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MODELLING AND STABILIZATION OF FLEXIBLE SPACECRAFT

by

Saroj Kanti Biswas

A thesis
presented to the University of Ottawa
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
Electrical Engineering,
Department of Electrical Engineering
Faculty of Science and Engineering.

OTTAWA, Ontario, 1985



UNIVERSITÉ D'OTTAWA
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ABSTRACT

A methodology has been developed for rigorous modelling of spacecraft having flexible appendages. It is shown that the complete dynamics of the system could be described by a coupled system of ordinary and partial differential equations. These equations indicate a very strong and intricate nature of interaction between the rigid and the flexible parts of the spacecraft. Stabilization of the system has been proved using Lyapunov's approach. Some simple and practically implementable feedback controls are suggested for stabilization.

Stability of flexible structures in the presence of distributed white noise has been investigated. It is shown that by application of velocity feedback a flexible beam perturbed by distributed white noise could be stabilized in the mean square sense and almost sure sense with respect to a ball in the energy space. An optimal damping coefficient has been deduced for obtaining the maximum decay rate and the minimum size of the attractor. It is also shown that a flexible spacecraft becomes unstable when subjected to random disturbances. Stability of system vibrations could be achieved by simple feedback controls.

Techniques of optimal control theory have been utilized in identifying the parameters of vibrating systems. Necessary conditions of optimality have been derived for identification of parameters in systems governed by second order evolution equations. These results are utilized to determine the parameters of a flexible beam.



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Chapter I
INTRODUCTION

With the ever increasing demand for high speed global communication links, the number of satellites in earth orbit is steadily increasing. However, the capacity of the geosynchronous orbit is not unlimited. It has been suggested that, in order to reduce interference, geostationary satellites must be at least four degrees apart from each other, so that there could be at most 90 satellites in the orbit. A possible solution to reduce congestion in the geosynchronous orbit is to deploy very large satellites with multiple beam antenna so that one satellite would meet the communication requirements of a large area on the earth.

In addition, large geosynchronous satellites are also required to harness the unlimited supply of energy available from the sun, to build space observatory or spacelab, to set up permanent orbital base for deep space exploration etc. In general, these spacecraft would be very large, hence structurally flexible, and highly complex structures. These are the so-called third generation spacecraft (the flexible spacecraft), and are expected to be deployed in space in the early nineties.

Typically these spacecraft would consist of a rigid main body along with several flexible appendages such as long beams, large solar panels and antennas etc. A schematic diagram of a communication satellite proposed by the Government of Canada is shown in Fig. 1.1. In Fig. 1.2 we show another proposed model of a NASA spacecraft.

The problem of modelling and control of these large flexible spacecraft has been a subject of considerable research over the last few years. Flexibility of various components of the spacecraft introduces many unforeseen complexities in the process of system modelling and controller design. To ensure satisfactory performance of the spacecraft it has become necessary to take into account distributed nature of the flexible members. The subject of this thesis concerns with the problem of developing a methodology for rigorous modelling of these large flexible spacecraft, taking into account the flexibility of their components, and design of stabilizing control schemes.

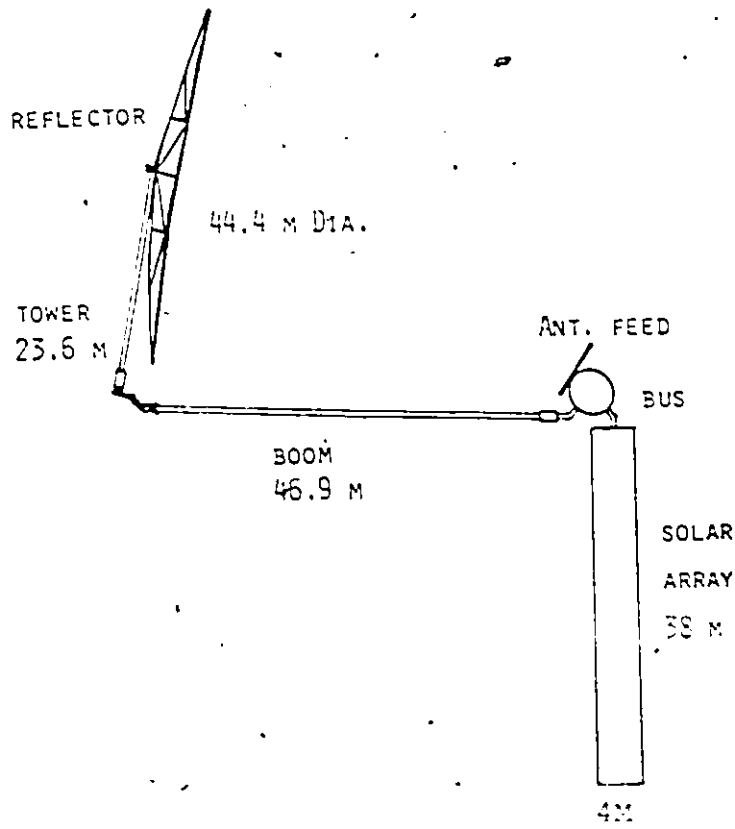


Fig. 1.1 Schematic Diagram of a Flexible Spacecraft (Canada)

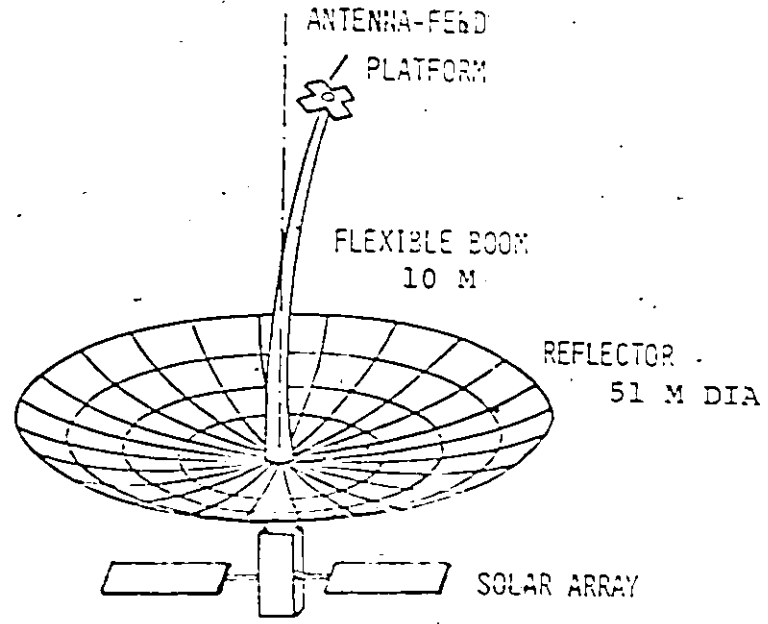


Fig. 1.2 Schematic Diagram of a NASA spacecraft

1.1 A BRIEF REVIEW OF PREVIOUS STUDIES

1.1.1 Modelling and Control

In the early stages of space exploration when spacecraft were built small and mechanically simple, the elastic deformations were relatively insignificant. The assumption of rigidity of the satellite was, thus, acceptable for all practical purposes. Numerous investigations have been carried out in the past on modelling of attitude and orbital dynamics of rigid body satellites, and design of active and passive controllers for attitude stabilization or orbital maneuver. A few representative studies, in this respect, are [1, 9,38,59,65,73]. A relatively detailed treatment of the subject could be found in [36,74].

On the contrary, a modern space vehicle containing rigid bus, long beams, large antennas and solar panels would be partly rigid and partly flexible. A need for rigorous investigation of the effects of structural flexibility on the spacecraft motion was felt as early as 1958 when some anomalous behavior of EXPLORER-I was observed. EXPLORER-I was passively spin stabilized about its principal axis of minimum moment of inertia - a configuration which was later proved to be unstable. Thomson and Reiter [74,75], followed by Meirovitch [48] attributed this behavior to two flexible antennas attached to the satellite. Attitude instability problems were also observed in EXPLORER-XX and ALOUETTE-I,

and were believed to be caused by flexibility of some of the components. These observations led to an extensive research, as documented in the survey paper [57], towards understanding the effects of structural flexibility on the satellite motion.

There are several methods available in the literature for mathematical modelling of flexible satellites. Many of these methods treat the elastic continuum in a discrete fashion. An earlier approach [39,60] is to regard the elastic system as being concentrated at certain points in the domain of extension of the continuum. Then the classical methods of rigid body dynamics is used in order to obtain the complete dynamics in the form of a set of ordinary differential equations.

Another commonly used approach [22,32,37] in modelling is to approximate the dynamics of the flexible parts by a finite number of modes. In this method the dependent variables, describing the motion of the elastic continuum, are described by finite sums of product of space dependent eigenfunctions and time dependent generalized coordinates known as the modal coordinates. The system dynamics, thus obtained, is frequently represented in the form of a linear differential equation such as

$$M \ddot{q} + G \dot{q} + K q = F. \quad (1.1)$$

The vector q in (1.1) consists of rigid body modes representing angular positions and velocities of the bus, and the modal coordinates associated with the natural vibration of the appendages.

A wide list of contributions could be found in the literature on the control of large flexible spacecraft represented by the modal equation (1.1). A number of papers in the proceedings of AIAA Symposium [50] deal with the modal dynamics and control system design using modern control theory. Rank conditions for controllability and observability have been derived in [40,41]. Techniques of optimal control theory have also been utilized [13,54,67] in designing optimal feedback regulators for stabilization of flexible spacecraft.

The modal method has the advantages that it gives rise to a set of linear ordinary differential equations which are readily understood by the practicing engineers and which can be treated with advantage using the known results from finite dimensional control theory. However, the number of modes of a flexible structure is actually infinite, and for a given system there are no clues as to how many modes should be included in the model to yield satisfactory results. It is also obvious that the number of modes producing satisfactory results is different for different structures, so that a generalization of the method is not

possible. Another important problem that is to be resolved is the effect of controls derived from this finite dimensional model on the actual system. This problem is commonly known as the 'Control Spillover' [14].

Studies have also been performed in the past towards rigorous modelling of flexible spacecraft in terms of partial differential equations [14,15,33,34,77]. In these studies large space structures are described by a system of hyperbolic partial differential equations of the form

$$M \frac{\partial^2 y}{\partial t^2} + D \frac{\partial y}{\partial t} + Ay = f(t,x), \quad (1.2)$$

where $y \equiv y(t,x)$ represents a vector of instantaneous displacements of the structure from its equilibrium position and f is the applied force. The internal restoring forces are represented by the time invariant, symmetric, positive (spatial differential) operator A . The operator D represents the structural and gyroscopic damping. Using infinite dimensional system theory for hyperbolic systems, optimal controllers and stabilizing feedback regulators have been suggested in these works. The dynamics of a large space structure consisting of multiple beams, and a stabilizing control scheme for the system using boundary feedback have been suggested in [30].

Representation of structural vibration (both transverse and torsional) of the spacecraft by equations of the form (1.2) is mathematically rigorous as compared to the modal method. These studies would be of practical significance if the elastic body dynamics could be decoupled from the rigid body dynamics. However, in general, equation (1.2) is only a partial representation of the complete dynamics of large flexible spacecraft. This is because of the fact that the dynamics of the rigid and the flexible members are actually nonlinearly coupled. Rigid body rotation of the structure is not reflected in the equation (1.2).

The most natural model describing the dynamics of the flexible spacecraft would be a set of ordinary differential equations for the rigid body and a set of partial differential equations for the flexible parts. This type of system model will be termed as "Hybrid system". A dynamic model of flexible spacecraft in the form of hybrid system has been introduced by Meirovitch [51,52]. Using Lagrangian mechanics, Hamilton's canonical equations in the form of hybrid system have been obtained in these papers. Defining rigid body angular rotations by θ_i , $i = 1,2,3$ and elastic body displacements by (u,v,w) , this hybrid dynamics is given by:

ODE for Attitude Motion

$$\frac{\partial L}{\partial \theta_i} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) = 0, \quad i = 1,2,3, \quad (1.3.a)$$

PDE for Elastic Motion

$$\frac{\partial \hat{L}}{\partial u} - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{u}} \right) + L_u(u, v, w) = 0 ,$$

$$\frac{\partial \hat{L}}{\partial v} - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{v}} \right) + L_v(u, v, w) = 0 , \quad (1.3.b)$$

$$\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + L_w(u, v, w) = 0 ,$$

where the Lagrangian L is defined as

$$L = \text{K.E.} - \text{P.E.} = \int_{\Omega} \hat{L}(t, u, v, w) d\Omega .$$

In (1.3), L represents an appropriate spatial differential operator for deformation of the elastic continuum. Lyapunov's approach have been utilized in these papers to obtain gravity gradient stabilization [51] and spin stabilization [52] of a flexible spacecraft.

Although this method provides a rigorous approach towards studying the dynamic behavior of flexible satellites, there are few criticisms that are worth mentioning. In these papers the deformation of the elastic members is described with respect to a coordinate system located at the mass center of the deformed body. Since this mass center varies with time because of vibrations, calculation of the total deformation of the body would produce computational complexities. In practice, one would, rather, be interested to measure the deformation with reference to the undeformed

state. The proof of stabilization in [51,52] requires the assumption that the reflections of the elastic body in the three directions are independent. It also makes critical use of the smallest eigenvalue of the elastic member which is very difficult to determine except for some simple cases. References [51,52] consider only passive stabilization of the system.

In view of the above discussion it is clear that, to ensure satisfactory performance, there is a need to take into account the flexibility of the different components of the spacecraft. In order to study the effects of structural flexibility on the satellite motion, it is necessary to develop a model which is mathematically rigorous yet simple enough for theoretical as well as numerical investigation. To facilitate design and analysis of control schemes, it is preferable to describe the deformation of the elastic parts with reference to the undeformed configuration. It is also essential to design active controllers for stabilization of the flexible spacecraft.

1.1.2 Stability in a Noisy Environment

Satellites in space are often subjected to random disturbances. Some common sources of such disturbances are meteorite collisions, variations in solar and magnetic pressure due to disturbances occurring in the sun,

aerodynamic forces due to atmospheric effects, thermal gradient, as well as on-board disturbances due to motors and pumps, fuel sloshing etc. These disturbances would produce random torques as well as spatially distributed random forces on the flexible spacecraft. Thus it is essential to study the stability properties of the spacecraft in the presence of random disturbances, and to develop appropriate control schemes for stabilization.

The questions of control and stabilization of the systems governed by the deterministic wave equation have been discussed by several authors, notably [3,18,26-29,44,62,63]. Using distributed and boundary feedback, Chen [26-29] proposes several control schemes for exponential stabilization of the wave equation in a bounded domain. Input output stability of a transmission line has been treated in [3].

The problem of stability of a class of parabolic partial differential equations in the presence of distributed and boundary noise have been discussed in [4,10]. Chow [31] has developed a Lyapunov type stability criterion for a general class of stochastic evolution equations on Hilbert space:

Application of stochastic filtering and control theory in satellite attitude control problems is an important area of research. Wong [79], and Wong and Ahmed [80] have suggested optimal and suboptimal controllers for stabilization of

rigid body satellites in the presence of random disturbances. Control system design for flexible structures in a noisy environment have also been reported in the literature, for example [47,64,66]. The system dynamics used in these studies is essentially the modal dynamics (1.1) with additive white noise, such as

$$\begin{aligned} \dot{x} &= Ax + Bu + v, & \text{(State equation)} \\ y &= Cx + w, & \text{(Observation equation)} \end{aligned} \quad (1.4)$$

where v and w are Gaussian white noise processes. The associated cost function for the optimization problem is taken as

$$J = E \int_0^T \{ \langle x, Qx \rangle + \langle u, Ru \rangle \} dt \quad (1.5)$$

Using Riccati formulation, optimal feedback regulators have been suggested in these studies.

However, to the knowledge of the author, no attempt has been made on stabilization of flexible spacecraft whose dynamics is governed by a hybrid system with additive distributed white noise.

1.1.3 Parameter Identification

An important and essential aspect of modelling any physical system is the identification of parameters in the model equation. For the flexible spacecraft, the pertinent parameters are moments of inertia of the rigid parts and the

elastic parameters for the flexible parts. There are several experimental methods available for measurement of inertia. On the other hand the elastic elements, such as beams, are frequently truss structures rather than a solid continuum of metal. For simplicity and reduction of computer cost in dynamic studies, it has been suggested [68,69] that a truss beam could be represented by an equivalent solid beam. Thus one needs to determine the parameters of the equivalent beam.

The problem of identification of unknown parameters in distributed systems have received considerable attention in the past as documented in the survey paper [43]. The question of exact identifiability of parameters in distributed systems have been considered in [42,58,70]. Trotter-Kato theorem have been used in [16,17] to prove the convergence of an approximating sequence of parameters which is obtained on the basis of finite dimensional approximation of the distributed system. Techniques of optimal control theory have also been utilized in identification of parameters in distributed systems. Optimality conditions for controls in the coefficients of a class of hyperbolic systems have been developed by Ahmed [5]. Necessary conditions of optimality for both controls and parameters combined together in a class of distributed systems arising from stochastic differential equations are presented in [6]. Identification of parameters for certain first and second

order partial differential equations have been discussed by Lions [45]. Chavent [24,25] has suggested a numerical technique for determination of parameters in distributed systems. A general formulation for identification of an operator in systems governed by parabolic, hyperbolic and structurally damped hyperbolic evolution equations have been discussed recently in [11].

The concept of structural damping is an important topic in the theory of vibrations. Experience shows that energy is dissipated in all vibrating systems, including those generally known as conservative, so that all vibrations eventually die out. (One should, however, note that the decay rate of energy due to this intrinsic damping is very small.) This type of damping is called "structural damping" and it is generally attributed to the hysteresis of the elastic material. The author is not aware of any investigation regarding the identification of structural damping coefficient.

1.2 OUTLINE OF THE THESIS

The thesis is organized as follows: Chapter I contains the motivation to the problem of modelling and stabilization of flexible spacecraft and a brief review of previous studies in this area.

In chapters II and III we present a methodology for developing a rigorous dynamic model for flexible satellites. For clarity of the method, a simple satellite configuration is considered in chapter II. The complete dynamics of the satellite in terms of hybrid system is developed following two different approaches. Some simple feedback controls are suggested for stabilization of the satellite. An algorithm for numerical integration of the hybrid dynamics, and illustrative numerical results are also presented in this chapter.

In chapter III, the results of chapter II are generalized towards modelling and stabilization of flexible spacecraft of more general structures. The results presented in this chapter covers flexible spacecraft with multiple beams, bend beams, and solar panel.

Stabilization of flexible structures in the presence of random disturbances is considered in Chapter IV. First, the question of stabilization of systems governed by the beam equation in the presence of distributed white noise is

considered. The technique developed for the beam equation is then extended to prove the stability of flexible spacecraft governed by hybrid dynamics.

Chapter V concerns with the problem of identification of parameters of a flexible beam. Techniques of optimal control theory have been utilized in this chapter for optimal identification of beam parameters, especially flexural rigidity and structural damping.

Concluding remarks and suggestions for further research are presented in Chapter VI.

Original contributions in this thesis include:

- i) A methodology for rigorous modelling of flexible spacecraft in terms of variables which are physically meaningful and are available for direct measurement; a direct proof of stabilization of the system using simple feedback controls; sections 2.2, 2.3, 2.4 [12,19,20], also sections 3.2 and 3.3.
- ii) Exponential and almost sure stabilization of vibratory systems in the presence of distributed white noise; sections 4.2.2 and 4.2.3 [21].
- iii) Stability of stochastic hybrid systems as applied to flexible spacecraft; section 4.3.
- iv) Identification of structural damping of a flexible beam ; section 5.3.1.

Chapter II
MODELLING AND STABILIZATION OF A FLEXIBLE
SPACECRAFT

The third generation spacecraft to be launched in the next decade would consist of a rigid main body which carries (most of) the control and instrumentation hardware, and several flexible appendages such as long beams, large solar panels, and antennas etc. Typical configuration of these satellites are shown in Fig. 1.1 and Fig. 1.2 in Chapter I. From the dimension of the various components in these figures, it is clear that these satellites are structurally very flexible. For practical applications such as communication, it is essential to maintain a high degree of pointing accuracy. Also it is necessary that the flexible parts of the spacecraft must not vibrate. This, in turn, requires representation of the complete system by an appropriate dynamic model and development of suitable control schemes for stabilization. In order to develop a methodology for modelling a spacecraft of arbitrary structure, in this chapter we consider a relatively simple spacecraft configuration, and derive an accurate dynamic model and several stabilizing control schemes.

