A Study of Asymptotic Expansions of Kernels of an Integral Transform

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Abstract

In this study we prove an asymptotic expansion of a pair of unsymmetrical Fourier kernels which can be expressed as Mellin-Barnes type integral, namely,

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\prod_{j=1}^{m} \Gamma(a_j + A_j s)}{\prod_{j=1}^{n} \Gamma(b_j + B_j s)} x^s \, ds ,$$

where γ is real, the A_j, B_j are real, and the path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integrand. These kernels give rise to a Watson type integral transform. The importance of this integral transform is due to its very general yet simple form from which many known as well as new integral transforms can be obtained as special cases.

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Introduction

The purpose of this thesis is to prove a theorem which is stated on pages 18-20. In this theorem we study the asymptotic behaviour of the kernels involved in the unsymmetrical Watson type transform introduced in example 8, section 2 of Chapter I. The kernels called H-functions are generalization of Meijer's G-function.

In Chapter I, we give a brief review of generalized Fourier transform and the definition of the H-function. In Chapter II, a definition of asymptotic expansion of analytic function and some of its properties are given. In Chapter III we develop nine lemmas and in Chapter IV we use these lemmas to prove our main theorem and finally we give some examples of Fourier type kernels which can be written in the form of H-functions.

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Chapter I

Preliminaries

1. Integral Transformations.

Let $K(x,y)$ be a $\mu \times \mu$ measurable function on the product set $R \times R$, where R is a μ-measurable set. Then, if $K(x,y)$, taken as a function of y belongs to the class $L_q(R,\mu)$ for almost every $x \in R$ and if $k(x) = ||K(x,y)||_q$ belongs to $L_p(R,\mu)$, where $1/p + 1/q = 1$, the transformation T, defined by the equation

$$g(x) = Tf = \int_R K(x,y)\ f(y)\ d\mu \tag{1.1.1}$$

is a linear transformation on $L_p(R,\mu)$ into $L_p(R,\mu)$. The transformation $g = Tf$ defined in this way is called [30, p. 177] an <u>integral transformation</u> with kernel $K(x,y)$. The function g is called the <u>integral transform</u> of f with respect to this kernel. The positive number

$$||k(x)||_p = \Big|\Big|\, ||K(x,y)||_q \Big|\Big|_p$$

is called the <u>double-norm</u> of the kernel $K(x,y)$ and it is sometimes denoted by $|||K|||$. The double norm is therefore given by the formula

$$|||K||| = \left\{ \int_R \left(\int_R |K(x,y)|^q \, d\mu \right)^{p/q} d\mu \right\}^{1/p} .$$

It may be easily shown [23, pp. 169-170] that if the $|||K|||<\infty$, the transformation (1.1.1) is bounded and that $||T|| \leq |||K|||$.

It is obvious that $K(x,y) = 0$ a.e. on $R \times R$ implies that $|||K||| = 0$ so that $||T|| = 0$, and T is the null transformation. If, conversely, we have

$$g(x) = \int_R K(x,y) \, f(y) \, d\mu = 0$$

a.e. on R for every $f(y) \varepsilon L_p(R,\mu)$, then $K(x,y) = 0$ a.e. on $R \times R$. If T^{-1} exists then [30, pp. 162-163] it is both linear and bounded.

For $1 \leq p \leq \infty$, the Banach-adjoint T^*g^* may be identified with the transformation

$$\int_R K(y,x) \, g^*(y) \, d\mu$$

on $L_q(R,\mu)$ into $L_q(R,\mu)$.

Methods connected with the use of integral transforms have gained wide importance in mathematical analysis in the recent years. These methods have been successfully applied to the solution of differential and integral equations, the study of special functions, the evaluation of integrals and many other areas. See [3], [31], [32].

In this work we intend to confine ourselves to the case where R is a subset of real line and μ a Lebesgue measure.

2. Fourier Kernels and Fourier Type Integral Transforms.

If an integral transformation has an inverse, then in certain circumstances, the inverse transformation may be of a kind similar to the original one. In other words it may be possible that the equation

$$g(x) = Tf = \int_0^\infty K(x,y)\ f(y)\ dy$$

regarded as an integral equation for the determination of $f(y)$ may have a solution of the form

$$f(x) = T^{-1}g = \int_a^b H(x,y)\ g(y)\ dy\ .$$

A result of this kind which expresses the function $f(x)$ in terms of its integral transform $g(x)$, is called an inversion theorem.

In the special case in which $a = 0$, $b = \infty$, and $K(x,y) = K(xy)$, $H(x,y) = H(xy)$, functions of xy alone, the relationship between the function and its transform is expressed by the equations

$$g(x) = \int_0^\infty K(xy)\ f(y)\ dy \tag{1.2.1}$$

$$f(x) = \int_0^\infty H(xy)\ g(y)\ dy\ . \tag{1.2.2}$$

The functions K, H which admit this kind of reciprocal relationship are said to form a pair of Fourier kernels. The kernels are said to be symmetrical if $K = H$ almost everywhere and unsymmetrical otherwise.

We shall give a few examples of Fourier kernels here. More examples of symmetrical and unsymmetrical Fourier kernels will be given elsewhere in this thesis.

1. $K(x) = H(x) = \sqrt{\frac{2}{\pi}} \cos x$ gives rise to Fourier Cosine transform [25, p. 3] .

2. $K(x) = H(x) = \sqrt{\frac{2}{\pi}} \sin x$ gives rise to Fourier Sine transform [25, p. 4] .

3. $K(x) = H(x) = x^{\frac{1}{2}} J_\nu(x)$, $\nu \geq -\frac{1}{2}$, gives rise to Hankel transform [25, p. 240] .

4. $K(x) = H(x) = \tilde{\omega}_{\mu,\nu}(x)$ gives rise to an integral transform studied by Watson [26, p. 308] .

5. $K(x) = x^{\frac{1}{2}} \mathbf{H}_\nu(x)$, $H(x) = x^{\frac{1}{2}} Y_\nu(x)$,

where $\mathbf{H}_\nu(x)$ denotes the Struve's function and $Y_\nu(x)$ the Bessel function of the second kind [27, pp. 328, 64] , gives rise to an integral transform studied by Hardy and Titchmarsh [12, p. 119] .

6. $K(x) = H(x) = \frac{1}{c} G^{q,\,p}_{2p,2q}\left(x^{\frac{1}{c}} \middle| \begin{matrix} a_1,\cdots,a_p,\ 1-c-a_1,\cdots,1-c-a_p \\ b_1,\cdots,b_q,\ 1-c-b_1,\cdots,1-c-b_q \end{matrix}\right)$,

where G is the Meijer's G-function, defined on page 13 gives rise to a Fourier type transform investigated by C. Fox [9, p. 396] .

7. The kernels

$$K(x) = Ax^{\frac{1}{2}} G^{m,\,p}_{p+q,m+n}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix} a_1, \cdots, a_p, b_1, \cdots, b_q \\ c_1, \cdots, c_m, d_1, \cdots, d_n \end{matrix}\right)$$

$$H(x) = A^{-1} x^{\frac{1}{2}} G^{n,\,q}_{p+q,m+n}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix} -b_1, \cdots, -b_q, -a_1, \cdots, -a_p \\ -d_1, \cdots, -d_n, -c_1, \cdots, -c_m \end{matrix}\right),$$

with certain restrictions on the parameters involved, gives rise to an unsymmetrical Fourier type transform investigated by R. Kesarwani [15, p. 953] .

8. Functions of the form*

$$(1.2.3) \quad K(x) = H^{(1)}(x) = \frac{1}{2\pi i} \int_L \frac{\prod^{m} \Gamma(c_j+\gamma_j(s-\frac{1}{2})) \prod^{p} \Gamma(a_j-\alpha_j(s-\frac{1}{2}))}{\prod^{n} \Gamma(d_j-\delta_j(s-\frac{1}{2})) \prod^{q} \Gamma(b_j+\beta_j(s-\frac{1}{2}))} x^{-s}\, ds ,$$

$$(1.2.4) \quad H(x) = H^{(2)}(x) = \frac{1}{2\pi i} \int_L \frac{\prod^{n} \Gamma(d_j+\delta_j(s-\frac{1}{2})) \prod^{q} \Gamma(b_j-\beta_j(s-\frac{1}{2}))}{\prod^{m} \Gamma(c_j-\gamma_j(s-\frac{1}{2})) \prod^{p} \Gamma(a_j+\alpha_j(s-\frac{1}{2}))} x^{-s}\, ds ,$$

which are a generalization of Meijer's G-function have recently occupied the attention of several authors. Under suitable conditions on the parameters involved, they have been proved to be a pair of unsymmetrical Fourier kernels [18, p. 362] . These are the kernels which we wish to investigate in this thesis.

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* For the sake of typing convenience the sums $\sum_{j=1}^{p} f(j)$ and the products $\prod_{j=1}^{p} G(j)$ in which the summation or the product starts from $j = 1$ will be denoted by $\sum^{p} f(j)$ and $\prod^{p} G(j)$, respectively.

3. Other Integral Transforms.

There are many other well-known integral transforms whose kernels are not Fourier type. We shall give a short list of them and their inversion formulae.

1. Laplace transform

$$g(s) = \int_0^\infty e^{-st} f(t)\, dt ,$$

with the inversion formula given by [28, p. 65]

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} g(s)\, ds , \quad t > 0 .$$

2. Mellin transform

$$f(s) = \int_0^\infty t^{s-1} F(t)\, dt ,$$

with the inversion formula given by [25, p. 46]

$$F(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} t^{-s} f(s)\, ds .$$

3. Meijer transform

$$g(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{st}\, K_\nu(st)\, f(t)\, dt ,$$

where $K_\nu(t)$ is Macdonald's function [27, p. 78]. Its inversion formula takes the form [3, p. 75]

$$f(t) = \frac{1}{\sqrt{2\pi} i} \lim_{\beta\to\infty} \int_{c-i\beta}^{c+i\beta} \sqrt{ts}\, I_\nu(ts)\, g(s)\, ds .$$

4. Exponential (complex) Fourier transform

$$g(u) = \int_{-\infty}^\infty e^{iut} f(t)\, dt ,$$

with the inversion formula given by [10, p. 16]

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu}\, g(u)\, du \, .$$

5. Kontorovich-Lebedev transform

$$g(x) = \int_{0}^{\infty} K_{i\tau}(x)\, f(\tau)\, d\tau \, ,$$

where $K_{\nu}(x)$ is Macdonald's function. Its inversion formula takes the form [3, p. 79]

$$f(\tau) = \frac{2}{\pi^2}\, \tau \sinh \pi\tau \int_{0}^{\infty} \frac{K_{i\tau}(x)}{x}\, g(x)\, dx \, .$$

6. Hilbert transform

$$g(x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt \, ,$$

where $⨍_{-\infty}^{\infty}$ denotes a principal value at $t = x$. Its inversion formula takes the form [25, p. 120]

$$f(t) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{g(x)}{x-t}\, dx \, .$$

4. A Necessary Condition for Two Functions to be a Pair of Fourier Kernels.

Before recording the inversion theorems appropriate to the kernels $K(xy)$ and $H(xy)$ we shall establish heuristically a necessary condition for a given pair of function to be Fourier kernels.

Let $K(x)$ and $H(x)$ be a pair of unsymmetrical Fourier kernels, i.e., if

$$G(x) = \int_0^\infty K(xy)\, F(y)\, dy\,, \tag{1.4.1}$$

then

$$F(x) = \int_0^\infty H(xy)\, G(y)\, dy\,. \tag{1.4.2}$$

Let $k(s)$, $h(s)$, $f(s)$ and $g(s)$ denote the Mellin transform of $K(x)$, $H(x)$, $F(x)$ and $G(x)$ respectively. Then, multiply both sides of equation (1.4.1) by x^{s-1} and integrate over $(0, \infty)$. We obtain formally

$$\begin{aligned} g(s) &= \int_0^\infty x^{s-1}\, G(x)\, dx \\ &= \int_0^\infty x^{s-1}\, dx \int_0^\infty K(xy)\, F(y)\, dy \\ &= \int_0^\infty F(y)\, dy \int_0^\infty x^{s-1}\, K(yx)\, dx \\ &= \int_0^\infty y^{-s}\, F(y)\, dy \int_0^\infty w^{s-1}\, K(w)\, dw \\ &= f(1-s)\, k(s). \end{aligned}$$

Similarly, (1.4.2) gives

$$f(s) = g(1-s)\, h(s)$$

and, if we change s into $1 - s$ in one of these equations, and multiply the corresponding sides, we deduce that

$$k(s)\, h(1-s) = 1\,. \tag{1.4.3}$$

Thus a necessary condition for $K(x)$, $H(x)$ to be a pair of unsymmetrical Fourier kernels is that their Mellin transforms satisfy the functional relation (1.4.3). In case of symmetrical Fourier kernel, the relation (1.4.3) reduces to

$$(1.4.4) \qquad k(s)\,k(1-s) = 1 .$$

Note also that, if $K(x)$, $H(x)$ are a pair of unsymmetrical Fourier kernels, so are the following two pairs

$$\sqrt{a}\,K(ax) \quad \text{and} \quad \sqrt{a}\,H(ax) \;;$$

$$\lambda x^{\frac{1}{2}(\lambda-1)}\,K(x^{\lambda}) \quad \text{and} \quad \lambda x^{\frac{1}{2}(\lambda-1)}\,H(x^{\lambda}) \;.$$

As an example, if we take $K(x) = x^{\frac{1}{2}}\,J_{\nu}(x)$, it is readily found [6, p. 326] that

$$k(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}$$

which obviously satisfies the functional equation (1.4.4).

As another example, take

$$K(x) = x^{\frac{1}{2}}\,\mathbf{H}_{\nu}(x) \;, \qquad H(x) = x^{\frac{1}{2}}\,Y_{\nu}(x) \;;$$

then [6, pp. 335,329]

$$k(s) = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}\,\tan \pi(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)$$

and

$$h(s) = -2^{s-\frac{1}{2}}\,\frac{1}{\pi}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\cos \pi(\tfrac{1}{4}-\tfrac{1}{2}\nu+\tfrac{1}{2}s) \;.$$

Now

$$k(s)\ h(1-s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)} \tan \pi(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)$$

$$\times\{ -2^{\frac{1}{2}-s} \frac{1}{\pi} \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\nu-\tfrac{1}{2}s) \cos \pi(\tfrac{3}{4}-\tfrac{1}{2}\nu-\tfrac{1}{2}s)\}$$

$$= - \csc \pi(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s) \tan \pi(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s) \cos \pi(\tfrac{3}{4}-\tfrac{1}{2}\nu-\tfrac{1}{2}s) = 1,$$

on using the relation [22, p. 21]

(1.4.5) $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$.

