

THE NATURE OF MATHEMATICAL ABSTRACTION

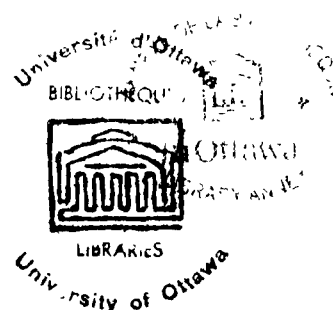
A DISSERTATION

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by

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PREFACE

An inestimable amount of specialization and self-criticism characterizes current mathematical research. In the field of mathematics proper, a great number of mathematical books and periodicals are being issued, containing countless new theorems, restatements and new, original solutions of old problems, and new methods of proof. It has been said that though the delegates to mathematical congresses may be quite competent mathematicians themselves, they cannot possibly comprehend all the over-specialized topics presented (1). According to the historian Cajori, if the history of mathematics in the nineteenth century alone were written with some detail, fourteen or fifteen large volumes would be required (2). The same situation obtains in the literature, and at the congresses dealing with applied mathematics. Even the non-mathematician feels the impact of mathematics on general culture through the increasing number of books and articles endeavoring to present mathematics in a simplified and popularized form.

Contemporary research in mathematics itself is nearly rivalled by an equal productivity in philosophical inquiries on mathematics. Questions of mathematical method and symbolism are being examined under the title of Symbolic Logic, while other investigations are concerned with the foundations of mathematics, with theories on number, functions and structure. Numerous philosophical discursions deal with the applicability of mathematics to physics, to chemistry and biology, and to psychological and economic situations. The more extended problems covering the speculative relations of mathematics to logic, psychology and philosophy in general -- part

(1) E. T. Bell, The Queen of the Sciences, Baltimore: Williams & Wilkin Company, 1931, p. 13; a similar remark can be found in George Sarton, The Study of the History of Mathematics, Harvard: Harvard University Press, 1936 p. 13.

(2) Florian Cajori, "Plans for a History of Mathematics in the Nineteenth Century," Science, n.s., 48 (1918), p. 279.

of a growing philosophy of science -- are being investigated by a growing number of thinkers.

The specific problem which is herein conceived as being persistent and basic to much -- indeed, if not all -- of the current discussions concerning the philosophy of mathematics is the nature of mathematical abstraction. It is with this question that the present inquiry is concerned as attempting to disclose the doctrinal pertinence and speculative value of mathematical abstraction for current philosophical theories on the nature of mathematics. A study of this kind naturally invokes many related problems, as an analysis of its historical antecedents, its relation to experience, its difference from other abstractive processes, and its pertinence to contemporary extensions of mathematics.

These closely related themes have provoked the following order of consideration. An introductory part of one chapter aims to clarify the problem by distinguishing and speculatively ordering the various possible views that can be held on mathematical thought. The next part considers some selected historical positions, and the last part contains the discussion of mathematical abstraction.

My thanks are due primarily to the Society of the Precious Blood to which I owe a cherished debt of gratitude. It was through its agency that I was given the opportunity to study and that I was directed into the philosophical aspects of mathematics and science. The influence of the late Rev. Joseph M. Marling, O.P.P.S., Ph.D., Auxiliary Bishop of Kansas City, Dr. Rudolf Allers of Georgetown University and Dr. Raymond L. Wilder of the University of Michigan -- all of them my teachers in the past -- is gratefully acknowledged. The direction and kindness of the Rev. Henri Gratton, O.M.I., Ph. L., C.T.L., of the University of Ottawa, Rev. Romeo Trudel, O.M.I., Ph.D. S.T.L., J. L.B., Dean of the Faculty of Philosophy of the University of Ottawa, and Dr. Thomas Greenwood of the Universities of Ottawa and Montreal, and of their colleagues at the University of Ottawa, were invaluable.

INTRODUCTORY.

Chapter 1. The Problem of Mathematical Abstraction

A fundamental confusion of terminology characterizes the literature bearing on the philosophical aspects of mathematics. Mathematical Logic, Mathematical Philosophy, Philosophical Mathematics and Metamathematics are but a few of the expressions currently used to designate the cognate aspects of mathematics and philosophy. This verbal disorder is but symptomatic of more basic disputes. A variety of opinionated attitudes is common today: mathematics is equated to logic and to all exact thought; the perfection of mathematical abstraction has supplanted and superseded metaphysics; the universe of mathematical discourse offers a unique though tentative view of the world of nature (1).

Various factors are responsible for these disparate views. Mathematics presently occupies a position of envious importance (2). Exaggerated opinions of its nature are readily occasioned by its strong influence on speculative and cultural thought and its numberless practical applications. Secondly, the philosophical background of our times and present philosophical trends are notorious for an excessive regard for mathematics and science, and a correlate distaste for metaphysics, ethics and theology. Mathematicians and scientists have philosophized in this antimetaphysical atmosphere

(1) A recent summary and survey of mathematical research and philosophical speculation on mathematics can be found in Louis O. Kattsoff, A Philosophy of Mathematics, Ames; Iowa State College Press, 1946. Kattsoff's book, to which frequent reference will be made, supplants an earlier, similar survey by Max Black, The Nature of Mathematics, London; Kogan Paul, 1933. For the contemporary situation regarding the philosophy of mathematics see Chapter VII, below.

(2) It is well known that mathematics, so successfully applied to physics, chemistry and the sciences of non-living matter is gradually being introduced into biology, psychology and the sciences concerned with life. For a broad, popularized and somewhat exaggerated survey of the significance of mathematics for other sciences, see Hollis R. Cooley, David Gans, Morris Klin and Howard E. Wahlert, Introduction to Mathematics: A survey emphasizing Mathematical Ideas and their Relations to other Fields of Knowledge, New York; Houghton Mifflin Company, 1937, Chapter 21, pp. 583-617.

since the days of the Cartesians (3). Thirdly, most philosophers are neglectful of mathematical doctrine. The difficulties inherent in its abstractive process and symbolism, the lack of time, interest or mathematical talent, and the comfort of relaxing in the view of one's own philosophical position are partial reasons for this dearth of philosophical response. Fourthly, the mathematician as a rule untutored in philosophy, scrutinise the universe from the view of mathematical abstraction alone (4). Neglectful of metaphysical implications, many mathematicians make indiscriminate generalisations, as mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true" (5); or, "From the intrinsic evidence of his creation, the Great Architect of the Universe now begins to appear as a pure mathematician" (6).

The general disregard for the doctrine of philosophical abstraction has contributed to and emphasized this want of speculative accord (7). The clarifications which it embodies regarding the nature of science and the distinction and

(3) F.-J. Thonnard, Précis d'histoire de la philosophie, nouvelle édition revue et corrigée, Tournai: Desclée & Cie, 1945, pp. 445-450; or see Charles Singer, A Short History of Science to the Nineteenth Century, Oxford: The Clarendon Press, 1941, "The Insurgent Century," pp. 185-247. A thorough study of the origins of modern science is contained in Edwin Arthur Burtt, The Metaphysical Foundations of Modern Physical Science, New York: Harcourt, Brace and Company, 1932, Chapter I, "Introduction," pp. 1-22 and passim; and in Etienne Gilson, The Unity of Philosophical Experience, London: Sheed and Ward, 1938, Part One, "The Mediaeval Experiment," pp. 3-122.

(4) Jacques Maritain, The Degrees of Knowledge, London: The Centenary Press, 1937, pp. 58-59.

(5) Bertrand Russell, "Recent Work on the Principles of Mathematics," International Monthly, 4(1901), p. 84.

(6) Sir James Jeans, The Mysterious Universe, new revised edition, New York: The Macmillan Company, 1942, p. 165.

(7) Through the influential thought of Leibniz and Kant, the word "abstraction" has suffered a falsification of its original meaning. It frequently refers, in the contemporary view, to a qualitative state or pose of the mind which lacks an extra-mental counterpart. For a use of the term in this falsified sense see Sir James Jeans, ibid., Chapter V, "Into the Deep Waters," pp. 137-186, or George D. Birkhoff, "Mathematics: Quantity and Order," in Science for a New World edited by J. G. Crowther, New York: Harper & Brothers, 1934, pp. 308-309. The word is used in this essay to designate that action of the mind which specifies speculative science, and will be explained below in Chapter VIII.

interrelation of sciences, are, however, fundamental in solving prevalent mathematical disputes. If a mathematician ignores or neglects the distinctions of logical, metaphysical, scientific and philosophical abstraction in his reflections on the universe, strange consequences result. The metaphysical teaching on the analogy of being, the distinction of substance, quantity and quality, matter and spirit, lose their significance. Logic, Cosmology and Psychology are credited with scientific status only to the extent that they are amenable to mathematical treatment, while the vast realm of probable and imperfectly scientific knowledge together with the opinions and knowledge of the ordinary person, are reputed as erroneous and unenlightened (8). That such is the present situation for many mathematician-philosophers is a generally accepted proposition.

In order to distinguish the point of view which is adopted throughout this essay, it is necessary to classify these divergent views, illustrate them, and adopt a definite and consistent terminology. Adopting abstraction itself as a basis for division, the possible views on mathematics fall under five broad categories ranging from a pre-scientific attitude towards mathematics to the extreme of identifying mathematical

(8) This neglect of the pre-scientific and other legitimate views besides that offered by mathematics and science is soundly criticized in Emily Meyerson's Identity and Reality (Authorized translation by Kate Lowenberg, London, George Allen and Unwin, 1930), passim; see especially Chapter II, "Common Sense," pp. 354-363. For a similar criticism see Leon Chwistek, The Limits of Science: Outline of the Methodology of the Exact Sciences (Translation by Helen Charlotte Brodie and Arthur F. Coleman, Introduction and Appendix by Helen Charlotte Brodie, New York: Harcourt, Brace and Company, 1948), Chapter I, "Introduction," pp. 1-23 and Chapter II, "The Limits of Sound Reason," pp. 24-52. In Etudes sur le rôle de la pensée médiévale dans la formation du système cartésien (Deuxième partie les "Etudes de Philosophie médiévale," Revue et considérablement augmentée, Paris; J. Vrin 1930), Etienne Gilson points out how Descartes' mathematicism eliminated other scientific speculation; see Deuxième Partie, "Descartes et la métaphysique médiévale," Chapitre I, pp. 141-190.

and metaphysical abstraction and thus equating all science to mathematics. The first is the non-abstractive or pre-scientific, which includes number-mysticism, the ordinary person's acquaintance with mathematics, the experimental view, and theories which do not credit mathematics with achieving scientific abstraction. The second is the abstractive view of the sciences and of the philosophy of nature which includes the historical view and that of applied mathematics. The third category is the abstraction proper to the science of mathematics. The fourth is the abstractive view of logic as distinct from mathematics (9). The fifth is the abstraction of metaphysics, more commonly called "critical" today (10). The attempt to place both legitimate and erroneous positions on mathematics within each of these classes, meets with a partial overlapping and clash among these divisions, especially in the relation of any class to that of metaphysical abstraction. This is attributable, though, not to the inadequacy of the divisions but to the unity of human science and knowledge and to the fact that the mind seeks synthesis and truth (11). Moreover, much of the concomitant confusion and many of the errors are directly attributable to the capriciousness of a given thinker and to his predisposition, from his mathematical viewpoint, towards hasty and unwarranted generalisations. For example, one historian may merely

(9) This is in contrast to the tenets of logicism, a current theory maintaining the identity of logic and mathematics. For a summary of this position, see Louis O. Kattsoff, *op. cit.*, pp. 93-115.

(10) *Ibid.*, pp. 1-7. Kattsoff views "metaphysics" or philosophy as having merely a critical function, which is but a modern restatement of Kant's transcendental criticism.

(11) Erwin Schrodinger, *What is Life?*, New York: The Macmillan Company, 1944, Preface. pp. vii-viii. Schrodinger points out the mind's desire for synthesis and unity. This tendency towards unification is also brought out strongly in Pierre Rousselot, *The Intellectualism of Saint Thomas*, Translated with a foreword by James F. O'Mahony, London: Sheed and Ward, 1935, Chapter V, "Third Substitute for the Pure Idea; Systems and Symbols," pp. 147-168.

catalogue dates, names and mathematical facts, considering this to be his proper function (12). Another may exhibit a delightful blend of humanism (13), or a curious socio-biological attitude to the historical development of mathematics and the influences which played upon its growth (14).

A brief consideration and elucidation of these five groups of attitudes towards mathematics is essential both to delimit the scope of this essay and to disclose the significant position occupied by the doctrine of mathematical abstraction. Essentially it is our intention merely to explicate mathematical abstraction itself, a speculative determination which is proper to the metaphysician. Even though it is the mathematician who "abstracts" in the science of mathematics, it is readily granted that the mathematician who initiates a self-conscious analysis of his mathematical operations is abstracting anew -- abstracting on mathematical abstraction -- and has entered the abstractive process of metaphysics. If such abstraction is performed without deliberate intent and conscious realization, metaphysical and mathematical abstraction are identified, and disaster to

(12) This is the general method adopted by historians of mathematics. As an example, see Florian Cajori, A History of Mathematics, second edition, New York: The Macmillan Company, 1931.

(13) The reference is to George Sarton who is largely responsible for the emphasis on studying the history of science and mathematics in the United States. Sarton is somewhat imbued with the positivism of Auguste Comte. For his humanist view, see, especially his The History of Science and the New Humanism, Cambridge: Harvard University Press, 1936; The Study of the History of Science, Cambridge: Harvard University Press, 1936. His monumental work inspired by the Comtian notion of history is the Introduction to the History and Method of Science, three volumes in five, Baltimore, Williams and Wilkins Company, 1927-1948.

(14) E. T. Bell, The Development of Mathematics, New York: McGraw Hill, 1940, *passim*. Bell's style and remarks are indicative of a frame of mind frequently found in the specialist who subjects the march of history to the limited scope of his science. The effects of such treatment, naturally disastrous as they are, increase for a mathematician accustomed as he is to certitude and deducibility, and frequently lacking, as he does, the care for detail and the insight into particulars and the complicated situations of history which characterize the good historian.

speculative truth is an inevitable consequence (15). This is but the inceptive stage of a current frame of mind, eventually begetting a mental distaste for other scientific abstraction and a cavalier treatment of the speculations of other sciences.

I. The Pre-Scientific View.

The four groups included under a non-abstractive acquaintance with mathematics, comprising number mysticism, the view of the ordinary man, the experimental view and theories not crediting mathematics with the perfection of science, are alike in being non-scientific. Consequently, for this view, mathematical knowledge is merely probable and opinionative. The pre-scientific statements and propositions which are held by people in the pre-scientific view, have not attained the perfection of science, cannot be supported by proofs and reasons, and lack the characteristics of certainty, necessity and universality proper to scientific knowledge.

From the abundant evidence of astrology and number mysticism it is easily seen that a quite fabulous and mystical -- really, a non-rational -- view of mathematics has frequently occurred in history (16). The notion of number or form was intriguing even in the early history of thought. Strange powers and divine influences were attributed to number or form either under the guise of some religious rite or mystical philosophy.

(15) "There, at any rate, lies the deepest root of the Cartesian philosophy. If anything can be truly said to express its innermost spirit, it is what I venture to call "Mathematicism," for Descartes' philosophy was nothing else than a recklessly conducted experiment to see what becomes of human knowledge when moulded into conformity with the pattern of mathematical evidence." Etienne Gilson, *op. cit.*, p. 133. This original error of Descartes has not as yet been repudiated by many contemporaries.

(16) See for example, the study of Vincent Foster Ropper, Medieval Number Symbolism, New York: Columbia University Press, 1938, or Lynn Thorndike, History of Magic and Experimental Science, six volumes, New York: Columbia University Press, 1923-1928.

This interpretation of mathematical entities derives from the vagaries of an uncontrolled or excessive use of the imagination. As a consequence, even though this attitude has occurred frequently in the history of thought, exerted much influence and even exists today, it can hardly be classed as rational and will not concern us here.

The pre-scientific acquaintance and use of mathematics by the ordinary person, since it is but imperfectly scientific, likewise belongs here. This view, however, is legitimate, even though it is confused and embraces many inaccuracies. For it is precisely this class of people who attend school, and when we attempt to lift up, by our teaching, to the level of mathematical abstraction. Ordinary acquaintanceship with the universe includes, as a rule, some imperfectly mathematical notions; on the strength of these notions teachers of mathematics attempt to induce the scientific consideration of mathematical entities in the imaginations and minds of their pupils, thus actualising in them the science of mathematics. Now this can be done most expertly and properly, of course, implicates a knowledge of what mathematical abstraction is; as, of course, a philosophy of teaching (17).

In the experimental approach to mathematics, as investigation is made of the process through which a child or person learns to count or learns about figures. From these experiments, which often furnish illuminating and helpful results, one can infer tentative theories on the psychological origins of mathematics. Such theories are useful in formulating beneficial

(17) In Mathematics' Teachers Views on Certain Issues in the Teaching of Mathematics (New York: Columbia Teachers College, 1941), Howard Homer gives the results of a survey concerned with philosophical aspects of the teaching of mathematics. See also Max Black, "The Relevance of Mathematical Philosophy to the Teaching of Mathematics," The Mathematical Gazette, 22(1938), pp. 149-153.

methods of teaching and shed some light on speculations concerning the foundations of mathematics (18). Quite of necessity, however, the investigator has some principle of hypothesis in mind whose veracity he wishes to establish, and frequently the results of such experimentation are colored and interpreted from this psychological or metaphysical point of view (19).

As an example of theories not crediting mathematics with being a speculative science, the materialistic view of Thomas Hobbes readily comes to mind (20). Granting universality only to the names of things, he had a radically empiricist view of all science, regarding mathematics as a frustrated idealization of sense impressions, and ~~denying~~^{limiting} the ability of human nature to consider lines or surfaces as being devoid of breadth or thickness. This view of mathematics, though rare, was adopted in a milder form by the British empiricists, Locke, Hume and John Stuart Mill.

II. The Sciences and the philosophy of nature.

~~This~~^{The} historical treatment of mathematics and the sciences which use mathematics comprise this second group. It is characteristic of the multiple sciences in this latter class -- physics, chemistry, biology, sociology and others -- merely to apply mathematical principles or conclusions

(18) Louis O. Kattsoff, op. cit., Chapter 13, "Kannoury's Signific Point of View," pp. 196-206.

(19) Arthur Fisher Bentley, Linguistic Analysis of Mathematics, Bloomington: Principia Press, 1937, passim.

(20) Carl B. Boyer, The Concepts of the Calculus, New York: Columbia University Press, 1939, pp. 176-178.

to their own subject matter (21). They do not, as a rule, question its validity or its method. The interaction between these sciences and mathematics in the course of their development is significant (22). New problems frequently arise in the sciences, requiring mathematical treatment and thus stimulating mathematical discoveries. Similarly, mathematical inventions are suggestive of new theories in science ^{and} predictive of new fields of research.

The historian of mathematics is concerned, first, with narrating the occurrence of mathematical facts. By investigating human records, historians look for the answers to the question, What contributions have been made to mathematics? The problems of when, where and by whom such items were invented and discovered and what these items are, lies in this domain. Properly speaking, the historian of mathematics should know what mathematics is in order to be able to classify the contributions.

Frequently a philosophical interpretation of these facts is coupled with the historical work (23). In investigating the early records of man,

(21) In the metaphysical view, these sciences are said to be subalternated to mathematics. Subalternation can be based on the principles, end or object. See Jacques Maritain, *op. cit.*, Chapter I, "Philosophy and Experimental Science," pp. 27-35, or see Vincent Edward Smith, *The Philosophical Frontiers of Physics*, Washington: Catholic University of America Press, 1947, Chapters II and III, pp. 22-55, or Bernard Mullaly, "Subalternation and Mathematical Physics," *Laval Theologique et Philosophique*, 2(1946), pp. 69-107.

(22) Sir William Cecil Dampier, *A History of Science and its Relations with Philosophy & Religion*, third edition, revised and enlarged, New York: The Macmillan Company, 1943. Despite occasional inaccuracies, Dampier well illustrates this interaction between science and mathematics; see Chapter IV, "The Newtonian Epoch," pp. 160-216.

(23) For a brief comparison of history and the philosophy of history, see Gilbert J. Garraghan, *A Guide to Historical Method*, edited by Jean Delangles, New York: Fordham University Press, 1946, Chapter 16, No. 8, "The Philosophy of History," pp. 367-380. On pages 370-371 Garraghan points out the error of historicism which would, from a positivistic bias, limit the historian to a mere recording of facts.

for example, the historian may approach these records with an evolutionistic or pragmatic attitude of mind, thus modifying his presentation and interpretation of these facts. This attitude is indicative of the gradual reintroduction of the historian as one who looks for the meaning in ~~the~~ history and considers the laws of historical growth. Recent histories of mathematics point in this direction (24). Some historians, however, betray the fallacy of comparison judgment in which the perfections of our age or some historical period are taken as a standard or measure of the greatness or the value of previous or future historical events (25).

Historico-philosophical accounts of mathematics present the following problem: has mathematics changed its nature as a science as it grew in content and methodology throughout history? If it is maintained that such an essential change is possible (for example, that it is now seen that mathematics and logic are identical (26)), then a true science of mathematics or a philosophy of mathematics never really exists or is possible. The science of mathematics would constantly be fluctuating not only in method, but in its very nature. One would never know, in any historical period, whether we had ever arrived at the degree of abstraction proper to mathematics, nor whether any contributions were properly scientific. There would be no reason in concerning oneself with the nature of the problems

(24) This is true especially of the works of George Sarton who inclines, however to a Comtian interpretation of science; cf. Footnote 13, above.

(25) This seems to be one of the faults of E. T. Bell's The Development of Mathematics; see footnote 14, above.

(26) "The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself." Bertrand Russell, The Principles of Mathematics, New York: W. W. Norton & Company, 2nd edition, 1937.

of a science which never attains proper existence. A strictly evolutionistic view of mathematics or of science is a contradiction (27).

On the other hand, it is possible to maintain that though the intrinsic nature of mathematics does not change, the abstractive process of the mathematician can become more precise and less dependent, for example, on imaginative intuitions or constructions. In this case, the historian would show the birth of mathematics as an independent science, complete in itself in the sense that it would have the necessary abstraction and perfection to be recognised as mathematics, yet have the possibilities of further development and of a more abstractive treatment within the limits of mathematical abstraction itself. It is maintained here that mathematics does not alter its nature so fundamentally as to become unrecognisable as such ^{AND} as to destroy mathematical continuity. The type of abstraction proper to mathematics was born and had achieved a recognisable content and method already by the time of Euclid (28). By their separation and distinction of number mysticism or allied contributions from mathematics,

(27) Louis O. Kattsoff, *op. cit.*, Chapter Twelve, "Gödel's Theorem and Formal Systems," pp. 182-195. Kattsoff writes about Gödel's famous theorem and its relation to formal systems: "If, as is agreed, the formal systems can be used to systematise fields of knowledge, and if the Gödel theorem applies to such applications, it would mean that no field of knowledge so systematized can ever be complete. At any state there will always be questions formulable in the system that need a wider system for their solution. Since it is the contention of the author of this book that logic and mathematics do tell us something about reality, the Gödel theorem tells us that there can never be a complete and final theory of reality, i.e., metaphysics. This does not lead to mysticism, but to an evolutionary scheme of reality." pp. 194-195.

(28) "...if we should study any subject properly, we must study it as something that is alive and growing and consider it with reference to its origins and its evolution in the past. In the case of mathematics, it is the Greek contribution which it is most essential to know, for it was the Greeks who first made mathematics a science." Sir Thomas L. Heath, A Manual of Greek Mathematics, Oxford: The Clarendon Press, 1931, p. 1.

historians of mathematics indicate their belief in the recognizable content of mathematics. Further discoveries after Euclid -- new methods and new problems, applications to other sciences and practical affairs -- are but a natural growth and ^{the} derived expansion of the fundamental abstractions achieved by the Greeks. What increases is our understanding and use of mathematical abstraction through the introduction, for example, of new methods, various systems of geometry and algebra.

The applications of mathematics to problems of chemistry, physics, biology and economics, are made by men who fall into the class of applied mathematicians. This application of the principles of mathematics to the objects of scientific abstraction was adumbrated by the early Pythagoreans; and, of course, the efforts of Archimedes in this direction are justly famous. The essential element in this process is the coordination and subordination of the manifold and mutable in nature under the unity and principles of mathematical abstraction. The history of mathematics memorializes the terrific struggle that thinkers faced in gradually establishing the relevance of a quantitative study of nature. For example, the many attempts to find a procedure for studying motion and change were finally rewarded with the algorithm of the calculus, clarifying, at the same time, the concepts of mathematical ^{approach} with the notions of number, function and the limit process (29).

Mathematics is being applied with increasing success to three groups of science. In the sciences concerned with non-living matter, astronomy, physics, chemistry, geology and crystallography, its use has been most

(29) An interesting account of this century-old struggle is given in Carl B. Boyer's The Concepts of the Calculus cited above.

striking and successful. The application of mathematical principles to the sciences concerned with living matter is of three types: mathematics is applied to biology and related subjects (30); to psychology (31); and, in statistical applications, an analytic attempt is made to place the predictable relatedness between contingent things in correspondence to the interrelatedness of mathematical entities (32). The mathematics of probability involved in statistics attempts to measure the fluctuation, motion and change associated with the world of non-living and living nature.

Examples of philosophical speculation by all types of applied mathematicians are numerous. Considering only those who deal with astronomy and physics, the names of Einstein, Jeans, Whitehead, Eddington and Schrödinger readily suggest themselves. Mentally overfamiliarised in viewing nature through the abstraction of mathematics or physics (and, possibly, never having really achieved the abstraction of metaphysics), these men readily adopt a reductionist attitude to reality. One thinker, viewing an ordinary desk with his scientific mental eye, proclaims that

It is a host of tiny electric charges darting hither and thither with inconceivable velocity. Instead of being solid substance my desk is more like a swarm of gnats (33).

(30) W. R. Thompson, Science and Common Sense; An Aristotelian Exposition, New York: Longmans Green and Co., 1937, pp. 111-124. See the more recent article on mathematics and biology by Thompson, "Mathématique et biologie," Laval intellectuelle et Philosophique, 3(1947), pp. 77-88.

(31) Joseph Fröbes, Compendium Psychologiae Experimentalis, Romae: Apud Aedes Universitatis Gregorianae, 1937, "Introductio, De Fine et viis psychologiae empiricae," pp. 8-18.

(32) Vincent Edward Smith, op. cit., pp. 42-55; or see Hollis R. Cooley, et. al., op. cit., pp. 595-599. For a popularized mathematical presentation see Harold T. Davis, A Course in General Mathematics, Bloomington: The Principia Press, 1935, Chapter XIV, "The Functions of Statistics," pp. 237-254.

(33) Sir Arthur Eddington, New Pathways in Science, New York: The Macmillan Company, 1935, p. 1.

In concluding a series of lectures on What is Life?, Schrodinger finds that although my body functions as a pure mechanism according to the laws of nature, I know that I direct it. This directive "I" has led, he believes, to the notion of a plurality of such "I's," to the invention of souls and similar problems. The better alternative which he suggests, is:

simply to keep to the immediate experience that consciousness is a singular of which the plural is unknown; that there is only one thing and that, what seems to be a plurality, is merely a series of different aspects of this one thing, produced by a deception (the Indian Maja); the same illusion is produced in a gallery of mirrors, and in the same way Gaurisankar and Mt. Everest turned out to be the same peak seen from different valleys (34).

Examples of such scientific extravaganzas can be called from the works of many leading astronomers, physicists, biologists and sociologists (35).

The relation of the philosophy of nature as the branches of cosmology and psychology to mathematics was an acute diagnostic problem for Plato, Aristotle, Descartes, Leibniz and Kant. Today, however, outside of those who adhere to the unity of the philosophical tradition (36), cosmologists are few in number and most psychologists consider their work to be merely

(34) Erwin Schrodinger, What is Life? The Physical Aspect of the Living Cell, New York: The Macmillan Company, 1946, p. 90.

(35) "Contrary to the view that psychology is hostile to religion we believe that if we are willing to modernize our conception of God we can employ the methods and results of biological psychology in defense of a view of the world which is essentially religious. But the god whom we here propose to substitute for the god of traditional theology may seem a strange god to many. And yet this god alone makes possible the only life we know of; this god is the source of all the energy on earth which sustains man and his institutions. In brief, I wish to argue that god is light; And as light god can do all those things which this most remarkable reality of modern physics can accomplish." Oliver L. Reiser, Philosophy and the Concepts of Modern Science, New York: The Macmillan Co., 1935, p. 194

(36) A number of fine studies deal with the relations of the philosophy of nature to mathematics, of which the following are representative: Jacques Maritain, La Philosophie de la Nature: Essai critique sur ses frontieres et son objet, Paris: Pierre Tequi, 3eme edition, n.d., and by the same author, Science and Wisdom, New York: Charles Scribner's Sons, 1940, Ch. I, "Science and Wisdom," pp. 3-33 and Ch. II, "The Philosophy of Nature, pp. 34-69, and the Degrees of Knowledge cited above; Yves Simon, Prevoir et savoir, Montreal: Editions de l'Arbre, 1944; W. R. Thompson, Science and Common Sense, cited above; Edward Vincent Smith, The Philosophical Frontiers of Physics cited above,

experimental and statistical. This current dearth of a philosophical appraisal of nature distinct from metaphysics is partially explicable by the huge success of the mathematical and physical sciences, and is a strong factor in explaining the many philosophical reflections stemming from scientists in biology, chemistry, physics and allied fields of investigation. In this relation of the sciences to the philosophy of nature, in fact, there is a confusion of terminology and thought parallel to that covering the relation of philosophy and mathematics. Though the vague designation "philosophy of science," is often used to cover psychological, moral, political and metaphysical remarks and views, it is, properly speaking, the function of metaphysical science (37).

III. Logical Abstraction

In the negative abstraction of logic, one considers an object which is related to something in nature, and which does not exist as such in nature but only in the mind. The objects of logical consideration are supplied, so to speak, by the reflective mind which is capable of subjecting these logical objects to a dialectical process without necessarily implicating a reference to an object in nature. The close connection and use of this type of abstraction and mental dialectical process to that of mathematics is readily seen. For the concepts of mathematical abstraction, though they have an extra-mental reference of origin to objects in nature,

(37) An able presentation of the relevance of metaphysics as properly being the "philosophy of science" can be found in any of the works cited in the previous footnote. That many contemporary works are sprinkled with moral psychological and metaphysical allusions under the broad cover of "philosophy of science" is evident in the works of Jeans, Whitehead, Planck, Eddington, Cassirer and others.

are considered by the mind as abstracted from these natural objects.

Many mathematicians, holding that mathematics and logic are identical, frequently propose the added equation of all scientific thinking to that of mathematics. Bertrand Russell is an outstanding contemporary example of this view which had previously been held by Leibniz, Bolff, Peano, Frege and Boole (38). Russell's definition of mathematics is well known:

Pure mathematics is the class of all proposition of the form 'p implies q' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth (39).

Similar definitions, equating logic and mathematics, are frequent (40), a sampling of which is contained in R  cher's article on "Fundamental

(38) See C. I. Lewis, A Survey of Symbolic Logic, Berkeley: University of California Press, 1918, for an historical study of mathematics and logic.

(39) Bertrand Russell, The Principles of Mathematics, p. 3.

(40) Samples of various definitions of mathematics have been collected and presented in the following works: Robert Edouard Moritz, Memorabilia Mathematica, New York: The Macmillan Company, 1914, Chapter I: "Definition and Objects of Mathematics, pp. 1-9; John Wesley Young, Lectures on Fundamental Concepts of Algebra and Geometry, New York: The Macmillan Company, 1911, Lecture XXI, "General Conclusions," pp. 215-225; E. T. Bell, Men of Mathematics, New York: Simon and Schuster, 1937, "They Say, What They Say, Let Them Say," pp. xix-xxi; Louis O. Kettsoff, op. cit., Chapter 2; "Definitions of Mathematics," pp. 8-15.

Conceptions and Methods of Mathematics" (41). Pierce, for example, says that "Mathematics is the science which draws necessary conclusions" (42), while Bôcher himself gives this definition of mathematics:

If we have a certain class of objects and a certain class of relations and if the only questions which we investigate are whether ordered groups of these objects do or do not satisfy the relations, the results of the investigations are called mathematics (43).

For this group of mathematicians, of which there is a large number, logical and mathematical abstraction are not clearly distinguishable. As a consequence, all sciences should be governed by mathematical logic -- the contemporary version of Descartes', Leibniz's and Comte's universal mathematics (44).

IV. Mathematical Abstraction

The mathematician as teacher or specialist -- the mathematical mathematician, so to speak -- fits into this class. The mathematician as such exists only when he is actually using his mind as a mathematician in that degree of science called mathematical abstraction, as he performs actual mathematical operations of the mind. At other times, he retains mathematical images in his memory, a certain mathematical adaptability in his imagination, and the scientific habit of

(41) M. Bôcher, "The Fundamental Conceptions and Methods of Mathematics," Bulletin of the American Mathematical Society, Vol. XI (1904), pp. 115-135.

(42) Ibid., p. 117.

(43) Ibid., p. 127.

(44) A presentation and criticism of the program of mathematical logic can be found in Harold L. Smart, The Philosophical Presuppositions of Mathematical Logic, Cornell Studies in Philosophy, no. 17, New York: Longmans Green & Company, 1925.

the conclusions of the science of mathematics in his mind (45).

Suppose, to illustrate this viewpoint, that a student is engaged in the study of the Calculus. It is not necessary for him to know its precise definition or its history in order to understand the demonstrations or to work the problems. The subject matter is presented to him, a method of demonstrating properties is used; and the student, if he is capable of mathematical abstraction, acquires the knowledge of the qualitative and quantitative characters of the Calculus. If the student questions the existence of a proper subject matter for mathematics, he cannot proceed any further in the science until its existence be shown him by construction or definition; if he questions the mode of demonstration, the logic of the procedure needs to be shown him. Supposing that he grants both of these and is capable of mathematical abstraction, he acquires, with the help of his imagination and by a number of acts of mental abstraction, the habit of the science of the Calculus.

Or consider the case of the mathematician as specialist. He is concerned with what is immediately given: the problem to be solved and the methods of solution. The mathematician considers structure and its relations, symbols and their laws of combination, or merely reflects on abstract relations which may materialise into mathematical conclusions. When, however, the mathematician wonders about the meaning of mathematics or questions its applicability to nature or its relation to logic, he has begun to consider mathematics itself from an extrapolated view, that of metaphysics. Unless he is conscious of this new viewpoint, he may confuse

(45) Chapter IX of this essay will attempt to clarify the notion of mathematical abstraction.

or identify the entities which he holds in mathematical abstraction with other concepts.

This defective understanding of the proper place and function of mathematical abstraction colors most of the current research and speculation concerned with the philosophy of mathematics. Such questions in mathematics as the notion of mathematical existence, the ultimate nature of number or function, though they can be given a strictly mathematical treatment -- the inner consistency and rigor of these concepts, for example -- are questions which are outside the abstraction of mathematics and belong to metaphysics or logic. Of the false and strange speculative conclusions which result from a neglect of the doctrine of mathematical abstraction, the antinomies and paradoxes which have arisen out of discussion on the foundations of mathematics are illustrative (46).

As an example of the limitation and uniqueness of mathematical abstraction, consider the relation of the imagination to mathematics. Suppose that a mathematician wishes to examine the psychology of invention in mathematics (47), or illustrate the relation of the imagination to mathematical abstraction (48). The existence and nature of the imagination, a question proper to psychology, is assumed, as is the nature and existence of mathematical entities as established in metaphysics. Such an investigation, consequently, can have neither a psychological

(46) Louis O. Kattsoff, *op. cit.*, pp. 87-92.

(47) As is done, for example, in Jacques Hadamard, The Psychology of Invention in the Mathematical Field, Princeton: Princeton University Press, 1945.

(48) For example, see Edward Kasner and James Newman, Mathematics and the Imagination, New York: Simon and Schuster, 1940, especially pp. 357-362.

nor a mathematical attitude, for it is outside the demonstrative scope of either science, and it is metaphysical because it orders the relations of a psychological subject to mathematics. One cannot do this successfully unless he admits the validity of both metaphysics and psychology, and understands the relation of the abstraction proper to both of these sciences in contrast to mathematics (49).

An example from one of the schools on the foundations of mathematics serves to point up the distinction between mathematical and non-mathematical abstraction. The theory known as formalism (50) holds, briefly, that mathematics is similar to a game played with arbitrary symbols according to arbitrary rules, internal consistency being the only requirement in this game. By this jou d'esprit, a mathematical theory arises which must be kept distinct from a theory about mathematics. Theories about mathematics are classified as metamathematics. Metamathematics deals with such problems as structure, categoricalness and consistency. Whether this view, Hilbert's formalism, be true or not (it hardly credits the science of mathematics as being scientific), it is evident that here is a distinction between the activity of the mathematician as such, and a more abstracted consideration of mathematics.

(49) For an extreme example of speculative error at its worst, see Lancelot Hogben, Mathematics for the Million, London: George Allen & Unwin, 1942, and, by the same author, Science for the Citizen, second edition, London: George Allen & Unwin, 1943. Hogben interprets the historical development of mathematics by reference to sorceries, fabulous stories and to the process of dialectical materialism. He masks himself alternately as theologian, metaphysician, philosopher of history and scientist. He is far outside the scope of mathematics or science proper and assumes the role primarily of the metaphysician who orders and relates the branches of knowledge to each other.

(50) Max Black, The Nature of Mathematics, "Supplement A: Formalism," pp. 147-158.

Unlike the logistic view of mathematics, formalism and intuitionism (51) seem to offer a basic distinction between mathematical abstraction proper and an alternate view of mathematics.

V. Metaphysical Abstraction

The ultimate in speculative science, metaphysics as a scientific habit informs the mind both with the knowledge of being and with a judging, critical habit (52). Its capacity as a scientific habit has been denied since the days of Kant; in its critical and systematizing role however, its validity is still quite universally acknowledged though often identified with logic or with philosophy itself. Bacon, Descartes, Leibniz and Kant, viewing metaphysics from mathematical, scientific and logical abstraction, bemoaned its miserable plight (53), and their attempts to make it mathematical, physical or logical are echoed today.

Before considering an outline of the legitimate view that metaphysics offers on current mathematical speculation, the pseudometaphysical view of mathematicism must be clarified. Mathematicism appears under various names as mathematical philosophy (54); mathematical Idealism (55); mathematico-

(51) Ibid., pp. 169-210.

(52) Joseph F. Casey, The Primacy of Metaphysics, Washington: The Catholic University of America Press, 1936, pp. 84-101.

(53) This is brought out in Etienne Gilson's The Unity of Philosophical Experience cited above.

(54) Writers in French usually use the term philosophie mathématique. See, for example, P. Gonthier, Philosophie mathématique, Paris: Hermann et Cie, 1939. Or see Cassius J. Keyser, Mathematical Philosophy: A Study of Fate and Freedom, New York: E. P. Dutton & Co., 1937. Keyser's books and numerous articles are tinged with mathematicism. Tobias Dantzig echoes a similar strain in his two books Aspects of Science, New York: The Macmillan Company, 1937, passim, and Number, the Language of Science: A Critical Survey Written for the Cultured Non-Mathematician third edition, revised and augmented, New York: The Macmillan Company, 1941.

(55) Gaston Milhaud, Les Philosophes Géomètres de la Grèce: Platon et ses prédécesseurs, Paris: J. Vrin, 1934, pp. 5-9.

metaphysical philosophy (56); universal mathematics (57); and, in a broad sense, scientism (58). Briefly, mathematicism maintains that only mathematical abstraction affords a legitimate outlook on the universe and consequently, that mathematics is the only science. Its adherents propose, in some form, that the abstract view and methodology of mathematics alone qualify in dealing with problems of scientific, cultural or general knowledge. Mathematics, in this view, is coterminous in some sense with all science; philosophy, biology, chemistry -- science in general -- either are mathematical or should aim to become such. This hyper-reductionist view of the universe submits all of reality to the measure of the mathematical yardstick, subsuming the diversity of things under one monotonous view, the mathematical.

Some general characteristics are common to the various distinguishable types of mathematicism (59). First of all, it is an extreme, ultimate, or final attitude, manifesting itself as philosophical rationalism and idealism, or positivism and scepticism (60). Secondly, by not distinguishing various levels or types of being, mathematicism is universalist and univocal (61). Because it is ultimate and universal, mathematicism is likewise reductionist. There is no distinction of quantity from quality, homogeneity from hetero-

(56) Frederick Copleston, A History of Philosophy: Volume I, Greece and Rome, Westminster: The Newman Bookshop, 1946, p. 32. Copleston speaks of the Pythagorean mathematico-metaphysical philosophy.

(57) Both Descartes and Leibniz spoke of a "Universal Mathesis;" see Joseph Maréchal, Précis d'histoire de la philosophie moderne. Tome premier: de la renaissance à Kant, Louvain: Museum Lessianum, 1953, pp. 75-80 on Descartes, and pp. 168-181 on Leibniz.

(58) John Wellmuth, The Nature and Origins of Scientism, Milwaukee: Marquette University Press, 1944, passim.

(59) Gaston Milhaud, op. cit., "Introduction Générale," pp. 1-49. In these opening pages of his book on the philosopher-geometers of Greece, Milhaud stresses the importance of a mathematical education in leading an individual to idealism and dogmatism.

(60) Ibid., pp. 2-9.

(61) James F. Anderson, "Mathematical and Metaphysical Analogy in St. Thomas," The Thomist, 3(1941), pp. 467-506.

generality, infinite from finite, in its view by comparison to these distinctions in metaphysics and philosophy as conformable to extra-mental reality (62). Thirdly, in view of the rigid determination with which mathematical abstraction regards reality, mathematicism evokes a propensity to dogmatize and to make categorical statements concerning the complex, detailed qualitative states which characterize the universe along with the quantitative.

As a class into which thinkers can be placed, mathematicism may be subdivided into three groups. The views of two of these subdivisions, of course, may be straddled by a given mathematician. Mathematicism may exhibit itself under the form of idealism. This peculiar attitude, an ancient habit of mind, was found in its perfect state in the Pythagoreans and Platonists, in Descartes and Spinoza; and today, in Jeans, Eddington, Russell and others. Mathematicism takes the form of empiricism in the claims of those who require mathematical certitude in all sciences, including sciences in which this goal has not, nor will likely be attained. Concepts or propositions which cannot be subjected to mathematical procedure are regarded as conjectural, superstitious or unscientific; qualities or substances not amenable to measuring devices are considered fabulous, mystical or supernatural. Protagoras' dictum, "man is the measure of all things" is parodied as "measure is the measure of all things (63)." Kepler, Galileo, Kelvin and some modern mathematical

(62) Gaston Milhaud, *op. cit.*, pp. 18-49.

(63) Robert Andrews Millikan, Electrons (γ and β), Protons, Photons, Neutrons, Mesotrons, and Cosmic Rays, revised edition, Chicago: The University of Chicago Press, 1947, p. 4.

physicists are members of this group (64). Thirdly, mathematicism assumes the form of positivism or logical positivism (65). As such, it proposes that only such items as are reducible to and verifiable by the procedures of linguistic analysis or symbolic logic are creditable with scientific status. Although mathematicism is frequently, in this view, identified with logic, it is an ultimate, metaphysical view. For these men, logic is little more than a symbolic language congealing or stratifying the results of empirical investigation.

Finally, there is a legitimate field of philosophical inquiry concerned with the nature of mathematical abstraction which is termed the philosophy of mathematics throughout this essay. This view bears some resemblance to a field of inquiry which appears under the guise of various names, as critical philosophy of mathematics, the epistemology of mathematics, the metaphysics of mathematics (66); metamathematics, intuitionism and logicism (67). In being thus connected with mathematics, however, the word philosophy is currently given a meaning and usage that is quite liberal and inexact; philosophy, as a rule, is credited with a mere critical function, granted solely a judging capacity, or denominated as contributing nothing further than a

(64) The extreme, sensualistic mathematicism of Thomas Hobbes belongs here likewise. Cf. Carl S. Boyer, op. cit., pp. 176-178.

(65) Louis O. Kattsoff, op. cit., p. 245.

(66) For a similar view, see Charles De Koninck, "Thomism and Scientific Indeterminism," in Proceedings of the American Catholic Philosophical Association, 12(1936), p. 75, and John F. Whittaker, "The Position of Mathematics in the Hierarchy of Speculative Science," The Thomist, 5(1941), pp. 467-506. Other attempts to clear the relations of mathematics and philosophy can be found in Louis O. Kattsoff, op. cit., pp. 1-2, and James Byrne Shew, Lectures on the Philosophy of Mathematics, Chicago: Open Court Publishing Company, 1918, pp. 1-18.

(67) These latter three views will be explained below in Chapter VII.

measure of restraint and caution in the analysis of mathematical knowledge (68).

The philosophy of mathematics which concerns us here, however, and of which the nature of mathematical abstraction is the basic and central problem, involves both a more exact and extensive use of the term. Metaphysics not only has a critical and defensive function (69); it is a genuine science similar to other sciences in that it inlays the mind with a qualitative habit, yet distinct and unique inasmuch as it is the supreme science (70). In its capacity as a science, then, it is correctly viewed as performing a double function; it actually confers scientific knowledge of things (71); and, secondly, it criticizes and defends the truth, value and validity of human knowledge both pre-scientific and scientific, and it arbitrates the relations between the sciences (72).

The contemporary situation in philosophical speculations on mathematics is of such a nature that both functions of metaphysics are employed throughout this essay. In its first function, that of contributing genuine scientific knowledge, metaphysics is the science of the very being of things

(68) Hans Reichenbach, *Experience and Prediction*, Chicago: The University of Chicago Press, 1938. In the Preface, (p. v), Reichenbach writes: "American pragmatists and behaviorists, English logistic epistemologists, Austrian positivists, German representatives of the analysis of science, and Polish logicians are the main groups to which is due the origin of that philosophic movement which we now call "logistic empiricism." ...all characterized by their common descent from a strict disavowal of the metaphor language of metaphysics..." In Chapter I, "Meaning," he attributes to what he calls epistemology the threefold task of being descriptive, critical and advisory. See pp. 3-16.

(69) A.-M. Pirotta, "De Metaphysicae Defensivae Natura secundum Doctrinam Aristotelico-Thomisticam," *Angelicum*, 4(1927), pp. 252-270 and pp. 430-453.

(70) Joseph T. Casey, *Op. Cit.*, pp. 84-101.

(71) George M. Buckley, *The Nature and Unity of Metaphysics*, Chapter II, "Definition and Division of the Subject-Genus of Metaphysics," pp. 37-80.

(72) *Ibid.*, pp. 206-210.

as such and deals with anything which is in any way whatsoever; in this way it answers the problem of the general nature of science, the unity and plurality of sciences, and the nature and functions of particular sciences. In its critical and defensive role metaphysics will be used as validating the truth and pertinence of other scientific abstractive processes than that of mathematical abstraction, and ^{AS} ~~by~~ ordering the complex relations with which mathematics is related to pre-scientific knowledge, and to the particular, multiple sciences that are being cultivated today.

PART I. AN HISTORICAL SURVEY OF MATHEMATICAL ABSTRACTION

Chapter II. The Historical Problem of Mathematical Abstraction

An historical account of the nature of mathematical abstraction shows that it is the central problem of any philosophical speculation on mathematics. From the time of Thales, or the first Greek thinker who abstracted the mathematical, intellectual content from the sensible, thinkers have confused or identified the objects held in mathematical abstraction with the objects of logic, of science or metaphysics, and even with the nature of things themselves. All the speculative questions whose correct answer hinges on an adequate grasp of abstraction in mathematics -- existence, certitude, truth, applicability to nature -- are unsuccessfully met by thinkers who neglected this problem.

The deductive certainty of mathematics, its tenuous relation to the sense-perceptible, its exactness and profundity, exercise an attraction that is difficult to resist (1). Mathematical abstraction had scarcely been achieved, when the ancient Pythagoreans, captivated by its excellence, initiated the first program of mathematicism; they attempted to subsume the diversity and fluctuation of things under the rigor and determination of number and form (2). Plato and Aristotle had to take account of the popularity of the Pythagorean doctrine; in fact, we already find Aristotle insisting against the Pythagoreans and the Platonists, in those early days, that it is impossible to attribute real existence to the objects held in

(1) Gaston Milhaud, *op. cit.*, "Introduction Generale," pp. 1-49.

(2) F. Copleston, A History of Philosophy, Volume I: Greece and Rome, Westminster: The Newman Bookshop, 1948, pp. 29-37.

mathematical abstraction (3). The Pythagorean experiment, though unsuccessful, is a frequently recurring theme in the history of thought and is often credited with exercising a strong influence on all subsequent scientific and philosophical thinking (4).

Of the many thinkers who concern themselves with mathematics in its philosophical aspect, few deal expressly and explicitly with mathematical abstraction as such. As a separated question of inquiry, the nature of mathematical abstraction is not a frequently occurring question; most thinkers assume that ^{THE} nature of mathematics is self-evident or never consider an understanding of mathematical abstraction as basic to all discussion on the nature of mathematics. For example, when Cartesian mathematics was popular, few thinkers broached the delicate but pertinent query, What is mathematics? At that time, the attitude of acceptability to mathematics which was inaugurated with the Renaissance, and the remarkable applicability to problems and the numberless discoveries that quickly followed Descartes' analytic geometry, precluded the prudence of questioning its nature (5). Hence it is necessary to ferret out the theories on mathematical abstraction from the implications which are

(3) More basically, of course, the Pythagorean attempt can be regarded as the inception of a long struggle to gain recognition for the validity of the abstraction proper to mathematics and the consequent admission of the importance of the quantitative element in nature. Cf. Thomas Little Heath, "Greek Mathematics and Astronomy," *Scripta Mathematica*, 5(1938), pp. 215-232.

(4) Modern thinkers pay tribute, sometimes excessive, to Pythagoras or flate for their program of mathematization of reality. See, for example, the following: Sir James Jeans, *The New Background of Science*, second edition, New York: The Macmillan Company, 1904, pp. 230-71 and, by the same author, *The Mysterious Universe*, second edition, Cambridge: The University Press, 1937, p. 117 and p. 132; or see Alfred North Whitehead, *Adventures of Ideas*, Cambridge: The University Press, 1933, pp. 194-195; Ernst Cassirer, *Substance and Function and Einstein's Theory of Relativity*, Chicago: Open Court, 1923, p. 138; Bertrand Russell, *A History of Western Philosophy*, New York: Simon and Schuster, 1948, pp. 36-37.

(5) E. G. Collingwood, *The Idea of Nature*, Oxford: The Clarendon Press, 1945, pp. 93-105.

latent in what such thinkers as Descartes have written on mathematics or from the role and the function mathematics is given in their philosophy and science.

The explicit secondary sources for a history of mathematical abstraction, or, more especially, for the relation of mathematics to philosophy are relatively few (6). There is, moreover, much disagreement and dispute on the extent, influence and relation of mathematics to the philosophy of individual thinkers. It is only recently, in fact, along with the self-consciousness of mathematics and science, that the influence of mathematics on the historical formation of philosophy is being sufficiently credited (7). Many scholars today are cautious about the relation of mathematics to philosophy, and some shie away from or reject the relevance of a philosophical viewpoint of mathematics (8). Others welcome a philosophical attitude as healthful and even provocative of mathematical discovery (9). These attitudes, important as they are in the study of an individual thinker, also have a pronounced effect on the future relation

(6) A classical study, in French, is Leon Brunschvicg, Les Etapes de la Philosophie Mathématique, Paris: Presses Universitaires, 3ème édition, 1947. Brunschvicg's work is, unfortunately, tinged with idealism and mathematicism. Edwin Arthur Burt's book, The Metaphysical Foundations of Modern Science, cited above, likewise seems to be an extreme interpretation. A contrary thesis to the one contained in Burt's book can be found in Edward W. Strong, Procedures and Metaphysics, Berkeley: University of California Press, 1936.

(7) A good example of this can be found in Joseph Maréchal, Précis d'Histoire de la Philosophie Moderne, Tome Premier: De la Renaissance à Kant, Louvain: Museum Lessianum, 1933. By the same author, see the study entitled Le Point de départ de la Métaphysique, 5 volumes, Paris: Desclée de Brouwer, 1944-1947, passim.

(8) "There will be less mysticism in the mathematics of the future than there has been in that of the past, and fewer grandiose claims to immortality and eternal youth. Mathematics will become less self-conscious, less introspectively critical, and more boldly creative. It will resign its soul to the metaphysicians for such tortures as they may choose to inflict, feeling nothing; for it will continue to serve with its living body the purposes of the man who create it to meet human needs rather than to be the plaything of sterile philosophies." E. T. Bell, The Development of Mathematics, New York: McGraw Hill, 1948, pp. 174-175.

(9) Carl B. Boyer, The Concepts of the Calculus, A Critical and Historical Discussion of the Derivative and the Integral, New York: Columbia University Press, 1939, Chapter VIII, "Conclusion", pp. 299-309.

of mathematics to philosophy.

A passing emphasis must be laid on the mutual influence of mathematics and philosophy on each other in the course of their development (10). As will be shown below, the evidence for their interaction, quite evident as it is in the works of philosopher-mathematicians as Pythagoras, Plato, Descartes and Leibniz, can likewise be detected in philosophers who were not mathematicians as Locke, Berkeley and Kant. They^{RE} are, naturally, obvious speculative issues about which both mathematics and philosophy are concerned: quantity, space, time, infinity and number. Even with regard to less apparent theoretical matters, however, both mathematics and philosophy have gravitated towards each other in their developmental history and are doing so today. For example, the type of knowledge and procedure furnished by mathematics in comparison to moral knowledge was the program of investigation to which the British empiricists set themselves. Analogous questions concerning the interrelatedness of mathematics and philosophy, mathematics and the sciences, have frequently appeared and are common speculative coin today (11).

The usual chronology of the history of mathematics proper is not

(10) Hollis R. Cooley, et al., op. cit., pp. 593-609.

(11) William Cecil Dampier (Formerly Whetham), A History of Science and its Relations with Philosophy and Religion, third edition, revised and enlarged, New York: The Macmillan Company, 1943, Chapter X, "Scientific Philosophy and its Outlook," pp. 445-492.

relevant to an historical survey of mathematical abstraction (12). Neither does the basis for demarcating the different historical periods nor do the guides for classifying types of contributions apply (13). Instead, a classification based on what are frequently called the three crises of mathematics will be followed (14). The first period witnesses the struggle of mathematical abstraction -- along with that of science and philosophy -- to gain and keep recognition. It comprises the ancient views which Pythagoras, Plato and Aristotle began and which continue throughout medieval times. Although a close relation between philosophy, science and mathematics characterizes this

(12) For a short survey of the history of mathematics, cf. Raymond Clare Archibald, Outline of the History of Mathematics, Fifth edition, revised and enlarged, Oberlin: The Mathematical Association of America, 1941. The history of mathematics is usually divided into seven periods. The first period, from early times to ancient Babylonia and Egypt inclusive, is but a period of beginnings. The Greek contribution, from about 600 B.C. to 500 A.D., comprises the second period which signalizes the important introduction of deductive reasoning into mathematics. From 500 to 1200 A.D., the third period comprises the Hindu, Arabic and Persian contributions and is significant for the transmission of Greek mathematics and the introduction of Hindu-Arabic numerals. The fourth period, that of European mathematics extending from 1200 to 1600, marks especially the development of symbolic algebra by the French and Italian mathematicians. The fifth period is the outstanding seventeenth century which includes such great names as Fermat, Descartes, Kepler, Galileo, Huygens, Newton and Leibniz, and such an important step as mathematical analysis from which analytic geometry, the calculus and other developments arose. Applications of mathematics to the physical sciences and astronomy are noteworthy in this period. The next period, the sixteenth, developed the mathematical tradition and created such subjects as actuarial science, the calculus of variation and different geometries and algebras. The final period, the nineteenth century to present times, is one of unbelievable fruitfulness and productivity in mathematics. Laplace and Lagrange, Labatschewsky, Gauss and Cauchy are but a few of the early great names in this period which includes many great contemporary mathematicians.

(13) Contributions to mathematics are generally classed under three broad headings. One of these is number with its implication of discreteness, exhibited at first as arithmetic and algebra. The concept of form and continuity, the second class, showed itself especially as geometry and later as structure in mathematical relations and theories. The third heading includes contributions to method, notation and similar devices. Cf. H. T. Bell, op. cit., pp. 3-15.

(14) Abraham A. Fraenkel, "The Recent Controversies about the Foundations of Mathematics," Scripta Mathematica 13(1947), pp. 17-18.

period, mathematical thought and discovery is often found completely divorced from philosophy; philosophy often is found neglectful of mathematics in its analysis (15). The second period, though it arises from a slow and scattered though coherent development, begins approximately with the publication of Descartes' analytic geometry in 1637, and his program of mathematicism. In this second period, a great rebirth in mathematics and in science, and a close connection frequently begetting a confusion of science, mathematics and philosophy, reappears. The invention of analytic geometry and the recognition of its fecundity as a means of invention in mathematics and physics, foster in Descartes the dream of a universal mathematics which he hopes will qualify in dealing with problems of all knowledge. The Cartesians follow Descartes in furthering this ideal science. Leibniz, by his calculus and mathematicism, actually initiates the steps towards its realization. Most of the philosophers of this period feel themselves bound to deal with mathematics in some form, or to explain how and why other knowledge, though it lacks the handsome qualities of mathematics, is yet valid (16). The problem of the distinction between philosophy and mathematics, posed by the program of mathematicising all knowledge at least with respect to mathematical method, appears frequently, leading to Christian von Wolff's proposal of two types of knowledge: Metaphysics and its ultimate causes, science and its proximate causes. The period concludes with the mathematically ultimate and absolute interpretations of Newton, Kant and Auguste

(15) This separability of mathematical from philosophical or scientific abstraction can be viewed as an historical argument favoring the distinction of these various types of abstractive processes.

(16) William Cecil Dampier, op. cit., pp. 206-216.

Conte.

The third and final ^{PERIOD,} ~~report~~, the contemporary scene, is one of growing self-consciousness for both mathematics and science. In the eighteenth century, discussions arise on the foundations of the calculus and the theory of infinite series. The program of arithmetization follows, and is succeeded by a number of antinomies and paradoxes. Schools of thought on the foundations of mathematics develop, and the analysis of the nature of mathematics, its relation to other sciences and philosophy again become paramount problems (17).

It will be easily seen that the survey embodied in this chapter is not a history of mathematical abstraction. For, properly speaking, an account of the results arising from mathematical abstraction is merely the history of the science of mathematics. The historical sketch which follows is more aptly termed a history of the metaphysics of mathematical abstraction or a history of the philosophy of mathematics. It arises in answering these two questions: (a) what metaphysical position have past thinkers adopted on the nature of mathematics? and (b) in their philosophical account of what mathematics is, have they sufficiently distinguished mathematics and its method from metaphysics, logic and science in general? A speculative inquiry of this sort implies, then, that though a mathematician frequently questions the nature of his science, he engages in this inquiry not as mathematician but as philosopher; and, secondly, that such an endeavor is not the proper function of the mathematician as such but that of the mathematicians as man and philosopher.

(17) Herman Weyl, The Open World, New Haven: Yale University Press, 1934, passim.

In answering the two questions posed above, the following historical resume, by placing a man's theory on mathematical abstraction within his systematic philosophical efforts, reveals two truths: firstly, the mutual interaction of mathematics with other types of scientific abstraction; secondly, the recurrence of an ancient, modern and recent confusion or identification of mathematical method or abstraction with that of other sciences frequently manifesting itself as the error of mathematicism. Thus the historical view argues that a proper understanding of mathematical abstraction is necessary for speculative truth in general; and, more pointedly, for the contemporary philosophical speculations stemming from the science of mathematics.

Chapter III. Ancient Conceptions.

Mathematics had its historical origins in practical necessities (1).

The counting and measuring processes of the ancient peoples, especially the Babylonians, Egyptians and Persians, grew out of their primitive engineering problems -- surveying, construction, building walls and digging dikes -- and out of their attempts to number the days of the year, to count sheep and compute interest. These countless submathematical facts, constituting an elementary arithmetic, a geometry and an analysis of a sort, and an algebra without symbolism, were the inheritance of the Greeks (2).

The Greeks did more than absorb these facts (3). Breaking with mythology and the current appeal to popular fancy, they achieved the abstraction proper to mathematics as well as that proper to science and philosophy and created the scientific outlook. In mathematics itself, they are to be credited not only for their contributions but especially for founding mathematics as a science by introducing the uni-

(1) A discussion of the views on this problem can be found in Carl B. Boyer, op. cit., pp. 14-16. Many recent books, popularizing mathematics, give a delightful but fanciful account of the origin of mathematics, regarding it as the offspring of convention, use, or as a result of the gradual perfecting of the human brain. Such accounts are quite fictitious. These authors interpret historical facts from a prejudiced and even mythological point of view. See, for example, Lancelot Hogben, Mathematics for the Millions, passim, and Gaylord M. Merriman, To Discover Mathematics, New York: John Wiley & Sons, 1942, pp. 1-3.

(2) Thomas L. Heath, A Manual of Greek Mathematics, Oxford: The Clarendon Press, 1941, Chapter II, "Numerical Notation and Practical Calculation," pp. 11-35. See also Oystein Ore, Number Theory and its History, Chapter I, "Counting and Recording of Numbers," pp. 1-24.

(3) "Even if we grant that the practical gardener-mathematics of the Egyptians and the astronomical observations of Babylonian astrologers influenced the Greeks and supplied them with preliminary material, this admission is in no way prejudicial to the originality of the Greek genius. Science and Thought, as distinct from mere practical calculation and astrological lore, were the result of the Greek genius and were due neither to the Egyptians nor to the Babylonians. F. Copleston, op. cit., pp. 15-16. For a similar testimony see John Burnet, Early Greek Philosophy, fourth edition, London: Adam and Charles Black, 1945,

pp. 15-30, and Thomas L. Heath, op. cit., Chapter I, "Introductory", pp. 1-10.

lying principle of deductive reasoning (4). In their attempt to relate structure to number, they are also responsible for the view that a mathematical concern with nature is quite relevant, thus opening the way for a program of applying mathematics to nature. Finally, the genius of the Greek mind extended to a speculative treatment of mathematical abstraction, and a consideration of the relations between mathematics and the other sciences which had but lately been born (5). In this way they initiated the first notions of a philosophy of mathematics (6).

I. The Pythagoreans

When the school of thinkers known as the Pythagoreans speculated in Greece, a number of opinions were being offered on the nature and principle of all things. These early pioneers, whom Aristotle called "physicists" or philosophers of nature, had achieved a high level of scientific and philosophical abstraction. The early Ionians held, with Thales, that water is the principles of all things. Others sided with Anaximander's teaching that the

(4) "They created the systematic mathematical theory based upon axioms and developed by means of logical deductions in the form of proofs, the standard pattern for mathematics, as it is formulated today," L. L. Foodruff, editor, The Development of the Sciences, Second Series, New Haven: Yale University Press, 1941, p. 4. See Gaston Milhaud, *op. cit.*, pp. 51-59.

(5) Most scholars pay tribute to the Greek genius. In The Development of Mathematics, E. T. Bell seems to take a contrary view on the worth of the Greek contribution to mathematics and to civilization. A comparison of the various places in which he speaks of the Greeks indicates, however, that his en passant remarks are to be interpreted as flippantly as they are made.

(6) It is beyond our purpose here to examine the exaggerated and mystical views of number or form, or their use in symbolism. A detailed study of these developments can be found in Lynn Thorndike's History of Magic and Experimental Science, cited above. For the symbolism and mysticism often attached to parts of mathematics, see Vincent Foster Hopper's Medieval Number Symbolism, cited above, passim.

changes we observe in nature originate from a conflict of opposites which arise within the original element called the indeterminate. Some followed Anaximenes, who taught that all things come from air through a process of condensation and reaction. This propensity to rationalize the ceaseless, almost unintelligible change in nature, was adopted by the Pythagoreans. Although they came of a curious mixture of mysticism and a bent for mathematics, the Pythagoreans were likewise scientific, and thrived in the new intellectual atmosphere (7).

To the current scientific and philosophical attempts towards rendering nature intelligible, the Pythagoreans contributed a new outlook: was there not a connection between the attitudes of the physicists and that offered by mathematics? All the physicists had tried to solve the problem of the many and the one; they had sought for identity in diversity. The Pythagoreans suggested that possibly the sameness in diversity could be accounted for by number and structure. We need no longer inquire what the primitive matter is (thus the Pythagorean mathematical outlook eliminates that of scientific and philosophical abstraction); we can say that all things are number and form. Aristotle writes that

The Pythagoreans, as they are called, devoted themselves to mathematics, they were the first to advance this study, and having been brought up in it, they thought that its principles were the principles of all things (8).

The application of the Pythagorean tenets to nature were successful. In acoustics, their doctrine pointed out that the qualitative differences

(7) John Burnet, *op. cit.*, Chapter VII, "The Pythagoreans," pp. 276-309.

(8) Metaphysics, 988 b 23-6; see also 986 a 15 and 1090 a 32. References to Aristotle are from The Works of Aristotle, Translated into English under the editorship of W. D. Ross, eleven volumes, Oxford: The Clarendon Press, 1908-1931.

between notes are due to the way in which the plucked musical string takes up different shapes. Likewise the concordance of musical tones was well explained by their ratio of numbers. By analyzing bodies into surfaces, surfaces into lines and lines into points, they applied their doctrine to nature and demonstrated, to the physicists, that mathematical concepts explained the world of nature (9). Each body expresses the number four; surfaces, three; lines, two; the point, one. If we add these together, we get ten, the perfect number. The whole universe seemed to submit to this interpretation by mathematical abstraction; the forms and figures of the objects of sense experience could be seen to have geometrical contours, while the constellations of the heavens were explicable both as a configuration and a number. And hence, as four dots can represent the configuration of a square, three of a triangle, two of a line and one of a point, the Pythagoreans readily

supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number (10).

Pythagorean mathematicism, then, attempted to identify the objects held in mathematical abstraction with all things and eliminated the propriety of the earlier abstracted views of the physicists. Nature, they agreed with the earlier physicists, is orderly and changes in harmonious proportions; but the very order and the proportion is number and form. And whereas the earlier physicists had thought of water, air, fire and

(9) Thomas L. Heath, op. cit., pp. 86-88.

(10) Metaphysics, 986 a 1-8.

earth as elements and principles, the Pythagoreans held numbers and forms to be the intrinsic principles of nature and to be immanent in things themselves (11).

The plausibility of the Pythagorean doctrines attracted many followers and the school flourished. Aristotle wrote of them and the Platonists:

mathematics has come to be identical with philosophy for modern thinkers, though they say that it should be studied for the sake of other things (12).

The "scandalous discovery" (13) of the irrationals in mathematics, however, became an insuperable obstacle to Pythagorean mathematics and their mathematicism. In the application of the side of a square to its diagonal, an incommensurable ratio was discovered, and no way of resolving this difficulty could be found. The attempt to apply mathematics to nature, and the endeavor to relate continuity as form to discreteness as number, was abandoned, although the Pythagorean

(11) R. G. Gellingwood, *op. cit.*, pp. 49-55.

(12) *Metaphysics*, 992 a, 32-4.

(13) Brunschwig speaks of the "scandalous discovery" of the irrationals as a factor in lessening the influence of the Pythagorean school; Leon Brunschwig, Le rôle du Pythagorisme dans l'évolution des Idées, Paris: Hermann & Cie., 1937, p. 21.

influence and outlook on these matters continued (14).

In the meantime, other schools of thought with highly metaphysical abstractions, flourished. Heraclitus opposed the static and atomic concept of nature advocated by his predecessors with a dynamic philosophy of becoming. He answered the question of identity in diversity by holding that the One is the principle of all things. The essence of the One lies in its being both the many and the One at the same time, so that all things are in a constant state of flux and change. Parmenides, on the contrary, taught a philosophy of being. There is no change; being is, and all things are. The doctrines were pitted against each other, and acute discussion arose through the efforts of Zeno ^{of} Elea, Parmenides' most prominent pupil. In support of his master and in opposition to the Pythagoreans and other schools, Zeno offered his famous dialectical abstractions which argued in favor of continuous as against discrete quantity. Zeno used his dialectical talents to show that Parmenides' static philosophy was the best; if one granted the ideas of the Pythagoreans or even of the Atomists with their "infinitely small" quantities (15), antinomies arise which are

(14) The Pythagorean influence passes through Plato and the Neo-Platonists to Saint Augustine, eventually reaching scholasticism where the Pythagorean notion of numbered order in geometry, arithmetic, music and astronomy, made these four subjects the quadrivium of medieval instruction; Sir W. C. Dampier, A History of Science, New York: MacMillan, 1943, pp. 17-18. On page twenty Dampier writes, "After the Renaissance, the idea of the importance of number was taken up by Copernicus and Kepler, who laid chief stress on the mathematical harmony and simplicity of the Heliocentric hypothesis as the best evidence of its truth. In our own day, Aston with his integral atomic weights, Moseley with his atomic numbers, Planck with his quantum theory, and Einstein with his claim that physical facts such as gravitation are exhibitions of local space-time properties, are revising ideas that, in older, cruder forms, appear in Pythagorean philosophy." Dampier refers to Whitehead as his authority for this statement. It should be pointed out that it is possible to separate procedure in science from such metaphysical implications as are found, for example, in Pythagorean mathematics. One can say that scientists as such are merely relating structure to number, without any metaphysical implications as to the existence of these numbers in things, which is closer to the Pythagorean view. This distinction between procedures and metaphysics in mathematics has been studied in Edward W. Strong, Procedures and Metaphysics, Berkeley: University of California Press, 1936, see Chapter I and Chapter II.

(15) H. D. P. Lee, Zeno of Elea: A text with translation and notes, Cambridge: The University Press, 1936; or see Carl B. Boyer, op. cit., pp. 21-22.

insoluble by the theory of number and form. Mathematics fails to rationalize nature as it fails to rationalize the incommensurable segments in the teaching of the Pythagoreans.

Another school, the Atomists, taught that an infinite number of indivisible units called atoms, differing only in size and shape are the principles of all things. From the various formations and configurations which these atoms subsume, the four elements and the world of nature are brought into being. This theory, however, was also unable to cope with Zeno's arguments. Hence the doctrine of the infinitesimally small and the infinite in number, which the Atomists proposed, was gradually abandoned (16).

II. Plato

Through their insistence on the intelligibility of nature, all these early thinkers exercised a profound influence on the philosophy of Plato. Socrates, Plato's teacher, advanced the thought of these men by introducing the maieutic method and the notion of definition; he accentuated the importance of ethics and displayed the necessity for clarifying the goal of all scientific endeavor, the Universal. Plato's respect and debt to all these thinkers is repaid in the work of his dialogues, where they often play the leading roles in presenting scientific, mathematical and philosophical points of view.

