Parametric Geometry of Numbers

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Abstract

This thesis is primarily concerned in studying the relationship between different *exponents of Diophantine approximation*, which are quantities arising naturally in the study of *rational approximation* to a fixed n-tuple of real irrational numbers.

As Khinchin observed, these exponents are not independent of each other, spurring interest in the study of the *spectrum* of a given family of exponents, which is the set of all possible values that can be taken by said family of exponents.

Introduced in 2009-2013 by Schmidt and Summerer and completed by Roy in 2015, the *parametric geometry of numbers* provides strong tools with regards to the study of exponents of Diophantine approximation and their associated spectra by the introduction of combinatorial objects called *n-systems*. Roy proved the very surprising result that the study of spectra of exponents is equivalent to the study of certain quantities attached to n-systems. Thus, the study of rational approximation can be replaced by the study of n-systems when attempting to determine such spectra.

Recently, Roy proved two new results for the case $n = 3$, the first being that spectra are semi-algebraic sets, and the second being that spectra are stable under the minimum with respect to the product ordering. In this thesis, it is shown that both of these results do not hold in general for $n \geq 4$, and examples are given.

This thesis also provides non-trivial examples for $n = 4$ where the spectra is stable under the minimum.

An alternate and much simpler proof of a recent result of Marnat-Moshchevitin proving an important conjecture of Schmidt-Summerer is also given, relying only on the parametric geometry of numbers instead. Further, a conjecture which generalizes this result is also established, and some partial results are given towards its validity. Among these results, the simplest, but non-trivial, new case is also proven to be true.

In a different vein, this thesis considers functions of the form

$$\theta(q) = \sum_{k=0}^{\infty} x_k q^{-p(k)},$$

where $p(n)$ is a quadratic polynomial with $p(\mathbb{N}) \subset \mathbb{N}$, and where $x_k > 0$ is rational for each $k \in \mathbb{N}$, thereby generalizing the theta function $T(q) = \sum_{k\in\mathbb{N}} q^{-k^2}$. We show under additional conditions on x_k that $[\mathbb{Q}(\theta(q)) : \mathbb{Q}] \geq 3$ for each integer $q \geq 2$.

Contents

Introduction

One can attach several *exponents of Diophantine approximation* to each non-zero vector $\boldsymbol{\xi} = (\xi_1, \dots \xi_n) \in \mathbb{R}^n$. They measure how well the point $\boldsymbol{\xi}$ can be approximated by rational subspaces of $\mathbb{R}^n$ for a given dimension $d \in \{1, \dots, n-1\}$, as in [1, 12, 23]. We adopt the following convention: Let $\|\cdot\|$ denote the standard Euclidean norm, and for a wedge product $\mathbf{x} \wedge \mathbf{y}$ with $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, the Euclidean norm of $\mathbf{x} \wedge \mathbf{y}$ is defined as the quantity

$$\|\mathbf{x} \wedge \mathbf{y}\| = \sqrt{\sum_{1 \le i < j \le n} (x_i y_j - x_j y_i)^2}.$$

An exposition on the topic of wedge products can be found in [13, Chapter 16].

For the case $d = 1$, one considers the exponent $\lambda(\boldsymbol{\xi})$, respectively $\hat{\lambda}(\boldsymbol{\xi})$, defined as the supremum of all real numbers $\lambda > 0$ such that

$$\|\mathbf{x}\| \le X \quad \text{and} \quad \|\mathbf{x} \wedge \boldsymbol{\xi}\| \le X^{-\lambda}$$

admits a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for arbitrarily large values of X, respectively for every sufficiently large value of X. For the case $d = n-1$, one considers instead the dual exponent $\tau(\boldsymbol{\xi})$, respectively $\hat{\tau}(\boldsymbol{\xi})$, defined as the supremum of all real numbers $\tau > 0$ such that

$$\|\mathbf{x}\| \le X \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \le X^{-\tau}$$

admits a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for arbitrarily large values of X, respectively for every sufficiently large value of X.

As Khinchin first observed in [10, 11], these exponents are not independent of each other. This led to the study of the *spectrum* of a given family of exponents, which is the set of all possibles values that can be taken by this family of exponents when restricting to $\boldsymbol{\xi}$ with $\xi_1, \dots, \xi_n$ linearly independent over $\mathbb{Q}$.

For instance, the spectrum of $(\lambda, \hat{\lambda}, \tau, \hat{\tau})$ is the set of points

$$(\lambda(\boldsymbol{\xi}), \hat{\lambda}(\boldsymbol{\xi}), \tau(\boldsymbol{\xi}), \hat{\tau}(\boldsymbol{\xi}))$$

where $\boldsymbol{\xi}$ runs through the non-zero points $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\xi_1, \dots, \xi_n$ linearly independent over $\mathbb{Q}$. Optimal constraints on these exponents were determined and shown to describe the full spectrum by Laurent for $n = 3$ in [12]. However, determining optimal constraints on these four exponents remains an open problem for $n \geq 4$. Nevertheless, several related results are known.

When $\xi_1, \ldots, \xi_n$ are linearly independent over $\mathbb{Q}$, Khinchin's transference principle [10, 11] yields the inequalities

$$\frac{\tau(\boldsymbol{\xi})}{(n-2)\tau(\boldsymbol{\xi}) + n - 1} \leq \lambda(\boldsymbol{\xi}) \leq \frac{\tau(\boldsymbol{\xi}) - n + 2}{n-1}.$$

Jarník proved in [6, 7] that these inequalities are best possible. Constraints on the pair $(\hat{\lambda}(\boldsymbol{\xi}), \hat{\tau}(\boldsymbol{\xi}))$ were also determined by Jarník in [8] for $n = 3$ and by German for $n \geq 4$ in [4]. Jarník proved that these constraints are best possible for $n = 3$ in [9], and Marnat proved they are best possible for $n \geq 4$ in [14].

The spectra of $(\lambda, \hat{\lambda})$ and $(\tau, \hat{\tau})$ for $n = 4$ were determined respectively by Moshchevitin in [17] and by Schmidt and Summerer in [27]. Recently, Marnat and Moshchevitin determined both of these spectra for $n \geq 5$ in [15].

The notion of *height* of a subspace V of $\mathbb{R}^n$ or $\mathbb{C}^n$ in relation to a fixed number field K was introduced by Schmidt in 1967 [23]. This allowed him to approximate a vector subspace V by vector subspaces S defined over K of a given dimension d. This thesis considers the case of real vector spaces with $K = \mathbb{Q}$. Given a non-zero subspace S defined over $\mathbb{Q}$, the height of S is defined by

$$H(S) = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\|,$$

where $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis of $S \cap \mathbb{Z}^n$ as a $\mathbb{Z}$-module, and where $\|\cdot\|$ is the usual Euclidean norm. Note that this is independent of the choice of basis. Given such a subspace S and a non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$, the *projective distance* between $\boldsymbol{\xi}$ and S is defined by

$$\mathrm{d}(\boldsymbol{\xi}, S) = \frac{\|\boldsymbol{\xi} \wedge \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\|}{\|\boldsymbol{\xi}\| H(S)},$$

noting again that this is independent of the choice of basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

In [12], Laurent considers for each $j \in \{1, \ldots, n-1\}$ the intermediate exponent $\omega_j(\boldsymbol{\xi})$, respectively $\hat{\omega}_j(\boldsymbol{\xi})$, defined as the supremum of all real numbers $\omega > 0$ such that

$$H(S) \leq Q \quad \text{and} \quad H(S)\mathrm{d}(\boldsymbol{\xi}, S) \leq Q^{-\omega} \tag{0.0.1}$$

is satisfied for some j-dimensional subspace S of $\mathbb{R}^n$ defined over $\mathbb{Q}$, for arbitrarily large values of Q, respectively for every sufficiently large value of Q. These exponents are more general than the four exponents $\lambda, \hat{\lambda}, \tau, \hat{\tau}$ since $\lambda, \hat{\lambda}$ are the same as $\omega_1, \hat{\omega}_1$, and $\tau, \hat{\tau}$ are the same as $\omega_{n-1}, \hat{\omega}_{n-1}$.

Laurent obtained constraints for the spectrum of $(\omega_1, \ldots, \omega_{n-1})$ in [12], and Roy proved that they describe the full spectrum in [21].

In order to study spectra of exponents in greater generality, Schmidt and Summerer introduced the *parametric geometry of numbers* in [25, 26], a theory which was completed by Roy in [20]. This theory is based on the *geometry of numbers*.

The geometry of numbers was introduced by Minkowski in 1910. This theory studies *convex bodies* $\mathcal{C}$ in $\mathbb{R}^n$, which are subsets $\mathcal{C}$ of $\mathbb{R}^n$ with non-empty interior such that $\mathcal{C}$ is compact, convex, and *centrally symmetric*, i.e. $\mathbf{0} \in \mathcal{C}$ and $-\mathcal{C} = \mathcal{C}$. The theory is concerned with the study of convex bodies in relation to lattices Λ in $\mathbb{R}^n$. Minkowski proved two fundamental theorems in this direction. The first is Minkowski's *first convex body theorem*, which states that a convex body $\mathcal{C}$ with

$$\mathrm{vol}(\mathcal{C}) \geq 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)$$

contains a non-zero point in Λ. The second is concerned with the *successive minima*

$$\lambda_1(\mathcal{C}, \Lambda), \dots, \lambda_n(\mathcal{C}, \Lambda)$$

of a convex body $\mathcal{C}$, where $\lambda_i = \lambda_i(\mathcal{C}, \Lambda)$ is defined as the infimum of all $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least i linearly independent points in Λ, for $i = 1, \dots, n$. It is easily verified that

$$0 < \lambda_1 \leq \cdots \leq \lambda_n < \infty.$$

Minkowski's *second convex body theorem* states that

$$\frac{2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)}{n!} \leq \lambda_1 \cdots \lambda_n \mathrm{vol}(\mathcal{C}) \leq 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda).$$

The geometry of numbers has lead to important advances in the study of *rational approximation* to a fixed n-tuple of real irrational numbers. For instance, let

$$\boldsymbol{\xi} = (\xi_1, \dots \xi_n) \in \mathbb{R}^n$$

with $\xi_1 = 1$ and $\xi_1, \dots, \xi_n$ linearly independent over $\mathbb{Q}$. For each $q > 0$, the set

$$\mathcal{C}(q) = \{\mathbf{x} \in \mathbb{R}^n \,|\, |x_1| \leq q \text{ and } |x_1\xi_i - x_i| \leq q^{-1/(n-1)} \text{ for } i = 2, \dots, n\}$$

is a convex body in $\mathbb{R}^n$. Since the volume of $\mathcal{C}(q)$ is 2^n for $q > 0$, then Minkowski's first convex body theorem implies that $\mathcal{C}(q)$ contains a non-zero point in $\mathbb{Z}^n$. Since $\xi_1 = 1$, then a non-zero point $\mathbf{x} \in \mathcal{C}(q) \cap \mathbb{Z}^n$ is such that that the line L passing through the origin and the point $\mathbf{x}$ is an approximation of the line L' passing through the origin and $\boldsymbol{\xi}$, in the sense that the angle between L and L' is small. Thus, letting q go to infinity yields increasingly better rational approximations to the point $\boldsymbol{\xi}$.

Along this line of reasoning, Schmidt and Summerer introduced the parametric geometry of numbers in order to study such convex bodies $\mathcal{C}(q)$ by studying the parametric family of convex bodies $(\mathcal{C}(q))_{q \geq 0}$. With respect to the theory, there are a number of ways of redefining the convex bodies $\mathcal{C}(q)$. Certain authors instead study the convex bodies defined by

$$\mathcal{C}^\star_{\boldsymbol{\xi}}(q) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le e^{(n-1)q} \text{ and } \|\mathbf{x} \wedge \boldsymbol{\xi}\| \le e^{-q}\} \quad (q > 0).$$

This thesis however adopts the convention used by Roy [22], which is to consider the parametric convex bodies given by

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le 1 \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \le e^{-q}\} \quad (q > 0).$$

The convex bodies $\mathcal{C}^\star_{\boldsymbol{\xi}}(q)$ and $\mathcal{C}_{\boldsymbol{\xi}}(q)$ are in some sense dual. In order to study the latter parametric family, define the map $\mathbf{L}_{\boldsymbol{\xi}} : \mathbb{R}^+ \to \mathbb{R}^n$ by

$$q \mapsto (\log \lambda_1(q), \ldots, \log \lambda_n(q)),$$

where $\lambda_i(q)$ is the i^{th} minimum of $\mathcal{C}_{\boldsymbol{\xi}}(q)$ in relation to $\mathbb{Z}^n$. The case where $\xi_1, \ldots \xi_n$ are linearly independent over $\mathbb{Q}$ is equivalent to the case where $L_{\boldsymbol{\xi},1}$ is unbounded.

Schmidt notes in [24] the importance of the maps $\mathbf{L}_{\boldsymbol{\xi}}$ with regards to Diophantine approximation. Furthermore, translated to the conventions of [22], he observes that

$$L_{\boldsymbol{\xi},1}(q) \le \cdots \le L_{\boldsymbol{\xi},n}(q) \quad (q \ge 0),$$

that Minkowski's second convex body theorem implies that

$$|L_{\boldsymbol{\xi},1}(q) + \cdots + L_{\boldsymbol{\xi},n}(q) - q| = \mathcal{O}(1) \quad (q \ge 0),$$

and that each $L_{\boldsymbol{\xi},i}$ is a continuous piecewise linear map with slopes either 0 or 1.

To see the relation between the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and the aforementioned exponents of Diophantine approximation, define for each $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ the quantities

$$\underline{\varphi}_i(\boldsymbol{\xi}) = \liminf_{q \to \infty} q^{-1} L_{\boldsymbol{\xi},i}(q) \quad \text{and} \quad \overline{\varphi}_i(\boldsymbol{\xi}) = \limsup_{q \to \infty} q^{-1} L_{\boldsymbol{\xi},i}(q),$$

for $i = 1, \ldots, n$, as well as the quantities

$$\underline{\psi}_j(\boldsymbol{\xi}) = \liminf_{q \to \infty} q^{-1} \sum_{i=1}^{j} L_{\boldsymbol{\xi},i}(q) \quad \text{and} \quad \overline{\psi}_j(\boldsymbol{\xi}) = \limsup_{q \to \infty} q^{-1} \sum_{i=1}^{j} L_{\boldsymbol{\xi},i}(q),$$

for $j = 1, \ldots, n$. Roy notes in [20] that

$$\underline{\varphi}_1(\boldsymbol{\xi}) = \frac{1}{1+\tau(\boldsymbol{\xi})}, \quad \overline{\varphi}_1(\boldsymbol{\xi}) = \frac{1}{1+\hat{\tau}(\boldsymbol{\xi})}, \quad \underline{\varphi}_n(\boldsymbol{\xi}) = \frac{\hat{\lambda}(\boldsymbol{\xi})}{1+\hat{\lambda}(\boldsymbol{\xi})} \quad \text{and} \quad \overline{\varphi}_n(\boldsymbol{\xi}) = \frac{\lambda(\boldsymbol{\xi})}{1+\lambda(\boldsymbol{\xi})},$$

for each non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$. Since

$$|q^{-1}L_{\boldsymbol{\xi},1}(q) + \cdots + q^{-1}L_{\boldsymbol{\xi},n}(q) - 1| = o(1) \quad (q \ge 0),$$

then

$$\underline{\psi}_{n-1}(\boldsymbol{\xi}) = 1 - \overline{\varphi}_n(\boldsymbol{\xi}) \quad \text{and} \quad \overline{\psi}_{n-1} = 1 - \underline{\varphi}_n(\boldsymbol{\xi}),$$

and so we also have that

$$\underline{\psi}_{n-1}(\boldsymbol{\xi}) = \frac{1}{1+\lambda(\boldsymbol{\xi})} \quad \text{and} \quad \overline{\psi}_{n-1}(\boldsymbol{\xi}) = \frac{1}{1+\hat{\lambda}(\boldsymbol{\xi})},$$

for each non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$. More generally, Roy shows in [21] that

$$\underline{\psi}_{j}(\boldsymbol{\xi}) = \frac{1}{1+\omega_{n-j}(\boldsymbol{\xi})} \quad \text{and} \quad \overline{\psi}_{j}(\boldsymbol{\xi}) = \frac{1}{1+\hat{\omega}_{n-j}(\boldsymbol{\xi})},$$

for each non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$, for $j = 1, \dots, n-1$, where the $\omega_{n-j}(\boldsymbol{\xi}), \hat{\omega}_{n-j}(\boldsymbol{\xi})$ quantities are defined as in (0.0.1). The spectra of pairs $(\underline{\psi}_j, \overline{\psi}_j)$ where $j \in \{1, \dots, n-1\}$ play an important role in this thesis.

We also consider *generalized* exponents of Diophantine approximation. These were introduced by Roy [22], and they are defined as follows.

For each linear map $T = (T_1, \dots, T_m) : \mathbb{R}^n \to \mathbb{R}^m$, define $\underline{\varphi}_T$ and $\overline{\varphi}_T$ to be the functions which take each non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ to the m-tuples

$$\underline{\varphi}_T(\boldsymbol{\xi}) = (\underline{\varphi}_{T_1}(\boldsymbol{\xi}), \dots, \underline{\varphi}_{T_m}(\boldsymbol{\xi})) \quad \text{and} \quad \overline{\varphi}_T(\boldsymbol{\xi}) = (\overline{\varphi}_{T_1}(\boldsymbol{\xi}), \dots, \overline{\varphi}_{T_m}(\boldsymbol{\xi})),$$

where

$$\underline{\varphi}_{T_i}(\boldsymbol{\xi}) = \liminf_{q\to\infty} q^{-1} T_i(\mathbf{L}_{\boldsymbol{\xi}}(q)) \quad \text{and} \quad \overline{\varphi}_{T_i}(\boldsymbol{\xi}) = \limsup_{q\to\infty} q^{-1} T_i(\mathbf{L}_{\boldsymbol{\xi}}(q)),$$

for $i = 1, \dots, m$. The functions $\underline{\varphi}_{T_i}$, $\overline{\varphi}_{T_i}$ are called *generalized exponents of Diophantine approximation*. The *spectrum* of $\underline{\varphi}_T$ is the set of points $\underline{\varphi}_T(\boldsymbol{\xi})$ where $\boldsymbol{\xi}$ runs through the points $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\xi_1, \dots, \xi_n$ linearly independent over $\mathbb{Q}$.

Observe that the preceding examples of exponents can be recovered through these generalized exponents, as verified by the identities

$$\underline{\varphi}_i = \underline{\varphi}_{\pi_i}, \quad \overline{\varphi}_i = \overline{\varphi}_{\pi_i} \quad \text{and} \quad \underline{\psi}_j = \underline{\varphi}_{\sigma_j}, \quad \overline{\psi}_j = \overline{\varphi}_{\sigma_j},$$

where $\pi_i(\mathbf{x}) = x_i$ and $\sigma_j(\mathbf{x}) = x_1 + \cdots + x_j$, for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

The notion of an n-system was proposed by Schmidt and Summerer in [26] as a model for the functions $\mathbf{L}_{\boldsymbol{\xi}}$. In that paper, they are called $(n, 0)$-systems. In [20], Roy gives a definition which is adapted for the context of parametric families of convex bodies $(\mathcal{C}_{\boldsymbol{\xi}}(q))_{q\geq 0}$. In this context, an *$n$-system* is a map $\mathbf{P} : I \to \mathbb{R}^n$ satisfying the following simple properties:

- The domain I is a closed subinterval of $\mathbb{R}$ with $\min I > 0$.

- For each $q \in I$, we have that

$$0 \le P_1(q) \le \cdots \le P_n(q) \quad \text{and} \quad P_1(q) + \cdots + P_n(q) = q.$$

  Moreover, by writing $S_i = P_1 + \cdots + P_i$ for $i = 1, \ldots, n$, we have the following properties for each $i \in \{1, \ldots, n\}$.

- The map S_i is a continuous piecewise linear map with slopes either 0 or 1.

- If $q \in I^\circ$ and $j \ge 0$ are such that $S_i'(q^-) = 1 \ne S_{i+j}'(q^+)$, then we have

$$P_i(q) = P_{i+j+1}(q).$$

An equivalent definition of an n-system is given in the first chapter.

Roy shows in [20] that, for each map $\mathbf{L}_{\boldsymbol{\xi}}$, there exists an n-system $\mathbf{P}$ such that $\mathbf{P}$ and $\mathbf{L}_{\boldsymbol{\xi}}$ have bounded difference. Conversely, Roy shows in [20] that, for each n-system $\mathbf{P}$, there exists a non-zero vector $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $\mathbf{P}$ and $\mathbf{L}_{\boldsymbol{\xi}}$ have bounded difference. Studying the maps $\mathbf{L}_{\boldsymbol{\xi}}$ is in this sense equivalent to studying n-systems. The benefit of studying n-systems instead of the maps $\mathbf{L}_{\boldsymbol{\xi}}$ comes from the simplicity in their description, and how it eliminates the need to consider the points $\boldsymbol{\xi}$ directly.

To see how n-systems can be used to study exponents, define for each linear map $T : \mathbb{R}^n \to \mathbb{R}$ and each n-system $\mathbf{P}$ the quantities

$$\underline{\varphi}_T(\mathbf{P}) = \liminf_{q\to\infty} q^{-1} T(\mathbf{P}(q)) \quad \text{and} \quad \overline{\varphi}_T(\mathbf{P}) = \limsup_{q\to\infty} q^{-1} T(\mathbf{P}(q)).$$

If $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ is such that $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{P}$ have bounded difference, then

$$\underline{\varphi}_T(\mathbf{P}) = \underline{\varphi}_T(\boldsymbol{\xi}) \quad \text{and} \quad \overline{\varphi}_T(\mathbf{P}) = \overline{\varphi}_T(\boldsymbol{\xi}).$$

Thus, the spectrum of $\underline{\varphi}_T$ is the set of points $\underline{\varphi}_T(\mathbf{P})$ where $\mathbf{P}$ runs through the n-systems $\mathbf{P}$ with P_1 unbounded. Such an n-system is called *proper*.

The following notation is often useful when studying spectra. To each n-system $\mathbf{P} : I \to \mathbb{R}^n$ with unbounded domain, one defines the set $\mathcal{F}(\mathbf{P})$ to be the set of all points $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y}$ is the limit of $q_i^{-1}\mathbf{P}(q_i)$ for some unbounded sequence $(q_i)_{i\in\mathbb{N}}$ in I. We also define the set $\mathcal{K}(\mathbf{P})$ to be the convex hull of $\mathcal{F}(\mathbf{P})$. Since convexity is preserved by linear maps, and since the infimum remains unchanged by taking the convex hull, it follows that

$$\underline{\varphi}_T(\mathbf{P}) = \inf T(\mathcal{F}(\mathbf{P})) = \inf T(\mathcal{K}(\mathbf{P}))$$

for each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$.

The first chapter of this thesis presents numerous notions and results which are related to the study of n-systems, following the work of Roy in [20, 22, 21]. We also develop certain results which are original to this thesis.

In the second chapter of this thesis, we study the spectra of pairs $(\underline{\psi}_j, \overline{\psi}_j)$. For the case $n \geq 3$, Marnat and Moshchevitin recently determined the set of all such pairs for $j = 1$ and $j = n-1$ in [15]. The proof of Marnat and Moshchevitin uses some results from the parametric geometry of numbers, but relies largely on a more traditional approach. This thesis reproduces their result, but relying only on the parametric geometry of numbers. The proof is also markedly simpler. The idea is motivated by a transformation of proper n-systems $\mathbf{P}$ into canonical n-systems $\tilde{\mathbf{P}}$ preserving the maps $S_j = P_1 + \cdots + P_j$ for a given j. This transformation is given in the first chapter, and is original to this thesis.

We also give a conjecture describing all possible values $(\underline{\psi}_j(\mathbf{P}), \overline{\psi}_j(\mathbf{P}))$ for a given pair (n, j), and we prove that the conjecture holds true for a special subclass of n-systems which we call *quasi-regular*. We also prove the case $(4, 2)$, which completes the case $n = 4$ in light of the result of Marnat and Moshchevitin. Though not included in this thesis, we also tested by computer, checking for several pairs (n, j), that the conjecture for j holds true for many n-systems. A non-trivial test case for the pair $(5, 2)$ is also proven to hold true using elementary algebra arguments.

In the third chapter, we prove the existence of spectra of n-systems which are not semialgebraic sets, for each $n \geq 4$. This is in contrast to a remarkable result of Roy proving that spectra of n-systems are semialgebraic sets for $n = 2, 3$ in [22]. Moreover, the existence of such spectra is made explicit for a certain family of linear maps $T : \mathbb{R}^n \to \mathbb{R}^{n+1}$.

The idea of the proof is to show, for certain linear maps

$$T = (T_0, \ldots, T_n) : \mathbb{R}^n \to \mathbb{R}^{n+1},$$

that the intersection of the spectra S_T of $\underline{\varphi}_T$ with the semialgebraic set

$$A_T = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, T_0(\mathbf{x}) > 0 \text{ and } T_i(\mathbf{x}) \geq 0 \text{ for } i = 1, \ldots, n\}$$

is not a semialgebraic set. Since the intersection of semialgebraic sets remains so itself, this allows us to conclude that S_T is not semialgebraic. The proof is done in two steps: First, we consider the set $\tilde{S}_T$ of points $\underline{\varphi}_T(\mathbf{P})$ where $\mathbf{P}$ runs through an important subclass of n-systems called *self-similar*. We then show that $\tilde{S}_T \cap A_T$ is an infinite discrete subset of $\mathbb{R}^{n+1}$, which in turn implies that $\tilde{S}_T$ is not semialgebraic. Second, we prove that every image point $\underline{\varphi}_T(\mathbf{P})$ in A_T of a proper n-system $\mathbf{P}$ is also the image point $\underline{\varphi}_T(\tilde{\mathbf{P}})$ in A_T for some self-similar n-system $\tilde{\mathbf{P}}$. Hence, we conclude that $S_T = \tilde{S}_T$, which completes the proof.

This reduction relies on proving that certain limits of sequences of n-systems remain n-systems themselves. To this end, in the first chapter we introduce maps called *prototypes*, which share many properties with n-systems. For a given integer $n \geq 2$, the class of prototypes is strictly larger than the class of n-systems, but has the advantage of being closed under uniform convergence. Under certain added conditions, we can show that certain limits of prototypes are in fact n-systems. Another important result in this direction is a natural generalization of the Arzelá-Ascoli theorem, which we will use to prove that an arbitrary sequence of n-systems defined on a compact interval I of $(0, \infty)$ has a subsequence which converges uniformly.

For each $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we define the minimum

$$\min\{\mathbf{x}, \mathbf{y}\} = (\min\{x_1, y_1\}, \ldots, \min\{x_m, y_m\}).$$

The fourth chapter gives a condition under which certain spectra $S \subseteq \mathbb{R}^m$ of 4-systems are stable under the minimum, i.e. for each $\mathbf{x}, \mathbf{y} \in S$, the point $\min\{\mathbf{x}, \mathbf{y}\} \in S$. This expands upon another result of Roy proving that spectra for n-systems are stable under the minimum for $n = 2, 3$ in [22].

The main result leading to this condition of stability under the minimum is the proof of the following result: For each proper 4-system $\mathbf{P}$, there exists a proper 4-system $\mathbf{R}$ such that $\mathcal{K}(\mathbf{R})$ is the convex hull of

$$\mathcal{F}(\mathbf{P}) \cup \{(1, 1, 1, 1)/4\}.$$

On the other hand, the fifth chapter proves the existence of spectra of 4-systems which are not stable under the minimum. The proof relies on the construction of two proper 4-systems $\mathbf{R}$ and $\mathbf{S}$ with the following property: If $\mathbf{T}$ is a proper 4-system, then $\mathcal{K}(\mathbf{T})$ is not equal to the convex hull of $\mathcal{F}(\mathbf{R}) \cup \mathcal{F}(\mathbf{S})$.

In a different vein, the sixth and final chapter considers functions of the form

$$\theta(q) = \sum_{k=0}^{\infty} x_k q^{-p(k)},$$

where $p(n)$ is a quadratic polynomial with $p(\mathbb{N}) \subset \mathbb{N}$, and where $x_k > 0$ is rational for each $k \in \mathbb{N}$, thereby generalizing the theta function $T(q) = \sum_{k \in \mathbb{N}} q^{-k^2}$. While it is known by an application of Nesterenko's theorem that $T(q)$ is transcendental for each algebraic $q \neq 0$ with $|q| > 1$, this thesis shows that $\theta(q)$ is not an algebraic number of degree less than 3 at integers $q \geq 2$, under certain height constraints on the polynomial p and the numbers x_i. The proof uses a gap argument relying on a result of Richards.

Chapter 1

Preliminaries on Parametric Geometry of Numbers

This chapter is divided into several sections. The first on *parametric geometry of numbers* gives a brief introduction to the theory, following conventions of Roy. The second introduces the notion of a *system*, the main object of study throughout this thesis, and intends to help the reader develop an intuition towards how these objects are used as tools in the study of *exponents of Diophantine approximation*, primarily with regards to their *spectra*. The remaining sections cover several results concerning systems, most of which are key in proving the results in the following chapters.

Throughout this chapter, fix an integer $n \geq 2$.

1.1 Parametric Geometry of Numbers

The contents of this section describing the parametric geometry of numbers are attributed to [22]. It is also worth noting that while the conventions in [22] differ than those used by Schmidt and Summerer [25, 26], the theory remains essentially equivalent.

A *convex body* $\mathcal{C}$ in $\mathbb{R}^n$ is a subset $\mathcal{C}$ of $\mathbb{R}^n$ which has non-empty interior, and which is compact, convex, and *centrally symmetric*, i.e. $\mathbf{0} \in \mathcal{C}$ and $-\mathcal{C} = \mathcal{C}$. Introduced by Minkowski [16], the *geometry of numbers* studies convex bodies in relation to lattices Λ in $\mathbb{R}^n$. A first important result in this direction is Minkowski's *first convex body theorem*, which states that a convex body $\mathcal{C}$ with $\mathrm{vol}(\mathcal{C}) \geq 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)$ contains a non-zero point in Λ. A second important result is concerned with the *successive minima* $\lambda_1(\mathcal{C}, \Lambda), \dots, \lambda_n(\mathcal{C}, \Lambda)$ of a convex body $\mathcal{C}$, where each $\lambda_i = \lambda_i(\mathcal{C}, \Lambda)$ is defined as the infimum of all $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least i linearly independent points in Λ. One readily verifies that $0 < \lambda_1 \leq \cdots \leq \lambda_n < \infty$, and Minkowski's *second convex body theorem* is thus stated

$$\frac{2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)}{n!} \leq \lambda_1 \cdots \lambda_n \mathrm{vol}(\mathcal{C}) \leq 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda).$$

In [25], Schmidt and Summerer go one step further by studying certain parametric families of convex bodies stemming from the study of rational approximation, thus initiating the *parametric geometry of numbers*.

For each non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$, define the parametric family $(\mathcal{C}_{\boldsymbol{\xi}}(q))_{q\geq 0}$ of convex bodies in $\mathbb{R}^n$ by defining for each $q \geq 0$ the convex body

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1 \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \leq e^{-q}\},$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \cdot \boldsymbol{\xi}$ denotes the standard scalar product of $\mathbf{x}$ and $\boldsymbol{\xi}$ in $\mathbb{R}^n$. The parametric geometry of numbers implicitly studies the quantities $\lambda_i(\mathcal{C}_{\boldsymbol{\xi}}(q), \mathbb{Z}^n)$ as functions of the parameter $q \in \mathbb{R}^+$ by taking their logarithms, in an effort to simplify the exposition and theory. To this end, define for each $i \in \{1, \ldots, n\}$ the function $L_{\boldsymbol{\xi},i} : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}$ by

$$L_{\boldsymbol{\xi},i}(q) = \log \lambda_i(\mathcal{C}_{\boldsymbol{\xi}}(q), \mathbb{Z}^n) \quad (q \geq 0),$$

and define the function $\mathbf{L}_{\boldsymbol{\xi}} : \mathbb{R}^+ \to \mathbb{R}^n$ by

$$\mathbf{L}_{\boldsymbol{\xi}}(q) = (L_{\boldsymbol{\xi},1}(q), \ldots, L_{\boldsymbol{\xi},n}(q)) \quad (q \geq 0).$$

Schmidt notes in [24] the importance of the maps $\mathbf{L}_{\boldsymbol{\xi}}$ with regards to Diophantine approximation. Further, translated to the conventions of [22], he observes that

$$L_{\boldsymbol{\xi},1}(q) \leq \cdots \leq L_{\boldsymbol{\xi},n}(q) \quad (q \geq 0),$$

that Minkowski's second convex body theorem implies that

$$|L_{\boldsymbol{\xi},1}(q) + \cdots + L_{\boldsymbol{\xi},n}(q) - q| = \mathcal{O}(1) \quad (q \geq 0),$$

and that each $L_{\boldsymbol{\xi},i}$ is a continuous piecewise linear map with slopes either 0 or 1.

Proposition 1.1.1. *Let* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$*, and let* $m \in \{0, \ldots, n-1\}$ *be maximal such that* $L_{\boldsymbol{\xi},i}$ *is bounded for each* i *with* $1 \leq i \leq m$*. It follows that*

$$n - m = \dim\langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}}.$$

Proof: Let $m' = n - \dim\langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}}$, and so there exists linearly independent points $\mathbf{a}_1, \ldots, \mathbf{a}_{m'} \in \mathbb{Z}^n$ such that $\mathbf{a}_i \cdot \boldsymbol{\xi} = 0$ for $i = 1, \ldots, m'$. By letting λ be such that $\mathbf{a}_1, \ldots, \mathbf{a}_{m'} \in \lambda\mathcal{C}_{\boldsymbol{\xi}}(q)$, it follows that $L_{\boldsymbol{\xi},m'} \leq \log \lambda$, and so $m' \leq m$. For the reverse inequality, note that $L_{\boldsymbol{\xi},m} \leq \log \lambda$ for some $\lambda > 0$, and so $\lambda\mathcal{C}_{\boldsymbol{\xi}}(t) \cap \mathbb{Z}^n$ is finite and contains at least m linearly independent points for each $t \geq 0$. Thus, as $\lambda\mathcal{C}_{\boldsymbol{\xi}}(s) \subseteq \lambda\mathcal{C}_{\boldsymbol{\xi}}(t)$ for each s, t with $s > t \geq 0$, it follows as $t \to \infty$ that the finite set $\lambda\mathcal{C}_{\boldsymbol{\xi}}(t) \cap \mathbb{Z}^n$ stabilizes at some $t = M$, that is $\lambda\mathcal{C}_{\boldsymbol{\xi}}(M) \cap \mathbb{Z}^n = \cap_{t\geq 0}\mathcal{C}_{\boldsymbol{\xi}}(t) \cap \mathbb{Z}^n$. By letting $\mathbf{c}_1, \dots, \mathbf{c}_m$ be linearly independent points in $\lambda\mathcal{C}_{\boldsymbol{\xi}}(M) \cap \mathbb{Z}^n$, it follows that $\mathbf{c}_i \cdot \boldsymbol{\xi} = 0$ for $i = 1, \dots, m$, and so $m \leq m'$. ∎

Finally, the work by Schmidt and Summerer in [25, 26], completed by Roy in [20], provides a complete characterization of all possible maps $\mathbf{L}_{\boldsymbol{\xi}} : [0, \infty) \to \mathbb{R}^n$, up to bounded difference in $\mathbb{R}^n$, and is described in the following section on n-systems.

1.1.1 Spectra of Families of Exponents

Of primary interest in this theory are quantities called *exponents of Diophantine approximation*, which are named as such because they manifest as critical exponents in problems of Diophantine approximation. Some common examples are the quantities

$$\underline{\varphi}_i(\boldsymbol{\xi}) = \liminf_{q\to\infty} q^{-1}L_{\boldsymbol{\xi},i}(q) \quad \text{and} \quad \overline{\varphi}_i(\boldsymbol{\xi}) = \limsup_{q\to\infty} q^{-1}L_{\boldsymbol{\xi},i}(q),$$

for $i = 1, \dots, n$, as well as the quantities

$$\underline{\psi}_j(\boldsymbol{\xi}) = \liminf_{q\to\infty} q^{-1}\sum_{i=1}^{j} L_{\boldsymbol{\xi},i}(q) \quad \text{and} \quad \overline{\psi}_j(\boldsymbol{\xi}) = \limsup_{q\to\infty} q^{-1}\sum_{i=1}^{j} L_{\boldsymbol{\xi},i}(q),$$

for $j = 1, \dots, n$, which will play an important role in this thesis. The following definition generalizes these exponents.

Definition 1.1.2. For each linear map $T = (T_1, \dots, T_m) : \mathbb{R}^n \to \mathbb{R}^m$, define $\underline{\varphi}_T$ and $\overline{\varphi}_T$ to be the functions which take each non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ to the m-tuples

$$\underline{\varphi}_T(\boldsymbol{\xi}) = (\underline{\varphi}_{T_1}(\boldsymbol{\xi}), \dots, \underline{\varphi}_{T_m}(\boldsymbol{\xi})) \quad \text{and} \quad \overline{\varphi}_T(\boldsymbol{\xi}) = (\overline{\varphi}_{T_1}(\boldsymbol{\xi}), \dots, \overline{\varphi}_{T_m}(\boldsymbol{\xi})),$$

where

$$\underline{\varphi}_{T_i}(\boldsymbol{\xi}) = \liminf_{q\to\infty} q^{-1}T_i(\mathbf{L}_{\boldsymbol{\xi}}(q)) \quad \text{and} \quad \overline{\varphi}_{T_i}(\boldsymbol{\xi}) = \limsup_{q\to\infty} q^{-1}T_i(\mathbf{L}_{\boldsymbol{\xi}}(q)),$$

for $i = 1, \dots, m$. The functions $\underline{\varphi}_{T_i}$, $\overline{\varphi}_{T_i}$ are called *generalized exponents of Diophantine approximation.*

Observe that the preceding examples of classical exponents can be recovered through these generalized exponents, as verified by the identities

$$\underline{\varphi}_i = \underline{\varphi}_{\pi_i}, \quad \overline{\varphi}_i = \overline{\varphi}_{\pi_i} \quad \text{and} \quad \underline{\psi}_j = \underline{\varphi}_{\sigma_j}, \quad \overline{\psi}_j = \overline{\varphi}_{\sigma_j},$$

where $\pi_i(\mathbf{x}) = x_i$ and $\sigma_j(\mathbf{x}) = x_1 + \cdots + x_j$, for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Given linear maps $T : \mathbb{R}^n \to \mathbb{R}^m$ and $T' : \mathbb{R}^n \to \mathbb{R}^{m'}$, it is in general difficult to determine the image of $(\underline{\varphi}_T, \overline{\varphi}_{T'})$. This leads to the following definition.

Definition 1.1.3. For linear maps $T : \mathbb{R}^n \to \mathbb{R}^m$ and $T' : \mathbb{R}^n \to \mathbb{R}^{m'}$, the *spectrum* of $(\underline{\varphi}_T, \overline{\varphi}_{T'})$ is the set of values $(\underline{\varphi}_T(\boldsymbol{\xi}), \overline{\varphi}_{T'}(\boldsymbol{\xi}))$ where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ runs through the points in $\mathbb{R}^n$ having $\mathbb{Q}$-linearly independent coordinates.

Remark that $\overline{\varphi}_{-T} = -\underline{\varphi}_T$ for each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, so that one can study spectra in general by instead studying spectra of functions of the form $\underline{\varphi}_T$.

In order to study spectra of maps $\underline{\varphi}_T$, it is important to know something about the maps $\mathbf{L}_{\boldsymbol{\xi}}$. Extending the notion of exponents to maps is useful in this regard.

1.1.2 Exponents of Maps

The following notation is often used throughout this thesis.

Definition 1.1.4. For each map $f : S \to \mathbb{R}^n$ defined on a subset S of $\mathbb{R}$, and for each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, define the functions $\varphi, \varphi_T : S\backslash\{0\} \to \mathbb{R}^m$ by

$$\varphi(q; f) = q^{-1} f(q), \quad \varphi_T(q; f) = q^{-1} T(f(q)) \quad (q \in S\backslash\{0\}).$$

Upon writing $\varphi = (\varphi_1, \ldots, \varphi_n)$, also define for each $j \in \{1, \ldots, n\}$ the map

$$\psi_j = \sum_{i=1}^{j} \varphi_i.$$

Moreover, for $i = 1, \ldots, m$, if S is bounded, then define the quantities

$$\underline{\varphi}_{T_i}(f) = \inf_{q \in S} \varphi_{T_i}(q; f) \quad \text{and} \quad \overline{\varphi}_{T_i}(f) = \sup_{q \in S} \varphi_{T_i}(q; f),$$

and if S contains an interval $[q_0, \infty)$ for some $q_0 \in S$, then define the quantities

$$\underline{\varphi}_{T_i}(f) = \liminf_{q \to \infty} \varphi_{T_i}(q; f) \quad \text{and} \quad \overline{\varphi}_{T_i}(f) = \limsup_{q \to \infty} \varphi_{T_i}(q; f).$$

The quantites $\underline{\varphi}_i, \overline{\varphi}_i, \underline{\psi}_j, \overline{\psi}_j$ associated to f are defined similarly. Finally, define

$$\underline{\varphi}_T(f) = (\underline{\varphi}_{T_1}(f), \ldots, \underline{\varphi}_{T_m}(f)) \quad \text{and} \quad \overline{\varphi}_T(f) = (\overline{\varphi}_{T_1}(f), \ldots, \overline{\varphi}_{T_m}(f)),$$

and the quantities $\underline{\varphi}(f), \overline{\varphi}(f)$ similarly.

Remark. $\underline{\varphi}_T(\mathbf{L}_{\boldsymbol{\xi}}) = \underline{\varphi}_T(\boldsymbol{\xi})$ for each $\boldsymbol{\xi} \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ and each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$.

It is worth mentioning that the distinction between maps defined on bounded domains and those whose domain contains an interval of the form $[q_0, \infty)$ stems naturally from considerations in this thesis. Specifically, when studying φ, φ_T for maps with unbounded domains, it is always the case that only the limiting behaviour is of interest when studying infima and suprema.

1.1.3 Approximation Criterion

The goal of this subsection is to determine a general criterion under which maps $\mathbf{L}$ and $\mathbf{P}$ satisfy $\underline{\varphi}_T(\mathbf{L}) = \underline{\varphi}_T(\mathbf{P})$ independently of the choice of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. The following definitions, observations, and lemma lead to this criterion.

Definition 1.1.5. Let $f : S \subseteq \mathbb{R} \to \mathbb{R}^m$ be a map, for some $m \in \mathbb{N}^+$, and let $q \in \mathbb{R} \cup \{\pm\infty\}$. A point $\mathbf{y} \in \mathbb{R}^m$ is said to be a *limit point of f at q* if there exists a sequence $(q_i)_{i\in\mathbb{N}}$ of elements in S converging to q and such that $f(q_i) \to \mathbf{y}$. The set of such $\mathbf{y}$ is denoted by $\lim(f, q) = \lim_x(f(x), q)$.

In particular, for a bounded map $f : [q_0, \infty) \to \mathbb{R}$, it follows that

$$\inf(\lim(f, \infty)) = \liminf_{q\to\infty} f(q) \quad \text{and} \quad \sup(\lim(f, \infty)) = \limsup_{q\to\infty} f(q). \tag{1.1.1}$$

It is also worth noting that if $g : \mathbb{R}^m \to \mathbb{R}^{m'}$ is a continuous map, then

$$g(\lim(f, q)) \subseteq \lim(g \circ f, q), \tag{1.1.2}$$

and that equality holds if the image of f is bounded on some neighbourhood of q.

Before stating the lemma, note for maps $L, h : [q_0, \infty) \to \mathbb{R}$ that $L = \mathcal{O}(h)$ means that there exists a positive constant C such that $|L(q)| \leq Ch(q)$ for every sufficiently large q, and note that $L = o(h)$ means that $h(q) > 0$ for every sufficiently large q and that $\lim_{q\to\infty} L(q)/h(q) = 0$. In practice, this thesis considers the cases where the map $h(q)$ is either the identity mapping q or the constant mapping 1.

Lemma 1.1.6. *Let $L, P, h : [q_0, \infty) \to \mathbb{R}$ be maps with $h(q) > 0$ for every sufficiently large q and such that $L - P = o(h)$. It follows that $\lim(L/h, \infty) = \lim(P/h, \infty)$. Moreover, if L/h and P/h are bounded, then*

$$\liminf_{q\to\infty} \frac{L(q)}{h(q)} = \liminf_{q\to\infty} \frac{P(q)}{h(q)} \quad \textit{and} \quad \limsup_{q\to\infty} \frac{L(q)}{h(q)} = \limsup_{q\to\infty} \frac{P(q)}{h(q)}.$$

Proof: Let x be a limit point of L/h at infinity, and so

$$\lim_{i\in\mathbb{N}} \frac{L(q_i)}{h(q_i)} = x$$

for some sequence $(q_i)_{i\in\mathbb{N}}$ with $q_i \to \infty$. It follows that

$$\lim_{i\in\mathbb{N}} \frac{P(q_i)}{h(q_i)} = \lim_{i\in\mathbb{N}} \frac{P(q_i)}{h(q_i)} + \lim_{i\in\mathbb{N}} \frac{L(q_i) - P(q_i)}{h(q_i)} = \lim_{i\in\mathbb{N}} \frac{L(q_i)}{h(q_i)} = x.$$

A symmetric argument completes the proof of the first claim.

If L/h and P/h are bounded, then (1.1.1) yields the second claim. ∎

Definition 1.1.7. For each $q_0 > 0$ and each map $f : [q_0, \infty) \to \mathbb{R}^n$, define the set $\mathcal{F}(f) = \lim_q(\varphi(q; f), \infty)$, and denote its convex hull by $\mathcal{K}(f)$.