2.1 NOTATIONS

Reference Frames

$F_b \equiv (i_b, j_b, k_b)$: Body frame of reference.

$F_r \equiv (i_r, j_r, k_r)$: Orbital reference frame.

$F_B \equiv (i_B, j_B, k_B)$: Beam reference frame.

Velocities

$\omega_r = (0, -\omega_0, 0)$: Angular velocity of the F_r frame with respect to the inertial space.

$\omega_b = (\omega_1, \omega_2, \omega_3)$: Angular velocity of the F_b frame with respect to the F_r frame.

ω_0 : Earth angular rate.

v : Velocity of mass element of the beam with respect to the inertial space.

v : Velocity of the mass element of the beam with respect to the F_b frame.

v_0 : Velocity of the origin of the F_b frame in the inertial space.

Displacements

$y \equiv y(x, t)$: Beam deflection in the j_B direction.

$z \equiv z(x, t)$: Beam deflection in the k_B direction.

$\delta \equiv (y, z)$: Beam deflection vector.

(x, y, z) : Position vector denoting the beam mass element in the beam frame.

$(R, 0, 0)$: Position of the origin of the beam frame in the body frame.

$r \equiv (R+x, y, z)$: Position vector denoting the beam element in the body frame.

Parameters

- $m \equiv m(x)$: Beam mass density per unit volume.
 $\rho \equiv \rho(x)$: Beam mass density per unit length.
 L : Length of the beam.
 $\Omega \equiv [0, L]$: Beam spatial domain.
 I^b : Bus inertia tensor (Constant).
 I^B : Beam inertia tensor (time varying).
 $I^T \equiv I^b + I^B$: Total satellite inertia.
 EI_Y (EI_Z) : Flexural rigidity of the beam for deflection in the j_B (k_B) direction.
 H : Angular momentum of the satellite with respect to the body frame.
 $T \equiv (T_1, T_2, T_3)$: Torque applied to the bus.
 $F \equiv (F_Y, F_Z)$: Force applied to the beam.

Mathematical symbols

- $A \cdot b$: Dot product between a tensor and a vector, or between two vectors.
 $a \times b$: Cross product between two vectors.
 $|a| = \left(\sum_{i=1}^3 a_i^2 \right)^{1/2}$: Norm of the vector a.
 $\frac{dA}{dt}$: Time derivative of a vector A in the inertial space.
 \dot{A} : Time derivative of a vector in the body frame.

Additional notations will be introduced in the text as required.

2.2 MODELLING

A flexible structure in an earth-orbit would undergo three types of motions, viz. a) rigid body translation perturbing the orbit, b) rigid body rotation perturbing the orientation of the satellite, and c) vibration of the elastic members causing elastic deformation of the structure. In this study, we shall be concerned only with the last two types of motions, while assuming that the attitude motion and the vibration of the elastic members have negligible effect upon the orbital motion of the spacecraft, or that the spacecraft is equipped with suitable control mechanism which maintains the desired orbit.

The spacecraft considered in this chapter consists of a rigid main body (commonly known as bus) and a long flexible beam rigidly attached to it as shown in Figure 2.1.

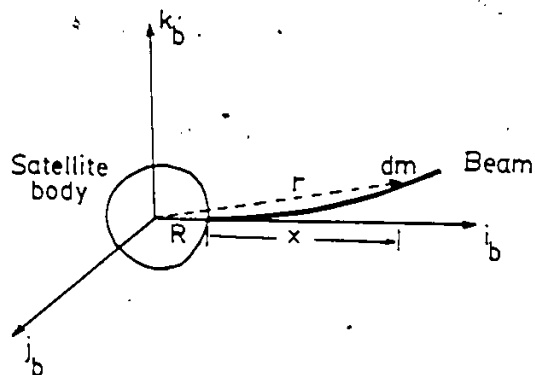


Fig. 2.1 Schematic Diagram of a Flexible Spacecraft

We first note that if the body* and the beam are considered separately, then the dynamics of each is well established. More specifically, the attitude motion of a rigid body satellite vehicle can be described [36] by a set of nonlinear ordinary differential equations, and the transverse vibrations of a flexible beam is governed [76] by the Euler equation, which is a hyperbolic partial differential equation involving fourth order spatial derivatives. However, in the present problem, since the beam is mounted on the bus which spins about its axes during perturbations, the complete dynamics of the system can not be described just by a collection of these two sets of equations; rather must be modified appropriately so as to take care of the interaction between the body and the beam. Following [19,20], in this section, we derive the attitude dynamics of the body considering the effects of vibrations of the beam, and the equations of transverse vibration of the beam taking into account the effects of body motions.

2.2.1 Reference Frames

In order to derive the dynamics of the flexible spacecraft, we need to introduce appropriate reference frames. We define a body coordinate system represented by the right handed unit vector triad $F_b = (i_b, j_b, k_b)$. This body coordinate system will be assumed to be fixed to the satellite bus. We suppose that the rest position of the

* We shall use the terms body and bus interchangeably.

beam is along the i_b axis, with one end of the beam rigidly fixed to the satellite body at a distance R from the center of the body coordinate system (see Fig. 2.1). For the study of beam vibrations, we need to define a beam coordinate system represented by the right handed unit vector triad $F_B \equiv (i_B, j_B, k_B)$ such that i_B (respectively j_B and k_B) is parallel to i_b (respectively j_b and k_b), and that the origin of the beam frame F_B is at $(R, 0, 0)$ with reference to the body frame F_b .

We consider the motion of the flexible spacecraft in a circular geo-synchronous orbit around earth. We shall suppose that the acceleration of the earth is negligible, so that for all practical purposes its center could be regarded as fixed in inertial space. For the study of attitude motion of the spacecraft in a circular orbit, we shall introduce an orbital reference system $F_r \equiv (i_r, j_r, k_r)$ with the origin coinciding with the origin of the body frame, the axis i_r tangent to the orbital path in the direction of travel, the axis k_r pointed towards the center of the earth, and the axis j_r normal to the orbital plane. Let $\omega_b \equiv (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the body frame with respect to the reference frame and ω_r be the angular velocity of the reference frame with respect to the inertial space. We shall denote $\omega \equiv \omega_b + \omega_r$. Letting ω_0 denote the angular velocity of the earth about its own axis, we have

$$\omega_r = (0, -\omega_0, 0). \quad (2.1)$$

2.2.2 Inertia Tensor

The inertia tensor of the satellite main body is denoted by I^b , which is not necessarily diagonal. However, we assume that I^b is not time varying and is positive definite.

We now compute the moments of inertia of the flexible beam with respect to the body coordinate system. Let a point P on the perturbed beam be denoted by the vector (x, y, z) in the F_B frame. Then, the point P can be represented by the vector

$$r = (x + R, y, z), \quad (2.2)$$

with reference to the body frame F_B . Letting dm denote the mass of a generic element on the beam, the beam inertia tensor I^B with respect to the body frame is given by

$$I^B = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad (2.3)$$

where

$$\begin{aligned} I_{xx} &= \int (y^2 + z^2) dm, & I_{xy} &= I_{yx} = \int (x+R)y dm, \\ I_{yy} &= \int ((x+R)^2 + z^2) dm, & I_{yz} &= I_{zy} = \int yz dm, \\ I_{zz} &= \int ((x+R)^2 + y^2) dm, & I_{zx} &= I_{xz} = \int (x+R)z dm. \end{aligned} \quad (2.4)$$

Unless otherwise specified, the integrals will be assumed to be taken over the whole body of the beam.

Let L be the length of the beam and $\rho \equiv \rho(x)$ be its mass per unit length. Then, assuming that the displacement of the beam from the rest position is small, (2.4) can be rewritten as:

$$\begin{aligned}
 I_{xx} &= \int_0^L \rho \{y^2 + z^2\} dx, & I_{xy} &= I_{yx} = \int_0^L \rho (x+R)y dx, \\
 I_{yy} &= \int_0^L \rho \{ (x+R)^2 + z^2 \} dx, & I_{yz} &= I_{zy} = \int_0^L \rho yz dx, \quad (2.5) \\
 I_{zz} &= \int_0^L \rho \{ (x+R)^2 + y^2 \} dx, & I_{xz} &= I_{zx} = \int_0^L \rho (x+R)z dx.
 \end{aligned}$$

We note that deflections of the beam could occur in the j_B and k_B directions, which are denoted by $y \equiv y(x,t)$, $z \equiv z(x,t)$, and that these deflections are governed by the beam dynamics to be discussed in the next section. Naturally, the inertia tensor of the beam varies with time. We, however, assume that the beam will not elongate in the axial or i_B direction.

2.2.3 Modelling using Newtonian Dynamics

With this preparation, we can now derive the dynamics of the complete system. We shall use Newton's laws of motion in order to derive the attitude dynamics of the satellite bus and the equations for the transverse vibration of the beam.

2.2.3.1 Bus Dynamics

Consider an incremental mass element dm of the beam located at r from the center of the body coordinate system F_b and let v denote the velocity of dm with respect to the inertial space. We also suppose that the center of mass of the bus coincides with the origin of the body frame. Then, the total angular momentum H of the system about the center of the body frame is given by

$$H = I^b \cdot (\omega_b + \omega_r) + \int (r \times v) dm, \quad (2.6)$$

where $I \cdot a$ denotes the usual dot product between a tensor and a vector. The velocity v of the incremental mass element dm with respect to the inertial space can be expressed [36,71] by

$$v = v_0 + v + (\omega_b + \omega_r) \times r, \quad (2.7)$$

where v_0 is the velocity of the origin of the body frame with respect to the inertial space and v is the velocity of dm with respect to the body frame. Assuming that the beam will not elongate in the i_b direction, we can write

$$v = (0, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) \quad (2.8)$$

The equation of attitude motion of the spacecraft is obtained by equating the external torque to the time derivative of the total angular momentum, viz:

$$\frac{dH}{dt} = T, \quad (2.9)$$

where T denotes the external torque applied to the satellite.

The dynamics of an earth orbiting satellite could be considered as being equivalent to the one rotating about an axis normal to the orbital plane with an angular velocity equal to the orbital angular velocity [53]. As mentioned earlier, we shall be concerned with the case in which the origin of the body frame describes a fixed circular orbit, i.e. radius $|R_0| = \text{constant}$, so that the term involving $\frac{dR_0}{dt} = v_0$ could be ignored (c.f. [51]). Then using (2.6) and (2.7) in (2.9), the equation of attitude motion of the flexible spacecraft is obtained as

$$\frac{d}{dt} (I^b \cdot \omega) + \frac{d}{dt} (I^B \cdot \omega) + \frac{d}{dt} \left(\int (r \times v) dm \right) = T. \quad (2.10)$$

Note that the angular momentum of the beam about the center of the body frame is given by

$$I^B \cdot \omega = I^B \cdot (\omega_b + \omega_r) = \int r \times ((\omega_b + \omega_r) \times r) dm, \quad (2.11)$$

where the vector r denotes the position of generic mass element dm on the beam as given in (2.2). It is interesting to note that the vibrational velocity of the beam has a direct influence on the attitude motion of the spacecraft; indeed, the integral in the left hand side of (2.10) gives the contribution of the beam velocities to the total angular momentum of the system.

In order to obtain the attitude dynamics in some convenient form, it is necessary to perform the differentiations indicated in (2.10). The time rate of

change of any vector A relative to the inertial space is denoted by " $\frac{d}{dt}$ " and that relative to the body frame by " $\dot{}$ ".

Then, as is well-known [35,71],

$$\frac{dA}{dt} = \dot{A} + \omega \times A . \quad (2.12)$$

Using the differentiation rule (2.12) in (2.10), we obtain the vector dynamic equation governing the attitude motion of the flexible spacecraft as

$$I^T \cdot \dot{\omega} + \omega \times (I^T \cdot \omega) + 2 \int r \times (\omega \times v) dm + \int r \times \dot{v} dm = T , \quad (2.13)$$

where $I^T = I^b + I^B$ is the total inertia tensor of the spacecraft. We recall that the beam inertia tensor I^B (given by (2.3)) is time varying, so that I^T is also time varying.

Another useful form of the attitude dynamics is obtained by performing the vector operations (see Appendix A) in (2.13) as given below:

$$\begin{bmatrix} I^T \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 - \omega_0 \\ \omega_3 & 0 & -\omega_1 \\ -(\omega_2 - \omega_0) & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I^T \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 - \omega_0 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} , \quad (2.14)$$

where

$$f_1 = \int_0^L \rho \left\{ y \frac{\partial^2 z}{\partial t^2} - z \frac{\partial^2 y}{\partial t^2} + 2 \omega_1 y \frac{\partial y}{\partial t} + 2 \omega_1 z \frac{\partial z}{\partial t} \right\} dx ,$$

$$f_2 = \int_0^L \rho \left\{ -(x+R) \frac{\partial^2 z}{\partial t^2} + 2(\omega_2 - \omega_0) z \frac{\partial z}{\partial t} - 2\omega_3 z \frac{\partial y}{\partial t} - 2\omega_1 (x+R) \frac{\partial y}{\partial t} \right\} dx , \quad (2.15)$$

$$f_3 = \int_0^L \rho \left\{ (x+R) \frac{\partial^2 y}{\partial t^2} - 2\omega_1 (x+R) \frac{\partial z}{\partial t} - 2(\omega_2 - \omega_0) y \frac{\partial z}{\partial t} + 2\omega_3 y \frac{\partial y}{\partial t} \right\} dx .$$

It is to be noted here that the total inertia matrix I^T varies only with the position of the beam, but f_1 , f_2 and f_3 depend also on the velocity and acceleration of the beam particles. It is clear that the attitude motion of the flexible spacecraft is highly affected by the vibrations of the beam. In the case of an (infinitely) rigid beam, the displacement y, z , velocity $\frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}$ and acceleration $\frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2}$ of the beam particles with respect to the body frame are all zero. Consequently, f_1, f_2 and f_3 reduce to zero and I^T becomes time invariant. The resulting form of (2.14) is the well-known attitude dynamics [36] of a rigid body satellite.

2.2.3.2 Beam Dynamics

The transverse vibration of a flexible beam with respect to a reference frame fixed in inertial space is given [76] by the well-known Euler equation:

$$\rho(x) \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2} \right] = f, \quad (2.16)$$

where ρ is the mass per unit length, EI is the flexural rigidity of the beam and f denotes the distribution of forces applied to the beam. In the present problem, although the beam reference frame is not fixed in the inertial space because of angular motions of the bus, the above equation could be appropriately modified to obtain the dynamics of the beam.

In the present problem, deflection of the beam could occur in two directions, viz: j_B and k_B . Denoting the corresponding deflections by $y \equiv y(x,t)$ and $z \equiv z(x,t)$ and under the assumption of small deflection and no "structural damping" the transverse vibration of the beam is governed by

$$\rho(x) \frac{\partial^2}{\partial t^2} \begin{bmatrix} y \\ z \end{bmatrix} + \frac{\partial^2}{\partial x^2} \left(\begin{bmatrix} EI_y & 0 \\ 0 & EI_z \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{bmatrix} y \\ z \end{bmatrix} \right) = \begin{bmatrix} \tilde{F}_y \\ \tilde{F}_z \end{bmatrix} \quad (2.17)$$

where \tilde{F}_y and \tilde{F}_z denote the force distribution acting on the beam, and EI_y and EI_z denote the flexural rigidity of the beam for deflections in the j_B and k_B directions respectively.

In a rotating frame of reference, a particle experiences some extraneous force [71] arising essentially due to the rotation of the coordinates. Since the beam is mounted on the main body, having the angular velocity $(\omega_b + \omega_r)$, each particle of the beam would be acted upon by such a force in addition to the externally applied forces, such as control forces. Considering an infinitesimal section of the beam of length dm located at a distance r from the center of body frame, the total force distribution acting on the beam is, then, given by

$$F_{\text{total}} = F_{\text{control}} - \rho \alpha \quad (2.18)$$

where

$$\alpha = (\dot{\omega}_b + \dot{\omega}_r) \times r + (\omega_b + \omega_r) \times ((\omega_b + \omega_r) \times r) + 2(\omega_b + \omega_r) \times v$$

The three terms in the above equation are commonly known [71,74] as the Euler acceleration, centrifugal acceleration and coriolis acceleration respectively.

Substituting the j_B and k_B component of (2.18) to the right hand side of (2.16), we obtain the equation governing the transverse vibration of the flexible beam.

$$\begin{aligned} \rho \frac{\partial^2}{\partial t^2} \begin{bmatrix} y \\ z \end{bmatrix} + \rho \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} + \frac{\partial^2}{\partial x^2} \left(\begin{bmatrix} EI_y & 0 \\ 0 & EI_z \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{bmatrix} y \\ z \end{bmatrix} \right) \\ + \rho \begin{bmatrix} -\omega_1^2 - \omega_3^2 & -\dot{\omega}_1 + (\omega_2 - \omega_0)\omega_3 \\ \dot{\omega}_1 + (\omega_2 - \omega_0)\omega_3 & -\omega_1^2 - (\omega_2 - \omega_0)^2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \\ + \rho \begin{bmatrix} \dot{\omega}_3 + \omega_1(\omega_2 - \omega_0) \\ -\dot{\omega}_2 + \omega_1\omega_3 \end{bmatrix} (x+R) = \begin{bmatrix} F_Y \\ F_Z \end{bmatrix} \quad (2.19) \end{aligned}$$

where F_Y and F_Z denote the control forces applied to the beam in the j_B and k_B direction respectively.

We note here that the i_B component of (2.18) represents an axial force acting on the beam which would tend to change the orbit of the spacecraft. However, this component does not come into consideration in this study, since we have assumed that the spacecraft is on a fixed geo-synchronous orbit.

The boundary conditions for (2.19) can be obtained by observing the cantilever nature of the beam. They are as follows:

$$\begin{aligned}
 y(0,t) &= 0, & z(0,t) &= 0, \\
 \frac{\partial y}{\partial x}(0,t) &= 0, & \frac{\partial z}{\partial x}(0,t) &= 0, \\
 EI_y \frac{\partial^2 y}{\partial x^2}(L,t) &= 0, & EI_z \frac{\partial^2 z}{\partial x^2}(L,t) &= 0, \\
 \frac{\partial}{\partial x} \left(EI_y \frac{\partial^2 y}{\partial x^2} \right) (L,t) &= 0, & \frac{\partial}{\partial x} \left(EI_z \frac{\partial^2 z}{\partial x^2} \right) (L,t) &= 0.
 \end{aligned}
 \tag{2.20}$$

It is interesting to note that the vibration of the beam in the two directions are not independent, rather is given by the coupled system of partial differential equations (2.19). Further, in the absence of any rotation of the main body, these equations reduce to the usual equation for transverse vibration of a cantilever beam.

2.2.4 Modelling by Lagrangian Method

The dynamics of the flexible spacecraft could also be derived using the principle of least action or the Hamilton's principle, according to which every mechanical system is characterized by a certain function $L(q, \dot{q}, t)$ known as the Lagrangian, and the motion of the system is such that the action S defined by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt,$$

is minimum. The Lagrangian L of the system is a function of the generalized coordinates q and the generalized velocities \dot{q} , and is given by

$$L = T - V,$$

where T is the total kinetic energy and V is the total potential energy of the system. The necessary condition for the action S to have a minimum is that its first variation

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0, \quad (2.21)$$

with the end conditions $\delta q = 0$ at $t=t_1$ and t_2 .

2.2.4.1 System Kinetic and Potential energy

For the derivation of the dynamics of the flexible spacecraft using the Hamilton's principle, we now introduce the kinetic and potential energy of the system.

The kinetic energy of the satellite bus is given by

$$T_b = \frac{1}{2} \int_{\text{bus}} \frac{d\tilde{r}}{dt} \cdot \frac{d\tilde{r}}{dt} d\tilde{m}, \quad (2.22)$$

where $\tilde{r} = R_0 + \tilde{D}$, denotes the absolute position of the incremental mass element $d\tilde{m}$ of the bus relative to the inertial space, with R_0 defining the position of the origin of the body frame in the inertial frame and \tilde{D} defining the position of the mass element $d\tilde{m}$ relative to the body frame.

