Elliptic Zeta Functions

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Abstract

The main goal of this thesis is to develop and study the theory of the so-called elliptic zeta functions. These are functions on $\mathbb{C} \times \mathcal{L}$, where $\mathcal{L}$ is the set of rank 2 lattices in the complex plane, satisfying a quasi-periodicity with respect to the first factor and a certain modular invariance property with respect to the second factor. The prototype is the Weierstrass zeta $\zeta-$function. We show how these elliptic zeta functions are closely connected to modular forms and to the theory of equivariant functions.

Acknowledgment

I cannot thank Prof. Abdellah Sebbar enough for his patience and support throughout the years. It has been a privilege to build my skills and knowledge on the elliptic function theory under his guidance. Beyond knowledge and insights, he also taught me perseverance and to explore off the beaten path, skills that will prove invaluable as I progress in my career. He has all my gratitude for making the completion of this thesis possible.

Dedications

I wish to dedicate this thesis to those who made it possible.

Thanks and praise be to Allah, who gives me the strength to achieve this work.

To my mother. Thank you for showering me with your blessings and prayers..

To my father, Allah rest his soul. I am honored to give you a higher degree to be proud of.

To my little blessings, Sarah, Sanad and Jasmine. You have made me stronger, better and more fulfilled than I could have ever imagined.

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To my family and extended family. Thank you for giving me the chance to come study in Canada.

Contents

Preface

This thesis studies the relationship between certain complex objects, filling in the gaps in the current literature. We show how the theory of elliptic zeta functions leads to two interesting classes of meromorphic functions on the upper half plane $\mathbb{H}$: equivariant functions, $Eq(\Gamma)$, and modular forms of weight two, $M_2(\Gamma)$. The interesting aspect is that there is a triangular commutative correspondence between the set of these elliptic zeta functions, $Eq(\Gamma)$ and $M_2(\Gamma)$, which encompasses the modular and elliptic nature of equivariant functions where Γ is any modular subgroup of $SL_2(\mathbb{Z})$.

We will review the theoretical background of these three sets and present classical results about the theory of elliptic functions for the full modular group $SL_2(\mathbb{Z})$. After showing how they are connected by a triangular relationship, we develop this material to obtain the first interesting result of this thesis, which is to generalize this relationship to any modular subgroup of $SL_2(\mathbb{Z})$. We then show how we can generate elliptic zeta functions from integrals of power of $\wp$ and test whether they yield rational equivariant functions.

Finally, we provide an application by constructing a non-linear differential operator, called the $S-$map, which maps the space of (meromorphic) weight 4 modular forms for a subgroup of the modular group into itself. We do so by using the Schwarz derivative of rational equivariant functions and we study its fixed points.

Notation and conventions

The following notation will be sometimes used tacitly throughout the thesis.

$$\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{C}$$

to represent the positive integers, rational numbers, integer numbers, complex numbers. Further, if $\mathbb{C}$ is the complex numbers, then $\mathbb{C}^\times$ is $\mathbb{C}/\{0\}$, $\mathbb{C}(\Lambda)$ is the set of all elliptic functions and $\hat{\mathbb{C}}$ is $\mathbb{C} \cup \infty$; and if $\mathbb{H}$ is the upper half plane then $\mathbb{H}^*$ is $\mathbb{H} \cup \infty$. The big-O notation has the usual meaning: given two quantities $A(t), B(t)$ parameterized by a variable t we say that $A = O(B)$ if there exists a constant $c > 0$ such that $|A(t)| \leq c|B(t)|$ for all t for which this makes sense.

Chapter 1

Introduction

This thesis deals with the topic of Elliptic Zeta Functions. The prototype is the Weierstrass $\zeta-$function defined as follows. Let Λ be a rank 2 lattice in $\mathbb{C}$ generated by two $\mathbb{R}-$linearly independent complex numbers ω_1 and ω_2, that is $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, $\omega_1/\omega_2 \notin \mathbb{R}$. We can suppose $\Im(\omega_2/\omega_1) > 0$. The Weierstrass $\wp-$function is defined by

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

It is an elliptic function with respect to Λ in the sense that

$$\wp(z+\omega, \Lambda) = \wp(z) \ \text{ for all } z \in \mathbb{C} \text{ and } \omega \in \Lambda.$$

In other words, $\wp$ is $\Lambda-$periodic. Similarly, the Weierstrass $\zeta-$function is defined by

$$\zeta(z, \Lambda) = \frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

It satisfies

$$\frac{d}{dz}\zeta(z, \Lambda) = -\wp(z, \Lambda) .$$

Since $\wp$ is $\Lambda-$periodic, ζ is actually quasi-periodic in the sense that

$$\zeta(z+\omega, \Lambda) = \zeta(z, \Lambda) + H_\Lambda(\omega), \quad \text{for all } z \in \mathbb{C} \text{ and } \omega \in \Lambda,$$

where $H_\Lambda(\omega)$ is independent of z. The function H is called the quasi-period of ζ. It is $\mathbb{Z}-$linear and completely determined by the values of $H_\Lambda(\omega_1)$ and $H_\Lambda(\omega_2)$.

In fact both $\wp$ and ζ are maps from $\mathbb{C} \times B$ into $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where B is the set of rank 2 lattices in $\mathbb{C}$. Furthermore, both ζ and its quasi-period map satisfy the following homogeneity property:

$$\zeta(\alpha z, \alpha\Lambda) = \alpha^{-1}\zeta(z, \Lambda) \quad \text{and} \quad H_{\alpha\Lambda}(\alpha\omega) = \alpha^{-1}H_\Lambda(\omega) .$$

Suppose we are given another basis (ω_1', ω_2') for Λ with $\Im(\omega_2'/\omega_1') > 0$, then there exists a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the modular group $SL_2(\mathbb{Z})$ such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$. As a consequence, we have

$$\frac{H_\Lambda(a\omega_1 + b\omega_2)}{H_\Lambda(c\omega_1 + d\omega_2)} = \frac{H_\Lambda(\omega_1)}{H_\Lambda(\omega_2)}$$

If we specialize to $\Lambda = \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ where τ is a complex number with positive imaginary part, then one can show that H_{Λ_τ}, as a function of τ, is a meromorphic function on the upper half-plane $\mathbb{H}$. In addition, if we set

$$h_0(\tau) = \frac{H_{\Lambda_\tau}(\tau)}{H_{\Lambda_\tau}(1)} \ ,$$

then h_0 is a meromorphic function on $\mathbb{H}$ satisfying

$$h_0\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{ah_0(\tau) + b}{ch_0(\tau) + d}, \quad \tau \in \mathbb{H}\, , \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}).$$

In other words, h is equivariant with respect to $SL_2(\mathbb{Z})$. In the meantime, the theory of equivariant functions, defined as the functions on $\mathbb{H}$ commuting with the action of the modular group, has been developed and investigated by my supervisor and his co-authors in several papers. In particular, there is a close connection between them and modular forms that can be illustrated in many ways. For example the equivariant functions are connected to weight zero modular functions through the cross-ratio, to weight 2 modular forms (and thus to differential forms on modular curves), and to weight 4 modular forms via the Schwarz derivative. Several applications have also been obtained for the critical points of modular forms as well as for automorphic differential equations among others.

The main objective of this thesis is to show that, along with their modular aspect, there is an elliptic aspect to equivariant functions. To be more precise, we will show that all equivariant functions arise from elliptic objects in a similar manner to the example h_0 above. To this end, we introduce the notion of elliptic zeta functions generalizing the Weierstrass zeta-function. Let

$$M = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : \Im(\omega_2/\omega_1) > 0\},$$

and let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$ acting on M in the following way:

$$\gamma \cdot (\omega_1, \omega_2) = (\omega_1, \omega_2)\gamma^t\, , \quad \gamma \in SL_2(\mathbb{Z})\, , \quad (\omega_1, \omega_2) \in B.$$

For $(\omega_1, \omega_2) \in M$, we denote by $[\omega_1, \omega_2]$ its orbit under this action, and $\Lambda_{(\omega_1,\omega_2)}$ the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. In the case $\Gamma = SL_2(\mathbb{Z})$, the quotient $\Gamma\backslash M$ coincides with the set of rank 2 lattices in $\mathbb{C}$.

Inspired by the work of Brady [2], we define an elliptic zeta function for Γ as a map

$$\mathcal{Z} : \mathbb{C} \times \Gamma\backslash M \longrightarrow \widehat{\mathbb{C}}$$

satisfying the following conditions.

1. For each class $[\omega_1, \omega_2] \in \Gamma\backslash M$, the map

$$\mathcal{Z}(\cdot, [\omega_1, \omega_2]) : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$$

   is quasi-periodic with respect to $\Lambda_{(\omega_1,\omega_2)}$, that is, for all $z \in \mathbb{C}$ and all $\omega \in \Lambda_{(\omega_1,\omega_2)}$ we have

$$\mathcal{Z}(z + \omega, [\omega_1, \omega_2]) = \mathcal{Z}(z, [\omega_1, \omega_2]) + H_{[\omega_1,\omega_2]}(\omega).$$

2. The map $\mathcal{Z}$ is homogeneous, that is, there exists an integer k, referred to as the weight of $\mathcal{Z}$, such that for all $\alpha \in \mathbb{C}^\times$, $[\omega_1, \omega_2] \in \Gamma\backslash M$ and $z \in \mathbb{C}$ we have

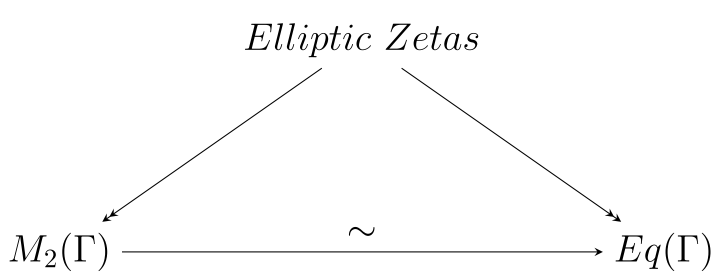
$$\mathcal{Z}(\alpha z, \alpha[\omega_1, \omega_2]) = \alpha^k \mathcal{Z}(z, [\omega_1, \omega_2]).$$

3. The maps

$$\tau \mapsto H_{[1,\tau]}(\tau) \text{ and } \tau \mapsto H_{[1,\tau]}(1)$$

   are meromorphic in $\mathbb{H}$.

The first property of an elliptic zeta function is that the quotient of its two quasi-period maps $\tau \mapsto H_{[1,\tau]}(\tau)$ and $\tau \mapsto H_{[1,\tau]}(1)$ is an equivariant functions for the subgroup Γ. Thanks to a structure theorem, to each elliptic zeta function one can associate a modular form of weight 2 for Γ, and this map from the set of elliptic zeta functions for Γ to the space of weight 2 modular forms $M_2(\Gamma)$ is surjective. On the other hand, there is a one-to-one correspondence between $M_2(\Gamma)$ and the set of equivariant functions $Eq(\Gamma)$ of geometric nature, realizing the equivariant functions as meromorphic sections of the canonical line bundle of the modular curve attached to Γ. Our investigation of the above three correspondences can be summarized in the following commutative diagram where all arrows are surjective.



The thesis is organized as follows. In Chapter 2, we recall the definitions and the important facts from the theory of elliptic and quasi-periodic functions. In Chapter 3, we do the same for modular forms and equivariant functions. We provide the structure theorem for the latter as well as their connection with the cross-ratio and the Schwarz derivative. In Chapter 4, we focus on the case of the modular group $SL_2(\mathbb{Z})$, define the notion of elliptic zeta functions on the set of rank 2 lattices and prove the first theorems with respect to equivariance of ratios of quasi-periods. Several examples are constructed from integrals of powers of the Weierstrass $\wp-$functions. In Chapter 5, we generalize our construction to an arbitrary finite index subgroup of the modular group. In Chapter 6, we treat the case of a particular class referred to as the rational equivariant functions. We investigate which elliptic zeta functions arising from integrals of power of $\wp$ yields rational equivariant functions. In Chapter 7, we provide an application by constructing a non-linear differential operator, called the $S-$map, from the space of weight 4 modular forms to itself. Using the Eisenstein series and the Jacobi theta functions, we construct fixed points to this map and as a consequence, we show that they satisfy the same degree four non-linear differential equation with constant coefficients.

Chapter 2

Elliptic and Quasi-Periodic Functions

In this chapter we give a concise exposition of classical material from the theory of elliptic and quasi-periodic functions which we shall use extensively in the rest of the thesis. Most of the content of this chapter can be found in Silverman's book on these functions [16]. We give the definition and properties of elliptic functions and we explain how to construct them. We will then introduce the notion of quasi-periodic functions.

Throughout the chapter, $\Lambda \subset \mathbb{C}$ is a rank 2 lattice in $\mathbb{C}$ with $\mathbb{R}-$ basis (ω_1, ω_2), that is, $\Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$. We suppose that this basis is oriented in the sense that $\Im(\omega_2/\omega_1) > 0$. Such lattice can be expressed with a different basis (ω_1', ω_2') if $\omega_1' = a\,\omega_1 + b\,\omega_2$ and $\omega_2' = c\,\omega_1 + d\,\omega_2$ with $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. In other words, $(\omega_1', \omega_2') = (\omega_1, \omega_2)\gamma^t$ where γ^t denotes the transpose of the matrix γ.

2.1 Elliptic Functions

In this section we go over some definitions and properties of elliptic functions.

Definition 2.1.1. *An elliptic function relative to the lattice* Λ *is a meromorphic function* $f(z)$ *on* $\mathbb{C}$ *which satisfies*

$$f(z+\omega) = f(z) \text{ for all } \omega \in \Lambda, z \in \mathbb{C}.$$

The group Λ acts additively on the set $\mathbb{C}$ and elliptic functions are $\Lambda-$invariant. We also say that they are $\Lambda-$periodic.

Definition 2.1.2 (Fundamental Parallelogram)**.** *A fundamental parallelogram for* Λ *is a set of the form*

$$D = \{a + t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\},$$

where $a \in \mathbb{C}$ *and* ω_1, ω_2 *form a basis for* Λ*.*

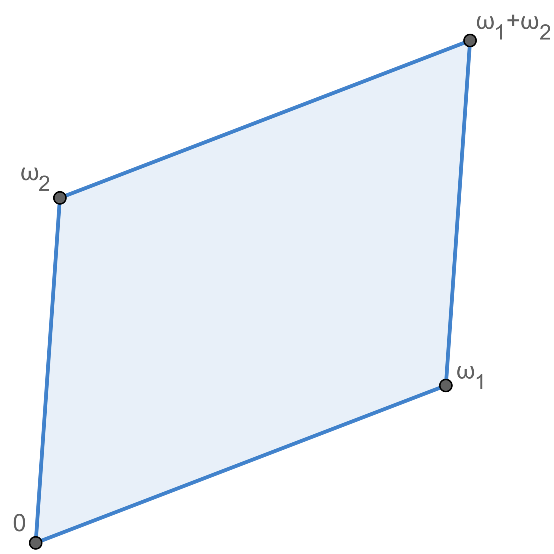


Figure 2.1: Fundamental domain for Λ.

Two points $z, z' \in \mathbb{C}$ are equivalent modulo Λ if $z' = z + \omega$ for some $\omega \in \Lambda$. The complex plane $\mathbb{C}$ is covered by the copies of D shifted by elements of Λ, so every point in $\mathbb{C}$ is equivalent to some point in D. In other words, the points of a fundamental parallelogram D correspond bijectively to the equivalence classes modulo Λ.

Now we consider some important theorems on zeros and poles of the elliptic functions.

Proposition 2.1.3. *An elliptic function without poles or zeros is constant.*

Proof: Here we consider 2 cases:

Case 1: If f has no poles. Suppose that $f(z)$ is an elliptic function and is holomorphic. Let D be a fundamental parallelogram for Λ. Then the periodicity of f implies that

$$\sup_{z\in\mathbb{C}} |f(z)| = \sup_{z\in\bar{D}} |f(z)|.$$

But f is continuous and $\bar{D}$ is compact, then $|f(z)|$ is bounded on $\bar{D}$ and hence bounded on all of $\mathbb{C}$. Therefore, f is constant by Liouville's theorem [1, ch.4, §2.3].

Case 2: If f has poles, but no zeros, then $1/f$ is an elliptic function with no poles and so it is constant. ∎

Consider an elliptic function f and $w \in \mathbb{C}$. Then we define $ord_w(f)$ to be the order of vanishing of f at w, and $res_w(f)$ to be the residue of f at w. Notice that the order and residue of f remain the same if z is replaced by $z + \omega$. This prompts the following convention: by $\sum_{w \in \mathbb{C}/\Lambda}$ we mean the sum over $w \in D$, where D is a fundamental parallelogram for Λ.

Theorem 2.1.4. *Let f be an elliptic function, then:*

$$(a) \sum_{w \in \mathbb{C}/\Lambda} res_w(f) = 0; \tag{2.1.1}$$

$$(b) \sum_{w \in \mathbb{C}/\Lambda} ord_w(f) = 0; \tag{2.1.2}$$

$$(c) \sum_{w \in \mathbb{C}/\Lambda} ord_w(f) w \in \Lambda. \tag{2.1.3}$$

Proof: Let D be a fundamental parallelogram for Λ such that $f(z)$ has no zeros or poles on the boundary ∂D of D. All three parts of the theorem are simple applications of the residue theorem.

(a) By the residue theorem,

$$\sum_{w \in \mathbb{C}/\Lambda} res_w(f) = \frac{1}{2\pi i} \int_{\partial D} f(z) dz.$$

The periodicity of f implies that the integrals on opposite sides of the parallelogram cancel each other, so the total integral around the boundary of D is zero.

(b) It follows from the periodicity of $f(z)$ that $f'(z)$ is also periodic, so applying (a) to the elliptic function $f'(z)/f(z)$ gives

$$\sum_{w \in \mathbb{C}/\Lambda} ord_w(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = 0.$$

(c) First, we apply the residue theorem to the function $zf(z)'/f(z)$

$$\sum_{w \in \mathbb{C}/\Lambda} ord_w(f) w = \frac{1}{2\pi i} \int_{\partial D} \frac{zf'(z)}{f(z)} dz$$

$$= \frac{1}{2\pi i}\left(\int_a^{a+\omega_1} + \int_{a+\omega_1}^{a+\omega_1+\omega_2} + \int_{a+\omega_1+w_2}^{a+\omega_2} + \int_{a+\omega_2}^{a} \right).$$

Next, we substitute the variable $z \mapsto z - \omega_1$ and $z \mapsto z - \omega_2$ in the second and third integrals respectively. The periodicity of f'/f implies

$$\sum_{w \in \mathbb{C}/\Lambda} ord_w(f)w = -\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1} \frac{f'(z)}{f(z)}dz + \frac{\omega_1}{2\pi i}\int_a^{a+\omega_2} \frac{f'(z)}{f(z)}dz.$$

We know that for any meromorphic function $g(z)$, the integral

$$\frac{1}{2\pi i}\int_a^b \frac{g'(z)}{g(z)}dz$$

is the winding number around 0 of the path in $\mathbb{C}$:

$$t \mapsto g((1-t)a + tb), \quad 0 \le t \le 1.$$

In particular, if $g(a) = g(b)$, then the integral is an integer. Hence the periodicity of $f(z)'/f(z)$ implies that $\sum_{w \in \mathbb{C}/\Lambda} ord_w(f)w$ belongs to Λ. ∎

The previous proposition implies that the number of poles of an elliptic function equals the number of its zeros in any fundamental parallelogram. We define the order of an elliptic function to be that number.

Corollary 2.1.5. *The order of a non-constant elliptic function is at least 2.*

Proof: If $f(z)$ has a single simple pole, then from Theorem 2.1.4(a) the residue at that pole is 0. Hence, f is holomorphic and we can apply Proposition 2.1.3. ∎

2.2 Construction of Elliptic Functions

In this section, we define some important elliptic functions. More precisely, we introduce the notion of the $\wp$ function.