In other words, the set $\mathcal{F}(f)$ is the set of limits in $\mathbb{R}^n$ of sequences $(q_i^{-1} f(q_i))_{i\in\mathbb{N}}$ where $(q_i)_{i\in\mathbb{N}}$ is a sequence of positive reals with $q_i \to \infty$.

Remark. If $\|f(q)\| = \mathcal{O}(q)$, then $\varphi(q; f)$ is a bounded function, and if $T : \mathbb{R}^n \to \mathbb{R}$ is a linear map, then $T \circ \varphi(q; f) = \varphi(q; T \circ f)$, and so (1.1.2) implies that

$$T(\mathcal{F}(f)) = \mathcal{F}(T \circ f). \tag{1.1.3}$$

This remark is key in proving the following proposition.

Proposition 1.1.8. *Let* $\mathbf{L}, \mathbf{P} : [q_0, \infty) \to \mathbb{R}^n$ *be maps such that* $\|\mathbf{L}(q)\| = \mathcal{O}(q)$ *and* $\|\mathbf{L}(q) - \mathbf{P}(q)\| = o(q)$*. It follows that* $\|\mathbf{P}(q)\| = \mathcal{O}(q)$*, and that*

$$\underline{\varphi}_T(\mathbf{L}) = \underline{\varphi}_T(\mathbf{P}) = \inf T(\mathcal{F}(\mathbf{P}))$$

for each linear map $T : \mathbb{R}^n \to \mathbb{R}$.

Proof: Note that $T(\mathbf{L}(q)) - T(\mathbf{P}(q)) = \|T\| o(q)$, and that $\|\mathbf{P}(q)\| = \mathcal{O}(q)$ since

$$\|\mathbf{P}(q)\| \le \|\mathbf{P}(q) - \mathbf{L}(q)\| + \|\mathbf{L}(q)\| = o(q) + \mathcal{O}(q) = \mathcal{O}(q).$$

Hence, $\varphi(q; \mathbf{L})$, $\varphi(q; \mathbf{P})$ are bounded, so Lemma 1.1.6 implies $\underline{\varphi}_T(\mathbf{L}) = \underline{\varphi}_T(\mathbf{P})$. Finally,

$$\underline{\varphi}_T(\mathbf{P}) = \liminf_{q\to\infty} \varphi_T(q; \mathbf{P}) = \inf \lim_q(\varphi(q; T \circ \mathbf{P}); \infty) = \inf \mathcal{F}(T \circ \mathbf{P})$$

by (1.1.1), and $\mathcal{F}(T \circ \mathbf{P}) = T(\mathcal{F}(\mathbf{P}))$ by (1.1.3), completing the proof. ∎

Note that the infimum is preserved by taking convex hulls, and that convexity is preserved by linear maps. Thus, as the maps $\|\mathbf{L}_{\boldsymbol{\xi}}(q)\|$ are known to be $\mathcal{O}(q)$, it follows that if $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\mathbf{P} : \mathbb{R}^+ \to \mathbb{R}^n$ satisfy $\|\mathbf{L}_{\boldsymbol{\xi}}(q) - \mathbf{P}(q)\| = o(q)$, then

$$\underline{\varphi}_T(\boldsymbol{\xi}) = \underline{\varphi}_T(\mathbf{P}) = \inf T(\mathcal{F}(\mathbf{P})) = \inf T(\mathcal{K}(\mathbf{P})), \tag{1.1.4}$$

for each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, where the infimum is taken with respect to the *product ordering* on $\mathbb{R}^m$, i.e. $(x_1, \ldots, x_m) \le (y_1, \ldots, y_m) \Leftrightarrow x_i \le y_i$ for $i = 1, \ldots, m$.

1.2 n-Systems

In this section, the definition of objects known as *n-systems* is given, and some important subclasses of these objects are defined. Two important theorems proven by Roy in [20], one of which implies that studying the functions $\mathbf{L}_{\boldsymbol{\xi}}$ is in some sense equivalent to studying n-systems, are stated. Some tools are also developed, many of which are key in proving results in the following chapters, and some examples and illustrations are given to help the reader develop an intuition towards better understanding n-systems and how they may be used in proofs.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the canonical basis elements of $\mathbb{R}^n$, and define the sets

$$\Delta^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, 0 \le x_1 \le \cdots \le x_n\} \quad \text{and} \quad \Sigma^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_1 + \cdots + x_n = 1\}.$$

Definition 1.2.1. A map $\mathbf{P} : I \to \mathbb{R}^n$ is called an *n-system* if I is a closed subinterval of $\mathbb{R}$ with non-zero length and $\min I > 0$, and if the following properties hold for each $q \in I$:

(S1) The point $\mathbf{P}(q)$ lies in Δ^n, that is $0 \le P_1(q) \le \cdots \le P_n(q)$.

(S2) The point $q^{-1}\mathbf{P}(q)$ lies in Σ^n, that is $P_1(q) + \cdots + P_n(q) = q$.

(S3) There exists k, l with $P_k(q) \ge P_l(q)$ and a neighbourhood U of q such that

$$\mathbf{P}(q') = \begin{cases} \mathbf{P}(q) + (q' - q)\mathbf{e}_k & \text{if } q' \le q, \\ \mathbf{P}(q) + (q' - q)\mathbf{e}_l & \text{if } q' \ge q; \end{cases} \qquad (q' \in U).$$

Thus, writing $\mathbf{P} = (P_1, \dots, P_n)$, the components $P_1, \dots, P_n$ are monotone increasing and continuous, and the pair (k, l) is unique when q is in the interior of I.

Moreover, it is *non-degenerate* if the following condition holds for each $q \in I$:

(S4) For any $i \in \{1, \dots, n-1\}$ such that $P_i(q) = P_{i+1}(q)$, it follows that

$$\begin{cases} P_i'(q^-) = 1 & \text{if } q > \inf I, \\ P_{i+1}'(q^+) = 1 & \text{if } q < \sup I, \end{cases}$$

where q^- (resp. q^+) indicates left (resp. right) differentiation at q.

Observe that if $P_1(q), \dots, P_n(q)$ are distinct, then (S4) holds automatically at q.

Figure 1.2.1 illustrates the combined graph of some non-degenerate 3-system $\mathbf{P} = (P_1, P_2, P_3)$, which is the graph obtained by superimposing the graphs of the individual components P_1, P_2 and P_3. Observe by property (S1) that there is no ambiguity in determining which line segments correspond to which component.

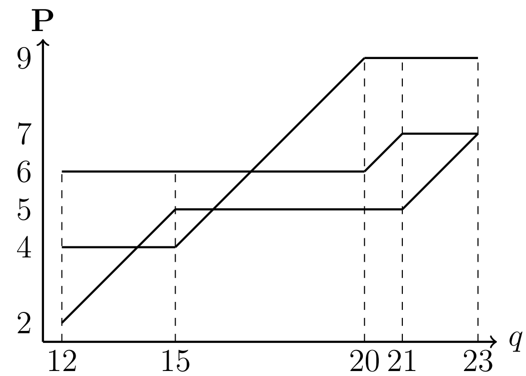

Figure 1.2.1: Some non-degenerate 3-system $\mathbf{P}$

Definition 1.2.2. A *system* refers to any m-system where $m \in \mathbb{N}$ with $m \geq 2$.

Verifying that a map is an n-system can be simplified via the following criteria.

Proposition 1.2.3. *Let $a, b, d_0, \ldots, d_k > 0$ with $a = d_0 < \cdots < d_k = b$, and suppose that $\mathbf{P} : [a, b] \to \mathbb{R}^n$ is a map which is affine on $[d_{i-1}, d_i]$ for $i = 1, \ldots, k$. If one of the properties (S1), (S2), (S3) holds at each d_i for $i = 0, \ldots, k$, then it holds on $[a, b]$.*

Proof: Let $i \in \{1, \ldots, k\}$, write $\mathbf{x} = \mathbf{P}(d_{i-1})$ and $\mathbf{y} = \mathbf{P}(d_i)$, and let $\mathbf{z} = \mathbf{P}(q)$ where $q = \lambda d_{i-1} + (1-\lambda)d_i$ for some $\lambda \in (0, 1)$. Hence, as $\mathbf{P}$ is affine on $[d_{i-1}, d_i]$, it follows that $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. Now, if (S1) holds at d_{i-1} and d_i, then one finds for each $l \in \{1, \ldots, n-1\}$ that

$$z_l = \lambda x_l + (1-\lambda)y_l \leq \lambda x_{l+1} + (1-\lambda)y_{l+1} = z_{l+1},$$

and that $z_1 = \lambda x_1 + (1-\lambda)y_1 \geq 0$, so that (S1) holds at q. On the other hand, if (S2) holds at d_{i-1} and d_i, then the sum of the coordinates of $\mathbf{z}$ is $\lambda d_{i-1} + (1-\lambda)d_i = q$, and so (S2) holds at q, completing the proof of the first implication. Finally, if property (S3) holds at d_i, then $\mathbf{P}'(q) = \mathbf{P}'(d_i^-)$ implies that it holds at q. ∎

Remark. The original definition of an n-system by Schmidt and Summerer was given in terms of the maps $S_i = P_1 + \cdots + P_i$, as seen through the present context in Section 2 of [20]. This interpretation can lead to certain simplifications, especially through the application of the following proposition in certain proofs.

Proposition 1.2.4. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, and write $S_i = P_1 + \cdots + P_i$ for each $i \in \{1, \ldots, n\}$. If $q \in I^\circ$ and $k, l \in \{1, \ldots, n\}$ are such that $k \leq l$ and $S_k'(q^-) = 1 \neq S_l'(q^+)$, then $P_k(q) = \cdots = P_{l+1}(q)$.*

Proof: Suppose that $S_k'(q^-) = 1 \neq S_l'(q^+)$, and so $S_l'(q^+) = 0$. Then, there exists k', l' such that $k' \leq k \leq l < l'$ and $P_{k'}'(q^-) = 1 = P_{l'}'(q^+)$, from which properties (S1) and (S3) yield $P_{k'}(q) = P_{l'}(q)$. Hence, property (S1) yields $P_k(q) = \cdots = P_{l+1}(q)$. ∎

This may all seem nebulous at first, especially properties (S3) and (S4), and so further discussion illustrating how one might better interpret these systems will be given in the sequel. However, before doing so, some terminology and a theorem giving a natural correspondence between n-systems and the points $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ are given.

1.2.1 Correspondence Theorem

In order to make sense of the following definition, note that properties (S1) and (S2) imply that the image of a system is unbounded if and only if its domain is unbounded. Hence, a system is called *unbounded* if either its domain or image is unbounded. Also, observe for an unbounded n-system $\mathbf{P} : I \to \mathbb{R}^n$ that if P_i is bounded for some $i \in \{1, \ldots, n\}$, then property (S1) implies that each $P_1, \ldots, P_{i-1}$ is bounded as well. In particular, it follows from property (S2) that P_n is unbounded.

Definition 1.2.5. Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, and let $m \in \{0, \ldots, n\}$ be maximal such that P_i is bounded for each i with $1 \leq i \leq m$, and so P_i is unbounded for each $i = m+1, \ldots, n$. The *dimension* of $\mathbf{P}$ is the quantity given and denoted by

$$\dim \mathbf{P} = n - m \in \{0, \ldots, n\},$$

and an n-system is said to be *proper* if $\dim \mathbf{P} = n$, i.e. if the dimension is maximal.

Remark. A system has dimension zero if and only if it is bounded.

The following proposition gives properties of components based on whether or not they are bounded. It is especially important in studying proper systems, which are important in the study of spectra, as Theorem 1.2.10 will show.

Proposition 1.2.6. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an unbounded n-system, let m be such that $\dim \mathbf{P} = n - m$, and let $i \in \{1, \ldots, n-1\}$. If $i \leq m$, then P_i is eventually constant, and if $i > m$, then $P_i(q) = P_{i+1}(q)$ for arbitrarily large values of $q \in I$.*

Proof: Write $S_i = P_1 + \cdots + P_i$, and suppose that P_i is not eventually constant. Property (S3) implies for each $N \in \mathbb{N}$ that $P_i'(q_N) = 1$ for some $q_N > N$. Hence, as P_n is unbounded, for each $N \in \mathbb{N}$ there exists t_N minimal with $t_N > q_N$ such that $S_i'(t_N^+) = 0$, and so $S_i'(t_N^-) = 1$. Proposition 1.2.4 then yields for each $N \in \mathbb{N}$ that

$$P_i(t_N) = P_{i+1}(t_N).$$

For each $N \in \mathbb{N}$ and each $q \leq t_N$, it follows that $P_{i+1}(q) \leq P_{i+1}(t_N) = P_i(t_N)$. Hence, if P_i is bounded, then P_{i+1} is bounded, since $t_N \to \infty$ as $N \to \infty$. Now, suppose that P_{i+1} is eventually constant, and so there exists M such that $P_i(t_N) = P_{i+1}(t_N) = P_{i+1}(t_M)$ for each $N > M$. As P_i is monotone, it follows that P_i is eventually constant, which is a contradiction. Hence, P_{i+1} is not eventually constant.

Now, if $i \leq m$, then as P_i is bounded, induction yields that $P_i, \ldots, P_n$ are all bounded, which is a contradiction. Therefore, if $i \leq m$, then P_i is eventually constant.

Finally, if $i > m$, then P_i is unbounded, and so is not eventually constant. Since $t_N \to \infty$ as $N \to \infty$, then $P_i(q) = P_{i+1}(q)$ for arbitrarily large values of $q \in I$. ∎

The following theorem is due to Roy in [20], stated differently but equivalently.

Theorem 1.2.7. *For each $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, there exists an n-system $\mathbf{P} : [q_0, \infty) \to \mathbb{R}^n$ such that $\|\mathbf{P} - \mathbf{L}_{\boldsymbol{\xi}}\| = \mathcal{O}(1)$. Conversely, for each n-system $\mathbf{P} : [q_0, \infty) \to \mathbb{R}^n$, there exists $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\|\mathbf{P} - \mathbf{L}_{\boldsymbol{\xi}}\| = \mathcal{O}(1)$. Moreover, the coordinates of $\boldsymbol{\xi}$ are linearly independent over $\mathbb{Q}$ if and only if $\mathbf{P}$ is proper, i.e. if $\dim \mathbf{P} = n$.*

The following corollary of Proposition 1.1.1 refines the last statement.

Corollary 1.2.8. *Let $\mathbf{P} : [q_0, \infty) \to \mathbb{R}^n$ be a map and let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a non-zero point in $\mathbb{R}^n$. If $\|\mathbf{P} - \mathbf{L}_{\boldsymbol{\xi}}\| = \mathcal{O}(1)$, then*

$$\dim \mathbf{P} = \dim \langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}},$$

i.e. the rank of $\{\xi_1, \ldots, \xi_n\}$ over $\mathbb{Q}$ is the dimension of $\mathbf{P}$.

Proof: Let $m \in \{0, \ldots, n-1\}$ be maximal such that $L_{\boldsymbol{\xi},i}$ is bounded for each i with $1 \leq i \leq m$. Since $\|\mathbf{P} - \mathbf{L}_{\boldsymbol{\xi}}\| = \mathcal{O}(1)$, then P_i is bounded if and only if $L_{\boldsymbol{\xi},i}$ is bounded, and so $\dim \mathbf{P} = n - m$. Therefore, as Proposition 1.1.1 yields $n - m = \dim \langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}}$, the conclusion follows. ∎

The following theorem is also due to Roy in [20], presented in a form which is weaker but sufficiently strong with regards to the scope of this thesis.

Theorem 1.2.9. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an unbounded n-system. There exists a non-degenerate n-system $\tilde{\mathbf{P}} : I \to \mathbb{R}^n$ such that $\|\mathbf{P} - \tilde{\mathbf{P}}\| = \mathcal{O}(1)$.*

Taking into consideration the approximation criterion given by Proposition 1.1.8, and combining the results of Theorems 1.2.7 and 1.2.9 with Corollary 1.2.8, the image and spectra of functions of the form $\underline{\varphi}_T$ can be described as in the following theorem. Before its statement, let $\mathcal{S}^n$ denote the set of unbounded n-systems.

Theorem 1.2.10 (Correspondence)**.** *For each $d \in \{1, \ldots, n\}$, write*

$$\mathbb{R}^n_d = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid \dim\langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}} = d\},$$

and

$$\mathcal{S}^n_d = \{\mathbf{P} \in \mathcal{S}^n \mid \dim \mathbf{P} = d\}.$$

It follows for each linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ that

$$\underline{\varphi}_T(\mathbb{R}^n_d) = \underline{\varphi}_T(\mathcal{S}^n_d) = \underline{\varphi}_T(\{\mathbf{P} \in \mathcal{S}^n_d \,|\, \mathbf{P} \text{ is non-degenerate}\}).$$

In particular, the spectrum of $\underline{\varphi}_T$ is the set of m-tuples $\underline{\varphi}_T(\mathbf{P})$ where $\mathbf{P}$ runs through the set of proper non-degenerate n-systems.

1.2.2 Division Points and Non-Degenerate Systems

As the correspondence theorem shows, it suffices to consider proper non-degenerate systems in order to determine the spectra of families of exponents. Incidentally, it is somewhat easier to work with non-degenerate systems as they have simpler pointwise properties. In order to see this, consider the following notion, and define the sets

$$\Delta_j = \Delta^n_j = \{\mathbf{x} \in \Delta^n \,|\, x_j = x_{j+1}\},$$

for each $j \in \{1, \ldots, n-1\}$.

Definition 1.2.11. Fix an n-system $\mathbf{P} : I \to \mathbb{R}^n$, and let $q \in I$. If k, l can be chosen to be distinct for the property (S3) at q, then q is called a *division number*, and $\mathbf{P}(q)$ is called a *division point.* If $k < l$, then q is called a *transition number* and $\mathbf{P}(q)$ is called a *transition point.* If $k > l$, then q is called a *switch number* and $\mathbf{P}(q)$ is called a *switch point.*

In particular, a number $q \in I^{\circ}$ is not a division number with respect to $\mathbf{P}$ if and only if the derivative of $\mathbf{P}$ exists at q, in which case $\mathbf{P}'(q) = \mathbf{e}_i$ for some $i \in \{1, \ldots, n\}$. Moreover, since the domain of an n-system is closed by definition, the set of division numbers has the following property.

Proposition 1.2.12. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, and D its set of division numbers. The set D is enumerable in increasing order, and $|D| = \infty \Leftrightarrow \sup D = \infty$.*

Proof: Assume without loss of generality that $|D| = \infty$. Now, suppose that the set D has a limit point q in $\mathbb{R}$, and so $q \in I$ as I is closed. By property (S3), there exists a punctured neighbourhood U of q such that $\mathbf{P}'$ is defined on U. Thus, $D \cap U$ is empty, which is a contradiction as q is a limit point of D. Hence, D is a closed discrete set, and so D is finite in each compact interval. ∎

Definition 1.2.13. Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, and let $(d_i)_{i\in J}$ be its increasing sequence of division numbers. The sequence $(\mathbf{P}(d_i))_{i\in J}$ is called the *ordered sequence of division points of* $\mathbf{P}$.

The following proposition highlights a property of non-degenerate systems.

Proposition 1.2.14. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a non-degenerate n-system, and let $q \in I$. If $\mathbf{P}(q) \in \Delta_i \cap \Delta_j$ for some $i, j \in \{1, \dots, n\}$, then $i = j$ and q is a transition number.*

Proof: Suppose that $q > \inf I$, and so property (S4) yields $P_i'(q^-) = 1 = P_j'(q^-)$, while property (S2) implies that $i = j$. The case $q < \sup I$ is proven similarly. ∎

For a non-degenerate n-system $\mathbf{P} : I \to \mathbb{R}^n$ and each $q \in I$, it follows that $\mathbf{P}(q)$ either has distinct coordinates or lies in Δ_j for a unique choice of $j \in \{1, \dots, n-1\}$.

Corollary 1.2.15. *If $\mathbf{P} : I \to \mathbb{R}^n$ is an n-system, then $\mathbf{P}$ is non-degenerate if and only if for all $q \in I$, the point $\mathbf{P}(q)$ has distinct coordinates when $\mathbf{P}'(q)$ is defined.*

Proof: If $\mathbf{P} : I \to \mathbb{R}^n$ is a non-degenerate n-system and $q \in I$, then Proposition 1.2.14 implies that $\mathbf{P}(q)$ has distinct coordinates if q is not a transition number. Conversely, if an n-system $\mathbf{P}$ has distinct coordinates wherever its derivative is defined, and if $\mathbf{P}(q) \in \Delta_i$ for some q, i, then property (S3) guarantees that $\mathbf{P}$ has distinct coordinates in a punctured neighbourhood of q. It follows that $P_i'(q^-) = 1$ if $q < \sup I$ and that $P_{i+1}'(q^+) = 1$ if $q > \inf I$, and so property (S4) holds at each $q \in I$. ∎

Combining Propositions 1.2.6, 1.2.12 and 1.2.14 yields the following theorem.

Theorem 1.2.16. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a non-degenerate n-system, and T its set of transition numbers. The set T can be partitioned by sets $T_1, \dots, T_{n-1}$ so that*

$$\mathbf{P}(q) \in \Delta_j \Leftrightarrow q \in T_j \quad (q \in I),$$

for $j = 1, \dots, n-1$. In particular, the set T_j is enumerable in increasing order and satisfies $|T_j| = \infty \Leftrightarrow \sup T_j = \infty \Leftrightarrow P_j$ is unbounded, for each $j \in \{1, \dots, n-1\}$.

The following proposition shows that an n-system is non-degenerate if and only if it is non-degenerate at some point of each affine segment, which is key in determining when Theorem 1.2.16 can be applied. In particular, it is non-degenerate if it is non-degenerate at each division point.

Proposition 1.2.17. *Let $\mathbf{P} : [a, b] \to \mathbb{R}^n$ be an n-system, let $d_0, \dots, d_k$ be its successive division points, and suppose that property (S4) is satisfied at some $t_i \in [d_{i-1}, d_i]$ for each $i = 1, \dots, k$. It follows that $\mathbf{P}$ is non-degenerate.*

Proof: Let $i \in \{1, \ldots, k\}$, let $q_\lambda = (1-\lambda)d_{i-1} + \lambda d_i$ for $\lambda \in [0,1]$, and let $\mu \in [0,1]$ be such that $q_\mu = t_i$. Since there are no division points in (d_{i-1}, d_i), there exists j such that $\mathbf{P}'(q_\lambda) = \mathbf{e}_j$ for each $\lambda \in (0,1)$. Hence, for each $\lambda \in [0,1]$, it follows that $P_m(q_\lambda) = P_m(d_{i-1}) = P_m(d_i)$, for $m \neq j$, and that

$$P_j(q_\lambda) = P_j(d_{i-1}) + \lambda(P_j(d_i) - P_j(d_{i-1})) = P_j(d_{i-1}) + \lambda(d_i - d_{i-1}).$$

Now, fix $\lambda \in (0,1)$, and suppose that $\mathbf{P}(q_\lambda)$ does not have distinct coordinates, and so there exists l such that $\mathbf{P}(q_\lambda) \in \Delta_l$. If $j = l$, then

$$P_j(d_i) > P_j(q_\lambda) = P_{j+1}(q_\lambda) = P_{j+1}(d_i) \geq P_j(d_i),$$

which contradicts (S1), and if $j = l+1$, then

$$P_j(d_{i-1}) \geq P_{j-1}(d_{i-1}) = P_{j-1}(q_\lambda) = P_j(q_\lambda) > P_j(d_{i-1}),$$

which contradicts (S1). Hence, $j \notin \{l, l+1\}$, and so $\mathbf{P}(q) \in \Delta_l$ for all $q \in [d_{i-1}, d_i]$. If $\mu \in (0,1)$, then $\mathbf{P}'(q_\mu)$ is defined and so Corollary 1.2.15 implies that $\mathbf{P}(q_\mu)$ has distinct coordinates, which is contradiction. Therefore, $\mu \in \{0,1\}$, and so combining property (S4) with $P_j(d_{i-1}^+) = 1 = P_j'(d_i^-)$ yields $j \in \{l, l+1\}$, which is also a contradiction. Thus, $\mathbf{P}(q)$ has distinct coordinates for each $q \in (d_{i-1}, d_i)$, and so Corollary 1.2.15 implies that (S4) holds everywhere on $[d_{i-1}, d_i]$. ∎

A nice consequence of Theorem 1.2.16 lies in how much simpler it is to deal with non-degenerate systems and their *combined graphs*.

1.2.3 Combined Graphs

The following notion provides a very useful visualization technique with regards to the study of systems, especially those which are non-degenerate.

Definition 1.2.18. Let $S \subseteq \mathbb{R}$ and let $\mathbf{P} = (P_1, \ldots, P_n) : S \to \mathbb{R}^n$ be a map. The *combined graph* of $\mathbf{P}$ is the union of the graphs of its components $P_1, \ldots, P_n : S \to \mathbb{R}$.

In view of property (S1), it is typically easy to recover a system from its combined graph, but this can be difficult locally, since multiple maps can coincide ambiguously. For instance, the 5-systems $\mathbf{P}, \mathbf{P}' : [10, 11] \to \mathbb{R}^5$ defined by

$$\mathbf{P}(10+t) = (1,2,2,2,3) + t\mathbf{e}_4 \quad \text{and} \quad \tilde{\mathbf{P}}(10+t) = (1,1,2,3,3) + t\mathbf{e}_3 \quad (t \in [0,1])$$

share the same combined graph as in Figure 1.2.2.

As Theorem 1.2.16 guarantees, this is not a problem for non-degenerate systems, since one can locally recover a non-degenerate system from its combined graph on any

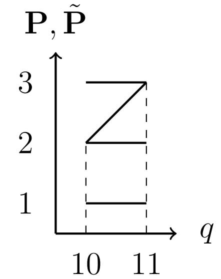

Figure 1.2.2

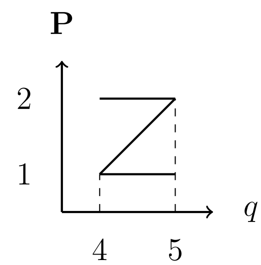

Figure 1.2.3

subinterval of I with positive length: This is easy for points with distinct coordinates as a consequence of property (S1). Hence, the only potentially problematic points are transition points, except that these are easily resolved as well, since it suffices to know any information on either the slopes to the right or the slopes to the left of a transition point in order to determine which two coordinates are coinciding. For instance, one observes that the 3-system $\mathbf{P} : [4, 5] \to \mathbb{R}^3$ defined by

$$\mathbf{P}(q) = (1, 1, 2) + (q - 4)\mathbf{e}_2 \quad (q \in [4, 5])$$

has its combined graph as in Figure 1.2.3, and that the slopes right of $\mathbf{P}(4)$ imply that $P_1 = P_2$ at 4, while the slopes left of $\mathbf{P}(5)$ imply that $P_2 = P_3$ at 5.

Meanwhile, property (S2) allows one to recover the abscissa of an image point by summing its coordinates, and allows one to recover a single coordinate if the other coordinates and the abscissa are known. In other words, the $n + 1$ data consisting of the n coordinates and the abscissa have n degrees of freedom.

Finally, property (S3) can be divided into three cases: Let k, l, U be as in property (S3) defining an n-system $\mathbf{P} : I \to \mathbb{R}^n$ at a point $q \in I$. If $k = l$, then the combined graph of $P_k = P_l$ on U is just a line segment with slope 1. If $k < l$, then properties (S1) and (S3) imply that $P_k(q) = P_l(q)$, and so the combined graph of (P_k, P_l) on U is as illustrated in Figure 1.2.4. Finally, if $k > l$, then (S1) implies that $P_k(q) \geq P_l(q)$. If $P_k(q) = P_l(q)$, then (S3) implies $P_k(q+\varepsilon) < P_l(q+\varepsilon)$ for some $\varepsilon \neq 0$. This contradicts (S1), and so $P_k(q) > P_l(q)$, and Figure 1.2.5 illustrates the combined graph of (P_k, P_l) on U. The combined graph of P_i on U is a flat line for $i \notin \{k, l\}$.

In particular, if $\mathbf{P}$ is non-degenerate, then $l = k + 1$ in Figure 1.2.4.

1.2.4 Construction of Systems

In general, given two n-systems $\mathbf{R} : [a, b] \to \mathbb{R}^n$ and $\mathbf{S} : [b, c] \to \mathbb{R}^n$ with $\mathbf{R}(b) = \mathbf{S}(b)$, it follows that the map $\mathbf{P} : [a, c] \to \mathbb{R}^n$ defined by $\mathbf{P} = \mathbf{R}$ on $[a, b]$ and $\mathbf{P} = \mathbf{S}$ on $[b, c]$ satisfies properties (S1) and (S2) which define an n-system. Moreover, if (S4) is

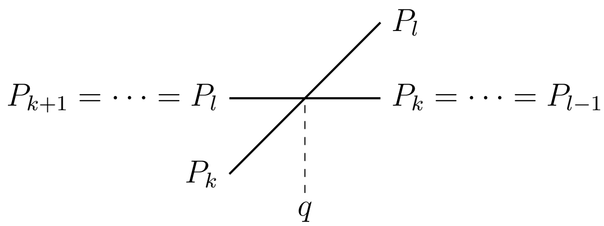

Figure 1.2.4: Case $k < l$

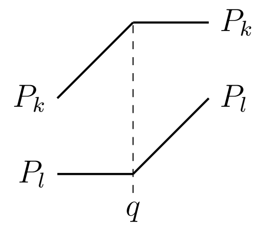

Figure 1.2.5: Case $k > l$

satisfied by $\mathbf{R}$ and $\mathbf{S}$, then it is satisfied by $\mathbf{P}$ as well. On the other hand, while (S3) must still hold at points $q \in [a, c]$ not equal to b, it is conceivable that it no longer holds at b. For instance, the map $\mathbf{P} : [2, 4] \to \Delta^2$ defined by the combined graph in Figure 1.2.6 is a 2-system on $[2, 3]$ and $[3, 4]$ but does not satisfy property (S3) at 3.

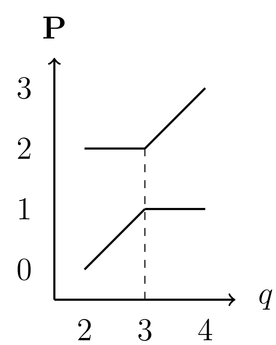

Figure 1.2.6

Hence, in order for $\mathbf{P}$ to be an n-system, it is necessary and sufficient to check that property (S3) holds at b. This leads to the following definition and result, which in particular provides a simple criterion for gluing non-degenerate n-systems.

Definition 1.2.19. Let $\mathbf{R} : [a, b] \to \mathbb{R}^n$ and $\mathbf{S} : [b, c] \to \mathbb{R}^n$ be n-systems with $\mathbf{R}(b) = \mathbf{S}(b)$, and define a map $\mathbf{P} : [a, c] \to \mathbb{R}^n$ by

$$\mathbf{P}(q) = \begin{cases} \mathbf{R}(q) & q \in [a, b], \\ \mathbf{S}(q) & q \in [b, c]; \end{cases} \quad (q \in [a, b])$$

If property (S3) holds at b for the map $\mathbf{P}$, then the pair $(\mathbf{R}, \mathbf{S})$ is said to *connect*, and the map $\mathbf{P} : [a, c] \to \mathbb{R}^n$ is an n-system called the *connection* of $(\mathbf{R}, \mathbf{S})$. Similarly, if a sequence $(\mathbf{R}^{(i)} : [a_i, a_{i+1}] \to \mathbb{R}^n)_{i \geq 0}$ is such that $(\mathbf{R}^{(i)}, \mathbf{R}^{(i+1)})$ connects for $i \geq 0$ and such that $(a_i)_{i \geq 0}$ has no accumulation point, then the sequence is said to connect, and the map $\mathbf{P} : \cup_{i \geq 0}[a_i, a_{i+1}] \to \mathbb{R}^n$ defined by

$$\mathbf{P}(q) = \mathbf{R}^{(i)}(q) \quad (q \in [a_i, a_{i+1}], i \geq 0)$$

is an n-system called the connection of $(\mathbf{R}^{(i)})_{i\geq 0}$.

Theorem 1.2.20. *Let $\mathbf{R} : [a,b] \to \mathbb{R}^n$, $\mathbf{S} : [b,c] \to \mathbb{R}^n$ be n-systems with $\mathbf{R}(b) = \mathbf{S}(b)$. If $\mathbf{R}$ and $\mathbf{S}$ are non-degenerate, and if $\mathbf{R}(b), \mathbf{S}(b) \in \Delta_j$ for some $j \in \{1,\ldots,n-1\}$, then the pair $(\mathbf{R},\mathbf{S})$ connects, and their connection is non-degenerate.*

Proof: By gluing $\mathbf{R}$ and $\mathbf{S}$ into a map $\mathbf{P}$, it suffices to check that (S3) holds at b. As $\mathbf{R}$ and $\mathbf{S}$ are non-degenerate, and as $\mathbf{R}(b), \mathbf{S}(b) \in \Delta_j$, then $P_j'(b^-) = 1 = P_{j+1}'(b^+)$. Thus, as $P_j(b) = P_{j+1}(b)$, it follows that (S3) holds at b, completing the proof. ∎

The following theorem gives criteria determining when a pair $(\mathbf{x},\mathbf{y}) \in \Delta_i \times \Delta_j$, for some $i,j \in \{1,\ldots,n-1\}$, are such that a non-degenerate n-system $\mathbf{P} : [a,b] \to \mathbb{R}^n$ can be constructed with $\mathbf{P}(a) = \mathbf{x}$ and $\mathbf{P}(b) = \mathbf{y}$. In this way, a simple criterion for gluing such systems together is given by Theorem 1.2.20.

Theorem 1.2.21. *Let $\mathbf{x},\mathbf{y} \in \Delta^n$ such that $\mathbf{x} \in \Delta_i$ and $\mathbf{y} \in \Delta_j$ for a unique choice of $i,j \in \{1,\ldots,n-1\}$. Suppose that either*

$$j \leq i \quad \textit{and} \quad \begin{cases} x_k < y_k & \textit{if } k \in \{j, i+1\}; \\ x_k \leq y_k & \textit{if } j < k < i+1; \\ x_k = y_k & \textit{else}, \end{cases} \tag{1.2.1}$$

or

$$j > i \quad \textit{and} \quad (x_1,\ldots,\widehat{x_i},\ldots,x_n) = (y_1,\ldots,\widehat{y_j},\ldots,y_n), \tag{1.2.2}$$

where the hat on a coordinate means that it is omitted. Then, there exists a non-degenerate n-system $\mathbf{P} : [a,b] \to \mathbb{R}^n$ with $\mathbf{P}(a) = \mathbf{x}$ and $\mathbf{P}(b) = \mathbf{y}$.

Proof: Suppose that (1.2.1) is satisfied, and write $\mathbf{z}_m = (x_1,\ldots,x_m,y_{m+1},\ldots,y_n)$, letting d_m be the sum of its coordinates, for $m = 0,\ldots,n$. It follows that $d_n,\ldots,d_0$ is an increasing sequence with $\mathbf{z}_{m-1} = \mathbf{z}_m + (y_m - x_m)\mathbf{e}_m = \mathbf{z}_m + (d_{m-1} - d_m)\mathbf{e}_m$ for $m = 1,\ldots,n$. Letting $a = d_n$ and $b = d_0$, define a map $\mathbf{P} : [a,b] \to \mathbb{R}^n$ by

$$\mathbf{P}(d_m + t) = \mathbf{z}_m + t\mathbf{e}_m \quad (t \in [0, d_{m-1} - d_m], m \in \{1,\ldots,n\}).$$

Note that (1.2.1) implies that $a = d_n = d_{i+1} < d_i$ and $d_j < d_{j-1} = d_0 = b$, and so the division points in (a,b) form the set $\{d_j,\ldots,d_i\}$. Also note that properties (S1) and (S2) hold at $a = d_n$ and $b = d_0$ by construction. Thus, Propositions 1.2.3 and 1.2.17 imply that it remains to verify that the properties hold at $d_i,\ldots,d_j$. Towards this, let $m \in \{j,\ldots,i\}$, and observe by construction that property (S2) holds at d_m. Also, note that

$$P_{m+1}'(d_m^-) = 1 = P_m'(d_m^+),$$

and so property (S3) holds at d_m. Moreover, as $j \le m \le i$, it follows that

$$x_1 < \cdots < x_m \quad \text{and} \quad y_{m+1} < \cdots < y_n.$$

Note that (1.2.1) yields $x_i = x_{i+1} < y_{i+1}$, and $x_m < x_{m+1} \le y_{m+1}$ if $m < i$. Thus, $\mathbf{z}_m$ has distinct coordinates, and so properties (S1) and (S4) hold at d_m, while (S4) holds at a, b by definition, completing the proof that $\mathbf{P}$ is a non-degenerate n-system.

Now, suppose that (1.2.2) is satisfied, and write $\mathbf{z}_m = (y_1, \dots, y_m, x_{m+1}, \dots, x_n)$, letting d_m be the sum of its coordinates, for $m = i, \dots, j$. It follows that $d_i, \dots, d_j$ is an increasing sequence with $\mathbf{z}_{m-1} = \mathbf{z}_m - (y_m - x_m)\mathbf{e}_m = \mathbf{z}_m - (d_m - d_{m-1})\mathbf{e}_m$ for $m = i+1, \dots, j$. Letting $a = d_i$ and $b = d_j$, define a map $\mathbf{P} : [a, b] \to \mathbb{R}^n$ by

$$\mathbf{P}(d_m - t) = \mathbf{z}_m - t\mathbf{e}_m \quad (t \in [0, d_m - d_{m-1}], m \in \{i+1, \dots, j\}).$$

As before, it suffices to verify that the properties hold at $d_{i+1}, \dots, d_{j-1}$. To this end, let $m \in \{i+1, \dots, j-1\}$, noting that property (S2) holds at d_m by construction. Also note that the choice of i, j being unique by hypothesis implies by (1.2.2) that $x_m < y_m$ and $x_j < y_j$ so that $d_m - d_{m-1} > 0$ and $d_j - d_{j-1} > 0$. Hence, it follows that

$$P'_m(d_m^-) = 1 = P'_{m+1}(d_m^+).$$

Note that (1.2.2) yields that $\mathbf{z}_m \in \Delta_m$ if $m = i, \dots, j-1$, while $y_j = y_{j+1} = x_{j+1}$ by hypothesis, so that $\mathbf{z}_j \in \Delta_j$. It follows that (S1), (S3), and (S4) hold at d_m, while (S4) holds at a, b by definition, completing the proof that $\mathbf{P}$ is a non-degenerate n-system. ∎

In providing examples of this construction, Figure 1.2.7 gives a combined graph for the case $j \le i$, while Figure 1.2.8 gives a combined graph for the case $j > i$.

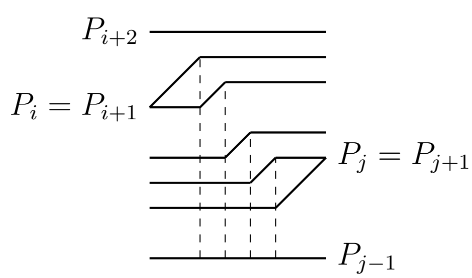

Figure 1.2.7: Case $j \le i$

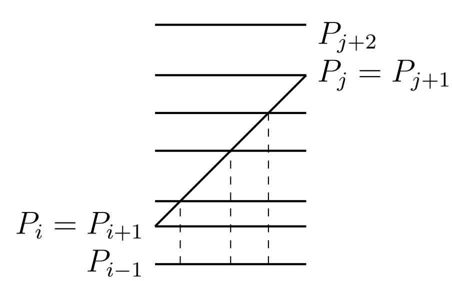

Figure 1.2.8: Case $j > i$

The following result generalizes Theorem 1.2.21 to sequences.

Theorem 1.2.22. *Let $N \subseteq \mathbb{N}$, and let $(\mathbf{x}_k)_{k\in N}$ be a sequence with $\mathbf{x}_k \in \Delta_{i_k}$ for a unique choice of i_k, for each $k \in N$. If each pair $(\mathbf{x}_k, \mathbf{x}_{k+1}) \in \Delta_{i_k} \times \Delta_{i_{k+1}}$ satisfies either (1.2.1) or (1.2.2) as in Theorem 1.2.21, then there exists a non-degenerate n-system $\mathbf{P} : \cup_{k\in\mathbb{N}}[a_k, a_{k+1}] \to \mathbb{R}^n$ with $\mathbf{P}(a_k) = \mathbf{x}_k$ for each $k \in \mathbb{N}$.*

Proof: This follows immediately by applying Theorems 1.2.20 and 1.2.21. ∎

Observe that in order for two n-systems $\mathbf{R} : [a, b] \to \mathbb{R}^n$ and $\mathbf{S} : [b', c'] \to \mathbb{R}^n$ to connect, it is required that $\mathbf{R}(b) = \mathbf{S}(b')$. However, as long as $\mathbf{R}(b)$ is proportional to $\mathbf{S}(b')$, one might be able to form a connection between $\mathbf{R}$ and a *rescaling* of $\mathbf{S}$.

1.2.5 Rescaling Systems

Before explaining what it means to *rescale* an n-system, the following general construction of a group action G on maps $X \to Y$ is given. In practice, this thesis considers the group $e^{\mathbb{R}} = (0, \infty)$ with standard multiplication, and maps $\mathbb{R} \to \mathbb{R}^n$.

Let G be a group which acts on sets X, Y, and let $F(X, Y)$ denote the set of maps $S \to T$ where $S \subseteq X$ and $T \subseteq Y$. Since G acts on X and Y, it follows that G acts naturally on the powersets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. In this way, the group G is seen to act on $F(X, Y)$ as follows. For each $g \in G$ and each $f : S \to T$ with $S \subseteq X$ and $T \subseteq Y$, define the map $g \cdot f : gS \to gT$ by

$$(g \cdot f)(q) = gf(g^{-1}q) \quad (q \in gS).$$

Indeed, if 1 is the identity for the group G, then $1 \cdot f = f$, and if $g, h \in G$, then

$$((gh) \cdot f)(q) = ghf(h^{-1}g^{-1}q) = g(h \cdot f)(g^{-1}q) = (g \cdot (h \cdot f))(q)$$

for each $q \in ghS$, and so $(gh) \cdot f = g \cdot (h \cdot f)$.

Definition 1.2.23. This action is called the *natural* group action of G on $F(X, Y)$.

Now, consider the action of $e^{\mathbb{R}} = (0, \infty)$ on $\mathbb{R}^k, \mathcal{P}(\mathbb{R}^k)$ for $k = 1, n$ by scalar multiplication, i.e. for each $\rho \in e^{\mathbb{R}}$, $\rho \cdot x = \rho x$ for all $x \in \mathbb{R}^k$, and $\rho S = \{\rho s \mid s \in S\}$ for all $S \subseteq \mathbb{R}^k$. Thus, for an n-system $\mathbf{P} : I \to \mathbb{R}^n$ and $\rho \in e^{\mathbb{R}}$, the natural group action of $e^{\mathbb{R}}$ on $F(\mathbb{R}, \mathbb{R}^n)$ induces a map $\rho \cdot \mathbf{P} : \rho I \to \mathbb{R}^n$ by

$$\rho \cdot \mathbf{P}(q) = \rho \mathbf{P}(\rho^{-1}q) \quad (q \in \rho I).$$

The map $\rho \cdot \mathbf{P}$ is called a *rescaling* of $\mathbf{P}$.

Proposition 1.2.24. *Any rescaling $\rho \cdot \mathbf{P} : \rho I \to \mathbb{R}^n$ is an n-system.*

Proof: Let $q \in I$ and write $q' = \rho q$. Since $\rho > 0$, it follows that $\rho \cdot \mathbf{P}(q') = \rho \mathbf{P}(q)$ satisfies property (S1). Moreover, the sum of the coordinates of $\rho \cdot \mathbf{P}(q')$ is the sum of the coordinates of $\rho \mathbf{P}(q)$, which is $\rho q = q'$, and so property (S2) holds at q'. Finally, by letting k, l with $P_k(q) \geq P_l(q)$ and $P'_k(q^-) = 1 = P'_l(q^+)$, one finds that if q is in the interior of I, then $\rho \cdot P_k(q) \geq \rho \cdot P_l(q)$ and

$$(\rho \cdot P_k)'(q'^-) = P'_k(q^-) = 1 = P'_l(q^+) = (\rho \cdot P_l)'(q'^+),$$

since $(\rho \cdot \mathbf{P})'(q') = \rho\rho^{-1}\mathbf{P}'(\rho^{-1}q') = \mathbf{P}'(q)$, thereby completing the proof. ∎

In particular, if two n-systems $\mathbf{R} : [a, b] \to \mathbb{R}^n$ and $\mathbf{S} : [b', c'] \to \mathbb{R}^n$ are such that $\varphi(b; \mathbf{R}) = \varphi(b'; \mathbf{S})$, that is $b^{-1}\mathbf{R}(b) = b'^{-1}\mathbf{S}(b')$, then one can define a map $\mathbf{P} : [b, bc'/b'] \to \mathbb{R}^n$ as in Definition 1.2.19 with $\mathbf{S}$ replaced by $(b'/b) \cdot \mathbf{S}$. Moreover, the pair $(\mathbf{R}, (b/b') \cdot \mathbf{S})$ connects if and only if property (S3) holds at b for the map $\mathbf{P}$.

An important notion related to rescaling is that of self-similar systems.

1.2.6 Self-Similar Systems

Definition 1.2.25. An n-system $\mathbf{P} : I \to \mathbb{R}^n$ is said to be *self-similar* if there exists $\rho > 1$ such that $\mathbf{P}(\rho q) = \rho\mathbf{P}(q)$ for each $q \in I$. Such ρ is called a *period* of $\mathbf{P}$.

In particular, if $\mathbf{P} : I \to \mathbb{R}^n$ is a self-similar n-system with period ρ, then $\rho I \subseteq I$, and so I is unbounded, and $\rho \cdot \mathbf{P} = \mathbf{P}|_{\rho I}$. Moreover, $\mathbf{P}$ is proper if and only if $P_1 > 0$.