In order to enter Plato's Academy, a certain amount of proficiency in mathematics was requisite. "Let none enter here ignorant of geometry" is

(16) F. Copleston, op. cit., pp. 72-75.

a well-known phrase which was, supposedly, the admissions standard to Plato's school. This respect and eulogization of mathematics which is found in Plato's writings, is principally due to Pythagorean influence. Plato's deference to mathematics likewise produced a number of graduates who made important contributions to the growing science of mathematics (17).

Plato's attempt to solve the problem of the one and the many consisted of a dual psychological process: a gradual purification of the shadows and sense images of the world of nature, and the process of recollection. His philosophy is dominated by the fundamental principle that the proper object of all true science is the world of ideas in which man pre-existed, and of which the sense-perceptible world is but a shadow or copy yielding only opinion. For Plato, in contrast to the earlier thinkers, held that essences, forms, numbers -- the intelligible as such -- transcended the world of nature and were not immanent within it.

A student is initiated into the process towards knowledge from his state of ignorance by beginning with the sensible world, of which he can garner only opinion. The lowest type of opinion is that acquaintanceship or mere conjecture a man has of the fleeting "images" in the world about us. Such a man judges wrongly that external nature is the true reality. A higher type of opinion, faith or belief, is achieved when a man realizes that those images are nothing more than reflections of something more basically real. He realizes that external nature is not what it purports to be; and, though he has not got it as yet, he is on the road towards scientific knowledge.

(17) See the interesting remarks Plato makes concerning Greek ignorance of geometry in Laws, 819 d - 820 e. References to Plato are taken from The Dialogues of Plato, Translated into English by B. Jowett, with an Introduction by Raphael Demos, two volumes, New York: Random House, 1937. /N/K ON 44 71001

The second stage, that of knowledge, also has two subdivisions. In the lower, a man begins from hypotheses and proceeds, not to a principle or form, but to a conclusion. Such men are "really endeavoring to behold those objects which a person can only see with the eye of thought" (18), or the objects of mathematics. Mathematical abstraction, then, though it has the function of rationalizing the fluctuation of nature, is only on the road to knowledge. The men who abstract in this way are classed as having understanding, which is the intermediate state of mind between opinion and the final stage of scientific knowledge -- pure reason, the understanding of the forms and ideas themselves. In order to arrive at the state of pure reason, one must begin with the hypotheses of the previous section (mathematics), and rise to the first principles, the world of forms and ideas. It is only here that the inner meaning of all things is grasped in the contemplation of the Good (19).

The problem of Plato's doctrine on the nature of mathematics and on the relation of mathematics to the Forms is a widely disputed one (20). It is agreed that Plato's search for rationality in nature was fruitless and that he thereby consigned the study of nature to the realm of uncertain, shifting opinion. The Pythagoreans had made the ensemble of numbers and things correspond. Plato lifted number out of the world

(18) Republic, 510 e - 511 a.

(19) For a good diagrammatic analysis of these various stages, see F. Copleston, op. cit., p. 162.

(20) See F. Copleston, op. cit., pp. 163-206 for a recent summary of the opinions on Plato's mathematicism. A study of the influence of the Pythagoreans on Plato is given in Thomas Greenwood, "The Platonic Theory of the Continuum," The Thomist, VII (April, 1944), pp. 179-203. See also F. Enriquez et G. De Santillana, Le Problème de la Matière: Pythagoriciens et Eleates, Paris: Hermann & Cie, 1936; these authors discuss the origin of Pythagorean mathematicism, pp. 11-17. See also, Gaston Milhaud, op. cit., pp. 327-366.

of nature and made it transcendent; it occupied an intermediary position between the sense world and the world of ideas. Seemingly, Plato -- and certainly later Platonists -- was not content with this scheme. According to Aristotle and many modern scholars, Plato taught that mathematics is not only the means by which a man reaches true knowledge, but even the ideal world itself can only be made rational and intelligible through mathematics. In this way Plato offered a different type of mathematicism from that of the Pythagoreans. The Pythagoreans had identified the scientific abstraction of the earlier thinkers with that of mathematics. Plato eliminates scientific abstraction (21), and identifies mathematical with metaphysical abstraction in idealistic mathematicism. In Plato's philosophy, it has been said, "God ever geometrizes (22)".

If Plato is not to be understood as proposing mathematicism, then his notions on mathematics were quite in harmony with current mathematical practice. Mathematics has unto itself a special faculty of the soul, called understanding (23). It is classed among the purely intellectual, as opposed to the practical sciences (24). Although the mathematician uses sensible things as symbols in his demonstrations, his purpose is to lay hold of the triangle or square in itself which he does by rationalizing nature; that is, by reducing the fluctuation of images and shadows to forms, for "in these perplexities the soul naturally summons to her aid

(21) "Plato's criterion of reality was not consistency in experience but reasonableness in thought. For him, as for the Pythagoreans, there was no necessary distinction between mathematics and science; both were the result of deduction from clearly perceived first principles." Carl B. Boyer, op. cit., p. 26.

(22) This quotation can be found in the delightful work of Robert Edouard Moritz, Memorabilia Mathematica, pp. 269-270; Moritz's book, though old, is an invaluable source of quotations on mathematics and the philosophy of mathematics.

(23) Republic, VI, 511 C; see Theaetetus, 184 D.

(24) Statesman, 258 D.

calculation and intelligence" (25). Mathematical abstraction makes much use of hypotheses, whose validity the mathematician does not question but simply uses in drawing his conclusions (26). Further, the existence of a square or a triangle is assumed by the mathematician; it is what is given to him, and furnishes his point of departure for mathematical demonstration. The usual division of mathematics into arithmetic and geometry is mentioned. Each of these branches, however, has two varieties, the impure (which seems to be mathematics itself), and the pure. The latter is the philosophical use of mathematics; the philosopher must be an arithmetician, "because he has to rise out of the sea of change and lay hold of true being" (27).

III. Aristotle

Aristotle owed much to his predecessors and especially to his teacher, Plato. There can be little doubt, however, of the originality and greatness of his contribution to the new scientific doctrine of the early Greeks ^{AND} ~~and~~ to the problems posed by the mathematical philosophers. In his logical treatises, we meet for the first time in the history of thought a complete and quite satisfying account of the nature of pre-scientific knowledge, and of the nature, divisions and extent of scientific knowledge (28). In both cases a man may possess knowledge; what especially distinguishes scientific knowledge is its universality, demonstrability and certainty.

(25) Republic, VII, 524.

(26) Menu, 87. Cf. Republic, VI, 510.

(27) Republic, VII, 525.

(28) Aristotle's logical treatises portray his scientific ideal. The Prior Analytics examines the formal validity of inference, or the valid derivation of conclusions from given premises. The Posterior Analytics analyses the characteristics of premises if our knowledge is to be the most perfect obtainable that is, truly scientific. In the Topics, Aristotle shows how we attain and justify the premises that are used in the arguments of daily life. The notion of science itself occupies the main part of the Posterior Analytics.

There is a realm of pre-scientific knowledge, Aristotle teaches, which corresponds roughly to Plato's theory that only opinion can be obtained of nature (29). It is the realm of the fancies of the poet which induce in us a state of mind called estimation about the truth of things; it is the realm of the persuasions of the rhetorician which breeds a suspicion of truth within us, inclining us to adopt one part of a contradiction; it is the realm of the probabilities of the dialectician which gives us opinion, whereby we accept one part of a contradictory proposition, with fear that we may be in error. Since some form of reasoning, man's distinctive activity, occurs in all these states of mind, Aristotle felt that the logician should examine their respective merits.

Dialectics, a favorite Aristotelian device, is a fertile source of argument, demonstration and methods of arriving at truth or approximations to the truth (30). Problems of any sort are its concern, and the arguments and discussions it provokes arise from propositions which either are, or seem to be generally accepted. Dialectics being an easy way to learn, it is of much service in the acquisition of scientific knowledge and plays an important role in all the sciences. Its main function occurs in dealing with the realm of the changeable and contingent where truth is difficult to establish. Dialectical arguments rest on what

(29) Posterior Analytics, 71 a 1-10.

(30) In Aristotle's conception, Dialectics played a large part as preparatory to the acquisition of each science. It performed an important function in the approach towards science through the consideration of doubts and difficulties, and the opinions offered by previous thinkers.

can be observed to be the case only in most instances. Inductive examples and the type of syllogism known as enthymemes are the main types of arguments used in dialectical knowledge.

Scientific knowledge, which is not limited merely to knowledge of forms or mathematics, is dealt with chiefly in the Posterior Analytics.

We suppose ourselves to possess unqualified scientific knowledge of a thing...when we think that we know the cause on which the fact depends, as the cause of that fact and of no other, and, further, that the fact could not be other than it is (31).

It is knowledge obtained only by strict demonstration through the syllogism of the first mood and first figure, having both premises universal and affirmative (32). This implies that the premises must be "true, primary, immediate, better known than and prior to the conclusion, which is related to them as effect to cause" (33). Scientific, in opposition to pre-scientific knowledge, is demonstrative (34), is knowledge of causes (35), of universals (36), of essence (37), of form (38). These strict requirements for knowledge to be classed as scientific, reduce much knowledge to the pre-scientific level, especially to that of Dialectics.

(31) Posterior Analytics, 71 b 8-12.

(32) This is a rigorous conception of science. "Of all the figures the most scientific is the first. Thus, it is the vehicle of the demonstrations of all the mathematical sciences, such as arithmetic, geometry, and optics, and practically of all the sciences that investigate causes..." Posterior Analytics, 79 a 18-21.

(33) Posterior Analytics, 71 b 20-22.

(34) Posterior Analytics, 74 b 27; 90 b 9; 87 b 35.

(35) Posterior Analytics, 71 b 9; 94 a 20. Cf. also Physics, 194 b 18. Here Aristotle has in mind his doctrine of the four causes. Causal knowledge implies necessity.

(36) Posterior Analytics, 87 b 28; see also Metaphysics, 1086 b 5; 1059 b 25; 1005 a 13. The individual thing as such is ineffable and cannot be the object of science.

(37) Posterior Analytics, 99 a 22; Metaphysics, 1031 b 7; 1031 b 20.

(38) On the Soul, 412 a 9; 414 a 4.

In the process of teaching, science is induced in the pupil by demonstration. It begins with the student's pre-existent knowledge of self-evident principles. It supposes the student's acquaintanceship with a fact, the comprehension of a term, or both. All proof ultimately rests on the grasp of self-evident principles which are known by an act of immediate understanding, incapable of demonstration and thus, self-evident. These principles are incapable of demonstration, Aristotle maintains, or we are committed to an infinite regress in the process of science. The teacher presents the students with facts, with the terms and the subject-matter of the science, with the varied opinions that have been offered on the matter under consideration, and with dialectical suppositions.

The actuality of scientific knowledge, consequently, arises in the mind of the student by inducing in him the process of abstraction from what is given in sense data. It is theoretical or speculative, as opposed to practical knowledge (39). Theoretical knowledge is interested in truth as such and is more noble than practical knowledge, which relates truth in the mind to human actions of an ethical or factive nature. Theoretical science arises in three ways. By the process of scientific abstraction of the first degree, a student studies nature itself under the aspect of mobile being. The concepts and definitions which arise in the student mind bear a direct relation to the world of nature. They are the mental grasp of the permanent, universal and the necessary lodged within nature in general, and in each thing. The entities which the mind holds in this

(39) Metaphysics, 993 b 20; 1025 b 1-1026^a 33; 1064^a 1-1064^b 14.

type of abstraction are seen to be of such a type that they are connected with matter for their existence, and in order to be known. They are not transcendent or apart from the world of nature. The whole realm of nature, living and non-living, is included under this abstractive process in which inductive and deductive methodology occur.

The second type of abstraction is that proper to mathematics, which likewise originates through the senses, though often from a minimum of experience (40). In his discussion of mathematics, Aristotle forcibly rejects the mathematicism of the Pythagoreans or Platonists or any theory which equates mathematical entities with a separately real world of nature or self-existing ideas (41). Mathematics is distinguished from the first abstractive process; there the mind abstracted from the individual sensible matter of each thing and arrive at the concept of common sensible matter. In mathematics, the mind abstracts from common sensible matter and arrives at the notion of "intelligible" matter. The mathematician

strips off all the sensible qualities, e.g., weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contraries, and leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of these qua quantitative and continuous...(42).

(40) "What has been said is confirmed by the fact that while young men become geometicians and mathematicians and wise in matters like these, it is thought that a young man or practical wisdom cannot be found...indeed one might ask this question too, why a boy may become a mathematician, but not a philosopher or a physicist. Is it because the objects of mathematics exist by abstraction, while the first principles of these other subjects come from experience, and because young men have no conviction about the latter but merely use the proper language, while the essence of mathematical objects is plain enough to them?" Nicomachean Ethics, 1142 a 10-19.

(41) Metaphysics, Books XIII and XIV; see Augustin Mansion, Introduction a la Physique Aristotelicienne, deuxieme edition, revue et augmentee, Paris, J. Vrin, 1945, Ch. V, No. 3, "Distinction de la physique et des mathematiques d'apres Aristote," pp. 143-186.

(42) Metaphysics, 1061 a 29-35.

What the mind achieves here, then, are the concepts proper to mathematics. The mathematician considers quantity either as the continuous extension of geometry or the discrete units of arithmetic. Though he summons the aid of the senses for illustration and makes much use of the imagination, the mathematician is not concerned with objects as the

limits of a physical body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated (43).

The application of mathematics to optics, harmonics and astronomy is achieved by sciences which lie between physics and mathematics. The objects held in the mind by scientific abstraction are dealt with by the abstraction of mathematics (44). Geometry, for example,

investigates physical lines but not qua physical, optics investigates mathematical lines, but qua physical, not qua mathematical (45).

In its applications to other sciences, mathematics is superior to them.

If one reaches impossible consequences in mathematics, they

will reproduce themselves when applied to physical bodies, but there will be difficulties in physics which are not present in mathematics; for mathematics deals with an abstract and physics with a more concrete object (46).

The methodology of mathematics, like that of all the sciences, is treated by logic. Mathematics, though it may use dialectics, is distinct from it (47).

(43) Physics, 193 b, 31-35.

(44) For a discussion of these "middle" sciences, see W. D. Ross, Aristotle, third edition revised, London: Methuen and Company, 1937, p. 70. See also Augustin Mansion, op. cit., pp. 186-185.

(45) Physics, 194 a, 9-11.

(46) De Coelo, 290 a, 13-16.

(47) Posterior Analytics, 77 b 27-33; 78 a 10-13.

Mathematical abstraction, though it is not the highest and noblest which the mind can achieve, is yet most proportionate to the human mind (48). It deals with the necessary (49), has the highest type of certitude (50) and offers conclusions that are timeless (51).

The third abstractive process does not arise from a remembrance or contemplation of a separated world of ideas, as Plato held, but from a consideration of the world of nature. This abstraction, metaphysics, is the highest, noblest and most difficult (52). It abstracts more deeply from the world of nature by not considering being as mobile or quantitative, but being as such. Its concepts and definitions, as substance, potentiality, actuality, are seen to have no necessary connection or dependence on matter in their consideration by the mind. Thus, not being fraught with change and contingency, these abstractions yield the highest type of speculative science. Besides this positive knowledge which is the fruit of this abstraction, metaphysics has a critical function: it orders the abstraction of the other sciences, determines the speculative relations between them, and establishes the speculative principles of all science. Metaphysics, the highest type of speculative science, is superseded by two higher states of mind. The first is understanding -- a habit of mind which sees the conclusions of science as related to the speculative principles of all thought; the

(48) Metaphysics, 1076^a 8-14.

(49) Physics, 200^a 15-24.

(50) De Coelo, 306^a 28.

(51) For a recent study of Aristotle's philosophy of mathematics, see Thomas Greenwood, Etudes sur la Connaissance Mathématique, Ottawa: Les Editions de l'Université d'Ottawa, 1942, "La Pensée Mathématique d'Aristote," pp. 33-62.

(52) Metaphysics, 993^a 30-993^b 30.

second is wisdom, which sees the order and relation of all things to God, its ultimate principle.

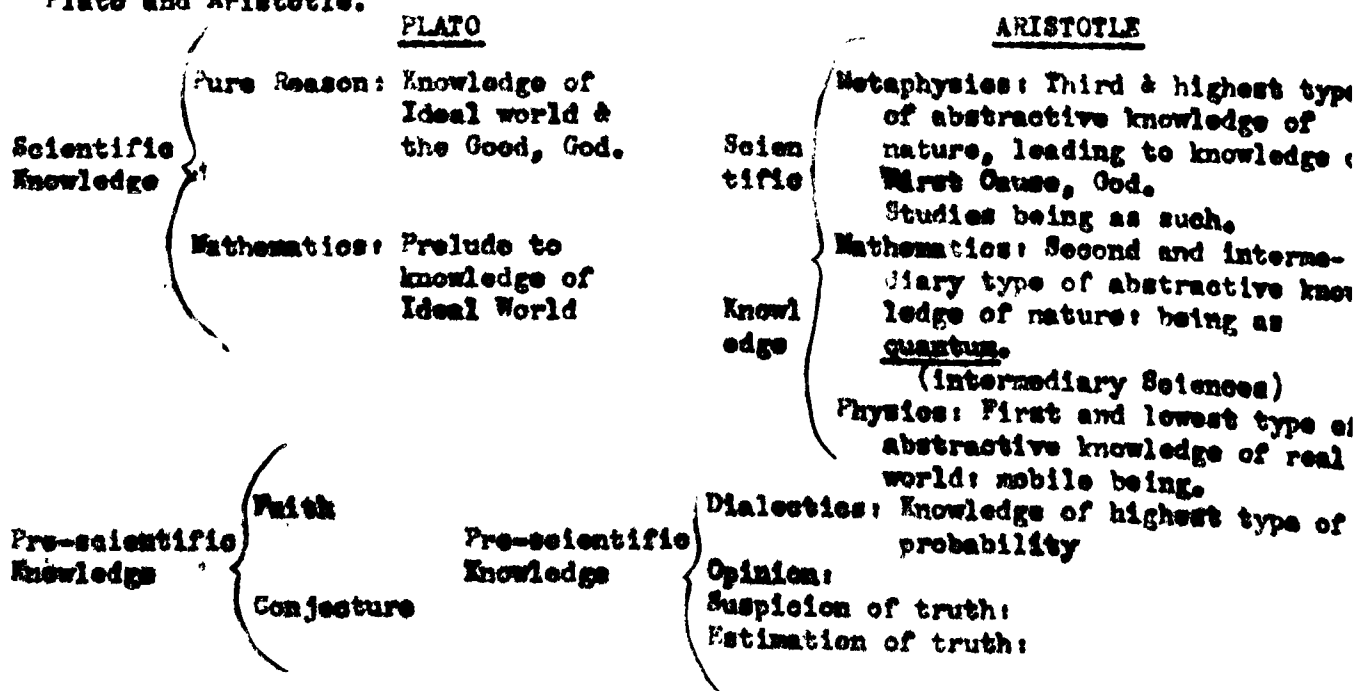
Although there are significant points of agreement between Aristotle and his predecessors, he did not merely harmonize their views. Rather, his attitude was that

It is just that we should be grateful, not only to those with whose views we may agree, but also to those who have expressed more superficial views; for these also contributed something, by developing before us the powers of thought (53).

By his theory of knowledge and his classification and distinction of the sciences he wished to vindicate many legitimate points of view. By eliminating the Platonic theory of a science concerned only with the recall and contemplation of a world of ideas; by denying the idealist identification of mathematical with metaphysical abstraction in the Platonist teaching or with scientific abstraction in the Pythagorean attitude, Aristotle hoped to establish many valid types of scientific knowledge (54). The transcendent ideas or forms become, so to speak,

(53) Metaphysics, 993 b, 12-15.

(54) The following diagram will serve to emphasize the difference between Plato and Aristotle.



immanent in the world of nature as the essences and accidents of things; through abstraction the mind conceptualizes the essence and the accidents of natural things in the various types of science. With regard to the Platonic mathematical ideas as such, they exist only in the mind which develops from them the pure or applied sciences of mathematics. Hence the essential, the quantitative and the qualitative aspects of nature are intelligible and the proper concern of various sciences.

The generations immediately succeeding the Pythagorean, Platonic and Aristotelian schools contributed little material that is germane to the problem of mathematical abstraction which is being considered here (55). With some exceptions, number mysticism, scepticism and lack of mathematical speculation followed in the decadence of Hellenic and Hellenistic thought. These intervening centuries are more properly viewed as a prolonged period of preparation for mathematics and science (56).

In fact, only in the Middle Ages is the problem of mathematical abstraction expressly given a philosophical consideration in the Aristotelian tradition, especially by St. Thomas Aquinas (57). But the medievals contributed towards the future growth of mathematics in a number of ways. Various mathematical compendiums were edited; change, variability of quantities, the infinite, infinitesimal and the continuum were subjected to lively theoretical discussion; Greek mathematics, as well as science and philosophy, were preserved and translated; the

(55) For the influence of Democritus and the sceptics, see Federigo Enriques The Historic Development of Logic: The Principles and Structure of Science in the Conception of Mathematical Thinkers, Authorized translation from the Italian by Jerome Rosenthal, New York: Henry Holt and Company, 1929, Chapter I, "The Logic of the Ancients," pp. 1-42; see also Leon Robin, La Pensée Grecque et les origines de l'esprit scientifique, avec une carte hors texte, Édition revue et corrigée avec une Bibliographie méthodique nouvelle, Paris: Albin Michel, 1948, pp. 377-388.

(56) George Sarton, Introduction to the History of Science, Volumes I, II; see also, A.N. Whitehead, Science and the Modern World, pp. 15-16.

(57) Thomas Greenwood, op. cit., "La Connaissance mathématique d'après Saint-Thomas," pp. 63-109.

relations and respective spheres of reason and faith were widely discussed and the appositeness of reason as a powerful instrument of speculative consideration was vindicated (58).

(58) Carl B. Boyer, op. cit., Chapter III. "Medieval Contributions," pp. 61-98.

Chapter IV. The Cartesian Era.

The founding of mathematics as a science and its accessory development by the Greeks marked the first golden era of mathematics. The seventeenth century, the second great period of mathematical history, signalized, by its emphasis on analysis, the beginnings of modern mathematics (1). This fruitful period comprised the founding of analytic geometry by Fermat and Descartes; the inception of probability theory in Fermat and Pascal and Fermat's higher arithmetic; the application of mathematics to dynamics by Galileo and the Newtonian theory of the universe; the discovery of algorithms for the calculus by Leibnis and Newton; the synthetic geometry of Desargues and Pascal; the notion of symbolic logic in Leibnis.

The rediscovery and progress of mathematical thought had a widespread effect on the current cultural outlook (2). An unbounded faith in the excellence and certainty of mathematics rose, and the application of its principles to astronomy, physics, and allied branches was initiated. A keen interest in the quantitative elements of nature slowly supplanted the study of nature in its qualitative aspects. Extreme views in favor of the new sciences, amounting even to contempt for theology, metaphysics and moral knowledge, led many thinkers to adopt a sheer mechanical view of nature.

The new mathematics had a direct influence on philosophical thought (3). In contrast to mathematics, philosophy was looked upon as empty speculation, and many attempts were made to remedy these supposed failings by introducing

(1) W. W. Rouse Ball, A Short Account of the History of Mathematics, third edition, New York: The Macmillan Co., 1901, pp. 269-276. See also F. T. Bell, op. cit., Chapter I, "The Beginning of Modern Mathematics," pp. 115-150.

(2) Charles Singer, A Short History of Science to the Nineteenth Century, Oxford: The Clarendon Press, 1941, "The Insurgent Century," pp. 185-247.

(3) For the impact of mathematics and science on sixteenth and seventeenth century thought see Edwin Burtt, op. cit., passim. See also Edward F. Strong, op. cit., passim.

mathematical abstraction and method into philosophy. In this way various forms of mathematicism appeared. The growing conviction that nature is essentially mathematical in its makeup, reestablished the Pythagorean-Platonic tradition in metaphysics. The successful adoption of mathematics in experimental science predisposed various thinkers to employ it in psychological, religious and ethical knowledge (4).

The general sentiment concerning mathematics before Descartes is epitomized in the views of Copernicus, Kepler and Galileo. With Copernicus (1473-1543), mathematics took a forward step by being applied anew to astronomy. Though his heliocentric theory of the universe was based on a small amount of empirical evidence, Copernicus proposed its acceptance for its evident arrangement of astronomical facts in a more simple and orderly mathematical frame (5). Proposing a theory of empirical mathematicism, he held mathematics to be the magic key to the study of nature, whose structure he considered as mathematical and quantitative. A strong anti-Aristotelian, Copernicus was an ardent enthusiast for the NeoPythagoreanism which had been ushered in with the Renaissance and had left such a marked influence on Cardinal Nicholas of Cusa. Both Cusa and Copernicus held extravagant views on the range of mathematical abstraction.

Kepler (1571-1630), had a deep love of astronomy almost equivalent to sun-worship (6). He eagerly read Copernicus' works and adopted his ideas.

(4) This will be seen to be true especially of Locke, Hume and the British empiricist school.

(5) Cf. E. A. Durr, *op. cit.*, Chapter II, "Copernicus and Kepler", pp. 23-43 and Edward W. Strong, *op. cit.*, Chapter VII, "Scientific Method and Metaphysics in Kepler," pp. 163-184.

(6) For a recent view somewhat akin to Kepler's fantastic position, see F.S.C. Northrop, *Science and First Principles*, New York: the Macmillan Co., 1932. "Hence, there are two gods. One, the macroscopic atom is a simple perfect substance. The other, the macroscopic unity of nature as a whole is a complex substance. Many of the inconsistencies in theological theory have arisen because of the failure to distinguish between these two divine objects." p. 282.

Convinced of the principles that nature does nothing in vain, that nature loves simplicity and chooses the easiest means for its ends, Copernicus claimed that his mathematical theories exhibited the deep connection of the mathematically-harmonious motions of nature. That the only true knowledge of nature is quantitative was the fundamental tenet of his empirical mathematicism. In Kepler's universe, mathematical knowledge was most perfect; its conclusions were certain and not subject to doubt, and it revealed to us the only real being, the quantitative.

The mathematicism of these scientists culminated in Galileo (1564-1642). A great and famous scientist; he was familiar with Greek geometry and was strongly influenced by the Pythagorean-Platonic line of thought. His scientific theories owed much to late medieval speculation on motion, the indivisibles, the infinite, the continuum and the notion of impetus. More than his predecessors, Galileo insisted on the mathematical character of nature and considered it as governed by immutable laws. Nature moved in perfect precision and order; not logic, but mathematics joined with experiment was the clue to nature's secrets. Observation yielded the general mathematical scheme of nature with which the mind became immediately informed; the mind then deduced a theory which must always be verifiable in actual events. Galileo was most clear in his emphasis on the reality of primary and the fictive character of secondary qualities;

But that external bodies, to excite in us these tastes, these odours, and these sounds, demand other than size, figure, number and slow or rapid motions, I do not believe; and I judge that, if the ears, the tongue, and the nostrils were taken away, the figure, the numbers, and the motions would indeed remain, but not the odours nor the tastes nor the sounds, which, without the living animal, I do not believe are anything else than names, just as tickling is precisely nothing but a name if the armpit and the nasal membrane be removed (7).

(7) As quoted in E. A. Burtt, op. cit., p. 75.

The new scientific movement fostered by Copernicus, Kepler and Galileo separated scientific from philosophical abstraction and led to the eventual disregard of the latter. Further, scientific abstraction became quasi-identified with mathematics. The quantitative and mathematical character of nature alone was important and mathematics slowly infiltrated into all scientific speculation.

I. Descartes

The favourisation of mathematics and science and the distaste for scholastic theology and philosophy are embodied in Descartes (1596-1650). The founder (with Fermat) of analytic geometry, Descartes was likewise the inaugurator of a mathematicized metaphysics. In both regards he was a child of the past and a man of his time (8). He displayed a certain mental coquettishness (9), however, by his admitted distaste and disavowal of historical and doctrinal influences on his speculations. His metaphysics, yet couched in scholastic terminology, discloses him as an habitue of their

(8) "In fact, though Descartes' antipathy to history is part of his philosophy, his philosophy in its turn has its historical setting. The two principal preoccupations of Descartes were his mathematical method and his personal approach to philosophy. In both there lurks a hostility to the historical outlook. The necessary truths of mathematics are independent of temporal changes. They are always necessary. The order in which they happen is irrelevant to them; their articulation is not that of events of time. On the other hand, the personal approach, placing as it does the whole onus of the search after truth on the individual, who must live through each turn of the argument he discovers, relieves him of the study of precedent and imposes simply the duty of reflecting methodically on the implications of his own experience." A. Boyce Gibson, The Philosophy of Descartes, London, Methuen and Co., 1932, pp. 3-4.

(9) Cf. Joseph Maréchal, Précis d'Histoire de la Philosophie Moderne, Tome Premier: De la Renaissance à Kant, Louvain: Massin Lessium, 1933, p. 62.

thought (10). Furthermore, his mental condition was due to a goodly number of his predecessors, Saint Augustine, the Renaissance Neo-Platonists and contemporary scientists (11).

A mental malcontent, Descartes considered his past experience in philosophy, science and letters merely as cumulative evidence for the sorry state in which he found knowledge and science to be;

But as soon as I had finished the entire course of study..., I found myself involved in so many doubts and errors, that I was convinced I had advanced no farther in all my attempts at learning, than the discovery at every turn of my own ignorance (12).

The excellence of mathematics and its formative influence on him, however, was given proper credit. Of his student days at La Fleche, he writes,

I was especially delighted with mathematics, on account of the certitude and evidence of their reasoning (13).

He looked to mathematics under its elementary forms of arithmetic and geometry as representative, but incipient models of truth;

(10) "En decidant d'étendre l'explication mathématique à la totalité du réel, Descartes vient de substituer le point de vue du sujet à celui de l'objet. S'engageant en effet à considérer la méthode mathématique comme le seul exercice normal et légitime de la raison, il remplace la science scolastique, diversifiée dans ses méthodes comme elle l'était dans ses objets, par une science comme l'est sa méthode même; il s'engage à procéder toujours a priori, en allant des idées aux choses, au lieu de procéder toujours a posteriori, comme le thomisme, en allant des choses aux concepts; il s'oblige enfin à substituer au réel, pris dans sa complexité concrète, un ensemble d'idées définies à chacune desquelles correspond une chose, au lieu de procéder, comme le thomisme, à une analyse conceptuelle du concret ou la complexité des concepts se modèle sur la complexité des choses." Etienne Gilson, Études sur le rôle de la pensée médiévale dans la formation du système cartésien, Paris: J. Vrin, 1950, p. 200.

(11) A. Boyce Gibson, op. cit., pp. 1-29.

(12) "Mais sitôt que j'en achève tout ce cours d'études...je me trouvois embarrassé de tant de fautes & erreurs, qu'il me sembloit n'avoir fait autre profit, en tâchant de m'instruire, sinon que j'avois decouvert de plus en plus mon ignorance." Oeuvres de Descartes, Charles Adam & Paul Tannery, Paris: Leopold Cerf, 1897-1913, Discours de la Methode, Vol. VI, p. 4. All quotations from Descartes are from this edition.

(13) "Je me plaisois surtout aux Mathematiques, a cause de la certitude & de l'evidence de leurs raisons." Discours de la Methode, Vol. VI, p. 7.

Arithmetica et Geometria...circa objectum ita purum et simplex versantur, ut nihil plane supponent, quod experientia reddiderit incertum, sed tota consistunt in consequentiis rationabiliter deducendis (14).

The inherent rigor of mathematical demonstration particularly gratified him, and he averred that those who seek truth

Circa nullum objectum debere occupari, de quo non possint habere certitudinem Arithmeticois et Geometricis demonstrationibus aequalem (15).

Current lack of mathematical development surprised Descartes; though mathematics had ^{such} strong foundations, it seemed strange that "they should have had no loftier superstructure reared on them (16)." Instead of being neatly compacted in a uniform doctrine, the parts of mathematics seemed disordered and of little use:

as to the analysis of the ancient and the algebra of the moderns, besides that they embrace only matters highly abstract, and, to appearance, of no use...there results an art full of confusion and obscurity calculated to embarrass, instead of a science fitted to cultivate the mind (17).

When Descartes regarded the philosophy of the Schoolmen, it was only to censure their syllogistic method as

that method of philosophizing which others have already discovered and those weapons of the schoolmen, probable syllogisms, which are so well suited for dialectical combat (18).

(14) "Ex quibus evidenter colligitur, quare Arithmetica & Geometria caeteris disciplinis longe certiores existant: quia scilicet hae solae circa objectum ita purum & simplex versantur, ut nihil plane supponent, quod experientia reddiderit incertum, sed totae consistunt in consequentiis rationabiliter deducendis." Regulae ad Directionem Ingenii, Vol. X, p. 365.

(15) "...rectum veritatis iter quaerentes circa nullum objectum debere occupari, de quo non possint habere certitudinem Arithmeticois & Geometricis demonstrationibus aequalem." Ibid., p. 366.

(16) "...mais je ne remarquois point encore leur vray usage, & passant qu'elles ne servoient qu'aux Arts Mechaniques, je m'estonnois de ce que, leurs fondemens estans si fermes & si solides, on n'avoit rien basti dessus de plus releve. Discours de la Methode, Vol. VI, p. 7.

(17) "Puis, pour l'Analyse des anciens & l'Algebre des modernes, autre qu'elles ne s'estendent qu'a des matieres fort abstraites, & qui ne semblent d'aucun usage...qu'on en a fait un art confus & obscur, qui embarrasse l'esprit, au lieu d'une science qui le cultive." Discours de la Methode, Vol. VI, pp. 17-18.

(18) "...damna illam, quam caeteri hactenus invenerunt, philosophandi rationem, & scholasticorum, optissima bellis, probabilium syllogismorum tormenta. Regulae ad Directionem Ingenii, Vol. X, p. 363.

The logic of the schoolmen is only able to effect a status of opinion and probable knowledge; it is concerned with empty problems and fails to guarantee the certitude of truth. Science, letters and learning in general were prescribed by Descartes, and he found no one adequately equipped with truth, no place where it was taught and no truth so certain that it could not be doubted.

Fraught with these doubts and with speculative dissatisfaction, Descartes conjectured that the prevalent barrenness of truth and fertility of error were due to a faulty and confused method. He desired a mode of procedure in reasoning that would dissolve his personal doubts, enable him to achieve certainty for the good of mankind, and possibly empower him with the reconstruction of the entire edifice of human knowledge. The method which he unearthed was disclosed to him in mathematics (19). It had two admirable traits: the simplicity and generality of its axioms, and the long chain of reasoned deductions enabling the mind to advance from the simple to the complex;

the long chain of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations, has led me to imagine that all things, to the knowledge of which man is competent, are mutually connected in the same way...(20).

(19) "Pour Descartes, la science est essentiellement une, puis qu'elle n'est rien d'autre que l'intelligence au travail, et qu'il n'y a qu'une seule manière de comprendre. Il doit donc y avoir une seule manière, une seule méthode pour arranger les données complexes d'un problème quelle qu'il soit, dans une chaîne de relations simples entre des éléments simples en dernière analyse. La relation initiale doit être la plus simple de toutes, afin que l'esprit puisse la saisir dans un seul acte, dans une intuition individuelle et indivise." Thomas Greenwood, "Origines de la géométrie analytique." Revue Trimestrielle Canadienne, 34 (1948), p. 177.

(20) "Ces longues chaînes de raisons, toutes simples & faciles, dont les Géomètres ont coutume de se servir, pour parvenir à leurs plus difficiles démonstrations, m'avoient donné occasion de m'imaginer que toutes les choses, qui peuvent tomber sous la connoissance des hommes, s'entresuivent en même façon..." Discours de la Méthode, Vol. VI, p. 19.

Mathematical procedure alone guarded the passage of truth and fitted all types of scientific inquiry, since "All sciences are interconnected and dependent on one another (21)."

We must not imagine that one kind of knowledge is more obscure than another, since all knowledge is of the same kind throughout, and consists solely in combining what is self evident (22).

Descartes' high regard for mathematics was not completely new or original (23). Favorable opinions on mathematics were prevalent, and Descartes himself speaks of his method as known to the ancients, "But my opinion is that these writers then with a low sort cunning, deplorable indeed, suppressed this knowledge" (24). To comprehend Descartes' contribution, it is necessary to understand his distinction between the literary form of writing geometrically and the geometrical method itself (25). The literary form or presenting doctrine is quite unimportant in itself, since it consists

merely in putting forward those things first that should be known without the aid of what comes subsequently, and arranging all other matters so that their proof depends solely on what precedes them (26).

(21) "...sunt enim omnes inter se conjunctae & a se invicem dependentes..." Regulae ad Directionem Ingenii, Vol. X, p. 361.

(22) "Denique colligitur quarte, ex dictis, nullas rerum cognitiones unas aliis obscuriores esse putandas, cum omnes ejusdem sint naturae, & in sola rerum per se notarum compositione consistent." Ibid., pp. 427-428.

(23) "But whether or not Descartes originated the high esteem in which mathematics was held, it was without doubt the conception which influenced him most profoundly in the development of his philosophy." Leon Roth, Spinoza, Descartes and Leibniz, Oxford: Clarendon Press, 1924, p. 12.

(24) "Hanc vero postea ab ipsis Scriptoribus perniciose quaedam astutia suppressam fuisse crediderim..." Regulae ad Directionem Ingenii, Vol. X, p. 376.

(25) See Harry Austryn Wolfson, The Philosophy of Spinoza: Unfolding the Latent Processes of his Reasoning, Cambridge: Harvard University Press, Volume I, Chapter I, "Behind the Geometrical Method," pp. 4-51.

(26) "Ordo in eo tantum consistit, quod ea, quae prima proponuntur, absque ulla sequentium ope debeant cognosci & reliqua deinde omnia ita disponi, ut ex praecedentibus facile demonstrantur." Secundae Responsiones, VII, p. 155.

The geometrical method proper is distinguished into proof by synthesis or by analysis. Euclid is a fine example of proof by synthesis; he begins with primary notions and common axioms, sets forth propositions and uses construction. Euclid proceeded from effect to cause, from the complex to the simple. It is this method which Descartes discards for proof by analysis. Thus the Cartesian originality lay in the universal application of a specific mathematical method to all scientific inquiry, the method of analysis (27).

Cartesian analysis reasons a prior. It is a movement from cause to effect, from the simple to the complex. Intuition is its point of departure (28). By annexing new materials to the original intuition, the mind employs deduction in unfolding what was germinally contained in intuition as a cause (29). Intuition is more certain than deduction, for it is simple. Intuition is its own explanation and justification; it is a mental ultimate which discloses irreducible, simple natures. Summarily, then, by intuition the mind lays hold of clear and distinct ideas, and supplements this act by deduction

through which we understand all those things which are concluded with necessity from certain other things which are certainly known (30).

Descartes' speculative conception, then, of mathematics and science in general, eliminates the necessity of any mental abstractive process

(27) "Now the characteristic mark of mathematical knowledge is, firstly, the simplicity and generality of the axioms with which it starts, and, secondly, the long chain of deductions by which we pass from the simplest truths to the most complex. Intuition and deduction, then, are the two complementary paths by which we may travel to the truth. But deduction requires a starting-point, and thus the centre of Descartes' epistemology is his theory of intuition." A. Boyce Gibson, *op. cit.*, p. 152.

(28) *Regulae ad Directionem Ingenii*, Vol. I, pp. 368-369.

(29) *Ibid.*, pp. 369-370.

(30) "...per quam intelligimus, illud esse quod ex quibusdam aliis certe cognitis necessarie concluditur." *Ibid.*, p. 369.

though it forced him to adopt a theory of innate ideas. The adoption of a clear and distinct idea as a beginning point was the first requisite. This initial point was then used as an instrument in the discovery of new truths by the mind's presenting new and unknown materials to this mental mirror. The mind then rationalizes each item of this new material by making each conformable, one after another, to the measure of the intuitive idea. From this unilateral inference of the mind, a series of deductions resembling a neatly linked chain of reasonings ensue (31). In this way,

passing little by little from one to the other, we may acquire in time a perfect knowledge of the whole of philosophy and ascend to the highest degree of wisdom (32).

Descartes gave a number of rules (33) to serve as guides for this mental process; to accept only what is clear and distinct (intuition); to separate each difficulty into all its component parts (analysis); to begin with the simplest and easiest and proceed to the unknown and complex (deduction); to be complete and omit nothing (hypothetical induction). The adoption of this method, Descartes believed, would perfect the habit of discovery, since

of all those who have hitherto sought truth in the sciences, the mathematicians alone have been able to find any demonstrations, that is, any certain and evident reasons, I did not doubt that such must have been the rule of their demonstrations (34).

(31) Esquisse de la verite, Vol. I, pp. 496-497.

(32) As quoted in Leon Roth, Spinoza, Descartes & Maimonides, Oxford, the Clarendon Press, 1924, p. 15; see Chapter I of Roth's work for a fine summary of Descartes' method. The book also contains a short but fine study of the relations between Descartes and Spinoza.

(33) Discours de la methode, Vol. VI, pp. 18-19.

(34) "...& considerent qu'entre tous ceux qui ont cy devant recherche la verite dans les sciences, il n'y a eu que les seuls Mathematiciens qui ont pu trouver quelques demonstrations, c'est a dire quelques raisons certaines & evidenttes, je ne doutois point que ce ne fust par les mesures qu'ils ont examinees..." Discours de la methode, Vol. VI, p. 19.

Descartes employed this method to all the sciences (35). In its application to metaphysics, his procedure and results are well known. He found himself forced to doubt anything that could be subject to doubt. The testimony of the external and internal senses, of the principles of reasoning, and specifically the existence of God, the external self and mathematical truths (whose method he retained), were included. Through this progressive withdrawal from all existence and evidence, nothing was left but his doubting, thinking self: Cogito, ergo sum arose as a clear and distinct intuition (36). The new materials which he found within the thinking self -- God, the world, his body -- were permitted to enter the domain of reality and intelligibility only by being made conformable to the basic intuition (37). Descartes' chain of reasoning has begun: the thinking self has an idea of God which only God could have placed there; God exists. God's veracity would not permit the thinking self to be deceived with regard to the impressions of an external world; the external

(35) "The mathematical interpretation of nature involves, in the first place, the application of general numerical relations to figures in space, which resulted in Descartes' own discovery of analytical geometry; in the second, the application of mathematical analysis as a whole to the study of moving bodies, as practiced (though not entirely to Descartes' satisfaction), by Galileo; and, in the third, the reduction to mechanical formulae of all physiological phenomena, including the instinctive reactions of animals; which appeared to be indicated by Harvey's new theory of the circulation of the blood." A. Boyce Gibson, *op. cit.*, p. 187.

(36) "Thus it is Descartes' mathematicism which leads to his psychologism. He posits the self for determination just as the geometer posits space; and misconceiving the fashion in which we find its properties in a triangle, he makes what is in fact an empirical enumeration of the modes of thinking substance." Marthinus Versfeld, An Essay on the Metaphysics of Descartes, London: Methuen and Co., 1940, p. 88.

(37) "En decretant l'universalite de la methode mathematique, il a fait un coup de force. On avait toujours su avant lui que les connaissances mathematiques etaient les plus certaines de tous, parce qu'elles portent sur l'objet concret le plus simple de tous, lui seul a songe a donner toute valeur aux connaissances autres que mathematiques. Il joue donc, du point de vue de la philosophie medievale, le role de l'indisciplinatus, celui qui met son point d'honneur a chercher dans n'importe quelle discipline un degre de certitude qu'elle ne comporte pas. R'un mot, il ne connaît plus d'intermediaire entre le vrai et le faux; c'est l'elimination radicale du probable." Etienne Gilson, *op. cit.*, pp. 234-235.

world exists.

The idealistic mathematicism which Descartes founded wreaked a retroactive revenge on its author and his successors. In return for the denial of pre-scientific knowledge and the proper views of abstractive processes other than the mathematical, disastrous speculative results followed (38). The Cartesian science lost sight of the analogy of beings; the existence of God was subjected to doubts and denials; the world of nature, denuded of qualities, was equated to extension, thus severing the delicate suture of matter to spirit and snuffing out the slightest bit of life in plant, animal and human bodies. Descartes' exaggerated dualism is evident in the metaphysical antithesis of matter and spirit, the psychological dichotomy of body and soul, the epistemological disjunction of thought and object (39).

The failure of Descartes' mathematicism was recognised by his successors (40). He had, however, successfully turned the direction of non-scholastic philosophical thought onto a road that it has since travelled: the emulation and imitation of mathematics and science, a distrust of metaphysics and a dislike for theology (41). Descartes'

(38) "Descartes commence en effet par éliminer toute qualité et toute forme du domaine de la nature; or la négation radicale de la qualité, c'est la négation radicale de l'analogie entre le matériel et l'intelligible, et comme l'analogie était la méthode scolastique par excellence pour aller du monde à Dieu, cette voie traditionnelle se ferme devant Descartes. Avec sa pénétration coutumière, il l'a bien vu, et il l'a dit avec franchise." *Ibid.*, p. 263.

(39) Marthinus Versfeld, *op. cit.*, Ch. XI "Concluding Remarks," pp. 148-170.

(40) Spinoza and Leibniz, among others, pointed this out. "So the rationalism of Descartes results in complete scepticism... Descartes' very insistence on the fact of the individual possession of truths has led him to the explicit denial of universal truth. In so far then as he was in search of, and believed himself to have found a logic which should help in the discovery of, and provide a theory for, truth, he must be proclaimed to have failed." Leon Roth, *op. cit.*, pp. 58-59.

(41) See the penetrating study of Descartes by Etienne Gilson, The Unity of Philosophical Experience, Part Two.

immediate and subsequent successors did not attempt to recast philosophical speculation but remained in the Cartesian tradition (42). Even those who opposed his thought and method fell under his influence and developed systems of rationalism, empiricism, pantheism, mechanism and scepticism.

II. Spinoza

Two of Descartes' immediate followers, Goulinex and Malebranche, developed the conclusions latent in his separation of mind from matter and in his notion of God. Spinoza (1632-1677) was apparently opposed to Descartes and criticized his doctrines (43). Yet his pantheism is directly based on Descartes' notion of substance and on the adoption of the Cartesian method. The very title of Spinoza's principal work, Ethica Ordine Geometrico Demonstrata (44), is an echo of Cartesian mathematicism. An ethical motive prompted Spinoza to write; he sought for the knowledge that would make men happy. By teaching men to look upon things in such wise that they would be led to great moral perfection, he hoped to lead them to eternal blessedness, to the knowledge of the "Summa bonum...cognitionem unionis, quam mens cum tota Natura habet" (45). The passage of the mind to this state is a hardy one. We must go from the mental condition of mere perception and inadequate experience to the state of rational knowledge with its adequate ideas; only from here can we be led to the final phase where man attains the intellectual love of God (46).

(42) Joseph Marechal, op. cit., pp. 83-84. Marechal summarizes the different estimates which students of Descartes' philosophy have made of his predominant characteristics, some regarding him primarily as a physicist and mathematician, others as a metaphysician or an apologist.

(43) A study of the numerous influences on Spinoza is found in Joseph Marechal op. cit., pp. 107-115.

(44) All quotations from Spinoza are taken from the following edition: J. Van Vloten et J. P. N. Land, Benedicti De Spinoza Opera Quotquot Reperta Sunt, Editio Tertia, Jussu Comitatus Apud Martinum Nijhoff, 1913.

(45) Treatatus De Intellectus Foundations, Vol. I, p. 6.

(46) Ibid., pp. 7-8.

To achieve this goal both for himself and others, seemed to Spinoza an impossible task. History revealed the many attempts men have made in this regard, and their tragic failures (47). Spinoza surmised that a faulty method was responsible, and he searched for an impersonal and certain procedure which would yet allow the freedom of unprejudiced inquiry. This method he found in Descartes. Distinguishing himself from Descartes, Spinoza complained that Descartes had erred in using this method. He had, first of all, treated God as an asylum ignorantiae.

Sed ille Mentem a Corpore adeo distinctam conceperat, ut nec hujus unionis, nec ipsius Mentis ullam singularem causam assignare potuerit, sed necesse ipsi fuerit, ad causam totius Universi, hoc est ad Deum, recurrere (48).

Descartes had likewise maintained the freedom of the will. Spinoza emphatically rejected both propositions, dissociating himself from Cartesianism as much as possible, but adopting the mathematical method.

The Principia Philosophiae Cartesianae, a work in which Spinoza recasts Descartes' Principia Philosophiae in synthetic form, contains an interesting preface by Meyer. It was written at Spinoza's request and approved by him (49). In his preface, Meyer eulogizes the mathematical method as being "tutissimam^{me}que veritatis indagandae atque docendi viam, omnibus, qui supra vulgus sapere volunt, unanimis est sententia (50)." He laments the fact that other sciences have not been treated mathematically, for most people judge it as merely applicable to mathematics.

(47) Cf. Ethica, V, Praef., p. 247.

(48) Amati Des Cartes Principiorum Philosophiae, Pars I et II, Praef., p. 103.

(49) Leon Roth, op. cit., pp. 40-61.

(50) Ibid., p. 103-4.

This has caused uncertain and unclear demonstrations in these sciences.

Descartes, discerning the miserable plight of philosophy,

nova Methodus e tenebris in lucem pretraxerat, Philosophias
fundamenta inconcussa eruit...quibus plurimas veritates ordine
ac certitudine Mathematica superstrui posso (51).

Meyer then describes the Cartesian method of analysis and contrasts it
with synthesis. Spinoza's purpose in presenting Descartes Principia in
synthetic form is to clarify the doctrine for those of meagre talent.

Meyer then mentions, significantly, that Spinoza's treatment does not
follow Descartes completely, nor does it imply that Spinoza accepts all
that follows. In fact, Meyer stresses the fact that Spinoza rejects the
Cartesian doctrine of final causality, and freedom of the will, and that
there are some things which surpass the capacity of human reason. Spinoza

Judicat enim ista omnia, ac etiam plura illa magis sublimia atque
subtilia, non tantum clare et distincte a nobis concipi, sed etiam
commode explicari posse, si modo humanus intellectus alia via,
quam quae a Cartesio aperta atque strata est, in veritatis investi-
gationem rerumque cognitionem deducatur; atque ideo scientiarum fun-
damenta a Cartesio eruta, et quae ab ipso superaedificata sunt, non
sufficere ad omnes ac difficillimas, quae in Metaphysicis occurrunt,
questiones emendandas atque solvendas, sed alia requiri, si ad illud
cognitionis fastigium intellectum nostrum cupimus evahere (52).

Spinoza's use of the mathematical method is impressive. Beginning with
the intuitive definition and notion of substance in the Ethics, he deduces
the real order. Spinoza's mathematicism differs from that of Descartes'
in point of departure, and by the rigorous correspondence of the real to
the mathematically ideal (53). Regarding the nature of God, Spinoza writes,

(51) Ibid., p. 104.

(52) Ibid., p. 108.

(53) Ethics, Part II., Prop. VII; see also John Caird, Spinoza,
Edinburgh and London: William Blackwood & Sons, 1868, pp. 115-120.

Verum ego me satis clare ostendisse puto, a summa rei potentia, sive infinita natura infinitis modis, hoc est, omnia necessario effluere, vel semper eadem necessitate sequi, eodem modo ac ex natura trianguli ab aeternae et in aeternum sequitur, etus tres annulos aequari duobus rectis (54).

With regard to his geometrical treatment of the nature, actions and emotions of man, Spinoza remarks

Hic sine dubio mirum videbitur, quod hominum vitia et ineptias more Geometrico tractare aggrediar, et certa ratione demonstrare velim ea, quae rationi repugnare, quaeque vana, absurda, et horrenda esse clamitant (55).

Spinoza replies that since only necessity and determination exist in nature, the geometrical treatment is valid;

De affectuum itaque natura et viribus, ac Mentis in eodem potentia, eadem Methodo agam, qua in praecedentibus de Deo et Mente egi, et humanas actiones atque appetitus considerabo perinde, ac si quaestio lineis planis, aut de corporibus esset (56).

Spinoza did not concern himself with the nature of mathematics. He was merely interested in its deductive certainty and exactness. His philosophy was but pure mathematicism (57).

Spinoza's development of Descartes' philosophy was not adopted by subsequent thinkers. Cartesianism, instead, adopted three major forms. One group of thinkers, of whom Leibniz is most prominent, emphasized the idealistic aspect in Descartes; another, typified by La Mettrie and the French philosophes, tended towards mechanism and materialism; the third, the British school, developed a pronounced empiricism. All three schools are alike in their adoption of some fundamental positions or tenets of Cartesian philosophy.

(54) Ethics, I, (Prop. XVII), Scholium, p. 53.

(55) Ethics, III, Pref., p. 120.

(56) Ibid., p. 121.

(57) Harry Austryn Wolfson, in The Philosophy of Spinoza, makes a strong case against Spinoza's mathematicism. See Chapter II, "The Geometrical Method," pp. 32-33.

III. Leibniz

Leibniz (1646-1716) is often called the last universal genius. His countless references to ancient, medieval and contemporary thinkers illustrate his wide learning and capacity for detail (58). They likewise reveal his basic trait and tendency: to compromise and synthesize, to remove individual differences, to harmonize speculative opposition in his own "centre de perspective" (59)."

Leibniz was fully appreciative of these numerous influences. His acquaintance with Aristotle, Plato and the Scholastics made him desire to combine their doctrines with the moderns (60). He was familiar with the doctrines of Francis Bacon, Hobbes, Galileo, Descartes, Spinoza, Malebranche and contemporary philosophers and scientists (61). Owing

(58) Joseph Marechal, *op. cit.*, pp. 154-161.

(59) "La consideration de ce systeme fait voir aussi que lorsqu'on entre dans le fonds des choses, on remarque plus de la raison qu'on ne croyait dans le plupart des sectes des philosophes. Le peu de realite substantielle des choses sensibles des Sceptiques; La reduction de tout aux harmonies ou nombres, ideas et perceptions des Pythagoristes et Platoniciens; l'un et meme un tout, de Parmenide et de Plotin, sans aucun Spinozisme; la connexion Stoicienne, compatible avec la spontaneite des autres; la philosophie vitale des Cabalistes et Hermetiques, qui mettent du sentiment par tout; les formes et Entelechies d'Aristote et des Scolastiques, et cependant l'explication mecanique de tous les phenomenes particuliers selon Democrite et les modernes, etc., se trouvent reunies comme dans un centre de perspective, d'où l'objet (embrouille en le regardant de tout autre endroit) fait voir sa regularite et la convenance de ses parties." *Lettre a Sammartini*, Erdmann, pp. 153-154. Similar passages occur throughout Leibniz's works; cf. *Nouveaux Essais*, Bk. I, Ch. I, Erdmann, p. 205.

(60) "J'ai toujours ete fort content, meme dans ma jeunesse de la Morale de Platon, et encore en quelque facon de sa Metaphysique; aussi ces deux sciences vont-elles de compagnie, comme la Mathematique et la Physique. Si quelqu'un reduisoit Platon en systeme, il rendrait un grand service au genre humain, et l'on verroit que j'en approche un peu." *Extrait d'une lettre a Mr. Remond de Montmort*, 1716, Erdmann, p. 725. In the same letter Leibniz writes, "Quand j'etois jeune garcon, les Etudiants de mon age chantoient: Summus Aristoteles, Plato et Euripides, occiderunt in profundum." *Ibid.*, p. 725.

(61) "Leibniz's philosophy was thus the attempt to reconcile two ways of looking at things, the ancient and the modern, the theological and the mechanical, the religious and the scientific. He desired to give all possible scope to scientific research, all possible breadth to knowledge; but not less did he wish to give all possible depth and reality to religious belief." John Theodore Mers, *Leibniz*, New York: Hacker Press, 1946.

to Descartes' influential position, Leibniz was prompted to make clear his own position by contrast to the Cartesians. Leibniz pronounced many anti-Cartesian sentiments (62). Maintaining that "in Cartesio ejus methodi tantum propositum tenet (63)," he proposed a dynamist view of nature, reinstated final causality and improved Descartes' mathematics as evidence of his opposition to the Cartesians. But his ambition to determine the simple elements of thought, his theory of innate ideas, and his mathematicism placed him in the Cartesian tradition.

Leibniz's notion of a universal characteristic and his metaphysical doctrine are two aspects which demonstrate the influence of mathematics on his thought (64). The first of these proposed an ultimate identification

(62) For example, Leibniz, "Il m'arriva un jour de dire, que le Cartesianisme en ce qu'il a de bon n'estoit que l'anti-chambre de la veritable philosophie." Lettre a un ami sur le Cartesianisme, 1695, Erdmann, p. 123.

(63) Epistola ad Jacobum Thomasium, III, 1689, Erdmann, op. cit.,

(64) The influence of mathematics on Leibniz's thought can be found in Leibniz: The Monadology and other Philosophical Writings, translated with Introduction and Notes by Robert Latta, London: Geoffrey Cumberlege, 1948; see Latta's Introduction, especially pp. 74-86. A similar study is prefixed to G.-W. Leibniz: Oeuvres Choisies, avec preface, notes, table par questions et table des noms propres, par L. Prenant, Paris: Librairie Garnier Freres, n.d.; see the editor's preface to the selections from Leibniz. An older, but still valuable work is Leibniz, The Philosophical Works of, Comprising the Monadology, New System of Nature, Principles of Nature and of Grace, Letters to Clarke, Refutation of Spinoza, and his other important philosophical opusculs, together with the Abridgement of the Theodicy and extracts from the New Essays on Human Understanding, Translated from the original latin and french with notes by George Martin Dumas, New Haven: Tuttle, Morehouse & Taylor, 1890. References to Leibniz are taken from the above three works, but principally from God, Guil. Leibnitii: Opera Philosophica Quae Exstant, Latina Gallica Germanica Latina, edita recognovit e temporum rationibus disposita pluribus ineditis auxit introductione critica atque indicibus, instruxit Joannes Eduardus Erdmann, Pars Prior et Pars Altera, Berolini: Sumptibus G. Reicheri, 1839.

of abstractive thinking with his symbolic logic (65). This idea was seen to be impracticable and was tentatively laid aside. In the metaphysics, which represents his mature thinking, the notion of a universal mathematical science played a formative part.

The ambition to unify all knowledge by a universal logic possessed Leibniz at the early age of eighteen, when it had been stimulated by the work of Raymond Lulle and others (66), who, he admits were his predecessors in this invention. He aimed to construe an "alphabet of human thoughts." What the Italian mathematician, Vieta, had done for Algebra, he hoped to do for all the sciences.

Si daretur vel lingua quaedam exacta (qualem quidam Adamicam vocant), vel saltem genus scripturae vere philosophicae, qua notiones reocarentur ad Alphabetum quoddam cogitationum humanarum, omnia, quae ex datis ratione assequi, inveniri possent quodam genere calculi, perinde ac resolvuntur problemata arithmetica aut geometrica (67).

Leibniz's idea of this Combinatory Art, Universal Characteristic, Universal Language, or Universal Mathematics was based on the notion that all concepts are themselves simple or reducible to a small number of simple concepts. If the simple concepts are designated by appropriate symbols, they can be combined into composite concepts with the laws of arithmetic. Summarily, then, the logical algebra aimed to replace concepts by symbols and their combinations, propositions by the relations between symbols,

(65) "Rendre tous les hommes en une famille qui ait la même science, la même langue: tel a été le but que Leibniz s'est proposé. Cette religion doit être le christianisme; et cette langue, la caractéristique universelle." Clodius Plat, Leibniz, Paris: Felix Alcan, 1918, p. 1. See also Louis Couturat, La logique de Leibniz d'après des documents inédits, Paris: Felix Alcan, 1901; and C.I. Lewis, A Survey of Symbolic Logic, Berkeley: University of California Press, 1918, pp. 5-18.

(66) For a study of these sources, see Joseph Iwanicki, Leibniz et les démonstrations mathématiques de l'existence de Dieu, Strasbourg: Librairie Universitaire d'Alsace, 1933, pp. 107-110.

(67) De Scientia Universali seu Caloulo Philosophico, Erdmann, op. cit., p. 83.

and thus substitute calculation for reasoning. Leibniz ventured this facetious remark as to future human argumentation,

Que factos, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos computistas. Sufficiet enim, calamos in manus sumere, sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: calculemus (68).

Besides eliminating barren argumentation, the combinatory art was to insure inventiveness in science and even permit a demonstration of God's existence with the certitude of mathematics.

Leibniz saw that two requirements were necessary to achieve this combinatory art: an encyclopedia as an inventory of all human knowledge and a universal characteristic or system of signs representing the concepts and the legitimate manners of combination. Realizing that the idea of an inventory of human thoughts was a vast undertaking and not the work of a man or a generation, Leibniz began to carry out the scheme himself and sought collaborators for the project. He even authorized a plan for a society which would work on the Encyclopedia, to be called "Ordre de la Charite," or "*Societas Theophilorum vel Amoris Divini*." Once the encyclopedia had begun to take shape, the symbolism and its application would be a relatively easy task. The choice of signs and their signification would be extremely important. They should be chosen for their usefulness in signifying and as an aid to reasoning. Leibniz proposed that the symbolism possess the suppleness and generality of his own notation of the differential calculus, thus obtaining controllable certitude and avoiding controversies.

(68) *Ibid.*, p. 84.

Two difficulties, not subject to the combinatory art, faced Leibniz. The one was the impracticability of the scheme; he was unable to get a sufficient number of people to carry out the work entailed in the encyclopedia. The other was that "ses caracteres presupposeroient la veritable philosophie, et ce n'est que presentement que j'oserois entreprendre de les fabriquer (69)." Accordingly, though he kept the project in mind, Leibniz turned his efforts to speculative thought and towards harmonising religious and political efforts (70).

Leibniz's numerous philosophical speculations can be centered, in a partial synthesis, about his doctrine of substance. Descartes had imputed reality to two opposed substances, matter and spirit, which Spinoza had reduced to a pantheist unity. Descartes' interpretation of matter as continuous, inert, geometrical extension capable of infinite divisibility, was opposed by the atomist conception of matter as discrete, active, arithmetical extension, ultimately indivisible. Leibniz scored both theories on their reduction of substance to quantity and the elimination of quality. Quantity alone does not differentiate; mere extension has no meaning; there must be something extended (71).

The differentiating character of Leibniz's substance lay in the self-active monad (72). These ultimate metaphysical units, monads, were real

(69) As quoted in Joseph Marechal, op. cit., p. 166.

(70) Harold R. Smart op. cit., Chapter Two, "Leibniz's Logic," pp. 12-35.

(71) "Ceux qui veulent que l'entendue meme soit une substance, renversent l'ordre des paroles aussi-bien que des pensees. Outre l'entendue il faut avoir un sujet qui soit etendu, c'est-a-dire, une substance a laquelle il appartienne d'etre repetee ou continuee. Car l'entendue ne signifie qu'une repetition ou multiplicité continuee de ce qui est repandu; une pluralite, continuee, et co-existence des parties; et par consequent elle ne suffit point pour expliquer la nature meme de la substance repandue ou repetee, dont la notion est anterieure a celle de sa repetition." Extrait d'une lettre, Erdmann, p. 114.

(72) Leibniz's doctrine on monads is synthesised in La Monadologie, Erdmann, pp. 705-711. La Monadologie, written in 1714, two years before his death, contains a condensed statement of his philosophy. In Systeme Nouveau de la Nature, (Erdmann, pp. 124-128), Leibniz gives a personal account of how he was led to his theory on the nature and communication of monads.

like the atom of physics but inextended like the mathematical point (73); they were immaterial in essence yet material in the sense that they made up an extended body (74). Since they were essentially self-active and activity has no parts, the monads were indestructible. Furthermore, since nothing can affect pure activity, the monads could only change from within; they were immutable *ab extra* (75). No two monads were alike, but each possessed proper individuality. The self-activity of each monad consisted in the constant mirroring, representing and reflecting of the entire monadic universe in varying degrees of continuous perfection (76). They were to be looked upon as mirroring, in microscopic fashion, the macroscopic universe.

This activity gave qualitative specification to each monad. Within the monad itself, desire and perception were distinguished (77). Desire was the self-active tendency of the monad as a part of a whole to realize itself; perception was its passive mirroring of the universe. Non-living matter -- bare monads -- had unconscious perception and mere impulse as desire; plants and animals had conscious perception with instinct as desire; man had self-conscious perception (*apperception*) and will as desire (78). Leibniz considered living bodies as composed of a central monad -- a soul

(74) *La Monadologie*, n. 6, Erdmann, p. 705.

(75) *Ibid.*, n. 7; see also Bertrand Russell, *A Critical Exposition of the Philosophy of Leibniz*, Cambridge: The University Press, 1900, p. 48.

(76) *La Monadologie*, n. 56-62, Erdmann, n. 708-710.

(77) *Ibid.*, n. 14-24, p. 706-707.

(78) "Mais la connoissance des verites necessaires et eternelles est ce qui nous distingue des simples animaux et nous fait voir la Raison et les sciences, & nous eleveant a la connoissance de nous memes et de Dieu. Et c'est ce qu'on appelle en nous ame raissonable ou Esprit." *Ibid.*, n. 29, Erdmann, p. 707. "The necessary and eternal truths are the first principles of all rational knowledge. They are innate in us. They are, in fact, the very principles of our nature, as of the universe, because it is of our essence to represent the whole universe. This consciousness or knowledge of these truths is knowledge of ourselves, and it is at the same time knowledge of God, who is the final reason of all things." Robert Latta, *Leibniz The Monadology and other Philosophical Writings*, Oxford: The Clarendon Press, 1896, pp. 233-234.

or entelechy -- surrounded by non-living monads which were subject to it. In this way Leibniz proposed his solution of Descartes' problem of matter and spirit.

The universe was composed of an infinity of such monads. With the law of indiscernibles, Leibniz held that no two monads are alike; there were no duplications and superfluities in nature. The law of analogy prevented contradictions and contraries in nature by viewing all monads as differing not in kind, but merely in degree. By the law of continuity, the monads could be seen as an infinite series of terms increasing in perfection from the least perfect monad to the limit of God, the perfect monad (79). Since there was an infinity of monads, no new ones were added nor taken away by death (80). All monads were created at the beginning of time and will perish together at time's end. Generation and corruption were merely other terms for the ceaseless evolution and involution of monads. Finally, by his principles of pre-established harmony and cosmological optimism, Leibniz considered the monadic universe as the best possible which God could have created, and in which all the inner monadic changes have been harmoniously pre-established and work constantly for the best.

With the aid of these ultimate units, Leibniz's two philosophies (81) present a somewhat coherent system. Essentially logical, idealist and mathematical in his general outlook, Leibniz based his philosophy on the adoption

(79) Extrait d'une lettre à Mr. Bayle, Erdmann, pp. 104-106.

(80) La Monadologie, n. 74-77, Erdmann, p. 711.

(81) Leibniz's mature philosophical thought was definitely formed by 1686 when he sent his Discours de Métaphysique to Arnauld. His latter philosophy is the logical development of the notion of substance. Cf. Bertrand Russell, A History of Western Philosophy, New York: Simon and Schuster, 1945, Chapter XI, Leibniz, pp. 581-596. Our understanding of Leibniz's thought stems from the efforts of Russell's and Couturat's works, cited above.

of mathematical method and two principles, contradiction and sufficient reason (82). In method he distinguished analysis from synthesis. The notion of analysis was not merely breaking down a complex whole into its component parts; analysis included the abstractive reduction of these parts (in which process Leibniz's mathematics prevents him from seeing that he thus destroys the identity of these parts) to a clear idea or a principle. In synthesis the process is reversed: the movement is from the simple to the composite. His monadology had already indicated this procedure, while his theory of a universal science illustrates this method more strikingly.

By his monadology, Leibniz was able to eliminate Descartes' radical dualism of matter and spirit with its consequent independence of self-consciousness. Though the mind is an individual, independent substance, its monadic nature qualifies it as one of an infinite number of substances, to which it is alike and all of which it reflects. Human self-consciousness, a more perfectly developed degree of perception, is due to the greater clarity and distinction of the ideas which it subjects to analysis. These ideas are innate and are the ultimate principles of all knowledge (83).

But Leibniz did not consider all ideas or knowledge as innate. The method of analysis had to be applied to propositions in order to make them

(82) "Dubius utor in demonstrando principiis, quorum unum est: falsum esse quod implicat contradictionem; alterum est: omnis veritatis (quae immediata sive identica non est) reddi posse rationem; hoc est, notionem praedicti semper notioni sui subjecti vel expresse vel implicite inesse, idque non minus in denominationibus extrinsecis quam intrinsecis, non minus in veritatibus continuentibus quam necessariis locum habere." *De Scientia Universali seu De Calculo Philosophico*, Erdmann, p. 83. Cf. *La Monadologie*, n. 31-32, Erdmann, p. 707. See also *De Natura et Usu Scientiae Generalis*, Erdmann, pp. 86-88.

(83) *Nouveaux Essais*, 3^e l., Chapitre I. "S'il y a des Principes inces dans l'esprit de l'homme," Erdmann, pp. 204-213.

clearly intelligible. Analysis was the reduction of all true propositions to identical ones (84). True propositions which do not predicate the mode of existing at a particular time are necessary and analytic, those which predicate particular existence are contingent and synthetic. The first type of propositions, called truths of reasoning, yield necessary truths; while the second, termed truths of fact, deal with contingent truths. Leibniz proposed the goal of human knowledge as the reduction of even contingent truths to identical propositions (85).

Since a proposition consists of a subject and a predicate, its reduction to identity required that all predicates were to be reducible to the subject of which they were predicated. This would be tantamount to demonstrating that all the requisites of an effect were comprised within its cause. This reduction, for the truths of reasoning, lay in the mind itself. The reduction was to be effected by aid of the principle of contradiction (or identity, or possibility) in virtue of which that was false which contained a contradiction and that true which was contradictory to the false. Primitive truths of reasoning were the identical propositions themselves; derivative were all others -- axioms, postulates, relations -- which were to be reduced to the primitive with the principle of contradiction (86). All these truths were at least virtually innate within us.

(84) Bertrand Russell, *op. cit.*, p. 4. Russell lists five principal premisses of Leibniz's philosophy: "I. Every proposition has a subject and a predicate. II. A subject may have predicates which are qualities existing at various times. (Such a subject is called a substance.) III. True propositions not asserting existence at particular times are necessary and analytic, but such as assert existence at particular times are contingent and synthetic. The latter depend upon final causes. IV. The ego is a substance. V. Perception yields knowledge of an external world, i.e., of existents other than myself and my states."

(85) *Ibid.*, p. 49.