As mentioned earlier, we shall assume that the origin of the body frame coincides with the center of mass of the satellite bus. Then, using (2.12) in (2.22) it follows that

$$T_b = \frac{1}{2} M^b v_0 \cdot v_0 + \frac{1}{2} \omega \cdot (I^b \cdot \omega), \quad (2.23)$$

where $v_0 = \frac{dR_0}{dt}$ is the velocity of the origin of the body frame relative to the inertial space, M^b is the mass of the satellite bus and I^b is its inertia tensor with respect to the body frame.

The kinetic energy of the beam is obtained in a similar way and is given by

$$T_B = \frac{1}{2} \int_{\text{beam}} \frac{d\tilde{r}}{dt} \cdot \frac{d\tilde{r}}{dt} dm, \quad (2.24)$$

with $\tilde{r} = R_0 + D + d$, where \tilde{r} denotes the absolute position of the incremental mass element dm of the beam relative to the inertial space, D defines the position of the mass element dm relative to the body frame in the undeformed state and d represents the deformation of dm relative to the body frame. Indeed, with reference to Fig. 2.1, one notes that $D = (R + x, 0, 0)$ and $d = (0, y, z)$. Clearly, $r = D + d$, where r denotes the position of the beam mass element dm relative to the body frame.

The system potential energy arises from two sources, namely elastic and gravitational. Since the aim of this study is stabilization by using active controllers, gravitational potential will be ignored. The elastic

potential energy due to deformation of the beam is a function of the partial derivatives of the displacements y, z with respect to the spatial variable x . Under the assumption of small deflections, the potential energy of the beam is given by

$$V_B = \frac{1}{2} \int_0^L \left(EI_Y \left(\frac{\partial^2 y}{\partial x^2} \right)^2 + EI_Z \left(\frac{\partial^2 z}{\partial x^2} \right)^2 \right) dx, \quad (2.25)$$

where $EI_Y \equiv EI_Y(x)$ (respectively EI_Z) is the flexural rigidity of the beam for deflections in the j_B direction (respectively k_B direction).

2.2.4.2 Complete System Dynamics

By Hamilton's principle, the dynamics of the flexible spacecraft in the absence of any control is determined [53]

by

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} (T_b + T_B - V_B) dt = 0, \quad (2.26)$$

subject to the end conditions

$$\delta \theta_1 = 0, \quad \delta \theta_2 = 0, \quad \delta \theta_3 = 0, \quad \delta y = 0, \quad \delta z = 0 \text{ at } t = t_1, t_2,$$

where $\theta_1, \theta_2, \theta_3$ are the Euler angles describing the orientation of the body frame relative to the reference frame.

Development of the system dynamics following this variational formulation of Hamilton's principle is very attractive because of its conceptual simplicity and universality of application. However, in the present problem, since the body frame rotates with respect to the inertial

space, one needs some extra care in taking the first variation of the Lagrangian L .

The virtual displacement of the elastic body relative to the inertial space can be given [23] by

$$\delta \hat{r} = \delta R_0 + \delta \theta \times (D+d) + \delta d, \quad (2.27)$$

where $\delta \theta = (\delta \theta_1, \delta \theta_2, \delta \theta_3)$ is the virtual rigid rotation of the body frame relative to the reference frame. Further, using (2.12) one obtains

$$\delta \left(\frac{d\hat{r}}{dt} \right) = \delta v_0 + \delta \omega \times (D+d) + \delta \theta \times \{ \omega \times (D+d) + \dot{d} \} + \frac{d}{dt} (\delta d), \quad (2.28)$$

where $\delta \omega = \frac{d}{dt} (\delta \theta)$ denotes the virtual rotation rate of the body frame.

As discussed earlier, for the case of a satellite in a fixed circular orbit, we have $v_0 = 0$. Then, taking the first variation of the Lagrangian as in (2.26) using the equations (2.23)-(2.28) and assuming small angle rotation, one obtains

$$\begin{aligned} \delta \int_{t_1}^{t_2} L dt &= \int_{t_1}^{t_2} \delta \omega \cdot (I^T \cdot \omega) dt + \int_{t_1}^{t_2} \delta \omega \cdot \int_0^L \rho ((D+d) \times \dot{d}) dx dt \\ &+ \int_{t_1}^{t_2} \int_0^L \rho \{ \omega \times (D+d) + \dot{d} \} \cdot \frac{d}{dt} (\delta d) dx dt \\ &- \int_{t_1}^{t_2} \int_0^L \left[\frac{\partial^2}{\partial x^2} (EI_Y \frac{\partial^2 y}{\partial x^2}) \delta y + \frac{\partial^2}{\partial x^2} (EI_Z \frac{\partial^2 z}{\partial x^2}) \delta z \right] dx dt \\ &- \int_{t_1}^{t_2} \left[EI_Y \frac{\partial^2 y}{\partial x^2} \delta \left(\frac{\partial y}{\partial x} \right) \Big|_0^L - \frac{\partial}{\partial x} (EI_Y \frac{\partial^2 y}{\partial x^2}) \delta y \Big|_0^L \right] dt \\ &- \int_{t_1}^{t_2} \left[EI_Z \frac{\partial^2 z}{\partial x^2} \delta \left(\frac{\partial z}{\partial x} \right) \Big|_0^L - \frac{\partial}{\partial x} (EI_Z \frac{\partial^2 z}{\partial x^2}) \delta z \Big|_0^L \right] dt, \end{aligned} \quad (2.29)$$

where $I^T = I^b + I^B$ is the total inertia tensor of the satellite with respect to the body axes and $\rho \equiv \rho(x)$ is the mass per unit length and L is the length of the beam. By assumption the beam is inextensible, so that $\delta d = (0, \delta y, \delta z)$. Integrating (2.29) by parts and noting that the variations $\delta \theta_1, \delta \theta_2, \delta \theta_3, \delta y, \delta z$ are arbitrary, we obtain the set of coupled differential equations for the dynamics of the flexible spacecraft as given below.

$$\frac{d}{dt} [(I^T \cdot \omega)] + \frac{d}{dt} \left[\int ((D+d) \times \dot{d}) \, dm \right] = 0, \quad (2.30)$$

and

$$\rho \frac{d}{dt} \left[\dot{d} + \omega \times (D+d) \right]_{j,k} + \frac{\partial^2}{\partial x^2} \left(\begin{bmatrix} EI_y & 0 \\ 0 & EI_z \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{bmatrix} y \\ z \end{bmatrix} \right) = 0, \quad (2.31)$$

with the boundary conditions (2.20), where we have used the notation $[A]_{j,k} = [A_i, A_j, A_k]_{j,k} = [A_j, A_k]$ to denote the j -th and k -th component of the vector A .

Using (2.12) in (2.30) and (2.31) and noting that $r = D+d$ and $v \equiv \dot{d} = (0, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t})$, we obtain the same dynamics for the controlled flexible spacecraft as given in section 2.2.3, i.e. (2.14) for the dynamics of the rigid body and (2.19) and (2.20) for the flexible beam.

2.2.5 Complete System Dynamics using Euler-Bernoulli Beam Theory

The complete dynamics of the flexible spacecraft consisting of a rigid main body and a flexible beam rigidly attached to it, therefore, is given by the set of ordinary differential (2.14) and the set of hyperbolic partial differential equations (2.19) with the boundary conditions (2.20). Defining $\phi \equiv (y, z)$ and noting that $\omega \equiv (\omega_1, \omega_2, \omega_3)$, this hybrid dynamics could be expressed in the following compact form:

$$I^T(\phi) \dot{\omega} + A(\omega) I^T(\phi) \omega + f(\omega, \phi) = T,$$

$$\rho \frac{\partial^2 \phi}{\partial t^2} + \rho B_1(\omega) \frac{\partial \phi}{\partial t} + \frac{\partial^2}{\partial x^2} \left(B_2 \frac{\partial^2 \phi}{\partial x^2} \right) + \rho B_3(\omega) \phi + (x+R) \rho B_4(\omega) = F, \quad (2.32)$$

for $t \geq 0$, $x \in \Omega \equiv [0, L]$, along with the boundary conditions

$$\begin{aligned} \phi(0, t) = 0, & \quad B_2 \frac{\partial^2 \phi}{\partial x^2}(L, t) = 0, \\ \frac{\partial \phi}{\partial x}(0, t) = 0, & \quad \frac{\partial}{\partial x} \left(B_2 \frac{\partial^2 \phi}{\partial x^2} \right)(L, t) = 0. \end{aligned} \quad (2.33)$$

The matrices A , B_1 , B_2 , B_3 and B_4 are defined appropriately so that (2.32) is compatible with (2.14) and (2.19). We note that the attitude dynamics of the spacecraft is nonlinear in the angular velocities ω , while the beam equation is only apparently linear in the deflection ϕ . But, since the coefficient matrices of the beam equation are functions of ω , which in turn depend nonlinearly on ϕ , the whole system of equations (2.32) is actually nonlinear. It is clearly observed that the two sets of equations are strongly

coupled. Any rotation of the body would induce vibrations in the beam and vice-versa. It is also important to note that the deflections of the beam are described with respect to the coordinate system fixed at the unperturbed position.

2.3 STABILIZATION

In this section we study the problem of stabilization of the flexible spacecraft. In order to use the spacecraft for any application, for example communication, it is desired that the spacecraft must not wobble and the beam must not vibrate. Accordingly, we consider the following rest state

$$\omega_1 = 0, \omega_2 = 0, \omega_3 = 0$$

and for almost all $x \in [0, L]$,

$$\begin{aligned} y = 0, \quad \frac{\partial y}{\partial t} = 0, \\ z = 0, \quad \frac{\partial z}{\partial t} = 0, \end{aligned} \tag{2.34}$$

and pose the following stabilization problem: If the system is perturbed from the above mentioned rest state, can we find a control that will eventually drive the system back to the rest state?

This problem of stabilization of the flexible spacecraft could be solved following Lyapunov's approach, with the total energy of the system serving as a natural choice of the Lyapunov function. In the following result, we show that the spacecraft could be stabilized in the asymptotic sense by application of simple feedback controls.

Theorem 2.1

Consider the system described by (2.14) and (2.19) with the boundary conditions (2.20). Suppose that the controls applied to the system are given by the feedback law

$$T = (-k_1 \omega_1, -k_2(\omega_2 - \omega_0), -k_3 \omega_3), \quad k_1, k_2, k_3 > 0,$$

and

$$F = (-d_1(x) \frac{\partial y}{\partial t}, -d_2(x) \frac{\partial z}{\partial t}), \quad d_1, d_2 > 0 \quad \text{a.e. on } \Omega. \quad (2.35)$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (2.34).

Proof

We shall denote $\omega \equiv (y, \frac{\partial y}{\partial t}, z, \frac{\partial z}{\partial t})$ and according to the previous notations $\omega \equiv (\omega_1, \omega_2 - \omega_0, \omega_3)$ and $\varphi \equiv (y, z)$. Scalar multiplying both sides of (2.13), (which is equivalent to (2.14)) by ω and noting that $I^T = I^b + I^B$, where I^B varies with time because of vibrations, we obtain

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} (I^T \cdot \omega) \cdot \omega \right] - \frac{1}{2} (\dot{I}^B \cdot \omega) \cdot \omega + 2 \int (r \times (\omega \times v)) \cdot \omega \, dm \\ + \int (r \times \dot{v}) \cdot \omega \, dm = T \cdot \omega. \end{aligned} \quad (2.36)$$

Now, it can be easily verified that

$$\begin{aligned} (\dot{I}^B \cdot \omega) \cdot \omega &= \int (r \times (\omega \times r)) \cdot \omega \, dm = \int |\omega \times r|^2 \, dm, \\ (\dot{I}^B \cdot \omega) \cdot \omega &= \int (v \times (\omega \times r)) \cdot \omega \, dm + \int (r \times (\omega \times v)) \cdot \omega \, dm \\ &= -2 \int (\omega \times (\omega \times r)) \cdot v \, dm. \end{aligned}$$

Using these results, (2.36) becomes

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} (I^b \cdot \omega) \cdot \omega \right] + \frac{d}{dt} \frac{1}{2} \int |\omega \times r|^2 \, dm - \int (\omega \times (\omega \times r)) \cdot v \, dm \\ + \int (r \times \dot{v}) \cdot \omega \, dm = T \cdot \omega. \end{aligned} \quad (2.37)$$

Equation (2.19) could be rewritten as

$$\rho \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2}{\partial x^2} (B_2 \frac{\partial^2 \phi}{\partial x^2}) + \rho Q (\dot{u}_x r + \omega_x (\omega_x r) + 2\omega_x v) = F, \quad (2.38)$$

where

$$B_2 = \begin{bmatrix} EI_Y & 0 \\ 0 & EI_Z \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, scalar multiplying the above equation by $\frac{\partial \phi}{\partial t}$ and integrating by parts over $x \in \Omega \equiv [0, L]$ and using the boundary conditions (2.20), we obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho \frac{\partial \phi}{\partial t}{}^2 dx + \frac{d}{dt} \frac{1}{2} \int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx \\ + \int_{\Omega} \rho (Q (\dot{u}_x r + \omega_x (\omega_x r) + 2\omega_x v)) \cdot \frac{\partial \phi}{\partial t} dx = \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx. \end{aligned} \quad (2.39)$$

Recalling that $v = (0, \frac{\partial v}{\partial t}, \frac{\partial z}{\partial t}) = (0, \frac{\partial \phi}{\partial t})$, (2.39) can also be written as

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} |v|^2 dm + \frac{1}{2} \int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx \right] + \int_{\Omega} (\dot{u}_x r) \cdot v dm \\ + \int_{\Omega} (\omega_x (\omega_x r)) \cdot v dm = \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx. \end{aligned} \quad (2.40)$$

Adding (2.37) and (2.40), one obtains

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} (I^b \cdot \omega) \cdot \omega + \frac{1}{2} \int_{\Omega} |v + \omega_x r|^2 dm + \frac{1}{2} \int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx \right] \\ = T \cdot \omega + \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx. \end{aligned} \quad (2.41)$$

Define $V(\omega, \phi; t)$ by

$$\begin{aligned} V(\omega, \phi; t) \equiv \frac{1}{2} (I^b \cdot \omega) \cdot \omega + \frac{1}{2} \int_{\Omega} EI_Y \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_{\Omega} EI_Z \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx \\ + \frac{1}{2} \int_{\Omega} \rho \left((\omega_2 - \omega_0) z - \omega_3 y \right)^2 dx + \frac{1}{2} \int_{\Omega} \rho \left(\frac{\partial v}{\partial t} + (x+R) \omega_3 - \omega_1 z \right)^2 dx \\ + \frac{1}{2} \int_{\Omega} \rho \left(\frac{\partial z}{\partial t} + \omega_1 y - (x+R) (\omega_2 - \omega_0) \right)^2 dx. \end{aligned} \quad (2.42)$$

Then from (2.41) and (2.42), we have

$$\frac{d}{dt}V(\omega, \psi; t) = T \cdot \omega + \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx. \quad (2.43)$$

For the given set of controls T and F as defined in the statement of the theorem, (2.43) becomes

$$\begin{aligned} \frac{d}{dt}V(\omega, \psi; t) = & -k_1 \omega_1^2 - k_2 (\omega_2 - \omega_0)^2 - k_3 \omega_3^2 \\ & - \int_{\Omega} d_1(x) \left(\frac{\partial y}{\partial t} \right)^2 dx - \int_{\Omega} d_2(x) \left(\frac{\partial z}{\partial t} \right)^2 dx. \end{aligned} \quad (2.44)$$

We claim that V is positive definite. In fact, positivity of V with respect to ω and $\frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}$ is immediate. Further, if $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x^2}$ are zero a.e. on Ω , then, using the boundary conditions (2.20), one could easily verify that y and z are necessarily zero on Ω . Hence, V is positive definite.

From (2.44), we see that $\frac{dV}{dt}$ is negative semidefinite for the given controls. Therefore, the rest state is stable in the sense of Lyapunov. We show that the rest state is actually asymptotically stable.

If $\frac{dV}{dt}$ were to vanish identically for all $t \in [t_1, t_2]$, ω must be zero and $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ must be zero a.e. on Ω for all $t \in [t_1, t_2]$ (see (2.44)). This requires that

$$\dot{\omega} = 0,$$

and

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 z}{\partial t^2} = 0 \text{ a.e. on } \Omega.$$

Then, from (2.38), it follows that

$$\frac{\partial^2}{\partial x^2} (B_2 \frac{\partial^2 \phi}{\partial x^2}) = 0. \quad (2.45)$$

The elliptic equation (2.45) along with the boundary conditions (2.20) implies that $\phi(\equiv(y,z))$ is zero on Ω . Therefore, $\frac{dV}{dt}$ is negative definite. Hence, the functional defined by (2.42) is, indeed, a Lyapunov functional and consequently, the rest state of the system (2.14) and (2.19) is asymptotically stable. This completes the proof. \square

Remark 2.1

The Lyapunov functional defined in (2.42) actually represents the total energy of the system. We note that the first term in (2.42) represents the kinetic energy of the satellite body; the second and third terms give the potential energy of the beam due to deformation caused by vibration. The last three integrals in (2.42) representing $\frac{1}{2} \int |v + \omega \times r|^2 dm$ is the total kinetic energy of the beam due to vibration and rotation of the body axes.

Remark 2.2

From (2.44), it is clear that in the absence of any control,

$$\frac{d}{dt} V(\omega, \psi; t) = 0,$$

so that $V(\omega, \psi; t) = V(\omega, \psi; 0)$ for all t . This implies that the system is conservative. However, interchange of energy between the body and the beam may take place leading to changes in the body velocity trajectories and in the amplitude of vibration of the beam.

The distributed control law given in Theorem 2.1 may be difficult or costly for practical implementation, since it involves application of damping throughout the length of the beam. Therefore, it is essential to consider stabilization of the flexible spacecraft using localized controls. We show that asymptotic stabilization the system could be obtained by application of localized damping on the beam along with the feedback torque on the bus. We will need the following result in the sequel.

Proposition 2.1

There exists a set $A \subset \Omega \equiv [0, L]$ of positive Lebesgue measure such that the velocity $\frac{\partial \phi}{\partial t}$ does not vanish identically for all t .

Proof

Let $\phi(x, 0) = \phi_0(x)$ and $\omega(0) = \omega_0$ be arbitrary. We prove the assertion by contradiction. Suppose that no such set A exists. Then, $\frac{\partial \phi}{\partial t}$ is zero a.e. on Ω for all t . Consequently, the second equation of (2.32) implies that

$$\frac{\partial^2}{\partial x^2} (B_2 \frac{\partial^2 \phi}{\partial x^2}) + \rho B_3(\omega) \phi + \rho(x+R) B_4(\omega) = 0,$$

which holds for arbitrary ω and $t \geq 0$. Scalar multiplying both sides of the above equation by ϕ and integrating by parts over Ω , we have

$$\int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx + \int_{\Omega} \rho B_3(\omega) \phi \cdot \phi dx + B_4(\omega) \int_{\Omega} \rho(x+R) \phi dx = 0.$$

Since ω_0 is arbitrary, ω can take arbitrary values in \mathbb{R}^3 .
Hence the above equation implies that

$$\int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx = 0 \quad \text{for all } t \geq 0.$$

By Poincaré's inequality

$$\int_{\Omega} \phi^2 dx \leq K \int_{\Omega} \left(\frac{\partial \phi}{\partial x} \right)^2 dx \quad \text{for some } K > 0,$$

$$\leq \frac{K}{\min\{EI_Y, EI_Z\}} \int_{\Omega} B_2 \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial x^2} dx = 0.$$

Hence ϕ is zero a.e. on Ω for all $t \geq 0$. This gives rise to a contradiction, since ϕ is continuous in t and ϕ_0 is arbitrary. \square

We now present the result on asymptotic stabilization of the system using localized controls.

