AN ORDER-THEORETIC PROOF
OF THE
SPECTRAL THEOREM
FOR SELF-ADJOINT OPERATORS IN HILBERT SPACE

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by

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INTRODUCTION

Here we offer a proof of John von Neumann's version of the spectral theorem for self-adjoint operators in Hilbert space making use of the theory of vector lattices. The theorem in question reads as follows:

For each self-adjoint operator B in a Hilbert space $\mathfrak{H}$ there is one and only one operator function E_t $(-\infty < t < \infty)$ such that

1) E_t is a projection operator;

2) $E_t E_s = E_t$ for $t \leq s$;

3) E_t permutes with each bounded operator which permutes with B;

4) $\lim_{t \to -\infty} E_t f = 0$ and

   $\lim_{t \to \infty} E_t f = f$ for all $f \in \mathfrak{H}$;

5) $E_t f$ is a left-continuous function for all $f \in \mathfrak{H}$;

6) f is in the domain of definition of B, i.e., $f \in \mathfrak{D}_B$, if and only if

$$\int_{-\infty}^{\infty} t^2 \, d \langle E_t f, f \rangle < \infty.$$

In this case

$$Bf = \int_{-\infty}^{\infty} t \, d E_t f$$

and

$$||Bf||^2 = \int_{-\infty}^{\infty} t^2 \, d \langle E_t f, f \rangle .$$

The operator function E_t satisfying conditions 1) to 6) is called the spectral function of B and the second last formula is called the spectral resolution of B.

The key to the present proof of the spectral theorem is the observation that every strongly closed ring $\mathfrak{A}$ of bounded self-adjoint operators is a complete vector lattice in the usual operator theoretic sense of partial ordering and that, if $\mathfrak{A}$ contains the identity operator, one can take the identity operator as unit of the vector lattice; moreover, the basis of the vector lattice in this case consists of all projection operators contained in $\mathfrak{A}$, and $\mathfrak{A}$ itself will be a complete vector lattice of bounded elements. This fact then allows the use of Hans Freudenthal's integral representation theorem for elements of a complete vector lattice with unit; the spectral resolution of a bounded self-adjoint operator in a Hilbert space turns out to be a direct consequence of Freudenthal's theorem. Furthermore, we obtain the spectral resolution of an unbounded self-adjoint operator in Hilbert space by use of the vector lattice theoretic union of complete vector lattices of bounded self-adjoint operators in Hilbert space. It is noteworthy that this approach to the spectral theorem succeeds with only a modest amount of information from operator theory in Hilbert space.

In the literature of the subject it has been appreciated for a long time that the theory of vector lattices constitutes a new foundation of classical spectral theory; in terms of this new approach the heavy apparatus of integration theory could moreover be jettisoned. However the case of unbounded self-adjoint operators did not seem to fit in readily and we therefore made a special effort to accommodate this crucial case in our treatment.

The study of vector lattices in functional analysis commenced with the address of F. Riesz at Bologna in 1928. A modernisation of spectral theory in terms of vector lattices began with H. Freudenthal and S.W.P. Steen in 1936. Starting in 1941, H. Nakano published a long series of papers in which he went substantially beyond his predecessors. Nakano wrote several treatises giving a systematic account of his findings. Other prominent contributors to the theory of vector lattices are L.V. Kantorovic, M.G. Krein, S. Kakutani and T. Ogasawara.

In our treatment we endeavoured to render a reasonably self-contained presentation throughout and, to aid the reader, we compiled a list of special symbols used in Chapter I.

Chapter I

A SURVEY OF THE THEORY OF VECTOR LATTICES

1. Let E denote a vector space over the real numbers. Suppose that in E there is given a certain set of elements of which it is asserted that they are "larger than zero"; this is signified by writing $x > \theta$, where θ denotes the zero vector in E. We call all elements of E which are larger than zero and the zero element θ positive and denote by E^+ the collection of all positive elements of E. We put $x > y$ if $x - y > \theta$. As usual, $x \geq y$ means that either $x > y$ or $x = y$.

The vector space E is said to be a vector lattice, if the ordering introduced above satisfies the requirements: A) if $x > \theta$, then $x \neq \theta$; B) if $x > \theta$ and $y > \theta$, then $x + y > \theta$; C) for any two elements $x, y \in E$ the supremum, $x \vee y$, exists; D) if $x > \theta$ and λ is a real number larger than 0, then $\lambda x > \theta$.

Another definition, equivalent to the foregoing one, is the following: A vector space E over the real numbers is termed a vector lattice, if E is a lattice and the relation $x > y$ implies that (a) $x + z > y + z$ for any $z \in E$ and (b) for any real λ larger than 0 we have $\lambda x > \lambda y$.

A vector lattice is called complete (resp. σ-complete) if every bounded (resp. bounded and countable) subset of E has a supremum.

2. Let E be a vector lattice, and let T be an indexing set. If $x_t \in E$ for all $t \in T$ and $y = \bigvee_{t \in T} (-x_t)$ exists, then $-y = \bigwedge_{t \in T} x_t$.

Proof: From $-x_t \leq y$ $(t \in T)$ we get $-y \leq x_t$ $(t \in T)$. If $u \in E$ and $u \leq x_t$ $(t \in T)$, then $-x_t \leq -u$ $(t \in T)$ and therefore $y \leq u$, or $u \leq -y$;

this means $-y = \bigwedge_{t \in T} x_t$.

Remark: From the foregoing follows that for any $x, y \in E$

$$x \wedge y = -[(-x) \vee (-y)]. \tag{2.1}$$

3. If E is a vector lattice and $x, y, z \in E$ then we have the identities

$$(x \vee y) + z = (x + z) \vee (y + z) \tag{3.1}$$

$$(x \wedge y) + z = (x + z) \wedge (y + z) \tag{3.2}$$

A special case of formula (3.1) is:

$$[(x - y) \vee \theta] + y = x \vee y \tag{3.3}$$

Proof: Let $u = x \vee y$. Then $x \leq u$ and $x + z \leq u + z$. Similarly we get $y + z \leq u + z$ and therefore $(x + z) \vee (y + z) \leq u + z$. If we let $v = (x + z) \vee (y + z)$, then $v \leq u + z$. Moreover $x + z \leq v$ or $x \leq v - z$. Similarly $y \leq v - z$. Thus $u \leq v - z$ or $u + z \leq v$. This means that $v = u + z$ and formula (3.1) is verified.

To show (3.2) we make use of (2.1) and (3.1)

$$(x \wedge y) + z = -[(-x) \vee (-y)] + z = -\{[(-x) \vee (-y)] + (-z)\} =$$
$$= -[(-x - z) \vee (-y - z)] = (x + z) \wedge (y + z).$$

4. Let E be a vector lattice. If $x_t \in E$ $(t \in T)$ and $\bigvee_{t \in T} x_t$ exists, then $y + \bigvee_{t \in T} x_t = \bigvee_{t \in T} (y + x_t)$ for any $y \in E$ and

$$\alpha \bigvee_{t \in T} x_t = \begin{cases} \bigvee_{t \in T} (\alpha x_t) & \text{for } \alpha \geq 0 \\ \bigwedge_{t \in T} (\alpha x_t) & \text{for } \alpha \leq 0 \end{cases}$$

Proof: Let $x = \bigvee_{t \in T} x_t$. We have that $y + x_t \leq y + x$ $(t \in T)$, hence $\bigvee_{t \in T} (y + x_t) \leq y + x$. If $u \in E$ such that $y + x_t \leq u$ $(t \in T)$, then

$x_t \leq u - y$ and therefore $x \leq u - y$ or $x + y \leq u$. Hence $y + x = \bigvee_{t \in T} (y + x_t)$.

If $\alpha = 0$ the statement is evident. Suppose that $\alpha > 0$. Then $\alpha x_t \leq x$ $(t \in T)$ and hence $\bigvee_{t \in T} (\alpha x_t) \leq \alpha x$. If $v \in E$ such that $x_t \leq v$ $(t \in T)$ then $x_t \leq \alpha^{-1} v$ or $x \leq \alpha^{-1} v$. Hence $\alpha x \leq v$ and therefore $\alpha x = \bigvee_{t \in T} (\alpha x_t)$ for $\alpha > 0$.

For the case when $\alpha < 0$, we note that $\bigvee_{t \in T} (|\alpha|\, x_t) = |\alpha| x$. By Section 2, $\bigwedge_{t \in T} (\alpha x_t) = \bigwedge_{t \in T} (-|\alpha|\, x_t) = -\bigvee_{t \in T} (|\alpha|\, x_t) = -|\alpha| \bigvee_{t \in T} x_t = \alpha x$.

Remark: Using (3.3), (2.1) and the first part of the statement above, we get

$x \vee y = [(x - y) \vee \theta] + y = [(-y) \vee (-x)] + x + y = -(x \wedge y) + x + y$.

Hence

$$(x \vee y) + (x \wedge y) = x + y \tag{4.1}$$

5. Let E be a vector lattice and $x \in E$, $A \subset E$. We define the following:

$x^+ = x \vee \theta$ (positive part of x); $A^+ = \{x^+ : x \in A\}$

$x^- = (-x) \vee \theta = -(x \wedge \theta)$ (negative part of x); $A^- = \{x^- : x \in A\}$

$|x| = x^+ + x^-$ (modulus of x); $|A| = \{|x| : x \in A\}$

It is clear that $x^+, x^-, |x| \geq \theta$ and $x^- = (-x)^-$.

6. If E is a vector lattice and $x \in E$, then $x = x^+ - x^-$. If $x = y - z$, where $y, z \in E^+$, then $x^+ \leq y$ and $x^- \leq z$.

Proof: Taking $y = \theta$ in (4.1) we get $x = (x \vee \theta) + (x \wedge \theta) = x^+ - x^-$. To prove the second assertion, we note that $y = x + z \geq x$ and $y \geq \theta$. It follows that $y \geq x \vee \theta = x^+$. Similarly $z = y - x \geq -x$ and $z \geq \theta$ thus $z \geq (-x) \vee \theta = x^-$.

7. It can readily be seen, that in a vector lattice E, the following relationships hold:

$(x + y)^{\pm} \leq x^{\pm} + y^{\pm}$;

$$(\alpha x)^{+} = \begin{cases} \alpha x^{+} & \text{if } \alpha \geq 0 \\ -\alpha x^{-} & \text{if } \alpha \leq 0 \end{cases}, \quad (\alpha x)^{-} = \begin{cases} \alpha x^{-} & \text{if } \alpha \geq 0 \\ -\alpha x^{+} & \text{if } \alpha \leq 0 \end{cases};$$

$-|x| \leq x \leq |x|$; $|x + y| \leq |x| + |y|$; $|\alpha x| = |\alpha||x|$;

$|x| = \theta$ only when $x = \theta$. If $x \leq y$, then $x^{+} \leq y^{+}$ and $x^{-} \geq y^{-}$. The set $\{x_t : x_t \in E, t \in T\}$ is bounded if and only if $\{|x_t| : x_t \in E, t \in T\}$ is bounded.

8. Let E be a vector lattice and $x_t \in E$ $(t \in T)$. If $x = \bigvee_{t \in T} x_t$ exists, then

$$x^{+} = \bigvee_{t \in T} x_t^{+}, \quad x^{-} = \bigwedge_{t \in T} x_t^{-} \tag{8.1}$$

<u>Proof</u>: To show that $x^{+} = \bigvee_{t} x_t^{+}$ we make use of the associative law for upper bounds: Let the least upper bound of a non-empty set X contained in a lattice L be denoted by sup X; if the set X can be represented in the form $X = \bigcup_{t \in T} X_t$ and for each $t \in T$ sup $X_t = y_t$ exists and if in addition $y = \bigvee_{t \in T} y_t$ exists, then $y = \sup X$. Indeed, if $x \in X$, then $x \in X_t$ for some $t \in T$ and thus $x \leq y_t \leq y$. Hence y is an upper bound of the set X. Since $X_t \subset X$ for any $t \in T$, and if u is an upper bound of X also, then u is an upper bound for X_t as well; therefore $y_t \leq u$, implying $y \leq u$. Hence it follows, if $y_t = \sup X_t$ and $y = \sup X$ exists, then $y = \bigvee_{t \in T} y_t$.

Thus $x^+ = x \vee \theta = (\bigvee_{t\in T} x_t) \vee \theta = \bigvee_{t\in T} (x_t \vee \theta) = \bigvee_{t\in T} x_t^+$. To show that $x^- = \bigwedge_{t\in T} x_t^-$ we note that $x \geq x_t$ implies that $x^- \leq x_t^-$ for any $t \in T$. Let $y \in E$ be such that $y \leq x_t^-$ for all $t \in T$. Then $-y \geq -x_t^-$ or $x_t^+ - y \geq x_t$. Passing to the upper bound on both sides of the inequality and observing that the upper bound on the left side exists by the first part of this proof and using the statement under Section 4, we get $x^+ - y \geq x = x^+ - x^-$. Hence $-y \geq -x^-$ or $y \leq x^-$. This completes the proof.

9. In a vector lattice E the infinite distributive laws

$$x \wedge (\bigvee_{t\in T} y_t) = \bigvee_{t\in T} (x \wedge y_t) \quad \text{and} \quad x \vee (\bigwedge_{t\in T} y_t) = \bigwedge_{t\in T} (x \vee y_t)$$

are satisfied.

<u>Proof</u>: Let $y = \bigvee_{t\in T} y_t$ exist, then $y - x = \bigvee_{t\in T} (y_t - x)$ and by (8) $(y - x)^- = \bigwedge_{t\in T} (y_t - x)^-$. By Section 2, $-(y - x)^- = \bigvee_{t\in T} (-(y_t - x)^-)$ or $(y - x) \wedge \theta = \bigvee_{t\in T} ((y_t - x) \wedge \theta)$. Adding x to both sides of the last equation we get $y \wedge x = \bigvee_{t\in T} (y_t \wedge x)$.

To prove the second formula we replace x by $-x$ and y_t by $-y_t$ in the first formula.

10. Let E be a vector lattice. Two elements $x, y \in E$ are said to be <u>disjunct</u>, denoted by $x \, d \, y$, if $|x| \wedge |y| = \theta$. Two sets $X_1, X_2 \subset E$ are said to be disjunct, denoted by $X_1 \, d \, X_2$, if $x_1 \, d \, x_2$ for arbitrary x_1, x_2 where $x_1 \in X_1$ and $x_2 \in X_2$. It is clear that $x \, d \, x = \theta$ only if

$x = \theta$; two disjunct subsets of E can therefore have only one common element, namely θ.

11. Let E be a vector lattice. For any $x \in E$ we have $x^+ \, d \, x^-$. If $x = y - z$, where $y, z \in E^+$ and $y \, d \, z$, then $y = x^+$ and $z = x^-$.

Proof: Let $u = x^+ \wedge x^-$. Since $x^+, x^- \in E^+$, we have that $u \in E^+$. We put $y = x^+ - u$ and $z = x^- - u$. Then $y, z \in E^+$ and by Section 6, $y-z = x$ we have, by Section 6 again $x^+ \leq y$ and therefore $u = x^+ - y \leq \theta$. Thus $u = \theta$, which proves the first part of the above statement.

Next let $x = y - z$ be a given representation, where $y, z \in E^+$ and $y \, d \, z$. By (6) $y = x^+ + v$ where $v \in E^+$. But then $z = x^- + v$ and $y \wedge z \geq v$. Since $y \, d \, z$, we get $v = \theta$ and the proof is complete.

Remarks: If E is a vector lattice, then $|x| = x \vee (-x)$ for any $x \in E$. If $-y \leq x \leq y$, then $|x| \leq y$. In fact, using (3.3), Section 6 and 11, we have

$x \vee (-x) = |x| + [(x - |x|) \vee (-x - |x|)] = |x| + [(-2x^-) \vee (-2x^+)] =$
$|x| - 2(x^- \wedge x^+) = |x|$.

If $-y \leq x \leq y$ then $-y \leq -x \leq y$, and since $|x| = x \vee (-x)$ we have $|x| \leq y$.

12. Let E be a vector lattice. Suppose that in E for some element u we have $\theta \leq u \leq x + y$ where $x, y \in E^+$. Then u can be represented in the form $u = v + w$, where $\theta \leq v \leq x$ and $\theta \leq w \leq y$.

Proof: Let $v = u \wedge x$ and $w = u - (u \wedge x)$. Then $v + w = u$, $\theta \leq v \leq x$ and $\theta \leq w$. To complete the proof it remains to show that $w \leq y$.
$w = u - v = u - (u \wedge x) = u + [(-u) \vee (-x)] = \theta \vee (u - x)$. Since $y \geq u - x$ and $y \geq \theta$, we have $\theta \vee (u - x) \leq \theta \vee y = y$, hence $w \leq y$.

Remark 1: If $x, y, z \in E^+$ then

$$x \wedge (y + z) \leq (x \wedge y) + (x \wedge z). \qquad (12.1)$$

Indeed, let $u = x \wedge (y + z)$ then $u \leq y + z$ and we can write $u = v + w$, where $\theta \leq v \leq y$, $\theta \leq w \leq z$. Moreover $v, w \leq u \leq x$ so that $v \leq y \wedge x$ and $w \leq x \wedge z$. Hence $x \wedge (y + z) \leq (x \wedge y) + (x \wedge z)$.