Remark. Let $\mathbf{P} : I \to \mathbb{R}^n$ be a self-similar n-system, and let ρ be the infimum of all periods of $\mathbf{P}$. Since $n > 1$, property (S3) implies that $\rho > 1$, and so ρ is a period of $\mathbf{P}$ by continuity. Moreover, if $\tilde{\rho}$ is a period of $\mathbf{P}$ with $\tilde{\rho} > \rho$, then there exists $m \in \mathbb{N}^+$ and $r \in [0, 1)$ such that $\tilde{\rho} = \rho^{m+r}$. Thus, as $\rho^m\mathbf{P}(\rho^r q) = \mathbf{P}(\tilde{\rho}q) = \rho^{m+r}\mathbf{P}(q)$ for each $q \in I$, it follows by the minimality of the period ρ that $r = 0$, and so $\tilde{\rho} = \rho^m$.

Now, if $\mathbf{P} : [a, \rho a] \to \mathbb{R}^n$ is an n-system such that $(\mathbf{P}, \rho \cdot \mathbf{P})$ connects, then $(\rho^i \cdot \mathbf{P})_{i\in\mathbb{N}}$ connects to form a self-similar n-system $\tilde{\mathbf{P}} : [a, \infty) \to \mathbb{R}^n$.

Definition 1.2.26. Let $\mathbf{P} : [a, \rho a] \to \mathbb{R}^n$ be an n-system such that $(\mathbf{P}, \rho\cdot\mathbf{P})$ connects. The connection $\tilde{\mathbf{P}} : [a, \infty) \to \mathbb{R}^n$ of $(\rho^i \cdot \mathbf{P})_{i\in\mathbb{N}}$ is said to be *generated by* $\mathbf{P}$.

Remark. For each $r \in (0, \infty)$, the map $r \cdot \mathbf{P}$ generates $\tilde{\mathbf{P}}$, up to rescaling.

These systems play a major role in simplifying the study of spectra. For instance, the following theorem is a somewhat weaker version of a result proven by Roy in [22].

Theorem 1.2.27. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The spectrum of $\underline{\varphi}_T$ is a compact and connected subset of $\mathbb{R}^m$, and the set of all points $\underline{\varphi}_T(\mathbf{P})$ where $\mathbf{P}$ is a proper non-degenerate self-similar n-system is dense in the spectrum of $\underline{\varphi}_T$.*

The following helps to compute the set $\mathcal{K}(\mathbf{P})$ of a self-similar n-system $\mathbf{P}$, which is useful in computing $\underline{\varphi}_T(\mathbf{P}) = \inf T(\mathcal{K}(\mathbf{P}))$ for linear maps $T : \mathbb{R}^n \to \mathbb{R}^m$.

Proposition 1.2.28. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a self-similar n-system with period ρ. Writing $\varphi(q) = \varphi(q; \mathbf{P})$, it follows for each $q_0 \in I$ that $\mathcal{F}(\mathbf{P}) = \varphi([q_0, \rho q_0])$.*

Proof: Since $\varphi(\rho q) = \varphi(q)$ for each $q \in I$, it follows that $\varphi([q_0, \rho q_0]) \subseteq \mathcal{F}(\mathbf{P})$. On the other hand, if $\mathbf{x} \in \mathcal{F}(\mathbf{P})$, then $\mathbf{x}$ is a limit point of the set $\varphi([q_0, \rho q_0])$. Since $[q_0, \rho q_0]$ is compact and φ is continuous, it follows that $\varphi([q_0, \rho q_0])$ is compact, and hence closed. Thus, $\mathbf{x} \in \mathcal{F}(\mathbf{P})$, and so $\mathcal{F}(\mathbf{P}) = \varphi([q_0, \rho q_0])$. ∎

The following weaker result is sometimes sufficient.

Corollary 1.2.29. *Let $\tilde{\mathbf{P}}$ be a self-similar n-system which is generated by some bounded n-system $\mathbf{P}$. Writing $\varphi(q) = \varphi(q; \mathbf{P})$, it follows if $d_0, \ldots, d_k$ are the ordered division numbers of $\mathbf{P}$, then $\mathcal{K}(\tilde{\mathbf{P}})$ is the convex hull of $\varphi(d_1), \ldots, \varphi(d_k)$.*

1.2.7 Regular Systems

Consider the following construction defined for strictly increasing sequences $(\alpha_m)_{m\in\mathbb{N}}$ of non-negative real numbers.

Define for each $m \in \mathbb{N}$ the points $\mathbf{x}_m, \mathbf{y}_m \in \Delta^n$ by

$$\mathbf{x}_m = (\alpha_m, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{m+n-2}) \quad \text{and} \quad \mathbf{y}_m = (\alpha_m, \ldots, \alpha_{m+n-3}, \alpha_{m+n-2}, \alpha_{m+n-2}),$$

and observe that $i = 1$ and $j = n - 1$ are the unique indices such that $\mathbf{x}_m \in \Delta_i$ and $\mathbf{y}_m \in \Delta_j$. Since $\mathbf{x} = \mathbf{x}_m$ and $\mathbf{y} = \mathbf{y}_m$ satisfy 1.2.2 and $\mathbf{x} = \mathbf{y}_m$ and $\mathbf{y} = \mathbf{x}_{m+1}$ satisfy 1.2.1 in Theorem 1.2.21, then Theorem 1.2.22 can be used to define an n-system $\mathbf{R}^{(m)} : [a_m, b_m] \cup [b_m, a_{m+1}] \to \mathbb{R}^n$ as in Figure 1.2.9 for each $m \in \mathbb{N}$, where a_m and b_m are the respective sums of the coordinates of $\mathbf{x}_m$ and $\mathbf{y}_m$.

Moreover, since $\mathbf{x}_m \in \Delta_1$ for each $m \in \mathbb{N}$, Theorem 1.2.20 implies that the sequence $(\mathbf{R}^{(m)})_{m\in\mathbb{N}}$ connects to form the map $\mathbf{R} : [a_0, \infty) \to \mathbb{R}^n$.

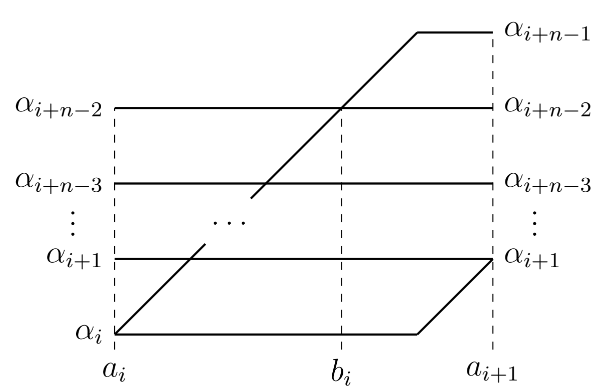

Figure 1.2.9: Combined graph of $\mathbf{R}^{(m)}$

Definition 1.2.30. Such a map $\mathbf{R}$ is said to be *generated* by $(\alpha_i)_{i\in\mathbb{N}}$. If there exist $\rho > 1$ and $m \in \mathbb{N}^+$ such that $\alpha_{m+i} = \rho\alpha_i$ for each $i \in \mathbb{N}$, then $\mathbf{R}$ is a self-similar n-system with period ρ and is said to be *quasi-regular* and generated by $(\alpha_0, \dots, \alpha_m)$. If there exists $\rho > 1$ such that $\alpha_{i+1} = \rho\alpha_i$ for each $i \in \mathbb{N}$, then $\mathbf{R}$ is called *regular*.

Remark. If ρ is minimal in the definition and $\tilde{\rho}$ and $\tilde{m}$ also satisfy $\alpha_{\tilde{m}+i} = \tilde{\rho}\alpha_i$ for each $i \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $\tilde{\rho} = \rho^k$, and so $\tilde{m} = km$. Hence, the polynomials $x^m = \rho$ and $x^{\tilde{m}} = \tilde{\rho}$ share the same unique positive root, say α.

Definition 1.2.31. The quantity α is called the *mean geometric reason* of such a map $\mathbf{R}$, and simply the *geometric reason* when $\mathbf{R}$ is regular.

Although these classes are sparse in the class of self-similar systems, they play an important role in the study of spectra of pairs $(\underline{\psi}_j, \overline{\psi}_j)$ where $j \in \{1, \dots, n-1\}$, especially with respect to the boundary of such spectra.

1.3 Qualitative Behaviour of ψ_j

This section provides for each $j \in \{1, \dots, n-1\}$ a construction which canonically transforms an n-system $\mathbf{P}$ into a simpler n-system $\tilde{\mathbf{P}}$ while preserving the sum of the first j components, i.e. $\tilde{P}_1 + \cdots + \tilde{P}_j = P_1 + \cdots + P_j$. Thus, the spectrum of $(\underline{\psi}_j, \overline{\psi}_j)$ is determined by these canonical systems.

1.3.1 Type of an Interval

This subsection introduces the notion of the *type* of an interval.

Definition 1.3.1. Let $\mathbf{P} : I \to \mathbb{R}^n$ be a map defined on an interval $I \subseteq \mathbb{R}$. An interval $J \subseteq \mathbb{R}$ is *of type* i (with respect to $\mathbf{P}$) if the component P_i *varies* on $J \cap I$, i.e. if $P_i|_{J\cap I}$ is not a constant function. The *type* of J is the set of indices i such that J is of type i, and is denoted $\mathrm{type}(J) = \mathrm{type}(J; \mathbf{P})$.

For the sake of compactness, define $\mathrm{type}(q, q') = \mathrm{type}([q, q'])$ for all $q, q' \in I$, and note that $\mathrm{type}(q, q') = \varnothing$ if $q' \leq q$.

The following proposition highlights some elementary properties of this map. These properties are key both in proving the results of this section, as well as in proving results in the section on limits of systems. Though there are numerous properties, each one follows easily from the definitions, and so their proofs are omitted.

Proposition 1.3.2. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a map, let J, K, L be subintervals of I, and let $q, q' \in I$.*

1. *If $J \subseteq K$, then* $\mathrm{type}(J) \subseteq \mathrm{type}(K)$.

2. *If* $L = J \cup K$ *and* $J \cap L \neq \varnothing$, *then* $\mathrm{type}(L) = \mathrm{type}(J) \cup \mathrm{type}(K)$.

3. *If* $\mathbf{P}$ *is nowhere locally constant, then* $\mathrm{type}(J) = \varnothing \Leftrightarrow J^{\circ} = \varnothing$.

4. *If* $\mathbf{P}$ *is continuous, then* $\mathrm{type}(J) = \mathrm{type}(\bar{J})$.

5. *If* $P_1, \ldots, P_n$ *are monotone on* $[q, q']$ *with* $q \leq q'$, *then*

$$\mathrm{type}(q, q') = \{i \in \{1, \ldots, n\} \mid P_i(q) \neq P_i(q')\}.$$

6. *If* $\mathbf{P}$ *is an* n*-system and* $k \in \mathrm{type}(q, q')$, *then* $\mathbf{P}'(t) = \mathbf{e}_k$ *for some* $t \in (q, q')$.

The next theorem provides insight into how the type of an interval is useful in studying the qualitative behaviour of systems.

Lemma 1.3.3. *Let* $\mathbf{P} : I \to \mathbb{R}^n$ *be an* n*-system, let* $S_k = P_1 + \cdots + P_k$ *for some* $k \in \{1, \ldots, n\}$, *and let* $r, s \in I$ *with* $r < s$. *The following statements hold:*

1. *If* $S_k'(r^+) = 1$ *and* $k < \max \mathrm{type}(r, s)$, *then* $\mathbf{P}(q) \in \Delta_k$ *for some* $q \in (r, s)$.

2. *If* $S_k'(s^-) = 0$ *and* $k \geq \min \mathrm{type}(r, s)$, *then* $\mathbf{P}(q) \in \Delta_k$ *for some* $q \in (r, s)$.

Proof: Let $m = \max \mathrm{type}(r, s)$, and suppose that $S_k'(r^+) = 1$ for some $k < m$. Since $m \in \mathrm{type}(r, s)$, there exists $t \in (r, s)$ such that $P_m'(t) = 1$. Thus, there exists $q \in (r, t)$ such that $S_k'(q^-) = 1 \neq S_k'(q^+)$, proving the first result by Proposition 1.2.4.

Let $l = \max \mathrm{type}(r, s)$, and suppose that $S_k'(s^-) = 0$ for some $k \geq l$. Since $l \in \mathrm{type}(r, s)$, there exists $t \in (r, s)$ such that $P_l'(t) = 1$. Thus, there exists $q \in (t, s)$ such that $S_k'(q^-) = 1 \neq S_k'(q^+)$, proving the second result by Proposition 1.2.4. ∎

Theorem 1.3.4. *Let* $\mathbf{P} : I \to \mathbb{R}^n$ *be a non-degenerate* n*-system. Suppose that there exist* $r, s \in I$ *with* $r < s$ *and* $i, j \in \{1, \ldots, n-1\}$ *such that* $\mathbf{P}(r) \in \Delta_i$ *and* $\mathbf{P}(s) \in \Delta_j$. *Letting* $l = \min \mathrm{type}(r, s)$ *and* $m = \max \mathrm{type}(r, s)$, *the following statements hold:*

1. *For each* $k = i, \ldots, m-1$, $P_{k+1}(r) \leq P_k(s)$ *and* $\mathbf{P}(q) \in \Delta_k$ *for some* $q \in [r, s)$. *Moreover, the integer interval* $\{i+1, \ldots, m\}$ *is contained in* $\mathrm{type}(r, s)$.

2. *For each* $k = l, \ldots, j$, $P_{k+1}(r) \leq P_k(s)$ *and* $\mathbf{P}(q) \in \Delta_k$ *for some* $q \in (r, s]$. *Moreover, the integer interval* $\{l, \ldots, j\}$ *is contained in* $\mathrm{type}(r, s)$.

In particular, $i+1, j \in \mathrm{type}(r, s)$, *and the following statements hold:*

1. *If* $j \geq i - 1$, *then* $P_{k+1}(r) \leq P_k(s)$ *for each* $k = l, \ldots, m-1$

2. *If* $j \geq i$, *then* $\mathrm{type}(r, s) = \{l, \ldots, m\}$.

Proof: Let $S_k = P_1 + \cdots + P_k$ for $k = 1, \ldots, n$, and note that

$$P_{i+1}(r) = P_i(r) \leq P_i(s) \quad \text{and} \quad P_{j+1}(r) \leq P_{j+1}(s) = P_j(s).$$

The first two results are proven by induction as follows:

Note that $m \in \text{type}(r,s)$. Suppose that $k+1 \in \text{type}(r,s)$ for some k with $i < k < m$. Since $k < m$ and $S_k'(r^+) = 1$, Lemma 1.3.3 yields $q \in (r,s)$ with $\mathbf{P}(q) \in \Delta_k$. As $i < k$, it follows that

$$P_k(r) < P_{k+1}(r) \leq P_{k+1}(q) = P_k(q) \leq P_k(s).$$

Hence, $k \in \text{type}(r,s)$ and $P_{k+1}(r) \leq P_k(s)$. Hence, induction yields $k \in \text{type}(r,s)$ and $P_{k+1}(r) \leq P_k(s)$ for $k = i+1, \ldots, m-1$.

Note that $l \in \text{type}(r,s)$. Suppose that $k \in \text{type}(r,s)$ for some k with $l \leq k < j$. Since $l \leq k$ and $S_k'(s^-) = 0$, Lemma 1.3.3 yields $q \in (r,s)$ with $\mathbf{P}(q) \in \Delta_k$. As $k < j$, it follows that

$$P_{k+1}(r) \leq P_{k+1}(q) = P_k(q) \leq P_k(s) < P_{k+1}(s).$$

Hence, $k+1 \in \text{type}(r,s)$ and $P_{k+1}(r) \leq P_k(s)$. Hence, induction yields $k \in \text{type}(r,s)$ and $P_{k+1}(r) \leq P_k(s)$ for $k = l, \ldots, j-1$. ∎

1.3.2 Computing $\underline{\psi}_j$ and $\overline{\psi_j}$

For the remainder of this section, suppose that $n \geq 3$ and fix $j \in \{1, \ldots, n-1\}$. Also fix a non-degenerate n-system $\mathbf{P} : [r,s] \to \mathbb{R}^n$ with the property that

$$\mathbf{P}(q) \in \Delta_j \Leftrightarrow q \in \{r, s\}.$$

Write $S_j = P_1 + \cdots + P_j$.

Lemma 1.3.5. *There exists $t \in (r,s)$ such that $S_j' = 0$ on (r,t) and $S_j' = 1$ on (t,s).*

Proof: Since $\mathbf{P}$ is non-degenerate, then $P_{j+1}'(r^+) = 1 = P_j'(s^-)$. It follows that $S_j'(r^+) = 0 \neq S_j'(s^-)$, so there exists maximal $t \in (r,s)$ such that $S_j' = 0$ on (r,t). Now, if $q > t$ is such that $S_j'(q^-) = 1 \neq S_j'(q^+)$, then $\mathbf{P}(q) \in \Delta_j$, which implies that $q = s$, and so $S_j' = 1$ on (t,s). ∎

Corollary 1.3.6. *The extrema of $\psi_j(q) = q^{-1}S_j(q)$ can be computed as follows.*

$$\overline{\psi}_j(\mathbf{P}) = \max\{\psi_j(r), \psi_j(s)\} \quad \textit{and} \quad \underline{\psi}_j(\mathbf{P}) = \frac{S_j(r)}{S_j(r) + (s - S_j(s))}.$$

Proof: By Lemma 1.3.5, noting that $\psi_j < 1$, one observes that ψ_j is decreasing on $[r, t]$ and that it is increasing on $[t, s]$, proving the first equality as well as showing that ψ_j is minimal at t. Now, since $S_j' = 0$ on (r, t), then $S_j(t) = S_j(r)$, and since $S_j' = 1$ on (t, s), then $t - S_j(t) = s - S_j(s)$, proving the second equality. ∎

Remark. If $\mathbf{R} : I \to \mathbb{R}^n$ is a proper non-degenerate n-system, then Theorem 1.2.16 implies that the set of points $q \in I$ with $\mathbf{R}(q) \in \Delta_j$ is infinite and enumerable in increasing order.

Thus, the following corollary can be stated and follows from Corollary 1.3.6.

Corollary 1.3.7. *Let $\mathbf{R} : I \to \mathbb{R}^n$ be a proper non-degenerate n-system, and let $(q_i)_{i\in\mathbb{N}}$ be the strictly increasing set of division numbers q_i with $\mathbf{R}(q_i) \in \Delta_j$ for each $i \in \mathbb{N}$. By writing $\psi_j(q) = \psi_j(q; \mathbf{R})$ and $T_j = R_1 + \cdots + R_j$, it follows that*

$$\overline{\psi}_j(\mathbf{R}) = \limsup_{i\in\mathbb{N}} \psi_j(q_i) \quad \textit{and} \quad \underline{\psi}_j(\mathbf{R}) = \liminf_{i\in\mathbb{N}} \frac{T_j(q_i)}{T_j(q_i) + (q_{i+1} - T_j(q_{i+1}))}.$$

Hence, one can compute the pair $(\underline{\psi}_j(\mathbf{R}), \overline{\psi}_j(\mathbf{R}))$ of a proper non-degenerate n-system $\mathbf{R}$ by simply knowing the values of $\mathbf{R}$ at each q with $\mathbf{R}(q) \in \Delta_j$. This result is key in the chapter devoted to the spectrum of $(\underline{\psi}_j, \overline{\psi}_j)$.

1.3.3 Canonical Transformation Preserving ψ_j

Let the notation be as in the previous subsection. The following theorem provides a canonical transformation of the non-degenerate n-system $\mathbf{P} : [r, s] \to \mathbb{R}^n$ into a non-degenerate n-system $\tilde{\mathbf{P}} : [r, s] \to \mathbb{R}^n$ with $S_j = P_1 + \cdots + P_j = \tilde{P}_1 + \cdots + \tilde{P}_j$.

Theorem 1.3.8. *Write $\mathbf{x} = \mathbf{P}(r)$ and $\mathbf{y} = \mathbf{P}(s)$, let t be as in Lemma 1.3.5, and let $l = \min \mathrm{type}(r, s)$ and $m = \max \mathrm{type}(r, s)$. Define the points*

$$\mathbf{x}' = (x_1, \dots, x_j, x_{j+2}, \dots, x_m, x_m, \dots, x_n) \in \Delta_{m-1}$$

and

$$\mathbf{y}' = (y_1, \dots, y_l, y_l \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \Delta_l,$$

and let r' and s' be the sums of the coordinates of $\mathbf{x}'$ and $\mathbf{y}'$, respectively. There exists a canonical non-degenerate n-system $\tilde{\mathbf{P}} : [r, s] \to \mathbb{R}^n$ with the following properties.

- $S_j = \tilde{P}_1 + \cdots + \tilde{P}_j$,
- $\tilde{\mathbf{P}}(q) \in \Delta_j \Leftrightarrow q \in \{r, s\}$,

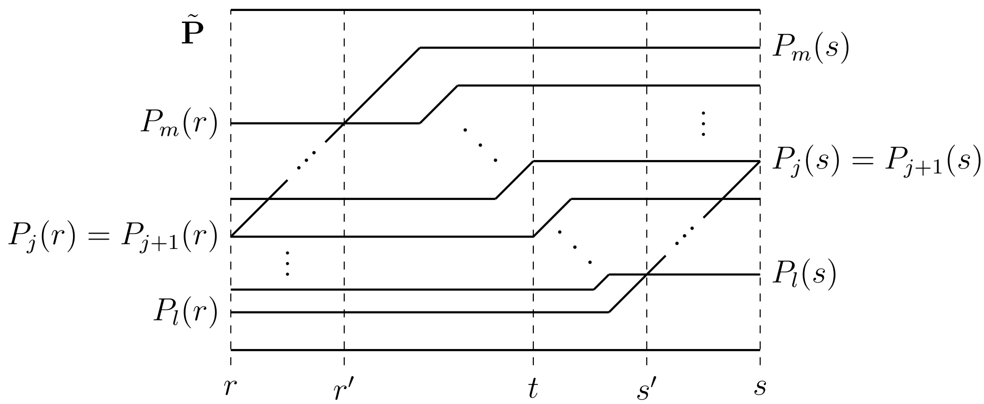

Figure 1.3.1: Combined Graph of $\tilde{\mathbf{P}}$

- $\tilde{\mathbf{P}}(r) = \mathbf{x}, \tilde{\mathbf{P}}(r') = \mathbf{x}', \tilde{\mathbf{P}}(t) = \mathbf{P}(t), \tilde{\mathbf{P}}(s') = \mathbf{y}', \tilde{\mathbf{P}}(s) = \mathbf{y}$.

Figure 1.3.1 gives the combined graph of $\tilde{\mathbf{P}}$.

Proof: Since the pairs $(\mathbf{x}, \mathbf{x}')$ and $(\mathbf{y}', \mathbf{y})$ satisfy (1.2.2) as in Theorem 1.2.21, it remains to show that $(\mathbf{x}', \mathbf{y}')$ satisfies (1.2.1) as in Theorem 1.2.21 in order to construct the map $\tilde{\mathbf{P}}$. Since $x_k < y_k$ for $k = l, m$, and since Theorem 1.3.4 yields $x_{k+1} \leq y_k$ for $k = l, \ldots, m-1$, it follows by the definition of l and m that $(\mathbf{x}', \mathbf{y}')$ satisfies (1.2.1). The map $\tilde{\mathbf{P}}$ is thus constructed by applying Theorem 1.2.22 to the sequence $(\mathbf{x}, \mathbf{x}', \mathbf{y}', \mathbf{y})$, and this map $\tilde{\mathbf{P}}$ satisfies

$$\tilde{\mathbf{P}}(q) \in \Delta_j \Leftrightarrow q \in \{r, s\}.$$

Finally, by writing $\tilde{S}_j = \tilde{P}_1 + \cdots + \tilde{P}_j$, the construction also implies that $\tilde{S}_j' = 0$ on (r, t) and $\tilde{S}_j' = 1$ on (t, s), and so Lemma 1.3.5 implies that S_j' and $\tilde{S}_j'$ are equal on (r, t) and (t, s). Since S_j and $\tilde{S}_j$ are equal at r, s, and t, it follows that $S_j = \tilde{S}_j$. ∎

1.4 Limits of Systems

Definition 1.4.1. If a sequence $(f_i : X \to Y)_{i \in \mathbb{N}}$ of functions converges to a function $f : X \to Y$ such that for each point $x \in X$, convergence is uniform on some neighbourhood of x then the convergence is said to be *locally uniform*, and f is called the *local uniform limit* of $(f_i)_{i \in \mathbb{N}}$.

Remark. If every point in X has a compact neighbourhood, e.g. $X = \mathbb{R}$, then local uniform convergence is equivalent to uniform convergence on every compact subset.

This section is devoted to the study of local uniform limits of sequences of systems. In particular, such a limit need not be a system in general, even when the domain is compact. In light of this, certain criteria establishing when such a limit is in fact a system are given, and are proven using some fairly technical notions and results. The notion of a *generalized system*, as introduced by Roy [21], is also provided, and shown to be a special subclass of the class of such limits.

Definition 1.4.2. For each closed subinterval I of $\mathbb{R}$ with non-zero length and $\min I > 0$, define $\mathcal{S}_I^n$ to be the set of n-systems defined on I, and define $\bar{\mathcal{S}}_I^n$ to be the set of maps $\mathbf{P} : I \to \mathbb{R}^n$ which can be realized as the local uniform limit of some sequence in $\mathcal{S}_I^n$.

Maps in $\bar{\mathcal{S}}_I^n$ need not have left and right derivatives at an interior point, and so it is important to study the way in which these maps vary locally without relying on differentiation. This observation leads to a class of maps called *prototypes* of systems, and this class is shown to contain the class of local uniform limits of systems. In this way, instead of characterizing such limits directly, properties can be deduced through the properties of prototypes.

1.4.1 Local Variation of Real-Valued Continuous Functions

This section introduces the notion of *local variation* of continuous maps, in contrast to the much stronger notion of continuous variation, the latter of which forms the basis of calculus. Local variation is significantly weaker than calculus in that its foundation lies in simply studying whether or not a continuous map is constant or not "near" a point. Though such considerations might appear trivial, its strength lies in studying whether or not a map is constant on a left or right neighbourhood of a point, similar to how a map can be left or right differentiable at a point.

Consider an $\mathbb{R}$-valued map f defined on some interval $J \subseteq \mathbb{R}$, and a point $q \in J$.

Definition 1.4.3. Such a map f is said to be *locally constant at* q if it is constant on a neighbourhood of q, and is said to be *constant to the left of* q if it is constant on a left neighbourhood of q. In contrast, f is said to *vary near* q if it is not locally constant at q, and is said to *vary to the left of* q if it is not constant to the left of q. One can similarly define *right* versions of these definitions replacing "left" by "right".

The following technical lemma is key in proving the next proposition.

Lemma 1.4.4. *Let $(x_\alpha)_{\alpha \in A}$ be a family of non-negative real numbers indexed by an uncountable set A. Define $\sum_{\alpha \in A} x_\alpha$ as the supremum of sums $\sum_{\alpha \in C} x_\alpha$ where C is an at most countable subset of A. Let $A' \subseteq A$ be the set of indices α satisfying $x_\alpha > 0$. If A' is uncountable, then $\sum_{\alpha \in A} x_\alpha = \infty$.*

Proof: Note that A' is the union of the countable family of sets $(B_m)_{m\in\mathbb{N}}$, where $B_m = \{\alpha \in A \,|\, x_\alpha > 1/m\}$. Now, suppose that A' is uncountable, and so B_m is uncountable for some $m \in \mathbb{N}$. Letting C be a countable subset of B_m, it follows that

$$\sum_{\alpha\in A} x_\alpha \geq \sum_{\alpha\in C} x_\alpha = \infty,$$

as $x_\alpha > 1/m$ for each $\alpha \in C$. ∎

Proposition 1.4.5. *Let $a, b \in J$ with $a < b$. If f is continuous and $f(a) \neq f(b)$, then f varies to the left and to the right of uncountably many $q \in (a, b)$.*

Proof: Let $Y = f((a, b))$. The intermediate value theorem implies that Y is uncountable. If $y \in Y$ is such that $f^{-1}(y) \cap (a, b)$ contains no interval with positive length, then f varies to the left and right of every point in $f^{-1}(y) \cap (a, b)$. By Lemma 1.4.4, a family of disjoint intervals contained in $[a, b]$ and having positive length is at most countable. Thus, there exists uncountably many $y \in Y$ such that f varies to the left and right of every point in $f^{-1}(y) \cap (a, b) \neq \varnothing$, proving the claim. ∎

Corollary 1.4.6. *If f is continuous and varies to the left (resp. right) of q, then f varies to the left and to the right of some arbitrarily close $t < q$ (resp. $t > q$).*

Proof: Suppose that f varies to the left (resp. right) of q, and so there exists arbitrarily close $q' < q$ (resp. $q' > q$) with $f(q') \neq f(q)$. Proposition 1.4.5 implies that f varies to the left and right of some $t \in (q', q)$ (resp. $t \in (q, q')$). ∎

1.4.2 Type of a Point

This subsection introduces the notion of the *type* of a point, which is useful in keeping track of which components of a map $\mathbf{P}$ vary locally, and which do not.

Definition 1.4.7. Let $\mathbf{P} : I \to \mathbb{R}^n$ be a map defined on an interval $I \subseteq \mathbb{R}$. A point $q \in I$ is *of type i* (with respect to $\mathbf{P}$) if the component P_i varies near q, i.e. if P_i is not constant in a neighbourhood of q. The *type* of q is the set of indices i such that q is of type i, and is denoted $\mathrm{type}(q) = \mathrm{type}(q; \mathbf{P})$. Similarly define the type of q^- and the type of q^+ by respectively replacing "varies near" by "varies to the left of" and "varies to the right of", and replacing "neighbourhood" by "left neighbourhood" and "right neighbourhood".

The type of a point can be seen as a limit notion of the type of an interval neighbourhood of a point, in that it coincides with the intersection of the sets $\mathrm{type}(J)$, where J runs through the interval neighbourhoods of that point, as follows.

Remark. For each $q \in I$, one finds that $\mathrm{type}(q) = \mathrm{type}(q^-) \cup \mathrm{type}(q^+)$, where $\mathrm{type}(q^-) = \{i \mid P_i \text{ varies left of } q\}$ and $type(q^+) = \{i \mid P_i \text{ varies right of } q\}$. Moreover, if $\mathcal{I}_q$ is the set of interval neighbourhoods of q, and $\mathcal{I}_q^-, \mathcal{I}_q^+$ are the sets of left/right interval neighbourhoods of q, respectively, then

$$\mathrm{type}(q) = \bigcap_{J \in \mathcal{I}_q} \mathrm{type}(J); \quad \mathrm{type}(q^-) = \bigcap_{J \in \mathcal{I}_q^-} \mathrm{type}(J) \quad \text{and} \quad \mathrm{type}(q^+) = \bigcap_{J \in \mathcal{I}_q^+} \mathrm{type}(J).$$

It is also worth noting that $\mathrm{type}(\{q\}) = \mathrm{type}([q,q]) = \mathrm{type}(q,q) = \varnothing$, in contrast to the set $\mathrm{type}(q)$ being empty if and only if $\mathbf{P}$ is locally constant at q.

1.4.3 Prototypes

The following notion of a *prototype* is a generalization of maps which are systems.

Definition 1.4.8. A continuous map $\mathbf{P} : I \to \mathbb{R}^n$ is called a *prototype* if I is a closed subinterval of $\mathbb{R}$ with non-zero length and $\min I > 0$, and if it satisfies the following:

(P1) For each $q \in I$, the point $\mathbf{P}(q)$ lies in Δ^n, that is $0 \le P_1(q) \le \cdots \le P_n(q)$.

(P2) For each $q \in I$, the point $q^{-1}\mathbf{P}(q)$ lies in Σ^n, that is $P_1(q) + \cdots + P_n(q) = q$.

(P3) The function $P_i : I \to \mathbb{R}$ is monotone increasing, for each $i \in \{1, \ldots, n\}$.

(P4) For each $q \in I$, if $(k,l) \in \mathrm{type}(q^-) \times \mathrm{type}(q^+)$, then $P_k(q) \ge P_l(q)$.

Denote by $\tilde{\mathcal{S}}_I^n$ the set of prototypes defined on I with values in $\mathbb{R}^n$.

In particular, each n-system $\mathbf{P} : I \to \mathbb{R}^n$ is also a prototype. The following proposition provides some elementary properties of prototypes.

Proposition 1.4.9. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$. The following properties hold:*

1. *If $a, b \in I$, with $a \le b$ then $0 \le P_i(b) - P_i(a) \le b - a$ for each $i = 1, \ldots, n$.*
2. *The map $\mathbf{P}$ is 1-Lipschitz continuous with respect to each p-norm on $\mathbb{R}^n$.*
3. *Let $q \in I$. If $q > \inf I$, then $\mathrm{type}(q^-) \neq \varnothing$, and if $q < \sup I$, then $\mathrm{type}(q^+) \neq \varnothing$.*
4. *For each $q \in I$, if k, l are such that $P_k(q) < P_l(q)$, then*

$$\begin{cases} k \in \mathrm{type}(q^-) & \Rightarrow i \notin \mathrm{type}(q^+) \text{ for } i = l, \ldots, n, \\ l \in \mathrm{type}(q^+) & \Rightarrow i \notin \mathrm{type}(q^-) \text{ for } i = 1, \ldots, k. \end{cases}$$

5. *For each $q \in I$, if $k \in \mathrm{type}(q^-)$, then $P_k(r) < P_k(q)$ for each $r \in I$ with $r < q$, and if $l \in \mathrm{type}(q^+)$, then $P_l(s) > P_l(q)$ for each $s \in I$ with $s > q$.*

Proof: The first property follows from (P3) and (P2). The second follows from

$$\|\mathbf{P}(b) - \mathbf{P}(a)\|_p \le \sum_{i=1}^{n} \|(P_i(b) - P_i(a))\mathbf{e}_i\|_p = \sum_{i=1}^{n} |P_i(b) - P_i(a)| \cdot \|\mathbf{e}_i\|_p = |b - a|,$$

having applied the first property. Since (P2) implies that $P_1 + \cdots + P_n$ varies to the left (resp. right) of each $q > \inf I$ (resp. $q < \sup I$), the third property follows. The fourth property follows from (P1) and (P4), and the fifth follows from (P3). ∎

1.4.4 Limit Theorem

As part 4 of Proposition 1.4.9 shows, if a component varies at a point, then another component might be forced to be constant on either some left or right interval neighbourhood of that point. The following lemma and corollary provide effective lower bounds on the lengths of these neighbourhoods, and is in particular key in proving the main result of this subsection.

Lemma 1.4.10. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$, and let $j, k \in \{1, \ldots, n-1\}$. For each $q \in I$:*

1. *If $k \ge j \in \mathrm{type}(q^-)$, then $i \notin \mathrm{type}(q, q + P_{k+1}(q) - P_k(q))$ for $i = k+1, \ldots, n$.*
2. *If $k < j + 1 \in \mathrm{type}(q^+)$, then $i \notin \mathrm{type}(q + P_k(q) - P_{k+1}(q), q)$ for $i = 1, \ldots, k$.*

Proof: Let $q \in I$, suppose that $k \ge j \in \mathrm{type}(q^-)$, and assume without loss of generality that $P_k(q) < P_{k+1}(q)$. Letting $T_{k+1} = P_{k+1} + \cdots + P_n$, property (P4) implies that T_{k+1} is constant to the right of q, and so there exists a maximal $t \in (q, \sup I]$ such that T_{k+1} is constant on $[q, t]$. Without loss of generality, assume that $t < \sup I$, and so T_{k+1} varies to the right of t. Thus, as T_{k+1} is constant to the left of t, part 3 of Proposition 1.4.9 implies that $P_{j'}(t) \ge P_l(t)$ for some $j' \le k < l$, and so (P1) implies that $P_k(t) = P_{k+1}(t) = P_{k+1}(q)$. Since $P_k(q) + t - q \ge P_k(t) = P_{k+1}(q)$, then

$$t \ge q + P_{k+1}(q) - P_k(q),$$

which implies the first claim. A similar argument yields the second claim. ∎

The following corollary extends the above result, and is far more practical.

Corollary 1.4.11. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$, and let $k, l \in \{1, \ldots, n\}$. For each $q \in I$:*

1. *If* $k \in \mathrm{type}(q^-)$*, then* $l \notin \mathrm{type}(q, q + P_l(q) - P_k(q))$.
2. *If* $l \in \mathrm{type}(q^+)$*, then* $k \notin \mathrm{type}(q + P_k(q) - P_l(q), q)$.

Proof: To prove the first claim, first observe that if $k \geq l$, then $P_k(q) \geq P_l(q)$, and so $\mathrm{type}(q, q + P_l(q) - P_k(q))$ is empty. Thus, assume without loss of generality that $k < l$. It will be shown by induction on $l = k+1, \ldots, n$ that

$$i \notin \mathrm{type}(q, q + P_l(q) - P_k(q)) \text{ for } i = l, \ldots, n. \tag{1.4.1}$$

The base case holds by Lemma 1.4.10. Now, suppose that (1.4.1) holds for some $l \in \{k+1, \ldots, n-1\}$. Hence, there exists j with $j < l$ such that P_j varies to the left of $t = q + P_l(q) - P_k(q)$. Hence, Lemma 1.4.10 implies that

$$i \notin \mathrm{type}(t, t + P_{l+1}(t) - P_l(t)) \text{ for } i = l+1, \ldots, n.$$

The induction hypothesis says that P_l and P_{l+1} are constant on $[q, t]$, and so

$$t + P_{l+1}(t) - P_l(t) = q + P_{l+1}(q) - P_k(q).$$

Therefore, the induction hypothesis also implies that

$$i \notin \mathrm{type}(q, q + P_{l+1}(q) - P_k(q)) \text{ for } i = l+1, \ldots, n.$$

Hence, (1.4.1) holds for $l+1$, and so it holds in general by induction. Hence, the first claim holds true. The second claim is proven similarly. ∎

The following theorem implies that $\mathcal{S}_I^n \subseteq \bar{\mathcal{S}}_I^n \subseteq \tilde{\mathcal{S}}_I^n$.

Theorem 1.4.12. *The set* $\tilde{\mathcal{S}}_I^n$ *is closed under local uniform convergence.*

Proof: Since the properties defining prototypes are local properties, it suffices to prove the theorem when I is a compact interval with non-empty interior. In this case, local uniform convergence is uniform convergence. Hence, let $(\mathbf{P}^{(m)})_{m \in \mathbb{N}}$ be a sequence of maps in $\tilde{\mathcal{S}}_I^n$ converging uniformly to a map $\mathbf{R} : I \to \mathbb{R}^n$. Since the properties (P1), (P2), and (P3) are preserved under pointwise convergence, it suffices to show for each $q \in I^\circ$ that (P4) holds for $\mathbf{R}$ at q.

To this end, let $q \in I^\circ$, and suppose that k, l are such that R_k varies to the left of q and that $R_l(q) > R_k(q)$. It remains to show that $l \notin \mathrm{type}(q^+)$. Let δ be such that

$$0 < 4\delta < R_l(q) - R_k(q) \quad \text{and} \quad q \pm \delta \in I. \tag{1.4.2}$$

Note by part 5 of Proposition 1.4.9 that $R_k(q - \delta) < R_k(q)$, and so there exists ε such that

$$0 < 2\varepsilon < R_k(q) - R_k(q - \delta) < \delta. \tag{1.4.3}$$

Now, let N be such that $\|\mathbf{P}^{(m)} - \mathbf{R}\|_\infty < \varepsilon$ for all $m > N$, and choose $m > N$. It follows from (1.4.3) that $R_k(q-\delta) + \varepsilon < R_k(q) - \varepsilon$, and so

$$P_k^{(m)}(q-\delta) < R_k(q-\delta) + \varepsilon < R_k(q) - \varepsilon < P_k^{(m)}(q).$$

Thus, Proposition 1.4.5 implies that $P_k^{(m)}$ varies to the left of some $q_m \in (q-\delta, q)$.

Since

$$P_l^{(m)}(q_m) \geq P_l^{(m)}(q) - (q - q_m) > (R_l(q) - \varepsilon) - \delta > R_l(q) - 3\delta/2,$$

and since

$$R_k(q) > P_k^{(m)}(q) - \varepsilon > P_k^{(m)}(q_m) - \delta/2,$$

then (1.4.2) implies $P_l^{(m)}(q_m) > R_l(q) - 3\delta/2 > R_k(q) + 5\delta/2 > P_k^{(m)}(q_m) + 2\delta$. Hence,

$$P_l^{(m)}(q_m) - P_k^{(m)}(q_m) > 2\delta.$$

Thus, as $P_k^{(m)}$ varies to the left of q_m, Corollary 1.4.11 yields that $P_l^{(m)}$ is constant on

$$[q_m, q_m + 2\delta] \cap I \supseteq [q_m, q + \delta] \supset [q, q + \delta].$$

Since this holds for all $m > N$, then R_l is constant on $[q, q+\delta]$. Therefore, R_l does not vary to the right of q. ∎

1.4.5 Relationships between Prototypes and n-Systems

The following theorem characterizes the n-systems among the prototypes.

Theorem 1.4.13. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$. The prototype $\mathbf{P}$ is an n-system if and only if for each $q \in I$, the sets* type(q^-) *and* type(q^+) *have cardinality at most* 1.

Proof: The condition is necessary by property (S3) defining an n-system. To prove that it is sufficient, note that $\mathbf{P}$ satisfies properties (S1) and (S2) by definition. To see that (S3) holds, let $q \in I$. By part 3 of Proposition 1.4.9, one may assume without loss of generality that type(q^-) has cardinality 1 if $q > \inf I$ and that type(q^+) has cardinality 1 if $q < \sup I$. Thus, by property (P4), there exists $k, l \in \{1, \ldots, n\}$ with $P_k(q) \geq P_l(q)$ such that type$(q^-) \subseteq \{k\}$ and type$(q^+) \subseteq \{l\}$. Now, there exists closed left and right neighbourhoods V^- and V^+ of q such that P_i is constant on V^- for $i \neq k$ and such that P_i is constant on V^+ for $i \neq l$, respectively. Hence, property (P2) implies for each $q \in V = V^- \cup V^+$ that

$$\mathbf{P}(q') = \begin{cases} \mathbf{P}(q) + (q' - q)\mathbf{e}_k & \text{if } q' \in V^-, \\ \mathbf{P}(q) + (q' - q)\mathbf{e}_l & \text{if } q' \in V^+. \end{cases}$$

Therefore, $\mathbf{P}$ satisfies property (S3) at q. ∎

In order to effectively use this characterization, it is important to have a better understanding of the sets $\operatorname{type}(q^-)$ and $\operatorname{type}(q^+)$, as in the following theorem.

Theorem 1.4.14. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$, and let $q \in I$. If $k, l \in \{1, \ldots, n\}$ satisfy either $k, l \in \operatorname{type}(q^-)$ or $k, l \in \operatorname{type}(q^+)$, then $P_k(q) = P_l(q)$. Moreover, the sets $\operatorname{type}(q^-)$ and $\operatorname{type}(q^+)$ are integer intervals.*

Proof: Suppose first that $k, l \in \operatorname{type}(q^-)$. Suppose further that $P_k(q) \neq P_l(q)$. By permuting k and l if necessary, one may assume that $l > k$. It follows that $P_l(q) > P_k(q)$, and so set $M = P_l(q) - P_k(q) > 0$. Corollary 1.4.6 implies the existence of some $t \in (q - M/2, q)$ such that $k \in \operatorname{type}(t^-)$, and so Corollary 1.4.11 implies that $l \notin \operatorname{type}(t, t + P_l(t) - P_k(t))$. As $P_k(t) \leq P_k(q)$, it follows that

$$P_l(t) - P_k(t) \geq (P_l(q) - M/2) - P_k(q) = M/2,$$

and so $l \notin \operatorname{type}(t, t + M/2) \supseteq \operatorname{type}(t, q)$, which implies that $l \notin \operatorname{type}(q^-)$, which is a contradiction, and so $P_k(q) = P_l(q)$. To see that $\operatorname{type}(q^-)$ is an integer interval, suppose that there exists $j \notin \operatorname{type}(q^-)$ with $k < j < l$. Thus, there exist $\varepsilon > 0$ such that $P_j(q-\varepsilon) = P_j(q)$. Since $P_k(q) = P_l(q)$, property (P1) implies that $P_l(q) = P_j(q)$. Since $l \in \operatorname{type}(q^-)$, part 5 of Proposition 1.4.9 yields $P_l(q - \varepsilon) < P_l(q)$, and so

$$P_l(q - \varepsilon) < P_l(q) = P_j(q) = P_j(q - \varepsilon),$$

which contradicts property (P1). The statements for $\operatorname{type}(q^+)$ are proven similarly. ∎

The following lemma will also prove useful in the theorems which follow.

Lemma 1.4.15. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$, let $[a, b] \subseteq I$, and suppose that there exists some $k \in \{1, \ldots, n-1\}$ such that $P_k < P_{k+1}$ on (a, b). The following statements hold:*

1. *If $k \in \operatorname{type}(a^+)$, then $i \notin \operatorname{type}(a, b)$ for $i = k+1, \ldots, n$.*
2. *If $k+1 \in \operatorname{type}(b^-)$, then $i \notin \operatorname{type}(a, b)$ for $i = 1, \ldots, k$.*

Proof: Suppose that $k \in \operatorname{type}(a^+)$ and let $\varepsilon \in (0, b-a)$. Corollary 1.4.6 implies that $k \in \operatorname{type}(c^-)$ for some $c \in (a, a+\varepsilon)$. Since $P_k(c) < P_{k+1}(c)$, there exists a maximal $t \leq b$ such that $P_{k+1} + \cdots + P_n$ is constant on $[c, t]$. Suppose that $t < b$, and so $P_k(t) < P_{k+1}(t)$. Hence, as $P_1 + \cdots + P_k$ varies to the left of t, property (P4) implies that $P_{k+1} + \cdots + P_n$ is constant to the right of t, contradicting the maximality of t. Thus, $t = b$, and so $i \notin \operatorname{type}(a+\varepsilon, b)$ for each $\varepsilon \in (0, b-a)$ and each $i = k+1, \ldots, n$. Therefore, P_i is constant on $(a, b]$, and on $[a, b]$ by continuity, for $i = k+1, \ldots, n$.

