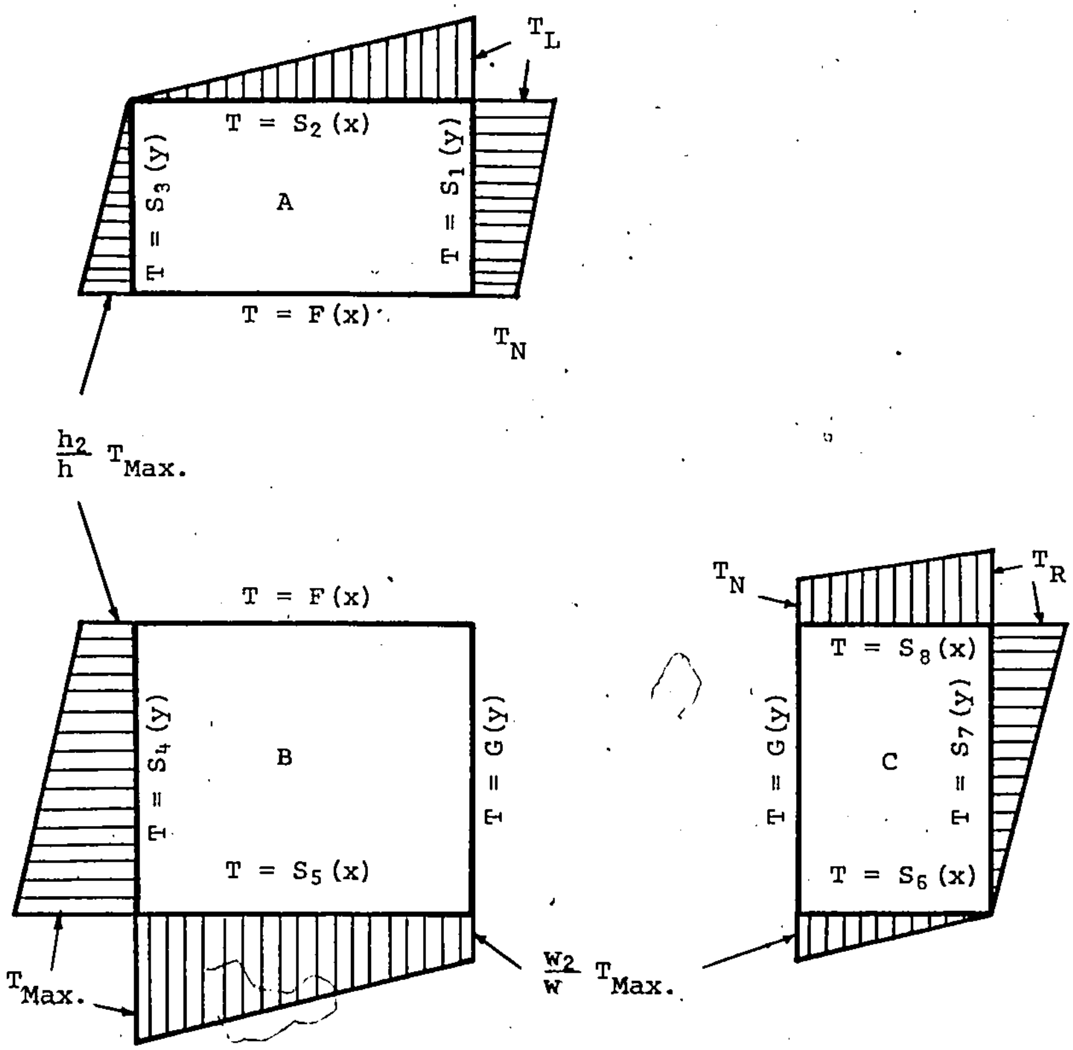


Fig. 16



T_L
T = S_2(x)
T = S_3(y)
A
T = S_1(y)
T = F(x)
T_N
h_2/h T_Max.
T = F(x)
T = S_4(y)
B
T = G(y)
T = S_5(x)
T_Max.
w_2/w T_Max.
T_N
T_R
T = S_8(x)
T = G(y)
C
T = S_7(y)
T = S_6(x)

Fig. 17

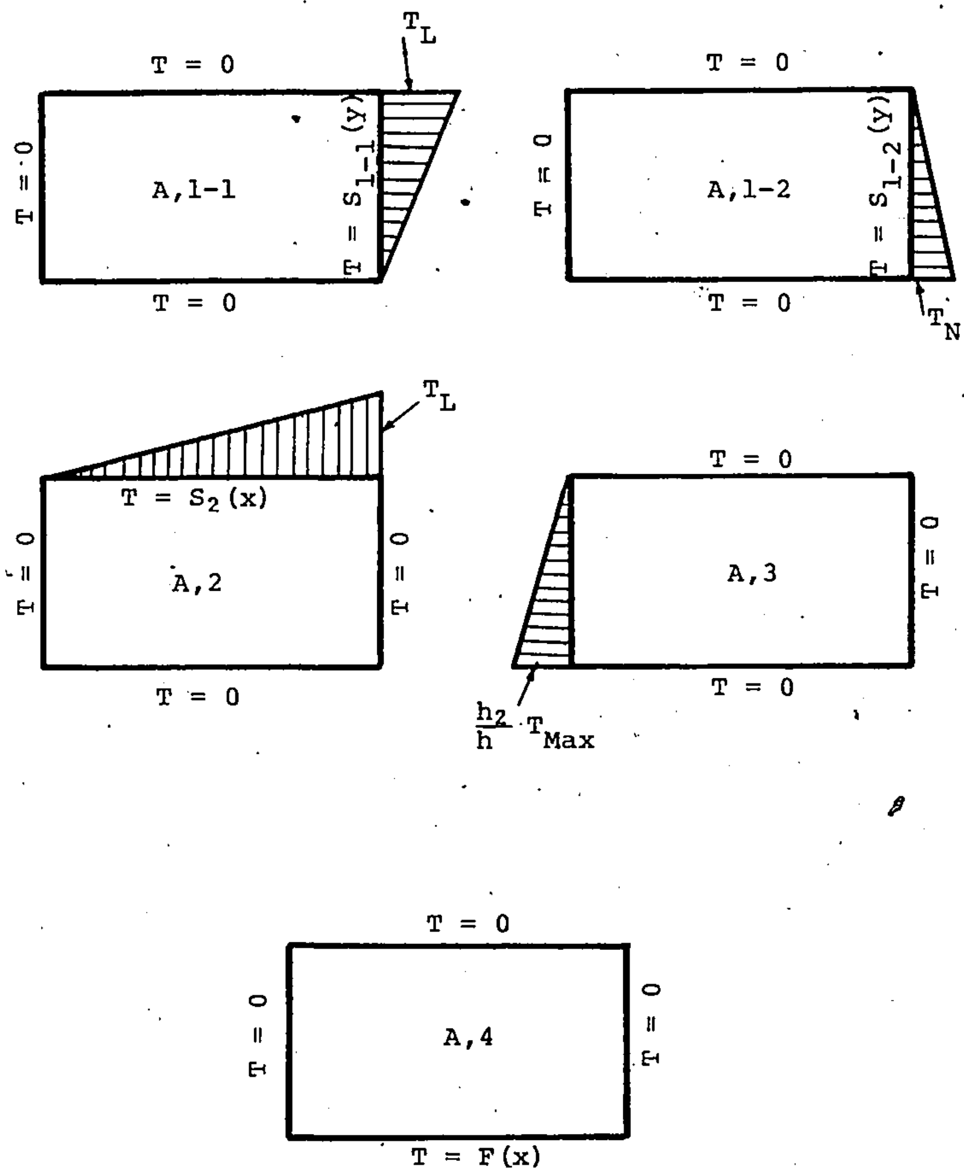

T = 0
T = 0
T = 0
T = S_{1-1}(y)
T_L
A,1-1
T = 0
T = 0
T = 0
T = S_{1-2}(y)
T_N
A,1-2
T = S_2(x)
T_L
T = 0
T = 0
T = 0
A,2
T = 0
T = 0
T = 0
A,3
$\frac{h_2}{h} T_{Max}$
T = 0
T = 0
T = 0
T = F(x)
A,4

Fig. 18

$$T_{A,1-1} = \frac{2}{h_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi (y-h_1)}{h_2} \int_{h_1}^{h} S_{1-1}(y)$$

$$\sin \frac{n\pi (y-h_1)}{h_2}\, dy \qquad (60)$$

But

$$S_{1-1}(y) = \frac{(y-h_1)}{h_2} T_L \qquad (61)$$

thus, $T_{A,1-1}$ reads

$$T_{A,1-1} = \frac{2\ T_L}{h_2^2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi (y-h_1)}{h_2} \int_{h_1}^{h} (y-h_1)$$

$$\sin \frac{n\pi (y-h_1)}{h_2}\, dy \qquad (62)$$

Setting $Y = y-h_1$ in the integral and changing the integration limits accordingly, we have

$$\int_0^{h_2} Y \sin \frac{n\pi Y}{h_2}\, dY = \left[\frac{h_2^2}{n^2\pi^2} \sin \frac{n\pi Y}{h_2} - \frac{h_2 Y}{n\pi} \cos \frac{n\pi Y}{h_2}\right]_0^{h_2},$$

$$= -\frac{h_2^2}{n\pi} \cos n\pi = -\frac{h_2^2(-1)^n}{n\pi} \qquad (63)$$

Substituting the result of the integration into Eq.(62), we get

$$T_{A,1-1} = -\frac{2\,T_L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi x}{h_2}}{n \sinh\frac{n\pi w_1}{h_2}} \sin\frac{n\pi(y-h_1)}{h_2} \tag{64}$$