Take $K(x) = H^{(1)}(x)$, $H(x) = H^{(2)}(x)$, where $H^{(1)}(x)$, $H^{(2)}(x)$ are functions as given on page 5, equations (1.2.3), (1.2.4) respectively. Then

$$k(s) = \frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))\ \prod^{p}\Gamma(a_j-\alpha_j(s-\frac{1}{2}))}{\prod^{n}\Gamma(d_j-\delta_j(s-\frac{1}{2}))\ \prod^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))}$$

$$h(s) = \frac{\prod^{n}\Gamma(d_j+\delta_j(s-\frac{1}{2}))\ \prod^{q}\Gamma(b_j-\beta_j(s-\frac{1}{2}))}{\prod^{m}\Gamma(c_j-\gamma_j(s-\frac{1}{2}))\ \prod^{p}\Gamma(a_j+\alpha_j(s-\frac{1}{2}))}$$

which obviously satisfy the functional equation (1.4.3).

5. Two Inversion Theorems.

Relation (1.4.3) is the formal necessary condition for $K(x)$ and $H(x)$ to be unsymmetrical Fourier kernels. However, we need only assume the existence of the functions $k(s)$ and $h(s)$ on the line $\sigma = \frac{1}{2}$ where $s = \sigma+i\tau$ so that (1.4.3) becomes

(1.5.1) $k(\frac{1}{2}+i\tau)\ h(\frac{1}{2}-i\tau) = 1$.

We also assume that $k(\frac{1}{2}+i\tau)$ and $h(\frac{1}{2}+i\tau)$ are both bounded as $|\tau| \to \infty$.

By

$$\underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} f(x,\alpha)\, dx$$

(limit in mean) we denote limit in L_2-sense, i.e., a function $\phi(\alpha)$ such that

$$\lim_{T\to\infty} \int_a^b \left|\phi(\alpha) - \int_{-T}^{T} f(x,\alpha)\, dx\right|^2 d\alpha = 0 ,$$

a,b having prescribed values.

Let $K_1(x)$ and $H_1(x)$ be so defined that $K_1(x)/x$ and $H_1(x)/x$ are the Mellin transforms of $k(\frac{1}{2}+i\tau)/(\frac{1}{2}-i\tau)$ and $h(\frac{1}{2}+i\tau)/(\frac{1}{2}-i\tau)$ respectively, that is,

$$(1.5.2) \qquad K_1(x)/x = \frac{1}{2\pi} \underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} \frac{k(\frac{1}{2}+i\tau)}{\frac{1}{2}-i\tau} x^{-\frac{1}{2}-i\tau}\, d\tau ,$$

$$(1.5.2') \qquad H_1(x)/x = \frac{1}{2\pi} \underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} \frac{h(\frac{1}{2}+i\tau)}{\frac{1}{2}-i\tau} x^{-\frac{1}{2}-i\tau}\, d\tau ,$$

the integrals being convergent in the mean square sense since on account of the boundedness of $k(\frac{1}{2}+i\tau)$ and $h(\frac{1}{2}+i\tau)$, $k(\frac{1}{2}+i\tau)/(\frac{1}{2}-i\tau)$ and $h(\frac{1}{2}+i\tau)/(\frac{1}{2}-i\tau)$ belong to $L_2(-\infty, \infty)$, and hence $K_1(x)/x$ and $H_1(x)/x$ belong to $L_2(0, \infty)$.

Theorem 1: [17, p. 272] *Let $k(\frac{1}{2}+i\tau)$ and $h(\frac{1}{2}+i\tau)$ be bounded functions of τ satisfying (1.5.1). Let $K_1(x)$ and $H_1(x)$ be defined by (1.5.2) and (1.5.2') and $F(x)$ be any function belonging to $L_2(0, \infty)$. Then the formulae*

(1.5.3) $$G_K(x) = \frac{d}{dx} \int_0^\infty K_1(xu)\ F(u)\ \frac{du}{u}\ ,$$

(1.5.4) $$G_H(x) = \frac{d}{dx} \int_0^\infty H_1(xu)\ F(u)\ \frac{du}{u}\ ,$$

define almost everywhere functions $G_K(x)$ *and* $G_H(x)$ *respectively both belonging to* $L_2(0, \infty)$. *Also the reciprocal formulae*

(1.5.5) $$F(x) = \frac{d}{dx} \int_0^\infty H_1(xu)\ G_K(u)\ \frac{du}{u}\ ,$$

(1.5.6) $$F(x) = \frac{d}{dx} \int_0^\infty K_1(xu)\ G_H(u)\ \frac{du}{u}\ ,$$

hold almost everywhere. And further,

(1.5.7) $$\int_0^\infty [F(x)]^2\ dx = \int_0^\infty G_K(x)\ G_H(x)\ dx.$$

A function $k(s)$ will be said to belong to k', $k(s) \varepsilon k'$, if it satisfies the following two conditions [12, p. 141].

(i) $k(s)$ is regular in a strip $\sigma_1 < \sigma < \sigma_2$, where $\sigma_1 < 0$, $\sigma_2 > 1$, except possibly for a finite number of simple poles on the imaginary axis, and

(ii) $k(s) = \{\alpha + \frac{\beta}{s} + 0(\frac{1}{|s|^2})\}\Gamma(s) \cos \frac{1}{2}s\pi$ for $\tau = -\infty$;

$k(s) = \{\alpha' + \frac{\beta'}{s} + 0(\frac{1}{|s|^2})\}\Gamma(s) \cos \frac{1}{2}s\pi$ for $\tau = -\infty$,

where α, β, α', β', are complex numbers. Using this notation, we have

Theorem 2: [16, p. 20] *If*

(i) $k(s) \varepsilon k'$,

(ii) $h(s) \varepsilon k'$,

(iii) $k(s)$ <u>and</u> $h(s)$ <u>satisfy the functional relation</u>

$$k(s)\ h(1-s) = 1\ ,$$

(iv) $F(y) \varepsilon L_1(0, \infty)$ <u>and is of bounded variation near</u> $y = x\ (x > 0)$, <u>then</u>

$$\int_0^\infty K(xu)\ du \int_0^\infty H(uy)\ F(y)\ dy = \tfrac{1}{2}\{F(x+0)+F(x-0)\}\ . \tag{1.5.8}$$

6. <u>Definitions of G- and H-function.</u>

A G-function is written as on the left of (1.6.1) below and defined [5, p. 207] by the integral on the right

$$G^{m,\,n}_{n+p,m+q}\left\{x\left|\begin{matrix}a_1,\cdots, a_n, c_1,\cdots, c_p\\ b_1,\cdots, b_m, d_1,\cdots, d_q\end{matrix}\right.\right\} = \frac{1}{2\pi i}\int_L \frac{\prod\limits^{m}\Gamma(b_j-s)\ \prod\limits^{n}\Gamma(1-a_j+s)}{\prod\limits^{q}\Gamma(1-d_j+s)\ \prod\limits^{p}\Gamma(c_j-s)}\, x^s\, ds, \tag{1.6.1}$$

where an empty product is interpreted as unity, m, n, p, q are non-negative integers, parameters a_j and b_j are such that no pole of $\Gamma(b_j-s)$, $j = 1,2,\cdots, m$, coincides with any pole of $\Gamma(1-a_j+s)$, $j = 1,2,\cdots, n$. Thus $a_k - b_j$ is not a positive integer. We retain this assumptions throughout.

There are three different paths L of integration:

(i) L runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_j-s)$, $j = 1,2,\cdots, m$, are to the right, and all the poles of $\Gamma(1-a_k+s)$, $k = 1,2,\cdots, n$, to the left, of L. The integral converges if $p + q < m + n$ and $|\arg x| < \frac{1}{2}\pi(m+n-p-q)$.

(ii) L is a loop begining and ending at $+\infty$ and encircling all poles of $\Gamma(b_j-s)$, $j = 1,2,\cdots, m$, once in the negative direction, but none of the poles of $\Gamma(1-a_k+s)$, $k = 1,2,\cdots, n$. The integral converges in $m+q \geq 1$ and either $n+p < m+q$ or $n+p = m+q$ and $|x| < 1$.

(iii) L is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma(1-a_k+s)$, $k = 1,2,\cdots, n$, once in the positive direction, but none of the poles of $\Gamma(b_j-s)$, $j = 1,2,\cdots, m$. The integral converges if $n+p \geq 1$ and either $n+p > m+q$ or $n+p = m+q$ and $|x| > 1$.

We shall always assume that the values of the parameters and of the variable x are such that at least one of the three definitions (i), (ii), (iii) make sense. In cases when more than one of these definitions make sense, they lead to the same result so that there will be no ambiguity involved [19, p. 145]. The G-function is an analytic function of x with a branch point at the origin. For more information on G-function one may refer to [19, Chapter 5].

A more general function than the G-function is the function known as H-function which is defined by the following equation [9, p. 408]

$$(1.6.2) \qquad H(x) = \frac{1}{2\pi i} \int_L \frac{\prod^{m}\Gamma(b_j+\beta_j s)\ \prod^{n}\Gamma(a_j-\alpha_j s)}{\prod^{q}\Gamma(d_j-\delta_j s)\ \prod^{p}\Gamma(c_j+\gamma_j s)}\, x^{-s}\, ds\ .$$

There are similar restrictions on the parameters as in case of G-functions. Here m, n, p, q are non-negative integers and parameters a_j and b_j are such that no pole of $\Gamma(b_j+\beta_j s)$,

$j = 1,2,\cdots, m$ coincides with any pole of $\Gamma(a_j-\alpha_j s)$, $j = 1,2,\cdots, n$. The path L of integration is as in case of G-function so that the poles of $\Gamma(b_j+\beta_j s)$, $j = 1,2,\cdots, m$ are on one side of L and those of $\Gamma(a_j-\alpha_j s)$, $j = 1,2,\cdots, n$ on the other side.

The H-function reduces to the G-function of (1.6.1) when $\alpha_j = 1$, $j = 1,2,\cdots, n$, $\beta_j = 1$, $j = 1,2,\cdots, m$, $\gamma_j = 1$, $j = 1,2,\cdots, p$, $\delta_j = 1$, $j = 1,2,\cdots, q$; a_j is replaced by $1 - a_j$, $j = 1,2,\cdots, n$, and d_j is replaced by $1 - d_j$, $j = 1,2,\cdots, q$ and the variable of integration s is replaced by $-s$.

7. Two Inversion Theorems for Integral Transforms with H-functions as Kernels.

The following two theorems with regard to integral transform with $H^{(1)}(x)$, $H^{(2)}(x)$, page 5, as kernels have been proved [18, pp. 360-366] .

Theorem 3: *If*

(i) $m - q = n - p$,

(ii) $\alpha_j > 0$, $j = 1,2,\cdots, p$; $\beta_j > 0$, $j = 1,2,\cdots, q$;
$\gamma_j > 0$, $j = 1,2,\cdots, m$; $\delta_j > 0$, $j = 1,2,\cdots, n$,

(iii) $\frac{1}{2}D = \sum^{m} \gamma_j - \sum^{q} \beta_j = \sum^{n} \delta_j - \sum^{p} \alpha_j > 0$,

(iv) $\sum^{m} c_j - \sum^{q} b_j = \sum^{n} d_j - \sum^{p} a_j$,

(v) $\mathrm{Re}(a_j) > 0$, $j = 1,2,\cdots, p$; $\mathrm{Re}(b_j) > 0$, $j = 1,2,\cdots, q$;
$\mathrm{Re}(c_j) > 0$, $j = 1,2,\cdots, m$; $\mathrm{Re}(d_j) > 0$, $j = 1,2,\cdots, n$,

(vi) $f(x) \in L_2(0, \infty)$,

then the formulae

(1.7.1) $$\frac{d}{dx}\int_0^\infty H_1^{(1)}(xu)\ f(u)\ \frac{du}{u} = g^{(1)}(x)\ ,$$

(1.7.2) $$\frac{d}{dx}\int_0^\infty H_1^{(2)}(xu)\ f(u)\ \frac{du}{u} = g^{(2)}(x)\ ,$$

define almost everywhere functions $g^{(1)}(x)$ and $g^{(2)}(x)$ respectively both belonging to $L_2(0,\infty)$. Also the reciprocal formulae

(1.7.3) $$\frac{d}{dx}\int_0^\infty H_1^{(2)}(xu)\ g^{(1)}(u)\ \frac{du}{u} = f(x)\ ,$$

(1.7.4) $$\frac{d}{dx}\int_0^\infty H_1^{(1)}(xu)\ g^{(2)}(u)\ \frac{du}{u} = f(x)\ ,$$

hold almost everywhere. And further

(1.7.5) $$\int_0^\infty [f(x)]^2\ dx = \int_0^\infty g^{(1)}(x)\ g^{(2)}(x)\ dx\ .$$

Theorem 4: If

(i) $m - q = n - p$,

(ii) $\alpha_j > 0,\ j = 1,2,\cdots, p;\quad \beta_j > 0,\ j = 1,2,\cdots, q;$
$\gamma_j > 0,\ j = 1,2,\cdots, m;\quad \delta_j > 0,\ j = 1,2,\cdots, n,$

(iii) $\frac{1}{2}D = \sum^m \gamma_j - \sum^q \beta_j = \sum^n \delta_j - \sum^p \alpha_j > 0$,

(iv) $\sum^m c_j - \sum^q b_j = \sum^n d_j - \sum^p a_j$,

(v) $\mathrm{Re}(a_j) > \frac{\alpha_j}{2D},\ j = 1,2,\cdots, p;\quad \mathrm{Re}(b_j) > \frac{\beta_j}{2D},\ j = 1,2,\cdots, q;$

(vi) $\mathrm{Re}(c_j) \geq \frac{\gamma_j}{2D},\ j = 1,2,\cdots, m;\quad \mathrm{Re}(d_j) \geq \frac{\delta_j}{2D},\ j = 1,2,\cdots, n,$

(vii) $y^{(1-D)/(2D)}\ f(y) \in L_1(0,\infty)$ and $f(y)$ is of bounded variation near $y = x$ $(x > 0)$, then

$$\int_0^\infty H^{(1)}(xu)\ du \int_0^\infty H^{(2)}(uy)\ f(y)\ dy = \tfrac{1}{2}\{f(x+0)+f(x-0)\}\ . \tag{1.7.6}$$

Chapter II

Asymptotic Expansion

1. The Theorem.

Let

$$(2.1.1)\quad H^{(1)}(x) = \frac{1}{2\pi i}\int_L \frac{\prod\limits^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))\ \prod\limits^{p}\Gamma(a_j-\alpha_j(s-\frac{1}{2}))}{\prod\limits^{n}\Gamma(d_j-\delta_j(s-\frac{1}{2}))\ \prod\limits^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))} x^{-s}\,ds ,$$

$$(2.1.2)\quad H^{(2)}(x) = \frac{1}{2\pi i}\int_L \frac{\prod\limits^{n}\Gamma(d_j+\delta_j(s-\frac{1}{2}))\ \prod\limits^{q}\Gamma(b_j-\beta_j(s-\frac{1}{2}))}{\prod\limits^{m}\Gamma(c_j-\gamma_j(s-\frac{1}{2}))\ \prod\limits^{p}\Gamma(a_j+\alpha_j(s-\frac{1}{2}))} x^{-s}\,ds ,$$

as in section 2, chapter I.