The second claim is proven similarly. ∎

The following notion and theorems are key in proving that there exist spectra which are not semialgebraic sets.

Definition 1.4.16. A prototype $\mathbf{P} \in \mathcal{S}_I^n$ is said to be *plain* if whenever $\mathbf{P}(q) \in \Delta_i \cap \Delta_j$ for some $q \in I$ and $i, j \in \{1, \ldots, n-1\}$, then $i = j$.

Theorem 1.4.17. *Let $\mathbf{P} \in \tilde{\mathcal{S}}_I^n \backslash \mathcal{S}_I^n$ be plain. There exists $q \in I$ and $k \in \{1, \ldots, n-1\}$ with $\mathbf{P}(q) \in \Delta_k$ such that the following property is satisfied. For each $\varepsilon > 0$, there exists a point $r \in I$ with $0 < |r - q| < \varepsilon$ satisfying*

$$\mathbf{P}(q) - \mathbf{P}(r) = \tfrac{1}{2}(\mathbf{e}_k + \mathbf{e}_{k+1})(q - r).$$

In particular, $\mathbf{P}(r) \in \Delta_k$.

Proof: Theorem 1.4.13 yields $q \in I$ and distinct indices k, l that are both contained in either type(q^-) or type(q^+). By Theorem 1.4.14 this implies that $P_k(q) = P_l(q)$. Since $\mathbf{P}$ is plain, it follows that $l = k+1$ and that either type(q^-) or type(q^+) is equal to the set $\{k, k+1\}$. Suppose that type$(q^-) = \{k, k+1\}$. Then, there exists a left neighbourhood V of q on which all P_j with $j \neq k, k+1$ are constant. It follows that

$$\mathbf{P}(q) - \mathbf{P}(q') = (P_k(q) - P_k(q'))\mathbf{e}_k + (P_{k+1}(q) - P_{k+1}(q'))\mathbf{e}_{k+1}$$

for each $q' \in V$. Now, let $\varepsilon > 0$ with $q - \varepsilon \in V$, and suppose that $P_k < P_{k+1}$ on $(q - \varepsilon, q)$. Lemma 1.4.15 implies that $k \notin$ type$(q - \varepsilon, q)$, which is a contradiction as $k \in$ type(q^-). Thus, there exist $r \in (q - \varepsilon, q)$ such that $P_k(r) = P_{k+1}(r)$. Hence,

$$\mathbf{P}(q) - \mathbf{P}(r) = (P_k(q) - P_k(r))(\mathbf{e}_k + \mathbf{e}_{k+1}),$$

and so property (P2) implies that $q - r = 2(P_k(q) - P_k(r))$. This proves the result when type$(q^-) = \{k, k+1\}$. The case when type$(q^+) = \{k, k+1\}$ is proven similarly. ∎

Theorem 1.4.18. *Let $\mathbf{P} \in \mathcal{S}_I^n$ be degenerate and plain. There exist $q \in I$ and $i \in \{1, \ldots, n-1\}$ with $\mathbf{P}(q) \in \Delta_i$ such that the following property is satisfied. There exists $j \in \{1, \ldots, n\}$ with $j \notin \{i, i+1\}$ and a neighbourhood U of q such that*

$$\mathbf{P}(q) - \mathbf{P}(r) = (q - r)\mathbf{e}_j.$$

In particular, $\mathbf{P}(r) \in \Delta_i$ for each $r \in U$.

Proof: As $\mathbf{P}$ is degenerate, Corollary 1.2.15 yields $q \in I^{\circ}$ such that $\mathbf{P}'(q)$ is defined and $\mathbf{P}(q) \in \Delta_i$ for some $i \in \{1, \ldots, n-1\}$. By property (S3), there exists $j \in \{1, \ldots, n\}$ and a neighbourhood U of q such that $\mathbf{P}(q') = \mathbf{P}(q) + (q' - q)\mathbf{e}_j$ for each $q' \in U$. If $j = i$ and $q' \in U$ with $q' > q$, then $P_i(q') > P_i(q) = P_{i+1}(q')$, and if $j = i+1$ and $q' \in U$ with $q' < q$, then $P_{i+1}(q') < P_{i+1}(q) = P_i(q')$. Either case is a contradiction, and so $j \notin \{i, i+1\}$. ∎

1.4.6 A Generalization of the Arzelà-Ascoli Theorem

This subsection shows that any sequence of prototypes defined on the same compact interval I has a subsequence which converges locally uniformly to some prototype defined on I. This result and the above Theorems 1.4.17 and 1.4.18 are key in proving that not all spectra are semialgebraic sets.

Throughout this subsection, let $I \subseteq \mathbb{R}$ be a compact subinterval, and let

$$f^{(m)} = (f_1^{(m)}, \ldots, f_n^{(m)}) : I \to \mathbb{R}^n$$

be a map for each $m \in \mathbb{N}$. The sequence $(f^{(m)})_{m \in \mathbb{N}}$ is said to be *equicontinuous* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f^{(m)}(x) - f^{(m)}(y)\| < \varepsilon$$

for all $x, y \in I$ with $|x - y| < \delta$, and each $m \in \mathbb{N}$. The sequence is said to be *uniformly bounded* if there exists $M \in \mathbb{R}$ such that for all $x \in I$, and each $m \in \mathbb{N}$,

$$\|f^{(m)}(x)\| \leq M.$$

If $(f^{(m)})_{m \in \mathbb{N}}$ is equicontinuous or uniformly bounded, then $(f_i^{(m)})_{m \in \mathbb{N}}$ is respectively equicontinuous or uniformly bounded, for $i = 1, \ldots, n$. The following lemma is a useful criterion for determining that a sequence is equicontinuous.

Lemma 1.4.19. *Let $K \in \mathbb{R}$ with $K > 0$, and suppose for each $m \in \mathbb{N}$ that the map $f^{(m)}$ is K-Lipschitz continuous, i.e.*

$$\|f^{(m)}(x) - f^{(m)}(y)\| \leq K|x - y|,$$

for all $x, y \in I$. It follows that the sequence $(f^{(m)})_{m \in \mathbb{N}}$ is equicontinuous.

Proof: Let $\varepsilon > 0$, let $\delta = \varepsilon / K$, and let $x, y \in I$ with $|x - y| < \delta$. It follows that

$$\|f^{(m)}(x) - f^{(m)}(y)\| < K\delta = \varepsilon,$$

for each $m \in \mathbb{N}$. Therefore, $(f^{(m)})_{m \in \mathbb{N}}$ is an equicontinuous sequence. ∎

The following theorem is a mild generalization of the Arzelà-Ascoli theorem. The original theorem corresponds to having maps with image in $\mathbb{R}$ instead of $\mathbb{R}^n$.

Theorem 1.4.20 (Arzelà-Ascoli)**.** *If $(f^{(m)})_{m\in\mathbb{N}}$ is equicontinuous and uniformly bounded, then there exists a subsequence of $(f^{(m)})_{m\in\mathbb{N}}$ converging uniformly.*

Proof: By Arzelà-Ascoli's original theorem, there exists a subsequence $f^{(g_1(m))}$ such that $f_1^{(g_1(m))}$ converges uniformly. Since $f^{(g_1(m))}$ is a subsequence of an equicontinuous and uniformly bounded sequence, it remains equicontinuous and uniformly bounded as well. Therefore, there exists a subsequence of this subsequence, $f^{(g_1(g_2(m)))}$, such that $f_2^{(g_1(g_2(m)))}$ converges uniformly. Since $f^{(g_1(g_2(m)))}$ is a subsequence of $f^{(g_1(m))}$, then $f_1^{(g_1(g_2(m)))}$ converges uniformly as well. By iterating this line of reasoning and noting that n is finite, there exists a subsequence $f^{(h(m))}$ such that $f_i^{(h(m))}$ converges uniformly to a function f_i for each $i = 1, \ldots, n$. Now, let $\varepsilon > 0$, and note that there exists $M_i \in \mathbb{N}$, for each $i \in \{1, \ldots, n\}$, such that

$$|f_i^{(h(m))}(x) - f_i(x)| < \varepsilon/n$$

for all $x \in [a, b]$, and each $m > M_i$. It follows that

$$\|f^{(h(m))}(x) - f(x)\| \leq |f_1^{(h(m))}(x) - f_1(x)| + \cdots + |f_n^{(h(m))}(x) - f_n(x)| < \varepsilon$$

for all $x \in [a, b]$, and each $m > \max\{M_1, \ldots, M_n\}$, proving the theorem. ∎

Theorem 1.4.21. *Suppose that I is compact and let $(\mathbf{P}^{(m)})_{m\in\mathbb{N}}$ be a sequence in $\tilde{\mathcal{S}}_I^n$. There exists a subsequence which converges uniformly to some $\mathbf{P} \in \tilde{\mathcal{S}}_I^n$.*

Proof: Part 2 of Proposition 1.4.9 implies that prototypes are 1-Lipschitz, and so Lemma 1.4.19 implies that the sequence is equicontinuous. Meanwhile, properties (P1) and (P2) imply that $\sup I$ is a uniform bound for the sequence. Therefore, Arzelá-Ascoli Theorem yields a subsequence converging uniformly to some map $\mathbf{P} : I \to \mathbb{R}^n$. Theorem 1.4.12 implies that $\mathbf{P} : I \to \mathbb{R}^n$ is a prototype. ∎

1.5 Generalized n-Systems

Define for each $i, j \in \{1, \ldots, n\}$ with $i \leq j$ the vector

$$\mathbf{e}_{i,j} = \frac{\mathbf{e}_i + \cdots + \mathbf{e}_j}{j - i + 1},$$

which is a unit vector with respect to the 1-norm.

Definition 1.5.1. A map $\mathbf{P} : I \to \mathbb{R}^n$ is called a *generalized n-system* if I is a closed subinterval of $\mathbb{R}$ with non-zero length and $\min I > 0$, and if the following properties hold for each $q \in I$:

(G1) The point $\mathbf{P}(q)$ lies in Δ^n, that is $0 \leq P_1(q) \leq \cdots \leq P_n(q)$.

(G2) The point $q^{-1}\mathbf{P}(q)$ lies in Σ^n, that is $P_1(q) + \cdots + P_n(q) = q$.

(G3) There exist $\underline{k}, \overline{k}, \underline{l}, \overline{l}$ with $P_{\underline{k}}(q) \geq P_{\overline{l}}(q)$ and a neighbourhood U of q such that

$$\mathbf{P}(q') = \begin{cases} \mathbf{P}(q) + (q'-q)\mathbf{e}_{\underline{k},\overline{k}} & \text{if } q' \leq q, \\ \mathbf{P}(q) + (q'-q)\mathbf{e}_{\underline{l},\overline{l}} & \text{if } q' \geq q. \end{cases} \qquad (q' \in U).$$

Denote by $\mathcal{G}_I^n$ the set of generalized n-systems defined on I, and denote by $\bar{\mathcal{G}}_I^n$ the set of maps which are the local uniform limits of sequences of maps in $\mathcal{G}_I^n$.

Observe that $\mathcal{S}_I^n \subseteq \mathcal{G}_I^n \subseteq \bar{\mathcal{G}}_I^n \subseteq \tilde{\mathcal{S}}_I^n$, and that the left and right derivatives are defined wherever relevant. The following theorem from Section 4 in [21] shows that unbounded generalized n-systems can be approximated by n-systems.

Theorem 1.5.2. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a proper generalized n-system. There exists a proper n-system $\mathbf{R} : I \to \mathbb{R}^n$ such that $\|\mathbf{P} - \mathbf{R}\| = \mathcal{O}(1)$, and so $\mathcal{F}(\mathbf{P}) = \mathcal{F}(\mathbf{R})$.*

One can also define division points of generalized n-systems analogously to the case of n-systems.

Definition 1.5.3. Fix an n-system $\mathbf{P} : I \to \mathbb{R}^n$, and let $q \in I$. If $(\underline{k}, \overline{k}), (\underline{l}, \overline{l})$ can be chosen to be distinct pairs for the property (G3) at q, then q is called a *division number*, and $\mathbf{P}(q)$ is called a *division point*.

In particular, the endpoints of I are division numbers. Moreover, a number $q \in I^{\circ}$ is not a division number with respect to $\mathbf{P}$ if and only if the derivative of $\mathbf{P}$ exists at q, in which case $\mathbf{P}'(q) = \mathbf{e}_{i,j}$ for some $i, j \in \{1, \ldots, n\}$. Moreover, like n-systems, the set of division numbers D of $\mathbf{P}$ is enumerable in increasing order, and $|D| = \infty \Leftrightarrow \sup D = \infty$ (cf. Proposition 1.2.12). The *ordered sequence of division points* is defined analogously to Definition 1.2.13.

1.5.1 Ordered Sequences of Division Points

This subsection considers the ordered sequences of division points coming from bounded generalized systems, and so in particular from bounded systems as well. These considerations extend naturally for the case where the generalized systems are unbounded.

The first observation is that a bounded generalized system is determined entirely by its ordered sequence of division points. Thus, an ordered sequence of division points for a bounded generalized system *defines* this generalized system.

Now, consider the following construction of certain piecewise linear maps.

Definition 1.5.4. Let $(\mathbf{x}_i)_{i\in J}$ be a sequence in $\mathbb{R}^n$, and let d_i denote the sum of the coordinates of $\mathbf{x}_i$ for each $i \in J$. If $(d_i)_{i\in J}$ is strictly increasing, then $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is called an *ordered sequence* in $\mathbb{R}^n$, in which case the piecewise linear map $\mathbf{P}$ *associated* to $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is defined as follows. Let

$$I = \bigcup_{i=\inf J}^{\sup J-1} [d_i, d_{i+1}],$$

and define $\mathbf{P} : I \to \mathbb{R}^n$ as the piecewise linear map such that

1. $\mathbf{P}(d_i) = \mathbf{x}_i$ for each $i \in J$,
2. $\mathbf{P}$ is affine on each $[d_i, d_{i+1}]$.

The quantities d_i and $\mathbf{P}(d_i)$ are called the *division numbers* and *division points* of $\mathbf{P}$, respectively. In particular, if $\lambda \in [0, 1]$ and $\inf J \le i < \sup J$, it follows that

$$\mathbf{P}(\lambda d_i + (1-\lambda)d_{i+1}) = \lambda\mathbf{P}(d_i) + (1-\lambda)\mathbf{P}(d_{i+1}).$$

The following proposition provides criteria determining when a sequence in $\mathbb{R}^n$ is an ordered sequence of division points for a generalized n-system. Its proof is omitted as it follows automatically from the properties (G1), (G2) and (G3) defining a generalized n-system.

Proposition 1.5.5. *Let $(\mathbf{x}_i)_{i\in J}$ be a finite ordered sequence in $\mathbb{R}^n$, let $\mathbf{P}$ be the piecewise linear map associated to it, and let $(d_i)_{i\in J}$ be its increasing sequence of division numbers. The sequence $(\mathbf{x}_i)_{i\in J}$ is an ordered sequence of division points for a generalized n-system if and only if for each $i \in \mathbb{N}$,*

1. *The point $\mathbf{x}_i$ is a non-zero point in Δ^n,*
2. *Property (G3) is satisfied for the map $\mathbf{P}$ at d_i,*

in which case $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is the ordered sequence of division points of $\mathbf{P}$.

It is worth noting that if moreover property (S3) is satisfied for the map $\mathbf{P}$ at each division number d_i, then $\mathbf{P}$ is an n-system.

Chapter 2

Results on the Spectra of $(\underline{\psi}_j, \overline{\psi}_j)$

This chapter provides several partial results towards determining the spectra of pairs $(\underline{\psi}_j, \overline{\psi}_j)$ for $n \geq 2$ and $j = 1, \ldots, n-1$. The cases $j = 1$ and $j = n-1$ were recently solved by Marnat and Moshchevitin in [15]. This thesis completes the case $n = 4$ by solving the case $j = 2$ when $n = 4$. An alternate proof for the cases $j = 1$ and $j = n-1$ when $n \geq 3$ is also given, relying on the parametric geometry of numbers and elementary algebra arguments. A conjecture for the general case is also established and demonstrated to hold true when restricting to quasi-regular n-systems.

2.1 Regular System Conjecture

This section establishes a conjectures generalizing the recent result in [15], and is given in terms of $n \geq 3$ and $j \in \{1, \ldots, n-1\}$. Specifically, if the conjecture holds true for the pair (n, j), then it completely determines the spectrum of $(\underline{\psi}_j, \overline{\psi}_j)$ for n-systems. Alternative versions of this conjecture are also established, and are shown to be equivalent to the original conjecture.

The following notation will be used throughout the remainder of this chapter.

Definition 2.1.1. For each integer $n \geq 2$, each $j \in \{1, \ldots, n-1\}$, and each n-system $\mathbf{P} : I \to \mathbb{R}^n$, write

$$\psi_j(q) = \psi_j(q; \mathbf{P}) = q^{-1}(P_1(1) + \cdots + P_j(q)),$$

and define a map $\chi_j : I \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\chi_j(q) = \chi_j(q; \mathbf{P}) = \frac{1 - \psi_j(q)}{\psi_j(q)} \quad (q \in I),$$

so that

$$\chi_j = \frac{P_{j+1} + \cdots + P_n}{P_1 + \cdots + P_j} \quad \text{and} \quad \psi_j = \frac{1}{1 + \chi_j}. \tag{2.1.1}$$

Also define the quantities

$$\underline{\chi}_j = \underline{\chi}_j(\mathbf{P}) = \frac{1 - \overline{\psi}_j(\mathbf{P})}{\overline{\psi}_j(\mathbf{P})} \quad \text{and} \quad \overline{\chi}_j = \overline{\chi}_j(\mathbf{P}) = \frac{1 - \underline{\psi}_j(\mathbf{P})}{\underline{\psi}_j(\mathbf{P})}. \tag{2.1.2}$$

Remark. If $\boldsymbol{\xi} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\mathbf{P} : I \to \mathbb{R}^n$ is an n-system such that

$$\|\mathbf{P} - \mathbf{L}_{\boldsymbol{\xi}}\| = \mathcal{O}(1),$$

then

$$\underline{\chi}_j(\mathbf{P}) = \hat{\omega}_{n-j}(\boldsymbol{\xi}) \quad \text{and} \quad \overline{\chi}_j(\mathbf{P}) = \omega_{n-j}(\boldsymbol{\xi})$$

for $j = 1, \ldots, n-1$

Thus, the notion of spectrum extends naturally for the pairs $(\underline{\chi}_j, \overline{\chi}_j)$ as follows.

Definition 2.1.2. For each integer $n \geq 2$ and each $j \in \{1, \ldots, n-1\}$, the *spectrum* of $(\underline{\chi}_j, \overline{\chi}_j)$ is the set of points $(\underline{\chi}_j(\mathbf{P}), \overline{\chi}_j(\mathbf{P}))$ where $\mathbf{P}$ runs through all proper n-systems. The spectrum of $\underline{\chi}_j$ and $\overline{\chi}_j$ are similarly defined.

2.1.1 First Version of the Conjecture

The first version of the conjecture uses the following function.

Definition 2.1.3. For each $n \geq 3$ and each $j \in \{1, \ldots, n-1\}$, define the function $\theta_j^n : \mathbb{R}^2 \to \mathbb{R}$ by

$$\theta_j^n(x, y) = \sum_{k=j-1}^{n-2} \left(\frac{y}{x}\right)^k - x \sum_{k=0}^{j-1} \left(\frac{y}{x}\right)^k.$$

Conjecture 2.1.4 (Version I)**.** *Let $n \geq 3$ and let $j \in \{1, \ldots, n-1\}$. The spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$ is the set $S \cup S'$, where*

$$S = \{(x, y) \in \mathbb{R}^2 \,|\, (n-j)/j \leq x \leq y \quad \textit{and} \quad \theta_j^n(x, y) \geq 0\},$$

and

$$S' = \begin{cases} [(n-j)/j, \infty] \times \{\infty\} & \textit{if } 1 \leq j \leq n-2, \\ [1/(n-1), 1] \times \{\infty\} & \textit{if } j = n-1. \end{cases}$$

The spectrum S_ψ of $(\underline{\psi}_j, \overline{\psi}_j)$ is related to the spectrum S_χ of $(\underline{\chi}_j, \overline{\chi}_j)$ as follows. Let $g : [0, 1] \to [0, \infty]$ and $h : [0, \infty] \to [0, 1]$ be the maps given by

$$g(x) = \frac{1 - x}{x} \quad \text{and} \quad h(x) = \frac{1}{1 + x}.$$

Observe that g is the inverse map of h, and that they are continuous for the usual topologies on $[0, 1]$ and $[0, \infty]$. Now, define the maps

$$G(x, y) = (g(y), g(x)) \quad \text{and} \quad H(x, y) = (h(y), h(x)).$$

Since G is inverse to H, then (2.1.2) implies that

$$G(S_\psi) = S_\chi \quad \text{and} \quad H(S_\chi) = S_\psi. \tag{2.1.3}$$

In particular, the topological subspace S_ψ of $[0, 1]^2$ is homeomorphic to the topological subspace S_χ of $[0, \infty]^2$.

2.1.2 Equivalent Form of the Conjecture

In order to state the second version of the conjecture, we define the following map.

Definition 2.1.5. For each $n \geq 3$ and each $j \in \{1, \ldots, n-1\}$, define the map $f_j : [0, \infty] \to [0, \infty]$ by

$$f_j(x) = \frac{x^{j-1} + \cdots + x^{n-2}}{1 + \cdots + x^{j-1}},$$

taking the appropriate limit when $x = \infty$.

Note that f_j is continuous for the usual topology on $[0, \infty]$. The map f_j is also strictly increasing, which can be seen as a consequence of the following proposition.

Proposition 2.1.6. *Let D, E be finite non-empty subsets of $\mathbb{R}$ with $\min D \geq \max E$. Let a_d, b_e be positive reals for each $d \in D$ and $e \in E$, and let $P, Q : (0, \infty) \to \mathbb{R}$ be the maps defined by*

$$P(x) = \sum_{d \in D} a_d x^d \quad \textit{and} \quad Q(x) = \sum_{e \in E} b_e x^e.$$

It follows that $(P/Q)' \geq 0$ on $(0, \infty)$. If moreover P/Q is not constant, then the inequality is strict on all of $(0, \infty)$, and P/Q is strictly increasing on $(0, \infty)$.

Proof: Let $M = \min D$ and $N = \max E$, and so as the coefficients a_d, b_e are positive, it follows for each $x \geq 0$ that $xP'(x) \geq MP(x)$ and $NQ(x) \geq xQ'(x)$. Hence, if $x > 0$, then $P(x), Q(x) > 0$, and so

$$P'(x)/P(x) \geq M/x \geq N/x \geq Q'(x)/Q(x).$$

It follows that $(P/Q)' = (P'Q - PQ')/Q^2 \geq 0$, on $(0, \infty)$. Moreover, if P/Q is not constant, then either $|D| > 1$, $|E| > 1$, or $M > N$, and so either $xP'(x) > MP(x)$, $NQ(x) > xQ'(x)$ or $M > N$. which implies that $(P/Q)' > 0$ on $(0, \infty)$. The fundamental theorem of calculus implies that P/Q is strictly increasing on $(0, \infty)$. ∎

The following corollary will be used later in the chapter.

Corollary 2.1.7. *Let D, E, P, Q be as in Proposition 2.1.6. Suppose that $P \neq Q$ and that $P - Q$ has a positive root z. Then, for all $x > 0$, it follows that*

$$\operatorname{sgn}(P(x) - Q(x)) = \operatorname{sgn}(x - z).$$

In particular, z is the unique positive root of $P - Q$. Moreover, if $\min E \geq 0$, then $P - Q$ is strictly increasing on $[z, \infty)$.

Proof: Since $P \neq Q$ and $P(z) = Q(z)$, it follows that P/Q is not constant, and so Proposition 2.1.6 implies that P/Q is strictly increasing on $(0, \infty)$. Thus, if $x > 0$, then $\operatorname{sgn}(x - z) = \operatorname{sgn}(P(x)/Q(x) - 1) = \operatorname{sgn}(P(x) - Q(x))$, as $Q(x) > 0$.

Now, suppose that $\min E \geq 0$. Thus, Q is monotonically increasing on $(0, \infty)$. Since P/Q is strictly increasing on $(0, \infty)$, then $P/Q - 1$ is strictly increasing on $(0, \infty)$. Since $P/Q - 1 \geq 0$ on $[z, \infty)$, it follows that $P - Q = Q(P/Q - 1)$ is strictly increasing on $[z, \infty)$, as Q is positive and monotonically increasing on $(0, \infty)$. ∎

As previously mentioned, Proposition 2.1.6 implies that f_j is strictly increasing. Hence, by computing $f_j(1)$ and $f_j(\infty)$, it follows that

$$f_j([1, \infty]) = \begin{cases} [(n-j)/j, \infty] & \text{if } j < n-1; \\ [(n-j)/j, 1] & \text{if } j = n-1. \end{cases} \tag{2.1.4}$$

The second version of the conjecture uses the following result.

Proposition 2.1.8. *If $\mathbf{P}$ is a proper n-system, then there exists a unique $\rho_j \in [1, \infty]$ such that $\underline{\chi}_j(\mathbf{P}) = f_j(\rho_j)$. In particular, the spectrum of $\underline{\chi}_j$ is contained in $f_j([1, \infty])$.*

Proof: Let $\mathbf{P} : I \to \mathbb{R}^n$ be a proper n-system. Since $P_1 \leq \cdots \leq P_n$, then by (2.1.1), we have that $(n - j)/j \leq \underline{\chi}_j$. By Proposition 1.2.6, we have $\mathbf{P}(q) \in \Delta_{n-1}$ for some arbitrarily large q. Since $\chi_{n-1}(q) \leq 1$ whenever $\mathbf{P}(q) \in \Delta_{n-1}$, it follows that $\underline{\chi}_{n-1} \leq 1$. Now, since f_j is strictly increasing, then by (2.1.4), there exists a unique $\rho_j \in [1, \infty]$ such that

$$f_j(\rho_j) = \underline{\chi}_j(\mathbf{P}).$$

It follows that the spectrum of $\underline{\chi}_j$ is contained in $f_j([1, \infty])$. ∎

Though we will later prove that $f_j([1, \infty])$ is exactly the spectrum of $\underline{\chi}_j$, only the above proposition is required to make sense of the following version of the conjecture.

Conjecture 2.1.9 (Version II)**.** *Let $n \geq 3$ and let $j \in \{1, \ldots, n-1\}$. The spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$ is the set*

$$\tilde{S} = \{(x, y) \in [0, \infty]^2 \mid \textit{There exists } \rho \in [1, \infty] \textit{ such that } x = f_j(\rho) \textit{ and } y \geq \rho x\}.$$

Theorem 2.1.10. *Let $n \geq 3$ and let $j \in \{1, \ldots, n-1\}$. The following are equivalent.*

1. *Conjecture 2.1.4 holds for the pair (n, j).*
2. *Conjecture 2.1.9 holds for the pair (n, j).*

Proof: Let S, S' be as in Conjecture 2.1.4, and let $\tilde{S}$ be as in Conjecture 2.1.9.

Let $(x, y) \in \mathbb{R}^2$ such that $y \geq x = f_j(\rho)$ for some $\rho \in [1, \infty]$. Suppose that $y < \infty$. Since f_j is strictly increasing, we have that $(x, y) \in \tilde{S}$ if and only if

$$y \geq \rho x \Leftrightarrow y/x \geq \rho \Leftrightarrow f_j(y/x) \geq f_j(\rho) = x.$$

On the other hand, we have that $(x, y) \in S$ if and only if

$$\theta_j^n(x, y) \geq 0 \Leftrightarrow \sum_{k=j-1}^{n-2} \left(\frac{y}{x}\right)^k \geq x \sum_{k=0}^{j-1} \left(\frac{y}{x}\right)^k \Leftrightarrow f_j(y/x) \geq x.$$

Hence, we have $(x, y) \in \tilde{S}$ if and only if $(x, y) \in S$, when $y < \infty$.

This is also true when $y = \infty$ because

$$(x, \infty) \in S \cup S' \Leftrightarrow x \in [(n-j)/j, f_j(\infty)] \Leftrightarrow (x, \infty) \in \tilde{S}.$$

Therefore, we have that $\tilde{S} = S \cup S'$, which proves the claim. ∎

A careful examination of the above proof yields the following corollary.

Corollary 2.1.11. *Let $n \geq 3$, let $j \in \{1, \ldots, n-1\}$, and let $\mathbf{P} : I \to \mathbb{R}^n$ be a proper n-system with $\overline{\chi}_j(\mathbf{P}) < \infty$. By letting $\rho \geq 1$ be such that $\underline{\chi}_j(\mathbf{P}) = f_j(\rho)$, we have that*

$$\theta_j^n(\underline{\chi}_j(\mathbf{P}), \overline{\chi}_j(\mathbf{P})) \geq 0 \iff \overline{\chi}_j(\mathbf{P}) \geq \rho \underline{\chi}_j(\mathbf{P}).$$

Moreover, we have $\theta_j^n(\underline{\chi}_j(\mathbf{P}), \overline{\chi}_j(\mathbf{P})) = 0$ if and only if $\overline{\chi}_j(\mathbf{P}) = \rho \underline{\chi}_j(\mathbf{P})$.

Moshchevitin proved the case $n = 4$ and $j = n - 1$ in [17]. The case $n = 4$ and $j = 1$ was later proved by Schmidt and Summerer in [27]. In [27, Section 3], they conjectured that the case $n \geq 3$ and $j = 1$ holds, and that the case $n \geq 3$ and $j = n - 1$ holds. This was recently proved by Marnat and Moshchevitin in [15].

Theorem 2.1.12 (Marnat-Moshchevitin)**.** *Conjecture 2.1.9 holds true for the case $n \geq 3$ and $j = 1$, as well as for the case $n \geq 3$ and $j = n - 1$.*

This thesis reproduces this result using only the parametric geometry of numbers, while the larger part of [15] is based on a more traditional approach.

This thesis will also prove the following case, thereby completing the case $n = 4$.

Theorem 2.1.13. *Conjecture 2.1.9 holds true for the case $n = 4$ and $j = 2$.*

This will be proven in Section 2.4.

2.1.3 Reduction of Conjecture 2.1.9

We show that Conjecture 2.1.9 is equivalent to the following conjecture.

Conjecture 2.1.14 (Version III). *Let $n \geq 3$ and let $j \in \{1, \ldots, n-1\}$. For each proper n-system $\mathbf{P} : I \to \mathbb{R}^n$, we have that*

$$\overline{\chi}_j(\mathbf{P}) \geq \rho_j \underline{\chi}_j(\mathbf{P}),$$

where $\rho_j \in [1, \infty]$ is characterized by $f_j(\rho_j) = \underline{\chi}_j$.

The proof that this statement is equivalent to Conjecture 2.1.9 uses the following result whose proof is based on a construction analogous to that of Marnat and Moshchevitin [15] for the cases $j = 1$ and $j = n-1$.

Theorem 2.1.15. *For each $j \in \{1, \ldots, n-1\}$, the set*

$$\{(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \,|\, \rho_j \in (1, \infty), \rho \in [1, \infty)\}$$

is contained in the spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$.

Proof: Let $j \in \{1, \ldots, n-1\}$, $\rho_j \in (1, \infty)$, and $\rho \in [1, \infty)$. We need to show that there exists a proper n-system $\mathbf{R}$ with

$$(\underline{\chi}_j(\mathbf{R}), \overline{\chi}_j(\mathbf{R})) = (f_j(\rho_j), \rho\rho_j f_j(\rho_j)).$$

Towards proving this, define

$$\mathbf{x} = (1, \ldots, \rho_j^{j-1}, \rho_j^{j-1}, \ldots, \rho_j^{n-2}), \quad \mathbf{x}' = (1, \ldots, \rho_j^{n-3}, \rho_j^{n-2}, \rho_j^{n-2}),$$

$$\mathbf{y}' = \rho(\rho_j, \rho_j, \rho_j^2, \ldots, \rho_j^{n-1}) \quad \text{and} \quad \mathbf{y} = \rho\rho_j \mathbf{x}.$$

Now, construct an n-system $\mathbf{P} : [r, s] \to \mathbb{R}^n$ by applying Theorem 1.2.22 to the sequence $(\mathbf{x}, \mathbf{x}', \mathbf{y}', \mathbf{y})$. Let $t \in (r, s)$ be such that $P_1 + \cdots + P_j$ has slope 0 on $[r, t]$ and slope 1 on $[t, s]$, as in Lemma 1.3.5. Figure 2.1.1 gives the combined graph of $\mathbf{P}$.

Since $P_1 + \cdots + P_j$ has slope 0 on $[r, t]$ and slope 1 on $[t, s]$, then

$$\underline{\chi}_j(\mathbf{P}) = \min\{\chi_j(r), \chi_j(s)\} \quad \text{and} \quad \overline{\chi}_j(\mathbf{P}) = \chi_j(t).$$

Thus, letting $\mathbf{R}$ be the self-similar n-system generated by $\mathbf{P}$, it follows that

$$\underline{\chi}_j(\mathbf{R}) = \frac{\rho_j^{j-1} + \cdots + \rho_j^{n-2}}{1 + \cdots + \rho_j^{j-1}} = f_j(\rho_j) \quad \text{and} \quad \overline{\chi}_j(\mathbf{R}) = \frac{\rho\rho_j^{j} + \cdots + \rho\rho_j^{n-1}}{1 + \cdots + \rho_j^{j-1}} = \rho\rho_j f_j(\rho_j),$$

thereby completing the proof. ∎

We deduce that the spectrum contains the following larger set.

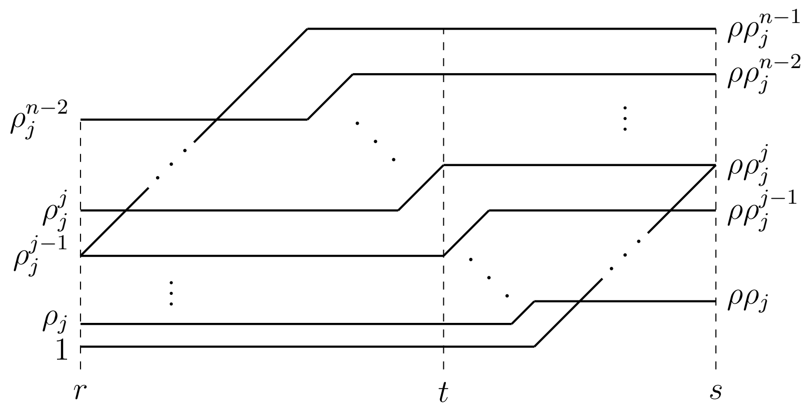

Figure 2.1.1: Combined Graph of **P**

Corollary 2.1.16. *For each $j \in \{1, \ldots, n-1\}$, the set*

$$S = \{(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in [1, \infty], \rho \in [1, \infty]\}$$

is contained in the spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$.

Proof: Let $j \in \{1, \ldots, n-1\}$, and define

$$\tilde{S} = \{(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in (1, \infty), \rho \in [1, \infty)\},$$

and

$$S = \{(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in [1, \infty], \rho \in [1, \infty]\}.$$

Theorem 2.1.15 yields that $\tilde{S}$ is contained in the spectrum S_χ of $(\underline{\chi}_j, \overline{\chi}_j)$.

Now, let G and H be as in (2.1.3) following Definition 2.1.2. It follows that

$$H(\tilde{S}) = \{H(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in (1, \infty), \rho \in [1, \infty)\}$$

is contained in the spectrum S_ψ of $(\underline{\psi}_j, \overline{\psi}_j)$. Theorem 1.2.27 implies that S_ψ is closed, and so as H and f_j are continuous maps, it follows that

$$H(S) = \{H(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in [1, \infty], \rho \in [1, \infty]\} \subseteq S_\psi.$$

Therefore, $S = G(H(S)) \subseteq G(S_\psi) = S_\chi$. ∎

Since the spectrum of $\underline{\chi}_j$ is contained in $f_j([1, \infty])$ for $j = 1, \ldots, n-1$, then Corollary 2.1.16 automatically implies the following result.

Corollary 2.1.17. *For each $j \in \{1, \ldots, n-1\}$, the spectrum of $\underline{\chi}_j$ is $f_j([1, \infty])$.*

The following theorem is a consequence of these last two corollaries.

Theorem 2.1.18. *Let S be as in Corollary 2.1.16, i.e.*

$$S = \{(f_j(\rho_j), \rho\rho_j f_j(\rho_j)) \mid \rho_j \in [1,\infty], \rho \in [1,\infty]\},$$

and let $H : [0,\infty] \times [0,\infty] \to [0,1] \times [0,1]$ be the map defined by

$$H(x,y) = \left(\frac{1}{1+y}, \frac{1}{1+x}\right).$$

The following conditions are equivalent for the pair (n,j).

1. *Conjecture 2.1.14 holds.*
2. *The set S is the full spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$.*
3. *The set $H(S)$ is the full spectrum of $(\underline{\psi}_j, \overline{\psi}_j)$.*

Proof: Suppose that the first condition holds true. By Corollary 2.1.17, the spectrum of $\underline{\chi}_j$ is equal to $f_j([1,\infty])$. Hence, if (a,b) is in the spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$, then there exists $\rho_j \in [1,\infty]$ such that $a = f_j(\rho_j)$. Conjecture 2.1.14 implies that $b \geq \rho_j a$, and so there exists $\rho \in [1,\infty]$ such that $b = \rho\rho_j a$. Thus, $(a,b) \in S$, and so Corollary 2.1.16 implies that S is the full spectrum of $(\underline{\chi}_j, \overline{\chi}_j)$. Therefore, the second condition holds true. Conversely, if the second condition holds true, then Conjecture 2.1.14 automatically holds true, which means that the first condition holds true. Finally, the second and third conditions are equivalent by (2.1.3). ∎

2.1.4 A Further Reduction

We use the following lemma to simplify Conjecture 2.1.4.

Lemma 2.1.19. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a proper n-system, and let $j \in \{1,\ldots,n-1\}$.*

1. *If $\overline{\chi}_j(\mathbf{P}) = \infty$, then Conjecture 2.1.14 holds for $\mathbf{P}$.*
2. *If $\underline{\chi}_j(\mathbf{P}) = f_j(\infty)$, then $\overline{\chi}_j(\mathbf{P}) = \infty$.*

Proof: The first claim is trivial. Now, assume without loss of generality that $\mathbf{P}$ is non-degenerate, and suppose that $\underline{\chi}_j(\mathbf{P}) = f_j(\infty)$. If $j < n-1$, then

$$\overline{\chi}_j \geq \underline{\chi}_j = f_j(\infty) = \infty.$$

Hence, suppose that $j = n-1$, and so $\underline{\chi}_j = 1$. To prove that $\overline{\chi}_j = \infty$, suppose otherwise. Then (2.1.2) implies that $\underline{\psi}_j > 0$, which implies that $\overline{\varphi}_n = 1 - \underline{\psi}_j < 1$. Now, since $1 - \varphi_n = \varphi_1 + \cdots + \varphi_{n-1} \leq (n-1)\varphi_{n-1}$, it follows that

$$0 < 1 - \overline{\varphi}_n = \liminf_{q\to\infty}(1-\varphi_n(q)) \leq (n-1)\liminf_{q\to\infty}\varphi_{n-1}(q) = (n-1)\underline{\varphi}_{n-1},$$

and so

$$\frac{\overline{\varphi}_n}{\underline{\varphi}_{n-1}} \leq \frac{(n-1)\overline{\varphi}_n}{1-\overline{\varphi}_n} = B < \infty.$$

Theorem 1.2.16 implies that there exists arbitrarily large $r, s \in I$ such that each $q \in [r, s]$ satisfies

$$\mathbf{P}(q) \in \Delta_{n-1} \quad \Longleftrightarrow \quad q \in \{r, s\},$$

and such that $n-2 \in \text{type}(r, s)$. Theorem 1.3.4 implies that $P_{n-1}(r) \leq P_{n-2}(s)$, and so by letting $t \in (r, s)$ be as in Lemma 1.3.5, it follows that

$$\frac{P_n(s)}{P_{n-2}(s)} = \frac{P_n(t)}{P_{n-2}(s)} \leq \frac{P_n(t)}{P_{n-1}(r)} = \frac{P_n(t)}{P_{n-1}(t)} = \frac{\varphi_n(t)}{\varphi_{n-1}(t)}.$$

Thus, since r, s, t are arbitrarily large, and since

$$\varphi_n(s) = \frac{P_n(s)}{P_1(s) + \cdots + P_{n-2}(s) + 2P_n(s)} \leq \frac{1}{P_{n-2}(s)/P_n(s) + 2} \leq \frac{1}{\varphi_{n-1}(t)/\varphi_n(t) + 2},$$

it follows that $\underline{\varphi}_n \leq 1/(B^{-1}+2) < 1/2$. However, a simple computation shows that

$$\underline{\chi}_{n-1} = 1 \quad \Longleftrightarrow \quad \overline{\psi}_{n-1} = 1/2 \quad \Longleftrightarrow \quad \underline{\varphi}_n = 1/2.$$

This is a contradiction since $\underline{\chi}_{n-1} = 1$. ∎

The second part of the above lemma implies that, for each proper n-system $\mathbf{P} : I \to \mathbb{R}^n$ and each $j \in \{1, \ldots, n-1\}$, we have

$$\overline{\chi}_j(\mathbf{P}) < \infty \quad \Longrightarrow \quad \underline{\chi}_j(\mathbf{P}) < f_j(\infty).$$

In view of the above, of Corollary 2.1.11, and of the first part of the above lemma, we deduce that Conjecture 2.1.14 is equivalent to the following conjecture.

Conjecture 2.1.20 (Version IV). *Let $n \geq 3$ and let $j \in \{1, \ldots, n-1\}$. For each proper n-system $\mathbf{P} : I \to \mathbb{R}^n$, with $\overline{\chi}_j(\mathbf{P}) < \infty$, we have that*

$$\theta_j^n(\underline{\chi}_j(\mathbf{P}), \overline{\chi}_j(\mathbf{P})) \geq 0. \tag{2.1.5}$$

2.2 The Conjecture for Quasi-Regular Systems

This section verifies that the conjecture holds when restricting to quasi-regular systems. Further, it is shown in this case that an even stronger result holds, and that this stronger result does not hold in general. It is also shown that those quasi-regular systems which are sharp with respect to (2.1.5) are in fact regular systems.

Fix $\mathbf{P}$ to be the quasi-regular n-system generated by some fixed strictly increasing sequence $(\alpha_i)_{i\in\mathbb{N}}$ of non-negative real numbers satisfying $\alpha_{m+k} = \rho\alpha_k$ for some $\rho > 1$, $m \in \mathbb{N}^+$, and for each $k \in \mathbb{N}$. Fix α to be the mean geometric reason of $\mathbf{P}$, i.e. the unique $\alpha \geq 0$ satisfying $\alpha^m = \rho$.

The following lemma is key in proving the main results in this section.

Lemma 2.2.1. *Let $a_1, \ldots, a_r, b_1, \ldots, b_s > 0$, let $k_1, \ldots, k_r, l_1, \ldots, l_s \in \mathbb{N}$, and let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a periodic sequence of positive real numbers with period T. Identifying numerators and denominators, consider the quantities*

$$\frac{A}{B} = \frac{a_1 + \cdots + a_r}{b_1 + \cdots + b_s} \quad \text{and} \quad \frac{A_j}{B_j} = \frac{\varepsilon_{j+k_1}a_1 + \cdots + \varepsilon_{j+k_r}a_r}{\varepsilon_{j+l_1}b_1 + \cdots + \varepsilon_{j+l_s}b_s} \quad (j \geq 0),$$

One of the following statements holds:

1. *$A_j/B_j = A/B$ for each $j \geq 0$.*
2. *There exist indices $k, l \geq M$ with $A_k/B_k < A/B < A_l/B_l$.*

Proof: Suppose that both (1) and (2) are false. Then, upon reversing fractions if necessary, assume without loss of generality that $A_j/B_j \leq A/B$ for all $j \geq 0$, and that $A_j/B_j < A/B$ for all $j \geq 0$ with $j \equiv k \mod T$ for some $k \geq 0$. By letting $E = \sum_{j=1}^{T} \varepsilon_j$, noting that $E = \sum_{j=N+1}^{N+T} \varepsilon_j$ for each $N \in \mathbb{N}$, it follows that

$$\begin{aligned} B\sum_{j=1}^{T} A_j &< A\sum_{j=1}^{T} B_j \\ B\sum_{j=1}^{T}\sum_{i=1}^{r} \varepsilon_{j+k_i}a_i &< A\sum_{j=1}^{T}\sum_{i=1}^{s} \varepsilon_{j+l_i}b_i \\ BE\sum_{i=1}^{r} a_i &< AE\sum_{i=1}^{s} b_i \\ BEA &< AEB, \end{aligned}$$

which is a contradiction. ∎

Corollary 2.2.2. *There exists $k, l \geq M$ with $A_k/B_k \leq A/B \leq A_l/B_l$.*

For the following proposition, recall the functions

$$f_j(x) = \frac{x^{j-1} + \cdots + x^{n-2}}{1 + \cdots + x^{j-1}} \quad (j = 1, \ldots, n-1),$$

as in Definition 2.1.5.