(86) "A savoir: une proposition necessaire est celle dont le contraire implique contradiction; telle est toute proposition identique ou derivee resoluble en identiques et telles sont les verites qu'on appelle necessites metaphysiques ou geometriques. Au effet, demontrer n'est pas autre chose que resoudre les termes d'une proposition et substituer au terme defini sa definition ou une de ses parties pour degager une sorte d'equation, c'est a dire la coincidence du predicat avec le sujet dans une proposition reciproque." De La Liberte, Prenant, p. 268.

Leibniz held that only God has a prior and adequate knowledge of the other class of judgments, those concerned with contingent matters, termed truths of fact (87). All these judgments involved an attribution of present existence to a subject. Our knowledge of these arose from our passive mirroring of the universe of monadic existence. Leibniz set up the reduction of truths of fact to identical propositions as his ideal aim; this implied that all the existential predicates which have or can ever be attributed to an individual -- for example, to Alexander the Great -- belong necessarily to him and make up his "notion" (88). It is this infinity of infinite monadic relations and complexities that God only knows, for each substance has infinite number of possible predicates and is an infinity of individuality (89). Men's minds were limited and thus unable properly to mirror this infinity which comes to us through our passive experience. We only know these truths with confused perception; they are obtained by induction through the sense (90). Were we able to grasp an adequate understanding of any existing subject, we would know all its possible predications (91).

In order for us to discern truths of fact, then, the principle of sufficient reason was to be applied (92). No fact was true unless there was a sufficient reason why it should be so and not otherwise. This would demonstrate the "compossibility" of an existent subject, showing its existence in the Divine pre-established harmony. All compossible things exist -- that is,

(87) La Monadologie, n. 33, 36-38. See also Nouveaux Essais, Chapitres I, II.

(88) Discours de Metaphysique, Prenant, pp. 92-93.

(89) Bertrand Russell, op. cit., pp. 60-61.

(90) Nouveaux Essais, "Introduction," passim.

(91) Bertrand Russell, op. cit., p. 47; see Discours de Metaphysique, Prenant, pp. 93-94.

(92) Nouveaux Essais, Bk. I, Chapitre I, 5, Erdmann, p. 209.

all these things which are harmonious with the surrounding totality of existence -- in keeping with the laws of finality and of cosmological optimism (83). Human knowledge of these contingent facts was called primitive when it referred to the facts of experience themselves. Derivative facts were those inferred from the primitive by using the principles of sufficient reason, thus determining their agreement with the world as a whole (84). "We know contingent facts only by probable arguments (85); they were merely stages of a series of terms which approached truths of reasoning as a limit (86).

Our knowledge of the truths of reasoning, had, for their proper sphere, certain knowledge, especially as in metaphysics (87). In metaphysical knowledge we properly attribute qualities to the monad which it really possesses. Metaphysics, for Leibniz, was founded on the method of traditional logic, and was predicative. The richness of the individual substance as a source of knowledge was evident, since it comprised within itself all past, present and future predicates which could ever be attributed to it. Metaphysics was thus the justification of every true proposition. It does not deal with pure abstractions. Leibniz censured any metaphysics, like the Cartesian, which wished to found itself on pure abstractions or mere relations. The perfect analysis of all conceivable predicates -- the infinitely infinite -- only was known by the Divine monad. Man, however, was able to reflect many of these predilections with the aid of the principles of contradiction.

(83) Bertrand Russell, op. cit., p. 66 ff.

(84) De la liberte, Prenant, pp. 288-291.

(85) Leibniz here envisioned the use of the calculus of probabilities; in Leibniz's well ordered world, the most probable combination must be achieved; see Joseph Marechal, op. cit., pp. 179-182.

(86) Nouveaux Essais, Bk. IV, Chapitre III, 14; Erdmann, pp. 343-345.

(87) See Prenant's Preface, pp. xxiv - xxxi, and Joseph Marechal, op. cit., pp. 186-193.

Holding to a metaphysics of predication, Leibniz maintained that the truths of fact properly belong to mathematics and mechanics (98). The proper concern of mathematics was relations (99). These relations he termed "abstractions" or "*phenomena bene fundata*," founded, that is, on the concrete reality of substance, but not metaphysical and predicable of substance as such. The mind thus achieved the universal, which was the principle of the relations between things. A logical or mathematical truth was but a glimpse of the system of infinitely possible relations which can characterize the monadic universe. Mathematical space and time are not things of substances. Space is merely our abstraction of one possible order of simultaneous existences; our "notions" of space are mere "abstractions" from the inner, monadic changes. Similarly, time is but our abstraction of the order of simultaneous existences. Likewise, extension is but an "abstraction" which neglects the properties of the rich, concrete existence of the monad, and extension lacks genuine reality as such (100).

The application of this distinction between a metaphysics of predication and a mathematics and science of mere relations would, possibly, have had many startling effects, though Leibniz did not carry out this program. The apparent rigor and necessity of mathematical science would, in such a case, be no longer

(98) Bertrand Russell, *op. cit.*, p. 49.

(99) *Ibid.*, pp. 13-14; See Remarques sur la lettre d'Arnauld, Prenant, p. 146.

(100) Prenant's Preface, p. xxvii.

a matter of much concern, since mathematics merely furnishes relative and not absolute or ultimate knowledge. Similarly, mechanics and other sciences merely deal with appearances, which, though well founded on substance, do not deal with things themselves. These conceptions of Leibniz (101), as those embodied in his universal characteristic (102), carried a pronounced influence on future speculation almost equal in weight to his invention of the calculus (103). Little attention, however, was given by successive thinkers to his insistence on philosophical tradition, or to his numerous attempts at speculative harmony.

(101) Ibid., pp. xxix-li; see also Joseph Marechal, op. cit., pp. 199-201.

(102) Harold E. Smart, The Philosophical Presuppositions of Mathematical Logic, Chapters II and III.

(103) Carl B. Boyer, The Concepts of the Calculus, Chapter V, "Newton and Leibniz," pp. 187-225.

Chapter V. British Empiricism

The school of rationalism which Descartes inaugurated on the European continent had a correlative but dissident rival in the era of British empiricism (1). In Mathematics and physics, Cartesian apriorism was offset by Newton's experimental-mathematical method, while in the formulation of philosophical problems, the British thinkers adopted the empirical and critical attitude of John Locke in contrast to continental dogmatism and idealism (2). This critical attitude was combined with a practical approach to philosophical problems, causing the British school to be more concerned with ethical and political problems while the rationalists emphasized systematic philosophy (3). The rationalist emphasis on intellectual and the empiricist concern with sense knowledge are the two threads of thought later gathered up by Kant, who attempted to knit them together without laying undue stress on either.

(1) For a brief contrast between these two schools of thought see C. A. Morris, Locke Berkeley Hume, Oxford: The Clarendon Press, 1931, "Conclusion," pp. 158-159.

(2) "The Essay was the first deliberate attempt, in modern philosophy, to engage in what might now be called epistemological inquiry, but mixed up by Locke with questions logical, psychological, and ontological; all subordinated in his design to 'what may be of use to us, in our present state,' and to 'our concerns as human beings.' Locke inaugurated the modern epistemological ⁵⁵ ~~55~~, characteristic of philosophy in the eighteenth century, which culminated in Kant..." Alexander Campbell Fraser, in An Essay Concerning Human Understanding, by John Locke, 2 vols., Oxford: the Clarendon Press, Fraser's "Prolegomena," p. iv.

(3) A comparison of the practical and empiricist aspects of British philosophy with the systematic philosophy of the continent is given in Edwin A. Burtt, The English Philosophers from Bacon to Mill, edited, with an introduction, New York: The Modern Library, 1939, "Introduction," pp. ix-xxii. Compare the analysis of the cultural background of British empiricism in Bertrand Russell, A History of Western Philosophy, New York: Simon and Schuster, 1945, Chapter XII, "Philosophical Liberalism," pp. 596-608.

Both the rationalist and empiricist schools treated the problem of scientific and mathematical abstraction somewhat cavalierly. The rationalist theory of science in Descartes, Spinoza and Leibniz, influenced by mathematics, inclined to an identification of all sciences, with no regard towards a careful mental process of abstraction. Certainty, exactness, clarity and intuitive ideas were the scientific measures. Fields of speculative research in which these goals were unattainable were either denied scientific status or "rationalized" to meet the ideal standard. This propensity to homogenize the probability and intricacy inherent in our knowledge of nature, of psychology morals and politics, was due to their mathematicism. The rationalists overlooked the composite nature of man and offered an "angelic" epistemology (4) which led them to misunderstand the morality of moral knowledge, to propose a non-ultimate attitude to ethics, psychology and politics, and to deny the appetiteness of any other sciences.

The same denial of abstraction, though on different grounds, characterized British empiricism. Attempting to model philosophical truth and method on the experimental-mathematical method of Newtonian science, the British empiricists applied a purely descriptive process to mental states. Though starting with metaphysical presuppositions (5) --

(4) Jacques Maritain, Three Reformers: Luther - Descartes - Rousseau, London: Sheed and Ward, 1947, "Descartes, or the Incarnation of the Angel," pp. 53-59.

(5) Gibson writes of Locke: "Thus, like many other thinkers, he was destined to prove an illustration of the truth that metaphysics has a way of avenging itself on those who slight or disregard it, and that its deepest entanglements are often reserved for those who think they have discovered a path, by following which its difficulties may be evaded." James Gibson, Locke's Theory of Knowledge and its Historical Relations, Cambridge: The University Press, 1931, p. 11.

yet denying the rightfulness of wondering about ultimate causes -- this approach, initiated by John Locke, culminated in the rabid scepticism of David Hume and inspired Kant to bemoan previous metaphysicians' imitation of mathematics and science (6).

I. Newton

Isaac Newton (1642-1727), who wielded a powerful influence on British empiricism (7), may well be looked upon as the reflection of the various new movements initiated with the Renaissance (8) and as their temporary culmination. He gathered unto himself the mathematical current that came through Copernicus, Kepler, Galileo and Descartes, the emphasis on observation of Bacon, Harvey, Gilbert, Boyle and Locke, and the tempering influence of religion and morals from the Cambridge Platonists. Locke, Berkeley, Hume and Kant, though they disagreed with the philosophical aspects in Newton -- especially his metaphysics of space and time -- admired the system wrought by his genius. His speculations were the springs of the cosmogony later proposed by Kant and Laplace. And although he was an anti-Cartesian, Newton continues the Cartesian spirit which reappears in the cosmological and evolutionary systems of the eighteenth and nineteenth centuries.

Newton's mathematical system of the universe, often referred to as the classical view, lasted until the more recent relativity and quantum theories. It had, in turn, dethroned the Cartesian physics and astronomy. Descartes, with characteristic mathematical apriorism, had proposed to

(6) See Chapter VI, below.

(7) For the interaction of Newton and Locke, Newton and his contemporaries, see James Gibson, *op. cit.*, Chapter I, "Locke and Contemporary English Philosophy," pp. 232-266.

(8) Edwin Arthur Burtt, *The Metaphysical Foundations of Physics*, Chapter VI, "Gilbert and Boyle," (1), *Summary of the Pre-Newtonian Development*, pp. 197-201.

establish the universal laws of nature on the basis of some arbitrarily chosen hypothesis according to which the facts were to become rationalized (9). For example, in explaining the origin and nature of the universe, he proposed the notion of homogeneous corporeal mass to which the Creator gave an original motion. In this way, vertical or whirlpool movements were set up and absolute extension brought into a manifold (10).

Newton opposed Descartes' scientific procedure and results with his own scientific contributions and his empirical-mathematical method which he constantly refers to as his own (11). His famous dictum, "hypotheses non fingo" (12), is the negative property of his method voicing disapproval of the choice of arbitrary principles and assumptions in the knowledge of nature. He proposed as a positive measure, that the proper orientation of science lay in an observational analysis of nature by experiments which yield facts and properties (13). Each property was then designated by an appropriate symbol and their mutual interdependence could be analysed

(10) A. Boyce Gibson, The Philosophy of Descartes, London: Methuen & Co., 1932, Chapter VIII, "The Attributes of God," pp. 264-270.

(11) For a study of Newton's method see Henry R. Burke, "Sir Isaac Newton's Formal Conception of Scientific Method," The New Scholasticism, 10(1936), pp. 86-115.

(12) "If any one offers conjectures about the truth of things from the mere possibility of hypotheses, I do not see how anything can be determined in any science....Wherefore I judged that one should abstain from considering hypotheses, as from a fallacious argument," Sir Isaac Newton, Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World, translated in English by Andrew Motte in 1729, edited by Florian Cajori, Berkeley: University of California Press, 1946, "General Scholium," p. 275; references to Newton are from this edition.

(13) "We no other way know the extension of bodies than by our sense, nor do these reach it in all bodies; but because we perceive extension in all bodies that are sensible, therefore we ascribe it universally to all others also. That abundance of bodies are hard we learn by experience...That all bodies are impenetrable we gather not from reason, but from sensation." Ibid., Rule III, p. 399.

mathematically (14). Through this synthetic mathematical combination one could predict and deduce further phenomena (15). Even for this process, experiments were constantly to guide and verify procedure (16).

Newton may be looked upon as a protagonist of the philosophy of science that is met today and as embodying the stress and strain between the two types of scientists, the mathematically-minded and the physically or empirically-minded (17). For the first, mathematics is the queen of the sciences and the validity of mathematical truths, grasped by a mental intuition or viewed as a free creation of the mind, is unquestionable. Pure mathematics is a frame into which all natural events are to be fitted, and the laws of nature as formulated in equations express the quantitative character of nature and allow further laws to be deduced.

(14) "...and therefore I offer this work as the mathematical principles of philosophy...by the propositions mathematically demonstrated in the first book, we then derive from the celestial phenomena the forces of gravity with which bodies tend to the sun and the several planets. Then, from these forces, by other propositions which are also mathematical, we deduce the motions of the planets, the comets, the moon, and the sea. I wish we could derive the rest of the phenomena of nature by the same kind of reasoning from mathematical principles..." Ibid., Preface, pp. xvii-xviii.

(15) "...the whole burden of philosophy seems to consist in this -- from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena..." Ibid., pp. xvii-xviii.

(16) "In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions. This rule we may follow, that the argument of induction may not be evaded by hypotheses." Ibid., Preface, Fourth Rule, p. 400.

(17) For a similar comparison, see Robert Blanche, La Science Physique et la Realite: Realisme, Positivisme, Mathematisme, Paris: Presses Universitaires, 1948, "Introduction", pp. 6-8 and passim.

For the empirically-minded scientist, mathematics is but a tool, the handmaiden of the sciences. Mathematical entities arise from observing nature and their validity is tested there, and mathematics is but a handy tool or symbolic language devised for interpreting physical facts. Strictly speaking, for this view, nothing is deducible, and we can only ascertain that nature is purely quantitative through observation. By his insistence on the priority of experiment, Newton seems to have favored the latter view (18). Moreover, in his attitude towards science as the exact mathematical formulation of the laws of the universe and by his positivistic notion that one can learn truths about nature without presupposing or implicating any hypothesis or theory on the ultimate nature of things, Newton fostered an outlook which is common today.

The spiritualistic and anti-mechanistic aspect of Newton is evident in his participation in current theological debates and in his religious writings (19). Science itself, as he writes in the *Optics*, leads us to God through our recognition of the order and adaptation of nature. God's chief attribute is His providence, which Newton construes in such fashion that God is a species of "cosmic plumber" who keeps the fixed stars from collapsing in the center of space (20), and patches up the irregularities of the universe that are constantly increasing. God is able to function in this wise because He is a perfect spirit, universally extended in space and time. This is Newton's ultimate explanation

(18) Edwin Arthur Burtt, *op. cit.*, pp. 216-223.

(19) J. W. N. Sullivan, *Isaac Newton, 1642-1727*, London: MacMillan and Company, 1938, Chapter V, pp. 188-170.

(20) Edwin Arthur Burtt, *op. cit.*, pp. 294-295.

of absolute space and time: they are -- with a tribute to Berkeley -- the omnipresence and eternity of God, the Divine sensorium.

The acceptance of this "scientific God" is paralleled, in Newton, with belief in the "scientific man." Newton accepts the current hyperdualism of Descartes and all its implications together with the notion that secondary qualities are mere modifications of the perceiver (21). Finally, the Newtonian world itself is "scientific." Only primary qualities subject to mathematical consideration are real. The universal theory of gravity, the reduction of all motion to three universal laws, atomism, the concept of absolute space and time which he tried to prove scientifically, and his notion of ether as a medium in which the forces and motions are propagated and recreated after being spent -- all add up to the picture of the universe, in Newton's whole view, as a large machine, with God introduced as a mechanical appendage (22).

II. Locke

Before the time of John Locke (1632-1704), two earlier English philosophers had held a somewhat pejorative view of mathematical abstraction. Francis Bacon had viewed it as an afterthought to Physics (23);

(21) Ibid., pp. 288-287.

(22) Ibid., pp. 280-299.

(23) See The Advancement of Learning, Book II, Vol. I, pp. 184-244 in The works of Francis Bacon, Lord Chancellor of England, With a Life of the Author by Basil Montagu, in three volumes, Philadelphia: M. Murphy, 1876. Bacon's ignorance of mathematics is well known. Despite the achievements of Kepler and Galileo, he subjected it to severe criticism. Cf. Novum Organum, Book I, Aphorisms, §. Vol. II, p. 345. In Book II of The Advancement of Learning, (Vol. I, p. 199) Bacon seems more interested in pointing out the benefits of mathematics for the mind, which he compares to the benefits of tennis for the body. Mathematics will sharpen the wit that is dull, make fit the wit that is wandering and make abstract the wit which is too inherent in sense.

and Thomas Hobbes, whose philosophy can be classed as sensualistic mathematicism, proposed a non-abstractive notion of mathematics tantamount to the denial of mathematical abstraction itself (24). In the meantime, mathematics had gained an enviable reputation. Newton's mathematics served but to intensify the high regard in which mathematics had come to be held in seventeenth century England. For most thinkers of this time mathematics, and especially geometry, was the ideal type and standard of all knowledge (25). Locke was subjugated to the influence of this view of mathematics, along with his training in Scholasticism, knowledge of Cartesianism and the currents of contemporary English thought (26). This varied philosophical conditioning stands out especially in his main work, An Essay Concerning Human Understanding.

The Essay grew out of a discussion in which Locke had engaged with five or six friends, the question at issue being the validity of

(24) Hobbes, the arch-materialist, is a unique example of sensualistic mathematicism. Reasoning for him was but a kind of addition and subtraction. Mathematics was almost akin to sensory experience, for he could not admit that lines and surfaces were deprived of slight dimensionality even in their conception; Carl B. Boyer, op. cit., p. 176.

(25) "The views of Hobbes, Locke and (later) Condillias are illustrative, in a general way, of the logical conceptions prevalent in the interim that separates Leibniz from Descartes. For these thinkers, just as for Descartes, the ideal type of all knowledge is that which they understood by mathematical. Outside the absolute certainty guaranteed by rigorous mathematical demonstration only probability remained. This however was not to limit all our knowledge to mathematics, since it was still held that the method employed so successfully in that field could also be adapted to other matter." Harold R. Smart, The Philosophical Presuppositions of Mathematical Logic, p. 6.

(26) Edwin Arthur Burtt, The English Philosophers from Bacon to Mill. On pp. xviii-xix of the introduction, Burtt points out how Locke reflects the practical attitude toward philosophy typical of his times.

the principles of morality and revealed religion (27), which the Cambridge Platonists had sought to put on a sound basis (28). It was this aim that prompted Locke in writing the Essay (29). Its purpose was,

to inquire into the original, certainty, and extent of human knowledge, together with the grounds and degrees of belief, opinion, and assent... (30).

The essay was conceived, however, as merely preliminary to the establishment of the validity of human knowledge with respect to morals and religion (31). For,

in an age that produces such masters as the great Huygenius, and the incomparable Mr. Newton, with some other of that strain, it is ambition enough to be employed as an under-laborer in clearing the ground a little, and removing some of the rubbish that lies in the way to knowledge (32).

(27) "Our business here is not to know all things, but those which concern our conduct." Essay, Introduction, Vol. I, p. 31.

(28) See pp. xvi-xvii of Fraser's prolegomena in An Essay Concerning Human Understanding by John Locke, (Collated and Annotated with Prolegomena, Biographical, Critical, and Historical by Alexander Campbell Fraser, two volumes, Oxford: The Clarendon Press, 1894). All quotations from Locke are taken from this edition.

(29) "We may now sum up our account of the primary and main problem of the Essay. In it Locke undertakes the investigation of the nature and conditions of a knowledge which is at once absolutely certain, strictly universal, 'instructive' or synthetical, and 'real'; the consequent determination of the possible extent of such knowledge; and the examination of its distinction from and relation to other forms of cognition, which are deficient in some of the respects enumerated. While the mathematical sciences furnish us with the typical example of such knowledge, its most important contents are held to refer to the objects of our moral and religious consciousness." James Gibson, Locke's Theory of Knowledge and its Historical Relations, Cambridge: at the University Press, 1931, p. 7. Gibson points out, in chapter one of his fine study, how Locke came to write the Essay. See also H. I. Aaron, John Locke, London: Oxford, 1937, p. 66.

(30) Essay, Introduction, 2 Vol. I, p. 28.

(31) "Less as was his interest in the scientific discoveries of his day, he leaves no room for doubt that in his opinion the knowledge which is of greatest importance for man is that which relates to his duty, and to the existence of the Divine Being, whose law he conceived this duty to be." James Gibson, op. cit., p. 6.

(32) Essay, Epistle to the Reader, Vol. I, p. 14.

Locke's theory of knowledge is gradually developed in the Essay. After showing, in the first book, that he holds no brief with a theory of innate ideas, he examines the origin of knowledge in the second book. All our ideas (33), from which our judgments or true knowledge are formed and which arise from the twofold experience of sensation and reflection, are either simple or complex. A man is not able to make up simple ideas; the mind is passive in getting them from experience. Simple ideas arise in diverse ways, from one sense or more, from reflection or reflection combined with sensation. (Locke then distinguished the simple ideas of primary or mathematical qualities which have real existence from those of secondary or imputed qualities which only virtually exist). By combining and extending simple ideas or "originals," the mind exhibits its power of producing new ideas in the activity of comparison, composition and abstraction, which yield complex ideas called modes, substances and relations. The third book deals with Locke's theory of the nature, meaning and use of language.

It is in the fourth book, the most important of the Essay, that Locke deals with knowledge in general. The judgment in which we have knowledge is "but the perception of the connection and agreement, or disagreement and repugnancy of any of our ideas" (34). Universal propositions are the makeup of knowledge which bears the distinctive badge of certainty and is

(33) Locke's wide use of "idea" is evident throughout his Essay. "Whatever is meant by phantasm, notion, species, or whatever it is which the mind can be employed about in thinking." Essay, Vol. I, I, p. 18. Gibson says, "as the universal implicate of cognition, ideas are involved alike in the sensible apprehension of a colour and in the thought of an abstract object or relation, which cannot be presented to the sense. For the word he disclaims any special partiality, and avows himself ready to 'change the term idea for a better' as soon as his critics can help him to one which will bear as well the required width of denotation." James Gibson, op. cit., p. 15.

(34) Essay, Book IV, Chapter 1, 2, Vol. II, p. 167.

referent to a reality independent of the mind; knowledge is "real." Of the two main kinds of knowledge, "demonstrative knowledge is much more imperfect than intuitive (35)." In the latter, "the mind perceives the agreement or disagreement of two ideas immediately by themselves, without the intervention of any other (36)," "and this kind of knowledge is the clearest and the most certain that human frailty is capable of. This part of knowledge is irresistible..." (37). In demonstrative knowledge, though its certainty is unquestionable, the mind must operate "by the intervention of other ideas (one or more, as it happens), to discover the agreement or disagreement which it searches..." (38). Freed by doubt, the stage or demonstrative knowledge is not as easy to obtain as intuitive; demonstrative knowledge is forced to rely on the ideas of intuitive knowledge which it uses as "proofs."

The most perfect type of demonstrative knowledge is mathematics. But although "it has been generally taken for granted that mathematics alone are capable of demonstrative certainty" (39), this is not the case. The reason for this, Locke says,

Has been not only the general usefulness of those sciences, but because, in comparing their equality or excess, the modes of numbers have every the least difference very clear and perceivable... But in other simple ideas...we have not so nice and accurate a distinction of their differences as to perceive or find ways to measure their just equality or the least differences (40).

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- (35) Essay, Book IV, Chapter 1, 9, Vol. II, p. 175.
 - (36) Essay, Book IV, Chapter 2, 1, Vol. II, p. 176.
 - (37) Essay, Ibid., p. 177.
 - (38) Essay, Book IV, Chapter 2, 9, Vol. II, p. 178.
 - (39) Essay, Book IV, Chapter 2, 9, Vol. II, p. 182.
 - (40) Essay, Book IV, Chapter 2, 10, Vol. II, p. 183.

Other sciences, then, are capable of demonstrative certainty.

Continuing, Locke examines the extent of human knowledge with regard to the four sorts of affirmation or negation we can make, identity, co-existence, relation and real existence. Under relation, he makes a strong case for our demonstrative knowledge of morality in comparison to mathematics. The complexity and sensible representations accompanying moral knowledge have made people think it incapable of demonstration. Mathematics, by comparison, has the advantage, since it uses sensible marks which are "not liable to the uncertainty that words carry in their signification (41)." Secondly, the moral ideas are more complex than those of mathematics, hence their names are of more uncertain signification and it is more difficult for the mind to retain these precise combinations. Thirdly, the weakness of human memory is not as evident in remembering mathematical as moral demonstrations, for

the numerical characters are helps to the memory, to record and retain the several ideas about which the demonstration is made, whereby a man may know how far his intuitive knowledge in surveying several of the particulars has proceeded; that so he may without confusion go on to what is yet unknown, and at last have in one view before him the result of all his perceptions and reasonings (42).

In concluding his comparison, Locke suggests that the difficulties of other demonstrative knowledge, such as morality, be resolved by accurate definitions, universally accepted and retained,

and what methods algebra or something of that kind may hereafter suggest, to remove the other difficulties...(43).

(41) Essay, Book IV, Chapter 3, 19, Vol. II, p. 209.

(42) Essay, Book IV, Ibid., p. 210.

(43) Essay, Book IV, Chapter 3, 20, Vol. II, p. 211.

Chapter four of the fourth book of the Essay deals with the reality of human knowledge. Simple ideas yield real knowledge, since they are but "the products of things operating on the mind in a natural way (44)." With regard to complex ideas, all of them, except for substance,

being archetypes of the mind's own making, not intended to be the copies of anything, not referred to the existence of anything, as to their originals, cannot want any conformity necessary to real knowledge (45).

Mathematical knowledge, since it is of this type, is of unquestionable reality and certainty, for

it is only of our own ideas. The mathematician considers, the truth and properties belonging to a ^{RECTANGULAR} rectangle or circle, only as they are in idea of his own mind. For it is possible he never found either of them existing mathematically, i.e., precisely true, in his life (46).

Locke's analysis of mathematical knowledge holds that it is the mere perception of agreement or disagreement of ideas; its demonstration is but the perception of this agreement through other ideas or mediums. Moral knowledge is of the same nature, and hence has similar validity. For, just as

all the discourses of the mathematicians about the squaring of a circle, conic sections, or any other part of mathematics, concern not the existence of any of those figures; but their demonstrations, which depend on their ideas, are the same, whether there be any square or circle existing in the world, or not...(47).

So it is likewise true that

the truth and certainty of moral discourses abstracts from the lives of men, and the existence of those virtues in the world whereof they treat...(48).

(44) Essay, Book IV, Chapter 4, 5, Vol. II, p. 230.

(45) Essay, Book IV, Chapter 4, 6, Vol. II, p. 231.

(46) Essay, Ibid.

(47) Essay, Book IV, Chapter 4, 8, Vol. II, p. 240.

(48) Essay, Ibid.

The methods of improving our knowledge are considered in chapter twelve. Not from maxims of "certain præcognita," but rather from the comparing of clear and distinct ideas to which we annex proper and constant names, is our knowledge improved. The best way to do this,

is to be learned in the schools of the mathematicians, who from very plain and easy beginnings, by gentle degrees and a continued chain of reasonings, proceed to the discovery and demonstration of truths...(49).

These strong and excessive claims for the excellence of mathematics, the surety of its method and the felicity of its symbols, Locke attributed to the elementary subjects of arithmetic and geometry. Both of these subjects, though they originated from the experience of primary qualities and could be applied to the world, were independent of the existence of objects conformable to them. Of the two, that of arithmetic or number offers greater precision and accuracy and is more perfect. The mind immediately, in fact, perceives the truth of simple arithmetical operations. The quantities of geometry, however, cannot be expressed with such precision and finish. Continuous extension does not lend itself as readily to every minute difference of quantity.

These characteristics of mathematics, then, signalise its distinction and value; though, Locke insists, knowledge of moral matters is more important and equally as certain. His frequent use of examples taken from mathematics or quantitative notions and his very account of knowledge itself, reveal a strong bias for mathematics and a preclivity to model

(49) Essay, Book IV, Chapter 12, V, Vol. II, p. 266.

all science and demonstration accordingly. It is these factors which lead one to believe that Locke not only had an inadequate view of mathematics as such, but that he inclined to a mild form of mathematicism.

III. Berkeley.

When but a young man of twenty at Trinity College, Dublin, George Berkeley (1685-1753) published two works, Arithmetica absque Algebra aut Euclide Demonstrata and Miscellanea Mathematica, which are indicative of his early interest and competence in mathematics. At the same college Berkeley was exposed to the new spirit in philosophy through the study of Bacon, Hobbes, Descartes, Locke, and John Toland the free-thinker, as well as to the scientific discoveries of Leibniz and Newton. How these philosophers and scientists affected him is recorded, fortunately enough, in his Commonplace Book, his diary from his twentieth to his twenty-fourth year. It was during this early period that he formed his famous rejection of substantiality to matter, his chief title to fame in philosophical annals. His opposition to the atheistic trends of his time and to free-thinkers in moral and mathematical matters, moreover, were an important factor in determining the progress of mathematics and science (80).

Locke's theory of primary and secondary qualities is the point of departure for the philosophy of Berkeley. Locke proposed a distinction between the reality of the mathematical and quantitative qualities of matter, which he called primary and which were real, from the hazy experience of the secondary qualities⁸⁰ of touch and taste, which were less

(80) Carl B. Boyer, Op. cit., pp. 224-225.

real. Locke also left an unsuccessful analysis of substance, equivalent to a denial of its reality,

...not imagining how these simple ideas can subsist by themselves, we accustom ourselves to suppose some substratum wherein they do subsist, and from which they do result; which therefore we call substance (51).

Berkeley, however, could see no reason for maintaining the first distinction. He points out, in A Treatise Concerning the Principles of Human Knowledge, that some of the primary qualities themselves are given through the same senses which yield secondary qualities. Further, primary qualities themselves only arise through combinations of secondary qualities; Berkeley confidently concluded,

in short, extension, figure, and motion, abstracted from all other qualities, are inseparable. Where therefore the other sensible qualities are, there must these be also, to wit, in the mind and nowhere else (52).

Once Berkeley has disposed of the first, Locke's weak defence of substance is no longer required. Berkeley rejects substance which is not perceived and retains that which is perceived: esse est percipere et percipi (53). This phrase contains both Berkeley's speculative position and the answer that he will thrust against the free-thinkers and materialists: materiality removed from matter, substantiality from substance, with matter or substance being but an exteriorisation or extrapolation of the Divine (54).

(51) John Locke, Essay, Book II, Chapter 23, 1.

(52) A Treatise Concerning the Principles of Human Knowledge, Part 1, 9, Vol. I, p. 268. All references to Berkeley are from The Works of George Berkeley, with Preface, Annotations, Appendices, and An Account of his Life, by Alexander Campbell Fraser, 4 volumes, Oxford: at the Clarendon Press, 1901.

(53) Ibid., n. 3, p. 269.

(54) "Some truths there are so near and obvious to the mind that a man need only open his eyes to see them. Such I take this important one to be, viz. that the choir of heaven and furniture of the earth, in a word all those bodies which compose the mighty frame of the world, have not any subsistence without a mind; that their being is to be perceived or known; that consequently so long as they are not actually perceived by men, or do not exist in my mind, or that of any other created spirit, they must either have no existence at all, or else subsist in the mind of some External Spirit: it being perfectly unintelligible, and involving all the absurdity of abstraction, to attribute to any single part of them an existence independent of a spirit. Principles, Part II, no. 6, Vol. I p. 260.

Armed with this doctrine, and with his pointed gifts of wit and sarcasm, Berkeley bravely wanders off into the camps of the free-thinkers of his time (55). He states that the acceptance of the doctrine in the Principles can achieve a refutation of scepticism; a liberation of thought from unnecessary and meaningless abstractions; a purification of natural philosophy which becomes but the interpretation of the ideas of sense, the Divine language of nature; a simplification of mathematics with its meaningless infinitesimals and acres discarded; an explanation of belief in the existence of others; a vindication of faith in God Who signifies and is signified in the world of nature.

This initial work of Berkeley's had been preceded, in 1708, by An Essay toward a New Theory of Vision (56), which was both a prelude to his main theory on substance and also an attack on Newton's conception of nature. The work contains his investigation of the relations of the sense of sight to space, distance, magnitude and place. Berkeley followed these two works with another in 1713, written in dialogue form, entitled Three Dialogues between Hylas and Philonous. In this work, Philonous represents Berkeley, while Hylas argues against Berkeley's position on substance. During this same year, Berkeley contributed a number of essays to an English magazine, the Guardian, whose main content is directed again towards the free-thinkers. In one of these essays he shows his wit and satire both in the content of the essays and in their title, "A visit to the Pineal Gland" of a free-thinker," and "The Pineal Gland of a Free-Thinker." Berkeley imagines himself a

(55) The Principles has the following subtitle: "Wherein the Chief Causes of Error and Difficulty in the Sciences, with the Grounds of Scepticism, Atheism and Irreligion, are Inquired Into." Vol. I, p. 211.

(56) "The Essay was a pioneer, meant to open the way for the disclosure of the secret with which he was burdened, lest the world might be shocked by an abrupt disclosure." Works, "Life of Berkeley by the editor," Vol. I, p. xviii.

hemunculus travelling about in the contents of the free-thinker's mind, and exposes its shameful contents.

Others of Berkeley's works are along the same strain. His De Motu (57) and Alciphron, or The Minute Philosopher aim to justify religion and oppose scepticism. With the appearance of The Analyst Or, A Discourse Addressed to an Infidel Mathematician (58) Berkeley's hitherto scattered remarks about the mathematics and mathematicians of his time were expanded. Fraser says of The Analyst,

It was followed by a controversy in which some of the most eminent mathematicians took part. Mathematica exsunt in mysteriis might have been the motto of the Analyst. The assumptions in mathematics, it is argued, are as mysterious as those of theologians and metaphysicians. Mathematics cannot translate into perfectly intelligible thought their own doctrines in fluxions (59).

Berkeley's criticism had a "ed effect on the inadequately formulated foundations of the Calculus in Newton and Leibniz, and was one of the factors that paved the way towards a rigorous formulation of its bases (60).

In addition to his direct criticism of mathematicians, Berkeley did some analysis of mathematics itself. In the Principles, for example, he proposed a species of mathematical nominalism, similar to Hobbes'

(57) Fraser writes (Life of Berkeley by the Editor, Vol. I, p. xlvi), "The material world, chiefly under the category of substance, inspired the Principles. The material world, under the category of cause or power, inspired the De Motu... Moles evolvit et agitat mentes might be taken as the formula of the materialism he sought to dissolve. Mens percipit et agitat molem significantem, cujus esse est percipi expresses what Berkeley would substitute for the materialistic formula.

(58) The subtitle of the Analyst reads: "Wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries." Works, Vol. III, p. 1.

(59) Works, "Life of Berkeley by the Editor," Vol. I, p. lxxiii-lxiv.

(60) For a fine presentation of Berkeley's criticisms and of their effect, see Carl B. Boyer, op. cit., Chapter VI, "The period of Indecision," pp. 224-230, passim.

sensualistic mathematics. Arithmetic, he writes, "hath been thought to have for its object abstract ideas of number (61)." But, argues Berkeley, number is but a collection of units, and since he has previously established

that there be no such thing as unity, or unit in abstract, there are no ideas of numbers in abstract, denoted by the numeral names and figures (62).

Numerical relations, therefore, are only realised in concrete experience.

For, in Arithmetic,

we regard not the things but the signs; which nevertheless are not regarded for their own sake, but because they direct us how to act with relation to things, and dispose rightly of them...it is evident from what has been said, those things which pass for abstract truths and theorems concerning numbers, are in reality conversant about no object distinct from particular numerable things....(63).

The same reasoning is applied to Geometry. Kettled by the loose use of "infinities" and "infinitesimals" of his time, he says,

The infinite divisibility of finite extension, though it is not expressly laid down either as an axiom or theorem in the elements of that science, yet is throughout the same everywhere supposed, and thought to have so inseparable and essential a connection with the principles and demonstrations in Geometry that mathematicians never admit it into doubt, or make the least question of it (64).

This notion of the infinite is the source of the paradoxes which militate against common sense, and which make mathematics difficult and tedious. Berkeley argues from the nature of our ideas that no finite extension either contains innumerable parts or is infinitely divisible. Berkeley imagines someone objecting to his elimination of the infinite that it

(61) Works, Principles, Part I, n. 119, Vol. I, p. 324.

(62) Ibid., n. 120, p. 325.

(63) Ibid., n. 122, p. 326.

(64) Ibid., n. 123, p. 327.

would destroy the very foundations of Geometry, and

those great men who have raised that science to so astonishing a height, have been all the while building a castle in the air (65).

He answers, significantly,

that whatever is useful in geometry, and promotes the benefit of human life, does still remain firm and unshaken on our Principles; that science considered as practical will rather receive advantage than any prejudice from what has been said (66).

For the rest, though it should follow that some of the more intricate and subtle parts of Speculative Mathematics may be pared off without any prejudice to truth, yet I do not see what damage will be thence derived to mankind (67).

In this, and in his other works, Berkeley, when he speaks of mathematics, inclines towards a nominalist view and appears to introduce a pragmatic test of usefulness for ascertaining the validity of mathematical notions.

IV. Hume.

David Hume (1711-1776) applied the procedures of the new sciences to the study ^{OF MAN} in his Treatise of Human Nature (68). Both this work and his Essay Concerning Human Understanding were merely introductory studies to morals and politics which claimed his foremost interest (69). With these

(65) Ibid., n. 181, p. 351.

(66) Ibid.

(67) Ibid., n. 181, p. 352.

(68) The Treatise was first published in 1738. An Inquiry Concerning Human Understanding, published in 1748, is a recasting of the first part of the Treatise. Similarly, An Inquiry Concerning Human Morals, published in 1752, is a re-edition of the second part of the Treatise. All quotations from Hume are taken from A Treatise of Human Nature by David Hume, Reprinted from the original edition and edited, with an Analytical Index, by L. A. Selby-Sigge, Oxford: The Clarendon Press, 1896.

(69) The subtitle of Hume's work reads, "An Attempt to introduce the Experimental Method of Reasoning into Moral Subjects." In this connection see A. D. Lindsay's remarks in his introduction to the modern library edition, A Treatise on Human Nature, Volume one, p. xxvi, (New York: E. P. Dutton and Co., 1954).

two works he hoped to show that neither mathematics nor physics had any greater claim to exactness than the science of man (70). A great admirer of Newton (71), Hume proposed to use the experimental method of observation in all the human sciences, without any attempt to deal with ultimate explanations, the use of hypotheses or preconceived theories. In this way, Hume helped to further interest and enthusiasm for the sciences of man which were extensively developed in the nineteenth century (72).

Locke and Berkeley had likewise attempted to justify moral and religious knowledge by equating them to mathematics and science. Hume, though within their tradition and greatly obligated to both (73), was

(70) "It is evident, that all the sciences have a relation greater or less, to human nature; and that however wide any of them may seem to run from it, they still return back by one passage or another. Even Mathematics, Natural Philosophy and Natural Religion, are in some measure dependent on the science of Man; since they lie under the cognizance of Man, and are guided of by their powers and faculties." Treatise, Introduction, p. xix.

(71) Hume, it seems hoped to be the Newton of mental science. For Newton's influence on Hume, see Norman Kemp Smith, The Philosophy of David Hume, A Critical Study of its Origins and Central Doctrines, London: MacMillan Co., 1941, Chapter III, "The Influence of Newton and of Locke," pp. 53-78.

(72) "Hume wrote them as a builder of the Science of Human Nature, and so it was as such that he exercised his main influence on his successors. He encouraged in men the hope of the successful extension of scientific progress to human nature itself; thus might come a reasoned, definitive and accurate answer to all problems -- religious, moral, social and political...Here was a rationalist indeed: the founder, with Butler and Hartley, of that belief in and enthusiasm for the Science of Human Nature which was so important in the philosophy and in the practical politics of the nineteenth century." C. E. Morris, Locke Berkeley Hume, Oxford: The Clarendon Press, 1931, p. 113.

(73) "The spiritual philosophy of Berkeley in Ireland was thus a development in one direction of elements latent in Locke's Essay. The next succeeding evolution of philosophy in these islands, of historical importance, occurred in Scotland, in an opposite direction. Hume's Treatise of Human Nature and his Inquiry Concerning Human Understanding, were published half a century after the Essay; the one some years before, and the other rather later than Siris. Locke's Essay was the new philosophical influence at work at Edinburgh in Hume's youth, and the negative side of Berkeley's new conception of the world of the senses was engaging attention there and then, as a sceptical paradox. Hume awoke into intellectual life in this atmosphere, with a natural disposition to doubt, and to apply sceptical paradox to the prevailing philosophy, as he found it in Locke's Essay. His agnostic criticism emptied the Essay of most of its fundamental elements, and in particular banished the 'propositions of real existence' that Locke took as presupposed in all knowledge by means of ideas." Alexander Campbell Fraser, editor, An Essay Concerning Human Understanding by John Locke, Vol. I, p. cxxxiv.

dissatisfied with their account and initiated a fresh start. Treating the mind as an object is treated in physical science and carefully analysing its contents, he reached a surprising and sceptical conclusion: all knowledge and science were explicable by the impressions of sense associated through custom or habit.

In order to hold to his original contention that the richness and complexity of scientific knowledge was due solely to the habitual association of primitive perceptions (74), and that no sciences go beyond experience (75), Hume was forced to do violence to the notion of abstraction. Hume quoted the opinion of Berkeley, "a great philosopher," approvingly;

...all general ideas are nothing but particular ones, annexed to a certain term, which gives them a more extensive signification, and makes them recall upon occasion other individuals, which are similar to them (76).

Abstract ideas were not universal or conceptual; their nature is such that they are

...in themselves individual, however they may become general in their representation. The image in the mind is only that of a particular object, tho' the application of it in our reasoning be the same, as if it were universal (77).

(74) "All the perceptions of the human mind resolve themselves into two distinct kinds, which I shall call impressions and ideas." Treatise, Bk. I, Part I, Sect. I, p. 1.

(75) "But if this impossibility of explaining ultimate principles should be esteemed a defect in the science of man, I will venture to affirm, that 'tis a defect common to it with all the sciences, and all the arts, in which we can employ ourselves, whether they be such as are cultivated in the schools of the philosophers, or practised in the shops of the meanest artisans. None of them can go beyond experience, or establish any principles which are not founded on that authority." Treatise, Introduction, p. xxii.

(76) Treatise, Book I, Part I, Sect. VII, p. 17. Hume was indebted to Locke for the principle that all our knowledge comes from experience and the concomitant problem of reconciling this principle with the doctrine of the generality of ideas. Locke concluded with the inconsistency of abstract ideas which Berkeley denied; Hume follows the position of Berkeley.

(77) Ibid., p. 20.

Hume then claimed that a repetition of experience is associated by custom and given a name. The utterance of that name "revives that custom" we have acquired and one or more of these originally experienced ideas. The mind then separates or partially considers those ideas which are demanded by occasions and by the purpose of life.

Hume considered mathematics and science under the complex ideas of relation (76). Complex ideas arose from simple ideas through the law of association, which Hume compared to the idea of attraction in physics (79). Of the seven types of relations (80) he admitted, resemblance, contrariety, degrees in quality and quantitative or numerical proportion constituted the realm of mathematics and science. These relations depended solely on ideas, and could be the objects of knowledge and certainty. The first three of these relations "are discoverable at first sight, and fall more properly under the province of intuition than demonstration (81)." It is under the fourth, proportions in quantity or number, that Hume considers mathematics.

(76) John Laird, Hume's Philosophy of Human Nature, London: Methuen and Co., 1932, Chapter IV, "Space, Time and External Existence," pp. 64-73.

(79) "Here is a kind of attraction, which in the mental world will be found to have an extraordinary effects as in the natural, and to show itself in as many and as various forms. Its effects are every where conspicuous; but as to its causes, they are mostly unknown, and must be resolv'd into original qualities of human nature, which I pretend not to explain. Nothing is more requisite for a true philosopher, than to restrain the intemperate desire of searching into causes, and having establish'd any doctrine upon a sufficient number of experiments, rest contented with that, when he sees a farther examination would lead him into obscure and uncertain speculations." Treatise, Bk. I, Part I, Sect. IV, pp. 12-13.

(80) The seven relations which Hume distinguishes are divided into two classes. The first comprises the four listed above. The second class includes identity, relations of time and place and causation. Treatise, Bk. I, Part III, Section I, pp. 69-73.

(81) Ibid., p. 72.

In his treatment of mathematics, Hume distinguished between geometry and arithmetic. In his view, geometry is but an empirical doctrine, founded on observation of the rough appearances of physical facts;

...geometry...tho' it much excels, both in universality and exactness, the loose judgments of the senses and imagination; yet never attains a perfect precision and exactness. Its first principles are still drawn from the general appearance of the objects; and that appearance can never afford us any security, when we examine the prodigious minuteness of which nature is susceptible (82).

Geometry lacks exactness; we only approximate the abstractness of its figures or propositions.

When geometry decides anything concerning the proportions of quantity, we ought not to look for the utmost precision and exactness. None of its proofs extend so far. It takes the dimensions and proportions of figures justly; but roughly, and with some liberty. Its errors are never considerable; nor wou'd it err at all, did it not aspire to such an absolute perfection (83).

The ideas then, of perfect lines, figures and angles have no real existence. In pretending to use abstract ideas, mathematicians are like philosophers who use them as a means to "cover many of their absurdities (84)."

Hume viewed arithmetic and algebra differently, for exactness and certainty could be achieved in these subjects. In them

We are possess'd of a precise standard, by which we can judge of the equality and proportions of numbers; and according as they correspond or not to that standard, we determine their relations, without any possibility of error. When two numbers are so combin'd, as that the one has always an unite answering to every unite of the other, we pronounce them equal; and 'tis for want of such a standard of equality in extension, that geometry can scarce be esteem'd a perfect and infallible science (85).

(82) Ibid., pp. 72-73.

(83) Ibid., Bk. I, Part II, Section IV, p. 45.

(84) Ibid., Bk. I, Part III, Section I, p. 72.

(85) Ibid., p. 71.

Hume seems to have held this view of the exactness of arithmetic and algebra because of his psychological atomism which proposed that each element of conscious experience is preserved as a unit. In Hume's view, geometry, arithmetic and algebra, in keeping with all demonstrative sciences, contained a fair amount of probability. No algebraist or mathematician is so expert or so confident, however, that he immediately believes truth on its discovery; he regards it rather as a mere possibility which gradually increases towards a certainty that is never perfectly attained (86).

(86) Ibid., Bk. I, Part IV, Section I, pp. 180-187.

Chapter VI. Idealism and Positivism.

I. Kant.

To Immanuel Kant (1724-1804) it appeared that Descartes, Locke and their followers had been seduced in metaphysics, for they had unsuccessfully attempted to introduce the method and abstractive processes of mathematics and science into philosophical research (1). The accent being laid, in the Cartesian tradition, on pure mathematics, various forms of idealism and rationalism ensued. Under Newton's influence, the British philosophers had, instead, stressed the physical and observational aspect of applied mathematics, concluding with empiricism, positivism and scepticism. Kant seconded the high regard with which both schools had looked upon mathematics and science, whose success made them self-justified. Consequently he directed his philosophical efforts towards demonstrating how they were true and at ascertaining what felicitous functionings of reason enabled them to reason so successfully (2). On the other hand, metaphysics, championed

(1) See *Kritik der reinen Vernunft*, "Vorrede zur zweiten Auflage," Band I, pp. 23-44, passim, and *Prolegomena zu einer jeden künftigen Metaphysik*, "Vorrede," Band II, pp. 1-11 of *Immanuel Kant's Sämtliche Werke*, Herausgegeben von F. Geden W. Kinkel, J. H. V. Kirchmann, F. E. Schiele, Th. Valentiner, K. Vorländer, 8 Bänden und Supplement-Band, Leipzig: Dörff'schen Buchhandlung, 1901-1905.

(2) "Or, il est frappe par un double fait. D'un cote, on trouve les sciences, au sens moderne du mot: les mathematiques, la geometrie et la physique mathematique qui, avec l'astronomie de Newton, s'imposaient dorénavant sans conteste: de celles-ci, la valeur es consacree par le succes, c'est a dire par l'accord de tous les penseurs, et le progres constant de leurs decouvertes. De l'autre cote, on voit la metaphysique au sens devenu traditionnel depuis Descartes: c'est a dire l'etude des trois principales substances: le monde (cosmologie), l'ame humaine (psychologie) et Dieu (theologie); et celle-ci n'est au contraire qu'une perpetuelle succession de systemes contradictoires qui s'entredétruisent." P.-J. Thomard, *Prezis d'Histoire de la Philosophie*, p. 623.

alternately by idealism and empiricism, had been embroiled in successive contradictory systems and had seemingly failed in the pursuit of truth. Kant aimed, then, to name the constitutive elements of mathematics and science whose certainty was unquestionable and to define the value and sphere of metaphysics by properly articulating the extreme of rationalism as typified in the Leibniz-Wolff philosophy with that of Hume's empiricism (3).

Kant's philosophy of transcendental criticism -- in which the respective roles of reason and experience were criticized and then reunited in a transcendent synthesis -- was formed over a number of years. His early career was marked by the study of Newtonian science and rationalist philosophy (4). In his "Thoughts on the True Estimate of *Vis Viva*" of 1747, he entered the current dispute of the Cartesians and Leibnizians with regard to the proper expression for the amount of a force. Newton's influence led to his "General History and Theory of the Heavens" of 1755, in which he applied Newton's conception of the solar system to the sidereal universe (5). His philosophical training during this period was that of the Leibniz-Wolff tradition, which presented Leibniz's philosophy

(3) For a short summary of Kant's relations to his predecessors and contemporaries, see William Wallace, Kant, Edinburgh and London: William Blackwood and sons, 1892, Chapter X: "The Philosophical Environment of Kant," pp. 156-158.

(4) For Newton's early influence on Kant see Joseph Maréchal, Le Point de Départ de la Métaphysique, Cahier III: La Critique de Kant, troisième édition: Paris: Desclée de Brouwer, 1944, pp. 37-39.

(5) The full title of this work is "General History of Nature and Theory of the Heavens: an Essay concerning the Constitution and the mechanical origin of the Universe treated on Newtonian Principles." See W. Hastie, Kant's Cosmogony: As in his Essay on the Retardation of the Rotation of the Earth and his Natural History and Theory of the Heavens, with Introduction, Appendices and a Portrait of Thomas Wright of Durham, edited and Translated by W. Hastie; consult "Translator's Introduction," for the influence of Newton and of science on Kant.

as a science of pure possibility (6). Kant was much concerned, in this early period, with attempts to compromise Newton and the Leibniz when he knew (7).

A gradual break with the Wolffian philosophy and a tendency towards Hume's empiricism marked the second period of Kant's formation. He wrote a number of treatise concerned with the question that was to become capital in his future speculations, the relation of mathematics and science to metaphysics. Noteworthy among these is the essay he submitted to the contest sponsored in 1763 by the Berlin Academy of Sciences on the question, "Is metaphysics capable of a certainty equal to mathematics?" In this essay, "On the Evidence of the Principles of Natural Theology and Morals," Kant censured those thinkers who had mistakenly attempted to make metaphysics like mathematics (8), claiming that they had misunderstood the nature of metaphysics, physics and mathematics. Mathematical certainty was due mainly to the mind; the mathematician begins with axioms and definitions, forming concepts by synthetic construction in a special intuition. Past physics had been too a priori; by beginning with postulates and definitions, it had attempted to model itself too strictly on mathematics. Recent physics, especially that of Newton, had begun rightly with observation and quantitative measurements. Physical method is more akin

(6) For Wolff's development of Leibniz's philosophy, see Edward Caird, The Critical Philosophy of Immanuel Kant, two volumes, Glasgow: James Maclehose & Sons, 1889, Volume one, pp. 84-104.

(7) A. D. Lindsay, Kant, London: Ernest Benn, 1934, Chapter II, "Kant's Pre-critical Writings and his relation to his predecessors."

(8) Joseph L. Schaef, op. cit., pp. 66-76.

to metaphysics, for the metaphysician must begin with such concepts as body, soul, space and time, which were complex items of experience. Mathematics, then, was synthetic; while physics and metaphysics were analytic (9). In this essay Kant had little hope for the future of metaphysics -- no one had as yet written a metaphysic; metaphysics must change its nature in order to attain truth.

Kant's inaugural dissertation of 1770 marked his first definitive synthesis between rationalism and empiricism (10). In this work, "A Dissertation on the Form and Principles of the Sensible and Intelligible World," Kant reaffirmed his faith in and hope for metaphysics (11).

(9) "Thus -- at the outset at any rate -- mathematics and philosophy proceed in opposite ways. The method of mathematics is synthetic or direct from the first, inasmuch as it provides its own definitions and proceeds straightway to formulate its axioms, and from these and its definitions to deduce their consequences. The method of metaphysics, on the other hand, must be analytic or inverse so long as its definitions and axioms are still to seek." James Ward, A Study of Kant, Cambridge: the University Press, 1922, p. 22.

(10) "It has often been noticed that Kant, from the beginning to the end of his career, shows a tendency to seek for some middle term or higher reconciling principles between opposite schools of thought. 'We are in a way defending the honour of human reason, when we reconcile it with itself in the persons of different writers of high intelligence, and discover the truth, which by such men is never entirely missed, even in their contradictory utterances.' This sentence quoted from the first published essay of Kant, in which he endeavors to find a way of combining the different views of Descartes and Leibniz upon vis viva, gives the keynote of most of his subsequent writing. To mediate between Leibniz and Newton was the aim of his first philosophical essays; to mediate between the English Empiricism and the German Rationalism may be said to be one of the main objects of the Critical Philosophy." Edward Caird, op. cit., Volume I, p. 161.

(11) Kant's interest in metaphysics was partly renewed by the reading of Leibniz's Nouveaux Essais, which had recently been published; see A. D. Lindsay, op. cit., pp. 29-32. On Kant's relation to Leibniz, see Robert Latta, op. cit., appendix I, "Kant on his Relation to Leibniz," pp. 208-211.

Maintaining that "sensitive cogitata esse rerum representationes uti apparent, intellectualia autem sicuti sunt" (12), Kant distinguished the phenomenal world of the passivity of sense from the noumenal world of the activity of intelligence. Within each, he subdistinguished matter and form. Though sensations were the matter of sensible knowledge, they were received in a certain spatial and temporal ordering (the intuitions of his latter Kritik) (13). With respect to active, intelligible knowledge, Kant spoke of a logical and a real use of intelligence. In its logical use, intelligence actively organised the appearances of sense into empirical knowledge as in physics for the external senses, and as in psychology for the internal sense. The real use of intelligence occurred in metaphysics which dealt with the intelligible world proper, the noumenal realm. The matter of the noumenal world lay in its ultimate, simple parts, the laws of our mental activity in relation to experience; the form was the interaction between the substances of which the noumenal world was constituted. Metaphysics was concerned with noumenal or ^{the} real world, science with the phenomenal world.

(12) Kant, Werke, "De Mundi Sensibilis atque Intelligibilis Forma et Principiis," Supplement-Band, p. 107, see also A.D. Lindsay, op. cit., pp. 44-45.

(13) "Il y a une autre question préliminaire, plus difficile à résoudre: c'est de savoir quelles sont les sciences que Kant a considérées comme faisant partie de la Mathématique pure, et quel est leur rapport aux deux formes a priori de la sensibilité qui en sont selon lui le fondement. La pensée de Kant est singulièrement flottante sur ces deux points, pourtant essentiels. Dans la Dissertatio de 1770, l'espace était l'objet de la Géométrie, le temps celui de la Mécanique pure; et ces deux sciences faisaient partie de la Mathématique pure." Louis Couturat, "La Philosophie des Mathématiques de Kant," Revue de Métaphysique et de Morale, 12(1904), n. 335.

Eleven years later, in 1781, Kant published the Kritik der reinen Vernunft, which embodied his ultimate synthesis concerning the possibility of speculative truth in metaphysics, mathematics and science (14). In this work Kant was not concerned with a doctrine of abstraction as the clue to the nature of science and metaphysics; in fact, he used the word "abstraction" to designate mental constructs having no absolute reference to things or their appearances. Kant was concerned, instead, with the nature of judgments, their origin and role in acquiring truth. Dismissing analytical judgments as mere tautologies for they "add nothing to the notion of the subject" (15), Kant limited the Kritik to synthetic judgments which "add to the subject a predicate not at all thought in it" (16); only these judgments were ampliative and increased knowledge (17).

(14) "La question fondamentale de la Critique de la Raison pure est: 'Comment des jugements synthétiques a priori sont-ils possible?' Qu'il existe de tels jugements, c'est ce dont Kant ne doute pas un instant, car ce sont de tels jugements qui constituent, selon lui, la métaphysique et la mathématique pure. Expliquer comment ces jugements sont légitimes en mathématique et illégitimes en métaphysique, tel paraît être le but de la Critique de la Raison pure; tel est en tout cas l'objet de la Methodologie transcendentale." Ibid., p. 321; see also Joseph Maréchal, Le Point de Départ de la Métaphysique, Cahier III, pp. 87-92.

(15) Kant, Werke, Kritik der reinen Vernunft, "Von dem Unterscheide Analytischer und synthetischer Urtheile," Band I, p. 25.

(16) Ibid., p. 66.

(17) "The belief in the power of analysis had been the result of reflexion on mathematical processes. The certain assurance in that field of new knowledge, apparently got by analysis, made analytical reasoning appear the true type of all a priori knowledge, Kant, however, insists that mathematical discovery is not a mere matter of analysis. It always involves a synthetical element: even the result of a simple sum is not got by analysis, but by counting or some kind of construction. He challenges, therefore, the rationalist assumption that a priori judgments are necessary analytic." A. D. Lindsay, op. cit., pp. 58-59.

Not all synthetic judgments were alike. Synthetic a priori judgments, concerned with accidental and contingent material, lack the notes of universality and necessity proper to true, scientific knowledge. These latter could not furnish universal and necessary truth. Hence the scientific character of these judgments arose from a source prior to experience; a pure contribution of the mind. They were a synthesis of the matter of sense data and the form which the mind contributed to these shapeless data of sense. Kant then proposed that a true theory of scientific knowledge had to inquire into the nature of this double contribution in sense perception (Transcendental Aesthetic), in pure understanding (Transcendental Analytic), and in pure reasoning (Transcendental Dialectic). Kant's aim, to show how synthetic a posteriori judgments were possible, was equivalent, in his eyes, to solving the question of the nature of metaphysics and demonstrating the possibility of mathematics, and science. The conclusion of the Kritik and of the Kantian synthesis was that only mathematics and science, which dealt with the phenomenal world, were marked as genuine science; metaphysical speculations was relegated to the realm of practical necessity, its objects -- God, nature and the world -- were merely noumenal unknowable realities which averted the absolute doom of metaphysics, morals and religion forecasted in the closing pages of the Kritik der reinen Vernunft.

The first part of Kant's Kritik der reinen Vernunft contains his non-abstractive solution to the nature of mathematics and science under

the title of Transcendental Aesthetic (18). While he regarded Newton's and Leibniz's notions of space and time as inadequate, he held fast to the finality of Newtonian physics (19) and was largely influenced by the results of Leibniz's mathematicism (20). Using arguments from geometry and arithmetic, Kant proposed that space and time were but

(18) "If we take the Aesthetic by itself, we may view it as directed against Scepticism, in so far as it shows that the determinations of our space and time are not discovered by analysis but developed by synthesis, and that they are not pure 'relations of ideas' but have objective reality. At the same time, as it proves that this a priori synthesis has reference to the forms of sense, it shows that the objective reality in question can be regarded only as phenomenal, and that, therefore, no inference can be drawn from the objective value of mathematical science to support the idea of the possibility of knowing things in themselves. The ultimate result of the Aesthetic, therefore, seems to fall on the side of Scepticism, in so far as it confines knowledge to phenomena, or at least gives no encouragement to the idea of its extension beyond phenomena; and on the side of Dogmatism, in so far as it proves that mathematical truth is due to a a priori synthesis, and yet objective. Edward Caird, op. cit., Volume I, p. 229.

(19) "He happily began his work by appropriating all the mathematical and physical science of his age, and he made it the stable foundation and criterion of all his subsequent thinking. He was a faithful disciple of Newton, to whose principles and method he owed most of his formative power. He laid it down that the 'genuine method of Metaphysics is one and the same in principle with that which Newton introduced into physical science.'" W. Hastie, op. cit., p. xvii.

(20) "From Leibniz also Kant derived the distinction between knowledge of things in themselves, which is given in pure intellectual concepts and knowledge of phenomena, and the conception that space and time are concerned with phenomena not with things in themselves." A. D. Lindsay, op. cit., p. 29.

the forms in which all appearances of things were manifested; they were were the subjective conditions of sensibility which rendered external and internal sensation possible (21). To be raised to the distinctive level of judgments, these resultants of sensibility must receive the character of universality and necessity from the a priori forms of intelligibility supplied by conception.

The process of conferring intelligibility on sensibility through the twelve categories -- the a priori forms of pure understanding yielding judgments -- was demonstrated in the Transcendental Analytic (22). In this realm, pure understanding mediated between the categories and space. Time was the schemata according to which the understanding functioned, each category possessing its own schemata which was somehow related to time and resulted in judgments (23).

In the Transcendental Dialectic Kant explained how the final unity of judgments arose and how the definitive stamp of science was given to mathematical and scientific knowledge. Pure reason was given this task through the three a priori forms, the ego, the world and God. These ideas did not contribute to, but merely regulated knowledge. With the aid of these three, reason forms the three sciences of psychology, cosmology and theology. It ultimate

(21) Kant, Werke, Kritik der reinen Vernunft, "Von der Zeit," Band I, pp. 91-95.

(22) "When we go to the Analytic...it is there shown that the a priori forms of conception cannot, any more than the a priori forms of perception, supply an instrument by which we may reach any knowledge of reality not given in sense; while they do supply, and are needed to supply, principles for empirical knowledge, i. e. for knowledge of phenomenal reality." Edward Caird, op. cit., Volume I, p. 230

(23) Louis Couturat, op. cit., pp. 348-352

referred all phenomena to the ego as their unknowable common ground, seeing all external events and phenomena as parts of a systematized, inexplicable universe, and thinking all things to be the result of a supreme, unknowable intelligence. The temptation to consider the three ideas as constitutive and not merely regulative of experience had been succumbed to by philosophers in the past. By the paralogisms of pure reason Kant attempted to show the errors of such views.

Thus the Kritik der reinen Vernunft concluded with the unknowability of the noumenal world for metaphysics and the quasi-knowledge of things as they appear in mathematics and science. Kant devoted his second critique, the Kritik der praktischen Vernunft, to show how the truths which he removed from the province of speculative reason, the freedom of the will, the immortality of the soul and the existence of God, arose; the obligatory weight of the moral law was the basis for their acceptance. In the third critique, Kritik der Urtheils Kraft, Kant employed the faculty of aesthetic experience as the mediating link between the abstract conceptions of pure reason and the practical view of the moral order in practical reason.

Kant had begun with this problem: the miserable plight of metaphysics as alternately subjected to the strains of dogmatism or empiricism and the contrasting success of mathematics and science. History, he saw, revealed that applying mathematical and scientific procedure to the spiritual world yielded barren, sceptical results. The only conclusion he could validate was to credit mathematics and science with knowledge of the appearances of sense, and to deny the possibility of speculative, metaphysical knowledge of the nature of noumenal things. In the Kritik

der reinen Vernunft (24), by comparing the method of mathematics with that of metaphysics, Kant showed why the method of the one did not obtain in the other. First, mathematics began with axioms or synthetic a priori principles (25), an initial procedure which was impossible for philosophy. Second, mathematical definitions were created by an arbitrary synthesis and thus were indisputable. Contrarily, were we even to attempt definition in philosophy, there was no assurance that a definition of an empirical or pure concept was adequate, since these definitions never revealed the noumenal thing itself; nor can philosophical concepts as substance or causality be necessarily and universally known either in perception or conception, for they were merely relative notions. Third, mathematics alone was capable of incontestable demonstrations; it presented, by its axioms and definitions, the measures of necessity and universality necessary for science. Metaphysics, by opposition, could never produce the intuitive representation of noumenal things in themselves. Metaphysical speculation, then, was radically impossible.

Kant's conception of human nature and the mind -- his primitive, unproved assumptions -- were such as to make true knowledge either in science, mathematics, metaphysics or theology unattainable. It was THE overbearing influence of mathematics and physics on Kant's philosophy

(24) See, in this connection the study of C. D. Broad, "Kant's Theory of Mathematical and Philosophical Reasoning," Proceedings of the Aristotelian Society, New Series, 42(1942), pp. 1-24.

(25) John J. Cooley, "Kant on the Propositions of Pure Mathematics," The New Scholasticism, 11(1937), pp. 140-157.

which helped bring out this deplorable solution (26). His successors, like the fellows of Cartesian idealism and Lockean empiricism, accepted his conclusions as the only instruments with which they could approach the questions of the possibility and of the types of speculative knowledge. The confusion of mathematical, logical, scientific and metaphysical abstraction with each other and their disastrous results in ancient and previous forms of mathematicism led Kant to the same mental impasse: the separation of God from His creation, the bifurcation of the intelligible and sensible realms, the denial of the analogy of being and the theoretical closure of man's efforts to answer the Kantian questions, what can I know? what can I hope for? and what ought I do?

II. Comte

In a more extensive sense than any of his predecessors -- Descartes, Newton, Kant or the English empiricists -- Auguste Comte (1798-1857) was fascinated with the new sciences (27). It is true that Descartes had, by his mechanism, made it fashionable to consider living things as devoid of life. The supposed impossibility of explaining the interaction of matter and spirit -- a problem posed by Descartes -- had engendered a progressive materialisation of psychological study and an increasing predilection to consider mental phenomena as governed by the laws of

(26) Etienne Gilson, The Unity of Philosophical Experience, Chapter IX, "The Physicism of Kant," pp. 227-252.

(27) For an analysis of Comte's reverence for the sciences and his attitude to them, see pp. 7-17 of Gouhier's introduction to Auguste Comte: Œuvres choisies (avec une introduction par Henri Gouhier, Paris: Aubier, 1948).

physical science. A host of materialists and pan-phenomenalists arose, in most of whom, however, strains of metaphysics were still discernible. The English empiricists, stirred by the ascendancy of mathematical and physical science, attempted to safeguard religious and moral knowledge by emulating the methods of science. Kant's strictures on intellectual knowledge, and his favorization of physics, mathematics and the exact sciences of experience, canonized this general trend. But all these multiple movements towards making knowledge mathematical and scientific were encompassed in the positive philosophy of Auguste Comte (28).

Comte had achieved some show of brilliance in mathematics when but a young man, and later earned his daily bread by teaching mathematics both privately and at the Ecole Polytechnique. His association as student and secretary to Saint-Simon gave him a vigorous impulse towards the social sciences. Disagreeing with the notes of urgency and immediacy that Saint-Simon connected with his social reform program, Comte left him to establish his own school of thought. Both Joseph De Maistre and Condorcet contributed greatly to the central idea of his positive philosophy: the gradual replacement of dogma and metaphysics by a unified mathematical science (29).

Comte's Cours de Philosophie Positive (30) embodies this new

(28) See Etienne Gilson, The Unity of Philosophical Experience, Chapter I, "The Sociologism of A. Comte," pp. 253-276, for a fine summary on Comte.

(29) Later in life Comte entered a "mystical" period. He aimed to unify not only human thought, but the entirety of human existence under his positive religion which is expounded in his Système de Politique Positive. See Henri Gouhier, op. cit., pp. 17-31.

(30) All quotations are taken from Cours de Philosophie Positive par Auguste Comte, Quatrième édition augmentée de la Préface d'un disciple et d'une étude sur les Progrès du positivisme par F. Littré, Six volumes, Paris: Librairie J.-B. Baillière et Fils, 1877.

doctrine in the form of sixty lectures. It opens with some brief considerations on the progress of the human race as exemplified by his law of the three stages (31). In the first, or theological era, man sought for an absolute and transcendent meaning of nature, richly indulging in the use of the imagination. This synthetic view proposed by theology had been, for a time, both successful and useful. Gradually the second or metaphysical stage, searching yet for an absolute though immanent answer to man's queries, supplanted the theological. It likewise was characterized by vain and imaginative solutions, but it proved itself to be neither resourceful nor satisfactory. The third, or final era, is the positive. Abandoning the quest for an absolute, it is content with the relative, and supplants the use of imaginary fictions by an increasing use of observation and by its growing collection of the facts of experience. The inner incompatibility of the three stages is not absolute. They can coexist in a single era or in a single man (32); more advanced peoples, however, will be recognisable by their ascription to positivistic doctrines, of which Comte proclaimed himself the high priest and to which he invited disciples.

To the question, what is positive philosophy? Comte answers that

(31) Cours, Vol. I, pp. 8-11.

(32) "Chacun de nous, en contemplant sa propre histoire, ne se souvient-il pas qu'il a été successivement, quant à ses notions les plus importantes, théologien dans son enfance, métaphysicien dans sa jeunesse, et physicien dans sa virilité? Cette vérification est facile aujourd'hui pour tous les hommes au niveau de leur siècle." Cours, Vol. I, pp. 11-12.

le caractère fondamental de la philosophie positive est de regarder tous les phénomènes comme assujettis à des lois naturelles invariables, dont la découverte précise et la réduction au moindre nombre possible sont le but de tous nos efforts, en considérant comme absolument inaccessible et vide de sens pour nous recherche de ce qu'on appelle les causes, soit premières, soit finales (33).

This is the originality, then, of the Comtian doctrine: a) the negation of philosophical abstraction or the search for final and first causes; b) the affirmation of the basic identity of all sciences. All other problems are to be regarded as insoluble and left to the imagination of the theologians and the subtleties of the metaphysicians (34).