Theorem 2.2

Consider the system (2.14) and (2.19) with the boundary conditions (2.20) and suppose that the controls are given by the localized feedback structure:

$$T = (-k_1 \omega_1, -k_2 (\omega_2 - \omega_0), -k_3 \omega_3), \quad k_1, k_2, k_3 > 0,$$

and

$$(2.46)$$

$$F = \left(-\tilde{d}_1(x) \frac{\partial v}{\partial t}, -\tilde{d}_2(x) \frac{\partial z}{\partial t} \right),$$

with $\tilde{d}_1(x) = \sum_{i=1}^N a_i x_{A_i}(x)$, $a_i > 0$,

$$\tilde{d}_2(x) = \sum_{i=1}^M b_i x_{B_i}(x) \quad b_i > 0,$$

where x_E denotes the indicator function of the set E and $\{A_i, i=1, 2, \dots, N\}$ and $\{B_i, i=1, 2, \dots, M\}$ are any two families of Lebesgue measurable subsets of Ω such that $\frac{\partial v}{\partial t}$ and $\frac{\partial z}{\partial t}$ do

not vanish identically on \mathcal{A}_1 and \mathcal{B}_1 respectively. Then, the system is asymptotically stable with respect to the rest state (2.34).

Proof

Follows from Proposition 2.1 and Theorem 2.1. \square

2.3.1 Controls applied to both the Bus and the Beam

We now present the results of numerical simulation of the flexible spacecraft governed by the set of ordinary differential equations (2.14) and the set of partial differential equations (2.19) with the boundary conditions (2.20). This hybrid set of equations were solved simultaneously using a combination of Runge-Kutta method for the ordinary differential equations and finite difference method for the partial differential equations. The flow chart for the algorithm is given in Appendix B.

For the purpose of simulation, we assume that the satellite bus inertia matrix is diagonal and that the beam is uniform along its length. The following parameters were used in the simulation:

Bus Data

$$I_{11}^b = 645 \text{ slug ft}^2$$

$$I_{22}^b = 100 \text{ "}$$

$$I_{33}^b = 669 \text{ "}$$

Beam Data

$$\text{Length } L = 18.8 \text{ ft}$$

$$\text{Flexural rigidity } EI = 3550.8 \text{ lb ft}^2$$

$$\text{Mass density } \rho = 2.86 \times 10^{-2} \text{ slug/ft}$$

$$\text{Location of Beam } R = 3 \text{ ft}$$

from the center of body frame.

where I_i^b , $i=1,2,3$ are the diagonal entries of the satellite bus inertia I^b , which is constant. However, note that the beam inertia $I^B = I^B(y,z)$ (see (2.3)) and hence the total inertia $I^T = I^b + I^B$ are implicit functions of time. The initial conditions were taken as:

$$\dot{\theta}_1(0) = 0.03 \text{ rad/sec}$$

$$\dot{\theta}_2(0) = 0.02 \text{ "}$$

$$\dot{\theta}_3(0) = 0.01 \text{ "}$$

and for $x \in [0, L]$

$$\frac{\partial y}{\partial t}(x,0) = 0, \quad \frac{\partial z}{\partial t}(x,0) = 0,$$

$$y(x,0) = z(x,0) = e^{c \left[(\cosh \lambda L + \cos \lambda L) (\sinh \lambda x - \sin \lambda x) - (\sinh \lambda L + \sin \lambda L) (\cosh \lambda x - \cos \lambda x) \right]},$$

where c satisfies $\cos \lambda L \cosh \lambda L + 1 = 0$, $c = -0.001071$.

2.3.1.1 Proportional Control

We first consider stabilization of the spacecraft by using proportional controls applied to both the bus and the beam as in (2.46). We consider

$$k_1 = 600, \quad k_2 = 100, \quad k_3 = 600,$$

$$d_1 = d_2 = 3.0 \chi_{[0.95, 1.0]}$$

where χ_E is the indicator function of the set E (expressed in normalized form). The beam damping coefficients d_1 and d_2 imply that the controls are applied only on a small section of 5% of the length of the beam located at the free end. Fig. 2.2-2.10 show the effectiveness of the chosen feedback law in stabilizing the flexible spacecraft.

As mentioned earlier (see Remark 2.2), the uncontrolled system is conservative. In other words, the sum of the absolute (i.e. with respect to the inertial space) bus energy and the absolute beam energy remains constant. Note that the absolute energy of the bus is given by the first term of the Lyapunov functional (2.42) and the rest of the terms of (2.42) represent the absolute energy of the beam. Since the beam is attached to the bus, interchange of energy between the bus and the beam may take place during perturbations leading to changes in the amplitude of vibrations of the beam and the wobbling of the bus. This can be clearly observed from Fig. 2.2, Case A. (A slight variation in the total energy of the uncontrolled system could be attributed to the truncation and discretization errors associated with numerical simulation). With application of the suggested controls the bus angular motions and the beam vibrations decay to zero as can be seen in Fig. 2.2, Case B.

Fig 2.2 describes the response of the flexible spacecraft in terms of energy. Additional information regarding the beam vibrations and the bus angular motions are given in details in Fig. 2.3-2.10.

In Fig. 2.3 and 2.5, we show the deflections of the beam at the free end, i.e. $y(L,t)$ and $z(L,t)$. The corresponding velocities are plotted in Fig 2.4 and 2.6 respectively. From these figures, it is clear that in absence of appropriate

controls (Fig. 2.3-2.6, Case A), the beam oscillations grow with time, thus making the spacecraft unsuitable for any application. However, with the application of the chosen feedback controls, these oscillations eventually decay to zero. This can be observed from Fig. 2.3-2.6, Case B.

The fact that the beam vibrations are reduced throughout its length can be observed from the beam 'relative energy' curve of Fig. 2.7. We define the beam 'relative energy' function as a measure of vibration of the beam with respect to the body axes by

$$E^B(t) = \int_0^L \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial z}{\partial t} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 y}{\partial x^2} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 z}{\partial x^2} \right)^2 \right] dx. \quad (2.47)$$

We note that $E^B(t)$ represents the total energy of the beam relative to the body axes. Clearly, $E^B(t)=0$ implies that all the points of the beam are at the rest position. In Fig. 2.7, Case A, we observe that in the uncontrolled system the beam relative energy increases with time, indicating growing oscillations of the beam with respect to the body axes. The applied feedback controls successfully eliminate all the beam vibrations throughout its length as indicated by the decayed beam energy in Fig. 2.7, Case B.

The effects of beam vibrations on the satellite bus angular velocities are shown in Fig. 2.8-2.10, Case A. It can be observed, from these figures, that the angular velocities ω_2 and ω_3 experience larger fluctuations as

compared to ω_1 . This is because of the fact that the beam vibrations in the j_b and k_b directions would contribute to the total angular momentum about the k_b and j_b axes respectively. Note that the beam is along the i_b axis, and that there is no deformation along the axial direction. The relative magnitude of the bus inertia and the beam inertia is another contributing factor. Stabilization of the angular velocities by the application of the feedback controls can be observed from Fig. 2.8-2.10, Case B.

It is to be noted here that the rate of decay of beam vibrations and also the bus angular velocities depend on the magnitude of the coefficients in the feedback law. A larger feedback would produce a faster stabilization of the flexible spacecraft. Furthermore, from Fig. 2.7, Case B it can be observed that the beam vibration increases initially, even though the controls are applied to the system. However, the magnitude of the controls can be suitably chosen in order that the vibrational energy is nonincreasing.

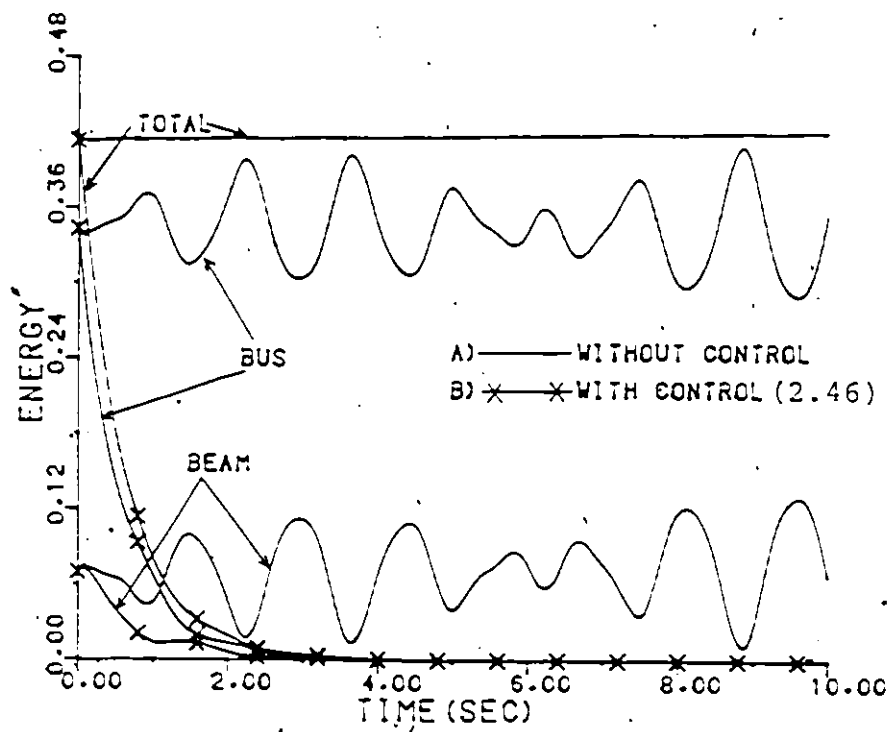
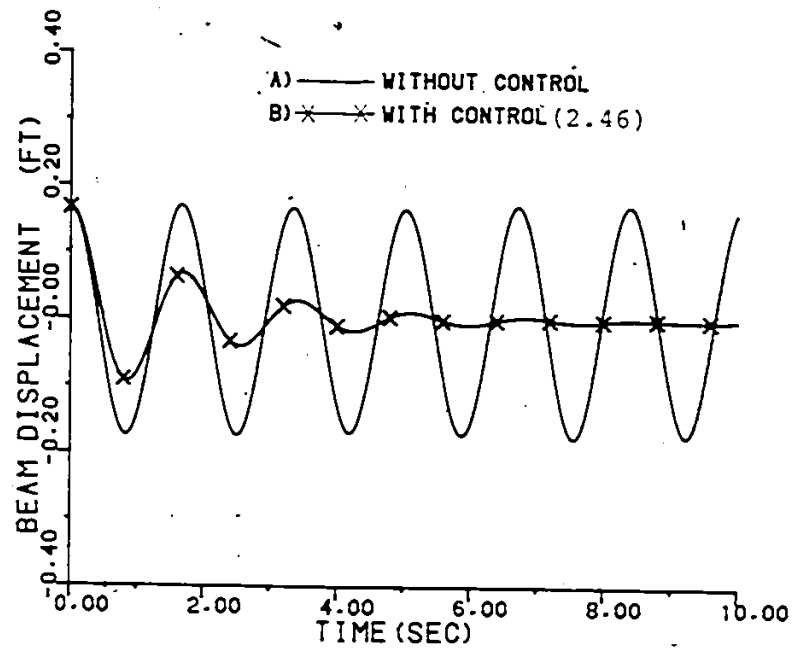
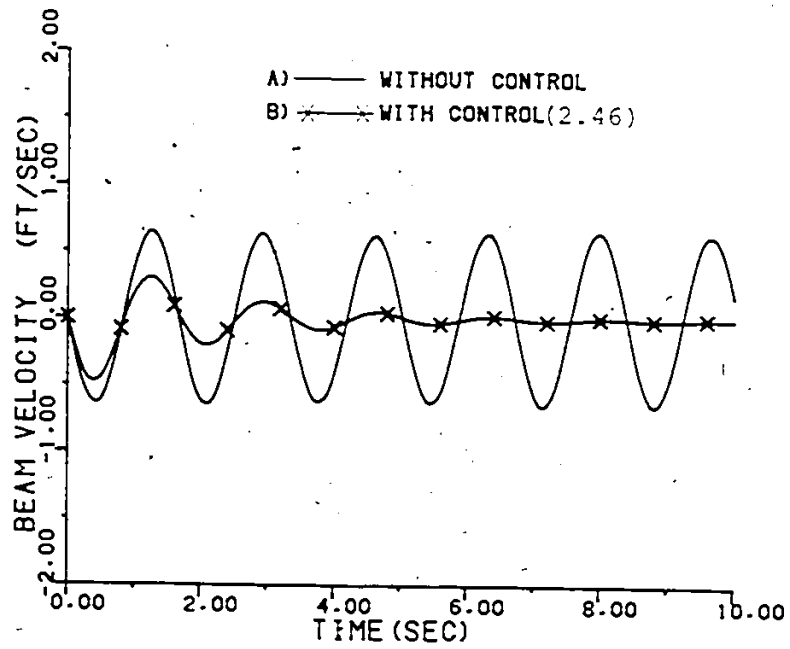


Fig. 2.2 Absolute Energy of the Bus and the Beam

Fig. 2.3 Beam Displacement $y(L,t)$ Fig. 2.4 Beam Velocity $\frac{\partial y}{\partial t}(L,t)$

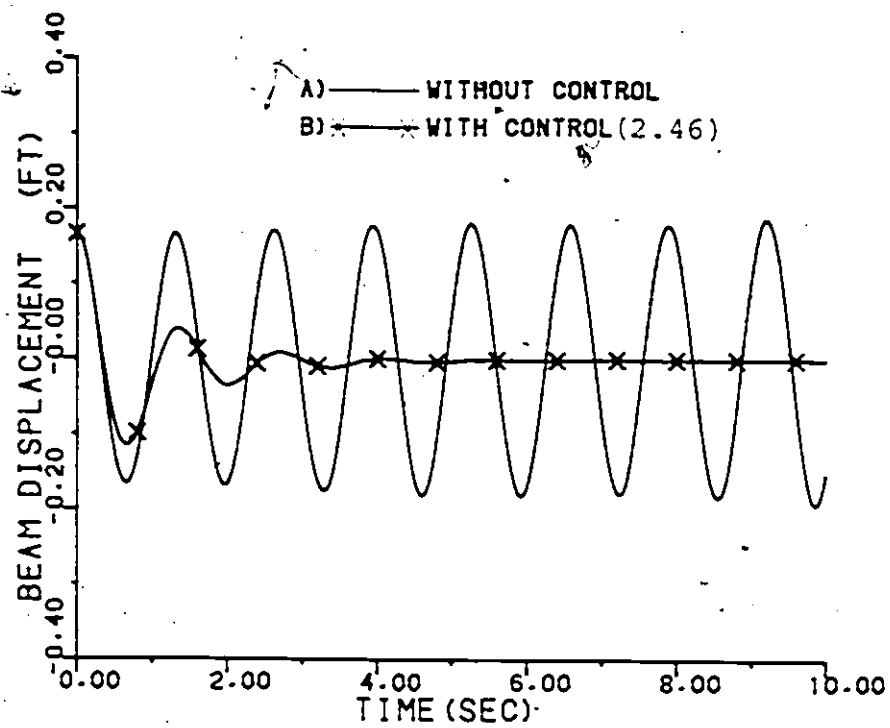


Fig. 2.5 Beam Displacement $z(L,t)$

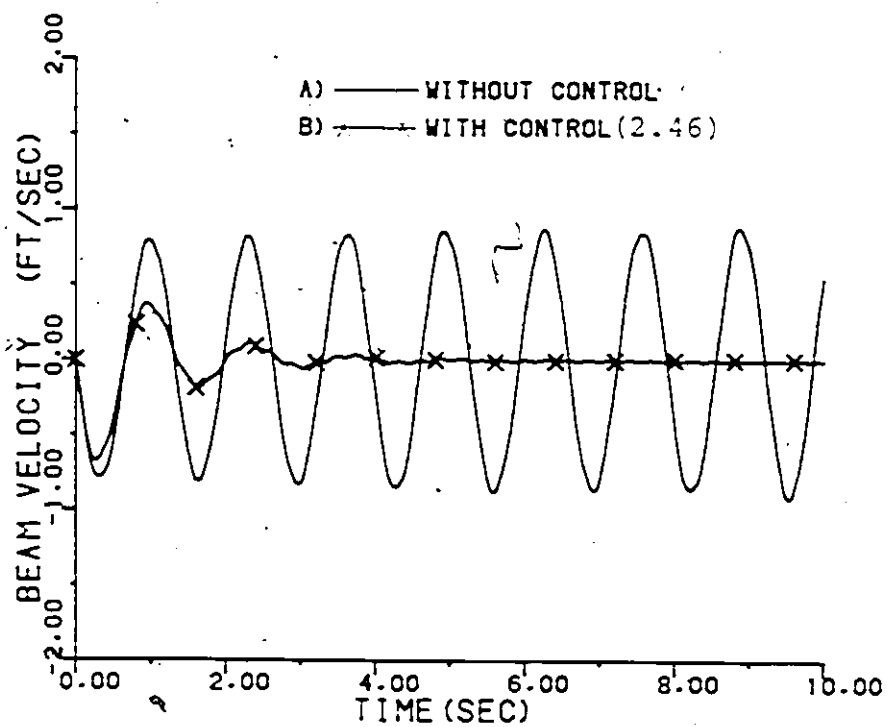


Fig. 2.6 Beam Velocity $\frac{\partial z}{\partial t}(L,t)$

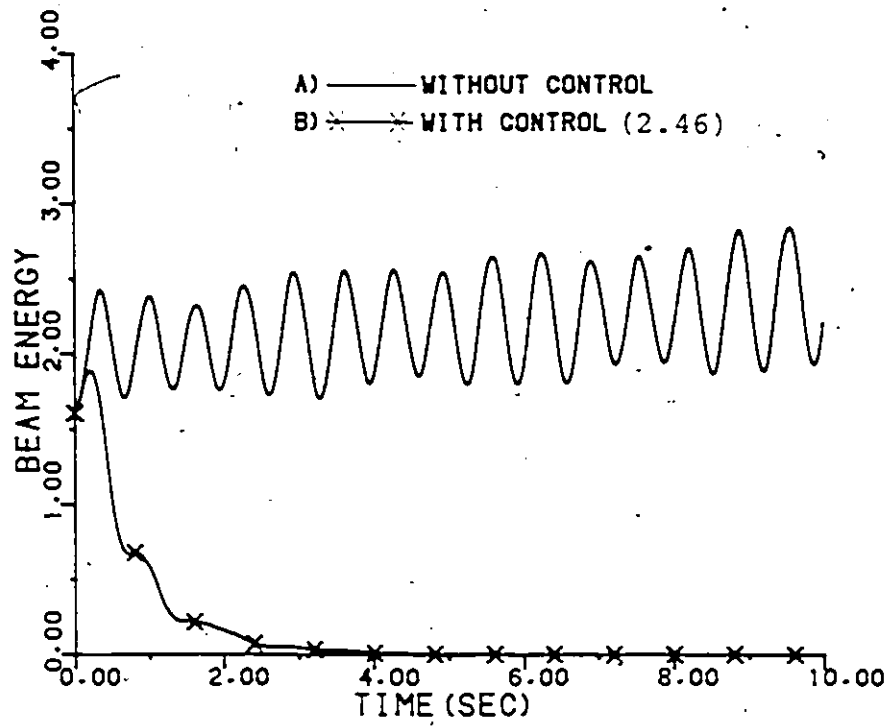


Fig. 2.7 Beam Relative Energy

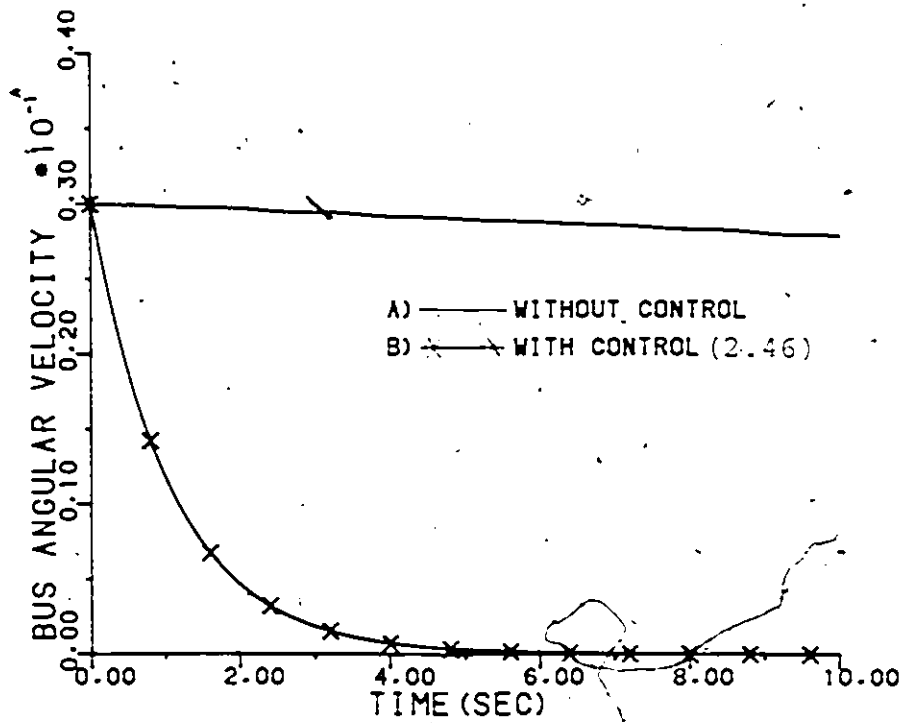


Fig. 2.8 Bus Angular Velocity ω_1 (rad/sec)