Definition 2.2.1. *Let $\Lambda \subset \mathbb{C}$ be a lattice. We define the Weierstrass $\wp$-function (relative to Λ) by the series*

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}. \tag{2.2.1}$$

Furthermore, we define the Eisenstein series of weight $2k$, $k \in \mathbb{N}$ by the series

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda \setminus 0} \omega^{-2k}. \tag{2.2.2}$$

Theorem 2.2.2. *Let $\Lambda \subset \mathbb{C}$ be a lattice.*

(a) The Eisenstein series G_{2k} for Λ is absolutely convergent for all $k > 1$.

(b) The series defining the Weierstrass $\wp$-function converges absolutely and uniformly on every compact subset of $\mathbb{C} - \Lambda$. It defines a meromorphic function on $\mathbb{C}$ having a double pole with residue 0 at each lattice point and no other poles.

(c) The Weierstrass $\wp$-function is an even elliptic function.

Proof:

(a) Because Λ is discrete in $\mathbb{C}$, there is a constant $c = c(\Lambda)$ so that for all $N \geq 1$, the number of lattice points in an annulus satisfies

$$\#\{\omega \in \Lambda : N \leq |\omega| < N + 1\} < cN.$$

For $|w| < 1$, there is only a finite number of terms, hence

$$\sum_{\substack{\omega \in \Lambda \\ |\omega| \geq 1}} \frac{1}{|\omega|^{2k}} \leq \sum_{N=1}^{\infty} \frac{\#\{\omega \in \Lambda : N \leq |\omega| < N + 1\}}{N^{2k}} < \sum_{N=1}^{\infty} \frac{c}{N^{2k-1}} < \infty.$$

(b) If $|\omega| \geq 2|z|$, then

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right| \leq \frac{10|z|}{|\omega|^2}.$$

Hence we see from (a) that the series for $\wp(z, \Lambda)$ is absolutely convergent for $z \in \mathbb{C} - \Lambda$, and uniformly convergent on every compact subset of $\mathbb{C} - \Lambda$. Therefore the series defines a holomorphic function on $\mathbb{C} - \Lambda$. Furthermore, it is clear from the series expansion that $\wp(z, \Lambda)$ has a double pole with residue 0 at each point of Λ.

(c) Clearly $\wp(z, \Lambda) = \wp(-z, \Lambda)$. Because the series for $\wp$ is uniformly convergent, we can compute its derivative $\wp'(z, \Lambda)$ by term-wise differentiation:

$$\wp'(z, \Lambda) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}.$$

From this expression we see that $\wp'$ is an elliptic function, thus integrating

$$\wp'(z + \omega, \Lambda) = \wp'(z, \Lambda)$$

yields

$$\wp(z+\omega,\Lambda)=\wp(z,\Lambda)+c(\omega)$$

for all $z \in \mathbb{C}-\Lambda$, where $c(\omega) \in \mathbb{C}$ is independent of z. Now let $z=-\omega/2$ and use the evenness of $\wp(z,\Lambda)$ to conclude that $c(\omega)=0$. ∎

The proof of the following theorem is given in details in [16].

Theorem 2.2.3. *Every elliptic function with respect to a lattice Λ is a rational combination of $\wp(z,\Lambda)$ and $\wp'(z,\Lambda)$ with complex coefficients. In other words, we have*

$$\mathbb{C}(\Lambda)=\mathbb{C}(\wp(z,\Lambda),\wp'(z,\Lambda)),$$

where $\mathbb{C}(\Lambda)$ is the set of all elliptic functions.

Remark 2.2.4. It is clear from Theorem 2.2.3 that every derivative of $\wp$ is a rational function of $\wp$ and $\wp'$. In the next section, we will see that this function is actually a polynomial, see equations 2.3.1 to 2.3.6.

2.3 Laurent Series Expansion

The aim of this section is to give some Laurent expansions of $\wp$ around its zeros and poles, from which we will deduce the fundamental algebraic relation satisfied by $\wp(z,\Lambda)$ and $\wp'(z,\Lambda)$, given in (b) below. We will refer to these later in the text.

Theorem 2.3.1. *We have*

(a) *The Laurent series for $\wp(z,\Lambda)$ around $z=0$ is given by*

$$\wp(z,\Lambda)=z^{-2}+\sum_{k=1}^{\infty}(2k+1)G_{2k+2}(\Lambda)z^{2k}.$$

(b) *For all $z \in \mathbb{C}$ with $z \notin \Lambda$,*

$$\wp'(z,\Lambda)^2=4\wp(z,\Lambda)^3-60G_4(\Lambda)\wp(z,\Lambda)-140G_6(\Lambda).$$

Proof:

(a) For $|z|<|\omega|$, we have

$$(z-\omega)^{-2}-\omega^{-2}=\omega^{-2}[(1-z/\omega)^{-2}-1]=\sum_{n=1}^{\infty}(n+1)z^n/\omega^{n+2}.$$

Substituting this result into the series for $\wp(z,\Lambda)$ and reversing the order of summation gives the desired result.

(b) Let us write the first few terms of the Laurent expansions we are interested in:

$$\begin{aligned}\wp'(z,\Lambda)^2 &= 4z^{-6} - 24G_4z^{-2} - 80G_6 + ...\\ \wp(z,\Lambda)^3 &= z^{-2} + 9G_4z^{-2} + 15G_6 + ...\\ \wp(z,\Lambda) &= z^{-2} + 3G_4z^2 + ...\end{aligned}$$

We can define the function

$$f(z) = \wp'(z,\Lambda)^2 - 4\wp(z,\Lambda)^3 + 60G_4\wp(z,\Lambda) + 140G_6$$

which is holomorphic around $z = 0$ and vanishes at $z = 0$. It is also elliptic relative to Λ, we know from 2.2.2 that it is holomorphic away from Λ, hence $f(z)$ is a holomorphic elliptic function. From 2.1.3, we conclude that $f(z)$ is identically zero. ∎

Now, we provide the Taylor expansion around one of the two zeros of the $\wp$ function in the fundamental domain $\mathbb{C}/\Lambda$. An exact formula for the zeros of $\wp$ function is complicated but can be found in [14, proposition 9.2]. Here we simply write z_0 to denote one of these zeros. From Theorem 2.3.1, we have

$$\wp'(z,\Lambda)^2 = 4\wp(z,\Lambda)^3 - g_2\wp(z,\Lambda) - g_3.$$

This gives us

$$2\wp'\wp'' = 12\wp^2\wp' - g_2\wp',$$

where $\wp$ is used instead of $\wp(z,\Lambda)$ to ease the notation. It follows that

$$\wp'' = 6\wp^2 - \frac{g_2}{2},$$

and then

$$\wp''' = 12\wp\wp', \tag{2.3.1}$$

$$\wp^{(4)} = 12(\wp')^2 + 12\wp\wp'', \tag{2.3.2}$$

$$\wp^{(5)} = 36\wp'\wp'' + 12\wp\wp''', \tag{2.3.3}$$

$$\wp^{(6)} = 36(\wp'')^2 + 48\wp'\wp''' + 12\wp\wp^{(4)}, \tag{2.3.4}$$

$$\wp^{(7)} = 12\wp\wp^{(5)} + 60\wp'\wp^{(4)} + 120\wp''\wp''', \tag{2.3.5}$$

$$\wp^{(8)} = 12\wp\wp^{(6)} + 72\wp'\wp^{(5)} + 180\wp''\wp^{(4)} + 120(\wp''')^2. \tag{2.3.6}$$

Now we calculate the values of $\wp$ and its derivatives at z_0 to get the first terms of the Taylor expansion:

$$\wp(z_0,\Lambda) = 0,$$

$$\begin{aligned}
\wp'(z_0,\Lambda) &= \sqrt{-g_3},\\
\wp''(z_0,\Lambda) &= -g_2/2,\\
\wp'''(z_0,\Lambda) &= 0,\\
\wp^{(4)}(z_0,\Lambda) &= -12g_3,\\
\wp^{(5)}(z_0,\Lambda) &= -18\sqrt{-g_3}g_2,\\
\wp^{(6)}(z_0,\Lambda) &= -9g_2^2,\\
\wp^{(7)}(z_0,\Lambda) &= -960\sqrt{-g_3}g_3,\\
\wp^{(8)}(z_0,\Lambda) &= 2376g_2g_3.
\end{aligned}$$

Therefore, we get the first few terms of the Taylor Series around z_0:

$$\wp(z+z_0,\Lambda) = \sqrt{-g_3}z+\frac{-g_2}{4}z^2+\frac{-g_3}{2}z^4+\frac{-3}{20}\sqrt{-g_3}g_2z^5+\frac{-1}{80}g_2^2z^6+\frac{-4}{21}\sqrt{-g_3}g_3z^7+O(z^8).$$

2.4 Quasi-periodic functions

In this section, we present the definition of the quasi-periodic functions and introduce an important classical example, the Weierstrass ζ function.

Let $\Lambda \subset \mathbb{C}$ be a lattice in $\mathbb{C}$.

Definition 2.4.1 (Quasi-periodic function)**.** *A meromorphic function f on $\mathbb{C}$ is quasi-periodic relative to Λ if for every $\omega \in \Lambda$, there exists a constant $H(\omega) \in \mathbb{C}$, such that*

$$f(z+\omega) = f(z) + H(\omega).$$

It follows from the definition that if f is quasi-periodic, then the quasi-period map H is $\mathbb{Z}-$linear.

Example 2.4.2. The Weierstrass $\zeta-$function is defined by the series

$$\zeta(z,\Lambda) \;=\; \frac{1}{z} + \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right). \tag{2.4.1}$$

It is absolutely and uniformly convergent on compact subsets of $\mathbb{C} \setminus \Lambda$. Moreover, it defines a meromorphic function on $\mathbb{C}$ with simple poles at the points of Λ and no other poles. Differentiating (2.4.1) gives, for all $z \in \mathbb{C}$,

$$\frac{d}{dz}\zeta(z,\Lambda) \;=\; -\wp(z,\Lambda)\ .$$

Since $\wp$ is $\Lambda-$periodic, we have $\wp(z+\omega)-\wp(z)=0$ for all $\omega \in \Lambda$ and for all $z \in \mathbb{C}$. Taking the primitive, we get

$$\zeta(z+\omega,\Lambda) \;=\; \zeta(z,\Lambda) \;+\; \eta_\Lambda(\omega), \tag{2.4.2}$$

where $\eta_\Lambda(\omega)$ is independent of z. Hence ζ is quasi-periodic.

We call $\eta_\Lambda : \Lambda \to \mathbb{C}$ the quasi-period map associated to ζ. It is clear that η_Λ is $\mathbb{Z}-$linear and thus it is completely determined by the values of $\eta_\Lambda(\omega_1)$ and $\eta_\Lambda(\omega_2)$, where (ω_1,ω_2) is a basis of Λ. Also, since ζ is an odd function, it follows that if $\omega \in \Lambda$ and $\omega \notin 2\Lambda$, then taking $z=-\omega/2$ in equation 2.4.2 gives

$$\eta_\Lambda(\omega) \;=\; 2\zeta(\frac{\omega}{2},\Lambda)\;. \tag{2.4.3}$$

Proposition 2.4.3. *Let f be a quasi-periodic function and let $H:\Lambda\to\mathbb{C}$ be the quasi-period map associated to f, then*

$$\omega_1 H(\omega_2) - \omega_2 H(\omega_1) = 2\pi i R, \tag{2.4.4}$$

where R is the sum of the residues of f in a fundamental domain D.

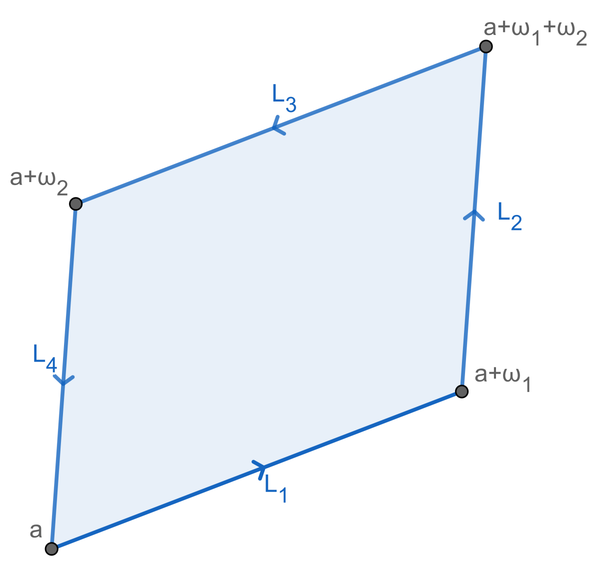


Figure 2.2: Integral around the fundamental domain.

Proof: We integrate f around a fundamental parallelogram offset slightly so as not to contain poles of f on its boundary. Thus let D be the region

$$D = \{a + t_1\omega_1 + t_2\omega_2, 0 \le t_1, t_2 \le 1\}$$

and let

$$\partial D = L_1 + L_2 + L_3 + L_4$$

be its boundary as illustrated in Figure 2.2. Now by the residue theorem we have $\int_{\partial D} f(z)dz = 2\pi i R$. If we compute the integrals of opposite lines, we find that they partially cancel each other:

$$\begin{aligned}\int_{L_1+L_3} f(z)dz &= \int_0^1 f(a+t\omega_2)\omega_2 dt + \int_1^0 f(a+\omega_1+t\omega_2)\omega_2 dt \\ &= \int_0^1 f(a+t\omega_2)\omega_2 dt - \int_0^1 f((a+t\omega_2)+H(\omega_1))\omega_2 dt \\ &= -H(\omega_1)\omega_2.\end{aligned}$$

Similarly,

$$\int_{L_2+L_4} f(z)dz = H(\omega_2)\omega_1.$$

Therefore

$$2\pi i R = \int_{\partial D} f(z)dz = H(\omega_2)\omega_1 - H(\omega_1)\omega_2.$$

∎

For $f = \zeta$, a fundamental domain D only has a simple pole at $z = 0$. Moreover, the residue $R = 1$, so we get the Legendre relation:

$$\eta_\Lambda(\omega_2)\omega_1 - \eta_\Lambda(\omega_1)\omega_2 = 2\pi i \tag{2.4.5}$$

Finally, let us illustrate the following homogeneity property of ζ and η, which will be useful in the following chapters.

Proposition 2.4.4. *If Λ is a lattice and $\alpha \in \mathbb{C}$ then*

$$\zeta(\alpha z, \alpha\Lambda) \;=\; \alpha^{-1}\zeta(z,\Lambda) \quad \textit{and} \quad \eta_{\alpha\Lambda}(\alpha\omega) \;=\; \alpha^{-1}\eta_\Lambda(\omega)\;. \tag{2.4.6}$$

Proof: The first relation follows from the expansion of equation 2.4.1. For $\alpha \in \mathbb{C}^\times$, we have

$$\begin{aligned}\zeta(\alpha z, \alpha\Lambda) &= \frac{1}{\alpha z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{\alpha z - \alpha\omega} + \frac{1}{\alpha\omega} + \frac{\alpha z}{(\alpha\omega)^2} \\ &= \frac{1}{\alpha}\left(\frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)\end{aligned}$$

$$= \alpha^{-1}\zeta(z, \Lambda).$$

The second relation follows from equation 2.4.3. ∎

We refer to equation 2.4.6 by saying that ζ and η are homogeneous of weight -1.

Chapter 3

Modular Forms and Equivariant Functions

The main reference for this chapter is another book by Silverman [15], as we will focus on the classical facts about modular forms and functions. We also discuss lattices of the form $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ where τ is in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We will use the notation $\eta_1(\tau), \eta_2(\tau)$ for $\eta_{\Lambda_\tau}(1), \eta_{\Lambda_\tau}(\tau)$.

3.1 Modular Forms

The modular group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by Möbius transformation: for $\tau \in \mathbb{H}$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, we have $\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$.

The region delimited by $\Re\tau = \frac{-1}{2}$, $\Re\tau = \frac{1}{2}$ and $|\tau| = 1$ depicted in fig. 3.1 is the fundamental domain for this action.

If f is a modular form for $SL_2(\mathbb{Z})$, then $f(\tau + 1) = f(\tau)$. In other words, any modular function is periodic and thus has a Fourier expansion that can be written as a power series in the form $q = e^{2\pi i\tau}$. This representation is called the q-expansion of f. By this property, f is meromorphic at ∞ if its q-expansion has only a finite number of negative powers of q, and f is holomorphic at ∞ if the limit

$$f(\infty) := \lim_{\Im\tau \to +\infty} f(\tau)$$

exists.

Equivalently, a modular function f is holomorphic at ∞ if its q-expansion has only non-negative powers of q. Finally, a cusp form is a holomorphic modular form that vanishes at ∞.

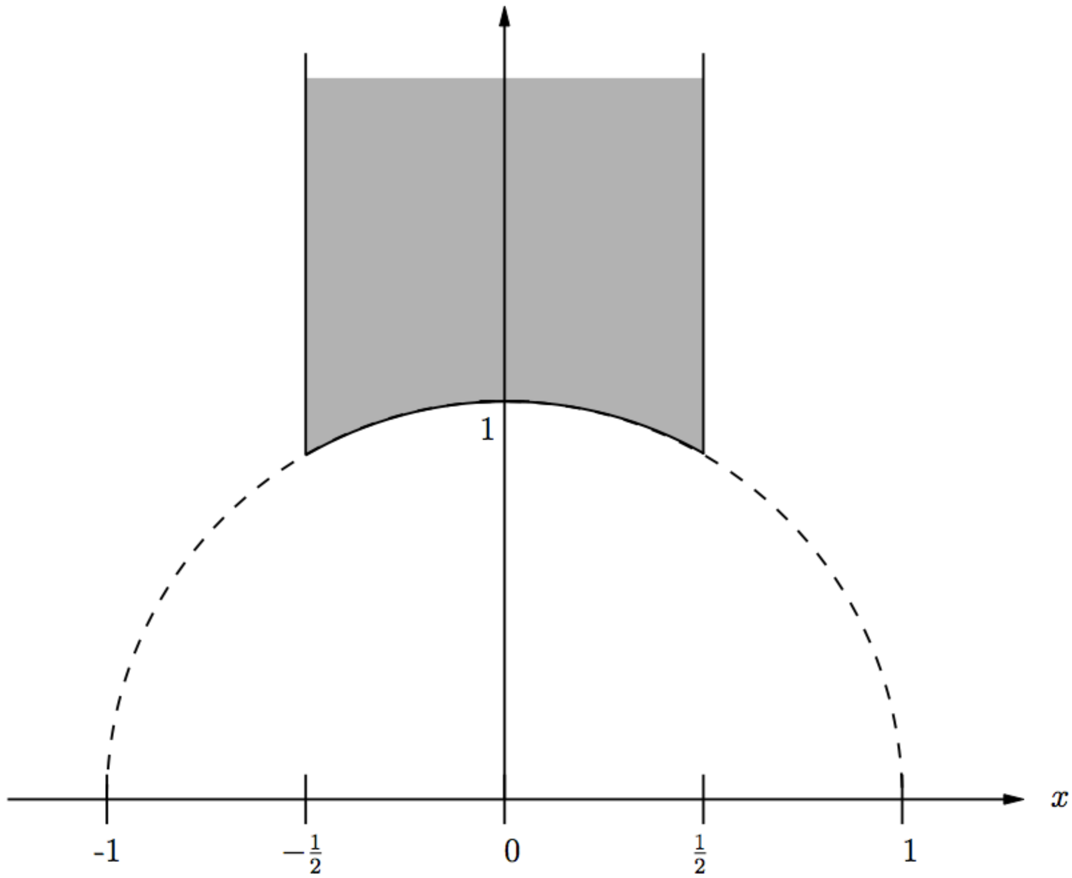


Figure 3.1: Fundamental domain

Modular forms play an important role in number theory and many areas of mathematics and physics. The Eisenstein series are interesting modular forms for our purpose.

Definition 3.1.1 (Meromorphic Modular Form)**.** *A (meromorphic) modular form is a function f that is meromorphic on the upper half plane $\mathbb{H}$ such that for all $\tau \in \mathbb{H}$ and all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, we have*

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$$

for some integer k, called the weight of f. f should also be meromorphic at ∞ in a sense that will be detailed below. If f is also holomorphic on $\mathbb{H}$ and at ∞, then we simply say that f is a holomorphic modular form.