Similarly,

$$S_{1-2}(y) = \frac{(h-y)}{h_2}\,T_N \tag{65}$$

which leads to

$$T_{A,1-2} = -\frac{2\,T_N}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi x}{h_2}}{n \sinh\frac{n\pi w_1}{h_2}} \sin\frac{n\pi(h-y)}{h_2} \tag{66}$$

Let us now introduce $T_{A,1}$ as the sum of $T_{A,1-1}$ and $T_{A,1-2}$ in order to adopt the procedures used in Chapter II

$$T_{A,1} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi x}{h_2}}{n \sinh\frac{n\pi w_1}{h_2}}\left(T_L \sin\frac{n\pi(y-h_1)}{h_2} + T_N \sin\frac{n\pi(h-y)}{h_2}\right) \tag{67}$$

Taking the first derivative with respect to y along the line $y = h_1$, we get

$$\left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} = -\frac{2\ T_L}{h_2} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}}$$

$$+ \frac{2\ T_N}{h_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \tag{68}$$

For $T_{A,2}$, an equation analogous to Eq. (60) is

$$T_{A,2} = \frac{2}{w_1} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi (y-h_1)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \int_0^{w_1} S_2(x) \sin \frac{n\pi x}{w_1}\, dx \tag{69}$$

But

$$S_2(x) = \frac{x}{w_1} T_L \tag{70}$$

and

$$\int_0^{w_1} x \sin \frac{n\pi x}{w_1}\, dx = -\frac{w_1^2 (-1)^n}{n\pi} \tag{71}$$

therefore

$$T_{A,2} = -\frac{2\ T_L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi (y-h_1)}{w_1}}{n \sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \tag{72}$$

Approaching the line $y = h_1$, the temperature gradient in y-direction is

$$\left(\frac{\partial T_{A,2}}{\partial y}\right)_{y=h_1} = -\frac{2\ T_L}{w_1}\sum_{n=1}^{\infty}\frac{(-1)^n \sin\frac{n\pi x}{w_1}}{\sinh\frac{n\pi h_2}{w_1}} \tag{73}$$

Going through the same steps with respect to $T_{A,3}$, we have

$$T_{A,3} = -\frac{2}{h_2}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi(w_1-x)}{h_2}}{\sinh\frac{n\pi w_1}{h_2}}\sin\frac{n\pi(h-y)}{h_2}\int_h^{h_1} S_3(y)$$

$$\sin\frac{n\pi(h-y)}{h_2}\,dy$$

$$S_3(y) = \frac{h-y}{h}\,T_{Max.}$$

$$\int_h^{h_1}(h-y)\sin\frac{n\pi(h-y)}{h_2}\,dy = \frac{h_2^2\ (-1)^n}{n\pi}$$

and, finally,

$$T_{A,3} = -\frac{2\ h_2\ T_{Max.}}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh\frac{n\pi(w_1-x)}{h_2}}{n\sinh\frac{n\pi w_1}{h_2}}\sin\frac{n\pi(h-y)}{h_2} \tag{74}$$

from which the temperature gradient in y direction at $y = h_1$ is

$$\left(\frac{\partial T_{A,3}}{\partial y}\right)_{y=h_1} = \frac{2\ T_{Max.}}{h} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(w_1-x)}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \tag{75}$$

We now have all the required expressions to be inserted into Eq.(59) and the temperature in region A finally reads

$$T_A = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \left(T_L \sin \frac{n\pi(y-h_1)}{h_2} + T_N \sin \frac{n\pi(h-y)}{h_2}\right)$$

$$- \frac{2\ T_L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(y-h_1)}{w_1}}{n \sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$- \frac{2\ h_2\ T_{Max.}}{h\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi(h-y)}{h_2}$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \tag{76}$$

Clearly, everything related to region A is equally applicable to region C with an appropriate transposition of symbols. Thus, x becomes y, h becomes w, w becomes h and T_L

becomes T_R. T_N and $T_{Max.}$ of course, remain the same. Consequently, we have (see Fig.19)

$$T_{C,1} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi y}{w_2}}{n \sinh\frac{n\pi h_1}{w_2}}\left[T_R \sin\frac{n\pi(x-w_1)}{w_2} + T_N \sin\frac{n\pi(w-x)}{w_2}\right] \tag{77}$$

$$\left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} = -\frac{2\,T_R}{w_2}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi y}{w_2}}{\sinh\frac{n\pi h_1}{w_2}} + \frac{2\,T_N}{w_2}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi y}{w_2}}{\sinh\frac{n\pi h_1}{w_2}} \tag{78}$$

$$T_{C,2} = -\frac{2\,T_R}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi(x-w_1)}{h_1}}{n \sinh\frac{n\pi w_2}{h_1}}\sin\frac{n\pi y}{h_1} \tag{79}$$

$$\left(\frac{\partial T_{C,2}}{\partial x}\right)_{x=w_1} = -\frac{2\,T_R}{h_1}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\frac{n\pi y}{h_1}}{\sinh\frac{n\pi w_2}{h_1}} \tag{80}$$

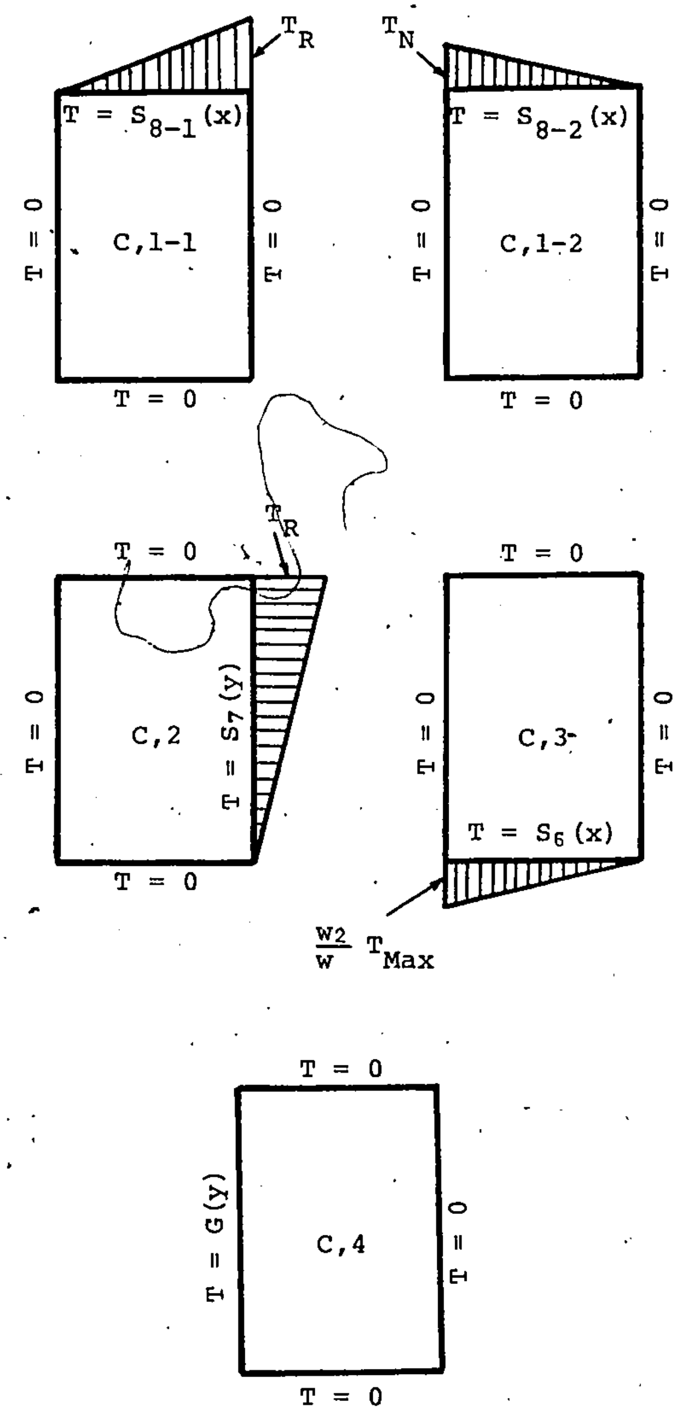


Fig. 19

$$T_{C,3} = -\frac{2\ w_2\ T_{Max.}}{w\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi(w-x)}{w_2} \tag{81}$$

$$\left(\frac{\partial T_{C,3}}{\partial x}\right)_{x=w_1} = \frac{2\ T_{Max.}}{w} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(h_1-y)}{w_2}}{\sinh \frac{n\pi h_1}{w_2}} \tag{82}$$

$$\begin{aligned} T_C = & -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi y}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \left[T_R \sin \frac{n\pi(x-w_1)}{w_2} \right. \\ & \left. + T_N \sin \frac{n\pi(w-x)}{w_2}\right] \\ & - \frac{2\ T_R}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(x-w_1)}{h_1}}{n \sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \\ & - \frac{2\ w_2\ T_{Max.}}{w\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi(w-x)}{w_2} - \\ & + \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi(w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \end{aligned} \tag{83}$$