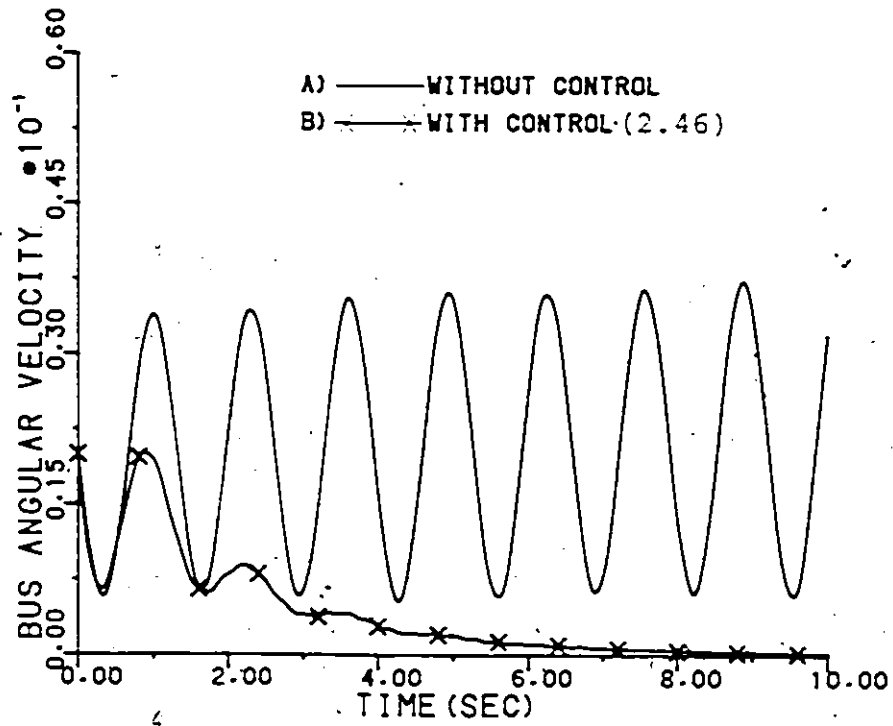


Fig. 2.9 Bus Angular Velocity ω_2 (rad/sec)

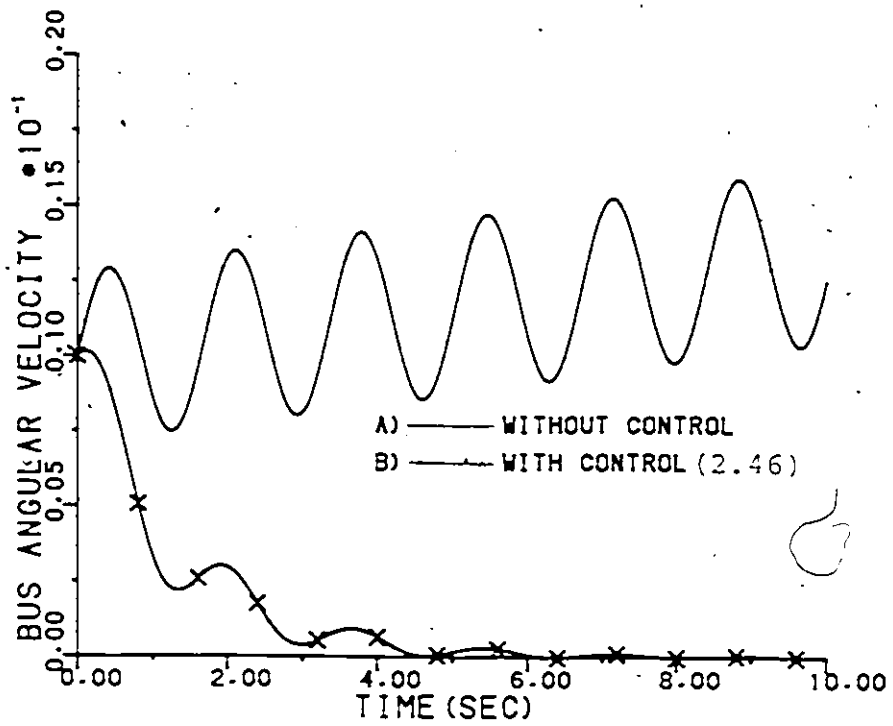


Fig. 2.10 Bus Angular Velocity ω_3 (rad/sec)

2.3.1.2 Bang Bang Control

Practical implementation of the proportional controls of Theorem 2.1 and Theorem 2.2, require throttleable devices. However, for satellite applications bang bang type of controls are more suitable as compared to the proportional controls. The following result on the stabilization of the flexible spacecraft using bang bang controls follows easily from Theorem 2.2.

Corollary 2.1

Let the function $\text{sign } s$ be defined by

$$\text{sign } s = \begin{cases} 1, & s > 0, \\ -1, & s < 0, \\ 0, & s = 0. \end{cases}$$

Then, for the feedback control

$$T = (-k_1 \text{sign } \omega_1, -k_2 \text{sign } (\omega_2 - \omega_0), -k_3 \text{sign } \omega_3), k_1, k_2, k_3 > 0, \quad (2.48)$$

and

$$F = (-d_1(x) \text{sign}(\frac{\partial Y}{\partial t}), -d_2(x) \text{sign}(\frac{\partial Z}{\partial t})), d_1, d_2 \geq 0 \text{ a.e. on } \Omega.$$

the rest state of the system described by equations (2.14) and (2.19) is asymptotically stable.

The effects of controls corresponding to the feedback law (2.48) are also illustrated by simulation results as shown in Fig. 2.11-2.19. In these figures, we have used the same initial conditions as in Section 2.3.1.1. For the control law (2.48) we take

$$k_1 = 3.0, \quad k_2 = 0.5, \quad k_3 = 2.0,$$
$$d_1 = d_2 = 0.05 \times [0.95, 1.0].$$

A comparison of the Figures 2.11-2.19 with the Figures 2.2-2.10 indicates that the overall response of the system under the action of the control law (2.48) is quite similar to those corresponding to the controls (2.46). The basic difference in the two sets of figures is in the nature of decay of various states of the system. In the former case, i.e. Figures 2.2-2.10, the decay rates become slower as the magnitude of various states become smaller, whereas in the later case, i.e. Figures 2.11-2.19, it remains more or less unchanged. This could be observed specially by comparing the pairs Fig. 2.8 - Fig. 2.17, Fig. 2.9 - Fig. 2.18, and Fig. 2.10 - Fig. 2.19.

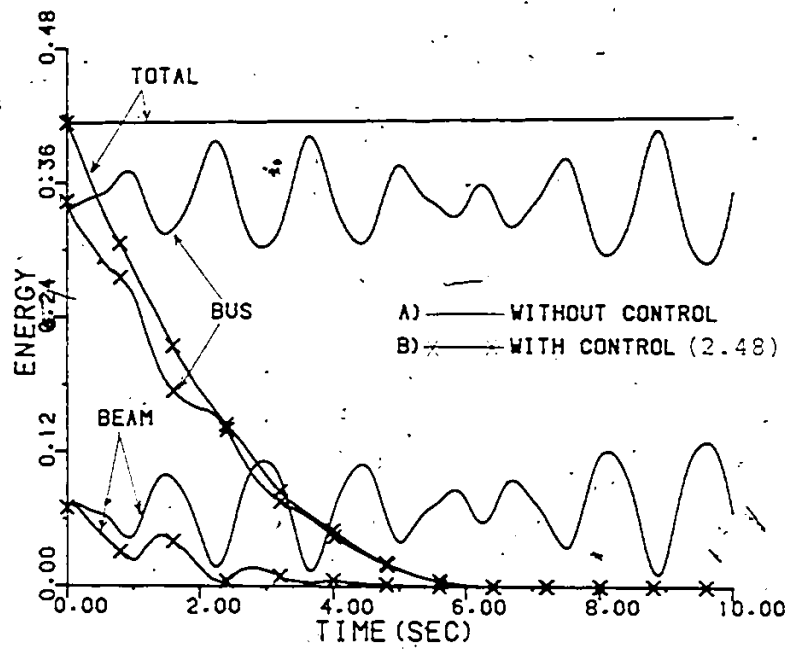


Fig. 2.11 Absolute Energy of the Bus and the Beam

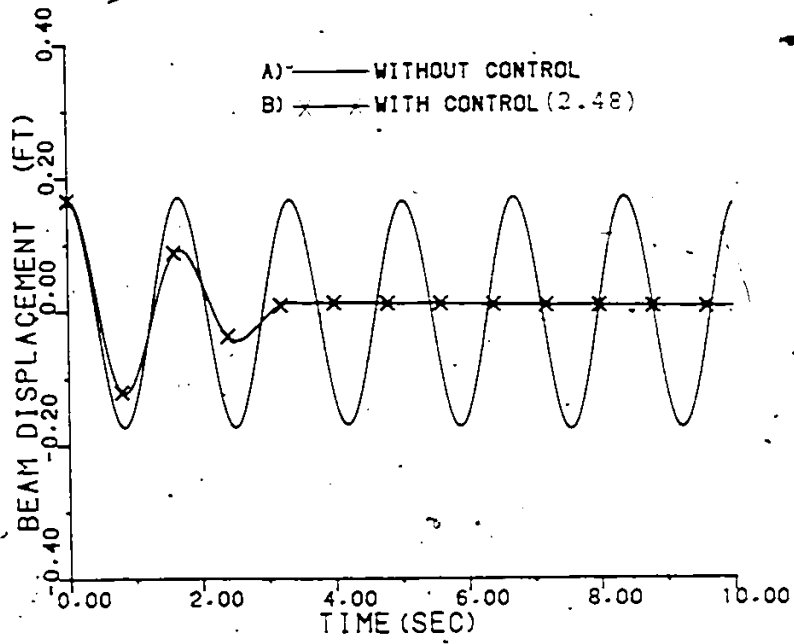


Fig. 2.12 Beam Displacement $y(L,t)$

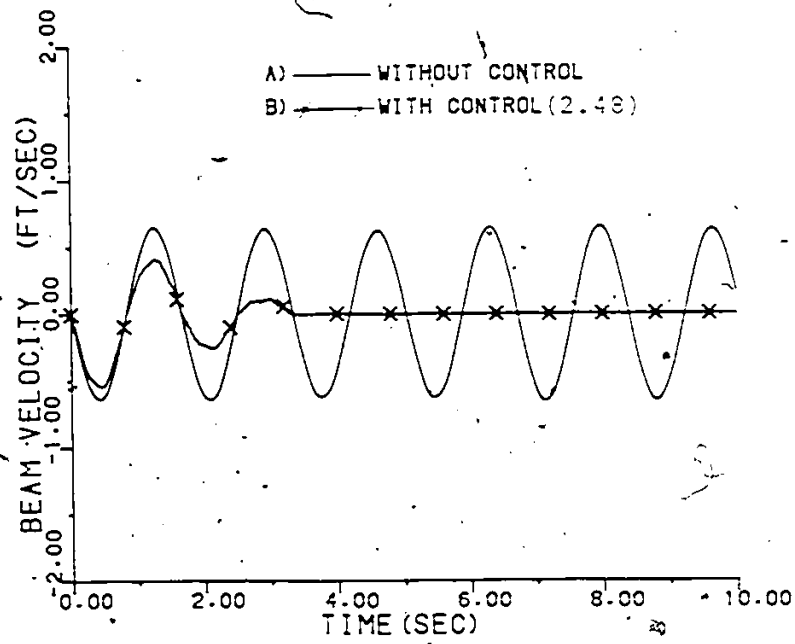
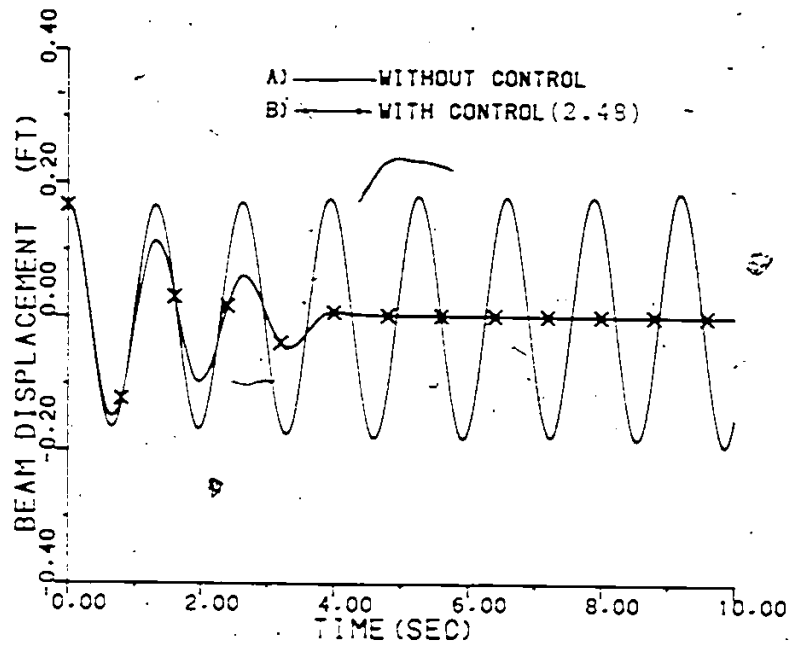
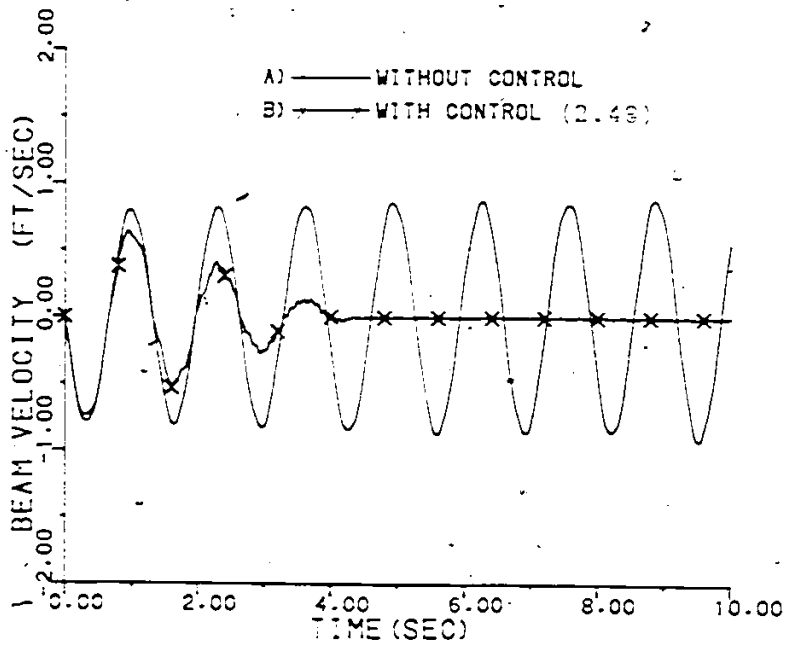


Fig. 2.13 Beam Velocity $\frac{\partial y}{\partial t}(L,t)$

Fig. 2.14 Beam Displacement $z(L,t)$ Fig. 2.15 Beam Velocity $\frac{dz}{dt}(L,t)$

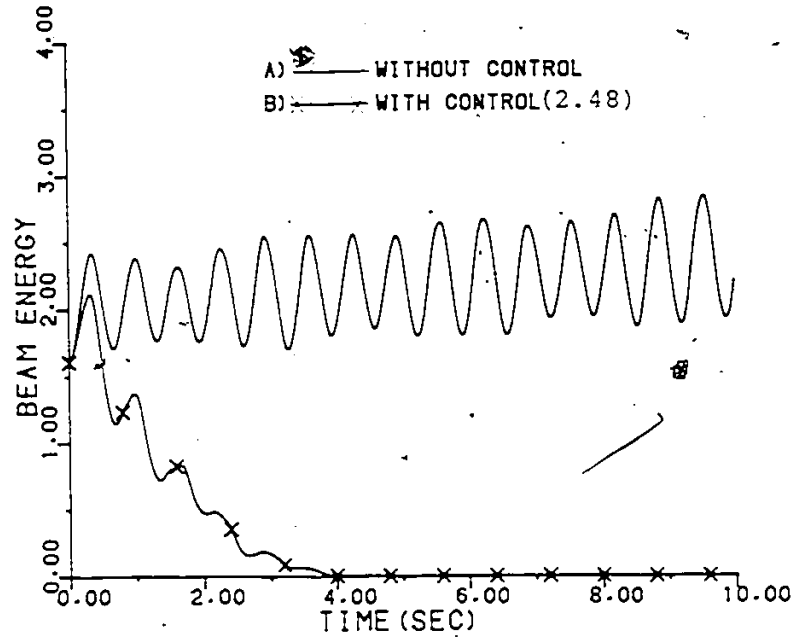


Fig. 2.16 Beam Relative Energy

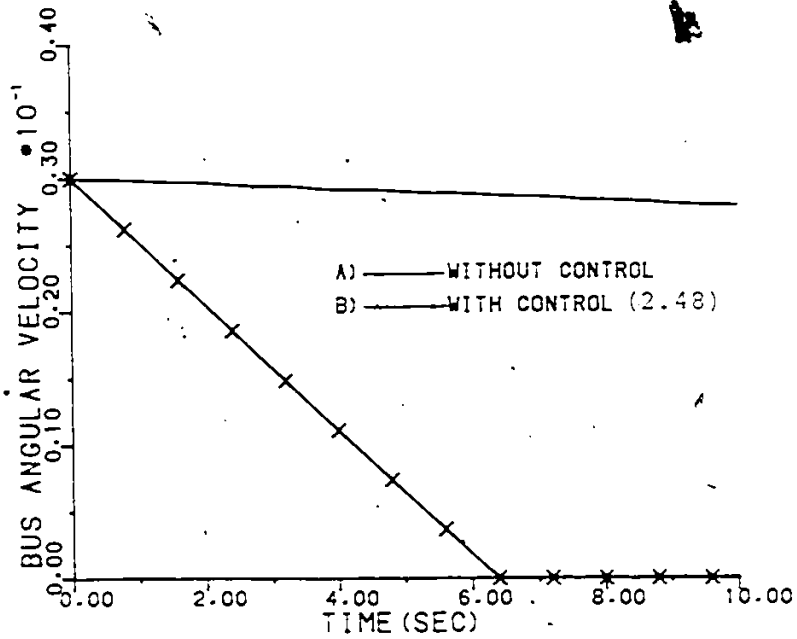


Fig. 2.17 Bus Angular Velocity ω_1 (rad/sec)

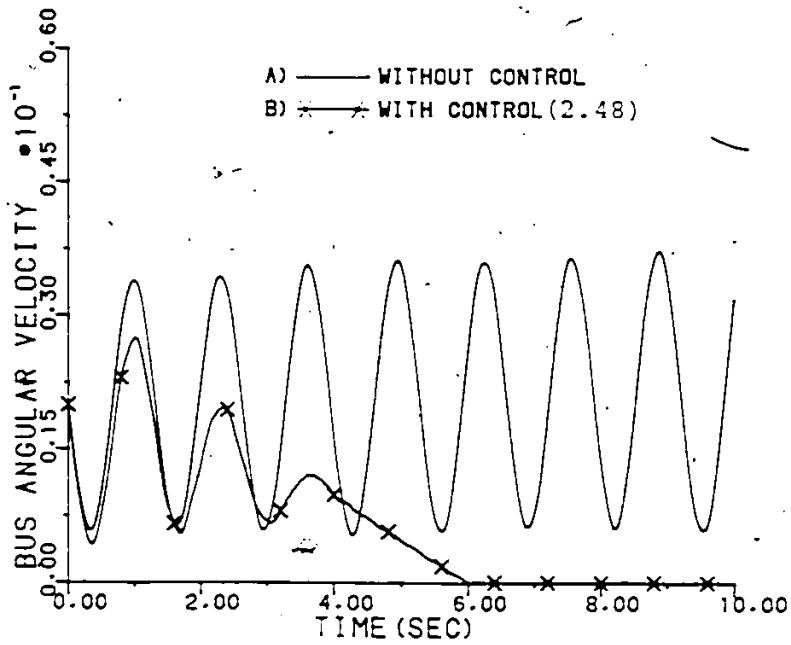


Fig. 2.18 Bus Angular Velocity ω_2 (rad/sec)

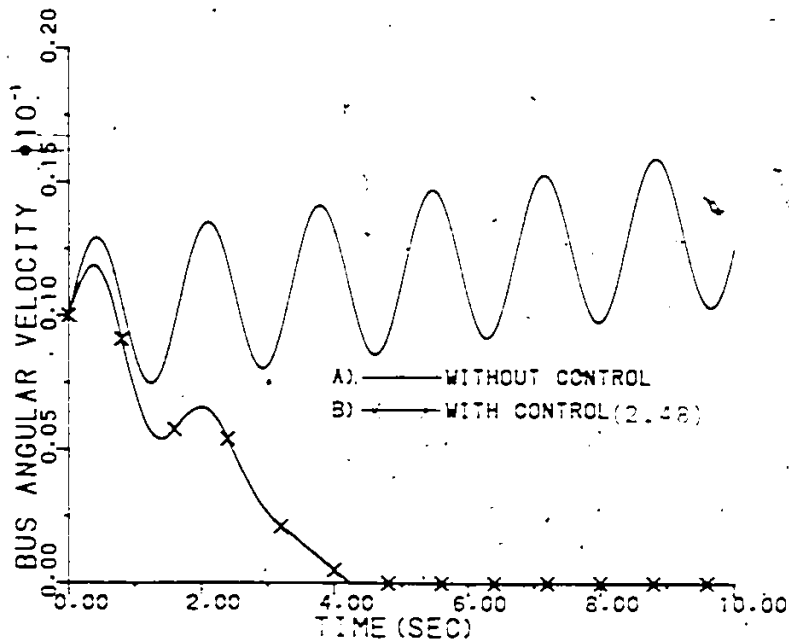


Fig. 2.19 Bus Angular Velocity ω_3 (rad/sec)

2.3.1.3 Deadzone Control

In some of the applications of geo-synchronous satellites, it is not absolutely essential to attain asymptotic stabilization of the system. Further, for the purpose of saving fuel, it suffices to obtain only an approximate stabilization in the sense that the perturbed system trajectory would converge to an ϵ -neighbourhood of the desired rest state by application of appropriate controls. For such cases, we have the following corollary.