Notice that a modular form of weight zero is invariant with respect to the modular group and it is called a modular function. Moreover, the modular forms of odd weight are all 0, as taking $\gamma = -I$ in definition 3.1.1 gives $f(\tau) = (-1)^{2k+1} f(\tau)$.

Proposition 3.1.1. *For $k \geq 2$ the Eisenstein series*

$$G_{2k}(\tau) := \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \frac{1}{(m+n\tau)^{2k}}$$

is a modular form of weight $2k$.

Proof: Let us revisit the definition of Eisenstein series for lattices, equation 2.2.2:

$$G_{2k}(\Lambda) = \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{\omega^{2k}}.$$

Notice that $G_{2k}(\tau) = G_{2k}(\Lambda_\tau)$ and $G_{2k}(c\Lambda) = c^{-2k} G_{2k}(\Lambda)$ for every $c \in \mathbb{C}^\times$. Moreover, we have

$$\begin{aligned}
\Lambda_{\gamma\tau} &= \mathbb{Z} + \mathbb{Z}\frac{a\tau + b}{c\tau + d} \\
&= \frac{1}{c\tau + d}(\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d)) \\
&= \frac{1}{c\tau + d}(\mathbb{Z}(a+c)\tau + \mathbb{Z}(b+d)) \\
&= \frac{1}{c\tau + d}(\mathbb{Z}\tau + \mathbb{Z}) \\
&= \frac{1}{c\tau + d}\Lambda_\tau.
\end{aligned}$$

Hence

$$G_{2k}(\gamma\tau) = G_{2k}(\Lambda_{\gamma\tau}) = G_{2k}(\frac{1}{c\tau + d}\Lambda_\tau) = (c\tau + d)^{2k} G_{2k}(\Lambda_\tau) = (c\tau + d)^{2k} G_{2k}(\tau).$$

Thus G_{2k} is a modular form of weight $2k$. ∎

Proposition 3.1.2. *For all $k \geq 2$, we have $G_{2k}(\infty) = 2\zeta(2k)$, where $\zeta(s)$ is the Riemann zeta function,*

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s}.$$

In particular, $G_{2k}(\tau)$ is a holomorphic modular form.

For a finite index subgroup Γ of $SL_2(\mathbb{Z})$, a modular form of weight $k \in \mathbb{N}$ is a meromorphic function on $\mathbb{H}$ such that for $\tau \in \mathbb{H}$ and $\gamma \in \Gamma$ we have

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau),$$

in addition to some meromorphic behavior at the cusps of Γ.

We already saw that G_{2k} are closely connected with elliptic functions as they appear as coefficients of the Laurent expansion of the $\wp$−function. They are also involved in elliptic invariants, which we define here.

Definition 3.1.2. *The elliptic invariants g_2, g_3 and $\Delta : \mathbb{H} \to \mathbb{C}$ are given by*

$$g_2(\tau) = 60G_4(\Lambda_\tau) \tag{3.1.1}$$
$$g_3(\tau) = 140G_6(\Lambda_\tau) \tag{3.1.2}$$
$$\Delta = g_2^3 - 27g_3^2, \tag{3.1.3}$$

for all $\tau \in \mathbb{H}$.

Here we provide the Fourier series of G_{2k} and use it to deduce various properties of the Fourier expansions for $\Delta(\tau)$. For $k \geq 2$ we have

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n \tag{3.1.4}$$

where

$$\zeta(s) = \frac{1}{n^s} \quad \text{and} \quad \sigma_k(s) = \sum_{0<d|n} d^k$$

are respectively the Riemann ζ-function and the k^{th}-power divisor function.

Definition 3.1.3. *We define the normalized Eisenstein series as follow:*

$$E_{2k}(\tau) = 1 + \frac{(2\pi i)^{2k}}{\zeta(2k)(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where $\sigma_k(n)$ is the sum of the k-th powers of the positive divisors of n.

We can now write $g_2(\tau)$ and $g_3(\tau)$ in terms of normalized Eisenstein series:

$$g_2(\tau) = 60G_4(\tau) = 120\zeta(4)E_4(\tau) = 2\pi^4 \frac{1}{2^2 3} E_4(\tau), \tag{3.1.5}$$

where $E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$, and

$$g_3(\tau) = 140G_6(\tau) = 280\zeta(6)E_6(\tau) = (2\pi)^6 \frac{1}{2^3 3^2} E_6(\tau) \tag{3.1.6}$$

where $E_6(\tau) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$.

The normalized Eisenstein series E_4 and E_6 are modular forms for $SL_2(\mathbb{Z})$ of weight 4 and 6 respectively.

Proposition 3.1.3. *[15] The graded algebra of holomorphic modular forms of weight $2k$ is isomorphic to $\mathbb{C}[g_2, g_3] = \mathbb{C}[E_4, E_6]$.*

For all $k \geq 2$, we define M_{2k} as the space of holomorphic modular forms of weight $2k$ for $SL_2(\mathbb{Z})$, and M_{2k}^0 as the space of cusp forms of weight $2k$ for $SL_2(\mathbb{Z})$. Note that both M_{2k} and M_{2k}^0 are $\mathbb{C}$-vector spaces.

Proposition 3.1.4. *[15] For all $k \geq 2$, we have that the Eisenstein series G_{2k} is in M_{2k} but is not in M_{2k}^0. Hence*

$$M_{2k} \cong M_{2k}^0 + \mathbb{C}G_{2k}$$

Proposition 3.1.5. *The modular forms g_2, g_3 and Δ, with weights $4, 6, 12$ respectively, satisfy*

$$g_2(\infty) = \frac{4}{3}\pi^4, \quad g_3(\infty) = \frac{8}{27}\pi^4, \quad \Delta(\infty) = 0.$$

Proof: Using theorem 3.1.2, definition 3.1.3 and the fact that

$$E_4(\infty) = E_6(\infty) = 1,$$

the proposition follows. ∎

The weight 12 cusp form for $SL_2(\mathbb{Z})$ is given by the following equation.

$$\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}. \tag{3.1.7}$$

We have only covered situations where $k \geq 2$ so far. We now look at $k = 1$ and will see that G_2 is not a modular form.

Definition 3.1.4. *Using $k = 1$ in proposition 3.1.1, we have*

$$G_2(\Lambda_\tau) = \lim_{N\to\infty}\sum_{-N\leq n\leq N}\lim_{M\to\infty}\sum_{\substack{-M\leq m\leq M\\(m,n)\neq 0}}\frac{1}{(m\tau+n)^2}.$$

As the above summation is not absolutely convergent , we cannot define it as a sum over $\mathbb{Z}^2$ as we did for G_{2k} when $k \geq 2$.

We also introduce the normalized Eisenstein series E_2 given by

$$E_2(\tau) = 1 - 24 \sum_{n=1} \sigma_1(n) q^n = \frac{6}{\pi^2} G_2(\tau) \tag{3.1.8}$$

where $\sigma_1(n)$ is the sum of positive divisors of n. E_2 is not a modular form but it is rather referred to as a quasimodular form of weight 2.

We are interested in the following representations of Eisenstein series. Note that we are using the short form notation of η introduced at the beginning of this chapter.

Proposition 3.1.6. *Let us define*

$$\begin{aligned} (a) \quad & \eta_1(\tau) = G_2(\tau), \\ (b) \quad & \eta_2(\tau) = \tau G_2(\tau) - 2\pi i, \\ (c) \quad & E_2(\tau) = \frac{1}{2\pi i} \frac{\Delta'(\tau)}{\Delta(\tau)}. \end{aligned}$$

Proof: (a) Using equations 2.4.3 and 2.4.1, we have

$$\begin{aligned} \eta_1 = 2\zeta(\frac{1}{2}, \Lambda_\tau) &= 2 \left[2 + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \frac{2}{1-2\omega} + \frac{1}{\omega} + \frac{1}{2\omega^2} \right] \\ &= 4 + 2 \sum_{\substack{(m+n\tau)\in\Lambda_\tau \\ (m+n\tau)\neq 0}} \left(\frac{2}{1-2(m+n\tau)} + \frac{1}{(m+n\tau)} + \frac{1}{2(m+n\tau)^2} \right) \\ &= 4 + 2 \sum_{n\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z} \\ (m,n)\neq 0}} \left(\frac{2}{(1-2m)-2n\tau} + \frac{2}{2m+2n\tau} + \frac{1}{2(m+n\tau)^2} \right) \\ &= 4 + 2 \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} \left(\frac{2}{(1-2m)-2n\tau} + \frac{2}{2m+2n\tau} + \frac{1}{2(m+n\tau)^2} \right) \\ &= 4 + 2 \lim_{N\to\infty} \lim_{M\to\infty} -2 + \sum_{n=-N}^{N} \frac{2}{1+2M-2n\tau} + \sum_{n=-N}^{N} \sum_{m=-M}^{M} \frac{1}{2(m+n\tau)^2} \end{aligned}$$

If we take the limit $M \to \infty$, each fraction $2/(1+2M-2n\tau)$ goes to 0 and what remains is $G_2(\tau)$.

(b) Follows from the Legendre relation (2.4.5).

(c) From equation 3.1.7, we get

$$\begin{aligned}\frac{\Delta'(\tau)}{\Delta(\tau)} &= \frac{d}{d\tau}\left(2\pi i\tau + 24\sum_{n+1}^{\infty}\log(1-q^n)\right)\\ &= 2\pi i\left(1-24\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}\right)\\ &= 2\pi i E_2.\end{aligned}$$

∎

We now study the effects of differentiation on modular forms. The following functions were studied by Ramanujan [7], who proved that they satisfy the following differential equations:

Proposition 3.1.7. *[16] If f is a modular function of weight k, then*

$$Df := (2\pi i)f' + k\eta_1 f \tag{3.1.9}$$

is a modular function of weight $k+2$.

Example 3.1.8. Ramanujan considered derivatives of Eisenstein series [7] and showed the following formulas:

$$(2\pi i)G_2' = 6(\eta_1)^2 - 4g_2; \tag{3.1.10}$$

$$(2\pi i)g_2' = 6g_3 - 4\eta_1 g_2; \tag{3.1.11}$$

$$(2\pi i)g_3' = \frac{1}{3}g_2^2 - 6g_3\eta_1; \tag{3.1.12}$$

$$(2\pi i)\Delta' = -12\eta_1\Delta; \tag{3.1.13}$$

These allow us to study a special modular function, the j invariant, in terms of which any other modular function of weight 0 can be expressed explicitly.

Definition 3.1.5. *The j invariant is defined as*

$$j(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)}, \quad \tau \in \mathbb{H}$$

Notice that j is the ratio of two modular forms of weight 12, hence it is a modular function of weight 0. Since Δ has a simple zero at infinity but vanishes nowhere else and $g_2(\infty) \neq 0$, j has a simple pole at infinity and is holomorphic on $\mathbb{H}$. Its derivative can be found by substituting terms using the values from 3.1.8:

$$(2\pi i)j' = \frac{1}{8}\frac{g_3}{g_2}j.$$

Proposition 3.1.9. *j induces a bijection between $SL_2(\mathbb{Z})\backslash\mathbb{H} \cup \{\infty\}$ and $\hat{\mathbb{C}}$.*

Proof: The cardinality of the preimage of ∞ is only one. Moreover, $SL_2(\mathbb{Z})\backslash\mathbb{H}^\star$ and $\mathbb{C} \cup \{\infty\}$ are two compact analytic spaces and the map j is analytic so the cardinality of the preimage is one for every point in the co-domain as j realizes a covering map between the two compact spaces. Hence, the map is bijective. ∎

As a consequence, we have

Corollary 3.1.10. *Every modular function f of weight 0 is a rational function of j, that is, $f \in \mathbb{C}(j)$.*

Proof: Since f is $SL_2(\mathbb{Z})$-invariant then it induces an analytic function from $SL_2(\mathbb{Z})\backslash\mathbb{H}^\star$ to $\hat{\mathbb{C}}$ and by proposition 3.1.9 we get that $f \circ j^{-1}$ is a meromorphic function on $\hat{\mathbb{C}}$. Hence, it is a rational function in $\hat{\mathbb{C}}$. ∎

3.2 Equivariant Functions

In this section, we introduce and study the notion of equivariant functions, which was first introduced in [14] for the modular group $SL_2(\mathbb{Z})$. These are meromorphic functions on $\mathbb{H}$ which commute with the action of the group $SL_2(\mathbb{Z})$. Moreover, these functions were studied by Brady [2] as quotients of quasi-periods of the Weierstrass quasi-periodic functions.

Definition 3.2.1 (Equivariant function)**.** *A function $h(\tau) : \mathbb{H} \to \hat{\mathbb{C}}$ is equivariant if for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have $h(\gamma(\tau)) = \gamma(h(\tau))$, where $\gamma(\tau) = \dfrac{a\tau + b}{c\tau + d}$ and $h(\tau) - \tau$ is meromorphic at ∞.*

Example 3.2.1. The identity function $h(\tau) = \tau$ is an equivariant function.

Proposition 3.2.2. *As was discussed in [2], the function h_0 defined by*

$$h_0(\tau) = \frac{\eta_2}{\eta_1}. \tag{3.2.1}$$

is an equivariant function.

Proof: Let $\gamma \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. We can use the linearity of η (again, using the short-form notation) and the fact that the lattices Λ_τ and $(c\tau + d)\mathbb{Z} + (a\tau + b)\mathbb{Z}$

are the same to find the following:

$$\begin{aligned}\eta_{\Lambda_{\gamma\tau}}(\gamma\tau) &= \eta_{\mathbb{Z}+\frac{a\tau+b}{c\tau+d}\mathbb{Z}}\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)\eta_{(c\tau+d)\mathbb{Z}+(a\tau+b)\mathbb{Z}}(a\tau+b) \quad \text{(by homogeneity of } \eta) \\ &= (c\tau+d)\eta_{\Lambda_\tau}(a\tau+b) \\ &= (c\tau+d)(a\eta_{\Lambda_\tau}(\tau)+b\eta_{\Lambda_\tau}(1))\end{aligned}$$

Also,

$$\begin{aligned}\eta_{\Lambda_{\gamma\tau}}(1) &= (c\tau+d)\eta_{(c\tau+d)\mathbb{Z}+(a\tau+b)\mathbb{Z}}(c\tau+d) \quad \text{(by homogeneity of } \eta) \\ &= (c\tau+d)\eta_{\Lambda_\tau}(c\tau+d) \\ &= (c\tau+d)(c\eta_{\Lambda_\tau}(\tau)+d\eta_{\Lambda_\tau}(1)).\end{aligned}$$

Therefore,

$$h(\gamma\tau) = \frac{a\eta_{\Lambda_\tau}(\tau)+b\eta_{\Lambda_\tau}(1)}{c\eta_{\Lambda_\tau}(\tau)+d\eta_{\Lambda_\tau}(1)} = \frac{ah(\tau)+b}{ch(\tau)+d} = \gamma h(\tau)\,.$$

∎

If h is equivariant for $SL_2(\mathbb{Z})$ then $h(\tau+1)=h(\tau)+1$. Hence $h(\tau)-\tau$ is also periodic of period one and thus has a Fourier expansion in q. The proper behavior of h at the cusp at infinity is that $h(\tau)-\tau$ is meromorphic in q (see [4]).

3.3 Rational Equivariant Functions

We will now study equivariant functions in connection with modular forms, leading to what we call rational equivariant functions. It is shown in [4] that the rational equivariant functions are only a small class among the general equivariant functions. Let us show that each modular form of any weight gives rise to an equivariant function.

Proposition 3.3.1. *From [4], if f is a modular form of weight k, then the function*

$$h_f(\tau) = \tau + k\frac{f(\tau)}{f'(\tau)} \tag{3.3.1}$$

is equivariant with respect to $SL_2(\mathbb{Z})$.

Proof: Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{C}$. We have

$$h(\gamma z) = \frac{az+b}{cz+d} + k\frac{(cz+d)^k f(z) ck(cz+d)^{k+1} f(z)}{(cz+d)^{k+2} f'(z)}$$

$$= \frac{(az+b)(ckf(z)+(cz+d)f'(z))+kf(z)}{(cz+d)(ckf(z)+(cz+d)f'(z))}$$

$$= \frac{akf(z)+(az+b)f'(z)}{ckf(z)+(cz+d)f'(z)}.$$

Meanwhile, we have

$$\gamma(h(z)) = \frac{ah(z)+b}{ch(z)+d}$$

$$= \frac{(az+b)f'(z)+akf(z)}{(cz+d)f'(z)+ckf(z)}.$$

Therefore h commutes with the action of $SL_2(\mathbb{Z})$ ∎

Note that not all the equivariant functions arise in this way from a modular form. We will see some examples in section 6.2.

Proposition 3.3.2. *[4] The function*

$$\hat{h}_f = \frac{1}{h_f(\tau)-\tau} = \frac{f'(\tau)}{kf(\tau)}$$

attached to a modular form f has rational residues at its poles which are all simple and $\frac{1}{2\pi i}\widehat{h_f}(\infty) \in \mathbb{Q}$.

Proof: Since f is meromorphic, it has a Laurent series at every $\tau_0 \in \mathbb{H}$, such that

$$f(\tau) = a_0(\tau-\tau_0)^m + a_1(\tau-\tau_0)^{m+1} + ...,$$

where $m \in \mathbb{Z}$ and so,

$$f'(\tau) = ma_0(\tau-\tau_0)^{m-1} + (m+1)a_1(\tau-\tau_0)^m + ...$$

Now,

$$\frac{f'(\tau)}{kf(\tau)} = \frac{(\tau-\tau_0)^{m-1}}{k(\tau-\tau_0)^m}\frac{ma_0+O(\tau-\tau_0)}{a_0+O(\tau-\tau_0)} = \frac{m}{k(\tau-\tau_0)} + O(\tau-\tau_0)$$

so $\hat{h}_f$ has only simple poles with rational residues. Since f is meromorphic at ∞, then it has a q expansion

$$f(\tau) = a_0q^m + a_1q^{m+1} + ...,$$

where $m \in \mathbb{Z}$ and $q = e^{2\pi i\tau}$ so,

$$f'(\tau) = 2\pi i m a_0 q^m + 2\pi i(m+1)a_1 q^{m+1} + ...$$

Now,

$$\frac{f'(\tau)}{kf(\tau)} = \frac{q^m}{kq^m}\frac{2\pi i m a_0 + O(q)}{a_0 + O(q)} = 2\pi i\frac{m}{k} + O(q)$$

so $\hat{h}_f(\infty) \in 2\pi i\mathbb{Q}$. ∎

Definition 3.3.1 (Rational Equivariant Function)**.** *An equivariant function h attached to a modular form f as in proposition 3.3.1, is called a rational equivariant function.*

It turns out that the conditions of the above proposition are also sufficient for an equivariant function to be rational.

Proposition 3.3.3. *[4] Let h be an equivariant function. If the poles of $\widehat{h} = \dfrac{1}{h(\tau)-\tau}$ in $\mathbb{H}$ are simple with rational residues and $\widehat{h}(\infty) \in 2\pi i\mathbb{Q}$, then h is rational, i.e. $h = h_f$ for some modular form f.*

Proof: See [4, theorem 7.4]. ∎

Proposition 3.3.4. *The function h_0 given by equation 3.2.1 is a rational equivariant function.*

Proof: We have

$$E_2(\tau) = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)}.$$

Therefore,

$$h_0(\tau) = \frac{\eta_2}{\eta_1} = \tau + 12\,\frac{\Delta}{\Delta'} = h_\Delta.$$

∎

3.4 Equivariant Functions and Weight 2 Modular Forms

The main goal of this section is to exhibit another relation between the equivariant functions and weight 2 modular functions. This type of relationship appeared in [14] where the terminology of equivariant functions was first coined. This point of view is more general than the previous one, because it takes into account all the equivariant functions, not only the rational ones. In fact, this new approach is related to that of the previous section, and in a sense it generalizes it in the following way.

Given a modular form f of weight k, the associated rational equivariant function h_f is given by

$$h_f = \tau + k\frac{f}{f'}$$

using equation 3.1.9. This can be rewritten as

$$h_f = \tau - \frac{2\pi i}{\eta_1 - Df/kf},$$

where D is the differential operator given by equation 3.1.9. Note that Df/kf is a modular form of weight 2. The idea now is to replace this term with a general modular form of weight 2.