The temperature in region B will be obtained by the superposition of six boundary conditions shown in Fig.20.

$$T_B = T_{B,1-1} + T_{B,1-2} + T_{B,2-1} + T_{B,2-2} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}}$$

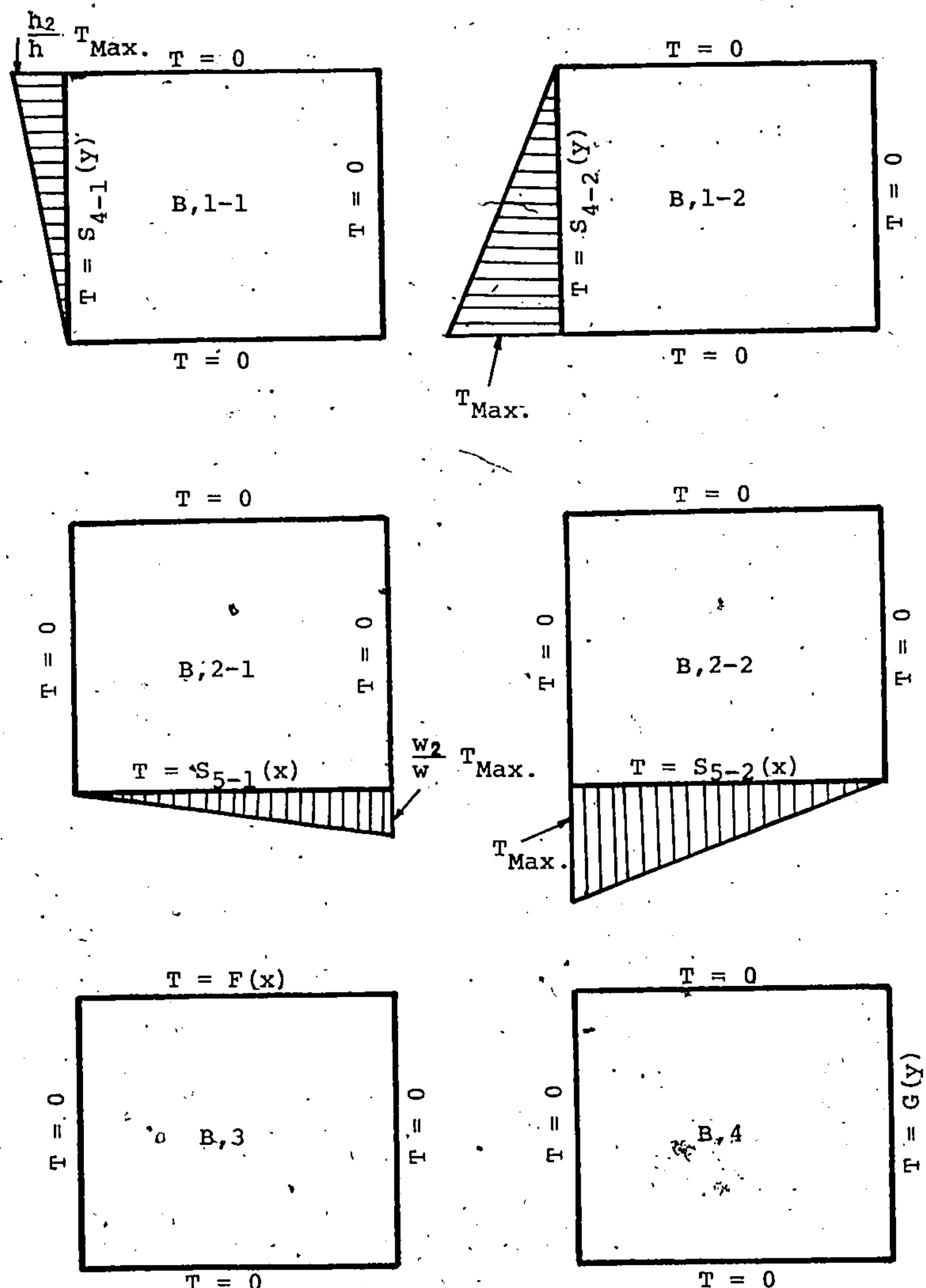
$$\sin \frac{n\pi x}{w_1} + \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (84)$$

$T_{B,1-1}$ is obtained by transforming Eq.(64) in the following manner: T_L becomes $h_2 T_{Max.}/h$, x becomes w_1-x, h_2 becomes h, and $y-h_1$ becomes y.

$$T_{B,1-1} = -\frac{2\, h_2 T_{Max.}}{h\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (85)$$

In a similar way, $T_{B,1-2}$ is obtained by transforming appropriately Eq.(66).

$$T_{B,1-2} = -\frac{2\, T_{Max.}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi(h_1-y)}{h_1} \qquad (86)$$



h2/h TMax.
T = 0
T = S4-1(y)
B,1-1
T = 0
T = 0
T = 0
T = S4-2(y)
B,1-2
T = 0
T = 0
TMax.
T = 0
T = 0
B,2-1
T = 0
T = S5-1(x)
W2/W TMax.
T = 0
T = 0
B,2-2
T = 0
T = S5-2(x)
TMax.
T = F(x)
T = 0
B,3
T = 0
T = 0
T = 0
T = 0
B,4
T = G(y)
T = 0

Fig. 20

As was done in $T_{A,1}$, we shall now introduce $T_{B,1}$ to represent the sum of $T_{B,1-1}$ and $T_{B,1-2}$:

$$T_{B,1} = -\frac{2\ h_2\ T_{Max.}}{h\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1}$$

$$- \frac{2\ T_{Max.}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi(h_1-y)}{h_1} \qquad (87)$$

The temperature $T_{B,2} = T_{B,2-1} + T_{B,2-2}$ can be calculated by a suitable transformation of Eq.(77) which refers to $T_{C,1} = T_{C,1-1} + T_{C,1-2}$. The similarity of the respective boundary conditions can be observed by comparing Figs.19 and 20. The result is

$$T_{B,2} = -\frac{2\ w_2\ T_{Max.}}{w\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_1}}{n \sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$- \frac{2\ T_{Max.}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_1}}{n \sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi(w_1-x)}{w_1} \qquad (88)$$

For the entire region B, Eqs.(84), (87) and (88) produce

$$T_B = -\frac{2\ h_2\ T_{Max.}}{h\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1}$$

$$-\frac{2\ T_{Max.}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{n \sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi(h_1-y)}{h_1}$$

$$-\frac{2\ w_2\ T_{Max.}}{w\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_1}}{n \sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$-\frac{2\ T_{Max.}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_1}}{n \sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi(w_1-x)}{w_1}$$

$$+\sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+\sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (89)$$

The derivatives of $T_{B,1}$ and $T_{B,2}$ with respect to y along $y = h_1$ are

$$\left(\frac{\partial T_{B,1}}{\partial y}\right)_{y=h_1} = -\frac{2\ h_2\ T_{Max.}}{hh_1}\sum_{n=1}^{\infty}\frac{\sinh\frac{n\pi(w_1-x)}{h_1}}{\sinh\frac{n\pi w_1}{h_1}}$$

$$+\frac{2\ T_{Max.}}{h_1}\sum_{n=1}^{\infty}\frac{(-1)^n\ \sinh\frac{n\pi(w_1-x)}{h_1}}{\sinh\frac{n\pi w_1}{h_1}} \tag{90}$$

and

$$\left(\frac{\partial T_{B,2}}{\partial y}\right)_{y=h_1} = \frac{2\ w_2\ T_{Max.}}{ww_1}\sum_{n=1}^{\infty}\frac{(-1)^n\ \sin\frac{n\pi x}{w_1}}{\sinh\frac{n\pi h_1}{w_1}}$$

$$+\frac{2\ T_{Max.}}{w_1}\sum_{n=1}^{\infty}\frac{(-1)^n\ \sin\frac{n\pi(w_1-x)}{w_1}}{\sinh\frac{n\pi h_1}{w_1}} \tag{91}$$

The derivatives of the same temperatures with respect to x at $x = w_1$ are

$$\left(\frac{\partial T_{B,1}}{\partial x}\right)_{x=w_1} = \frac{2\ h_2\ T_{Max.}}{hh_1}\sum_{n=1}^{\infty}\frac{(-1)^n\ \sin\frac{n\pi y}{h_1}}{\sinh\frac{n\pi w_1}{h_1}}$$

$$+\frac{2\ T_{Max.}}{h_1}\sum_{n=1}^{\infty}\frac{(-1)^n\ \sin\frac{n\pi(h_1-y)}{h_1}}{\sinh\frac{n\pi w_1}{h_1}} \tag{92}$$

and

$$\left(\frac{\partial T_{B,2}}{\partial x}\right)_{x=w_1} = -\frac{2 w_2 T_{Max.}}{w w_1} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(h_1-y)}{w_1}}{\sinh \frac{n\pi h_1}{w_1}}$$

$$+ \frac{2 T_{Max.}}{w_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(h_1-y)}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \quad (93)$$

Now everything is ready to produce Eqs.(20) and (21) in their final form for the particular boundary conditions considered in this chapter.