Remark 2: Let $x \wedge y = \theta$, and $z \geq \theta$ then

$$z \wedge (x + y) = (z \wedge x) + (z \wedge y)$$

Proof: From $x \wedge y = 0$ follows

$$x + y = x \vee y$$

hence $z \wedge (x + y) = z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$

Finally, from $x \wedge y = \theta$ follows

$$(z \wedge x) \wedge (z \wedge y) = \theta$$

hence $z \wedge (x + y) = (z \wedge x) + (z \wedge y)$.

13. If the sets F and G in a vector lattice E are disjunct, then so are their linear hulls.

Proof: We show that, if $x \in F$ and $y, z \in G$, then $x \, d \, (y + z)$ and $x \, d \, (\alpha y)$ for any real α. Using (7) and (12.1) we get

$$\theta \leq |x| \wedge |y + z| \leq |x| \wedge (|y| + |z|) \leq (|x| \wedge |y|) + (|x| \wedge |z|) = \theta.$$

Hence $|x| \wedge |y + z| = \theta$.

Let $\beta = \max \{1, |\alpha|\}$. Then

$$\theta \leq |x| \wedge |\alpha y| \leq |x| \wedge |\alpha| |y| \leq \beta (|x| \wedge |y|) = \theta.$$

Hence $|x| \wedge |\alpha y| = \theta$.

14. If E is a vector lattice, $x_j \in E^+$ $(j = 1, 2, \ldots, n)$ and $x_j \, d \, x_k$ for $j \neq k$ then

$$\sum_{i=1}^{n} x_i = \bigvee_{i=1}^{n} x_i \qquad (14.1)$$

<u>Proof:</u> For $n = 2$ (4.1) implies (14.1). Suppose (14.1) holds for some n; we show that it is also true for $n + 1$.

Let $y = \bigvee_{i=1}^{n} x_i$, then by hypothesis $y = \sum_{i=1}^{n} x_i$. Since $x_{n+1} \wedge x_i = \theta$ $(i = 1, \ldots, n)$ it follows by Section 13, that $x_{n+1} \wedge y = \theta$ and hence

$$\bigvee_{i=1}^{n+1} x_i = (\bigvee_{i=1}^{n} x_i) \vee x_{n+1} = y \vee x_{n+1} = y + x_{n+1} = \sum_{i=1}^{n+1} x_i.$$

15. If E is a vector lattice, $x, y \in E$ and $x \; d \; y$, then

$$(x + y)^{\pm} = x^{\pm} + y^{\pm} \quad \text{and} \quad |x + y| = |x| + |y|.$$

<u>Proof:</u> Since $|x| \wedge |y| = \theta$ we have that $x^+ \wedge y^- = x^- \wedge y^+ = \theta$. Moreover by Section 11 $x^+ \; d \; x^-$ and $y^+ \; d \; y^-$. By Section 13 $(x^+ + y^+) \; d \; (x^- + y^-)$. But $x + y = (x^+ + y^+) - (x^- + y^-)$. Again by Section 11 $(x + y)^+ = x^+ + y^+$ and $(x + y)^- = x^- + y^-$ and therefore $|x + y| = (x + y)^+ + (x + y)^- = x^+ + y^+ + x^- + y^- = |x| + |y|$.

16. If $x = x_1 + x_2 + \ldots + x_n$, where x_j belongs to the vector lattice E and $x_j \; d \; x_k$ for $j \neq k$ and $x \in E^+$, then $x_j \in E^+$ for $j = 1, 2, \ldots, n$.

<u>Proof:</u> By Section 13 and 15 we have that $x^- = x_1^- + \ldots + x_n^-$. Thus if we had for at least one index j that $x_j^- > \theta$, then we would have $x^- > \theta$ and this would contradict the assumption $x \in E^+$. Therefore $x_j^- = \theta$ and $x_j \geq \theta$ for $j = 1, 2, \ldots, n$.

Remark 1: If $x_1 + x_2 + \ldots + x_n = \theta$ and $x_j \; d \; x_k$ $(k \neq j)$, then $x_j = \theta$ for $j = 1, 2, \ldots, n$. Indeed in this case $-x_1 - x_2 - \ldots - x_n = \theta$ and by what has just been shown we obtain $x_j, - x_j \in E^+$ for $j = 1, 2, \ldots, n$; this means however that $x_j = \theta$ for $j = 1, 2, \ldots, n$.

Remark 2: If $x_1, x_2, \ldots, x_n \neq \theta$ and $x_j \; d \; x_k$ for $j \neq k$, then $x_1, x_2, \ldots, x_n$ are linearly independent. In fact, let $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \theta$. Since $(\alpha_j x_j) \; d \; (\alpha_k x_k)$ for $j \neq k$, then by Remark 1 $\alpha_j x_j = \theta$ for $j = 1, 2, \ldots, n$ and therefore in view of our assumption $\alpha_j = 0$ for $j = 1, 2, \ldots, n$.

17. Let E be a vector lattice. If $x_t \in E$ $(t \in T)$, $y \in E$ and $x_t \; d \; y$ for all $t \in T$ and if $x = \bigvee_t x_t$ (or $x = \bigwedge_t x_t$) exists, then $x \; d \; y$.

Proof: Let $x = \bigvee_t x_t$. Assume first that $x_t \geq \theta$ for all $t \in T$. By Section 9, $x \wedge |y| = (\bigvee_t x_t) \wedge |y| = \bigvee_t (x_t \wedge |y|) = \theta$.

In the general case replace the element x_t by $x_t - x_{to}$, where $t_o \in T$ is some fixed index. Here $\bigvee_t (x_t - x_{to}) = x - x_{to} \geq \theta$ and by Section 8 $x - x_{to} = \bigvee_t (x_t - x_{to})^+$. By Section 13, $(x_t - x_{to}) \; d \; y$ holds and therefore $(x_t - x_{to})^+ \; d \; y$ holds for all $t \in T$. Using now what has already been verified about positive elements, it follows that $(x - x_{to}) \; d \; y$ and therefore $x \; d \; y$ where $x = (x - x_{to}) + x_{to}$.

When $x = \bigwedge_t x_t$, the proof is similar, namely we replace x_t by $-x_t$.

18. Let L be a lattice. A sequence $(x_n)_{n=1}^{\infty}$ of elements of L is said to be order-convergent ((o)-convergent for short) to the limit $x \in L$, if there exist two monotone sequences $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$ of elements of L, which are, respectively, decreasing and increasing such that $x = \bigwedge_n y_n = \bigvee_n z_n$ and $z_n \leq x_n \leq y_n$ for n = 1, 2, ...

<u>Remarks</u>: The (o)-limit is unique provided it exists. The algebraic and lattice theoretic operations in a vector lattice E are (o)-continuous (this can be seen by Section 4 and 9):

$x_n \overset{(o)}{\to} x$ and $x'_n \overset{(o)}{\to} x' \Rightarrow x_n + x'_n \overset{(o)}{\to} x + x'$, $\alpha x_n \overset{(o)}{\to} \alpha x$ (for any real number α), $x_n \vee x'_n \overset{(o)}{\to} x \vee x'$ and $x_n \wedge x'_n \overset{(o)}{\to} x \wedge x'$.

From the (o)-continuity of the operations in a vector lattice it readily follows that

(i) $x_n \overset{(o)}{\to} x \Rightarrow x_n^+ \overset{(o)}{\to} x^+$, $x_n^- \overset{(o)}{\to} x^-$, $|x_n| \overset{(o)}{\to} |x|$;

(ii) $|x_n| \overset{(o)}{\to} \theta \Rightarrow x_n \overset{(o)}{\to} \theta$;

(iii) $x_n \overset{(o)}{\to} x$ is equivalent with $|x_n - x| \overset{(o)}{\to} \theta$;

(iv) $x_n \overset{(o)}{\to} x$, $x'_n \overset{(o)}{\to} x'$ and $x_n \mathrm{d}\, x'_n$ for all n implies $x \mathrm{d}\, x'$.

(v) $x_n \overset{(o)}{\to} x$ if and only if there exists a monotone sequence $v_n \downarrow \theta$ in the sense of (o)-convergence such that $|x_n - x| \leq v_n$ for all n.

19. A subset F of a vector lattice E is said to be <u>normal</u> if it is stable under the formation of linear combinations of its elements and if the normality condition holds, i.e.

$$x \in F,\ |y| \leq |x| \Longrightarrow y \in F.$$

If F is a vector subspace of a vector lattice E satisfying the conditions of normality, then F is a sublattice of E.

<u>Proof:</u> Clearly $x, y \in F \Longrightarrow |x|, |y| \in F$ and $|x| + |y| \in F$. Since

$$|x \wedge y|,\ |x \vee y| \leq |x| \vee |y| \leq |x| + |y|,$$

we have $$x \wedge y,\ x \vee y \in F.$$

Thus a normal subset of a vector lattice is itself a vector lattice.

20. Let E be a vector lattice and let F be normal in E. If $x_t \in F$ $(t \in T)$ and $x = \bigvee_t x_t$, $(x = \bigwedge_t x_t)$ exists in F, then x is the supremum (infimum) of the set $\{x_t \in E : t \in T\}$ in E.

<u>Proof:</u> Let $x = \bigvee_t x_t$ in F, $y \in E$ and $y \geq x_t$ for all $t \in T$. Then $z = x \wedge y \leq x$ and $z \geq x_t$ for all $t \in T$. From $x_t \leq z \leq x$ we get $|z| \leq |x| + |x_t|$ because $z^+ \leq x^+$ and $z^- \leq x_t^-$ and hence

$$|z| = z^+ + z^- \leq x^+ + x_t^- \leq |x| + |x_t|.$$

Therefore $z \in F$ and since $x = \bigvee_t x_t$ is in F, we have $z \geq x$. Thus $z = x$ and hence $x \leq y$.

21. Let E be a vector lattice and let F be normal in E. Then $x_n \overset{(o)}{\to} x$ in F, where $x_n \in F$ for all n, if and only if $x_n \overset{(o)}{\to} x$ in E and $(x_n)_{n=1}^{\infty}$ is bounded in F.

Proof: If $x_n \xrightarrow{(o)} x$ in F, then there exists a monotone (O)-convergent sequence $v_n \downarrow \theta$, where $v_n \in F$ such that $|x_n - x| \leq v_n$ for all n. By Section 20 $x_n \to x$ in E. That $(x_n)_{n=1}^{\infty}$ is bounded in F follows from its (o)-convergence.

Next, let $x_n \xrightarrow{(o)} x$ in E, where $(x_n)_{n=1}^{\infty}$ is bounded in F. Then there exists some $y \in F$ such that $|x_n| \leq y$ for all n. Also $|x_n - x| \leq v_n \downarrow \theta$, where $v_n \in E$. By (i) of Section 18, $|x| \leq y$. Thus $x \in F$ and $|x_n - x| \leq 2y$. Putting $u_n = v_n \wedge 2y$ we get $u_n \in F$ for all n, $u_n \downarrow \theta$ in F and $|x_n - x| \leq u_n$. Hence $x_n \xrightarrow{(o)} x$ in F.

22. A vector lattice E is called Archimedean if from $x \in E^+$ and boundedness of the set $\{nx\}_{n=1}^{\infty}$ it follows that $x = \theta$.

Remark: If E is an Archimedean vector lattice, then it is easy to see that:

1. If $nx \leq y$ for all $n = 1, 2, \ldots$, then $x \leq \theta$
2. If $\alpha = \sup \alpha_t$ ($\alpha = \inf \alpha_t$) where α_t are real numbers, and $x \in E^+$, then $\alpha x = \sup \alpha_t x$ ($\alpha x = \inf \alpha_t x$).
3. If $|x_n| \leq y$ $(n = 1, 2, \ldots)$ and $\alpha_n \to 0$ then $\alpha_n x_n \xrightarrow{(o)} \theta$.
4. If $x_n \xrightarrow{(o)} x$ and $\alpha_n \to \alpha$ then $\alpha_n x_n \to \alpha x$.

23. In an Archimedean vector lattice E we say that the sequence $(x_n)_{n=1}^{\infty}$ converges in the sense of regulated convergence ((r)-convergence

for short) to the element $x \in E$, if there is an element $u \in E^+$, called the <u>regulator</u> of the convergence, having the property that to any $\varepsilon > 0$ there corresponds an $n_0 = n_0(\varepsilon)$ such that $|x_n - x| \leq \varepsilon u$ for $n \geq n_0$.

<u>Remarks</u>: $x_n \xrightarrow{(r)} x$ implies $x_n \xrightarrow{(o)} x$ because we can take $y_n = x + \frac{1}{n} u$ and $z_n = x - \frac{1}{n} u$; then $y_n \downarrow x$ and $z_n \uparrow x$ in the sense of the (o)-limit.

The (r)-limit is continuous w.r.t. the algebraic and lattice theoretic operations:

If $x_n \xrightarrow{(r)} x$, $x'_n \xrightarrow{(r)} x'$ and $\alpha_n \to \alpha$, then $x_n + x'_n \xrightarrow{(r)} x + x'$, $\alpha_n x_n \xrightarrow{(r)} \alpha x$, $x_n \vee x'_n \xrightarrow{(r)} x \vee x'$ and $x_n \wedge x'_n \xrightarrow{(r)} x \wedge x'$.

24. A positive element of a vector lattice E is called <u>unit</u> and is denoted by 1, if $x \wedge 1 > \theta$ for any $x > \theta$ and $x \in E$. In other words, if $x \in E$ and $x \, d \, 1$, then $x = \theta$.

If 1 is a unit of a vector lattice E and if E contains elements different from θ, then $1 > \theta$; if E consists of θ only, then θ is the unit element.

25. Let E be a vector lattice with unit 1. The element $e \in E$ is called <u>unitary</u> if $e \wedge (1 - e) = \theta$. The set of all unitary elements is called the <u>basis</u> of the vector lattice E and is denoted by $\mathscr{B}(E)$.

<u>Remark</u>: From the definition of unitary elements it follows that $e \geq \theta$ and $1 - e \geq \theta$; this means that $\theta \leq e \leq 1$. Hence θ and 1 belong to the basis. If $e \in \mathscr{B}(E)$, then $1 - e \in \mathscr{B}(E)$ and conversely.

26. The basis $\mathcal{B}(E)$ of a vector lattice E is a sublattice of E. If for any set of unitary elements e_t $(t \in T)$ $e = \bigvee_t e_t$ or $e = \bigwedge_t e_t$ exists, then $e \in \mathcal{B}(E)$. The basis, with the ordering already existing in it, forms a Boolean algebra and for any $e \in \mathcal{B}(E)$, $e' = 1 - e$ is the complement of e.

<u>Proof</u>: Let $e = \bigvee_t e_t$. Then $\theta \leq 1 - e \leq 1 - e_t$ for any $t \in T$. Since $e_t \wedge (1 - e_t) = \theta$, we have $e_t \wedge (1 - e) = \theta$ for any $t \in T$, hence by distributivity $e \wedge (1 - e) = (\bigvee_t e_t) \wedge (1 - e) = \bigvee_t [e_t \wedge (1 - e)] = \theta$. If $e = \bigwedge_t e_t$ then $-e = \bigvee_t (-e_t)$ and $1 - e = \bigvee_t (1 - e_t)$. Since $(1 - e_t) \in \mathcal{B}(E)$, it follows by what has been shown, that $(1-e) \in \mathcal{B}(E)$ and therefore $e \in \mathcal{B}(E)$.

Since $\mathcal{B}(E)$ is sublattice of E and E is a distributive lattice, $\mathcal{B}(E)$ is a distributive lattice as well. The elements 1 and θ are the largest and the smallest elements, respectively; $e \wedge (1 - e) = \theta$ by definition of unitary elements. From Section 14 follows $e \vee (1 - e) = e + (1 - e) = 1$ and so $1 - e = e'$.

<u>Remark</u>: Let $e_1, e_2 \in \mathcal{B}(E)$. If $e_1 \,d\, e_2$ then $e_1 + e_2 \in \mathcal{B}(E)$. If $e_1 \geq e_2$, then $e_1 - e_2 \in \mathcal{B}(E)$. The element e_2 is disjunct from the element $[e_1 - (e_1 \wedge e_2)]$.

Indeed, $e_1 + e_2 = e_1 \vee e_2 \in \mathcal{B}(E)$. To prove the second statement we observe that $(1 - e_1) \,d\, e_1$ implies $(1 - e_1) \,d\, e_2$; therefore $1 - e_1 + e_2 \in \mathcal{B}(E)$ and consequently $e_1 - e_2 = 1 - (1 - e_1 + e_2) \in \mathcal{B}(E)$.

To prove the third statement we note that

$$e_1 - (e_1 \wedge e_2) = e_1 + ((-e_1) \vee (-e_2)) = \theta \vee (e_1 - e_2) \leq 1 - e_2$$

and $(1 - e_2)\ d\ e_2$.

27. If E is a vector lattice with unit 1, then every element $x \in E$ for which $|x| \leq \delta 1$ for a certain real δ (depending on x) is called bounded. A vector lattice with unit all elements of which are bounded, is called a vector lattice of bounded elements.

All bounded elements of a vector lattice with unit form a normal set in the vector lattice. The notion of bounded element depends on the choice of unit in a given vector lattice.

28. For any Boolean algebra $\mathfrak{B}$ there exists an Archimedean vector lattice of bounded elements, the basis of which is isomorphic to the algebra $\mathfrak{B}$.