Six sciences comprise the corps of positivist doctrine. They are based on Comte's law of the encyclopedic division of the sciences, that of decreasing generality and increasing complexity. Mathematics is the highest and the greatest science and knows bodies as such (35). It has the simplest objects and the most universal laws; it functions as the universalizing and unifying doctrine. The other sciences merely consider special groups of bodies; astronomy deals with the stars, physics with our planet, chemistry with terrestrial and mineral bodies, and physiology with organized or living bodies. Sociology, or social physics, deals with human facts; its objects are the most complex, and its laws, the least universal. The order listed above is, Comte claims, both a rational order and the order of the historical appearance of these sciences.

(33) Course, Première Leçon, Vol. I, p. 16.

(34) Ibid., p. 17.

(35) Comte definitely excludes metaphysics from his purview, and attributes a metaphysical knowledge to mathematics; Etienne Gilson, op. cit., pp. 261-266.

The aims of positive philosophy, a special and general one, are inseparable. The special aim, "le plus grand et le plus pressant besoin de notre intelligence", is to terminate the system of observation^{at} sciences by founding social physics (36). The general aim identifies the Cours as one of positive philosophy; it is to inculcate a knowledge of general scientific relations and laws, to coordinate previous knowledge as branches of the great tree of positivistic science (37). This ambitious program warrants a division of labor among scientists, for which Comte pleads by pointing out the salutary effects to the positive program for society: the general laws of logic will be clarified; a needed reform in education will ensue, and the six sciences will be studied in the proper order of procedures from the simple to the complex; the progress of positive science will be promoted; a social reorganization will originate (38).

The unifying science, in the Comtian scheme, and the ideal to which the others were to be made conformable, was mathematics. Comte had really aspired to constitute an objective synthesis of the sciences through mathematics; but, as he complains, an insufficient development of positivistic philosophy made him be content with a subjective,

(36) "Maintenant que l'esprit humain a fondé la physique céleste, la physique terrestre, soit mécanique, soit chimique; la physique organique, soit végétale, soit animale, il lui reste à terminer le système des sciences d'observation en fondant la physique sociale." Cours, p. 22.

(37) Ibid., p. 23.

(38) "La grande crise politique et morale des sociétés actuelles tient, en dernière analyse, à l'anarchie intellectuelle...le désordre actuel des intelligences tient, en dernière analyse, à l'emploi simultané des trois philosophies radicalement incompatibles: la philosophie théologique, la philosophie métaphysique et la philosophie positive." Ibid., p. 41.

personal synthesis. His mathematicism, then, is evidently a bold and extensive program. It is the most ambitious of all types of mathematicism, uniquely envisaging all orders of truth as comprised under the fundamental notions of mathematical abstraction: figure, movement, quantity and number. Only in mathematics do we achieve universality and accuracy. It renders precise and exact the indefiniteness of observable phenomena (39).

Before presenting his important division of mathematics as concrete and abstract, Comte considers mathematics in general. It is to be viewed as "n'étant autre chose qu'une immense extension admirable de la logique naturelle à un certain ordre de déductions (40)." Comte pays warm tributes to its unity and its ancient character, but is piqued at its former definitions and divisions, vague and insignificant as they were. The first problem, then, since mathematics is now sufficiently perfected, is to coordinate its parts and to prepare it for progress.

Mathematics has been ill-defined as the "science des grandeurs" or "la science qui a pour but la mesure des grandeurs" (41); for

(39) "Dans l'état actuel de développement de nos connaissances positives, il convient, je crois de regarder la science mathématique, moins comme une partie constituante de la philosophie naturelle proprement dite, que comme étant, depuis Descartes et Newton, la vraie base fondamentale de toute cette philosophie, quoique, à parler exactement, elle soit à la fois l'une et l'autre. Aujourd'hui, en effet, la science mathématique est bien moins importante par les connaissances, très-réelles et très-précieuses néanmoins, qui la composent directement, que comme constituant l'instrument le plus puissant que l'esprit humain puisse employer dans la recherche des lois des phénomènes naturels." *Ibid.*, Deuxième Leçon, p. 36.

(40) *Ibid.*, p. 87.

(41) *Ibid.*, Troisième Leçon, p. 90.

"cet aperçu scolastique" is only basically correct. It merely implies a direct and naive comparison of object to object-measured. Such measurement is often impossible, say, in the case of celestial distances. From this quasi-impracticability, Comte claims, an impetus was given to form the mathematical sciences by instituting a search for indirect measurement; this solution was found by mathematicians. They substituted indirect measures which are directly measurable; then, by diagnosing the relations between the indirect but known measures, the direct and unknown measures could be found. In this way mathematics supplied the power for the measurement of all of nature, and its exact definition is given, "en lui assignant pour but la mesure indirecte des grandeurs, et disant qu'on s'y propose constamment de determiner les grandeurs les unes par les autres, d'après les relations précises qui existent entre elles" (42). But this definition, Comte argues, is equivalent to the Greek ideas of science in general, and not the modern designation of mathematics as "la science par excellence (43)." For science has no other purpose but to determine the relations of phenomena and coordinate facts (44). Mathematical research,

(42) "Cet enseignement, au lieu de donner seulement d'idées d'un art, comme le font jusqu'ici toutes les définitions ordinaires, caractérise immédiatement une véritable science, et la montre sur-le-champ composée d'un immense enchaînement d'opérations intellectuelles, qui pourront évidemment devenir très-complicées, à raison de la suite d'intermédiaires qu'il faudra établir entre quantités inconnues et celles qui comportent une mesure directe, du nombre des variables coexistantes dans la question proposée, et de la nature des relations qui fourniront entre toutes ces diverses grandeurs les phénomènes considérés." Ibid., p. 88.

(43) Ibid., p. 98.

(44) "...de déterminer des phénomènes les uns par les autres, d'après les relations qui existent entre eux. Toute science consiste dans la coordination des faits; si les diverses observations étaient entièrement isolées, il n'y aurait pas de science. On peut même dire que la science est essentiellement destinée à dispenser, autant que le comportent les divers phénomènes, de toute observation directe, en permettant de déduire du plus petit nombre possible de données immédiates le plus grand nombre possible de résultats." Ibid., p. 99.

in fact, deals with both the quantitative and the qualitative in a higher and more perfect way than each lower science considers its sphere of objects. It is the perfect science and is the highest abstraction attainable in the positivist era (45).

Comte gives a twofold division of mathematics into abstract and concrete. This principal division (the secondary division is into astronomy, physics, chemistry, physiology and social physics) is seen to arise from the consideration of any mathematical problem. The order of research constituting concrete mathematics aims to know the precise relations which hold between quantities under consideration. Once this has been accomplished, abstract mathematics is simply a question of determining the unknown numbers from the precise relations of the known. For example, Galileo used concrete mathematics to establish the law of falling bodies; once he had found the law, then, knowing the height of the body, he could learn the unknown time of fall, or vice versa. Both branches, Comte concludes, are equivalent in point of difficulty and understanding (46).

Concrete mathematics is the basis of all science. It considers the phenomena of the universe either in their static aspect as general geometry, or in their dynamic aspect as in mechanics. Both branches of concrete mathematics aim to discover the equations which unify phenomena, and they comprise

(45) "C'est donc par l'étude des mathématiques, et seulement par elle, que l'on peut se faire une idée juste et approfondie de ce que c'est qu'une science. C'est la uniquement qu'on doit chercher à connaître avec précision la méthode générale que l'esprit humain emploie constamment dans toutes ses recherches positives...C'est là également que notre entendement a donné les plus grandes preuves de sa force, parce que les idées qu'il y considère sont du plus haut degré d'abstraction possible dans l'ordre positif. Toute éducation scientifique qui ne commence point par une telle étude pèche donc nécessairement par sa base." *Ibid.*, pp. 99-100, (italics inserted).

(46) *Ibid.*, p. 102.

the two fundamental sciences (47). Concrete mathematics, under both aspects is basically and essentially experimental, founded on direct observation. The simplicity of the phenomena which it considers, yields a high degree of systematisation, "qui a pu quelque fois faire reconnaître le caractère expérimental de leurs premières principes (48)." Unlike abstract mathematics, concrete mathematics necessarily varies its laws when new phenomena are introduced. It is, therefore, special, and immediately concerned with phenomena. In its present development, however, the program of concrete mathematics has only been completely applied to astronomical phenomena and to a part of terrestrial physics.

Abstract mathematics is general in nature. It is purely logical, rational, consists of "une série de deductions rationnelles plus ou moins prolongées" (49), and is composed of what is called the calculus or mathematical analysis. The terms of concrete mathematics, equations, are its point of departure, and abstract mathematics has for its purpose "déduire toujours les valeurs des quantités inconnues par celles des quantités connues (50)." If we compare abstract to concrete mathematics, we find that analytical ideas are more abstract, general and simple than geometrical or mechanical ideas (51). Mathematical analysis is, in fact, the basis of positivist knowledge; its ideas are most universal, abstract

(47) "Ainsi la géométrie et la mécanique constituent, par elles-mêmes, les deux sciences naturelles fondamentales, en ce sens, que tous les effets naturels peuvent être vus comme de simples résultats nécessaires, ou des lois de l'étendue, ou des lois du mouvement." *Ibid.*, p. 106.

(48) *Ibid.*, p. 87.

(49) *Ibid.*, p. 104.

(50) *Ibid.*, p. 108.

(51) "Les idées analytiques sont évidemment à la fois plus abstraites, plus générales et plus simples que les idées géométriques ou mécaniques." *Ibid.*, pp. 108-109.

and simple (52). Mathematical abstraction is the highest man is capable of. If one attempts to probe more deeply, he will fall "inevitablement dans les reveries metaphysiques" (53). This high perfection of mathematical analysis is not due, as Condillac and the metaphysicians argue, to the nature of the concise and general signs employed as instruments in reasoning. This, Comte says, is an exaggerated idea. All the great conceptions of analysis, he says, were discovered without recourse to signs; they were due to the extreme simplicity of the ideas of analysis which can be expressed in any kind of signs.

The domain of both concrete and abstract mathematics is, from the logical viewpoint, necessarily and rigorously universal, and ultimately reducible to a question of numbers (54). That all our researches are ultimately pointed to a question of numbers Comte illustrates by an example taken from living bodies -- a class of phenomena least accessible to "l'esprit mathematique (55)." Similar to the Pythagorean conception of a harmony in the body, Comte's illustration holds that in every pathological case the therapeutic questions involved can be envisaged as a mere determination of quantities which are necessary for the normal existence of the organism (56). In practice, Comte admits, the phenomena

(52) "La veritable base rationnelle du systeme entier de nos connaissances positives. Elle constitue la premiere et la plus parfaite de toutes les sciences fondamentales. Les idees dont elle s'occupe sont les plus universelles, les plus abstraites et les plus simples que nous puissions reellement concevoir." *Ibid.*, p. 109.

(53) *Ibid.*, p. 109.

(54) "Car il n'y a pas de question quelconque qui ne puisse finalement etre conue comme consistant a determiner des quantites les unes par les autres d'apres certaines relations, et, par consequent, comme reductible, en derniere analyse, a une simple question de nombres." *Ibid.*, p. 111.

(55) *Ibid.*, p. 112.

(56) "...dans le cas pathologique, n'est-il pas manifeste que toutes les questions de therapeutique peuvent etre envisagees comme consistant a determiner les quantites de tous les divers modificateurs de l'organisme qui doivent agir sur lui pour le ramener a l'etat normal, en admettant, suivant l'usage des peometres, les valeurs nulles, negatives, ou meme contradictoires, pour quelques-unes de ces quantites dans certain cas?" *Ibid.*, p. 112.

are so complex that we are presented with unsurmountable difficulties in using this method, but it is an aim worthy of attainment (57).

Comte's mathematicism and sociologism had a widespread and telling effect on the future study of science, psychology and sociology. Three Comtian directions were followed explicitly: first, we only know sense phenomena and their laws; second, hypotheses can be utilized to coordinate, conciliate and interpret these phenomena, but without seeking into deeper causes; third, the mathematico-scientific emphasis and method was adopted. Biological science, under Darwin and Spencer, was garbed in a positivist interpretation, leading to the theories of socialism and Marxism. A gradual mathematization of psychology resulted in the foundation of the psychophysics of Wundt and the dynamic psychology of Ribot. Finally, the sociology of Durkheim and his school, applied the positive philosophy to Comte's dream of a social physics (58).

(57). "Parce que il y a evidemment impossibilite totale d'obtenir jamais de veritables lois mathematiques, ce n'est pas neanmoins qu'on doive cesser, d'apres cela, de concevoir, en these philosophique generale, les phenomenes de tous les ordres comme necessairement soumis par eux-memes a des lois mathematiques." *Ibid.*, p. 117. ...C'est par les mathematiques que la philosophie positive a commence a se fermer: c'est d'elles que nous vient la methode. Il etait donc naturellement inevitable que, lorsque la meme maniere de proceder a du s'entendre a chacune des autres sciences fondamentales, on s'efforçait d'y introduire l'esprit mathematique a un plus haut degre que ne le comportaient les phenomenes correspondants..." *Ibid.*, p. 122.

(58) For a fine summary of the nature and influence of positivism see F.-J. Thouard, *op. cit.*, Chapitre IV, "Le positivisme," pp. 592-594.

Chapter VII. Contemporary Directions.

Only two other historical periods, that of the Greeks and that of the resurgence of mathematics in the seventeenth to eighteenth centuries, witnessed the close conjunction of mathematics to philosophy that exists today (1). Soon after its birth as a science, mathematics successively ran the gauntlet of Pythagorean realism and Platonic idealism; it became an abstractive science for Aristotle and then passed into the irresponsible hands of mysticism, electionism and scepticism (2). The rise of mathematics in the seventeenth and eighteenth centuries shows similar trends in the philosophical view on mathematics expressed by Fermier, Galileo, Descartes and Leibniz, by the British empiricists and Kant. After this period the philosophy of mathematics was given comparatively little attention by either philosophers or mathematicians; while mathematicians busied themselves with furthering their science and in establishing the calculus on a sound basis, philosophers, for the most part, turned their efforts in the direction of the social and psychological sciences (3).

Of the large number of historical factors responsible for the present self-consciousness of mathematicians, the following can be

(1) John Wesley Young, Lectures on Fundamental Concepts of Algebra and Geometry, New York: The Macmillan Company, 1911, p. 37. See also Abraham A. Fraenkel, "The Recent Controversies about the Foundations of Mathematics," Scripta Mathematica, 13(1947), pp. 17-19.

(2) Leon Robin, La Pensée Grecque et les origines de l'esprit scientifique, Livre IV, Chapitre Premier, "Le Scepticisme," pp. 377-386.

(3) For a summary view of the major philosophical tendencies during this period see P.-J. Thomard, op. cit., Chapitre IV, "Le Positivisme."

signalized as being mathematically the most significant and formative. Already in the eighteenth century, the acute discussions which had arisen on the foundations of the infinitesimal calculus (4) led to difficulties with the theory of infinite series and stimulated a more rigorous approach to mathematics. The gradual rise of various algebras and geometries after an uninterrupted reign of Euclid for thirteen centuries occasioned a further, critical wonder concerning the nature of deductive reasoning and of mathematics itself (5). Weierstrass' quite successful attempt to found mathematical analysis on arithmetical basis, Cantor's introduction of the theory of infinite sets, and Peano's, Frege's and Dedekind's aim (6) to base mathematics on pure logic followed each other in rapid succession (7). Kronecker vigorously objected to these views, especially to those of Cantor, maintaining that all valid mathematics must be expressible in terms of the natural numbers, but his objections were passed over and Cantor's ideas, fruitful for

(4) An excellent study of the part the infinitesimal calculus played in fostering a critical views of mathematics can be found in Carl B. Boyer's The Concepts of the Calculus, already referred to. For a shorter study, see Jean Taton, Histoire du calcul, Paris: Presses Universitaires, 1948, Chapitre V, "L'analyse mathématique," pp. 110-123.

(5) William Cecil Dampier, A History of Science, pp. 451-455.

(6) Richard Dedekind, Essays on the Theory of Numbers: I, Continuity and Irrational Numbers; II, The Nature and Meaning of Numbers, authorized translation of Wooster Woodruff Beman, Chicago: Open Court, 1901. In the preface to the first edition of The Nature and Meaning of Numbers, Dedekind speaks of the number concept as an immediate inference from the laws of thought (p. 31), and of the possibility of bringing our notions of space and time into harmony with the number-domain which is thus created; see pp. 31-40 for this interesting preface.

(7) Leon Brunschwig, Les étapes de la philosophie mathématique, pp. 310-324. See also E. T. Bell, The Development of Mathematics, pp. 301-313, and Pierre-L. Marshal, Histoire de la Géométrie, Paris: Presses Universitaires, 1948, Chapitres VI et VII.

mathematical discoveries as they were found to be, prevailed (8).

With the introduction of the antinomies and paradoxes which Cantor's theory of sets provoked (9), however, discussion became more acute; mathematicians became more severely critical of the nature of mathematical abstraction, and a number of schools of thought arose. Poincaré demanded that all mathematical notions be finitely definable and that definitions be verifiable by a finite number of operations (10). The Paris School, headed by such men as Borel, Lebesgue, Baire and Hadamard, though primarily concerned with functions and measure theory, joined the current philosophical speculations on the finite and the infinite. Russell and Whitehead's Principia Mathematica, ultimately derived from the work of Peano and Frege, attempted to establish the validity and consistency of mathematics on the basis of primitive logical concepts (11). The school of Hilbert, known as formalism, emphasized postulational procedure and attempted to retain the whole of formal mathematics by proving its freedom from contradiction through introduction of the

(8) In the introduction to Contributions to the Founding of the Theory of Transfinite Numbers by Georg Cantor (Translated, and provided with an Introduction and Notes, by Philip E. Jourdain, LaSalle: Open Court, 1947), Jourdain points out the influence of previous work on Cantor, especially that of Weierstrass (pp. 10-23). Jourdain also summarizes Cantor's philosophical attempts towards justifying the introduction of the transfinite numbers (pp. 67-71), and summarizes the development of the theory (pp. 73-82).

(9) See Louis O. Kattsoff, op. cit., pp. 87-92, for a summary account of these antinomies.

(10) H. Poincaré, The Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method, authorized translation by George Bruce Halsted, with a special preface by Poincaré, and an introduction by Josiah Royce, New York: The Science Press, 1929, pp. 448-485.

(11) Bertrand Russell and A. N. Whitehead, Principia Mathematica, second edition, three volumes, Cambridge: The University Press, 1927. For a summary of the arguments of Principia Mathematica see Max Black, The Nature of Mathematics: A Critical Survey, London: Methuen & Co., 1933, pp. 19-118; see also the criticism of Black on pp. 139-144.

notion of metamathematics. Brouwer, whose views trace back to Kant and Kronecker, proposed his intuitionist view which regards all mathematics as constructible from a basal mathematical intuition and which would eliminate a large part of the infinite in mathematics.

The self-critical analysis of mathematics which arose, in part, from these and other factors, is significant inasmuch as it is a field of research principally the concern of mathematicians. Initially concerned with what is generally termed a criticism of the foundations of mathematics, the investigations carried on by mathematicians have spread into other sciences -- history, logic psychology and metaphysics -- and are rightfully viewed as a confrontation of mathematics to philosophy (12).

In attempting to present a partial but synthetic view of contemporary speculation stemming from mathematics or radiating towards it, the classification offered in chapter one of this essay will be followed. Under the pre-scientific view, the many attempts to popularize mathematical and scientific doctrine will be briefly summarized. Since the use of mathematics, especially in the physical sciences, has likewise bred speculation on the nature of mathematics, the views of the scientists and philosophers of nature will form the second group.

(12) This fact is increasingly being brought to the fore by mathematicians. For examples of conceptions of the philosophy of mathematics, see Max Black, *op. cit.*, "Introduction," pp. 1-6; Hermann Weyl, *The Open World*, Chapter III "Infinity," pp. 57-64; Bertrand Russell, *Our Knowledge of the External World*, New York: W. W. Norton & Company, 1929, Chapter II, "Logic as the Essence of Philosophy," pp. 33-44.

The view of logical abstraction, forming a third group, will summarize those views which still maintain a distinction of logic from mathematics. The fourth group will deal with speculations offered by mathematicians themselves; due to variation of opinion, the three schools of logicism, formalism and intuitionism are still distinguishable and will be treated in that order (13). Inasmuch as a number of philosophers have concerned themselves with the nature of mathematics as they have of science in general, their theories, forming a fifth group, will be briefly outlined in conclusion to this historical survey.

Each of the theories on the nature of mathematical abstraction which are presented below necessarily implicate a metaphysical position on mathematics. It is only metaphysics which has, as its proper concern, a view which is completely outside that of any particular science, for, if a mathematical physicist claims that mathematics is pure thought, his theory expresses a view of the entire edifice of mathematics; it is a mental stance completely outside the domain, content and meaning of either mathematics or physics. It is frequently conceded that metaphysical theories have some bearing on mathematics, but merely a critical one (14). But no theory merely

(13) For an introduction to current theories on the foundations of mathematics, see E. N. Stabler, "An Interpretation and Comparison of Three Schools of Thought in the Foundations of Mathematics," Mathematics Teacher, 28(1935), pp. 5-35; Everett H. Lagueier, "Schools of Thought in Modern Mathematics," Thought 12(1937), pp. 225-240; Abraham A. Fraenkel, "The Recent Controversies about the Foundations of Mathematics," Scripta Mathematica, 13(1937), pp. 17-36; Adolf Fraenkel, Einleitung in die Mengenlehre, dritte ungewandelte und stark erweiterte Auflage, New York: Dover Publications, 1938. Viertes Kapitel, "Erschütterungen der Grundlagen und ihre Folgen," pp. 209-268.

(14) Louis O. Kattsoff, op. cit., p. 4 and Max Black, op. cit., pp. 1-6. Hans Reichenbach, in Experience and Prediction (Chicago: The University of Chicago Press, 1938), merely credits "epistemology" with three tasks, that of being descriptive, critical and advisory; pp. 3-16.

voices critical comments without being accompanied by a doctrinal, expository position. If I say that mathematics is not to be identified with logic, I usually lend reasons in support of this view implicating me in some doctrinal position on the nature of both of these sciences (15).

A clarifying remark should be put here on the relations between mathematics and metaphysics. In presenting the theories below, no claim is being made here that the mathematician is not to develop mathematics at top and bottom; that is to say, the mathematician is not only to develop his science, but should even exhibit a rightful concern qua mathematician with mathematical ultimates and with the methods and bases of his science (16). The mathematician turned philosopher, however, may maintain that beyond the mathematical ultimate nothing more is to be sought; or he may freely indulge in metaphysical speculation on mathematics with little regard to the implications involved. Consequently, the views presented below aim

(15) P. Gonseth, Philosophie Mathématique, avec cinq déclarations de MM. A. Church, W. Ackermann, A. Heyting, E. Bernays et J. Quine, Paris: Hermann & Cie, 1939, pp. 5-6.

(16) "We are here concerned with the nature of the positive integers which are known also as natural numbers. Contemporary mathematical opinion follows that of previous centuries in regarding these numbers as the keystone of the mathematical structure... However, unlike our predecessors of only half a century ago, we do not think of mathematics as beginning with the natural numbers as its basic elements and proceeding thence to the development of its various branches. The latest views tend to assign to these numbers a middle position in the structure of this science. The lower portions are devoted to the foundations of mathematics which are established in the general theories of relation, order, sets, and groups as well as of logic. Above the level of the natural numbers have been erected the various specific mathematical disciplines such as the theory of numbers, algebra and the various theories of functions, as well as most of the geometrical studies." Abraham A. Fraenkel, "Natural Numbers as Cardinals," Scripta Mathematica, 6(1939), p. 69.

to compress, with some measure of completeness and exactitude, a metaphysical presentation of the leading theories on the nature of mathematics, whether offered by a logician, mathematician, psychologist or philosopher.

I. Pre-Scientific Influences

rescinding from current number-mysticism and astrological practices which seem an inevitable counterpart of civilisation, the relation of mathematics to the pre-scientific man can be seen in the growing number of books which attempt to popularise mathematics. The first class of these books merely aim to teach mathematics as such. Filled with useful information and delightful examples, aiming to unveil the mystery which man imagines is cast about mathematics and to popularise its ideas, these books boast of a large sale and a broad influence (17). A careful perusal of some of these works, however, shows that besides a presentation of mathematical doctrine, their authors are likewise interested in presenting definite commitments on the nature of society, philosophy, religion and science -- on the whole field of human culture (18). Historical inaccuracies and aprioristic views neither proved nor capable of proof are mingled with the presentation of mathematical facts, procedures and content (19).

(17) A few examples of this type of book, some of which have been referred to previously, are: Edward Kasner and James Newman, Mathematics and the Imagination; E. Richardson, Fundamentals of Mathematics; Arnold Dresden, An Invitation to Mathematics, New York: Henry Holt and Co., 1936. New York: Macmillan Company, 1941.

(18) See, for example, Morris R. Cooley et al, Introduction to Mathematics, Chapter I and Chapters XVIII-XXI; see also E. Richardson, Fundamentals of Mathematics, Chapters I, II, XVIII, or Gaylord E. Herriman, To Discover Mathematics, Chapters I, 13 and passim.

(19) As has been indicated above, Lancelot Hogben's Mathematics for the Millions and Science for the Citizen are noteworthy examples of an extreme and biased interpretation of the entire realm of human culture.

In this regard, these books are similar to another class of books about mathematics which have a more philosophical purpose. Convinced of the delights of mathematics, its historical and cultural importance, mathematicians and speculative thinkers generally will present reflected views on what mathematics is and how it is related as a social force to civilisation and culture (20). Here, likewise, little regard is often had for truth, and there seems to be more concern in presenting one's private, arm-chair musings on "mathematics as a culture clue" (21).

II. Scientists and Philosophers of Nature

The second group of speculations forming the contemporary mathematical pattern, comprises the speculations of a number of scientists, cosmologists and historians on the nature of mathematics. A half-expressed tendency towards idealism or scepticism is frequently discernible among scientists, a tendency which may be described as the belief that reality as such can be placed into partial or complete correspondence to mathematical ideas or mathematical thinking; consequently, all available knowledge is coextensive to mathematics and

(20) See the recent Mathematics: Our Great Heritage, edited by William L. Schaaf, and referred to previously, for a typical example of a group of conflicting views on the nature of mathematics and its value as a social force.

(21) The essays and books of Cassius J. Keyser (See Bibliography at the end of this essay) are good illustrations of this free, liberal view; see, for example, Lectures I, IX, and Lectures XVIII-XXI of this book, Mathematical Philosophy: A Study of Fate and Freedom; Lectures for Educated Laymen, New York, E.P. Dutton & Co., 1937. For a similar example, see the two books of Tobias Dantzig, Aspects of Science (New York, The Macmillan Company, 1937), Number: The Language of Science; A Critical Survey Written for the Cultured Non-Mathematician (Third edition, revised and enlarged, New York: The Macmillan Company, 1941). The series Que Sais-je? issued by Presses Universitaires of Paris, France, has a similar purpose, but is an attempt to cover the whole field of learning.

mathematical method is the only reliable procedure for thought (22).

This theory is disguised in various expressions: the mathematician creates his science (23), he exercises an activity of pure thought (24), mathematical laws are the laws of all pure and exact thought (25), the mathematician seizes mathematics independently of all experience (26). The congeniality of nature to a mathematical interpretation has been partly responsible, it seems, for the views that point in this direction (27), although it is a commonplace but necessary warning to state that an absolute proposal of this type of mathematicism is not frequently made nor clearly

(22) A. Cornelius Benjamin, op. cit., Chapter XX, "The Nature of Reality," pp. 436-442; see also Etienne Gilson, The Unity of Philosophical Experience, pp. 132-132, and Paul Duoy, "Les mathématiques et l'idéalisme philosophique," in Les grands courants de la pensée mathématique, présentés par P. de Lichtenais, Paris: Bellesand A Fontenay-aux-roses, 1948, pp. 370-377.

(23) "We shall see more clearly the important bearing of this latter remark when we come to discuss more closely in another chapter how the universe is largely the construction of each individual mind." Karl Pearson, The Grammar of Science, Part I: Physical, London: Adam and Charles Black, 1911, p. 16.

(24) "La science mathématique a pour essence de jouer perpétuellement sur la mutation de forme-loi en forme-objet et réciproquement." R. Davalet G.-T. Guilbaud, Le raisonnement mathématique, Paris: Presses Universitaires, p. 139; see also pp. 135-136 and pp. 143-145.

(25) "Likewise the mathematician, these many centuries, has been searching for logical structure and finding it everywhere...the vast domain of purely mathematical thought forms and for irrefutable testimony that the subject as well as the objective world is mathematical. Thus with Descartes he will declare *Omnia prout se mathematica fiunt* -- with me everything turns into mathematics." George D. Birkhoff, "Mathematics: Quantity and Order," in Science for a New World, edited by J. G. Growther, New York: Harper and Brothers, 1934, p. 257.

(26) "Today mathematics is unbound; it has cast off its chains. Whatever its essence, we recognise it to be as free as the mind, as prehensible as the imagination. Non-Euclidean geometry is proof that mathematics, unlike the music of the spheres, is man's own handiwork, subject only to the limitations imposed by the laws of thought." Edward Kasner and James Newman, Mathematics and the Imagination, p. 359; cf. also pp. 361-362.

(27) J. W. N. Sullivan, The Limitations of Science, London: Chatto and Windus, 1933, p. 231.

formulated, and may later be revoked (28).

The position of Sir James Jeans, for example, is well known and his doctrine has been frequently presented and criticised (29). Jeans maintains that mathematics enters the universe from above -- through the Divine creative process -- and all our knowledge is but a gradual intussusception of the mathematical universe created by the Divine Mathematician (30). Though he has given no definite formulation of his views, Einstein, in a number of places, seems to lead towards idealism (31). A number of other thinkers show a similar tendency in this direction (32).

That mathematics is to be regarded as merely an extrinsic tool or instrument distinct from physical ideas which the mind employs in

(28) Bertrand Russell, The Principles of Mathematics, "Introduction," pp. ix-x.

(29) See A. Cornelius Benjamin, op. cit., pp. 448-451; L. Susan Stebbing, Philosophy and the Physicists, London, Methuen and Co., 1937, Chapter II, "The Escape of Sir James Jeans," pp. 19-44.

(30) Sir James Jeans, The New Background of Science, Oxford: Cambridge University Press, 1933, p. 296-297; The Mysterious Universe, second edition, Cambridge: The University Press, 1937, pp. 111-122. In Physics and Philosophy (New York: The Macmillan Company, 1943), he seems to waver a bit on this position; see pp. 45-49 and pp. 195-217.

(31) Albert Einstein, The Method of Theoretical Physics, Oxford: The Clarendon Press, 1933, pp. 11-13 and The World as I see it, New York: Coviel Press, 1934, pp. 33-34. See also Albert Einstein and Leopold Infeld, The Evolution of Physics, New York: Simon and Schuster, 1942, pp. 235-249, and pp. 310-311, and his article, "Remarks on Bertrand Russell's Theory of Knowledge," in The Philosophy of Bertrand Russell, Volume V of The Library of Living Philosophers, Edited by Paul Arthur Schilpp, Evanston and Chicago: Northwestern University, 1944, pp. 276-291.

(32) For a summary account of idealistic trends in physics from a positivist view, see Phillip Frank, Between Physics and Philosophy, Cambridge: Harvard University Press, 1941, Chapter IV, "Is there a Trend toward idealism in Physics?" pp. 104-126.

physical science if also a popular view entertained by mathematical physicists (33). In fact, by generally failing to grant that the truth of human knowledge lies in its conformableness to the real, the major tendency of mathematical physics seems oriented toward a sceptical attitude towards the truth and validity of both mathematics and the physical sciences (34). Disallowing for the truth of either ordinary and philosophical knowledge, many scientists hold that physical science itself beggars truth and yields nothing but mere conjecture or plausibility (35) -- to which they add postscripted tributes to the mystical experience (36) or the reality of mathematical universals or relations (37).

Though he has frequently been accused of mathematical idealism, A. S. Eddington seemingly supports the conception of mathematics as being a tool or instrument for physical science (38), even considering geometry as merely an experimental science (39). Physical science, in

(33) "Mathematics is a tool specially suitable for dealing with abstract concepts of any kind and there is a limit to its power in this field. For this reason a book on the new physics; if not purely descriptive of experimental work, must be essentially mathematical. All the same the mathematics is only a tool and one should learn to hold the physical ideas in one's mind without reference to the mathematical form." P. Dirac, The Principles of Quantum Mechanics, New York: Oxford University Press, 1935, "Preface," p. vii.

(34) Of the many statements that could be adduced in support of this tendency towards scepticism, see, for example, Bertrand Russell, Human Knowledge: Its Scope and Limits, New York: Simon and Schuster, 1948, Part Three, Chapter IV, "Physics and Experience," pp. 195-209, and Part Four, passim.

(35) A. S. Eddington, The Nature of the Physical World, Chapter XIV, "Causation," 293-315.

(36) Ibid., Chapter XV, "Science and Mysticism," pp. 316-342.

(37) For a brief presentation of this view, see Antonio Aliotta, "Recent Developments in Logic," in Science for a New World, edited by J. G. Crowther, pp. 334-337.

(38) Sir Arthur Eddington, The Philosophy of Physical Science, p. 74 and pp. 137-140.

(39) "I do not think that it can be disputed that, both etymologically and traditionally, geometry is the science of measurement of the space around us; and however much the mathematical superstructure may now overweigh the observational basis, it is properly speaking an experimental science." A. S. Eddington, The Nature of the Physical World, p. 142.

in Eddington's view, has escaped the tyranny of the engineer only to be subjected to the yoke of the mathematician; neither mathematics nor physics, he concludes, give us a true knowledge of the world, a knowledge which is only to be had through either ordinary or the mystical experience (40).

The term cosmologists, as here employed, refers to the relatively small number of thinkers who adhere to the distinction and pertinence of a philosophical study of nature distinct from science and from metaphysics (41). Abstracting from the soundness of their cosmological doctrine, there is discernible, in this group, a tendency to consider the abstractive nature of mathematics from a cosmological instead of a metaphysical point of view (42). For example, it may be proposed that the mathematician properly receives his understanding of infinity, motion and other speculative conceptions from the cosmologist (43). Similarly, one easily finds an unsatisfactory understanding of the nature of such mathematical abstractions

(40) See footnote 36, above; see also "Conclusion," (pp. 343-353) of The Nature of the Physical World.

(41) For an historical-critical study of the gradual displacement of the philosophy of nature by science, see Jacques Maritain, La Philosophie de la Nature, Chapitres I et II; for a statement on the connection and relation of the philosophy and the sciences of nature, see Chapitre III, ibid. A recent attempt to relate physical science and philosophy is that of Jean Dautat, L'oeuvre de l'intelligence en physique, referred to previously.

(42) See, for example, Pridericus Saintonge, Summa Cosmologiae seu Philosophia Naturalis Generalis, Montreal: Imprimerie du Messager, 1941, Caput V et VI, or Jacobus Gustavus Moran, Cosmologia, Mexico: Buena Prensa, 1944, Caput II. These authors seem to confuse a physical and a mathematical view of quantity.

(43) See, for example, Jacques Maritain, The Degrees of Knowledge, pp. 92-93 and p. 343; see also, his La Philosophie de la Nature, pp. 80-83, where he speaks of the philosophy of nature as being a wisdom secundum quid, imitative of metaphysical wisdom.

as number (44), or the mistaken attribution of a cosmological conception of quantity and its determinations to the quantitative consideration proper to mathematics (45).

History, being a science concerned with speculatively knowing the universal causes and reasons of the past, contingent mobility and change of man and his cultural efforts, lies on the same level of speculative science as physics, cosmology and psychology (46). Among historians, a large number, perhaps unwittingly, adopt this view and succeed, as much as is possible, in re-actualizing the ancestral conditioning of mankind. Thus, the work of T. L. Heath (47) and Carl S. Boyer (48), among others, is a frank portrayal and appraisal of what mathematics was in the past, and of the necessary reasons why it developed as it did. The predominant influence of Comte's positivism, however, is seen in many historians of mathematics and science. Even the work of the noted historian George Sarton (49), admirable and painstaking

(44) This seems to be the basis for the criticism of Marcel Lallemand, *Le Transfinit: Sa logique et sa métaphysique*, Paris: Desclée de Brouwer et Cie, 1934.

(45) This, unfortunately, seems to characterize the good pioneer work of W. H. W. Rouse Ball, in *Geometriae* (Rome: Apud Aedes Pont. Universitatis Gregorianae, 1945), Nota VII, "De Spatiis Metageometricis," pp. 453-463, and in a series of articles appearing in the *Gregorianum* (see Bibliography to this essay). His attitude and that of other authors in the philosophical tradition may be due to their proposal of three degrees of formal abstraction for speculative science, in which a certain kind of continuity holds between these three levels of speculative science.

(46) See Chapter IX, below, for a presentation of the distinction of speculative sciences.

(47) T. L. Heath's work has been done especially in the history of Greek mathematics; see Bibliography at the end of this essay.

(48) Carl S. Boyer's *The Concepts of the Calculus* is an accurate and quite complete study of the many factors which helped or retarded the invention of the calculus.

(49) The influence of Comte on Sarton is discernible in scattered remarks in his *Introduction to the History of Science*, in *The History of Science and the New Humanism*, Cambridge: Harvard University Press, 1937, and in *Essays in Science and the New Humanism*.

research though it is, is interspersed by comments which show Comte's influence. Other historians, such as E. T. Bell, judge past mathematics on the present merits of the science, and make free, interpretative remarks on the historical nature of mathematics that have little factual or theoretical foundation (50). Similarly, there is a number of historical-critical essays and books in which an attempt is made to establish the continuous unity of mathematical thought, sometimes misusing or restraining past conceptions to fit a privately adopted philosophical pattern (51).

III. Logicians

Although many contemporary mathematicians view mathematics as identical with logic, as will be shown below, and although many scientists (52) and others have adopted this view (53), there is a growing number of speculative thinkers from all fields who deny that this proposition of equivalency has been established (54). In various

(50) See, for example, The Development of Mathematics or The Magic of Numbers, New York: McGraw-Hill, 1946.

(51) As an example, see Chapters I-III of Tobias Dantzig's Aspects of Science.

(52) "Logic in its purest form, which is mathematics, only coordinates and articulates one truth with another. It gives harmony to the superstructure of science, but it cannot provide the foundation or the building-stones." Max Planck, Where Is Science Going?, Prologue by Albert Einstein, translation and biographical notes by James Murphy, New York: W. W. Norton & Company, 1932, p. 65.

(53) For example, in the conclusion to The Concepts of the Calculus, Carl B. Boyer writes: "Materialistic and idealistic philosophies have both failed to appreciate the nature of mathematics, as accepted at the present time. Mathematics is neither a description of nature nor an explanation of its operation; it is not concerned with physical motion or with the metaphysical generation of quantities. It is merely the symbolic logic of possible relations, and as such is concerned with neither approximate nor absolute truth, but only with hypothetical truth. That is, mathematics determines what conclusions will follow logically from given premises." p. 308.

(54) See, for example, Morris L. Cohen and Ernest Nagel, An Introduction to Logic and Scientific Method, Preface; see also John C. Cooley, A Primer of Formal Logic, New York: The Macmillan Company, 1942, Preface.

critical studies, it has been pointed out that the tenets of logicism in Leibniz, Boole, Peano, Frege, Russell and others (55), is based on an implicit or explicit metaphysical position whose falsity is undeniable. Further, the supposed difference of Aristotelian and Symbolic Logic (56) or the proposed supplanting of Aristotelian and classical logic by a non-Aristotelian system (57), are merely speculative positions whose validity is questionable (58). In fact, in a number of writers dealing with the problem of Aristotelian and Symbolic Logic, there is discernible a basic solution for reuniting classical and symbolic logic, as there is for validating the unity of Logic and its distinction from mathematics (59).

IV. Mathematicians

The first of the theories currently proposed by the mathematicians

(55) See, for example, Andre Carbon, La Philosophie des Mathematiques, for a recent, critical study of logicism; see also Hermann Weyl, "Mathematics and Logic; A Brief Survey serving as a preface to a review of The Philosophy of Bertrand Russell," The American Mathematical Monthly, 53(1946), pp. 2-13.

(56) For an introductory description and analysis of Symbolic Logic and its points of difference from Aristotelian Logic, see Karl Menger, "The New Logic," Philosophy of Science, 4(1937), pp. 299-336.

(57) Alfred Korzybski, Science and Sanity: An Introduction to Non-Aristotelian Systems and General Semantics, third edition with new preface, Lakeville: The International Non-Aristotelian Library Publishing Company.

(58) See the discussion on logicism in Thomas Greenwood, Les fondements de la logique symbolique, Tome I: Critique du nominalisme logistique, Paris: Hermann et Cie, 1938.

(59) Thomas Greenwood, op. cit., Tome II: Justifications des calculs logiques; and, by the same author, "The Unity of Logic," The Thomist, 8(1945), pp. 457-470; the work of Morris R. Cohen and Ernest Nagel, An Introduction to Logic and Scientific Method, is likewise pointed toward reuniting Aristotelian and Symbolic Logic; see also, I. M. Bochenski, "On Analogy," The Thomist, 11(1946), pp. 424-447.

who form the fourth group under consideration is logicism (60), in which emphasis is placed on the close affinities that hold between mathematics and logic. Harking back to a dream of Raymond Lulle, Leibniz had already hoped to establish a universal characteristic for all thought which would be analogous to the symbolism of mathematics, enabling man to think and reason merely with symbols. His efforts in this direction were taken up and abetted by such men as Augustus De Morgan (1806-1876) and George Boole (1815-1864), the latter being usually credited with founding symbolic logic (61). The more effective direction from which current logicism sprang, however, was due to the work of Frege and Peano (62), to whose inspiration Russell and Whitehead were indebted for Principia Mathematica (63). Contemporary investigators, of whom there is a large

(60). For an introductory account of logicism, see Louis C. Kattsoff, op. cit., Chapter Eight, "The Foundations of Mathematics: I. Logicistic Foundations," pp. 93-115, or see the article referred to above by Karl Menger. For a quite complete picture of the contemporary scene in logic, see Encyclopedie Francaise, Tome I; L'Outillage Mental: Pensee, Langage, Mathematiques, Paris: Librairie Larousse, 1937, Section B, La Pensee Logique, Chapitre III, "La Critique Contemporaine," p. 1° 16-1 to p. 1° 20-11, which also contains a sketch of the effect of the new developments on mathematics and experimental knowledge.

(61). For a brief historical summary of logicism see H. T. Bell, op. cit., pp. 516-536. A strong presentation of the case for logicism is given in Bertrand Russell, The Principles of Mathematics, Chapter II, "Symbolic Logic," pp. 10-32, and in Frank Plumpton Ramsey, The Foundations of Mathematics and Other Logical Essays, edited by R. B. Braithwaite, with a preface by G. E. Moore, New York: Harcourt, Brace and Company, 1931. Leon Chwistek's The Limits of Science, contains a metaphysical and cultural plea for logicism, see especially pp. 19-23.

(62). Max Black, op. cit., pp. 18-19, and Louis C. Kattsoff, op. cit., pp. 24-31.

(63). See Harold A. Smart, The Philosophical Presuppositions of Mathematical Logic, Chapters III-VI, pp. 36-96 for a summary account of the development of mathematical logic and an able criticism of its proposals.

number, include such men as Chwistek (64), Tarski (65) and Lukasiewicz in the Polish School (66); C. I. Lewis (67), Alonso Church (68), Max Black (69), and Quine (70) in the United States, and a growing number of contemporaries (71).

(64) Leon Chwistek's work, The Limits of Science, contains a fine "Introduction" (pp. xxi-lvii) to his work by the translator. For a summary of his technical contributions, see Louis O. Kattsoff, op. cit., pp. 107-115. Chwistek's work is one of the most recent in which a metaphysical critique accompanies a technical contribution to the field of Symbolic Logic.

(65) A summary view of the work of Alfred Tarski is available in his Introduction to Logic and to the Methodology of Deductive Sciences, translated by Glaif Helmer, enlarged and revised edition, New York: Oxford University Press, 1941, see Author's prefaces.

(66) J. Jordan, The Development of Mathematical Logic and of Logical Positivism in Poland between the two Wars, Polish Science and Learning, New York: Oxford University Press, 1939.

(67) C. I. Lewis is author of A Survey of Symbolic Logic (Berkeley: University of California Press, 1918), which traces the history of symbolic logic from the time of Leibniz.

(68) For a summary account of Church's work see Louis O. Kattsoff, op. cit., Chapter Ten, "Church's Elementaristic Mathematics: Combinatory Foundations," pp. 136-155.

(69) Max Black, author of The Nature of Mathematics frequently cited here, has recently published Critical Thinking on Introduction to Logic and Scientific Method, New York: Prentice-Hall, 1946, in which he speaks of the value of mathematics as due to its symbolic and deductive character; pp. 355-356. Its distinction from other sciences is, however, adopted: "Respect and admiration for mathematics should, however, always be tempered by the obvious reflection that scientists using mathematics must have something to count; however important, mathematics is but an instrument of science, and without the measurement of the products of experience, the mathematical quantities occurring in scientific theory would have no link with the real world." p. 356.

(70) W. V. Quine, Mathematical Logic, New York: M. N. Norton Company, 1940.

(71) It is beyond our purpose here to attempt to list the growing amount of research being done in logicism. For a broad, summary picture, see H. T. Bell, The Development of Mathematics, pp. 516-536. In the United States, The Association for Symbolic Logic, issued a bibliography of Symbolic Logic for 1938-1939 in its official publication, The Journal of Symbolic Logic, 1(1938), pp. 121-216; this bibliography is being supplemented by reviews and indexes, and can be seen for a list of the more prominent contributors to both logicism and symbolic logic.

The proposition fundamental to all these investigations is that mathematics has a closer relationship to logic than had previously been realized and that mathematics is ultimately based on logic. The aim of logicism is to erect mathematics as a closed deductive system on the basis of logical ultimates to which all mathematical propositions and concepts, it is claimed, are reducible (72). Although the exact speculative relationship between these two sciences is not frequently stressed, the views on their interaction vary from Bertrand Russell's overt identification of them (73) to that of merely analysing the speculative proximity of their nature and methods (74). The use of symbolism in this reductive process of mathematics to logic is of utmost importance, and the problem of mathematics and logic is looked upon as a part of a more global concern which is the formal structure of deductive reasoning as investigated by the symbolic method (75).

Two basic concerns of logicism are the elucidation of the notion of number and the concept of the continuum. In attempting to reduce these two problems to logical nature and formulation, logicism encountered the problem of the finite and the infinite in mathematics

(72) Max Black, *op. cit.*, pp. 7-8.

(73) Cf. Bertrand Russell, *Introduction to Mathematical Philosophy*, London: George Allen & Unwin, 1938, pp. 194-206.

(74) "I do not feel that much of a case can be made for the necessity of reducing mathematics to pure logic..." John C. Cooley, *op. cit.*, "Preface," p. vii; see also his presentation of the Peano postulates for the natural number system, pp. 271-281 and of the class definition of number, pp. 281-294.

(75) Alonzo Church, "A Bibliography of Symbolic Logic," *The Journal of Symbolic Logic*, 1(1936), p.121.

with the result that a number of paradoxes, inconsistencies and apparent contradictions developed (76). To solve these paradoxes, various extra-logical axioms and postulates were proposed, which, however, failed to absolve logicism from failure (77).

The theory now known as formalism can properly be viewed as having its historical genesis in the method of axioms and postulates that traces back to Euclid's Elements (78). Its distinction as a theory, however, lies more immediately in the use of non-Euclidean geometries and in the rise of algebras through the efforts of Lobatschewsky, Riemann, George Peacock, Hamilton and De Morgan. Formalism today, though closely allied to these historical antecedents and to logicism, aims to combat the over-emphasis placed on logic or a recourse to intuition by a stress on pre-logical or pre-mathematical symbols. Also known as axiomatics or postulationism, formalism is principally due to David Hilbert (79) (1862-1934), and his collaborators, Von Neumann, Bernays and Ackerman (80).

(76) For a summary of these, see Clarence Irving Lewis and Cooper Harold Langford, Symbolic Logic, Chapter XIII, "The Logical Paradoxes," pp. 438-486; or, for a simple statement see Louis O. Kattsoff, op. cit., pp. 87-92.

(77) "The relation between mathematics and logic is neither identity nor that of conclusions to premisses, but consists in the fact that mathematics must be used in the systematic development of logic (as of all organized systems); and the similarities between logic and mathematics spring from the fact that the first, in its 'philosophic' aspect, is the syntax of possible states of affairs, while the second is the syntax of all organized systems." Max Black, op. cit., p. 144. This remark is preceded, pp. 138-144, by a short summary of arguments against the proposed identification of logic with mathematics.

(78) E. T. Bell, op. cit., Chapter 9, "Towards Mathematical Structure: 1801-1910," pp. 170-201; see especially pp. 170-175 and pp. 195-6.

(79) See Hermann Weyl, "David Hilbert and his Mathematical Work," Bulletin of the American Mathematical Society, 50(1944), pp. 612-666, for a fine presentation of Hilbert's work. For a technical presentation of Formalism, see Louis O. Kattsoff, op. cit., Chapter nine, "The Foundations of Mathematics: II. Hilbert's Formalism," pp. 116-135.

(80) P. Godeaux, Philosophie mathématique, "Le formalisme Hilbertien," pp. 49-51.

Moved by the many paradoxes to which logicism and Cantor's theory of sets had given occasion, Hilbert wished to erect both classical mathematics and set theory as a complete and non-contradictory system under the form of a sheer calculus devoid of interpretation and content. Formalism claims that mathematics comprises two divisions, mathematics and metamathematics (81). Mathematics itself is composed of arbitrary marks or symbols. Of these we have a pre-mathematical insight, which we are able to manipulate almost mechanically without reference to any actual mathematical entities. Though sometimes presented as though the symbols were meaningless, formalism really holds that one can manipulate the symbols into a consistent system without any concern, in so doing, as to whether or not the symbols have meaning or an application. Metamathematics deals with the theory of proof for mathematics itself; it is concerned with proving consistency, independence and categoricalness of a set of mathematical axioms (82).

In a broader view, Hilbert seems to hold that mathematics expresses the ideal conditions of any science, and that metamathematics is a description of the rules according to which all rational thinking proceeds (83). Hence it is interesting to note that despite its success in postulate-systems and axiomatics, the two aims which characterize formalism -- that of

(81) "Formalism is a technique first, and only secondarily a philosophy: a technique for the investigation of the logical interrelation of branches of mathematics and a philosophy to account for the success of that technique." Max Black, *op. cit.*, p. 149.

(82) M. P. Bernays, "Sur les questions methodologiques actuelles de la theorie Hilbertienne de la demonstration," in Les Entretiens de Zurich sur les fondements et la methode des sciences mathematiques, 6-9 decembre 1938, Exposes et discussions publies par F. Gonseth, Zurich: S.A. Loemann, 1941, pp. 144-161.

(83) Max Black, *op. cit.*, pp. 147-162.

formalising all mathematics and logic and of demonstrating the formal consistency of this formalisation -- led to the interesting theorem of Gödel. Gödel demonstrated that though everything mathematical can be formalised, all of mathematics cannot be exhausted in any one system; consequently, he showed that there exist undecidable formulae which cannot be formalised within the system itself, for each formal system requires a wider and more inclusive system to demonstrate its own formalisations (84).

The postulational technique embodied in Hilbert's formalism and the emphasis on logical analysis were two factors which encouraged the rise of twentieth century logical positivism. Wittgenstein and Hans Reichenbach, for example, maintained that mathematics is nothing more than a vast tautology (85). Wittgenstein's views had a strong influence on the positivism of the Vienna Circle, which had been organised under

(84) *Ibid.*, pp. 167-168. "If, as is agreed, the formal systems can be used to systematise fields of knowledge, and if the Gödel theorem applies to such applications, it would mean that no field of knowledge so systematized can ever be complete. At any stage there will always be questions formulable in the system that need a wider system for their solution. Since it is the contention of the author of this book that logic and mathematics do tell us something about reality, the Gödel theorem tells us that there can never be a complete and final theory of reality, i.e., metaphysics. This does not lead to mysticism, but to an evolutionary scheme of reality." Louis C. Rattsoff, *op. cit.*, pp. 184-185. A non-technical presentation of the theorem can be found in J. B. Rosser, "An informal exposition of Proofs of Gödel's theorems and Church's theorems," *Journal of Symbolic Logic*, 4(1939), pp. 35-60.

(85) L. Wittgenstein, *Tractatus Logico-Philosophicus*, New York: Harcourt, Brace and Company, 1932, *passim*; Hans Reichenbach, *Logique, mathématique et connaissance de la réalité*, introduction de M. Boll, Paris: Hermann et Cie, 1938.

Schlick and which is presently advocated by Carnap (86), Reichenbach (87), Frank (88) and others (89). The aim of the logical positivists is not restricted to mathematics, but is concerned with an analysis of all languages and their semantics, and with a completely negative attitude towards metaphysics (90).

(86) For Rudolf Carnap's views see The Logical Syntax of Language, New York, Harcourt, Brace and Company, 1937, and Foundations of Logic and Mathematics, Volume I, Number 3 of International Encyclopedia of Unified Science, Chicago: The University of Chicago Press, 1938.

(87) Hans Reichenbach, in Experience and Prediction (Chicago: The University of Chicago Press, 1938), speaks of the movement of logistic empiricism as being both a strict disavowal of the metaphorical language of metaphysics and a submission to the postulates of intellectual discipline; Preface, p. v. Similar views are expressed in his Philosophic Foundations of Quantum Mechanics, Berkeley: University of California Press, 1944, Preface, pp. v-viii.

(88) Philip Frank, Between Physics and Philosophy, Cambridge: Harvard University Press, 1941, *passim*. See his remarks in his biography of Einstein called Einstein: His Life and Times, translated by George Rosen, edited and revised by Shuichi Kusaka, New York: Alfred A. Knoff, 1947, pp. 42-48.

(89) A summary view of positivism can be found in Yves Simon, Prevoir et savoir, Montreal: Editions de l'arbre, 1944, Chapitre III, "L'école de Vienne," pp. 117-139, and A. Cornelius Benjamin, An Introduction to the Philosophy of Science, Chapter VIII, "Theories of Scientific Concepts," pp. 147-172.

(90) A growing movement called the Unity of Science Movement is concerned with unifying science especially through the analysis of languages, meta-languages, logics and meta-logics. See P. Conseth, *op. cit.*, "VI. Le mouvement de l'unité de la science," pp. 60-62, and the publications being issued from the University of Chicago Press under the general title of International Encyclopedia of Unified Science.

The intuitionism of Brouwer, sometimes called finitism or neo-intuitionism (91) can be viewed as developing from the views of Kant, the mathematician Kronecker, Poincare, and the Paris School of mathematicians (92). The brilliant efforts of Weierstrass towards the arithmetisation of all of mathematics by orienting mathematical analysis towards an arithmetical basis instead of its previous geometric foundation, was soon followed by Cantor's theory of infinite sets. These two discoveries led to a fruitful development of the theory of functions and stimulated efforts towards greater rigor through abstractness and generality. Weierstrass' efforts were favorably viewed by Kronecker (1823-1891), who made the extreme demand that all mathematical results are and must be shown to be results issuing from the natural numbers, and who forcibly rejected the Cantor theory of infinite sets (93).

Kronecker's position had an undoubted influence on many mathematicians among whom was Henri Poincare (1854-1912). Though Poincare held to a formalist attitude in geometry (94), he vigorously opposed both logicism and formalism (95). Holding to a finitist attitude towards mathematics, he insisted that everything mathematical except the

(91) Abraham A. Fraenkel, *The Recent Controversies about the Foundations of Mathematics*, p. 25.

(92) *Ibid.*, pp. 23-24.

(93) E. V. Bell, *op. cit.*, Chapter 9, "Towards Mathematical Structure: 1801-1910," pp. 170-201.

(94) "The axioms of geometry therefore are neither synthetic a priori judgments nor experimental facts. They are conventions; our choice among all possible conventions is guided by experimental facts; but it remains free and is limited only by the necessity of avoiding all contradiction. Thus it is that the postulates can remain rigorously true even though the experimental laws which have determined their adoption are approximative. In other words, the Axioms of geometry (I do not speak of those of arithmetic) are merely disguised definitions." Henri Poincare, *op. cit.*, p. 472-488.

(95) See, for example, Chapter V, "The Latest Efforts of the Logicians," *Ibid.*, pp. 472-485.

set of natural numbers be finitely definable, and proposed that the body of mathematics is ultimately rooted in the principle of mathematical induction, whose validity is intuitively ascertained (96). Identifying mathematical existence with any demonstration of non-contradiction, he rejected the choice axiom (97) and claimed that it merely begged the question and did not demonstratively yield mathematical existence.

The Paris School of pure mathematics, comprised of Borel, Baire and Lebesgue, together with Hadamard -- who differs from the first three -- was concerned mainly with function theory and its establishment on a sound basis (98). In a series of letters to each other (99), these men entered into a discussion concerning the validity of Zermelo's multiplicative or choice axiom. The interest of this school centers mainly in function and set theory in relation of the foundations of mathematics.

Brouwer (1902-) and the Dutch school of intuitionism which

(96) "Mathematical induction, that is, demonstration by recurrence, on the contrary, imposes itself necessarily because it is only the affirmation of a property of the mind itself." *Ibid.*, p. 40, see also p. 41, 220 and pp. 483-485.

(97) *Ibid.*, p. 483-485.

(98) *Max Black, op. cit.*, pp. 169-172.

(99) These letters were published in various places; see Emile Borel, *Leçons sur la théorie des fonctions*, 3ème édition, Paris: 1928, "Appendix." See also H. H. Lebesgue, "Les controverses sur la théorie des ensembles et la question des fondements," in F. Gonseth, *Les entretiens de Zurich*, pp. 109-124.

includes Heyting (100), Kolmogoroff (101), and Mannoury (102), proposed a quite complete and systematic basis for the foundations and nature of mathematics (103). In order to get a clear understanding of the opposing theories on the foundations, Brouwer maintains that one must understand what we mean by *intuition* as a social force, and relate mathematics to its psychological and sociological origins. The mathematical attitude, for Brouwer, is based ultimately on man's intuitive recognition of structure akin to the early man's recognition of structure in the family or in society. Mathematics, as a branch of scientific thought concerned with the structure of phenomena, arises from man's intuition of time as abstracted from all emotional content and yields the fundamental phenomena of mathematical thinking: the intuition of two-oneness (104). From such a basic intuition, this school

(100) A statement of the intuitionist position by Arend Heyting can be found in F. Gonseth, Philosophie mathématique, "Les Fondements des mathématiques du point de vue intuitionniste" pp. 73-75.

(101) Ibid., pp. 62-64.

(102) A presentation of Mannoury's position is given in Louis O. Kattsoff, op. cit., Chapter Thirteen, "Mannoury's Signific Point of view: A return to Psychologism," pp. 196-206.

(103) L. E. J. Brouwer's inaugural address at the University of Amsterdam in 1912 announcing the intuitionist program has been reprinted as "Intuitionism and Formalism," Bulletin of the American Mathematical Society, 20(1913), pp. 81-96. See also Arnold Dresden, "Brouwer's Contributions to the Foundations of Mathematics," Bulletin of the American Mathematical Society, 20(1913), pp. 31-40.

(104) Max Black, op. cit., pp. 188-189 and p. 193. See also Everett N. Lagueier, "Brouwerian Philosophy of Mathematics," Scripta Mathematica, 7(1940), pp. 69-78.

claims to develop not merely the numbers one and two and all the natural numbers; but, by a modification and repeated application of this same method, it proposes to derive all other legitimate mathematical entities.

Hence, for Brouwer, the origin of mathematics is directly connected with the psychological intuition of two-oneness, while mathematical existence is coincident with constructibility, and mathematical truth is likewise assured by constructibility in a finite number of steps. Brouwer would not admit that a mere proof of freedom from contradiction gives the status of existence to a mathematical concept or theorem. Brouwer was led, by the logical consequences of his theory to deny the validity of the tertium non datur, or principle of excluded middle, though not in the generally accepted understanding of the logical principle itself. Emphasizing, from his outlook, the inability of mathematicians to give a rule whereby the necessary construction of proposed mathematical entities can be carried out, he concludes that there are mathematical theorems which apparently can be neither proved nor disproved. Thus he denies the solvability of all mathematical problems (105).

(105) "So the dispute between Brouwer and more orthodox philosophers with respect to the validity of the tertium non datur is seen to be one as to the nature of mathematical existence rather than as to the validity of the logical principle...Brouwer, indeed, is not denying the tertium non datur in the generally accepted interpretation of that logical principle, but rather emphasizing that existence in mathematics is synonymous with constructibility, and that the truth, and indeed significance, of mathematical theorems is conditional on the possibility of constructing the entities which occur in their formulation." Max Black, op. cit., p. 166.

In the larger view of his philosophy, Brouwer holds that mathematics is equivalent to all exact and precise formulation of thought and of itself bears no necessary relation to experience. In this regard, Brouwer indicates his indebtedness to Kant (106). For though Brouwer rejected Kant's notion of intuitive space, he retained his teaching that time is a pure intuition given a priori. Likewise, his denial of the principle of the excluded middle appears to be little more than a reinterpretation of Kant's insistence on the necessity of constructing mathematical concepts. Brouwer's insistence of the independence of mathematics (intuitive mathematics) from mathematical speech (its symbolic expression) indicates his leaning towards Kantian idealism; for mathematical speech or language, he says, is merely needed for communication and for man's memory (107).

V. Metaphysicians

Professional philosophers, generally regarding mathematics as the most certain and yet the most difficult of sciences, have, as rule, left untouched the vastness of modern mathematical research and speculation. Those who have not partaken in an outright professionally philosophical study of mathematics, generally view it, as did Kant, as unimpeachable evidence of man's ability to achieve certainty, and generally

(106) Ibid., pp. 187-191. In Philosophie Mathématique, F. Gonseth suggests that Brouwer's views hold a stronger kinship to Spinoza than to Kant; see p. 68.

(107) Abraham A. Fraenkel, "The Recent Controversies about the Foundations of Mathematics," pp. 24-30.

speaking, philosophers accept statements from mathematicians without criticism. Others, of course, adopt an extreme, anti-mathematical attitude. However, the number of philosophers representative of a properly metaphysical view on the nature of mathematics in the recent past is sufficiently large to dwarf the supposed prominence of any one thinker. Consequently, a broad classification of philosophical speculation on mathematics will be adopted as showing directions towards scepticism, idealism, anti-intellectualism and a final group of distinct and diverse tendencies.

Though not frequently nor openly advocated, scepticism, as a philosophical explanation of mathematical science, has its modern as well as its ancient defenders. Being a proposal which appears in various disguises, it can generally be recognized as negating the possibility of mathematics as a science concerned with real being and as affirming that the perfection of mathematics is but a refined derivative of sense impressions (108). Mathematics, in this view, is frequently viewed as a mere language or a facile instrument of research (109), as dealing with no specific subject matter (110), or as being reducible to the abstraction of lower sciences (111). Theories which may have a surface as sets of idealism or logicism,

(108) C.-A. Laisant, La Mathématique: Philosophie, Enseignement, deuxième édition, revue et corrigée, Paris: Gauthier-Villars, 1907; in the "Preface", Laisant proposes that the mathematical sciences are experimental, p. 12; see also Louis O. Kattsoff, op. cit., Chapter Five, "Pasci's Empirico-postulational definition of number," pp. 48-49.

(109) This seems to be the view of Ernst Mach; see his The Science of Mechanics: A Critical and Historic Account of its Development, translated from the German by T. J. McCormack, fifth edition, La Salle: Open Court, 1945.

(110) This view is generally advanced by those who over-emphasize the formal character of mathematical reasoning.

(111) In same way, scientists and cosmologists generally incline to this "physical" view of mathematics.

or which, lay undue emphasis on logic or thought (112), often lead, by the principle of the interreduction of error, to a sceptical view of mathematical sciences.

Both in his Le probleme de l'abstraction and in L'idee de necessite, Jean Laporte ably presents the sceptical position. In the latter work, he undertakes an analysis of the property of hypothetico-deductive necessity attributed to mathematics by such men as Serrus, Bertrand Russell, Pierre Boutroux, Louis Couturat, Cavaillès, and of the current proposals by formalists and logicists that mathematical demonstration is equivalent to construction (113). Since both proposals, he points out, ultimately maintain that necessity in mathematics and its mode of demonstration are due ultimately to mere mental conventions of some sort, necessity in mathematics is but a pseudo-idea and mathematics itself can hardly be viewed as a genuine science (114). To counteract these views, he proposes that what has ordinarily been termed the process of abstraction be viewed as reducible to mere tendencies or sentiments regarding the real. These subjective sentiments have some resemblances or analogy to the real; and, though not yielding speculative knowledge, enable us to deal with the real, existing sensible

(112) Marcel Boll and Jacques Reinhardt, Les Etapes de la logique, Paris: Presses Universitaires, 1944; see Chapitre IX, "L'unité de la science," (pp. 117-123), where the authors view logic as the primordial science, and mathematics as applied logic.

(113) Jean Laporte, L'idee de necessite, Paris: Presses Universitaires, 1941. In the introduction (p. xi), Laporte claims that there are but two types of rational necessity; the first, based on existence, is causal or physical, while the second, based on essence, is ideal or logic-mathematical. In part one he concludes, with Malebranche and Kuma, that causal necessity must be abandoned. Part two, pp. 75-148, contains a summary of current views on the nature of demonstration and necessity in mathematics, and on the first principles of logic.

(114) Ibid., "Conclusion," pp. 149-156.

world (115).

An undue stress on the creative ability of the mind is portrayed by a group of philosophers who are either outright idealists or whose basic tenets and final philosophical decisions are unmistakably marked by an inclination in this direction. Prominent among these is Leon Brunschvicg (116), who has done much commendable work on the philosophy of mathematics, of importance equal to that of Brunschvicg. While Meyerson's acute analysis of science, his sound criticism of the history of mathematics and science, of positivism, and of the philosophical tenets proposed by scientists is, unfortunately, tinged with a vesture of idealism (117). The mathematician and philosopher Hermann Weyl, who presently holds to intuitionism on the foundations of mathematics, is an outstanding example of the delights of metaphysics presented from the mathematical view (118). A special group under idealism is that of logicism as a broader view, comprising a number of thinkers for whom logic is both an art of reasoning and a theory of reality, as Jean Cavailles and F. Gonseth (119); for others the field of science and thought is largely self-created, and mathematics is presented as being the science of pure creation by

(115) "La pensée abstraite demeure confinée aux alentours du réel; elle ne mord pas sur lui. Elle nous aide à le manier; elle ne nous le fait pas connaître." Jean Laporte, Le problème de l'abstraction, Presses Universitaires, 1940, p. 129.

(116) For a short study of Brunschvicg, see A.-J. Sertillanges, Le Christianisme et les philosophes, deux tomes, deuxième édition, Paris: Aubier, 1941, tome deuxième, pp. 488-504.

(117) Thomas R. Kelly, Explanation and Reality in the Philosophy of Emile Meyerson, Princeton: Princeton University Press, 1937, p. 118.

(118) See, for example, Weyl's The Open World, Chapter III, "Infinity."

(119) F. Gonseth has shown a continued interest in the philosophy of mathematics and has proposed a theory on the foundations of mathematics which he calls ideonism; see his Qu'est-ce que la logique?, Paris: Hermann & Cie, 1937.

the mind (120).

Anti-intellectualist theories, which are widespread today, are similar in their denial of an intellectual truth-content to mathematical as to all scientific knowledge. By pinning the validity of all scientific knowledge on actionary, affective or volitional bases, these theories must be regarded as being little else than complicated expressions of scepticism with regard to the attainment of speculative science. Mathematics, in these views, is regarded as being merely symbolic (121), or it is subsumed under a pragmatic (122), an instrumentalist (123), or a phenomenological attitude (124).

In the final group of those who have devoted themselves to speculations on mathematics in the total view of philosophy are those who, while remaining within the traditional sobriety of philosophy itself, have attempted to establish a speculative rapproch between metaphysics and mathematics. Among mathematicians, for example, a large number give evidence of a sound philosophical attitude on the nature, function and teaching of mathematics. Among scientists, the view of Jean Dujat and W. R. Thompson, and the views of such professional philosophers as Jacques Maritain, Vincent Edward Smith, Thomas Greenwood, Robert Feys, Harold A. Smart, Gervald des Lauriers, Louis O. Katscoff and Max Black are indicative of a thoughtful consideration of the

(120) For a recent example of this view, see F.S.C. Northrop's The Logic of the Sciences and the Humanities, (New York: The Macmillan Company, 1947); Preface and Chapter XXI.

(121) Alfred Nersyski, op. cit., Book III, Part VIII, "On the Structure of Mathematics."

(122) William James, The Principles of Psychology, two volumes, London: Macmillan and Company, 1901, volume two, pp. 652-659.

(123) See Ratner's Introduction to Intelligence in the Modern World: John Dewey's Philosophy (edited, and with an introduction by Joseph Ratner, New York: The Modern Library, 1939).

(124) Werner Brock, An Introduction to Contemporary German Philosophy, Cambridge: The University Press, 1935, "Husserl," pp. 15-20.

many problems disclosed by present mathematical research, and point towards the possibility of establishing a speculative harmony between metaphysics and mathematics.

PART II: THE NATURE OF MATHEMATICAL ABSTRACTION

Chapter VIII. The Distinction of Speculative Sciences.

The preceding historical sketch argues for the existence of two antithetical tendencies in philosophical views of mathematics and its position in the wider domain of science (1). Of frequent occurrence and popular today, the monist view of idealistic mathematicism argues for the generic and specific identity of speculative science and the legality of but one science, mathematics (2). Resurrecting the ancient Pythagorean-Platonic outlook, the tradition of Ousa, Kepler and Descartes became a powerful influence in speculation and fathered many attempts at mathematical monism (3). By overemphasizing the role and freedom of the mind in mathematical science which recognises no intermediary stages between the mathematically true or false, this view engenders an absolute demand -- unequivocally certain and exact knowledge or none -- and a reductionist

(1) "We find, accordingly, that idealists have tended more and more to regard all mathematics as dealing with mere appearance, while empiricists have held everything mathematical to be approximation to some exact truth about which they had nothing to tell us." Bertrand Russell. The Principles of Mathematics, p. 4.

(2) "The whole philosophy of Descartes was virtually contained in that initial decision, for the I think, hence I am is the first principle of Descartes' philosophy, but it is his pledge to mathematical evidence that led Descartes to the I think. This, I am afraid, was one of those initial decisions, which beget systems of philosophy where everything is conclusively justified, except their very principle. If we need a philosophy whose certitude is equal to that of mathematics, our first principle will have to be the I think; but do we need such a philosophy? And supposing we do, can we have it? In other words, are we sure that everything that is is susceptible of a mathematically evident interpretation? The answer, of course, is arbitrary. You have a full right to bet on the affirmative, but it is gambling, and if by any chance you happen to be wrong, you will be playing a losing game from beginning to end. Everything will be mathematically proved in your philosophy, save only this, that everything can, and must be, mathematically proved." Etienne Gilson, The Unity of Philosophical Experience, pp. 132-133.