Corollary 2.2

Consider the system described by (2.14) and (2.19) with the boundary conditions (2.20). Let the feedback controls be given by

$$T = (-k_1 dz_1(u_1), -k_2 dz_2(u_2 - u_0), -k_3 dz_3(u_3)), k_1, k_2, k_3 > 0, \quad (2.49)$$

and

where

$$F = (-d_1(x) dz_4(\frac{\partial y}{\partial t}), -d_2(x) dz_5(\frac{\partial z}{\partial t})), d_1, d_2 > 0 \text{ a.e.,}$$

$$dz_i(t) = \begin{cases} 1 & \text{if } t \in I_i \\ -1 & \text{if } t \in J_i \\ 0 & \text{if } t \notin I_i \cup J_i \end{cases}$$

Then the system is (approximately) asymptotically stable with respect to the ϵ -neighbourhood N_ϵ of the desired rest state, defined by

$$N_\epsilon = \{(x, t) : |u_1| < \epsilon_1, |u_2 - u_0| < \epsilon_2, |u_3| < \epsilon_3, \frac{\partial y}{\partial t} < \epsilon_4, \frac{\partial z}{\partial t} < \epsilon_5\}$$

where $\epsilon_i > 0, i=1,2,\dots,5$, and $\gamma, z, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 z}{\partial x^2}$ as determined by equations (2.14) and (2.19).

Simulation results corresponding to the deadzone control (2.49) are presented in Fig. 2.20-2.28, where, for comparison, we have used the same initial conditions as in the case of bang bang controls (see page 56). The feedback coefficients in the control law (2.49) were taken as

$$k_1 = 3.0, \quad k_2 = 0.5, \quad k_3 = 2.0,$$

$$d_1(x) = d_2(x) = 0.05 \times [0.95, 1.0]$$

and the deadzone limits as

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.001 \quad \text{and} \quad \epsilon_4 = \epsilon_5 = 0.05.$$

Energy decay characteristic of the system corresponding to the deadzone controls is shown in Fig. 2.20. It is observed that the system vibrational energy is reduced to a small value determined by the size of the deadzone. In the simulation results this 'deadzone level' energy was obtained as 7.0×10^{-4} , which, however, is not visible in Fig. 2.20.

The response of the system in the presence of the deadzone controls is further illustrated in the beam vibrational velocity curves of Fig. 2.22 and Fig. 2.24 and in the bus angular velocity curves of Fig. 2.26-2.28. From these figures, it is clear that all the beam vibrations and the bus angular motions are reduced to the small neighbourhood, as permissible by the size of the deadzone, of the zero state. In practice, the size of the deadzone is determined on the basis of stability requirements of the particular application. Comparing the Figures 2.20-2.28

with those for the bang bang controls, i.e. Fig. 2.11-2.19, we observe that the response of the system in the two cases are very similar outside the deadzone region. In case the deadzone limits, i.e. ϵ_i , $i=1,2,\dots,5$, are reduced to zero, the system response would coincide with that for bang bang controls.

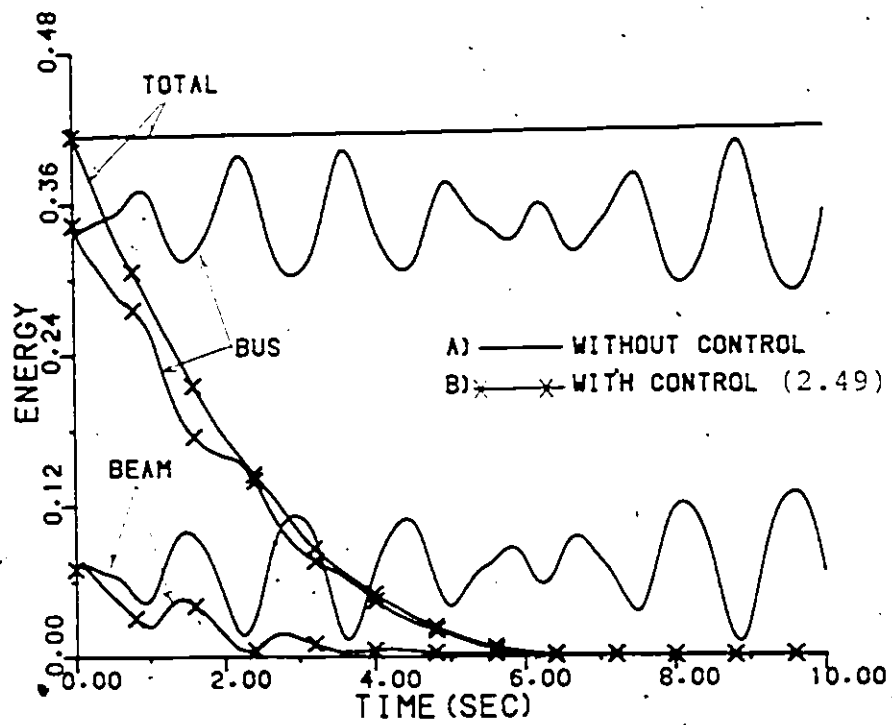
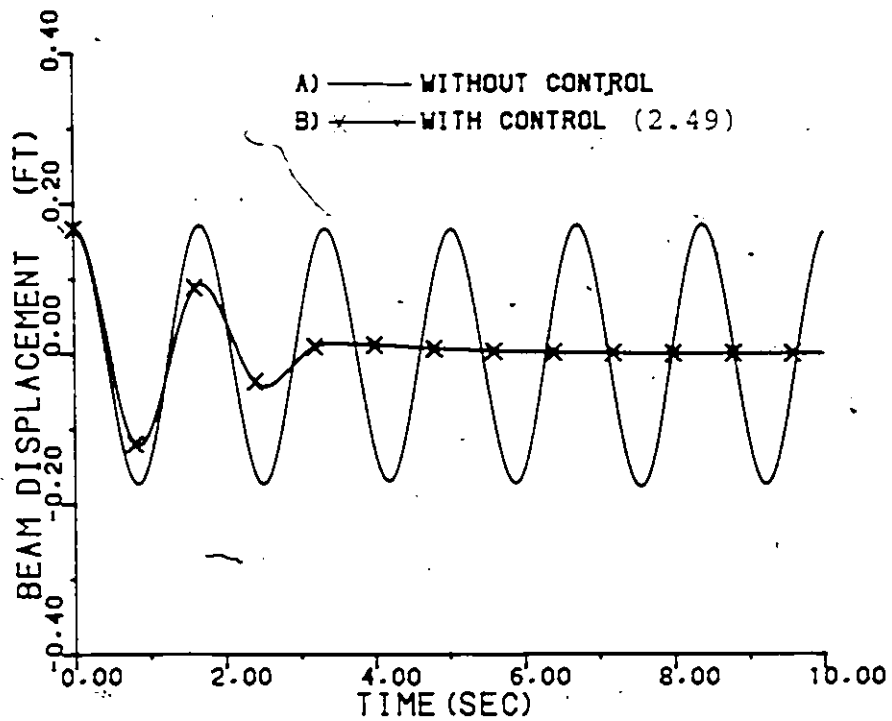
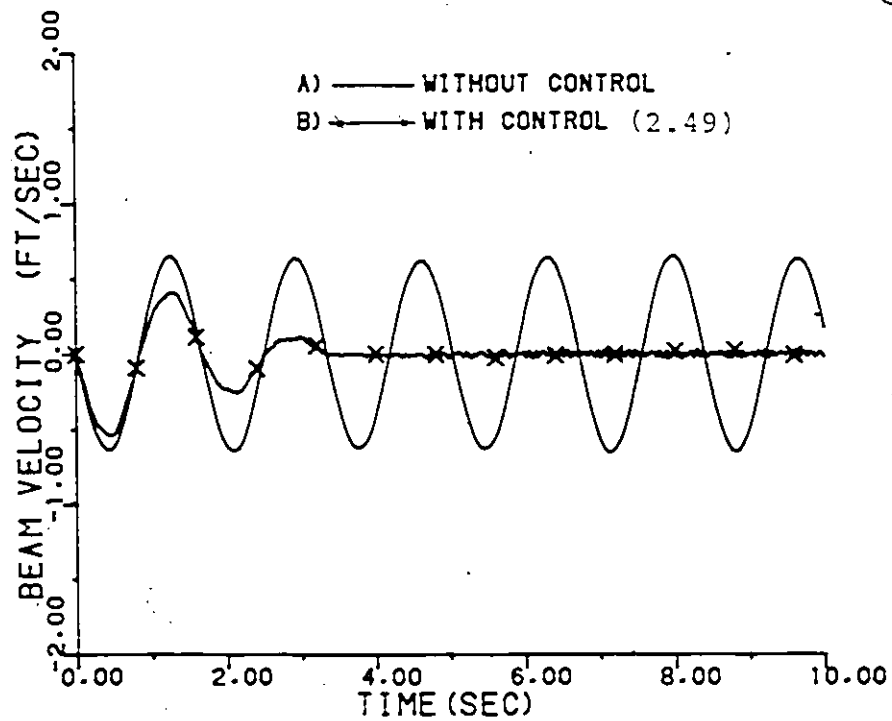


Fig. 2.20 Absolute Energy of the Bus and the Beam

Fig.2.21 Beam Displacement $y(L,t)$ Fig. 2.22 Beam Velocity $\frac{\partial y}{\partial t}(L,t)$