Substituting Eqs.(68), (73), (75), (90) and (91) into the left-hand side of Eq.(20), we obtain

$$\frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \, n \left(\coth \frac{n\pi h_1}{w_1} + \coth \frac{n\pi h_2}{w_1}\right) \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n \, n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} = -\frac{2 T_L}{h_2} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}}$$

$$+ \frac{2 T_N}{h_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} - \frac{2 T_L}{w_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi x}{w_1}}{\sinh \frac{n\pi h_2}{w_1}}$$

$$+ \frac{2\ T_{Max.}}{h} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(w_1-x)}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} + \frac{2\ h_2\ T_{Max.}}{hh_1} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(w_1-x)}{h_1}}{\sinh \frac{n\pi w_1}{h_1}}$$

$$- \frac{2\ T_{Max.}}{h_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi(w_1-x)}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} - \frac{2\ w_2\ T_{Max.}}{ww_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi x}{w_1}}{\sinh \frac{n\pi h_1}{w_1}}$$

$$- \frac{2\ T_{Max.}}{w_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi(w_1-x)}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \qquad (20c)$$

Similarly, from Eqs.(21), (78), (80), (82), (92) and (93), we obtain

$$\frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n\ n\ (\coth \frac{n\pi w_1}{h_1} + \coth \frac{n\pi w_2}{h_1}) \sin \frac{n\pi y}{h_1}$$

$$+ \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n \frac{(-1)^n\ n \sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} = \frac{2\ T_{Max.}}{w} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi(h_1-y)}{w_2}}{\sinh \frac{n\pi h_1}{w_2}}$$

$$- \frac{2\ T_R}{h_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi y}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} - \frac{2\ T_R}{w_2} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi y}{w_2}}{\sinh \frac{n\pi h_1}{w_2}}$$

$$+ \frac{2\ T_N}{w_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi y}{w_2}}{\sinh \frac{n\pi h_1}{w_2}} - \frac{2\ h_2\ T_{Max.}}{hh_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi y}{h_1}}{\sinh \frac{n\pi w_1}{h_1}}$$

$$- \frac{2\ T_{Max.}}{h_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi (h_1-y)}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} + \frac{2\ w_2\ T_{Max.}}{ww_1} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi (h_1-y)}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} -$$

$$- \frac{2\ T_{Max.}}{w_1} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \frac{n\pi (h_1-y)}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \qquad (21c)$$

Equations (20c) and (21c) are again written for different values of x and y, respectively, in the manner already described for other boundary conditions, and the resulting linear system is solved for the coefficients A_n and C_n which are listed in Table 9.

With these coefficients, we use Eqs.(76), (83) and (89) to evaluate the temperatures in the different regions of the plate. It remains now to compare these values with the exact ones obtained from Eq.(58). This comparison is presented in Table 10.

It can be seen that, even in the case of diverse boundary conditions being applied to all segments of the boundary, the results provide a good practical approximation in all regions with the accuracy deteriorating mainly in the vicinity of the inner corner N (see Fig.15). This is not an important defect, because in the temperature map the correct end of the isothermal lines at the boundary is known and each one can be faired between that known point and the other more distant from the boundary thus providing a degree of correction to the least accurate values. Remains perhaps to repeat that we were limited to 2x37 undetermined coefficients and with a larger number the results would be more accurate.

Figure 21 represents the temperature map for this case.

Table 9. The Coefficients Obtained Using Eqs.(20c) and (21c) for the Bounday Conditions Shown in Fig.16 with $T_L = 120°$, $T_N = 102°$, $T_R = 117°$ and $T_{Max.} = 195°$

n	A_n	C_n
1	1.07489×10^2	1.06350×10^2
2	-4.57405×10^0	-5.67924×10^0
3	3.26442×10	3.24361×10
4	-1.67022×10^0	-2.21464×10^0
5	1.79936×10	1.79794×10
6	-8.53148×10^{-1}	-1.19810×10^0
7	1.18163×10	1.18712×10
8	-5.03976×10^{-1}	-7.44629×10^{-1}
9	8.43635×10^0	8.52086×10^0
10	-3.24199×10^{-1}	-5.00548×10^{-1}
11	6.31821×10^0	6.41440×10^0
12	-2.20670×10^{-1}	-3.53922×10^{-1}
13	4.87411×10^0	4.97304×10^0
14	-1.56669×10^{-1}	-2.59087×10^{-1}
15	3.83170×10^0	3.92810×10^0
16	-1.14602×10^{-1}	-1.94440×10^{-1}
17	3.04736×10^0	3.13803×10^0
18	-8.57728×10^{-2}	-1.48432×10^{-1}
19	2.43827×10^0	2.52123×10^0
20	-6.51890×10^{-2}	-1.14605×10^{-1}
21	1.95313×10^0	2.02729×10^0
22	-5.00636×10^{-2}	-8.89216×10^{-2}
23	1.55884×10^0	1.62359×10^0
24	-3.84359×10^{-2}	-6.89527×10^{-2}
25	1.23278×10^0	1.28788×10^0
26	-2.94207×10^{-2}	-5.30635×10^{-2}
27	9.59078×10^{-1}	1.00457×10^0
28	-2.21028×10^{-2}	-4.01193×10^{-2}
29	7.26319×10^{-1}	7.62423×10^{-1}
30	-1.62259×10^{-2}	-2.94400×10^{-2}
31	5.25957×10^{-1}	5.53108×10^{-1}
32	-1.11723×10^{-2}	-2.04222×10^{-2}
33	3.51694×10^{-1}	3.70316×10^{-1}
34	-6.94545×10^{-3}	-1.26749×10^{-2}
35	1.98442×10^{-1}	2.09181×10^{-1}
36	-3.25761×10^{-3}	-5.93356×10^{-3}
37	6.25125×10^{-2}	6.58919×10^{-2}

Table 10. Comparison of the Result of Temperature Calculations for the Case Shown in Fig.16 with $T_L = 120°$, $T_N = 102°$, $T_R = 117°$ and $T_{Max.} = 195°$

x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq.(58) (Exact solution)	x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq.(58) (Exact solution)
1	1	168.85205	169.0	3	1	142.54997	143.0
1	2	157.70453	158.0	3	2	135.11401	136.0
1	3	146.55519	147.0	3	3	127.68951	129.0
1	4	135.39815	136.0	3	4	120.25926	122.0
1	5	124.21880	125.0	3	5	112.80721	115.0
1	6	112.97812	114.0	3	6	105.33347	108.0
1	7	101.54015	103.0	3	7	97.87494	101.0
1	8	89.36879	92.0	3	8	90.54515	94.0
1	9	76.30739	81.0	3	9	83.54427	87.0
1	10	67.49457	70.0	3	10	77.00334	80.0
1	11	57.77542	59.0	3	11	70.70314	73.0
1	12	47.29680	48.0	3	12	64.40602	66.0
1	13	36.59436	37.0	3	13	58.01477	59.0
1	14	25.81166	26.0	3	14	51.53256	52.0
2	1	155.70331	156.0	4	1	129.37903	130.0
2	2	146.41039	147.0	4	2	123.80370	125.0
2	3	137.11815	138.0	4	3	118.26666	120.0
2	4	127.81622	129.0	4	4	112.72685	115.0
2	5	118.48799	120.0	4	5	107.15359	110.0
2	6	109.10565	111.0	4	6	101.55800	105.0
2	7	99.61885	102.0	4	7	96.02473	100.0
2	8	90.01198	93.0	4	8	90.73233	95.0
2	9	80.74802	84.0	4	9	85.87602	90.0
2	10	72.28162	75.0	4	10	81.47820	85.0
2	11	64.11708	66.0	4	11	77.31834	80.0
2	12	55.76993	57.0	4	12	73.15195	75.0
2	13	47.25835	48.0	4	13	68.86732	70.0
2	14	38.65089	39.0	4	14	64.46608	65.0

Table 10 (Continued)

x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq.(58) (Exact solution)	x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq.(58) (Exact solution)
5	1	116.14990	117.0	7	1	88.55214	91.0
5	2	112.44943	114.0	7	2	89.39673	92.0
5	3	108.84872	111.0	7	3	90.15820	93.0
5	4	105.23457	108.0	7	4	90.63733	94.0
5	5	101.53456	105.0	7	5	90.79298	95.0
5	6	97.73793	102.0	7	6	90.48575	96.0
5	7	93.94232	99.0	7	7	89.36610	97.0
5	8	90.44881	96.0	7	8	86.83311	98.0
5	9	87.68501	93.0	7	9	86.43167	99.0
5	10	85.71373	90.0	7	10	94.45439	100.0
5	11	83.96841	87.0	7	11	98.66953	101.0
5	12	82.03542	84.0	7	12	100.84891	102.0
5	13	79.84338	81.0	7	13	102.40689	103.0
5	14	77.46638	78.0	7	14	103.74243	104.0
6	1	102.72200	104.0	8	1	73.50809	78.0
6	2	100.99068	103.0	8	2	78.02614	81.0
6	3	99.45305	102.0	8	3	81.12558	84.0
6	4	97.83737	101.0	8	4	83.74281	87.0
6	5	96.02823	100.0	8	5	85.98657	90.0
6	6	93.95236	99.0	8	6	87.74156	93.0
6	7	91.60114	98.0	8	7	88.55339	96.0
6	8	89.35432	97.0	8	8	86.38310	99.0
6	9	88.63664	96.0	9	1	62.66023	65.0
6	10	89.76736	95.0	9	2	67.62115	70.0
6	11	90.87437	94.0	9	3	72.52016	75.0
6	12	91.19684	93.0	9	4	77.18184	80.0
6	13	91.00513	92.0	9	5	81.61700	85.0
6	14	90.55522	91.0	9	6	85.81575	90.0