Proposition 2.2.3. *Let $j \in \{1, \ldots, n-1\}$. With the above notation, we have that*

$$\underline{\chi}_j = \underline{\chi}_j(\mathbf{P}) \leq f_j(\alpha) \quad \textit{and} \quad \overline{\chi}_j = \overline{\chi}_j(\mathbf{P}) \geq \alpha f_j(\alpha).$$

In particular, the conjecture is satisfied for $\mathbf{P}$.

Observe that in this statement, the number α depends only on $\mathbf{P}$ and not on the choice of j. This result is stronger than Conjecture 2.1.9 in that it implies a non-trivial property for the spectrum of $(\underline{\psi}_1, \overline{\psi}_1, \ldots, \underline{\psi}_{n-1}, \overline{\psi}_{n-1})$ for quasi-regular systems, as will be shown later.

Proof: (**Proposition 2.2.3**)
Define for each $i \in \mathbb{N}$ the quantity $\varepsilon_i = \alpha_i/\alpha^i > 0$, and note that

$$\alpha^{m+i}\varepsilon_{m+i} = \alpha_{m+i} = \rho\alpha_i = \alpha^m\alpha^i\varepsilon_i \Rightarrow \varepsilon_{m+i} = \varepsilon_i, \tag{2.2.1}$$

for all $i \in \mathbb{N}$. Corollary 2.2.2 yields $k, l \in \mathbb{N}$ such that

$$f_j(\alpha) = \frac{\alpha^{j-1} + \cdots + \alpha^{n-2}}{1 + \cdots + \alpha^{j-1}} \geq \frac{\alpha^{j-1}\varepsilon_{k+j-1} + \cdots + \alpha^{n-2}\varepsilon_{k+n-2}}{\varepsilon_k + \cdots + \alpha^{j-1}\varepsilon_{k+j-1}},$$

and

$$\alpha f_j(\alpha) = \frac{\alpha^j + \cdots + \alpha^{n-1}}{1 + \cdots + \alpha^{j-1}} \leq \frac{\alpha^j\varepsilon_{l+j} + \cdots + \alpha^{n-1}\varepsilon_{l+n-1}}{\varepsilon_l + \cdots + \alpha^{j-1}\varepsilon_{l+j-1}}.$$

Hence, (2.2.1) implies that

$$f_j(\alpha) \geq \frac{\alpha_{k+j-1} + \cdots + \alpha_{k+n-2}}{\alpha_k + \cdots + \alpha_{k+j-1}} \geq \underline{\chi}_j,$$

and that

$$\alpha f_j(\alpha) \leq \frac{\alpha_{l+j} + \cdots + \alpha_{l+n-1}}{\alpha_l + \cdots + \alpha_{l+j-1}} \leq \overline{\chi}_j,$$

proving the claim. ∎

Corollary 2.2.4. *Let $j \in \{1, \ldots, n-1\}$. Suppose that $\theta_j(\underline{\chi}_j, \overline{\chi}_j) = 0$. Then $\mathbf{P}$ is a regular system.*

Proof: Corollary 2.1.11 implies $f_j(\alpha') = \underline{\chi}_j$ and $\alpha' f_j(\alpha') = \overline{\chi}_j$ for some $\alpha' > 1$. Proposition 2.2.3 yields

$$f_j(\alpha') = \underline{\chi}_j \leq f_j(\alpha) \quad \text{and} \quad \alpha' f_j(\alpha') = \overline{\chi}_j \geq \alpha f_j(\alpha),$$

from which it follows that $\alpha' \leq \alpha \leq \alpha'$, as $f_j(x)$ and $x f_j(x)$ are increasing on $[0, \infty]$. This implies that

$$f_j(\alpha) = \underline{\chi}_j \quad \text{and} \quad \alpha f_j(\alpha) = \overline{\chi}_j.$$

Thus, for each $k \geq 0$, it follows that

$$\frac{\alpha^{j-1} + \cdots + \alpha^{n-2}}{1 + \cdots + \alpha^{j-1}} = \underline{\chi}_j \leq \frac{\alpha_{k+j-1} + \cdots + \alpha_{k+n-2}}{\alpha_k + \cdots + \alpha_{k+j-1}} = \frac{\alpha^{j-1}\varepsilon_{k+j-1} + \cdots + \alpha^{n-2}\varepsilon_{k+n-2}}{\varepsilon_k + \cdots + \alpha^{j-1}\varepsilon_{k+j-1}};$$

$$\frac{\alpha^{j} + \cdots + \alpha^{n-1}}{1 + \cdots + \alpha^{j-1}} = \overline{\chi}_j \geq \frac{\alpha_{k+j} + \cdots + \alpha_{k+n-1}}{\alpha_k + \cdots + \alpha_{k+j-1}} = \frac{\alpha^{j}\varepsilon_{k+j} + \cdots + \alpha^{n-1}\varepsilon_{k+n-1}}{\varepsilon_k + \cdots + \alpha^{j-1}\varepsilon_{k+j-1}}.$$

Applying Lemma 2.2.1 to each of these two inequalities yields equality throughout for each k. Thus, since $f_j(\alpha) = \underline{\chi}_j$ and $\alpha f_j(\alpha) = \overline{\chi}_j$, it follows that

$$f_j(\alpha) = \frac{\alpha_{k+j-1} + \cdots + \alpha_{k+n-2}}{\alpha_k + \cdots + \alpha_{k+j-1}} \quad \text{and} \quad \alpha f_j(\alpha) = \frac{\alpha_{k+j} + \cdots + \alpha_{k+n-1}}{\alpha_k + \cdots + \alpha_{k+j-1}},$$

for each k. It follows that $\alpha_{k+1} + \cdots + \alpha_{k+j} = \alpha(\alpha_k + \cdots + \alpha_{k+j-1})$, for each k. This is a linear recurrence relation, and its characteristic polynomial is

$$x^j + \cdots + x - \alpha(x^{j-1} + \cdots + 1) = (x - \alpha)(x^{j-1} + \cdots + 1) = (x - \alpha)\prod_{l=1}^{j-1}(x - e^{2li\pi/j}).$$

A standard result implies the existence of $c, c_l \in \mathbb{C}$ such that

$$\alpha_k = c\alpha^k + \sum_{l=1}^{j-1} c_l e^{2kli\pi/j} \quad (k \in \mathbb{N}).$$

Since $\lim_{k\to\infty} \alpha_k = \infty$, then $c \neq 0$. Hence, as $|e^{2li\pi}| = 1 < \alpha$ for each $l = 1, \ldots, j-1$, it follows that

$$\lim \alpha_{k+1}/\alpha_k = c\alpha^{k+1}/(c\alpha^k) = \alpha.$$

Meanwhile, as $\alpha_{k+m} = \rho\alpha_k$ for all $l \in \mathbb{N}$, it follows for each $l = 0, \ldots, m-1$ that

$$\frac{\alpha_{km+l+1}}{\alpha_{km+l}} = \frac{\rho^k \alpha_{k+1}}{\rho^k \alpha_k} = \frac{\alpha_{k+1}}{\alpha_k} \quad (k \in \mathbb{N}).$$

Thus, the sequence $(\alpha_{k+1}/\alpha_k)_{k\in\mathbb{N}}$ is periodic and converges to α, which implies that it is the constant sequence $(\alpha)_{k\in\mathbb{N}}$. Therefore, $\alpha_{k+1} = \alpha\alpha_k$ for each $k \in \mathbb{N}$, which means that $\mathbf{P}$ is a regular system. ∎

Observe by Proposition 2.2.3 that the conjecture restricted to quasi-regular systems not only holds true, but is quite stronger. Indeed, by defining β_j, γ_j such that $f_j(\beta_j) = \underline{\chi}_j$ and $\gamma_j f_j(\gamma_j) = \overline{\chi}_j$, the inequality $\theta_j(\underline{\chi}_j, \overline{\chi}_j) \geq 0$ is equivalent to the claim that the interval $[\beta_j, \gamma_j]$ is non-empty. In the case of quasi-regular systems, one deduces further that the intersection of these intervals for $j = 1, \ldots, n-1$ is non-empty, following from the fact that α is contained in each of them.

However, while the regular system conjecture might still be true in general, this stronger variant is not. For example, consider the self-similar 5-system generated by the 5-system $\mathbf{P} : [24.5, 73.5] \to \mathbb{R}^5$ with period 3 as in Figure 2.2.1.

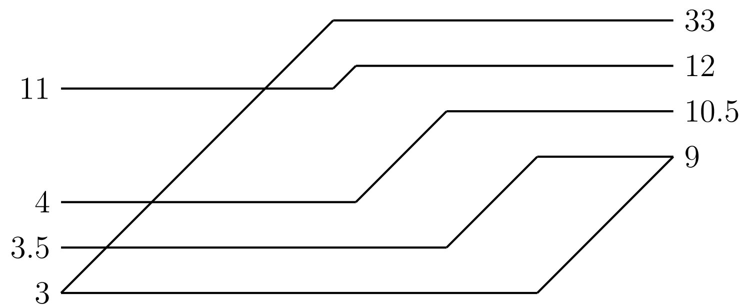

Figure 2.2.1: Combined Graph of $\mathbf{P}$

Some straightforward computations yield

$$\underline{\chi}_1 = \frac{43}{6}; \quad \underline{\chi}_2 = \frac{37}{13}; \quad \underline{\chi}_3 = \frac{10}{7}; \quad \underline{\chi}_4 = \frac{22}{43},$$

and that $\overline{\chi}_j = 3\underline{\chi}_j$ for each $j = 1, 2, 3, 4$. Meanwhile, one finds numerically that

$$\beta_1 = 1.40645...; \quad \gamma_1 = 1.80346...; \quad \beta_2 = 1.49878...; \quad \gamma_2 = 1.93646...;$$

$$\beta_3 = 1.73082...; \quad \gamma_3 = 2.21118...; \quad \beta_4 = 1.89324...; \quad \gamma_4 = 2.49529....$$

Since $\gamma_1 < \beta_4$, the intersection of the intervals $[\beta_j, \gamma_j]$ is empty.

2.3 Marnat-Moshchevitin Theorem

This section provides an alternate proof of the recent result of Marnat and Moshchevitin in [15]. It relies only on the parametric geometry of numbers, while the larger part of [15] is based on a more traditional approach. The novelty of the proof also lies in its simplicity. In particular, the number of points considered in the proof is in some sense minimal.

2.3.1 Preliminaries

Fix $n \geq 3$, fix $\mathbf{P}$ be a proper non-degenerate n-system, and define $S_i = P_1 + \cdots + P_i$ for $i = 1, \ldots, n-1$. Fix $j \in \{1, \ldots, n-1\}$, fix $(q_i)_{i\in\mathbb{N}}$ to be the unbounded and increasing sequence of division numbers of $\mathbf{P}$ with $\mathbf{P}(q_i) \in \Delta_j$, and define

$$M_k = S_j(q_k) \quad \text{and} \quad N_k = q_k - S_j(q_k),$$

as well as

$$\alpha_k = \inf_{i>k}(N_i/M_i), \quad g_k = \sup_{i>k}(M_{i+1}/M_i) > 1 \quad \text{and} \quad h_k = \sup_{i>k}(N_{i+1}/N_i) > 1,$$

for each $k \in \mathbb{N}$. Observe by Corollary 1.3.7 that

$$\overline{\chi}_j = \frac{1}{\underline{\psi}_j} - 1 = \limsup_{i\in\mathbb{N}} \frac{S_j(q_i) + q_{i+1} - S_j(q_{i+1})}{S_j(q_i)} - 1 = \limsup_{i\in\mathbb{N}} \frac{N_{i+1}}{M_i},$$

and that

$$\underline{\chi}_j = \frac{1}{\overline{\psi}_j} - 1 = \liminf_{i\in\mathbb{N}} \frac{q_i}{S_j(q_i)} - 1 = \liminf_{i\in\mathbb{N}} \frac{N_i}{M_i} = \lim_{k\in\mathbb{N}} \alpha_k. \tag{2.3.1}$$

The following lemma is based on a crucial observation by Marnat and Moshchevitin in [15]. It is key in proving the regular system conjecture for $j = 1$ and $j = n-1$.

Lemma 2.3.1. *If $g = \lim_{k\in\mathbb{N}} g_k$ and $h = \lim_{k\in\mathbb{N}} h_k$, then $\overline{\chi}_j/\underline{\chi}_j \geq \max\{g, h\}$.*

Proof: There exist unbounded subsets $I, J \subseteq \mathbb{N}$ such that

$$g = \lim_{i\in I} \frac{M_{i+1}}{M_i} \quad \text{and} \quad h = \lim_{i\in J} \frac{N_{i+1}}{N_i}.$$

Hence,

$$\overline{\chi}_j = \limsup_{i\in\mathbb{N}} \frac{N_{i+1}}{M_i} \geq \limsup_{i\in I} \frac{N_{i+1}}{M_i} = \limsup_{i\in I} \frac{gN_{i+1}}{M_{i+1}} \geq \liminf_{i\in\mathbb{N}} \frac{gN_{i+1}}{M_{i+1}} = g\underline{\chi}_j,$$

$$\overline{\chi}_j = \limsup_{i\in\mathbb{N}} \frac{N_{i+1}}{M_i} \geq \limsup_{i\in J} \frac{N_{i+1}}{M_i} = \limsup_{i\in J} \frac{hN_i}{M_i} \geq \liminf_{i\in\mathbb{N}} \frac{hN_i}{M_i} = h\underline{\chi}_j,$$

proving the claim. ∎

2.3.2 Spectrum of $(\underline{\psi_1}, \overline{\psi_1})$ for $n \geq 3$

Suppose that $j = 1$, with the notation of Subsection 2.3.1.

For this subsection, an integer $m \in \mathbb{N}$ is said to be *of type* I if $I \in \text{type}(q_m, q_{m+1})$. By Theorem 1.2.16, there exist infinitely many $m \in \mathbb{N}$ of type n. Moreover, by hypothesis, every m is of type 1 and of type 2. Hence, Theorem 1.3.4 implies that if m is of type k, then it is of type i for $i = 1, \dots, k$. Hence, fix $N \in \mathbb{N}$ arbitrarily large. There exist $m_1, \dots, m_{n-2} \in \mathbb{N}$ with $N < m_1 < \cdots < m_{n-2}$ satisfying the following properties.

$$m_{n+1-I} \text{ is of type } I \quad (I = 3, \dots, n), \tag{2.3.2}$$

$$m \text{ is not of type } I \text{ for each } m \in (m_{n-I}, m_{n+1-I}) \quad (I = 3, \dots, n-1). \tag{2.3.3}$$

Define the quantities

$$a_i^{(k)} = P_i(q_{m_k}), \quad b_i^{(k)} = P_i(q_{m_k+1}),$$

and

$$A_k = N_{m_k} = a_2^{(k)} + \cdots + a_n^{(k)}, \quad B_k = N_{m_k+1} = b_2^{(k)} + \cdots + b_n^{(k)},$$

for each $k = 1, \dots, n-2$ and each $i = 2, \dots, n$. Define also

$$h = h_N \quad \text{and} \quad \alpha = \alpha_N > 1,$$

and note that $hA_k \geq B_k$, $A_k \geq \alpha a_2^{(k)}$ and $B_k \geq \alpha b_2^{(k)}$ for $k = 1, \dots, n-2$.

The following proposition provides a key inequality in the proof of the result.

Proposition 2.3.2. $\alpha(B_1 + \cdots + B_{n-3}) + B_{n-2} \geq (\alpha - 1)(A_1 + \cdots + A_{n-2})$.

Proof: For $I = 4, \dots, n$, condition (2.3.3) implies for $k = I, \dots, n$ that

$$a_k^{(n+2-I)} = b_k^{(n+1-I)},$$

while condition (2.3.2) and Theorem 1.3.4 imply for $k = 2, \dots, I-1$ that

$$a_{k+1}^{(n+1-I)} \leq b_k^{(n+1-I)},$$

and that $a_3^{(n-2)} \leq b_2^{(n-2)}$. Hence, it follows for $I = 4, \dots, n$ that

$$B_{n+1-I} \geq \left(a_3^{(n+1-I)} + \cdots + a_I^{(n+1-I)}\right) + \left(a_I^{(n+2-I)} + \cdots + a_n^{(n+2-I)}\right),$$

and that $B_{n-2} \geq \alpha b_2^{(n-2)} \geq \alpha a_3^{(n-2)}$. Thus,

$$B_{n-2} + \alpha \sum_{I=4}^{n} B_{n+1-I} \geq \alpha a_3^{(n-2)} + \alpha \sum_{I=4}^{n} \left(a_3^{(n+1-I)} + \cdots + a_I^{(n+1-I)} + a_I^{(n+2-I)} + \cdots + a_n^{(n+2-I)}\right).$$

Reordering the summation on the right yields

$$B_{n-2} + \alpha \sum_{I=4}^{n} B_{n+1-I} \geq \alpha a_3^{(n-2)} + \left(\alpha \sum_{I=3}^{n} \left(a_3^{(n+1-I)} + \cdots + a_n^{(n+1-I)} \right) - \alpha a_3^{(n-2)} \right),$$

and the right hand side is equal to

$$\alpha \sum_{k=1}^{n-2} \left(A_k - a_2^{(k)} \right) \geq (\alpha - 1) \sum_{k=1}^{n-2} A_k,$$

completing the proof. ∎

Theorem 2.3.3. $\alpha - 1 \leq h + \cdots + h^{n-2}$.

Proof: Define $F_k = h + \cdots + h^k$,

$$C_k = A_1 + \cdots + A_k \quad \text{and} \quad D_k = B_1 + \cdots + B_k,$$

for $k = 1, \ldots, n-2$, and let $C_0 = D_0 = 0$. Define the inequality T_k by

$$T_k : \alpha D_{n-2-k} + h^{-1} F_k B_{n-1-k} \geq (\alpha - 1) C_{n-1-k},$$

for $k = 1, \ldots, n-2$. Since $F_1 = h$, then Proposition 2.3.2 states that T_1 holds. Now, if $F_k > \alpha - 1$ for some $k \in \{1, \ldots, n-3\}$, then $F_{n-2} > \alpha - 1$, which proves the theorem. Thus, assume without loss of generality that $F_k \leq \alpha - 1$ for $k = 1, \ldots, n-3$.

Now, let $k \in \{1, \ldots, n-3\}$ and suppose that T_k holds. Having assumed that $F_k \leq \alpha - 1$, the following statements automatically hold true.

1. $F_k \leq \alpha - 1$
2. $C_{n-1-k} = C_{n-2-k} + A_{n-1-k}$
3. $D_{n-2-k} = D_{n-3-k} + B_{n-2-k}$
4. $A_{n-1-k} \geq B_{n-2-k}$
5. $hA_{n-1-k} \geq B_{n-1-k}$
6. $F_k + 1 = h^{-1} F_{k+1}$

Using (5) on T_k yields

$$\alpha D_{n-2-k} + F_k A_{n-1-k} \geq (\alpha - 1) C_{n-1-k}.$$

By (2), it follows that

$$\alpha D_{n-2-k} + (F_k - (\alpha - 1))A_{n-1-k} \geq (\alpha - 1)C_{n-2-k}.$$

Since (1) implies that $F_k - (\alpha - 1) \leq 0$, then (4) yields

$$\alpha D_{n-2-k} + (F_k - (\alpha - 1))B_{n-2-k} \geq (\alpha - 1)C_{n-2-k}.$$

Substituting by (3) yields

$$\alpha D_{n-3-k} + (F_k + 1)B_{n-2-k} \geq (\alpha - 1)C_{n-2-k}.$$

Finally, (6) implies that T_{k+1} holds. It follows by induction that T_{n-2} holds, and so

$$F_{n-2}B_1 \geq (\alpha - 1)hA_1 \geq (\alpha - 1)B_1 \Rightarrow \alpha - 1 \leq F_{n-2},$$

completing the proof. ∎

Corollary 2.3.4. *Let* $h = \lim_{k\in\mathbb{N}} h_k$. *It follows that* $\underline{\chi}_1 \leq 1 + \cdots + h^{n-2}$.

Proof: As $k \to \infty$, $h_k \to h$, and $\alpha_k \to \underline{\chi}_1$, proving the result by continuity. ∎

Corollary 2.3.5. $\underline{\chi}_1 \leq 1 + \cdots + (\overline{\chi}_1/\underline{\chi}_1)^{n-2}$.

Proof: Let $P(x) = 1 + \cdots + x^{n-2}$ and $Q(x) = \underline{\chi}_1$, noting that P, Q satisfy the conditions of Proposition 2.1.6. Corollary 2.3.4 yields that $P(h) - Q(h) \geq 0$, and so Corollary 2.1.7 implies that $P - Q$ is strictly increasing on $[h, \infty)$. Thus, as Lemma 2.3.1 yields $\overline{\chi}_1/\underline{\chi}_1 \geq h$, it follows that

$$P(\overline{\chi}_1/\underline{\chi}_1) - Q(\overline{\chi}_1/\underline{\chi}_1) \geq 0,$$

from which the conclusion follows. ∎

2.3.3 Spectrum of $(\underline{\psi_{n-1}}, \overline{\psi_{n-1}})$ for $n \geq 3$

Suppose that $j = n - 1$ with the notation of Subsection 2.3.1.

As in Subsection 2.3.2, an integer $m \in \mathbb{N}$ is said to be *of type* I if $I \in \text{type}(q_m, q_{m+1})$. By Theorem 1.2.16, there exist infinitely many $m \in \mathbb{N}$ of type 1. Moreover, by hypothesis, every m is of type n and of type $n-1$. Hence, Theorem 1.3.4 implies that if m is of type k, then it is of type i for $i = k, \ldots, n$. Hence, fix $N \in \mathbb{N}$ arbitrarily large.

There exist $m_1, \dots, m_{n-2} \in \mathbb{N}$ with $N < m_1 < \cdots < m_{n-2}$ satisfying the following properties.

$$m_{n-1-I} \text{ is of type } I \quad (I = 1, \dots, n-2), \tag{2.3.4}$$

$$m \text{ is not of type } I \text{ for each } m \in (m_{n-1-I}, m_{n-I}) \quad (I = 2, \dots, n-2). \tag{2.3.5}$$

Define the quantities

$$a_i^{(k)} = P_i(q_{m_k}), \quad b_i^{(k)} = P_i(q_{m_k+1}),$$

and

$$A_k = M_{m_k} = a_1^{(k)} + \cdots + a_{n-1}^{(k)}, \quad B_k = M_{m_k+1} = b_1^{(k)} + \cdots + b_{n-1}^{(k)},$$

for each $k = 1, \dots, n-2$ and each $i = 1, \dots, n-1$. Define also

$$g = g_N \quad \text{and} \quad \alpha = \alpha_N < 1,$$

and note that $B_k \le gA_k$, $\alpha A_k \le a_{n-1}^{(k)}$ and $\alpha B_k \le b_{n-1}^{(k)}$ for $k = 1, \dots, n-2$.

The following proposition provides some key inequalities in the proof of the result.

Proposition 2.3.6. $\alpha A_i + A_{i+1} + \cdots + A_{n-2} \le (1-\alpha)(B_i + \cdots + B_{n-2})$, *for each* $i \in \{1, \dots, n-2\}$.

Proof: Since $B_i \le A_{i+1}$ and $\alpha < 1$, then

$$-\alpha A_i - A_{i+1} + \alpha A_{i+1} \le -(1-\alpha)B_i,$$

for $i = 1, \dots, n-3$. Hence, if the claim holds for $i = 1$, then the claim holds for each $i = 1, \dots, n-2$ by a simple recurrence argument. Thus, it suffices to prove the claim when $i = 1$.

For $I = 1, \dots, n-3$, condition (2.3.5) implies for $k = 1, \dots, I$ that

$$a_k^{(n-1-I)} = b_k^{(n-2-I)},$$

while condition (2.3.4) and Theorem 1.3.4 imply for $k = I, \dots, n-2$ that

$$a_{k+1}^{(n-1-I)} \le b_k^{(n-1-I)},$$

and that $a_{n-1}^{(1)} \le b_{n-2}^{(1)}$. Hence, it follows for $I = 1, \dots, n-3$ that

$$A_{n-1-I} \le \left(b_1^{(n-2-I)} + \cdots + b_I^{(n-2-I)}\right) + \left(b_I^{(n-1-I)} + \cdots + b_{n-2}^{(n-1-I)}\right).$$

and that $\alpha A_1 \le a_{n-1}^{(1)} \le b_{n-2}^{(1)}$. Thus,

$$\alpha A_1 + \sum_{I=1}^{n-3} A_{n-1-I} \le b_{n-2}^{(1)} + \sum_{I=1}^{n-3} \left(b_1^{(n-2-I)} + \cdots + b_I^{(n-2-I)} + b_I^{(n-1-I)} + \cdots + b_{n-2}^{(n-1-I)}\right).$$

Reordering the summation on the right yields

$$\alpha A_1 + \sum_{I=1}^{n-3} A_{n-1-I} \leq b_{n-2}^{(1)} + \left(\sum_{I=1}^{n-2} \left(b_1^{(n-1-I)} + \cdots + b_{n-2}^{(n-1-I)}\right) - b_{n-2}^{(1)} \right),$$

and the right hand side is equal to

$$\sum_{k=1}^{n-2} \left(B_k - b_{n-1}^{(k)}\right) \leq (1-\alpha) \sum_{k=1}^{n-2} B_k,$$

completing the proof. ∎

Theorem 2.3.7. $(1-\alpha)g^i \geq \alpha(g^{i-1} + \cdots + 1)$, *for* $i = 1, \ldots, n-2$.

Proof: Define $F_k = 1 + \cdots + g^{k-1}$ for $k = 1, \ldots, n-2$, let

$$C_i = A_i + \cdots + A_{n-2} \quad \text{and} \quad D_i = B_i + \cdots + B_{n-2},$$

for $i = 1, \ldots, n-2$, and let $C_{n-1} = D_{n-1} = 0$. Define the inequality $T_{i,k}$ by

$$T_{i,k} : \alpha F_i A_k + g^{i-1} C_{k+1} \leq (1-\alpha) g^{i-1} D_k,$$

for $i = 1, \ldots, n-3$ and $k = i, \ldots, n-2$. Since $F_1 = 1$, then Proposition 2.3.6 states that $T_{1,k}$ holds for $k = 1, \ldots, n-2$.

Now, let $i \in \{1, \ldots, n-3\}$ and suppose that $T_{i,k}$ holds for each $k = i, \ldots, n-2$. In particular, $T_{i,n-2}$ implies that

$$\alpha F_i A_{n-2} \leq (1-\alpha) g^{i-1} B_{n-2} \leq (1-\alpha) g^i A_{n-2} \Rightarrow \alpha F_i \leq (1-\alpha) g^i. \tag{2.3.6}$$

Hence, the following statements automatically hold true.

1. $\alpha F_i \leq (1-\alpha) g^i$
2. $C_{k+1} = C_{k+2} + A_{k+1}$, when $k \leq n-3$
3. $D_k = D_{k+1} + B_k$
4. $B_k \leq A_{k+1}$
5. $B_k \leq g A_k$
6. $F_{i+1} = F_i + g^i$

Using (5) on $T_{i,k}$ yields

$$\alpha F_i B_k + g^i C_{k+1} \leq (1-\alpha) g^i D_k.$$

By (3), it follows that

$$(\alpha F_i - (1-\alpha) g^i) B_k + g^i C_{k+1} \leq (1-\alpha) g^i D_{k+1}.$$

Since (1) implies that $\alpha F_i - (1-\alpha) g^i \leq 0$, then (4) yields

$$(\alpha F_i - (1-\alpha) g^i) A_{k+1} + g^i C_{k+1} \leq (1-\alpha) g^i D_{k+1}.$$

Substituting by (2) if $k \leq n-3$ yields

$$(\alpha F_i + \alpha g^i) A_{k+1} + g^i C_{k+2} \leq (1-\alpha) g^i D_{k+1}.$$

Finally, (6) implies that $T_{i+1,k+1}$ holds for $k = i, \ldots, n-3$. Hence, $T_{i+1,k}$ holds for $k = i+1, \ldots, n-2$. It follows by induction that $T_{i,n-2}$ holds for each $i = 1, \ldots, n-2$. Thus, (2.3.6) yields

$$(1-\alpha) g^i \geq \alpha F_i,$$

for $i = 1, \ldots, n-2$, completing the proof. ∎

Corollary 2.3.8. *Let $g = \lim_{k \in \mathbb{N}} g_k$. It follows that $g^{n-2} \geq \underline{\chi}_{n-1}(g^{n-2} + \cdots + 1)$.*

Proof: As $N \to \infty$, $g_N \to g$, and $\alpha_N \to \hat{\lambda}$, which proves the result by continuity. ∎

Corollary 2.3.9. $(\overline{\chi}_{n-1}/\underline{\chi}_{n-1})^{n-2} \geq \underline{\chi}_{n-1}((\overline{\chi}_{n-1}/\underline{\chi}_{n-1})^{n-2} + \cdots + 1)$.

Proof: Let $P(x) = x^{n-2}$ and $Q(x) = \underline{\chi}_1(x^{n-2} + \cdots + 1)$, noting that P, Q satisfy the conditions of Proposition 2.1.6. Corollary 2.3.8 yields that $P(g) - Q(g) \geq 0$, and so Corollary 2.1.7 implies that $P - Q$ is strictly increasing on $[g, \infty)$. Thus, as Lemma 2.3.1 yields $\overline{\chi}_1/\underline{\chi}_1 \geq g$, it follows that

$$P(\overline{\chi}_1/\underline{\chi}_1) - Q(\overline{\chi}_1/\underline{\chi}_1) \geq 0,$$

from which the conclusion follows. ∎

2.4 Spectrum of $(\underline{\psi_2}, \overline{\psi_2})$ for $n = 4$

This section is devoted to proving that the regular system conjecture holds in the case $j = 2$ when $n = 4$, thereby completing the proof of the conjecture for $n = 4$. The conjecture for this case is the claim that each proper 4-system $\mathbf{P}$ satisfies

$$(\overline{\chi}_2/\underline{\chi}_2)^2 + (\overline{\chi}_2/\underline{\chi}_2) \geq \underline{\chi}_2((\overline{\chi}_2/\underline{\chi}_2) + 1).$$

Since this inequality is equivalent to

$$\overline{\chi}_2/\underline{\chi}_2 \geq \underline{\chi}_2,$$

the following theorem is equivalent to the conjecture in the case $n = 4$ and $j = 2$.

Theorem 2.4.1. *If $\mathbf{P} : I \to \mathbb{R}^4$ is a proper 4-system, then $\overline{\chi}_2 \geq \underline{\chi}_2^2$.*

Proof: Let $\mathbf{P}$ be a non-degenerate proper 4-system, and let $(q_i)_{i\in\mathbb{N}}$ be the increasing sequence of division numbers with $\mathbf{P}(q_i) \in \Delta_2$. For each $k = 1, 2, 3, 4$, let $J_k \subseteq \mathbb{N}$ be the set of indices i such that $P_k(q_i) < P_k(q_{i+1})$, noting that J_k is infinite as $\mathbf{P}$ is proper. Furthermore, $J_2 = J_3 = \mathbb{N}$ since $\mathbf{P}$ is non-degenerate.

Let $N \in \mathbb{N}$, recall that $\psi_2 = (1 + \chi_2)^{-1}$ and $\chi_2 = (P_3 + P_4)/(P_1 + P_2)$, and define

$$\alpha = \inf_{q \geq q_N} \chi_2 \quad \text{and} \quad \beta = \sup_{q \geq q_N} \chi_2.$$

Note that there exist $i \in J_4$ and $j \in J_1$ with $N \leq i \leq j$. Fixing such j, assume without loss of generality that i is maximal. Let $a, b, c, d, e, f \in \mathbb{R}$ be such that

$$\mathbf{P}(q_i) = (a, b, b, c) \quad \text{and} \quad \mathbf{P}(q_{j+1}) = (d, e, e, f).$$

Let $t \in [q_i, q_{i+1}]$ be as in Lemma 1.3.5 for $\mathbf{P}|_{[q_i,q_{i+1}]}$, i.e. $P_1 + P_2$ is constant on $[q_i, t]$ and $P_3 + P_4$ is constant on $[t, q_{i+1}]$. Hence, as $k \notin J_4$ when $i < k \leq j$, it follows that P_4 is constant on $[t, q_{j+1}]$, and so $P_4(t) = P_4(q_{j+1})$. Thus, there exists $c' \in \mathbb{R}$ such that

$$\mathbf{P}(t) = (a, b, c', f).$$

Assume without loss of generality that the form of $\mathbf{P}$ on $[q_i, q_{i+1}]$ is canonical, as in Theorem 1.3.8. The rest of the proof is done by dividing in two cases.

For the first case, suppose that $i = j$. Figure 2.4.1 gives the combined graph of $\mathbf{P}$ on $[q_i, q_{j+1}]$.

Define the quantites

$$A' = a + b; \quad A = b + c; \quad B' = d + e; \quad B = e + f,$$

and note that $A \leq B'$, and by the definitions of α and β that

$$\alpha \leq \min\{A/A', B/B'\} \quad \text{and} \quad \beta \geq \chi_2(t) = B/A'.$$

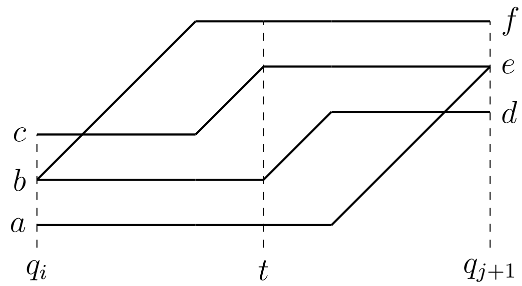

Figure 2.4.1: Case $i = j$

It follows that

$$\alpha^2\beta^{-1} \leq \frac{AB}{A'B'} \cdot \frac{A'}{B} = \frac{A}{B'} \leq 1.$$

For the second case, suppose that $i < j$. Figure 2.4.2 gives the combined graph of $\mathbf{P}$ on $[q_i, q_{j+1}]$. Since $\chi_2 = (P_3 + P_4)/(P_1 + P_2)$, it follows by the definitions of α and β that

$$\alpha^2 \leq \frac{f+e}{d+e} \cdot \frac{c+b}{a+b} \quad \text{and} \quad \beta \geq \chi_2(t) = \frac{f+c'}{a+b} \geq \frac{f+c}{a+b}.$$

Now, since $b \leq c \leq d \leq e \leq f$, it follows that

$$\alpha^2\beta^{-1} \leq \frac{f+e}{d+e} \cdot \frac{c+b}{a+b} \cdot \frac{a+b}{f+c} = \frac{f+e}{d+e} \cdot \frac{c+b}{f+c} \leq \frac{f+e}{d+e} \cdot \frac{c+d}{f+c} \leq \frac{f+d}{d+d} \cdot \frac{d+d}{f+d} = 1.$$

Thus, $\alpha^2 \leq \beta$ in general. As N can be arbitrarily large, it follows that $\underline{\chi}_2^2 \leq \overline{\chi}_2$. ∎

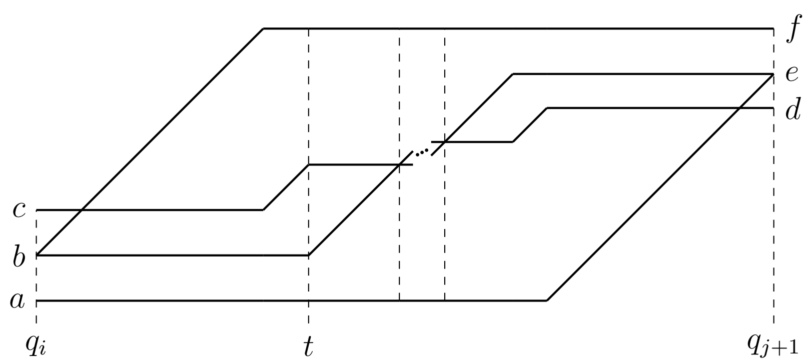

Figure 2.4.2: Case $i < j$

2.5 A Test Case for $(\underline{\psi_2}, \overline{\psi_2})$ in Dimension $n = 5$

This section shows that the conjecture holds for a small subclass of those n-systems which are canonical with respect to the spectrum of $(\underline{\psi}_j, \overline{\psi}_j)$ as in Theorem 1.3.8.

Consider a 5-system with the switch points

$$(a, b, \underline{b}, c, d) \to (a, \underline{b}, c, d, \overline{f}) \to (\underline{a}, c, d, \overline{e}, f) \to (c, \overline{d}, d, e, f),$$

with $0 < a < b < c < d < e < f$ and with $(c, d, d, e, f) = \rho(a, b, b, c, d)$ for some $\rho > 1$. Figure 2.5.1 gives the combined graph of this 5-system.

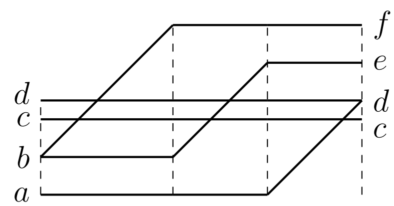

Figure 2.5.1

This 5-system generates a self-similar 5-system $\mathbf{P}$ depending on the choice of a, b, c, d, e, f. Note that $\rho = c/a$, that d, e, f are fixed by any choice of a, b, c, and that the block is well-defined for any choice of a, b, c with $0 < a < b < c$. Hence, write $\mathbf{P}^{(a,b,c)} = \mathbf{P}$, and consider the subset

$$\mathcal{S}' = \{\mathbf{P}^{(a,b,c)} \mid a, b, c \in \mathbb{R} \text{ with } 0 < a < b < c\},$$

of the set of all proper 5-systems. The main result of this section is the following.

Theorem 2.5.1. *Let $\mathbf{P} \in \mathcal{S}'$, and write $\alpha = \underline{\chi}_2$, $\beta = \overline{\chi}_2$ and $g = \beta/\alpha$. We have*

1. $g^2 + 2g \geq 2\alpha$,
2. $g^3 + g^2 + g \geq \alpha(g + 1)$.

In particular, $\mathbf{P}$ *satisfies the regular system conjecture.*

Proof: To prove the first result, let a, b, c with $0 < a < b < c$ be such that

$$\mathbf{P} = \mathbf{P}^{(a,b,c)}.$$

By the definition of α, the first switch point of $\mathbf{P}$ implies that

$$\alpha \leq \frac{b + c + d}{a + b} = \frac{b/a + c/a + (b/a)(c/a)}{1 + b/a}.$$

By the definition of β, the second switch point of $\mathbf{P}$ implies that

$$\beta \geq \frac{c+d+f}{a+b} = \frac{c/a + (b/a)(c/a) + (b/a)(c/a)^2}{1+b/a}.$$

Since scaling a, b, c does not change the value of α and β, assume without loss of generality that $a = 1$. Hence,$1 < b < c$, and

$$\alpha \leq A = \frac{b+c+bc}{1+b} \quad \text{and} \quad \beta \geq \frac{c+bc+bc^2}{1+b}.$$

Let $g = \beta/\alpha > 1$, and note that

$$g \geq G = \frac{c+bc+bc^2}{b+c+bc}.$$

Now, define $f(x,y)$ as the polynomial

$$(y+xy+xy^2)^2(1+x)+2(y+xy+xy^2)(x+y+xy)(1+x)-2(x+y+xy)(x+y+xy)^2,$$

and note that

$$G^2 + 2G \geq 2A \Leftrightarrow f(b,c) \geq 0.$$

Now, the polynomial $f(b,y)$ is equal to

$$(y+by+by^2)^2(1+b)+2(y+by+by^2)(b+y+by)(1+b)-2(b+y+by)(b+y+by)^2$$

which can be rewritten, following a series of computations, as

$$(b^3+b^2)y^4 + 2(b^3+b^2-b-1)y^3 - (b^3+b^2-3b-3)y^2 - 2(2b^3+b^2-b)y - 2b^3.$$

Since $b > 1$, then the coefficients of y^3 and y^4 are positive, while the coefficients of 1 and y are negative. Hence, Corollary 2.1.7 implies that $f(b,y)$ has a unique non-negative root y_0 in y, and that

$$\operatorname{sgn}(f(b,y)) = \operatorname{sgn}(y - y_0) \quad (y \geq 0). \tag{2.5.1}$$

After another of series of computations, one finds the polynomial

$$h(x) = f(x,x)/x^2 = x^5 + 3x^4 + x^3 - 7x^2 - 3x + 5.$$

Observe that $h(1) = 0$, and note that the derivative of h is given by

$$h'(x) = 5x^4 + 12x^3 + 3x^2 - 14x - 3.$$

Corollary 2.1.7 yields a unique non-negative root for the polynomial $h'(x)$, say x_0, and that x_0 satisfies $\operatorname{sgn}(h'(x)) = \operatorname{sgn}(x - x_0)$ for $x \geq 0$. Since $h'(1) = 3 > 0$, it follows that $x_0 < 1$, and so $h'(x) > 0$ for all $x \geq 1$. It follows that $h(x) \geq h(1) = 0$ for all $x \geq 1$, and so $f(x, x) \geq 0$ for all $x \geq 1$. Thus, $f(b, b) \geq 0$, and so (2.5.1) implies that $c > b \geq y_0$, which in turn implies that $f(b, c) > 0$, and so $G^2 + 2G \geq 2A$. Since $g \geq G$ and $A \geq \alpha$, then

$$g^2 + 2g \geq G^2 + 2G \geq 2A \geq 2\alpha,$$

proving the first result. Now, since $g > 1$, then the first result yields

$$g^2 \geq 2(\alpha - g) \Rightarrow g^3 \geq 2g(\alpha - g) \Rightarrow g^3 \geq (g + 1)(\alpha - g).$$

It follows that

$$g^3 + g(g + 1) \geq \alpha(g + 1),$$

proving the second result. Hence, **P** satisfies the regular system conjecture. ∎

Chapter 3

Existence of Non-Semialgebraic Spectra

This chapter provides a brief introduction to the theory of *semialgebraic geometry*, whose primary objects of study are *semialgebraic* sets, defined in the following section. This section also introduces some terminology relevant to the study of such sets, with a focus on convex *polytopes*, which are a generalization of convex polyhedra in higher dimensions. The Tarski-Seidenberg theorem is also stated, and is key in the proof of the main result in this section.

The main result in this chapter is the existence for each $n \geq 4$ of a family of spectra of n-systems which are not semialgebraic sets. Although not as natural as the exponents associated to the maps φ_i and ψ_j, they remain simple to define using certain linear maps $T : \mathbb{R}^n \to \mathbb{R}^{n+1}$. The proof relies on a relatively simple construction and characterization of systems $\mathbf{P}$ with $T(\mathbf{P}) \subseteq [0, \infty)^{n+1}$, and uses theorems from Section 1.4, in particular the Arzèla-Ascoli theorem. This is in contrast to a recent and remarkable result of Roy in [22] proving that spectra of n-systems are always semialgebraic sets for $n = 2, 3$.

3.1 Semialgebraic Geometry

Throughout this section, fix $m \in \mathbb{N}^+$. For a more thorough exposition of the theory which follows, the reader can refer to [28], [29] and [3].

Definition 3.1.1. A subset X in $\mathbb{R}^m$ is said to be *semialgebraic* if it can be realized as a finite combination of intersections, unions and complements of sets which are defined by systems of polynomial equalities and inequalities with real coefficients.

In this way, semialgebraic sets are related to the problem of determining existence of a real solution to a system of polynomial equalities and inequalities with real coefficients. A result in [28] shows that any semialgebraic set can be written as follows.

Theorem 3.1.2. *Let $X \subseteq \mathbb{R}^m$ be semialgebraic. There exists $k, s_1, \dots, s_k \in \mathbb{N}^+$ and polynomials $P_i(\mathbf{x}), Q_{i,j}(\mathbf{x})$ in $\mathbb{R}[x_1, \dots, x_m]$ such that*

$$X = \bigcup_{i=1}^{k} \{\mathbf{x} \in \mathbb{R}^m \mid P_i(\mathbf{x}) = 0 \text{ and } Q_{i,j}(\mathbf{x}) > 0 \text{ for } j = 1, \dots, s_i\}.$$

3.1.1 Tarski-Seidenberg Theorem

The Tarski-Seidenberg theorem is an important theorem in semialgebraic geometry. It is often stated as follows, though different versions exist (cf. [28], [3].)

Theorem 3.1.3. *Let $X \subseteq \mathbb{R}^{m+k}$ be semialgebraic, and let $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^m$ be the projection onto the first m-coordinates. The set $\pi(X)$ is a semialgebraic set in $\mathbb{R}^m$.*

This is a key theorem in the proof that spectra of 3-systems are always semialgebraic [22], and is used in this chapter to deduce that a certain family of spectra consists of sets which are not semialgebraic.

There also exists a version providing an algorithm, as well as a version focused on systems of polynomial equalities and inequalities allowing quantifiers [3]. However, neither of these are used in this thesis, and their statements are omitted.

3.1.2 Positive Loci of $\mathbb{R}^m$-Valued Maps

Recall that $\mathbb{R}^m$ is partially ordered by the product ordering, i.e. if $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ are points in $\mathbb{R}^m$, then

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, \dots, m.$$

In particular, $\mathbf{x} \geq \mathbf{0}$ if and only if $x_i \geq 0$ for $i = 1, \dots, m$. The inequality is said to be *strict*, denoted $\mathbf{x} < \mathbf{y}$, if $\mathbf{x} \neq \mathbf{y}$, i.e. $\mathbf{x} < \mathbf{y} \Leftrightarrow \mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Whereas algebraic geometry can be said to study the zero loci of polynomials, i.e. the set of zeroes of polynomials, semialgebraic geometry can be said to study the *positive loci* of polynomials. In general, consider the following definition.

Definition 3.1.4. The *positive locus* of a map $f : X \to \mathbb{R}^m$ is the set P of points $x \in X$ such that $f(x) \geq \mathbf{0}$ for each $x \in P$. If the inequality is instead strict, the resulting set P is instead called the *strictly positive locus* of f. The *negative locus* and *strictly negative locus* are defined similarly.