Theorem 5. [9, p. 417], [18, p. 367]. Assumptions:

(i) $m - q = n - p$,

(ii) $\alpha_j > 0,\ j = 1,\cdots, p,\quad \beta_j > 0,\ j = 1,\cdots, q,$

$\gamma_j > 0,\ j = 1,\cdots, m,\quad \delta_j > 0,\ j = 1,\cdots, n,$

(iii) $\frac{1}{2}D = \sum^{m}\gamma_j - \sum^{q}\beta_j = \sum^{n}\delta_j - \sum^{p}\alpha_j > 0$,

(iv) $\frac{1}{2}L = \sum^{m}c_j - \sum^{q}b_j = \sum^{n}d_j - \sum^{p}a_j$,

(v) x is real and positive,

(vi) $\mathrm{Re}(a_j) \geq 0,\ j = 1,\cdots,p,\quad \mathrm{Re}(b_j) \geq 0,\ j = 1,\cdots,q,$

(vii) $\mathrm{Re}(c_j) \geq \frac{1}{2}\gamma_j,\ j = 1,\cdots,m,\quad \mathrm{Re}(d_j) \geq \frac{1}{2}\delta_j,\ j = 1,\cdots,n,$

(viii) given $\tilde{\omega} > \frac{1}{2} + \tilde{\omega}_0$,

where $\tilde{\omega}_0 = \max(\mathrm{Re}(a_1)/\alpha_1,\cdots,\mathrm{Re}(a_p)/\alpha_p,\ \mathrm{Re}(b_1)/\beta_1,\cdots,\mathrm{Re}(b_q)/\beta_q)$,

N *denotes the greatest integer less than* $\{ D(\tilde{\omega} - \frac{1}{2}) + 3/2 \}$, M_j *denotes the greatest integer less than* $\{ \alpha_j(\tilde{\omega} - \frac{1}{2}) - \mathrm{Re}(a_j)\}$, $j = 1,\cdots,p$ *and* M_j' *denotes the greatest integer less than* $\{ \beta_j(\tilde{\omega} - \frac{1}{2}) - \mathrm{Re}(b_j)\}$, $j = 1,\cdots,q$,

(ix) $m - q = n - p$ *is an odd integer.*

Conclusions:

$$(2.1.3)\quad H^{(1)}(x) = \frac{1}{D}\left(\frac{x}{\beta}\right)^{(1-D)/(2D)} \sum_{j=0}^{N} \nu_j \left(\frac{x}{\beta}\right)^{-j/D} \sin\left\{\frac{\pi}{2}(L+\tfrac{1}{2}-j) - \left(\frac{x}{\beta}\right)^{1/D}\right\}$$

$$+ \sum^{p} x^{-\frac{1}{2}-a_j/\alpha_j} \left(A_j + B_j x^{-\frac{1}{\alpha_j}} + C_j x^{-\frac{2}{\alpha_j}} + \cdots + U_j x^{-\frac{M_j}{\alpha_j}}\right) + O(x^{-\tilde{\omega}})$$

and

$$(2.1.4)\quad H^{(2)}(x) = \frac{1}{D}\left(\frac{x}{\beta}\right)^{(1-D)/(2D)} \sum_{j=0}^{N} \nu_j' \left(\frac{x}{\beta}\right)^{-j/D} \sin\left\{\frac{\pi}{2}(L+\tfrac{1}{2}-j) - \left(\frac{x}{\beta}\right)^{1/D}\right\}$$

$$+ \sum^{q} x^{-\frac{1}{2}-b_j/\beta_j} \left(A_j' + B_j' x^{-\frac{1}{\beta_j}} + C_j' x^{-\frac{2}{\beta_j}} + \cdots + U_j' x^{-\frac{M_j'}{\beta_j}}\right) + O(x^{-\tilde{\omega}}),$$

where

(2.1.5)
$$\beta = \mu^D$$

(2.1.6)
$$\mu = \frac{\prod^{m} \{\gamma_j/D\}^{\gamma_j/D} \prod^{n} \{\delta_j/D\}^{\delta_j/D}}{\prod^{p} \{\alpha_j/D\}^{\alpha_j/D} \prod^{q} \{\beta_j/D\}^{\beta_j/D}} \equiv \frac{1}{D}\left\{\frac{\prod^{m} \gamma_j^{\gamma_j} \prod^{n} \delta_j^{\delta_j}}{\prod^{p} \alpha_j^{\alpha_j} \prod^{q} \beta_j^{\beta_j}}\right\}^{1/D}$$

and
$$\nu_j,\ \nu_j,\ j = 0,1,2,\cdots, N,$$

$$A_j,\ B_j,\ C_j,\cdots,\ U_j,\ j = 1,2,\cdots, p,$$

$$A_j',\ B_j',\ C_j',\cdots,\ U_j',\ j = 1,2,\cdots, p,$$

are constants which depend on the parameters

$$a_j,\ \alpha_j,\ j = 1,2,\cdots, p,\quad b_j,\ \beta_j,\ j = 1,2,\cdots, q,$$
$$c_j,\ \gamma_j,\ j = 1,2,\cdots, m,\quad d_j,\ \delta_j,\ j = 1,2,\cdots, n,$$

but are independent of x. *The constants* ν_j, ν_j', $j = 0,1,\cdots, N$, *are computed by a method specified in Lemma 5, in Chapter III, and the constants* A_j, B_j, $C_j,\cdots$, U_j, A_j', B_j', $C_j',\cdots$, U_j', $j = 1,2,\cdots, p$, *are computed by means of residues as specified in section 1, chapter IV*.

If $m - q = n - p$ *is an even integer (instead of the condition (ix) above) then in the right hand side of* (2.1.3) *and* (2.1.4) *we must replace* sin *by* cos.

Conditions (vi) and (vii) of the theorem enable us to use (2.1.1) and (2.1.2) as the definitions of $H^{(1)}(x)$ and $H^{(2)}(x)$ with the line $\sigma = \sigma'$, where $0 < \sigma' < \frac{1}{2}$ as the contour L of integration and the integrals will then converge by virtue of the condition (v).

Since $H^{(1)}(x)$ and $H^{(2)}(x)$ are quite similar, we will obtain the asymptotic expansion for $H^{(1)}(x)$ only. The case of $H^{(2)}(x)$ can be treated in a similar manner. We now write (2.1.1) in the form

$$H^{(1)}(x) = \frac{1}{2\pi i} \int Q^{(1)}(s) \left(\frac{x}{\beta}\right)^{-s} ds \,, \tag{2.1.7}$$

using the contour $\sigma = \sigma'$ just mentioned above and with

$$Q^{(1)}(s) = \frac{\prod\limits^{m} \Gamma(c_j+\gamma_j(s-\frac{1}{2})) \prod\limits^{p} \Gamma(a_j-\alpha_j(s-\frac{1}{2}))}{\prod\limits^{n} \Gamma(d_j-\delta_j(s-\frac{1}{2})) \prod\limits^{q} \Gamma(b_j+\beta_j(s-\frac{1}{2}))} \mu^{-sD} \,, \tag{2.1.8}$$

and β as in (2.1.5).

The proof of this theorem is based on the lemmas proved in the following chapter. First we give a brief account of asymptotic expansion of an analytic function and some of their properties needed in our work.

2. Asymptotic Expansion of Analytic Functions.

Let $f(z)$ and $g(z)$ be two functions defined on a set S in the complex plane, and let z_0 be a limit point of S. By a neighbourhood of z_0, we mean an open disc $|z - z_0| < \delta$ if z_0 is at a finite distance from the origin, a region $|z| > \delta$ if z_0 is the point at infinity. We write $f = O(g)$ if there exists a constant A such that $|f| \leq A|g|$ for all z in S. We also write $f = O(g)$ as $z \to z_0$ if there exists a constant A and a neighbourhood U of z_0 such that $|f| \leq A|g|$ for all points in the intersection of U and S; and $f = o(g)$ as $z \to z_0$ if, for any positive number ε, there exists a neighbourhood U of z_0 such that $|f| \leq \varepsilon|g|$ for all points z of the intersection of U and S.

It is obvious that

(2.2.1) $$O(f)\ O(g) = O(fg)\ .$$

The formal series

(2.2.2) $$S(z) = \sum_{k=0}^{\infty} a_k z^{-k}$$

is said to be an asymptotic expansion of an analtytic function $f(z)$ for $|z| > R$ and for a given interval of $\arg z$ independent of $|z|$, if for each fixed n,

(2.2.3) $$f(z) - S_n(z) = o(z^{-n}) \qquad \text{as} \quad |z| \to \infty$$

where

$$S_n(z) = \sum_{k=0}^{n} a_k z^{-k}\ .$$

If this is true, we write $f(z) \sim S(z)$. A partial sum, $S_n(z)$, of this formal series will often be called an asymptotic approximation to $f(z)$. The coefficients are given successively by

(2.2.4) $$a_m = \lim_{|z|\to\infty} \left\{ f(z) - \sum_{k=0}^{m-1} \frac{a_k}{z^k} \right\} / z^m\ .$$

A power series in $1/z$ that converges for $|z| > \delta$ satisfies the definition of an asymptotic expansion. For there is an M such that the remainder after the term z^{-n} has an absolute value less than $\frac{M|z|^{-n}}{|z|/\delta - 1}$, for all values of $\arg z$.

It may happen that $f(z)$ itself has no asymptotic expansion in the sense of the foregoing definition. We, then, write

$$f(z) \sim h(z) + g(z) \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{as} \quad |z| \to \infty \tag{2.2.5}$$

whenever

$$\frac{f(z) - h(z)}{g(z)} \sim \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{as} \quad |z| \to \infty. \tag{2.2.6}$$

Asymptotic series are usually divergent, though there is no reason to insist on this point in the definition.

2.1. Asymptotic expansions are unique; that is, if

$$f(z) \sim \sum_{k=0}^{\infty} a_k z^{-k} \qquad \text{as} \quad |z| \to \infty$$

and

$$f(z) \sim \sum_{k=0}^{\infty} b_k z^{-k} \qquad \text{as} \quad |z| \to \infty$$

in $\alpha \leq \arg z \leq \beta$, then $a_k = b_k$ for all k. For the statements imply that for any $\varepsilon > 0$ and for some R and $|z| > R$ in $\alpha \leq \arg z \leq \beta$

$$\left| z^n \left\{ \sum_{k=0}^{n} a_k z^{-k} - f(z) \right\} - z^n \left\{ \sum_{k=0}^{n} b_k z^{-k} - f(z) \right\} \right| \leq 2\varepsilon ,$$

that is,

$$\left| z^n \sum_{k=0}^{n} (a_k - b_k) z^{-k} \right| \leq 2\varepsilon .$$

If we make $|z| \to \infty$ this will be false for every n unless all $a_k = b_k$.

2.2. The converse is not true; the same expansion in a given region may be an asymptotic expansion of several functions provided their difference $f(z) - g(z)$ satisfy for every n

$$\lim_{|z|\to\infty} z^n\{ f(z) - g(z)\} = 0 .$$

2.3. Asymptotic expansions valid in the same sector can be multiplied unconditionally. For if

$$S_n(z) = a_0+\frac{a_1}{z} +\cdots+\frac{a_n}{z^n} ,$$

$$T_n(z) = b_0+\frac{b_1}{z} +\cdots+\frac{b_n}{z^n} ,$$

are asymptotic approximations respectively of $f(z)$, $g(z)$ for $\alpha \leq \arg z \leq \beta$, we can choose z so that

$$|z^n\{ f(z) - S_n(z)\}| , \quad |z^n\{ g(z) - T_n(z)\}|$$

are arbitrarily small. Now

$$S_n(z)\ T_n(z) = c_0+\frac{c_1}{z}+\cdots+\frac{c_n}{z^n}+o(z^{-n}) = U_n(z)+o(z^{-n})$$

where $c_m = a_0b_m+a_1b_{m-1}+\cdots+a_mb_0$ and $z^n\{ f(z)\ g(z)-U_n(z)\}$ is the sum of such terms that tend to zero as $|z| \to \infty$ for all n. Hence $U_n(z)$ is the asymptotic approximation of $f(z)\ g(z)$.

2.4. Asymptotic approximations can also be divided provided the divisor contains at least one non-zero coefficient. If

$$f(z) = a_0+\frac{a_1}{z}+\cdots+\frac{a_n}{z^n}+\frac{\omega}{z^n} = S_n(z)+\frac{\omega}{z^n} ,$$

where $a_0 \neq 0$, $\omega \to 0$ as $|z| \to \infty$, we have for $|z|$ large enough

$$\frac{1}{f(z)} = \frac{1}{S_n(z)} - \frac{\omega}{z^n S_n^2(z)} + \frac{\omega^2}{z^{2n} S_n^3(z)} - \dots .$$

For $|z|$ large enough the terms after the first form a series whose sum has modulus less than $K|\omega z^{-n}|$, where K is a constant, and the first term can be expanded in a convergent series if z is such that

$$|a_0| > \sum_{k=1}^{n} a_k z^{-k} .$$

Stopping at z^{-n} then gives an error $O(z^{-n-1})$.

If $a_0 = 0$ we need only consider, instead of $f(z)$, $z^m f(z)$, where a_m is the first non-zero coefficient.

Then if $g(z)$ has an asymptotic expansion the product of those of $g(z)$ and $1/f(z)$ gives that of $g(z)/f(z)$.

2.5. An asymptotic expansion can be integrated term-by-term unconditionally. For any positive $\varepsilon > 0$, take R so that for $\alpha \leq \arg z \leq \beta$, $|z| > R$, $f(z)$ is analytic, and

$$|f(z) - S_n(z)| < \varepsilon |z^{-n}| .$$

Since any z_1, z_2 in the sector can be connected by a path of constant $\arg z$ followed by one of constant $|z|$. So

$$\int_{z_1}^{z_2} \{ f(z) - S_n(z) \} dz = \int_{z_1}^{z_2'} \{ f(z) - S_n(z) \} dz + \int_{z_2'}^{z_2} \{ f(z) - S_n(z) \} dz ,$$

where the path of integration from z_1 to z_2' is $\arg z =$ constant and the path of integration from z_2' to z_2 is $|z| =$ constant.