Table 10 (Continued)

x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq. (58) (Exact solution	x	y	T obtained using the method of undetermined coefficient with p=37	T obtained using Eq. (58) (Exact solution
9	7	89.82045	95.0	11	4	64.57333	66.0
9	8	94.44212	100.0	11	5	73.37819	75.0
10	1	50.92998	52.0	11	6	82.25995	84.0
10	2	57.41989	59.0	11	7	91.37343	93.0
10	3	64.13560	66.0	11	8	100.94495	102.0
10	4	70.83989	73.0	12	1	25.75281	26.0
10	5	77.48204	80.0	12	2	36.55489	37.0
10	6	84.11739	87.0	12	3	47.40990	48.0
10	7	90.98050	94.0	12	4	58.29800	59.0
10	8	98.71835	101.0	12	5	69.21477	70.0
11	1	38.44601	39.0	12	6	80.18958	81.0
11	2	47.05345	48.0	12	7	91.28581	92.0
11	3	55.79008	57.0	12	8	102.56490	103.0

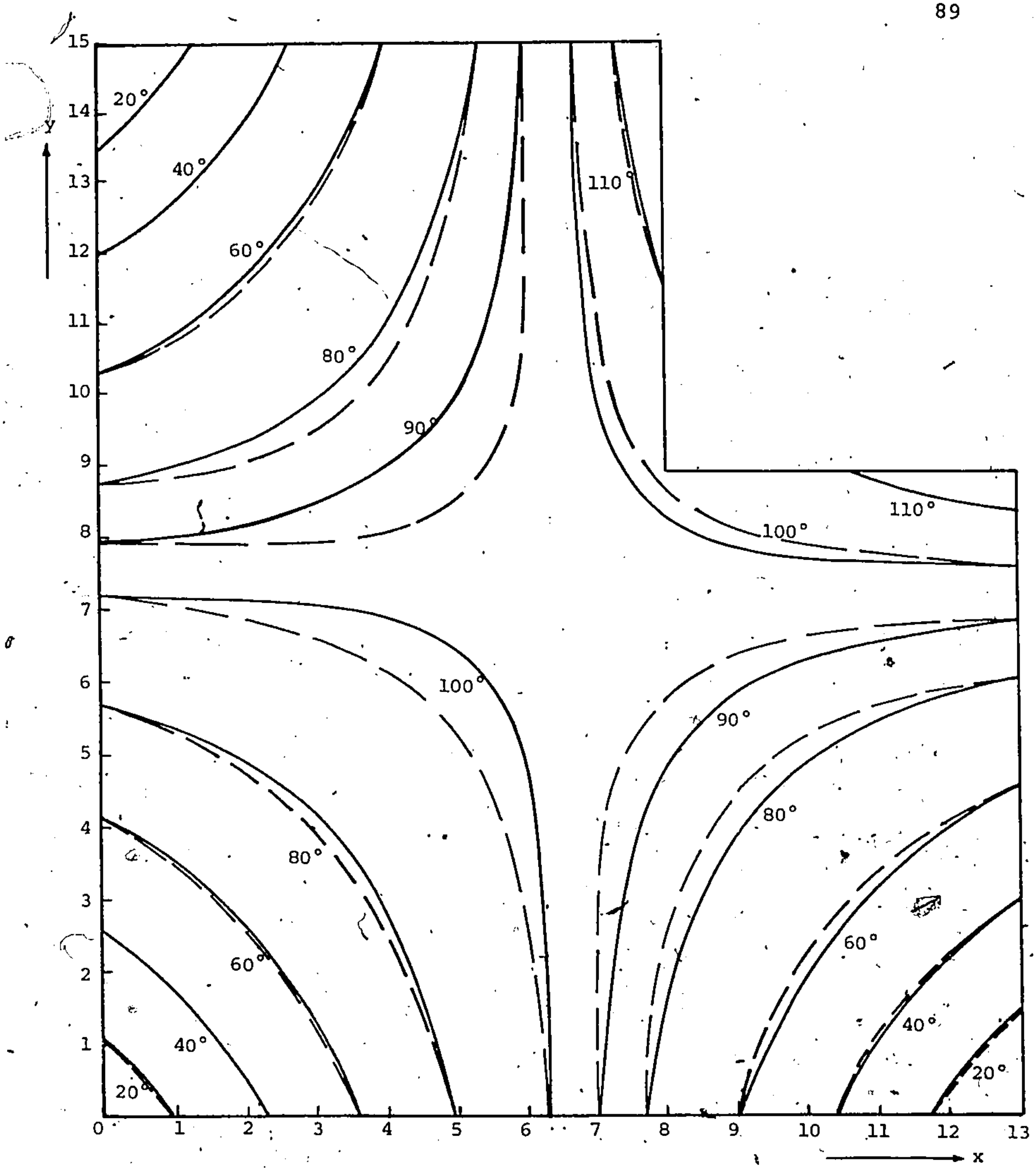


Fig. 21 The Family of Isothermal Curves for the Boundary Conditions Shown in Fig. 16, with $T_L = 120°$, $T_N = 102°$, $T_R = 117°$ and $T_{Max.} = 195°$

——— Exact solution obtained from Eq. (58)

- - - - - - Solution obtained by the method of undetermined coefficients

IX. Practical Application

We shall now develop an interesting corollary which will extend the direct application of our study of L-shaped plates to a symmetrical hollow rectangle which occurs in a variety of practical situations. It may, e.g., represent a section through a thick-walled furnace.

Consider the shaded area in Fig.22. The temperature distribution along the outer and the inner boundary of this area is arbitrary but symmetrical with respect to x' and y' axes. In order to determine the temperature distribution throughout the entire area, it is enough to study one quarter of it. We shall however, examine the L-shaped region bounded by the heavy line ABCDEF. The reason for so doing will soon become obvious. The temperatures along the lines AB, AF, DC and DE are given, while those along BC and EF are, because of the symmetry, the same as those expressed by our Fourier series along MD and DN respectively. There is, thus, no difficulty in applying the method of undetermined coefficients with the appropriate modifications to accommodate these particular boundary conditions. We have

$$T_{A,2} = \frac{2}{w_1} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi (y-h_1)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{m\pi x}{w_1} \int_0^{w_1} S_2(x) \sin \frac{m\pi x}{w_1} \, dx \tag{94}$$

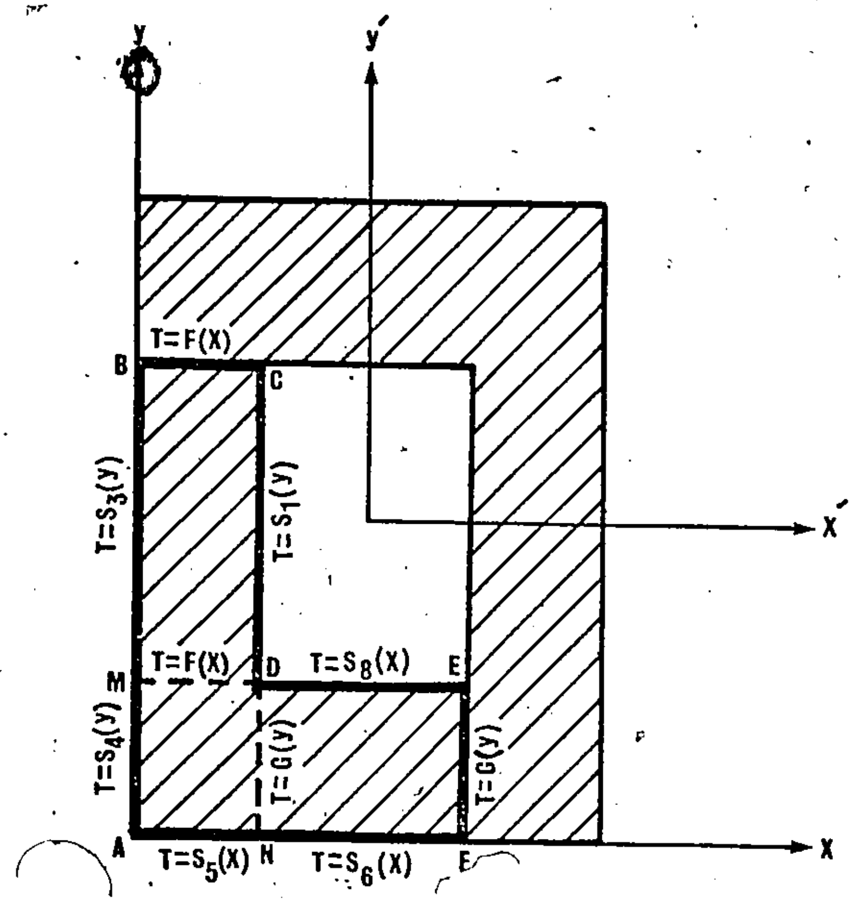


Fig. 22

but

$$S_2(x) = F(x)$$

$$F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{w_1} \quad (2)$$

and

$$\int_0^{w_1} \sin \frac{n\pi x}{w_1} \sin \frac{m\pi x}{w_1}\, dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{w_1}{2} & \text{when } m = n \end{cases}$$

therefore

$$T_{A,2} = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi(y-h_1)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \quad (95)$$

and the temperature gradient in y direction at $y = h_1$ is

$$\left(\frac{\partial T_{A,2}}{\partial y}\right)_{y=h_1} = \frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n\, n \frac{\sin \frac{n\pi x}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \quad (96)$$