Proof: By the Stone representation theorem (see Appendix), every Boolean algebra $\mathfrak{B}$ is isomorphic with the Boolean algebra of all open-closed subsets of a certain totally disconnected compact Hausdorff space H. Considering the subset E of the vector lattice $C(H)$, consisting of all finite valued continuous functions, where the functions of E are all possible finite linear combinations of characteristic functions of the open-closed sets of H. Under the customary ordering and the usual definition of the algebraic operations, E is an Archimedean vector lattice. For unit in E we take the function $x(t) \equiv 1$. Then the basis $\mathfrak{B}(E)$ coincides with the family of characteristic functions of all open-closed sets of H and will be isomorphic with the given abstract Boolean algebra $\mathfrak{B}$. In addition, all elements in E will be bounded.

29. If in a vector lattice E each non-empty bounded set of positive elements has a supremum, then E is a complete vector lattice.

Proof: Every non-empty set in E which is bounded from above has a supremum. Indeed, let $x_t \in E$ $(t \in T)$ and $x_t \leq u$ for all $t \in T$. Put $x'_t = x_t \vee x_{t_0}$, where t_0 is some fixed index in T. We have that $x'_t \leq u$ for all $t \in T$. Moreover $x'_t - x_{t_0} \geq \theta$ for all $t \in T$. Since $x'_t - x_{t_0} \leq u - x_{t_0}$ it follows from the hypothesis that $y' = \bigvee_t (x'_t - x_{t_0})$ exists. By Section 4 $y' + x_{t_0} = \bigvee_t x'_t$. By the associative law of upper bounds (see Section 8) $\bigvee_t x'_t = \bigvee_t (x_t \vee x_{t_0}) = \bigvee_t x_t$ and the existence of $\bigvee_t x_t$ is shown.

To complete the proof we have to show that if in a vector lattice E each non-empty subset bounded from above has a supremum, then it also has an infimum, i.e. the lattice is conditionally complete in the usual terminology of lattice theory. Indeed it is enough to show that for any non-empty set bounded from below there exists an infimum: If $X \subset E$ is non-empty and bounded from below, then the set I of its lower bounds is also non-empty and bounded from above by an element of X. Then $y = \sup I$ exists. By definition $y \leq x$ for any $x \in X$, i.e. y is one of the lower bounds of the set X, which means that $y \in I$. Since $y = \sup I$, y is the greatest of the lower bounds of the set X, so that $y = \inf X$.

30. A normal subset F of a complete vector lattice E is a complete vector lattice.

Proof: By Section 19, F is a sublattice of E and it remains to show that F is a conditionally complete vector lattice. Let $x_t \in F$, $x_t \geq \theta$ $(t \in T)$ and $x_t \leq u \in F$. Then $y = \bigvee_t x_t \leq u$ exists in E. Since F is normal and $y \in F$, y is the least upper bound of the set $(x_t)_{t \in T}$ in F. By Section 29, F is then a complete vector lattice.

31. If in a vector lattice E for every countable bounded set of positive elements there exists a least upper bound, then E is a σ-complete vector lattice.

Proof: Repeating the proof in Section 29 we show that every countable set of elements of E bounded from above has a supremum.

To show the existence of the infimum, let $x_n \in E$ $(n = 1, 2, \ldots)$ and $x_n \geq u \in E$. Then $-x_n \leq u \in E$ and the set $\{-x_n\}_{n=1}^{\infty}$ is bounded from above and $y = \bigvee_{n=1}^{\infty} (-x_n)$ exists. By Section 2, $-y = \bigwedge_{n=1}^{\infty} x_n$.

32. A normal subset of a σ-complete vector lattice is a σ-complete vector lattice.

Proof: See proof in Section 30.

33. Any σ-complete vector lattice is Archimedean.

Proof: Let $\theta \leq x$ and $nx \leq y$ for all $n = 1, 2, \ldots$. Put $u = \bigvee_n (nx)$. By Section 4 $u + x = \bigvee_n (nx + x) = u$ and thus $x = \theta$.

34. The basis $\mathfrak{B}(E)$ of a complete (σ-complete) vector lattice E with unit 1 is a complete (σ-complete) Boolean algebra.

Proof: By Section 26, $\mathfrak{B}(E)$ is a Boolean algebra in which 1 and θ are the largest and smallest elements and therefore any set of unitary elements is bounded. If E is a complete vector lattice, then any set of its unitary elements has a bound which lies in $\mathcal{L}(E)$ (see Section 26). The completeness of $\mathcal{L}(E)$ is therefore shown in case E is a complete vector lattice. The verification of the above statement in case E is a σ-complete vector lattice is similar.

35. If E is a σ-complete vector lattice with unit 1, then for any $x \in E^+$

$$\bigvee_{n=1}^{\infty} (n1 \wedge x) = x \tag{35.1}$$

<u>Proof:</u> Since $(n1 \wedge x) \leq x$ for all $n = 1, 2, \ldots$, $\bigvee_n (n1 \wedge x) = y \leq x$ exists. Using the associative and distributive laws, $\bigvee_n (n1 \wedge y) = \bigvee [(n1 \wedge x) \wedge y] = (\bigvee (n1 \wedge x)) \wedge y = y$. Put $u = (x - y) \wedge 1$. Then $u \in E^+$. Here $(n1 \wedge y) + u = (n1 + u) \wedge (y + u) \leq (n + 1)1 \wedge x$, and therefore $y = \bigvee (n1 \wedge x) = \bigvee [(n + 1)1 \wedge x] \geq \bigvee (n1 \wedge y) + u = y + u$. Consequently $u \leq \theta$. We have that $u = \theta$ and $(x - y) \wedge 1 = \theta$. From the definition of unit we get $x - y = \theta$ or $x = y$.

36. A normal subset F of a σ-complete vector lattice E is called a <u>component</u> of E provided that the following condition is satisfied: if $X \subset F$ and $\sup X$ ($\inf X$) exists in E, then $\sup X \in F$ ($\inf X \in F$).

<u>Remark 1:</u> Since F is a linear space, the fore-going condition amounts to the requirement that any subset X of F, having a least upper bound in E, satisfy $\sup X \in F$. The corresponding statement concerning greatest lower bound follows from Section 2.

<u>Remark 2:</u> The concept of component of a complete vector lattice E is defined in the same manner.

37. If X is an arbitrary subset of the σ-complete vector lattice

E, then the set X^d, consisting of all $x \in E$ which are disjunct from the set X is called the disjunct complement of X.

38. The disjunct complement of any $X \subset E$ is a component of the σ-complete vector lattice E.

Proof: By Section 13, X^d is a linear space. If $x \in X^d$ and $|y| \leq |x|$, $y \in E$ then $y\ d\ X$. Thus X^d is normal in E. By Section 17, X^d is then a component of E.

39. Let F be a component of a σ-complete vector lattice E and $x \in E^+$. If in the set of all elements $y \in F$ satisfying the inequality $\theta \leq y \leq x$, there exists a largest element, then this element is called the projection of the element x onto the component F and is denoted by $[F]x$. For an arbitrary element $x \in E$ we define the projection of x onto F by the formula

$$[F]x = [F]x^+ - [F]x^-,$$

provided that $[F]x^+$ and $[F]x^-$ exist. According to the foregoing definition, if $x \in E^+$ and $[F]x$ exists, then $\theta \leq [F]x \leq x$. For an arbitrary $x \in E$, if the projection exists we have

$$\theta \leq [F]x^+ \leq x^+ \quad \text{and} \quad \theta \leq [F]x^- \leq x^-, \text{ therefore}$$

$$|[F]x| = |[F]x^+ - [F]x^-| \leq [F]x^+ + [F]x^- \leq x^+ + x^- = |x|. \qquad (39.1)$$

40. Let F be a component of a σ-complete vector lattice E and F^d the disjunct complement of F. In order that $[F]x$ exist for an element $x \in E$, it is necessary and sufficient that x be representable in the form $x = y + z$, where $y \in F$ and $z \in F^d$. If such a representation exists, then it is unique and we have

$$x = [F]x + [F^d]x. \qquad (40.1)$$

Therefore, if the projection of the element x onto F exists, then its projection onto the disjunct complement F^d exists as well.

Proof: First we show the necessity. It is enough to consider the case when $x \in E^+$. Let $y = [F]x$ exist. We put $z = x - y$ (then $z \geq \theta$) and show that $z \, d \, F$ holds. Take an arbitrary $u \in F$ and put $v = z \wedge |u|$. Then $v \geq \theta$, $v \in F$, since F is a component and $y + v \leq x$. By definition of projection it follows that $y + v \leq y$ and therefore $v = \theta$ and $z \, d \, F$. The required representation $x = y + z$ results, where $y \in F$ and $z \in F^d$.

We next show the sufficiency. Let $x = y + z$ where $y \in F$ and $z \in F^d$. We consider first the case when $x \in E^+$. By Section 16, $y, z \geq \theta$. If $\theta \leq u \leq x$ and $u \in F$ then $u \, d \, z$ and by inequality (12.1) $u = u \wedge x \leq (u \wedge y) + (u \wedge z) = u \wedge y$ i.e. $u \leq y$. Thus $y = [F]x$. If x is arbitrary, then by Section 15 $x^+ = y^+ + z^+$, $x^- = y^- + z^-$ and by what has just been proved $y^+ = [F]x^+$ and $y^- = [F]x^-$, i.e. $y = [F]x$.

In a similar way we get $z = [F^d]x$, and the representation of formula 40.1 is clear.

41. If E is a complete vector lattice, then for any $x \in E$ the projection onto any component F exists.

Proof: It is enough to consider the case when $x \in E^+$. Since the set of elements $y \in F$ satisfying the inequality $\theta \leq y \leq x$ is bounded in E the supremum of this set exists and belongs to F because F is a component of E. This supremum is the greatest among the y's.

Remark: The foregoing statement does not hold in a general σ-complete vector lattice.

42. The component F of a σ-complete vector lattice E is called

essential component, if F is a σ-complete vector lattice with unit.

43. In a σ-complete vector lattice E the projection of any $x \in E$ onto any essential component exists.

Proof: Let F be an essential component of E and let the element $u \in F$ be a unit, i.e. $x \wedge u > \theta$ for any $x \in F$ and $x > \theta$. Let $x \in E$, $x \geq 0$ and put $z = \bigvee_n (x \wedge nu)$ $(n = 1, 2, \ldots)$. Then $\theta \leq z \leq x$, and since $nu \in F$ for all n, we have $x \wedge nu \in F$ because F is a component. Thus $z \in F$. Now take an arbitrary $y \in F$, such that $\theta \leq y \leq x$. By section 35, $y = \bigvee_n (nu \wedge y) \leq \bigvee_n (nu \wedge x) = z$. Thus z is the largest among the elements $y \in F$ satisfying the inequality $\theta \leq y \leq x$, and therefore $z = [F]x$, that is for $x \geq \theta$

$$[F]x = \bigvee_n (x \wedge nu) \tag{43.1}$$

The existence of the projection for arbitrary $x \in E$ is now clear.

44. Suppose that X is an arbitrary subset of a σ-complete vector lattice E. We can form X^{dd} in the usual way. By Section 38, X^{dd} $(=(X^d)^d)$ is a component and it is clear that $X \subset X^{dd}$ holds. The component X^{dd} is called the component generated by the set X. If the component is generated by a single element u, we write E_u $(=\{u\}^{dd})$.

45. X^{dd} is the smallest component of the complete vector lattice E, containing the set X; if X is itself a component, then $X = X^{dd}$.

Proof: Let X be a component. Take any $x \in X^{dd}$; it can be represented in the form $x = y + z$, where $y \in X$ and $z \in X^d$. By Section 15, $|x| = |y| + |z|$ and therefore $|z| \leq |x|$ and $z \in X^{dd}$. Thus $z \, d \, z$, implying $z = \theta$ and $x = y \in X$. Hence $X^{dd} \subset X$ and we have $X^{dd} = X$.

Let X be an arbitrary set in E and let F be the component generated by X. Then $F^d \subset X^d$ and $F^{dd} \supset X^{dd}$. By what has already been shown $F^{dd} = F$ and therefore $X^{dd} \subset F$.

46. If the sets $X_1, X_2 \subset E$ and $X_1 \; d \; X_2$, where E is a complete vector lattice, then the components generated by X_1 and X_2 are also disjunct.

Proof: We have $X_1 \subset X_2^d$. By Section 38, X_2^d is a component and since X_1^{dd} is the smallest component containing X_1 it follows that $X_1^{dd} \subset X_2^d$ and hence $X_1^{dd} \; d \; X_2^{dd}$.

47. For a σ-complete vector lattice the notion of essential component and component generated by an element are equivalent; in the component E_u the element $|u|$ plays the role of unit.

Proof: Let F be an essential component and let u be its unit. Then $F^d \subset \{u\}^d$. We show that $x \; d \; u$ implies $x \; d \; F$, i.e. $\{u\}^d \subset F^d$, hence $F^d = \{u\}^d$ and by Section 45, $F = \{u\}^{dd}$. Indeed suppose that for some $y \in F$, x is not disjunct with y i.e. $z = |x| \wedge |y| > \theta$. Since $z \leq |y|$ $z \in F$ and $z \wedge u > \theta$ because u is a unit in F. Clearly $|z| \wedge u > \theta$. From $|z| \leq |x|$ and $|x| \wedge u = \theta$ follows $|z| \wedge u = \theta$ which is a contradiction. Thus $x \; d \; F$, and the proof of the first part is complete.

For the converse we show that in each component of the form $\{u\}^{dd}$ the element $|u|$ plays the role of unit. In fact, if $x \in \{u\}^{dd}$ and $x \geq \theta$, and at the same time $x \wedge |u| = \theta$, then $x \in \{u\}^d$. Thus $x \; d \; x$ and this means that $x = \theta$.

Remark: We have shown that if the element $u > \theta$ plays the role of the unit in the essential component F, then $F = E_u$. It follows therefore that if the σ-complete vector lattice E is itself generated by the unit 1, then the component E_1, generated by the unit, coincides with E. Since $\{1\}^d$ consists of θ only, we have $\{1\}^{dd} = E$.

48. For the projection onto an essential component we use new notation; if E_u is the component generated by the element u, then we write $[u]x$ for $[E_u]x$. We see that $[1]x = x$ for all $x \in E$.

By (43.1) the projection onto the component E_u for $x \in E^+$ becomes

$$[u]x = \bigvee (x \wedge nu). \qquad (48.1)$$

49. In a complete vector lattice E with unit 1, all components are essential.

Proof: Let F be an arbitrary component of E. We put $u = [F]1$ and show that u acts as unit in F. Let $x \in F$ such that $x > \theta$ and put $y = x \wedge 1$. Clearly $\theta < y \leq x$ and $y \in F$. By the definition of projection $y \leq u$, hence

$$x \wedge u \geq y \wedge u = y > \theta.$$

50. Let E be a σ-complete vector lattice and F a component such that for any $x \in E$ the projection $[F]x$ exists. In Section 43 it was shown that the projection exists if F is an essential component. If E is a complete vector lattice then F can be any of its components, see Section 41. The projection mapping satisfies the following properties:

(i) $[F](\alpha x + \beta y) = \alpha[F]x + \beta[F]y$;

Remark: We have shown that if the element $u > \theta$ plays the role of the unit in the essential component F, then $F = E_u$. It follows therefore that if the σ-complete vector lattice E is itself generated by the unit 1, then the component E_1, generated by the unit, coincides with E. Since $\{1\}^d$ consists of θ only, we have $\{1\}^{dd} = E$.

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$$x \wedge u \geq y \wedge u = y > \theta.$$

50. Let E be a σ-complete vector lattice and F a component such that for any $x \in E$ the projection $[F]x$ exists. In Section 43 it was shown that the projection exists if F is an essential component. If E is a complete vector lattice then F can be any of its components, see Section 41. The projection mapping satisfies the following properties:

(i) $[F](\alpha x + \beta y) = \alpha[F]x + \beta[F]y$;

(ii) $([F]x)^{\pm} = [F]x^{\pm}$, $|[F]x| = [F]|x|$;

(iii) $[F]x = \begin{cases} x \text{ if and only if } x \in F \\ \theta \text{ if and only if } x \, d \, F; \end{cases}$

(iv) if $x \geq y$ then $[F]x \geq [F]y$;

(v) if $x_n \xrightarrow{(0)} x$ then $[F]x_n \xrightarrow{(0)} [F]x$

Proof:

(i) It is only necessary to verify the additivity of the projection mapping, since homogeneity is trivial. Let $u = x + y$ then by Section 40 $x = x_1 + x_2$, $y = y_1 + y_2$ where $x_1, y_1 \in F$ and $x_2, y_2 \in F^d$ with $x_1 = [F]x$ and $y_1 = [F]y$. Therefore $u = (x_1 + y_1) + (x_2 + y_2)$ and since F and F^d are linear subspaces, $x_1 + y_1 \in F$ and $x_2 + y_2 \in F^d$. Again by Section 40, $x_1 + y_1 = [F]u$.

(ii) Clearly $[F]x = [F]x^+ - [F]x^-$, $[F]x^+$, $[F]x^- \geq \theta$ and $[F]x^+ \, d[F]x^-$ hence by Section 11 $[F]x^+ = ([F]x)^+$ and $[F]x^- = ([F]x)^-$. For the second part we use the linearity of the projection operator

$|[F]x| = ([F]x)^+ + ([F]x)^- = [F]x^+ + [F]x^- = [F](x^+ + x^-) = [F]|x|$.

(iii) Result follows from (40.1) and the uniqueness of representation proved in (40).

(iv) $\theta \leq x - y$ implies $\theta \leq [F](x - y) = [F]x - [F]y$ implies $[F]x \geq [F]y$.

(v) Using (39.1) we have $|[F]x_n - [F]x| = |[F](x_n - x)| \leq |x_n - x|$.