But

$$\left|\int_{z_1}^{z_2'}\{ f(z)-S_n(z)\}\, dz\right| \le \int_{|z_1|}^{|z_2'|} |f(z)-S_n(z)|\, dz$$

$$\le \int_{|z_1|}^{\infty} |f(z)-S_n(z)|\, dz \le \int_{|z_1|}^{\infty} \frac{\varepsilon}{r^n}\, dr$$

$$= \frac{\varepsilon}{(n-1)|z_1|^{n-1}},$$

and on a path of constant $|z|$ of length $L(\le 2\pi|z|)$

$$\left|\int_{z_2'}^{z_2}\{ f(z)-S_n(z)\}\, dz\right| \le \int_{z_2'}^{z_2}|f(z)-S_n(z)|\, dz \le L\,\frac{\varepsilon}{|z_2|^n} \le \frac{2\pi\varepsilon}{|z_2|^{n-1}}$$

It follows that

$$\left|\int_{z_1}^{z_2}\{ f(z)-S_n(z)\}\, dz\right| \le \frac{K\varepsilon}{\{\min(|z_1|,|z_2|)\}^{n-1}}$$

where K is a constant, i.e.,

$$\left|\int_{z_1}^{z_2} f(z)\, dz - \left\{a_0(z_2-z_1)+a_1 \log\frac{z_2}{z_1} + \sum_{p=2}^{n}\frac{a_p}{(p-1)}\left(\frac{1}{z_1^{p-1}} - \frac{1}{z_2^{p-1}}\right)\right\}\right|$$

$$< \frac{K\varepsilon}{\{\min(|z_1|,|z_2|)\}^{n-1}}.$$

Chapter III

Lemmas

1. In this chapter, we develop a few lemmas some of which are analogous to results of Fox [9, pp. 418-424]. The results will be used in the next chapter.

1.1. *Lemma 1.* (The asymptotic expansion of the gamma function [29, p. 278]) *For large* $|s|$, $|\arg s| < \pi - \delta$, $\delta > 0$ *we have*

$$\Gamma(s+a) = F\cdot\exp\{(s+a-\tfrac{1}{2})\log s-s\}, \tag{3.1.1}$$

where

$$F = A+Bs^{-1}+Cs^{-2}+\cdots+Rs^{-r}+O(|s|^{-r-\frac{1}{2}}), \tag{3.1.2}$$

the constants $A, B, C, \cdots, R$ *being all independent of* s. *It is important to note that* $A \neq 0$; *in the present case* $A = (2\pi)^{\frac{1}{2}}$.

Any expression, such as F of (3.1.2), starting with a constant, not equal to zero, and continuing with negative powers of s until an algebraical order term is reached, will be referred to as a function of type F.

1.2. *Lemma 2.* (The first amalgamation lemma) *A group of finite number of functions of type* F *which are combined by multiplication and division can be amalgamated into one function of type* F.

The lemma is obvious for the case of multiplication. It is also obvious in the case of division since the constant term of the F

type function in the denominator does not vanish.

1.3. *Lemma 3*. (The second amalgamation lemma) *Let* $s = \sigma + i\tau$, σ *and* τ *real*. *Then for fixed* σ *and large positive or negative* τ

$$\frac{\prod^{n} \sin \pi(d_j - \delta_j(s-\tfrac{1}{2}))}{\prod^{p} \sin \pi(a_j - \alpha_j(s-\tfrac{1}{2}))} = \begin{matrix} \{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_1 + O(e^{-c|\tau|})\} \\ \{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_2 + O(e^{-c|\tau|})\} \end{matrix} \tag{3.1.3}$$

where $c = 2\pi \min(\delta_1, \cdots, \delta_n, \alpha_1, \cdots, \alpha_p)$, *the upper expression on the right side of* (3.1.3) *is to be used when* $n - p$ *is an odd integer and the lower expression when* $n - p$ *is an even integer*. *Also* $\lambda_1 = (2i)^{p-n+1}$ *and* $\lambda_2 = 2(2i)^{p-n}$, *both being real*.

Write $s = \sigma + i\tau$. For large positive τ, we have

$$\begin{aligned} \sin \pi(d_j - \delta_j(s-\tfrac{1}{2})) &= \frac{1}{2i}\{\exp(i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 - \exp(-2i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\} \\ &= \frac{1}{2i}\{\exp(i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 + O(e^{-c\tau})\} \\ &= \frac{1}{2i}\{\exp(i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 + O(e^{-c|\tau|})\}, \quad \tau > 0, \end{aligned} \tag{3.1.4}$$

and for large negative τ we have

$$\begin{aligned} \sin \pi(d_j - \delta_j(s-\tfrac{1}{2})) &= -\frac{1}{2i}\{\exp(-i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 - \exp(2i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\} \\ &= -\frac{1}{2i}\{\exp(-i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 + O(e^{c\tau})\} \\ &= -\frac{1}{2i}\{\exp(-i\pi(d_j - \delta_j(s-\tfrac{1}{2})))\}\{1 + O(e^{-c|\tau|})\}, \quad \tau < 0. \end{aligned} \tag{3.1.5}$$

There are corresponding formulae for $\sin \pi(a_j - \alpha_j(s-\tfrac{1}{2}))$.

First, consider the case for large positive τ. Writing

$$(3.1.6) \qquad A = \frac{\prod^{n} \sin \pi(d_j-\delta_j(s-\frac{1}{2}))}{\prod^{p} \sin \pi(a_j-\alpha_j(s-\frac{1}{2}))} ,$$

and using (3.1.4), we have

$$(3.1.7) \qquad A = \frac{\prod^{n}(\frac{1}{2i})\exp\{i\pi(d_j-\delta_j(s-\frac{1}{2}))\}}{\prod^{p}(\frac{1}{2i})\exp\{i\pi(a_j-\alpha_j(s-\frac{1}{2}))\}}\{1+O(e^{-c|\tau|})\}$$

$$= (\frac{1}{2i})^{n-p}\{\exp(i\pi(\sum^{n} d_j-\sum^{p} a_j-(s-\tfrac{1}{2})(\sum^{n}\delta_j-\sum^{p}\alpha_j)))\}\{1+O(e^{-c|\tau|})\}$$

$$= (\frac{1}{2i})^{n-p}\{\exp(i\pi(\tfrac{1}{2}L-\tfrac{1}{2}D(s-\tfrac{1}{2})))\}\{1+O(e^{-c|\tau|})\} ,$$

by (iii) and (iv) of the Theorem. When $n - p$ is an odd integer, (3.1.7) finally reduces to

$$A = (2i)^{p-n+1}\{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{1+O(e^{-c|\tau|})\}$$

$$= \{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_1+O(e^{-c|\tau|})\} ;$$

and when $n - p$ is an even integer, (3.1.7) becomes

$$A = 2(2i)^{p-n}\{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{1+O(e^{-c|\tau|})\}$$

$$= \{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_2+O(e^{-c|\tau|})\} .$$

Secondly, for large negative τ, by using (3.1.5)

$$(3.1.8) \qquad A = \frac{\prod^{n}(-\frac{1}{2i})\{\exp(-i\pi(d_j-\delta_j(s-\frac{1}{2})))\}}{\prod^{p}(-\frac{1}{2i})\{\exp(-i\pi(a_j-\alpha_j(s-\frac{1}{2})))\}}\{1+O(e^{-c|\tau|})\}$$

$$= \left(-\frac{1}{2i}\right)^{n-p}\{\exp(-i\pi(\sum^{n} d_j - \sum^{p} a_j - (s-\tfrac{1}{2})(\sum^{n}\delta_j - \sum^{p}\alpha_j)))\}\{1+O(e^{-c|\tau|})\}$$

$$= \left(-\frac{1}{2i}\right)^{n-p}\{\exp(-i\pi(\tfrac{1}{2}L-\tfrac{1}{2}D(s-\tfrac{1}{2})))\}\{1+O(e^{-c|\tau|})\} .$$

In case of $n - p$ is an odd integer, (3.1.8) reduces to

$$\begin{aligned} A &= (2i)^{p-n+1}\{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{1+O(e^{-c|\tau|})\} \\ &= \{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_1+O(e^{-c|\tau|})\} ; \end{aligned}$$

and when $n - p$ is an even integer, (3.1.8) simplifies to

$$\begin{aligned} A &= 2(2i)^{p-n}\{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{1+O(e^{-c|\tau|})\} \\ &= \{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_2+O(e^{-c|\tau|})\} . \end{aligned}$$

1.4. *Lemma 4*. (The asymptotic expansion for $Q^{(1)}(s)$ *of* (2.1.8)) *If* $s = \sigma+i\tau$, σ *is fixed and* τ *is large and either positive or negative then*

$$Q^{(1)}(s) = \begin{cases} s^{\frac{1}{2}-\frac{1}{2}D}\Gamma(Ds)\{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}F, & n-p \ \underline{\text{odd}}, \\ s^{\frac{1}{2}-\frac{1}{2}D}\Gamma(Ds)\{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}F, & n-p \ \underline{\text{even}}, \end{cases} \tag{3.1.9}$$

where F *is a function of type* F *defined in Lemma 1*.

Applying the identity (1.4.5) to each of the gamma functions on the right of (2.1.8) whose argument involves s with a negative coefficient, we get

$$Q^{(1)}(s) = \frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))\ \prod^{n}\Gamma(1-d_j+\delta_j(s-\frac{1}{2}))}{\prod^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))\ \prod^{p}\Gamma(1-a_j+\alpha_j(s-\frac{1}{2}))}\ \pi^{n-p}A\mu^{-sD}$$

where A is given by (3.1.6). So by Lemma 3,

$$(3.1.10)\quad Q^{(1)}(s)=\frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))\ \prod^{n}\Gamma(1-d_j+\delta_j(s-\frac{1}{2}))}{\prod^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))\ \prod^{p}\Gamma(1-a_j+\alpha_j(s-\frac{1}{2}))}\mu^{-sD}\{\sin\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}\{\lambda_1+0(e^{-c}$$

when $n-p$ is odd. By applying Lemma 1 and Lemma 2, we get

$$\frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))}{\prod^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))}=\frac{\prod^{m}\{\exp((c_j-\frac{1}{2}\gamma_j+\gamma_j s-\frac{1}{2})\log\gamma_j s-\gamma_j s)\}}{\prod^{q}\{\exp((b_j-\frac{1}{2}\beta_j+\beta_j s-\frac{1}{2})\log\beta_j s-\beta_j s)\}}F$$

$$=F\exp\{(\sum^{m}c_j-\sum^{q}b_j+(s-\tfrac{1}{2})(\sum^{m}\gamma_j-\sum^{q}\beta_j)-\tfrac{1}{2}(m-q))\log s-s(\sum^{m}\gamma_j-\sum^{q}\beta_j)\}$$
$$\times\exp\{\sum^{m}((c_j-\tfrac{1}{2}\gamma_j-\tfrac{1}{2})\log\gamma_j+\gamma_j s\log\gamma_j)+\sum^{q}((\tfrac{1}{2}-b_j+\tfrac{1}{2}\beta_j)\log\beta_j-\beta_j s\log\beta_j)\}$$

$$=F\{\exp((\tfrac{1}{2}L+\tfrac{1}{2}D(s-\tfrac{1}{2})-\tfrac{1}{2}(m-q))\log s-\tfrac{1}{2}Ds)\}\{\exp(s(\sum^{m}\gamma_j\log\gamma_j-\sum^{q}\beta_j\log\beta_j))\},$$

where F is a function of type F defined in Lemma 1, and similarly

$$\frac{\prod^{n}\Gamma(1-d_j+\delta_j(s-\frac{1}{2}))}{\prod^{p}\Gamma(1-a_j+\alpha_j(s-\frac{1}{2}))}$$

$$=F\{\exp((\tfrac{1}{2}(n-p)-\tfrac{1}{2}L+\tfrac{1}{2}D(s-\tfrac{1}{2}))\log s-\tfrac{1}{2}Ds)\}\{\exp(s(\sum^{n}\delta_j\log\delta_j-\sum^{p}\alpha_j\log\alpha_j))\}.$$

If F is of type F, Lemma 1, σ is finite and $|\tau|$ is large, then $\{\lambda_1+0(e^{-c|\tau|})\}F$ or $\{\lambda_2+0(e^{-c|\tau|})\}F$ amalgamates into a function of type F. Therefore in case $n-p$ is odd, (3.1.10) reduces to

$$Q^{(1)}(s)=F\{\exp(D(s-\tfrac{1}{2})\log s-Ds)\}(\mu D)^{sD}\mu^{-sD}\sin\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))$$

on using (2.1.6) of the Theorem. Hence

$$Q^{(1)}(s)=F\{\exp(Ds\log s-Ds-\tfrac{1}{2}D\log s)\}D^{sD}\sin\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))$$

$$= F\{\exp((Ds-\tfrac{1}{2})\ \log s-Ds)\}\{\exp((\tfrac{1}{2}-\tfrac{1}{2}D)\ \log s)\}D^{sD-\frac{1}{2}}D^{\frac{1}{2}}\ \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))$$

$$= F\ s^{\frac{1}{2}-\frac{1}{2}D}\{\exp((Ds-\tfrac{1}{2})\ \log s-Ds)\}\{\exp((Ds-\tfrac{1}{2})\ \log D)\}\ \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))$$

$$= F\{\exp((Ds-\tfrac{1}{2})\ \log Ds-Ds)\}s^{\frac{1}{2}-\frac{1}{2}D}\ \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))$$

$$= s^{\frac{1}{2}-\frac{1}{2}D}\Gamma(Ds)\{\sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}F$$

if $n - p$ is odd integer. Similarly

$$Q^{(1)}(s) = s^{\frac{1}{2}-\frac{1}{2}D}\Gamma(Ds)\{\cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}F$$

when $n - p$ is even integer. This completes the proof of this lemma.

1.5. Lemma 5: Let $s = \sigma+i\tau$, where σ is fixed and τ may be large and either positive or negative. Then we can choose $N + 1$ constants ν_j, $j = 0,1,\cdots, N$, which are independent of s such that

$$(3.1.11)\quad Q^{(1)}(s) - \begin{matrix}\sin\\ \\ \cos\end{matrix}\ \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\sum_{j=0}^{N} \nu_j\Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D) = 0(|s|^{D\sigma-N-\frac{1}{2}-\frac{1}{2}D}),$$

where on the left hand side we read sin or cos according as $n - p$ is odd or even.