Proposition 3.4.1. *[5] Let f be a modular function of weight 2, then*

$$h(\tau) = \tau - \frac{2\pi i}{\eta_1 + 2\pi i f(\tau)}. \tag{3.4.1}$$

is an equivariant function.

Proof: Let f be a modular form of weight 2 and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. We get

$$h(\gamma\tau) = \gamma\tau - \frac{2\pi i}{\eta_1(\gamma) + 2\pi i f(\gamma\tau)} \tag{3.4.2}$$

$$= \frac{a\tau + b}{c\tau + d} - \frac{2\pi i}{\eta_1(\tau)(c\tau + d)^2 - 2\pi i c + 2\pi i(c\tau + d)^2 f(\tau)} \tag{3.4.3}$$

Since $ad - bc = 1$, we get

$$h(\gamma\tau)\frac{(a\tau + b)\eta_1(\tau) + 2\pi i f(\tau)(a\tau + b) - 2\pi i a}{(c\tau + d)\eta_1(\tau) - 2\pi i c + 2\pi i f(\tau)}$$

On the other hand, we have

$$\gamma.h(\tau) = \frac{ah(\tau)+b}{ch(\tau)+d} \tag{3.4.4}$$

$$= \frac{a\tau - \dfrac{2\pi i a}{\eta_1 + 2\pi i f(\tau)} + b}{c\tau - \dfrac{2\pi i c}{\eta_1 + 2\pi i f(\tau)} + d} \tag{3.4.5}$$

$$= \frac{(a\tau+b)(\eta_1 + 2\pi i f(\tau)) - 2\pi i a}{(c\tau+d)(\eta_1 + 2\pi i f(\tau)) - 2\pi i c}. \tag{3.4.6}$$

The identity $h(\gamma\tau) = \gamma h(\tau)$ follows. ∎

Conversely, we have

Theorem 3.4.2. *[5] All equivariant functions where $h \neq \tau$ on $\mathbb{H}$ can be written as in* (3.4.1) *for some modular form f of weight 2.*

Proof: Let h be an equivariant function. We will prove that

$$2\pi i f(\tau) = \frac{1}{h(\tau)-\tau} + \frac{\eta_1(\tau)}{2\pi i}$$

is a modular form of weight 2. We have

$$\begin{aligned}
2\pi i f(\gamma\tau) &= \frac{1}{h(\gamma.\tau) - \gamma.\tau} + \frac{\eta_1(\gamma.\tau)}{2\pi i} \\
&= \frac{(ch(\tau)+d)(c\tau+d)}{(ah(\tau)+b)(c\tau+d) - (ch(\tau)+d)(a\tau+b)} + \frac{\eta_1(\tau)}{2\pi i}(c\tau+d)^2 - 2\pi i c(c\tau+d) \\
&= \frac{(ch(\tau)+d)(c\tau+d)}{h(\tau)-\tau} - c(c\tau+d) + (c\tau+d)^2\frac{\eta_1(\tau)}{2\pi i} \\
&= (c\tau+d)^2\left(\frac{1}{h(\tau)-\tau} + \frac{\eta_1(\tau)}{2\pi i}\right).
\end{aligned}$$

The meromorphic behavior of f at ∞ follows from the definition of an equivariant function and the behavior of η_1. Thus f is a modular form of weight 2. ∎

3.5 Modular Functions and the Cross-ratio of Equivariant Functions

We now establish a one-to-one correspondence between the set of all equivariant functions on $SL_2(\mathbb{Z})$ except the identity, $Eq(SL_2(\mathbb{Z}))$, and the set $M_0(SL_2(\mathbb{Z}))$ of all modular functions of weight 0. This construction uses the cross-ratio, which is an important invariant in projective geometry. The cross-ratio is defined for four distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)}{(z_3 - z_2)}\frac{(z_3 - z_4)}{(z_1 - z_4)}.$$

A well-known property of the cross-ratio is the invariance under Möbius transformations. Indeed, we know $\mathrm{GL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ by Möbius transformations, in the sense that $\gamma(z) = \dfrac{az+b}{cz+d}$ if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Then

$$[\gamma(z_1), \gamma(z_2), \gamma(z_3), \gamma(z_4)] = \left(\frac{\dfrac{az_1+b}{cz_1+d} - \dfrac{az_2+b}{cz_2+d}}{\dfrac{az_3+b}{cz_3+d} - \dfrac{az_2+b}{cz_2+d}}\right)\left(\frac{\dfrac{az_3+b}{cz_3+d} - \dfrac{az_4+b}{cz_4+d}}{\dfrac{az_1+b}{cz_1+d} - \dfrac{az_4+b}{cz_4+d}}\right).$$

We simplify the denominators in the right-hand side to get

$$\frac{\Big[(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)\Big]\Big[(az_3+b)(cz_4+d) - (az_4+b)(cz_3+d)\Big]}{\Big[(az_3+b)(cz_2+d) - (az_2+b)(cz_3+d)\Big]\Big[(az_1+b)(cz_4+d) - (az_4+b)(cz_1+d)\Big]}.$$

Now expanding the expressions in the square brackets and using the fact that $ad - bc \neq 0$, we obtain

$$[\gamma(z_1), \gamma(z_2), \gamma(z_3), \gamma(z_4)] = \frac{(z_1 - z_2)}{(z_3 - z_2)}\frac{(z_3 - z_4)}{(z_1 - z_4)} = [z_1, z_2, z_3, z_4].$$

This equation proves that the cross-ratio is a projectively invariant, in the sense that it is preserved by the projective (Möbius) transformations. We can now prove the following theorem.

Theorem 3.5.1 ([2, 5.3.7])**.** *Fix three mutually distinct equivariant functions h_1, h_2 and h_3. Then the cross ratio given by*

$$g(\tau) = [h_1(\tau), h_2(\tau), h_3(\tau), h(\tau)] \tag{3.5.1}$$

is a modular function of weight 0 for every equivariant function $h(\tau)$ such that $h(\tau) \neq h_1(\tau)$. Moreover, this defines a bijection between the set of the equivariant functions $h(\tau) \neq h_1(\tau)$ and the set of modular functions of weight 0.

Proof: As $h_1, h_2, h_3, h,$ are equivariant functions for $SL_2(\mathbb{Z})$ and as the cross ratio is invariant under any Möbius transformations, the function g is invariant under $SL_2(\mathbb{Z})$. The inverse $g \mapsto h$ is given by the formula

$$h(\tau) = \frac{r(\tau)h_3(\tau) - g(\tau)h_1(\tau)}{r(\tau) - g(\tau)},$$

where $r = (h_1 - h_2)(h_3 - h_2)$. ∎

Example 3.5.2. Consider the following three equivariant functions,

$$e_1 := \tau, \quad e_2 := \tau + 4\frac{g_2}{g_2'} \quad \textit{and} \quad e_3 := \frac{\eta_2}{\eta_1}.$$

Then theorem 3.5.1 provides, via the cross-ratio, a one-to-one correspondence between the equivariant functions $h(\tau)$ and the field of weight zero modular functions $g(\tau)$ for $SL_2(\mathbb{Z})$:

$$g(\tau) = \frac{e_1(\tau) - e_2(\tau)}{e_3(\tau) - e_2(\tau)} \frac{e_3(\tau) - h(\tau)}{e_1(\tau) - h(\tau)}. \tag{3.5.2}$$

3.6 The Schwarz Derivative

The goal of this section is to introduce an important differential operator and to use it to show another relationship between equivariant functions and modular forms.

Definition 3.6.1. *The Schwarz derivative of a meromorphic function h is defined by:*

$$\{h, z\} = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 = \frac{2h'h''' - 3h''^2}{2h'^2}, \tag{3.6.1}$$

where $h'(z), h''(z), h'''(z)$ are respectively the first, second and third derivatives of h.

This notion appeared in connection with equivariant functions and modular functions in [11, 13, 14]. More generally, quoting from [6]: "[Equation 3.6.1] is ubiquitous and tends to appear in seemingly unrelated fields of mathematics: classical complex analysis, and differential equations, as well as, more recently, Teichmüller theory, integrable systems, and conformal field theory.". A fundamental property of the Schwarz derivative is that it is a projective invariant: $\{\gamma(f), z\} = \{f, z\}$ for every Möbius transformation γ. More precisely:

Lemma 3.6.2. *We have $\{f, z\} = \{g, z\}$ if and only if each function is a linear fraction of the other, i.e, $g(z) = (af(z) + b)/(cf(z) + d)$, where a, b, c, d are constants* *with $ad-bc \neq 0$. In particular we have $\{f,z\}=0$ if and only if f is a linear-fractional transformation:*

$$f(z)=\frac{az+b}{cz+d} \tag{3.6.2}$$

The Schwarz derivative is sometimes considered as the differential counterpart of the cross-ratio. In fact, the Schwarz derivative is related to infinitesimal deformations of the cross-ratio in the following way

Lemma 3.6.3. *[6] Suppose h is holomorphic at τ, and $\tau_i=\tau+\epsilon a_i$, then we have*

$$\frac{[h(\tau_1),h(\tau_2),h(\tau_3),h(\tau_4)]}{[\tau_1,\tau_2,\tau_3,\tau_4]}=1+\frac{\epsilon^2}{6}(a_1-a_2)(a_3-a_4)\{h,\tau\}+O(\epsilon^3). \tag{3.6.3}$$

From this formula we can make an important observation in the case where h is an equivariant function.

Proposition 3.6.1. *[6] Suppose h is an equivariant function, then the Schwarz derivative $F(\tau)=\{h,\tau\}$ is a weight 4 modular form.*

Proof: Let τ be any point of $\mathbb{H}$ where h is holomorphic, fix four distinct complex numbers $a_1,\dots,a_4$ and denote $\tau_i=\tau+\epsilon a_i$ for all $\epsilon \geq 0$. If $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, then $\gamma(\tau_i)=\gamma(\tau)+\epsilon a_i'+O(\epsilon^2)$, where

$$a_i'=\frac{a_i}{(c\tau+d)^2}.$$

If we denote $\tau_i'=\gamma(\tau_i)$ then by lemma 3.6.3 we get

$$\frac{[h(\tau_1'),h(\tau_2'),h(\tau_3'),h(\tau_4')]}{[\tau_1',\tau_2',\tau_3',\tau_4']}=1+\frac{\epsilon^2}{6}\frac{(a_1-a_2)(a_3-a_4)}{(c\tau+d)^4}F(\gamma(\tau))+O(\epsilon^3).$$

Since the cross-ratio is $SL_2(\mathbb{Z})$-invariant and h is equivariant, we can compare this formula with the one in lemma 3.6.3 and get

$$F(\gamma(\tau))=(c\tau+d)^4F(\tau).$$

∎

We remark that the modularity of the Schwarz derivative of an equivariant function can also be proved using lemma 3.6.2 and the following chain rule: if w is a function of z, then

$$\{f,z\}=\{f,w\}(dw/dz)^2+\{w,z\}.$$

Chapter 4

Elliptic Zeta Functions

In this chapter we introduce the notion of elliptic zeta functions generalizing a construction by Brady in [2]. These are functions on $\mathbb{C} \times \mathcal{L}$, where $\mathcal{L}$ is the set of rank 2 lattices in $\mathbb{C}$, satisfying certain homogeneity and quasi-periodicity properties. The prototype is the Weierstrass $\zeta-$function. It turns out that the quasi-periods have interesting modular properties and that their quotients are equivariant functions. Interesting examples are obtained by taking integrals of powers of the Weierstrass $\wp-$function [12].

4.1 Elliptic Zeta Functions

In this section we study the theory of elliptic zeta functions with respect to the modular group $SL_2(\mathbb{Z})$. We denote by $\mathcal{L}$ the set of all lattices Λ in $\mathbb{C}$.

Definition 4.1.1 (Elliptic Zeta Function)**.** *A function $\mathcal{Z} : \mathbb{C} \times \mathcal{L} \to \hat{\mathbb{C}}$ is called an elliptic zeta function of weight m, $m \in \mathbb{Z}$, if:*

(a) For every fixed $\Lambda \in \mathcal{L}, \mathcal{Z}$ is a meromorphic function of z on $\mathbb{C}$ which is quasi-periodic, i.e, for each $\omega \in \Lambda$, there exists a unique constant $H(\omega, \Lambda) \in \mathbb{C}$ such that

$$\mathcal{Z}(z + \omega, \Lambda) = \mathcal{Z}(z, \Lambda) + H(\omega, \Lambda).$$

(b) $\mathcal{Z}$ is homogenous of weight m, i.e, for all $\lambda \in \mathbb{C}^{\times}$ we have

$$\mathcal{Z}(\lambda z, \lambda\Lambda) = \lambda^m \mathcal{Z}(z, \Lambda).$$

(c) $H_1(\tau) := H(1, \Lambda_\tau)$ and $H_2(\tau) := H(\tau, \Lambda_\tau)$ are meromorphic functions of $\tau \in \mathbb{H}$, where $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$.

The functions H_1 and H_2 above are called the *quasi-periods* of the elliptic zeta function $\mathcal{Z}$.

The classical example, which motivates our definition, is given by the Weiestrass zeta function defined by $\zeta' = -\wp$

Theorem 4.1.1. *The Weierstrass zeta function $\zeta : \mathbb{C} \times \mathcal{L} \to \hat{\mathbb{C}}$ is an elliptic zeta function of weight -1.*

Proof: Condition (a) was proved in Example 2.4.2 and condition (b) was proved in theorem 2.4.4. The two quasi-periods of ζ are η_1 and η_2. The fact that η_1 is meromorphic can be seen from equation 2.4.3, from theorem 3.1.6(a) or from equation 3.1.13. By Legendre's relation, the second quasi-period is $\eta_2(\tau) = \tau\eta_1(\tau) + 2\pi i$, so it is meromorphic as well. ∎

It follows from definition 4.1.1 that for each Λ, the quasi-period function H_Λ is $\mathbb{Z}-$linear, and therefore it is completely determined by $H_\Lambda(\omega_1)$ and $H_\Lambda(\omega_2)$. Moreover, we have the following result generalizing theorem 2.4.4.

Proposition 4.1.2. *Let $\mathcal{Z}$ be an elliptic zeta function of weight k and let H_Λ be the quasi-period function for each lattice Λ. Then for all $\alpha \in \mathbb{C}^\times$ and $\omega \in \Lambda$, we have*

$$H(\alpha\omega, \alpha\Lambda) = \alpha^k H(\omega, \Lambda). \tag{4.1.1}$$

Proof: On one hand we have:

$$\begin{aligned} \mathcal{Z}(\alpha(z+\omega), \alpha\Lambda) &= \mathcal{Z}(\alpha z, \alpha\Lambda) + H_{\alpha\Lambda}(\alpha\omega) \\ &= \alpha^k \mathcal{Z}(z, \Lambda) + H(\alpha\omega, \alpha\Lambda). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathcal{Z}(\alpha(z+\omega), \alpha\Lambda) &= \alpha^k \mathcal{Z}(z+\omega, \Lambda) \\ &= \alpha^k (\mathcal{Z}(z, \Lambda) + H(\omega, \Lambda)), \end{aligned}$$

and the proposition follows. ∎

Notice that two elliptic zeta functions having the same quasi-period function must differ by an elliptic function. Simple examples are given by the identity map z, or the Weierstrass zeta function $\zeta(z, \Lambda)$. We will see below that these two examples will, in a way, generate all the other elliptic zeta functions. Also, since for a fixed lattice the derivative of an elliptic zeta function with respect to z is an elliptic function for the lattice, we have a way to construct infinitely many of them by taking integrals of elliptic functions.

In the following we will simplify the notation of $H(\omega_i, \Lambda)$ by $H(\omega_i)$ and $\eta(\omega_i, \Lambda)$ by $\eta(\omega_i)$ for $i = 1$, 2 when there is no confusion.

Let ω_1 and ω_2 be such that $\Im(\omega_2/\omega_1) > 0$ and set

$$M_{(\omega_1,\omega_2)} = \begin{bmatrix} \omega_2 & \eta(\omega_2) \\ \omega_1 & \eta(\omega_1) \end{bmatrix},$$

where η is the quasi-period map of the Weierstrass zeta function $\zeta(z, \omega_1\mathbb{Z} + \omega_2\mathbb{Z})$. Using the Legendre relation (2.4.5), we have

$$\det M_{(\omega_1,\omega_2)} = -2\pi i.$$

Let $\mathcal{Z}$ be an elliptic zeta function of weight k with the two quasi-periods $H(\omega_1)$ and $H(\omega_2)$ and set

$$\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = M^{-1}_{(\omega_1,\omega_2)} \begin{bmatrix} H(\omega_2) \\ H(\omega_1) \end{bmatrix}.$$

In other words, the relations (4.1.2) and (4.1.3) are the generalizations for an elliptic zeta function of the Legendre relation 2.4.5 for the Weierstrass zeta function. The following proposition is straightforward.

Proposition 4.1.3 (Generalized Legendre relation)**.**

$$2\pi i\Phi = \eta(\omega_2)H(\omega_1) - \eta(\omega_1)H(\omega_2) \tag{4.1.2}$$

$$2\pi i\Psi = \omega_1 H(\omega_2) - \omega_2 H(\omega_1) \tag{4.1.3}$$

Note that for $\Phi = 1, \Psi = 0$, we find the classical Legendre relation.

Proposition 4.1.4. *The quantities Φ and Ψ do not depend on the choice of the basis (ω_1, ω_2), and, as functions of the lattice $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, they are homogeneous of respective weights $k - 1$ and $k + 1$.*

Proof: Let a, b, c and d be integers such that $ad - bc = 1$. The expressions in equations 4.1.2 and 4.1.3 are invariant if we change the basis (ω_1, ω_2) to the basis $(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$. Indeed, using the linearity of η and H, we have for the expression of $2\pi i\Phi$:

$$\begin{aligned} &\eta(c\omega_1 + d\omega_2)H(a\omega_1 + b\omega_2) - \eta(a\omega_1 + b\omega_2)H(c\omega_1 + d\omega_2) \\ &\quad = [c\eta(\omega_1) + d\eta(\omega_2)][aH(\omega_1) + bH(\omega_2)] - [a\eta(\omega_1) + b\eta_(\omega_2)][cH(\omega_1) + dH(\omega_2)] \\ &\quad = H(\omega_1)\eta(\omega_2) - H(\omega_2)\eta(\omega_1)\ . \end{aligned}$$

Similar calculations hold for the expression of $2\pi i\Psi$. The values of the weights are straightforward knowing that the weight is k for H, -1 for η and 1 for both ω_1 and ω_2. ∎

We can therefore denote Φ and Ψ by Φ_Λ and Ψ_Λ as they depend only on the lattice Λ and not on the oriented basis.

Proposition 4.1.5. *Let $\mathcal{Z}$ be an elliptic zeta function of weight k and quasi-period function H, and let Φ_Λ and Ψ_Λ be as above. Then for each lattice Λ, there exists an elliptic function E_Λ such that*

$$\mathcal{Z}(z,\Lambda) \;=\; \Phi_\Lambda\, z \;+\; \Psi_\Lambda \zeta(z) \;+\; E_\Lambda(z). \tag{4.1.4}$$

Proof: It is clear by construction of Φ and Ψ that the map $\Phi_\Lambda z + \Psi_\Lambda \zeta$ satisfies the conditions of a weight k elliptic zeta function. Moreover, for each $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, the quasi-periods of $\Phi_\Lambda\, z \;+\; \Psi_\Lambda\, \zeta(z)$ are $\Phi_\Lambda\, \omega_i \;+\; \Psi_\Lambda\, \eta(\omega_i)$, $i = 1,\, 2$, which coincide with the quasi-periods $H(\omega_1)$ and $H(\omega_2)$ of $\mathcal{Z}$ as we have $\begin{bmatrix} H(\omega_2) \\ H(\omega_1) \end{bmatrix} = \begin{bmatrix} \omega_2 & \eta(\omega_2) \\ \omega_1 & \eta(\omega_1) \end{bmatrix} \begin{bmatrix} \Phi_\Lambda \\ \Psi_\Lambda \end{bmatrix}$. Therefore the two elliptic zeta functions differ by an elliptic function for the lattice Λ. ∎

It is clear that the expression (4.1.4) for an elliptic zeta function is unique up to the elliptic function $E_\Lambda(z)$ since Φ_Λ and Ψ_Λ are uniquely determined.