Since $|x_n - x| \xrightarrow{(0)} \theta$, the result follows.

51. If E is a vector lattice and the elements $x_t \in E(t \in T)$ are mutually disjunct and $\bigvee x_t^+$ and $\bigvee x_t^-$ exist in E, then the element $\underset{t}{S}\, x_t$ defined by

$$\underset{t}{S}\, x_t = \bigvee x_t^+ - \bigvee x_t^-$$

is called the union of the elements x_t $(t \in T)$.

52. The union of a finite number of elements x_k $(k = 1,2, \ldots, n)$ of a vector lattice E is defined. Moreover

$$\overset{n}{\underset{k=1}{S}}\, x_k = \sum_{k=1}^{n} x_k$$

holds. In fact by Section 14 we have

$$\overset{n}{\underset{k=1}{S}}\, x_k = \bigvee_{k=1}^{n} x_k^+ - \bigvee_{k=1}^{n} x_k^- = \sum_{k=1}^{n} x_k^+ - \sum_{k=1}^{n} x_k^- = \sum_{k=1}^{n} (x_k^+ - x_k^-) = \sum_{k=1}^{n} x_k.$$

If A is a complete vector lattice, then the union of any bounded set of pairwise disjunct elements $x_t (t \in T)$ exists because in this case $\bigvee x_t^+$ and $\bigvee x_t^-$ exist. In a σ-complete vector lattice the union of any bounded countable set of pairwise disjunct elements exists.

53. Let E be a vector lattice and suppose that $\underset{t}{S}\, x_t$ exists for $x_t \in E$ $(t \in T)$. For any $t, s \in T$ and $t \neq s$ we have $x_t \, d \, x_s$ and also $x_t^+ \, d \, x_s^-$. Since $x_t^+ \, d \, x_t^-$ for any $t \in T$ it follows from Section 17 that $\bigvee x_t^+ \, d \bigvee x_t^-$. From the definition of union of elements (see Section 51) we derive by using Section 11, that $(\underset{t}{S}\, x_t)^+ = \bigvee x_t^+$ and $(Sx_t)^- = \bigvee x_t^-$.

Hence $|\underset{t}{S}\, x_t| = (\underset{t}{S}\, x_t)^+ + (\underset{t}{S}\, x_t)^- = \bigvee_t x_t^+ + \bigvee_t x_t^- = (\bigvee_t x_t^+) \vee (\bigvee_t x_t^-) =$

$\bigvee_t (x_t^+ \vee x_t^-) = \bigvee_t |x_t|$. Thus for positive disjunct elements the union coincides with the supremum.

54. If $S\, x_t$ exists and $x_t\, d\, y$ for all x_t $(t \in T)$ in a vector lattice E, then $y\, d\, \underset{t}{S}\, x_t$.

Proof: By Section 17 $y\, d\, |x_t|$ implies $y\, d \bigvee_t |x_t|$. But by Section 53 $\bigvee_t |x_t| = |\underset{t}{S}\, x_t|$, hence $y\, d\, S\, x_t$.

55. The associative law holds for unions. In a complete vector lattice E the associative law is formulated in the following manner: If the set $\{x_t : t \in T\}$ is bounded and consists of pairwise disjunct elements and the set T of indices t is represented in terms of pairwise non-overlapping subsets T_a $(a \in A)$ such that $T = \bigcup_{a \in A} T_a$, then

$$\underset{t \in T}{S}\, x_t = \underset{a \in A}{S}\, (\underset{t \in T_a}{S}\, x_t). \tag{55.1}$$

Proof: If $|x_t| \leq y$ for all $t \in T$, then $|\underset{t \in T_a}{S}\, x_t| \leq y$ for all $a \in A$. In addition, if $a \neq b$, then we have $(\underset{t \in T_a}{S}\, x_t)\, d\, (\underset{t \in T_b}{S}\, x_t)$ so that the iterated union on the right hand side of (55.1) has meaning. Using the associative law for upper bounds (see Section 8), we obtain

$$\underset{a \in A}{S}\, (\underset{t \in T_a}{S}\, x_t) = \bigvee_a (\underset{t \in T_a}{S}\, x_t)^+ - \bigvee_a (\underset{t \in T_a}{S}\, x_t)^- = \bigvee_a \bigvee_{t \in T_a} x_t^+ - \bigvee_a \bigvee_{t \in T_a} x_t^- =$$

$$= \bigvee_{t \in T} x_t^+ - \bigvee_{t \in T} x_t^- = \underset{t \in T}{S}\, x_t.$$

56. Let F be a subset of a vector lattice E and let $A = (x_t)_{t \in T}$, $A \subset F$. A is said to be complete in F if $A^d \cap F = \theta$, i.e. if there is no element (except θ) in F which is disjunct from all $x_t \in A$.

A set $(E_t)_{t \in T}$ of components is said to be complete in F if the set $A = \bigcup_t E_t$ is complete in F.

57. In a σ-complete vector lattice E a set $(E_t)_{t \in T}$ of mutually disjunct components complete in E, is said to form a decomposition of E if for any $x \in E$ the projection onto each of the components E_t exists.

Remark: An example of a set of components forming a decomposition of a σ-complete vector lattice is any essential component and its disjunct complement.

58. If two components F and G form a decomposition of a σ-complete vector lattice E, then $G = F^d$.

Proof: Since $F \, d \, G$, we have that $G \subset F^d$. On the other hand we can represent any $x \in F^d$ in the form $x = [G]x + y$, where $y = [G^d]x$. By (39.1) $|y| \leq |x|$ and therefore $y \in F^d$. Moreover $y \in G^d$. By virtue of the completeness of the system $\{F,G\}$, $y = \theta$ and $x = [G]x \in G$. Thus $F^d \subset G$ and the proof is complete.

59. If in a σ-complete vector lattice there is given a complete set $(x_t)_{t \in T}$ of pairwise disjunct elements x_t then the components E_t, generated by these elements $(E_t = E_{x_t})$, form a decomposition of the σ-complete vector lattice E.

Proof: By Section 46 the components E_t are pairwise disjunct. Moreover, if some $x \in E$ is disjunct from all E_t, then it is of course also disjunct from all x_t and because of the completeness of the set $(x_t)_{t \in T}$, $x = \theta$. This implies that the set of components $(E_t)_{t \in T}$ is complete. By Section 47 all components E_t are essential and therefore projections onto each of them can be realized.

60. If the components E_t $(t \in T)$ form a decomposition of a complete vector lattice E, then any element $x \in E$ is representable uniquely in the form of the union

$$x = \underset{t \in T}{S}\ x_t \qquad (60.1)$$

where $x_t \in E_t$ and $x_t = [E_t]x$.

Proof: We verify the formula

$$x = \underset{t}{S}\ [E_t]x. \qquad (60.2)$$

Taking $x \in E^+$, then (60.2) becomes $x = \bigvee_t [E_t]x$. Since $\theta \leq [E_t]x \leq x$ for all $t \in T$, we get $y = \bigvee_t [E_t]x \leq x$. If $(x - y)\ d\ E_t$ for all $t \in T$ then we conclude from the completeness of the given set of components that $x - y = \theta$, and (60.2) is verified for $x \in E^+$ and hence also for arbitrary $x \in E$. We now show that $(x - y)\ d\ E_t$ is in fact true for any $t \in T$. Assuming the contrary, then for some $s \in T$ there would exist an element $z \in E_s$ such that $u = (x - y) \wedge z > \theta$. In this case we would have $u \in E_s$, $[E_s]x + u \in E_s$ and $\theta \leq [E_s]x + u \leq y + u \leq x$, which contradicts the definition of $[E_s]x$, and the assertion is proved.

The uniqueness of the representation (60.1) is seen as follows. Fixing $t \in T$ and using the associative law for unions (see Section 55) and Section 52, we have $x = \underset{t}{S}\, x_t = S(x_t, \underset{s \neq t}{S}\, x_s) = x_t + \underset{s \neq t}{S}\, x_s$. Clearly $x_t \, d \, \underset{s \neq t}{S}\, x_s$ and by Section 40, x_t is unique for every $t \in T$.

61. Given a family $(E_t)_{t \in T}$ of complete vector lattices, let E be the cartesian product of the sets E_t; define addition and scalar multiplication in E componentwise and let $E^+ = \{(x_t)_{t \in T} : x_t \in E^+$ for all $t \in T\}$. With these definitions E is a complete vector lattice, called the <u>union of</u> the <u>vector lattices</u> E_t, and denoted by $\underset{t \in T}{S}\, E_t$. Note that if $x = (x_t)_{t \in T} \in E$, then $|x| = (|x_t|)_{t \in T}$.

If $t_o \in T$ is fixed, then the set Z_{t_o} of all $x \in E$ for which

$$x_t = \begin{cases} x_{t_o} & \text{if } t = t_o \\ \theta & \text{if } t \neq t_o \qquad (x_{t_o} \in E_{t_o}) \end{cases}$$

will be a component in E. This component is algebraically and lattice theoretically isomorphic to E_{t_o}. Moreover, if each element $x_{t_o} \in E_{t_o}$ is identified with the family $(x_t)_{t \in T}$, where $x_t = \delta_{tt_o} x_{t_o}$ (Kronecker delta), then E_{t_o} is identified with Z_{t_o}. Therefore the E_t's form a decomposition of E and $[E_t]x = x_t$ for $x = (x_t)_{t \in T}$. We can write the elements of E as $x = \underset{t \in T}{S}\, x_t$ and call x the union of the elements x_t.

62. The union $E = SE_t$ of the complete vector lattices E_t with units 1_t is a complete vector lattice with unit; as unit in E we can take $S1_t$.

Proof: Let $y = S1_t$, then $y \in E^+$. If $x = Sx_t \in E$ and $x > \theta$, then $x_t > \theta$ for some t. But then $x_t \wedge 1_t > \theta$; hence $x \wedge y > \theta$, which means that y acts as a unit in E.

63. If the complete vector lattice E is decomposed into components E_t, then E is algebraically and lattice theoretically isomorphic to a certain normal sublattice of the complete vector lattice $Y = SE_t$.

Proof: By Section 60 each $x \in E$ has the form $x = Sx_t$ where $x_t = [E_t]x$. We associate with every element $x \in E$ the element $y \in Y$ which by formula introduced above has the form $y = Sx_t$, i.e. is the set $([E_t]x)$ of projections of the element x. The set of all elements $y \in Y$ which are the image elements of $x \in E$ we denote by Y'. Since each $x \in E$ by (60.2) is uniquely determined by it's projections onto E_t, the correspondence among E and Y' is one-to-one. Since the projection mapping is linear, we have that the algebraic operations on the elements of E correspond to the same operations on their images in Y'. If $x \geq \theta$, then $[E_t]x \geq \theta$ for all $t \in T$ and therefore the corresponding set $([E_t]x) \geq \theta$. Conversely if $y \in Y'$, $y=([E_t]x) \geq \theta$ then all $[E_t]x \geq \theta$ and by (60.2) we have $x \geq \theta$. This means that E and Y' are algebraically and lattice theoretically isomorphic.

It remains to verify that Y' is a normal sublattice in Y. Let

$y = (x_t) \in Y'$, $z = (z_t) \in Y$ and $|z| \leq |y|$. From the definition of Y' follows that there exists $x \in E$ for which $[E_t]x = x_t$ for all $t \in T$. Then $|z_t| \leq |x_t| = |[E_t]x| \leq |x|$. Thus the pairwise disjunct elements z_t of E are in a bounded set and $x' = Sz_t$ exists in E. By definition, the image of the element x' in the set Y' will be z. Hence $z \in Y'$ and Y' is a normal sublattice of Y.

64. Let E be a σ-complete vector lattice with unit 1 and $\mathcal{B}(E)$ as basis. We verify the following statements:

(i) If $1 = \underset{t}{S}\, e_t$ $(e_t \in \mathcal{B}(E), t \in T)$, then the set (e_t) is complete in E.

(ii) For any $e \in \mathcal{B}(E)$, $E_e^d = E_{1-e}$

(iii) If $e = \underset{t}{S}\, e_t$ $(e_t \in \mathcal{B}(E), t \in T)$, then for any $x \in E$ we have $[e]x = \underset{t}{S}\, [e_t]x$

(iv) If $e_1, e_2 \in \mathcal{B}(E)$, then for any $x \in E$

$$[e_1]([e_2]x) = [e_2 \wedge e_1]x \tag{64.1}$$

(v) If $e_n \downarrow \theta$ $(e_n \in \mathcal{B}(E))$, then $[e_n]x \xrightarrow{(o)} \theta$ for any $x \in E$.

Proof:

(i) Let $x \in E$ and $x \, d \, e_t$ for all $t \in T$. Then by Section 54 $x \, d \, 1$ and this is only possible if $x = \theta$.

(ii) Since $e \, d \, (1-e)$, we have by Section 52 that $1 = S(e, 1 - e)$. Thus by statement (i) above, the system of elements e and $1-e$ is complete in E. By Section 59 the components E_e and E_{1-e} form a decomposition of the σ-complete vector lattice E and by Section 58 $E_e^d = E_{1-e}$.

(iii) It is sufficient to prove this formula for $x \in E^+$. By (48.1)
$[e]x = \bigvee_n (x \wedge ne) = \bigvee_n (x \wedge \bigvee_t ne_t) = \bigvee_n \bigvee_t (x \wedge ne_t) = \bigvee_t \bigvee_n (x \wedge ne_t) =$
$\bigvee_t [e_t]x = S[e_t]x$.

(iv) Since $e_1 = (e_2 \wedge e_1) + (e_1 - (e_2 \wedge e_1))$ and the terms on the right hand side are disjunct and belong to the basis, we get by Section 3 that $[e_1]([e_2]x) = [e_2 \wedge e_1]([e_2]x) + [e_1 - (e_2 \wedge e_1)]([e_2]x)$. By the remark in (26) $e_2 d (e_1 - (e_1 \wedge e_2))$ and therefore $E_{e_2} d [e_1 - (e_1 \wedge e_2)]$ which implies that $[e_1 - (e_2 \wedge e_1)]([e_2]x) = \theta$. Similarly $[e_2 \wedge e_1]([1-e_2]x) = \theta$. Since $x = [e]x + [1 - e]x$ and the projection mapping is additive we derive $[e_1]([e_2]x) = [e_2 \wedge e_1]x$.

(v) The sequence $\{[e_n]|x|\}$ is decreasing. Putting $y = \inf \{[e_n]|x|\}$, we get $\theta \leq y \leq [e_n]|x|$ and hence $y \in E_{e_n}$ and $y \; d \; (1 - e_n)$ for all n. By Section 17 $y \; d \bigvee_n (1 - e_n) = 1$; but this occurs only for $y = \theta$. Therefore $[e_n]|x| \xrightarrow{(o)} \theta$ and therefore $[e_n]x \xrightarrow{(o)} \theta$.

65. Let E be a σ-complete vector lattice with unit 1 and basis $\mathcal{L}(E)$. Then every unitary element is a projection of the unit 1 onto a certain essential component. Conversely, the projection of the unit element 1 onto any essential component is a unitary element, acting in this component as a unit element.

Proof: If $e \in \mathcal{L}(E)$, then $1 = e + (1 - e)$ is a decomposition into disjunct summands of the unit. But $e \in \{e\}^{dd} = E_e$ and $1 - e \in E_{1-e} = E_e^d$

and by (40) it follows that $e = [e]1$.

Conversely, let $x \in E$ and $e = [x]1$. Then by (40.1) the unit decomposes into disjunct summands $1 = [x]1 + z$, where $z = 1 - [x]1 = [1 - e]1$. Thus $e \, d \, (1 - e)$ and hence $e \in \mathcal{L}(E)$. To see that e acts as a unit in the component E_x we use an argument similar to Section 49.

66. The basis $\mathcal{L}(E)$ is isomorphic with the set of all essential components of a σ-complete vector lattice E with unit 1 and ordered by inclusion. If E is a complete vector lattice with unit, then the basis $\mathcal{L}(E)$ is isomorphic with the set of all its components.

Proof: We associate each essential component with the projection of the unit onto this component. Since we get in this way the unitary elements generating the same component, the correspondence is one-to-one. By Section 65 the image set of all essential components is the entire basis $\mathcal{L}(E)$. Moreover, the inequality $e_1 \leq e_2$ is equivalent with the inclusion $E_{e_1} \subset E_{e_2}$. The second part of the above statement follows from Section 49.

67. Let E be a σ-complete vector lattice with unit 1 and let $\mathcal{L}(E)$ denote its basis. For any $e_1, e_2 \in \mathcal{L}(E)$ we have

$$[e_2]e_1 = e_1 \wedge e_2 \qquad (67.1)$$

Proof: In Section 65 we showed that $e = [e]1$ for any $e \in \mathcal{L}(E)$. From (64.1) follows (67.1) by substituting $x = 1$.

68. Let E be a σ-complete vector lattice with unit 1 and basis $\mathcal{L}(E)$. The projection of the unit 1 onto the essential component E_x, $x \in E$, is called the trace of the element x and is denoted by e_x; thus

$$e_x = [x]1 = \bigvee_n (1 \wedge n|x|).$$

Remarks: By Section 65 the trace $e_x \in \mathcal{B}(E)$ for any $x \in E$ is a unit in the component E_x, that is $E_{e_x} = E_x$. Thus, if $x \neq \theta$, then $e_x \neq \theta$. Moreover $e_x = [x]1 = [|x|]1 = e_{|x|}$.