(3) Ibid., Part Two, "The Cartesian Experiment," pp. 125-223. Gilson points out how Cartesian mathematicism was succeeded by the physicism of Kant and the sociologism of Comte.

program -- all sciences must conform to mathematical rigor. How impatient is the thought of Plato, Descartes, Spinoza and Leibniz to brand all matters of speculative thought with the deductive character of mathematics, with its certainty, clarity and abstractness. Once an initial proposition is found as a point of departure, their reasoning, like an infinite series, continues precipitately in the unilateral direction of sheer deduction (4). With regard to mathematics itself, idealism apparently holds that the mind, severed from all dependence or relevance to sense data or the world, generates mathematical natures in virginal fashion (5). When pressed for an ultimate response concerning the origin of mathematical entities, the idealist can only have recourse to innate ideas, intuitions or other inexplicable ultimates (6).

The empiricist trend which issues finally in scepticism and materialism and of which Hobbes, Hume and some contemporaries are typical, has not been as often upheld. Laying undue stress on the element of sensation which occurs in mathematics and in all science, this sceptical view endangers or

(4) Harold R. Smart, The Philosophical Presuppositions of Mathematical Logic, pp. 6-8.

(5) "It is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention." P.W. Bridgman, The Logic of Modern Physics, New York: The Macmillan Company, 1928, p. 60. "Today mathematics is unbound; it has cast off its chains. Whatever its essence, we recognize it to be as free as the mind, as prehensile as the imagination. Non-Euclidean geometry is proof that mathematics, unlike the music of the spheres, is man's own handiwork, subject only to the limitations imposed by the laws of thought." Edward Kasner and James Newman, Mathematics and the Imagination. New York: Simon and Schuster, 1940, p. 389.

(6) For a brief analysis of mathematical idealism see C. E. M. Joad, Guide to Modern Thought, London: Faber & Faber, 1933, Section VI, "Reality conceived as mental or spiritual," pp. 94-100; or A. Cornelius Benjamin, op. cit., pp. 448-451.

denies the abstract character of mathematical reasoning. An earthy philosophy, it usually undertakes a piecemeal analysis of ideas, aiming to reduce them to mere impressions. Loath to grant the marks of universality, necessity and stability to human knowledge; it is slow to credit man's abstractive efforts in mathematics with any exactness or precision (7). A one-sided and disjointed explanation of mathematical science results. Human nature is conceived as a sponge or blank wall upon which impressions somehow impinge, leaving traces termed ideas. The existence of mind or its active importance in forming, retaining and refining mathematical concepts is denied (8). In response to questioning, the sceptic may merely repeat his opening position, "All I know is sensation," or "All ideas are but copies of impressions," or humbly and apologetically aver that man can know little.

Apparently proficiency in mathematics is of such a nature that the mathematician who philosophizes seems to be easy prey to the extremes of idealism or materialism (9). As will be pointed out below, mathematics

(7) The logical positivists apparently deny any relation between mathematics and reality. "For the positivists, mathematics is a branch of logic, and logic is merely a language, all of whose statements are tautologies. In so far as mathematics is a language, it is a construct of symbols formed by the investigator for the purpose of preserving his empirical data." Louis O. Kattsoff, *op. cit.*, p. 245.

(8) For example, in *The Logical Syntax of Language*, (New York: Harcourt, Brace and Company, 1937), Rudolf Carnap maintains, "Metaphysical philosophy tries to go beyond the empirical scientific questions of a domain of science and to ask questions concerning the nature of the objects of the domain. These questions we hold to be pseudo-questions." p. 331. The formalist view of the nature of mathematics seems to point to a sceptical or nominalist attitude; see J. Jørgensen, *Treatise of Formal Logic*, volume three, p. 85, Copenhagen: 1931.

(9) "But if the entities of which the universe is on a naively realistic view supposed to consist: substance and space-time, turn out to be mathematical, that is completely resolvable into mathematical formulae, and if to be mathematical is to be mental, more will be implied by the various statements asserting the mathematical nature of things than that the universe is describable in terms of mathematics; it will be implied that the universe somehow is mathematics. And, since mathematics is thought, to be mathematical will also be to be mathematical thought." G.E.M. Joad, *Philosophical Aspects of Modern Science*, New York: The Macmillan Company, 1932, p. 63. See also Gaston Milhaud, *op. cit.*, "Introduction," pp. 1-49 for the study of the influence of mathematical science on philosophy.

is an abstract science; it may consider concepts in their very state of abstraction with a quasi-indifference to their ultimate origin from sensation or their reapplicability to reality. Consequently the distinction of mathematical from logical and metaphysical concepts is precise and delicate, making it easy for the mind¹⁰ to fall into idealism and identify them. On the other hand, mathematical concepts retain a trace of their origin from sense data and are greatly dependent on the imagination. Hence any application to the real world which assumes the existence of a real mathematical order is necessarily materialist; and mathematicism can, as in the case of Hobbes, lead to brute materialism.

While it is proper to develop mathematical science and to apply it wherever possible; while it is of speculative import to study and discern its relation to metaphysics, chemistry and the numerous sciences, idealism supports an erroneous extreme in maintaining that all sciences are mathematics, just as does materialism in considering the universe mathematical in essence. Such statements, often cast off as speculative pearls of wisdom, have no meaning (10). No man is a daily-bread mathematician or scientist; the same mathematician who may write that God is a mathematician, or the scientist who maintains in a lecture-environment that a table is a whirlpool of electrons, complains that his electronic coffee is cold or distasteful. The whole of life and the life of thought is greater and richer than mathematics.

(10) In commenting on Book II of Aristotle's *Metaphysics*, Saint Thomas Aquinas points out how custom and training can influence a man's attitude to truth: "Here he shows how in the consideration of truth men accept it in various ways, saying that some do not accept what is said to them except it be put in a mathematical manner. Indeed, due to custom, this attitude is characteristic of those who were nourished on mathematics." *In II Metaphysicorum*, Lect. V, n. 334.

The proposition contrasted to idealism and materialism that there is a plurality of sciences;-- however the distinction between them is specified -- appears in some form in Aristotle, the medieval schoolmen, Locke, Berkeley, Kant and many contemporaries (11). Concerned as they were with the inroads made by mathematics and science and its encroachments on the validity of other views of the universe, these thinkers aimed to place mathematics within the wider domain of speculative and practical science. To this purpose, they investigated the nature, conditions and methods of knowledge in general and in particular, hoping to establish a correspondence between the unity of multiplicity in the universe with that of a number of distinct sciences. Many contemporary philosophers, mathematicians and scientists exhibit a like concern towards substantiating the authenticity and legality of different branches of knowledge, corroborating, in this way, the cogency of different viewpoints on reality (12).

It is this larger view of things, then, with which the present chapter is concerned. By placing mathematics in its proper place

(11) For a sketch of such theories, see J. Arthur Thomson, Introduction to Science, London: Williams & Norgate, 1911, Chapter IV, "Classification of the Sciences," pp. 82-124; but Thomson concludes (pp. 123-4) that it is a mere matter of discussion whether science is one or manifold. See also A. Cronelius Benjamin, An Introduction to the Philosophy of Science, Chapter XVIII, "The Classification of the Sciences," pp. 399-417, or Jean Fourastié, Note sur la philosophie des sciences, Paris: Presses Universitaires, 1948.

(12) This appears to be the trend of the philosopher-scientists, as Eddington, Jeans, Whitehead, Planck and others; and of the philosopher-mathematicians as Weyl, Russell, Poincaré. For an inclusive study of this type, see Jean Dautat, L'Oeuvre de l'Intelligence en Physique, Paris: Presses Universitaires, 1948; see also E. W. Hobson, The Domain of Natural Science, Cambridge: The University Press, 1923, Chapter I, "Introduction," pp. 1-23; and Chapter XIX, "Natural Science and General Thought," pp. 452-477.

within the domain of knowledge and thus showing that, in common with all sciences, it arises through the mental action of abstraction from sense data, the theory of a plurality of sciences which is here proposed validates the existence of different sciences each with its own principles, questions and methods, unified by metaphysics. Mathematics, in this view, occupies a unique position; it has its own fitness in the range of human knowledge and a singular pertinence in man's study of reality. The validity of this enquiry, however, takes into account the truth that the unity of human nature arises from a twofold principle, soul and matter. This metaphysical dualism, once poorly understood and rejected by Descartes and by many today, alone gives proper emphasis to the germ of truth contained in the empiricist's emphasis on sensation and the idealist's stress on mind. True philosophy takes into account things as they are, letting its thought of reality be measured by the realities about which it thinks and reasons in mathematics, philosophy and in all knowledge (13).

Knowledge in general

The question here proposed on the nature of mathematical abstraction can only be satisfactorily answered when it is viewed as part of a more comprehensive problem, the nature of knowledge in general. For, in order to

(13) For an earnest and patient study of the function of philosophy in dealing with the problems of reality inclusive of the questions proposed by contemporary science and mathematics see Jacques Maritain, The Degrees of Knowledge, Part One, "The Degrees of Rational Knowledge," pp. 27-301.

point out how pure mathematics is a speculative science which originates from the mental action of formal abstraction from things, it is necessary, first, to show the relation of all scientific knowledge to pre-scientific knowledge by reviewing the nature of knowledge in general and pointing out that both actual and habitual science is but a more perfect possession of knowledge. Once the meaning of the elusive term "science" has been resolved in this way (14), the manner of distinguishing speculative from practical science and the distinction of three specific types of speculative science -- metaphysical, mathematical and physical science -- can be more readily ascertained (15).

In the larger comprehension of knowledge, it is apparent that not only scientists or philosophers possess true knowledge, but every man who enjoys the use of reason possesses some truth and is able to enunciate something true as signifying the interior quality of consideration and reflection (16). When we commonly assert that a man shows something --

(14) It is generally admitted that the word science is currently used with different meanings. For a summary analysis of five current theories on the nature of science and of philosophy, see A. Cornelius Benjamin, op. cit., pp. 3-40.

(15) The fundamental tenets on which the above inquiry rests, that of the truth of human knowledge, the unity of human nature and the distinction of faculties in man, though they lie beyond the scope of this inquiry, are propositions without which one cannot philosophize and attain truth. The historical reasons for denying these truths are summarized in various places; for example, see Etienne Gilson, The Unity of Philosophical Experience, Chapter XI, "The Breakdown of Modern Philosophy," pp. 277-301, and Chapter XII, "The Nature and Unity of Philosophical Experience," p. 306-327.

(16) Aristotle, Metaphysics, 995^a 30 - 995^b 14. For an analysis of the nature of pre-scientific knowledge and its relation to science and philosophy, see W. H. Kane, "Introduction to Philosophy," The Thomist, 1(1939), pp. 193-212.

as when we helplessly say "I can do nothing more with him; he knows what he is doing" -- we attribute a content of some kind to that person which is distinctly his and is directive of his actions. All our dealings with our fellow men assume that this content resides, in some way, within him; for whether we wake the professor of astronomy at night to question him about the stars or rouse some doctor or mechanic for professional purposes, we are confident that their ordinary and technical knowledge is part of their makeup. That all knowledge, whether pre-scientific or scientific, speculative or practical, sense or intellectual, is habitual to a person is a matter of common belief and a foundation for social organization and action (17).

The knowledge which a man holds within him is the union of his power to know with what is knowable in things; it is his mental possession of the truth about things of all sorts; numbers, waves, motors, horses, symbols, and the whole of extra-mental reality. Philosophers, in this connection, make the essential nature of knowledge clear when they say that the essence of an actual act of knowledge lies in the identification of the intellect with the thing known (18). The man who is asked to bespeak for us his

(17) Alan J. McSweeney, The Social Role of Truth, Washington: The Catholic University of America Press, 1943, pp. 187-188. Or see William A. Gerard, "Public Opinion, Determinism, and Free Will," The Thomist, 9(1946), pp. 172-185.

(18) "The most general idea of knowledge is that of the extension of a being, or rather of the to-be of the knower, beyond itself. In order to know, one must somehow become another; for to know is to be another. It is a sort of participation in the to-be of another, not inasmuch as this other, the object, is an individual, existing distinct and separate from the knowing subject, but inasmuch as the subject by knowing becomes in the intentional order and lives the idea, the reality, which is the object," Henri Lanson, "The Problem of knowledge in General," The Modern Schoolman, 24(1946), p. 1.

knowledge of radium is psychologically aware of the knowledge he possesses and of how he orders the intelligible notes of the nature and properties of radium so that he can present them as being in one to one correspondence with that which he knows of radium itself. Here, of course, as in all human matters, the differences between men and their possession of truth may be strikingly diverse; the knowledge which an expert has achieved is vastly superior and more perfect than that of a tyro, while heredity, early education and environment, individual talents, the nature and difficulty of the objects we aim to know and various psychological factors help partially in accounting for these differences (19).

True philosophy has always argued with idealists and sceptics and maintained that our minds know things and not merely impressions or ideas of things. Were we to hold seriously that we do not know things and that we know naught but our own images, then everything we perceived would be true; in this case man would be the measure of all things and their first principle, and Socrates' ancient snarl at Protagoras for his theory of knowledge would apply: "I wonder that he did not begin his book on Truth with a declaration that a pig... which has sensation is the measure of all things (20)." It has been the

(19) A thorough and studied explanation of the reasons for these differences can be found in Leo M. Bond, "The Effect of Bodily Temperament on Psychological Characteristics," *The Thomist*, 10(1947), pp. 483-501, and 11(1948), pp. 28-104.

(20) "I am charmed with his doctrine, that what appears is to each one, but I wonder that he did not begin his book on Truth with a declaration that a pig or dog-faced baboon, or some other yet stranger monster which has sensation is the measure of all things... For if truth is only sensation, and no man can discern another's feelings better than he, or has any superior right to determine whether his opinion is true or false, but each, as we have several times repeated, is to himself the sole judge, and everything that he judges is true and right, why, my friend should Protagoras be preferred to the place of wisdom and instruction, and deserve to be well paid, and we poor ignoramuses have to go to him, if each one is the measure of his own wisdom." Plato, *The Dialogues of*, pp. 163-164. See also Saint Thomas Aquinas, *Summa Theologiae*, q. 85, art. 2 c.; and, by the same author, *In XI Metaphysicorum*, Lect. 5, n. 2224.

philosophic sport of idealism and scepticism, through an erroneous analysis of the judgment and through their neglect of either mind or reality in explaining knowledge, to deny the existence of true knowledge conformable to things, thus leaving a see-saw effect in the history of philosophy (21).

Since all intellectual knowledge is, in some way, an identification of that which knows with the thing known and the human mind has true knowledge of things, an unassailably necessary condition for pre-scientific or scientific knowledge is immateriality (22). When I say "I know tobacco" or "I am a connoisseur of wines" I do not mean that I have peppered my mind with tobacco or drowned it in wine, but I do mean that my judgment of what each of these things is and which I hold in an intellectual manner within me -- the nature and essence of each -- has a real, objective counterpart in tobacco or in wine. Similarly, when I chide a student at an incorrect answer or refuse to give him a passing mark for an inept definition of a function, I do not merely assume that he has not memorized the definition but, more specifically, that he has not assimilated that which is knowable. Knowledge as embodied in judgment is thus a quality which delineates and specifies the human mind with the knowable being of real, extra-mental being; the knowable nature, quantity, qualities and properties of things immanently exist within the mind, and a relation of similitude

(21) For a short sketch of the opposition between idealism and scepticism, see Paul Janet & Gabriel Seailles, A History of the Problems of Philosophy, translated by Ada Menahan, edited by Henry Jones, Vol. II, Chapter IV, "The Relations Between Mind and Matter," pp. 214-243.

(22) Maurice R. Holloway, "Abstraction from Matter in Human Cognition According to St. Thomas," The Modern Schoolman, 23(1946), pp. 120-130.

between these notes as known by the mind and as existentially resident in things, obtains (23).

Immateriality, the condition of all intellectual knowledge, is not however immediately attainable by the human mind which is bound to matter and must act through man's sensory makeup; nor is removal from matter, which is the necessary property of man's knowledge, immediately given to the mind from things as they exist in existential concretion (24). Hence we find that in the actions of acquiring and perfecting knowledge, man's mind exercises the complex combination of mental actions which are traditionally familiar as simple apprehension, judgment and reasoning (25). It is through the exercise of these three actions of his mind that man becomes the knower, subsuming the rich content

(23) "This becoming another without losing one's own individuality may seem an implicit denial of the principle of contradiction. In point of fact, however, there is no such denial here for the simple reason that the one knowing becomes the object in the order of intention (the intentional order) while remaining what he is -- man -- in the order of nature. Now the order of intention is the order of the mind, the order of knowledge, and, for man, is the order of immaterial representation. Consequently, to the end that the subject may become the object in the order of intention while remaining itself in the order of nature, it is necessary that the corporal object be received not physically, as it exists in reality, but intentionally, that is by means of an immaterial representation in the subject." Henri Renard, *op. cit.*, p. 6.

(24) "Knowledge and sensation are divided to correspond with the realities, potential knowledge and sensation answering to potentialities, actual knowledge and sensation to actualities. Within the soul the faculties of knowledge and sensation are potentially these objects, the one what is knowable, the other what is sensible. They must be either the things themselves or their forms. The former alternative is of course impossible; it is not the stone which is present in the soul but its form." Aristotle, *On the Soul*, 431^b 24-30.

(25) George Barry O'Toole, "Truth is in the Judgment," *The New Scholasticism*, 17(1943), pp. 1-15.

of things unto himself and acquiring an immaterial counterpart,
within his mind, of the truth about things (26).

Through the aid of internal and external sensory powers, man's mind is able to effectuate a liaison with reality in the phantasm through which the thing is preferred to the mind and, on the basis of which, the mind realises for itself the knowability in the thing (27). This obtention of the requisite immateriality is initially achieved through simple apprehension. For, although the truth or falsity of human knowledge resides in judgments which are mental pronouncements on the very existence of things, simple apprehension is the first actualisation of the mind's potentiality to know (28). In achieving immateriality through this complex psychological action, man's mind is concerned merely with the simple natures of things as they are presented through the auxiliary imagery of the external and internal

(26)* Most thinkers recognize the fact that the human mind has an unlimited potentiality to know; this seems to be the meaning of the phrase of Hermann Weyl, for example, in The Open World (New Haven: Yale University Press, 1934), "the infinite is accessible to the mind intuitively in the form of the field of possibilities open into infinity... p. 83.

(27) In the Summa Theologica (q. 84, art. VII, c.), Saint Thomas Aquinas shows that man's sensory faculties are required for acquiring knowledge and for utilising knowledge already possessed.

(28) However they may have terminated their reasoning on this question, philosophers have recognized that the problem of human knowledge ultimately rests on two pivotal questions: 1) Is truth a conformity of mind to thing? and 2) In what operation of the mind does truth reside? A decision on either of these questions implicates a decision on the other; for example, if truth is said to be conformity of mind to thing, then it must be decided in what acts of the mind this conformity occurs. Skepticism seems to answer that there is no mind, but that man, being merely receptive, achieves some kind of conformity; idealism seems to lodge the ultimate truth of human knowledge in the acts of simple apprehension in which the mind creates or produces ideas. For a statement of the median solution to this question, see Gerald B. Phelan, "Verum Sequitur Esse Rerum," Mediaeval Studies, 1(1939), pp. 11-22; see also Saint Thomas Aquinas, op. cit., q. 85, art. 2, c.

sensory powers. The mind does not consider that with which a simple nature may be joined; it rather prescind from spatio-temporal characteristics. By giving due attention and consideration, it merely apprehends or perceives the nature of the thing present to the mind. For example, if I were to present ^{an} ~~the~~ odd-shaped, covered object to my class, they would merely, in simple apprehension, perceive it as a "thing"; after removing its cover, they would perceive a clock, face, or minute-hand -- dependent, among other factors, on whether they direct their attention and consideration. Each of these mental actions is a simple ~~abstruse~~ ^{abstraction} by my mind of a nature, of that which the thing is. It is a universal concept abstracted from the particular thing, which, though it may be joined with other things in reality, is capable of being separately considered and conceived by the mind. Summarily, then, though the thing itself is the point of departure in my pursuit of knowledge, the first terminal point which I achieve is the immaterial similitude that the mind has thus formed unto itself (29).

Having once acquired, through simple apprehension, concepts which are mental presentations of the natures of things, the mind is able to enunciate a judgment by which it possesses the truth of things or discerns their falsity (30). In this act of mental composition or

(29) For a good explanation of this relation of mind to being, see Etienne Gilson, *Le Thomisme*, Deuxieme Partie: La Nature, Chapitre VII, "Connaissance et verite," pp. 313-331.

(30) "In fact, it is our judgments alone that are capable of being true or false. Percepts and concepts, on the other hand, can never be false. Their function is simply to represent -- to make mentally present what is really present." George Barry O'Toole, *op. cit.*, pp. 1-2.

division, the mind affirms or denies the correspondence of what is within the mind to what is true of things themselves. For example, having once shown to me some object which I am told is a clock, and having apprehended its nature, I am able in the future to make a judgment affirming or denying the nature of clock as I understand it of any object shown to me. It is, therefore, in the action of judging and the mental act of reasoning which ultimately issues in judgments, that the truth of human knowledge lies. Thus both simple apprehension and reasoning are only related to truth to the extent that I make a mental or spoken affirmational or denial concerning extra-mental reality in the judgment (31).

Therefore by judgments, then, my mental knowledge identifies itself with what is true or false in things themselves, and is exhibited as having or lacking such conformity. If I am taking a course in electrical measurements and falsely judge the positive to be the negative pole with disastrous results, or if a mechanic judges that my automobile will not function properly because it lacks oil, or if I state that the space of our experience is curved -- I am making mental pronouncements and decisions which either have an exact counterpart in things and are true, or lack this counterpart and are false.

(31) "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true; so that he who says of anything that it is, or that it is not, will say either what is true or what is false..." Aristotle, *Metaphysics*, 1011^b 28-29. See also Gerald B. Phelan, *op. cit.*, pp. 17-22.

But all judgments, though they may be true, are not qualified by the same degree of certitude or firmness, and it is this variable relation to truth as doubtfully, probably or necessarily true which the mind may possess that differentiates human knowledge into the two large classes of prescientifically and of scientifically true judgments (32). For example, a mechanic is ignorant or doubtful of why my automobile does not function when he arrives in answer to my call for assistance; he may, however, have a large number of possible judgments that his experience tells him may probably apply; after I have informed him of what occurred, he may finally and certainly judge that the automobile lacks gasoline or some other necessary element. The same range of the plausible and the certain, the contingent and necessary, holds for other practical affairs, as it does for speculative science itself. I may truly judge that certain reactions can be explained if I suppose that light exhibits a wave-like motion, while a corpuscle-like action may explain others and a quantum judgment will explain some or most observable phenomena (33).

(32) The distinction of pre-scientific and scientific knowledge and the validity of the former is gradually being recognized. See for example, the interesting remarks made by George Boole (Volume II: The Laws of Thought in George Boole's Collected Logical Works, edited by F. E. S. Jourdain, Chicago: Open Court, 1940), on the nature of science, pp. 418-445. Or see Chwistek's concern with demarcating the distinction between these two types of knowledge, and in combatting the excessive claims of scientists in Leon Chwistek, The Limits of Science, Chapter I, "Introduction," pp. 1-23 and Chapter II, "The Limits of Sound Reason," pp. 24-52. Emil Meyerson's tendencies in this regard are well known; see for example his article, "Le sens commun vis-à-vis la Connaissance?" Revue de Métaphysique et de Morale, 30(1923), pp. 13-21, and Emil Meyerson, Identity and Reality, Chapter XI, "Common Sense."

(33) Thomas Greenwood, Prolegomenes à la Théorie des Quanta, (Ottawa: Les Éditions de l'Université d'Ottawa, 1942, Section VII, "Connaissance et Causalité Physiques," pp. 49-61.

In all these cases I am making true judgments; their difference lies in the degree of certainty and extent of applicability to real effects, having as its lower bound perplexity, uncertainty or sheer doubt, and for its upper bound the absolute certitude, necessity and demonstrability of science itself (34).

It is important, then, in any philosophical view of science, to consider that the realm of pre-scientific knowledge is an area in which truth has been attained and is attainable and likewise that much of what is frequently paraded as certain is later seen to be merely conjectural or probable. The distinguishing features, in fact, of scientific knowledge are its comprehension of a universal necessity in nature itself, and a consequent certitude and firmness of adherence to a judgment on the part of the mind itself. Is it not, in fact, true, that a man usually progresses through various stages before he acquires scientific knowledge (35)? Our senses are the principles from which knowledge originates; through the acquisition of

(34) The failure to recognize the distinction of pre-scientific and scientific knowledge and the consequent inability to distinguish the nature of certainty and truth is found in many writers on the philosophical aspects of mathematics and science. The various stages may be outlined as follows: the judgments of pre-scientific knowledge may go from doubt through suspicion and probability to the certainty of science. In all these cases except doubt, the mind can make true judgments. Truth, on the other hand, formally considered as such, does not admit of degrees; however, it can be said to be more perfect as knowing an object completely or in piecemeal fashion, or by the firmness to which the mind assents to the true judgment. For an account of the likenesses and differences of pre-scientific and scientific knowledge, see W. R. Thompson, Science and Common Sense, Chapter I, "Introduction," pp. 1-8 and Chapter II, "Common Sense and Science," pp. 9-16.

(35) It is perhaps a petulant remark to recall that many writers forget or neglect the psychological origins of science from extra-mental reality, through the aid of parents, teachers and various other influences. In this connection see Vincent Herr, How We Influence One Another: The Psychology of Social Interaction, Milwaukee: Bruce Publishing Company, 1945, Chapter IV, "Social Learning," pp. 60-81.

experience, art, and the aid of memory, a man advances towards the goal of scientific possession of truth (36).

This vast realm of pre-scientific knowledge or imperfect possession of truth is presumbatory and propaedeutic to that of science and arranges itself in a hierarchical order (37). Estimation, the lowest type, is the relation of the mind to truth usually associated with poetry which emphasizes the appeal to the senses or the delights of regular movements, and is merely representative of truth itself. The mind is not overly concerned with the truth or falsity of that about which the poet sings, but is pleased to accept and enjoy the images and ideas which are suggested. The mind is more definitely related to truth in the state of doubt, for it has begun to ruminate and reason about possible judgments, though it actually embraces none of them for lack of evidence. Rhetoric, be it that of the mathematician, orator, politician or friend, stirs the mind and inclines it to accept one, rather than another part of a contradiction and breeds a state of mind called suspicion of truth. The fourth imperfect relation to truth by which the mind charts its way towards science is the state of opinion, belief or probability. Actually adopting one part of a contradictory, the mind yet fears it may be in error and runs back and forth in cogitative fashion, seeking for truth in the matter concerned.

(36) Aristotle, Metaphysics, 980^b 22-983^a 23.

(37) Aristotle, Posterior Analytics, 71^a 1-71^b 8;

The sources of opinion or belief are probable propositions which are true in the most cases, generally accepted, or which seem to be generally accepted but are not really such (38). Probability is investigated in a domain of reasoning which philosophers call *Dialectics* (39); it is in this realm that, cumulatively speaking, common public opinion is formed on matters of politics and prudence, advertisement of manufactured products molds opinion, and popularized information on mathematical and scientific matters is presented (40).

(38) The history of all the sciences is illustrative of how conjectures and generalisations of a high degree of probability and acceptability in explaining a number of effects were helpful in furthering knowledge or in achieving scientific certitude. Even the progress of science today is much assisted by hypotheses, conjectures and approximations. See, for example, Sir William Cecil Dampier, *op. cit.*, Chapter V, "Nineteenth-Century Physics," pp. 217-268. This same point is brought out strikingly in Thomas Greenwood, Prolegomene a la Theorie des Quantes, *passim*.

(39) The term is used here as denominating that part of logic, frequently neglected, concerned with the method of reasoning from probable propositions which are based on common knowledge, the use of induction and hypotheses. All science, in its progress, uses dialectical material which it may later discard or revamp into a more accurate approximation. For an explanation and illustration of the term quite close to its usage here, see Mortimer J. Adler, Dialectic, New York: Harcourt, Brace and Company, 1927, "Introductory," pp. 1-25. R. D. Carmichael's book, The Logic of Discovery, (Chicago: Open Court: 1930) is an attempt in dialectics, as is also Jacques Hadamard, The Psychology of Invention in the Mathematical Field.

(40) The question of the extent to which a thoughtful person should allow himself to be influenced by public opinion is studied in Max Guszowski, "Public Opinion and Thomistic Principles," The New Scholasticism, 19(1945), pp. 156-160. Leon Chwistek in The Limits of Science, soundly criticises the erroneous views which are current today and propagated as scientific truths; pp. 47-52.

In the latter realm of probability, moreover, much of the true knowledge that science proposes is to be allocated. For it is being more generally admitted that the truth of many scientific proposals, theories and conclusions are not certain science but probably or conjecturally true (41). In the history of mathematics, for example, the scientific demonstration of the mathematical truth of the calculus was not at all evident to its inventors; a comparatively long period of time elapsed, in fact, before the notions of function, limit and number had themselves become sufficiently clear to mathematicians to be subsequently utilized in establishing the calculus as mathematically scientific doctrine (42). Or, as another illustration, it can be pointed out that both the theory of Newton and of Einstein truly apply to the universe; the degree of their application, however, is limited not so much by our mathematics as by our imperfect and hence probable knowledge of the nature of the universe (43). For, as will be pointed out later on, a mathematical scheme or system is but one way in which the mathematician may conjecture that the continuity of the universe is ordered; the application of a mathematical scheme to extra-mental reality of living or non-living nature discloses, consequently, our limited and merely probable knowledge of its actual

(41) Of the frequent utterances made by scientists in this regard, see, for examples, A. S. Eddington, The Nature of the Physical World, pp. 303-315 and, by the same author, The Philosophy of Physical Science, pp. 89-105 and *passim*. Many scientists however, expect to find absolute certitude on all matters and are led, as a consequence, to apparently sceptical conclusions on matters of science; for example, see the conclusion in Bertrand Russell, Human Knowledge; Its Scope and Limits, New York: Simon and Schuster, 1948, p. 501.

(42) Carl B. Boyer's The Concepts of the Calculus excellently illustrates this dialectical development of the calculus; see his remarks in Chapter VIII, "Conclusion," pp. 299-309.

(43) In The Evolution of Physics: The Growth of Ideas from Early Concepts to Relativity and Quanta, (New York: Simon and Schuster, 1942) Albert Einstein and Leopold Infeld illustrate the use of probability and conjecture in the progress of physics; the authors, however, frequently represent this progress as due to a "creation" of the mind; see, for example, the concluding remarks, pp. 310-313.

existence.

II. Scientific Knowledge

When we consider the domain of science or perfect knowledge, we are no longer concerned with a fluctuating or unstable, but a firm and unshakable adoption by the mind of one part of a contradictory proposition (44). The mind no longer has fear of error; after much doubt, uncertainty and difficulty, it has finally assimilated the truth of things. The mind now knows that the truth of its judgment has been demonstrated with unshakable certitude, that the judgment deals with what is necessarily such in extra-mental reality and is capable of demonstration (45). If we compare the distinction between the knowledge, say, of a professional mathematician and a mechanic, a baker and a politician, it is seen that each is concerned with the things of experience, but each in his own way. Here the baker, who might be skilled in his practical art, to maintain that democracy is the best form of government, he would be unable, as a rule, to demonstrate this proposition or to support his assertion by sufficiently scientific reasons; and a mechanic who had a passing, popularised acquaintance with relativity theory would, no doubt, fail an examination on the mathematics of relativity. Education and personal initiative are two of the many factors which accentuate the genuine

(44) Owen Bennett, The Nature of Demonstrative Proof, Washington: The Catholic University of America Press, 1943, Chapter II, "The Beginnings of Intellectual Knowledge," pp. 3-13.

(45) Aristotle, Posterior Analytics, 72^b 5-73^a 20.

difference between pre-scientific and scientific knowledge, and which enable us to understand how science is marked by a firm tenure of the mind to the truth it knows (46).

The many roads which lead to scientific knowledge as a certain and necessary cognition of what we had vaguely known, or as leading from the unknown to the known, all originate, consequently, in prior, pre-scientific knowledge (47). Science is acquired on the basis of this pre-cognition by a demonstration of the existence of a thing or of a property of the thing. Whether actually taught us by another as a body of doctrine or whether we ourselves acquire it as a discipline, science arises from pre-cognitions of various sorts ultimately classifiable as three: 1) the complex principles or sources of demonstration, 2) the subject of demonstration, which is that which the demonstration concerns, and 3) the properties we demonstrate as true of the subject matter under consideration (48).

The unprovable complex principles, needless to say, may be general axioms common to all scientific endeavor or proper to a certain field of investigation, and their number for science as such or for a particular

(46) "We suppose ourselves to possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends, as the cause of that fact and of no other, and, further, that the fact could not be other than it is. Now that scientific knowing is something of this sort is evident -- witness both those who falsely claim it and those who actually possess it, since the former merely imagine themselves to be, while the latter are also actually, in the condition described." Aristotle, *Ibid.*, 71^b 8-14.

(47) Aristotle, *ibid.*, 71^a 1-71^b 7; see also Owen Bennett, *op. cit.*, Chapter III, "The Pre-Existing Knowledge Necessary for the Demonstrative Syllogism," pp. 14-33.

(48) George E. Buckley, *The Nature and Unity of Metaphysics*, Chapter I, "The Nature of Science," pp. 1-36.

science may be large. Frequently, of course, what may be assumed as a principle early in an investigation, may later be seen either to be a false assumption or capable of proof (49). While each particular science has its own principles, the principles of identity, contradiction, sufficient reason, causality, and uniformity of nature are general axioms (50).

With regard to the pre cognition of a proper subject matter for a given science, it is manifest that no teacher or student can proceed unless, in some way, the subject matter is foreknown before actual demonstration (51). Were a student of plane geometry to question the existence of points, lines or planes, I might credit him with metaphysical astuteness but would likewise endeavor to demonstrate, by definition and construction, the existence of a circle or square which form part of the proper subject matter of geometry. In physical science, moreover, due to the spatio-temporal existence of the subject, a demonstration of the existence of subject matter is frequently done more immediately by experiment. Much of physical science is, in fact, concerned with sense-

(49) For discussion on this point, see Morris R. Cohen and Ernest Nagel, An Introduction to Logic and Scientific Method, New York: Harcourt, Brace and Company, 1936, Chapter VII, "The Nature of a Logical or Mathematical System," pp. 129-133.

(50) Morris R. Cohen and Ernest Nagel, op. cit., Chapter II, "Some Problems of Logic," see especially pp. 161-187, where the authors show the basis of logical principles in extra-mental reality. See also James Macen Sullivan, An Examination of First Principles in Thought and Being, Washington: Catholic University of America Press, 1939; see Chapter I (pp. 1-34) for a study of general principles, and Chapter II (pp. 35-96) for particular principles.

(51) In Procedures and Metaphysics, Edward W. Strong analyses this point of view for mathematical demonstration in the "Appendix," pp. 221-245.

immediate demonstration of subjects of demonstration as when I show, by various experiments, the existence of atoms or of electricity to a doubting pupil.

^{FINALLY}
~~Firstly~~, what I endeavor to do with the subject matter, say, of geometry, is to demonstrate the properties of any of its continuous quantities, as of a circle or a triangle. Again, before I can proceed to demonstrate properties of equality, similarity or collinearity as genuinely attributable to the subject matter, I must first inform the student what these properties are; only then can their proper inference in the subject be established (52).

It is in this manner, by acts of scientific demonstration based on previous knowledge and combined with illustrations and various imaginative insights, that science is self-acquired or taught, and likewise reaches out to include the previously unknown. The teacher of mathematical science, for instance, has guiding principles for his reasoning process which enable him habitually to reason with facility, confidence and surety about the existence or non-existence of a countless number of quantitative relations and properties attributable to continuous or discrete quantities. Similarly, the existence of a micro-universe of unique, uni-cellular animals opened for biologists a new subject matter of investigation which has grown both in complexity and vastness while revealing, at the same time, a realm in which much of our knowledge is

(52) Ibid., pp. 242-245.

not as yet scientific, but conjectural, opinionative and probable (53).

Consequently, demonstrative science, concerned with principles, subject matter and properties, shows the intimate connectedness of scientific to pre-scientific knowledge and ^{REVEALS} ~~shows~~ both how the habit of science is acquired and that it is either an individual mental action or a habit of mind (54), which characterises and differentiates some knowledge as certain, capable of demonstration and concerned with what is necessary and universal in extra-mental reality. Our cultural belief and intercommunication substantiate these characteristics of scientific knowledge (55). If I perform a number of experiments on light and publish the scientifically certain results in a treatise or scientific journal, I have no fear that the universal and certain knowledge contained in my experiment can be understood and verified, if account is made of various conditions that may exist in different parts of the world.

(53) Ralph Stayer Lillie, General Biology and Philosophy of Organism, Chicago: The University of Chicago Press, 1945; see Chapter I, "Living and Nonliving: Biological Evolution and Development," pp. 11-24, and Chapter II, "Living and Nonliving Systems," pp. 26-50.

(54) The "habit" of science here refers to an intellectual virtue or to what is more commonly but loosely called the "scientific outlook", the "scientific habit" or l'esprit scientifique. These current expressions, though vague and undetermined in their use, are surely indicative of some internal view, aspect or content within man -- an intellectual habit or virtue as here maintained -- which a man may or does possess. For example see Jean Ulma, "La position moderne du débat: esprit géométrique, esprit de finesse," (pp. 331-338) or René Dugas, "La mathématique, objet de culture et outil de travail," (pp. 339-345) in Les grands courants de la pensée mathématique, présentées par F. Leclercq. For an explanation of the term as here used, see Etienne Gilson, op. cit., Deuxième Partie, Chapitre VI, "L'Intellect et la connaissance rationnelle," pp. 291-312.

(55) Alan J. McGroarty, op. cit., pp. 137-140.

True, it is not the same, identical experiment; but the sameness and the identity of the universality, certainty and necessity which it contains are the properties of scientific knowledge which my publication purports to express and which furnish the basis for historical study (56).

The argument holds equally well with regard to mathematical science. Suppose, for example, that I publish the results of a proposition in number theory or formulate a mathematical system hitherto not invented. Were I even to use words instead of symbolism to express my mathematically scientific conclusions, a contemporary or future mathematician would be capable of understanding the meaning of my scientific efforts and results (57). As in other sciences, it is apparent that mathematical concepts are universal, have unequivocal certainty, are characterized by necessity and are capable of demonstration. Mathematical doctrines, in fact, possess such indifference to reality, that they do not, as a rule, provoke the heated arguments that sociological or psychological judgments or propositions usually incite.

(56) Gilbert J. Carraghan, A Guide to Historical Method, Chapter Three, "Certainty in History," pp. 70-80. For a fine discussion on the notions of universality and contingency as characteristic of human knowledge, see A. Dalbès, "Discussion du rapport de M. Buytendjik," in Cahiers de philosophie de la nature, publiés par Amy Collin et Roland Dalbès, IV: Vues sur la psychologie animale, par Hans Andre, "—J.—J. Buytendjik, George Welshauvers, Maurice Manquant, Avant-propos; discussion, notes par Amy Collin, Roland Dalbès, Jacques Maritain, Paris: J. Vrin, 1930, pp. 95-103.

(57) The use of words before symbols, for example, in the early history of algebra is well known: For a short sketch see Henry B. Fine, The Luller-System of Algebra treated Theoretically and Historically, New York: G. F. Stechert & Co., 1947, Part II: "Historical"; or see John Wesley Young, Lectures on Fundamental Concepts of Algebra and Geometry, "Note: The Growth of Algebraic Symbolism," by S. G. Mitchell, pp. 226-239. For a more comprehensive presentation, see F. Cajori, A History of Mathematical Notations, two volumes. Chicago: Open Court, 1928-9, Volume I.

Based on a difference of purpose or principle, habitual science is commonly looked upon as being either pure science -- theoretical or speculative -- or applied science (58). The end for which a man seeks speculative science is to acquire truth -- mathematical, biological, metaphysical or any sort; the purpose of practical knowledge is some good to be achieved, the proper functioning of any action or work outside the mind itself (59). When a pupil grasps the notion of function, number, limit, group or operator, he has acquired speculative mathematics; the application of these truths to practical problems and other sciences constitutes applied mathematics.

The principles, or ultimate sources from which speculative or practical science arises is the second means of distinguishing them (60).

(58) There are, of course, many current theories which negate the distinction between speculative (pure) and practical (applied) science. Positivism, operationalism, instrumentalism and existentialism are generally alike in denying the existence or validity of speculative knowledge and emphasise the volitional, active or affective aspect in man as the source of knowledge. The two books of Felix Klein, Elementary Mathematics from an Advanced Standpoint, (translated from the third German edition by E. F. Hedrick and C. A. Noble, Volume I: Arithmetic, Algebra, Analysis, New York: Dover Publications, 1945, and Volume II: Geometry, New York: The Macmillan Company, 1939), are excellent illustrations of the difference between pure and applied mathematics. For a recent philosophical study of the distinction between speculative and practical knowledge in general, see Jean Petrin, Connaissance Speculative et Connaissance Pratique: Fondements de leur distinction, Ottawa: Les Editions de l'Université d'Ottawa, 1948.

(59) This does not deny the possibility of an ordering or construction within the mind as a speculative art which is done in logic and mathematics, as will be pointed out below. See, in this connection, Saint Thomas Aquinas, De Veritate, q. 3, art. 3, c.

(60) Jean Petrin, op. cit., pp. 23-54.

For speculative knowledge, things themselves or extra-mental reality are the principles from which, in final analysis, speculative science arises and by which it is measured. The movement can be viewed as going from things to the mind; for the incipient reaction of man on impressions from objects results, under proper conditions, in his acquiring knowledge of what these things are. In practical knowledge, on the contrary, the principles are within the mind and the movement is from the mind to things; in phraseology analogous to mathematics, the mind is measured in speculative science by things, while the mind is the measure of things in practical or productive science (61).

This manner of distinguishing these two species of science is readily verified in mathematics. In the teaching of mathematics at the elementary, secondary, college and university levels, we corroborate the difference which we hold to exist between pure and applied mathematics in the same fashion that we do for science in general (62). As the history of science points out, the application or applicability of mathematics to relativity, to quantum theory and to the subject matter of other sciences which has resulted in such combinational sciences as mathematical physics, physical chemistry, mathematical biology and so forth, further validates the distinction

(61) Henri Pichette, "Considération sur quelques principes fondamentaux de la doctrine du spéculatif et du pratique," Laval Théologique et Philosophique, 1(1945), pp. 53-58.

(62) There is, of course, a tendency in those occupied with the practical application of mathematics to speak of physical geometry or chemical mathematics as there is frequently a tendency to neglect or despise the applications of mathematics. For an illustration of the tension between these two attitudes, see Pierre Duhem, op. cit., Chapitre IV, "Les théories abstraites et les modèles mécaniques," pp. 77-81.

of pure from applied science (63).

Leaving aside, for the present, the realm of practical science especially as applicable to mathematics (64), we ask whether there is but one science or many: is the truth of things speculatively knowable in uniquely different modes which exhibit themselves as specifically distinct habits of speculative science? This question -- science as both a unity and a multiplicity, or how the same thing can be known in different sciences -- requires that both the common character of science and its principle of diversification be considered. The habit of science, while being similar to pre-scientific knowledge as a mental possession by way of identification with the knowable, is yet different from pre-scientific knowledge by being causal and certain knowledge concerned with what is universal and necessary in extra-mental reality. To answer the question regarding the nature and distinction of speculative sciences, then, due attention must be paid both to the manner in which man holds scientific knowledge and to the part which things themselves play in its acquisition (65).

(63) This is brought out well in E. T. Bell's The Development of Mathematics, Chapter 16, "The Impulse from Science," pp. 336-343 and Chapter 17, "From Mechanics to Generalized Variables," pp. 344-355. In the following Chapter, "From Applications to Abstractions," (pp. 356-373), Bell points up the interaction between mathematics and the sciences to which it is applied.

(64) For a fine study of the application of mathematics to physics, see Jean Dautat, L'oeuvre de l'intelligence en physique, especially Chapters IV and V. On the relation of mathematics to biology, see H. R. Thompson, Science and Common Science, Chapters 5 and 6.

(65) Thomas Von Aquinas In Librum Beati de Trinitate Questiones Quinta et Sexta, Nach dem Autograph Cod. Vat. lat. 9850 mit Einleitung herausgegeben von Paul Feyer, Fribourg: Société philosophique, 1948; see Quaestio V, Art. 1, pp. 23-31. All references or quotations from the Commentary of Saint Thomas Aquinas on the De Trinitate will be taken from this edition.

The basic condition of all knowledge and hence of science is immateriality. Unable to know things in their ephemeral subjection to the unpredictable vicissitudes of change and motion which typify their actual existences, the mind, by dematerialising things, constitutes them as scientific objects (60). Since, moreover, science deals with what is necessarily such in reality and not with what is contingent or capable of being other than it is, immobility or removal from change is an additional requirement for the judgments of scientific knowledge (67). Both requisites, immateriality and immobility or removal from matter and motion, are achieved by the mind in simple apprehension through abstraction from sense data or through the second operation of the mind in which a judgment of separation is pronounced (68). In simple apprehension the mind forms unto itself a scientific concept as formally signifying the immaterially and immobily conceived nature of a thing, while in the judgment it indicates separation (69). Were it not for the fact that an objective stratification in reality corresponds to this abstractive or separative action of the mind, knowledge might be considered idealist -- a falsification of reality. Consequently, in what follows, a consideration of abstraction as total or formal, and of the judgment of separation, will be seen to disclose a triple

(60) Thomas Von Aquin, *Ibid.*, p. 26.

(67) L.-S. Greiger, "Abstraction et separation d'après S. Thomas," *Revue des Sciences Philosophiques et Théologiques*, 31(1947), pp. 3-20.

(68) *Ibid.*, pp. 21-40.

(69) J.-D. Robert, "La Métaphysique, science distincte de toute autre discipline philosophique selon Saint Thomas d'Aquin," *Divus Thomas*, 24(1947), pp. 208-222.

distinction of what is immaterial, necessary and immobile in reality itself, and yield three types of sciences: physical science arising from total abstraction; mathematical, from formal abstraction; metaphysical, from the judgment of separation (70).

As the history of science and our own experience assure us, the mind of man is not constantly considering reality in a state of scientific abstraction, nor does the mind readily do so (71). Although man is constantly being affected in his sensory makeup by matter and motion of all sorts, he is not constantly exercising an actual scientific action on these many influences and sense impressions; no scientist is constantly engaged in issuing scientific judgments. In fact, before achieving science, man has already arrived at much pre-scientific knowledge through the actions of judgments and reasonings on extra-mental reality; in this process he is stimulated, of course, by books, conversations and arguments, lectures, his own musings -- and any number of extra-mental impressions. The immobile conception and consideration of objects necessary for scientific knowledge is attained either through a more perfect, sustained and patient consideration of reality in simple apprehension in which the mind actually

(70) Thomas Von Aquin, op. cit., Quaestio V, Art. 3, pp. 36-43. The question of the specification of sciences outlined above, adheres closely to the thought of Saint Thomas Aquinas. For a similar view, see L.-V. Regis, "La Philosophie de la nature," and Etude et Recherches Publiées par le Collège Dominicain d'Ottawa, Cahier I: Philosophie, pp. 127-156, Ottawa: L.-B. Geiger, La Participation dans la philosophie de S. Thomas d'Aquin, Paris: J. Vrin, 1942, Chapitre XII, "L'etre", pp. 315-341; L.-B. Geiger, "Abstraction et separation d'après S. Thomas," cited above, and J.-D. Robert, op. cit., Some contemporary writers, following Cajetan and John of St. Thomas, speak of three degrees of formal abstraction; for a presentation of this view, see, for example, Jacques Maritain, The Degrees of Knowledge, Part One, passim.

(71) This plea for the distinction of the scientist as actually engaged in scientific abstraction or the details attendant upon scientific work from the scientist as a man is frequently ignored despite its relevance in recognizing a distinction of science from the pre-scientific and science from life. The part that pre-scientific knowledge of mathematics plays in learning the science of mathematics is pointed out to some extent, for example, in Cooley, Hollis R. et al., Introduction to Mathematics, passim, and Felix Klein, op. cit., passim.

engages in total or formal abstraction, or through the close, scientific realization of reality which the mind can acquire in the separation of metaphysics. Confused concepts and the probable truth of judgments become clear and fixed in the mind in such wise that the scientific judgments made are perfectly conformable to reality. Accordingly, steady attention, prolonged consideration and a complex set of psychological factors -- dependent on individual talents and makeup -- are required, so that the mind may form unto itself a formal signification of the nature presented to it in the phantasm and achieve scientific judgments (72).

Since it is the indeterminacy and potentiality of matter which is the source of change and movement of every type, the difficulty of a scientific appraisal of things invested with matter and subject to motion and change lies in this very fact: the relation of dependence or independence which things bear to matter for their very existence (73). A scientific consideration of things show that they can be considered

(72) Jacques Maritain, Distinguer pour unir ou les degres du Savoir, nouvelle edition, revue et augmentee, Paris: Desclée de Brouwer et Cie, 1946, Annexe I: "A Propos du Concept," pp. 769-819.

(73) This entire question of matter and its relevance for the nature of mathematical abstraction has received comparatively little attention by philosophers or philosophizing mathematicians, possibly because of a biased attitude towards ontological and epistemological dualism. Basically Aristotelian, the distinction of the various types of matter that is used here is also brought out, in various places, by Saint Thomas Aquinas: see, for example, In VII Metaphysicorum, Lect. 11, n. 1801-1836 where he points out its use in distinguishing speculative sciences; see also In III de Anima, Lect. 8, n. 705-717, and Summa Theologiae, q. 88, art. 1, ad 2.

(a) in the sciences and philosophy of nature as dependent on matter for their existence and in order to be known; (b) in mathematical science as dependent on matter for their existence but not in order to be known; (c) in metaphysical science as independent of matter for existence and in order to be known.

The first scientific study of things in the philosophy and in the sciences of nature is one which is based on the intrinsic and necessary relation which things have to be joined to matter for their very existence and, accordingly, to be known. Trixie, the cat with whom the children play, an elm tree which shades the yard, a huge mastiff which frightens the children out of the yard and out of their wits -- exist in sensible matter (74). Extra-mental reality as considered under this first scientific point of view is the reality of animals, men, trees, chemicals, and existing things which are concentered in individual sensible matter; as, for example, the privately possessed fur, claws, flesh and teeth of a dog. The scientific mind, however, in total abstraction, constitutes these things as scientific objects by precluding from their individual traits and forms the universal: it abstracts the universal, potential whole from the parts in which the thing exists as subject. Thus, to know scientifically the nature of dog, I cannot

(74) The term sensible matter is here used as being either individual or common. Individual sensible matter refers to the personal, private matter of flesh and bones which an individual, existing substance possesses in its spatio-temporal existence; common sensible matter refers to the mind's abstracted, universal conception of individual sensible matter as freed by the mind from the determinations of dimensions, time, and individualizing, private qualities. The speculative sciences of total abstraction abstract from individual sensible matter, and judge, reason and define in terms of common sensible matter.

scientifically consider him without reference to matter at all, but the matter in which I scientifically envision him is the matter common to all material things and technically known as common sensible matter; the dog has legs, fur, claws, teeth. Or, in the scientific study of electricity, light, skin or hydrogen, although I use individual instances of these as my point of departure, my interest in these objects is their common nature, properties and accidents as the terminal point of science (75).

In order to have the genus of science, this first point of departure of abstraction -- that of relinquishing things in their concreteness -- is requisite. Inasmuch as matter is the basis of motion, concomitant with this abstraction from matter is the correlative requirement of abstraction from particular movements which are featured in the flux and variation of individual things. That which is subject to change and movement can be what it is one instant, and be changed the next; thus it is required for the attainment of the scientific level that natures, essences, common properties -- the necessary and unchanging reasons of that which in itself is subject to mobility and contingency -- be achieved (76). Relinquishing, then,

(75) Emil Meyerson, Identity and Reality, Chapter XII, "Conclusion," pp. 384-440.

(76) ^{SOURCE} Though the ~~science~~, meaning and nature of science is disputed, it is commonly granted that science is concerned with the general, the abstract, and the universal. Even logical positivists allow some type of generalisation and temporary synthesis. For some references to the idea of abstractness and generality in mathematics, see G. H. Hardy, A Mathematician's Apology, Cambridge: The University Press, pp. 28-52, or the remarks of Julian Lowell Coolidge in the "Epilogue" to A History of Geometrical Methods (Oxford: Clarendon Press, 1940), pp. 422-423.

the private, personal, individuating characteristics of sensible matter, the mind, in total abstraction, considers the universal nature which it has actualised unto itself from the phatasm, as abstracted from the parts in which it is actually the subject. Once the mental terminus is thus achieved for scientific consideration, the mind is able to utter scientific judgments bearing on extra-mental reality (77).

All speculative sciences are characterised by the first degree of abstracting the universal from particular things which is called total abstraction since it abstracts a universal whole -- a totum -- from its subjective parts. While familiar with and knowing the individual things of experience in daily life, man has no strict science of the concrete individual thing which is ineffable: mentally and verbally the individual thing is, for science, an unspeakable and undescribable heterogeneity (78). Though the mind does not directly of itself attain a knowledge of the singular thing in its concretion, it ^{RE} reflects on the phantasm or image supplied by its internal powers of memory and imagination, and thus has a reflective knowledge of individual things. If, for example, I ask you if you liked the sketch of Napoleon Bonaparte which I showed you yesterday, memory and imagination supply you with the image upon which the mind reflects and reaches out to attain

(77) The entire psychological process by which the mind is led to scientific judgments is a pertinent part of the growing philosophy of science and philosophy of mathematics. Felix Klein, in the two volumes of Elementary Mathematics from an Advanced Standpoint, Jacques Hadamard, in his Psychology of Invention in the Mathematical Field, and Henri Poincaré, in Science and Hypothesis, among others, have concerned themselves with the mathematical aspects of this question.

(78) W. L. Thompson, op. cit., Chapter III, "The Nature of Science," pp. 17-27, and Thomas Greenwood, op. cit., pp. 49-50.

a knowledge of the singular thing. The immediate sense experience of the astronomer at the telescope, for example, enables him to identify a star; but his means of identification are mainly sensory and his sensory faculties fulfill these functions by supplying the necessary imagery. This reflective knowledge is manifestly different from the knowledge he exhibits in a lecture on the nature and constitution of the stars (79).

Our knowledge, then, in this first level of science and in all science, is that of a mind joined with matter and dependent on matter for its information about things. The mind deals with universal natures, accidents and properties, and with such general conceptions as culture lag, minimum threshold of experience; but, for full and complete knowledge (and in order to add to our comprehension of things and re-embrace them in wider and more accurate generalities) we must, as men, be concerned with the individual and the singular; in this way we compensate for the mental retreat into abstractness and generality by a correlative movement towards the singular which we aim, ultimately, to know as it is (80).

(79) "Our mind cannot directly know the singular, but the singular is directly known by us through the sensitive powers, which are receptive of forms from things in a corporeal organ: thus our sensitive powers receive these forms under determinate dimensions and in this way lead to the knowledge of the singular material thing." Saint Thomas Aquinas, *De Veritate*, q. 10, art. 5. c.

(80) "...Science is concerned with something in two ways: in one way primarily and principally; and thus science is founded on the universal reasons of which it deals; in another way science deals with some things secondarily and by a certain reflection, and this it concerns those things of which these are the reasons, inasmuch as it applies the universal reasons of particular things even to particular things by the help of the inferior powers. For the knower uses the universal reason both as the thing known and as a means of knowing. I am able, for example, through the universal reason of man, to judge about this or that man. All the universal reasons of things, however, are immobile; consequently, in this regard, every science deals with the necessary. Some of the things, however, of which these are the reasons are necessary and immobile, and some are contingent and mobile; in this way there are said to be sciences of contingent and mobile things." Thomas Von Aquin, *op. cit.*, Quaestio V, art. 2, ad 4; p. 38. See also Harold P. Smart, *The Logic of Science*, pp. 80-81.

The sciences contained under this first degree of total abstraction are generally called physical sciences or the sciences and philosophy of nature, and include many particular branches -- physics, chemistry geology, psychology, ethics, sociology and others. Though each has its own questions, methods and principles, they are alike specifically in abstracting from sensible matter and movement, considering reality in terms of common matter, and requiring, for the full realisation of knowledge, the aim to know things in their ultimate concretion. Though much of our knowledge in these sciences has not as yet been crowned with the certitude of science, our aim in knowing is, evidently, to achieve this perfect certitude. Granting the distinction of the ordinary man's knowledge from highly probable judgments and from the certain judgments of science, and conceding the distinction between these speculative sciences in each of which certitude is attainable, the dialectical movement of knowledge towards science and the tension and strain which frequently arise between the sciences and scientists is readily explicable (81).

Philosophers have rightly contended that in this lowest, yet perfect type of scientific knowledge, we first find man confronted the first scientific determination of with the proper formal object of his mind: the quiddity or nature of sensible things as given him in simple apprehension whose truth

(81) Emile Meyerson's Identity and Reality is a close study of this inter-scientific tension and the relation of scientific to pre-scientific knowledge.

is attained in judgments on the very existence and nature of extra-mental reality (82). The difficulty of attaining knowledge of reality in this type of scientific endeavor was already understood by Plato who condemned all these sciences to the realm of probability; he was unable to explain how man could achieve rationality from the multiple, irrational movements and changes which characterize things in this level of science. Like many a subsequent thinker, he failed to grasp that here the scientific mind, though faced with mobility and materiality, is well able immaterially and immobily to conceive and judge of reality; and, by reflection on phantasms, compare its true knowledge with things themselves and know the nature of mobile being (83).

The judgments and definitions of the objects of sciences which lay on this first level of science, are in terms of common, sensible matter. Abstracting from their mathematical formulation, electricity, atoms, nitrogen and skin tissue, if they are sufficiently known to man to be defined, are defined in terms of common matter. Our difficulty in defining these things lies in the fact that we may as yet be ignorant or have merely probable knowledge of their natures, and both the rise and the multiplication of sciences on this first level is due to the fact that nature, being highly determined and actual, contains much

(82) "Natural things are known through abstraction from individual matter, though not through complete abstraction from sensible matter. For man is known as composed of flesh and bones; through abstraction, however, from this flesh and these bones. Thus it is that not the intellect but the senses or imagination directly know singulars." Saint Thomas Aquinas, In III De Anima, Lect. 8, n. 716; see also his Summa Theologiae, I, q. 85, art. 5, ad 3.

(83) Thomas Von Aquin, Quaestio V, Art. 2, p. 34.

that man can know. Further, our manner of knowing through abstraction adds to the difficulties attendant on knowing immaterially and immobily, those things which are being subjected to ceaseless motion and change. The ultimate justification of the principles which are used as bases of demonstration or guiding principles in these sciences, for example, the uniformity of nature or the principle of causality, is due to metaphysics. Any questioning of the validity of these principles, or any attempt to validate their use, while a frequent indulgence of scientists in this realm, is properly a metaphysical view outside the scope of these first sciences (84).

In formal abstraction, the second type of scientific abstraction proper to mathematical sciences, the activity of simple apprehension is uniquely distinct in its point of termination within the mind, although alike in its points of departure or origin. Mathematical natures originate by abstraction from the data of sense, but in a different manner from the physical natures of total abstraction. In total abstraction, as has been seen, the mind prescind from the singular characteristics and concentrates on the universal nature, thus yielding a whole with its subjective, potentially predicable parts. On the other hand, in mathematical abstraction, we are asked to consider, for example, a regular polygon or a line drawn on a blackboard or imperfectly realised as the limits of some physical body. In this mental

(84) See George M. Duckley, op. cit., pp. 208-210, where the author briefly shows the relationship of metaphysics to the particular sciences.

action, we neglect in either or both cases, all the sense qualities, the color, weight, texture of the object presented to us and merely consider the order of the parts of the given object; the order, namely, which gives the thing a hexagonal, circular, ellipsoidal or other structure. In this case the mind abstracts form from the matter to which it is essentially joined in reality and from which it is seized -- through the aid of the sensory powers -- and presented by the imagination to the mind (85).

The termination of this abstraction with a mathematical nature, whether it be a circle, a set, a group or a number is vastly different from the first. In total abstraction the mind is concerned with a formal representation of the nature of the thing; in formal abstraction, the mind prescind from what the thing is -- a circular apple or quasi-continuous pieces of chalk forming a hexagonal figure -- and forms a signification of the thing merely as indicative of the order of its parts (86).

The scientific consideration of the universe in mathematical science, consequently, is not concerned with the reality of plants, horses and sounds; but, by prescinding from all the qualities of these material objects and forming a conceptual signification of them only as in some way quantified things, the mind forms an abstracted universe unto itself. Reality, then, as the basis

(85) Thomas Von Aquin, Quaestio VI, Art. 2, c., p. 43.

(86) Ibid., Quaestio V, Art. 5, ad 5, p. 41.

and the measure of my knowledge in speculative mathematical science, is not reality as such but reality only as quantified. The things with which mathematics deals are dependent on sensible matter to exist but not to be known: if anything circular, square, or ellipsoidal is to exist, it does so with sensible matter in the real, extra-mental world; in mathematical abstraction, however, I prescind from sensible matter both as individual and as common.

I do not, however, relinquish all reference to matter in mathematical science. Quantity, being an accident of bodies, cannot exist or be conceived apart from some reference to matter. In mathematical abstraction I abstract quantified substance. In this case, though, substance is merely considered as the subject in which quantity properly and really resides in reality -- as that in which quantity or the order of the parts of substance are -- and as the subject in which I abstractively consider it with the mind. Inasmuch as the first level has spoken of sensible matter -- that is, matter as perceptible to the senses -- as individual or common, the achievement of mathematical abstraction, not attaining complete freedom from all reference to matter as such, is technically referred to as intelligible matter (87). Intelligible matter, or matter

(87) Intelligible matter refers to substance as abstracted from all qualities and merely affected by quantity; inasmuch as we retain quantified substance, we retain matter, though intelligible matter is not directly attainable as such by the senses except through the qualities and by prescinding from the qualities. Intelligible matter is individual or common. Individual intelligible matter refers to single instances of substance as affected by quantity as visualized in the imagination, for example, three concentric circles; common intelligible matter is the common nature -- for example, of concentric circles -- as known by the mind. The mathematical sciences are concerned with common intelligible matter in its qualitative state of discreteness or continuity, through the aid of the individuation and multiplication of quantified substances in the imagination. See in this connection, footnotes 73 and 75, above. Metaphysical science, as will be pointed out, abstracts from all matter.

which is the principle of indetermination and potentiality as made quasi-intelligible to the mind, is likewise individual or common. The mind attains the common nature of mathematical entities by conceptualising from the individual phantasms of the imagination; the multiplication of mathematical natures in this "mathematical universe" is done by acts of the imagination. For example, I may ask my class to consider three circles lying on their common tangent; the multiplication of circles is a task which man achieves through the aid of the imagination, while the nature of the circle and its definition are limited merely by this imaginative content, and hence, in a sense, as unlimited as the imagination (18).

Since the mathematical universe bears a furtive likeness to the physical universe, it is important to lay emphasis on this notion of intelligible matter. The fact that my abstract knowledge of man is seen to be true of each man has given occasion to some philosophers and scientists to conclude that there is a human nature which exists apart from all men and in which they share; the apparent incongruity of being able to multiply mathematical instances of circles, numbers and other entities and of being able quite successfully to apply mathematical knowledge to reality has also led thinkers and mathematicians to

(18) It should be pointed out that the above explanation does not grant that mathematics is a free creation of the mind or that the laws of mathematical thought are imposed by the mind on reality. Instead, it is here maintained that the laws of mathematical thinking are imposed on the mind from the very nature of quantified substance as given to the mind as it abstracts from the phantasms given by the sense from extra-mental reality. A more detailed explanation on this point is offered in the following chapter.

substantiate mathematical natures and to speak of the externality of mathematical relations (89). This speculative error can be obviated only if it is born in mind that in mathematical abstraction and judgments, the mathematician considers that which is in reality -- substance and the accident quantity -- not as it is in reality where it is always joined to sensible matter, qualities motion and change, but as abstracted from reality by a mental removal of these features; the mathematician considers substance and quantity in terms of intelligible matter with the help of the imagination. Hence, though it has happened in the past that mathematicians have erred in identifying the imaginative 'presentation' of space with its intelligible content, it is clear that such judgments bearing on extra-mental reality are not the proper function of speculative mathematical science as such (90). Much of the recent progress in mathematical science is due to the progressive realization that imaginative entities are not to be identified with the intelligibility of the quantity of mathematics; just as much of the speculative confusion retarding progress is due to a continued lack in understanding the nature of abstraction in mathematics (91).

(89) This is the metaphysical implication of the theory of logicism, especially that of Bertrand Russell; see André Darbon, La Philosophie des Mathématiques: Etude sur la logistique de Russell, publiée par Madeleine La Grace-Darbon avec la collaboration de François Chatelet, Preface de René Poirier, Paris: Presses Universitaires, 1949, seizième Leçon, "Le Réalisme Logique," pp. 177-197.

(90) The question of applying mathematical principles to other sciences and consequently to reality is, of course, based on the fact that quantified substance is abstracted from extra-mental reality and re-applies there if due account is taken of qualities, motion and change.

(91) Carl B. Boyer, op. cit., Chapter VIII, "Conclusion," pp. 298-309.

Certain points of similarity and difference between total and formal abstraction will help to clarify them. Both types of abstraction and the judgments of the speculative sciences developed from them are ultimately measured by reality; however, it is the familiar world of things with sounds, colors and tastes whose scientific nature is ultimately sought in the physical sciences, and all ^aits judgments ultimately terminate there; while in mathematics, the nature of the real world only as abstractively considered as quantified substance is sought, and its judgments terminate in the imagination (22). In both cases, the ultimate point of departure for the abstraction peculiar to each science is sensible matter; the point of arrival, however, or the manner in which the formal object of each science is constituted, though alike in being universal, is different inasmuch as the universal of total abstraction is a universal whole based on the composition in reality of a whole with its parts, while that of mathematics is the universal of quantified form based on the composition in reality of matter and form (23). Further the process of total or formal abstraction is alike inasmuch as what is abstracted and considered abstractively by the mind, is really joined in reality: whether I abstract the nature of man from an individual man, or the nature of circle from a circular object, I am considering things which are not so separated

{22} Thomas Von Aquin, op. cit., Quaestio V, Art. 2, pp. 63-65.
 {23} Ibid., Quaestio V, Art. V, pp. 38-41.

in reality, but merely as constituted as scientific objects.

Finally, there is a difference in the intelligibility which each of these sciences yields. The movement of the mind in total abstraction is an ascension from the particular to the universal and thence to the more universal, as when the mind proceeds from the nature of brute and man to that of animal, going towards what is more general in extension and less in comprehension. A similar case is the progressive generalisation of physical theories, which, though capable of including more phenomena, contain less intelligibility and comprehensibility in themselves. In total abstraction, as a consequence, the farther the mind becomes removed from the determination and actuality of extra-mental reality, the more congeniality and intelligibility there is for the scientist. In the case of formal abstraction, however, the abstracted natures of mathematics are more intelligible to us in the abstractive state than in their extra-mental existence. For example, the nature of a circle or an algebraic group is more knowable to us than is an extra-mental circular object or some imperfect realisation of an algebraic group (94).

The distinction of the movement of the mind as being simply apprehensive in which the mind obtains simple quiddities, natures

(94) Georges Bouligand et Jean Desbats, La Mathématique et son unité: introduction aux éléments de l'analyse et à la philosophie des sciences déductives, avec cent quatre-vingt-cinq figures, Paris: Payot, 1947, Troisième Partie, Chapitre II, pp. 230-253.

or essences is vastly different from that which is proper to the judgment which directly attains existence. Since the mind is concerned in metaphysical science with the very being of things, it no longer abstracts that which is joined as such in reality, for that would be equivalent to a separation of the very existence of things and the mind would be in error (95). In the judgments of metaphysics one can only separate that which is really separated in reality, as when I say ~~a man is not a mountain~~, or being is not equivalent to non-being; consequently, metaphysics is more properly said to be based on a process of separation rather than on a process of abstraction (96).

The scientific judgments of metaphysical science, then, are pronouncements on the very texture of reality itself primarily through separation and negation. For example, it is only by knowing the separation of a being from other beings that I am able to discern that being is; similarly, it is only by grasping the actual division of a thing from other things and the lack of division in the thing itself that I am able to affirm that being has unity (97). In the total or formal abstraction of simple apprehension the mind is only concerned with a particular aspect of being; it abstracts one thing without knowing and considering another to which the thing may be joined,

(95) Thomas Von Aquin, op. cit., Quaestio V, Art. 3, c., p. 39.

(96) For a more detailed examination of the notions of abstraction and separation see J.-D. Robert, op. cit., pp. 209-222, or L.-B. Geiger, op. cit., pp. 15-40.

(97) Saint Thomas Aquinas, De Veritate, q. 1, art. 1.

and is concerned with judgments about this particular aspect of reality. In metaphysical science the mind aims to know the very existence of things and the ontological texture of reality itself; its initial processes, in consequence, are concerned in pronouncing on the very separation of things in order to make consequent affirmations on the nature of reality.

Metaphysics, like all sciences, originates ultimately from the senses; the things with which it concerns itself by constituting them as scientific objects, however, being independent of matter in their existence, are likewise independent of matter in being conceived, known and defined (98). In this immaterial manner I know those things which in some cases do exist in matter as substance, being, potentiality and actuality, but to whose essence it is not necessary that they be joined to matter. Furthermore, I am also capable by a process of negation, removal of imperfections and analogy, to attain the knowledge of God, the ultimate principle of all being (99). Metaphysics is properly said to deal with whatever is, inasmuch as it is, or, as is frequently said, it deals with being as such (100).

The inherent excellence of metaphysics, then, is due to the fact that it is a scientific attainment of the very being of reality, and that

(98) Thomas Von Aquin, op. cit., Quaestio VI, Art. 2, c., p. 64.

(99) Ibid., Quaestio VI, Art. 3, c., pp. 67-70, and Quaestio VI, Art. 4, c., pp. 74-75.