4.2 Modular Forms and Elliptic Zeta Functions

In this section we will investigate the connection between elliptic zeta functions and modular forms for $SL_2(\mathbb{Z})$. As we will see in the following theorem, each elliptic zeta function gives rise to a weight 2 (meromorphic) modular form for $SL_2(\mathbb{Z})$, and conversely, each weight 2 modular form yields an elliptic zeta function.

Theorem 4.2.1. *Let $\mathcal{Z}$ be an elliptic zeta function with Φ_Λ and Ψ_Λ as in equation* (4.1.4) *and suppose Ψ_{Λ_τ} is not identically zero as a function of τ. Then the map*

$$\mathcal{Z} \longmapsto \frac{\Phi_{\Lambda_\tau}}{\Psi_{\Lambda_\tau}} \tag{4.2.1}$$

is well defined between the set of elliptic zeta functions and the space of weight 2 modular forms $M_2(SL_2(\mathbb{Z}))$. In addition, this map is surjective.

Proof: Let $k \in \mathbb{Z}$ be the weight of $\mathcal{Z}$ and set

$$f(\tau) \;=\; \frac{\Phi_{\Lambda_\tau}}{\Psi_{\Lambda_\tau}}\,,\;\; \tau \in \mathbb{H}.$$

Because Φ_{Λ_τ} and Ψ_{Λ_τ} are meromorphic in τ, so is $f(\tau)$. Now let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Also, as Φ_Λ and Ψ_Λ are homogeneous of weights $k-1$ and $k+1$ respectively, we have

$$\Phi_{\Lambda_{\gamma\tau}} \;=\; (c\tau+d)^{-k+1}\Phi_{(a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}} \;=\; (c\tau+d)^{-k+1}\Phi_{\Lambda_\tau}$$

and

$$\Psi_{\Lambda_{\gamma\tau}} \;=\; (c\tau+d)^{-k-1}\Psi_{(a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}} \;=\; (c\tau+d)^{-k-1}\Psi_{\Lambda_\tau}.$$

Therefore

$$f(\gamma\tau) \;=\; (c\tau+d)^2\, f(\tau)\,.$$

Hence the map is well defined as Φ_Λ and Ψ_Λ are uniquely determined by $\mathcal{Z}$.
We now prove that the map is surjective. Let $f \in M_2(SL_2(\mathbb{Z}))$ and set, for $\Lambda = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$,

$$\Phi_\Lambda \;=\; \frac{1}{\omega_1^2}\, f\left(\frac{\omega_2}{\omega_1}\right)\,,\;\; \Psi_\Lambda \;=\; 1\,.$$

The map Φ_Λ is well defined in the sense that it is independent of the choice of the basis (ω_1, ω_2). Indeed, if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, then, because f has weight 2,

$$\begin{aligned}\frac{1}{(a\omega_1+b\omega_2)^2}\, f\left(\frac{c\omega_1+d\omega_2}{a\omega_1+b\omega_2}\right) \;&=\; \frac{1}{(a\omega_1+b\omega_2)^2}\, f\left(\frac{d\,\frac{\omega_2}{\omega_1}+c}{b\,\frac{\omega_2}{\omega_1}+a}\right)\\ &=\; \frac{(b\,\frac{\omega_2}{\omega_1}+a)^2}{(a\omega_1+b\omega_2)^2}\, f\left(\frac{\omega_2}{\omega_1}\right)\\ &=\; \frac{1}{\omega_1^2}\, f\left(\frac{\omega_2}{\omega_1}\right)\,.\end{aligned}$$

Thus, we have an elliptic zeta function

$$\mathcal{Z}(z,\Lambda) \;=\; \frac{1}{\omega_1^2}\, f\left(\frac{\omega_2}{\omega_1}\right)\, z+\zeta(z)$$

of weight -1 that is sent to $f(\tau)$ by the map (4.2.1). ∎

4.3 Equivariant Functions and Elliptic Zeta Functions

In this section we study the relationship between the elliptic zeta functions and equivariant functions that were introduced earlier.

Proposition 4.3.1. *Let $\mathcal{Z}$ be an elliptic zeta function with quasi-period maps H_1, H_2. Suppose that H_1 is not identically zero, then the meromorphic function $h(\tau) = H_2(\tau)/H_1(\tau)$ is equivariant.*

Proof: As h is the ratio of two functions that are meromorphic on $\mathbb{H}$, and whose denominator is not identically 0, it defines a meromorphic function on $\mathbb{H}$. Now for $\gamma \in SL_2(\mathbb{Z})$, using the linearity of H, we have

$$\begin{aligned}
\gamma(h) &= \frac{a\dfrac{H(\tau,\Lambda_\tau)}{H(1,\Lambda_\tau)} + b}{c\dfrac{H(\tau,\Lambda_\tau)}{H(1,\Lambda_\tau)} + d} \\
&= \frac{aH(\tau,\Lambda_\tau) + bH(1,\Lambda_\tau)}{cH(1,\Lambda_\tau) + dH(1,\Lambda_\tau)} \\
&= \frac{H(a\tau+b,\Lambda_\tau)}{H(c\tau+d,\Lambda_\tau)} \\
&= \frac{H(a\tau+b,(c\tau+d)\Lambda_{\gamma(\tau)})}{H(c\tau+d,(c\tau+d)\Lambda_{\gamma(\tau)})} \\
&= \frac{H(\gamma\tau,\Lambda_{\gamma\tau})}{H(\tau,\Lambda_{\gamma\tau})} \\
&= h(\gamma(\tau)).
\end{aligned}$$

∎

The results of this section and of the previous one provide maps from the set of elliptic zeta functions to both the set M_2 of modular functions of weight 2 and the set Eq of equivariant functions. We also recall from the previous chapter that we have a one-to-one correspondence between M_2 and $Eq - \{\tau\}$. This situation is summarized in the diagram in Figure 4.1.

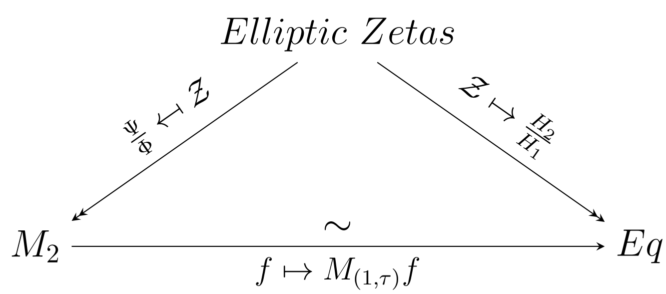


Figure 4.1: The triple correspondence

In fact, it is not difficult to show that this diagram is commutative. We observe that the identity equivariant function $h(\tau) = \tau$ is the image through the map

$$\mathcal{Z} \longmapsto \frac{H_\tau}{H_1} \tag{4.3.1}$$

of the 'identity' elliptic zeta function $\mathcal{Z}(z,\Lambda) = z$. In particular, it is easy to see from the commutativity of the triple-correspondence diagram that the map defined in proposition 4.3.1 is surjective.

4.4 Examples of Elliptic Zeta Functions

Here we will study an important class of elliptic functions given by integrals of the powers $\wp^n$ of the Weierstrass $\wp-$function. These integrals were treated in [14]. For a non-negative integer n, the power $\wp^n(z)$ can be written as a linear combination of 1, $\wp$ and successive derivatives of $\wp$.

Theorem 4.4.1 ([17], page 108). *For $n \in \mathbb{Z}$, we have*

$$\wp^n(z,\Lambda) \;=\; \Phi_n(\Lambda) - \Psi_n(\Lambda)\wp(z,\Lambda) + \sum_{k=1}^{n-1} \alpha_k \wp^{(2k)}, \tag{4.4.1}$$

where the coefficients α_k are polynomials in g_2 and g_3 with rational coefficients. For example, we have

$$\Phi_0 = 1, \quad \Psi_0 = 0, \quad \Phi_1 = 0 \quad \text{and} \quad \Psi_1 = -1. \tag{4.4.2}$$

For each lattice Λ and $z \in \mathbb{C}$, a primitive $\displaystyle\int \wp^n(u)du$ of $\wp^n$ has the form

$$\Phi_n(\Lambda)\, z + \Psi_n(\Lambda)\zeta(z,\Lambda) + E_n(z,\Lambda) \tag{4.4.3}$$

where for each Λ, $E_n(z,\Lambda)$ is a $\Lambda-$elliptic function.

Next, we define

$$\mathcal{Z}_n(z,\Lambda) \;:=\; \Phi_n(\Lambda)\, z + \Psi_n(\Lambda)\zeta(z,\Lambda).$$

It is clear that for each Λ, $\mathcal{Z}_n(z,\Lambda)$ is quasi-periodic with the quasi-period map given by

$$H_n(\omega,\Lambda) = \Phi_n(\Lambda)\, \omega + \Psi_n(\Lambda)\eta(\omega),$$

where η is the quasi-period map for the Weierstrass $\zeta-$function. If there is no confusion, we will write Φ_n for $\Phi_n(\Lambda)$ and Ψ_n for $\Psi_n(\Lambda)$.

Proposition 4.4.1. *The maps Φ_n, Ψ_n, and thus H_n satisfy the same three-term recurrence relation*

$$u_{n+1} = r(n)g_2u_{n-1} + s(n)g_3u_{n-2}.$$

where $r(n) = \frac{2n-1}{4(2n+1)}$, $s(n) = \frac{n-1}{2(2n+1)}$, with the initial conditions stated in equation 4.4.2 and $\Phi_{-1} = \Psi_{-1} = 0$.

Proof: See [17], page 109, and [14, §9]. ∎

Theorem 4.4.2. *The map $\mathcal{Z}_n(z,\Lambda)$ is an elliptic zeta function of weight $-2n+1$.*

Proof: One can easily see that when $\Lambda = \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, H_n is polynomial in g_2, g_3, η_1 and η_2, and thus $H_n(1)$ and $H_n(\tau)$ are meromorphic functions of τ. ∎

Let $\Phi_n(\tau) = \Phi_n(\Lambda_\tau)$, $\Psi_n(\tau) = \Psi_n(\Lambda_\tau)$, $\eta_1 = \eta(1)$ and $\eta_2 = \eta(\tau)$. Then $\Phi_2(\tau) = \dfrac{1}{12}\, g_2(\tau)$, $\Phi_3(\tau) = \dfrac{1}{10}\, g_3(\tau)$, $\Psi_2(\tau) = 0$ and $\Psi_3(\tau) = \dfrac{-3}{20}\, g_2(\tau)$. More generally, we can use the above proposition to show the following proposition by induction.

Proposition 4.4.2. *For each positive integer n, Φ_n and Ψ_n are weighted homogeneous polynomials in g_2 and g_3 with rational coefficients and of degrees $2n$ and $2(n-1)$ respectively, and these degrees are also their weights as holomorphic modular forms.*

Proof: For $\gamma(\tau) = \dfrac{a\tau+b}{c\tau+d}$, where $\gamma \in SL_2(\mathbb{Z})$, we have

$$\Phi_n(\Lambda_{\gamma\tau}) = \phi_n\Big(\frac{1}{c\tau+d}\Lambda_\tau\Big) = (c\tau+d)^{2n}\Phi_n(\Lambda_\tau)$$

and

$$\Psi_n(\Lambda_{\gamma\tau}) = \psi_n\Big(\frac{1}{c\tau+d}\Lambda_\tau\Big) = (c\tau+d)^{2n-2}\Psi_n(\Lambda_\tau).$$

∎

In light of proposition 4.3.1, the elliptic zeta function corresponds to the following equivariant function:

$$h_n(\tau) = \frac{H_n(\tau)}{H_n(1)} = \frac{\Phi_n(\tau)\,\tau + \Psi_n(\tau)\eta_2}{\Phi_n(\tau) + \Psi_n(\tau)\eta_1}.$$

Applying Legendre's relation, we obtain

$$h_n(\tau) \;=\; \tau + \frac{2\pi i}{f_n(\tau)+\eta_1}, \qquad (4.4.4)$$

where $f_n = \frac{\Phi_n}{\Psi_n}$ is a modular function of weight 2. The following table gives Φ_n, Ψ_n, f_n and h_n for $1 \le n \le 10$

n	Φ_n	Ψ_n	f_n
1	0	1	0
2	$\frac{g_2}{12}$	0	-
3	$\frac{g_3}{10}$	$\frac{3g_2}{20}$	$\frac{2g_3}{3g_2}$
4	$\frac{5g_2^2}{336}$	$\frac{g_3}{7}$	$\frac{5g_2^2}{48g_3}$
5	$\frac{g_2g_3}{30}$	$\frac{7g_2^2}{240}$	$\frac{8g_3}{7g_2}$
6	$\frac{15g_2^3}{4928}+\frac{g_3^2}{55}$	$\frac{87g_2g_3}{1540}$	$\frac{25g_2^2}{464g_3}+\frac{28g_3}{87g_2}$
7	$\frac{433g_2^2g_3}{43680}$	$\frac{77g_2^3}{12480}+\frac{5g_3^2}{182}$	$\frac{866g_2^2g_3}{539g_2^3+2400g_3^2}$
8	$\frac{13g_2^4}{19712}+\frac{7}{660}g_3^2g_2$	$\frac{167g_2^2g_3}{9240}$	$\frac{195g_2^2}{5344g_3}+\frac{98g_3}{167g_2}$
9	$\frac{383g_3g_2^3}{136136}+\frac{7g_3^3}{1870}$	$\frac{77g_2^4}{56576}+\frac{6021g_3^2g_2}{340340}$	$\frac{61280g_3g_2^3+81536g_3^3}{29645g_2^4+385344g_3^2g_2}$
10	$\frac{2873g_2^5+86848g_3^2g_2^2}{19475456}$	$\frac{3251g_3g_2^3+3520g_3^3}{608608}$	$\frac{2873g_2^5+86848g_3^2g_2^2}{104032g_3g_2^3+112640g_3^3}$

Chapter 5

Elliptic Zeta Functions With Respect to a Modular Subgroup

In this chapter, we establish a close connection between three notions attached to a modular subgroup [12]. Namely, the set of weight two meromorphic modular forms, the set of equivariant functions on the upper half-plane commuting with the action of the modular subgroup, and the set of elliptic zeta functions, generalizing the case of $SL_2(\mathbb{Z})$ of the previous chapter. In particular, we show that the equivariant functions for a modular subgroup can be parameterized by modular objects as well as by elliptic objects.

5.1 $\Gamma-$Elliptic Zeta Functions

Here we generalize the construction of elliptic zeta functions from the full modular group $SL_2(\mathbb{Z})$ to any finite index subgroup.

The elliptic zeta functions $\mathcal{Z} : \mathbb{C} \times \mathcal{L} \to \hat{\mathbb{C}}$ that we studied in Chapter 3 are related to the modular group $SL_2(\mathbb{Z})$ as follows. Set

$$\mathbb{M} = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : \Im(\omega_2/\omega_1) > 0\}.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{M}$ in the following way: if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\gamma(\omega_1, \omega_2) = (c\omega_2 + d\omega_1, a\omega_2 + b\omega_1) = (\omega_1, \omega_2)\gamma^t.$$

This action is defined so that it commutes with the natural map $\mathbb{M} \to \mathbb{H}$ given by

$$(\omega_1, \omega_2) \mapsto \tau = \frac{\omega_2}{\omega_1}$$

and with the usual action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ by Möbius transformations.

Theorem 5.1.1. *The map $f : SL_2(\mathbb{Z})\backslash\mathbb{M} \to \mathcal{L}$ that takes the orbit of a pair (ω_1, ω_2) to the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ is an isomorphism.*

Proof:

1) f is well-defined: Let (ω_1, ω_2), and (ω_1', ω_2') be two pairs in $\mathbb{M}$ such that $(\omega_1', \omega_2') = (\omega_1, \omega_2)\gamma^t$ for some $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. If $\Lambda_1 = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $\Lambda_2 = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$, we need to show that $\Lambda_1 = \Lambda_2$. Indeed

$$\begin{aligned} \Lambda_2 &= \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z} \\ &= (a\omega_1 + b\omega_2)\mathbb{Z} + (c\omega_1 + d\omega_2)\mathbb{Z} \\ &\subseteq a\omega_1\mathbb{Z} + b\omega_2\mathbb{Z} + c\omega_1\mathbb{Z} + \omega_2\mathbb{Z} \\ &\subseteq \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \end{aligned}$$

so $\Lambda_2 \subseteq \Lambda_1$. Analogously we have $\Lambda_1 \subseteq \Lambda_2$ and so $\Lambda_1 = \Lambda_2$.

2) f is surjective: Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \in \mathcal{L}$. Then ω_1 and ω_2 are complex numbers linearly independent over $\mathbb{R}$, so $\Im(\omega_2/\omega_1) \neq 0$. If $\Im(\omega_2/\omega_1) > 0$, then $(\omega_1, \omega_2) \in \mathbb{M}$ and $f([\omega_1, \omega_2]) = \Lambda$, where $[\omega_1, \omega_2]$ denotes the class of (ω_1, ω_2) modulo $SL_2(\mathbb{Z})$. If instead $\Im(\omega_2/\omega_1) < 0$, then $(\omega_2, \omega_1) \in \mathbb{M}$ and $f([\omega_2, \omega_1]) = \Lambda$.

3) f is injective: Let $[\omega_1, \omega_2]$ and $[\omega_1', \omega_2']$ be two classes in $SL_2(\mathbb{Z})\backslash\mathbb{M}$ such that $f([\omega_1, \omega_2]) = f([\omega_1', \omega_2']) = \Lambda$, that is

$$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \Lambda = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'.$$

In particular, $\omega_2' = a\omega_2 + b\omega_1$ and $\omega_1' = c\omega_2 + d\omega_1$ for some integers a, b, c, d. Thus we have

$$\gamma \cdot \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$$

for some matrix γ with integer entries. Inverting the roles of (ω_1, ω_2) and (ω_1', ω_2'), we find another matrix γ' with integer entries so that $\gamma\gamma'$ is the identity. Therefore $\gamma \in GL_2(\mathbb{Z})$, that is $\det\gamma \in \{\pm 1\}$. However by the formula

$$\Im\left(\frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1}\right) = \frac{(ad - bc)|\omega_1|^2}{|c\omega_2 + d\omega_1|^2}\Im(\omega_2/\omega_1)$$

we see that $\det\gamma > 0$ and so $\gamma \in SL_2(\mathbb{Z})$. ∎

Now we can see how the elliptic zeta function

$$\mathcal{Z} : \mathbb{C} \times SL_2(\mathbb{Z})\backslash\mathbb{M} \to \hat{\mathbb{C}}$$

is related to the modular group $SL_2(\mathbb{Z})$. In the notion of modular forms and equivariant functions, the group $SL_2(\mathbb{Z})$ can be restricted to any subgroup Γ of finite index. Thus, we need to define the notion of elliptic zeta functions for any such modular subgroup by replacing $SL_2(\mathbb{Z})$ by Γ. Denote by Ω_Γ the quotient $\Gamma\backslash\mathbb{M}$ and the class of (ω_1, ω_2) by $[\omega_1, \omega_2]$. Also, $\mathbb{C}^\times$ acts on $\mathbb{M}$ in the usual way and this action descends to Ω_Γ as:

$$\alpha[\omega_1, \omega_2] = [\alpha\omega_1, \alpha\omega_2].$$

If $\Gamma = SL_2(\mathbb{Z})$, then $[\omega_1, \omega_2]$ is identified with the lattice $\Lambda_{(\omega_1,\omega_2)} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. However, for an arbitrary finite index subgroup Γ, the situation is different. Following the ideas in [3], Ω_Λ is identified with the set of pairs of lattices (Λ, Λ') with Λ' being a finite index sub-lattice of Λ fixed by Γ and Λ' is the smallest such lattice (and thus defined as the intersection of all such sub-lattices that are $\Gamma-$invariant). If such pair (Λ, Λ') is given, and as $SL_2(\mathbb{Z})$ acts by automorphisms of Λ by a change of basis, Γ would be defined by

$$\Gamma = \{\gamma \in SL_2(\mathbb{Z}) : \gamma\Lambda' \subseteq \Lambda'\}.$$

For example, if $\Gamma = \Gamma(N)$ is the principal congruence subgroup of level $N \geq 1$, then $\Lambda' = N\omega_1\mathbb{Z} + N\omega_2\mathbb{Z}$, which is a sub-lattice of $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ of index N^2 . If $\Gamma = \Gamma_0(N)$, then $\Lambda' = \omega_1\mathbb{Z} + N\omega_2\mathbb{Z}$ of index N in $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. However, we will not need this identification in what follows.