Equations (6) and (95) yield

$$T_A = T_{A,1} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi (y-h_1)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} + T_{A,3}$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi (h-y)}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \sin \frac{n\pi x}{w_1} \qquad (97)$$

Having in mind that

$$S_7(y) = G(y)$$

and

$$G(y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{h_1} \qquad (10) \text{ rep.}$$

we derive $T_{C,2}$ in a manner similar to what we did for $T_{A,2}$, but using an appropriate transformation of coordinates:

$$T_{C,2} = \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi (x-w_1)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (98)$$

The temperature gradient in x direction approaching the line $x = w_1$ is

$$\left(\frac{\partial T_{C,2}}{\partial x}\right)_{x=w_1} = \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n\, n \frac{\sin \frac{n\pi y}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \qquad (99)$$

Combining Eqs. (13) and (98), we have

$$T_C = T_{C,1} + \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi(x-w_1)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} + T_{C,3}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi(w-x)}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \sin \frac{n\pi y}{h_1} \quad (100)$$

For region B, recall Eq. (15)

$$T_B = T_{B,1} + T_{B,2} + \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1}$$

Substituting Eq. (96) into Eq. (20) and rearranging, we get

$$\frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n\, n \left(\coth \frac{n\pi h_1}{w_1} + \coth \frac{n\pi h_2}{w_1} - \frac{1}{\sinh \frac{n\pi h_2}{w_1}}\right) \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n\, n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} = \left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} + \left(\frac{\partial T_{A,3}}{\partial y}\right)_{y=h_1}$$

$$- \left(\frac{\partial T_{B,1}}{\partial y}\right)_{y=h_1} - \left(\frac{\partial T_{B,2}}{\partial y}\right)_{y=h_1} \quad (20d)$$

In a similar manner, Eqs.(20) and (99) give

$$\frac{\pi}{h_1}\sum_{n=1}^{\infty} C_n\, n\left(\coth\frac{n\pi w_1}{h_1} + \coth\frac{n\pi w_2}{h_1} - \frac{1}{\sinh\frac{n\pi w_2}{h_1}}\right)\sin\frac{n\pi y}{h_1}$$

$$+ \frac{\pi}{w_1}\sum_{n=1}^{\infty} A_n \frac{(-1)^n\, n \sinh\frac{n\pi y}{w_1}}{\sinh\frac{n\pi h_1}{w_1}} = \left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} + \left(\frac{\partial T_{C,3}}{\partial x}\right)_{x=w_1}$$

$$- \left(\frac{\partial T_{B,1}}{\partial x}\right)_{x=w_1} - \left(\frac{\partial T_{B,2}}{\partial x}\right)_{x=w_1} \qquad (21d)$$

Equations (20d) and (21d) are now ready for the application of the method of undetermined coefficients.

As an example, we shall consider a simple case of the outer surface temperature, $T_{o,s}$, being zero, while the inner surface temperature, $T_{i,s}$, is maintained at a constant value. Thus,

$$T_{A,3} = T_{B,1} = T_{B,2} = T_{C,3} = 0 \qquad (101)$$

and, therefore, all the derivatives of these temperatures are also zero.

Accordingly, all that is now required are the expressions for $T_{A,1}$, $T_{C,1}$, $(\partial T_{A,1}/\partial y)_{y=h_1}$ and $(\partial T_{C,1}/\partial x)_{x=w_1}$.

In this case

$$T_{A,1} = \frac{2}{h_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi (y-h_1)}{h_2} \int_{h_1}^{h} T_{i,s} \sin \frac{n\pi (y-h_1)}{h_2}\, dy$$

$T_{i,s}$ being constant, this reduces to

$$T_{A,1} = \frac{2\ T_{i,s}}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n) \sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi (y-h_1)}{h_2}$$

but $$(1-(-1)^n) = \begin{cases} 0 & \text{when n is even} \\ 2 & \text{when n is odd} \end{cases}$$

therefore,

$$T_{A,1} = \frac{4\ T_{i,s}}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi (y-h_1)}{h_2} \qquad (102)$$

and the temperature gradient in y direction along $y = h_1$ is

$$\left(\frac{\partial T_{A,1}}{\partial y}\right)_{y=h_1} = \frac{4\ T_{i,s}}{h_2} \sum_{n=1,3,5...}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}} \qquad (103)$$

For the temperature distribution in region A, Eqs. (97), (101) and (102) give

$$T_A = \frac{4\ T_{i,s}}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{n \sinh \frac{n\pi w_1}{h_2}} \sin \frac{n\pi(y-h_1)}{h_2}$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi x}{w_1}}{\sinh \frac{n\pi h_2}{w_1}} \left(\sinh \frac{n\pi(y-h_1)}{w_1} + \sinh \frac{n\pi(h-y)}{w_1}\right) \quad (104)$$

Similarly,

$$T_{C,1} = \frac{2}{w_2} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi y}{w_2}}{\sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi(x-w_1)}{w_2} \int_{w_1}^{w} T_{i,s} \sin \frac{n\pi(x-w_1)}{w_2}\, dx$$

Because in our case $T_{i,s}$ is constant, this becomes

$$T_{C,1} = \frac{4\ T_{i,s}}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sinh \frac{n\pi y}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi(x-w_1)}{w_2} \quad (105)$$

and the required derivative is

$$\left(\frac{\partial T_{C,1}}{\partial x}\right)_{x=w_1} = \frac{4\ T_{i,s}}{w_2} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sinh \frac{n\pi y}{w_2}}{\sinh \frac{n\pi h_1}{w_2}} \quad (106)$$

The temperature distribution in the entire region C is obtained from Eqs. (100), (101) and (105):

$$T_C = \frac{4\,T_{i,s}}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{\sinh \frac{n\pi y}{w_2}}{n \sinh \frac{n\pi h_1}{w_2}} \sin \frac{n\pi (x-w_1)}{w_2}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sin \frac{n\pi y}{h_1}}{\sinh \frac{n\pi w_2}{h_1}} \left(\sinh \frac{n\pi (x-w_1)}{h_1} + \sinh \frac{n\pi (w-x)}{h_1}\right) \qquad (107)$$

Equations (15) and (101) give the temperature T_B:

$$T_B = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{w_1}}{\sinh \frac{n\pi h_1}{w_1}} \sin \frac{n\pi x}{w_1}$$

$$+ \sum_{n=1}^{\infty} C_n \frac{\sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} \sin \frac{n\pi y}{h_1} \qquad (108)$$

Returning to Eqs. (20d) and (21d), we now insert the appropriate expressions for the partial derivatives to obtain

$$\frac{\pi}{w_1} \sum_{n=1}^{\infty} A_n\, n \left(\coth \frac{n\pi h_1}{w_1} + \coth \frac{n\pi h_2}{w_1} - \frac{1}{\sinh \frac{n\pi h_2}{w_1}}\right) \sin \frac{n\pi x}{w_1}$$

$$+ \frac{\pi}{h_1} \sum_{n=1}^{\infty} C_n \frac{(-1)^n\, n \sinh \frac{n\pi x}{h_1}}{\sinh \frac{n\pi w_1}{h_1}} = \frac{4\,T_{i,s}}{h_2} \sum_{n=1,3,5...}^{\infty} \frac{\sinh \frac{n\pi x}{h_2}}{\sinh \frac{n\pi w_1}{h_2}}$$

(20e)

and

$$\frac{\pi}{h_1}\sum_{n=1}^{\infty} C_n\, n \left(\coth\frac{n\pi w_1}{h_1} + \coth\frac{n\pi w_2}{h_1} - \frac{1}{\sinh\frac{n\pi w_2}{h_1}}\right)\sin\frac{n\pi y}{h_1}$$

$$+\frac{\pi}{w_1}\sum_{n=1}^{\infty} A_n \frac{(-1)^n\, n \sinh\frac{n\pi y}{w_1}}{\sinh\frac{n\pi h_1}{w_1}} = \frac{4\, T_{i,s}}{w_2}\sum_{n=1,3,5\ldots}^{\infty}\frac{\sinh\frac{n\pi y}{w_2}}{\sinh\frac{n\pi h_1}{w_2}} \tag{21e}$$

The calculations were performed for $T_{o,s} = 0$ and $T_{i,s} =$ 200 degrees using Eqs.(20e) and (21e). The dimensions of one quarter of the furnace cross-section are the same as those used in previous examples. The resulting 2x37 coefficients are presented in Table 11. The temperatures in various regions were calculated from Eqs.(104), (107) and (108). The temperature map for this case is shown in Fig.23.