In particular, the positive locus of f is the disjoint union of its zero locus Z and its strictly positive locus S. It is also worth noting that the zero locus of f is the intersection of the positive loci P and $\tilde{P}$ of the maps f and $-f$. In this way,

$$Z = P \cap \tilde{P} \quad \text{and} \quad S = P \backslash Z = P \backslash \tilde{P},$$

and so zero loci and strictly positive loci can be studied using only positive loci.

3.1.3 Convex Polytopes in $\mathbb{R}^m$

This thesis considers a generalization of convex polyhedra in $\mathbb{R}^3$ called *convex polytopes*. Before stating its definition, consider the following proposition and corollary.

Proposition 3.1.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^m$. If $T(\mathbf{x}), T(\mathbf{y}) \geq \mathbf{z}$, then $T(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \mathbf{z}$ for each $\lambda \in [0, 1]$.*

Proof: If $\lambda \in [0, 1]$, then $T(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda T(\mathbf{x}) + (1-\lambda)T(\mathbf{y}) \geq \mathbf{z}$. ∎

Corollary 3.1.6. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then its positive locus P is convex.*

Definition 3.1.7. A *convex polytope* in $\mathbb{R}^m$ is a non-empty, bounded subset P of $\mathbb{R}^m$ which can be realized as the positive locus of a map $L = (L_1, \ldots, L_k) : \mathbb{R}^m \to \mathbb{R}^k$ where each L_i is a polynomial of degree 1.

Convex polytopes are indeed convex sets by Proposition 3.1.5. Moreover, the definition remains equivalent if the maps L_i are allowed to be of degree at most 1. Indeed, if all the maps are constant, then the set of solutions is either $\mathbb{R}^m$, which is unbounded, or the emptyset. Moreover, adding a constant map $L_{k+1}(\mathbf{x}) = c$ either gives the emptyset when $c < 0$, or does not change the set of solutions when $c \geq 0$. This leads to the following observation.

Remark. If $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^k$, then the set of solutions $\mathbf{x} \in \mathbb{R}^m$ to the inequation $A\mathbf{x} \geq \mathbf{b}$ is a convex polytope if and only if it is non-empty and bounded.

Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, recalling that one can compute for an unbounded n-system $\mathbf{P}$ the quantity $\underline{\varphi}_T(\mathbf{P})$ as being equal to $\inf T(\mathcal{K}(\mathbf{P}))$, the following definition will prove useful.

Definition 3.1.8. If $f : X \to Y$ and $F : Y \to \mathbb{R}^m$ are maps such that $f(X)$ is contained in the positive locus of F, then f is said to be *positively oriented* by F.

In other words, f is positively oriented by F if and only if $F(f(x)) \subseteq [0,\infty)^n$. For example, in light of Corollary 1.2.29, the bounded set $\mathcal{K}(\mathbf{P})$ for a self-similar n-system $\mathbf{P}$ is the convex hull of finitely many points. Hence, linear algebra can be used to determine a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{K}(\mathbf{P})$ is the positive locus of T, and so $\mathbf{P}$ is positively oriented by T by self-similarity. Conversely, if an unbounded n-system $\mathbf{P}$ is positively oriented by some linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, then $\mathcal{K}(\mathbf{P})$ is contained in the positive locus of T by continuity.

3.2 A Family of Non-Semialgebraic Spectra

Throughout the remainder of this chapter, fix an integer $n \geq 4$, fix $\alpha \in (1,\infty)$, and define the linear map $T = (T_0, T_1, \ldots, T_n) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ by

$$T_0(\mathbf{x}) = x_1; \quad T_1(\mathbf{x}) = x_n - \alpha^{n-2}x_1; \quad T_n(\mathbf{x}) = x_n - \alpha^{n-3}x_2,$$

and by

$$T_j(\mathbf{x}) = \alpha x_j - x_{j+1} \quad (j = 2, \ldots, n-1).$$

The goal of this section is to prove the following theorem.

Theorem 3.2.1. *The spectrum of $\underline{\varphi}_T$ is not a semialgebraic set.*

Its proof relies on several results, among which the following lemma is key.

Lemma 3.2.2. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a map with $P_n > 0$ and positively oriented by T. Suppose for some $q \in I$ and $j \in \{1, \ldots, n-1\}$ that $P_j(q) = P_{j+1}(q)$. It follows that*

$$P_{i+1}(q) = \alpha P_i(q) \quad \text{for each } i \in \{2, \ldots, n-1\} \backslash \{j\}.$$

In particular, if k is such that $P_j(q) = P_{j+1}(q)$ and $P_k(q) = P_{k+1}(q)$, then $k = j$.

Proof: If $j = 1$, then $P_1(q) = P_2(q)$, and the hypothesis yields

$$\alpha^{n-2}P_1(q) = \alpha^{n-2}P_2(q) \geq \alpha^{n-3}P_3(q) \geq \cdots \geq P_n(q) \geq \alpha^{n-2}P_1(q),$$

which implies equality throughout. If $j \geq 2$, then $P_j(q) = P_{j+1}(q)$, and so

$$\alpha^{n-3}P_2(q) \geq \cdots \geq \alpha^{n-j-1}P_j(q) = \alpha^{n-j-1}P_{j+1}(q) \geq \cdots \geq P_n(q) \geq \alpha^{n-3}P_2(q),$$

which implies equality throughout. The second claim follows automatically. ∎

In particular, Lemma 3.2.2 applies to n-systems $\mathbf{P}$ as well as prototypes $\mathbf{P}$, since these always have $P_n > 0$. Moreover, Lemma 3.2.2 can be interpreted as follows.

Corollary 3.2.3. *With the notation and hypotheses of Lemma 3.2.2, it follows that*

$$\begin{cases} \mathbf{P}(q) = (c, c, c\alpha, \dots, c\alpha^{n-2}) & \text{if } j = 1; \\ \mathbf{P}(q) = (d, e, \dots, e\alpha^{j-2}, e\alpha^{j-2}, \dots, e\alpha^{n-3}) & \text{if } j \geq 2, \end{cases} \tag{3.2.1}$$

for some $c, d, e > 0$. In particular, $e \geq \alpha d$.

Proof: This first claim follows automatically from Lemma 3.2.2. To conclude that $e \geq \alpha d$, note that $T_1(\mathbf{P}(q)) = e\alpha^{n-3} - \alpha^{n-2} d \geq 0$ when $j \geq 2$. ∎

The following notation will be useful.

Definition 3.2.4. For each $\mathbf{x} \in \mathbb{R}^n$, define the quantity $|\mathbf{x}|$ to be the 1-norm of $\mathbf{x}$, i.e. $|\mathbf{x}| = |x_1| + \cdots + |x_n|$. If $\mathbf{x}$ is also non-zero, then define the quantity $\bar{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$.

In particular, this notation is consistent with the absolute value notation in the case where $n = 1$. Moreover, recalling the notation

$$\Delta^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, 0 \leq x_1 \leq \cdots \leq x_n\} \quad \text{and} \quad \Sigma^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_1 + \cdots + x_n = 1\},$$

it follows that if $\mathbf{x} \in \Delta^n$, then $|\mathbf{x}| = x_1 + \cdots + x_n$, and $\bar{\mathbf{x}} \in \Delta^n \cap \Sigma^n$ when $\mathbf{x} \neq 0$. Thus, if $\mathbf{P} : I \to \mathbb{R}^n$ is an n-system, then properties (S1) and (S2) imply for each $q \in I$ that $|\mathbf{P}(q)| = q$ and $\varphi(q; \mathbf{P}) = \overline{\mathbf{P}(q)} \in \Delta^n \cap \Sigma^n$, as $\min I > 0$. In view of this, define the set

$$\bar{\Delta}^n = \Delta^n \cap \Sigma^n,$$

and define for each $j \in \{1, \dots, n-1\}$ the sets

$$\bar{\Delta}_j = \bar{\Delta}_j^n = \Delta_j^n \cap \Sigma^n,$$

recalling that

$$\Delta_j = \Delta_j^n = \{\mathbf{x} \in \Delta^n \,|\, x_j = x_{j+1}\}.$$

3.2.1 Preliminary Results

The following lemmas and corollaries will prove useful in the main construction.

Lemma 3.2.5. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, let $r, s \in I$ with $r < s$, and let $k, l \in \{1, \dots, n\}$. If $k \in \operatorname{type}(r, s)$ and $l \notin \operatorname{type}(r, s)$, then*

$$\frac{P_k(r)}{P_l(r)} < \frac{P_k(s)}{P_l(s)},$$

Proof: Since P_k is monotone, it follows that $P_k(r) < P_k(s)$, while $P_l(r) = P_l(s)$. ∎

Lemma 3.2.6. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be a non-degenerate n-system, and let $r, s \in I$ be transition numbers with $r < s$ such that there are no transition numbers in (r, s). If i, j are such that $\mathbf{P}(r) \in \Delta_i$ and $\mathbf{P}(s) \in \Delta_j$, then the following statements hold:*

1. $\min \operatorname{type}(r, s) = j$ *and* $\max \operatorname{type}(r, s) = i + 1$.
2. *If $j > i$, then $j = i + 1$ and $\mathbf{P}(s) = \mathbf{P}(r) + (P_{i+2}(r) - P_{i+1}(r))\mathbf{e}_j$.*
3. *There exists $t_0, \ldots, t_n \in [r, s]$ with $r = t_n \le \cdots \le t_0 = s$ such that*

$$P_k'(q) = 1 \Leftrightarrow q \in (t_k, t_{k-1})$$

for $k = 1, \ldots, n$.

Proof: Since there are no transition numbers in (r, s), the non-degeneracy of $\mathbf{P}$ and Theorem 1.3.4 imply the first result. The second result follows from the first. Now, if $q \in (r, s)$ with $P_k'(q^-) = 1 = P_l'(q^+)$, then $k \ge l$, since there are no transition numbers in (r, s), from which the third result follows. ∎

Corollary 3.2.7. *Under the hypotheses of Lemma 3.2.6, the following hold:*

1. *If $i + 1 < n$, then $P_{i+2}(s)/P_k(s) < P_{i+2}(r)/P_k(r)$ for $k = j, i + 1$.*
2. *If $j > 1$, then $P_k(r)/P_{j-1}(r) < P_k(s)/P_{j-1}(s)$ for $k = j, i + 1$.*

Proof: Both statements follow automatically by Lemmas 3.2.5 and 3.2.6. ∎

The following proposition will be useful in the proof of non-semialgebraicity.

Proposition 3.2.8. *Let $\mathbf{P} : I \to \mathbb{R}^n$ be an unbounded and non-degenerate n-system, let $i \in \{1, \ldots, n-1\}$, and let T_i be the set of transition numbers q such that $\mathbf{P}(q) \in \Delta_i$. Suppose that P_i is unbounded and that there exists $\delta > 0$ such that $\underline{\varphi}_i(\mathbf{P}) \ge \delta$. There exists $B > 1$ such that $T_i \cap (q, Bq] \ne \varnothing$ for each $q \in T_i$.*

Proof: Note by Theorem 1.2.16 that $\sup T_i = \infty$. Now, let $A \in I$ be such that

$$q^{-1} P_i(q) = \varphi_i(q; \mathbf{P}) > \delta/2$$

for all $q \ge A$, and let r, s be successive numbers in T_i with $A < r < s$. Lemma 1.3.5 implies that there exists $t \in (r, s)$ such that $P_1 + \cdots + P_i$ has slope 0 on (r, t) and slope 1 on (t, s). Hence, $P_i(r) = P_i(t)$ and $P_{i+1}(t) = P_{i+1}(s)$. Proposition 1.4.9 yields $P_{i+1}(t) - P_{i+1}(r) \leq t - r$, and so $s^{-1}P_i(s) > \delta/2$ implies

$$(\delta/2)s < P_i(s) = P_{i+1}(s) = P_{i+1}(t) \leq P_{i+1}(r) + t - r = P_i(r) + t - r,$$

Since $P_i(r) = P_i(t) > (\delta/2)t$, it follows that $t < (2/\delta)P_i(r)$, and so

$$s < \frac{2}{\delta}\left(\left(1 + \frac{2}{\delta}\right)P_i(r) - r\right) < \frac{2}{\delta}\left(\left(1 + \frac{2}{\delta}\right)r - r\right) < \frac{4}{\delta^2}r.$$

It follows that $4/\delta^2 > 1$, and that if $B = 4A/\delta^2$, then the claim holds for each $q \in T_i$ with $q > A$. Hence, $B = (4A/\delta^2)(A/\inf I)$ satisfies the claim holds for each $q \in T_i$. ∎

3.2.2 Main Construction

This subsection provides a construction for each $k \in \mathbb{N}^+$ of an n-system $\mathbf{P}^{(k)}$ which is positively oriented by a certain linear map $T^{(k)}$, and it is shown that $\mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(k)}$ are in some sense the only n-systems which are positively oriented by $T^{(k)}$.

Recall that α has been fixed in $(1, \infty)$. Define

$$\alpha_0 = (1, 1, \alpha, \ldots, \alpha^{n-2}) \in \Delta_1,$$

$$a = |\alpha_0| = 2 + \alpha + \cdots + \alpha^{n-2}, \quad \text{and} \quad b_k = \alpha^k a,$$

and define the points

$$\begin{aligned} \alpha_{k,2} &= (1, \alpha^k, \alpha^k, \alpha^{k+1}, \ldots, \alpha^{k+n-3}) \in \Delta_2, \\ \alpha_{k,3} &= (1, \alpha^k, \alpha^{k+1}, \alpha^{k+1}, \alpha^{k+2}, \ldots, \alpha^{k+n-3}) \in \Delta_3, \\ &\vdots \\ \alpha_{k,n-1} &= (1, \alpha^k, \ldots, \alpha^{k+n-4}, \alpha^{k+n-3}, \alpha^{k+n-3}) \in \Delta_{n-1}, \\ \alpha_{k,n} &= (1, \alpha^k, \ldots, \alpha^{k+n-2}) \in \Delta^n, \end{aligned}$$

for each $k \in \mathbb{N}^+$. Writing $\beta_l = \alpha_{l,2}, \ldots, \alpha_{l,n}$ for $l \in \mathbb{N}^+$, Theorem 1.2.22 implies that

$$\alpha_0, \beta_1, \ldots, \beta_k, \alpha^k\alpha_0$$

provides the division points for an n-system $\mathbf{P}^{(k)} : [a, b_k] \to \mathbb{R}^n$. Moreover, as each successive division point changes value in exactly one coordinate, then these are the only division points, and so the map $\mathbf{P}^{(k)}$ is determined uniquely by this sequence.

Define for each $k \in \mathbb{N}^+$ the linear map $T_0^{(k)} : \mathbb{R}^n \to \mathbb{R}$ by

$$T_0^{(k)}(\mathbf{x}) = \alpha^k x_1 - x_2,$$

and the linear map $T^{(k)} = (T_0^{(k)}, T_1, \ldots, T_n) : \mathbb{R}^n \to \mathbb{R}^{n+1}$, recalling that

$$T_1(\mathbf{x}) = x_n - \alpha^{n-2}x_1; \quad T_n(\mathbf{x}) = x_n - \alpha^{n-3}x_2,$$

and

$$T_j(\mathbf{x}) = \alpha x_j - x_{j+1} \quad (j = 2, \ldots, n-1).$$

The following proposition will be useful in computing part of the spectra of $\underline{\varphi}_T$.

Proposition 3.2.9. *Let $k \in \mathbb{N}^+$. It follows that $\underline{\varphi}_T(\mathbf{P}^{(k)}) = (y_k, 0, \ldots, 0)$, where*

$$y_k = \frac{1}{1 + \alpha^k(1 + \alpha + \cdots + \alpha^{n-2})} > 0,$$

Moreover, for each $m \in \mathbb{N}^+$ with $m \leq k$, it follows that

$$\underline{\varphi}_{T^{(k)}}(\mathbf{P}^{(m)}) \geq \mathbf{0},$$

with equality if and only if $m = k$. In particular, $\mathbf{P}^{(m)}$ is positively oriented by $T^{(k)}$.

Proof: By Proposition 3.1.5, it suffices to consider the division points of $\mathbf{P}^{(k)}$. It is straightforward to check that each division point of $\mathbf{P}^{(m)}$ is in the positive locus of T and in the positive locus of $T_0^{(k)}$ for each $m \in \mathbb{N}^+$ with $m \leq k$. Moreover, one finds that $T_j(\alpha_0) = 0$ for $j = 1, \ldots, n-1$ and that $T_n(\alpha_{1,2}) = 0$. Letting $u \in [a, b_k]$ be such that $\mathbf{P}^{(k)}(u) = \alpha_{k,n}$, the construction of $\mathbf{P}^{(k)}$ implies that $P_1' = 0$ on (a, u) and $P_1' = 1$ on (u, b_k). Hence, $y_k = \bar{\alpha}_{k,n}$ is the global minimum of $\varphi_1(u; \mathbf{P}^{(m)})$, completing the proof of the first claim. Observe that $T_0^{(k)}(\alpha_{k,n}) = 0$, and that each division point of $\mathbf{P}^{(m)}$ is in the strictly positive locus of $T_0^{(k)}$ for each $m \in \mathbb{N}^+$ with $m \leq k$. ∎

This result is key in the proof that the spectrum S_T of $\underline{\varphi}_T$ is not semi-algebraic, as it will be shown that S_T intersected by $\{(y, \mathbf{y}) \in \mathbb{R}^{n+1} \mid y > 0, \mathbf{y} \geq \mathbf{0}_n\}$ is exactly the set $\{(y_k, \mathbf{0}_n) \mid k \in \mathbb{N}\}$. Since this set is easily shown to be not semialgebraic, it will follow that S_T is not semialgebraic as well.

3.2.3 Characterization for the Main Construction

Throughout this subsection, let α and a be as in subsection 3.2.2, and fix $b > a$ and a non-degenerate n-system $\mathbf{P} : [a, b] \to \mathbb{R}^n$ such that

$$\mathbf{P}(q) \in \Delta_1 \iff q \in \{a, b\}.$$

Suppose that $\mathbf{P}$ is positively oriented by $T^{(k)}$ for some fixed $k \in \mathbb{N}^+$. The following theorem fully characterizes such n-systems $\mathbf{P}$, and is proven after the next lemmas.

Theorem 3.2.10. *With the above notation,* $\mathbf{P} = \mathbf{P}^{(m)}$ *for some* $m \in \{1, \ldots, k\}$.

For the following two lemmas, let $r, s \in [a, b]$ be transition numbers of $\mathbf{P}$ with $r < s$ such that there are no transition numbers in (r, s), and let $i, j \in \{1, \ldots, n-1\}$ be such that $\mathbf{P}(r) \in \Delta_i$ and $\mathbf{P}(s) \in \Delta_j$.

Note that $j = i+1$ if and only if there are no switch numbers in (r, s).

Lemma 3.2.11. *If* $i < n-1$, *then* $j = i+1$.

Proof: Suppose that $j \neq i+1$, and so Lemma 3.2.6 implies that $j < i+1$, from which Lemma 3.2.2 yields $P_{i+2}(q) = \alpha P_{i+1}(q)$ for each $q \in \{r, s\}$. Since $i+1 < n$ Corollary 3.2.7 implies

$$\alpha = \frac{P_{i+2}(s)}{P_{i+1}(s)} < \frac{P_{i+2}(r)}{P_{i+1}(r)} = \alpha,$$

which is a contradiction. ∎

Lemma 3.2.12. *If* $i = n-1$, *then* $j \in \{1, 2\}$, *and there is exactly one switch number* t *in* (r, s). *Further,* $(P_2(t), P_2(t), \ldots, P_n(t))$ *is proportional to* α_0.

Proof: Note that $j \leq n-1 = i$, and suppose that $j \geq 3$, from which Lemma 3.2.2 yields $P_j(q) = \alpha P_{j-1}(q)$ for each $q \in \{r, s\}$. Since $j > 1$, Corollary 3.2.7 yields

$$\alpha = \frac{P_j(r)}{P_{j-1}(r)} < \frac{P_j(s)}{P_{j-1}(s)} = \alpha,$$

which is a contradiction, and so $j \leq 2$. Now, let $t_0, \ldots, t_n \in [r, s]$ as in Lemma 3.2.6. Since $\mathbf{P}$ is non-degenerate, $P_n'(r^+) = P_{i+1}'(r^+) = 1$ and $P_j'(s^-) = 1$, which implies

$$r = t_n < t_{n-1} \quad \text{and} \quad t_j < t_{j-1} = s.$$

Thus, t_{n-1} and t_j are switch numbers, possibly the same, and there are no switch numbers in $(r, t_{n-1}) \cup (t_j, s)$. Hence, it remains to show that $t = t_{n-1} = t_j$ and that $(P_2(t), P_2(t), \ldots, P_n(t))$ is proportional to $\alpha_0 = (1, 1, \alpha, \ldots, \alpha^{n-2})$.

Suppose that $t_{n-1} < t_2$, and so there exists $k \in \{3, \ldots, n-1\}$ such that $t_k < t_{k-1}$. Since P_k is constant on $[r, t_k]$ and P_{k-1} is constant on $[r, t_{k-1}]$, Lemma 3.2.2 yields

$$P_k(t_{k-1}) > P_k(t_k) = P_k(r) = \alpha P_{k-1}(r) = \alpha P_{k-1}(t_{k-1}). \tag{3.2.2}$$

Since $\mathbf{P}$ is positively oriented by $T_{k-1}(\mathbf{x}) = \alpha x_{k-1} - x_k$, then $\alpha P_{k-1}(t_{k-1}) \geq P_k(t_{k-1})$, which contradicts (3.2.2), and so $t_{n-1} = t_2$. Hence, $P_3, \ldots, P_{n-1}$ are constant on $[r, s]$.

Now, since $j \leq 2 < n-1$, Corollary 3.2.3 yields $P_n(s) = \alpha P_{n-1}(s)$ and

$$\mathbf{P}(r) = (c, d, \ldots, d\alpha^{n-3}, d\alpha^{n-3}) \in \Delta_{n-1},$$

for some c, d with $0 < c < d$. Since P_n is constant on $[t_{n-1}, s]$, Lemma 3.2.2 yields

$$P_n(t_{n-1}) = P_n(s) = \alpha P_{n-1}(s) = \alpha P_{n-1}(r),$$

and so

$$\mathbf{P}(t_{n-1}) = (c, d, \ldots, d\alpha^{n-3}, d\alpha^{n-2}).$$

Letting $t = t_{n-1}$, it follows that $(P_2(t), P_2(t), \ldots, P_n(t))$ is proportional to α_0.

Now, if $j = 2$, then $t_{n-1} = t_2$ implies that t is the only switch number in (r, s), completing the proof. On the other hand, if $j = 1$, then

$$\mathbf{P}(s) = (P_2(t_1), P_2(t_1), d\alpha, \ldots, d\alpha^{n-2}).$$

Corollary 3.2.3 implies that $\mathbf{P}(s)$ is proportional to α_0, and so $P_2(s) = d = P_2(r)$. Hence, $t_2 = t_1$, and so t is the only switch number in (r, s). ∎

Theorem 3.2.10 can now be proven using Lemmas 3.2.11 and 3.2.12.

Proof: (**Proof of Theorem 3.2.10**)
Lemma 3.2.2 implies that $\mathbf{P}(a) = \alpha_0$, and so Lemma 3.2.11 implies that $\alpha_{1,2}$ is the following division point of $\mathbf{P}$.

Now, if $\alpha_{m,2}$ is a division point of $\mathbf{P}$ for some $m \in \mathbb{N}$, then a simple recurrence argument shows that $\alpha_{m,3}, \ldots, \alpha_{m,n-1}$ are the following division points. If $\alpha_{m,n-1}$ is a division point of $\mathbf{P}$, then Lemma 3.2.12 implies that $\alpha_{m,n}$ is the following division point, and that either $\alpha_{m+1,2}$ or $\alpha^m \alpha_0$ is the division point following $\alpha_{m,n}$.

Therefore, writing $\beta_l = \alpha_{l,2}, \ldots, \alpha_{l,n}$ for $l \in \mathbb{N}^+$, the sequence of division points of $\mathbf{P}$ is given by

$$\alpha_0, \beta_1, \ldots, \beta_m, \alpha^m \alpha_0$$

for some $m \in \mathbb{N}^+$, and so $\mathbf{P} = \mathbf{P}^{(m)}$. Letting $u \in [a, b]$ be such that $\mathbf{P}(u) = \alpha_{m,n}$, and recalling that $\mathbf{P}$ is positively oriented by $T_0^{(k)}$, it follows that

$$\alpha^m P_1(u) = P_2(u) \leq \alpha^k P_1(u),$$

which implies that $m \leq k$. ∎

3.2.4 Main Result

Let S_T denote the spectrum of $\underline{\varphi}_T$, noting that $S_T \subseteq \mathbb{R}^{n+1}$, and define the set

$$\mathcal{A} = \{(a, \mathbf{a}) \in S_T \mid a > 0, \mathbf{a} \geq \mathbf{0}_n\}.$$

Proposition 3.2.13. *The set* $\{(y_m, \mathbf{0}_n) \,|\, m \in \mathbb{N}^+\}$ *is contained in* $\mathcal{A}$.

Proof: Observe for each $m \in \mathbb{N}^+$ that $\mathbf{P}^{(m)}$ generates a self-similar n-system $\mathbf{P} : [a, \infty) \to \mathbb{R}^n$, which is proper as $P_1^{(m)} > 0$. Proposition 1.2.28 and 3.2.9 imply that $\underline{\varphi}_T(\mathbf{P}) = \underline{\varphi}_T(\mathbf{P}^{(m)}) = (y_m, \mathbf{0}_n)$. ∎

The following lemma will be useful in showing that S_T is not semialgebraic.

Lemma 3.2.14. *Suppose that* $\mathbf{R} : I \to \mathbb{R}^n$ *is a prototype with* $\mathbf{R}(\sup I) \in \Delta_1$*, which is positively oriented by* T*. Then* $\mathbf{R}$ *is a non-degenerate* n*-system.*

Proof: Corollary 3.2.3 yields that $\mathbf{R}$ is plain, i.e. if $\mathbf{R}(q) \in \Delta_i \cap \Delta_j$ for some $q \in I$ and some $i, j \in \{1, \ldots, n-1\}$, then $i = j$. Let r, s be distinct points in I with $r < s$, and suppose that there exists $j \in \{1, \ldots, n-1\}$ such that $\mathbf{R}(r), \mathbf{R}(s) \in \Delta_j$.

Consider two cases: If $j = 1$, then Corollary 3.2.3 shows that $\mathbf{R}(r)$ and $\mathbf{R}(s)$ are proportional, which implies that $R_l(r) < R_l(s)$ for $l = 1, \ldots, n$. Now, if $j > 1$, then Corollary 3.2.3 implies that either $R_l(r) < R_l(s)$ for $l = 2, \ldots, n$ or that $R_l(r) = R_l(s)$ for $l = 2, \ldots, n$. To prove that $R_l(r) < R_l(s)$ for $l = 2, \ldots, n$, suppose otherwise. Hence, $R_l(r) = R_l(s)$ for $l = 2, \ldots, n$, and so $R_1' = 1$ on (r, s). Since $\mathbf{R}(\sup I) \in \Delta_1$, there exists a minimal $t \in I$ with $t > s$ such that $\mathbf{R}(t) \in \Delta_1$. It follows that $R_1 < R_2$ on $[s, t)$. Now, if $R_1(q) = R_2(q)$ for some $q \in (r, s)$, then $R_1'(q) = 1$ implies that $R_2 < R_1$ on $(q, s]$, which is a contradiction. Thus, $R_1 < R_2$ on (r, t). Since $R_1'(r^+) = 1$, Lemma 1.4.15 implies that $R_l(r) = R_l(t)$ for $l = 2, \ldots, n$. It follows that $\mathbf{R}(t) \in \Delta_1 \cap \Delta_j$, which is a contradiction as $j > 1$ and $\mathbf{R}$ is plain. Therefore, $R_l(r) < R_l(s)$ for $l = 2, \ldots, n$.

In both cases, $R_l(r) \neq R_l(s)$ for $l = 2, \ldots, n$. Thus, as $n \geq 4$, Theorem 1.4.17 implies that $\mathbf{R}$ is an n-system, and Theorem 1.4.18 implies that $\mathbf{R}$ is non-degenerate. ∎

That S_T is not semialgebraic is a consequence of the following theorem.

Theorem 3.2.15. *The set* $\mathcal{A}$ *is not semialgebraic. Indeed,* $\mathcal{A} = \{(y_m, \mathbf{0}_n) \,|\, m \in \mathbb{N}^+\}$.

Proof: Let $\mathbf{y} \in \mathcal{A}$, and so $\mathbf{y} = (y, \mathbf{y}')$ for some $y > 0$ and for some $\mathbf{y}' \in \mathbb{R}^n$ with $\mathbf{y}' \geq \mathbf{0}_n$. Since $\mathbf{y} \in S_T$, there exists a proper, non-degenerate n-system $\mathbf{P} : I \to \mathbb{R}^n$ with

$$\underline{\varphi}_T(\mathbf{P}) = \mathbf{y} \geq \mathbf{0}.$$

As $\mathbf{P}$ is proper and non-degenerate, Theorem 1.2.16 implies that there exists an increasing sequence $(d_i)_{i \in \mathbb{N}}$ satisfying $\mathbf{P}(d_i) \in \Delta_1$. Moreover, since $\underline{\varphi}_1(\mathbf{P}) = y > 0$, Proposition 3.2.8 yields $B > 1$ such that $d_{i+1} \leq B d_i$, for each $i \in \mathbb{N}$. Define for each $i \in \mathbb{N}$ the n-system $\mathbf{R}^{(i)} : [1, B] \to \mathbb{R}^n$ by

$$\mathbf{R}^{(i)}(q) = d_i^{-1} \mathbf{P}(d_i q) \quad (q \in [1, B]).$$

Lemma 1.3.5 yields $t_i \in (d_i, d_{i+1})$ such that $P_1' = 0$ on (d_i, t_i) and $P_1' = 1$ on (t_i, d_{i+1}). Since $t_i \to \infty$ and the local minima of $\varphi_1(q; \mathbf{P})$ occur only at those points $q = t_i$, there exists a subsequence $(d_{i_j})_{j \in \mathbb{N}}$ such that $t_{i_j}^{-1} P_1(t_{i_j}) \to \underline{\varphi}_1(\mathbf{P}) = y$. By applying the Bolzano-Weierstrass theorem twice, there exists a subsequence of this subsequence such that $t_{i_{j'}} d_{i_{j'}}^{-1} \to t$ and $d_{i_{j'}+1} d_{i_{j'}}^{-1} \to d$ as $j' \to \infty$, for some $t, d \in [1, B]$. By the Arzelà-Ascoli theorem, there exists a subsequence of $\mathbf{R}^{(i_{j'})}|_{[1,d]}$ converging uniformly to some prototype $\mathbf{R} : [1, d] \to \mathbb{R}^n$. This prototype satisfies

$$t^{-1} R_1(t) = \lim_{j' \to \infty} t_{i_{j'}}^{-1} d_{i_{j'}} R_1^{(i_{j'})}(t_{i_{j'}} d_{i_{j'}}^{-1}) = \lim_{j \to \infty} t_{i_j}^{-1} P_1(t_{i_j}) = y,$$

while similar limit arguments show that $R_1(d) = R_2(d)$, and that R_1 is constant on $[1, t]$ and has slope 1 on $[t, d]$. Moreover, the continuity of T implies that $\mathbf{R}$ is positively oriented by T, as $\mathbf{R}([1, d]) \subseteq \mathcal{F}(\mathbf{P})$ and $\underline{\varphi}_T(\mathbf{P}) \geq \mathbf{0}$. Hence, Lemma 3.2.14 implies that $\mathbf{R}$ is a non-degenerate n-system. Hence, since $\mathbf{R}(1), \mathbf{R}(d) \in \Delta_1$ and since $R_1' = 0$ on $(1, t)$ and $R_1' = 1$ on (t, d), then

$$\mathbf{R}(q) \in \Delta_1 \Leftrightarrow q \in \{1, d\}.$$

Now, since $(y_m)_{m \geq 1}$ is a sequence tending to 0, there exists $j \in \mathbb{N}^+$ such that $y \geq y_j$. It follows that $\mathbf{R}$ is positively oriented by $T^{(j)}$, and so Theorem 3.2.10 yields $m \leq j$ such that $a \cdot \mathbf{R} = \mathbf{P}^{(m)}$, recalling that $a = |\alpha_0|$. Thus,

$$y = t^{-1} R_1(t) = \underline{\varphi}_1(\mathbf{R}) = \underline{\varphi}_1(\mathbf{P}^{(m)}) = y_m.$$

This shows that

$$\mathcal{A} = \{(y_i, \mathbf{0}_n) \mid i \in \mathbb{N}^+\}.$$

If $\mathcal{A}$ is semialgebraic, then by the Tarski-Seidenberg theorem, its projection

$$\mathcal{B} = \{y_i \in \mathbb{R} \mid i \in \mathbb{N}^+\}$$

on the first coordinate is semialgebraic. By Theorem 3.1.2, a semialgebraic subset of $\mathbb{R}$ is a finite union of points and intervals. This is a contradiction since $\mathcal{B}$ is an infinite discrete set. Thus, $\mathcal{A}$ is not semialgebraic. ∎

Hence, Theorem 3.2.1 is proven by the following reformulation.

Corollary 3.2.16. *The set S_T is not semialgebraic.*

Proof: Suppose that S_T is semialgebraic. Then the set $\mathcal{A}$ is the intersection of two semialgebraic sets. Hence, $\mathcal{A}$ is semialgebraic, contradicting Theorem 3.2.15. ∎

Chapter 4

Spectra which Are Stable under the Minimum

A set S in $\mathbb{R}^m$ is said to be *stable under the minimum* if $\min\{\mathbf{x}, \mathbf{y}\} \in S$ for each $\mathbf{x}, \mathbf{y} \in S$, where the minimum is taken with respect to the product ordering on $\mathbb{R}^m$. A result by Roy in [22] shows that spectra of n-systems are stable under the minimum for $n = 2, 3$. This is not true in general for $n = 4$, as we will prove in the next chapter. Nevertheless, in this chapter we give a condition on linear maps $T : \mathbb{R}^4 \to \mathbb{R}^m$ under which the spectrum of $\underline{\varphi}_T$ is stable under the minimum.

4.1 Main Result

Define the point

$$\mathbf{E}_1 = \frac{(1,1,1,1)}{4} \in \bar{\Delta}^4.$$

The main result of this chapter is the following theorem.

Theorem 4.1.1. *Let $T = (T_1, \dots, T_m) : \mathbb{R}^4 \to \mathbb{R}^m$ be a linear map such that*

$$T_i(\mathbf{x}) \leq T_i(\mathbf{E}_1) \quad (i = 1, \dots, m)$$

for each $\mathbf{x} \in \bar{\Delta}^4$. The spectrum of $\underline{\varphi}_T$ is stable under the minimum.

This is proven by using the next two theorems, both of which are proven later in this chapter.

Theorem 4.1.2. *If $\mathbf{P}$ is a proper 4-system, then there exists a proper 4-system $\mathbf{R}$ such that $\mathcal{K}(\mathbf{R})$ is the convex hull of $\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$.*

Theorem 4.1.3. *Let $\mathbf{S}$ and $\mathbf{T}$ be proper 4-systems with $\mathbf{E}_1 \in \mathcal{K}(\mathbf{S}) \cap \mathcal{K}(\mathbf{T})$. There exists a proper 4-system $\mathbf{R}$ with $\mathcal{F}(\mathbf{R}) = \mathcal{F}(\mathbf{S}) \cup \mathcal{F}(\mathbf{T})$.*

Taking these results for granted, the main result can be proven as follows.

Proof: (**Theorem 4.1.1**)
For each proper 4-system $\mathbf{P}$, Theorem 4.1.2 yields a proper 4-system $\tilde{\mathbf{P}}$ such that $\mathcal{K}(\tilde{\mathbf{P}})$ is the convex hull of $\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$. Since

$$\underline{\varphi}_{T_i}(\mathbf{P}) = \inf T_i(\mathcal{F}(\mathbf{P})) \leq T_i(\mathbf{E}_1) \quad (i = 1, \ldots, m),$$

it follows that

$$\underline{\varphi}_T(\tilde{\mathbf{P}}) = \inf T(\mathcal{K}(\tilde{\mathbf{P}})) = \inf T(\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}) = \min\{\underline{\varphi}_T(\mathbf{P}), T(\mathbf{E}_1)\} = \underline{\varphi}_T(\mathbf{P}).$$

Now, for $i = 1, 2$, let $\mathbf{x}^{(i)}$ be contained in the spectrum of $\underline{\varphi}_T$, and so there exists a proper 4-system $\mathbf{P}^{(i)}$ with $\mathbf{x}^{(i)} = \underline{\varphi}_T(\mathbf{P}^{(i)})$. Since $\mathbf{E}_1 \in \mathcal{K}(\tilde{\mathbf{P}}^{(1)}) \cap \mathcal{K}(\tilde{\mathbf{P}}^{(2)})$, then Theorem 4.1.3 yields a proper 4-system $\mathbf{R}$ such that $\mathcal{K}(\mathbf{R})$ is the convex hull of the set $\mathcal{F}(\tilde{\mathbf{P}}^{(1)}) \cup \mathcal{F}(\tilde{\mathbf{P}}^{(2)})$. Therefore,

$$\underline{\varphi}_T(\mathbf{R}) = \inf T(\mathcal{F}(\tilde{\mathbf{P}}^{(1)}) \cup \mathcal{F}(\tilde{\mathbf{P}}^{(2)})) = \min\{\underline{\varphi}_T(\mathbf{P}^{(1)}), \underline{\varphi}_T(\mathbf{P}^{(2)})\} = \min\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\},$$

and so $\min\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ is contained in the spectrum of $\underline{\varphi}_T$. ∎

The spectrum of $(\underline{\psi}_1, \ldots, \underline{\psi}_{n-1})$ for the dimension n was determined by Roy in [21]. The fact that it is stable under the minimum for the dimension $n = 4$ can be seen as a consequence of Theorem 4.1.1, as in the following corollary.

Corollary 4.1.4. *The spectrum of* $(\underline{\psi}_1, \underline{\psi}_2, \underline{\psi}_3)$ *in dimension* $n = 4$ *is stable under the minimum.*

Proof: Let $T = (T_1, T_2, T_3) : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map defined by

$$T_1(\mathbf{x}) = x_1, \quad T_2(\mathbf{x}) = x_1 + x_2, \quad \text{and} \quad T_3(\mathbf{x}) = x_1 + x_2 + x_3,$$

so that the spectrum of $(\underline{\psi}_1, \underline{\psi}_2, \underline{\psi}_3)$ is the spectrum of $\underline{\varphi}_T$. Now, let $\mathbf{x} \in \bar{\Delta}^4$. Since

$$x_1 + x_2 + x_3 + x_4 = 1 \quad \text{and} \quad 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4,$$

then

$$T_1(\mathbf{x}) \leq 1/4, \quad T_2(\mathbf{x}) \leq 1/2, \quad \text{and} \quad T_3(\mathbf{x}) = 3/4.$$

It follows that

$$T(\mathbf{x}) \leq (1/4, 1/2, 3/4) = T(1/4, 1/4, 1/4, 1/4) = T(\mathbf{E}_1).$$

Theorem 4.1.1 implies that $\underline{\varphi}_T$ is stable under the minimum. ∎

4.2 Proof of Theorem 4.1.3

The following notation and results are used to prove Theorem 4.1.3. Define

$$\mathbf{E}_1^n = \frac{\mathbf{e}_1 + \cdots + \mathbf{e}_n}{n} \in \bar{\Delta}^n.$$

Definition 4.2.1. For each unbounded n-system $\mathbf{P}$, the set $\mathcal{F}(\mathbf{P}, \mathbf{e}_i)$ is the set of limits of sequences $(q_i^{-1}\mathbf{P}(q_i))_{i\in\mathbb{N}}$ such that $(q_i)_{i\in\mathbb{N}}$ is unbounded and such that either $\mathbf{P}'(q_i^-) = \mathbf{e}_i$ or $\mathbf{P}'(q_i^+) = \mathbf{e}_i$ for each $i \in \mathbb{N}$.

Lemma 4.2.2. *Let $\mathbf{P}$ be an n-system with $\mathbf{E}_1^n \in \mathcal{F}(\mathbf{P})$. It follows that $\mathbf{E}_1^n \in \mathcal{F}(\mathbf{P}, \mathbf{e}_1)$.*

Proof: Since $\mathbf{E}_1^n \in \mathcal{F}(\mathbf{P})$, there exists an unbounded sequence $(q_i)_{i\in\mathbb{N}}$ such that $q_i^{-1}\mathbf{P}(q_i) \to \mathbf{E}_1^n$ as $i \to \infty$. For each $i \in \mathbb{N}$, let $r_i \leq q_i$ be maximal such that $P_1'(r_i^-) = 1$, and so $P_1(r_i) = P_1(q_i)$. Since $r_i^{-1}P_1(r_i) \geq q_i^{-1}P_1(q_i)$ for each $i \in \mathbb{N}$, then

$$\liminf_{i\to\infty} r_i^{-1}P_1(r_i) \geq 1/n.$$

Since $r_i^{-1}\mathbf{P}(r_i) \in \bar{\Delta}^n$, then $r_i^{-1}P_1(r_i) \leq 1/n$ for each $i \in \mathbb{N}$, and so

$$\lim_{i\to\infty} r_i^{-1}P_1(r_i) = 1/n.$$

Since $\mathbf{E}_1^n$ is the only point in $\bar{\Delta}^n$ whose first coordinate is $1/n$, and since $\bar{\Delta}^n$ is compact, it follows that $r_i^{-1}\mathbf{P}(r_i) \to \mathbf{E}_1^n$. Hence, $\mathbf{E}_1^n \in \mathcal{F}(\mathbf{P}, \mathbf{e}_1)$. ∎

The next two theorems are special cases of results due to Roy [20, Theorem 1.3], [22, Corollary 8.3].

Theorem 4.2.3. *Let $\mathbf{P}$ be an unbounded n-system. There exists an unbounded n-system $\mathbf{R}$ whose division points lie in $\mathbb{N}^n$ such that $\|\mathbf{R} - \mathbf{P}\| = \mathcal{O}(1)$.*

Theorem 4.2.4. *Let $\mathbf{S}$ and $\mathbf{T}$ be n-systems whose division points lie in $\mathbb{N}^n$. Suppose that $\mathcal{F}(\mathbf{S}, \mathbf{e}_1) \cap \mathcal{F}(\mathbf{T}, \mathbf{e}_1)$ is not empty. Then, there exists a proper n-system $\mathbf{R}$ such that $\mathcal{F}(\mathbf{R}) = \mathcal{F}(\mathbf{S}) \cup \mathcal{F}(\mathbf{T})$.*

Theorem 4.1.3 is the special case of the following corollary for $n = 4$.

Corollary 4.2.5. *Let $\mathbf{S}$ and $\mathbf{T}$ be proper n-systems with $\mathbf{E}_1^n \in \mathcal{K}(\mathbf{S}) \cap \mathcal{K}(\mathbf{T})$. There exists a proper n-system $\mathbf{R}$ with $\mathcal{F}(\mathbf{R}) = \mathcal{F}(\mathbf{S}) \cup \mathcal{F}(\mathbf{T})$.*

Proof: First suppose that the division points of $\mathbf{S}$ and $\mathbf{T}$ lie in $\mathbb{N}^n$. Since $\mathbf{E}_1^n$ is an extreme point of $\bar{\Delta}^n$, it follows that $\mathbf{E}_1^n$ is an extreme point of $\mathcal{K}(\mathbf{S})$ and $\mathcal{K}(\mathbf{T})$. Thus, $\mathbf{E}_1^n \in \mathcal{F}(\mathbf{S}) \cap \mathcal{F}(\mathbf{T})$, and so Lemma 4.2.2 implies that

$$\mathbf{E}_1^n \in \mathcal{F}(\mathbf{S}, \mathbf{e}_1) \cap \mathcal{F}(\mathbf{T}, \mathbf{e}_1).$$

Theorem 4.2.4 provides the desired n-system $\mathbf{R}$. In general, Theorem 4.2.3 implies that $\mathbf{S}$ and $\mathbf{T}$ can be replaced by n-systems $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$ with division points in $\mathbb{N}^n$ such that $\mathcal{F}(\mathbf{S}) = \mathcal{F}(\tilde{\mathbf{S}})$ and $\mathcal{F}(\mathbf{T}) = \mathcal{F}(\tilde{\mathbf{T}})$. The above argument provides an n-system $\mathbf{R}$ with $\mathcal{F}(\mathbf{R}) = \mathcal{F}(\tilde{\mathbf{S}}) \cup \mathcal{F}(\tilde{\mathbf{T}}) = \mathcal{F}(\mathbf{S}) \cup \mathcal{F}(\mathbf{T})$. ∎

4.3 Limits of Sets

This section introduces the notion of limits of sets in $\mathbb{R}^n$, and establishes a fundamental property for such limits. This notion is key in proving Theorem 4.1.2.

Definition 4.3.1. For non-empty bounded subsets $A, B \subseteq \mathbb{R}^n$, define the distance $\mathrm{d}(A, B)$ between A and B as the infimum of real numbers $\varepsilon \geq 0$ such that

$$A \subseteq B + [-\varepsilon, \varepsilon]^n \quad \text{and} \quad B \subseteq A + [-\varepsilon, \varepsilon]^n.$$

The function d satisfies the triangle inequality, and $\mathrm{d}(A, B) = 0$ implies that $A = B$ when A and B are closed. Hence, the set of all non-empty compact subsets of $\mathbb{R}^n$ together with d form a metric space. Moreover, if A, B, C are non-empty compact subsets of $\mathbb{R}^n$, then

$$\mathrm{d}(A \cup C, B \cup C) \leq \mathrm{d}(A, B) \quad \text{and} \quad \mathrm{d}(K_A, K_B) \leq \mathrm{d}(A, B),$$

where K_A and K_B are the convex hulls of A and B, respectively.