Definition 5.1.1. *A $\Gamma-$elliptic zeta function with respect to Γ is a map*

$$\mathcal{Z} : \mathbb{C} \times \Omega_\Gamma \longrightarrow \hat{\mathbb{C}}$$

satisfying the following properties:

1. *For each $[\omega_1, \omega_2] \in \Omega_\Gamma$, the map*

$$\mathcal{Z}(\cdot, [\omega_1, \omega_2]) : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$$

   is quasi-periodic with respect to $\Lambda_{(\omega_1,\omega_2)}$, that is, for all $z \in \mathbb{C}$ and all $\omega \in \Lambda_{(\omega_1,\omega_2)}$, we have

$$\mathcal{Z}(z + \omega, [\omega_1, \omega_2]) = \mathcal{Z}(z, [\omega_1, \omega_2]) + H_{[\omega_1,\omega_2]}(\omega).$$

2. *The map $\mathcal{Z}$ is homogeneous, that is, there exists an integer k, referred to as the weight of $\mathcal{Z}$, such that for all $\alpha \in \mathbb{C}^\times$, $[\omega_1, \omega_2] \in \Omega_\Gamma$ and $z \in \mathbb{C}$, we have*

$$\mathcal{Z}(\alpha z, \alpha[\omega_1, \omega_2]) = \alpha^k \mathcal{Z}(z, [\omega_1, \omega_2]).$$

3. *The maps*

$$\tau \mapsto H_{[1,\tau]}(\tau) \text{ and } \tau \mapsto H_{[1,\tau]}(1)$$

   are meromorphic in $\mathbb{H}$.

From this definition, it is clear that the quasi-period map $H_{[\omega_1,\omega_2]}$ is $\mathbb{Z}-$linear on the lattice $\Lambda_{(\omega_1,\omega_2)}$ and thus it is completely determined by its values on ω_1 and ω_2. It is also homogeneous of weight k:

$$H_{[\alpha\omega_1,\alpha\omega_2]}(\alpha\omega) = \alpha^k H_{[\omega_1,\omega_2]}(\omega)\,,\ \omega \in \Lambda_{(\omega_1,\omega_2)}\,,\ \alpha \in \mathbb{C}^\times.$$

Using the same arguments as in §4.1, one can easily establish the following.

Proposition 5.1.2. *Let $\mathcal{Z} : \mathbb{C} \times \Omega_\Gamma \longrightarrow \hat{\mathbb{C}}$ be a $\Gamma-$elliptic zeta function. There exists unique maps $\Phi_{[\omega_1,\omega_2]}$ of weight $k-1$ and $\Psi_{[\omega_1,\omega_2]}$ of weight $k+1$ such that for all $[\omega_1,\omega_2] \in \Omega_\Gamma$ and $z \in \mathbb{C}$, we have*

$$\mathcal{Z}(z,[\omega_1,\omega_2]) = \Phi_{[\omega_1,\omega_2]}\, z + \Psi_{[\omega_1,\omega_2]}\, \zeta(z) + E_{[\omega_1,\omega_2]}(z)\,,$$

where $E_{[\omega_1,\omega_2]}(z)$ is a $\Lambda_{(\omega_1,\omega_2)}-$elliptic function.

Notice that $\Phi_{[\omega_1,\omega_2]}$ and $\Psi_{[\omega_1,\omega_2]}$ can be shown to be independent of the choice of the representative of the class $[\omega_1,\omega_2]$ in the same way as in Proposition 4.1.4 using transformations from Γ instead of $SL_2(\mathbb{Z})$.

5.2 $\Gamma-$Modular Forms Versus $\Gamma-$Equivariant Functions

We now review the notion of equivariant functions. We will see that they arise from elliptic objects and weight two modular forms with respect to Γ and will establish a correspondence between them.
For a finite index subgroup Γ of $SL_2(\mathbb{Z})$, an equivariant function is a meromorphic function on the upper half-plane $\mathbb{H}$ which commutes with the action of Γ on $\mathbb{H}$. Namely, for $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$. We have

$$f(\gamma\tau) = \gamma f(\tau)\,,$$

where γ acts by linear fractional transformations on both sides. These were studied extensively in connection with modular forms in [5, 4, 14] and have important applications to modular forms and vector-valued modular forms [9, 10].

Recall that if f is a meromorphic function and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, we denote

$$\gamma f(\tau) \;=\; \frac{af(\tau)+b}{cf(\tau)+d}\,.$$

Now recall from formula (4.1) the matrix

$$M_{(1,\tau)} \;=\; \begin{bmatrix} \tau & \eta_\tau \\ 1 & \eta_1 \end{bmatrix},$$

which is invertible thanks to the Legendre relation. Let $M_2(\Gamma)$ denote the space of meromorphic weight two modular forms and $Eq(\Gamma)$ be the set of equivariant functions, both with respect to Γ. Although $h(\tau) = \tau$ is trivially equivariant, it will be excluded from $Eq(\Gamma)$.

Theorem 5.2.1. *The map from $M_2(\Gamma)$ to $Eq(\Gamma)$ given by*

$$f \longmapsto M_{(1,\tau)}\, f$$

is a bijection. The inverse map is given by

$$h \longmapsto M^{-1}_{(1,\tau)}\, h.\;.$$

Proof: Let $f \in M_2(\Gamma)$ and set

$$h(\tau) \;=\; M_{(1,\tau)}\, f(\tau) \;=\; \frac{\tau f(\tau) + \eta(\tau)}{f(\tau) + \eta(1)}.$$

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, we have

$$h(\gamma\tau) \;=\; \frac{\gamma\tau f(\gamma\tau) + \eta_{\Lambda_{\gamma\tau}}(\gamma\tau)}{f(\gamma\tau) + \eta_{\Lambda_\tau}(1)}\;.$$

Since

$$\eta_{\Lambda_{\gamma\tau}}(\gamma\tau) = (c\tau + d)(a\eta(\tau) + b\eta(1)),$$
$$\eta_{\Lambda_\tau}(1) = (c\tau + d)(c\eta(\tau) + d\eta(1))$$

and

$$f(\gamma\tau) = (c\tau + d)^2 f(\tau),$$

we have

$$h(\gamma\tau) = \frac{(a\tau + b)f(\tau) + a\eta(\tau) + b\eta(1)}{(c\tau + d)f(\tau) + c\eta(\tau) + d\eta(1)} = \gamma h(\tau).$$

Similarly, one can prove that if $h \in Eq$, then $M^{-1}_{(1,\tau)}\, h \in \Gamma$. ∎

5.3 The Triple Relationship Between $\Gamma-$elliptic Zeta Functions, $\Gamma-$ Modular Forms and $\Gamma-$Equivariant Functions

In this chapter we explored three distinct notions attached to a modular subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$: the set of equivariant functions for Γ, the space of weight 2 meromorphic modular forms $M_2(\Gamma)$ and the Γ-elliptic zeta functions. We will now study the relationship between these three notions.

We will start with the connection between elliptic zeta functions and modular forms for Γ. Let us show that each elliptic zeta function gives rise to a weight 2 (meromorphic) modular form for Γ, and conversely, each weight 2 modular form yields an elliptic zeta function.

Theorem 5.3.1. *Let $\mathcal{Z}$ be an elliptic zeta function with $\Phi_{[1,\tau]}$ and $\Psi_{[1,\tau]}$ as in equation 5.1.1 and suppose $\Psi_{[1,\tau]}$ is not identically zero as a function of τ. Then the map*

$$\mathcal{Z} \longmapsto \frac{\Phi_{[1,\tau]}}{\Psi_{[1,\tau]}} \tag{5.3.1}$$

is well defined between the set of elliptic zeta functions and the space of weight 2 modular forms $M_2(\Gamma)$. In addition, this map is surjective.

Proof: Let $k \in \mathbb{Z}$ be the weight of $\mathcal{Z}$ and set

$$f(\tau) = \frac{\Phi_{[1,\tau]}}{\Psi_{[1,\tau]}}, \ \tau \in \mathbb{H}.$$

Since $\Phi_{[1,\tau]}$ and $\Psi_{[1,\tau]}$ are meromorphic in τ, so is $f(\tau)$. Now let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. As $\Phi_{[1,\tau]}$ and $\Psi_{[1,\tau]}$ are homogeneous of weights $k-1$ and $k+1$ respectively, we have

$$\Phi_{[1,\gamma\tau]} = (c\tau + d)^{-k+1}\Phi_{[a\tau+b,c\tau+d]} = (c\tau + d)^{-k+1}\Phi_{[1,\tau]}$$

and

$$\Psi_{[1,\gamma\tau]} = (c\tau + d)^{-k+1}\Psi_{[a\tau+b,c\tau+d]} = (c\tau + d)^{-k+1}\Psi_{[1,\tau]}.$$

Therefore

$$f(\gamma\tau) = (c\tau + d)^2 f(\tau) .$$

Hence the map is well defined as $\Phi_{[1,\tau]}$ and $\Psi_{[1,\tau]}$ are uniquely determined by $\mathcal{Z}$. We now prove that the map is surjective. For $f \in M_2(\Gamma)$ we have

$$\Phi_{[\omega_1,\omega_2]} = \frac{1}{\omega_1^2} f\left(\frac{\omega_2}{\omega_1}\right) , \ \Psi_{[\omega_1,\omega_2]} = 1 .$$

The map $\Phi_{[\omega_1,\omega_2]}$ is well defined in the sense that it is independent of the choice of the basis (ω_1, ω_2). Indeed, if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, then

$$\begin{aligned}
\frac{1}{(a\omega_1 + b\omega_2)^2}\, f\left(\frac{c\omega_1 + d\omega_2}{a\omega_1 + b\omega_2}\right) &= \frac{1}{(a\omega_1 + b\omega_2)^2}\, f\left(\frac{d\,\frac{\omega_2}{\omega_1} + c}{b\,\frac{\omega_2}{\omega_1} + a}\right) \\
&= \frac{(b\,\frac{\omega_2}{\omega_1} + a)^2}{(a\omega_1 + b\omega_2)^2}\, f\left(\frac{\omega_2}{\omega_1}\right) \\
&= \frac{1}{\omega_1^2}\, f\left(\frac{\omega_2}{\omega_1}\right).
\end{aligned}$$

Thus, we have an elliptic zeta function

$$\mathcal{Z}(z, [\omega_1, \omega_2]) = \frac{1}{\omega_1^2}\, f\left(\frac{\omega_2}{\omega_1}\right) z + \zeta(z)$$

of weight -1 that is sent to $f(\tau)$ by the map (5.3.1). ∎

$H(\tau)$ turns out to hold important information and is used to construct equivariant functions as well as elements of $M_2(\Gamma)$.

Theorem 5.3.2. *Let $\mathcal{Z}$ be an elliptic zeta function with quasi-period map H as in definition 5.1.1 and suppose $H_{[1,\tau]}(1)$ is not identically zero. Then the map*

$$\mathcal{Z} \longmapsto \frac{H_{[1,\tau]}(\tau)}{H_{[1,\tau]}(1)} \tag{5.3.2}$$

is well defined between the set of $\Gamma-$elliptic zeta functions and the space of equivariant functions $Eq(\Gamma)$. In addition, this map is surjective.

Proof: Set $h = \dfrac{H_{[1,\tau]}(\tau)}{H_{[1,\tau]}(1)}$. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, we have

$$\begin{aligned}
h(\gamma\tau) &= \frac{H_{[1,\gamma\tau]}(\gamma\tau)}{H_{[1,\gamma\tau]}(1)} \\
&= \frac{H_{[1,\frac{a\tau+b}{c\tau+d}]}(\frac{a\tau+b}{c\tau+d})}{H_{[1,\frac{a\tau+b}{c\tau+d}]}(1)} \\
&= \frac{H_{[a\tau+b,c\tau+d]}(a\tau+b)}{H_{[a\tau+b,c\tau+d]}(c\tau+d)}
\end{aligned}$$

$$= \frac{aH_{[1,\tau]}(\tau) + b}{cH_{[1,\tau]}(1) + d}$$
$$= \gamma h(\tau).$$

∎

We have now established a triangular correspondence between the set of equivariant functions for Γ, the space of weight 2 meromorphic modular forms $M_2(\Gamma)$ and the Γ-elliptic zeta functions. This correspondence is summarized in the commutative diagram 5.1 below.

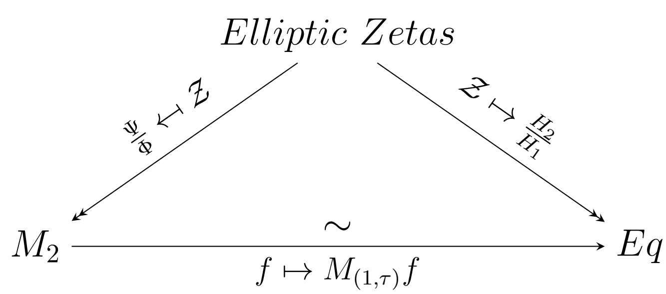


Figure 5.1: The triple correspondence

In this figure, H_1 and H_2 are the quasi-periods of the elliptic zeta function $\mathcal{Z}$, Φ and Ψ are such that $\mathcal{Z}(z, \Omega_\Gamma) = \Phi\, z + \Psi\zeta(z) + E$, E is elliptic and $M_{(1,\tau)}$ as defined in the above section. Of course, this diagram is commutative and each map is surjective.

Notice that using the bijection between $M_2(\Gamma)$ and $Eq(\Gamma)$ and the surjective map (5.3.1), one can also see that the map (5.3.2) is surjective. Thus we have shown that each $\Gamma-$equivariant function arises from a $\Gamma-$elliptic zeta function. Notice also that the trivial equivariant function τ is the quotient of the quasi-periods of the trivial $\Gamma-$elliptic zeta function $\mathcal{Z}(z) = z$.

Chapter 6

Rational Equivariant Functions

In the previous chapters, we showed that equivariant functions are parameterized by modular forms of weight 2. In §3.3, we saw how one can construct an equivariant function from an arbitrary modular form which yields an important infinite family of such functions called rational equivariant functions. It turns out that the rationality property is connected to the analytic behavior at the fixed points of the function. We will investigate whether the equivariant functions arising from the quasi-periods of integrals of the powers $\wp^n$ are rational or not.

To this aim we prove a criterion which examines the rationality of equivariant functions constructed from ratios of modular functions of low weight. This has the following consequence: the equivariant functions h_n which we introduced in previous chapter are rational for all $n \leq 12$ and for $n = 14$.

Next we turn our attention to the problem of establishing the non-rationality of given equivariant functions. In particular we prove that h_{13} does not belong in the set of rational equivariant functions. We also conjecture that all h_n for $n \geq 15$ are non-rational, and we provide evidence for this conjecture.

Recall that if f is a modular form of weight k for a subgroup Γ of $SL_2(\mathbb{Z})$, then

$$h_f(\tau) = \tau + k\,\frac{f(\tau)}{f'(\tau)}$$

defines an equivariant function for Γ.

We examine two different techniques to prove that an equivariant function h is not rational. One approach, that uses the classification of rational equivariant functions, is to prove that $\hat{h}$ has irrational residue at some pole. For instance, the residues at the poles of $\hat{h_{13}}$ can be computed explicitly, and we found that they are quadratic irrationals. The disadvantage of this approach is that to find the poles and the residues of given equivariant functions such as h_n, we often need to find the roots of polynomials. This can be difficult for polynomials of higher degree.

$M_0 =$	$\mathbb{C}$
$M_2 =$	0
$M_4 =$	$\mathbb{C}g_2$
$M_6 =$	$\mathbb{C}g_3$
$M_8 =$	$\mathbb{C}g_2^2$
$M_{10} =$	$\mathbb{C}g_2g_3$
$M_{12} =$	$\mathbb{C}g_2^3 + \mathbb{C}g_3^2$
$M_{14} =$	$\mathbb{C}g_2^2g_3$
$M_{16} =$	$\mathbb{C}g_2^4 + \mathbb{C}g_2g_3^2$
$M_{18} =$	$\mathbb{C}g_2^3g_3 + \mathbb{C}g_3^3$
$M_{20} =$	$\mathbb{C}g_2^5 + \mathbb{C}g_2^2g_3^2$
$M_{22} =$	$\mathbb{C}g_2^4g_3 + \mathbb{C}g_2g_3^3$
$M_{24} =$	$\mathbb{C}g_3^4 + \mathbb{C}g_2^3g_3^2 + \mathbb{C}g_2^6$
$M_{26} =$	$\mathbb{C}g_2^5g_3 + \mathbb{C}g_2^2g_3^3$
$M_{28} =$	$\mathbb{C}g_2g_3^4 + \mathbb{C}g_2^4g_3^2 + \mathbb{C}g_2^7$

Table 6.1: Table of M_{2n} for $n \leq 14$.

Then, we propose another criterion for non-rationality of equivariant functions, which is based on the notion of irreducible polynomials. There are criteria to test if a polynomial is irreducible, such as Eisenstein's criterion. The advantage of this approach in proving the irrationality of h_n is most evident for large n values, because this criterion requires only the analysis of coefficients of large polynomials, not of their roots.

6.1 Examples of Rational Equivariant Functions

We show in this section that quotients of modular forms of low weight can be used to produce functions with rational residues yielding various examples of rational equivariant functions.

Proposition 6.1.1. *Let $N, D \in \mathbb{Q}[g_2, g_3]$, with $N \in M_{2n}$ and $D \in M_{2n-2}$ where $n \leq 14$ and $n \neq 13$. Suppose also that D is not a cusp form, $ord_i D \leq 1$ and $ord_\rho D \leq 2$. Then the meromorphic function $\phi = \frac{N}{2\pi i D}$ has only simple poles in $\mathbb{H}$ with rational residues and $\frac{1}{2\pi i}\phi(\infty)$ is rational.*

Proof: First, we know from [15, Table 1.1] that $g_2(\tau)$ has a simple zero at $\tau = \rho$ and no other zero in the fundamental domain, while $g_3(\tau)$ has a simple zero at $\tau = i$

and no other zero in the fundamental domain. Since $ord_\rho D \leq 2$, the function D cannot be divisible by g_2^3, when considered as an element of $\mathbb{Q}[g_2, g_3]$. Analogously, D is not divisible by g_3^2, because $ord_i D \leq 1$. Now, we notice that the function $\frac{N}{2\pi i D}$ can be written as

$$\frac{N}{2\pi i D} = \frac{ag_3^4 + bg_2^3 g_3^2 + cg_2^6}{(2\pi i)g_2 g_3(dg_2^3 + eg_3^2)} \tag{6.1.1}$$

for some suitable $a, b, c, d, e \in \mathbb{C}$. This can be seen from table 6.1 using the fact that $N \in M_{2n}$ and $D \in M_{2n-2}$ and then multiplying or dividing by suitable factors to match the denominator in equation 6.1.1. Because $N, D \in \mathbb{Q}[g_2, g_3]$, we may choose a, b, c, d, e to be in $\mathbb{Q}$ and $d, e \neq 0$ because D is not divisible by g_2^3 or g_3^2. Moreover we can choose d, e such that $dg_2(\infty) + eg_3(\infty) \neq 0$ because D is not a cusp form.