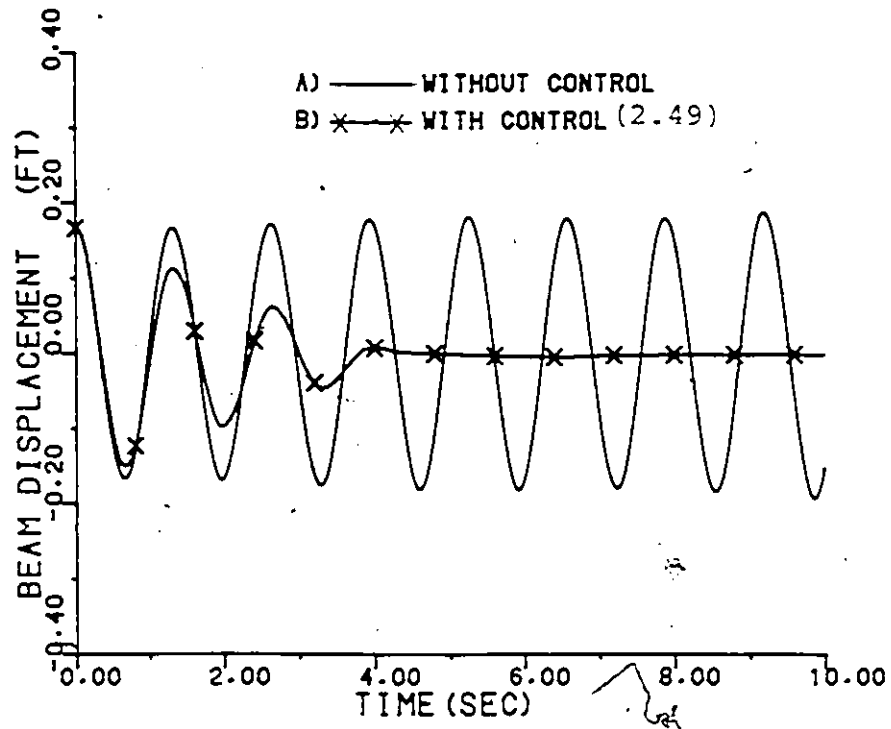


Fig.2.23 Beam Displacement $z(L,t)$

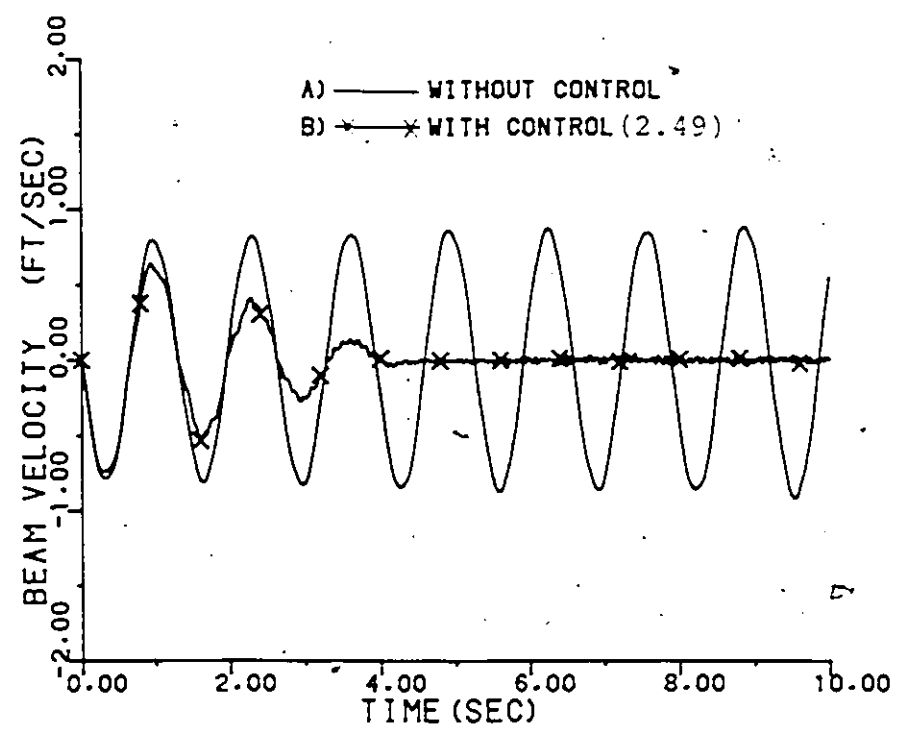


Fig. 2.24 Beam Velocity $\frac{\partial z}{\partial t}(L,t)$

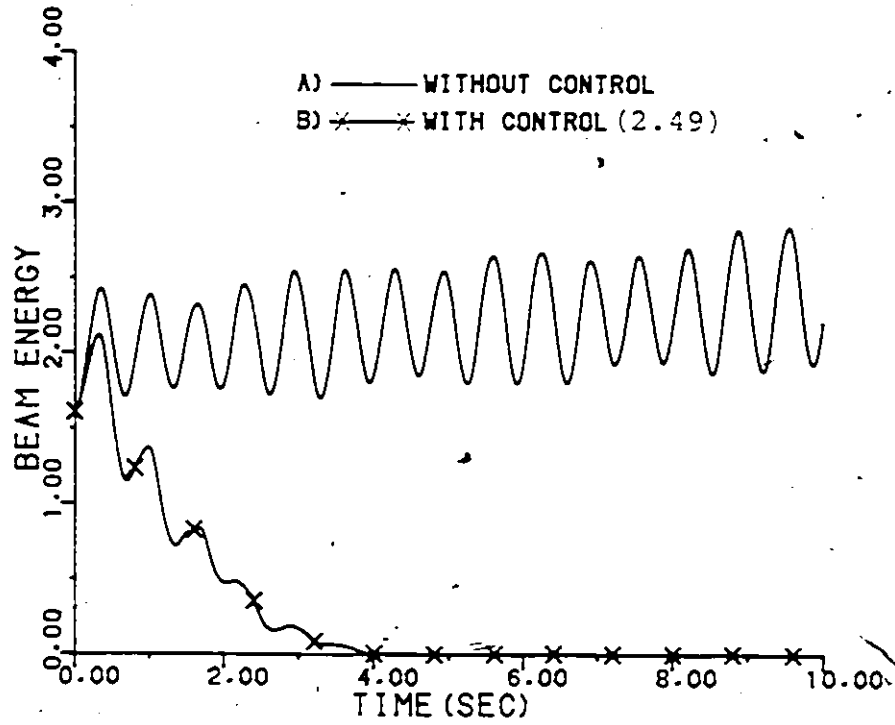


Fig. 2.25 Beam Relative Energy

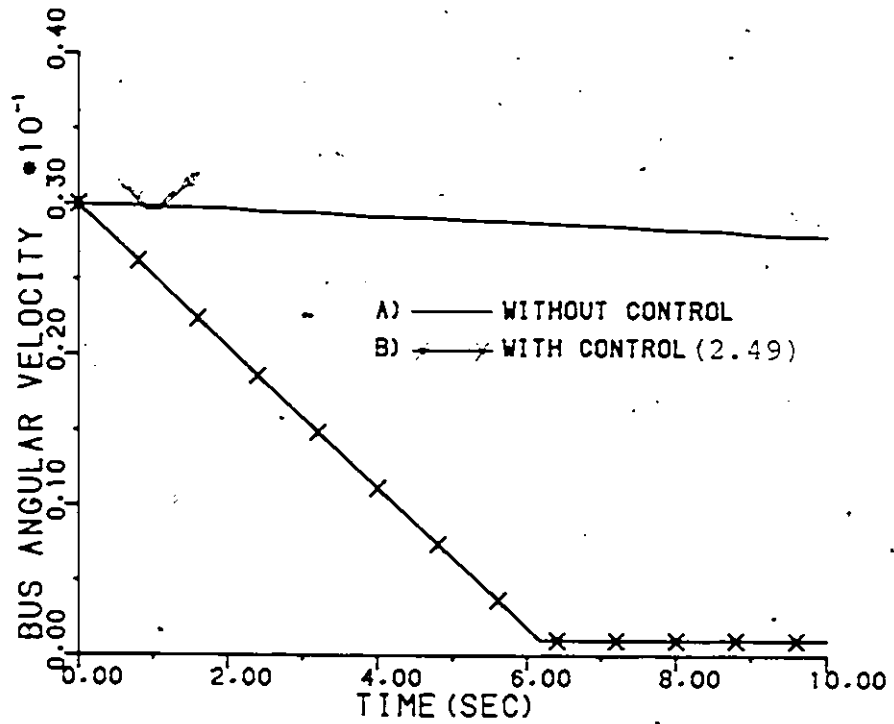
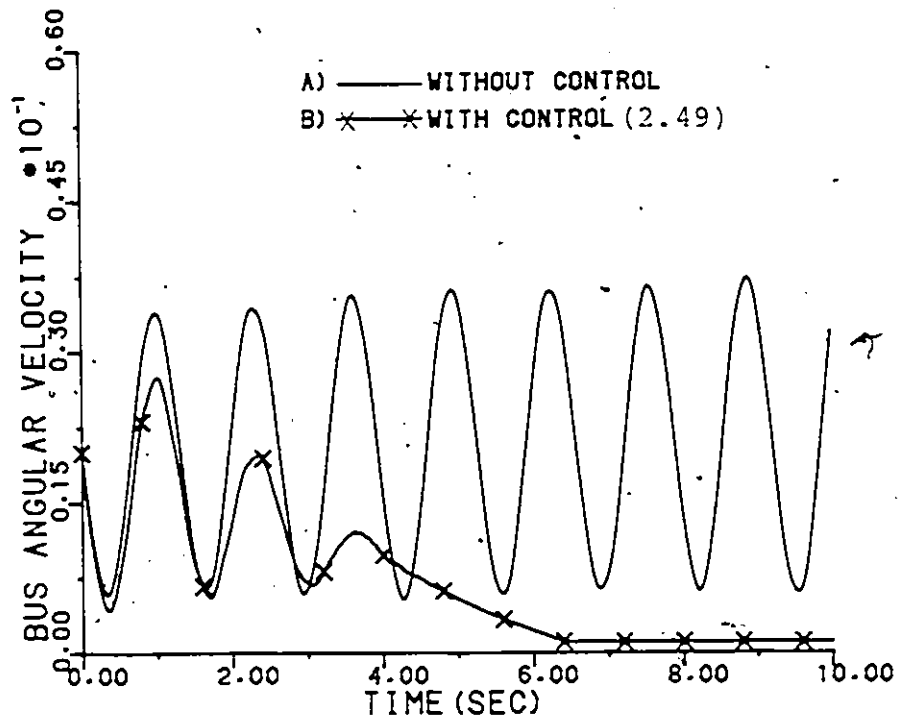
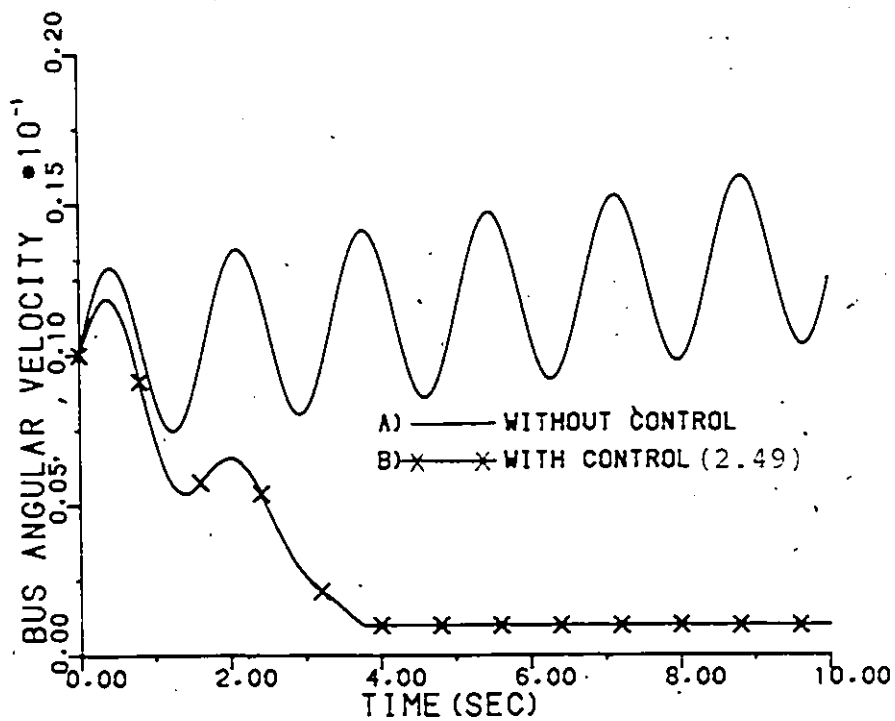


Fig. 2.26 Bus Angular Velocity ω_1 (rad/sec)

Fig. 2.27 Bus Angular Velocity ω_2 (rad/sec)Fig. 2.28 Bus Angular Velocity ω_3 (rad/sec)

2.3.2 Controls applied to the Bus only

In this section, we consider the stabilization of the flexible spacecraft by application of controls applied only on the bus. We shall consider the control structure given by

$$T = (-k_1\omega_1, -k_2(\omega_2 - \omega_0), -k_3\omega_3), \quad k_1, k_2, k_3 > 0,$$

and

$$(2.50)$$

$$F = (0, 0).$$

From (2.43), it is clear that when controls are applied only on the bus, the Lyapunov functional V (see (2.44)) has a negative semidefinite derivative. Consequently, one concludes that the system is merely stable, rather than asymptotically stable.

Simulation results of the hybrid dynamics (2.14) and (2.19) with the above control law are presented in Fig. 2.29 and Fig. 2.30. The feedback coefficients k_1, k_2 and k_3 were taken as in Section 2.3.1.1. From Fig. 2.29, it is clear that the given control stabilizes all the bus angular velocities quickly. Further, it was observed that the beam vibrations are also reduced to some extent (Fig. 2.30) by the action of the bus velocity feedback. This can be explained from energy consideration of the system. As discussed in Section 2.2, the beam vibrations have the effect of producing some motions of the bus. These bus angular velocities would then be reduced by the action of the (bus feedback) control. This in effect means flow of energy from the beam to the bus and eventual dissipation. Consequently, the beam vibrations would eventually decrease.

We note, however, that the rate of decay of the beam energy is slow as compared to the case when controls are applied to both the bus and the beam (Section 2.3.1.1) and becomes even slower as amplitude of vibrations becomes smaller, thereby taking longer time for stabilization. Therefore, the control scheme with both the bus and the beam feedback (Section 2.3.1.1) is more effective than the one with the bus feedback only.

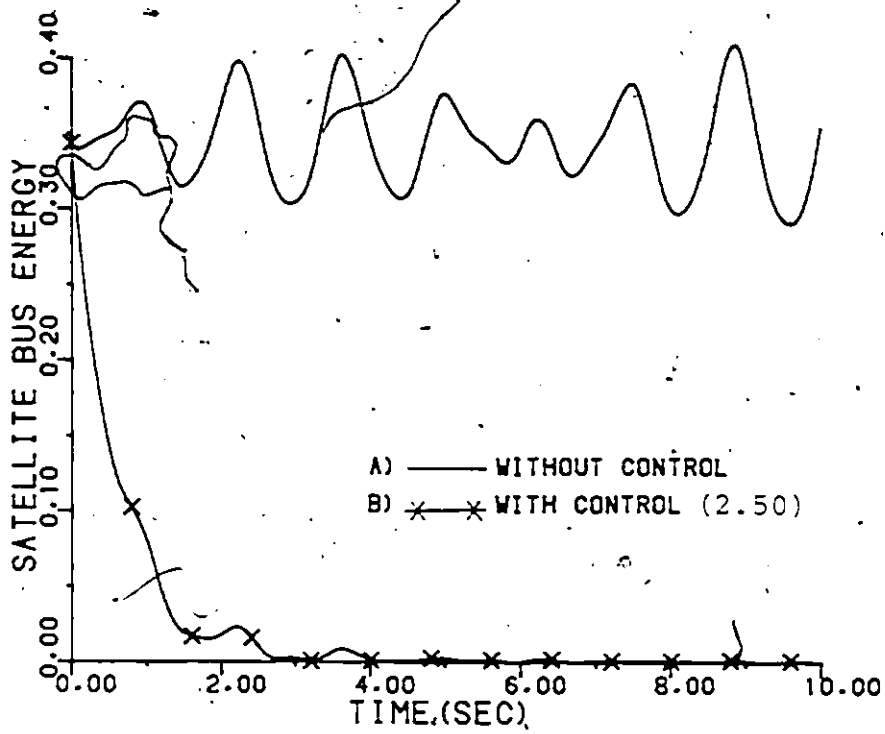


Fig.2.29 Satellite Bus Energy

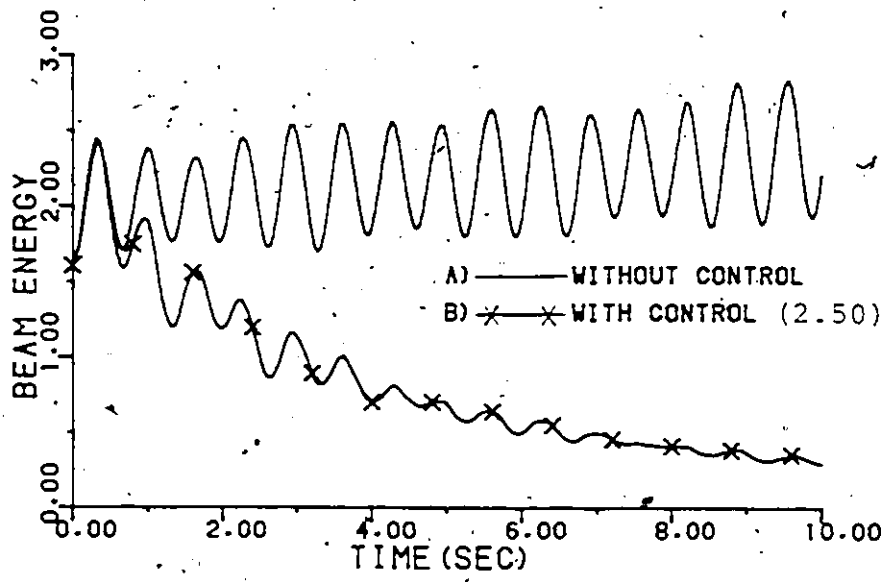


Fig.2.30 Beam Relative Energy

2.3.3 Controls applied on the Beam only

We consider the beam feedback given by

$$\text{and } T = (0, 0, 0) \quad (2.51)$$

$$F = \left(-\bar{d}_1(x) \frac{\partial v}{\partial t}, -\bar{d}_2(x) \frac{\partial z}{\partial t} \right), \quad \bar{d}_1, \bar{d}_2 \geq 0 \text{ a.e. on } \Omega.$$

where \bar{d}_1 and \bar{d}_2 are as given in (2.46). Then, it follows from (2.43) that $\frac{\partial V}{\partial t}$ is only negative semidefinite, indicating that the system is merely stable.

Simulation results with this control law are presented in Fig. 2.31 and Fig. 2.32. The beam energy curve of Fig. 2.31 clearly illustrates the fact that all the vibrations of the beam (with respect to the body axes) are eliminated by the action of the control applied to the beam. However, this control has very little effect in stabilizing the rotations of the satellite bus as can be observed from Fig. 2.32. An explanation similar to that of Section 2.3.2 can be given from energy consideration of the system and the fact that any rotation of the bus gives rise to vibration of the beam. The only difference is that the direction of flow of energy is reversed in this case. However, since the rate of dissipation of the (bus) energy is extremely slow as observed in Fig. 2.32, this control scheme would be unsuitable for stabilization of the spacecraft.

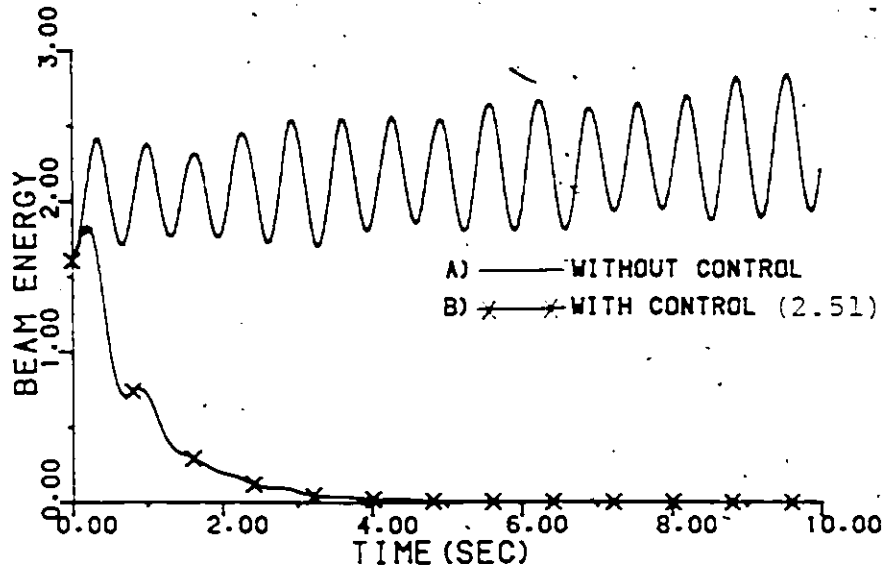


Fig. 2.31 Beam Relative Energy

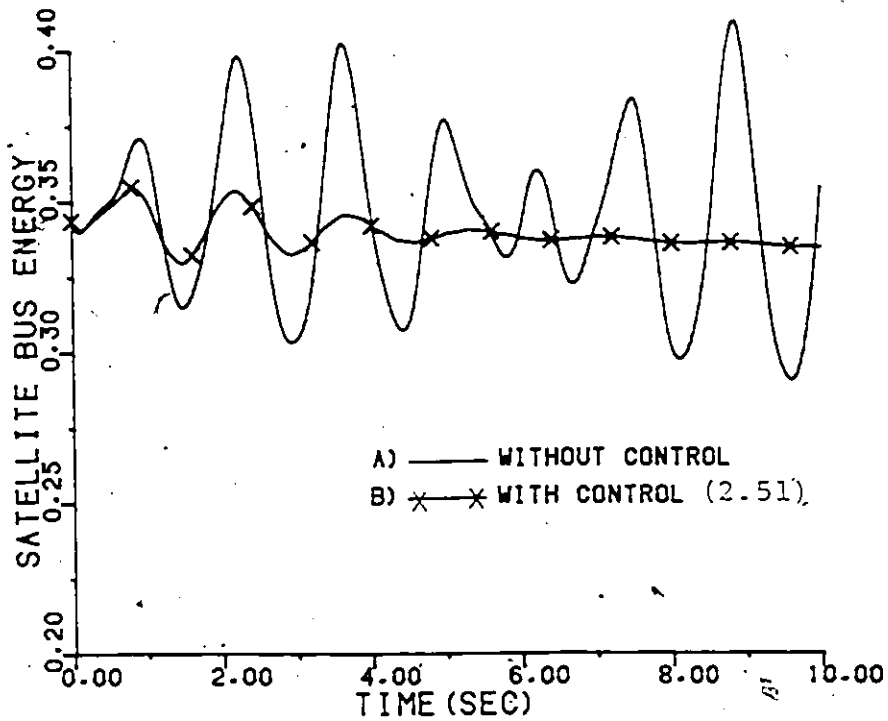


Fig. 2.32 Satellite Bus Energy

2.4 SYSTEM DYNAMICS USING TIMOSHENKO BEAM EQUATION

In the formulation of the beam dynamics (Section 2.2) associated with the satellite, we have assumed that the beam vibration is governed by the Euler-Bernoulli equation. The assumptions implicit in the theory are that the bending wavelength is several times larger than the cross-sectional dimension of the beam, and that the rotary inertia and shear displacement are negligible. The Euler-Bernoulli theory is fairly accurate in the case of a thin beam. However, a more detailed dynamics of the beam, which overcomes the above-mentioned limitations, is given by the Timoshenko beam equation. Using the Timoshenko model for the beam, one could obtain an improved dynamics for the flexible spacecraft as discussed below. Further the results in this section show that the control law of Section 2.3 is robust in the sense that it remains equally effective in stabilizing the modified system.

2.4.1 Timoshenko Beam Equation.

Considering rotary inertia and shear displacement of an incremental section, an improved model for the transverse vibration of a flexible beam is given by the Timoshenko beam equation [76]:

$$\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} \left[KGA \left(\frac{\partial v}{\partial x} - \theta \right) \right] = f, \quad (2.52)$$

$$\frac{\rho I}{A} \frac{\partial^2 \theta}{\partial t^2} - KGA \left(\frac{\partial v}{\partial x} - \theta \right) - \frac{\partial}{\partial x} \left(EI \frac{\partial \theta}{\partial x} \right) = 0$$

- ρ = mass per unit length,
 A = area of cross section,
 K = Timoshenko Shear constant,
 G = Shear modulus,
 I = moment of area of cross section,
 f = force distribution acting on the beam,
 y = transverse deflection,
 θ = angle of rotation of the mass element
 on the beam due to bending,
 E = Young's modulus.

Defining the vector $p \equiv (p_1, p_2, p_3, p_4)$ by

$$\begin{aligned}
 p_1 &= \frac{\partial v}{\partial t}, & p_2 &= KGA \left(\frac{\partial v}{\partial x} - \theta \right), \\
 p_3 &= \frac{\partial \theta}{\partial t}, & p_4 &= EI \frac{\partial \theta}{\partial x}.
 \end{aligned}
 \tag{2.53}$$

the system of equations (2.52) can be written in the first order hyperbolic form as

$$M(x) \frac{\partial p}{\partial t} = B \frac{\partial p}{\partial x} + C p + F, \tag{2.54}$$

$$M(x) = \text{diag} \left(\rho, \frac{1}{KGA}, \frac{\rho I}{A}, \frac{1}{EI} \right), \tag{2.55}$$

with the diagonal entries of M being strictly positive and

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.56}$$

2.4.2 Complete System Dynamics

As discussed in Section 2.2, the complete dynamics of the flexible spacecraft could be derived following Newtonian dynamics or Lagrangian formulation. In this section, we shall use Newtonian method for the derivation. First, we note that the bus dynamics and its derivation are completely identical to those discussed in Section 2.2.3.1. So we shall discuss the derivation of the beam dynamics only.

Corresponding to the deflection in the j_B direction, we define the vector p as in (2.53). Similarly, we define the vector q for the deflection in the k_B direction. Following the procedure as discussed in Section 2.2.3.2, we note that the effective force distribution acting on the beam in the j_B and k_B direction is given by

$$\tilde{F}_Y = F_Y - \rho \alpha_j, \quad \tilde{F}_Z = F_Z - \rho \alpha_k, \quad (2.56)$$

where F_Y and F_Z are the control forces applied to the beam in the j_B and k_B directions respectively, and α_j and α_k are the j_B and k_B components of the acceleration α given by (as in (2.18))

$$\alpha = \dot{\omega} \times r + \omega \times (\omega \times r) + 2\omega \times v, \quad (2.18)$$

Using (2.56), the dynamics of the flexible beam mounted on the satellite bus is, therefore, given by

$$\begin{bmatrix} M_Y & 0 \\ 0 & M_Z \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + F \quad (2.57.a)$$

$$(2.57.b)$$

where $\tilde{F} = (F_Y - \rho \alpha_j, 0, 0, F_Z - \rho \alpha_k, 0, 0, 0)$.

The matrix M_y (respectively M_z) in (2.57) denote the diagonal matrix M (as defined in (2.55)) corresponding to deflection in the j_B (respectively k_B) direction. We note that M_y and M_z are not necessarily the same.

The boundary conditions for (2.57) are given by

$$\begin{aligned}
 p_1(0,t) = 0, & \quad q_1(0,t) = 0, \\
 p_2(L,t) = 0, & \quad q_2(L,t) = 0, \\
 p_3(0,t) = 0, & \quad q_3(0,t) = 0, \\
 p_4(L,t) = 0, & \quad q_4(L,t) = 0.
 \end{aligned}
 \tag{2.58}$$

The complete dynamics of the flexible spacecraft considering the more detailed dynamics of the beam is, therefore, given by the set of ordinary differential equations (2.14) in conjunction with the partial differential equations (2.57) with the boundary conditions (2.58). Once again, we note that these two sets of equations are very strongly coupled. Further, the vibrations of the beam in the j_B and k_B directions are interdependent through the forcing term which is a function of the angular velocity of the bus. Clearly, in the absence of any rotation of the bus, these equations reduce to the usual equations for the transverse vibration of the beam.