(100) Gerald B. Phelan, "A Note on the Formal Object of Metaphysics," The New Scholasticism, 18(1944), pp. 197-201; see also Vincent Edward Smith, "On the 'Being' of Metaphysics," The New Scholasticism, 20(1946), pp. 72-84.

it is the most perfect manner of knowing of which the human mind is capable (101). Inasmuch as it deals with whatever is in any manner, metaphysics is not only a science concerned with extra-mental reality as the universe of substance, accidents, actuality and potentiality, but it has a legitimate concern with the being and nature of the particular sciences and the relation between sciences; it is, in fact, the universal science of which thinkers have dreamed and whose knowledge is sought for by philosophizing scientists and mathematicians (102). Metaphysics offers a weltanschauung and a lebensanschauung that is only limited by the limits of whatever is or can be; and, as a corollary to this scientific, doctrinal function which it exercises, metaphysics has a defensive and conciliatory role which is often spoken of as epistemological, the philosophy of science or the philosophy of mathematics (103). If a speculative thinker, however, loses its critical edge while denying the existence of metaphysics, he thereby places himself in the awkward, contradictory position of denying the relevance of a view which he himself has adopted (104).

(101) Aristotle, Metaphysics, 982^a 8-983^a 24, and 993^a 30-993^b 30.

(102) "There is a science which investigates being as being and the attributes which belong to this in virtue of its own nature. Now this is not the same as any of the so-called special sciences; for none of these others treats universally of being as being. They cut off a part of being and investigate the attribute of this part; this is what the mathematical sciences for instance do." Ibid., 1003^a 21-26; see also 1003^a 26-1003^b 34, passim.

(103) A.-L. Dierckx, "De Metaphysicae Defensivae Natura secundum Doctrinam Aristotelico-Thomisticam." Angelicum, 4(1927), pp. 252-270 and pp. 430-453.

(104) "You say, wrote Aristotle in a celebrated dilemma, one must philosophise. Then you must philosophise. You say one should not philosophise. Then (to prove your contention) you must philosophise. In any case you must philosophise." Jacques Maritain, An Introduction to Philosophy, translated by E. I. Watkin, London: Sheed and Ward, 1946. p. 76.

Logic, which has supplanted metaphysical fare for many a contemporary thinker, is not a speculative science of total or formal abstraction nor of separation. Logical consideration is traditionally designated as dealing with second intentions (105), though it has a close affinity with metaphysical, mathematical and physical science. For example, I can scientifically know the nature of man; on the basis of this knowledge my mind can reflect on its own action, consider man as known by my mind and realize that this nature with its abstraction from the conditions of matter can be attributed to, and really is, in other individuals. In this way I form the second intention, the universal as predicable of many, and it is thus that the mind, acting in mode logice, supplies the notions of genus, species, ^{DIFFERENCE}~~difference~~, accident and property (106).

Logic, in the sense of being a science and an art of the mind, considers these second intentions absolutely, regardless of their materiality or immateriality, corruptibility or incorruptibility in extra-mental reality, and is not concerned with the essences of things as such (107). The lack of properly distinguishing a logical and a dialectical reasoning about things which need not apply except as sheer possibility to the real, from a physical,

(105) The first intentions of the mind are the direct universals; by reflection, the mind sees these direct universals as predicable of a group, and thus forms unto itself a second intention or the logical universal as species or genus.

(106) Aristotle, *Metaphysics*, 1004^a 33-1004^b 28.

(107) Thomas Von Aquin, *op. cit.*, Quaestio VI, Art. 3, c., p. 62.

mathematical or metaphysical consideration of reality, readily leads to speculative confusion (108). In this connection, not only the open claim that mathematics and logic are identical, but the frequent logical investigation of reality and ordering of the sciences is due to a failure to recognise the oblique nature of logic. Here again, of course, the multiple aspects open to science are revealed: there is a metaphysical, physical and mathematical science of reality, and there is a metaphysics of science and mathematics as well as a logic of science and mathematics (109).

The hierarchy of three speculative sciences, based as they are on the nature of the mind and of things, is consonant to a similar gradation in man's powers to know. Each of the sciences originates with the senses, and, ultimately considered, terminates there (110), though more immediately at other sense powers. Inasmuch as the concepts and definitions of total abstraction are in terms of common sensible matter, the ultimate test of their veracity -- as the entire mode of procedure in the sciences and philosophy of nature indicates -- lies in the evidence furnished by naive experience or the more refined experience of scientific

(108) This has been pointed out in Etienne Gilson's The Unity of Philosophical Experience.

(109) Thomas Von Aquin, op. cit., Quaestio VI, Art. 1, c., pp. 56-57.

(110) Saint Thomas Aquinas, De Veritate, c. 12, art. 3, ad 2.

instruments (111). Since, in the formal abstraction of mathematical science, the mind is free of sensible matter in its conceptions and definitions, it does not necessarily terminate in the external senses (112). However, in its retention of intelligible matter or substance as quantified mathematics demands verification in the imagination which reproduces individual mathematical entities (113). Finally, in metaphysics, though the mind begins with external sense evidence, it achieves immateriality as such, and the truth of metaphysical judgments lies within the first principles of being as known by the mind.

In summary, then, as a certain and evident knowledge of what a thing is and why it cannot be other than it is, speculative science is the habit, the qualitative faculty of obtaining true conclusions in matters of scientific concern about extra-mental reality. This quality of mind, the scientific, is already induced though merely in its inchoative state, by the first scientific act of demonstration which an individual mind realises; but this is an imperfect, defective and relative state of the scientific habit. Such a man can, perhaps, induce some scientific conclusions but only with difficulty and through a laborious process. As a perfect habit of mind, science actualises

(111) "For the truth of the conclusions of physical science, observation is the supreme Court of Appeal...I do not think that anyone -- least of all, those who are critical of the modern tendencies of physics -- will disagree with the first axiom of scientific epistemology, namely that the knowledge obtained by the methods of physical science is limited to observational knowledge in the sense explained above." Sir Arthur Eddington, The Philosophy of Physical Science, pp. 9-10; see also Thomas Von Aquin, op. cit., Quaestio VI, Art. 2, c. pp. 63-64.

(112) It is sometimes suggested that the ultimate validity of mathematics rests on its applicability to external objects. This theory, however, seems to deny the speculative nature of mathematics and reduces it to practical mathematics. For an expression of this view, see Louis O. Kattsoff, op. cit., pp. 250-255.

(113) Thomas Von Aquin, op. cit., Quaestio VI, Art. 2, c., p. 63.

the mind's potentiality to know in such wise that the mind is easily able to ratiocinate concerning those matters which appertain to a full scientific knowledge of extra-mental reality and to point out errors in scientific matters. Even in a long or difficult demonstration, the man who perfectly enjoys the scientific habit can proceed surely and confidently, not needing teachers or books as an aid to his reasoning process."

This distinction of a perfect and imperfect habit of science is one which we frequently find validated in daily life; among the pupils who present themselves to us each year, there are always some who grasp the matter quickly and without difficulty, and who soon become adept at reasoning in a given subject, while others, we know, never seem to develop beyond an imperfect adaptation to scientific truth. These latter pupils achieve a high degree of faith and probability but lack the habit of science. Even among men of scientific calibre there are the weak and the strong, and not all teachers or habitues of science achieve the perfect possession of habitual science. Acquired as it is through personal acts of demonstration, and disposing its possessor to demonstrate easily and surely the truth he possesses, the habit of science can be intensively increased by repeated acts and extensively by the wider scope of objects to which it is seen to apply, as it can, by lack of use, gradually lose its force.

One of the happy results of possessing the scientific habit is the ability to order, arrange and systematize the concepts which we possess into a system of science. The principles of the science, its definitions,

divisions and proofs are coordinated, arranged in a logical order, and thus presented to the pupil or recorded in articles and books. This important aspect of science is all important inasmuch as it has a bearing on the current endeavor of mathematicians to deduce the whole content of mathematical science from a basic, given content, and on similar efforts toward systematising other sciences. This logical arrangement does not always conform to the historical order of the invention of the science, nor to the best pedagogical method of teaching; it aims, rather, to effect a logical order of inferring and demonstrating the unknown from the previously known.

Perfect science, the complete realisation of similitude between the mind and the total knowable content of things, is an ideal towards which an individual man may tend but never realises. An unaccountable number of facts militate against the individual's mind becoming all things, or perfectly reflecting the total order of the universe; by our combined efforts, however, we are able to contribute greatly towards the advance of metaphysical, physical and mathematical science (114).

(114) Aristotle, Metaphysics, 993^a 30-993^b 30.

Chapter IX. The Nature of Mathematical Abstraction

It has been pointed out in the preceding analysis of the distinction of speculative science into the three classes of metaphysics, mathematics and physics, that the inherent supremacy of metaphysics -- dealing, as it does with anything that is in any way at all -- qualifies it to study the nature of the being which is investigated by any science (1). Consequently, the question concerning the nature of mathematical abstraction, if asked by a speculative thinker as logician, cosmologist, psychologist or mathematician is not properly posed in this manner and can only lead to speculative error. Evidently if I ask What is mathematics? I have stepped outside of the confines of mathematics; holding it at bay and at a reflective distance, I scrutinize its nature and envision it in the relationship which coordinates it to the other sciences and to all that is (2).

(1) "It is philosophy, for example, not mathematics, which tells us whether irrational numbers and indefinite numbers are real beings or only rational beings, whether the non-euclidean geometries are rational constructions built on euclidean geometry and which leave the latter its privileged position, or if, on the contrary, they constitute a much greater system of which euclidean geometry is only one specimen; whether mathematics and logic are divided or not by immovably drawn frontiers, etc. In a word, it is philosophy which assigns the order which reigns between the sciences: scientia est ordinata," Jacques Maritain, The Degrees of Knowledge, p. 42.

(2) "In general one might raise the question, to what kind of science it belongs to discuss the difficulties about the matter of the objects of mathematics. Neither to physics (because the whole inquiry of the physicist is about the things that have in themselves a principle of movement and rest), nor yet to the science which inquires into demonstration and sciences; for this is just the subject which it investigates. It remains then that it is the philosophy which we have set before ourselves that treats of those subjects." Aristotle, Metaphysics, 1029^b 18-20.

The first distinction, then, that is involved in any attempted investigation of current mathematical research is that which holds between the mathematician and the metaphysician (3). In the domain of mathematics itself, one meets such mathematical ultimates as point, number, unit and order, whose meaning and use the mathematician investigates as possible points of departure for mathematical demonstration; aided by these foundational elements and by other influences, the mathematician develops mathematical science. This development of mathematics at top and bottom may, for the sake of clarity, be analogously termed a uni-dimensional concern with mathematics (4); moreover, since

(3) This distinction seems to be in the background of the following quotation: "However, unlike our predecessors of only half a century ago, we do not think of mathematics as beginning with the natural numbers as its basic elements and proceeding thence to the development of its various branches. The latest views tend to assign to these numbers a middle position in the structure of this science. The lower portions are devoted to the foundations of mathematics which are established in the general theories of relations, order, sets, and groups as well as of logic. Above the level of the natural numbers have been erected the various specific mathematical disciplines such as the theory of numbers, algebra, and the various theories of functions, as well as most of the geometrical studies." Abraham A. Fraenkel, "Natural Numbers as Cardinals, Scripta Mathematica, 6(1939), p. 69. The distinction between the metaphysician and the mathematician is brought out in Edward W. Strong, Procedures and Metaphysics, see Chapters I and II; books twelve and thirteen of Aristotle's Metaphysics contain the résumé of his arguments for this distinction in opposition to the Pythagoreans and Platonists.

(4) "We must state whether it belongs to one or to different sciences to inquire into the truths which are in mathematics called axioms, and into substances. Evidently, the inquiry into these also belongs to one science, and that the science of the philosopher; for these truths hold good for everything that is, and not for some special genus apart from others. And all men use them, because they are true of being qua being and each genus has being. But men use them just so far as to satisfy their purposes; that is, as far as the genus to which their demonstrations refer extends. Therefore since these truths clearly hold good for all things qua being (for this is what is common to them), to him who studies being qua being belongs the inquiry into these as well. And for this reason no one who is conducting a special inquiry tries to say anything about their truth or falsity -- neither the geometer nor the arithmetician." Aristotle, Metaphysics, 1005^a 16-31.

they only deal with a particular aspect of being and not with being as such, a similar uni-dimensional and limited development characterizes all other sciences but metaphysics. Any failure to recognize the limitations intrinsic to the very nature of each of these special sciences -- chemistry, geology, mathematics and psychology, for example -- implies that a man has unwittingly succumbed to uttering ultimate pronouncements on the nature of things from the specialized view of his science, and is hard put to escape speculative errors (5).

The second, broader domain of current research on mathematics comprises any further questioning on the nature of these mathematical ultimates or on the foundations and nature of mathematics as a speculative science; here the mathematician or speculative thinker, having recourse to a reflective analysis (6) of mathematics, has turned philosopher and metaphysician (7). It is mathematics as currently envisioned in

(5) This attempted deduction of reality according to the limited measure of some particular science, is, of course, a chronic speculative sickness of mind. Etienne Gilson's The Unity of Philosophical Experience illustrates how metaphysics became a logicism, a mathematicism, a physicism and a sociologism -- a process which is being repeated today. The mind's desire for truth and unity can only be satisfied with the ultimate answers of metaphysics.

(6) Reflective as used here does not mean private opinion or arm-chair speculation, but rather refers to a redirection of the mind on the content of its own actions; see Jacques Maritain, The Degrees of Knowledge, Part One: The Degrees of Rational Knowledge, Chapter IV, "Metaphysical Knowledge," pp. 246-301.

(7) Of course, a prevalent anti-metaphysical and anti-philosophical attitude would deny that the Mathematician is currently engaged in metaphysical speculation. The only retort that can be made to such a denial is that of Aristotle: "You say, wrote Aristotle in a celebrated dilemma, one must philosophize. Then you must philosophize. You say one should not philosophize. Then (to prove your contention) you must philosophize. In any case you must philosophize." Jacques Maritain, An Introduction to Philosophy, p. 78.

this second domain -- a reflective study of its nature, position in the wide domain of knowledge and culture -- which is that proper to the philosophy of mathematics and which may be termed its multi-dimensional aspect, with which this essay and a great amount of current speculation is primarily concerned (8).

Viewed from this multi-dimensional aspect afforded by metaphysics, it is readily seen that the full meaning of current self-consciousness in mathematics is really but a confrontation of mathematics to metaphysics. Disavowing the existence or relevance of metaphysics, however, speculative thinkers variously incline to logicism, physicism, mathematicism, and, in the breadth of their own metaphysical position, to idealism or to scepticism (9). If, for example, I write that whatever philosophy there be is merely critical, controversial and persuasive, and that mathematics, as equivalent to logic, is contrastingly deductive and constructive, I propose an ultimate view (10); though erroneous, this

(8) Louis O. Kattseff, op. cit., Chapter Sixteen, "Mathematics and Reality," pp. 244-253.

(9) See the discussion of G. Bouligand, "Le mathématicien au seuil de la métaphysique," Revue de Métaphysique et de Morale, 52(1947), pp. 308-321. Bouligand concludes, "...je crois pouvoir conclure que chaque mathématicien se réclame d'une métaphysique; elle varie d'une varie d'une personne à l'autre, et dans l'ensemble elle balance entre la pôle de la simplicité jointe à la généralité maxima, qui nous sollicitait à l'instant, et celui de la constructibilité, qui fascinera toujours les praticiens." p. 321. See also the interesting preface of Jacques Hadamard (pp. v-xii) in P. Cousset, Les Fondements des mathématiques.

(10) "The distinction of philosophy and mathematics is broadly one of point of view; mathematics is constructive and deductive, philosophy is critical, and in a certain impersonal sense controversial. Wherever we have deductive reasoning, we have mathematics; but the principles of deduction, the recognition of indefinable entities, and the distinguishing between such entities are the business of philosophy. Philosophy is, in fact, mainly a question of insight and perception." Bertrand Russell, The Principles of Mathematics, p. 129.

view tempers and colors my consideration of both real and logical existence and I am caught, alternately, in denominating the nature of reality with logical fiction or mathematical rigor (11). The principles, methods; the very abstractive and limited view of some special science is, by an act of the will (12), thus confounded with the view of metaphysics, and even reality itself is unable to rectify the adoption of such an erroneous view (13).

The aforementioned distinction between a metaphysician and a mathematician readily explains the contemporary metaphysical situation in regard to mathematics. The contemporary mathematician, having fallen heir to a highly developed science, is attempting to reassemble and reestablish this inherited body of mathematical doctrine into a systematic and independent science. Though mathematics has frequently been prompted and stimulated in its historical or individual development by practical needs, by problems from physics, chemistry and other

(11) "...philosophical argument, strictly speaking, consists mainly of an endeavor to cause the reader to perceive what has been perceived by the author. The argument, in short, is not of the nature of proof, but of exhortation. Thus the question of the present chapter: Is there any indefinable set of entities commonly called numbers, and different from the set of entities above defined? is an essentially philosophical question, to be settled by inspection rather than by accurate chains of reasoning. *Ibid.*, p. 130.

(12) The influence of the will in adopting erroneous views is brought out in Aloysius J. Reicher, Truth and Error: A Study in Critical Logic, St. Louis: B. Herder, 1914, Chapter Sixth, "The Will in Relation to Error," pp. 104-117.

(13) For example, if a solipsist is asked to explain how he can get along with the reality of knives and forks, dogs, cats and telephone poles, he is unable to adequately include these things in his outlook; see, in this regard, Bertrand Russell, Human Knowledge: Its Scope and Limits, Part Three, Chapter II, "Solipsism," pp. 176-181.

sciences, by influences suggested through experience and intuitions of all sorts (14), these factors are looked upon as extraneous. Aiming to reduce the complexity of mathematical doctrine to a bare minimum of elements (15), the mathematician methodically directs his efforts towards a constant resimplification and reduction of principles by a recourse to logical, psychological or symbolical ultimates, or by denying the need of metaphysical or extra-mathematical notions and appeals (16). Summarily, then, the metaphysical endeavors of many mathematicians can be described as comprising 1) a previous acceptance of the entire corpus of current mathematical doctrine as valid; 2) an attempt to determine the simple, ultimate mathematical natures; 3) a consequent rebuilding of this initially accepted mathematical truth with the hope of being able to denominate mathematics as an articulate, coherent, self-consistent, systematic and self-contained science (17).

(14) See, in this connection, the interesting analysis of the difference between invention and description in mathematics and its effect on a sound view of the nature of mathematics in Thomas Greenwood, "Invention and Description in Mathematics," Proceedings of the Aristotelian Society, n.s., 30(1930), pp. 77-90.

(15) This attempt to attain the minimum elements of mathematics is common to all philosophical excursions on the foundations of mathematics; see Louis O. Kattsoff, op. cit., Chapter Three, "The Nature of the Objects of Mathematics," pp. 16-23.

(16) In The Limits of Science, Leon Chrstok argues strongly against a metaphysical reference for mathematics. See Chapters II and III.

(17) In The Nature of Mathematics, Max Black points out how the formalist and logicist theory on the nature of mathematics, presenting mathematics as something which has been achieved, is a static view; he mentions the dynamic aspect of intuitionism as one of the points in its favor; pp. 169-173.

In his endeavor to re-erect the corps of mathematical doctrine, the mathematician has frequently bounded into extra-mathematical territory and found himself confronted with logic, history, psychology, metaphysics and reality. His efforts, in consequence, have a philosophical significance, for they show first, that in his philosophical conditioning the mathematician has fallen heir to either a pronounced idealist or empiricist outlook (18). By proposing that the whole content of mathematics is virtually and solely resident in a basic, simple intuition or in a small number of simple, mental natures immediately given and known -- out of which the whole complex content of mathematics is weaved by the mind -- the mathematician adheres to a philosophical theory on the nature of mathematics which is frequently referred to as mathematical atomism (19). His philosophical theory on the nature of mathematical reasoning is, in this connection, similar to the Idealist Rationale of Descartes; he proposes something as "given" for his point of departure a movement of deduction by the mind towards a desired, foreknown goal, and a point of arrival which is really synonymous with the initial starting point (20).

(18) "We find accordingly, that idealists have tended more and more to regard all mathematics as dealing with mere appearance, while empiricists have held everything mathematical to be approximation to some exact truth about which they had nothing to tell us." Bertrand Russell, op. cit., p. 4. See also Phillip Frank, Between Physics and Philosophy, Chapter IV, "Is there a Trend toward idealism and Physics?" pp. 104-128.

(19) Carl B. Boyer, op. cit., p. 23 and p. 50.

(20) Speaking of Hobbes, Locke and Condillac, Harold R. Smart writes: "The fundamental conception common to all of the thinkers mentioned, and perhaps inherent in their mutual insistence upon the supremacy of mathematics, together with the implied emphasis on the quantitative aspect of things, is what has come to be called, in our own day, the 'linear' conception of inference. Such differences as that between Hobbes' advocacy of the syllogism and Locke's partial rejection of it, or between the various absolutely fixed starting-points of knowledge, such as 'simple ideas' or 'intuitions' did not prevent these thinkers from adopting as their own, and uncritically accepting as an adequate account of inference, the Cartesian schema of a demonstrative chain of reasoning, mechanically 'compounded' or added and subtracted, link by link, from premises or definitions to the conclusion. So much these philosophers gleaned from a logical study of Euclidean and Cartesian geometry."

Harold R. Smart, The Philosophical Presuppositions of Mathematical Logic, pp. 6-7.

The adoption of this philosophical theory on the nature of mathematical reasoning (21), seems to underly the frequent statements which are made accepting the natural numbers as mathematically "given" and the accompanying claim that an historical, psychological or metaphysical reference for the individual mathematician, or for the science in general, is irrelevant (22). As another example, in the development of function theory from the natural numbers as point of departure, it is sometimes claimed that the mind builds up the whole complex doctrine of mathematics in this fashion (23). Is this not rather a simple case of the scientific mathematician possessing the

(21) This is not the place for a detailed philosophical or psychological analysis of mathematics. Only a few points should be stressed. There seems to be a great danger in the prevailing overemphasis on the deductive-postulational character of mathematics. True, the element of constructive invention, of directing and motivating intuition, is apt to elude a simple philosophical formulation; but it remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces. A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules and syllogisms, without motive or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth." Richard Courant and Herbert Robbins, What is Mathematics?, p. xvii; see also Harold A. Smart, The Logic of Science, p. 71.

(22) For a summary of these recent views, see Leon Chwistek, op. cit., Chapter III, "The Development of the Concept of Number," pp. 53-82.

(23) Current textbooks on function theory illustrate this claim. See, for example, I. M. Mitchell and W.H. Bells, The Elements of Mathematical Analysis, two volumes, London: Macmillan and Company, 1937, Volume II, Chapter XXI, "Arithmetic Basis of the Infinitesimal Calculus."

habit of mathematical science, using this habitual ability, employing memory and imagination, introducing lemmas to include extraneous matter, and employing various illustrations to demonstratively teach the rich content of the subject matter under consideration? (24).

A further, second point of philosophical significance in current speculation lies in the evident failure to distinguish the multiple aspects of the problems under consideration (25). The question, what are the foundations of mathematics? is but part of a more global question, requiring, for a complete response, that we be concerned with 1) the nature of mathematical abstraction itself as a speculative science, which is properly the metaphysical aspect of the problem; 2) its psychological origins, which includes its order of learning and order of invention; 3) its historical origins; 4) its logic, or the nature of proof, manner of demonstration, the problems of consistency, freedom from contradiction, as well as its dialectical aspects, as the notion of probability in mathematics. Only after such distinctions are made can, for example, such a question as the notion of mathematical "existence" -- so frequently confused with tests for its validity and

(24) This view is aptly illustrated in both volumes of Felix Klein's Elementary Mathematics from an Advanced Standpoint. Klein's presentation, his critical remarks and his efforts to offset excessive formalism in the teaching of mathematics, are well known.

(25) Chapters three and four on the mathematical sciences in Harold R. Smart's The Logic of Science argue for the merit of these distinct consideration of mathematics.

constructibility -- be seen to mean the habit of science in an individual mathematician, or the abstraction of quantified substance as the proper concern of mathematics, or the historical records of the science in books and articles (26).

The distinction of the metaphysician from the mathematician and the distinction of the many parts of the question on the nature of mathematics are especially neglected in discussions on its foundations (27). Though much concerned with ultimates, statements of an individual investigator are sometimes unclear as to whether he believes there is an ultimate in mathematics which mathematics alone can find and beyond^{which} no further questions are possible (28), or whether he grants the pos-

(26) See, for example, Aristotle's discussion on the notion of mathematical existence in Book Thirteen of the Metaphysics, especially 1076^a 17-1078^b 6. In his discussions with Pythagoreans and Platonists, Aristotle faced many problems on the nature of mathematics that are discussed today.

(27) It has been brought out in a number of places that each of these theories on the foundations -- logicism, formalism and intuitionism -- are characterized both by a method and a philosophical theory; for logicism, see Thomas Greenwood, Les fondements de la logique symbolique, tome I, *passim*, and, by the same author, Etudes sur la Connaissance Mathématique, pp. 1-7 and pp. 30-32, or see Andre Barban, La philosophie des mathématiques, pp. 1-6; for formalism, see Max Black, *op. cit.*, 147-162 and Louis O. Kattsoff, *op. cit.*, pp. 116-119 and pp. 134-135; for intuitionism, see Max Black, *op. cit.*, pp. 186-196 and Louis O. Kattsoff, *op. cit.*, pp. 156-158 and pp. 174-177. A broad, critical view of the three schools can be found in Abraham A. Fraenkel's article, "The Recent Controversies about the Foundations of Mathematics."

(28) This seems to be the basic assumption of Leon Chrístek; his impassioned comments and critical remarks on all sciences but mathematical logic are scattered throughout The Limits of Science.

sibility of an ultimate within the science -- for which he seeks -- without denying the relevance of metaphysics as supplying and as pointing out the relations of mathematics to experience, existence and to the sciences (29). Formalism, insisting on marks and symbols as ultimate natures, brings out, it is true, the formal character of mathematical method, the dialectical aspect of mathematics and the importance of symbolism; it fails, however, to answer adequately what mathematics itself is. Formalism does not consider the origin and relation of mathematics to non-mathematical sciences, nor does it answer how a mere repetition or arrangement of supposedly meaningless marks is mathematical science or allows us to count real objects (30). Logicism has brought to the front the fact that mathematics is a close ally to logic, and, under the form of mathematical logic has opened a wide

(29) "It has turned out that (under the assumption that modern mathematics is consistent) the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception. Furthermore it seems likely that for deciding certain propositions of abstract set theory and even for certain related questions of the theory of real numbers new axioms based on some hitherto unknown idea will be necessary. Perhaps also the apparently insurmountable difficulties which some other mathematical problems have been presenting for many years are due to the fact that the necessary axioms have not yet been found." Kurt Godel, "Russell's Mathematical Logic," in The Philosophy of Bertrand Russell, Volume V, The Library of Living Philosophers, Edited by Paul Arthur Schilpp, Evanston and Chicago: Northwestern University, 1944, pp. 127-128. In this connection, see the interesting article by the same author on the continuum problem: "What is Cantor's Continuum problem?" The American Mathematical Monthly, 54(1947), pp. 515-525, where he concludes that possibly a new set of axioms or a new mathematical outlook is required to solve the problem of the continuum.

(30) See the brief criticism of formalism in Bertrand Russell, op. cit., "Introduction to the Second Edition," pp. v-vi, or that in Frank Plumpton Ramsey, The Foundations of Mathematics and other Logical Essays, pp. 2-5 and pp. 62-81.

breach in the whole problem of deductive reasoning, symbolism and mathematical methodology. Its assertion of mathematical and logical equivalence and its proposed logical ultimates, however, are as yet unproved and logicism has left untouched a whole series of genuine speculative problems (31). Even though it reintroduces the close relationship of psychology and experience to mathematics, intuitionism, by holding that the mathematical intuition is the original experience of man and by attempting to equate mathematics with constructibility, proposes an incomplete solution that bristles with metaphysical, historical and psychological assumptions (32).

Taken, then, as philosophical proposals explaining the complete nature of mathematics, these theories appear to implicate a metaphysics of knowledge and incline towards idealism or empiricism. They are but common indications of an outspoken contemporary metaphysical bias (33), a prejudicial reward for all philosophical speculation as such (34), and

(31) Louis O. Kattsoff, op. cit., pp. 106-107.

(32) Max Black, op. cit., pp. 186-200; see the interesting description of the mutual criticism of formalism and intuitionism in F. T. Bell, The Development of Mathematics, pp. 625-628, and pp. 630-632.

(33) Richard Courant and Herbert Robbins, op. cit., pp. xvii-xviii.

(34) "In the epilogue to The Foundations of Mathematics and Other Logical Essays," Frank Plumpton Ramsey, referring to a lecture by Bertrand Russell on "What I Believe," writes: "Of course his main statement about value might be disputed, but most of us would agree that the objectivity of good was a thing we had settled and dismissed with the existence of God. Theology and Absolute Ethics are two famous subjects which we have realized to have no real objects." pp. 288-289. A little later he concludes: "This brings me back to Russell and 'What I believe.' If I was to write a Weltanschauung I should call it not 'What I believe' but 'What I feel.' This is connected with Wittgenstein's view that philosophy does not give us beliefs, but merely relieves feelings of intellectual discomfort." pp. 290-291.

a lack of metaphysical training and insight (35), -- all of which have influenced the realm of speculative reflection on mathematics. At the same time, however, this global examination of the foundations of mathematics and related questions has again disclosed the multi-dimensional and manifold relations which inseparably bind mathematics to metaphysics, logic psychology and the sciences, and, in a broader view, to civilisation and culture itself (36).

It will be our purpose in this section and in the following chapter, to indicate, as much as possible, how these recent discussions have, from the metaphysical point of view, served but to polarize the intrinsic relatedness by which mathematics is secured in its proper place within the order of speculative sciences (37). To propose,

(35) "Philosophy can only contemplate, examine, and clarify the forms of coordinate thought and principles by means of which a coordinate sphere of thought gains systematic unity. The special sciences provide the empirical filling which, by controlling these principles, makes them coordinate with actual events. It helps to give an integrated view of our knowledge as regards direction and ideal; it is in this sense that it may be called the science of sciences," Irwin Biser, A General Schema for Natural Systems (Nature Considered as a Function of Types of Selectivity and of Modes of Selection), Philadelphia: Westbrook Publishing Co., 1938, p. 116. This remark is later followed by "It is hoped that on the appropriate occasion the contours of this view will be filled out with richer concrete detail and that the identity of philosophy and science may be more conclusively shown." *Ibid.*, p. 140.

(36) Richard Courant and Herbert Robbins, *op. cit.*, pp. xv-xix; see also Chapter Twenty-one, "The Further Significance of Mathematics for Other Fields of Knowledge," in Morris R. Cooley et al., Introduction to Mathematics, pp. 583-615.

(37) Impossible as it would be here to disclose the existence and validity of the philosophical sciences of metaphysics, cosmology and psychology, their pertinence for mathematics and relatedness to it is claimed throughout this essay. The only alternative to this view is to claim that all sciences must ultimately harken to the pronouncements of mathematics or logic or mere private opinion. While it is here claimed that metaphysics is effectively able to attain speculative truth and that it determines the nature of sciences and orders them with respect to each other, at the same time each science has a legitimate concern with its own principles, questions, methods, consistency, sources of truth and problems of error. For a presentation of this view see Jacques Maritain, *op. cit.*, Part One: The Degrees of Rational Knowledge.

consequently, as is here done, that speculative mathematics is an abstractive, speculative science is not to take a position that runs counter to all contemporary proffered by mathematicians (38). Many of the theories commonly proposed, it will be seen, are partial emphases and aspects of the total nature of mathematics as a speculative science; they frequently deal with matters on the fringes and periphery of mathematics, and need to be properly ordered to indicate how, as partial truths or partial errors, they lie within the common ground of metaphysics and mathematics (39).

The resolution of both the basic and the peripheral problems which comprise the metaphysical investigation on the nature of mathematics are here seen to cluster about the centripetal problem; an adequate specification and determination of the intellectual habit of mathematics as an abstractive science. Since speculative science is measured by the truth of things, an objective, extra-mental foundation for mathematics in reality must be pointed out 1) as revealing, in some way, the laws of mathematical being to the mind; 2) as being the point of departure for each man who acquires mathematical science; 3) as containing an objective content concerning which the mind reasons and to which the mind becomes identified by judgments in the habit of mathematical science;

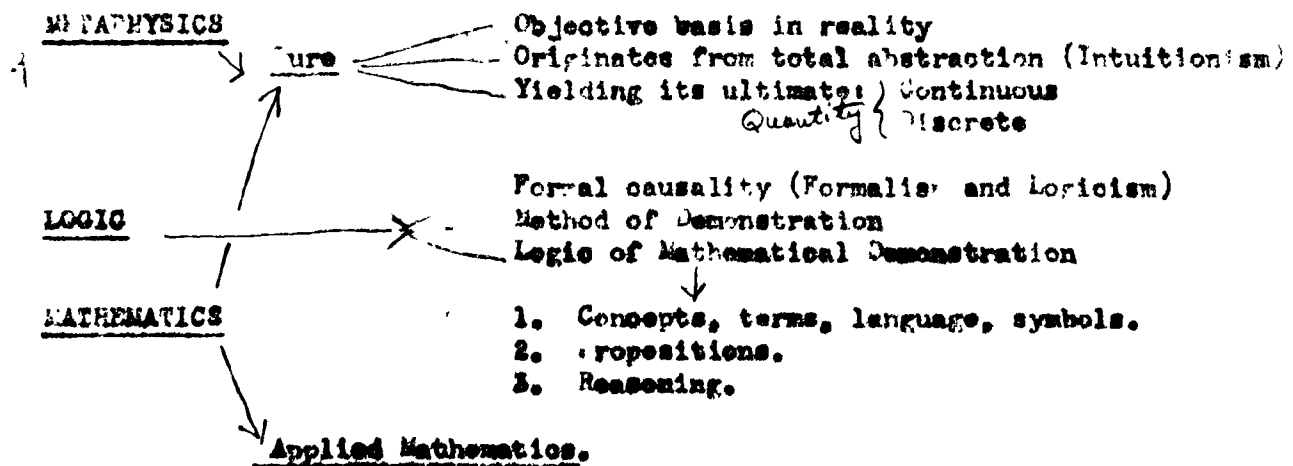
(38) For example, mathematics is generally conceded to be a science of abstractions or as dealing with generalities; see, for example, the following selected references: Eric T. Bell, The Development of Mathematics, p. 8, Chapters 8 and 9, passim, and, by the same author, The Queen of the Sciences, p. 14; Julian Lowell Coolidge, A History of Geometrical Methods, "Epilogue", pp. 422-423; George David Birkhoff, Mathematics: Quantity and Order, p. 308.

(39) Logicism, for example, has merely considered the relation of mathematics to logic; intuitionism, to psychology and formalism has emphasized mathematical method.

4) as insuring the objective "givenness" upon which the mathematician invents and elaborates mathematical science in various ways. The aim of this chapter is to determine the mathematical outlook, and to summarily point out its distinction from and relatedness to the other speculative sciences (40)

In the process of simple apprehension, when a mathematical figure is pointed out for the first time and called a "hexagon," the mind merely abstracts mathematical natures from sense data (39). This psychic state is manifestly different from one in which a peculiarly complex looking curve is drawn, astonishingly called a simple curve, and a mathematical judgment is uttered: "a simple curve is one which nowhere crosses upon itself". (41). In the case of simple apprehension the mind abstracts

(40) A complete philosophy of mathematics as here envisioned would necessitate a consideration of problems suggested in this outline.



(41) "Here there are really two distinct categories of things of which an account must be given -- the ideas or concepts of mathematics, and the propositions of mathematics. This distinction is neither artificial nor unnecessary, for the great majority of writers on the subject have concentrated their attention on the explanation of one or other of these categories, and erroneously supposed that a satisfactory explanation of the other would immediately follow." Frank Plumpton Ramsey, op. cit., p. 2.

natures which are not abstracted or separated in reality; or, as is frequently and correctly said, the perfect circle, line or group of mathematical abstraction does not exist as such in reality but only as held in abstraction by the mind. Hence it follows that we cannot lift this abstraction as such to a judgment and say "The mathematical circle exists in this circle drawn on the board," or "This object is the number one." Nor, on the other hand, can we judge that "the circle exists apart from all matter," or the natural numbers really exercise a separate existence." It can only be said that with the aid of the imagination, the mind, by abstracting from sensible matter, is able to consider and judge the essence of a mathematical nature in abstracto. The objects, therefore, which are given to the mathematician for his consideration in the process of simple apprehension, are seen to depend on sensible matter for their extra-mental, imperfect existence as mere terminal points of the extension of any sense-perceptible thing, but do not depend on sensible matter for being conceived, imagined, defined and considered in mathematical science; they are conceived and defined in terms of intelligible matter (42).

(42) The use of the technical term "intelligible matter," will surely meet with adverse reaction. Simply stated, the necessity of defining or conceiving of mathematical entities in terms of intelligible matter means: 1) I only know singular imaginative or constructed instances of a circle by knowing the universal nature of circle; 2) to define or adequately designate the universal nature of circle I must place circle in the larger class of figure or infinite set of points (in the continuum of intelligible matter) and determine its formal nature by distinguishing it from the class or set in which it belongs. Intelligible matter, then, ultimately is continuity itself, or a determined kind of continuity, as when I limit a mathematical investigation to what determination of the infinitely potential line from zero to one. See, in this connection, Aristotle's Metaphysics, 1034b 20-1036^a 25 and the commentary of Saint Thomas Aquinas on this passage.

Thus, in mathematical abstraction, a form which philosophers have traditionally designated as quantified substance, is abstracted from sensible matter. Technically, philosophers have said that this form is abstracted from sensible matter to which it is necessarily united secundum esse -- for its very being -- but not necessarily jointed to it secundum intellectum -- for it to be known (43). This is equivalent to asserting that mathematical natures are seen not to be realisable as such in reality, though it likewise means that if anything circular or hexagonal is to exist, it only exists in nature as the imperfect dimensional terminations of a physical body.

The attribution of quantity as the formal object of mathematical abstraction and as the ultimate subject matter of mathematical demonstration requires some elucidation because of the frequent denial that this denomination is no longer valid for contemporary or modern mathematics. From the many criticisms voiced in this regard (44), one would almost conclude that quantity in the sense of sheer matter, weight, dimensions and measurements were being proposed as the proper concern of mathematicians. It is said, for example, that quantity can be considered the object of Greek or classical mathematics (45); modern mathematics has no subject

(43) Aristotle, Metaphysics, 1077^b 11-33.

(44) See, for example, Bertrand Russell, Introduction to Mathematical Philosophy, London: George Allen & Unwin, 1938, pp. 125-126; and, by the same author, The Principles of Mathematical Philosophy, Chapter XIX, and Chapter XXIII, where Russell insists on the notion of order, and presents a view commonly shared by logicians. See also Alfred Korsybeki, Science and Sanity, Part V, Chapter XVIII, "Mathematics as a Language of a Structure Similar to the Structure of the World," pp. 247-267.

(45) Vincent Edward Smith, The Philosophical Frontiers of Physics, pp. 50-51.

matter but pure form (48), or the mere deduction of any premises (47).

On the other hand, an equally frequent denomination of mathematics as dealing with the discrete and continuous (48), as concerned with form and structure (49), or as investigating order and number (50), point towards the recognition of quantity as the basic consideration of mathematics.

The notion of quantity which metaphysics designates as the formal object of mathematics and in which the laws of mathematical being are contained means the order of the parts of substance (51). In this sense, quantity does not mean the bulk or weight of an object whose impact we feel upon physical contact. Rather, quantity here responds to the ultimate quest of the mathematician: the scientific knowledge of the order of the parts of quantified substance as abstracted from nature, through possession of a scientific intellectual habit which is specifically mathematical. Hence, in contrast to idealist or empiricist notions, mathematics is not created from nothing by the mind nor is it reducible to mere exact sense-impressions (52). Quantity as "given" through the

(46) See the definition under "Group 2" in Louis O. Kattssoff, op. cit., pp. 11-12.

(47) Ibid., "Group 3," pp. 12-14.

(48) E.Y. Bell, The Development of Mathematics, pp. 12-13; Abraham A. Fraenkel, "The Recent Controversies About the Foundations of Mathematics," passim.

(49) George David Birkhoff, "Mathematics: Quantity and Order," pp. 295-297; the author's position on the nature of mathematics is not consistent throughout the entire article.

(50) The notions of order and number are currently emphasized as being ultimate notions; see Bertrand Russell, The Principles of Mathematics, passim.

(51) Thomas Von Aquin, op. cit., Quaestio V, Art. 3 ad 3, p. 41; see also A. Cornelius Benjamin, op. cit., Chapter XIII, "Order, Number and Quantity," pp. 254-278.

(52) Thomas Greenwood, "Invention and Description in Mathematics," pp. 88-90.

mental process of scientific abstraction -- though not reflectively known, analysed or considered by the mathematician as quantity itself which is a metaphysical consideration -- is substance considered solely as quantified and defined as the order of the parts of substance. Hence, metaphysically speaking, both the ontological ultimate in nature and the psychological ultimate which is given through simple apprehension and in the first scientific act of mathematical abstraction, is not a simple but a complex thing, indefinitely rich in content (53). The essential determination of the scientific habit of mathematical abstraction lies in having the mind become conformable and identifiable to the nature of substance solely as quantified; in this way the mathematician possesses the readiness and facility to judge and to reason further concerning the order of the parts of substance with the aid of his sensory makeup and many other extra-mathematical suggestions and insights (54). The main difficulty of understanding quantity as the formal object of mathematics, of course, lies in realising that substance and quantity alone do not

(53) "But the mind can also consider objects abstracted and purified from matter in so far as it is the general ground of the sensible properties, whether active or passive, of bodies. In this case it considers only one property which it detaches from bodies -- what remains when all the sensible is removed -- quantity, number and extension taken as such: an object of thought which cannot exist without sensible matter, but which can be conceived without it, e.g., nothing sensible or experimental enters into the definition of an ellipse or a square-root. This is the great kingdom of Mathematics: the knowledge of quantity as such, in its rightful relations of order and measurement." Jacques Maritain, The Degrees of Knowledge, pp. 48-49.

(54) See Thomas Greenwood, "Invention and Description in Mathematics," for a description of the habit of mathematical reasoning. The two volumes of Felix Klein's Elementary Mathematics from an Advanced Standpoint are an excellent illustration of the many, complex factors which are concerned in mathematical inventiveness and demonstration.

exist as such and as separated from qualities in nature, but only as conceivable and abstractible by the mind (55).

One of the reasons why quantity was previously accepted by mathematicians as their proper consideration and why it is not so regarded today no doubt lies in the fact that pure mathematics has been more clearly distinguished from applied mathematics (56) and also in the fact that philosophical discussions on the foundations of mathematics, lodging the mathematical ultimate in logic, symbolism or intuition, have had a pronounced influence on mathematical thought (57). A more detailed presentation of the processes of metaphysicians on quantity, however, will show how mathematics is the study of quantified substance even in its present emphasis on correspondence, order and measure (58).

Quantity is analogously divided by metaphysicians into transcendental and predicamental. They point out that transcendental quantity, a field of metaphysical inquiry, is used to signify the amount of perfection or entitative being which a thing possesses (59). The being of a giraffe,

(55) "...for in mathematical science the mind lays hold on the constituent elements of entities and constructs and reconstructs in its own right what it has drawn from sensible data or built upon them, treating what in the real (when they are entia realia) are the accidents or properties of bodies as if they were substantial beings and as if the notions which it holds of them were free of any experimental origin..." Jacques Maritain, op. cit., p. 41.

(56) T. T. Bell, The Development of Mathematics, Chapter 18, "From Applications to Abstractions," pp. 386-378.

(57) For a short, summary account of this influence see Harold A. Smart, The Logic of Science, Chapter III, "The Mathematical Sciences," pp. 64-94.

(58) L.-B. Guerard des Lauriers, "Analyse de l'etre mathematique," Revue des Sciences Philosophiques et Theologiques, 22(1933), pp. 386-431 and pp. 585-639. The author presents a summary of the present emphasis on cardinality, order and limit by mathematicians, and their philosophical relevance. His method, approach and conclusions have been summarized and criticized by D. Salmon in Bulletin Thomiste, 4(1934-6), pp. 359-368.

(59) Saint Thomas Aquinas, In V Metaphysicorum, Lect. 16, n. 954-976.

for example, is intrinsically greater than the being of a stone. Transcendental quantity, as based on the substantial or accidental perfection of a being -- on its amount or plenitude of perfection -- is spoken of as virtual quantity, while that based on a consideration of a plurality of such beings is spoken of as transcendental quantity or transcendental number. This quantity is denominated as transcendental because it deals with the very being, truth, unity and goodness of things which are properties transcending the categories of substance and accidents (60). Predicamental quantity or that proper to the categories, is the designation of quantity as the order of parts outside of parts in a quantified whole. The word order here does not signify the relation of order itself, but the basis of this relation or the fact that parts are arranged according to priority and posteriority (61). Predicamental quantity, then, contains both the notion of a multitude, class, or set of parts, and the order of these parts according to some measure or standard of reference.

Predicamental quantity, of which mathematics is an abstractive study, is either that of continuity and content or discreteness and

(60) See, in this connection, Gaston Isaye, La theorie de la mesure et d'existence d'un maximum selon Saint Thomas: Archives de Philosophie, XVI, Paris: Beauchesne et ses fils, 1940, Introduction.

(61) A. Cornelius Benjamin, An Introduction to the Philosophy of Science, pp. 254-260; for a mathematical analysis of the notions of order, sets and the continuum, see Edward V. Huntington, The Continuum and Other Types of Serial Order, With an Introduction to Cantor's Transfinite Numbers, Second edition, Cambridge: Harvard University Press, 1942, Chapter II, "General Properties of Simply Ordered Classes or Series," pp. 10-18.

determination (62). Continuity in quantity is that signified by the juxtaposed continuation of its parts; it is often called magnitude and is reputed especially for the intrinsic potentiality and indetermination which it possesses as well as for its divisibility or separability into parts of like nature. The geometer or topologist determines this potentiality of the continuum (63) -- this intelligible matter -- in various qualitative ways; it is the field, domain or potential background for various figures, their qualitative states, their fictive movements of translation and rotation, their transformations, mutual relations, equivalence and mutual reducibility within this infinitely potential content (64). Continuous quantity when thus

(62) "Now it is obvious that in these remarkably similar pronouncements, by scientists and philosophers of the greatest diversity of outlook and training, there is implicit something of the utmost importance for the logician to consider. These thinkers actually confirm each other in the view that homogeneity and externality -- or, better, continuity and discreteness -- belong, as complementary formal characteristics, to all mathematically constituted objects, to the entire content of mathematical science." Harold R. Smart, The Logic of Science, p. 82; see pp. 87-94 for Smart's discussion on this point.

(63) "Nous dirons encore quelque mots d'un troisième grand type de structure, les structures topologiques (ou topologies): elle fournissent une formulation mathématique abstraite des notions intuitives de voisinage, de limite et de continuité, auxquelles nous conduit notre conception de l'espace." Nicolas Bourbaki, "L'Architecture des mathématiques: La mathématique, ou les Mathématiques?" in Les Grands Courants de la pensée mathématique, p. 42; see Bourbaki's whole article (pp. 35-47) for an interesting analysis of how the determination of the continuum may be regarded.

(64) "The reason for this is that matter, which is the principle of individuation, is not known in itself and is known only through form, from which the universal reason is taken. And therefore singular things are not known in their absence except through universals. Matter, however, is not only a principle of individuation in singular sensible things, but also in mathematical matters. One kind of matter is sensible, another is intelligible. Bronze and wood are examples of sensible matter, or any kind of mobile matter as fire and water and all others of this kind; by such matter singular sensible things are individuated. Intelligible matter, however, is that which is in sensible things not inasmuch as they are sensible -- as are mathematical things. For just as the form of man is in this matter which is an organic body, so the form of a circle or triangle is in this matter, which is either continuity, or a surface or a body." Saint Thomas Aquinas, in VII Metaphysicorum, lect. 10, n. 1496. In this seventh lesson on the metaphysics, Saint Thomas frequently points out that continuity itself, or as a determined type of continuity, is the background and basis of mathematical actions and operations.

determined in some way by a formal causality (65) of some type, may be determined mathematically in any number of directions (66).

Emphasizing, in contrast, the units of continuity, discrete quantity is at once more determined than continuous and yet arises, in diverse manners, from the continuous. Although number, like form, arises from abstraction (67), not all numbers arise in this way; to the mathematician, in fact, all numbers can be regarded as mathematically alike and arise, in some way, from the abstractive consideration of the needs of the continuum (68). Discreteness involves the separateness

(65) The traditional view, that mathematics is primarily concerned with formal causality, can be seen as being substantiated today by the current emphasis which logicism, formalism and even intuitionism place on the formal aspect in mathematical science. See Thomas Von Aquin, op. cit., Quaestio VI, Art. 1, c., n. 57.

(66) The notion of dimensionality in mathematics, though analogously derived from the pre-scientific and scientific experience of the mathematician is unique, and is not a study of extra-mental dimensionality. Philosophers, in judging the results, worth and merit of mathematics, frequently fail to realize that the mathematical universe only bears a likeness to the extra-mental universe. For a quite accurate presentation of this notion, see Victor F. Lenzen, The Nature of Physical Theory: A Study in Theory of Knowledge, New York: John Wiley & Sons, 1951, Chapter II, "Euclidean Geometry," pp. 50-74.

(67) A. Cornelius Benjamin, op. cit., pp. 260-271. The author quite clearly points out the relevance of pre-scientific experience for the mathematician's notion of number, and stresses the difference of mathematical numbers from those which are used in ordinary processes of counting and measuring.

(68) "All measurement which obtains in continuous quantities is derived, in some way, from number. Consequently the relations which hold of continuous quantity, also are attributable to number." Saint Thomas Aquinas, In V Metaphysicorum, Lect. 17, n. 1007. The close relationship and inter-action between form and number held by Aristotle (see, for example, Metaphysics, 1002^b 20-21) and by Saint Thomas Aquinas, is quite in keeping with current thought on the numerical determination of the continuum. For a philosophico-mathematical study exemplifying these notions, see Thomas Greenwood, In Natura du Transfinito, Ottawa: Les Editions de l'Université d'Ottawa, 1946, passim; see especially "Conclusions Philosophiques," pp. 55-68.

or separability of a multitude of continuities, each of which is viewed as a determined continuum and whose amount is measured by adopting some unit as a measure. For example, by imbedding numbers and points in the continuum we determine its potential, indeterminate parts (69); in so doing, of course, the correspondence or matching of parts may be emphasised, or their juxtaposition, position, or other type of order. The ultimate units of discreteness which arise from the measurement of the continuum are numbers of some sort -- natural, real or even groups of numbers as in transfinite consideration -- for any of these numbers is but one type of ordering and measuring the indeterminacy of the continuum (70).

Both continuous and discrete quantity, then, admit of various qualitative, formal determinations (71). A non-terminating line or a

(69) For an introductory account of the determination of the continuum numerically, see John Wesley Young, Lectures on Fundamental Concepts of Algebra and Geometry, Lectures VI-XII.

(70) See Maelaw Sierpinski, Hypothese du Continu, Warsaw: Lwow, 1934, "Preface," (pp. iii-v) for a recent statement on both the hypothesis and the problem of the continuum; Sierpinski approaches the hypothesis of the continuum from the notion of quantity and from that of order. Throughout he is concerned with the problem of effectively determining continuous potentiality with the discrete determination of number and series of numbers. For a more recent statement of the continuum problem, see Kurt Gödel, "What is Cantor's Continuum Problem," The American Mathematical Monthly, 54(1947), pp. 515-525; Gödel conjectures, in concluding, that possibly a new set of axioms may help resolve the present difficulties on the continuum.

(71) The insistence on quantity as the proper object of mathematics should not be taken to exclude the vast realm of the qualitative which obtains in mathematics. That with which mathematics is concerned is quantified substance (without sensible qualities); the determinations of this quantified substance are obtained through mathematically qualitative attributions. This can be seen especially in Topological considerations and in projective geometry. For a fine discussion on the interrelation between the mathematically quantitative and qualitative, see Leibniz: The Monadology and other philosophical Writings, translated by Robert Latta, Introduction, Part III, "Detailed Statement of the Philosophy of Leibniz," (pp. 74-80), where Latta discusses Leibniz's mathematics in relation to his philosophy.

line from zero to one, the qualities of obliqueness, angularity, curvature, denseness, collinearity, connectivity and dimensionality are but a random number of these formalities which denominatively lie in the domain of intelligible matter. These various determinations, in fact, give the lie to both idealistic and sceptical attitudes to mathematics (72). The very names of these mathematical formalities indicate their importation from the world of sensation, as do the names for the mathematical operations of adding, imbedding, making a set dense about a point. At the same time, the creativeness of man the mathematician is strikingly illustrated by this mode of procedure in the mathematical realm; in this realm, stimulated by his own previous investigations, the pleasures of mathematical reasoning, the desire for unity, the urge to solve problems and by various needs and insights, the mathematician specifies and determines the unlimited potentiality of intelligible matter (73).

(72) "We have found that the mathematician shares with other natural scientists the content supplied by the world of existential phenomena. But he studies this world only in its quantitative aspects, i. e., only in so far as the objects of that world may be regarded as at the same time more or less completely homogeneous with, and external or indifferent to, one another. And we know now, as logicians and philosophers, what to think of the doctrine that pure mathematics is a completely a priori science, constructed by pure thought, functioning apart from all experience. That doctrine turns out to be the product, not of an analysis of actual mathematical conceptions and methods, but of a gratuitous metaphysical hypothesis. This hypothesis, to the effect that thought and experience are separate somewhat, landing us, as it does, in all sorts of impossible dualisms and irresolvable contradictions, so far from being confirmed by any science, is the enemy of all science and of all philosophy." Harold R. Smart, op. cit., p. 93.

(73) See Henri Poincaré's presentation of the creation of the mathematical continuum in The Foundations of Science (pp. 46-54); abstracting from the philosophical insertions, a somewhat accurate picture of the inventiveness of mathematical reasoning is given.

Mathematics also treats of other entities which, though not quantitative of themselves, have an accidental correlation with quantity as being somehow dependent on it. Time and motion, as viewed by the mathematician, are capable of quantitative treatment because that to which time and motion belong, are themselves quantified and divisible (74). Time is both divisible and continuous because of the successive continuity of motion; time is motion conceived as a uni-dimensional current. Motion is divisible and measurable not primarily due to itself, but ultimately because of the continuous magnitude or medium in which motion occurs. Similarly, space as studied in mathematics is not the real space of our experience, though ultimately real space is the foundation for the mathematician's study (75). The mathematician regards space by viewing it under the formality of simultaneous continuity; in this multi-dimensional context, he regards the continuum of intelligible matter as possessing limitless potentiality and content which he variously determines by the mathematical concept of dimensionality (76).

Another group of entities to which a mathematical treatment is

(74) See A. Cornelius Benjamin, *op. cit.*, Chapter XIV, "Space, Time," pp. 279-311, for a quite accurate analysis of the notions of space and time in ordinary and in scientific considerations.

(75) The mathematical notions of space, time and dimension, reductively and retroactively viewed, arise from experience; in the process of mathematical abstraction, however, only an analogous likeness to extra-mental space and time are retained. This conception helps explain the reason why philosophers and mathematicians frequently make confusing statements about mathematical space and time, and, at the same time, explains how these mathematical space or time systems are applicable, with varying measures of exactness, to extra-mental dimensionality in space and time.

(76) For an introduction to the mathematical consideration of dimensionality, see John Wesley Young, *op. cit.*, Lectures XIII-XVII.

accidentally or connectedly attributable are qualities, such as the color of an object, or the quality of being a musician or mathematician. Here, of course, there is an indirect measurement of the qualitative accidents of some subject who possesses these modifications; various methods of measuring these qualitative perfections are invented and properly lie in the realm of applied mathematics, which attempts a correlation of the principles and conclusions of mathematics to quantity as subject to qualities, matter and motion on the first level of science (77). In passing, it should be remarked, that whatever is joined, in some way, to quantity, as dependent essentially or accidentally for its existence and operation upon quantity, can somehow be given an analogously quantitative consideration (78).

In quantity as given in mathematical abstraction, then, both the notion of continuity and discreteness are potentially contained; quantity can be abstractly viewed as an undivided and indivisible unity -- a number, or as an actually undivided form or whole, though potentially divisible ad infinitum (79). Even in a simple consideration of a circle we can distinguish the abstracted substance of the circle as given, and as capable of being viewed in its discrete determination as a unit or one circle, or in

(77) For a recent study of the applications of mathematics, see Vincent Edward Smith, The Philosophical Frontiers of Physics, Chapters II and III.

(78) See, however, in this connection, the discerning remarks on the limits of mathematical applications in Jean Dujat, L'Oeuvre de l'intelligence en physique, Chapitre V, "Les limites de l'intelligibilité mathématique," pp. 144-181.

(79) "Now motion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous -- that is how it comes about that 'infinite' is often used in definitions of the continuous ('what is infinitely divisible is continuous')." Aristotle, Physics, 200^b 17-19.

its fluctuating indetermination as a continuous form or figure.

In this connection, frequent mention is made in mathematical literature of the ever-recurrent problem of the discrete and the continuous as the focal problem of mathematics; it is pointed out how the tension between these two qualitative states of mathematical quantity has been a puzzling question, yet, withal, a fruitful source of mathematical inventiveness' (80).

The three types of speculative science, then, deal with quantity in a different manner, and there is an objective basis for this difference. In the first degree of total abstraction, the mind considers quantity as 'the accident proper to mobile, quantified bodies; in the philosophy and the physical sciences of nature, bodies are considered in their existential determination in space-time, and as subjected to genuine quantitative and qualitative change and motion (81). On the other hand, mathematics considers bodies as merely quantified substances through abstracting from the qualities and changes which are part and parcel of sensibly existent things; while applicable to the extra-mental world with some measure of exactness, mathematical conceptions and systems of space, time and the order of the parts

(80) "Abraham A. Fraenkel, "The Recent Controversies About the Foundations of Mathematics," *passim*; the author seems to conclude (p. 36), by admitting some type of "pre-existence" of the continuum. This interplay of the continuous and discrete not only occurs throughout the history of mathematics (see F. Y. Bell, *The Development of Mathematics, passim*), but also in physics; see, for example, Louis de Broglie, "Le rôle des mathématiques dans le développement de la physique théorique contemporaine," in *Les Grands Courants de la pensée mathématique*, présentés par F. Le Lionnais, pp. 388-412.

(81) "This homogeneity and externality of part and whole which obtains when objects are considered mathematically may be contrasted with the relative diversity of heterogeneity and interpenetration of part and whole when objects are considered more concretely. In the latter case the part is never in any important sense of the same nature as the whole into which it enters, and both part and whole undergo essential modification of nature as a result of any change in the one relatively to the other." Harold E. Smart, *op. cit.*, p. 80.

of quantified bodies never correspond exactly to extra-mental reality where qualities, mobility and change characterize real existence (82).

This diversity of consideration between total and formal abstraction is necessarily emphasized in order to safeguard these two legitimate views of quantity, confused as they frequently are. For example, if a scientist, philosopher or cosmologist maintains that it is absurd to speak of four, five or n -dimensions or transfinite numbers (83), he is confounding the quantity of mathematics and that of the sciences and the philosophy of nature. Pure mathematics does not pronounce its scientific judgments on the extra-mental real, but only on abstracted, quantified substance -- a state in which things do not exist except in mathematical abstraction (84).

In contrast to the view of quantity afforded either in total or formal abstraction, metaphysics regards predicamental quantity as a principle of being, the ontological accident which belongs to the categories, and likewise studies the analogous notion of transcendental quantity. With regard to predicamental quantity, metaphysicians point out that though substance is prior to all the accidents of being, quantity has a priority of nature over the other accidents. Frequently termed, for this reason, the first accident, quantity orders the parts of material substance and is the basis of extension and other corporeal properties (85). All sensible qualities are directly rooted in quantity as their immediate principle of being; through quantity, these sensible accidents are rooted in the individual substance. This is the ultimate

(82) This trait of mathematics -- its partial and approximative application to extra-mental reality -- is quite generally admitted, so that it would be futile to insist on this point.

(83) See, for example, the conclusions of Marcel Lallemand in La Transfini: Sa logique et sa metaphysique, pp. 201-207.

(84) Thomas Von Aquin, op. cit., Quaestio VI, Art. 1, c., pp. 56-57.

(85) Saint Thomas Aquinas, In V Metaphysicorum, Lect. 15, n. 577-588.

reason why whatever is qualitative or rooted in quantity can be given an analogously quantitative consideration; through various mathematical measures, the principles of mathematics are applied to qualitative states (86).

This ontological juxtaposition of quantity to substance gives quantity a prominence and pre-eminence over the other accidents which is unique and distinct. In fact, many thinkers have been led to substantify the various mathematical abstractions of form, number and solids, by attributing an extra-mental existence to the abstractions of mathematical science (87), or to deny the existence of secondary, qualitative characteristics. One of the reasons for this ready attribution of an extra-mental existence to mathematical abstractions lies in the fact that though quantity is an accident, it is rightly treated as the subject matter of mathematical science with its own definable qualitative properties (88). Unless some sound, metaphysical reflection points out that quantity is here considered as intelligible matter -- substance joined merely to quantity -- the jump from the mathematical to the real order can readily occur.

Quantified substance, then, as dealt with by the mathematician differs from quantity as reflectively known, and viewed as a principle of being by the metaphysician, as it does from extra-mental quantity of physical science. The same or similar questions about quantity, moreover, may be

(86) Thomas Von Aquin, op. cit., Quaestio V, Art. 3, ad 3, n. 42.

(87) As has been frequently pointed out, the school of logicism has been led to this view; see, for example, Thomas Greenwood, Les Fondements de la logique symbolique, Tome I: Critique du nominalisme logistique, passim, and Haskell W. Smart, The Philosophical Presuppositions of Mathematic Logic, passim.

(88) Mathematicians of past and present have looked upon the natural numbers, the continuum, and mathematical relations as being somehow "real." A collection of these attributions of extra-mental reality of some sort to mathematical entities would be both interesting and instructive.

asked, in each of these sciences (89). The problem of infinity in quantity may be considered by each, though the three responses to the question of infinity in quantity cannot be contradictory inasmuch as ultimately they refer to extra-mental quantity as abstractly viewed under three distinct, scientific aspects. In the view of total abstraction, corporeal substance or its whole summation may be looked upon as actually finite and potentially infinite; from the mathematical view, the question of actual infinity in quantity is of little import compared to the potential infinity of quantified substance abstractively considered; in the metaphysical view, the very nature and being of infinity and its relation to the infinity of physical and mathematical science is considered (90)

The above comparison between three views of quantity illustrates how the greater immateriality of a science gives it a speculative hegemony over the sciences below it. Metaphysics is the most immaterial science and needs the least speculative principles. Mathematics, inasmuch as it adds quantity to substance is less immaterial, and not only needs the principles of metaphysics but additional ones; physical science, dealing not merely with substance and quantity, but adding quality and mobility, and aiming to know mobile being, is the most complex of the three level of science (91). While both mathematics and physics receive their principles of speculation from metaphysics, mathematics exercises a speculative function over the conclusions

(89) Zeno's famous paradoxes of motion, for example, are frequently looked upon as solved only through the aid of recent mathematics. (See Bertrand Russell *Our Knowledge of the External World*, New York: W. W. Norton & Company, 1929, Lecture VI, "The Problem of Infinity Historically Considered," pp. 169-198.) It is here maintained, that these paradoxes can be considered and solved on all three levels of speculative science.

(90) It is interesting to note that many mathematicians have considered the problem of infinity under the physical, mathematical and metaphysical aspect even though they may conclude that only the mathematical answer has validity. See the remarks of Philip E. B. Jourdain in *Contributions to the Founding of the Theory of Transfinite Numbers* by Georg Cantor, "Introduction," p. 52-82.

(91) A summary of this doctrine on the number and dependence of the principles of speculative science can be found in Thomas Von Aquin, *op. cit.* *Quaestio V, Art. 3, ad 6, p. 42.*

of physical science. If, for example, it is mathematically certain that no contradiction results from considering quantified substance as potentially divisible ad infinitum, then it will not be possible for the philosophy or the science of nature to disprove certainly the possibility of an infinite multitude (92).

It should be emphasized that quantity as the formal object of mathematics is not to be confused or identified with its analysis in the philosophy or sciences of nature; not is mathematical quantity to be looked upon as identical to one or all of the common sensibles or secondary qualities (93). This "physical" view of mathematics, though frequently found, is due either to a lack of grasping the nature of mathematical abstraction, or perhaps to the fact that the quantity of mathematics is psychologically attained through the sensible qualities of things (94). These and similar difficulties can only be obviated if the legitimacy of both total and formal abstraction and of metaphysics is conceded.

Somewhat in contrast, then, to current philosophical theories on the existence and nature of the mathematical ultimate, it is here maintained that quantity as given to the mathematician is not a simple but a complex subject, indefinitely rich in content. There is, in things, the real accident of quantity inseparably joined to an existing substance with concomitant accidents of color, taste, weight and shape; in mathematics, the mind actualises only

(92) Saint Thomas Aquinas, In III^o De Caelo et Mundo, Lect. 3, n. 4.

(93) Joseph Maréchal, Le Point de départ de la métaphysique, Cahier V: Le Thomisme devant la Philosophie critique, pp. 118, seq.; see also Jacques Maritain, op. cit., p. 175.

(94) Jacques Maritain, op. cit., p. 175 and p. 250.

the formality of substance and quantity by abstraction, thus making the potential formality in the thing an intelligible formality and denominating quantified substance as a scientific object (95). The specification of the habit of mathematical science consists, accordingly, in having the mind become conformable to quantified substance considered abstractly as denuded of qualities, action and change; possession of this habit places the mind not only with a conformity to quantified substance but also with the readiness and facility to reason further concerning the order of the parts of substance (96).

Similar to all knowledge and specifically to speculative science, mathematics as a science consists in judgments which are the result of the second operation of the mind and not of simple apprehension. The nature, origin, necessity and truth of these judgments, is, of course, a matter closely allied to one's view on the nature of both science and mathematics itself (97). For example, Plato, holding an innatist theory, laid the origin of the mathematical judgment in the process of recalling what had previously been known in a pre-existent contemplation of the world of ideas, while the Pythagoreans located the surety of mathematical judgments in their applicability to reality itself. Descartes idealistically pinned the certainty of mathematical judgments on the badge of clarity and self-evidence which they possessed, and was forced to resort to an innatist view; while the British empiricists, incli-

(95) See, in this connection, Saint Thomas Aquinas' description of the manner of procedure in total and formal abstraction and in metaphysics; Thomas Von Aquin, *op. cit.*, *Quaestio VII, Art. I, c.*, pp. 55-61.

(96) See the following Chapter for a presentation of the mode of reasoning in mathematical science.

(97) Emile Meyerson, *Le cheminement de la pensée*, 3 tomes, Paris: Felix Alcan, 1931, Livre III, Le raisonnement mathématique, Chapitre premier, "Les Deux Conceptions du raisonnement mathématique," pp. 217-320.

ning towards the extremes of the Pythagorean outlook, made reality itself the measure of the exactness of our mathematical ideas. For Immanuel Kant, the indubitable certainty of mathematical judgments was due to special categories of the mind which impressed the fluctuating impressions of sense, while for Comte the rival of the mathematical era was, in itself, the historico-sociological guarantee of their certainty. In the contemporary sense, this vacillation towards idealism or scepticism on the nature of the mathematical judgment is seen in logicism, intuitionism and formalism (98).

A realian solution to the origin and nature of the mathematical judgment is tended by viewing mathematics as an abstractive speculative science. Like all science and knowledge, the habit of mathematics arises originally, by abstraction from the senses, though the preciseness of this point of origin may differ in manner for each individual (99). Those who have the mysterious and unknown combination of gifts which are called mathematical talent or genius, frequently require little previous experience with

(98) Louis O. Kattsoff, op. cit., Chapter Sixteen, "Mathematics and Reality," pp. 244-253; Kattsoff gives a broad summary of the directions toward which contemporary philosophical thought on mathematics tends.

(99) "Nous avons, d'autre part, pris a tache autrefois de montrer qu'en ce qui concerne leur nature intime, les etres qui sont censés composer le monde de notre perception directe celui dont le sens commun nous impose la presence quand nous ouvrons les yeux le matin, ne sont en rien differentes de ceux des theories scientifiques, s'assimilant au contraire entierement a ces derniers: le sens commun, en son ensemble, n'est qu'une premiere esquisse de theorie scientifique et philosophique, d'elaboration entierement inconsciente, mais confectionnee selon des procedes tres analogues a ceux que suit la pensee consciente, aussi bien en science qu'en philosophie." Emile Meyerson, op. cit., Livre III, p. 356. Though experience is the point of departure for the formal abstraction of mathematics, it plays a suggestive and not a direct, regulatory role in mathematics. This notion will be briefly considered in the following chapter.

reality, or little tutoring to acquire the mathematical habit; in fact, one act of scientific demonstration may be sufficient. Others, who may even later become capable mathematicians, require a more close and more frequent liaison with the mathematical content of extra-mental reality, before the habit is induced within them (100). In all cases, however, pre-scientific knowledge and the use of the senses and the imagination are the principles from which mathematical abstraction arises; the speculative thinker who may adopt an idealistic or sceptical position in this regard is but denying the history of his own experience, and that of mathematics itself (101).

Since it is here claimed that mathematical judgments are neither pronouncements self-evident to the mind as in an idealist theory, nor yet merely such as to be verified only if applicable to reality, what is the mathematical judgment, and in what does its truth consist? It has been pointed out that unlike total abstraction, the formal abstraction of mathematics considers things as independent of sensible matter for being conceived, but dependent on intelligible matter; and that it is in the imagination that quantified substance is visualized as individual intelligible matter. The mathematical judgments, accordingly, terminate in the

(100) See the off-hand remarks of Henri Poincaré on this point in The Foundations of Science, Part I: Number and Magnitude, Chapter I, "On the Nature of Mathematical Reasoning," pp. 31-42.

(101) The main point on which we wish to insist is that mathematics is not innate or sheerly a self-creative affair; except for the views of absolute idealists, there is quite general agreement, it seems, that experience furnishes at least the point of departure for mathematics. In The Development of Mathematics, H. T. Bell brings this point out in a number of places; see, for example, pp. 3-13, pp. 144-150, p. 190 and p. 194.

imagination, where their truth is seen to be resident in their capability of being imaginatively realized or actualized; once this has been achieved, and the judgment verified, a further validation is no longer necessary, and a more abstract formulation of the judgment may ensue (102).