Now we take a closer look at $\psi = dg_2^3 + eg_3^2$, which is a modular form of weight 12. By [15, Cor 3.8] we have the following relation known as the valence formula:

$$\frac{1}{2} ord_i(\psi) + \frac{1}{3} ord_\rho(\psi) + ord_\infty(\psi) + \sum_{\substack{\tau \in \Gamma(1)\backslash\mathbb{H}^* \\ \tau \neq i, \rho, \infty}} ord_\tau(\psi) = 1. \tag{6.1.2}$$

We notice that $ord_i(\psi) = ord_\rho(\psi) = ord_\infty(\psi) = 0$, and because ψ is not a cusp form, $\psi(i) = dg_2(i) \neq 0$ and $\psi(\rho) = eg_3(\rho) \neq 0$. Moreover, both g_2 and g_3 are holomorphic on all $\mathbb{H}^*$, so $ord_\tau(\psi)$ is a nonnegative integer for all $\tau \in \Gamma(1)\backslash\mathbb{H}^*$. We deduce that ψ has exactly one simple zero at some $\tau_0 \in \Gamma(1)\backslash\mathbb{H}^*$ with $\tau_0 \notin \{i, \rho, \infty\}$.

Coming back to the function ϕ, we see from equation 6.1.1 that it can have poles only in the $SL_2(\mathbb{Z})$-orbits of i, ρ and τ_0. On these points ϕ is either holomorphic or it has a simple pole. Since the denominator in equation 6.1.1 has simple zeros, we can compute the residues of ϕ via

$$res_\tau \phi = \frac{(ag_3^4 + bg_2^3 g_3^2 + cg_2^6)(\tau)}{(2\pi i)(g_2 g_3 \psi)'(\tau)}$$

for any $\tau \in \mathbb{H}$. So in the next computations we need the derivatives of this denominator:

$$(g_2 g_3 \psi)' = g_2' g_3 (dg_2^3 + eg_3^2) + g_2 g_3' (dg_2^3 + eg_3^2) + g_2 g_3 (3dg_2^2 g_2' + 2eg_3 g_3').$$

Now we find the residue of ϕ at i by using Ramanujan's formula 3.1.12:

$$res_i \phi = \frac{cg_2^6(i)}{(2\pi i)dg_2^4(i)g_3'(i)} = \frac{cg_2^6(i)}{\frac{1}{3}dg_2^6(i)} = \frac{3c}{d}.$$

Similarly, we use equation 3.1.11 to compute the residue of ϕ at ρ:

$$res_\rho \phi = \frac{ag_3^4(\rho)}{(2\pi i)eg_2' g_3^3(\rho)} = \frac{ag_3^4(\rho)}{6eg_3^4(\rho)} = \frac{a}{6e}.$$

Next, we now compute the residue of ϕ at τ_0. Because ψ is zero at τ_0, we can write $g_2^3(\tau_0) = -\frac{e}{d}g_3^4(\tau_0)$, and use it to simplify the numerator in the next computation. In order to keep the formulas simple, we will write g_2, g_3 in place of $g_2(\tau_0), g_3(\tau_0)$:

$$\begin{aligned}
res_{\tau_0}\phi &= \frac{ag_3^4 + bg_2^3g_3^2 + cg_2^6}{g_2g_3(3dg_2^2g_2' + 2eg_3g_3')} \\
&= \frac{-a\frac{d}{e}g_2^3g_3^2 + bg_2^3g_3^2 - c\frac{e}{d}g_2^3g_3^2}{g_2g_3[3dg_2^2(6g_3 - 4\eta_1g_2) + 2eg_3(\frac{1}{3}g_2^2 - 6g_3\eta_1)]} \\
&= \frac{g_2^3g_3^2\frac{-ad^2+bed-ce^2}{ed}}{-12\eta_1g_2g_3[dg_2^3 + eg_3^2] + g_2^3g_3^2[18d + \frac{2}{3}e]} \\
&= \frac{g_2^3g_3^2\frac{-ad^2+bed-ce^2}{ed}}{0 + \frac{2}{3}g_2^3g_3^2[27d + e]} \\
&= \frac{3(-ad^2 + bed - ce^2)}{2de(27d + e)},
\end{aligned}$$

which is a rational number. Finally, we compute the value of ϕ at infinity,

$$\phi(\infty) = \frac{ag_3^4(\infty) + bg_2^3(\infty)g_3^2(\infty) + cg_2^6(\infty)}{(2\pi i)g_2(\infty)g_3(\infty)(dg_2^3(\infty) + eg_3^2(\infty))}.$$

Using $g_2(\infty) = \frac{1}{12}(2\pi i)^4$ and $g_3(\infty) = \frac{-1}{216}(2\pi i)^6$ we get the following:

$$\frac{1}{2\pi i}\phi(\infty) = \frac{\frac{a}{(216)^6} + \frac{b}{12^3(216)^2} + \frac{c}{12^6}}{\frac{1}{12}\frac{-1}{216}\left(\frac{d}{12^3} + \frac{e}{216^2}\right)} \in \mathbb{Q}.$$

This concludes the proof. ∎

Recall from §4.4 that h_n is the equivariant function obtained as a ratio of quasi-periods of integrals of $\wp^n$. The above theorem has the following consequence.

Corollary 6.1.2. *For all $n = 1, \ldots, 12$ and $n = 14$ the function h_n is rational.*

Proof: By equation 4.4.4 we have $h_n = \tau + \dfrac{2\pi i}{f_n(\tau) + \eta_1}$ with $f_n = \dfrac{\Phi_n}{\Psi_n}$, and so

$$\widehat{h_n} = \frac{1}{h_n - \tau} = \frac{f_n + \eta_1}{2\pi i}.$$

Recall from proposition 4.4.2 that $\Phi_n \in M_{2n}$, $\Psi_n \in M_{2n-2}$ and $\Phi_n, \Psi_n \in \mathbb{Q}[g_2, g_3]$.

n	Φ_n	Ψ_n
3	$\frac{g_3}{10}$	$\frac{3g_2}{20}$
4	$\frac{5g_2^2}{336}$	$\frac{g_3}{7}$
5	$\frac{g_2g_3}{30}$	$\frac{7g_2^2}{240}$
6	$\frac{15g_2^3}{4928} + \frac{g_3^2}{55}$	$\frac{87g_2g_3}{1540}$
7	$\frac{433g_2^2g_3}{43680}$	$\frac{77g_2^3}{12480} + \frac{5g_3^2}{182}$
8	$\frac{13g_2^4}{19712} + \frac{7g_2g_3^2}{660}$	$\frac{167g_2^2g_3}{9240}$
9	$\frac{383g_2^3g_3}{136136} + \frac{7g_3^3}{1870}$	$\frac{77g_2^4}{56576} + \frac{6021g_2g_3^2}{340340}$
10	$\frac{2873g_2^5 + 86848g_2^2g_3^2}{19475456}$	$\frac{3251g_2^3g_3\vert + 3520g_3^3}{608608}$
11	$\frac{20327g_2^4g_3}{26138112} + \frac{7g_2g_3^3}{2244}$	$\frac{209g_2^5}{678912} + \frac{134g_2^2g_3^2}{17017}$
12	$\frac{663g_2^6}{19689472} + \frac{775529g_2^3g_3^2}{475931456} + \frac{7g_3^4}{8602}$	$\frac{2884469g_2^4g_3 + 9834816g_2g_3^3}{1903725824}$
13	$\frac{2623663g_2^5g_3 + 21088240g_2^2g_3^3}{12415603200}$	$\frac{4807g_2^6}{67891200} + \frac{44139g_2^3g_3^2}{14780480} + \frac{11g_3^4}{8645}$
14	$\frac{1221025g_2^7 + 86159616g_2^4g_3^2 + 138098688g_2g_3^4}{156649439232}$	$\frac{1367889g_2^5g_3 + 9613504g_2^2g_3^3}{3263529984}$

Table 6.2: Table of Φ_n and Ψ_n for $n \leq 14$.

We now prove that $ord_i\Psi_n \leq 1$ and $ord_\rho\Psi_n \leq 2$. In table 6.2, we list the values of Φ_n and Ψ_n for $n \leq 14$, which are computed recursively from the definitions. From this table we see that Ψ_n is not divisible by g_3^2 and so $ord_i\Psi_n \leq 1$.

Similarly, Ψ_n is not divisible by g_2^3 and so it has no triple zeros at ρ, meaning $ord_\rho\Psi_n \leq 2$.

Now we need to show that Ψ_n is not a cusp form, that is $\Phi_n(\infty) \neq 0$. Equivalently, we need to verify that Ψ_n is not divisible by $\Delta = g_2^3 + 27g_3^2$ when considered as an element of $\mathbb{C}[g_2, g_3]$. This requirement is fulfilled, as we can easily verify by looking at table 6.2.

By proposition 6.1.1, we get that $\dfrac{f_n}{2\pi i}$ has only simple poles in $\mathbb{H}$ with rational residues and that $\dfrac{f_n(\infty)}{(2\pi i)^2}$ is rational. As η_1 is holomorphic on $\mathbb{H}$, we have

$$\frac{\eta_1(\infty)}{(2\pi i)^2} = \frac{\pi^2 E_2(i\infty)}{3(2\pi i)^2} = \frac{1}{12} \in \mathbb{Q}.$$

In conclusion, $\widehat{h_n}$ has simple poles with rational residues and $\frac{1}{2\pi i}\widehat{h_n}(\infty)$ is rational. This proves that h_n is a rational equivariant function by the classification stated in proposition 3.3.3. ∎

6.2 A Non-rational Equivariant Function

In this section we prove the non-rationality of h_{13}. The strategy here is to compute explicitly the residue at some pole of $\widehat{h_{13}}$ and verify that it is an irrational number.

Theorem 6.2.1. h_{13} *is not rational.*

Proof: By proposition 3.3.3, to prove that h_{13} is not a rational equivariant function, it suffices to show that there exists a pole $z_0 \in \mathbb{H}$ of $\widehat{h}_{13}$ such that $res_{z_0}(\widehat{h_{13}}) \notin \mathbb{Q}$. We begin by recalling that h_{13} satisfies

$$h_{13} = \frac{\Phi_{13}z - \Psi_{13}\eta_2}{\Phi_{13}z - \Psi_{13}\eta_1}.$$

Therefore, we have

$$\widehat{h_{13}} = \frac{\Phi_{13}}{2\pi i\Psi_{13}} - \frac{\eta_1}{2\pi i},$$

with $\frac{\eta_1}{2\pi i}$ being holomorphic. The values of Φ_{13} and Ψ_{13} are listed in table 6.2:

$$\Psi_{13} = \alpha g_2^6 + \beta g_2^3 g_3^2 + \gamma g_3^4,$$
$$\Phi_{13} = \delta g_2^5 g_3 + \epsilon g_2^2 g_3^3,$$

where

$$\alpha = \frac{4807}{67891200}, \quad \beta = \frac{44139}{14780480}, \quad \gamma = \frac{11}{8645},$$
$$\delta = \frac{2623663}{12415603200}, \quad \text{and} \quad \epsilon = \frac{21088240}{12415603200}.$$

These modular functions are written as polynomials in g_2 and g_3. In order to deal with polynomials in one variable only, it is useful to introduce the modular function $x : \mathbb{H} \to \hat{\mathbb{C}}$ of weight 0 given by the formula $x = \frac{g_2^3}{g_3^2}$. Then we have

$$\Psi_{13} = g_3^4(\alpha x^2 + \beta x + \gamma), \tag{6.2.1}$$
$$\Phi_{13} = g_2^2 g_3^3(\delta x + \epsilon). \tag{6.2.2}$$

For the computations of the residue of $\widehat{h_{13}}$ we need the derivative of Ψ_{13}.

$$(2\pi i)\Psi_{13}' = 6\alpha g_2^5(2\pi i)g_2' + 3(2\pi i)\beta g_2^2 g_3^2 g_2' + 2(2\pi i)\beta g_2^3 g_3 g_3' + 4\gamma(2\pi i)g_3^3 g_3'. \tag{6.2.3}$$

By the Ramanujan identities (equation 3.1.8), after some simplification, equation 6.2.3 becomes

$$= -\eta_1 g_3^4(24\alpha x^2 + 24\beta x + 24\gamma) + (36\alpha g_2^5 g_3 + 18\beta g_2^2 g_3^3 + 2/3\beta g_2^5 g_3 + 4/3\gamma g_2^2 g_3^3),$$

and so

$$(2\pi i)\Psi_{13}' = 24\eta_1\Psi_{13} + g_2^2 g_3^3\left[\left(36\alpha + \frac{2}{3}\beta\right)x + \left(18\beta + \frac{4}{3}\gamma\right)\right]. \tag{6.2.4}$$

By equation 6.2.1, we have that $\tau \in \mathbb{H}$ is a zero of Ψ_{13} if and only if $g_3(\tau) = 0$ or

$$\alpha x(\tau)^2 + \beta x(\tau) + \gamma = 0. \tag{6.2.5}$$

Notice that $g_3(\tau) = 0$ if and only if $\tau = i$. Let us now apply the following lemma.

Lemma 6.2.2. *The modular function $x : \mathbb{H}^* \to \hat{\mathbb{C}}$ given by $x = \frac{g_2^3}{g_3^2}$ is surjective.*

Proof: In fact, any non-constant modular function of weight zero is surjective as it realizes a non-constant covering between compact Riemann surfaces. Alternatively, for our case, we can give a direct proof as follows. First it is well-known that the j-function induces a bijection between $\Gamma(1)\backslash\mathbb{H}^*$ and $\hat{\mathbb{C}}$ [15, Theorem 4.1]. Since the projection $\mathbb{H}^* \to \Gamma(1)\backslash\mathbb{H}^*$ is surjective, we have that $j : \mathbb{H}^* \to \hat{\mathbb{C}}$ is surjective. Now, we have that $x = r \circ j$, where $r : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is given by $r(t) = 27^{-1}(1 - 1728t^{-1})$. But r is bijective with inverse $r^{-1}(t) = 1728(1 - 27t)^{-1}$. Therefore x is surjective. ∎

We now resume the main proof. By Lemma 6.2.2, there exists $\tau_0 \in \mathbb{H}^*$ such that

$$x(\tau_0) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{6.2.6}$$

Then equation 6.2.5 is satisfied by $\tau = \tau_0$. Moreover we can see from an explicit calculation that $\tau_0 \neq i$, $\tau_0 \neq \rho$ and $\tau_0 \neq \infty$. In particular this means that $g_2(\tau_0) \neq 0$, $g_3(\tau_0) \neq 0$ and $\Psi_{13}(\tau_0) = 0$. Thus τ_0 is a simple pole of $\widehat{h_{13}}$.

It is clear that the residue of $\widehat{h_{13}}$ at τ_0 is the same as the residue of Φ_{13}/Ψ_{13} as η_1 is holomorphic everywhere. By the usual formula, this residue is equal to

$$res_{\tau_0}(\widehat{h_{13}}) = \frac{\Phi_{13}(\tau_0)}{(2\pi i)\Psi_{13}'(\tau_0)}$$

as long as $\Phi_{13}(\tau_0) \neq 0$ and $\Psi_{13}'(\tau_0) \neq 0$. The values of $\Phi_{13}(\tau_0)$ and $\Psi_{13}(\tau_0)$ are calculated from equations 6.2.2 and 6.2.4. Performing all simplifications we finally get:

$$res_{\tau_0}(\widehat{h_{13}}) = \frac{A + B\sqrt{55211205}}{C},$$

for some nonzero integers A, B, C. Since this is not a rational number, we get by proposition 3.3.3 that h_{13} is not a rational equivariant function. ∎

6.3 Irreducibility of Denominators Implies Non-rationality

From the previous proof, we claim that the rationality of the equivariant functions h_n for $n \leq 12$ or $n = 14$ is more the exception than the rule. More precisely, we make the following conjecture.

Conjecture 6.3.1. *For every $n \geq 15$ the equivariant function h_n is not a rational equivariant function.*

Unfortunately, we are not able to prove this conjecture. In this section we replace the ad-hoc arguments of the previous section with considerations that are valid for every h_n.

First, in the previous proof we rewrote Φ_{13} and Ψ_{13}, up factors of the form $g_2^a g_3^b$, as polynomials in the modular function $x = g_2^3/g_3^2$ which is of weight 0 and degree 1. There is another natural choice we can make, namely to rewrite everything in terms of the modular function j.

The functions h_n are constructed in terms of the modular functions Φ_n and Ψ_n of weight 2. The next proposition shows that each ratio Φ_n/Ψ_n belongs to a specific class of modular forms of weight 2, namely those that can be written as $R(j)j'$, where R is some rational function with rational coefficients. This expression is particularly useful to compute the residues.

Proposition 6.3.1. *For every $n \in \mathbb{N}$ with $n \geq 3$ we have $\dfrac{\Phi_n}{2\pi i \Psi_n} = R_n(j)j'$ where $R_n \in \mathbb{Q}(t)$.*

Proof: The key observation is that $g_2\Phi_n$ and $g_3\Psi_n$ are modular forms of the same weight which can be written as polynomials in g_2 and g_3 with rational coefficients. This is clear from the recursion in the definition of Φ_n and Ψ_n. If $x = g_2^3/g_3^2$ then we can write

$$g_2\Phi_n = g_2^a g_3^b P(x)$$
$$g_3\Psi_n = g_2^a g_3^b Q(x)$$

for some $a \in \{0, 1, 2\}$, some $b \in \{0, 1\}$ and some polynomials P, Q with rational coefficients. The proposition is then proved by noticing the formulas

$$x = \frac{1}{27}(1 - 1728j)$$

and

$$\frac{g_3}{(2\pi i)g_2} = \frac{1}{18j}j'.$$

We will now use the following lemma.

Lemma 6.3.2. *For every $n \in \mathbb{N}$ with $n \geq 5$, we have*

$$\frac{g_2}{g_3}\frac{\Psi_n}{\Psi_{n-1}} = \tilde{R}_n(j) \quad \textit{and} \quad \frac{\Psi_n}{g_2\Psi_{n-2}} = \tilde{\tilde{R}}_n(j)$$

for some $\tilde{R}_n, \tilde{\tilde{R}}_n \in \mathbb{Q}(t)$.