To prove this we first note that from Lemma 1 we have

$$\Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D) = \{\exp((Ds-j-\tfrac{1}{2}D)\ \log Ds-Ds)\}F_j$$

$$= \{\exp((Ds-\tfrac{1}{2})\ \log Ds-Ds)\}\{\exp((\tfrac{1}{2}-\tfrac{1}{2}D-j)\ \log Ds)\}F_j$$

$$= s^{\frac{1}{2}-\frac{1}{2}D}\{\exp((Ds-\tfrac{1}{2})\ \log Ds-Ds)\}s^{-j}\ F_j\ ,$$

since any constant multiple of a function of type F is a function of type F. Denote the left hand side of (3.1.11) by $R(s)$, then by Lemma 4

$$(3.1.12)\quad R(s) = s^{\frac{1}{2}-\frac{1}{2}D}\{\exp((Ds-\tfrac{1}{2})\log Ds-Ds)\}\begin{matrix}\sin\\ \cos\end{matrix}\ \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\{F-\sum_{j=0}^{N}\nu_j s^{-j}F_j\}$$

where in using (3.1.9) we have replaced $\Gamma(Ds)$ by its asymptotic expansion (3.1.1). The terms F and F_j, $j = 0,1,\cdots, N$, are functions of type F defined in Lemma 1. Each starts with a constant not equal zero and continues through negative powers of s until an order term is reached. In the case of F the coefficients of the various negative powers of s depend upon a_j, α_j, $j = 1,2,\cdots, p$, b_j, β_j, $j = 1,2,\cdots, q$, c_j, γ_j, $j = 1,2,\cdots, m$, and d_j, δ_j, $j = 1,2,\cdots, n$, only and are independent of s. For F_j, $j = 0,1,\cdots, N$, these coefficients depend upon D and j only. These coefficients are therefore also independent of s.

Consider now the expression $\{F - \sum_{j=0}^{N}\nu_j s^{-j}\ F_j\}$. We start by assuming that each of the F type functions terminates with the same order term, i.e., $O(|s|^{-N-\frac{1}{2}})$. We can now choose the constants ν_j, $j = 0,1,\cdots, N$, successively so that the constant term and the coefficients of s^{-j}, $j = 0,1,\cdots, N$, all vanish. Evidently we have $N + 1$ equations for the ν_j, $j = 0,1,\cdots, N$. It is also evident that these ν_j computed in this manner, will depend upon a_j, α_j, $j = 1,2,\cdots, p$, b_j, β_j, $j = 1,2,\cdots, q$, c_j, γ_j, $j = 1,2,\cdots, m$, and d_j, δ_j, $j = 1,2,\cdots, n$ only and will be independent of s. The constants can be computed successively since on equating the first term of $\{F - \sum_{j=0}^{N}\nu_j s^{-j}\ F_j\}$ to zero we obtain an equation involving ν_0 only, on equating the coefficients of s^{-1} to zero we obtain an equation involving ν_0 and ν_1 and so on for higher powers of s^{-1}. So that we have, for large $|s|$,

$$(3.1.13) \qquad F - \sum_{j=0}^{N} \nu_j s^{-j} F_j = O(|s|^{-N-\frac{1}{2}}).$$

This is the method of computation for the constants ν_j, $j = 0,1,\cdots, N$, prescribed in the statement of the Theorem 5.

We now turn to the other factors of $R(s)$. On writing $s = \sigma+i\tau$, with σ fixed and large $|\tau|$, we claim that

$$(3.1.14) \qquad \{\exp((Ds-\tfrac{1}{2}) \log Ds-Ds)\} \begin{matrix}\sin\\ \\ \cos\end{matrix} \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2})) = O(|s|^{D\sigma-\frac{1}{2}}).$$

For

$$\begin{aligned}\exp((Ds-\tfrac{1}{2}) \log Ds-Ds) &= \{\exp((Ds-\tfrac{1}{2}) \log Ds)\}\exp(-Ds)\\ &= e^{-Ds}(Ds)^{Ds-\frac{1}{2}} = e^{-D(\sigma+i\tau)}(Ds)^{D(\sigma+i\tau)-\frac{1}{2}}\\ &= e^{-D\sigma-iD\tau}(Ds)^{D\sigma-\frac{1}{2}} \exp(iD\tau \log(D\sigma+iD\tau))\\ &= e^{-D\sigma}(Ds)^{D\sigma-\frac{1}{2}} \exp(iD\tau(\log D\sqrt{\sigma^2+\tau^2} -1)-D\tau \tan^{-1} \tfrac{\tau}{\sigma})\\ &= e^{-D\sigma}(Ds)^{D\sigma-\frac{1}{2}} \exp(iD\tau(\log D\sqrt{\sigma^2+\tau^2} -1)-D|\tau|(\tfrac{1}{2}\pi+O(\tfrac{1}{|\tau|}))\\ &= O(|s|^{D\sigma-\frac{1}{2}} \exp(-\tfrac{1}{2}\pi D|\tau|)) \;;\end{aligned}$$

and

$$(3.1.15) \quad \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2})) = \frac{1}{2i}\{\exp(\tfrac{1}{2}\pi i(L+\tfrac{1}{2}D-D\sigma)+\tfrac{1}{2}\pi D\tau)-\exp(-\tfrac{1}{2}\pi i(L+\tfrac{1}{2}D-D\sigma)-\tfrac{1}{2}\pi D\tau)\}$$

$$= O(e^{\frac{1}{2}\pi D|\tau|});$$

and

$$(3.1.15') \qquad \cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2})) = O(e^{\frac{1}{2}\pi D|\tau|}) .$$

Hence our claim (3.1.14) is established in view of (2.2.1). So by using (3.1.13) and (3.1.14), (3.1.12) reduces to

$$R(s) = s^{\frac{1}{2}-\frac{1}{2}D}\, O(|s|^{-N-\frac{1}{2}})(|s|^{D\sigma-\frac{1}{2}}) = O(|s|^{D\sigma-N-\frac{1}{2}-\frac{1}{2}D}),$$

again by virtue of (2.2.1). Thus the proof of this lemma is complete.

1.6. Let

$$(3.1.16)\quad I_1 = \frac{1}{2\pi i}\int \{Q^{(1)}(s) - \sum_{j=0}^{N} \nu_j \Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D) \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}(\tfrac{x}{\beta})^{-s}\, ds$$

when $n - p$ is odd and

$$(3.1.16')\quad I_2 = \frac{1}{2\pi i}\int \{Q^{(1)}(s) - \sum_{j=0}^{N} \nu_j \Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D) \cos \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}(\tfrac{x}{\beta})^{-s}\, ds$$

when $n - p$ is even. The path of integration of I_1 and I_2 is the straight line $\sigma = \tilde{\omega}$, $\tilde{\omega} > \frac{1}{2}+\tilde{\omega}_0$, $\tilde{\omega}_0$ being defined as in (viii) of the Theorem 5.

Lemma 6: *If constants* ν_j, $j = 0,1,\cdots, N$, *are chosen so that* (3.1.11) *is valid, and a positive integer* N *is chosen so that*

$$(3.1.17)\qquad -D(\tilde{\omega}-\tfrac{1}{2})+N+\tfrac{1}{2} > 1\ ,$$

then

$$(3.1.18)\qquad I_1 = O(x^{-\tilde{\omega}})\ ,\qquad I_2 = O(x^{-\tilde{\omega}}).$$

By Lemma 5, we conclude that the modulus of the integrand of either I_1 or I_2 is $O(|s|^{D(\tilde{\omega}-\frac{1}{2})-N-\frac{1}{2}})(x/\beta)^{-\tilde{\omega}}$. Hence, by (3.1.17), as a function of s either of these integrands is $0(|s|^{-1-\varepsilon})$ where $\varepsilon > 0$ and so both integrals are absolutely convergent. Consequently considered as a function of x, each integral must be $0(x^{-\tilde{\omega}})$. This completes the proof of this lemma.

1.7. The straight line $\sigma = \tilde{\omega}$ divides the circle whose center is the origin and whose radius is ρ into two arcs. Denoting the arc on the left of $\sigma = \tilde{\omega}$ by $\mathcal{L}$, let

$$(3.1.19)\quad J_1 = \frac{1}{2\pi i}\int_{\mathcal{L}}\{Q^{(1)}(s) - \sum_{j=0}^{N}\nu_j\Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D)\ \sin\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}(x/\beta)^{-s}\,ds$$

when $n - p$ is odd, and

$$(3.1.19')\quad J_2 = \frac{1}{2\pi i}\int_{\mathcal{L}}\{Q^{(1)}(s) - \sum_{j=0}^{N}\nu_j\Gamma(Ds-j+\tfrac{1}{2}-\tfrac{1}{2}D)\ \cos\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}(x/\beta)^{-s}\,ds$$

when $n - p$ is even.

<u>Lemma 7</u>: $\lim_{\rho\to\infty} J_1 = 0, \quad \lim_{\rho\to\infty} J_2 = 0.$

We assume that as $\rho \to \infty$ the arc $\mathcal{L}$ avoids crossing any poles of either of the integrands of J_1 and J_2. This is always possible because all these poles are simple and isolated.

Let (ρ, θ), $-\pi \le \theta \le \pi$, be the polar coordinates of any point on the arc $\mathcal{L}$. On writing $s = \rho e^{i\theta}$ in the asymptotic expansion of $\Gamma(s+a)$, (3.1.1), it follows that for $|\theta| < \pi-\delta$, $\delta > 0$, $|\theta| \neq \frac{\pi}{2}$

$$\Gamma(s+a) = \{\exp((s+a-\tfrac{1}{2})\log s - s)\}F$$

$$= \{\exp((\rho\cos\theta+i\rho\sin\theta+a-\tfrac{1}{2})(\log\rho+i\theta)-\rho\cos\theta-i\rho\sin\theta)\}F$$

$$= \exp\{(\rho\cos\theta + \mathrm{Re}(a) - \tfrac{1}{2})\log\rho - \theta(\rho\sin\theta + \mathrm{Im}(a)) - \rho\cos\theta\}$$
$$\times \exp\{i(\theta(\rho\cos\theta + \mathrm{Re}(a) - \tfrac{1}{2}) + (\rho\sin\theta + \mathrm{Im}(a))\log\rho - \rho\sin\theta)\}F\ .$$

Therefore the dominant factor for this case is

$$\exp(\rho\cos\theta\log\rho)\ .$$

The case $|\theta-\pi| < \delta < \pi/2$ can be settled easily as follows. By (1.4.5)

$$\Gamma(s+a) = \frac{\pi\csc\pi(s+a)}{\Gamma(1-s-a)} = \frac{\pi\csc\pi(s+a)}{\Gamma(1+u-a)}$$

where $u = -s = -\rho e^{i\theta} = \rho e^{i(\theta-\pi)} = \rho e^{i\phi}$ so that $|\phi| = |\theta-\pi| < \pi-\delta$. Then

$$\Gamma(s+a) = F\,\frac{1}{\exp((\rho e^{i\phi}-a+\frac{1}{2})(\log\rho+i\phi)-\rho e^{i\phi})} \times \frac{\pi}{\frac{1}{2i}\{\exp(i\pi(s+a))-\exp(-i\pi(s+a))\}}\ .$$

Therefore the dominant factor in $\Gamma(s+a)$ in this case is

$$\frac{1}{\exp(\rho\cos\phi\log\rho+\pi\rho|\sin\theta|)} \le \frac{1}{\exp(\rho\cos\phi\log\rho)}$$
$$= \exp(-\rho\cos\phi\log\rho) = \exp(\rho\cos\theta\log\rho)\ .$$

Thus the dominant factor in $\Gamma(s+a)$ is

$$\exp(\rho\cos\theta\log\rho) \qquad -\pi \le \theta \le \pi \quad \text{and } |\theta| \ne \pi/2. \tag{3.1.20}$$

On applying this result to the various terms of the integrands of J_1 and J_2, the dominant factor of $Q^{(1)}(s)$ can be found out as

$$\frac{\prod^{m}\exp(\gamma_j\rho\cos\theta\log(\gamma_j\rho))\prod^{p}\exp(\alpha_j\rho\cos(\theta-\pi)\log(\alpha_j\rho))}{\prod^{n}\exp(\delta_j\rho\cos(\theta-\pi)\log(\delta_j\rho))\prod^{q}\exp(\beta_j\rho\cos\theta\log(\beta_j\rho))} \tag{3.1.21}$$
$$= \exp\{D\rho(\log\rho+\log D\mu)\cos\theta\},$$

which behaves like $\exp(D\rho\log\rho\cos\theta)$ as $\rho\to\infty$, $\pi \ge |\theta| \ne \pi/2$, since

$\log D\mu$ is negligible compared to $\log\rho$ as $\rho\to\infty$; and the dominant factor of $\Gamma(Ds-j-\frac{1}{2}-\frac{1}{2}D)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\frac{1}{2}\pi(L-D(s-\frac{1}{2}))$ is $\exp(D\rho\cos\theta\log\rho)$, $|\theta|\neq\pi/2$. Note also that $\frac{1}{2}\pi$ is negligible in comparison with $\cos\theta\log D\rho$ when $\rho\to\infty$. Since $(x/\beta)^{-s}=\exp(-s\log(x/\beta))=O(\exp(-\rho\cos\theta\log(x/\beta)))$, the dominant factors of the integrands of J_1 and J_2 are

$$\begin{aligned}&(\exp(D\rho\cos\theta\log\rho))O(\exp(-\rho\cos\theta\log(x/\beta)))\\ &=O(\exp(\rho\cos\theta\,(D\log\rho-\log(x/\beta))))=O(\exp(D\rho\cos\theta\log\rho)).\quad |\theta|\neq\frac{\pi}{2}.\end{aligned}$$

We now break the path $\mathcal{L}$ into three parts

(i) $\mathcal{L}_1$, the upper arc of the circle $|s|=\rho$ which lies between $\sigma=\tilde{\omega}$ and $\sigma=\sigma_0$, where $\sigma_0<0$ and $D\sigma_0<-1$;

(ii) $\mathcal{L}_2$, the arc of the circle $|s|=\rho$ which lies to the left of $\sigma=\sigma_0$;

(iii) $\mathcal{L}_3$, the lower arc of the circle $|s|=\rho$ which lies between $\sigma=\tilde{\omega}$ and $\sigma=\sigma_0$.