Table 11. The Coefficients Obtained Using Eqs.(20e) and (21e) for a Furnace Cross-section with $T_{o,s}$ = 0° and $T_{i,s}$ = 200°

n	A_n	C_n
1	8.99030×10	8.67995×10
2	-4.57968×10	-4.57522×10
3	2.96223×10	2.99467×10
4	-2.13166×10	-2.17418×10
5	1.62897×10	1.67451×10
6	-1.29363×10	-1.31961×10
7	1.05510×10	1.10030×10
8	-8.77490×10^{0}	-9.21317×10^{0}
9	7.40655×10^{0}	7.82782×10^{0}
10	-6.32426×10^{0}	-6.72652×10^{0}
11	5.44990×10^{0}	5.83223×10^{0}
12	-4.73122×10^{0}	-5.09327×10^{0}
13	4.13220×10^{0}	4.47366×10^{0}
14	-3.62669×10^{0}	-3.94771×10^{0}
15	3.19569×10^{0}	3.49664×10^{0}
16	-2.82489×10^{0}	-3.10609×10^{0}
17	2.50129×10^{0}	2.76532×10^{0}
18	-2.22239×10^{0}	-2.46580×10^{0}
19	1.97550×10^{0}	2.20089×10^{0}
20	-1.75717×10^{0}	-1.96523×10^{0}
21	1.56307×10^{0}	1.75442×10^{0}
22	-1.38973×10^{0}	-1.56505×10^{0}
23	1.23405×10^{0}	1.39412×10^{0}
24	-1.09377×10^{0}	-1.23919×10^{0}
25	9.66810×10^{-1}	1.09821×10^{0}
26	-8.51428×10^{-1}	-9.69504×10^{-1}
27	7.46204×10^{-1}	8.51512×10^{-1}
28	-6.49853×10^{-1}	-7.43051×10^{-1}
29	5.61393×10^{-1}	6.43016×10^{-1}
30	-4.79861×10^{-1}	-5.50502×10^{-1}
31	4.04432×10^{-1}	4.64615×10^{-1}
32	-3.34441×10^{-1}	-3.84692×10^{-1}
33	2.69399×10^{-1}	3.10129×10^{-1}
34	-2.08575×10^{-1}	-2.40337×10^{-1}
35	1.51607×10^{-1}	1.74825×10^{-1}
36	-9.81122×10^{-2}	-1.13185×10^{-1}
37	4.76702×10^{-2}	5.50178×10^{-2}

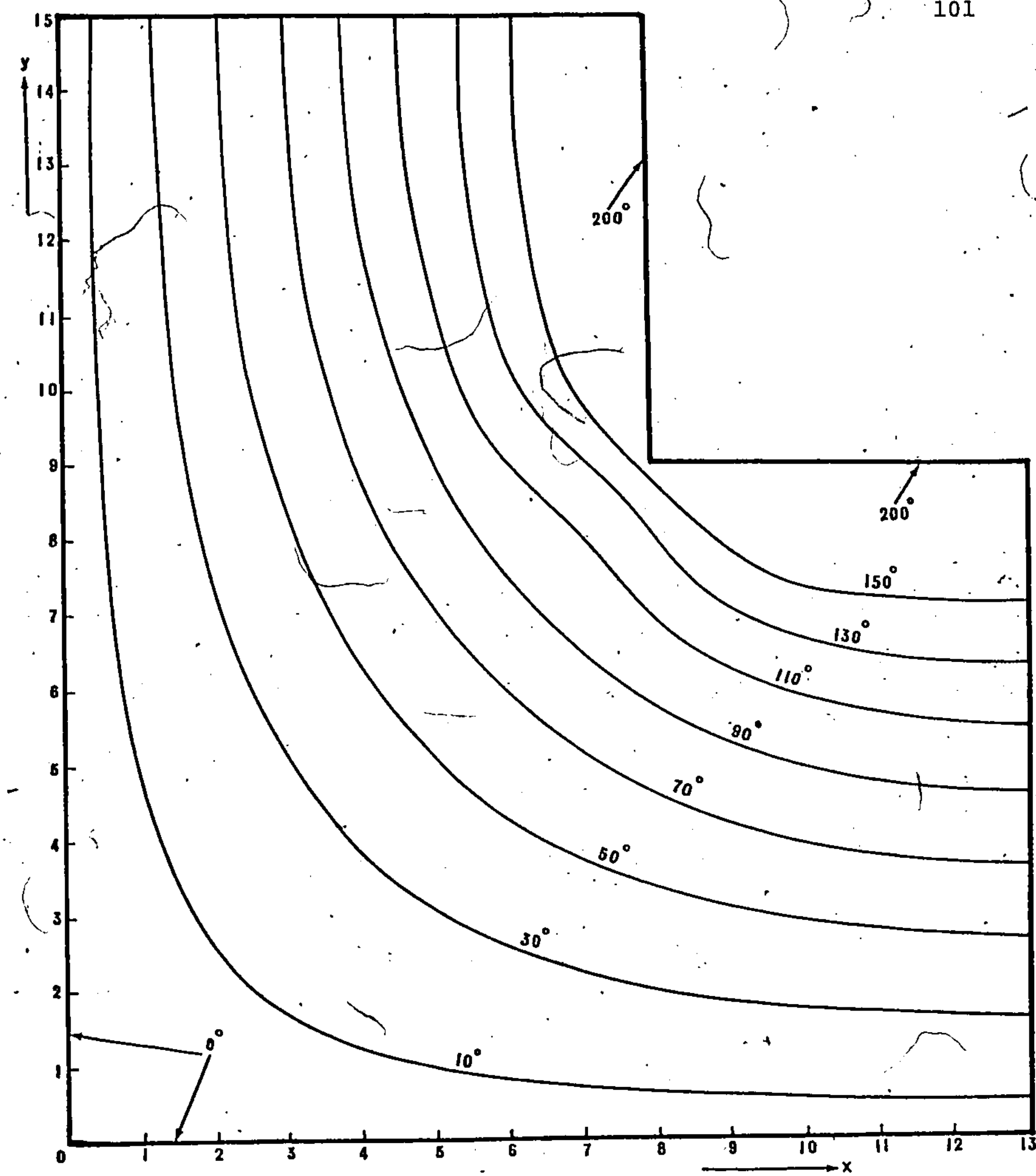


Fig. 23 The Temperature Map for One Quarter of the Furnace Cross-section with $T_{o,s} = 0°$ and $T_{i,s} = 200°$

X. Conclusion

It should be pointed out that the new method is not limited to L-shaped plates. It can, indeed, within obvious practical limitations, be applied to any area divisible into rectangles. Thus, for instance, the area shown in Fig.24 can be divided into five rectangles by means of four separation lines, along which we have four temperature functions expressible in the form of Fourier series with undetermined coefficients. The rest of the process is identical with the one employed for L-shaped plates, the only difference being that we shall end with a system of 4p rather than 2p equations.

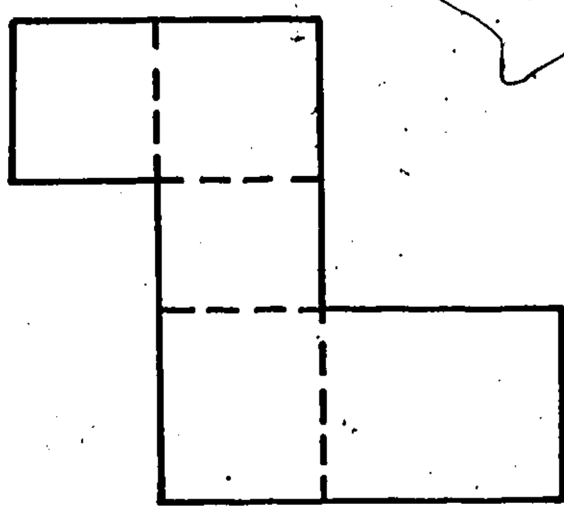


Fig. 24

XI. Appendices

A.1 Consider a thin adiabatic plate shown in Fig.25 with the boundary conditions

$$
\begin{aligned}
T &= 0 && \text{at } x = 0 \\
T &= 0 && \text{at } x = a \\
T &= 0 && \text{at } y = 0 \\
T &= F(x) && \text{at } y = b
\end{aligned}
\tag{109}
$$

The solution T(x,y) must satisfy the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{110}$$

and the four boundary conditions [4].

Assume the existence of a product solution

$$T = XY$$

where X is a function of x alone and Y is a function of y. Laplace equation becomes

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Separating the variables, we get

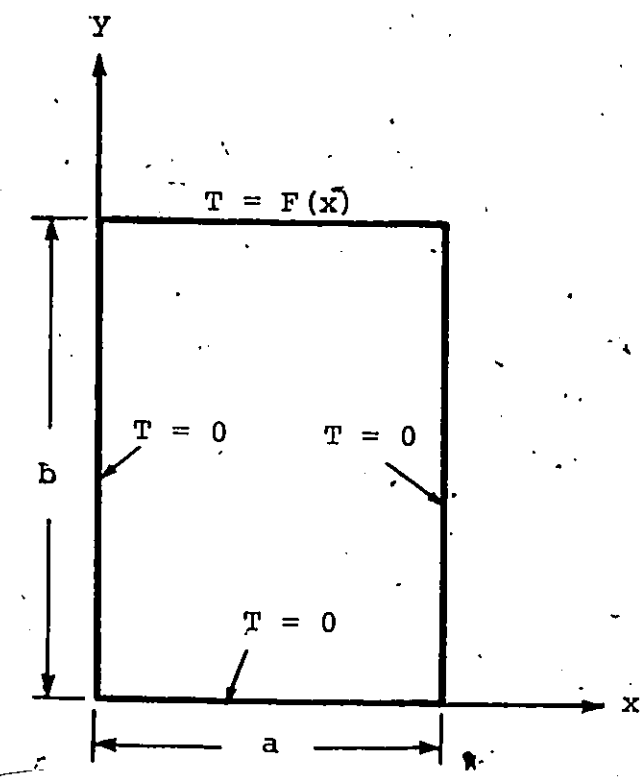


Fig. 25

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \mu \tag{111}$$

where μ is the separation constant.