69. The trace has the following properties:

(i) $x \, d \, y$ and $e_x \, d \, e_y$ are equivalent;

(ii) $|x| \leq |y|$ implies $e_x \leq e_y$;

(iii) $[e_x]x = x$;

(iv) $[e_{x^+}]x = x^+$; $[e_{x^-}]x = -x^-$; $[1 - e_{x^+}]x = -x^-$; $[1 - e_{x^-}]x = x^+$;

(v) $e_x = e_{x^+} + e_{x^-}$;

(vi) If $x = \bigvee x_t$, where $x_t \geq \theta$ $(t \in T)$ then $e_x = \bigvee e_{x_t}$.

(If $x = \bigwedge x_t$, where $x_t \geq \theta$ $(t \in T)$, then $e_x \leq e_{x_t}$ holds for all t).

Proof:

(i) If $x \, d \, y$ then by Section 46 $E_x \, d \, E_y$ and therefore $e_x \, d \, e_y$.

The proof of the converse is similar.

(ii) From $|x| \leq |y|$ follows $\{y\}^d \subset \{x\}^d$ or $E_x = \{x\}^{dd} \subset \{y\}^{dd} = E_y$;

this together with $e_x = [x]1 = \sup \{x \in E_x : \theta \leq x \leq 1\}$ and

$e_y = [y]1 = \sup \{y \in E_y : \theta \leq y \leq 1\}$ implies $e_x \leq e_y$

(iii) Since $x \in E_x = E_{e_x}$ it follows from Section 50 (iii) that

$[e_x]x = x$.

(iv) $[e_{x^+}]x = [e_{x^+}]x^+ - [e_{x^+}]x^- = [x^+]x^+ - [x^+]x^-$. But $x^- \, d \, x^+$, hence $x^- \, d \, \{x\}^{+\,dd}$ and $[x^+]x^- = \theta$. Thus $[e_{x^+}]x = x^+$. From Section 64 (iii) follows $[1 - e_{x^+}]x = [1]x - [e_{x^+}]x = x - x^+ = - x^-$. The other two equations are proved similarly.

(v) $e_x = e_{|x|} = [|x|]1 = \bigvee_n (1 \wedge n|x|) =$

$$= \bigvee_n (1 \wedge (nx^+ \vee nx^-)) = \bigvee_n ((1 \wedge nx^+) \vee (1 \wedge nx^-)) =$$

$$= (\bigvee_n (1 \wedge nx^+)) \vee (\bigvee_n (1 \wedge nx^-)) = [x^+]1 \vee [x^-]1 =$$

$$= e_{x^+} \vee e_{x^-} = e_{x^+} + e_{x^-}.$$

(vi) $e_x = \bigvee_n (1 \wedge nx) = \bigvee_n (1 \wedge n \bigvee_t x_t) =$

$$= \bigvee_n \bigvee_t (1 \wedge nx_t) = \bigvee_t \bigvee_n (1 \wedge nx_t) = \bigvee_t e_{x_t}.$$

70. Let E be a σ-complete vector lattice with unit 1. For each $x \in E$ and each real λ we put

$$e_\lambda^x = e_{(\lambda 1 - x)^+} ;$$

e_λ^x is the trace of the positive part of the element $\lambda 1 - x$ and $e_\lambda^x \in \mathcal{L}(E)$. For fixed $x \in E$ the system of elements (e_λ^x), where λ runs through $(-\infty, \infty)$ is called <u>resolution</u> of x.

<u>Remark</u>: From the definition of trace it follows that $e_\lambda^x = [(\lambda 1 - x)]1$. By Section 69 (iv) $[e_\lambda^x](\lambda 1 - x) = [e_{(\lambda 1 - x)^+}](\lambda 1 - x) = (\lambda 1 - x)^+ \geq \theta$. On the other hand by the linearily of the projection mapping and (67.1)

we have $[e^x_\lambda](\lambda 1 - x) = \lambda[e^x_\lambda]1 - [e^x_\lambda]x = \lambda e^x_\lambda - [e^x_\lambda]x$. Hence $\lambda e^x_\lambda - [e^x_\lambda]x \geq \theta$

or

$$[e^x_\lambda]x \leq \lambda e^x_\lambda. \tag{70.1}$$

Similarly, $[1 - e^x_\lambda](\lambda 1 - x) = -(\lambda 1 - x)^- \leq \theta$ and hence

$$[1 - e^x_\lambda]x \geq \lambda(1 - e^x_\lambda). \tag{70.2}$$

71. Let E be a σ-complete vector lattice with unit 1. For each $x \in E$ the resolution of the element x has the following properties:

(i) $e^x_\mu \geq e^x_\lambda$ for $\mu \geq \lambda$

(ii) $\bigvee_\lambda e^x_\lambda = 1$

(iii) $\bigwedge_\lambda e^x_\lambda = \theta$

(iv) $\bigvee_{\mu<\lambda} e^x_\mu = e^x_\lambda$ (the resolution is left-continuous for each λ).

(v) If $\mu_1 \geq \mu_2 \geq \lambda_1 \geq \lambda_2$, then $(e^x_{\mu_1} - e^x_{\mu_2}) \, d \, (e^x_{\lambda_1} - e^x_{\lambda_2})$.

For increasing $\lambda(-\infty < \lambda < \infty)$ e^x_λ increases from θ to 1.

Proof:

(i) If $\mu \geq \lambda$, then $\mu 1 - x \geq \lambda 1 - x$ and hence $(\mu 1 - x)^+ \geq (\lambda 1 - x)^+$.

By Section 69 (ii) $e^x_\mu \geq e^x_\lambda$

(ii) Let $e = \bigvee_\lambda e^x_\lambda$, then $1 - e \leq 1 - e^x_\lambda$ for any real λ. Now for any $\lambda > \theta$ we obtain by using (39.1) and (70.2)

$$\lambda(1 - e) \leq \lambda(1 - e^x_\lambda) \leq [1 - e^x_\lambda]x \leq |x|.$$

Recalling that every σ-complete vector lattice is Archimedean (see Section 33) it follows that $1 - e = \theta$, or $e = 1$.

(iii) The verification is similar; for negative λ we use the inequality $|\lambda| e^x_\lambda \geq [e^x_\lambda] x$ which obtains from (70.1)

(iv) We have $\lambda 1 - x = \bigvee_{\mu<\lambda} (\mu 1 - x)$ for every real λ. By Section 8 $(\lambda 1 - x)^+ = \bigvee_{\mu<\lambda} (\mu 1 - x)^+$ and by Section 69 (vi) $e^x_\lambda = \bigvee_{\mu<\lambda} e^x_\mu$.

(v) Since the resolution is monotone, we have $e^x_{\mu_1} - e^x_{\mu_2} \leq 1 - e^x_{\lambda_1}$ and $e^x_{\lambda_1} - e^x_{\lambda_2} \leq e^x_{\lambda_1}$ and therefore $(e^x_{\mu_1} - e^x_{\mu_2}) \, d \, (e^x_{\lambda_1} - e^x_{\lambda_2})$.

72. Let E be a σ-complete vector lattice with unit 1. For any $\mu \geq \lambda$ we have the formula

$$\lambda(e^x_\mu - e^x_\lambda) \leq [e^x_\mu - e^x_\lambda] x \leq \mu(e^x_\mu - e^x_\lambda). \tag{72.1}$$

Proof: First we verify the formula

$$e^x_\mu \wedge (1 - e^x_\lambda) = e^x_\mu - e^x_\lambda. \tag{72.2}$$

In fact $1 - e^x_\lambda = (1 - e^x_\mu) + (e^x_\mu - e^x_\lambda) = (1 - e^x_\mu) \vee (e^x_\mu - e^x_\lambda)$; by distributivity $e^x_\mu \wedge (1 - e^x_\lambda) =$

$(e^x_\mu \wedge (1 - e^x_\mu)) \vee (e^x_\mu \wedge (e^x_\mu - e^x_\lambda)) = e^x_\mu - e^x_\lambda$. Now by applying to both sides of (70.2) the projection mapping $[e^x_\mu]$ we obtain

$$\lambda [e^x_\mu](1 - e^x_\lambda) \leq [e^x_\mu][1 - e^x_\lambda] x.$$

Using (67.1) and (72.2) on the left-hand side and (64.1) and (72.2) on the right-hand side of this inequality we obtain the first inequality in (72.1). To obtain the second inequality in (72.1) we substitute in 70.1

μ for λ and apply the projection mapping $[1 - e_\lambda^x]$ on both sides. Finally we apply (64.1) and (72.2) on the left-hand side and (67.1) and(72.2) on the right hand side.

73. In a σ-complete vector lattice E we can, with the help of (o)-convergence, introduce the notion of infinite series $\sum_{n=0}^{\infty} x_n$ $(x_n \in E)$, where $(o)\text{-}\lim_{p\to\infty} (\sum_{n=0}^{p} x_n)$ is taken to be the sum of the series under consideration. It is easy to see that the existence of $(o)\text{-}\lim_{p\to\infty} (\sum_{n=0}^{p} x_n)$ implies $x_n \xrightarrow{(o)} \theta$. For (o)-convergence of a series with positive elements $(x_n \in E^+)$ it is necessary and sufficient that the sequence of its partial sums be bounded; in this case the sum of the series coincides with the supremum of it's partial sums. If $\theta \le x_n \le y_n$ for all $n > n_0$, then form the (o)-convergence of $\sum_{n=0}^{\infty} y_n$ follows the (o)-convergence of $\sum_{n=0}^{\infty} x_n$. If the series $\sum_{n=0}^{\infty} |x_n|$ is (o)-convergent, then $\sum_{n=0}^{\infty} x_n$ is (o)-convergent as well; in this case the latter series is said to be absolutely convergent.

Let all terms of the series $\sum_{n=0}^{\infty} x_n$ be pairwise disjunct and suppose that the set $(x_n)_{n=1}^{\infty}$ is bounded; then $\underset{n}{S}\, x_n$ exists. We verify that $\underset{n=0}{\overset{\infty}{S}}\, x_n = \sum_{n=0}^{\infty} x_n$; this generalizes Section 52. Indeed,

$$\underset{n}{S}\, x_n = \bigvee_n x_n^+ - \bigvee_n x_n^- = (o) - \lim_p \bigvee_{n\le p} x_n^+ - (o) - \lim_p \bigvee_{n\le p} x_n^- =$$

$$= (o) - \lim_p \left(\sum_{n=0}^{p} x_n^+ - \sum_{n=0}^{p} x_n^- \right) = (o) - \lim_p \sum_{n=0}^{p} x_n = \sum_{n=0}^{\infty} x_n.$$

Two-sided in infinite series $\sum_{-\infty}^{+\infty} x_n$ offer no difficulty; we define

$$\sum_{-\infty}^{\infty} x_n = (o) - \lim \sum_{-\infty}^{\infty} x_n = (o) - \lim \sum_{n=1}^{\infty} x_n + (o)\text{-}\lim \sum_{n=0}^{\infty} x_{-n}.$$

74. <u>The integral representation theorem of H. Freudenthal:</u>

Let E be a σ-complete vector lattice with unit 1. Then each element $x \in E$ is representable in the form

$$x = \int_{-\infty}^{\infty} \lambda \, de_\lambda^x ,$$

where the integral sign signifies the (o)-limit of the integral sums

$$\mathfrak{J} = \sum_{-\infty}^{\infty} t_n \, (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x)$$

formed for an arbitrary partition of the real line $(-\infty,\infty)$ with the points $-\infty < \ldots \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots < \infty$, where $\lambda_{n-1} \le t_n \le \lambda_n$ for all n and the (o)-limit is taken under the condition that $\varepsilon = \sup_n (\lambda_n - \lambda_{n-1}) \to 0.$

<u>Proof</u>: By (71.(vi)) : $(e_{\lambda_n}^x - e_{\lambda_{n-1}}^x) \, d \, (e_{\lambda_m}^x - e_{\lambda_{m-1}}^x)$ for $n \neq m$. Moreover all elements of the form $(e_{\lambda_n}^x - e_{\lambda_{n-1}}^x)$ are bounded by 1; hence their union is defined and using Section 71: $\underset{n}{S} \, (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x) = \bigvee_n (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x) =$

$$(o) = \lim_p \bigvee_{-p \le n \le p} (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x) - (o)\text{-}\lim_p \sum_{n=-p}^{p} (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x) =$$

$$(o)\text{-}\lim_p (e_{\lambda_p}^x - e_{\lambda_{-p-1}}^x) = 1.$$

It follows from Section 64 (iii) that

$$x = \underset{n}{S}\,[e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x \tag{74.1}$$

Taking an arbitrary t_n satisfying the inequality $\lambda_{n-1} \leq t_n \leq \lambda_n$ we consider the pairwise disjunct elements $v_n = t_n(e^x_{\lambda_n} - e^x_{\lambda_{n-1}})$ for $n = 0, \pm 1, \pm 2, \ldots$ From inequality (72.1) follows $v_n - [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x \leq$

$$\leq (\lambda_n - \lambda_{n-1})(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) \leq \varepsilon(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) . \tag{74.2}$$

Using formula (39.1) we obtain $|x| + \varepsilon 1 \geq [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x - \varepsilon(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) \geq v_n \geq [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x - \varepsilon(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) \geq - |x| - \varepsilon 1$. Thus the elements v_n form a bounded set and their union is defined. By Section 73 $Sv_n = \sum_{-\infty}^{\infty} v_n$ and hence the integral sum $\mathfrak{J}$ has meaning. Formula (74.1) can be written as

$$x = \sum_{-\infty}^{\infty} [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x.$$

Using inequality (74.2) and properties of infinite series σ-complete vector lattices we get

$$|\mathfrak{J} - x| = \Big| \sum_{-\infty}^{\infty} (t_n(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) - [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x)\Big| = \Big| \sum_{-\infty}^{\infty} (v_n - [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x)\Big| \leq$$

$$\leq \sum_{-\infty}^{\infty} |v_n - [e^x_{\lambda_n} - e^x_{\lambda_{n-1}}]x| \leq \sum_{-\infty}^{\infty} \varepsilon(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) = \varepsilon\, S(e^x_{\lambda_n} - e^x_{\lambda_{n-1}}) = \varepsilon 1.$$

Therefore $x = (o)\text{-}\lim_{\varepsilon \to 0} \mathfrak{J}$.

<u>Remark</u>: If two elements $x, y \in E$ have the same resolution, i.e. $e^x_\lambda = e^y_\lambda$ for all λ (or even all rational λ), then $x = y$. Also, the integral representation is unique since the resolution is left-continuous.

75. A σ-complete vector lattice E with a binary multiplication is called a partially ordered ring provided the following conditions hold:

(A): $(xy)z = x(yz)$;

(B): $x(y + z) = xy + xz$, $(y + z)x = yx + zx$;

(C): $(\alpha x)y = x(\alpha y) = \alpha(xy)$ for every real number α;

(D): There is a multiplicative unit 1 $(>\theta)$ such that for all $x \in E$ we have $x1 = 1x = x$ (therefore there is only one unit);

(E): $x \geq \theta$ and $y \geq \theta$ implies $xy \geq \theta$;

(F): $x \wedge 1 = \theta$ implies $x = \theta$.

Remark: If $x \geq y$ and $z \geq \theta$, then $xz \geq yz$ and $zx \geq zy$.

76. Let e_1 and e_2 be unitary elements in a partially ordered ring E. Then we have the formula

$$[e_2]e_1 = [e_1]e_2 = e_1e_2 = e_2e_1$$

Proof: We substitute the identity $e_1 = e_1e_2 + e_1(1 - e_2)$ into formula (67.1): $[e_2]e_1 = e_2 \wedge e_1 = e_2 \wedge (e_1e_2 + e_1(1 - e_2))$. We shall show that $e_1e_2 \wedge e_1(1 - e_2) = \theta$ and hence by remark 2 in Section 12

$$[e_2]e_1 = (e_2 \wedge e_1e_2) + (e_2 \wedge (e_1(1 - e_2))).$$

Indeed since $\theta \leq e_1 \leq 1$, $\theta \leq 1 - e_2 \leq 1$ we have by remark in Section 75 that $\theta \leq e_1(1 - e_2) \leq 1 - e_2$ hence

$\theta \leq e_1e_2 \wedge (e_1(1 - e_2)) \leq e_2 \wedge (e_1(1 - e_2)) \leq e_2 \wedge (1 - e_2) = \theta$ i.e.

$e_1e_2 \wedge (e_1(1 - e_2)) = \theta$. This string of inequalities also shows that $e_2 \wedge (e_1(1 - e_2)) = \theta$, which together with $e_2 \wedge e_1e_2 = e_1e_2$ establishes

$[e_2]e_1 = e_1e_2$. The remaining equalities follow from (67.1).

77. If E is a partially ordered ring, $x \in E^+$ and $y \in E$, then $|xy| \leq x|y|$ and $|yx| \leq |y|x$.

Proof: Indeed, $|xy| = |xy^+ - xy^-| \leq |xy^+| + |xy^-| = xy^+ + xy^- = x|y|$. Similarly $|yx| \leq |y|x$.

78. Let x and y belong to a partially ordered ring E and $\theta \leq x \leq \beta\, 1$, where β is some positive real constant. Then for fixed x the products xy and yx are continuous.