Now

$$\begin{aligned}\left|\int_{\mathcal{L}_2}\right|&=\int_{\mathcal{L}_2}O(\exp(D\rho\cos\theta\log\rho))\,ds\leq\int_{\mathcal{L}_2}O(\exp D\sigma_0\log\rho)\,ds\\ &=O(\rho^{D\sigma_0}\int_{\mathcal{L}_2}ds)=O(\rho^{D\sigma_0+1})\to 0\quad\text{as}\quad\rho\to\infty.\end{aligned}$$

On the path $\mathcal{L}_1$, the modulus of the integrand of the integral J_1 is

$$O(\rho^{D(\sigma-\frac{1}{2})-N-\frac{1}{2}})(x/\beta)^{-\rho\cos\theta},$$

where $\sigma_0\leq\sigma\leq\tilde{\omega}$, by chosing the constants ν_j, $j=0,1,\cdots,N$, in such the way that Lemma 5 is valid. Hence the order of the integrand of J_1 is

$$O(\rho^{D(\tilde{\omega}-\frac{1}{2})-N-\frac{1}{2}})(x/\beta)^{-\rho\cos\theta} ,$$

and this cannot exceed

$$O(\rho^{-1})(x/\beta)^{-\rho\cos\theta}$$

in view of the inequality (3.1.17). Since

$$(x/\beta)^{-\rho\cos\theta} \leq \max\{(x/\beta)^{-\sigma_0}, (x/\beta)^{\tilde{\omega}}\} = O(1) .$$

We have

$$\left| \int_{\mathcal{L}_1} \right| = \int_{\mathcal{L}_1} O(\rho^{-1})O(1)\, ds = O(\rho^{-1}) \int_{\mathcal{L}_1} ds = O(\rho^{-1}) \to 0 \quad \text{as} \quad \rho \to \infty .$$

Similarly $|\int_{\mathcal{L}_3}| \to 0$ as $\rho \to \infty$. i.e.,

$$\lim_{\rho\to\infty} J_1 = 0.$$

Similarly we can prove $\lim_{\rho\to\infty} J_2 = 0$. This completes the proof of this lemma.

1.8. Lemma 8: The contour L of the integral $H^{(1)}(x)$, (2.1.1), may be closed by a semicircle whose certre is the origin and radius ρ on the left of the contour L as $\rho \to \infty$.

Let the semicircle $\mathcal{L}$ be made up of

(i) $\mathcal{L}_1$, the upper arc between $\sigma = \sigma'$, where $0 < \sigma' < \frac{1}{2}$ and $\sigma = \sigma_0$, where $\sigma_0 < 0$ and $D\sigma_0 < -1$;

(ii) $\mathcal{L}_2$, the arc which lies to the left of $\sigma = \sigma_0$;

(iii) $\mathcal{L}_3$, the lower arc which lies between $\sigma = \sigma_0$ and $\sigma = \sigma'$.

Let $s = \rho e^{i\theta}$. On paths $\mathcal{L}_1$ and $\mathcal{L}_3$, $|\theta| \le \pi-\delta$, $\delta > 0$, and for large $|s|$, by Lemma 1, page 27,

$$\Gamma(c_j+\gamma_j(s-\tfrac{1}{2})) = F \exp((c_j+\gamma_j(s-\tfrac{1}{2})-\tfrac{1}{2}) \log \gamma_j s-\gamma_j s),$$

where F is a function of type F, defined in Lemma 1.

$$|\Gamma(c_j+\gamma_j(s-\tfrac{1}{2}))| = F \exp((c_j-\tfrac{1}{2}-\tfrac{1}{2}\gamma_j+\gamma_j\rho \cos \theta)\log \rho-\gamma_j\rho\theta \sin \theta-\gamma_j\rho(1-\log \gamma_j)\cos \theta).$$

Since $\gamma_j\rho(1-\log \gamma_j)\cos \theta$ is negligible compared to $(\gamma_j\rho \cos \theta)\log \rho$ as $\rho \to \infty$,

$$|\Gamma(c_j+\gamma_j(s-\tfrac{1}{2}))| = F \exp((c_j-\tfrac{1}{2}-\tfrac{1}{2}\gamma_j+\gamma_j\rho \cos \theta) \log \rho-\gamma_j\rho\theta \sin \theta).$$

Applying relation (1.4.5) to each of the gamma functions whose argument involves s with a negative coefficient, Lemma 3 and (3.1.15) we get

$$\left| \frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\tfrac{1}{2}))\ \prod^{p}\Gamma(a_j-\alpha_j(s-\tfrac{1}{2}))}{\prod^{n}\Gamma(d_j-\delta_j(s-\tfrac{1}{2}))\ \prod^{q}\Gamma(b_j+\beta_j(s-\tfrac{1}{2}))}\, x^{-s} \right|$$

$$= F \exp((-\tfrac{1}{2}D+D\rho \cos \theta) \log \rho-\rho \cos \theta \log x-D\rho\theta \sin \theta+\tfrac{1}{2}\pi D\rho|\sin \theta|).$$

Let (ρ, θ') denote the polar coordinates of the point of intersection of $\mathcal{L}_1$ and the straight line $\sigma = \sigma'$. On the path $\mathcal{L}_1$,

$$-D\rho\theta \sin \theta+\tfrac{1}{2}\pi D\rho|\sin \theta| = D\rho(\tfrac{1}{2}\pi-\theta) \sin \theta \le D\rho(\tfrac{1}{2}\pi-\theta') \le D\rho\tfrac{1}{2}\pi \sin(\tfrac{1}{2}\pi-\theta')$$

$$= \tfrac{1}{2}\pi D\rho \cos \theta' = \tfrac{1}{2}\pi D\sigma',$$

by using relation [21, p. 31]

$$\theta \le \tfrac{1}{2}\pi \sin \theta, \quad \text{for } 0 \le \theta \le \tfrac{1}{2}\pi,$$

and

$$-\tfrac{1}{2}D+D\rho\cos\theta \le -\tfrac{1}{2}D+D\sigma' = D(\sigma'-\tfrac{1}{2}) < 0 .$$

Let

$$a = \max_{\sigma_0\le\sigma\le\sigma'} \exp(-\sigma\log x) ,$$

$$\left|\int_{\mathcal{L}_1}\right| = O\left(\int_{\mathcal{L}_1} \exp(D(\sigma'-\tfrac{1}{2})\log\rho-\sigma\log x+\tfrac{1}{2}\pi D\sigma')\,ds\right)$$

$$= O\left\{\exp(D(\sigma'-\tfrac{1}{2})\log\rho)\int_{\mathcal{L}_1} ds\right\}$$

$$= O(\rho^{D(\sigma'-\frac{1}{2})}) \to 0 \qquad \text{as } \rho\to\infty .$$

Similarly

$$\left|\int_{\mathcal{L}_3}\right| = 0 .$$

Let (ρ, θ'') denote the polar coordinates of the point of intersection of $\mathcal{L}_1$ and the straight line $\sigma = \sigma_0$. On the path $\mathcal{L}_2$, by relation (3.1.21)

$$\left|\int_{\mathcal{L}_2}\right| = O\left\{\int_{\mathcal{L}_2} \exp(D\rho\cos\theta\log\rho-\rho\cos\theta\log x)\,ds\right\}$$

$$= O(\exp(D\rho\cos\theta''\log\rho))\,O(\rho)$$

$$= O(\exp((D\rho\cos\theta''+1)\log\rho)) \to 0 \qquad \text{as } \rho\to\infty ,$$

since on the path $\mathcal{L}_2$, $D\rho\cos\theta''+1 = D\sigma_0+1 < 0$. Thus

$$\left|\int_{\mathcal{L}}\right| = 0 \qquad \text{as } \rho\to\infty .$$

1.9. Lemma 9: For any θ such that $|\theta|$ is finite,

$$\sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!}\sin(\tfrac{1}{2}\pi(\phi+k)) = \sin(\tfrac{1}{2}\pi\phi-\theta)$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!}\cos(\tfrac{1}{2}\pi(\phi+k)) = \cos(\tfrac{1}{2}\pi\phi-\theta) .$$

Let

$$C = \sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!}\cos(\tfrac{1}{2}\pi(\phi+k)) ,$$

$$S = \sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!}\sin(\tfrac{1}{2}\pi(\phi+k)) .$$

Since the series $\sum_{k=0}^{\infty}\theta^k/k!$ converges absolutely for all finite $|\theta|$, so the series $\sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!}\begin{matrix}\sin\\ \cos\end{matrix}(\tfrac{1}{2}\pi(\phi+k))$ are convergent. Therefore

$$\begin{aligned} C+iS &= \sum_{k=0}^{\infty}(-1)^k \frac{\theta^k}{k!} e^{\frac{1}{2}\pi(\phi+k)i} \\ &= e^{\frac{1}{2}\pi\phi i}\exp(-\theta i) \\ &= \exp(i(\tfrac{1}{2}\pi\phi-\theta)) \\ &= \cos(\tfrac{1}{2}\pi\phi-\theta)+i\sin(\tfrac{1}{2}\pi\phi-\theta). \end{aligned}$$

Equating real and imaginary parts from the two sides of this equation we get the results.

Chapter IV

Proof of the Main Theorem

1. Proof of Theorem 5.

We can now find [9, pp. 426-428] the asymptotic expansion of $H^{(1)}(x)$ and $H^{(2)}(x)$ with the prescribed error term $O(x^{-\tilde{\omega}})$. We shall assume that $\tilde{\omega} > \frac{1}{2}+\tilde{\omega}_0$. The proof is based on two integrals I_1 and I_2, page 35, one dealing with the case when $n-p$ is odd and the other when $n-p$ is even. The proofs are exactly the same for both cases and so we shall only discuss the case when $n-p$ is odd. The integral we have to investigate is then

$$(4.1.1)\quad I_1 = \frac{1}{2\pi i}\int_{\sigma=\tilde{\omega}}\Big\{Q^{(1)}(s) - \sum_{j=0}^{N}\nu_j\Gamma\Big(Ds-j+\frac{1-D}{2}\Big)\sin\tfrac{1}{2}\pi\big(L-D(s-\tfrac{1}{2})\big)\Big\}(x/\beta)^{-s}\,ds\ ,$$

where $Q^{(1)}(s)$ is defined in (2.1.8), page 21.

We assume temporarily that the path of integration does not cross any pole of the integrand of I_1. Since $\tilde{\omega} > \frac{1}{2}+\tilde{\omega}_0$ this means that $\tilde{\omega} \neq \frac{1}{2}+(\mathrm{Re}(a_j)+k)/\alpha_j$, $j = 1,2,\cdots,p$, $k = 0,1,2,\cdots$, and also $\tilde{\omega} \neq \{j-k-\frac{1}{2}(1-D)\}/D$, $j = 0,1,\cdots, N$, $k = 0,1,2,\cdots$. We shall prove at the end of this section that this restriction is unessential.

Given $\tilde{\omega}$ we now choose N to be the greatest integer less than

$\{D(\tilde{\omega}-\frac{1}{2})+3/2\}$, so that we have

$$1 < -D(\tilde{\omega}-\tfrac{1}{2})+N+\tfrac{1}{2} < 2 . \tag{4.1.2}$$

The left hand inequality of (4.1.2) is the same as the inequality (3.1.17), Lemma 6, and therefore the constants ν_j, $j = 0,1,\cdots, N$, can be chosen in such a way that (3.1.11) is valid. Hence according to Lemma 6, page 35, we have $I_1 = 0(x^{-\tilde{\omega}})$.

From Lemma 7, page 36, we know that the straight line path $\sigma = \tilde{\omega}$ of (4.1.1) can be completed to a closed contour by means of an arc of the circle $|s| = \rho$ on the left of $\sigma = \tilde{\omega}$, whose radius tends to infinity. Hence I_1 is also equal to the sum of all the residues of the integrand of (4.1.1) lying to the left of the line $\sigma = \tilde{\omega}$. Hence the sum of all residues of (4.1.1) due to the poles of the integrand on the left of $\sigma = \tilde{\omega}$ is $0(x^{-\tilde{\omega}})$.

We now proceed to compute all these residues. It will be convenient to divide these residues into three groups.

Group 1 contains all the residues of $Q^{(1)}(s)(x/\beta)^{-s}$ which arise from the poles of $\Gamma(c_j+\gamma_j(s-\frac{1}{2}))$, $j = 1,2,\cdots, m$. All these poles are to the left of $\sigma = \tilde{\omega} > \frac{1}{2}+\tilde{\omega}_0$ by condition (vii) of the Theorem 5. Evidently these residues are the same as those of the integral in (2.1.7) which defines $H^{(1)}(x)$. Hence the sum of the residues in this group is $H^{(1)}(x)$.

Group 2 contains all the residues of $Q^{(1)}(s)(x/\beta)^{-s}$ which arise from those poles of $\Gamma(a_j-\alpha_j(s-\frac{1}{2}))$, $j = 1,2,\cdots, p$, lying to the left of $\sigma = \tilde{\omega}$. According to condition (vi) of the Theorem 5 all these poles lie to the right of the line $\sigma = \frac{1}{2}$ so that all the poles of this group lie between $\sigma = \frac{1}{2}$ and $\sigma = \tilde{\omega}$.

Let M_j denote the greatest integer less than $\{\alpha_j(\tilde{\omega}-\frac{1}{2})-\mathrm{Re}(a_j)\}$, $j=1,2,\cdots,p$, then the poles of this group are at $s=\frac{1}{2}+(a_j+k)/\alpha_j$, where $k=0,1,\cdots,M_j$ and $j=1,2,\cdots,p$. The sum of all residues of this group is then

$$\sum_{j=1}^{p}\sum_{k=0}^{M_j}\lim_{s\to\frac{1}{2}+(a_j+k)/\alpha_j}\{s-\tfrac{1}{2}-(a_j+k)/\alpha_j\}\frac{\prod^{m}\Gamma(c_j+\gamma_j(s-\frac{1}{2}))\prod^{p}\Gamma(a_j-\alpha_j(s-\frac{1}{2}))}{\prod^{n}\Gamma(d_j-\delta_j(s-\frac{1}{2}))\prod^{q}\Gamma(b_j+\beta_j(s-\frac{1}{2}))}\mu^{-sD}(x/\beta)^{-s}$$

$$=-\sum^{p}x^{-\frac{1}{2}-a_j/\alpha_j}\{A_j+B_jx^{-1/\alpha_j}+C_jx^{-2/\alpha_j}+\cdots+U_jx^{-M_j/\alpha_j}\}, \tag{4.1.3}$$

where A_j, B_j, $C_j,\cdots$, U_j are respectively the residues of $-Q^{(1)}(s)\beta^s$ at the poles $s=\frac{1}{2}+(a_j+k)/\alpha_j$, $k=0,1,\cdots,M_j$. Since $Q^{(1)}(s)$ is a ratio of products of gamma functions, these residues can easily be obtained. The expression (4.1.3) gives us all the algebraic terms of the asymptotic expansion of $H^{(1)}(x)$ in the Theorem 5.