Three cases are possible:

i. $\mu = \lambda^2 > 0$ leading to

$$X = c_1 \sinh\lambda x + c_2 \cosh\lambda x$$
$$Y = c_3 \sin\lambda y + c_4 \cos\lambda y \tag{112}$$

ii. $\mu = \lambda^2 < 0$ leading to

$$X = c_1 \sin\lambda x + c_2 \cos\lambda x$$
$$Y = c_3 \sinh\lambda y + c_4 \cosh\lambda y \tag{113}$$

iii. $\mu = \lambda^2 = 0$ for which the general solution is in the form of

$$X = c_1 + c_2 x$$
$$Y = c_3 + c_4 y \tag{114}$$

It is clear that the solution (112) and (114) cannot satisfy the conditions shown in Fig.25 so the product solution in this case can only be

$$T = (c_1 \sin\lambda x + c_2 \cos\lambda x)(c_3 \sinh\lambda y + c_4 \cosh\lambda y)$$

Applying the first boundary condition, we get

$$c_2 = 0$$

and the third boundary condition yields

$$c_4 = 0$$

This reduces the solution to

$$T = c \sin\lambda x \sinh\lambda y$$

With the second boundary condition, this becomes

$$0 = c \sin\lambda a \sinh\lambda y$$

Because c can clearly not be zero, this can only be satisfied for all values of y when

$$\lambda = \frac{m\pi}{a}$$

This gives an infinite number of values of λ with a different value of c corresponding to each of them.

Because any linear combination of solutions is itself a solution of the given problem, we obtain the series

$$T = \sum_{m=1}^{\infty} c_m \sinh \frac{m\pi y}{a} \sin \frac{m\pi x}{a} \tag{115}$$

Applying the fourth boundary condition, we get

$$F(x) = \sum_{m=1}^{\infty} c_m \sinh \frac{m\pi b}{a} \sin \frac{m\pi x}{a}$$

If we expand the left-hand side into a Fourier series, we get

$$c_m \sinh \frac{m\pi b}{a} = \frac{2}{a} \int_0^a F(x) \sin \frac{m\pi x}{a}\, dx$$

$$c_m = \frac{2}{a \sinh \frac{m\pi b}{a}} \int_0^a F(x) \sin \frac{m\pi x}{a}\, dx \tag{116}$$

Substituting Eq.(116) into Eq.(115), we obtain

$$T = \frac{2}{a}\sum_{m=1}^{\infty}\frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}}\sin\frac{m\pi x}{a}\int_0^a F(x)\sin\frac{m\pi x}{a}\,dx \qquad (117)$$

A.2 [illegible]s consider another set of boundary conditions [illegible]ig.26

[illegible] at y = 0

[illegible] at y = b

T = 0 at x = 0

T = G(y) at x = a

(118)

Clearly, the boundary conditions in this case are equivalent to those in Appendix A.1 with a simple exchange of x and y and a and b while G(y) takes the place of F(x).

Thus, the solution must be

$$T = \frac{2}{b}\sum_{m=1}^{\infty}\frac{\sinh\frac{m\pi x}{b}}{\sinh\frac{m\pi a}{b}}\sin\frac{m\pi y}{b}\int_0^b G(y)\sin\frac{m\pi y}{b}\,dy \qquad (119)$$

If instead of boundary conditions given by Eq.(109) we had

$$\begin{aligned} T &= 0 && \text{at } x = 0 \\ T &= 0 && \text{at } x = a \\ T &= F(x) && \text{at } y = 0 \\ T &= 0 && \text{at } y = b \end{aligned} \qquad (120)$$

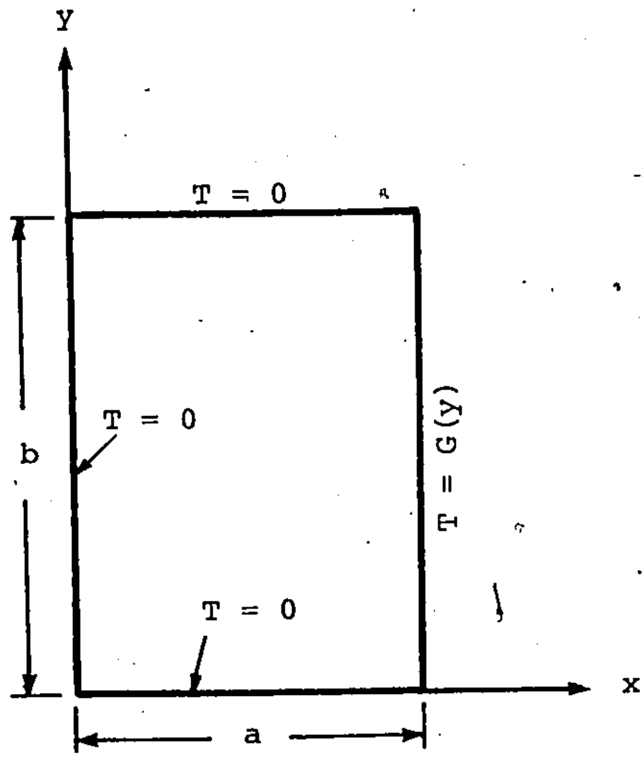


Fig. 26

this would be tantamount to a shift of x axis and y in Eq.(117) would be replaced by (b-y).

Similarly, with boundary conditions

$$\begin{aligned} T &= 0 && \text{at } y = 0 \\ T &= 0 && \text{at } y = b \\ T &= G(y) && \text{at } x = 0 \\ T &= 0 && \text{at } x = a \end{aligned} \qquad (121)$$

x in Eq.(119) would be replaced by (a-x).

A.3 $\displaystyle\int_0^{w_1} \sin\frac{n\pi x}{w_1}\sin\frac{m\pi x}{w_1}\,dx$

The integration of an indefinite integral of this form produces [5]

$$\int \sin kx \sin px\, dx = \frac{\sin(k-p)x}{2(k-p)} - \frac{\sin(k+p)x}{2(k+p)}$$

when $k \neq p$ (otherwise the first term on the right-hand side becomes indefinite). Thus for $m \neq n$ we shall have

$$\int_0^{w_1} \sin\frac{n\pi x}{w_1}\sin\frac{m\pi x}{w_1}\,dx = \left[\frac{\sin\left(\frac{n\pi}{w_1} - \frac{m\pi}{w_1}\right)x}{2\left(\frac{n\pi}{w_1} - \frac{m\pi}{w_1}\right)} - \frac{\sin\left(\frac{n\pi}{w_1} + \frac{m\pi}{w_1}\right)x}{2\left(\frac{n\pi}{w_1} + \frac{m\pi}{w_1}\right)}\right]_0^{w_1}$$

$$= \frac{\sin(n\pi - m\pi)}{2\left(\frac{n\pi}{w_1} - \frac{m\pi}{w_1}\right)} - \frac{\sin(n\pi + m\pi)}{2\left(\frac{n\pi}{w_1} + \frac{m\pi}{w_1}\right)} = 0$$

When m = n, we have

$$\int_0^{w_1} \sin^2 \frac{n\pi x}{w_1}\, dx = \left[- \frac{w_1}{2\pi} \cos \frac{n\pi x}{w_1} \sin \frac{n\pi x}{w_1} + \frac{x}{2} \right]_0^{w_1}$$

$$= - \frac{w_1}{2\pi} \cos n\pi \sin n\pi + \frac{w_1}{2} = \frac{w_1}{2}$$

Summing up:

$$\int_0^{w_1} \sin \frac{n\pi x}{w_1} \sin \frac{m\pi x}{w_1}\, dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{w_1}{2} & \text{when } m = n \end{cases} \qquad (122)$$

A.4 $$\int_h^{h_1} \sin \frac{n\pi (h-y)}{h_2} \sin \frac{m\pi (h-y)}{h_2}\, dy$$

With the transformation

$Y = h - y$

we have

$$\int_h^{h_1} \sin \frac{n\pi (h-y)}{h_2} \sin \frac{m\pi (h-y)}{h_2}\, dy = - \int_0^{h_2} \sin \frac{n\pi Y}{h_2} \sin \frac{m\pi Y}{h_2}\, dY$$

Therefore, according to Appendix A.3,

$$\int_{h}^{h_1} \sin \frac{n\pi(h-y)}{h_2} \sin \frac{m\pi(h-y)}{h_2} \, dy = \begin{cases} 0 & \text{when } m \neq n \\ -\frac{h_2}{2} & \text{when } m = n \end{cases} \qquad (123)$$

XII. References

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