Definition 4.3.2. Consider a sequence $(A_i)_{i\in\mathbb{N}}$ of non-empty subsets of $\mathbb{R}^n$. Its *inner limit* is the set of points $\mathbf{x} \in \mathbb{R}^n$ which are limits of sequences $(x_i)_{i\in\mathbb{N}}$ with $x_i \in A_i$ for each $i \in \mathbb{N}$, and its *outer limit* is the set of points $\mathbf{x} \in \mathbb{R}^n$ which are limits of subsequences of such sequences. This inner limit and outer limit are respectively denoted

$$\liminf_{i\to\infty} A_i \quad \text{and} \quad \limsup_{i\to\infty} A_i.$$

In particular, the inner limit of a sequence is contained in the outer limit.

Proposition 4.3.3. *Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of non-empty compact sets in $\mathbb{R}^n$, and suppose that it converges to some compact subset A of $\mathbb{R}^n$. It follows that*

$$\liminf_{i\to\infty} A_i = A = \limsup_{i\to\infty} A_i.$$

Proof: Let $\varepsilon_i = \mathrm{d}(A_i, A)$ for each $i \in \mathbb{N}$, and so $\varepsilon_i \to 0$ as $i \to \infty$. Let $a \in A$, and note that there exists $a_i \in A_i$ with $a - a_i \in [-\varepsilon_i, \varepsilon_i]^n$, for each $i \in \mathbb{N}$. Hence, $a_i \to a$ as $i \to \infty$, and so $a \in \liminf_{i\to\infty} A_i$. Now, let $a \in \limsup_{i\to\infty} A_i$, and so there exists a sequence $(a_i)_{i\in\mathbb{N}}$ such that $a_i \in A_i$ for each $i \in \mathbb{N}$, and that has a subsequence $(a_{i_j})_{j\in\mathbb{N}}$ converging to a. For each $j \in \mathbb{N}$, there exists $b_j \in A$ with $a_{i_j} - b_j \in [-\varepsilon_{i_j}, \varepsilon_{i_j}]^n$. Hence, $b_j \to a$ as $j \to \infty$, and so $a \in A$, since A is closed. Therefore,

$$A \subseteq \liminf_{i\to\infty} A_i \quad \text{and} \quad \limsup_{i\to\infty} A_i \subseteq A,$$

and so both inclusions are equalities. ∎

4.4 Main Construction

The main goal of this section is to show that for any proper 4-system $\mathbf{P}$, there exists a proper 4-system $\mathbf{R}$ such that $\mathcal{K}(\mathbf{R})$ is the convex hull of $\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$. The result is first proven for the case when $\mathbf{P}$ is self-similar, and then for the general case.

Recall that $\bar{\Delta}^n = \Delta^n \cap \Sigma^n$, where

$$\Delta^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, 0 \le x_1 \le x_2 \le \cdots \le x_n\} \quad \text{and} \quad \Sigma^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_1 + \cdots + x_n = 1\},$$

and recall that $\bar{\Delta}_i = \bar{\Delta}_i^n = \Delta_i^n \cap \Sigma^n$, where

$$\Delta_i = \Delta_i^n = \Delta^n \cap \{\mathbf{x} \in \mathbb{R}^n \,|\, x_i = x_{i+1}\},$$

for each $i = 1, \ldots, n-1$.

Also recall that $|\mathbf{x}| = x_1 + \cdots + x_n$ and $\bar{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ for each non-zero $\mathbf{x} \in \Delta^n$, and define a map $\pi : \Delta^n \backslash \{\mathbf{0}\} \to \bar{\Delta}^n$ by

$$\pi(\mathbf{x}) = \bar{\mathbf{x}} \quad (\mathbf{x} \in \Delta^n \backslash \{\mathbf{0}\}).$$

Definition 4.4.1. An n-system $\mathbf{P}$ is said to *project* in a set $S \subseteq \bar{\Delta}^n$ if $\pi(\mathbf{P}) \subseteq S$.

The following proposition due to Roy [22, Proposition 3.4] is key in what follows.

Proposition 4.4.2. *Let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function, and define*

$$\nu(x_1, \ldots, x_n) = (\nu(x_1), \ldots, \nu(x_n)) \quad (x_1, \ldots, x_n \in \mathbb{R}^+).$$

Let $\mathbf{P} : I \to \mathbb{R}^n$ be an n-system, and let $q_1, \ldots, q_m \in I$ be finitely many consecutive division numbers of $\mathbf{P}$. The sequence $\nu(\mathbf{P}(q_1)), \ldots, \nu(\mathbf{P}(q_m))$ is the ordered sequence of division points of some n-system $\mathbf{R} : J \to \mathbb{R}^n$.

4.4.1 Self-Similar Case

Let $\mathbf{P} : I \to \mathbb{R}^4$ be a self-similar, non-degenerate, proper 4-system with period $\rho > 1$, and let $\mathcal{K}$ be the convex hull of $\mathcal{F} = \mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$. The goal is to prove the existence of a proper 4-system $\mathbf{R}$ with $\mathcal{K}(\mathbf{R}) = \mathcal{K}$.

Since $\mathbf{P}$ is proper and non-degenerate, there exist division numbers $r, s \in I$ with $r < s$ such that P_1, P_4 are constant on $[r, s]$ and such that $P_1'(r^-) = 1 = P_4'(s^+)$. In particular, it follows that

$$\mathbf{P}(r) \in \Delta_1 \quad \text{and} \quad \mathbf{P}(s) \in \Delta_3.$$

Let $(q_i)_{i\in\mathbb{N}}$ be the ordered sequence of division numbers of $\mathbf{P}$ on $[r, \infty)$, with $q_0 = r$. Let t be the index such that $s = q_{t+1}$. Also let

$$A_i = \mathbf{P}(q_i) \quad (i \in \mathbb{N}),$$

noting that $\mathbf{P}(r) = A_0$ and $\mathbf{P}(s) = A_{t+1}$. Further, let $m \in \mathbb{N}^+$ be such that $q_m = \rho q_0$. Thus, $q_{Nm+i} = \rho^N q_i$ for each $i, N \in \mathbb{N}$.

Since $\mathbf{P}$ is non-degenerate, the hypotheses on r, s imply that

$$A_0 = (x, x, y, w), A_1 = (x, y, y, w), A_t = (x, z, z, w), A_{t+1} = (x, z, w, w),$$

for some $x, y, z, w \in \mathbb{R}$ with $0 < x < y \leq z < w$. Define the point

$$\mathbf{E} = (w, w, w, w),$$

noting that $\pi(\mathbf{E}) = \mathbf{E}_1$.

Now, for each $\lambda \in [0, 1]$, define

$$a^\lambda = \lambda a + (1 - \lambda)w \quad (\forall a \in \mathbb{R}),$$

and

$$A^\lambda = (a_1^\lambda, a_2^\lambda, a_3^\lambda, a_n^\lambda) = \lambda A + (1 - \lambda)\mathbf{E} \quad (\forall A = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4).$$

Observe for each $A \in \Delta^4$ and $\lambda, \mu \in [0, 1]$ that

$$(A^\lambda)^\mu = (\lambda A + (1 - \lambda)\mathbf{E})^\mu = \mu(\lambda A + (1 - \lambda)\mathbf{E}) + (1 - \mu)\mathbf{E} = A^{\lambda\mu},$$

and that if $\pi(A) \in \mathcal{F}(\mathbf{P})$, then $\pi(A^\lambda)$ is in the convex hull $\mathcal{K}$ of $\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$. Moreover, if $\lambda > 0$, then the map $a \mapsto a^\lambda$ is strictly increasing. Thus, if $B_1, \ldots, B_k$ is the ordered sequence of division points of some n-system, then Proposition 4.4.2 yields for each $\lambda > 0$ that $B_1^\lambda, \ldots, B_k^\lambda$ is the ordered sequence of division points of another n-system. The idea of the proof is to use this transformation to construct an n-system $\mathbf{R}$ such that

$$\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\} \subseteq \mathcal{F}(\mathbf{R}) \subseteq \mathcal{K},$$

where $\mathcal{K}$ is the convex hull of $\mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$.

4.4.2 Approaching the Point $\mathbf{E}_1$

Since $z \in (x, w)$, there exists $l \in (0,1)$ such that $z = lx + (1-l)w$. Let $u = ly + (1-l)w$, noting that $x < y < w$ implies that $u \in (z, w)$. Now, define

$$B = (x, z, u, w),$$

and note that

$$A_0^l = (z, z, u, w) = B + (z - x)\mathbf{e}_1.$$

Since $u \in (z, w)$, it follows that B is a convex combination of A_t and A_{t+1}, and so $\pi(B) \in \mathcal{K}(\mathbf{P})$. Hence, the sequence

$$(A_0, A_1, \ldots, A_t, B, A_0^l)$$

is the ordered sequence of division points for some non-degenerate 4-system $\tilde{\mathbf{P}}$ which projects in $\mathcal{K}$. Figure 4.4.1 gives the combined graph of $\tilde{\mathbf{P}}$.

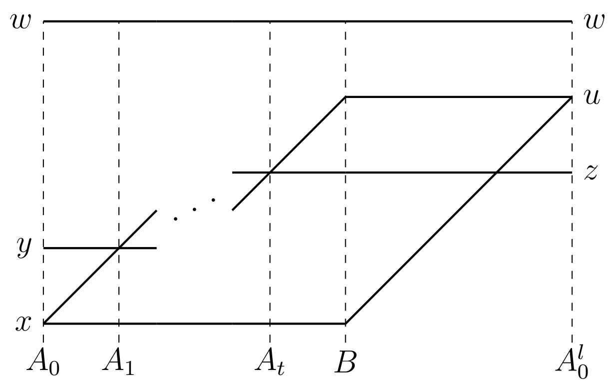

Figure 4.4.1: Combined Graph of $\tilde{\mathbf{P}}$

Proposition 4.4.2 yields for each $\lambda > 0$ that

$$(A_0^\lambda, A_1^\lambda, \ldots, A_t^\lambda, B^\lambda, A_0^{l\lambda})$$

is the ordered sequence of division points for some non-degenerate 4-system $\tilde{\mathbf{P}}^{(\lambda)}$ which also projects in $\mathcal{K}$. Thus, as $A_0 \in \Delta_1$, the sequence $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}^{(l)}, \ldots, \tilde{\mathbf{P}}^{(l^{N-1})})$ connects to form an n-system for each $N \in \mathbb{N}^+$, and the ordered sequence of division points of this n-system is

$$\left(A_0, \ldots, A_t, B, A_0^l, \ldots, A_t^l, B^l, \ldots, A_0^{l^{N-1}}, \ldots, A_t^{l^{N-1}}, B^{l^{N-1}}, A_0^{l^N}\right).$$

Proposition 4.4.2 yields for each $N \in \mathbb{N}^+$ and each $\lambda > 0$ that

$$\mathbf{A}_N^\lambda = \left(A_0^\lambda, \ldots, A_t^\lambda, B^\lambda, A_0^{l\lambda}, \ldots, A_t^{l\lambda}, B^{l\lambda}, \ldots, A_0^{l^{N-1}\lambda}, \ldots, A_t^{l^{N-1}\lambda}, B^{l^{N-1}\lambda}, A_0^{l^N\lambda}\right),$$

is the ordered sequence of division points of some n-system.

Now, note for each $\lambda \in (0,1]$ that $l^N\lambda \to 0$ as $N \to \infty$. It follows that

$$\lim_{N\to\infty} A_0^{l^N\lambda} = \mathbf{E},$$

and so the n-systems obtained from A_N^λ take values approaching $\mathbf{E}$ as $N \to \infty$.

4.4.3 Approaching the Set $\mathcal{F}(\mathbf{P})$

For each $i, N \in \mathbb{N}$ and each $\lambda \in [0,1]$, observe that there exist $c_N(\lambda), e_N(\lambda)$ such that

$$A^\lambda_{Nm+i} = (\rho^N A_i)^\lambda = \lambda\rho^N A_i + (1-\lambda)\mathbf{E} = c_N(\lambda)A_i^{e_N(\lambda)}.$$

Indeed, a simple computation shows that

$$c_N(\lambda) = \lambda\rho^N + (1-\lambda) \quad \text{and} \quad e_N(\lambda) = \lambda\rho^N/c_N(\lambda).$$

For each $\lambda > 0$ and each $N \in \mathbb{N}^+$, Proposition 4.4.2 yields that

$$\mathbf{B}_N^\lambda = \left(A_0^\lambda, A_1^\lambda, \ldots, A_m^\lambda \ldots, A^\lambda_{Nm}, A^\lambda_{Nm+1}, \ldots, A^\lambda_{Nm+m}\right)$$

is the ordered sequence of division points for some n-system. For each $\lambda > 0$, observe that

$$\lim_{N\to\infty} e_N(\lambda) = \lim_{N\to\infty} \frac{\lambda\rho^N}{\lambda\rho^N + 1 - \lambda} = 1.$$

It follows that

$$\lim_{N\to\infty} A^\lambda_{Nm+j}/c_N(\lambda) = \lim_{N\to\infty} A_j^{e_N(\lambda)} = A_j$$

for $j = 0, \ldots, m$.

4.4.4 Connecting Sequences of Division Points

Theorem 4.4.3. *Let $\mathbf{P}$ be a self-similar, non-degenerate, proper 4-system, and let $\mathcal{K}$ be the convex hull of $\mathcal{F} = \mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$. There exists a proper 4-system $\mathbf{R}$ such that*

$$\mathcal{F} \subseteq \mathcal{F}(\mathbf{R}) \subseteq \mathcal{K}.$$

In particular, it follows that $\mathcal{K}(\mathbf{R}) = \mathcal{K}$.

Proof: Let the notation be as in the preceding subsections. Let $\mathbf{N} = (N_i)_{i\in\mathbb{N}}$ be a sequence of positive integers, and define iteratively two sequences $(\lambda_i)_{i\in\mathbb{N}}$ and $(\rho_i)_{i\in\mathbb{N}}$ of real numbers starting with $\lambda_0 = \rho_0 = 1$ using the following recurrence formulas.

$$\lambda_{i+1} = \begin{cases} l^{N_i}\lambda_i & \text{if } i \equiv 0 \mod 2 \\ e_{N_i+1}(\lambda_i) & \text{if } i \equiv 1 \mod 2, \end{cases}$$

and

$$\rho_{i+1} = \begin{cases} \rho_i & \text{if } i \equiv 0 \mod 2 \\ \rho_i c_{N_i+1}(\lambda_i) & \text{if } i \equiv 1 \mod 2, \end{cases}$$

By construction, one has that

$$\begin{cases} \rho_i A_0^{l^{N_i}\lambda_i} = \rho_{i+1} A_0^{\lambda_{i+1}} & \text{if } i \equiv 0 \mod 2 \\ \rho_i A_{N_i m+m}^{\lambda_i} = \rho_{i+1} A_0^{\lambda_{i+1}} & \text{if } i \equiv 1 \mod 2. \end{cases}$$

It follows for each $i \in \mathbb{N}$ that the last point of $\rho_{2i}\mathbf{A}_{N_{2i}}^{\lambda_{2i}}$ is the first point of $\rho_{2i+1}\mathbf{B}_{N_{2i+1}}^{\lambda_{2i+1}}$, and that the last point of $\rho_{2i+1}\mathbf{B}_{N_{2i+1}}^{\lambda_{2i+1}}$ is the first point of $\rho_{2i+2}\mathbf{A}_{N_{2i+2}}^{\lambda_{2i+2}}$. Thus, let $\mathbf{C_N}$ be the sequence obtained by connecting the sequences

$$\rho_0\mathbf{A}_{N_0}^{\lambda_0}, \rho_1\mathbf{B}_{N_1}^{\lambda_1}, \rho_2\mathbf{A}_{N_2}^{\lambda_2}, \ldots$$

so that the connecting points are equal.

Note that $A_0 \in \Delta_1$, and that all ordered sequences of division points involved in the construction of $\mathbf{C_N}$ yield 4-systems which start with increasing 2^{nd} coordinate and which end with increasing 1^{st} coordinate. It follows that $\mathbf{C_N}$ is the ordered sequence of division points of some 4-system $\mathbf{R}^{(\mathbf{N})}$ with

$$\mathcal{F}(\mathbf{R}^{(\mathbf{N})}) \subseteq \mathcal{K}.$$

It remains to show that there exists a choice of $\mathbf{N}$ for which $\mathcal{F} \subseteq \mathcal{F}(\mathbf{R}^{(\mathbf{N})})$. To this end, recall that $l^N, e_N(\lambda) \in (0,1)$, for each $N \in \mathbb{N}$ and $\lambda \in (0,1)$, and that $l^N\lambda \to 0$ and $e_N(\lambda) \to 1$ as $N \to \infty$. Thus, the integers N_i can be defined iteratively so that

$$\begin{cases} l^{N_i}\lambda_i < 2^{-i} & \text{if } i \equiv 0 \mod 2 \\ e_{N_i}(\lambda_i) > 1 - 2^{-i} & \text{if } i \equiv 1 \mod 2. \end{cases}$$

Let $\mathbf{R} = \mathbf{R}^{(\mathbf{N})}$ for this choice of $\mathbf{N}$. Since $\lim_{i\to\infty} \pi(\rho_{2i}A_0^{l^{N_{2i}}\lambda_{2i}}) = \mathbf{E}_1$, it follows that

$$\mathbf{E}_1 \in \mathcal{F}(\mathbf{R}).$$

Now, for each $i \in \mathbb{N}$, let $I_i = [d_i, d_i']$ where

$$d_i = |\rho_{2i+1}A_{N_{2i+1}m}^{\lambda_{2i+1}}| \quad \text{and} \quad d_i' = |\rho_{2i+1}A_{N_{2i+1}m+m}^{\lambda_{2i+1}}|,$$

so that the ordered sequence of division points of $\mathbf{R}|_{I_i}$ is

$$\rho_{2i+1}\Big(A_{N_{2i+1}m}^{\lambda_{2i+1}}, \ldots, A_{N_{2i+1}m+m}^{\lambda_{2i+1}}\Big).$$

Since

$$\lim_{i\to\infty} \pi(\rho_{2i+1}A_{N_{2i+1}m+j}^{\lambda_{2i+1}}) = \lim_{i\to\infty} \pi\Big(A_j^{e_{N_{2i+1}}(\lambda_{2i+1})}\Big) = \pi(A_j),$$

for $j = 0, \ldots, m$, it follows that

$$\lim_{i\to\infty} \pi(\mathbf{R}(I_i)) = [\pi(A_0), \pi(A_1)] \cup [\pi(A_1), \pi(A_2)] \cup \cdots \cup [\pi(A_{m-1}), \pi(A_m)],$$

where $[A, B]$ denotes the line segment between any two points $A, B \in \mathbb{R}^4$. Since the left hand side is contained in $\mathcal{F}(\mathbf{R})$ while the right hand side is equal to $\mathcal{F}(\mathbf{P})$, then

$$\mathcal{F} = \mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\} \subseteq \mathcal{F}(\mathbf{R}) \subseteq \mathcal{K}.$$

Moreover, since $\mathcal{K}$ is the convex hull of $\mathcal{F}$, then $\mathcal{K}(\mathbf{R}) = \mathcal{K}$. ∎

4.4.5 General Case

The next two theorems, due to Roy [22, Theorem 7.1 and Corollary 8.4], are used to prove Theorem 4.1.2

Theorem 4.4.4. *Let $\mathbf{P}$ be a proper n-system. For any $\varepsilon > 0$, there is a self-similar, non-degenerate, proper n-system $\mathbf{S}$ with $\mathrm{d}(\mathcal{F}(\mathbf{P}), \mathcal{F}(\mathbf{S})) \le \varepsilon$.*

Theorem 4.4.5. *Let $(\mathbf{R}^{(i)})_{i\ge1}$ be a sequence of proper n-systems. Then, there is a proper n-system $\mathbf{R}$ such that*

$$\liminf_{i\to\infty} \mathcal{F}(\mathbf{R}^{(i)}) \subseteq \mathcal{F}(\mathbf{R}) \subseteq \limsup_{i\to\infty} \mathcal{F}(\mathbf{R}^{(i)}).$$

Theorem 4.1.2 can now be proven.

Proof: (**Theorem 4.1.2**)
Let $\mathbf{P}$ be a proper 4-system. Theorem 4.4.4 yields for each $i \in \mathbb{N}$ a self-similar, non-degenerate, proper 4-system $\mathbf{P}^{(i)}$ with $\mathrm{d}(\mathcal{F}(\mathbf{P}), \mathcal{F}(\mathbf{P}^{(i)})) \le 2^{-i}$. For each $i \in \mathbb{N}$, let $\mathcal{F}_i = \mathcal{F}(\mathbf{P}^{(i)}) \cup \{\mathbf{E}_1\}$, and let $\mathcal{K}_i$ be its convex hull. Theorem 4.4.3 yields for each $i \in \mathbb{N}$ a proper 4-system $\mathbf{R}^{(i)}$ such that $\mathcal{F}_i \subseteq \mathcal{F}(\mathbf{R}^{(i)}) \subseteq \mathcal{K}_i$.

Now, let $\mathcal{F} = \mathcal{F}(\mathbf{P}) \cup \{\mathbf{E}_1\}$ and let $\mathcal{K}$ denote its convex hull, noting that

$$\mathrm{d}(\mathcal{K}, \mathcal{K}_i) \le \mathrm{d}(\mathcal{F}, \mathcal{F}_i) \le \mathrm{d}(\mathcal{F}(\mathbf{P}), \mathcal{F}(\mathbf{P}^{(i)})) \le 2^{-i}$$

for each $i \in \mathbb{N}$. It follows that

$$\mathcal{F} = \lim_{i\to\infty} \mathcal{F}_i \quad \text{and} \quad \mathcal{K} = \lim_{i\to\infty} \mathcal{K}_i.$$

Proposition 4.3.3 implies that

$$\mathcal{F} = \liminf_{i\to\infty} \mathcal{F}_i \subseteq \liminf_{i\to\infty} \mathcal{F}(\mathbf{R}^{(i)}) \quad \text{and} \quad \limsup_{i\to\infty} \mathcal{F}(\mathbf{R}^{(i)}) \subseteq \limsup_{i\to\infty} \mathcal{K}_i = \mathcal{K}.$$

Theorem 4.4.5 yields a proper 4-system $\mathbf{R}$ with $\mathcal{F} \subseteq \mathcal{F}(\mathbf{R}) \subseteq \mathcal{K}$. ∎

Chapter 5

Spectra which Are Not Stable under the Minimum

The main result of this chapter is the existence of a family of spectra S of 4-systems which are not stable under the minimum, i.e. there exists $\mathbf{x}, \mathbf{y} \in S$ such that $\min\{\mathbf{x}, \mathbf{y}\} \notin S$. The proof relies on a construction of proper generalized 4-systems $\mathbf{R}, \mathbf{S}$ satisfying the following property: There does not exist a proper 4-system $\mathbf{P}$ such that $\mathcal{K}(\mathbf{P})$ is the convex hull of $\mathcal{K}(\mathbf{R}) \cup \mathcal{K}(\mathbf{S})$.

5.1 Main Construction

Fix $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta > 1$, and define

$$A_1 = (1, 1, 1, \alpha), \quad A_2 = (1, 1, \alpha, \alpha) \quad A_3 = (1, 1, \alpha, \alpha^2),$$

and

$$B_1 = (1, \beta, \beta, \beta), \quad B_2 = (1, \beta, \beta^2, \beta^2) \quad B_3 = \beta(1, 1, \beta, \beta).$$

The sequences

$$(A_1, A_2, A_3, \alpha A_1) \quad \text{and} \quad (B_1, B_2, B_3, \beta B_1),$$

are sequences of division points of generalized 4-systems $\tilde{\mathbf{R}}^\alpha$ and $\tilde{\mathbf{S}}^\beta$, respectively. Moreover, $\tilde{\mathbf{R}}^\alpha$ and $\tilde{\mathbf{S}}^\beta$ generate self-similar generalized 4-systems $\mathbf{R}^\alpha$ and $\mathbf{S}^\beta$, respectively. The figures below give the combined graphs of $\tilde{\mathbf{R}}^\alpha$ and $\tilde{\mathbf{S}}^\beta$.

The next result will be proven later in this chapter.

Theorem 5.1.1. *Let $\mathbf{R}^\alpha, \mathbf{S}^\beta$ be as above, with $1 < \alpha < \beta$, and let $\mathcal{K}$ be the convex hull of $\mathcal{F}(\mathbf{R}^\alpha) \cup \mathcal{F}(\mathbf{S}^\beta)$. There does not exist a proper 4-system $\mathbf{P}$ such that $\mathcal{K}(\mathbf{P}) = \mathcal{K}$.*

Remark. With the notation in Theorem 5.1.1, if instead $1 < \beta < \alpha$, then on the contrary there exists a proper 4-system $\mathbf{P}$ such that $\mathcal{K}(\mathbf{P}) = \mathcal{K}$. We have proven this in draft form, but it is not included in thesis.

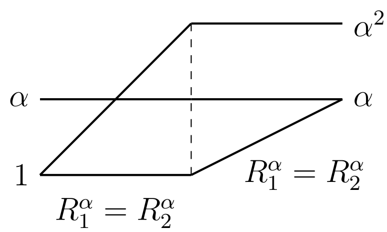

Figure 5.1.1: Graph of $\tilde{\mathbf{R}}^\alpha$

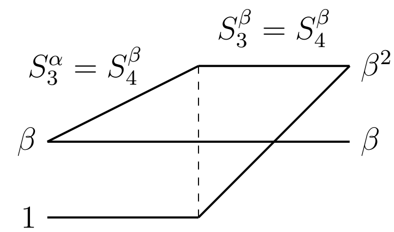

Figure 5.1.2: Graph of $\tilde{\mathbf{S}}^\beta$

5.2 Main Result

Recall that $\bar{A} = A/|A|$ for all non-zero points $A \in \mathbb{R}^4$, as in Definition 3.2.4.

Suppose throughout this section that $1 < \alpha < \beta$, with the notation of the previous section, and let $\mathcal{K}$ be the convex hull of $\mathcal{F}(\mathbf{R}^\alpha) \cup \mathcal{F}(\mathbf{S}^\beta)$.

Define the linear maps $L_0, L_1, L_2, L_3 : \mathbb{R}^4 \to \mathbb{R}$ by

$$\begin{aligned} L_0(\mathbf{x}) &= (\alpha-1)\alpha\beta x_2 - (\alpha^2-\beta)x_3 - (\beta-\alpha)x_4, \\ L_1(\mathbf{x}) &= (\beta-1)x_4 - (\alpha-1)\beta x_1 - (\beta-\alpha)x_2, \\ L_2(\mathbf{x}) &= (\alpha-1)\beta x_1 + (\beta-1)\alpha x_3 - (\alpha-1)\beta x_2 - (\beta-1)x_4, \\ L_3(\mathbf{x}) &= (\alpha-1)\alpha\beta x_1 + (\beta-1)x_3 - (\alpha-1)\alpha x_2 - (\beta-1)x_4, \end{aligned}$$

for each $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Also define the half-spaces

$$F_i = \{\mathbf{x} \in \mathbb{R}^4 \mid L_i(\mathbf{x}) \geq 0\} \quad (i = 0, 1, 2, 3).$$

Since $\mathcal{K}$ is the convex hull of $\{\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{B}_1, \bar{B}_2, \bar{B}_3\}$, some rather lengthy but straightforward linear algebra shows that

$$\mathcal{K} = F_0 \cap F_1 \cap F_2 \cap F_3 \cap \bar{\Delta}^4.$$

Moreover, $\mathcal{K}$ is a three-dimensional convex polyhedron with 6 faces. Two of its faces are the quadrilaterals

$$\bar{A}_1\bar{A}_2\bar{B}_3\bar{A}_3 \subseteq \bar{\Delta}_1 \quad \text{and} \quad \bar{B}_1\bar{B}_2\bar{B}_3\bar{A}_2 \subseteq \bar{\Delta}_3.$$

The other four faces are the triangles

$$\bar{A}_3\bar{B}_3\bar{B}_2 \subseteq \partial F_0, \qquad \bar{A}_1\bar{A}_2\bar{B}_1 \subseteq \partial F_1,$$

$$\bar{B}_1\bar{A}_1\bar{A}_3 \subseteq \partial F_2, \quad \text{and} \quad \bar{B}_1\bar{B}_2\bar{A}_3 \subseteq \partial F_3,$$

where ∂F_i denotes the boundary of F_i for $i = 0, 1, 2, 3$. However, the results in this chapter do not make use of these facts concerning the geometry of $\mathcal{K}$. Instead, define

$$\tilde{\mathcal{K}} = F_1 \cap F_2 \cap F_3 \cap \bar{\Delta}^4 = \{\mathbf{x} \in \bar{\Delta}^4 \mid L_i(\mathbf{x}) \geq 0 \text{ for } i = 1, 2, 3\}. \tag{5.2.1}$$

As mentioned above, a series of straightforward computations yields

$$\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{B}_1, \bar{B}_2, \bar{B}_3 \in \tilde{\mathcal{K}}. \tag{5.2.2}$$

It follows that $\mathcal{K} \subseteq \tilde{\mathcal{K}}$. In particular, as $\bar{B}_1 \in \mathcal{F}(\mathbf{S}^\beta)$, it follows that

$$\bar{B}_1 \in \mathcal{K} \subseteq \tilde{\mathcal{K}}. \tag{5.2.3}$$

The following theorem will be proven later in this chapter.

Theorem 5.2.1. *Let* $\mathbf{P}$ *be a proper* 4*-system. If* $\bar{B}_1 \in \mathcal{K}(\mathbf{P}) \subseteq \tilde{\mathcal{K}}$*, then* $\mathcal{K}(\mathbf{P}) \subseteq \Delta_3$*.*

Theorem 5.1.1 is a particular case of the following corollary.

Corollary 5.2.2. *If* $\mathbf{P}$ *is a proper* 4*-system, then* $\mathcal{K}(\mathbf{P}) \neq \mathcal{K}$ *and* $\mathcal{K}(\mathbf{P}) \neq \tilde{\mathcal{K}}$*.*

5.2.1 Spectra which Are Not Stable under the Minimum

This subsection defines a certain family of spectra, and proves that these spectra are not stable under the minimum. The proof takes Theorem 5.2.1 for granted.

Recall that $L_1, L_2, L_3 : \mathbb{R}^4 \to \mathbb{R}$ are the linear maps associated to the half-spaces F_1, F_2, F_3, respectively, and let $L_4 : \mathbb{R}^4 \to \mathbb{R}$ be the linear map $L_4(\mathbf{x}) = x_4 - x_3$. Define

$$\mathcal{L} = \left\{ \sum_{i=1}^{4} \lambda_i L_i \,\middle|\, \lambda_i > 0 \text{ for } i = 1, 2, 3, 4. \right\}.$$

Lemma 5.2.3. *Let* $L \in \mathcal{L}$ *and* $\mathbf{x} \in \tilde{\mathcal{K}}$*. It follows that* $L(\mathbf{x}) \geq 0$*, with equality if and only if* $\mathbf{x} = \bar{B}_1 = \overline{(1, \beta, \beta, \beta)}$*.*

Proof: Since $\tilde{\mathcal{K}} = F_1 \cap F_2 \cap F_3 \cap \bar{\Delta}^4$, then $L_i(\mathbf{x}) \geq 0$ for $i = 1, \ldots, 4$. This implies that $L(\mathbf{x}) \geq 0$. Further, some computations show that $L_i(1, \beta, \beta, \beta) = 0$ for $i = 1, \ldots, 4$, and so $L(\bar{B}_1) = 0$. Now, suppose that $L(\mathbf{x}) = 0$. Since $L_i(\mathbf{x}) \geq 0$ for $i = 1, \ldots, 4$, it follows that $L_i(\mathbf{x}) = 0$ for $i = 1, \ldots, 4$. Hence, $x_3 = x_4$, and

$$0 = L_1(\mathbf{x}) + L_2(\mathbf{x}) = \alpha(\beta - 1)x_3 - \alpha(\beta - 1)x_2 \Rightarrow x_2 = x_3 = x_4.$$

It follows that

$$0 = L_1(\mathbf{x}) = (\beta - 1)x_4 - (\beta - \alpha)x_2 - (\alpha - 1)\beta x_1 = (\alpha - 1)(x_2 - \beta x_1),$$

and so $x_2 = \beta x_1$. Thus, $\mathbf{x} = \overline{(1, \beta, \beta, \beta)} = \bar{B}_1$. ∎

Taking Theorem 5.2.1 for granted, the main result can be proven as follows.

Theorem 5.2.4. *Let $L \in \mathcal{L}$, and let $T = (L_1, L_2, L_3, L, -L_4)$. The spectrum of $\underline{\varphi}_T$ is not stable under the minimum.*

Proof: Theorem 1.5.2 implies that $\underline{\varphi}_T(\mathbf{R}^\alpha), \underline{\varphi}_T(\mathbf{S}^\beta)$ are in the spectrum of $\underline{\varphi}_T$. Lemma 5.2.3 yields $L(\bar{B}_1) = 0$, and so $L_i(\bar{B}_1) = 0$ for $i = 1, \ldots, 4$. It follows that

$$\underline{\varphi}_T(\mathbf{S}^\beta) = (0, 0, 0, 0, 0).$$

A straightforward series of computations shows that

$$L_1(A_1) = L_2(A_3) = L_3(A_3) = 0, \quad \text{and} \quad -L_4(A_1) = 1 - \alpha < 0.$$

Hence, since $\mathcal{K}(\mathbf{R}^\alpha) \subseteq \tilde{\mathcal{K}}$, then there exists $a \geq 0$ and $b > 0$ such that

$$\underline{\varphi}_T(\mathbf{R}^\alpha) = (0, 0, 0, a, -b).$$

Now, suppose that there exists a proper 4-system $\mathbf{P}$ such that

$$\underline{\varphi}_T(\mathbf{P}) = \min\{\underline{\varphi}_T(\mathbf{R}^\alpha), \underline{\varphi}_T(\mathbf{S}^\beta)\} = (0, 0, 0, 0, -b).$$

The first three coordinates being zero implies that $\mathcal{K}(\mathbf{P}) \subseteq \tilde{\mathcal{K}}$. Hence, as the fourth coordinate is zero, and since $\mathcal{K}(\mathbf{P})$ is closed, Lemma 5.2.3 implies that $\bar{B}_1 \in \mathcal{K}(\mathbf{P})$. Theorem 5.2.1 implies that $\mathcal{K}(\mathbf{P}) \subseteq \bar{\Delta}_3$, and so $b = 0$, which is a contradiction. Therefore,

$$\min\{\underline{\varphi}_T(\mathbf{R}^\alpha), \underline{\varphi}_T(\mathbf{S}^\beta)\}$$

is not in the spectrum of $\underline{\varphi}_T$. ∎

For example, consider the case $\alpha = 2$ and $\beta = 3$. Then,

$$L_1(\mathbf{x}) = -3x_1 - x_2 + 2x_4, \quad L_2(\mathbf{x}) = 3x_1 - 3x_2 + 4x_3 - 2x_4, \quad L_3(\mathbf{x}) = 6x_1 - 2x_2 + 2x_3 - 2x_4.$$

Letting

$$L(\mathbf{x}) = \tfrac{1}{2}L_1(\mathbf{x}) + \tfrac{1}{2}L_2(\mathbf{x}) + \tfrac{1}{2}L_3(\mathbf{x}) + L_4(\mathbf{x}) = 3x_1 - 3x_2 + 2x_3$$

and

$$T = (L_1, L_2, L_3, L, x_3 - x_4),$$

the above theorem implies that the spectrum of $\underline{\varphi}_T$ is not stable under the minimum. Moreover, the proof shows that

$$\underline{\varphi}_T(\mathbf{S}^3) = (0, 0, 0, 0, 0) \quad \text{and} \quad \underline{\varphi}_T(\mathbf{R}^2) = (0, 0, 0, 4, -2),$$

but that

$$(0, 0, 0, 0, -2) = \min\{\underline{\varphi}_T(\mathbf{S}^3), \underline{\varphi}_T(\mathbf{R}^2)\}$$

is not contained in the spectrum of $\underline{\varphi}_T$.

5.3 Proof of Theorem 5.2.1

Now, suppose that there exists a proper 4-system $\mathbf{P} : I \to \mathbb{R}^4$ such that

$$\bar{B}_1 \in \mathcal{K}(\mathbf{P}) \subseteq \tilde{\mathcal{K}}.$$

The goal is to show that $\mathcal{K}(\mathbf{P}) \subseteq \bar{\Delta}_3$, thereby proving Theorem 5.2.1. Since $\mathcal{K}(\mathbf{P}) = \mathcal{K}(\tilde{\mathbf{P}})$ for some non-degenerate 4-system $\tilde{\mathbf{P}}$, one may assume without loss of generality that $\mathbf{P}$ is non-degenerate.

Now, write

$$\varphi(q) = \varphi(q; \mathbf{P}) = q^{-1}\mathbf{P}(q) \quad (q \in I).$$

A result by Roy [22, Lemma 4.1] yields for each $\delta > 0$ some $Q_\delta \in I$ such that

$$\{\varphi(q) \,|\, q \geq Q_\delta\} \subseteq \mathcal{F}(\mathbf{P}) + [-\delta, \delta]^4.$$

Moreover, the points of $\mathcal{F}(\mathbf{P}) + [-\delta, \delta]^4$ are contained in the convex sets

$$\tilde{\mathcal{K}}^\delta = \tilde{\mathcal{K}} + [-\delta, \delta]^4 \subseteq \big(F_1 + [-\delta, \delta]^4\big) \cap \big(F_2 + [-\delta, \delta]^4\big) \cap \big(F_3 + [-\delta, \delta]^4\big).$$

Now, define for each $\delta > 0$ the half-spaces

$$F_i^\delta = \{\mathbf{x} \in \mathbb{R}^4 \,|\, L_i(\mathbf{x}) + c_i\delta \geq 0\} \quad (i = 1, 2, 3),$$

where each c_i is the sum of the absolute values of the coefficients of L_i. Since

$$F_i + [-\delta, \delta]^4 \subseteq F_i^\delta \quad (i = 1, 2, 3)$$

for each $\delta > 0$, then it follows that

$$\{\varphi(q) \,|\, q \geq Q_\delta\} \subseteq F_1^\delta \cap F_2^\delta \cap F_3^\delta, \tag{5.3.1}$$

for each $\delta > 0$.

5.3.1 Key Results

The proof of Theorem 5.2.1 uses the maps $\kappa_1, \kappa_2, \kappa_3 : I \to \mathbb{R}$ defined by

$$\kappa_1(q) = \beta\varphi_1(q) - \varphi_2(q), \quad \kappa_2(q) = \varphi_4(q) - \varphi_2(q) \quad \text{and} \quad \kappa_3(q) = \varphi_4(q) - \varphi_3(q). \tag{5.3.2}$$

The idea of the proof is to consider the ordered sequence $(w_i)_{i\in\mathbb{N}}$ of points in I such that $\mathbf{P}(w_i) \in \Delta_2$ for each $i \in \mathbb{N}$, and to show that the following property: If $\kappa_1, \kappa_2, \kappa_3$ are small at some w_N, then they remain small for each w_i with $i \geq N$. Moreover, κ_3 remains small on all of $[w_N, \infty)$. This will be made precise in the next subsection.

This property will allow us to conclude that

$$\limsup_{q\to\infty} \kappa_3(q) = 0,$$

thereby proving that $\mathcal{K}(\mathbf{P}) \subseteq \Delta_3$, which in turn proves Theorem 5.2.1. To this end, the following lemma will be useful.

Lemma 5.3.1. *The following statements hold for all $\delta > 0$ and $q, r \in I$ with $q < r$:*

1. *If $P_1(q) = P_1(r)$, then $r\kappa_1(r) \leq q\kappa_1(q)$.*
2. *If $P_4(q) = P_4(r)$, then $r\kappa_i(r) \leq q\kappa_i(q)$, for $i = 2, 3$.*
3. $0 \leq \kappa_3(q) \leq \kappa_2(q)$.
4. *If $q \geq Q_\delta$, then $(\alpha - 1)\kappa_1(q) \leq (\beta - 1)\kappa_2(q) + c_1\delta$.*
5. *If $q \geq Q_\delta$ and $P_2(q) = P_3(q)$, then $(\beta - 1)\kappa_3(q) \leq (\alpha - 1)\kappa_1(q) + c_2\delta$.*
6. *If $q \geq Q_\delta$, then $(\beta - 1)\kappa_3(q) \leq \alpha(\alpha - 1)\kappa_1(q) + c_3\delta$.*

Proof: The maps $P_1, \ldots, P_4$ are monotone increasing and satisfy $P_1 \leq \cdots \leq P_4$, from which the first three statements follow. Now, suppose that $q \geq Q_\delta$, and let $\mathbf{x} = (x_1, x_2, x_3, x_4) = \varphi(q)$. Thus, (5.3.2) yields

$$\kappa_1(q) = \beta x_1 - x_2, \quad \kappa_2(q) = x_4 - x_2 \quad \text{and} \quad \kappa_3(q) = x_4 - x_3.$$

The inequality coming from F_1^δ yields

$$(\alpha - 1)\beta x_1 + (\beta - \alpha)x_2 \leq (\beta - 1)x_4 + c_1\delta.$$

Subtracting $(\beta - 1)x_2$ from both sides yields

$$\begin{gathered}(\alpha - 1)\beta x_1 + (1 - \alpha)x_2 \leq (\beta - 1)x_4 - (\beta - 1)x_2 + c_1\delta \\ (\alpha - 1)\kappa_1(q) \leq (\beta - 1)\kappa_2(q) + c_1\delta,\end{gathered}$$

proving the fourth result. Now, the inequality coming from F_2^δ yields

$$(\alpha - 1)\beta x_2 + (\beta - 1)x_4 \leq (\alpha - 1)\beta x_1 + (\beta - 1)\alpha x_3 + c_2\delta.$$

Since $P_2(q) = P_3(q)$, then $x_2 = x_3$, and so subtracting $\alpha\beta x_2 = \alpha\beta x_3$ from both sides yields

$$-\beta x_2 + (\beta - 1)x_4 \leq (\alpha - 1)\beta x_1 - \alpha x_3 + c_2\delta.$$

Adding $(1 - \beta)x_3 + \beta x_2 = \alpha x_3 + (1 - \alpha)x_2$ to both sides yields

$$\begin{gathered}(1 - \beta)x_3 + (\beta - 1)x_4 \leq (\alpha - 1)\beta x_1 + (1 - \alpha)x_2 + c_2\delta, \\ (\beta - 1)\kappa_3(q) \leq (\alpha - 1)\kappa_1(q) + c_2\delta,\end{gathered}$$

proving the fifth result. Now, the inequality coming from F_3^δ yields

$$(\alpha - 1)\alpha x_2 + (\beta - 1)x_4 \leq (\alpha - 1)\alpha\beta x_1 + (\beta - 1)x_3 + c_3\delta.$$

Finally, subtracting $(\beta-1)x_3$ and $(\alpha-1)\alpha x_2$ from both sides yields

$$(\beta-1)\kappa_3(q) \leq (\alpha-1)\alpha\kappa_1(q) + c_3\delta,$$

proving the sixth result. ∎

Before stating the next lemma, the following notation is established. Recall that $1<\alpha<\beta$. Define a variable $\varepsilon_1 \geq 0$, and the dependent variables

$$\delta = \frac{(\beta-\alpha)\varepsilon_1}{2c_4\beta} \geq 0 \quad \text{with} \quad c_4 = \frac{c_1+c_3}{\alpha-1}, \tag{5.3.3}$$

and

$$\varepsilon_2 = \varepsilon_3 = \frac{\alpha(\alpha-1)\varepsilon_1 + c_3\delta}{\beta-1} \geq 0. \tag{5.3.4}$$

Note that

$$\varepsilon_1\varepsilon_2\varepsilon_3\delta = 0 \quad \Longleftrightarrow \quad \varepsilon_1=\varepsilon_2=\varepsilon_3=\delta=0,$$

and that

$$\lim_{\varepsilon_1\to 0}\delta = \lim_{\varepsilon_1\to 0}\varepsilon_2 = \lim_{\varepsilon_1\to 0}\varepsilon_3 = 0. \tag{5.3.5}$$

Now, define the quantity

$$\Omega(\varepsilon_1) = \frac{1+2\varepsilon_3+3(c_2+\alpha c_4)\delta}{\beta-2\beta\varepsilon_2-(1+3\alpha^2)\varepsilon_1}\alpha + \frac{c_4\delta}{\varepsilon_1}. \tag{5.3.6}$$

It follows from (5.3.3) and (5.3.5) that

$$\lim_{\varepsilon_1\to 0}\Omega(\varepsilon_1) = \frac{\alpha}{\beta} + \lim_{\varepsilon_1\to 0}\frac{c_4\delta}{\varepsilon_1} = \frac{\alpha}{\beta} + \frac{\beta-\alpha}{2\beta} = \frac{\alpha+\beta}{2\beta} < 1.$$

Thus, there exists $\varepsilon_0 > 0$ such that

$$\varepsilon_1 \in (0,\varepsilon_0) \qquad \Longrightarrow \qquad \Omega(\varepsilon_1) < 1 \quad \text{and} \quad \beta - 2\beta\varepsilon_2 - (1+3\alpha^2)\varepsilon_1 > 0.$$

The following lemma provides the basis for a recurrence argument on the sequence of points $\mathbf{P}(q) \in \Delta_2$, which will be used to show that $\mathcal{K}(\mathbf{P})$ is contained in $\bar{\Delta}_3$.

Lemma 5.3.2. *Let $\varepsilon_1 \in (0,\varepsilon_0)$, so that $\varepsilon_2,\varepsilon_3,\delta > 0$ and $\Omega(\varepsilon_1) < 1$. Let $q \geq Q = Q_\delta$, and suppose that $\kappa_i(q) < \varepsilon_i$ for $i = 1,2,3$. Let $w \in I$ be minimal with $w > q$ such that $P_2(w) = P_3(w)$. Then we have $\kappa_i(w) < \varepsilon_i$ for $i = 1,2$, and*

$$\kappa_3(q') < \varepsilon_3$$

for all $q' \in [q,w]$.