Proof: Let $y_o = (o)\text{-}\lim\limits_{n\to\infty} y_n$, then by Section 18.(v) there exists a monotone sequence $v_n \downarrow \theta$ in the sense of (o)-convergence such that $|y_n - y_o| \leq v_n$ for all n. By Sections 75 and 77 we have

$$|xy_n - xy_o| = |x(y_n - y_o)| \leq \beta|y_n - y_o| \leq \beta\, v_n$$

$$|y_n x - y_o x| = |(y_n - y_o)x| \leq \beta|y_n - y_o| \leq \beta\, v_n.$$

Therefore $xy_o = (o)\text{-}\lim\limits_{n\to\infty} xy_n$ and $y_o x = (o)\text{-}\lim\limits_{n\to\infty} y_n x$.

79. For any unitary element e in a partially ordered ring E we have for any bounded element $x \in E$ the equation

$$[e]x = ex = xe \tag{79.1}$$

Proof: Using the integral representation theorem by Freudenthal Section 74 we have

$$x = \int_{-\infty}^{\infty} \lambda \, d\, e_\lambda^x.$$

Hence $x = (o)\text{-}\lim \sum\limits_{-\infty}^{\infty} t_n (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x)$. By Section 76 we get

$$[e]x = (o)\text{-}\lim \sum_{-\infty}^{\infty} t_n\, ([e]e_{\lambda_n}^x - [e]e_{\lambda_{n-1}}^x) =$$

$$= (o)\text{-}\lim \sum_{-\infty}^{\infty} t_n (e^x_{\lambda_n} - e^x_{\lambda_{n-1}})e =$$

$$= (o)\text{-}\lim e \left(\sum_{-\infty}^{\infty} t_n (e^x_{\lambda_n} - e^x_{\lambda_{n-1}})\right) =$$

$$= (o)\text{-}\lim \left(\sum_{-\infty}^{\infty} t_n (e^x_{\lambda_n} - e^x_{\lambda_{n-1}})\right)e .$$

Formula (79.1) follows then from Section 78.

Chapter II

AN APPLICATION OF VECTOR LATTICE THEORY.

We now turn to the consideration of linear operators in Hilbert space and endeavour to complete the task described in the Introduction.

Let $\mathcal{B}$ denote the set of all bounded linear operators in a Hilbert space $\mathfrak{H}$. If $f_1, \ldots, f_n \in \mathfrak{H}$ and $\varepsilon > 0$, then $\mathcal{U}(A_0; f_1, \ldots, f_n; \varepsilon)$ is to denote the set of operators A in $\mathcal{B}$ satisfying the system of inequalities

$$||(A - A_0)f_j|| < \varepsilon \qquad (j = 1, 2, \ldots, n)$$

for a given A_0 in $\mathcal{B}$. The set $\mathcal{U}(A_0; f_1, \ldots, f_n; \varepsilon)$ is called a strong neighborhood of the operator A_0. The set of strong neighborhoods generates a topology of $\mathcal{B}$. The convergence of a sequence (A_n), $n = 1, 2, \ldots$, to an operator A in this topology means that for every $f \in \mathfrak{H}$ $||A_n f - Af|| \to 0$ as $n \to \infty$; A is called the strong limit of the sequence (A_n).

The operations αA, $A + B$, AB (in the product one factor is kept fixed) are continuous relative to the strong operator convergence. Passage from A to A* is not necessarily continuous with respect to the strong operator convergence.

We recall that an arbitrary linear operator A in a Hilbert space is said to permute with a bounded linear operator B in $\mathfrak{H}$ if $BA \subset AB$ holds, that is if $BA = AB$ on $\mathcal{D}_{BA}$. If both A and B are bounded linear operators, then we say that they permute if they commute. We shall use the symbol AvB to signify that the operator A permutes with the operator B.

A self-adjoint operator A is said to be positive, if $\langle Af,f\rangle \geq 0$ holds for all $f \in \mathcal{D}_A$; we signify this by writing $A \geq 0$. If A is a bounded self-adjoint operator, then A^2 is a self-adjoint positive operator.

We now take up several elementary propositions which will turn out to be useful at a later stage of the discussion.

I: If the bounded linear operators A_1 and A_2 are self-adjoint, if $A_2 \geq 0$ and $A_1 v A_2$, then the self-adjoint operator $A_1{}^2A_2$ is positive.

Proof: $\langle A_1{}^2A_2f, f\rangle = \langle A_1A_2f, A_1f\rangle = \langle A_2A_1f, A_1f\rangle \geq 0.$

II: If the bounded linear self-adjoint operators A and B are positive and permutable, then the self-adjoint operator AB = BA is also positive.

Proof: We may assume that $A \neq 0$ and hence $||A|| > 0$. We define a sequence (A_n), n = 1, 2, . . . , of self-adjoint operators by putting

$$A_1 = \frac{A}{||A||}, \quad A_{n+1} = A_n - A_n{}^2 \qquad (n = 1, 2, \ldots)$$

and we show that $0 \leq A_n \leq E$ for every n; here E denotes the identity operator. For n = 1 we have

$$0 \leq \langle A_1f, f\rangle = \langle Af, f\rangle / ||A|| \leq \langle f, f\rangle$$

and therefore $0 \leq A_1 \leq E$. Since

$$E - A_{n+1} = (E - A_n) + A_n{}^2$$

and

$$A_{n+1} = A_n - A_n^2 = A_n^2 - A_n^3 + A_n - 2A_n^2 + A_n^3$$

we have that

$$A_{n+1} = A_n^2 (E - A_n) + A_n(E - A_n)^2.$$

By the inductive assumption $A_n \geq 0$ and $E - A_n \geq 0$ so that by proposition I above we obtain $E - A_{n+1} \geq 0$ and $A_{n+1} \geq 0$ and therefore $0 \leq A_{n+1} \leq E$. From

$$A_1 = A_1^2 + A_2^2 + \dots + A_n^2 + A_{n+1} \text{ and } A_{n+1} \geq 0$$

it follows that

$$\sum_{j=1}^{n} \langle A_j f, A_j f \rangle = \sum_{j=1}^{n} \langle A_j^2 f, f \rangle = \langle A_1 f, f \rangle - \langle A_{n+1} f, f \rangle \leq \langle A_1 f, f \rangle$$

holds; this means that the series $\sum ||A_j f||^2$ converges and therefore $\lim ||A_j f|| = 0$. It follows that

$$\lim \left(\sum_{j=1}^{n} A_j^2 \right) f = \lim (A_1 f - A_{n+1} f) = A_1 f.$$

Since BvA by assumption, we see that BvA_j for every j; hence

$$\langle BAf, f \rangle \ / \ ||A|| = \langle BA_1 f, f \rangle = \lim \sum_{j=1}^{n} \langle BA_j^2 f, f \rangle =$$

$$= \lim \sum_{j=1}^{n} \langle A_j BA_j f, f \rangle = \lim \sum_{j=1}^{n} \langle BA_j f, A_j f \rangle \geq 0$$

and this shows that BA is positive.

III: If all operators of an increasing sequence (A_n) of bounded linear self-adjoint operators are mutually permutable and if there exists a self-adjoint bounded linear operator B such that $A_n v B$ and $A_n \leq B$ for all n, then the sequence (A_n) converges to a self-adjoint operator A satisfying $A_n v A$ and $A_n \leq A \leq B$. A similar statement holds for decreasing sequences.

Proof: We consider the decreasing sequence of positive operators $H_n = B - A_n$. They are mutually permutable; for $m > n$ the operators $H_m(H_n - H_m)$ and $(H_n - H_m)H_n$ are positive by proposition II above.

Hence for every $f \in \mathfrak{H}$:

$$\langle H_m^2 f, f \rangle \leq \langle H_m H_n f, f \rangle \leq \langle H_n^2 f, f \rangle ,$$

from which we infer that the non-increasing sequence of non-negative numbers $\langle H_m^2 f, f \rangle = ||H_m f||^2$ has a limit, dependent on f, and that $\langle H_m H_n f, f \rangle$ has the same limit for $m, n \to \infty$. Thus we have

$$\lim ||(H_m - H_n) f||^2 = \lim \langle (H_m - H_n)^2 f, f \rangle =$$

$$= \lim [\langle H_m^2 f, f \rangle - 2 \langle H_m H_n f, f \rangle + \langle H_n^2 f, f \rangle] = 0.$$

By the completeness of the space $\mathfrak{H}$ the sequence $H_n f$ converges for every $f \in \mathfrak{H}$ so that the same is true of the sequence $A_n f$. We define $Af = \lim A_n f$ and note that A is linear and self-adjoint. The relations $A_n \leq A_{n+1}$ and $A_n \leq B$, for all n, imply $A_n \leq A \leq B$ and the relations $A_n v A_j$ and $\lim A_j = A$ imply $A_n v A$.

IV: If A is a bounded linear self-adjoint positive operator, then there exists a unique bounded self-adjoint positive operator B such that $B^2 = A$.

<u>Proof</u>: Evidently we may assume that $A \leq E$, since we may replace A by $A/||A||$. We define a sequence of self-adjoint operators B_n as follows

$$B_0 = 0, \quad B_{n+1} = B_n + (1/2)(A - B_n^2) \qquad (n = 0, 1, 2, \ldots).$$

It is seen by induction that if a bounded self-adjoint operator C permutes with A, then it permutes with all B_n; hence we have $A v B_n$ (because AvA). Next, $B_m vA$, for all m, implies $B_m v B_n$ for all m and n.

Since

$$E - B_{n+1} = E - B_n - (1/2)(A - B_n^2) = (1/2)(E - B_n)^2 + (1/2)(E - A)$$

we have that $E - B_n \geq 0$ for all n. Since $B_n v B_{n-1}$ we have

$$B_{n+1} - B_n = B_n + (1/2)(A - B_n^2) - B_{n-1} - (1/2)(A - B_{n-1}^2) =$$

$$= B_n - B_{n-1} - (1/2)(B_n^2 - B_{n-1}^2) = (B_n - B_{n-1})(E - (1/2)(B_n + B_{n+1})) =$$

$$= (1/2)\,[(E - B_n) + (E - B_{n-1})]\,(B_n - B_{n-1})$$

which means that $B_{n+1} \geq B_n$. Indeed, for $n = 0$ we have

$$B_1 = (1/2)A \geq 0 = B_0.$$

By the inductive assumption $B_n - B_{n-1} \geq 0$, so that by proposition II above $B_{n+1} - B_n \geq 0$ holds as well because $B_{n+1} - B_n$ is the product of the permutable and positive operators $(1/2)\,[(E - B_n) + (E - B_{n-1})]$ and $B_n - B_{n-1}$. The sequence (B_n), satisfying therefore $0 = B_0 \leq B_1 \leq \cdots \leq E$, converges by force of proposition III to a positive operator B. Hence, taking $n \to \infty$ in $B_{n+1} = B_n + (1/2)(A - B_n^2)$, we get that

Proof: Evidently we may assume that $A \leq E$, since we may replace A by $A/||A||$. We define a sequence of self-adjoint operators B_n as follows

$$B_0 = 0, \quad B_{n+1} = B_n + (1/2)(A - B_n^2) \qquad (n = 0, 1, 2, \ldots).$$

It is seen by induction that if a bounded self-adjoint operator C permutes with A, then it permutes with all B_n; hence we have AvB_n (because AvA). Next, $B_m vA$, for all m, implies $B_m vB_n$ for all m and n.

Since

$$E - B_{n+1} = E - B_n - (1/2)(A - B_n^2) = (1/2)(E - B_n)^2 + (1/2)(E - A)$$

we have that $E - B_n \geq 0$ for all n. Since $B_n vB_{n-1}$ we have

$$B_{n+1} - B_n = B_n + (1/2)(A - B_n^2) - B_{n-1} - (1/2)(A - B_{n-1}^2) =$$

$$= B_n - B_{n-1} - (1/2)(B_n^2 - B_{n-1}^2) = (B_n - B_{n-1})(E - (1/2)(B_n + B_{n+1})) =$$

$$= (1/2)\ [(E - B_n) + (E - B_{n-1})]\ (B_n - B_{n-1})$$

which means that $B_{n+1} \geq B_n$. Indeed, for $n = 0$ we have

$$B_1 = (1/2)A \geq 0 = B_0.$$

By the inductive assumption $B_n - B_{n-1} \geq 0$, so that by proposition II above $B_{n+1} - B_n \geq 0$ holds as well because $B_{n+1} - B_n$ is the product of the permutable and positive operators $(1/2)\ [(E - B_n) + (E - B_{n-1})]$ and $B_n - B_{n-1}$. The sequence (B_n), satisfying therefore $0 = B_0 \leq B_1 \leq \cdots \leq E$, converges by force of proposition III to a positive operator B. Hence, taking $n \to \infty$ in $B_{n+1} = B_n + (1/2)(A - B_n^2)$, we get that

$$B = B + (1/2)(A - B^2) \quad \text{or} \quad B^2 = A.$$

To see that B is uniquely determined, we observe first that every bounded self-adjoint operator C, permutable with A, is also permutable with B since CvA implies CvB_n, and this in turn implies CvB. If therefore C is bounded and positive, and $C^2 = A$, we get that CvB holds because $AC = CA = C^3$. Let $B^{1/2}$ and $C^{1/2}$ be two positive operators satisfying $(B^{1/2})^2 = B$ and $(C^{1/2})^2 = C$ (their existence is assured by the first part of the proof), let $f \in \mathfrak{H}$ be arbitrary and $g = (B - C)f$. Then we get, since CvB,

$$||B^{1/2}g||^2 + ||C^{1/2}g||^2 = \langle Bg, g\rangle + \langle Cg, g\rangle =$$

$$= \langle (B + C)(B - C)f, g\rangle = \langle (B^2 - C^2)f, g\rangle = 0;$$

Hence $B^{1/2}g = C^{1/2}g = 0$ and therefore $Bg = 0$ and $Cg = 0$. But this implies

$$||(B - C)f||^2 = \langle (B - C)^2 f, f\rangle = \langle (B - C)g, f\rangle = 0$$

or $Bf = Cf$. Since f is arbitrary, we have that $B = C$.

V: Given an unbounded self-adjoint operator A in a Hilbert space $\mathfrak{H}$. Then the positive bounded self-adjoint operator $B = (E + A^2)^{-1}$ exists, A permutes with B, and $||B|| \leq 1$.

Proof: Let T denote a linear operator whose domain of definition is dense in $\mathfrak{H}$ (a self-adjoint operator satisfies this condition). We show that the operator $B = (E + T^*T)^{-1}$ and the operator $C = T(E + T^*T)^{-1}$ are defined everywhere and $||B|| \leq 1$ and $||C|| \leq 1$; moreover B is symmetric and positive.

The graph of T, denoted by G_T, is the set of all pairs (f, Tf), where f runs through the elements of $\mathfrak{D}_T$, in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$. In $\mathfrak{H} \oplus \mathfrak{H}$ we consider the following two mappings

$$U(f, g) = (g, f), \qquad V(f, g) = (g, -f).$$

We note that these mappings are unitary operators and that

$$UV = -VU \quad \text{and} \quad U^2 = -V^2 = I,$$

where I stands for the identity mapping on $\mathfrak{H} \oplus \mathfrak{H}$. With this notation the equation $\langle Tf, g \rangle = \langle f, g^* \rangle$, which defines the adjoint operator $T^*g = g^*$, can be put into the form

$$\langle V(f, Tf), (g, g^*) \rangle = 0.$$

This means that G_{T^*} is made up of those elements of $\mathfrak{H} \oplus \mathfrak{H}$ which are orthogonal to VG_T. G_{T^*} is therefore a subspace of $\mathfrak{H} \oplus \mathfrak{H}$, namely the orthogonal complement of the closure of the set VG_T (that is $\overline{VG_T}$). Since $\overline{VG_T} = V\,\overline{G_T}$, we can write

$$G_{T^*} = (\mathfrak{H} \oplus \mathfrak{H}) \ominus V\,\overline{G_T}.$$

The graph of a linear operator is closed if and only if the operator is closed. Thus, if T is a closed linear operator with dense domain of definition in $\mathfrak{H}$, then G_T and G_{T^*} are orthogonal complements of each other in $\mathfrak{H} \oplus \mathfrak{H}$. Let h be an arbitrary element of $\mathfrak{H}$. One then can decompose the element $(h, 0)$ of $\mathfrak{H} \oplus \mathfrak{H}$ into the sum of an element of G_T and an element of VG_{T^*} in a unique manner:

$$(h, 0) = (f, Tf) + (T^*g, -g).$$

By passing to the components, we can write the system of equations

$$h = f + T^*g \quad \text{and} \quad 0 = Tf - g$$

with unique solution f in $\mathfrak{D}_T$ and g in $\mathfrak{D}_{T^*}$. By putting

$$f = Bh \quad \text{and} \quad g = Ch$$

we define two mappings on $\mathfrak{G}$ into itself which are linear. We can write the system of equations in the form

$$E = B + T^*C \quad \text{and} \quad 0 = TB - C,$$

whence $C = TB$ and $E = B + T^*TB = (E + T^*T)B$. From the decomposition of (h, 0) we get that

$$||h||^2 = ||(h, 0)||^2 = ||(f, Tf)||^2 + ||(T^*g, -g)||^2 =$$
$$= ||f||^2 + ||Tf||^2 + ||T^*g||^2 + ||g||^2$$

so that

$$||Bh||^2 + ||Ch||^2 = ||f||^2 + ||g||^2 \leq ||h||^2.$$

Thus $||B|| \leq 1$ and $||C|| \leq 1$.