The frequent statement that modern mathematics has shed intuitive appeals, seems, of course, to deny this reference to intelligible matter. However, as in the case of imagining non-euclidean geometry, it must be recalled that the imagination and memory supply other images than visual. If I mention the delights of consuming a choice steak or the touch of a cool breeze on a warm, clammy night, such word images are sufficient for my statement to be understood, even though memory and imagination may, in the case of a steak, have taste or visual images or both, and in the case of a breeze, touch or visual images, or both (103). Consequently, in non-euclidean terminology, when it is said that the "sum of three angles is greater than a straight angle" it is not necessary that this judgment be pictorially visualized and verified in the imagination just as it is not

(102) In "Invention and Description in Mathematics," Thomas Greenwood points out that this close dependence of mathematics on the imagination is especially seen in the process of invention and that the results of invention are, however, usually presented in a simple, logical form. See, in this connection, Henri Poincaré, *op. cit.*, Part I: The Mathematical Sciences, Chapter I, "Intuition and Logic in Mathematics," pp. 210-222. For an expression of the traditional view on this matter, see Thomas Von Aquin, *op. cit.*, *Quaestio VI, Art. 2, c.*, pp. 63-64. Within the framework of speculative science itself, of course, the mathematician is free to set up mathematical criteria which must be fulfilled in order to validate the existence of mathematical entities; it is on this point that much of the current discussion on the foundations of mathematics is lodged.

(103) Mark Aloysius Gaffney, *The Psychology of the Interior Senses*, St. Louis: B. Herder, 1942, Chapter III, "The Imagination," pp. 30-107.

necessary for the proposition of the sum of three angles equalling a straight angle (104). Or if I say that the first transfinite number is the number we have reached by summation of all denumerable infinities. I may attempt to illustrate this judgment with a line as continuum, but it is not necessary that this be done. In these cases, it is sufficient that the meaning of the words and symbols used be known, and that the order of the parts of substance be conceptually and imaginatively attainable, as being the ordinal character of quantified substance -- the ultimate subject of attribution in these cases (105).

When it is asked whether there is a correspondence of mathematical judgments to extra-mental reality, it is clear from what has been said, that no one-to-one correspondence exists. Mathematics is not scientifically concerned with the determinate, extra-mental qualitative quantity of individual things. This question, in fact, can best be answered by comparing its judgments with those of metaphysical science and with those of the philosophy and sciences of nature (106). In the latter two classes of science, a direct correspondence holds between the way things objectively exist and the way in which man utters scientific judgments upon them. For example,

(104) The frequent statement that a non-euclidean geometry cannot be imaginatively visualized seems to imply that the imagination is limited to visualizing in three-dimensional space. This seems to contain an implicit confusion of visual with imaginative representation, and an attribution of dimensionality to the imagination not consonant with its nature. Strictly speaking, according to such a view, one should not be able to "imagine" a uni-dimensional line or a two-dimensional plane.

(105) A good illustration of the close dependence of topology on the imagination can be found in Richard Courant and Herbert Robbins, What is Mathematics?, pp. 235-243. This section helps illustrate the remarkable capabilities of the imagination, as it representatively deforms and transforms mathematical natures.

(106) See, in this connection, the close analysis in Thomas Von Aquin, op. cit., Quaestio VI, Art. 1, c., pp. 55-61.

considering either the judgment that man is a matter-soul composite, or the judgment that man is a rational animal in contrast to the judgment that every part of a circle is equidistant from a point within called the center, I see that the first judgment is in the first degree of abstraction; it prescind from the matter and soul proper to an individual man, knows the specific nature of a man, and considers the nature as predicable of inferiors. In the second judgment, the metaphysical definition, the judgment terminates in the intelligible as such without matter; we are emphasizing the immaterial, though extra-mental points of intelligibility in man. In the mathematical judgment, no extra-mental circle fulfills the definition; however, if we prescind from sensible matter completely and consider form or quantified matter, the circle can be conceived, defined and imagined as fulfilling this definition in terms of intelligible matter alone (107).

Mathematical judgments, then, lying between those of metaphysics and the lower sciences, are unique. In fact, the nature of the mathematical judgment is such that it also lies between those of logic, which exclude existence, and those of mere possibility which retain some inner relation to existence; the mathematical judgment is indifferent to real, logical or possible existence. While the habit of mathematical science is a perfection of the mind as is any other scientific habit, the mode in which existence of mathematical judgments is formulated, guaranteed and verified is unique.

(107) In the eleventh lesson of his commentary on the seventh book of Aristotle's *Metaphysics* (*In VII Metaphysicorum*, Lect. II, nn. 1501-1536), Saint Thomas Aquinas explains the close analogy that obtains between the definitions and judgments of the three levels of speculative science.

Mathematical judgments are in a universe, as it were, by themselves, for the mathematician treats abstract things as abstract (108). Yet, the subject matter of mathematics is not purely mind-made nor completely discovered in nature as we discover the properties of radium; the mind receives quantified substance from nature, and, aided by the past pre-scientific and scientific experience of the mathematician, inducts mathematics with the aid of the imagination (109).

The intermediary position which the mathematical judgment holds between logical and possible being begets a propensity in the mathematician to reach out into both directions. On the one hand, mathematics, like all science, makes a rich use of the logical intentions which it reflectively forms from its consideration with the direct study of mathematical natures

(108) "The mathematician cannot, however, take much satisfaction in this immunity from being proved wrong by the facts of nature, for the price of this immunity is that his science has no necessary application to nature. That mathematics has been so widely applicable is due to the fact that the postulates and axioms were first chosen in the light of what experience had disclosed nature to be. But mathematics itself affords no guarantee that the choice of postulates represents an absolutely correct description of the mathematical properties of nature to all times and places. The mathematician is on certain ground only so long as he stays within a closed system and asserts merely that his theorems follow from given premises without making any claim for the factual truth of those premises." Albert G. Ramsperger, *Philosophies of Science*, New York: F. S. Crofts & Co., 1942, p. 188; with the next sentence, however, the author concludes, "Mathematics is a purely analytical science."

(109) While mathematics is not to be viewed as an experimental science, it should be pointed out that being a science of quantified substance (abstracting from sensible qualities), it is a direct science, concerned with real and not logical being. This is pointed out well in the following quotation: "...The mathematician, abstracting, does not consider a thing other than it is. For he does not understand the line to be without sensible matter, but he considers the line and its properties without the consideration of sensible matter; and thus there is no dissonance between the intellect and the thing, because that which is of the nature of line secundum rem, does not depend on that which makes matter sensible, but rather the converse obtains..." Thomas Von Aquin, *op. cit.*, *Quaestio V, Art. 3, ad 1*, p. 41.

within the imagination; it will speak of series of numbers and groups of systems, which may appear to be as real to the mathematician as a number of an equation (110). The proximity of mathematical to possible being, on the other hand, and its intimate connection with sense experience, will tempt the mathematician to attribute an extra-mental reality to his conceptions and judgments. Ultimately considered, in fact, any true mathematical judgment or self-coherent mathematical system contains the inner possibility of being re-applicable, with varying degrees of success, to extra-mental qualified substance (111).

Truth, then, and the scientific characters of necessity, universality and certainty are not, for the mathematician, a matter of correspondence with

(110) The use of logical beings in the invention and development of mathematical science will be considered in the following chapter. It should be pointed out here, however, that dialectics (*logica utens*) plays a large part in mathematical inventiveness, development and demonstration, as it does in all speculative sciences. That in fact may at first appear merely to be a logical being in mathematics, will, in the course of time, turn out to be a mathematical being. In this connection, for example, the frequent denomination of transfinite numbers or non-euclidean geometry as purely logical beings, is too hasty a generalization. It is here maintained that non-euclidean geometry is no more nor less real than euclidean to the mathematician, nor are transfinite numbers more or less real than imaginaries, irrationals or transcendentals. The main point at issue, it seems, is that thus far the mathematician has not as yet been able to give a mathematical demonstration of their existence, and the proof for the existence of these new mathematical entities is still in the dialectical stage.

(111) Probably because he has not approached the nature of mathematical abstraction from the view of metaphysics, Louis C. Kattsoff, in his fine summary of current theories on the foundations of mathematics, concludes: "From what has been said, it follows that there is no real distinction between pure and applied mathematics. Pure mathematics is pure only in so far as it can be applied, and applied mathematics can be applied only as it is pure. If we wish to keep such a distinction we might say that the pure mathematician is interested in mathematical propositions as tautologies, the applied mathematician in the nontautological aspect of these same propositions." Louis C. Kattsoff, *op. cit.*, p. 250; see also pp. 250-253. The position which is here maintained is that the principles of quantified substance, having been scientifically ascertained by abstraction from mobile reality, are re-applicable to their ultimate source with a measure of success that is limited precisely by the addition of the principles of mobile being. See, in this connection, Thomas von Aquin, *op. cit.* Quaestio V, Art. 3, ad 3, ad 5 and ad 6, pp. 41-42.

reality as such -- in its mobile being (112). Dealing with a realm of knowable natures and beings, mathematics is, in truth, a direct science; neglecting the qualitative changes and motions in extra-mental reality, however, mathematics construes its own world within the imagination which is modelled, through the architectonic art of the mathematician, on the real world from his past experience and creative insights. Consequently, the mathematical judgement is one to which the conformity notion of truth applies as an ultimate criterion: the mind becomes conformable to quantified substance in mathematical science (113). A more immediate criterion with which the mathematicians are largely concerned at present, is the coherence notion of truth; it is here, especially that the mathematician demands the notion of constructibility, or requires consistency, categoricalness and interpretation of axiom-sets (114).

(112) "And similarly with regard to any one type of geometry; Euclidean geometry was never absolutely true in the sense in which certain philosophers conceived it to be, owing mainly, perhaps, to its seeming stability. Neither is it now correctly interpreted as a merely 'conventional' system of definitions, postulates, and their formal consequences, telling us nothing about the real nature of space. The truth lies between these extremes. The researches of mathematicians and other scientists are simply demonstrating that no one system of geometry can exhaust the richness of content included in the general field of their investigations. Old elements prove capable of entering into new and hitherto unsuspected relationships, and new elements are naturally brought to light in the same process of development," Harold Smart, *op. cit.*, pp. 110-111.

(113) See the quotation from Saint Thomas Aquinas in footnote 109, above.

(114) For an introductory example of this notion, see Louis O. Kattsoff, *op. cit.*, Chapter Fourteen, "The Structure of the Mathematical System," pp. 207-222 and Chapter Fifteen, "Postulational Methods," pp. 223-243. For a simple, introductory account, see John Wesley Young, Lectures on Fundamental Concepts of Algebra and Geometry, Lectures IV-V, pp. 36-57. It is possible, of course, to regard this study of consistency and categoricalness as referring to the minimum requirements for dialectical propositions, a contention which will not be developed in this essay.

In these endeavors, the coherence notion of truth is a matter of setting up the minimum requirements for the admission of a mathematical conception, judgment or system; and, in this connection, it is obvious that not all of mathematics is certain, universal and demonstrable from the moment of its initial invention and that a long period of time may ensue before its scientific certainty is demonstrated (115).

The adaptability of the mind to mathematics, the readiness and facility with which it considers and reasons in this science, are likewise explicable by its intermediary position. Metaphysical science is difficult for the mind; joined as is the mind to sensible matter and dependent as it is upon the senses, the immateriality of metaphysics and its speculative distance from the world of the sense increase the difficulties of its acquisition (116). There are difficulties likewise in acquiring a perfect scientific knowledge in the physical sciences and in the philosophy of nature; the very things which we wish to know being immersed in mobility and materiality, speculative patience, detailed observation and renewed experience are required for knowledge of them (117). The sciences of politics, sociology and ethics, as well as the science of geology, physics and chemistry, only achieve increasing perfection and scientific status by a constant and studied concern with the very mobility and unpredictability of natural things themselves. Both difficulties are avoided in mathematics; on the one hand, mathematics

(115) As will be explained in the following chapter, it is here maintained that there is a dialectical tissue or periphery surrounding every science as the principal means by which science progresses. This fact, and the close connection of mathematical development on the imagination and the past experience of mathematician, has been aptly presented with regard to the development of the calculus in Carl B. Boyer's work, The Concepts of the Calculus, frequently referred to above.

(116) Jacques Maritain, The Degrees of Knowledge, pp. 258-268.

(117) The whole history of these sciences of total abstraction bears out the fact that a constant rapport with extra-mental reality is a requisite for acquiring and for developing knowledge of mobile being.

does not achieve complete immateriality as does metaphysics, while, on the other hand, though experience is necessary and frequently helpful, it is not the minute and studied observation of mobile things in their very mobility. In fact, the mathematical natures are most knowable in themselves and also most knowable to us; we have induced them within our imaginations and exercise a quasi-creative control over them (118). Having once adequately determined the nature of mathematical entities, we can, with the aid of a few principles, exhibit their inner coherence by showing how they follow from a minimum number of principles (119).

The metaphysical view of mathematics as a speculative science arising from abstraction, consequently, opens to view a new scope for what is commonly called mathematical philosophy and which is more properly termed the philosophy of mathematics (120). Disagreeing with the simplicist answer of "mind" offered

(118) In commenting on a remark of Aristotle's about the accuracy of mathematics, (Aristotle, Metaphysics, 995^a 15-17), Saint Thomas Aquinas writes that the minute accuracy of mathematics "is not to be required in all things about which there are sciences, but only in those which do not have (sensible) matter. For those things which have (sensible) matter are subject to motion and variation; consequently every type of certitude cannot be had in all of them. What is sought in them is not that which necessarily and always is, but that which obtains in most instances. Immaterial things, indeed, are, in themselves most certain for they are immovable. But those things which are immaterial in their very nature are not certain to us because, as has been said, of the nature of our intellect. But mathematical beings are abstracted from matter and yet are not above our intellect; consequently the most certain reasons are found in them." Saint Thomas Aquinas, In II Metaphysicorum, Lect. 2, n. 336.

(119) This deductive nature of mathematics should not be over-emphasized as it imply that deduction is the only method found in mathematics. One can present mathematics and other sciences deductively, inductively, analytically or synthetically. What makes mathematics peculiarly amenable to a deductive presentation is due to the small number of principles needed in mathematics, the small amount of experience requisite and its concern with formal causality; see Harold R. Smart, The Logic of Sciences, Chapter IV, pp. 95-109.

(120) The speculative analysis of the relation of mathematics and metaphysics which follows is based on the philosophical conception of analogy. For an introductory account of analogy and its place in philosophy, see Gerald B. Phelan, Saint Thomas and Analogy, Milwaukee: Marquette University Press, 1941, and James F. Anderson, "Two Studies in Metaphysics," The Thomist, 5(1941), pp. 564-587.

by idealists who would make a divine of human mathematics, and with the uninteresting reply of "impression" offered by empiricists who would reduce human to animal mathematics, the metaphysical answer afforded by mathematical abstraction is complex. But in its very complexity, in its insistence on the many relations which bind mathematics to reality, to other sciences and to the mind, in its insistence that the truth can be known and in its humble confession that there is much that is yet unknown, the nature of mathematics as an abstractive science opposes any conception of mathematics as regressive or static and insists on the progressive and dynamic view. Man the mathematician, in this view, does build up mathematics because of pragmatic needs, of demands from other sciences and from various other occasional stimulants; at the same time, by pointing out that the laws of mathematical thought derive from those of mathematical being, by insisting that mathematics is concerned with a definite content and that the mathematician does and has attained speculative mathematical truth, and by pointing out that there are other sciences and cultural endeavors open for speculation and enjoyment -- the view of mathematics within the hierarchy of sciences and within the wider domain of human endeavor sets no limitations or bounds to human mathematical effort (121).

The interesting complexity and manifold relatedness of the philosophy

(121) One of the reasons why an abundance of speculative confusion obtains in current discussions on the foundations of mathematics is the neglect or denial of other sciences. If psychology, logic and metaphysics are not admitted to be genuine sciences and a mere mathematical analysis of mathematics is attempted, speculative confusion and error is unavoidable. Any reflective or speculative consideration of mathematics is metaphysical -- it is a further abstraction on mathematical abstraction itself.

of mathematics is pointed up when we compare the universe of mathematical discourse with that of metaphysics and the lower sciences. The role of similitudes and analogies is both important and significant. Natural things, for example, which are studied in the first degree of science are composed of matter and form; through the aid of efficient and final causality, the realm of sensible matter is constantly shifting and changing. The mathematician, though, is not concerned with existence, with what comes to be or passes away; for, in relinquishing sensible matter, he has abstracted also from efficient and final causality, retained a mere likeness of material causality and is mainly concerned with formal causality (122). Yet the likeness to natural things are striking and readily deceitful. As there are many men under one species, so there are many triangles under the one species of triangle, many figures and groups, rings and ideals (123). In fact, intelligible matter plays the part of the material element in

(122) "...to act and to be acted upon does not pertain to beings as they exist in our consideration, but rather as they are in existence. The mathematician, however, considers abstract things only according to his consideration, and thus those things, to the extent that they come under the consideration of the mathematician, cannot be the principle and the term of motion; consequently, the mathematician does not demonstrate through efficient or final causes..." Thomas Von Aquin, op. cit., Questio V, Art. 4, ad 7, p. 61.

(123) "There are some forms which, though not needing matter under the determined disposition of sensible qualities, nevertheless require matter existing in quantity -- as a triangle, a square, and similar things -- and these are called mathematical; they abstract from sensible matter, but not from intelligible matter, inasmuch as continuous quantity, abstracted from any sensible quality, remains within the intellect. Thus it appears that as natural things have form in matter, so do mathematical things; because of this, both in natural and in mathematical things there is a difference between the thing and that which it is; consequently, in both there are found many individuals under one species. For just as there are many men under one species, so there are many triangles under one species." Saint Thomas Aquinas, In III Ethicorum, Lect. VIII, n. 708.

mathematical conceptions and definitions analogous to the role of sensible matter in the first degree (124). Though mathematical natures are not generated and do not, strictly speaking, come into being or exist in a state of becoming, there is a fanciful resemblance in the imaginary movements and qualitative states which characterize ^{AND OF MATTER} THE mode of existence to that of the world of natural things.

The fact that there are no singular things in the mathematical realm but merely individual resemblances, indicates another unique property of mathematical science. The mathematical natures, though not capable of extra-mental existence as such, are not purely fictitious; it is the natural state of mathematical natures to be so abstracted and abstractly considered and mathematical natures are still ens naturae and not merely ens rationis (125). Without adding anything further to determine or individuate it, a particular circle of any kind may be considered the universal circle and retains all the properties of circle as such. Consequently, in mathematics the general cases hold alike for the special; and, once mathematical natures have been properly denominated and understood, we are able to deduce the special cases from the universal formula (126).

(124) Saint Thomas Aquinas, In VII Metaphysicorum, Lect X, passim. This whole lesson should be consulted as a source for Saint Thomas Aquinas' philosophy of mathematics.

(125) "There seems to be a tendency in those who speculate within the bounds of traditional philosophy, to attribute a merely logical being to such mathematical natures as have recently been invented. It is here maintained that though these are mathematical beings -- entia realia -- mathematicians as yet do not sufficiently understand their mathematical nature, and are only capable of establishing them dialectically.

(126) "Thus we can proceed from a general mathematical formula, -- for example, from the formula of a curve of the second order, -- to the special geometrical forms of the circle, the ellipse, etc., by considering a certain parameter which occurs in them and permitting it to vary through a continuous series of magnitudes. Here the more universal concept shows itself also the more rich in content; whoever has it can deduce from it all the mathematical relations which concern the special problems not as isolated but as in continuous connection with each other, thus in their deeper systematic connections. The individual case is not excluded from consideration, but is fixed and retained as a perfectly determinate step in a general process of change." Ernst Cassirer, Substance and Function and Einstein's Theory of Relativity, authorized translation by William Curtis Swabey and Marie Collins Swabey, Chicago: Open Court, 1923, p. 20.

As mathematics increases in abstractness, it likewise increases in generality, and the more universal mathematical nature is also more rich in deductive content; from the formulation of higher forms of space, the nature of Euclidean space is more clearly understood. Similarly, from the formulation of transcendental and transfinite numbers, the nature of mathematical number itself is clarified. Mathematics, dealing with abstract things as such, naturally gravitates and develops towards increasing abstraction (127).

In the multiplication of mathematical forms in the potential continuum of intelligible matter, we can distinguish the thing -- the imaginative presentation of a parabola or line-segment -- from what it is, its nature (128). The mathematical nature is treated as if it were a substantial thing; we attribute qualities to it, and subject it to various types of mathematical operations and actions. Mathematical actions likewise bear but a fictive likeness to other actions; extracting roots, multiplying, imbedding numbers in the mathematical continuum, making a set dense -- while they are physical actions of man's mind and imagination -- considered in their nature as multiplying or dividing are merely actions by a kind of similitude (129). Even the use of potentiality and infinity in mathematics is analagous (130).

(127) See, in this connection, Alfred North Whitehead, Science and the Modern World, Cambridge: The University Press, 1926, Chapter II, "Mathematics as an Element in the History of Thought," pp. 28-54.

(128) Saint Thomas Aquinas, In III De Anima, Lect. VIII, n. 714.

(129) "For those things which obtain according to number, are not actions except according to similitude, as to multiply, divide and others, as has already been said in other places, namely, in the second book of the Physics, where Aristotle shows that mathematics abstracts from motion, and consequently there cannot be in them actions of this kind, which are according to motion." Saint Thomas Aquinas, In V Metaphysicorum, Lect. 17, n. 1024.

(130) Ibid., In VIII Metaphysicorum, Lect. I, n. 1774.

The infinity of points that I mathematically discern between zero and one, or the equivalence of geometrical systems or sets of postulates, are indicative of how metaphorically, imaginatively and quasi-creatively man is able to order and bring forth the mental universe of mathematical discourse.

The very analogy, metaphoricalness and similitude of this mathematical world, however, betrays both its dependence on and distinction from sense data, physical science, the philosophy of nature and metaphysics. The mathematician could not speak of the density of sets, of the inclusiveness of a group or ring, of the higher order of a differential equation, had he not himself learned and borrowed the meaning of density, groups and orders from his own previous experience of the universe. Nor could the mathematician respond to questionings on the validity of his mathematical actions and operations, unless he had recourse to some metaphysical and ultimately defensive doctrine.

The mathematical universe, though apparently forming a world of thought unto itself (131), occupies a median position within the field of speculative science. Above it lies the metaphysical realm of more perfect knowledge and of knowledge more satisfying in itself; for metaphysics by achieving immateriality as such, attains the very being of reality. Inasmuch as the ultimate determinations of the nature of mathematics are metaphysical, the mathematician is subject to metaphysical truths. In his multi-dimensional questionings on the nature of his science and in

(131) Aristotle, *Metaphysics*, 1076^a 8-1078^b 8. In this passage, Aristotle is concerned with determining the precise meaning in which mathematical objects can be said to "exist."

his very operations and reasonings as scientist, he makes frequent admission, at least by implication, of this metaphysical supremacy. Whenever he questions the difference between continuity and discreteness, the ultimate validity of geometrical reasoning, the origin of the natural numbers or number as such, the nature and types of infinite -- the mathematician is concerned with the very being and nature of his science.

The mathematician exercises a similar hegemony over the group of sciences in the first level of abstraction. Since his main concern is the study of quantity under the relations of order and measure, the mathematician is rightfully an authority in any domain where the quantitative is concerned. He has made a science, a wisdom, and an art of dealing with quantitative, to which the first level of science, interested as it is in knowing and adjudicating things in their mobility and materiality, looks for means of determining and dealing with the quantitative as invested in change and matter (132).

(132) "...Whatever impossibilities hold concerning mathematical bodies, necessarily hold for natural bodies; this is so because mathematical things arise through abstraction from natural things; mathematical things, however, are related appositely to the mathematical (for they add sensible nature and motion to them, from which mathematical beings abstract); and thus it is evident that those things which hold of mathematics are preserved in natural things, but not the converse. Consequently, whatever inconvenient reasons are against mathematical beings are also against natural, but not the converse." Saint Thomas Aquinas, In III De Coelo et Mundo, Lect. 3, n. 4. In The Degrees of Knowledge, Jacques Maritain seems to argue that the philosophy of nature governs mathematical concepts; see p. 38, and Part One: The Degrees of Rational Knowledge, passim.

Chapter X. Mathematical Abstraction and Contemporary Mathematics.

History testifies to the fact that many philosophers, by introducing current mathematical proposals or supposedly certain mathematical statements as illustrative or confirmative of their doctrine, were sometimes requited for this obeisance by a later condemnation (1). In fact, a mathematical condemnation of this sort is frequently viewed as being both a demonstration and an adequate refutation of the falsity of the philosophical proposals. In this apparent process of refuting philosophical teachings, however, there is not only found a certain amount of precipitancy in judging the whole philosophical system by some casual, illustrative example, but also a certain species of scientific dogmatism which may actually involve an uncertain, unproved or incompletely understood proposition (2).

(1) See Bertrand Russell's criticism of Immanuel Kant in The Principles of Mathematics, Chapter LII, "Kant's Theory of Space," pp. 456-461. or Leon Chwistek, The Limits of Science, Chapter IX, "Problems of the Methodology of the Exact Sciences," pp. 217-267, where Chwistek's use of a metaphysical view is combined with denials of the relevance of metaphysics and with warnings of its dangers.

(2) Writing about the new system of logic which was inaugurated with Russell and Whitehead's Principia Mathematica, Leon Chwistek maintains:

"When this new system is completely worked out, we will be able to say that we have at our disposal an infallible apparatus which sets off exact thought from other forms of thought.

The old dream of the logicians concerning a consistent logical apparatus will no longer be a mirage. Just as we have calculating machines, in time we will have the apparatus necessary to derive the general theorems of semantics.

However, I think that there is no reason to wait until this ideal has been achieved.

The very confirmation of such a possibility offers weapons which are adequate to combat the attacks of the anti-rationalists and to free us from any possibility of attack by them.

A science which is based upon an infallible system of logic and which involves no irrational assumptions will be able to fulfill the mission towards society which Professor Natanson requires of it.

Such a science will not fall into error and will not be brought to a standstill as a result of its own illusions.

Such a science will be able to say to the nations: Construct new concepts if nothing else, but guard against operating arbitrarily with them." Leon Chwistek, The Limits of Science, pp. 22-23. Chwistek's whole book should be consulted as an instance of a defence of mathematical logic and as an example of an attack, sometimes virulent, against all other systems.

I may say, for example, that mathematics and logic are identical, and, on the basis of this postulated position, believe that I am able to refute all previous philosophical analyses of mathematical problems (3); at the same time, my very position or the scientific statement I employ as the measure of all truth, may itself be unproved or, perhaps, even incapable of proof (4). The error of Immanuel Kant's philosophical position, for instance, is not at all pointed out by showing that his notion of absolute space is now mathematically untenable, though this may be employed as a confirmatory argument (5). The mathematician is not, as has frequently been pointed out, concerned with philosophy in his capacity as mathematician. Consequently, the philosophical error of Kant can only be

(3) This dogmatic position was adopted by Bertrand Russell, and expounded in a number of his writings, especially in The Principles of Mathematics; it is, further, the position of Leon Chwistek and others. F. S. C. Northrop, for example, in The Logic of the Sciences and the Humanities (New York: The Macmillan Company, 1947), substitutes logic for metaphysics throughout the book, with speculative fiction as a result.

(4) The supposed identity of mathematics with logic is slowly losing adherents. For a recent, thorough discussion of logicism and its metaphysical implications, see Andre Derbon, La philosophie des mathematiques.

(5) As has been pointed out in Chapter VIII above, metaphysics, mathematics, the philosophy and the sciences of nature are mutually harmonious and closely related. All these sciences are concerned, in their own ways, with true, scientific knowledge of extra-mental reality. Contradictions and disagreements between them are generally explicable by our inability, in a given historical period, of scientifically demonstrating the certitude of the truth we possess. Hence, conflicts between metaphysics and science cannot be real, but merely apparent. In this connection, one should not be speculatively precipitate and expect to garb each new, scientific proposal with philosophical adequacy.

refuted on philosophical grounds by showing, among other points, the error of his fundamental postulate of an apriority of categories in the mind. Similarly, to cast out Aristotle's, Descartes' or Leibniz's philosophy on the basis of some inaccurate mathematical statements is merely indulging in speculation which is dictated by an unprovable assumption: science and mathematics are the determinants of what is true (6).

Consequently, it is not the aim of this chapter to support or to establish the metaphysical conclusions that have been offered by giving them a mathematical blessing, nor to attempt the even more difficult task of establishing a minute correspondence of metaphysics to mathematics (7). The present chapter, rather, aims to show an inner, basic harmony between metaphysics and current directions in mathematics and in the philosophy of mathematics. In this attempt, the general truths of metaphysics and the truths particularly relevant to the nature of mathematical abstraction serve as speculative guides in ordering the content and discussing the nature of the multiple questions which lie between the disputed frontiers of metaphysics and mathematics.

It should be noted, in passing, that the pronouncements of even great mathematicians on the method or on the nature of mathematics are to be judged on their own merits rather than on the authority from which they

(6) In his most recent book, Human Knowledge: Its Scope and Limits, Bertrand Russell proposes a philosophy which takes as its point of departure the commonly accepted scientific knowledge of our times; see "Introduction," pp. xi-xvi, and Part One: The World of Science, Chapter I, "Individual and Social Knowledge," pp. 3-8.

(7) Jacques Maritain, The Degrees of Knowledge, pp. 58-63 and pp. 76-81.

come (8). The common intellectual disease of requesting and subsequently venerating philosophical, religious and sociological pronouncements from mathematicians who may have had no previous training in those fields is, no doubt, incurable (9). However, as one naturally patronizes and trusts a doctor for medical treatment but not necessarily for the history of physics, so it is proper to judge metaphysical pronouncements both with regard to their source and on their own merits (10).

(8) "No more than any other man does the scientist see himself thinking. Doubtless, if he has peculiarly powerful gifts, he may succeed, by a slow and patient analysis, in sometimes recognizing the true path which his thought has followed; but the fact of being a scientist, nay, even a great scientist, has nothing to do with it. Indeed, the distinctive quality of a great scientist is a powerful scientific instinct, a sort of divination which allows him to touch the high places only. The discovery, it has often been noticed, comes to him suddenly -- after long labor, of course; it is a flash, a revelation: is it astonishing that he has not been able to trace its genesis? And so it follows that we must not look to the scientist for the principles which have really guided his thought; we must not even believe him on his word when he tries to state them. He may have discovered these principles in almost any other way than by a patient analysis of his own thought. He has most often found them already fashioned in some book; they penetrated him without his knowledge, because they pervaded the intellectual atmosphere surrounding him." Hille Meyer-son, Identity and Reality, p. 387.

(9) "This explains how one can go astray in searching for the principles of science, even on the supposition that one is following closely the scientists; their methodical ways are cheerfully accepted, without questioning whether the aforesaid methods had ever really been applied by the scientists themselves." Ibid., pp. 387-388.

(10) No one has succeeded in pointing this out as well as Hille Meyer-son in his historical-critical works. A study similar to that of Meyer-son's, though concerned only with mathematics, is Pierre Boutroux, L'Idéal scientifique des mathématiciens (Paris: Felix Alcan, 1920); Boutroux, however, offers some conclusions from his study with which it is difficult to agree.

The search for the unity of mathematics (11), a popular theme today, is akin to the search for the unity of any particular science, or of science in general. What we seek in our questionings on the unity of science is a principle of unity and of diversity; the principle must help to order the diverse parts of the science, and, at the same time, serve as the measure of both the perfections and limitations of the science, thus enabling us to adopt or discard matters, questions, problems, theories and judgments that are relevant or alien to the science itself. In their quest for the unity of mathematics, thinkers have been led to logic (12), to language (13), to symbolism (14), to intuition (15), to

(11) See, for example, Albert Lautman, Essais sur l'unité des sciences mathématiques dans leur développement actuel, Paris: Hermann & Cie, 1938, or the general tenor of the articles in Les grands courants de la pensée mathématique, présentés par P. Le Lionnais.

(12) "En s'attachant ainsi chaque fois, non pas à la quantité des éléments, mais à l'existence ou à l'armature des êtres que l'on compare, on découvre ainsi entre le fini et l'infini des analogies de structure et des adaptations réciproques, d'où il résulte que l'unité des mathématiques est essentiellement celle des schémas logiques qui président à l'organisation de leurs édifices. Nous retrouvons ainsi des considérations qui rejoignent celles que nous avons développées dans notre thèse principale. Notre Essai sur les notions de structure et d'existence en mathématiques tend en effet à montrer qu'il est possible de retrouver au sein des théories mathématiques, des idées logiques incarnées dans le mouvement même de ces théories." Albert Lautman, op. cit., pp. 57-58.

(13) Rudolf Carnap, "Foundations of Logic and Mathematics," Volume I, Number 3 in the International Encyclopedia of Unified Science, Chicago: University of Chicago Press, 1939, passim.

(14) See the presentation of Hilbert's formalism in Louis O. Katssoff, op. cit., Chapter Nine, pp. 116-135.

(15) Quite a number of mathematicians and philosophers refer to "intuition" as a species of ultimate for mathematics, without accurately designating what the term means. This seems to be a mysterious answer or, at the least, a confession of facing an insoluble problem or of ignorance.

the natural numbers (16), to order (17), to the mind (18), and to reality itself (19); and, in all of these directions, a partial unity has been found (20). The problem of the unity of mathematics, however, approached through the insights offered by mathematical abstraction, is ultimately seen to reside in the notion of quantified substance. With this principle as the main guide, an attempt will be made, first, to point out the unity of the nature of mathematics; second, to illustrate this unity in the origin and development of mathematics; third, to correlate this unity with the logic of mathematics.

The nature of mathematical science is unified and specified by its ultimate subject matter, quantity, or the order of the parts of quantified substance. The historical development of mathematics (21), and the

(16) A short sketch of the adherents to natural numbers as an ultimate for mathematics, is found in Leon Chwistek, op. cit., Chapter III, "The Development of the Concept of Number," pp. 53-82.

(17) See the important part that the conception of order plays in Bertrand Russell, The Principles of Mathematics, Part IV: Order, pp. 199-256.

(18) The denomination of mind as the ultimate, unifying principle of mathematics generally stems from idealistic philosophy which conceives of the mind as somehow "creating" mathematics ex nihilo. See, for example, James Byrne Shaw, op. cit., pp. 166-168.

(19) Reality is designated as the unifying principle of mathematics by empiricist-minded speculations which would, for example, even make geometry a physical science.

(20) It should be pointed out, again, that man as mathematician has a rightful concern with strictly mathematical ultimates. Objections arise only when these mathematical ultimates are given a metaphysical status.

(21) If the distinction between mathematics and the philosophy of mathematics is granted, then the history of mathematics can be seen to polarize about the notions of quantity, continuity and discreteness, number and form -- on the ordering of the parts of quantified substance. See, for example, the remarks in J. T. Bell, The Development of Mathematics, Chapter I, "General Prospectus," pp. 3-24, or the remarks in Felix Klein, Elementary Mathematics from an Advanced Standpoint. Volume I: Arithmetic, Algebra, Analysis, pp. 6-16 and pp. 77-86.

testimony of mathematicians themselves (22), illustrate and help to confirm the consonance of mathematics itself to the metaphysical position on its nature. From the extra-polated view of mathematics which metaphysics affords, the alternating stress on number and form, the search for new methods and algorithms for treating number and form together or separately, are viewed as being a conflict intrinsic to the nature of quantity itself (23).

The unity of nature resident in mathematics requires not a static, but a developmental and dynamic unity which can account for the diverse considerations and emphases in the history of mathematics. It is sometimes contended that there is a dichotomy of some sort between ancient or classical and recent or modern mathematics (24). Not merely refer-

(22) One must likewise distinguish the mathematician proper from the philosophizing mathematician to see the full effect of this claim. There are, literally, hundreds of mathematicians who are concerning themselves with arithmetic, geometry, the calculus, and the various branches of mathematical investigation into continuous or discrete quantity. Historians of mathematics, mathematical congresses and mathematical journals are cumulative evidence of what mathematics is concerned with in its existential status today, as well as in the past.

(23) "The opposition of formal principles exhibits itself in the science of mathematics as well. Ever since Pythagoras, this science has been facing the issue between continuity and discontinuity. This issue has exhibited itself in different theories on the relation between geometry and arithmetic. At different times in the history of science one of these sciences has been supposedly reduced to the other. In the modern world, the tendency has been to reduce geometry to arithmetic. But the reductions have never been decisive" V. S. C. Northrop, Science and First Principles, p. 241. Some view this conflict as being inherent in, and as an expression of occidental or oriental culture. See, for example, Oswald Spengler, The Decline of the West, Authorized translation with notes by Charles Francis Atkinson, 2 volumes, New York: Alfred A. Knopf, 1926, Volume I, "Introduction." Spengler holds that the occidental mind emphasizes form and the oriental, number.

(24) The announcement of a new mathematics is usually accompanied by remarks which belittle past conceptions of mathematics as being somehow concerned with quantity. Alfred North Whitehead has been one of the able defenders of this new view; his proposals are summarized in The Philosophy of Alfred North Whitehead, Volume III: The Library of Living Philosophers, edited by Paul Arthur Schilpp, Evanston and Chicago: Northwestern University Press, 1941, "Whitehead's Philosophical Development," by Victor Lowe, see pp. 18-52.

ring to the fact that historians date the period of modern mathematics as having been initiated through the introduction of analytical geometry and the calculus, this conception of mathematics seems to imply either two mathematics, or that a new mathematics has been invented to supersede the old. As has been previously indicated, it is here maintained that the Greeks founded mathematics as a science; its consequent development and expansion are but evidence of a deeper understanding and clarification, on the part of man, of the science of quantity (25). If the unique nature of the judgments of mathematical abstraction had been properly understood by previous mathematicians and metaphysicians, there might possibly have been no misunderstanding concerning geometries of various dimensions of different methods in algebra, as there is no present need for doubting the reality of transfinite numbers. These inter-scientific clashes, however, may be beneficial to the scientists concerned and prevent misguided reasoning, loose terminology and extreme speculative statements; in this way, for example, the judgments of pure mathematics, though closely aligned, in their initial invention, to experience and to practical mathematics, eventually are seen to be capable of an abstract formulation, and generally such a formulation is fashioned (26).

That ancient and recent mathematics are but earlier and later manifestations of a unified mathematical science can be discerned by comparing

(25) In his Differential and Integral Calculus (translated by P. J. Mohr, new revised edition, 2 volumes, New York: Interscience Publishers, 1946), Richard Courant points out that elementary or ancient mathematics is characterized by its association with and proclivity for geometry, whereas modern mathematics, besides having benefited by new methods, emphasizes number; see Volume I, pp. 1-3.

(26) This is the basic thesis of Carl B. Boyer's The Concepts of the Calculus; see his discussion in Chapter I, "Introduction," pp. 1-13.

them first, with respect to content, and second, with regard to method. The content of mathematics for the ancients included arithmetic, euclidean geometry, number theory; in addition, they had the essential elements of trigonometry in the conic sections and, in Archimedes especially, a rudimentary calculus (27). It is evident that the content of mathematics has immeasurably increased. This development, however, has not been of such a nature as to demand a new definition of the science in the sense that it no longer deals with quantity, but has become, instead, the logical deduction of premises of any sort, or the free and completely unlimited domain of human creativeness.

It is possible, in fact, to regard the content of ancient and recent mathematics reductively as being ultimately arithmetic -- or reducible to the nature and to the science of discreteness; or, ultimately geometric -- or reducible to the nature and to the science of continuity (28). In this view, more advanced branches of mathematics are elaborations or combinations of these two fundamental sciences. The striking differences between

(27) For a short summary of Greek mathematics, see Thomas Little Heath, "Greek Mathematics and Astronomy," Scripta Mathematica, 5 (1938), pp. 215-231, or the same author's A Manual of Greek Mathematics.

(28) In the article, "L'Architecture des mathématiques: la mathématique, ou les mathématiques?" (in Les grands courants de la pensée mathématique, présentés par F. Le Lionnais), Nicholas Bourbaki writes: "Bien que plus actuelle que jamais, il ne faudrait pas croire que cette question soit nouvelle; elle s'est posée presque dès les premiers pas de la science mathématique. C'est qu'en effet, même en négligeant les mathématiques appliquées, il subsiste, entre la géométrie et l'arithmétique (du moins sur leur aspect élémentaire) une évidente dualité d'origine, celle-ci étant initialement science du discret, celle-là de l'étendue continue, deux aspects qui s'opposent radicalement depuis la découverte des irrationnelles." p. 36. See, in this connection, the collection of statements on arithmetic, algebra and geometry in Robert Mucard Moritz, Memorabilia Mathematica, Chapters XVI-VIII, pp. 261-321.

modern and previous mathematics is to be explained especially by the invention of new methods, of a more perfect and supple symbolism, and of an increased comprehension of the nature of continuity and discreteness (29).

By relinquishing sensible matter, all the branches of mathematics lie on the same level of abstraction; they are all concerned with abstract, quantified substance. Two specifically distinct sciences, however, by considering quantified substance as continuous or as discrete, comprise the unified corps of mathematical sciences. This bifurcation of quantity into the two irreducible types of arithmetic and geometry is based on the very nature of quantity itself as given in mathematical abstraction. Since, in mathematical abstraction, the laws of mathematical being are given to the mind, the mutual irreducibility of these two branches of mathematics resides, in last analysis, in the very nature of reality itself (30).

Of the two sciences, arithmetic is commonly credited with achieving a higher type of immateriality, for discrete quantity is, by its very nature, more determined, while continuous quantity is more indeterminate and potential (31). With continuous quantity there is given to the mind a subject which has an infinite potentiality for determination within its very

(29) E. T. Bell, The Development of Mathematics, Chapter I, "General Prospectus," pp. 3-24.

(30) Louis de Broglie has insisted on this irreducible character of the continuous and discrete in physics, mathematics and other sciences; see his Matter and Light: The New Physics, Translated by L. H. Johnston, New York: Dover Publications, 1946, Part V: Philosophical Studies on Quantum Physics, pp. 217-272, and his article, "Le rôle des mathématiques dans le développement de la physique théorique contemporaine," in Les grands courants de la pensée mathématique, présentés par F. Le Lionnais, pp. 398-412.

(31) Aristotle, Posterior Analytics, 87^a 31-35.

nature; that which is continuous may be const^{RU}ured as a line, plane, or even a multi-dimensional domain (32). The discreteness of number, however, though it can be increased ad infinitum, is a potential infinity which is more properly viewed as lying outside the number being added to or increased, as is the case, for example, in transfinite numbers which can be viewed as groups of numbers determinately given (33).

Number, then, as a determination of discrete quantity, is definite and determined, while continuous quantity, though determined to a certain extent as form, contains an intrinsic indetermination in its very nature. This enables us to understand why it is that arithmetic is, in a sense, less dependent than geometry on the imagination. Though phantasms are needed to represent the conceptions of both, almost any phantasm qualifies to represent a number, whereas a specific phantasm is required to represent a parabola, a triply-connected domain or a torus (34). The indetermination of continuity in comparison to discreteness is emphasized and supported by the fact that geometry has more of the qualitative attached to it (35); in fact, it may readily be that this indeterminateness and qualitative character of geometry explains why geometry is sometimes regarded as a physical

(32) For a mathematical and philosophical presentation of geometry similar to the view adopted here, see Thomas Greenwood, Essais sur la Pensée Géométrique, Ottawa: Les éditions de l'Université d'Ottawa, 1943.

(33) Thomas Greenwood, La nature du transfini, pp. 60-61.

(34) This is not meant to deny the possibility of giving a formal definition to these geometrical entities; it is maintained here, however, that a definition of a mathematical entity or subject must be given under some form of intelligible matter -- surface, line, plane, and so forth, or continuity itself.

(35) Pierre Duhem, La Théorie Physique: son objet -- sa structure, pp. 167-168.

science of real space, and further explain how philosophers in the past have not credited it with the more perfect exactness of number (36).

Since these two sciences are specifically distinct and irreducible, in a sense, they may have their own principles, use their own methods and ask their own questions, for their opposed nature is rooted within the very nature of quantity itself. At the same time, since quantity is the ultimate subject of consideration, the methods of the one can be successfully and fruitfully applied to the other; even in this case, the methods of arithmetic are more readily applicable to geometry than the converse. The merit and utility of analytical geometry resides more in the fact that figures can be dealt with by numbers than the converse. To the mathematician who, we might say, is more "physical" minded and who for many reasons prefers geometry precisely because it relies more on the imagination, the study of continuity will be preferred to that of discreteness (37).

Recognition of the primacy of arithmetic over geometry, of number over the continuum follows not only from the metaphysical considerations of quantity, but can likewise be gleaned from the actual development of mathematics (38), and from the testimony of mathematicians themselves (39). The present, pronounced emphasis on number and the recognition of its priority, however, is matched with an equal recognition of the importance of the con-

(36) Emile Meyerson, *Du cheminement de la pensee*, Livre III, pp. 298-306.

(37) This notion of preference for one aspect of mathematics than another is probably based on hereditary, environmental and psycho-physical factors. Henri Poincare has given some attention to this problem in *The Foundations of Science*, pp. 219-222, and Pierre Boutroux's *L'Ideal scientifique des mathematiciens* strikingly illustrates this preference for various branches of mathematics as well as for analytic, synthetic, deductive and inductive methods of presenting mathematical science.

(38) Carl B. Boyer's *The Concepts of the Calculus* and E. T. Bell's *The Development of Mathematics* illustrate this recognition of the primacy of number over form which gradually replaced the Greek emphasis of form over number.

(39) For a short summary of some of these opinions, see Leon Chwistek, *The Limits of Science*, Chapter III, pp. 53-82.

tinuum as a mathematical ultimate (40). The alternating stress on either of these two aspects of quantity illustrates the speculative dependance of mathematics on metaphysics; mathematicians of both past and present, frequently have recourse to metaphysical considerations for a more accurate determination of their conceptions (41). An interesting instance of the dependence of mathematics on metaphysics is illustrated in current discussion on the nature and origin of the "natural" numbers, which have been frequently signalized as mathematically ultimate. The speculative confusion which is presently attached to these discussion, however, can be partially cleared by recognising that though mathematics deals with numbers, it only considers them as elements of mathematical abstraction. The natural numbers, in the mathematical view, are no more nor less real than transcendental numbers, equations, or complex roots. Before the mathematician abstracts, he has already learned to count with numbers; consequently the numbers with which man counts and the counting process itself are not given to children or to mankind by the mathematician, but the converse obtains. For it is on the basis of previous experience that the existence of mathematical numbers are ascertained and further developed in the scientific process

(40) See, for example, the article by Kurt Gödel, "What is Cantor's Continuum Problem?" (The American Mathematical Monthly, 54 (1947), pp. 515-525, where the author shows the present points of tension between number and the continuum.

(41) Mathematicians, of course, will deny that any recourse to metaphysics is ever made. In questioning the ultimate origin of the natural numbers or the continuum, however, and in lodging these notions in logic or in attributing some sort of pre-existence to these numbers, mathematicians are seeking extra-mathematical answers which can only be supplied by metaphysics.

of mathematical abstraction (42).

The other issue of comparison between ancient and recent mathematics is that denominated by method. It is here, especially, that mathematics has made undeniable and significant progress which, besides improvement in notations and symbolism, includes the methodology of analytical geometry, the calculus, and the recent introduction of mathematical logic as an aid in formalizing and facilitating mathematical operations (43). Notable improvements in method are generally accompanied by an increased understanding of mathematical abstraction, even though the individual mathematician who fosters a major advance in method may not be reflectively cognizant of this fact. At the same time, though new methods increase the extensive domain of mathematics as well as clarifying its content (44), they are not sufficient in themselves to alter the nature of mathematics; properly viewed, in fact, they but enhance its unity as the science of dis-

(42) Number for the mathematician is a mathematical being; consequently mathematical numbers lie on the second level of abstraction. At the same time, the mathematician, before he engages in mathematical abstraction, has already learned how to count and has been given the so-called "natural" numbers in this way. All the mathematical numbers which are developed arise from mathematical abstraction with the help of various insights and experience of all sorts. This dependence of science on pre-scientific experience has been brought out, partially, in Bertrand Russell, Human Knowledge: Its Scope and Limits, pp. 237-239.

(43) A fine study of the advances in mathematical method, especially emphasizing notation and symbolism, is Florian Cajori's A History of Mathematical Notations.

(44) For example, the recent success of formalizing much of mathematics can be regarded, quite accurately, as pointing up the character of formal causality proper to mathematics. That discoveries in method have clarified the content and increased the extensive domain of mathematics can be seen from a consideration of the quick development that followed the introduction of analytical geometry and the calculus; their methodology was instrumental in revealing hitherto unknown curves and relationships between number and form; see E. T. Bell, The Development of Mathematics, Chapter 7.

erateness and continuity.

One of the first differences of method usually emphasized as differentiating recent mathematics is its freedom from imaginative schemes, diagrams or constructions. This freedom from constructibility, however, is a claim made from the logicist or formalist view, and is inescapably connected with the doctrinal position of each of these schools -- their denial of content to mathematics and their over-emphasis on the formal nature of mathematical reasoning. In opposition to logicism and formalism, intuitionism has insisted on the connectedness of mathematics with constructibility, and has contributed greatly towards a renewed consideration of mathematical method and rigor, especially in the current researches on the mathematical infinite (45). Mathematics of all times is heavily indebted to imaginative construction and schematic representation in its very nature, its development, process of invention and in teaching; it is to the merit of intuitionism that it has re-emphasized the connectedness of mathematics to pre-scientific experience and to psychology (46). In connection with constructibility, it should be pointed out that it is not quite true to say that the ancients used the synthetic method in contrast to the present emphasis on analytical methods; here, again, the history of mathematics and current practice show the employment of both methods freely (47).

(45) Louis O. Katssoff, op. cit., pp. 174-177.

(46) Harold P. Mart, op. cit., p. 113.

(47) See in this connection, the study of Pierre Boutroux, L'Idéal scientifique des mathématiciens. Boutroux studies the remarks which mathematicians themselves have made on mathematical methodology and on the nature of mathematics. He points out how one mathematician may merely prefer -- an inexplicable fact -- the analytic to the synthetic method, and how differently mathematicians have attempted to define each of these methods.

The closeness of early mathematics to experience and to practical needs has not been changed at all in recent mathematics; if anything, this close and suggestive relation of origin in experience, and the sphere of applicability to nature has immeasurably increased and helped further mathematical development (48). At the same time, the school of formalism has helped to stress the nature and use of deduction in mathematics, as well as the importance and fertility of symbolism and formal causality in mathematical demonstration. Though all the sciences deal with formal causality, their concern with material, efficient and final causality is not found in mathematics except analogously, and the notion of formal cause in mathematics is viewed as universal in itself and as having a deducible content. Once a given perfection has been achieved in mathematics, one can rearrange the order of invention into a logical order where the whole mathematical content or system is viewed as flowing from a few principles. This logical presentation of results, however, is but methodical; it need not be adopted except for purposes of rigor, logical coherence and mathematical elegance (49).

Differing not in ultimate content but mainly in method, ancient and recent mathematics exhibit the structural and developmental unity of the science concerned with ordering the parts of quantified substance. Even the

(48) The mutual interaction of mathematics and the sciences of total abstraction has benefited both. At the same time, quantitative methods are being introduced into almost all of these sciences. As an example of such an application prefaced by a philosophical discussion on its rightfulness, see John Von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton: Princeton University Press, 1944, Chapter I, "Formulation of Economic Problems," pp. 1-45.

(49) "The obvious conclusion is that mathematics cannot be identified in with their own description, or even with the deductive method which justifies this operation -- no more than physics can be identified with the descriptive structure of a theory. However far is carried a deductive process applied to a group of mathematical notions, it is never possible for deduction to exhaust the potentialities of the notions considered." Thomas Greenwood, "Invention and Description in Mathematics," p. 84.

discussions on the foundations of mathematics serve to emphasize this unity which is vouchsafed to mathematics by metaphysics. The distinction of logic from mathematics is gradually being admitted anew, and a number of efforts are being made to align symbolic logic with traditional logic (50). This research reopens the possibility of removing the metaphysical assumptions which were implied in the early and later investigations in the field of symbolic logic, and suggests the further possibility of pointing out how mathematical logic is reducible in part to mathematical method and symbolism, and to the expression of quantitative order (51). By its striking distinction of mathematics from metamathematics, formalism can be viewed as insisting on distinguishing mathematics as such from metaphysical theories on mathematics and from other sciences; in fact, by distinguishing mathematics and a theory about mathematics, formalism itself is a metamathematics -- a metaphysical theory on its nature (52). Aside from being rooted in idealistic philosophy, intuitionism, by insisting on the psychological, historical and sociological aspects of mathematics, has prevented the static conception of mathematics intrinsic to logicism and formalism from being overly influential, and has emphasized the intrinsic connectedness of man as

(50) See, for example, Thomas Greenwood, Les fondements de logique symbolique, Tome II: Justification des calculs logiques, and I. M. Bochenski, "On Analogy," The Thomist, 11 (1948), pp. 424-447.

(51) Thomas Greenwood, Etudes sur la connaissance Mathematique, "Les Principes de la logique mathematique," pp. 1-32.

(52) This seems to be the meaning contained in Kurt Godel's famous theorem; see Louis O. Kattsoff, op. cit., pp. 194-195.

mathematician to man as pre-scientific, scientific and cultural (53).

A second mode in which mathematics manifests its unity as a science concerned with quantity is in its origin, in its relation to pre-scientific knowledge and to experience, and in its relation to the imagination. Of the two main questions which naturally suggest themselves, the origin of mathematical science from previous experience will be considered first, and then its psychology of invention. The basis for a distinct consideration is that since the first is the order of learning and the second is the order of invention, some differences exist between these two manners of acquiring mathematical science (54).

While mathematicians are much concerned with proofs, symbols, constructions and judgments, mathematics, like all sciences, has its origin in pre-scientific knowledge and in sense experience (55). While it is true that mathematics is a science, that it is abstractive, it is also true that no man is constantly exercising acts of mathematical abstraction. Mathematical abstraction as a scientific act may either be self-induced -- as in mathematical genius, perhaps -- or stimulated by parent, teacher or book. But even the ability to count or the use of the numbers commonly employed in the counting process are not acts of mathematical abstraction; numbering and counting in mathematics are merely operations analogous to the physical acts of counting real objects. A recognition of this pre-scientific basis would possibly prevent attribution of a mystical quality to the natu-

(54) Henri Poincaré, The Foundations of Science, Book II: Mathematical Reasoning, Chapter II, "Mathematical Definitions and Teaching," pp. 430-447.

(55) Quite a number of testimonies could be adduced to support this conclusion; see, for example, C. E. M. Joad, Guide to Philosophy, London: Victor Gollancz, 1936, Chapter V, "Logic and the Laws of Thought," pp. 146-152, pp. 113-114. Henri Poincaré's The Foundations of Science, S. T. Bell's The Development of Mathematics, Carl B. Boyer's The Concepts of the Calculus, and Felix Klein's two volumes, Elementary Mathematics from an Advanced Standpoint are but a random number of studies which insist on the part of experience in mathematics.

ral numbers or arid, metaphysical speculation on how they are obtained. Questions on mathematical number are distinct from questions on number itself; the former are speculatively referable to metaphysics with regard to their nature and scientific content, and only reductively reducible in their origin to pre-scientific knowledge. In fact, the full meaning of the historical origins of mathematics indicates that (a) there was, as far as we can discover, a time when no man had as yet actually engaged in mathematical abstraction, and (b) whatever man it was who first, say, abstractly conceived of triangle and proved a necessary and universal property of triangle is the founder of mathematical science (56).

This historical lesson is analogous to the lesson that can be learned from daily life; without possibly ever having achieved mathematical abstraction, people count, add, multiply and subtract. A parallel case holds for abstraction in physical, and separation in metaphysical science (57). The lower bound of mathematics, consequently, both in its

1 (56) "In the case of mathematics, it is the Greek contribution which it is most essential to know, for it was the Greeks who first made mathematics a science. As Kant once wrote, 'a light broke upon the first man who demonstrated the property of the isosceles triangle (whether his name was Thales or what you will)'; since which time, thanks to 'that wonderful people the Greeks', mathematics has travelled 'the safe road of a science.' The Greeks in fact laid down the first principles in the shape of the indemonstrable axioms or postulates to be assumed, framed the definitions, fixed the terminology and invented the methods ab initio; and this they did with such unerring logic that, in the centuries which have since elapsed, there has been no need to reconstruct, still less to reject as unsound, any essential part of their doctrine." Thomas L. Heath, A Manual of Greek Mathematics, p. 1.

(57) That scientific knowledge is dependent in its origins on pre-scientific knowledge and closely connected with it, is being more widely admitted. While Meyerson has eloquently insisted on this fact; in his latest book, Human Knowledge: Its Scope and Limits, Bertrand Russell establishes a close tie between pre-scientific and scientific experience -- though both are viewed as yielding merely probable knowledge.

historical and psychological origins, lies in pre-scientific knowledge and experience; in this sense, intuitionism has directed the philosophy of mathematics in its proper direction, as formalism and logicism have pointed out metaphysics and logic as the upper bound of mathematical development (58).

In all processes of mathematical reasoning, and in embodying the results of such reasoning in lectures, articles, or books, the mathematician and all scientists necessarily use phantasms, words, sensible signs and symbols -- a vindication of the sense origins and sense-dependence of human mathematics and science. The idealist who confusedly claims that mathematics is somehow a pure creation of the mind, denies his own experience and the testimony of the history of mathematics just as the empiricist, by denying the mental process of scientific abstraction, confounds pre-scientific and scientific knowledge. Mathematics is not made more palatable or attractive by being thus accoutered in idealistic or empiricist habit, and such erroneous positions on its nature are not the explicative causes for its growth and development, and, as a matter of fact, may not be followed in mathematical practice at all (59).

(58) It is here claimed that any attempt to write about mathematics -- whether done by a physicist, historian or cosmologist -- is basically and reductively a metaphysical view of mathematics. Though acquaintance with the sciences on the first level of abstraction may help the mathematician further the development of his science and enable him to understand better the nature of mathematics, speculative remarks on mathematics or discussions on its relation to other sciences are the function of the metaphysician. For example, most of the essays in Les grands courants de la pensée mathématique (presentes par F. Le Lionnais), and in Mathematics: Our Great Heritage (edited by William L. Schaef), are really written from the metaphysical view.

(59) This point is brought out in Pierre Boutroux's L'Idéal scientifique des mathématiciens, and has been strongly emphasized in Mile Meyerson's works.

To present mathematics as originating ultimately from previous experience and pre-scientific knowledge is not, of course, to present fully-developed mathematics. Any originative theory, harking back to humble beginnings, is, in a sense, regressive. Nor does this theory aim to place restrictions on mathematical freedom and inventiveness. In fact, it is true that by comparison with the experimentation necessary for the physicist or the acquaintance with detail proper to practical and political wisdom, a minimum of sense experience may suffice for the acquisition of mathematics (60). The history of mathematics is replete with examples of men -- some of them even pioneers -- who had reached mathematical maturity and creative power at an early age. At twenty-two, Descartes had founded analytical geometry. Pascal's precociousness permitted him, at the age of fourteen, to be admitted to the scientific discussions out of which the French Academy of Sciences developed. Newton laid the foundations for all his subsequent work in mathematics from the age of twenty-one to twenty-three. At fifteen, Leibniz had already possessed the idea of his combinatory art, and many other mathematicians, Abel Galois, Lobatschewsky and others, did good mathematical work about the age of twenty (61).

(60) This fact was already recognized by Aristotle; see his Posterior Analytics, 81^b 1-6, and Nicomachean Ethics, 1142^a 12-19.

(61) See, in this regard, C. T. Bell, Men of Mathematics (New York: Simon and Schuster, 1937). Bell brings out, in his historical sketches, this trait of an early adaptability to mathematics.

This ready adaptability to mathematics at an early age, while it may be true of men of genius and, in a restricted sense, of any man who acquires the habit of mathematical science, must be properly understood. Though mathematics may arise from a minimum quota of sense experience, it is inevitably bound up with the imagination and the past experience of the individual mathematician. Consequently, while the physicist must be in constant liaison with extra-mental reality and while the mathematician need not have this intimate relationship, the mathematician relies, proportionately, on his imagination. Though a comparison of mathematician and physicist with respect to the origin of both sciences is frequently made, it must be recalled that mathematical natures are not existent as such in extra-mental reality, but merely in the imagination; hence the universe of the mathematician is one of imaginative construction and invention with the materials of his past experience. The physicist is more definitely limited by the complexity and mobility of extra-mental reality; in such median sciences as mathematical physics and mathematical biology, the physicist or biologist, after sufficient experimentation, reaches up to mathematical principles and conclusions as an aid in knowing and measuring extra-mental reality (62).

Thus mathematics, like all speculative sciences, has a double dependence on induction and experience. In the first place, the mathematician and all the scientists abstract the universal from singular things. Secondly, all sciences are concerned with the process of judging and reasoning, in which universal and necessary predicates are attributed to subjects. In both

(62) For an explanation of how the mathematician and the physicist are jointly concerned with knowing extra-mental reality, see Jean Dujat, *op. cit.*, Chapitre IV, "L'Intelligibilité mathématique de la théorie physique," pp. 108-143.

senses, mathematics and the other sciences can be said to be dependent on experience and induction. However, once mathematical abstraction has been achieved, a minute and detailed analysis of extra-mental reality may no longer be necessary, and the riches of mathematical experience lie open within the imagination, where the indetermination of intelligible matter is the infinitely potential support for qualitative determinations of all sorts.

The ultimate origin of mathematical entities from extra-mental experience does not necessarily imply that a reduction of mathematics to physics is being proposed (63). But the theory does claim that in its history, in its origin within each person and in its development, mathematics is bounden to sense experience. The very concepts which are employed in mathematics -- structure, order, relation, domains, inclusiveness, greater, upper bound and countless others -- are eloquent testimony to the fact that unless one had personally or vicariously learned what structure, order, relatedness, inclusiveness and so forth, are, he would not be able inventively to attribute these qualitative states to the mathematical natures in the mathematical universes. Mathematical conceptions likewise carry with them their relatedness and intimate conjunction with practical needs, and their familiarity with the suggestions, insights and methods of all the sciences of total abstraction (64).

The intimate affiliation of mathematics to previous pre-scientific,

(63) In his Lectures on the Philosophy of Mathematics, James Byrne Shaw labels as an empiricist or physical view (p. 10, pp. 154-155), any theory which attributes the origin of mathematics to the natural world of phenomena.

(64) See, in this connection, Robert Deltheil, "L'Analogie en mathématiques," in Les grands courants de la pensée mathématique, présentés par F. Le Lionnais, pp. 48-53, where the author gives some illustrations of the analogical insights suggested to mathematics from other sciences.

physical, mathematical, and even metaphysical knowledge is especially discernible in mathematical invention (65). Numerous examples abound in which analogies to physics (66), to previous mathematics (67), and even to metaphysical conceptions (68), afforded imaginative insight into mathematical inventiveness. Not serving as a hindrance but as a positive help and stimulant, extra-mathematical suggestions and references have proved to be aids to the greater fruition and powerful development of mathematical science; possibly because of the intermediary nature of mathematics and of its analogical character, mathematicians have, often unwittingly, transported the mathematical realm to that of extra-mental reality itself, to that of the physical and metaphysical sciences (69).

(65) This has been pointed out in Thomas Greenwood's article, "Invention and Description in Mathematics," passim.

(66) Louis De Broglie, "Le rôle des mathématiques dans le développement de la physique théorique contemporaine," in Les grands courants de la pensée mathématique, présentés par F. Le Lionnais, pp. 298-411.

(67) Henri Poincaré, The Foundations of Science: Science and Method, Book I, Science and the Scientist, Chapter III, "Mathematical Creation," pp. 383-394.

(68) Georg Cantor is the best example of this; see his Contributions to the Founding of the Theory of Transfinite Numbers, translated by Philip E. B. Jourdain, Jourdain's Introduction, pp. 52-82.

(69) Jacques Maritain, The Degrees of Knowledge, pp. 170-177.

In this regard, the presentation of a mathematical invention is rarely given in the multiplex and complex form in which its invention occurred, and most great mathematicians have, upon excoagitating a mathematical contribution, presented it in a logical, neat and deductive form (70). In fact, it would be impossible, in most cases, to retrace all the steps, processes and ruminations through which the mind was carried and moved about in the process of invention. The attempts at outlining this process, complex though it is, form part of a psychology and logic of invention on which increasing attention is being focused(71).

The third mode in which mathematical abstraction serves to unify the apparent heterogeneity of present mathematical science is in its relation to logic. Not being a direct but a reflective science, logic is not, as such, concerned with extra-mental reality; it is wholly and directly dependent for the existence of its conceptions and determinations of its nature on metaphysics, and only partially and imperfectly on the other sciences. Theories on the nature of logic which disavow its dependence on metaphysics and attempt a logical analysis of reality, are, as a result, extremely complicated speculative conceptions; yet they can still be shown to be little else than apparent substitutions of logical for metaphysical pronouncements.

(70) Thoms Greenwood, "Invention and Description in Mathematics," pp. 80-81.

(71) Henri Poincare was one of the first scientists to give his attention to this subject; see his The Foundations of Science. See the article, "L'Invention scientifique: la mathematique," by Jacques Hadamard (in L'Invention, exposes et discussions par Charles Blondel, d. Claparede, J. Hadamard, L. De Broglie, Ed. Bauer, Paul Valery, Louis Breguet et M. Guyenot, Paris: Felix Alcan, 1938), or the more recent work of Jacques Hadamard, An Essay on the Psychology of Invention in the Mathematical Field. R. D. Carminichael has given some attention to these problems in his The Logic of Discovery (Chicago and London: Open Court, 1930).

Both logicism and formalism, for example, profess to be non-metaphysical in announcing the quasi-identity of mathematics with logic. However, as has been mentioned above, various studies have pointed out that logicism and formalism comprise both a doctrine and a method; not only has the metaphysical doctrine of each been successfully challenged and impugned, but a realignment of their method and symbolism into that of the traditional logical form gives promise of a future benefit to logic itself and to other sciences.

The partial formalization of mathematics, which is the meritorious contribution of both logicism and formalism, has, however, led to various extreme positions. Quite in contrast to the bland compliments of absolute necessity and rigor which were formerly attributed to it, mathematics is, at present, often viewed as offering merely probable truth (72). Stimulated by the recognition of treating mathematical demonstration in a completely formal manner, variegated opinions have arisen^N from discussion on the foundations of mathematics. The wide assortment of views includes a denial that mathematical reasoning has any content whatever and that it is but tautology (73), that it is but a hypothetico-deductive system (74), that it is based on the pragmatic choice of many possible systems (75), and that

(72) Harold R. Smart, The Logic of Science, pp. 64-72.

(73) Frank Plumpton Ramsey and L. Wittgenstein, close adherents to the logicism of Bertrand Russell, proposed the tautological nature of mathematical propositions; see Frank Plumpton Ramsey, The Foundations of Mathematics and Other Logical Essays, *passim*.

(74) Morris R. Cohen and Ernest Nagel, op. cit., Chapter VII, "The Nature of a Logical or Mathematical System," pp. 129-130.

(75) See, for example, the Introduction (pp. 27-30) to Science and Hypothesis in Henri Poincaré's The Foundations of Science; Poincaré has consistently argued for this position.

it arises from the fiat of the mathematical creator (76). In the view of mathematical abstraction here adopted, each of these and other current theories emphasize but one, limited aspect of the nature or method of mathematics, and it is only because these theories generally are erected as the unique, ultimate explanation of the whole of mathematics, that their partial emphases on the truth are frequently lost sight of. For, while it is true that one can scientifically discourse with purely formal logic in mathematics as in other sciences, the subject matter of the demonstration, while temporarily suspended or not considered, returns again when there is question of estimating the merits of a mathematical system which may have thus been invented (77). Consequently, while there may be no limit to human inventiveness and indulgence in erecting formal systems, the relation of dependence of such systems on previous experience and imaginative adumbrations and recombinations, as well as the question of the consistency of the system and the subject with which it is concerned in applicability or use, are extra-formal and indispensable influences (78).

Only a true conception of logic can exhibit not only the unity of mathematics in itself, but also its close kinship with both natural and scientific logic. The aptitude and natural inclination which man possesses to reason orderly in his discursive processes, though it has frequently been ill-used, is inherent in the very nature of man; logic itself, as the science of speculatively ascertaining how man reasons correctly, and as the

(76) James Byrnie Shaw, *op. cit.*, pp. 166-168.

(77) Louis O. Kattsoff, *op. cit.*, Chapters 14-15.

(78) Jacques Hadamard, An Essay on the Psychology of Invention in the Mathematical Field, Chapter VII, "Different Kinds of Mathematical Minds," pp. 100-115.

finished art of properly conducting reason, is but a development of this native ability (79). Logic as a science considers the reasoning process not only in mathematics, but in all sciences, for the science of logic is but the ^{EMBODIMENT} ~~embodiment~~ of the mind's reflective consideration of its own actions. Mathematical abstraction, consequently, considered as a method of arriving at true and scientifically certain mathematical conclusions, is unified to the method of other sciences and philosophy by the unity of the mind itself; whatever diversity from other types of scientific reasoning mathematics may possess, arises from the fact that mathematics is concerned with a different subject matter of demonstration.

Formal Logic is the resolution of the mind's discursive movements into its formal principles; it denominates and explains the mental motions which the mind is to take in order to preserve the right directions in reasoning (80). Formal logic, and its use in all the sciences, is not as such interested in truth; it is interested in the rightness and consistency of the mind's movements; it encourages a simplicity of movements through the syllogism and through symbolic expression and abbreviation (81). Dealing with the study of the three main movements of the mind, simple apprehension, judgment and reasoning, formal logic does not as such include the imaginative and inventive ability of man, which is more proper to logic as dialectical, and to logic as an art combined with psychology (82). Just

(79) Angelo M. Pirotta, Curia Philosophiae Aristotelico-Thomisticae: Volumen I. Philosophia Rationalis, pp. 17-18.

(80) Ibid., p. 20.

(81) Albert G. Rasmussen, op. cit., pp. 169-171.

(82) Jacques Hadamard, op. cit., Chapter I, "General Views and Inquiries," pp. 1-20. This chapter contains an interesting, short account of various theories on human invention.

as the formal rules of logic apply to any type of formal ratiocination, so, too, the general rules of material logic apply to any type of matter or subject with which reasoning is concerned, that of mathematics as that of other sciences. Material logic, however, introduces a new consideration; by resolving reasoning or mental discourse into its matter or that about which reasoning is concerned, material logic considers the qualitative nature of the subject matter in order that a true conclusion may be formulated, whether its truth be scientifically certain, probable, or merely opinionative.

The fact that man can arrive at true conclusions which he may only hold as opinionative or as probable, is paralleled by a division of the body of logical science into demonstrative or doctrinal logic, and dialectical or useful logic (83). The logic of strictly scientific demonstration, concerned with the manner in which the mind may arrive at a certain, universal, and necessary knowledge, points out, among other features, the need and use of various pre-scientific and indemonstrably certain principles as pre-cognitions from which scientific demonstration arises. Doctrinal logic likewise includes a study of the nature of science and its distinctive divisions; in this latter as in its other functions, it is wholly and completely dependent on metaphysics to guide its reflective consideration.