Proof: First, suppose that $P_1(q) = P_1(w)$. For each $q' \in [q, w]$, it follows that $P_1(q') = P_1(q)$, and so part 1 of Lemma 5.3.1 yields

$$\kappa_1(q') \le (q/q')\kappa_1(q) \le \kappa_1(q) < \varepsilon_1. \tag{5.3.7}$$

Hence, part 6 of Lemma 5.3.1 implies that

$$\kappa_3(q') \le \frac{\alpha(\alpha-1)\kappa_1(q') + c_3\delta}{\beta - 1} < \frac{\alpha(\alpha-1)\varepsilon_1 + c_3\delta}{\beta-1} = \varepsilon_3, \tag{5.3.8}$$

for all $q' \in [q, w]$. Since $P_2(w) = P_3(w)$, then $\kappa_2(w) = \kappa_3(w)$. Hence, $\kappa_2(w) < \varepsilon_3 = \varepsilon_2$, completing the proof when $P_1(w) = P_1(q)$.

Now, suppose that $P_1(q) \neq P_1(w)$. The goal is to first determine some key division numbers in (q, w). Since $P_2 \neq P_3$ on (q, w), there exists $r \in [q, w)$ such that $P_1 + P_2$ has slope 0 on (q, r) and slope 1 on (r, w). Since $P_1(q) \neq P_1(w)$, there is a minimal $s \in [r, w)$ such that $P_1'(s^+) = 1$. Then, there exist $t \in (s, w)$ minimal such that $\mathbf{P}(t) \in \Delta_1$ and $u \in [t, w)$ maximal such that $\mathbf{P}(u) \in \Delta_1$. It follows that the division numbers $r, s, t, u \in (q, w)$ satisfy $r < s < t < u$, and we have

$$\mathbf{P}(s) = (P_1(q), P_2(s), P_3(r), P_4(r)), \qquad \mathbf{P}(t) = (P_2(s), P_2(s), P_3(r), P_4(r)),$$

$$\mathbf{P}(u) = (P_1(u), P_1(u), P_3(r), P_4(r)), \quad \text{and} \quad \mathbf{P}(w) = (P_1(u), P_3(r), P_3(r), P_4(r)).$$

For each $q' \in [q, s]$, as $P_1(q) = P_1(s)$, part 1 of Lemma 5.3.1 yields (5.3.7), and so $\kappa_1(q') < \varepsilon_1$. Moreover, part 6 of Lemma 5.3.1 implies (5.3.8), and so $\kappa_3(q') < \varepsilon_3$. Now, since $P_4(r) = P_4(w)$, part 2 of Lemma 5.3.1 yields

$$\kappa_3(q') \le (r/q')\kappa_3(r) \le \kappa_3(r) < \varepsilon_3 \quad (q' \in [r, w]).$$

Since $r \le s$, it follows that $\kappa_3(q') < \varepsilon_3$ for each $q' \in [q, w]$. Since

$$\kappa_2(w) = \kappa_3(w) < \varepsilon_3 = \varepsilon_2,$$

it remains to show that $\kappa_1(w) < \varepsilon_1$. Applying successively part 4 and part 2 of Lemma 5.3.1, using $\kappa_2(w) = \kappa_3(w)$ and $P_4(r) = P_4(w)$, we find

$$(\alpha-1)\kappa_1(w) \le (\beta-1)\kappa_2(w) + c_1\delta = (\beta-1)\kappa_3(w) + c_1\delta \le (\beta-1)\kappa_3(r)(r/w) + c_1\delta.$$

Applying successively part 6 and part 1 of Lemma 5.3.1, using $P_1(q) = P_1(r)$, we also find

$$(\beta-1)\kappa_3(r) \le \alpha(\alpha-1)\kappa_1(r) + c_3\delta \le \alpha(\alpha-1)\kappa_1(q)(q/r) + c_3\delta.$$

Combining the above two inequalities yields the following first bound on $\kappa_1(w)$:

$$\kappa_1(w) \le \alpha(q/w)\kappa_1(q) + \frac{c_3(r/w) + c_1}{\alpha - 1}\delta < \alpha(q/w)\varepsilon_1 + c_4\delta. \tag{5.3.9}$$

Now, since $w\kappa_3(w) = P_4(w) - P_3(w) = P_4(r) - P_3(r)$, then

$$w = P_1(u) + 2P_3(r) + P_4(r) = P_1(u) + 3P_4(r) - 2w\kappa_3(w). \tag{5.3.10}$$

Since $P_2(w) = P_3(w)$, then $\kappa_3(w) - \kappa_1(w) = \varphi_4(w) - \beta\varphi_1(w)$. Hence, part 5 of Lemma 5.3.1 yields

$$-c_2\delta \leq (\alpha-1)\kappa_1(w) - (\beta-1)\kappa_3(w) = \alpha\kappa_1(w) - \beta\kappa_3(w) + \varphi_4(w) - \beta\varphi_1(w). \tag{5.3.11}$$

Since $P_1(u) = P_1(w)$ and $P_4(r) = P_4(w)$, then multiplying (5.3.11) by w yields

$$P_4(r) \geq \beta(P_1(u) + w\kappa_3(w)) - w(\alpha\kappa_1(w) + c_2\delta) > \beta P_1(u) - w(\alpha\kappa_1(w) + c_2\delta).$$

Hence, (5.3.10) yields

$$w \geq P_1(u)(1+3\beta) - w(2\kappa_3(w) + 3\alpha\kappa_1(w) + 3c_2\delta).$$

Since $P_1(u) \geq P_2(s) \geq P_2(q) = \beta P_1(q) - q\kappa_1(q)$, it follows that

$$w \geq q(\beta\varphi_1(q) - \kappa_1(q))(1+3\beta) - w(2\kappa_3(w) + 3\alpha\kappa_1(w) + 3c_2\delta). \tag{5.3.12}$$

Now, since $1 = \varphi_1(q) + \cdots + \varphi_4(q)$, it follows that

$$1 + 3\kappa_1(q) - 2\kappa_2(q) + \kappa_3(q) = (1+3\beta)\varphi_1(q).$$

Hence, by noting that $\kappa_3(q) \geq 0$, (5.3.12) yields

$$\begin{aligned}(1 + 2\kappa_3(w) + 3\alpha\kappa_1(w) + 3c_2\delta)w/q &\geq (1+3\beta)(\beta\varphi_1(q) - \kappa_1(q)) \\ &\geq \beta(1 + 3\kappa_1(q) - 2\kappa_2(q)) - (1+3\beta)\kappa_1(q) \\ &= \beta - 2\beta\kappa_2(q) - \kappa_1(q) \\ &> \beta - 2\beta\varepsilon_2 - \varepsilon_1,\end{aligned}$$

i.e.

$$(1 + 2\kappa_3(w) + 3\alpha\kappa_1(w) + 3c_2\delta)w/q > \beta - 2\beta\varepsilon_2 - \varepsilon_1. \tag{5.3.13}$$

Now, (5.3.9) implies that $(w/q)\kappa_1(w) < \alpha\varepsilon_1 + (w/q)c_4\delta$, and so

$$(1 + 2\kappa_3(w) + 3\alpha\kappa_1(w) + 3c_2\delta)(w/q) < (1 + 2\varepsilon_3 + 3(\alpha c_4 + c_2)\delta)(w/q) + 3\alpha^2\varepsilon_1.$$

Thus, (5.3.13) implies that

$$\frac{q}{w} < \frac{1 + 2\varepsilon_3 + 3(c_2 + \alpha c_4)\delta}{\beta - 2\beta\varepsilon_2 - (1+3\alpha^2)\varepsilon_1},$$

since $\beta - 2\beta\varepsilon_2 - (1+3\alpha^2)\varepsilon_1 > 0$ as $\varepsilon_1 \in (0, \varepsilon_0)$. Substituting this inequality in (5.3.9) yields

$$\kappa_1(w) < \frac{1 + 2\varepsilon_3 + 3(c_2 + \alpha c_4)\delta}{\beta - 2\beta\varepsilon_2 - (1+3\alpha^2)\varepsilon_1}\alpha\varepsilon_1 + c_4\delta = \Omega(\varepsilon_1)\varepsilon_1,$$

by (5.3.6). Since $\varepsilon_1 \in (0, \varepsilon_0) \Rightarrow \Omega(\varepsilon_1) < 1$, then $\kappa_1(w) < \varepsilon_1$. ∎

The following theorem will be used to prove Theorem 5.2.1.

Theorem 5.3.3. *Let $\varepsilon > 0$. There exists $q \in I$ such that $\kappa_3(q') < \varepsilon$ for all $q' > q$.*

Proof: Assume without loss of generality that $\varepsilon < \varepsilon_0$. Let $\varepsilon_3 = \varepsilon$, and set $\varepsilon_1, \varepsilon_2, \delta > 0$ as in (5.3.3), (5.3.4). Among the points $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{B}_1, \bar{B}_2, \bar{B}_3$, the point $\bar{B}_1 = \overline{(1, \beta, \beta, \beta)}$ is the only one which is contained in $\Delta_2 \cap \Delta_3$. Since $\mathcal{K}$ is the convex hull of these points, it follows that $\bar{B}_1$ is an extremal point of $\mathcal{K}$. Thus, as $\bar{B}_1 \in \mathcal{K}(\mathbf{P}) \subseteq \mathcal{K}$ by the choice of $\mathbf{P}$, it follows that $\bar{B}_1$ is an extremal point of $\mathcal{K}(\mathbf{P})$. Hence, $\bar{B}_1 \in \mathcal{F}(\mathbf{P})$, and so there exists $q \geq Q_\delta$ such that

$$\varphi(q) \in \frac{(1, \beta, \beta, \beta)}{1 + 3\beta} + [-\varepsilon', \varepsilon']^4,$$

where $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}/(\beta + 1)$. It follows that

$$|\beta\varphi_1(q) - \beta/(1 + 3\beta)| < \beta\varepsilon' \quad \text{and} \quad |\varphi_j(q) - \beta/(1 + 3\beta)| < \varepsilon' \quad (j = 2, 3, 4).$$

Hence, as $2 < \beta + 1$, it follows that

$$\kappa_1(q) < (1 + \beta)\varepsilon' \leq \varepsilon_1, \quad \kappa_i(q) < 2\varepsilon' < \varepsilon_i \quad (i = 2, 3). \tag{5.3.14}$$

Now, since $\mathbf{P}$ is proper and non-degenerate, the numbers $w \in I$ with $w > q$ such that $P_2(w) = P_3(w)$ form an unbounded sequence $(w_i)_{i \in \mathbb{N}}$. To complete the proof, it remains to show by induction on $i \in \mathbb{N}$ that we have

$$\kappa_1(w_i) < \varepsilon_1, \quad \kappa_2(w_i) < \varepsilon_2, \quad \text{and} \quad \kappa_3(q') < \varepsilon_3 = \varepsilon \quad (q' \in [q, w_i]). \tag{5.3.15}$$

In light of (5.3.14), the base case $i = 0$ holds by Lemma 5.3.2 since $q \geq Q_\delta$. Now, suppose that (5.3.15) holds for some $i \in \mathbb{N}$. Since $w_i \geq Q_\delta$, and since w_{i+1} is minimal among the points $w > w_i$ with $P_2(w_{i+1}) = P_3(w_{i+1})$, Lemma 5.3.2 gives that (5.3.15) holds for $i + 1$. As $(w_i)_{i \in \mathbb{N}}$ is unbounded, induction gives $\kappa_3(q') < \varepsilon$ for all $q' > q$. ∎

Theorem 5.2.1 is a consequence of the following result.

Corollary 5.3.4. *If $\mathbf{x} \in \mathcal{K}(\mathbf{P})$, then $x_3 = x_4$.*

Proof: For each $\varepsilon > 0$, Theorem 5.3.3 yields $q \in I$ such that $\varphi_4(q') - \varphi_3(q') \in [0, \varepsilon)$ for all $q' \geq q$. Hence, letting $\mathbf{x} \in \mathcal{F}(\mathbf{P})$, it follows that $x_4 - x_3 \in [0, \varepsilon)$ for each $\varepsilon > 0$. Thus, $x_3 = x_4$, and so $\mathcal{F}(\mathbf{P}) \subseteq \Delta_3$. Since Δ_3 is convex, it follows that $\mathcal{K}(\mathbf{P}) \subseteq \Delta_3$. ∎

Chapter 6

On Values of Theta-Like Functions

This chapter is separate from the rest of the thesis, presenting a result in transcendence theory. The theta function

$$T(q) = \sum_{n \in \mathbb{N}} q^{-n^2}$$

is transcendental for each algebraic number q with $|q| > 1$. As noted in [2], the proof relies on Nesterenko's theorem [18]. In this chapter we consider a family of functions $\theta(q)$ generalizing the function $T(q)$, and we show that $\theta(q)$ is not an algebraic number of degree less than 3 for each integer $q \geq 2$. The proof relies on a gap method, using a result of Richards [19]. The main result is the following theorem, recalling our convention that $\mathbb{N} = \{0\} \cup \mathbb{N}^+$.

Theorem 6.0.1. *Let $a, b, c \in \mathbb{Z}$ with $a \geq 1$, and let $p \in \mathbb{Q}[x]$ be the polynomial given by*

$$p(x) = \frac{a}{2}(x^2 + x) + bx + c.$$

Define a map $\theta : (1, \infty) \to \mathbb{R}$ by

$$\theta(q) = \sum_{n=0}^{\infty} x_n q^{-p(n)} \quad (q > 1)$$

where $(x_n)_{n \in \mathbb{N}}$ is a fixed sequence of positive rational numbers such that the right hand side converges for each $q > 1$. Define maps $\chi : \mathbb{R}^+ \to \mathbb{R}$ and $\Delta : \mathbb{R}^+ \to \mathbb{R}$ by

$$\chi(t) = \max\{x_0, \ldots, x_{\lfloor t \rfloor}\} \quad (t \geq 0),$$

and

$$\Delta(t) = \min\{d \in \mathbb{N}^+ \mid dx_i \in \mathbb{N} \text{ for } i = 0, \ldots, \lfloor t \rfloor\} \quad (t \geq 0),$$

and define $H(t) = \Delta(t)\chi(t)$. *Suppose that there exists* $q_0 > 1$ *such that*

$$H(t) \ll t^{\gamma(q_0)},$$

where $\gamma(q_0) = (\log q_0)/(64a)$. *Then* $1, \theta(q), \theta(q)^2$ *are linearly independent over* $\mathbb{Q}$, *i.e.*

$$[\mathbb{Q}(\theta(q)) : \mathbb{Q}] > 2,$$

for each $q \in \mathbb{N}$ *with* $q > q_0$

In particular, we automatically have the following corollary.

Corollary 6.0.2. *Let* $p, (x_n)_{n\in\mathbb{N}}, \theta$, *and* H *be as in Theorem 6.0.1. Suppose that*

$$H(t) \ll t^{o(1)}.$$

Then $1, \theta(q), \theta(q)^2$ *are linearly independent over* $\mathbb{Q}$, *i.e.*

$$[\mathbb{Q}(\theta(q)) : \mathbb{Q}] > 2,$$

for each $q \in \mathbb{N}$ *with* $q \geq 2$.

Observe that a polynomial $p \in \mathbb{R}[x]$ is such that

$$p(\mathbb{Z}) \subseteq \mathbb{Z} \quad \text{with} \quad p(t) \to \infty \text{ as } t \to \infty$$

if and only if p is of the same form as in Theorem 6.0.1.

6.1 Estimates on Gaps of Representations

For this section, fix the polynomial p as in Theorem 6.0.1. The proof of Theorem 6.0.1 will require that we better understand gaps among the set of integers which can be represented by sums $p(n_1) + p(n_2)$ with $n_1, n_2 \in \mathbb{N}$. To this end, define the functions $\mathrm{r}_m, \mathrm{r}'_m : \mathbb{Z} \to \mathbb{N}$ by

$$\mathrm{r}_m(n) = |\{(n_1, \ldots, n_m) \in \mathbb{N}^m \,|\, p(n_1) + \cdots + p(n_m) = n\}|,$$

and

$$\mathrm{r}'_m(n) = |\{(n_1, \ldots, n_m) \in \mathbb{N}^m \,|\, n_1^2 + \cdots + n_m^2 = n\}|.$$

In other words, $\mathrm{r}_m(n)$ counts the number of representations of n as a sum of m numbers $p(k)$ with $k \in \mathbb{N}$. The m^{th} power of the formal q-series $\sum_{n=0}^{\infty} q^{-p(n)}$ is then given by

$$\left(\sum_{n=0}^{\infty} q^{-p(n)}\right)^m = \sum_{n\in\mathbb{Z}} \mathrm{r}_m(n) q^{-n}.$$

Only the cases $m = 1$ and $m = 2$ are used in the proof of the main result.

Definition 6.1.1. For each $N \in \mathbb{Z}$, define $k(N)$, respectively $k'(N)$, as the smallest integer $k > 0$ such that $\mathrm{r}_2(N+k)$, respectively $\mathrm{r}_2'(N+k)$, is not equal to 0.

In particular, the integers $N+1, \ldots, N+k(N)-1$ cannot be represented by sums of the form $p(n_1) + p(n_2)$ with $n_1, n_2 \in \mathbb{N}$, for each $N \in \mathbb{Z}$.

The following result and proof comes from modifying a result and proof of Richards in [19]. For its statement, let $\mathbb{P}$ denote the set of prime numbers. It is worth noting that the original result of [19] is a stronger result for the case $M = 0$. The general case $M \in \mathbb{N}$, however, is original, although its proof stems from only slightly modifying the original proof in [19].

Theorem 6.1.2 (Richards + ε). *For each $n \in \mathbb{N}^+$, let*

$$\mathcal{P}_n = \{q \in \mathbb{P} \mid q \equiv 3 \mod 4 \text{ and } q \leq 4n\},$$

and write $\beta(q) = \max\{\alpha \in \mathbb{N} \mid q^\alpha \leq 4n\}$. Also define

$$P_n = \prod_{q \in \mathcal{P}_n} q^{\beta(q)+1} \in 1 + 2\mathbb{Z},$$

and define $y_n \in \{1, \ldots, P_n - 1\}$ as the solution to the congruence

$$4y_n \equiv -1 \mod P_n.$$

There exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$n \geq \frac{\log P_n}{4(1+\varepsilon_n)},$$

for each $n \in \mathbb{N}^+$. Moreover, for each $M \in \mathbb{N}$, we have that

$$k'(MP_n + y_n) > n,$$

that is the numbers $MP_n + y_n + 1, \ldots, MP_n + y_n + n$ are not sums of two squares.

Observe that if y_n is as above, then

$$4y_n = \begin{cases} P_n - 1 & \text{if } P_n \equiv 1 \mod 4, \\ 3P_n - 1 & \text{if } P_n \equiv 3 \mod 4. \end{cases} \tag{6.1.1}$$

The proof below uses the following classical fact [5, Theorem 366]: A number N is the sum of two squares if and only if each prime factor q of N with $q \equiv 3 \mod 4$ has even exponent in the standard prime factorization of N.

Proof: (**Theorem 6.1.2**)
The prime number theorem for arithmetic progressions implies that

$$|\mathcal{P}_n| \sim \frac{2n}{\log 4n}.$$

Thus, for each $n \in \mathbb{N}^+$, there exists $\varepsilon_n \geq 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$|\mathcal{P}_n| \leq (1+\varepsilon_n)\frac{2n}{\log 4n}.$$

Now, let $n \in \mathbb{N}^+$. Since $q^{\beta(q)+1} \leq (4n)^2$ for each $q \in \mathcal{P}_n$, then

$$P_n \leq (4n)^{2(1+\varepsilon_n)2n/\log 4n} = e^{4n(1+\varepsilon_n)}.$$

Thus,

$$n \geq \frac{1}{4(1+\varepsilon_n)} \log P_n,$$

proving the first result. Now, let $M \in \mathbb{N}$. For each $j \in \{1, \ldots, n\}$, note that

$$4(MP_n + y_n + j) \equiv 4j - 1 \mod P_n.$$

Since $4j - 1 \equiv 3 \mod 4$, then $4j - 1$ has a prime divisor $q \equiv 3 \mod 4$ such that

$$q^\alpha | 4j - 1 \quad \text{and} \quad q^{\alpha+1} \not| 4j - 1,$$

for some $\alpha \in \mathbb{N}$, with α odd. Since $4j - 1 < 4n$, then $\alpha \leq \beta(q)$, and so $q^\alpha | P_n$. It follows that

$$q^\alpha | 4(MP_n + y_n + j).$$

Since $q^{\alpha+1} | P_n$ and $q^{\alpha+1} \not| 4j - 1$, then

$$q^{\alpha+1} \not| 4(MP_n + y_n + j).$$

Since q is coprime to 4, it follows that

$$q^\alpha | (MP_n + y_n + j) \quad \text{and} \quad q^{\alpha+1} \not| (MP_n + y_n + j).$$

Since $q \equiv 3 \mod 4$ and α is odd, it follows that $MP_n + y_n + j$ is not a sum of two squares for $j = 1, \ldots, n$, proving the second claim. ∎

We deduce the following analogue for sums of the form $p(n_1) + p(n_2)$.

Corollary 6.1.3. *For each $n \in \mathbb{N}^+$, let P_n, y_n be as in Theorem 6.1.2, and let*

$$y'_n = \left\lfloor \frac{y_n + 2c'}{8a} \right\rfloor, \text{ where } c' = 8ac - (a+2b)^2.$$

Then, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$k(MP_n + y'_n) > \frac{\log P_n}{32a(1+\varepsilon_n)},$$

for each $n \in \mathbb{N}^+$ and each $M \in \mathbb{N}$.

Proof: Let $b' = a + 2b$ and recall that $c' = 8ac - (a+2b)^2$, so that

$$p(n) = \frac{a}{2}(n^2+n) + bn + c = \frac{(2an+b')^2 + c'}{8a}.$$

Let $n \in \mathbb{N}^+$, $M \in \mathbb{N}$ and $j \in \mathbb{N}^+$. Suppose that $\mathrm{r}_2(MP_n + y'_n + j) > 0$. Then, there exists $m_1, m_2 \in \mathbb{N}$ such that $p(m_1) + p(m_2) = MP_n + y'_n + j$. Hence, we have that

$$(2am_1 + b')^2 + (2am_2 + b')^2 = 8aMP_n + 8ay'_n + 8aj - 2c'.$$

Since $a, b', m_1, m_2 \in \mathbb{Z}$, it follows from the last equality that

$$\mathrm{r}'_2(8aMP_n + 8ay'_n + 8aj - 2c') > 0.$$

Hence, as

$$8ay'_n + 8aj - 2c' \geq 8a(y'_n + 1) - 2c' \geq y_n + 1,$$

Theorem 6.1.2 implies that

$$y_n + n < 8ay'_n - 2c' + 8aj \leq y_n + 8aj,$$

and so $j > n/(8a)$. Hence, Theorem 6.1.2 yields $\varepsilon_n \geq 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$k(MP_n + y'_n) > \frac{n}{8a} \geq \frac{\log P_n}{32a(1+\varepsilon_n)},$$

which completes the proof. ∎

The following lemma provides an upper bound on the gaps between integers of the form $p(n_1) + p(n_2)$.

Lemma 6.1.4. *For each sufficiently large $N \in \mathbb{N}$ and each $A \in [p(N), p(N+1))$, we have $k(A) \leq 2a\sqrt{N}$.*

Proof: Let $N \in \mathbb{N}$ with $p(N) < p(N+1)$, and let $A \in [p(N), p(N+1))$. Let $m'_N, m_N \in \mathbb{N}$ be minimal such that

$$A < p(N) + p(m'_N) \quad \text{and} \quad p(N+1) < p(N) + p(m_N).$$

Note that $m'_N \leq m_N$. By definition, we have that $A + k(A) \leq p(N) + p(m'_N)$, and so

$$k(A) \leq p(m'_N).$$

The choice of m'_N gives $p(N) + p(m'_N - 1) \leq A < p(N) + p(m'_N)$ when $m'_N > 0$. Hence, in general, we have

$$k(A) \leq \max\{p(0), p(m'_N) - p(m'_N - 1)\}.$$

Since $p(x) - p(x-1) = ax + b$ is increasing, and since $m'_N \leq m_N$, we deduce that

$$k(A) \leq \max\{p(0), p(m_N) - p(m_N - 1)\}. \tag{6.1.2}$$

By construction, we have that $p(m_N) > p(N+1) - p(N)$. Since $p(x) - p(x-1)$ is unbounded, it follows that m_N tends to infinity with N. Thus, the right hand side of (6.1.2) is bounded above by $p(m_N) - p(m_N - 1)$ for each sufficiently large N, i.e.

$$k(A) \leq p(m_N) - p(m_N - 1) = am_N + b \quad (N \gg 1). \tag{6.1.3}$$

Now, for each sufficiently large N, we have $m_N > 0$, and so the choice of m_N implies that

$$p(m_N - 1) \leq p(N+1) - p(N) = aN + a + b.$$

Since $p(m_N - 1) \sim am_N^2/2$, then $m_N^2 \leq 3N$ for each sufficiently large N. By (6.1.3), we get

$$k(A) \leq 2a\sqrt{N},$$

for each sufficiently large $n \in \mathbb{N}$. ∎

6.2 Proof of Theorem 6.0.1

For the remainder, fix an integer $q \geq 2$, fix $p, (x_n)_{n \in \mathbb{N}}, \theta, \chi, \Delta, H$ as in Theorem 6.0.1, and write $\theta = \theta(q)$. Also set

$$b' = a + 2b, \quad c' = 8ac - (a + 2b)^2,$$

so that

$$p(n) = \frac{a}{2}(n^2 + n) + bn + c = \frac{(2an + b')^2 + c'}{8a}. \tag{6.2.1}$$

Theorem 6.0.1 is an automatic consequence of the following result.

Theorem 6.2.1. *Suppose that there exists* $\gamma < (\log q)/(64a)$ *such that*

$$H(t) \ll t^{\gamma}.$$

Then $1, \theta, \theta^2$ *are linearly independent over* $\mathbb{Q}$, *i.e.* $[\mathbb{Q}(\theta) : \mathbb{Q}] > 2$.

Since $\Delta(t) \geq 1$ and $\chi(t) \geq x_0 > 0$, the hypothesis of the theorem yields

$$\max\{\Delta(t), \chi(t)\} \ll \Delta(t)\chi(t) = H(t) \ll t^{\gamma}. \tag{6.2.2}$$

6.2.1 Preliminary Lemmas

We give some notation and lemmas which will be used in the proof of Theorem 6.2.1.

Definition 6.2.2. For each $x \in \mathbb{R}$, let $\|x\|$ denote the distance between x and a nearest integer.

We prove Theorem 6.2.1 by contradiction, assuming that θ is an algebraic number of degree at most 2. Hence, there exist $c_0, c_1, c_2 \in \mathbb{Z}$ not all zero such that

$$c_2\theta^2 + c_1\theta + c_0 = 0.$$

In particular, at least one of the constants c_1, c_2 is non-zero.

Now, for each $D \in \mathbb{N}$, we have that

$$\|Dc_2\theta^2 + Dc_1\theta\| = 0,$$

which in turn implies that

$$\|Dc_2\theta^2\| = \|Dc_1\theta\|. \tag{6.2.3}$$

We will arrive at a contradiction by constructing a sequence of integers D_l with the property that (6.2.3) does not hold for $D = D_l$ for each sufficiently large l.

Definition 6.2.3. Let P_n, y'_n be as in Corollary 6.1.3, for each $n \in \mathbb{N}^+$. For each $l \in \mathbb{N}^+$, define

$$A_l = 8aP_l^2 + y'_l.$$

Denote by $N(l) \in \mathbb{N}$ the smallest integer such that $A_l < p(N(l) + 1)$, and define

$$D_l = \Delta(N(l))^2 q^{A_l}.$$

We will show that $\|D_l c_1\theta\|$ and $\|D_l c_2\theta^2\|$ both tend to zero as l tends to infinity, but at a different rate. Thus, (6.2.3) does not hold with $D = D_l$, for each sufficiently large $l \in \mathbb{N}$.

The following lemma will be key in studying how rapidly $\|D_l c_1\theta\|$ tends to zero.

Lemma 6.2.4. *For each sufficiently large $l \in \mathbb{N}$, we have that*

$$p(N(l)+1) - A_l > N(l)/4.$$

Proof: Let $C_1 = 2a + b'$. Multiplying $p(N(l)+1)$ by $8a > 0$ yields

$$(2aN(l) + C_1)^2 + c' = 8ap(N(l)+1) > 8aA_l = (8aP_l)^2 + 8ay_l'.$$

By (6.1.1), we have $y_l \geq (P_l - 1)/4$, and so $y_l' \to \infty$ as $l \to \infty$. Hence, we have $8ay_l' > c'$ for each sufficiently large $l \in \mathbb{N}$. Combining this observation with the above inequality yields

$$8aP_l < 2aN(l) + C_1, \tag{6.2.4}$$

for each sufficiently large $l \in \mathbb{N}$. Since both sides of (6.2.4) are integers, then

$$8a\big(p(N(l)+1) - A_l\big) = (2aN(l) + C_1)^2 + c' - \big((8aP_l)^2 + 8ay_l'\big)$$

is bounded below by

$$(2aN(l) + C_1)^2 + c' - (2aN(l) + C_1 - 1)^2 - 8ay_l' \geq 4aN(l) - 8ay_l' + C_2,$$

for some constant C_2 which does not depend on l. Hence, we have

$$p(N(l)+1) - A_l \geq N(l)/2 - y_l' + C_2/8a, \tag{6.2.5}$$

Since $8ay_l' \leq y_l + c' < 3P_l/4 + 2c'$, then (6.2.4) implies that

$$-8ay_l' + C_2 > -8aP_l + C_1 > -2aN(l),$$

for each sufficiently large $l \in \mathbb{N}$. Dividing this inequality by $8a > 0$, by (6.2.5) we get that

$$p(N(l)+1) - A_l > N(l)/4,$$

for each sufficiently large $l \in \mathbb{N}$. ∎

In order to estimate $D_l\theta^2$, we need to better understand θ^2. To this end, define for each $n \in \mathbb{Z}$ the quantity

$$X_n = \sum_{p(n_1)+p(n_2)=n} x_{n_1}x_{n_2}.$$

Thus, we have that

$$\theta^2 = \left(\sum_{n=0}^{\infty} x_n q^{-p(n)}\right)^2 = \sum_{n\in\mathbb{Z}} X_n q^{-n}.$$

To estimate the coefficients X_n, we will use the following lemma.

Lemma 6.2.5. *There exists a constant $C_3 > 0$ such that*

$$X_n \ll n^{o(1)} \chi(C_3\sqrt{n})^2,$$

where the implicit constant coming from $\ll$ does not depend on n.

Proof: Define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(n) = \max\{\mathrm{r}_2'(i) \mid i \in \mathbb{N} \text{ with } 0 \le i \le n\} \quad (n \in \mathbb{R}^+).$$

By [5, Chapter 18, Section 7, Analogue of Theorem 317], we have

$$f(n) \sim n^{o(1)}. \tag{6.2.6}$$

Now, if $n \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$ are such that $p(n_1) + p(n_2) = n$, then (6.2.1) yields

$$(2an_1 + b')^2 + (2an_2 + b')^2 = 8an - 2c'.$$

In particular, since the map $x \mapsto 2ax + b'$ is injective, then accounting for sign yields

$$\mathrm{r}_2(n) \le 4\mathrm{r}_2'(8an - 2c') \le 4f(8an - 2c').$$

Hence, by (6.2.6) we have that

$$\mathrm{r}_2(n) \ll n^{o(1)}. \tag{6.2.7}$$

Now, there exists $C_3 > 0$ such that, for each $n \ge 1$, we have

$$p(n_1) + p(n_2) = n \quad \Longrightarrow \quad n_i \le C_3\sqrt{n} \quad (i = 1, 2),$$

for each sufficiently large $n \in \mathbb{N}$. Combining this fact with (6.2.7) yields

$$X_n = \sum_{p(n_1)+p(n_2)=n} x_{n_1} x_{n_2} \ll n^{o(1)} \chi(C_3\sqrt{n})^2,$$

which proves the claim. ∎

The following lemma gives a bound on the denominators of the coefficients X_n.

Lemma 6.2.6. *There exists $C_4 \ge 0$ and $N_0 \in \mathbb{N}$ such that for each $n, N \in \mathbb{Z}$ with $N \ge N_0$ and $n \le p(N+1) - C_4$, we have*

$$\Delta(N)^2 X_n \in \mathbb{N}.$$

Proof: Since $a > 0$, there exists $N_0 \in \mathbb{N}$ such that p is increasing on $[N_0, \infty)$, and there exists $C_4 \geq 0$ such that $p(n) > -C_4$ for each $n \in \mathbb{N}$. Hence, for each $N \geq N_0$ and each $m, k \in \mathbb{N}$, we have

$$p(N+1+k) + p(m) > p(N+1) - C_4.$$

Thus, if $n_1, n_2 \in \mathbb{N}$ are such that $p(n_1) + p(n_2) \leq p(N+1) - C_4$, then $n_1, n_2 \leq N$. Hence, we have

$$\Delta(N)^2 X_n = \sum_{p(n_1)+p(n_2)=n} \big(\Delta(N)x_{n_1}\big)\big(\Delta(N)x_{n_2}\big) \in \mathbb{Z},$$

for each $n \in \mathbb{N}$ with $n \leq p(N+1) - C_4$. ∎

6.2.2 Proof of Theorem 6.2.1

Let $A_l, N(l)$, and D_l be as in Definition 6.2.3. Recall that the hypothesis of the theorem yields

$$\max\{\Delta(t), \chi(t)\} \ll \Delta(t)\chi(t) = H(t) \ll t^{\gamma},$$

as noted by (6.2.2), where $\gamma < (\log q)/(64a)$. We will prove Theorem 6.2.1 using the following two lemmas.

Lemma 6.2.7. *For all sufficiently large $l \in \mathbb{N}$, we have*

$$0 < \|D_l\theta\| \ll q^{-N(l)/5}.$$

Proof: For each $n \leq N(l)$, we have $p(n) \leq A_l$, which implies that $q^{A_l - p(n)} \in \mathbb{N}$ and that $\Delta(N(l))x_n \in \mathbb{N}$. Hence, we have

$$D_l\theta = \sum_{n=0}^{\infty} \Delta(N(l))^2 x_n q^{A_l - p(n)} \equiv \sum_{n=N(l)+1}^{\infty} \Delta(N(l))^2 x_n q^{A_l - p(n)} \mod 1. \tag{6.2.8}$$

The right hand side is bounded above by

$$\sum_{n=N(l)+1}^{\infty} \Delta(n)^2 \chi(n) q^{A_l - p(n)} \ll \sum_{n=N(l)+1}^{\infty} H(n)^2 q^{A_l - p(n)} \ll \sum_{n=N(l)+1}^{\infty} n^{2\gamma} q^{A_l - p(n)}.$$

Since

$$\left(\frac{n+1}{n}\right)^{2\gamma} q^{p(n)-p(n+1)} = \left(\frac{n+1}{n}\right)^{2\gamma} q^{-(an+a+b)} < 1/2$$

for each sufficiently large $n \in \mathbb{N}$, then the previous inequality implies that

$$\sum_{n=N(l)+1}^{\infty} \Delta(n)^2 \chi(n) q^{A_l - p(n)} \ll (N(l)+1)^{2\gamma} q^{A_l - p(N(l)+1)} = q^{\mathcal{O}(\log N(l))} q^{A_l - p(N(l)+1)}.$$

Lemma 6.2.4 yields $A_l - p(N(l)+1) < -N(l)/4$, and so

$$0 < \sum_{n=N(l)+1}^{\infty} \Delta(N(l))^2 x_n q^{A_l - p(n)} \ll q^{-N(l)/5}.$$

By (6.2.8), the middle term above is equal to $\|D_l \theta\|$ for each sufficiently large $l \in \mathbb{N}$, because $q^{-N(l)/5}$ tends to zero as l tends to infinity. ∎

Lemma 6.2.8. *There exists $\delta > 0$ such that, for all sufficiently large $l \in \mathbb{N}$, we have*

$$q^{-2a\sqrt{N(l)}} \ll \|D_l \theta^2\| \ll N(l)^{-\delta}.$$

Proof: Since $A_l < p(N(l)+1) - N(l)/4$, Lemma 6.2.6 implies that $\Delta(N(l))^2 X_n \in \mathbb{N}$ for each $n \leq A_l$, so

$$D_l \theta^2 = \sum_{n=0}^{\infty} \Delta(N(l))^2 X_n q^{A_l - n} \equiv \sum_{n=A_l+1}^{\infty} \Delta(N(l))^2 X_n q^{A_l - n} \mod 1,$$

for each sufficiently large $l \in \mathbb{N}$. Since $X_n = 0$ for $A_l < n < A_l + k(A_l)$, this yields

$$D_l \theta^2 \equiv \sum_{n=A_l+k(A_l)}^{\infty} \Delta(N(l))^2 X_n q^{A_l - n} \mod 1, \tag{6.2.9}$$

for each sufficiently large $l \in \mathbb{N}$.

Now, let $C_3 > 0$ be as in Lemma 6.2.5. Since $A_l \asymp N(l)^2$, then $N(l) \leq C_5 \sqrt{A_l}$ for some $C_5 > C_3$. Hence, for each $n \geq A_l$, we have

$$\Delta(N(l))^2 X_n \leq \Delta(N(l))^2 \chi(C_3 \sqrt{n})^2 n^{o(1)} \leq H(C_5 \sqrt{n})^2 n^{o(1)} \ll n^{\gamma + o(1)}.$$

Lemma 6.1.4 gives $k(A_l) \leq 2a\sqrt{N(l)}$ for each sufficiently large $l \in \mathbb{N}$. Hence, we have $A_l + k(A_l) \ll A_l$, and so the above inequality yields

$$\sum_{n=A_l+k(A_l)}^{\infty} \Delta(N(l))^2 X_n q^{A_l - n} \ll \sum_{n=A_l+k(A_l)}^{\infty} n^{\gamma + o(1)} q^{A_l - n} \ll A_l^{\gamma + o(1)} q^{-k(A_l)}.$$

Since $N(l)^2 \asymp A_l \asymp P_l^2$, Corollary 6.1.3 implies that

$$k(A_l) \geq \frac{\log N(l)}{32a(1+o(1))},$$

for each sufficiently large $l \in \mathbb{N}$. Thus, we have that

$$A_l^{\gamma+o(1)} q^{-k(A_l)} \ll N(l)^{2\gamma+o(1)} N(l)^{-(\log q)/(32a(1+o(1)))}.$$

Since $2\gamma < (\log q)/(32a)$, then there exists $\delta > 0$ such that

$$\sum_{n=A_l+k(A_l)}^{\infty} \Delta(N(l))^2 X_n q^{A_l-n} \ll N(l)^{-\delta}. \tag{6.2.10}$$

Now, since $A_l < p(N(l)+1) - N(l)/4$ and since $k(A_l) \leq 2a\sqrt{N(l)}$, then

$$A_l + k(A_l) \leq p(N(l)+1) - C_4,$$

with the constant $C_4 > 0$ from Lemma 6.2.6, for each sufficiently large $l \in \mathbb{N}$. Hence, Lemma 6.2.6 gives that $\Delta(N(l))^2 X_{A_l+k(A_l)} \in \mathbb{N}$, which in turn implies that

$$\sum_{n=A_l+k(A_l)}^{\infty} \Delta(N(l))^2 X_n q^{A_l-n} \geq q^{-k(A_l)},$$

for those l. Assuming l large enough so that $k(A_l) \leq 2a\sqrt{N(l)}$, the above inequality combined with (6.2.10) yields

$$q^{-2a\sqrt{N(l)}} \leq \sum_{n=A_l+k(A_l)}^{\infty} \Delta(N(l))^2 X_n q^{A_l-n} \ll N(l)^{-\delta}.$$

By (6.2.9), the middle term above is equal to $\|D_l\theta\|$ for each sufficiently large $l \in \mathbb{N}$, because $N(l)^{-\delta}$ tends to zero as l tends to infinity. ∎

We can now prove Theorem 6.2.1.

Proof: (**Theorem 6.2.1**)
Suppose that $[\mathbb{Q}(\theta), \mathbb{Q}] \leq 2$, and so there exists $c_0, c_1, c_2 \in \mathbb{Z}$ not all zero such that

$$c_2\theta^2 + c_1\theta + c_0 = 0.$$

In particular, at least one of the constants c_1, c_2 is non-zero. As previously noted, we have that

$$\|c_1 D_l\theta\| = \|c_2 D_l\theta^2\|$$

for each l. By Lemmas 6.2.7 and 6.2.8, we have

$$|c_2| \cdot \|D_l\theta^2\| = \|c_2 D_l\theta^2\| = \|c_1 D_l\theta\| = |c_1| \cdot \|D_l\theta\| \tag{6.2.11}$$

for each sufficiently large $l \in \mathbb{N}$. Suppose that $c_2 = 0$. Then, $\|D_l\theta\| = 0$ for each sufficiently large $l \in \mathbb{N}$, which contradicts Lemma 6.2.7. Hence, $c_2 \neq 0$, and so Lemmas 6.2.7 and 6.2.8 yield

$$|c_1| \cdot \|D_l\theta\| \ll q^{-N(l)/5} \quad \text{and} \quad q^{-2a\sqrt{N(l)}} \ll |c_2| \cdot \|D_l\theta^2\|,$$

respectively. Hence, by (6.2.11), we have

$$q^{-2a\sqrt{N(l)}} \ll q^{-N(l)/5}.$$

This is a contradiction, as the right hand side converges to zero too rapidly compared to the left hand side. Therefore, we have

$$[\mathbb{Q}(\theta) : \mathbb{Q}] > 2,$$

proving the claim. ∎

Bibliography

[1] Y. Bugeaud and M. Laurent. On transfer inequalities in Diophantine approximation II. *Mathematische Zeitschrift*, 265:249–262, 2010.

[2] P. Bundschuh. Arithmetical properties of functions satisfying linear q-difference equations: a survey. *Sūrikaisekikenkyūsho Kōkyūroku*, 1219:110–121, 2001.

[3] M. Coste. An introduction to semialgebraic geometry. *Institut de Recherche Mathématique de Rennes*, pages 1–78, 2002.

[4] O.N. German. Intermediate Diophantine exponents and parametric geometry of numbers. *Acta Arithmetica*, 154:79–101, 2012.

[5] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 2008.

[6] V. Jarník. Über einen Satz von A. Khintchine. *Pràce Mat.-Fiz.*, 43:151–166, 1935.

[7] V. Jarník. Über einen Satz von A. Khintchine II. *Acta Arithmetica*, 2:1–22, 1936.

[8] V. Jarník. Zum Khintchineschen 'Übertragungssatz'. *Trav. Inst. Math. Tbilissi*, 3:193–212, 1938.

[9] V. Jarník. Contributions à la théorie des approximations Diophantiennes linéaires et homogènes. *Czechoslovak Mathematical Journal*, 4:330–353, 1954.

[10] A.Y. Khinchin. Über eine Klasse linearer diophantischer Approximationen. *Rendiconti Del Circolo Matematico Di Palermo*, 50:170–195, 1926.

[11] A.Y. Khinchin. Zur metrischen Theorie der diophantischen Approximationen. *Mathematische Zeitschrift*, 24:706–714, 1926.

[12] M. Laurent. On transfer inequalities in Diophantine approximation, in: Analytic Number Theory in Honour of Klaus Roth. *Cambridge University Press*, pages 306–314, 2009.

[13] S. MacLane and G. Birkhoff. *Algebra*. AMS Chelsea Publishing, 1999.

[14] A. Marnat. About Jarník's-type relation in higher dimension. *arXiv*, (1510.06334):1–25, October 2015.

[15] A. Marnat and N.G. Moshchevitin. An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation. *arXiv*, (1802.03081v1):1–28, February 2018.

[16] H. Minkowski. *Geometrie der Zahlen*. R. G. Teubner, Leipzig and Berlin, 1910.

[17] N.G. Moshchevitin. Exponents for three-dimensional simultaneous Diophantine approximations. *Czechoslovak Mathematical Journal*, 62:127–137, 2012.

[18] Y.V. Nesterenko. On the linear independence of numbers. *Moscow University Mathematics Bulletin*, 40:69–74, 1985.

[19] I. Richards. On the gaps between numbers which are sums of two squares. *Advances in Mathematics*, 46:1–2, 1982.

[20] D. Roy. On Schmidt and Summerer parametric geometry of numbers. *Annals of Mathematics*, 182:739–786, 2015.

[21] D. Roy. Spectrum of the exponents of best rational approximation. *Mathematische Zeitschrift*, 283:143–155, 2016.

[22] D. Roy. On the topology of Diophantine approximation spectra. *Compositio Mathematica*, 153:1512–1546, 2017.

[23] W.M. Schmidt. On heights of algebraic subspaces and Diophantine approximations. *Annals of Mathematics*, 85(3):430–472, 1967.

[24] W.M. Schmidt. Open problems in Diophantine approximation, in: Approximations diophantiennes et nombres transcendants (Luminy 1982). *Progress in Mathematics*, 31:271–287, 1983.

[25] W.M. Schmidt and L. Summerer. Parametric geometry of numbers and applications. *Acta Arithmetica*, 140:67–91, 2009.

[26] W.M. Schmidt and L. Summerer. Diophantine approximation and parametric geometry of numbers. *Monatshefte fur Mathematik*, 169:51–104, 2013.

[27] W.M. Schmidt and L. Summerer. Simultaneous approximation to three numbers. *Moscow Journal of Combinatorics and Number Theory*, 3:84–107, 2013.

[28] A. Seidenberg. A new decision method for elementary algebra. *Annals of Mathematics*, 60(2):365–374, 1954.

[29] A. Tarski. *A decision method for elementary algebra and geometry.* Berkeley, 1951.