For an element u in the domain of definition of T^*T we have

$$\langle (E + T^*T)u, u \rangle = \langle u, u \rangle + \langle Tu, Tu \rangle \geq \langle u, u \rangle$$

and therefore $(E + T^*T)u = 0$ implies $u = 0$. Consequently, the inverse $(E + T^*T)^{-1}$ exists. It is clear that $(E + T^*T)^{-1}$ is defined everywhere and equals B because of the equation $E = B + T^*TB = (E + T^*T)B$ which appeared further above. Since

$$\langle Bu, v \rangle = \langle Bu, (E + T^*T)Bv \rangle = \langle Bu, Bv \rangle + \langle Bu, T^*TBv \rangle =$$
$$= \langle Bu, Bv \rangle + \langle T^*TBu, Bv \rangle = \langle (E + T^*T)Bu, Bv \rangle = \langle u, Bv \rangle$$

and

$$\langle Bu, u \rangle = \langle Bu, (E + T^*T)Bu \rangle = \langle Bu, Bu \rangle + \langle TBu, TBu \rangle \geq 0$$

it follows that B is symmetric and positive.

Finally, we consider the self-adjoint operator A and let $B = (E + A^2)^{-1}$ and $C = AB = A(E + A^2)^{-1}$. By what has been proved already, B is symmetric and bounded; in fact $0 \leq B \leq E$. Multiplying both members of the equation $A(E + A^2) = (E + A^2)A$ by B on the left and on the right and keeping in mind that $(E + A^2)B = E$ and $B(E + A^2) \subset E$ we get at once that $BA \subset AB$; this means that A permutes with B.

VI: Let $M_1, M_2, \ldots, M_j, \ldots$ be a sequence of subspaces of a Hilbert space $\mathfrak{H}$ which are mutually orthogonal and whose direct sum is the entire space $\mathfrak{H}$. For an arbitrary element f in $\mathfrak{H}$ let f_j denote the projection of f onto M_j. Let $A_1, A_2, \ldots, A_j, \ldots$ be a sequence of given linear operators such that the component of A_j in M_j is a bounded self-adjoint operator mapping M_j into itself. Under these conditions there exists one and only one self-adjoint operator A on $\mathfrak{H}$, in general not bounded, whose component on M_j is A_j for $j = 1, 2, \ldots$. $\mathfrak{D}_A$ consists of all elements f for which the series

$$\sum_{j=1}^{\infty} ||A_j f_j||^2$$

converges, and for these f we have

$$Af = \sum_{j=1}^{\infty} A_j f_j.$$

<u>Proof:</u> First we note that the mapping A defined above is linear. $\mathfrak{D}_A$ is dense in $\mathfrak{H}$ because it contains all elements of the form $f_1 + f_2 + \ldots + f_n$. In addition, A is symmetric since for all elements

$f, g \in \mathcal{D}_A$ we have

$$\langle Af, g\rangle = \sum \langle A_j f_j, g_j\rangle = \sum \langle f_j, A_j g_j\rangle = \langle f, Ag\rangle .$$

Let $g \in \mathcal{D}_{A^*}$, then for all $f \in \mathcal{D}_A$ we get $\langle Af, g\rangle = \langle f, A^*g\rangle$

and therefore $\sum_{j=1}^{\infty} \langle A_j f_j, g_j\rangle = \sum_{j=1}^{\infty} \langle f_j, (A^*g)_j\rangle$.

Choosing as f an arbitrary element of M_i, then $f_j = 0$ for $j \neq i$

and

$$\langle A_i f_i, g_i\rangle = \langle f_i, (A^*g)_i\rangle .$$

When A_i is assumed to be self-adjoint in M_i, then

$$(A^*g)_i = A_i g_i.$$

We obtain

$$\sum_{i=1}^{\infty} ||A_i g_i||^2 = \sum_{i=1}^{\infty} ||(A^*g)_i||^2 = ||A^*g||^2;$$

thus g also belongs to $\mathcal{D}_A$ and we have

$$Ag = \sum_{i=1}^{\infty} A_i g_i = \sum_{i=1}^{\infty} (A^*g)_i = A^*g.$$

This shows that $A^* \subseteq A$. But A is symmetric, hence $A^* = A$. It remains to verify that A is unique. Let A' be any self-adjoint operator which has component A_j in M_j. Since A' is closed, it is necessarily defined for all elements f for which the series $\sum_{j=1}^{\infty} A'f_j$ converges; the sum of this series will also be equal to A'f. Since $A'f_j = A_j f_j$, and since the convergence of a series of orthogonal elements is equivalent with the convergence of the series of squares of the norms, the set of these elements f coincides with $\mathfrak{D}_A$ and for these f one has $A'f = Af$; thus $A' \supseteq A$. But A is selfadjoint and therefore maximally symmetric; hence $A' = A$.

Let $\mathfrak{T}$ denote the set of all linear bounded self-adjoint operators in a Hilbert space $\mathfrak{H}$. If $A, B \in \mathfrak{T}$, then the product $AB \in \mathfrak{T}$ if and only if $AB = BA$. Let $A \in \mathfrak{T}$ and α be a scalar, then $\alpha A \in \mathfrak{T}$ if and only if α is a real number. Motivated by these observations, we call a set $\mathfrak{A}$ of bounded linear operators in $\mathfrak{H}$ a <u>ring</u>, if for any $A, B \in \mathfrak{A}$ and any real number α the operators $A + B$, AB, and αA also belong to $\mathfrak{A}$.

In the set $\mathfrak{T}$ we can introduce a partial order by writing $A \geq 0$ if and only if A is positive; $A > 0$ means that $\langle Af, f \rangle \geq 0$ for all $f \in \mathfrak{H}$ and $A \neq 0$. R.V. Kadison [4] showed that under this ordering the set $\mathfrak{T}$ does not form a lattice (since $A \vee B$ exists in $\mathfrak{T}$ if and only if $A \leq B$ or $B \leq A$).

We give some definitions before going on with the considerations.

A partially ordered set S is said to be <u>directed upward</u> (downward), if for any two elements $a, b \in S$ there exists an element $c \in S$ such that $c \geq a$ and $c \geq b$ ($c \leq a$ and $c \leq b$). An upward directed set S is called a <u>path</u> if for any $a \in S$ there is an element $b \in S$ such that $b > a$ (i.e., S has no greatest element). By a path of operators will be meant a family of operators $(A_t)_{t \in T}$, where T is a path. A path of self-adjoint operators will be called increasing (decreasing) if and only if $s \leq t$ implies $A_s \leq A_t$ ($A_s \geq A_t$). A path $(A_t)_{t \in T}$ of bounded operators is said to <u>converge</u> to the operator A in the sense of the strong operator topology, if given any neighborhood U of A in the topological space $\mathcal{B}$ (see the first page of this chapter), then A_t is eventually in U, that is, if there exists an index t_o such that $A_t \in U$ for each $t \geq t_o$. A subset of the set $\mathcal{T}$ is said to be strongly closed, if it is closed in the sense of the strong operator topology, that is the limit of any strongly convergent path of operators of the subset also belongs to the subset. If $A_t \to A$ in the sense of the strong operator topology, A_t and A belong to $\mathcal{T}$ and if A_t permutes with $B \in \mathcal{T}$ for each $t \in T$, then A permutes with B as well. Indeed, for any $f \in \mathcal{D}_B$ we have $BA_tf = A_tBf \to ABf$. But $A_tf \to Af$ and, since B is closed, $Af \in \mathcal{D}_B$ and $BAf = ABf$.

The proof of the following proposition is completely analogous to the proof of proposition III further above.

VII : Let $(A_t)_{t \in T}$ be an increasing path of pairwise permuting self-adjoint operators such that $A_t \leq B$ for all $t \in T$ and $B \in \mathcal{T}$. Then $\sup A_t = A$ exists in $\mathcal{T}$ and A is the strong operator limit of $(A_t)_{t \in T}$.

<u>Proof:</u> Let $H_t = B_1 - A_t$ $(t \in T)$, where $B_1 = ||B||E$ and E is the identity operator on $\mathcal{G}$. It is clear that $H_t \geq 0$ and that the operators H_t form a decreasing path. If $s \geq t$ for $s, t \in T$, then $H_s \leq H_t$ and therefore $(H_t - H_s)H_t$ and $H_s(H_t - H_s)$ are positive by proposition II above. Just as in the proof of proposition III we obtain

$0 \leq \langle H_s^2 f, f \rangle \leq \langle H_s H_t f, f \rangle \leq \langle H_t^2 f, f \rangle$ for any $f \in \mathcal{G}$.

Thus $\lim \langle H_t^2 f, f\rangle = \lim \langle H_s H_t f, f\rangle$. Therefore

$$\lim ||H_t f - H_s f||^2 = \lim \langle H_t f - H_s f, H_t f - H_s f\rangle =$$

$$= \lim [\langle H_t^2 f, f\rangle - 2 \langle H_s H_t f, f\rangle + \langle H_s^2 f, f\rangle] = 0;$$

this means that for each $f \in \mathcal{H}$ the strong limit of $H_t f$ exists and therefore the strong limit of $A_t f$ exists. In other words, there exists a bounded linear operator A such that $A_t f \to Af$ for all $f \in \mathcal{H}$. The operator A is symmetric and therefore $A \in \mathcal{T}$. Since the path $(A_t)_{t \in T}$ is increasing, for any $f \in \mathcal{H}$ we have $\langle A_t f, f\rangle \leq \langle Af, f\rangle \leq$ $\leq \langle Bf, f\rangle$. Hence $A_t \leq A$ for all $t \in T$ and $A \leq B$. Since B can be taken as any upper bound of the set $\{A_t : t \in T\}$ we have that

$A = \sup A_t$.

VIII: Let $\mathcal{A}$ be any strongly closed ring of bounded self-adjoint operators, then $C \vee 0$ exists for any $C \in \mathcal{A}$.

<u>Proof</u>: Let $C \in \mathcal{A}$. Since $\mathcal{A}$ is a ring, $C^2 \in \mathcal{A}$. It is clear that $C^2 \geq 0$. From the proof of proposition IV above concerning the existence and uniqueness of the square root of a positive bounded self-adjoint linear operator we know that this square root can be represented as the strong limit of a certain sequence of polynomials in terms of the "radical" operator without free term. We recall that the sequence of approximating polynomials for the square root of the operator A was of the form:

$$B_0 = 0, \quad B_{n+1} = B_n + (1/2)(A - B_n^2),$$

where we assumed that $||A|| \leq 1$. Since $C^2 \in \mathcal{A}$, then any polynomial in terms of C^2 without free term is also contained in $\mathfrak{A}$, and since $\mathfrak{A}$ is strongly closed, $B = (C^2)^{1/2} \in \mathfrak{A}$. We put

$$A = (1/2)(B + C) \qquad (A \in \mathfrak{A})$$

and show that $A = C \vee 0$.

Let $D = A - C = (1/2)(B - C) \in \mathfrak{A}$. Let $\mathfrak{M}_1$ denote the null-space of the operator D, that is the subspace consisting of all f such that $Df = 0$. Let $\mathfrak{M}_2$ denote the orthogonal complement of $\mathfrak{M}_1$ in $\mathfrak{H}$. It is easy to see that $\mathfrak{M}_1$ is invariant for any operator $Q \in \mathfrak{A}$; indeed, if $f \in \mathfrak{M}_1$, then

$$D(Qf) = QDf = Q0 = 0$$

(because $\mathfrak{A}$ is a commutative ring) and therefore $Qf \in \mathfrak{M}_1$. From the invariance of $\mathfrak{M}_1$ also follows the invariance of $\mathfrak{M}_2$ for any operator $Q \in \mathfrak{A}$. From the definition of $\mathfrak{M}_1$ we have that $C = B = A$ on $\mathfrak{M}_1$. Moreover,

$$DA = (1/4)(B - C)(B + C) = (1/4)(B^2 - C^2) = 0.$$

Thus $DAf = 0$ for any $f \in \mathfrak{H}$ and therefore $Af \in \mathfrak{M}_1$ for all $f \in \mathfrak{H}$. On the other hand, if $f \in \mathfrak{M}_2$, then $Af \in \mathfrak{M}_2$ because of the invariance of the subspace $\mathfrak{M}_2$. Thus $A = 0$ on $\mathfrak{M}_2$. But $D = B - A$ and therefore $B = D$ on $\mathfrak{M}_2$.

Any $f \in \mathfrak{H}$ can be represented in the form $f = f_1 + f_2$, where $f_k \in \mathfrak{M}_k$ $(k = 1, 2)$. By the invariance of the subspace $\mathfrak{M}_k$ $(k = 1, 2)$ we have for any operator $Q \in \mathfrak{A}$:

$$\langle Qf, f \rangle = \langle Qf_1, f_1 \rangle + \langle Qf_2, f_2 \rangle . \tag{1}$$

But $Af_2 = Df_1 = 0$. We therefore get

$$\langle Af, f \rangle = \langle Af_1, f_1 \rangle = \langle Bf_1, f_1 \rangle \tag{2}$$

and

$$\langle Df, f \rangle = \langle Df_2, f_2 \rangle = \langle Bf_2, f_2 \rangle .$$

By the definition of square root $B \geq 0$; thus $A \geq 0$ and $D \geq 0$ implying $A \geq C$.

Suppose now that H is an arbitrary operator of $\mathfrak{A}$ satisfying $H \geq 0$ and $H \geq C$. Then for any $f \in \mathfrak{H}$ we have by virtue of the fact that $C = B$ on $\mathfrak{M}_1$ and by force of the equations (1) and (2) that $\langle Af, f \rangle = \langle Bf_1, f_1 \rangle = \langle Cf_1, f_1 \rangle \leq \langle Hf_1, f_1 \rangle \leq \langle Hf_1, f_1 \rangle + \langle Hf_2, f_2 \rangle = \langle Hf, f \rangle$. This means that $A \leq H$ and consequently $A = C \vee 0$.

<u>Remark</u>: It is easy to see that $\mathfrak{A}$ is a vector lattice; we only need to verify that $A, B \in \mathfrak{A}$ implies the existence of $A \vee B$. The foregoing proposition VIII gives that $(A - B) \vee 0$ exists. But it can be seen that $[(A - B) \vee 0] + B = A \vee B$.

IX: Any strongly closed ring $\mathfrak{A}$ of bounded self-adjoint linear operators on a Hilbert space forms a complete vector lattice.

<u>Proof</u>: By the foregoing remark it remains to show that an arbitrary subset $\mathcal{C}$ of $\mathfrak{A}$ which is bounded from above has a supremum. We adjoin to $\mathcal{C}$ the suprema of all its finite subsets. We can then regard $\mathcal{C}$ as an upward directed set. If in $\mathcal{C}$ there is a largest operator, then it will be the supremum of the set $\mathcal{C}$. If however there is no such operator in

$\mathcal{C}$, then the operators making up $\mathcal{C}$ form an increasing path and by proposition VII above this path has a strong limit A, where $A = \sup \mathcal{C}$ in $\mathcal{T}$. Since the ring $\mathcal{A}$ is strongly closed, $A \in \mathcal{A}$, and $A = \sup \mathcal{C}$ in the lattice $\mathcal{A}$ as well. This completes the proof.

Let $C^+ = C \vee 0$, $C^- = (-C) \vee 0$ and $|C| = C^+ + C^-$. In the proof of proposition VIII we showed that $C^+ = A = (1/2)(B + C)$. Thus $C^- = C^+ - C = (1/2)(B - C) = D$, $|C| = C^+ + C^- = B$, $|C| = (C^2)^{1/2}$, $C^- = 0$ on $\mathfrak{M}_1$ and $C^+ = 0$ on $\mathfrak{M}_2$; the space $\mathfrak{H}$ decomposes into the orthogonal subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Each operator of $\mathcal{A}$ is invariant on $\mathfrak{M}_1$ and on $\mathfrak{M}_2$. This means that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ reduce $\mathcal{A}$. $C = |C| = C^+$ on $\mathfrak{M}_1$ and $C = -|C| = -C^-$ on $\mathfrak{M}_2$. Therefore for any $f \in \mathfrak{H}$ we get $Cf = C^+f_1 - C^-f_2$, where $f_k \in \mathfrak{M}_k$ for $k = 1, 2$ ($f_k = \mathrm{pr}_{\mathfrak{M}_k} f$; $k = 1, 2$) and $C^+f_1 \in \mathfrak{M}_1$ and $C^-f_2 \in \mathfrak{M}_2$. On the other hand $|C| f = C^+f_1 + C^-f_2$ and therefore for any $f \in \mathfrak{H}$

$$\| \, |C| f \, \| = \| Cf \|. \tag{3}$$

By (3) we get that the null-spaces of the operators A and $|A|$ coincide and, moreover, that for any $A \in \mathcal{A}$

$$\| A \| = \| \, |A| \, \|. \tag{4}$$

X: Let $A_j \in \mathcal{A}$ $(j = 1, 2, \ldots, n)$ and $A = A_1 \vee A_2 \vee \ldots \vee A_n$. Then $\mathfrak{H}$ can be decomposed into mutually orthogonal subspaces $\mathfrak{H}_j$ reducing $\mathcal{A}$, such that $A = A_j$ on $\mathfrak{H}_j$ for each j.

Proof: First we verify the statement for $n = 2$. Then

$$A = A_2 + (A_1 - A_2)^+ = A_2 + (1/2)[(A_1 - A_2) + |A_1 - A_2|] =$$
$$= (1/2)[(A_1 + A_2) + |A_1 - A_2|].$$

By what has been said above, $\mathfrak{H}$ decomposes into the orthogonal subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, reducing $\mathfrak{A}$, such that $|A_1 - A_2| = A_1 - A_2$ on $\mathfrak{H}_1$ and $|A_1 - A_2| = A_2 - A_1$ on $\mathfrak{H}_2$. Then $A = A_1$ on $\mathfrak{H}_1$ and $A = A_2$ on $\mathfrak{H}_2$.