Group 3 contains all the residues of

$$-\{\sum_{j=0}^{N}\nu_j\Gamma(Ds-j+\tfrac{1}{2}(1-D))\sin\tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))\}(x/\beta)^{-s} \tag{4.1.4}$$

arising from the poles of this expression which lie to the left of the line $\sigma=\tilde{\omega}$. All the poles of (4.1.4) are simple and isolated and occur when $s=(j-\frac{1}{2}(1-D)-k)/D$, where $j=0,1,\cdots,N$ and $k=0,1,2,\cdots$. The pole at the extreme right to this set of points occurs when $j=N$ and $k=0$, that is, at $s=(N-\frac{1}{2}(1-D))/D$. But by the left hand inequality of (4.1.2) we know that

$$s=(N-\tfrac{1}{2}(1-D))/D>\tilde{\omega}\ . \tag{4.1.5}$$

Hence this pole is outside the contour. The pole immediately on the left of (4.1.5) occur when $j=N$ and $k=1$ or when $j.=N-1$

and $k = 0$. This pole is therefore at $s = (N-\frac{1}{2}(1-D)-1)/D < \tilde{\omega}$ by the right hand inequality of (4.1.2). Consequently, of all the poles of (4.1.4) one is to the right of the line $\sigma = \tilde{\omega}$ and so is outside our contour while all the others are inside the contour.

The residue due to the pole (4.1.5) which is outside the contour, is a constant multiple of x^{-s}, where $\mathrm{Re}(s) > \tilde{\omega}$. Hence this residue is not larger than $O(x^{-\tilde{\omega}})$, so that if this residue is added to the others we shall only include a term which is of an order not greater than the error term we were allowed in the Theorem 5. Thus no harm is done even if we do not ignore the contribution of the pole (4.1.5) to the residue of this group.

We now compute these residues as follows treating all the poles of (4.1.4) as lying to the left of $\sigma = \tilde{\omega}$.

Let R_{jk} denote the residues at $s = (j-\frac{1}{2}(1-D)-k)/D$, $j = 0,1,\cdots, N$; $k = 0,1,2,\cdots$, then

$$R_{jk} = \lim_{s\to(j-\frac{1}{2}(1-D)-k)/D} \{s-(j-\tfrac{1-D}{2}-k)/D\}\{-\nu_j\Gamma(Ds-j+\tfrac{1-D}{2})\} \sin \tfrac{1}{2}\pi(L-D(s-\tfrac{1}{2}))(x/\beta)^{-s}.$$

By (1.4.5), we have

$$\Gamma(Ds-j+\tfrac{1}{2}(1-D)) = \frac{\pi}{\sin \pi(Ds-j+\frac{1}{2}(1-D))\Gamma(\frac{1}{2}(1+D)+j-Ds)}$$

$$= \frac{(-1)^k\ \pi}{\sin \pi(Ds-j+k+\frac{1}{2}(1-D))\Gamma(\frac{1}{2}(1+D)+j-Ds)}$$

Hence

$$R_{jk} = \lim_{s\to(j-\frac{1}{2}(1-D)-k)/D} \{s-(j-\tfrac{1-D}{2}-k)/D\}\{-\frac{\nu_j(-1)^k\pi \sin \frac{1}{2}\pi(L-D(s-\frac{1}{2}))}{\sin(\pi(Ds-j+k+\frac{1-D}{2}))\Gamma(\frac{1+D}{2}+j-Ds)}\}(x/\beta)^{-s}$$

$$= -(-1)^k \frac{\nu_j}{D\Gamma(1+k)}\{\sin(\tfrac{1}{2}\pi(L-j+\tfrac{1}{2})+\tfrac{1}{2}k\pi)\}(x/\beta)^{-\{j-k-\frac{1}{2}(1-D)\}/D}$$

$$= -(x/\beta)^{-\{j-\frac{1}{2}(1-D)\}/D}\,\frac{\nu_j}{D}\,(-1)^k\,\frac{1}{k!}\,(x/\beta)^{k/D}\,\sin\tfrac{1}{2}\pi(L-j+\tfrac{1}{2}+k)\ .$$

Let $R_j = \sum_{k=0}^{\infty} R_{jk}$, $j = 0,1,\cdots, N$, then

$$R_j = -(x/\beta)^{-\{j-\frac{1}{2}(1-D)\}/D}\,\frac{\nu_j}{D}\sum_{k=0}^{\infty}(-1)^k\,\frac{1}{k!}\,(x/\beta)^{k/D}\,\sin\tfrac{1}{2}\pi(L-j+\tfrac{1}{2}+k)$$

$$= -(x/\beta)^{-\{j-\frac{1}{2}(1-D)\}/D}\,\frac{\nu_j}{D}\,\sin\{\tfrac{1}{2}\pi(L-j+\tfrac{1}{2})-(x/\beta)^{1/D}\}\ ,$$

by Lemma 9. So the residues due to this group is

$$\sum_{j=0}^{N} R_j = -\frac{1}{D}(x/\beta)^{\frac{1-D}{2D}}\sum_{j=0}^{N}\nu_j(x/\beta)^{-j/D}\,\sin\{\tfrac{1}{2}\pi(L-j+\tfrac{1}{2})-(x/\beta)^{1/D}\}\ .$$

These are all the trigonometrical terms of the asymptotic expansion of $H^{(1)}(x)$.

The sum of the residues in these three groups is, by Cauchy theorem, equal to the value of the integral of (4.1.1), and this, in turn, we know to be $O(x^{-\tilde{\omega}})$ by Lemma 6, page 35. This establishes the asymptotic expansion of $H^{(1)}(x)$ as given in the Theorem 5 for the case when $n - p$ is an odd integer. The case when $n - p$ is even integer is dealt with in the same way by considering I_2 of (3.1.16'), page 35, instead of I_1 of (3.1.16).

Suppose now that $\tilde{\omega}$ is prescribed in such a way that the path of integration of I_1 does pass through a pole, P say, of the integrand. In this case we must insert a small semicircular indentation in the line $\sigma = \tilde{\omega}$, with P as centre, and drawn so that P is either inside or outside the modified contour. Then the residue is a constant multiple of x^{-s}, where $\mathrm{Re}(s) = \tilde{\omega}$. Hence this residue is $O(x^{-\tilde{\omega}})$. Since there are at most p number of poles of this kind in group 2 and at most $N + 1$ number of poles of this kind in group 3 and hence is equal to $O(x^{-\tilde{\omega}})$. Consequently, the asymptotic expansions still remain true if $\tilde{\omega}$ takes either any of the values $\{\frac{1}{2}+(\mathrm{Re}(a_j)+k)/\alpha_j\}$, $j = 1,2,\cdots,p$, $k = 0,1,2,\cdots$, or any of $\{j-k-\frac{1}{2}(1-D)\}/D$, $j = 0,1,\cdots, N$, $k = 0,1,2,\cdots$. The restriction imposed in the beginning of the proof on the straight line path of integration is therefore unessential. This completes the proof of Theorem 5 for $H^{(1)}(x)$. The case of $H^{(2)}(x)$, as pointed out earlier, is similar.

2. Remark.

The classical Fourier kernels such as $\sin x$, $\cos x$ and $x^{\frac{1}{2}}J_\nu(x)$ all behave in a very similar manner as $x \to \infty$. When x is complex they all tend to infinity with exponential rapidity and when x is real they all oscillate finitely as $x \to \infty$ just like cosine x. It is in fact the behaviour of a function on the real axis which decides whether or not it is a Fourier kernel.

If we look at equations (2.1.3) and (2.1.4), then, by the conditions (ii) and (vi) of the Theorem 5, the algebraic terms in the asymptotic expansions of $H^{(1)}(x)$ and $H^{(2)}(x)$ tend to zero as $x \to \infty$. But the trigonometric terms contain a factor $x^{(1-D)/2D}$. Hence, when $D < 1$, $H^{(1)}(x)$ and $H^{(2)}(x)$ cannot oscillate finitely, as $x \to \infty$. However, if we perform a change of variables

$$x = X^D , \qquad y = Y^D , \qquad u = (U\mu)^D$$

the equation (1.7.6) of Theorem 4, page 17, takes the form

$$\int_0^\infty K^{(1)}(XU)\ dU \int_0^\infty K^{(2)}(YU)\ f^*(Y)\ dY = \tfrac{1}{2}\{f^*(X+0)+f^*(X-0)\} ,$$

where

$$K^{(1)}(X) = H^{(1)}\{(X\mu)^D\}(X\mu)^{\frac{1}{2}(D-1)}\ \mu^{\frac{1}{2}}D,$$

$$K^{(2)}(X) = H^{(2)}\{(X\mu)^D\}(X\mu)^{\frac{1}{2}(D-1)}\ \mu^{\frac{1}{2}}D ,$$

$$f^*(Y) = Y^{\frac{1}{2}(D-1)}\ f(Y^D) .$$

Our main theorem clearly show that the kernels $K^{(1)}(X)$ and $K^{(2)}(X)$, contain algebraic terms which tend to zero as $X \to \infty$, but that the trigonometric terms are of the type

$$\mu^{D/2} \sum_{j=0}^{N} \nu_j X^{-j}\{\sin \tfrac{1}{2}\pi(L+\tfrac{1}{2}-j)-X\} . \tag{4.2.1}$$

Evidently, the terms in (4.2.1) for $j \geq 1$ all tend to zero as $X \to \infty$ but the term corresponding to $j = 0$ oscillates finitely as $X \to \infty$. Hence the kernels $K^{(1)}(X)$ and $K^{(2)}(X)$ behave much like the Fourier kernels of classical theory and so much of this theory can be applied to them.

3. Particular Cases.

The importance of $H^{(1)}(x)$ and $H^{(2)}(x)$ as Fourier kernels is due to their very general yet simple form from which many known as well as new kernels can be deduced as special cases. Let us consider a few examples to illustrate this.

(a)

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}(s-\frac{1}{2}))\Gamma(-\frac{1}{2}\nu-\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds \equiv 2x^{\frac{1}{2}}\, \mathbf{H}_{\nu}(2x) ,$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}(s-\frac{1}{2}))\Gamma(-\frac{1}{2}\nu+\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds \equiv 2x^{\frac{1}{2}}\, Y_{\nu}(2x) ,$$

where $-\frac{1}{2} \leq \nu \leq 0$, $Y_{\nu}(x)$ denotes the Bessel function of the second kind and $\mathbf{H}_{\nu}(x)$ the Struve's function. These unsymmetrical Fourier kernels are equivalent to the kernels which were studied by Titchmarsh [24, p. xxxiv] by the fact that, if $K(x)$ and $H(x)$ are a pair of Fourier kernels, so are the pair $a^{\frac{1}{2}}K(ax)$ and $a^{\frac{1}{2}}H(ax)$.

(b) The functions

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(1/8 +(s-\frac{1}{2})/4)\Gamma(7/8 +(s-\frac{1}{2})/4)}{\Gamma(3/8 -(s-\frac{1}{2})/4)\Gamma(5/8 -(s-\frac{1}{2})/4)}\, x^{-s}\, ds \equiv \frac{2}{\sqrt{\pi}}(\cos 4x+\sin 4x+e^{-4x}),$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(3/8 +(s-\frac{1}{2})/4)\Gamma(5/8 +(s-\frac{1}{2})/4)}{\Gamma(1/8 -(s-\frac{1}{2})/4)\Gamma(7/8 -(s-\frac{1}{2})/4)}\, x^{-s}\, ds \equiv \frac{2}{\sqrt{\pi}}(\cos 4x+\sin 4x-e^{-4x}),$$

are unsymmetrical Fourier kernels which are equivalent to the kernels obtained by Guinand [11, p. 192] by the same reasoning as given at the end of case (a)

(c)

$$H^{(1)}(x) = H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}\mu_1+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\mu_n+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\mu_1+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\mu_n+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))}$$

$$\times\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\nu_m+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\nu_m+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$m < n,\ \mu_j \geq -\tfrac{1}{2},\ j = 1,2,\cdots, n;$$

$$\equiv 2^{\frac{1}{2}(n-m)}\, \tilde{\omega}^{\nu_1,\cdots,\nu_m}_{\mu_1,\cdots,\mu_n}(2^{n-m}x), \qquad \nu_j \geq -1,\ j = 1,2,\cdots, m,$$

which is equivalent to the kernel given by Mitra [20, p. 700] .

When $m = 0$, we have

$$H^{(1)}(x) = H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}\mu_1+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\mu_n+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\mu_1+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))\cdots\Gamma(\frac{1}{2}\mu_n+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2^{n/2}\, \tilde{\omega}_{\mu_1,\cdots,\,\mu_n}(2^n x), \qquad \mu_j \geq -\tfrac{1}{2},\ j = 1,2,\cdots, n.$$

This is equivalent to the kernel studied by Bhatnagar [1, p. 109] .

When $n = 2$, we have

$$H^{(1)}(x) = H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds \equiv 2\tilde{\omega}_{\mu,\nu}(4x),$$

where $\mu,\nu \geq -\frac{1}{2}$. This symmetrical Fourier kernel is equivalent to the kernel which was studied by Watson [26, p. 308] .

When $n = 1$, this reduces to

$$H^{(1)}(x) = H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds \equiv 2x^{\frac{1}{2}}\, J_\nu(2x), \quad \nu \geq -\tfrac{1}{2}$$

which is equivalent to the kernel of Hankel.

(d)

$$H^{(1)}(x) = H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(1-\frac{\nu}{2k}-\frac{s-\frac{1}{2}}{2k})}{\Gamma(\frac{1}{2}+\frac{(2k-1)\nu}{2k}-\frac{(s-\frac{1}{2})}{2k})}\prod_{n=0}^{k-1}\frac{\Gamma(\frac{1}{2}+\frac{n+\nu}{2k}+\frac{s-\frac{1}{2}}{2k})}{\Gamma(1-\frac{n+\nu}{2k}-\frac{s-\frac{1}{2}}{2k})}\, x^{-s}\, ds$$

$$\equiv 2kx^{\frac{1}{2}}\, J_{\nu,k}(2kx) \quad ,$$

where $J_{\nu,k}(x)$ is defined by Everitt [7, p. 271] and k being a positive integer such that $k = 1$, $2 \geq \nu \geq -\frac{1}{2}$ or $k > 1$, $\nu = 0$ and $\nu = \frac{1}{2}$ only. The functions $J_{\nu,k}(x)$ satisfy certain differential equations of even order, greater than two, and have properties similar to the Bessel functions. When $k = 1$, $J_{\nu,k}(x) = J_{\nu}(x)$.

(e) In the following examples, the coefficients of s are not all equal.