Proof: We are going to use mathematical induction using direct computation, which gives us

$$
\begin{aligned}
\frac{g_2}{g_3}\frac{\Psi_4}{\Psi_3} &= \frac{g_2}{g_3}\frac{\frac{-2}{14}g_3}{\frac{-3}{20}g_2} = \frac{20}{21} \\
\frac{g_2}{g_3}\frac{\Psi_5}{\Psi_4} &= \frac{g_2}{g_3}\frac{\frac{7}{240}g_2^2}{\frac{1}{7}g_3} = \frac{49g_2^3}{240g_3^2} = \frac{49x}{240} \\
\frac{\Psi_5}{g_2\Psi_3} &= \frac{\frac{-7}{240}g_2^2}{\frac{-3}{20}g_2^2} = \frac{7}{36},
\end{aligned}
$$

where x was introduced in the second section of this chapter. Now we suppose that both statements of the lemma are true up to n and we prove they are true for $n+1$. Using the recursion formula 4.4.1 we have

$$
\begin{aligned}
\frac{g_2}{g_3}\frac{\Psi_{n+1}}{\Psi_n} &= g_2\frac{r(n)g_2\Psi_{n-1} + s(n)g_3\Psi_{n-1}}{g_3\Psi_n} \\
&= \frac{g_2}{g_3}\left[r(n)g_2\frac{\Psi_{n-1}}{\Psi_n} + s(n)g_3\frac{\Psi_{n-2}}{\Psi_n}\right] \\
&= r(n)\frac{x}{\tilde{R}_n(j)} + s(n)\frac{1}{\tilde{\tilde{R}}_n(j)}.
\end{aligned}
$$

Since $r(n), s(n) \in \mathbb{Q}$ and $x = g_2^3/g_3^2 = r(j) \in \mathbb{Q}(j)$, then $\frac{g_2}{g_3}\frac{\Psi_{n+1}}{\Psi_n} \in \mathbb{Q}(j)$. Moreover

$$
\begin{aligned}
\frac{\Psi_{n+1}}{g_2\Psi_{n-1}} &= \frac{r(n)g_2\Psi_{n-1} + s(n)g_3\Psi_{n-2}}{g_2\Psi_{n-1}} \\
&= r(n) + s(n)\frac{g_3\Psi_{n-2}}{g_2\Psi_{n-1}} \\
&= r(n) + s(n)\frac{1}{\tilde{R}_{n-1}(j)}
\end{aligned}
$$

and so $\frac{\Psi_{n+1}}{g_2\Psi_{n-1}} \in \mathbb{Q}(j)$. ∎

Now we resume our proof of proposition 6.3.1. For $n = 3, 4, 5, 6$, we have

$$
\begin{aligned}
\frac{\Phi_3}{2\pi i\Psi_3} &= \frac{2g_3}{(2\pi i)3g_2} = \frac{1}{27j}j', \\
\frac{\Phi_4}{2\pi i\Psi_4} &= \frac{5g_2^2}{(2\pi i)48g_3} = \frac{5}{32(-1728+j)}j', \\
\frac{\Phi_5}{2\pi i\Psi_5} &= \frac{8g_3}{(2\pi i)7g_2} = \frac{4}{63j}j',
\end{aligned}
$$

$$\frac{\Phi_6}{2\pi i\Psi_6} = \frac{8g_2^2}{(2\pi i)7g_3} + \frac{28g_3^2}{(2\pi i)87g_2} = \frac{75}{(928(-1728+j))}j' + \frac{14}{783j}j'.$$

Now by using the recursion formula (equation 4.4.1), we find $R_n(j)$:

$$\begin{aligned}
\frac{\Phi_n}{2\pi i\Psi_n} =& \frac{r(n)g_2\Phi_{n-2} + s(n)g_3\Phi_{n-3}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} \\
=& \frac{r(n)g_2\Phi_{n-2}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} + \frac{s(n)g_3\Phi_{n-3}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} \\
=& \left(\frac{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})}{r(n)g_2\Phi_{n-2}}\right)^{-1} \\
&+ \left(\frac{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})}{s(n)g_3\Phi_{n-3}}\right)^{-1} \\
=& \left(\frac{2\pi i\Psi_{n-2}}{\Phi_{n-2}} + \frac{2\pi i(s(n)g_3\Psi_{n-2}\Psi_{n-3})}{r(n)g_2\Psi_{n-2}\Phi_{n-2}}\right)^{-1} \\
&+ \left(\frac{2\pi i r(n)g_2\Psi_{n-2}\Psi_{n-3}}{s(n)g_3\Psi_{n-3}\Phi_{n-3}} + \frac{2\pi i\Psi_{n-3}}{\Phi_{n-3}}\right)^{-1} \\
=& \left(\frac{1}{R_{n-2}(j)j'} + \frac{s(n)}{r(n)\tilde{R}_{n-2}(j)R_{n-2}(j)j'}\right)^{-1} \\
&+ \left(\frac{r(n)\tilde{R}_{n-2}(j)}{s(n)R_{n-3}(j)j'} + \frac{1}{R_{n-3}(j)j'}\right)^{-1} \\
=& \underbrace{\left(\frac{r(n)\tilde{R}_{n-2}(j)R_{n-2}(j)}{s(n) + r(n)\tilde{R}_{n-2}(j)} + \frac{s(n)R_{n-3}(j)}{s(n) + r(n)\tilde{R}_{n-2}(j)}\right)}_{R_n(j)} j'.
\end{aligned}$$

This concludes the proof. ∎

Table 6.3 lists the expressions of $R_n(j)$ for $n \leq 14$. For every $n \in \mathbb{N}$ with $n \geq 3$ we introduce the polynomials $p_n, q_n \in \mathbb{Z}[t]$ so that $R_n = p_n/q_n$ and the fraction p_n/q_n is written in reduced form.

Notice that the rational function R_n for $n \leq 12$ and for $n = 14$ decomposes as a sum of fractions that have a linear denominators with rational coefficients. On the contrary the denominator of R_{13} is an irreducible polynomial of the second degree. This is the motivation for the following criterion of non-rationality.

Theorem 6.3.3. *Let $n \in \mathbb{N}$ with $n \geq 3$ and suppose q_n is irreducible in $\mathbb{Q}[t]$ with degree of at least 2 and $p_n \neq cq_n'$ for some $c \in \mathbb{Q}$. Then h_n is an irrational equivariant function.*

n	$R_n(j)$
3	$\frac{1}{27j}$
4	$\frac{5}{32(-1728+j)}$
5	$\frac{4}{63j}$
6	$\frac{75}{(928(-1728+j))} + \frac{14}{783j}$
7	$\frac{433}{(-1382400+5651j)}$
8	$\frac{585}{(10688(-1728+j))} + \frac{49}{1503j}$
9	$\frac{637}{54189j} + \frac{165599575}{(162567(-8220672+14639j))}$
10	$\frac{8619}{208064(-1728+j)} + \frac{12760776}{3251(-6082560+91297j)}$
11	$\frac{637}{28944j} + \frac{55120391}{3216(-237109248+282053j)}$
12	$\frac{439569}{(13186144(-1728+j))} + \frac{12103}{1383021j} + \frac{769103818509693\,59}{(63321923823(-5664854016+29238493j))}$
13	$\frac{4(-36440478720+91927141j)}{(125791622922240-4758534328320j+4420585843j^2)}$
14	$\frac{1221025}{43772448(-1728+j)} + \frac{22477}{1351899j} + \frac{371958496352913151}{205471974579(-16612134912+46546507j)}$

Table 6.3: Table of $R_n(j)$

Proof: We are going to use the following lemma.

Lemma 6.3.4. *Let $p, q \in \mathbb{Q}[t]$ such that $\deg p < \deg q$, p is not identically zero and q is irreducible. Then p and q do not have common roots.*

Suppose that h_n has rational residues at all its poles and let j_0 be a zero of q_n. Because j is surjective then there exists $\tau_0 \in \mathbb{H}$ such that $j_0 = j(\tau_0)$. Thus, $q_n(j(\tau_0)) = 0$ $p_n(j(\tau_0)) \neq 0$ and $q_n'(j(\tau_0)) \neq 0$ by applying lemma 6.3.4 to both p_n and q_n'. Then τ_0 is a simple pole of h_n. Let

$$r = res_{\tau_0} h_n = \frac{p_n(j(\tau_0))}{q_n'(j(\tau_0))}$$

for some $r \in \mathbb{Q}$ by assumption, which gets us

$$p_n(j_0) = r q_n'(j_0).$$

Now let $p_n - r q_n' = m(j_0)$. We have $m(j_0) = 0$. But the degree of m is at most the maximum between the degrees of p_n, q_n' and $\deg m < \deg q_n$, so by lemma 6.3.4 we get $m(j_0) \neq 0$ which is a contradiction. ∎

6.4 Examples of Irrational Equivariant Functions via Irreducibility

There is a well-known criterion to test if a polynomial is irreducible.

Lemma 6.4.1 (Eisenstein Criterion)**.** *Let p be a prime number and $Q(x)$ be a polynomial with integer coefficients such that*

$$Q(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

If we find p such that:

- *p divides every a_i, and $i \neq n$;*
- *p does not divide a_n; and*
- *p^2 does not divide a_0,*

then Q is irreducible over $\mathbb{Q}$.

This criterion of irreducibility, in conjunction with our criterion of irrationality (proposition 6.3.3), can be used to prove more examples of irrational equivariant functions. As an easy example, let us prove again that h_{13} is irrational.

Corollary 6.4.1. *h_{13} is irrational.*

Proof: First let's prove that q_{13} is irreducible. From table 6.3 we know that

$$q_{13} = 125791622922240 - 4758534328320t + 4420585843t^2.$$

Then we can apply Eisenstein's criterion with the prime $p = 5$ and get that q_{13} is irreducible. So h_{13} is irrational by theorem 6.3.3. ∎

Chapter 7

The $S-$ map

In this chapter, we introduce a map from $M_4(\Gamma)$, the space of meromorphic weight 4 modular forms for a subgroup of the modular group, into itself. This map is built using tools from previous chapters, namely the Schwarz derivative and the rational equivariant functions. The fixed points of this map turn out to satisfy the same nonlinear differential equations of order 4 with constant coefficients. We will provide 4 examples of fixed points using the Eisenstein series and the Jacobi theta functions.

7.1 The map S

Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$ and let f be a modular form of weight 4 for Γ. As we have seen earlier, the function defined on the upper half-plane by

$$h_f(\tau) \,=\, \tau \,+\, 4\,\frac{f(\tau)}{f'(\tau)}$$

is an equivariant function for Γ which we refer to as a rational equivariant function. If we take the Schwarz derivative of h_f, then we obtain a weight 4 modular form. This defines a map

$$S : M_4(\Gamma) \longrightarrow M_4(\Gamma)$$

by

$$S(f)(\tau) \,=\, \{h_f, \tau\}\,,$$

Let N be the cusp width at ∞ of Γ defined as the smallest positive integer such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^N = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} \in \Gamma.$$

It exists because Γ has finite index inside $SL_2(\mathbb{Z})$. If f is a modular form for Γ then $f(\tau + N) \,=\, f(\tau)$, $\tau \in \mathbb{H}$. Hence f has a Fourier expansion in $q = \exp(2\pi i\tau/N)$.

In fact q is the local parameter at ∞ and since f is supposed to be meromorphic at ∞, then the Fourier expansion has only finitely many negative powers of q. In particular, the function $h_f(\tau) = \tau + 4f(\tau)/f'(\tau)$ has a logarithmic singularity at ∞ as $\tau = \log(q)/2\pi i$.

Proposition 7.1.1. *The map S is almost one-to-one in the sense that if $S(f) = S(g)$ then $f = cg$ for some $c \in \mathbb{C}$. Moreover, S is not onto.*

Proof: Suppose that $S(f) = S(g)$ for f and g in $M_4(\Gamma)$, then h_f is a linear fraction of h_g. But since both h_f and h_g have expansions $\tau+$ series in q, they must be equal. Thus we have $f/f' = g/g'$. We deduce that the derivative of f/g is zero where it is defined. Meanwhile, the poles of f/g are among the poles of g which are in finite number in a fundamental domain of Γ. If the poles are removed, we get a connected domain on which f/g is constant and the poles are removable singularities. Therefore $f = cg$ for some constant c.

For the second part, notice that from the expression of the Schwarz derivative

$$\{f, \tau\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

we see that if $\{f, \tau\}$ has a pole, then it must be a double pole. However, one can construct weight 4 modular forms having poles with multiplicity other than 2 and so they are not in the image of S. For instance, Δ/j'^4 has a pole of order 4 at i and a pole of order 8 at the cubic root of unity $exp(2\pi i/3)$. ∎

Remark 7.1.1. *It is worth noting that the poles of $\{f, \tau\}$ in $\mathbb{H}$ occur exactly at the zeros of f'.*

7.2 The case of the modular group

In this section we consider $\Gamma = SL_2(\mathbb{Z})$. We are interested in finding a fixed point for the map S. Recall that the space of weight 4 holomorphic modular forms is spanned by E_4 and we have

$$E_4(\tau) \;=\; 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)\, q^n\,, \quad q = e^{2\pi i\tau}\,,$$

where $\sigma_3(n)$ is the sum of the cubes of the positive divisors of n.

Theorem 7.2.1. *We have*

$$S(E_4) \;= 2\pi^2\, E_4.$$

In other words, $2\pi^2 E_4$ is a fixed point of S.

Proof: First, we show that $S(E_4)$ is holomorphic. To this end we need to show that

$$h'_{E_4} = \frac{5E_4'^2 - 4E_4E_4''}{E_4'^2}$$

does not vanish. Using the Ramanujan identities, one can easily show that $5E_4'^2 - 4E_4E_4'' = -3840\pi^2\Delta$. As the denominator $E_4'^2$ is holomorphic and Δ is never zero on $\mathbb{H}$, h'_{E_4} does not vanish and its Schwarz derivative is holomorphic on $\mathbb{H}$. If we analyze the $q-$expansion of $S(E_4)$, we see that $S(E_4) = 1 + \mathrm{O}(q)$ and so it is holomorphic at ∞ as well.

As the space of weight 4 holomorphic forms is spanned by E_4, and both E_4 and $S(E_4)$ take the value $2\pi^2$ at ∞, we conclude that $S(E_4) = 2\pi^2 E_4$. ∎

The following is straightforward.

Corollary 7.2.1. *The map S is constant on the subspace $M_4(SL_2(\mathbb{Z}))$ of holomorphic weight four modular forms for $SL_2(\mathbb{Z})$.*

Notice that if $h(\tau) = \tau + k\, f(\tau)/f'(\tau)$ where f is a weight k modular form, then

$$h' \;=\; \frac{(k+1)f'^2 - k f f''}{f'^2}\,.$$

The expression in the numerator is a special case of the Rankin-Cohen bracket of order n defined for two modular forms f of weight k and g of weight l by

$$[f,g]_n \;=\; \sum_{r+s=n} \binom{k+n-1}{s}\binom{l+n-1}{r} \frac{d^r f}{d\tau^r}\frac{d^s g}{d\tau^s},$$

which is a cusp form of weight $k+l+2n$. This allows us to replace the numerator like so

$$h' \;=\; -\frac{[f,f]_2}{f'^2}\,.$$

Applying this to our case, we obtain

$$[E_4, E_4]_2 = -3840\pi^2\Delta.$$

7.3 Level 2 subgroups

Let $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Define the subgroups of $SL_2(\mathbb{Z})$

$$\Gamma_U(2) = \{A \in SL_2(\mathbb{Z}) : \ A \equiv I_2 \text{ or } U \mod 2\} \ ,$$
$$\Gamma_V(2) = \{A \in SL_2(\mathbb{Z}) : \ A \equiv I_2 \text{ or } V \mod 2\} \ ,$$
$$\Gamma_W(2) = \{A \in SL_2(\mathbb{Z}) : \ A \equiv I_2 \text{ or } W \mod 2\} \ .$$

These groups are congruence groups of level 2 and index 3 in $SL_2(\mathbb{Z})$. In fact $\Gamma_U(2)$ is simply the classical congruence subgroup $\Gamma_0(2)$ defined by

$$\Gamma_0(2) \ = \left\{ \begin{bmatrix} q & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \ c \text{ is even} \right\}.$$

$\Gamma_W(2)$ is $\Gamma_0(2)$ with a similar definition but with d even. These three groups are all conjugate. We now define the Jacobi theta functions θ_i for $2 \leq i \leq 4$. Here $q = \exp(\pi i \tau)$.

$$\theta_2(\tau) = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \ ,$$

$$\theta_3(\tau) = 1 + 2 \sum_{n=0}^{\infty} q^{n^2} \ ,$$

and

$$\theta_4(\tau) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2} \ .$$

The function θ_2 (resp. θ_3 and θ_4) is a modular form of weight $1/2$ and a multiplier system u (resp. v and w) for Γ_U (resp. Γ_V and Γ_W). Moreover, the multiplier systems are eighth root of unity. In addition, they satisfy the Jacobi identity

$$\theta_3^4 \ = \ \theta_2^4 + \theta_4^4,$$

and the following important property

$$\Delta \ = \ \left(\frac{1}{2} \, \theta_2 \theta_3 \theta_4 \right)^8 \ .$$

Also, the three theta functions do not vanish on $\mathbb{H}$. We have the following important information on the spaces of holomorphic modular forms for the groups Γ_U, Γ_V and Γ_W.

Theorem 7.3.1. *[8, Thm. 7.1.4] Let k be a positive integer, then*

$$\dim M_{\frac{k}{2}}(\Gamma_U, u^k) = \dim M_{\frac{k}{2}}(\Gamma_V, v^k) = \dim M_{\frac{k}{2}}(\Gamma_W, w^k) = 1 + \left[\frac{k}{8}\right].$$

Furthermore, a basis for $M_{\frac{k}{2}}(\Gamma_U, u^k)$ (resp $M_{\frac{k}{2}}(\Gamma_V, v^k)$ and $M_{\frac{k}{2}}(\Gamma_W, w^k)$) is given by $\theta_2^{k-8r}(\theta_3\theta_4)^{4r}$, (resp. $\theta_3^{k-8r}(\theta_2\theta_4)^{4r}$ and $\theta_4^{k-8r}(\theta_2\theta_3)^{4r}$), $0 \le r \le k/8$.

7.4 Level 2 $S-$maps

In this section, we will determine a fixed point for each of the three level 2 groups. While in the case of $SL_2(\mathbb{Z})$, the space of weight 4 holomorphic modular forms is one-dimensional, this space has dimension 2 in the case of each of our level 2 groups (according to the above theorem).

Proposition 7.4.1. *We have the following identities between weight 10 cusp forms:*

$$\begin{aligned}
[\theta_2, \theta_2]_2 &= \frac{-\pi^2}{64}\theta_2^2(\theta_3\theta_4)^4, \\
[\theta_3, \theta_3]_2 &= \frac{-\pi^2}{64}\theta_3^2(\theta_2\theta_4)^4, \\
[\theta_4, \theta_4]_2 &= \frac{-\pi^2}{64}\theta_4^2(\theta_2\theta_3)^4.
\end{aligned}$$

Proof: We will prove the first identity and the others can be dealt with in the same way. The cusp form $[\theta_2, \theta_2]$ has weight 5. Meanwhile, the space $M_5(\Gamma_U(2), u^{10})$ has dimension 2, and its subspace of cusp forms is one-dimensional generated by $\theta_2^2(\theta_3\theta_4)^4$. Therefore, $[\theta_2, \theta_2] = c\theta_2^2(\theta_3\theta_4)^4$ for some constant c. Computing the leading coefficients of the $q-$expansions of both sides yields $c = -1/64$. ∎

Corollary 7.4.2. *For $2 \le i \le 4$, $S(\theta_i^8)$ is a holomorphic weight 4 modular form.*

Proof: Indeed, we have

$$h_{\theta_i^8} = -\frac{[\theta_i, \theta_i]}{\theta_i'^2}$$

which does not vanish on $\mathbb{H}$ according to the above proposition as each theta function does not vanish. Therefore, the Schwarz derivative of h_{θ_i} is holomorphic on $\mathbb{H}$. As it is always holomorphic at the cusps, we deduce that $S(\theta^8)$ is a holomorphic modular form. ∎

We come to the following conclusion

Theorem 7.4.3. *For $2 \le i \le 4$, $\frac{\pi}{2}\,\theta_i^8$ is a fixed point of the $S-$map.*

Proof: We begin with

$$\theta_2^8(\tau) = 256q^2 + 2048q^4 + \ \mathrm{o}(q^4),$$

$$\theta_3^8(\tau) = 1 + 16q + 112q^2 + \ \mathrm{o}(q^2),$$
$$\theta_4^8(\tau) = 1 - 16q + 112q^2 + \ \mathrm{o}(q^2).$$

For $i = 2$, one can see that that $S(\theta_2^8)$ is actually a cusp form, and so it is a constant multiple of θ_2^8. The constant turns out to be $\pi^2/2$. Therefore, $\frac{\pi^2}{2}\theta_2^8$ is fixed by S. As for $i = 3, 4$, one needs to compute the first two coefficients of $S(\theta_i^8)$ which can be easily done and compare to the above $q-$expansions to conclude that $\frac{\pi^2}{2}\theta_i^8$, $i = 3, 4$, is also fixed by the map S. ∎

Notice that $S(y) = y$ is simply a non-linear differential equation of order 4 with constant coefficients, which is then satisfied by $2\pi^2\, E_4$ and $\frac{\pi^2}{2}\theta_i^8$, $2 \le i \le 4$. Explicitly, we have

Proposition 7.4.4. *The differential equation takes the complicated form*

$$y^{(4)}(20yy'^2 - 16y^2y'') + 24y'''^2y^2 + 16y^3y''^2 + 40y^{(3)}y'^3 - y'^2(40y^2y'' + 36y''^2)$$
$$= y(-72y''^3 - 25y'^4 + 104y'''y'y'').$$

It should be noted that the existence of such differential equation of this low degree for the four functions is not known to the best of our knowledge.

It will be interesting to study the iterates of the $S-$map on a single modular form and investigate the behavior of its orbit. Moreover, one can easily prove the uniqueness of the four fixed points for the $S-$ map encountered in this chapter. The question one can ask is whether the uniqueness is a general aspect for any subgroup of the modular group, or is it simply related to the fact that the spaces of holomorphic modular forms is low-dimensional in our cases.

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