We now proceed by induction and suppose that the proposition is true for n operators. Let $A = A_1 \vee \dots \vee A_n \vee A_{n+1}$ $(A_i \in \mathfrak{A})$. We put $B = A_1 \vee \dots \vee A_n$ so that $A = B \vee A_{n+1}$. As in the case of two operators, we decompose $\mathfrak{H}$ into two subspaces $\mathfrak{L}$ and $\mathfrak{M}$, reducing $\mathfrak{A}$, such that

$$A = \begin{cases} B \text{ on } \mathfrak{L} \\ A_{n+1} \text{ on } \mathfrak{M}. \end{cases}$$

For each operator $Q \in \mathfrak{A}$, we signify by Q' its component in the subspace $\mathfrak{L}$. The operators Q' also form a strongly closed ring which we denote by $\mathfrak{A}'$ and by proposition IX $\mathfrak{A}'$ forms a complete vector lattice. The modulus of each operator Q' of $\mathfrak{A}'$ is defined by $((Q')^2)^{1/2}$, that is the component of the operator $|Q|$ in the subspace $\mathfrak{L}$. Then $(Q')^+$ and $(Q')^-$ are the components of Q^+ and Q^-, respectively. We see that the analogous statement for bounds of finite sets of operators of $\mathfrak{A}'$ holds as well in $\mathfrak{A}'$ we have the equality $B' = A_1' \vee \dots \vee A_n'$.

By the inductive assumption there exists a decomposition of the subspace

$\mathcal{L}$ into mutually orthogonal subspaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$, reducing $\mathfrak{A}'$, and therefore also reducing $\mathfrak{A}$, such that $B' = A_j$ on $\mathfrak{H}_j$. Thus we have that $B = A_j$ on $\mathfrak{H}_j$. If we put $\mathfrak{H}_{n+1} = \mathfrak{M}$, then we see that the subspaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ and $\mathfrak{H}_{n+1}$ form the required decomposition.

Remark: A similar proposition holds for infimum.

XI: In order that the operators $A, B \in \mathfrak{A}$ be disjunct as elements of the complete vector lattice $\mathfrak{A}$, it is necessary and sufficient that the space $\mathfrak{H}$ decomposes into orthogonal subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $A = 0$ on $\mathfrak{H}_1$ and $B = 0$ on $\mathfrak{H}_2$.

Proof: Since the null-space of any operator A of $\mathfrak{A}$ coincides with the null-space of the modulus $|A|$, it is enough to consider the case when $A, B \geq 0$.

If $A \wedge B = 0$, then by the above remark the condition is seen to be necessary.

Conversely, suppose that the condition holds. We put $C = A \wedge B$. We observe that $U, V \in \mathfrak{A}$ and $0 \leq U \leq V$ imply $||Uf|| \leq ||Vf||$ for any $f \in \mathfrak{H}$. Using this fact, we see that from $0 \leq C \leq A$ we get at once $C = 0$ on $\mathfrak{H}_1$. In the same manner we show that $C = 0$ on $\mathfrak{H}_2$. Thus $Cf = 0$ for all $f \in \mathfrak{H}$ and therefore $A \wedge B = 0$.

XII: If the strongly closed ring $\mathfrak{A}$ contains the identity operator E of $\mathfrak{H}$, then one can take E as unit of the complete vector lattice $\mathfrak{A}$. In this case the basis $\mathcal{B}(\mathfrak{A})$ of the complete vector lattice will consist of all projection operators, contained in $\mathfrak{A}$, and $\mathfrak{A}$ itself will be a complete vector lattice of bounded elements.

<u>Proof:</u> First we verify that E acts as unit in $\mathfrak{A}$. Let $A \in \mathfrak{A}$, $A \geq 0$ and $A \wedge E = 0$. By proposition XI above there exist mutually orthocomplemented subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $A = 0$ on $\mathfrak{H}_1$ and $E = 0$ on $\mathfrak{H}_2$. But the latter means that $\mathfrak{H}_2$ contains the zero element of $\mathfrak{H}$ only. Hence $\mathfrak{H}_1 = \mathfrak{H}$ and $Af = 0$ on all of $\mathfrak{H}$.

Let E be taken as unit in the complete vector lattice $\mathfrak{A}$. If A is an arbitrary operator in $\mathfrak{A}$ and $m = \inf\limits_{||f||=1} \langle Af, f \rangle$ and $M = \sup\limits_{||f||=1} \langle Af, f \rangle$, then for any $f \in \mathfrak{H}$ we have

$$\langle mEf, f \rangle = m \langle f, f \rangle \leq \langle Af, f \rangle \leq M \langle f, f \rangle = \langle MEf, f \rangle .$$

Thus $mE \leq A \leq ME$ which means that A is a bounded element of the complete vector lattice $\mathfrak{A}$.

Let the operator $P \in \mathfrak{A}$ project the space $\mathfrak{H}$ onto the subspace $\mathfrak{L}$. Then $E - P$ projects $\mathfrak{H}$ onto the subspace $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{L}$. Hence, by proposition XI above, we get that $P \wedge (E - P) = 0$; this means that P belongs to the basis $\mathfrak{B}(\mathfrak{A})$.

Conversely, let P be an arbitrary unitary element of $\mathfrak{A}$, $\mathfrak{H}_1$ the null-space of P and $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$. Since $P \wedge (E - P) = 0$, then by proposition XI above we get that $E - P = 0$ on $\mathfrak{H}_2$, or $P = E$ on $\mathfrak{H}_2$, and $P = 0$ on $\mathfrak{H}_1$. This means that P is the projection operator of $\mathfrak{H}$ onto $\mathfrak{H}_2$. The proof of the proposition is finished.

Let A denote an arbitrary (not necessarily bounded) self-adjoint linear operator in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{E}$ be the set of all bounded self-adjoint linear operators in $\mathfrak{H}$ which permute with A. The set $\mathfrak{E}$ is strongly closed, but might not be a ring because the operators permutable with A might not commute with each other. Select from $\mathfrak{E}$ the subset $\mathfrak{A}$, consisting of all operators contained in $\mathfrak{E}$ and permuting with any operator from $\mathfrak{E}$. It is clear that $\mathfrak{A}$ is not empty because it contains the identity operator E. $\mathfrak{A}$ is a ring. In fact, it is enough to verify that multiplication does not lead outside of $\mathfrak{A}$. If $U, V \in \mathfrak{A}$, then $U, V \in \mathfrak{E}$ and therefore $UV = VU$ and this product belongs to $\mathfrak{A}$. $\mathfrak{A}$ is strongly closed. By proposition IX $\mathfrak{A}$ is a complete vector lattice. We take E as unit in this vector lattice (see proposition XII).

If the operator A is bounded, then $A \in \mathfrak{A}$ and by the integral representation theorem of H. Freudenthal we have

$$A = \int_{-\infty}^{\infty} t \, d\, E_t,$$

where $E_t = e_t^A$, that is (E_t), $-\infty < t < \infty$, is the resolution of the operator A. By proposition XII the resolution (E_t) consists of projection operators. From Section 71, Ch. I we get the basic properties of the resolution. In particular, the left-continuity of the resolution means that

$$E_t = \lim_{s \to t-0} E_s \tag{5}$$

is in the sense of strong convergence. Indeed, by proposition VII we have that $\lim_{s<t} E_s$ in $\mathfrak{T}$ is the strong limit of the path (E_s), and since all $E_s \in \mathfrak{A}$, this limit also belongs to $\mathfrak{A}$, because the ring $\mathfrak{A}$ is strongly closed. This limit has to be the supremum of the set $\{E_s : s < t\}$ in $\mathfrak{A}$: by Section 71, Ch. I it coincides with E_t and we have formula (5). The resulting representation of the operator A forms the content of the spectral theorem for bounded self-adjoint operators.

We now suppose that A is unbounded. By proposition V the positive bounded self-adjoint linear operator $B = (E + A^2)^{-1}$ exists, A permutes with B, and $||B|| \leq 1$. We show that $B \in \mathfrak{A}$. Let C be any operator in $\mathcal{E}$. Then $CA^2 \subset A^2C$ and therefore $C(E + A^2) \subset (E + A^2)C$. From the latter we obtain $BC(E + A^2)B \subset B(E + A^2)CB$. But $(E + A^2)B = E$ and $B(E + A^2) \subset E$. Thus $BC \subset CB$. Since $\mathcal{D}_{BC} = \mathfrak{H}$, we get $BC = CB$. Thus $B \in \mathcal{E}$ and B commutes with any $C \in \mathcal{E}$; this means that $B \in \mathfrak{A}$.

Let (E_t) be the resolution of the operator B. Since $0 \leq B \leq E$, we have that $E_t = 0$ for $t \leq 0$ and $E_t = E$ for $t > 1$. We verify that the resolution (E_t) is continuous at $t = 0$ and to the right. Let $E_0 = \inf_{t>0} E_t$. Then E_0 is a projection operator and by formula (70.1) (see Ch. I) it follows that $0 \leq [E_0]B \leq tE_0$ for $t > 0$; but this is true for any $t > 0$ and therefore $[E_0]B = 0$. By Section 79, Ch. I the projection onto a component is equivalent with multiplication by the unitary element generating this component. Thus $E_0B = 0$, that is $B = 0$ on the subspace $\mathfrak{L}$ onto which the operator E_0 projects $\mathfrak{H}$. Since, however, the operator B has an inverse, $\mathfrak{L}$ has to consist of the zero element of $\mathfrak{H}$ only and therefore $E_0 = 0$.

Let $P_0 = E - E_1$ and $P_k = E_{1/k} - E_{1/(k+1)}$ for $k = 1, 2, \ldots$. By the continuity of the resolution (E_t) at $t = 0$ we have

$$\underset{k=0}{\overset{\infty}{S}} P_k = \sum_{k=0}^{\infty} P_k = E. \tag{6}$$

We denote $\mathfrak{A}_k$ $(k = 0, 1, 2, \ldots)$ the component of the complete vector lattice $\mathfrak{A}$ generated by the operator P_k and by $\mathfrak{H}_k$ the subspace of the Hilbert space $\mathfrak{H}$, on which the projection operator P_k is realized. By properties of the resolution, the operators P_k $(k = 0, 1, 2, \ldots)$ are pairwise disjunct; from proposition XI it follows that the subspaces $\mathfrak{H}_k$ $(k = 0, 1, 2, \ldots)$ are pairwise orthogonal. Formula (6) implies that the system $(\mathfrak{H}_k)$ is complete in $\mathfrak{H}$ and that the components $\mathfrak{A}_k$ generate a decomposition of the complete vector lattice $\mathfrak{A}$ (see Section 59, Ch. I). For, if we suppose that the system of subspaces $\mathfrak{H}_k$, generating the subspace $\mathfrak{L}$, were different from $\mathfrak{H}$ and P be the projection operator onto $\mathfrak{L}$, then by proposition VII it would follow that

$$P = \sum_{k=0}^{\infty} P_k$$

in the sense of strong convergence and thus $P \in \mathfrak{A}$ and $P = \sup P_k$ in $\mathfrak{A}$. But this would contradict formula (6).

Since $P_k \in \mathfrak{A}$, all subspaces $\mathfrak{G}_k$ are invariant relative to the operator A and relative to any operator of $\mathfrak{A}$.

The range of the operator B coincides with the domain of definition of the operator A^2 and $\mathfrak{D}_{A^2} \subset \mathfrak{D}_A$. We have $BP_k = [P_k]B \geq$ $\geq (1/(k + 1))P_k$; therefore the operator BP_k has on $\mathfrak{G}_k$ a positive infimum and thus maps $\mathfrak{G}_k$ onto $\mathfrak{G}_k$ which means that $\mathfrak{G}_k \subset \mathfrak{D}_A$. Thus the operator A is symmetric on $\mathfrak{G}_k$, mapping all of $\mathfrak{G}_k$ into itself. Hence the operator A is bounded and self-adjoint on $\mathfrak{G}_k$ and the operator $A_k = AP_k$ is bounded and self-adjoint on all of $\mathfrak{G}$.

It is clear that all $A_k \in \mathcal{E}$. We verify that $A_k \in \mathfrak{A}$. Let $C \in \mathcal{E}$, then taking into account that $P_k \in \mathfrak{A}$, we have $CA_k = CAP_k$ $\subset ACP_k = AP_kC = A_kC$. This means that A_k commutes with any $C \in \mathcal{E}$ and therefore $A_k \in \mathfrak{A}$.

The operators A_k are pairwise disjunct as elements of the complete vector lattice $\mathfrak{A}$, and A_k is disjunct from P_j for $k \neq j$. From the latter we get that $A_k \in \mathfrak{A}_k$ for each k.

We form the union X of the complete vector lattices $\mathfrak{A}_k$. Each element $x \in X$ has the form

$$x = \mathop{S}_{k=0}^{\infty} Q_k,$$

where $Q_k \in \mathfrak{A}_k$. From the set of bounded self-adjoint operators Q_k we construct the self-adjoint operator Q in the space $\mathfrak{G}$, coinciding on each $\mathcal{G}_k$ with the operator Q_k (see proposition VI). Let Y be the set of all self-adjoint operators obtained by this method. We have $\mathfrak{A} \subset Y$. Moreover, if $x = S\ A_k$, then the corresponding operator coincides with A and therefore $A \in Y$.

By this construction one establishes a one-to-one correspondence between the elements of the complete vector lattice X and the operators of Y, and therefore, with the usual definitions for the algebraic operations and ordering. Y is turned into a complete vector lattice. By Section 62, Ch. I we note that E can be taken as unit in Y and then by Section 63, Ch. I we get that the two bases $\mathcal{L}(\mathfrak{A})$ and $\mathcal{L}(Y)$ coincide.

Thus we have an imbedding of the operator A into a certain complete vector lattice with unit, the basis of which consists of projections. Then the spectral resolution of the operator A is gotten in the same way as for bounded self-adjoint operators further above. From the very method for getting the spectral resolution it directly follows that the spectral family of a self-adjoint operator A consists of projection operators E_t, permuting with any operator $C \in \mathfrak{T}$ which permutes with A.

APPENDIX

Stone's Representation Theorem: Every Boolean ring with unit is isomorphic with the Boolean ring of all open and closed subsets of a totally disconnected compact Hausdorff space.

For a proof consult item # 2 in the Bibliography.

Addendum to Section 75 of Chapter I:

A vector lattice E is called a generalized partially ordered commutative ring with a multiplicative unit if for certain pairs of elements $x, y \in E$ there is defined a product $xy \in E$ such that

1) if xy exists, then yx exists and $xy = yx$;
2) if xy, $(xy)z$ and yz exist, then $x(yz)$ exists and $(xy)z = x(yz)$;
3) if xz and yz exist, then $(x + y)z$ exists and $(x+y)z = xz+yz$;
4) in E there is an element i, called multiplicative unit, such that the product xi exists for any $x \in E$ and $xi = x$;
5) if xy exists and $|x_1| \leq |x|$ and $|y_1| \leq |y|$, where $x, x_1, y, y_1 \in E$, then x_1y_1 exists.
6) if $x, y \geq \theta$ and xy exist, then $xy \geq \theta$;
7) if xy exists, then $(\lambda x)y$ exists for any real λ and $(\lambda x)y = \lambda(xy)$.

In any σ-complete vector lattice E with unit 1 one can define a multiplication such that E becomes a generalized partially ordered commutative ring in which 1 coincides with i (multiplicative unit).

Let E be a σ-complete vector lattice of bounded elements and 1 be a unit in E. Suppose that F is a function mapping $E \times E$ into E as follows

1) $F(x + y, z) = F(x, z) + F(y, z)$, $F(z, x + y) = F(z, x) + F(z, y)$;

2) $F(x, 1) = F(1, x) = x$;

3) if $xy \geq \theta$, then $F(x, y) \geq \theta$.

Then $F(x, y) = xy$, where xy is the product defined further above.

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List of Special Symbols

(The enumeration refers to the section number in Chapter I)

x^+	positive part of x, 5
x^-	negative part of x, 5
$\|x\|$	modulus of x, 5
A^+	positive part of a set A, 5
A^-	negative part of a set A, 5
$\|A\|$	modulus of a set A, 5
$x \; d \; y$	disjunctness of two elements, 10
$X_1 \; d \; X_2$	disjunctness of two sets, 10
(o)-lim	order-convergent limit, 18
(r)-lim	regulated convergence limit, 23
1	unit of a vector lattice, 24
θ	zero of a vector lattice, 3
$\mathcal{L}(E)$	basis of a vector lattice E, 25
e'	complement of a basis element e, 26
X^d	disjunct complement of X, 37
$[E_1]x$	projection of x onto the component E_1, 39
X^{dd}	smallest component containing X, 44
E_u	component generated by element u, 44
$[u]x$	projection of x onto component generated by u, 44
$\underset{t}{S}\, x_t$	lattice theoretic union of elements, 51

$\underset{t}{S} E_t$ lattice theoretic union of sets, 61

e_x trace of x, 68

e_λ^x trace of $(\lambda 1 - x)^+$, 70

$\{e_\lambda^x\}_{\lambda \in (-\infty,\infty)}$ resolution of x, 70

$(o)\text{-}\lim_{p\to\infty} \sum_{n=0}^{p} x_n$ order convergent limit of a series, 73