(i)

$$\frac{1}{2\pi i}\int \frac{\Gamma(1/6\ +(s-\frac{1}{2})/3)\Gamma(\frac{1}{2}+(s-\frac{1}{2})/3)\Gamma(\frac{1}{4}-(s-\frac{1}{2})/6)}{\Gamma(1/6\ -(s-\frac{1}{2})/3)\Gamma(\frac{1}{2}-(s-\frac{1}{2})/3)\Gamma(\frac{1}{4}+(s-\frac{1}{2})/6)}\, x^{-s}\, ds$$

$$\equiv 2^{1/3}(3/\pi)^{\frac{1}{2}}\left\{-e^{-(3^{\frac{3}{2}}/2^{\frac{4}{3}})x}\sin\frac{3x}{2^{\frac{4}{3}}} + \cos\left[\frac{3x}{2^{\frac{1}{3}}} - \frac{\pi}{6}\right]\right\} .$$

(ii)

$$\frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}+(s-\frac{1}{2}))\Gamma(\frac{1}{12}\ -(s-\frac{1}{2})/6)\Gamma(\frac{1}{4}-(s-\frac{1}{2})/6)\Gamma(\frac{5}{12}\ -(s-\frac{1}{2})/6)}{\Gamma(\frac{1}{2}-(s-\frac{1}{2}))\Gamma(\frac{1}{12}\ +(s-\frac{1}{2})/6)\Gamma(\frac{1}{4}+(s-\frac{1}{2})/6)\Gamma(\frac{5}{12}\ +(s-\frac{1}{2})/6)}\, x^{-s}\, ds$$

$$\equiv (3\pi)^{-\frac{1}{2}}\left\{2e^{-\frac{\sqrt{3}}{12}x}\sin\left(\frac{x}{12}+\frac{\pi}{6}\right) - \cos\frac{x}{6}\right\}.$$

(iii)

$$\frac{1}{2\pi i}\int \frac{\Gamma(\tfrac{1}{2}+(s-\tfrac{1}{2}))\Gamma(\tfrac{7}{12}-(s-\tfrac{1}{2})/6)\Gamma(\tfrac{3}{4}-(s-\tfrac{1}{2})/6)\Gamma(\tfrac{11}{12}-(s-\tfrac{1}{2})/6)}{\Gamma(\tfrac{1}{2}-(s-\tfrac{1}{2}))\Gamma(\tfrac{7}{12}+(s-\tfrac{1}{2})/6)\Gamma(\tfrac{3}{4}+(s-\tfrac{1}{2})/6)\Gamma(\tfrac{11}{12}+(s-\tfrac{1}{2})/6)}\, x^{-s}\, ds$$

$$\equiv (3\pi)^{-\frac{1}{2}}\left\{ 2e^{-\frac{\sqrt{3}}{12}x}\cos\left(\frac{x}{12}+\frac{\pi}{6}\right) - \sin\frac{x}{6}\right\} .$$

These symmetrical kernels are equivalent to the symmetrical Fourier kernels $(2/\pi)^{\frac{1}{2}}\{-e^{-\frac{1}{2}x\sqrt{3}}\sin\frac{x}{2}+\cos(x-\frac{\pi}{6})\}$, $(2/\pi)^{\frac{1}{2}}\{2e^{-\frac{1}{2}x\sqrt{3}}\sin(\frac{x}{2}+\frac{\pi}{6})-\cos x\}$, $(2/\pi)^{\frac{1}{2}}\{2e^{-\frac{1}{2}x\sqrt{3}}\cos(\frac{x}{2}+\frac{\pi}{6})-\sin x\}$ respectively, which were given by Guinand [11, p. 193] .

(f) Let $\phi = \alpha_1-\rho_1-\rho_2+\frac{1}{2}$ and let ${}_p\mathcal{F}_q$ denote the hypergeometric function defined by Fox [8, p. 401] . The functions†

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{4}-\frac{1}{2}\phi+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\alpha_1-\frac{1}{4}+\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\rho_1-\frac{1}{4}+\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\rho_2-\frac{1}{4}+\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds \equiv 2x^{-\phi}\, {}_1\mathcal{F}_2\left\{ {\alpha_1 \atop \rho_1 \quad \rho_2};\ -x^2\right\}$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\rho_1+\frac{1}{2}\phi-\frac{1}{4}+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\rho_2+\frac{1}{2}\phi-\frac{1}{4}+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(-\frac{1}{2}\phi+\frac{1}{4}-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\alpha_1+\frac{1}{2}\phi-\frac{1}{4}+\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2\sum_{\rho_1,\rho_2}\frac{\sin(\alpha_1-\rho_1)\pi}{\sin(\rho_2-\rho_1)\pi}\, x^{2\rho_1+\phi-1}\ {}_1\mathcal{F}_2\left\{ {1-\alpha_1+\rho_1 \atop 1-\rho_2+\rho_1,\rho_1};\ -x^2 \right\} ,$$

3 $\mathrm{Re}(\alpha_1) \geq \mathrm{Re}(\rho_1+\rho_2)$, $\mathrm{Re}(\alpha_1) \geq \frac{1}{2}+|\rho_1-\rho_2|$, $\mathrm{Re}(\phi) \leq 0$, form kernels of the special case $p = 1$ of Fox's Theorem 1 [8, p. 402] .

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† The symbol $\sum_{\rho_1,\rho_2}$ denotes that to the expression following it a similar expression with ρ_1, ρ_2 interchanged is to be added.

With $\alpha_1 = 1$, $\rho_1 = 1+\alpha$, $\rho_2 = 1+\alpha+\nu$, we have

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\tfrac{1}{2}+\alpha+\tfrac{1}{2}\nu+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\tfrac{1}{2}-\alpha-\tfrac{1}{2}\nu-\tfrac{1}{2}(s-\tfrac{1}{2}))}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}(s-\tfrac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{\frac{1}{2}+2\alpha+\nu}\ {}_1\mathcal{f}_2\left\{{1 \atop 1+\alpha,1+\alpha+\nu};\ -x^2\right\},$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}(s-\tfrac{1}{2}))}{\Gamma(\tfrac{1}{2}+\alpha+\tfrac{1}{2}\nu-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\tfrac{1}{2}-\alpha-\tfrac{1}{2}\nu+\tfrac{1}{2}(s-\tfrac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{\frac{1}{2}}\{\cos\alpha\pi\, J_\nu(2x) + \sin\alpha\pi\, Y_\nu(2x)\},\quad -\tfrac{1}{4} \le \mathrm{Re}(\alpha+\nu/2) \le \tfrac{1}{2},\quad |\mathrm{Re}(\nu)| \le \tfrac{1}{2}.$$

When $\alpha = 0$, these formulae reduces to $H^{(1)}(x) = H^{(2)}(x) = 2x^{\frac{1}{2}}J_\nu(2x)$ and when $\alpha = \frac{1}{2}$, they reduce to the case (a).

(g) Let $\phi = \alpha_1+\alpha_2-\rho_1-\rho_2-\rho_3+\frac{1}{2}$. The functions[#]

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\alpha_1-\tfrac{1}{4}+\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\alpha_2-\tfrac{1}{4}+\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))}{\Gamma(\rho_1-\tfrac{1}{4}+\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\rho_2-\tfrac{1}{4}+\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\rho_3-\tfrac{1}{4}+\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{-\phi}\ {}_2\mathcal{f}_3\left\{{\alpha_1,\alpha_2 \atop \rho_1,\rho_2,\rho_3};\ -x^2\right\},$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\rho_1-\tfrac{1}{4}+\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\rho_2-\tfrac{1}{4}+\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\rho_3-\tfrac{1}{4}+\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))}{\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\phi-\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\alpha_1-\tfrac{1}{4}+\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))\Gamma(\alpha_2-\tfrac{1}{4}+\tfrac{1}{2}\phi+\tfrac{1}{2}(s-\tfrac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{\phi-1}\sum_{\rho_1,\rho_2,\rho_3}\frac{\sin(\alpha_1-\rho_1)\pi\,\sin(\alpha_2-\rho_1)\pi}{\sin(\rho_2-\rho_1)\pi\,\sin(\rho_3-\rho_1)\pi}\, x^{2\rho_1}\ {}_2\mathcal{f}_3\left\{{1-\alpha_1+\rho_1,1-\alpha_2+\rho_1 \atop 1-\rho_2+\rho_1,1-\rho_3+\rho_1,\rho_1};-x^2\right\}$$

$\mathrm{Re}(\alpha_i+\phi/2) \ge \frac{1}{4}$, $i = 1,2$; $2\,\mathrm{Re}(\rho_i) \ge 1+\mathrm{Re}(\phi)$, $i = 1,2,3$; $\mathrm{Re}(\phi) \le 0$,

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[#] The symbol $\sum_{\rho_1,\rho_2,\rho_3}$ denotes the sum of three terms in which each term is obtained from the preceding one by cyclically interchanging ρ_1, ρ_2, ρ_3.

form the kernels of the special case $p = 2$ of Fox's Theorem 1 [8, p. 402] .

With $\alpha_1 = 1$, $\alpha_2 = 3/2+a+\nu$, $\rho_1 = 1+a$, $\rho_2 = 1+a+\nu$, $\rho_3 = 1+a+2\nu$, we have

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{4}+\nu+a+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}-\nu-a-\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4\ -\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{3}{4}-\nu-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}+\nu-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{2a+2\nu}\ {}_2f_3\left\{\begin{matrix}1 & \nu+a+3/2 & \\ a+1, & \nu+a+1, & 2\nu+a+1\end{matrix}\ ;\ -x^2\right\}$$

and

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{3}{4}-\nu+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}+\nu+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{4}+\nu+a-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{3}{4}-\nu-a+\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4\ +\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv -\pi^{\frac{1}{2}}x\{\sin a\pi\, J^2_{-\nu}(x) - 2\sin(\nu+a)\pi\, J_{-\nu}(x) + \sin(2\nu+a)\pi\, J^2_{\nu}(x)\}\csc^2\nu\pi$$

$$\tfrac{3}{4} \geq \mathrm{Re}(\nu+a) \geq 0, \quad |\mathrm{Re}(\nu)| \leq \tfrac{1}{2}\ .$$

(h) Take $\phi = \frac{1}{2}+\sum^{p}\alpha_r-\sum^{p+1}\rho_s$. The functions

$$H^{(1)}(x) = \frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{4}-\frac{1}{2}\phi+\frac{1}{2}(s-\frac{1}{2}))\ \prod^{p}\Gamma(\alpha_j-\frac{1}{4}+\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))}{\prod^{p+1}\Gamma(\rho_j-\frac{1}{4}+\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{-\phi}\ {}_pf_{p+1}\left\{\begin{matrix}\alpha_1, \cdots, \alpha_p\\ \rho_1, \cdots, \rho_{p+1}\end{matrix}\ ;\ -x^2\right\}$$

$$H^{(2)}(x) = \frac{1}{2\pi i}\int \frac{\prod^{p+1}\Gamma(\rho_j-\frac{1}{4}+\frac{1}{2}\phi+\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{4}-\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))\ \prod^{p}\Gamma(\alpha_j-\frac{1}{4}+\frac{1}{2}\phi+\frac{1}{2}(s-\frac{1}{2}))}\, x^{-s}\, ds$$

$$\equiv 2x^{\phi-1}\sum_{h=1}^{p+1}\frac{\prod_{j=1}^{p}\sin\pi(\alpha_j-\rho_h)}{\prod_{j=1}^{p+1}{}'\sin\pi(\rho_j-\rho_h)}\,x^{2\rho_h}\,{}_p\mathcal{F}_{p+1}\left\{\begin{matrix}1+\rho_h-\alpha_1,\cdots,1+\rho_h-\alpha_p\\1+\rho_h-\rho_1,\cdots,*,\cdots,1+\rho_h-\rho_{p+1},\rho_h\end{matrix};-x^2\right\},$$

where the prime in Π' indicates the omission of the factor $\sin\pi(\rho_h-\rho_h)$; the asterisk in $\mathcal{F}$ the omission of the parameter $1-\rho_h-\rho_h$; $\mathrm{Re}(\alpha_j+\frac{1}{2}\phi)\geq\frac{1}{4}$, $j=1,2,\cdots,p$; $\mathrm{Re}(2\rho_j+\phi)\geq 1$, $j=1,2,\cdots,p+1$; $\mathrm{Re}(\phi)\leq 0$, form a pair of unsymmetrical Fourier kernels of which one is a hypergeometric function of the type ${}_p\mathcal{F}_{p+1}$ while the other is a combination of $p+1$ hypergeometric functions of the same type.

(i) Let $\phi=\alpha_1+\alpha_2-\rho_1-\rho_2-\rho_3+\frac{1}{2}$. The functions

$$H^{(1)}(x)=\frac{1}{2\pi i}\int\frac{\Gamma(\frac{1}{4}-\frac{1}{2}\phi+\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4-\frac{1}{2}\phi-\rho_1+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\alpha_1-\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\rho_2-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\rho_3-\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4-\frac{1}{2}\phi-\alpha_2+\frac{1}{2}(s-\frac{1}{2}))}\,x^{-s}\,ds$$

$$\equiv 2x^{-\phi}\left[\frac{\sin\alpha_2\pi}{\sin\rho_1\pi}\,{}_2\mathcal{F}_3\left\{\begin{matrix}\alpha_1,\alpha_2\\\rho_1,\rho_2,\rho_3\end{matrix};-x^2\right\}-\frac{\sin(\rho_1-\alpha_2)\pi}{\sin\rho_1\pi}\,x^{2-2\rho_1}\,{}_2\mathcal{F}_3\left\{\begin{matrix}1-\rho_1+\alpha_1,1-\rho_1+\alpha_2\\2-\rho_1,1-\rho_1+\rho_2,1-\rho_1+\rho_3\end{matrix};-x^2\right\}\right]$$

$$H^{(2)}(x)=\frac{1}{2\pi i}\int\frac{\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\rho_2+\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\rho_3+\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4-\frac{1}{2}\phi-\alpha_2-\frac{1}{2}(s-\frac{1}{2}))}{\Gamma(\frac{1}{4}-\frac{1}{2}\phi-\frac{1}{2}(s-\frac{1}{2}))\Gamma(5/4-\frac{1}{2}\phi-\rho_1-\frac{1}{2}(s-\frac{1}{2}))\Gamma(\frac{1}{2}\phi-\frac{1}{4}+\alpha_1+\frac{1}{2}(s-\frac{1}{2}))}\,x^{-s}\,ds$$

$$\equiv 2\sum_{\rho_2,\rho_3}\frac{\sin(\alpha_1-\rho_2)\pi}{\sin(\rho_3-\rho_2)\pi}\,x^{2\rho_2+\phi-1}\,{}_2\mathcal{F}_3\left\{\begin{matrix}1+\rho_2-\alpha_1,1+\rho_2-\alpha_2\\1+\rho_2-\rho_3,1+\rho_2-\rho_1,\rho_2\end{matrix};-x^2\right\},$$

where $\mathrm{Re}(\frac{1}{2}\phi+\alpha_1)\geq\frac{1}{4}$; $\mathrm{Re}(\frac{1}{2}\phi+\alpha_2)\leq 5/4$; $\mathrm{Re}(\phi)\leq 0$; $\mathrm{Re}(\frac{1}{2}\phi+\rho_1)\leq 1$; $\mathrm{Re}(\phi+2\rho_3)\geq 1$; $\mathrm{Re}(\phi+2\rho_2)\geq 1$, form a pair of unsymmetrical Fourier kernels.

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