While the study of demonstrative or doctrinal logic has been generally presented in teaching and in discussions on logic, the nature and use of dialectical logic has been widely neglected. Yet it is this reflective study that both modern science and mathematics have especially used to such advantage. In fact, there is a measure of dialectical reasoning in all the

(83) Angelo M. Pirotta, *op. cit.*, pp. 160-164; see, also Saint Thomas Aquinas, *In I Posteriorum /analyticorum*, *Lect.* 30.

sciences. The practice of Aristotle, though frequently dubbed as being too dogmatic, and the procedure of Saint Thomas Aquinas and the medievals, though often categorized as involvement in vapid and arid dialectical disputations, are the very methods that any man uses in the various routes and devious detours which his mind follows towards the goal of scientific truth, and these methods are employed in successful teaching (84).

In its dialectical movement, the mind is in inquisitive motion and in inventive research. It moves about within the set and limited sphere of discourse which it has outlined for itself, and, analogously speaking, runs back and forth in cogitative fashion from one element to another (85). It may be regarding a number of conceptions or propositions which have but a likeness of, and similitude to truth; the mind alternately but truly adopts this verisimilitude of what appears to be true -- checking, in various ways, the plausibility of each proposition. These propositions, the springs of dialectical reasoning, are of three main kinds: 1) they may be composed of purely logical terms which merely signify the second intentions of the mind, as genera, species and the other predicable, or analogously constructive inventions as group, system, set, or set of sets; 2) they may be principles which are not native to the science under consideration, but are borrowed from another science, or common to several sciences, as statements on mathematical continuity applied to physics or biology for purposes of comparison and insight; 3) they may be principles which, though probable

(84) Aristotle's use of dialectics can be seen especially in his Metaphysics; his theory of dialectical reasoning is scattered in his logical works, especially in the Posterior Analytics. The above presentation is based on Aristotle and on Saint Thomas Aquinas' commentaries on Aristotle.

(85) Saint Thomas Aquinas, In IV Metaphysicorum, Lect. IV, passim.

in themselves, are accepted as points of departure for reasoning as if they were certain premises, and only yield probable conclusions, as in the decisions of practical life and in mathematical probability (86). This interrogative dialectic of the mind can, in the first two instances, yield either probable or demonstrative conclusions, for in this case the principles of reasoning themselves are not probable, but the mind doubts their applicability to the matter under consideration. In the third case, since the very propositions themselves are not certain, only opinionative judgments can ensue -- though these opinions, it must be emphasized, are true, probable judgments (87).

Dialectical reasoning plays a large part in mathematical demonstration and especially in its developmental invention. In the first type of dialectical reasoning, where purely logical terms come into consideration, the mind forms logical conceptions bearing only a furtive likeness to mathematical entities, and cogitates upon them in a close, inventive fashion. The description of Cantor's process of inventing transfinite numbers (88), for example, indicates how these logical beings, reflectively formed for consideration by the mind, led to his contribution. In fact, inasmuch as the rigorous proof and mathematical demonstration of transfinite numbers has not, it seems, been as yet formulated, the transfinite numbers -- though the truth

(86) In his latest book, Human Knowledge: Its Scope and Limits; Bertrand Russell denies that we have certain, demonstrative knowledge. His analysis of the probability and dialectical reasoning in pre-scientific and scientific knowledge -- abstracting from his philosophical interpretations -- is quite generally sound.

(87) Saint Thomas Aquinas, op. cit., n. 574.

(88) Georg Cantor, op. cit., Jourdain's Introduction, pp. 52-52.

about them is most likely known -- are only probably true and their certainly true existence as mathematical entities has not as yet been established (89).

In like fashion, the second manner of dialectical reasoning is frequently employed in mathematics with fruitful results. One may take arithmetical principles and apply them to the continuum, or the converse (90); one may even adopt physical or metaphysical principles and, with the insight thus afforded, induce or invent new mathematical natures. The demands of physics on mathematics (91), or Cantor's case can be referred to as two of the many instances in which the principles of other sciences may be wittingly or unwittingly employed in mathematics. Here again, however, a mathematical demonstration of the results effected by borrowing principles of other sciences may not be immediately forthcoming, and the mathematical conclusion may be truly held, but only with probability and a degree of certainty.

The first two methods of dialectical reasoning, then, are used by the mathematician, the scientist and the philosopher. The third type, that based on propositions which in themselves are only probable, is also used in all the sciences, but does not yield more than a probable conclusion, and seems to be capable of both a regressive and a progressive use. In the

(89) This contention -- that mathematical natures may at first only be capable of a dialectical existence -- needs further development, of course. In An Essay on the Psychology of Invention in the Mathematical Field, Jacques Hadamard seems to have this notion in mind; see pp. 43-55.

(90) Georg Cantor, op. cit., Jourdain's Introduction, pp. 1-10.

(91) Albert Lautman, Essai sur l'unité des sciences mathématiques dans leur développement actuel, p. 7.

first, the regressive use, for example, it may appear to be quite probable that mathematics is completely expressible in logical or pre-logical symbols (92); in fact, in entering into the domain of mathematics with this principle and attempting to make mathematical natures lose their identity and be transformed into purely logical or pre-logical symbols, I may even achieve a phenomenal amount of success. However, there are two obstacles which I must yet overcome: 1) the proof of my initial proposition is lacking, and 2) I have only succeeded in formalizing that part of mathematics which has thus far been given historical and scientific status; consequently, though my method may produce some fruitful results, it has an empty, barren result as goal, and is a static conception of mathematical science. This method is likewise adaptable in other sciences: in philosophy, for example, I may adopt the position that I most likely can explain everything in terms of mathematics as did Descartes and as do many contemporaries, or in terms of spirit, as did Hegel.

In the second, or progressive use of a probable proposition, which is an employment of the method which may lead to new results, I adopt a certain number of probable propositions and then use this method of mental discourse to yield a conclusion which may have a greater amount of probability. For instance, all the sciences of total abstraction, aiming as they do to know nature in its mobile indetermination, make frequent and large use of this mental dialectic. Mathematics may analogously imitate this process. On the basis of some previously known, though probable propositions, I may construe further propositions which have a less, or equal probability. I may, for instance, take the existence of transfinite numbers as my point

(92) This attempt is true especially of logicism and formalism; see Harold R. Smart, The Logic of Science, pp. 96-99.

of departure, and launch out into new combinations of numbers, and leading me, perhaps, to new axioms and principles.

Viewing mathematics, then, as an abstractive science within the wider and richer domain of speculative science and the logic of science, its unity and quantitative character appears more plausible. For, in this conception, the mathematician is not viewed as in an idealist fashion, as merely contemplating quantitative substance and cut off from the richness of his previous pre-scientific and scientific experience, and yet his inventiveness and creativeness is sufficiently accounted for; neither is the mathematician viewed as in an empiricist analysis, as rooted to nature and limited by the sensations which are channelled into his sensory powers, where they may leave some limited impressions. In the view afforded by the metaphysician, the nature of mathematical abstraction and its relation to metaphysics, to logic, to psychology, to history and to pre-scientific experience, as well as its applicability to the sciences, are seen to related parts of the complex, yet true and adequate determination of mathematics. While the essence of mathematics is its freedom, it can only purchase this freedom and make limitless progress by recognizing the speculative bounds and limitations inherent in its very nature. Though it has been said that God made the natural numbers and man made the rest, it is more correct, metaphysically and mathematically, to say that God made the mathematician and the universe, and the mathematician has abstractively and inventively construed the mathematics of the divinely fashioned universe.

Chapter XI. Conclusion

It has been well said that in the consideration of truth, men look upon it in various ways because of custom and training (1). The philosophizing mathematician, for example, accustomed to the freedom of mathematical reasoning and to the apparent rigor in which mathematics can be presented, does not generally look with speculative sympathy at the efforts of the sciences of total abstraction, ^{and} or at metaphysical demonstrations. Similarly, philosophers and philosophizing scientists in the first abstract level of science, frequently regard the precision of pure mathematics as exaggerated or as being somewhat too refined to be of practical use. Some men, looking upon mathematics, see it as a logical or aesthetic art, while others view it as the work of man the maker and practitioner. It is likewise true, however, that a diversity of speculative sciences requires that truth and certitude be sought in proportion to the things about which each science is concerned.

With regard to the nature of mathematical abstraction, diverse attitudes, stemming from an artistic, logical, absolutist, physical or practical bent of mind, have, throughout history, made of mathematics an aesthetic art, a sheer logic, a creative knowledge, a physical science and a system of computation and measurement. These positions on the nature of mathematics can be partly accounted for, it is true, by psychological reasons. However, the very nature of mathematical abstraction, dealing as it does with abstract, quantified substance, and bearing such close speculative analogies to metaphysics, to logic, to physics and to extra-mental reality, seems to be such

(1) Aristotle, Metaphysics, 994^b 30-995^a 20.

as to lead itself readily to erroneous positions and half-truths.

A first type of confusion to which mathematics has readily lent itself, is one which tends to identify the very nature of mathematics with the nature of other sciences. Already in the infancy of mathematics, Greek philosophical conceptions of mathematics included the views that mathematical conception was synonymous with physical or metaphysical, that mathematical conception revealed the very nature of the universe, that mathematical natures exercised a separately real existence, and even the sceptical view that mathematical abstraction was a falsification of reality from which ethical abstention was urged as a form of virtue.

A second kind of confusion on the nature of mathematical abstraction stems from a lack of distinguishing the question of its nature from its methodological procedure. Mathematical natures are somewhat alike and somewhat different from metaphysical, logical, dialectical, and merely possible natures, and these various types of conceptions may be confused due to the similarity of the scientific or dialectical reasoning process employed. Descartes himself, forgetting that mathematics was mainly concerned with formal causality and that other sciences used material, final and efficient causality because the nature of these sciences demanded such causal treatment, viewed the invention of analytical geometry as a prophetic and introductory gesture to employing the same method of formal mathematical deduction in all the sciences. The Cartesian habit of mind became a speculative fault that tainted consequent mathematical speculation; Descartes' followers did not properly distinguish mathematical from physical and metaphysical science, and seemed unable to rid themselves of this inherited, aprioristic bent of mind.

The partial truth which each of these historical positions on the nature of mathematical abstraction portrays and their inadequacy in yielding a complete explanation and a true judgment on the nature of mathematics, can only be ascertained when mathematics is viewed within the total pattern of speculative science and human knowledge. The contemporary schools of logicism, formalism and intuitionism have defaulted in their consideration of the nature of mathematics especially because their views derive from idealistic, empiricist or logicist philosophical ancestry. Further, since they generally consider speculations in mathematics as being non-metaphysical and purely mathematical questions or questions of logic, most contemporary views are tinged with mathematicism.

Within the total view of human knowledge, the proper makeup of mathematical abstraction is seen to involve 1) an extra-mental foundation for mathematics and its consequent connectedness to the precognitions of pre-scientific knowledge -- a truth which is partially brought out by intuitionism; 2) the mental movement of scientific abstraction, and the accompanying distinction of mathematical abstraction from the abstraction proper to the philosophy and the sciences of nature -- a truth largely neglected in current speculations; 3) the intimate connection and, at the same time, distinction of mathematical abstraction from the reflectivity and symbolism of logical abstraction -- a truth over-emphasized by logicism and formalism; 4) the close relatedness of mathematical abstraction to all human forms of endeavor in both practical and speculative science.

BIBLIOGRAPHY

Books

- Actes du congres internationale de philosophie scientifique. Volume VI: Philosophie des mathematiques. Paris: Hermann et Cie, 1936.
- Adler, Mortimer. What Man has made of Man. New York: Longmans Green and Company, 1938.
- Albert, A. Adrian. Introduction to Algebraic Theories. Chicago: The University of Chicago Press, 1942.
- Aquinas, Saint Thomas. In Librum Boethii De Trinitate Quaestiones Quinta et Sexta. Nach dem Autograph Cod. Vat. Lat. 9850 mit Einleitung herausgegeben von Paul Nyser. Fribourg: Societe Philosophique, 1948.
- _____. Opera Omnia. 32 volumes in 16. Paris: Apud Ludovicum Vives, 1871-1880.
- Archibald, Raymond Clare. Outline of the History of Mathematics. Fifth edition revised, and enlarged. Oberlin: The Mathematical Association of America, 1941.
- Aristotle. The Works of Aristotle. Translated into English under the editorship of W. D. Ross. 11 volumes. Oxford: The Clarendon Press, 1931.
- Bacon, Francis. The Works of Francis Bacon Lord Chancellor of England. With a life of the Author by Basil Montagu. three volumes. Philadelphia: M. Murphy, 1876.
- Ball, W. W. Rouse. A Short Account of the History of Mathematics. London: The Macmillan Co., 1908.
- Bandas, Rudolph G. Contemporary Philosophy and Thomistic Principles. With an Introduction by J. Zyburn. Milwaukee: Bruce Publishing Co., 1932.
- Barrow, Isaac. The Geometrical Lectures of Isaac Barrow. Translated with notes and proofs, and a discussion on the advances made therein on the work of his predecessors in the infinitesimal calculus, by J. M. Child. Chicago and London: Open Court, 1916.
- Barry, Frederick. The Scientific Habit of Thought: An Informal Discussion of the Source and Character of Dependable Knowledge. New York: Columbia University Press, 1927.
- Bell, Eric T. Men of Mathematics. New York: Simon and Schuster, 1937.
- _____. Numerology. Baltimore: Williams and Wilkins Co., 1933.
- _____. The Development of Mathematics. Revised edition. New York: McGraw-Hill, 1945.
- _____. The Handmaiden of the Sciences. New York: McGraw-Hill, 1937.
- _____. The Magic of Numbers. New York: McGraw-Hill, 1946.
- _____. The Queen of the Sciences. Baltimore: Williams and Wilkins, 1931.

- Benjamin, A. Cornelius. An Introduction to the Philosophy of Science. New York: The Macmillan Company, 1937.
- _____. The Logical Structure of Science. London: Kegan Paul, 1936.
- Bennett, Owen. The Nature of Demonstrative Proof According to the Principles of Aristotle and St. Thomas Aquinas. Washington: The Catholic University of America Press, 1945.
- Bentley, Arthur Fisher. A Linguistic Analysis of Mathematics. Bloomington: Principia Press, 1932.
- Berkeley, George. The Works of George Berkeley. With Preface, Annotations, Appendices, and an Account of his Life, by Alexander Campbell Fraser. 4 volumes. Oxford: The Clarendon Press, 1901.
- Bidney, D. The Psychology and Ethics of Spinoza. New Haven: Yale University Press, 1940.
- Birkhoff, Garrett and MacLane, Saunders. A Survey of Modern Algebra. New York: The Macmillan Company, 1944.
- Biser, Irwin. A General Schema for Natural Systems (Nature Considered as a Function of Types of Selectivity and of Modes of Selection). Philadelphia: Westbrook Publishing Co., 1938.
- Black, Max. Critical Thinking: An Introduction to Logic and Scientific Method. New York: Prentice-Hall, 1946.
- _____. The Nature of Mathematics: A Critical Survey. London: Kegan Paul, 1933.
- Blanche, Robert. La Science Physique et la Realite: Realisme, Positivisme, Mathematisme. Paris: Presses Universitaires, 1946.
- Bocher, H. Introduction to Higher Algebra. Prepared for publication with the cooperation of J. P. R. Duval. New York: The Macmillan Company, 1907.
- Boll, Marcel. Les Etapes des Mathematiques. Paris: Presses Universitaires, 1947.
- _____, and Reinhart, Jacques. Les Etapes de la Logique. Paris: Presses Universitaires, 1946.
- Bonola, Roberto. Non-Euclidean Geometry. Translated, and additional appendices by H. S. Carslaw. Chicago: Open Court, 1912.
- Boole, George. Collected Logical Works: Volume II, The Laws of Thought (1854). Chicago and London: Open Court, 1940.
- Borel, E. Lecons sur la theorie des fonctions. Troisieme edition. Paris: Gauthiers-Villars, 1926.
- Born, Max. Experiment and Theory in Physics. Cambridge: The University Press, 19
- Bouligand, Georges. La Causalite des theories mathematiques. Paris: Hermann et Cie, 1934.
- _____, et Desbats, Jean. La Mathematique et son unite: Introduction aux elements de l'analyse et a la philosophie des sciences deductives. Paris: Payot, 1947.

- Bourtroux, P. Les Mathématiques. Paris: Albin Michel, 1922.
 ———, L'Idéal scientifique des mathématiciens dans l'Antiquité et dans les Temps Modernes. Paris: Felix Alcan, 1920.
- Boyer, Carl B. The Concepts of the Calculus: A Critical and Historical Discussion of the Derivative and the Integral. New York: Columbia University Press, 1939.
- Boyer, Lee S. Mathematics: A Historical Development. New York: Henry Holt, 1945.
- Bridgman, Percy W. The Nature of Physical Theory. Princeton: Princeton University Press, 1936.
 ———, The Logic of Modern Physics. New York: The Macmillan Co., 1928.
- Brook, Werner. An Introduction to Contemporary German Philosophy. Cambridge: The University Press, 1935.
- Brunschvicg, Leon. Le rôle du Pythagorisme dans l'évolution des idées. Paris: Hermann et Cie, 1937.
 ———, Les étapes de la philosophie mathématique. Paris: Presses Universitaires, 1947.
- Buchanan, Scott. Poetry and Mathematics. New York: John Day, 1929.
- Buckley, George M. The Nature and Unity of Metaphysics. Washington: Catholic University of America Press, 1946.
- Burnett, John. Early Greek Philosophy. Fourth edition. London: Adam and Charles Black, 1945.
- Burt, Edwin Arthur. The English Philosophers from Bacon to Mill. Edited, with an Introduction. New York: The Modern Library, 1939.
 ———, The Metaphysical Foundations of Modern Physical Science: A Historical and Critical Essay. Revised edition. New York: Harcourt, Brace and Company, 1932.
- Caird, Edward. The Critical Philosophy of Immanuel Kant. two volumes. Glasgow: James Maclehose & Sons, 1889.
- Caird, John. Spinoza. Edinburgh and London: William Blackwood and Sons, 1888.
- Cajori, Florian. A History of Mathematical Notations. 2 volumes. Chicago: Open Court, 1928-1929.
 ———, A History of Mathematics. Second edition. New York: The Macmillan Company, 1931.
- Cantor, Georg. Contributions to the Founding of the Theory of Transfinite Numbers. Translated, and Provided with an Introduction and Notes, by Philip E. Jourdain. La Salle: Open Court, 1941.
- Carimichael, R. D. The Logic of Discovery. Chicago: Open Court, 1930.
- Carnap, Rudolf. "Foundations of Logic and Mathematics." International Encyclopedia of Unified Sciences. Volume I, Number 3. Chicago: The University of Chicago Press, 1939.
 ———, The Logical Syntax of Language. New York: Harcourt, Brace and Company, 1937.

- Garnot, Lazare. Reflexions sur la metaphysique du calcul infinitesimal. 2 volumes. Paris: Gauthier-Villars, 1921.
- Casey, Joseph J. The Primacy of Metaphysics. Washington: The Catholic University of America Press, 1936.
- Cassirer, Ernst. An Essay on Man. New Haven: Yale University Press, 1944.
 _____, Substance and Function and Einstein's Theory of Relativity. Authorized translation by William Curtis Swabey and Marie Collins Swabey. Chicago: Open Court, 1923.
- Cavaillès, Jean. Sur la logique et la theorie de la science. Paris: Presses Universitaires, 1947.
- Church, Alonzo. Introduction to Mathematical Logic. Notes by G. A. Truesdell. Princeton: Princeton University Press, 1944.
- Chrastek, Leon. The Limits of Science: Outline of Logic and of the Methodology of the Exact Sciences. Translated from the Polish by Helen Charlotte Brodie and Arthur P. Coleman. Introduction and Appendix by Helen Charlotte Brodie. New York: Harcourt, Brace and Company, 1948.
- Clifford, William Kingdon. The Common Sense of the Exact Sciences. Edited, and with a preface, by Karl Pearson. New York: Alfred A. Knopf, 1946.
- Coffey, P. Epistemology or the Theory of Knowledge. 2 volumes. New York: Peter Smith, 1938.
 _____, The Science of Logic: An Inquiry into the Principles of Accurate Thought and Scientific Method. 2 volumes. New York: Peter Smith, 1938.
- Cohen, Morris R. and Nagel, Ernest. An Introduction to Logic and Scientific Method. New York: Harcourt, Brace and Company, 1936.
- Comte, Auguste. Cours de philosophie positive par Auguste Comte. Quatrieme edition augmentee de la preface d'un disciple et d'une etude sur le progres du positivisme par E. Littré. 6 volumes. Paris: J.-B. Bailliere et Fils, 1877.
 _____, Oeuvres Choisies. Avec une introduction par Henri Gouhier. Paris: Aubier, 1943.
- Collingwood, R. G. An Essay on Philosophical Method. Oxford: The Clarendon Press, 1933.
 _____, The Idea of Nature. Oxford: The Clarendon Press, 1945.
- Cooley, John C. A Primer of Formal Logic. New York: The Macmillan Company, 1942.
- Cooley, Hollis R., et al., Introduction to Mathematics: A Survey Emphasizing Mathematical Ideas and their Relations to other Fields of Knowledge. New York: Houghton Mifflin Company, 1937.
- Coolidge, Julian Lowell. A History of Geometrical Methods. Oxford: Clarendon Press, 1940.
- Copleson, F. A History of Philosophy. Volume I: Greece and Rome. Westminster: The Newman Bookshop, 1946.

- Courant, Richard. Differential and Integral Calculus. New revised edition. Translated by A. J. McShane. 2 volumes. New York: Interscience Publishers, 1946.
- _____, and Robbins, Herbert. What is Mathematics? An Elementary Approach to Ideas and Methods. New York: Columbia University Press, 1943.
- Couturat, Louis. La logique de Leibniz d'apres des documents inedits. Paris: Felix Alcan, 1901.
- Crowther, J. G. (editor). Science for a New World: The Scientific Outlook on World Problems Explained by Leading Exponents of Modern Scientific Thought. Planned and arranged by the late J. Arthur Thomson. New York: Harper & Brothers, 1934.
- Dampier, Sir William Cecil (Formerly Whetham). A History of Science and its Relations with Philosophy & Religion. Third edition, revised and enlarged. New York: The Macmillan Company, 1945.
- Dantzig, Tobias. Aspects of Science. New York: The Macmillan Company, 1937.
- _____, Number the Language of Science: A Critical Survey Written for the Cultured Non-Mathematician. Third edition, revised and augmented. New York: The Macmillan Company, 1941.
- Daujat, Jean. L'Oeuvre de l'Intelligence en physique. Paris: Presses Universitaires, 1946.
- Darbois, Andre. La philosophie des mathematiques: etude sur la logistique de Russell. Publiee par Madeleine Lagarde Darbois avec la collaboration de Francois Chatelet, preface de Rene Poirier. Paris: Presses Universitaires, 1949.
- Daval, R. and Guillebaud, G.-T. Le Raisonnement mathematique. Paris: Presses Universitaires, 1945.
- Davis, Harold T. A Course in General Mathematics. Bloomington: Principia Press, 1935.
- _____, Philosophy and Modern Science. Bloomington: The Principia Press, 1931.
- De Broglie, Louis. Matter and Light: The New Physics. Translated by W. E. Johnston. New York: Dover Publications, 1946.
- De Morgan, Augustus. On the Study and Difficulty of Mathematics. New edition. Chicago: Open Court, 1898.
- Dedekind, Richard. Essays on the Theory of Numbers: I. Continuity and Irrational Numbers. II. The Nature and Meaning of Numbers. Authorized translation by Wooster Woodruff Beman. Chicago: Open Court, 1901.
- Descartes, Rene. Oeuvres de Descartes. Publiees par Charles Adam & Paul Tannery. 11 volumes and Supplement: General Index. Paris: Leopold Cerf, 1897-1913.
- _____, The Philosophical Works of Rene Descartes. Rendered into English by Elizabeth S. Haldane and G. V. T. Ross in two volumes. Cambridge: The University Press, 1931-1934.
- Dewey, John. Intelligence in the Modern World: John Dewey's Philosophy. Edited, and with an Introduction by Joseph Ratner. New York: The Modern Library, 1939.
- _____, Logic: The Theory of Inquiry. New York: Henry Holt, 1938.

- Dickson, L. E. History of the Theory of Numbers. 3 volumes. Washington: The Carnegie Institute, 1919-1923.
- _____. Linear Algebras. Cambridge: The University Press, 1916.
- _____. Studies in the History of the Theory of Numbers. Chicago: The University of Chicago Press, 1930.
- Dirac, P. The Principles of Quantum Mechanics. New York: Oxford University Press, 1935.
- Dotterer, R. H. Philosophy by way of the Sciences. New York: The Macmillan Company, 1929.
- Dresden, Arnold. An Invitation to Mathematics. New York: Henry Holt, 1936.
- Duhem, Pierre. La Theorie Physique: Son objet -- Sa structure. Deuxieme edition, revue et augmentee. Paris: Marcel Riviere, 1914.
- Edington, Sir Arthur. New Pathways in Science. New York: The Macmillan Company, 1935.
- _____. Space, Time and Gravitation: An Outline of the General Relativity Theory. Cambridge: The University Press, 1921.
- _____. The Nature of the Physical World. New York: The Macmillan Company, 1940.
- _____. The Philosophy of Physical Science. New York: The Macmillan Company, 1939.
- Einstein, Albert. The Meaning of Relativity. Princeton: Princeton University Press, 1946.
- _____. The Method of Theoretical Physics. Oxford: The Clarendon Press, 1933.
- _____. The World as I see It. Translated from the German by Alan Harris. New York: Covici, 1934.
- _____. and Infeld, Leopold. The Evolution of Physics: The Growth of Ideas From Early Concepts to Relativity and Quanta. New York: Simon and Schuster, 1942.
- Encyclopedie Francaise. Tome I. L'Outillage Mental: Pensee. Langage. Mathematique. Paris: Librairie Larousse, 1937.
- Enriques, Federigo. The Historic Development of Logic: The Principles and Structure of Science in the Conception of Mathematical Thinkers. Authorized translation from the Italian by Jerome Rosenthal. New York: Henry Holt, 1929.
- _____. and De Santillana, G. Le Probleme de la matiere: Pythagoriciens et Eleates. Paris: Hermann & Cie, 1936.
- _____. Problems of Science. Authorized translation by Katherine Royce with an Introductory note by Josiah Royce. Chicago: Open Court, 1914.
- Fine, H. B. The Number System of Algebra Treated Theoretically and Historically. (Reprint from 1890 edition). New York: G. E. Stechert & Co., 1937.
- Fourastie, Jean. Note sur la philosophie des sciences. Paris: Presses Universitaires, 1948.
- Fraenkel, Adolf. Einführung in die Mengenlehre. Dritte umgearbeitete und stark erweiterte auflage. New York: Dover Publications, 1946.
- Fröbes, Joseph. Compendium Psychologiae Experimentalis. Romae: Apud Aedes Universitatis Gregoriana, 1937.

- Frank, Phillip. Between Physics and Philosophy. Cambridge: Harvard University Press, 1941.
- Gaffney, Mark Aloysius. The Psychology of the Interior Senses. St. Louis: W. Herder Book Co., 1948.
- Garraghan, Gilbert J. A Guide to Historical Method. Edited by Jean Delanglez. New York: Fordham University Press, 1946.
- Gaultier, Paul. La pensee contemporaine: Les grands problemes. Paris: Hachette et Cie, 1911.
- Gauss, Carl Friedrich. Inaugural Lecture on Astronomy and Papers on the Foundations of Mathematics. Translated and edited by G. Waldo Dunnington. Baton Rouge: Louisiana State University Press, 1937.
- Geiger, L.-B. La participation dans la philosophie de S. Thomas d'Aquin. Paris: J. Vrin, 1942.
- Gibson, A. Royce. The Philosophy of Descartes. London: Methuen & Co., 1932.
- Gibson, James. Locke's Theory of Knowledge in its Historical Relations. Cambridge: The University Press, 1931.
- Gilson, Etienne. "Etudes sur le role de la pensee medievale dans la formation du systeme cartésien." Deuxieme partie des Etudes de philosophie medievale. Revue et considerablement augmentee. Paris: J. Vrin, 1930.
- _____. God and Philosophy. New Haven: Yale University Press, 1941.
- _____. Le Thomisme: Introduction a la philosophie du Saint Thomas d'Aquin. cinquieme edition, revue et augmentee. Paris: J. Vrin, 1944.
- _____. The Unity of Philosophical Experience. London: Sheed and Ward, 1938.
- Gödel, Kurt. Consistency of the Axiom of Choice. Princeton: Princeton University Press, 1940.
- _____. On Undecidable Propositions of Formal Mathematical Systems. Princeton: Princeton University Press, 1934.
- _____. The Consistency of the Continuum Hypothesis. Princeton: Princeton University Press, 1942.
- Göseth, F. Les Entretiens de Zurich sur les fondements et la methode des sciences mathematiques. Publies par F. Göseth. Zurich: E. A. Leeman freres et Cie, 1941.
- _____. Les fondements des mathematiques: de la geometrie d'Euclide a la relativite generale et a l'intuitionisme. Preface de L. Jacques Hadamard. Paris: Albert Blanchard, 1926.
- _____. Philosophie mathematique. Avec cinq declarations de MM. A. Church, W. Ackermann, A. Heyting, F. Brouwers, et L. Chwistek. Paris: Hermann & Cie, 1939.
- _____. Qu'est-ce que la logique? Paris: Hermann et Cie, 1937.
- Greenwood, Thomas. Essais sur la pensee geometrique. Ottawa: Les editions de l'Universite d'Ottawa, 1945.
- _____. Etudes sur la connaissance mathematique. Ottawa: Les editions de l'Universite d'Ottawa, 1942.
- _____. La Nature du transfini. Ottawa: Les editions de l'Universite d'Ottawa, 1946.
- _____. Prolegomenes a la theorie des quantas. Ottawa: Les editions de l'Universite d'Ottawa, 1942.

- _____. Les fondements de la logique symbolique: Tome I. Critique du nominalisme logistiqu. Tome II. Justification du calculs logiques. Paris: Hermann et Cie, 1938.
- Hadamard, Jacques. An Essay on the Psychology of Invention in the Mathematical Field. Princeton: Princeton University Press, 1949.
- Hahn, Hans. Logique mathématique et connaissance de la realité. Introduction de W. Boll. Paris: Hermann et Cie, 1935.
- Hardy, G. H. A Course of Pure Mathematics. Eighth edition. New York: The Macmillan Company, 1943.
- _____. A Mathematician's Apology. Cambridge: The University Press, 1941.
- _____, and Wright, E. M. An Introduction to the Theory of Numbers. Second edition. Oxford: The Clarendon Press, 1945.
- Hastie, W. Fant's Cosmogony as in his Essay on the Retardation of the Rotation of the Earth and his Natural History and Theory of the Heavens. With Introduction, Appendices, and a portrait of Thomas Wright of Durham, edited and translated by W. Hastie. Glasgow: James Maclehose and Sons, 1900.
- Heath, Thomas Little. A Manual of Greek Mathematics. Oxford: The Clarendon Press, 1931.
- Helmholtz, H. L. F. Von. Counting and Measuring. New York: Von Nostrand, 1930.
- Herr, Vincent. How We Influence One Another: The Psychology of Social Interaction. Milwaukee: Bruce Publishing Company, 1945.
- Hilbert, David. The Foundations of Geometry. Authorized translation by E. J. Townsend. Third edition. LaSalle: Open Court, 1938.
- Hobson, E. W. The Domain of Natural Science: The Gifford Lectures Delivered in the University of Aberdeen in 1921 and 1922. Cambridge: The University Press, 1923.
- _____. The Ideal Aim of Physical Science. Cambridge: The University Press, 1923.
- _____. Theory of Functions of a Real Variable. second edition. 2 volumes. Cambridge: The University Press, 1927.
- Hoenen, P. Cosmologia. Mitio tertia aucta et emendata. Romae: Apud Aedes Universitatis Gregoriana, 1945.
- Hogben, Lancelot. Mathematics for the Million. London: George Allen & Unwin, 1942.
- _____. Science for the Citizen. second edition. London: George Allen & Unwin, 1943.
- Hooper, Alfred. Makes of Mathematics. New York: Random House, 1949.
- Hopper, V. F. Medieval Number Symbolism. New York: Columbia University Press, 193
- Howard, Homer. Mathematics' Teachers Views on Certain Issues in the Teaching of Mathematics. New York: Columbia University Press, 1941.
- Hume, David. A Treatise of Human Nature by David Hume. Reprinted from the original edition in three volumes and edited, with an analytical index, by L. A. Selby-Bigge. Oxford: The University Press, 1896.

- _____. A Treatise on Human Nature. 2 volumes. New York: A. P. Dutton and Co., 1939.
- Huntington, Edward V. The Continuum and Other Types of Serial Order with an Introduction to Cantor's Transfinite Numbers. Second edition. Cambridge: Harvard University Press, 1941.
- Isaye, Gaston. La theorie de la mesure et l'existence d'un maximum selon Saint Thomas. Paris: Beauchesne et ses fils, 1940.
- Iwanicki, Joseph. Leibniz et les demonstrations mathematiques de l'existence de Dieu. Strassbourg: Librairie Universitaire d'Alsace, 1933.
- James, William. The Principles of Psychology. 2 volumes. London: Macmillan and Co., 1901.
- Joans, Sir James. Physics and Philosophy. New York: The Macmillan Company, 1943.
- _____. The Growth of Physical Science. Cambridge: The University Press, 1947.
- _____. The Mysterious Universe. Cambridge: The University Press, 1930.
- _____. The New Background of Science. Cambridge: The University Press, 1933.
- _____. Through Space and Time. Cambridge: The University Press, 1934.
- Joachim, Harold H. A Study of the Ethics of Spinoza. Oxford: The Clarendon Press, 1901.
- Joad, C. E. M. Guide to Modern Thought. London: Faber & Faber, 1933.
- _____. Guide to Philosophy. London: Victor Gollancz, 1936.
- _____. Philosophical Aspects of Modern Science. New York: The Macmillan Company, 1932.
- _____. Philosophy for our Times. London: Thomas Nelson & Sons, 1940.
- Jørgensen, J. A Treatise on Formal Logic: Its Evolution and Main Branches with its Relation to Mathematics and Philosophy. 3 volumes in one. Copenhagen: Levin & Munksgaard, 1931.
- Jordan, Z. The Development of Mathematical Logic and of Logical Positivism in Poland between the two Wars: Polish Science and Learning. New York: Oxford University Press, 1945.
- Joseph, H. W. B. An Introduction to Logic. Cambridge: The University Press, 1916.
- Jourdain, P. A. B. Nature of Mathematics. London: T.C. & E.C. Jack, n.d.
- Kant, Immanuel. Immanuel Kant's sämtliche werke. Herausgegeben von P. Gedan, W. Kinkel, J.H.V. Kirchmann, F. A. Schiele, Th. Vautantiner, K. Vorländer. 8 volumes and supplement volume. Leipzig: Verlag Dörff'schen Buchhandlung, 1901-1905.
- Kasner, Edward and Newman, James. Mathematics and the Imagination. With drawings and diagrams by Rufus Isaacs. New York: Simon and Schuster, 1940.
- Katzoff, Louis O. A Philosophy of Mathematics. Ames: The Iowa State College Press, 1948.
- Kelly, Thomas R. Explanation and Reality in the Philosophy of Emil Meyerson. Princeton: Princeton University Press, 1937.

- Keyser, Cassius J. Mathematical Philosophy: A Study of Fate and Freedom. New York: E. P. Dutton & Co., 1937.
- _____, Mathematics and the Question of the Cosmic Mind. New York: Scripta Mathematica, 1935.
- _____, Pastures of Wonder. New York: Columbia University Press, 1929.
- _____, The Human Worth of Rigorous Thinking. New York: Columbia University Press, 1925.
- _____, Thinking about Thinking. New York: E. P. Dutton & Co., 1926.
- Klein, Felix. Elementary Mathematics from an Advanced Standpoint: Arithmetic. Algebra. Analysis. Translated from the German edition by E. F. Hedrick and C. A. Noble. New York: Dover Publications, 1945.
- _____, Elementary Mathematics from an Advanced Standpoint: Geometry. Translated from the third German edition by E. F. Hedrick and C. A. Noble. New York: The Macmillan Company, 1939.
- Korzybski, Alfred. Science and Sanity: An Introduction to Non-Aristotelian Systems and General Semantics. Third edition with new preface. Lakeville: The International Non-Aristotelian Library Publishing Company, 1948.
- Kruse, Vinding. Hume's Philosophy in his Principal Work: A Treatise on Human Nature and his Essays. Translated by P. T. Federspiel. New York: Oxford University Press, 1939.
- Laird, John. Hume's Philosophy of Human Nature. London: Methuen and Co., 1932.
- _____, Recent Philosophy. London: Thornton Butterworth, 1936.
- Laisant, G.-A. La Mathématique: Philosophie. Enseignement. Deuxième édition, revue et corrigée. Paris: Gauthier-Villars, 1907.
- Lallemand, Marcel. Le Transfini: Sa logique et sa métaphysique. Paris: Desclée et Brouwer, 1934.
- Langer, Susanne K. A Modern Introduction to Logic. London: George Allen & Unwin, 1937.
- Lautman, Albert. Essai sur les notions de structure et d'existence en mathématiques. Paris: Presses Universitaires, 1938.
- _____, Essai sur l'unité des sciences mathématiques dans leur développement actuel. Paris: Hermann et Cie, 1938.
- Lebesgue, Henri. Leçons sur l'intégration et la recherche des fonctions primitives professées au collège de France. Paris: Gauthier-Villars, 1928.
- Lecomte du Nuy, Pierre. Human Destiny. New York: Longmans Green and Co., 1947.
- _____, The Road to Reason. Translated and edited by Mary Lecomte du Nuy. New York: Longmans Green and Co., 1948.
- Lee, H. D. P. Zeno of Elea: A text, with translation and notes. Cambridge: The University Press, 1936.
- Leibniz. Oeuvres Choies. Avec préface, notes, table par questions et table des noms propres, par L. Prenant. Paris: Garnier Frères, n.d.
- _____, The Early Mathematical Manuscripts of Leibniz. Translated from the Latin texts published by Carl Immanuel Gerhardt, with Critical and Historical Notes by J. M. Child. Chicago: Open Court, 1920.

- _____. Opera Philosophica Quae Exstant: Latina, Gallica, Germanica Omnia. Edita recognovit e temporum rationibus disposita pluribus ineditis auxit introductione critica atque indicibus, instruxit Joannes Erdmann. Pars Prior et Altera. Berolini: Sumptibus G. Reihleri, 1839.
- _____. The Monadology and other Philosophical Writings. Translated with Introduction and Notes by Robert Latta. London: Geoffrey Cumberlege, 1948.
- _____. The Philosophical Works of Leibniz Comprising the Monadology, New System of Nature, Principles of Nature and of Grace, Letters to Clarke, Refutation of Spinoza, and his other important philosophical opuscles, together with the Abridgement of the Theodicy and Extracts from the New Essays on Human Understanding. Translated from the original latin and french with notes by George Martin Duncan. New Haven: Tuttle, Morehouse and Taylor, 1890.
- Le Lionnais, F. Les grands courants de la pensee mathematique. Seine: Imprimeries Bellenaud, 1948.
- Le Vasson, R. Philosophie des nombres. Paris: Desoee de Brouwer, 1932.
- Lenzen, Victor F. The Nature of Physical Theory: A Study in Theory of Knowledge. New York: John Wiley & Sons, 1931.
- La Porte, Jean. L'Idée de necessite. Paris: Presses Universitaires, 1941.
- _____. Le probleme de l'abstraction. Paris: Presses Universitaires, 1940.
- Lewis, C. I. A Survey of Symbolic Logic. Berkeley: University of California Press, 1918.
- _____. and Langford, Cooper Harold. Symbolic Logic. New York and London: The Century Co., 1932.
- Lillie, Ralph Stayner. General Biology and Philosophy of Organisms Chicago: University of Chicago Press, 1945.
- L'Invention. Exposes et discussions par Charles Blondel, Ed. Claparede, J. Hadamard, L. De Broglie, Ed. Bauer, Paul Valery, Louis Breguet, Em. Guyenot. Paris: Felix Alcan, 1938.
- Lindsay, A. D. Kant. London: Ernest Benn, 1934.
- Lobachevski, Nicholas. Geometrical Reseraches on the Theory of Parallels. Translated from the original by George Bruce Halstead. new edition. Chicago: Open Court, 1914.
- Locke, John. An Essay Concerning Human Understanding by John Locke. Collated and Annotated, with Prolegomena, Biographical, Critical and Historical, by Alexander Campbell Fraser. 2 volumes. Oxford: The Clarendon Press, 1894.
- Lowinger, Armand. The Methodology of Pierre Duham. New York: Columbia University Press, 1941.
- Mach, Ernst. Popular Scientific Lectures. Fourth edition. Chicago: Open Court, 1910.
- _____. The Science of Mechanics: A Critical and Historic Account of its Development. Translated from the German by T. J. McCormack. fifth edition. LaSalle: Open Court, 1945.

- Mansion, Augustin. Introduction a la physique Aristotelicienne. Deuxieme edition, revue et augmentee. Paris: J. Vrin, 1945.
- Marchal, Joseph. Le Point de depart de la metaphysique: Lecons sur le developpement historique et theorique du probleme de la connaissance. 5 volumes. Paris: Felix Alcan, 1928.
- _____. Precis d'Histoire de la Philosophie Moderne. Tome Premier: De la Renaissance a Kant. Louvain: Musseum Lessianum, 1933.
- Maritain, Jacques. An Introduction to Philosophy. Translated by E. I. Watkin.
- _____. La Philosophie de la nature: Essai critique sur ses frontieres et son objet. third edition. Paris: Pierre Tequi, n.d.
- _____. Distinguer pour unir ou les degres du savoir. nouvelle edition, revue et augmentee. Paris: Desclée de Brouwer, 1946.
- _____. Science and Wisdom. New York: Charles Scribner's Sons, 1940.
- _____. The Degrees of Knowledge. London: The Centenary Press, 1937.
- _____. Three Reformers: Luther -- Descartes -- Rousseau. London: Sheed and Ward, 1947.
- Marchal, Pierre. Histoire de la geometrie. Paris: Presses Universitaires, 1948.
- Marling, Joseph M. The Order of Nature in the Philosophy of St. Thomas Aquinas. Washington: The Catholic University of America Press, 1934.
- McSweeney, Alan J. The Social Role of Truth. Washington: The Catholic University of America Press, 1943.
- Merriman, Gaylord M. To Discover Mathematics. New York: John Wiley, 1942.
- Merz, John Theodore. A History of European Thought in the Nineteenth Century. 3 volumes. Chicago: The University of Chicago Press, 1924.
- Meyer, Hans. The Philosophy of Saint Thomas Aquinas. Translated by Frederic Kerkhoff. St. Louis: B. Herder Book Co., 1945.
- Meyerson, Emile. Du cheminement de la pensee. 3 volumes. Paris: Felix Alcan, 1931/
- _____. Identity and Reality. Authorized translation by Kate Loewenberg. London: George Allen & Unwin, 1930.
- Michel, J. H. and Belz, M. H. The Elements of Mathematical Analysis. 2 volumes. London: Macmillan and Company, 1937.
- Milhaud, Gaston. Le Rationnel. Paris: Felix Alcan, 1939.
- _____. Les philosophes geometres de la Grece: Platon et ses predecesseurs. Deuxieme edition. Paris: J. Vrin, 1934.
- Millikan, Robert Andrews. Electrons (- and -), Protons, Photons, Neutrons, Mesotrons, and Cosmic Rays. Revised edition. Chicago: The University of Chicago Press, 1947.
- Moran, J. G. Cosmologia. Mexico: Buena Prensa, 1944.

- More, P. A. Platonism. Princeton: Princeton University Press, 1931.
- Moritz, Robert Eduard. Memorabilia Mathematica or the Philomath's Quotation-Book. New York: The Macmillan Company, 1914.
- Morris, C. R. Locke, Berkeley, Hume. Oxford: The Clarendon Press, 1931.
- Morris, Charles W. "Foundations of the Theory of Signs." International Encyclopedia of Unified Science. Volume I, Number 2. Chicago: University of Chicago Press, 1940.
- Newton, Sir Isaac. Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World. Translated into English by Andrew Motte in 1729. Edited by Florian Cajori. Berkeley: University of California Press, 1946.
- _____. Sir Isaac Newton: 1727-1927. A Bicentenary Evaluation of his work. A Series of Papers prepared under the auspices of the History of Science Society. London: Valliere Tindall & Cox, 1928.
- Nicod, Jean. Foundations of Geometry and Induction: Containing Geometry in the Sensible World and the Logical Problems of Induction. With prefaces by Bertrand Russell and Andre Lalande. New York: Harcourt, Brace and Company, 1930.
- Northrop, F. T. C. Science and First Principles. New York: The Macmillan Company, 1932.
- _____. The Logic of the Sciences and the Humanities. New York: The Macmillan Company, 1947.
- _____. The Meeting of East and West: An Inquiry Concerning World Understanding. New York: The Macmillan Company, 1946.
- O'Neill, John. Cosmology: An Introduction to the Philosophy of Matter. Volume I. The Greeks and the Medieval Schoolmen. London: Longmans, Green and Co., 1923.
- Ore, Oystein. Number Theory and its History. New York: McGraw-Hill, 1948.
- Ouspensky, A. Tertium Organum. The Third Crown of Thought: A Key to the Riddles of the World. Translated from the Russian by Nicholas Bessaraboff and Claude Bragdon. Second American edition. New York: Alfred Knopf, 1931.
- _____. The New Model of the Universe. New York: Alfred Knopf, 1942.
- Pacotte, J. La pensee mathematique contemporaine. Paris: Felix Alcan, 1925.
- Philosophie et Sciences. Journees d'etudes de la Societe Thomiste. Louvain: Les editions du Cerf, 1935.
- Piaget, Jean. Classes, Relations et Nombres: Essai sur les groupements de la logistique et sur la reversibilite de la pensee. Paris: J. Vrin, 1942.
- Planck, Max. The Philosophy of Physics. Translated by W. H. Johnston. New York: W. W. Norton & Company, 1936.
- _____. Where is Science Going? Prologue by Albert Einstein. Translation and biographical note by James Murphy. New York: W. W. Norton & Company, 1932.

- Fiat, Cledius. Leibniz. Paris: Felix Alean, 1915.
- Plato. The Dialogues of Plato. Translated into English by B. Jowett, with an introduction by Raphael Fernow. 2 volumes. New York: Random House, 1930.
- Pledge, H. T. Science Since 1500: A Short History of Mathematics, Physics, Chemistry, Biology. London: His Majesty's Stationery Office, 1939.
- Poincare, Henri. The Foundations of Science: Science and Hypothesis. The Value of Science. Science and Method. Authorized Translation by George Bruce Halstead, with a special preface by Poincare, and an introduction by Josiah Royce. New York: The Science Press, 1929.
- Poirier, Rene. Le Nombre. Paris: Felix Alean, 1938.
- Quine, Willard Van Orman. Elementary Logic. Boston: Winn and Company, 1941.
Mathematical Logic. New York: W. W. Norton & Co., 1940.
- Rainich, G. Y. Mathematics of Relativity: Lecture Notes. Ann Arbor: Edwards Brothers, 1943.
- Ramsey, Frank Plumpton. The Foundations of Mathematics and Other Logical Essays. Edited by R. B. Braithwaite, with a Preface by G. E. Moore. New York: Harcourt, Brace and Company, 1931.
- Randall, John Herman. The Making of the Modern Mind: A Survey of the Intellectual Background of the Present Age. Revised edition. New York: Houghton Mifflin Company, 1940.
- Regis, L.-M. "La Philosophie de la nature." in Etudes et Recherches Publiées par le College Dominicain d'Ottawa. Cahier I: Philosophie. Ottawa: College Dominicain, 1936.
- Reichenbach, Hans. Experience and Prediction: An Analysis of the Foundations and Structure of Knowledge. Chicago: The University of Chicago Press, 1938.
Philosophic Foundations of Quantum Mechanics. Berkeley: The University of California Press, 1944.
- Reiser, Olive L. Philosophy and the Concepts of Modern Science. New York: The Macmillan Company, 1935.
- Renoirte, Fernand. Elements de Critique des sciences et de cosmologie. Louvain: Editions de l'Institute Supérieure de Philosophie, 1948.
- Richardson, M. Fundamentals of Mathematics. New York: The Macmillan Company, 1941.
- Richardson Robert P. and Landis, Edward H. Fundamental Conceptions of Modern Mathematics. Chicago: Open Court, 1916.
- Riezler, Kurt. Physics and Reality: Lectures of Aristotle on Modern Physics at an International Congress of Science. New Haven: Yale University Press, 1940.

- Robin, Leon. La pensée Grecque et les origines de l'esprit scientifique. Avec une carte hors texte. Edition revue et corrigée, avec une bibliographie méthodique nouvelle. Paris: Albin Michel, 1948.
- Robinson, Gilbert de B. The Foundations of Geometry. Toronto: The University of Toronto Press, 1940/
- Roth, Leon. Spinoza, Descartes & Maimonides. Oxford: The Clarendon Press, 1924.
- Rother, Aloysius J. Truth and Error: A Study in Critical Logic. St. Louis: B. Herder, 1914.
- Rousselet, Pierre. The Intellectualism of Saint Thomas. Translated, with a Foreword by James J. O'Mahony. London: Sheed and Ward, 1935.
- Russell, Bertrand. A Critical Exposition of the Philosophy of Leibniz. Cambridge: The University Press, 1900.
- _____. A History of Western Philosophy And its Connection with Political and Social Circumstances from the Earliest Times to the Present Day. New York: Simon and Schuster, 1945.
- _____. Human Knowledge: Its Scope and Limits. New York: Simon and Schuster, 1948.
- _____. Introduction to Mathematical Philosophy. New York: The Macmillan Company, 1938.
- _____. Mysticism and Logic. New York: W. W. Norton & Company, 1929.
- _____. Our Knowledge of the External World. New York: W. W. Norton & Company, 1929.
- _____. The Analysis of Matter. London: Kegan Paul, 1927.
- _____. The Principles of Mathematics. Second edition. New York: W. W. Norton & Company, 1937.
- _____. The Scientific Outlook. London: George Allen & Unwin, 1931.
- _____. The Philosophy of Bertrand Russell. Volume V: The Library of Living Philosophers. Edited by Paul Arthur Schilpp. Evanston and Chicago: Northwestern University, 1944.
- _____. and Whitehead, A. N. Principia Mathematica. Second edition. 3 volumes. Cambridge: The University Press, 1927.
- Saintonge, P. Fridericus. Summa Cosmologiae seu Philosophia Naturalis Generalis. Montreal: Imprimerie du Messager, 1941.
- Sanford, Vera. A Short History of Mathematics. Under the editorship of John Wesley Young and With an Introduction by David Eugene Smith. New York: Houghton Mifflin Company, 1930.
- Sarton, George. Essays in Science and the New Humanism. Cambridge: Harvard University Press, 1948.
- _____. Introduction to the History and Method of Science. 3 volumes in 5. Baltimore: Williams and Wilkins, 1927-1948.
- _____. The History of Science and the New Humanism. Cambridge: Harvard University Press, 1937.
- _____. The Study of the History of Mathematics. Cambridge: Harvard University Press, 1936.
- _____. The Study of the History of Science. Cambridge: Harvard University Press, 1936.

Schaaf, William L. (editor). Mathematics Our Great Heritage: Essays on the Nature and Cultural Significance of Mathematics. New York: Harper & Brothers, 1948.

Science Philosophy and Religion: A Symposium. New York: Conference on Science, Philosophy and Religion, 1940.

Schrödinger, Erwin. What is Life? The Physical Aspect of the Living Cell. New York: The Macmillan Company, 1946.

Shaw, James Byrne. Lectures on the Philosophy of Mathematics. Chicago and London: Open Court, 1918.

Sheen, Fulton J. Philosophy of Science. Milwaukee: The Bruce Publishing Company, 1934.

Sertillanges, A. D. Le Christianisme et les Philosophes. 2 volumes. second edition. Paris: Aubier, 1941.

_____, St. Thomas d'Aquin. second edition. 2 volumes. Paris: J. Vrin, 1912.

Simon, Yves. Prevoir et Savoir: Etudes sur l'idée de la nécessité dans la pensée scientifique et en philosophie. Montreal: Editions de l'Arbre, 1944.

Singer, Charles. A Short History of Science to the Nineteenth Century. Oxford: The Clarendon Press, 1941.

Sierpinaki, Macław. Hypothese du Continu. Warsaw: Lwow, 1934.

Smith, David Eugene. A History of Mathematics. 2 volumes. New York: Ginn & Company, 1923.

_____, The Poetry of Mathematics and Other Essays. New York: Scripta Mathematica, 1934.

Smart, Harold R. The Philosophical Presuppositions of Mathematical Logic. Cornell Studies in Philosophy, n. 17. New York: Longmans, Green and Co., 1925.

_____, The Logic of Science. New York: D. Appleton and Company, 1931.

Smith, Norman Kemp. The Philosophy of David Hume: A Critical Study of its Origins and Central Doctrines. London: The Macmillan Company, 1941.

Smith, Vincent Edward. The Philosophical Frontiers of Physics. Washington: The Catholic University of America Press, 1947.

Spengler, Oswald. The Decline of the West. Authorized translation with notes by Charles Francis Atkinson. New York: Alfred A. Knopf, 1926.

Spinoza. Benedicti de Spinoza Opera Quotquot Reperta Sunt. Hagae Comitum: Apud Martinum Nijhoff, 1913.

Stebbing, L. Susan. A Modern Elementary Logic. Second edition. London: Methuen & Co., 1946.

_____, Philosophy and the Physicists. London: Methuen & Co., 1937.

- Stephen, Sir Leslie. History of English Thought in the Eighteenth Century. Third edition. 2 volumes. London: Smith, Elder & Co., 1902.
- Strong, Edward F. Procedures and Metaphysics: A Study in the Philosophy of Mathematical-Physical Science in the Sixteenth and Seventeenth Centuries. Berkeley: University of California Press, 1936.
- Struik, Dirk J. A Concise History of Mathematics. 2 volumes. New York: Dover Publications, 1948.
- Sullivan, James Bacon. An Examination of First Principles in Thought and Being in the light of Aristotle and Aquinas. Washington: Catholic University of America Press, 1939.
- Sullivan, J. W. W. Aspects of Science. Second Series. London: W. Collins and Co., 1926.
- _____, Aspects of Science. 2 volumes. London: Richard Cobden, 1923.
- _____, The Bases of Modern Science. London: Ernest Benn, 1928.
- _____, The Limitations of Science. London: Chatto and Windus, 1933.
- _____, Science: A New Outline. London: Thomas Nelson & Sons, 1935.
- _____, Issac Newton: 1642-1729. London: Macmillan and Co., 1938.
- Tarski, Alfred. Introduction to Logic and to the Methodology of Deductive Sciences. Enlarged and revised edition. Translated by Olaf Helmer. New York: Oxford University Press, 1941.
- Taton, Rene. Histoire du calcul. Paris: Presses Universitaires, 1948.
- Taylor, A. C. Philosophical Studies. London: Macmillan and Co., 1934.
- Thompson, W. B. Science and Common Sense: An Aristotelian Excursion. With a Preface by Jacques Maritain. London: Longmans, Green and Co., 1937.
- Thomson, J. Arthur. Introduction to Science. London: Williams & Norgate, 1911.
- Thonnard, P.-J. Precis d'histoire de la philosophie. Nouvelle edition, revue et corrigee. Tournai: Societe de S. Jean de l'Evangeliste, 1945.
- Thorndike, Lynn. History of Magic and Experimental Science. six volumes. New York: Columbia University Press, 1923-1938.
- Tukey, John W. (conference reporter). The Princeton University Bicentennial Conference on the Problems of Mathematics. Princeton: Princeton University Press, 1947.
- Versfeld, Marthinus. An Essay on the Metaphysics of Descartes. London: Methuen and Co., 1940.
- Von Neumann, John and Morgenstern, Oskar. Theory of Games and Economic Behavior. Princeton: Princeton University Press, 1944.
- Wallace, William. Kant. Edinburgh and London: William Blackwood and Sons, 1892.

- Ward, James. A Study of Kant. Cambridge: The University Press, 1922.
- Weinberg, J. R. An Examination of Logical Positivism. New York: The Macmillan Company, 1936.
- Weiss, Charles. The Metaphysical Implications of Modern Physics. New York: New York University, 1931.
- Wellmuth, John. The Nature and Origins of Scientism. Milwaukee: Marquette University Press, 1944.
- Werkmeister, William Henry. A Philosophy of Science. New York: Harper & Co., 1940.
- Weyl, Hermann. Mind and Nature. Philadelphia: University of Pennsylvania Press, 1934.
- _____. Philosophie der mathematik und naturwissenschaft. Handbuch der philosophie, Band II. Munich: Oldenbourg, 1926.
- _____. Space - Time - Matter. Translated from the German by Henry L. Brose. London: Methuen and Company, 1922.
- _____. The Open World: Three lectures on the Metaphysical Implications of Science. New Haven: Yale University Press, 1934.
- Whewell, William. History of the Inductive Sciences from the Earliest to the Present Time. Third edition. 3 volumes. London: The Macmillan Company, 1847.
- Whitehead, Alfred North. An Introduction to Mathematics. London: Thornton Butterworth, 1939.
- _____. Adventures of Ideas. Cambridge: The University Press, 1933.
- _____. Process and Reality: An Essay in Cosmology. Cambridge: The University Press, 1929.
- _____. Science and the Modern World. Cambridge: The University Press, 1926.
- _____. The Philosophy of Alfred North Whitehead. Volume III: The Library of Living Philosophers. Edited by Paul Arthur Schilpp. Evanston and Chicago, 1941.
- Wittgenstein, L. Tractatus Logico-Philosophicus. With an introduction by Bertrand Russell. New York: Harcourt, Brace and Company, 1932.
- Wolf, A. A History of Science Technology and Philosophy in the Eighteenth Century. New York: The Macmillan Company, 1939.
- Wolfe, H. P. Introduction to Non-Euclidean Geometry. Ann Arbor: Edwards Brothers, 1941.
- Wolfson, H. A. The Philosophy of Spinoza. 2 volumes. Cambridge: Harvard University Press, 1934.
- Woodruff, L. L. (editor). The Development of the Sciences. Second series. New Haven: Yale University Press, 1941.

Young, John Wesley. Lectures on Fundamental Concepts of Algebra and Geometry. Prepared for publication with the cooperation of William Wells Denton, with a Note on the Growth of Algebraic Symbolism by Ulysses Grant Mitchell. New York: The Macmillan Company, 1911.

Young, J. W. A. Monographs on Topics of Modern Mathematics Relevant to the Elementary Field. New York: Longmans Green and Company, 1911.
 ———, The Teaching of Mathematics in the Elementary and the Secondary School. New York: Longmans, Green and Company, 1937.

Articles.

Anderson, James. "Two studies in Metaphysics." The Thomist, 3 (1941), pp. 467-506.

Ashton, J. "Mathematicians and the Mysterious Universe." Thought, 6 (1931), pp. 258-274.

Birkhoff, G. D. "Principle of Sufficient Reason." Rice Institute Pamphlets, 28 (1941), pp. 24-59.

Black, Max. "The Relevance of Mathematical Philosophy in the Teaching of Mathematics." The Mathematical Gazette, 22 (1938), pp. 149-163.

Blake, R. M. "Sir Isaac Newton's Theory of Scientific Method." The New Scholasticism, 17 (1933), pp. 453-486.

Bochenski, I. M. "On Analogy." The Thomist, 12 (1948), pp. 424-447.

Bocher, M. "Brouwer's Contribution to the Foundations of Mathematics." Bulletin of the American Mathematical Society, 30 (1924), pp. 31-40.
 ———, "Fundamental Conceptions and Methods of Mathematics." Bulletin of the American Mathematical Society, 2 (1904), pp. 115-135.

Bond, Leo M. "The Effect of Bodily Temperament on Psychical Characteristics." The Thomist, 10 (1947), pp. 423-501 and 11 (1948), pp. 28-104.

Bouligand, Georges. "Le mathématicien au seuil de la métaphysique." Revue de Métaphysique et de Morale, 52 (1947), pp. 308-321.

Boyer, Carl B. "Zero: the Symbol, the Concept, the Number." National Mathematics Magazine, 18 (1944), pp. 323-330.

Boyer, Charles. "Le sens d'un texte de St. Thomas: De Veritate, q.1, art., 9." Gregorianum, 5 (1924), pp. 424-443.

Broad, C. D. "Went's Theory of Mathematical and Philosophical Reasoning." Proceedings of the Aristotelian Society, n.s., 42 (1941), pp. 1-24.

Brouwer, L. E. J. "Intuitionism and Formalism." Translated by Arnold Dresden. Bulletin of the American Mathematical Society, 20 (1913), pp. 81-96.

- Burke, Henry R. "Sir Isaac Newton's Formal Conception of Scientific Method." The New Scholasticism, 10 (1936), pp. 93-115.
- Cajori, Florian. "Plans for a History of Mathematics in the Nineteenth Century." Science, n.s., 48 (1919), pp. 279-285.
- Conway, A. W. "Whither Mathematics?" Studies, 34 (1945), pp. 158-162.
- Court, H. A. "Geometry and Experience." Scientific Monthly, 60 (1945), pp. 63-66.
- Couturat, Louis. "La Philosophie des mathématiques de Kant." Revue de Métaphysique et de Morale, 12 (1904), pp.
- Church, Alonzo. "A Bibliography of Symbolic Logic." The Journal of Symbolic Logic, 1 (1936), pp. 121-218.
- Dresden, Arnold. "Brouwer's Contributions to the Foundations of Mathematics." Bulletin of the American Mathematical Society, 30 (1924), pp. 31-40.
- Fraenkel, Abraham. "Natural Numbers as Cardinals." Scripta Mathematica, 6 (1939), pp. 69-79.
- _____. "Natural Numbers as Ordinals." Scripta Mathematica, 7 (1940), pp. 9-20.
- _____. "Problems and Methods in Modern Mathematics." Scripta Mathematica, 9 (1943), pp. 245-255.
- _____. "The Recent Controversies about the Foundations of Mathematics." Scripta Mathematica, 13 (1947), pp. 17-36.
- Geiger, L.-B. "Abstraction et séparation d'après S. Thomas." Revue des Sciences Philosophiques et Théologiques, 31 (1947), pp. 3-40.
- Gerhard, William A. "Public Opinion, Determinism and Free Will." The Thomist, 9 (1946), pp. 170-185.
- Gödel, Kurt. "What is Cantor's Continuum Problem?" The American Mathematical Monthly, 54 (1947), pp. 513-525.
- Goodstein, R. L. "Mathematical Systems." Mind, n.s., 49 (1939), pp. 58-73.
- Greenwood, Thomas. "Invention and Description in Mathematics." Proceedings of the Aristotelian Society, n.s., 30 (1930), pp. 79-90.
- _____. "Origines de la géométrie analytique." Revue Trimestrielle Canadienne, 34 (1948), pp. 166-179.
- _____. "Plato and Aristotle, A Contrast between their Mathematical Outlook." The New Scholasticism, 18 (1944), pp. 262-269.
- _____. "The Platonic Theory of the Continuum." The Thomist, 7 (1944), pp. 179-203.
- _____. "The Unity of Logic." The Thomist, 8 (1945), pp. 457-470.
- Guerard des Lauriers, L.-B. "Analyse de l'atome mathématique." Revue des Sciences Philosophiques et Théologiques, 22 (1933), pp. 385-431 and pp. 585-639.
- Guzibowski, Max. "Public Opinion and Thomistic Principles." The New Scholasticism, 19 (1945), pp. 136-160.

- Hempel, C. G. "On the Nature of Mathematical Truth." American Mathematical Monthly, 52 (1945), pp. 543-556.
- Hoeneke, P. "De Origine primorum principiorum scientiae." Gregorianum, 14 (1933), pp. 153-184.
- _____, "De Philosophiae Scholasticae cognitione geometriae." Gregorianum, 19/ (1938), pp. 498-514.
- _____, "De Problemate constitutionis geometriae." Gregorianum, 24 (1943), pp. 171-234.
- _____, "De problemate necessitatis geometriae." Gregorianum, 20 (1939), pp. 19-64.
- Holloway, Maurice A. "Abstraction from Matter in Human Cognition." The Modern Schoolman, 23 (1945), pp. 120-130.
- Jeans, Sir James. "The Mathematical Aspect of the Universe." Philosophy, 7(1932), pp. 3-14.
- Joseph, H. W. B. "Logic and Mathematics." Journal of Philosophical Studies, 3 (1928), pp. 3-14.
- Kane, V. H. "Introduction to Philosophy." The Thomist, 1 (1939), pp. 193-212.
- Lafleur, Laurence J. "A Functionalistic Interpretation of Mathematics." Scripta Mathematica, 8 (1941), pp. 27-53.
- Larguer, Everett H. "A Theory of Mathematical Reality." The Modern Schoolman, 16 (1939), pp. 88-91.
- _____, "Brouwerian Philosophy of Mathematics." Scripta Mathematica, 3 (1940), pp. 69-78.
- _____, "Concerning Some Views on the Structure of Mathematics." The Thomist, 4 (1942), pp. 431-445.
- _____, "Schools of Thought in Modern Mathematics." Thoughts 12 (1937), pp. 225-240.
- Lawler, D. "Influence of Descartes on Modern Science." Thought, 12 (1937), pp. 637-652.
- Lawton, H. "A Rational Basis for Mathematical Physics." Thought, 13 (1938), pp. 416-432.
- _____, "Physical Research and Metaphysical Reality." Thought, 12 (1937), pp. 458-475.
- MacLaughlin, P. J. "Centenary of Discovery of Quaternions." Studies, 32 (1943), pp. 441-456.
- McWilliams, James A. "Mathematics and Metaphysics in Science." The New Scholasticism, 2 (1937), pp. 358-373.
- Menger, Karl. "The New Logic." Translated by H. B. Gottlieb and J. K. Senior. Philosophy of Science, 4 (1937), pp. 299-336.
- Meyerson, Paul. "Le sens commun vise-t-il la connaissance?" Revue de Métaphysique et de Morale, 30 (1923), pp. 13-21.

- Moorman, R. H. "The Influence of Mathematics on the Philosophy of Leibniz." National Mathematics Magazine, 19 (1944), pp. 131-140.
- _____, "The Influence of Mathematics on the Philosophy of Spinoza." National Mathematics Magazine, 18 (1943), pp. 108-115.
- O'Toole, G. B. "Physical Mathematics and Mathematical Physics." Catholic Educational Review, 42 (1944), pp. 257-270.
- _____, "Truth is in the Judgment." The New Scholasticism, 17 (1943), pp. 1-15.
- Phelan, Gerald B. "A Note on the Formal Object of Metaphysics." The New Scholasticism, 18 (1944), pp. 197-201.
- _____, "Verum Sequitur Esse Rerum." Mediaeval Studies, 1 (1939), pp. 11-22.
- Pichette, Henri. "Considerations sur quelques principes fondamentaux et la doctrine du speculatif et du pratique." Laval Theologique et Philosophique, 1 (1947), pp. 52-70.
- Pierpont, J. "History of Mathematics in the 19th Century." Bulletin of the American Mathematical Society, 11 (1904), pp. 136-159.
- _____, "Mathematical Rigor, Past and Present." Bulletin of the American Mathematical Society, 34 (1928), pp. 25-53.
- Pirotta, A.-M. "De Metaphysicae Defensivae Natura secundum Doctrinam Aristotelico-Thomisticam." Angelicum, 4 (1927), pp. 257-270 and pp. 437-453.
- Renard, Henri. "The Problem of Knowledge in General." The Modern Schoolman, 24 (1946), pp. 1-11.
- Richardson, M. "The Place of Mathematics in a Liberal Education." National Mathematics Magazine, 19 (1945), pp. 349-356.
- Robert, J.-D. "La metaphysique, science distincte de toute autre discipline philosophique, selon Saint Thomas d'Aquin." Divus Thomas, 24 (1947), pp. 206-222.
- Russell, Bertrand. "Recent Work on the Principles of Mathematics." International Monthly, 4 (1901), pp. 83-101.
- Stabler, E. R. "An Interpretation and Comparison of Three Schools of Thought in the Foundations of Mathematics." Mathematics Teacher, 28 (1935), pp. 5-35.
- Thompson, W. R. "Mathematiques et biologie." Laval Theologique et Philosophique, 3 (1947), pp. 77-88.
- Toohy, John J. "Kant on the Propositions of pure Mathematics." The New Scholasticism, 11 (1937), pp. 146-157.
- Wellmuth, John J. "Some Comments on the Nature of Mathematical Logic." The New Scholasticism, 16 (1942), pp. 9-15.
- Weyl, Hermann. "David Hilbert and his Mathematical Work." Bulletin of the American Mathematical Society, 50 (1944), pp. 612-645.
- _____, "Mathematics and Logic: A brief survey serving as preface to a review of The Philosophy of Bertrand Russell." American Mathematical Monthly, 53 (1946), pp. 2-13.

White, L. A. "Locus of Mathematical Reality: An Anthropological Footnote." Philosophy of Science, 14 (1947), pp. 298-303.

Whittaker, John F. "The Position of Mathematics in the Hierarchy of Speculative Science." The Thomist, 3 (1941), pp. 467-506.

Wilder, R. L. "The Nature of Mathematical Proof." American Mathematical Monthly, 51 (1944), pp. 309-323.

BIOGRAPHICAL NOTE

Edward Anthony Haziarz was born in Milwaukee, Wisconsin, March 6, 1915. His elementary studies were pursued at St. Casimir's parochial school in Milwaukee and at St. Mary's high school in Burkettsville, Ohio. His college education was obtained at Saint Joseph's College, Collegeville, Indiana. His courses in theology were obtained at Saint Charles Seminary, Carthage, Ohio, where, on September 8, 1940, he was ordained to the priesthood as a member of the Society of the Precious Blood. In that same month, he registered as a graduate student at the Catholic University of America, and received the degree of Master of Arts in philosophy in June, 1941. After teaching at Marian College, Fond du Lac, Wisconsin, for the scholastic year, 1941-1942, he returned to Saint Joseph's College from which he received a Bachelor of Science degree in mathematics and physics, in June, 1944. After attending the 1944 summer session at the University of Virginia, he registered as a graduate student at the University of Michigan, where he obtained a Master of Science degree in mathematics, in September, 1945. He was appointed to an assistant professorship at Saint Joseph's College in 1945, and was appointed Director of Student Welfare in 1946. In 1948 he registered as a graduate student at the University of Ottawa.