2.4.3 Stabilization

We now consider the problem of stabilization of the flexible spacecraft described by the detailed model (2.14) and (2.57) with the boundary conditions (2.58). We show that the same feedback controls considered in Section 2.3 are equally effective also in this case.

Theorem 2.3

Consider the system described by (2.14) and (2.57) with the boundary conditions (2.58). Suppose that the controls applied to the system are given by the feedback law

$$T = (-k_1\omega_1, -k_2(\omega_2 - \omega_0), -k_3\omega_3), \quad k_1, k_2, k_3 > 0, \quad \text{and} \quad (2.59)$$

$$F = (-d_1(x)\frac{\partial v}{\partial t}, -d_2(x)\frac{\partial z}{\partial t}), \quad d_1, d_2 > 0 \text{ a.e. on } \Omega.$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state

$$\begin{aligned} \omega_1 &= 0, \quad \omega_2 = \omega_0, \quad \omega_3 = 0, \\ p &= 0, \quad q = 0. \end{aligned} \quad (2.60)$$

Proof

Scalar multiplying (2.57) by (p, q) and integrating by parts over $x \in \Omega = [0, L]$ and using the boundary conditions (2.58), we obtain

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} (M_y p \cdot p + M_z q \cdot q) dx \right] + \int_{\Omega} (\omega \times r) \cdot v \, dm \\ + \int_{\Omega} (\omega \times (\omega \times r)) \cdot v \, dm = \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} \, dx, \end{aligned} \quad (2.61)$$

where $F = (F_y, F_z)$, $\phi = (y, z)$ and $v = (0, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t})$.

Adding (2.61) and (2.37), one obtains

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} (I^b \cdot \omega) \cdot \omega + \frac{1}{2} \int_{\Omega} |\dot{\omega} \times r|^2 dm + \frac{1}{2} \int_{\Omega} M_y p \cdot p dx + \frac{1}{2} \int_{\Omega} M_z q \cdot q dx \right] \\ + \int (\dot{\omega} \times r) \cdot \omega dm + \int (r \times \dot{\omega}) \cdot \omega dm = T \cdot \omega + \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx . \end{aligned}$$

Using (2.55), the above equation could be rewritten as

$$\frac{d}{dt} V(\omega, p, q; t) = T \cdot \omega + \int_{\Omega} F \cdot \frac{\partial \phi}{\partial t} dx \quad (2.62)$$

where the Lyapunov function $V(\omega, p, q; t)$ is defined as

$$\begin{aligned} V(\omega, p, q; t) \equiv & \frac{1}{2} (I^b \cdot \omega) \cdot \omega + \frac{1}{2} \int_{\Omega} \left\{ \frac{\partial z}{\partial t} + \omega_1 y - (x+R)(\omega_2 - \omega_0) \right\}^2 dx \\ & + \frac{1}{2} \int_{\Omega} \left\{ \rho \left(\frac{\partial y}{\partial t} + (x+R)\omega_3 - \omega_1 z \right)^2 dx + \frac{1}{2} \int_{\Omega} \left\{ \rho \left((\omega_2 - \omega_0) z - \omega_3 y \right)^2 dx \right. \\ & + \frac{1}{2} \int_{\Omega} \left\{ \left(\frac{\rho I}{A} \right)_y p_3^2 + \left(\frac{\rho I}{A} \right)_z q_3^2 \right\} dx \\ & + \frac{1}{2} \int_{\Omega} \left\{ \left(\frac{1}{KGA} \right)_y p_2^2 + \left(\frac{1}{KGA} \right)_z q_2^2 \right\} dx + \frac{1}{2} \int_{\Omega} \left\{ \left(\frac{1}{EI} \right)_y p_4^2 + \left(\frac{1}{EI} \right)_z q_4^2 \right\} dx . \end{aligned} \quad (2.63)$$

Clearly, V as defined in (2.63) is positive definite and for the control structure (2.59) given in the statement of the theorem, $\frac{dV}{dt}$ is apparently negative semidefinite. Therefore, the rest state of the system is stable in the sense of Lyapunov. We show that the system is actually asymptotically stable.

If $\frac{dV}{dt}$ were to vanish identically for all $t \in [t_1, t_2]$, ω must be zero and p_1, q_1 must be zero a.e. on Ω for all $t \in [t_1, t_2]$. This requires that

$$\dot{\omega} = 0,$$

and

$$\dot{p}_1 = 0, \dot{q}_1 = 0.$$

Consequently, from the first equation in (2.57.a) for p_1 together with the boundary condition $p_1(0,t) = 0$, it follows that $p_2 = 0$ a.e. for all $t \in [t_1, t_2]$. Then the second equation of (2.57.a) implies that $p_3 = 0$ a.e. for all $t \in [t_1, t_2]$ and using the third equation of (2.57) we have $p_4 = 0$ a.e. Similar conclusions hold for q . Hence $\frac{dV}{dt}$ is negative definite. Consequently, the rest state of the system (2.14) and (2.57) is asymptotically stable. \square

Remark 2.3

The Lyapunov functional (2.63) represents the total energy of the system described by (2.14) and (2.57). The first term in (2.63) represents the kinetic energy of the bus and the next three terms give the kinetic energy of beam due to vibration and rotation about the body axes. The kinetic energy due to rotational displacement of the beam particles is given by the fifth term in (2.63). The last two terms of (2.63) represent the potential energy of the beam due to shear displacement and bending respectively.

Remark 2.4

Results similar to Theorem 2.2, Corollary 2.1 and Corollary 2.2 remain valid also for the modified model of the flexible spacecraft given by equations (2.14) and (2.57).

Remark 2.5

From Theorem 2.1 and Theorem 2.3, it is clear that whether or not the beam is represented by a detailed dynamics, the flexible spacecraft can be asymptotically stabilized by application of simple feedback controls.

Remark 2.6

The feedback controls suggested in this chapter are practically implementable. This involves measurement of bus angular velocities and beam vibrational velocities; and could be done using accelerometers or similar devices. The control schemes could be realized using thruster jets located on the bus and on the beam at appropriate positions.

2.5 SUMMARY

In this chapter, we develop an accurate model of a flexible spacecraft consisting of a rigid bus and a beam rigidly attached to it. It is shown that the complete dynamics of the system could be described by a coupled system of ordinary and partial differential equations. These equations indicate a very strong and intricate nature of interaction between the bus angular motions and the beam vibrations.

An important aspect of this model is that the system is described in terms of variables which can be physically

measured. This is extremely important in the analysis of system behavior and also in the design of stabilizing controls.

It has been shown that the flexible spacecraft can be stabilized by application of simple feedback controls. A combination of bus velocity feedback and beam damping has been found to be most effective in stabilizing the system. These controls can be practically implemented. It is also shown that these control laws are equally effective irrespective of order of accuracy in modelling the beam dynamics.

An algorithm has been developed for simultaneous solution of the hybrid dynamics of the flexible spacecraft. No special attention was given to improve the efficiency of the program in this study. Yet the CPU time was found to be very small. A typical run for the solution of the system equations took 20 seconds of CPU time and a memory of 100K on a AMDAHL 470/V 8 computer. It is expected that the CPU time could be further reduced by improving the efficiency of the FORTRAN code.

Chapter III

MODELLING AND STABILIZATION OF FLEXIBLE SPACECRAFT OF MORE GENERAL STRUCTURE

In general, a flexible spacecraft is expected to be very large and highly complex structure. In order to develop a methodology for modelling such spacecraft, in this chapter we generalize the dynamics presented in Chapter II. We also consider the question of stabilization of the system and develop simple stabilizing control schemes.

We consider a spacecraft consisting of a bus, two beams and a solar panel. We shall suppose that one of the beams is rigidly attached to the bus; but its position and angle with respect to the body are completely arbitrary. The second beam is rigidly attached to the end of the first one with the angle between them being arbitrary. We shall call this two-beam-structure a bend beam. Similarly, the solar panel is located at an arbitrary position with respect to the bus. We shall assume that the satellite is in a fixed geosynchronous orbit and develop the complete dynamic equations describing the angular motions of the bus and the transverse vibrations of the beams. For simplicity of presentation, we shall discuss the results in two subsections: spacecraft consisting of 1) the bus and the

bend beam, and ii) the bus and the solar panel. These results will then be combined together to give the complete dynamics of a spacecraft consisting of a bus, a bend beam, and a solar panel.

3.1 NOTATIONS

Subscript Notations:

b for bus; 1 for boom; 2 for tower; 3 for solar panel.

Reference Frames

$F_b = (i_b, j_b, k_b)$: Body frame of reference.

$F_r = (i_r, j_r, k_r)$: Orbital reference frame.

$F_c = (i_c, j_c, k_c)$: Reference frames for Boom ($c=1$),
Tower ($c=2$) and Solar panel ($c=3$).

Velocities

$\omega_r = (0, -\omega_0, 0)$: Angular velocity of the F_r frame with respect to the inertial space.

$\omega_b = (\omega_1, \omega_2, \omega_3)$: Angular velocity of the F_b frame with respect to the F_r frame.

v_c : Velocity of the mass element dm_c with respect to the F_c frame.

\hat{v}_c : Velocity of the mass element dm_c with respect to the F_b frame.

Displacements

$r_c = (x_c, y_c, z_c)$: Position vector of the mass element dm_c with respect to the F_c frame.

\hat{r}_c : Position vector of the mass element dm_c with respect to the F_b frame.

- $\rho_1 \equiv (y_1, z_1)$: Boom displacement from the rest state.
 $\rho_2 \equiv (y_2, z_2)$: Tower displacement from the rest state.
 R_0 : Position of the origin of the F_1 frame in the body frame.

Parameters

- $\rho_0 = \rho_0(x_0)$: Mass density per unit length of the boom or the tower.
 L_1 (L_2) : Length of the boom (tower).
 L_3^x (L_3^y) : Length (width) of the solar panel.
 I_b : Bus inertia tensor (Constant).
 I_1 (I_2, I_3) : Boom (tower, plate) inertia tensor (time varying).
 EI_1 (EI_2) : Flexural rigidity of the boom (tower).
 T : Torque applied to the bus.
 F_1 (F_2, F_3) : Force applied to the boom (tower or plate).

Mathematical Symbols

- $A \cdot b$: Dot product between a tensor and a vector, or between two vectors.
 $a \times b$: Cross product between two vectors.
 $|a| = \left(\sum_i a_i^2 \right)^{1/2}$: Norm of the vector a .
 $\frac{dA}{dt}$: Time derivative of a vector in the inertial space.
 \dot{A} : Time derivative of a vector in a rotating frame.

Additional notations will be introduced in the text as required.

3.2 SPACECRAFT CONSISTING OF A BUS AND A BEND BEAM

We first consider the problem of modelling and stabilization of a spacecraft consisting of a rigid bus and two beams (or a bend beam) as shown in Fig. 3.1. We shall use the term boom to denote the beam which is directly attached to the bus, and the term tower to denote the other beam. It is assumed that the beams are flexible but the joint between the boom and the tower is rigid. Following the procedure discussed in Chapter II, we derive the dynamic equations describing the attitude motion of the bus considering the effects of vibrations of the boom and tower. Similarly, we develop the equations for transverse vibration of the boom and the tower taking into account the effects of motion of the bus.

3.2.1 Reference Frames and Inertia Tensors

Let $F_b \equiv (i_b, j_b, k_b)$ denote the body reference frame and $F_r \equiv (i_r, j_r, k_r)$ the orbital reference frame as introduced in Section 2.2.1. We also require reference frames to describe the vibration of the two beams. We shall use subscript 1 to denote any variable related to the boom, and subscript 2 to denote any variable related to the tower. We define the boom coordinate system $F_1 \equiv (i_1, j_1, k_1)$, such that i_1 is along the direction of the rest position of the boom and that the origin of F_1 frame is located at the joint between

the bus and the boom. The direction of j_1 can be suitably chosen to obtain other computational advantages. This will be further clarified later in this chapter. Then the direction of k_1 is given by $i_1 \times j_1$.

Similarly we define the tower coordinate frame $F_2 \equiv (i_2, j_2, k_2)$ with the origin located at the joint between the boom and the tower at rest. The axis i_2 is along the direction of the rest position of the tower and the axis j_2 is suitably chosen. We shall assume that the F_1 and F_2 frames are stationary with respect to the F_b frame. A schematic diagram of the spacecraft along with the reference frames is shown in Fig. 3.1 :

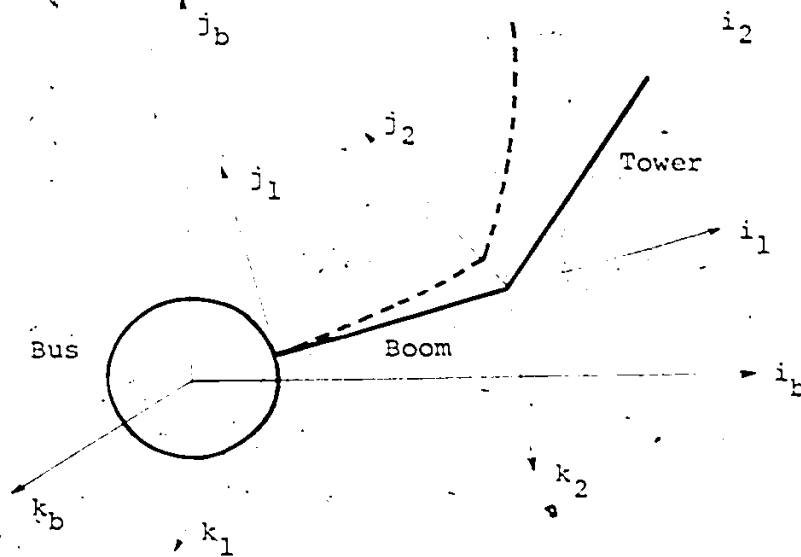


Fig. 3.1 Schematic Diagram of a Spacecraft with a Bend Beam.

Let a point P on the perturbed boom be described by the vector $r_1 = (x_1, y_1, z_1)$ in the boom frame of reference F_1 . Then, this point could also be described with respect to the body frame F_b , by

$$\bar{r}_1 = R_1 + M_1 r_1, \quad (3.1)$$

where R_1 is the vector denoting the origin of the F_1 frame with respect to the F_b frame and M_1 is the Euler transformation matrix [74] given by

$$M_1 = \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -S\phi C\psi - S\phi C\theta C\psi & S\phi S\psi \\ C\phi S\psi + S\phi C\theta C\psi & -S\phi S\psi + C\phi C\theta C\psi & -S\phi C\psi \\ S\phi S\psi & S\phi C\psi & C\phi \end{bmatrix}, \quad (3.2)$$

where $C\theta \equiv \cos\theta$ and $S\theta \equiv \sin\theta$.

The angles ϕ , ψ , and θ are the Euler angles between the F_b frame and the F_1 frame of reference. We note that M_1 is orthogonal i.e. $M_1^* = M_1^{-1}$, and reduces to identity when the F_b and F_1 frames are parallel.

Let v_1 denote the velocity of the particle P with respect to the F_1 frame. Then the velocity of P in the F_b frame is given by

$$\bar{v}_1 = M_1 v_1 \quad (3.3)$$

Similarly, let M_2 denote the Euler transformation matrix between the F_b and F_2 frames. Then, the displacement and the velocity of a particle in the tower with respect to the F_b and F_2 frames are related by:

$$\begin{aligned}\hat{r}_2 &= R_2 + M_2 r_2 \\ \hat{v}_2 &= M_2 v_2\end{aligned}\quad (3.4)$$

where \hat{r}_2 and \hat{v}_2 (respectively r_2 and v_2) denote the position and velocity of the particle in the F_b frame (respectively F_2 frame) and R_2 defines the position of the origin of F_2 frame with respect to the F_b frame.

It is to be noted that since F_1 and F_2 frames are stationary with respect to the F_b frame, the transformation matrices M_1 and M_2 are independent of time and can be determined if the unperturbed positions of the boom and the tower are given.

The inertia tensor of the satellite bus will be denoted by I^b . We shall assume that I^b is not time varying. The boom and the tower inertia tensors (with respect to the body frame) will be denoted by I_1 and I_2 respectively.

Let L_1 and ρ_1 denote the length and the mass per unit length of the boom. Then the inertia tensor I_1 is given by

$$I_1 = \int_0^{L_1} \rho_1 [(r_1 \cdot r_1) U_3 - r_1 r_1'] dx_1$$

where U_3 is the identity matrix of dimension 3×3 . Further, it can be verified that

$$I_1 \cdot \omega = \int_0^{L_1} \rho_1 (\hat{r}_1 \times (\omega \times \hat{r}_1)) dx_1$$

Similarly, letting L_2 and ρ_2 denote the length and mass per unit length of the tower, we have

$$I_2 = \int_0^{L_2} \rho_2 ((\hat{r}_2 \cdot \hat{r}_2) U_3 - \hat{r}_2 \hat{r}_2^T) dx_2$$

and

$$I_2 \cdot \omega = \int_0^{L_2} \rho_2 (\hat{r}_2 \times (\omega \times \hat{r}_2)) dx_2$$

3.2.2 System Modelling

As mentioned in Chapter II, the complete dynamics of a flexible spacecraft could be derived using the Lagrangian method or the Newtonian dynamics. In this chapter we shall follow the Lagrangian method. For this purpose, we first introduce the total kinetic energy and the total potential energy of the system.

Let M^b denote the total mass of the satellite bus and I^b its moment of inertia with respect to the body frame. Then, assuming that the origin of the body frame coincides with the center of mass of the bus, the bus kinetic energy is given by (as in (2.23))

$$T_b = \frac{1}{2} M_b v_0 \cdot v_0 + \frac{1}{2} \omega \cdot (I_b \cdot \omega) \quad (3.5)$$

where v_0 is the velocity of the origin of the body frame and $\omega (= \omega_b + \omega_r)$ denotes the angular velocity of the body frame, all with respect to the inertial space. Note that the first term of (3.5) represents the translational kinetic energy and the second term gives the rotational kinetic energy of the bus.

The kinetic energy of the boom is given by

$$T_1 = \frac{1}{2} \int_{\text{boom}} \frac{d\tilde{r}_1}{dt} \cdot \frac{d\tilde{r}_1}{dt} dm_1 \quad (3.6)$$

$$\text{with } \tilde{r}_1 = R_0 + R_1 + M_1 r_1 \quad (3.7)$$

where \tilde{r}_1 denotes the absolute position of the incremental mass element dm_1 of the boom relative to the inertial space, the vector R_0 defines the position of the origin of the body frame in the inertial space and $R_1 + M_1 r_1 \equiv \tilde{r}_1$ defines the position of the mass element dm_1 with respect to the body frame (see Section 3.2.1 and equation 3.1). We note that the position of the mass element dm_1 with respect to the boom frame is given by the vector r_1 .

The kinetic energy of the tower is obtained in a similar way and is given by

$$T_2 = \frac{1}{2} \int_{\text{tower}} \frac{d\tilde{r}_2}{dt} \cdot \frac{d\tilde{r}_2}{dt} dm_2 \quad (3.8)$$

$$\text{with } \tilde{r}_2 = R_0 + R_2 + M_2 r_2 \quad (3.9)$$

where \tilde{r}_2 denotes the absolute position of the incremental mass element dm_2 of the tower with respect to the inertial

space. The position of dm_2 is defined by the vector r_2 relative to the tower frame, and by $R_2 + M_2 r_2$ relative to the body frame (see equation 3.4).

The elastic potential energy of the system arises from transverse and torsional deformations of the boom and the tower. In this study torsional deformations will be neglected. The deformations of the boom could occur in the two transverse directions, i.e. j_1 and k_1 , with the corresponding deflections denoted by $y_1 = y_1(x_1, t)$ and $z_1 = z_1(x_1, t)$ respectively. We shall, however, assume that the boom will not elongate in the i_1 direction. Then under the assumption of small deflections, the elastic potential energy of the boom is given by:

$$V_1 = \frac{1}{2} \int_0^{L_1} EI_1 \frac{\partial^2}{\partial x_1^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} \cdot \frac{\partial^2}{\partial x_1^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} dx_1 \quad (3.10)$$

where $EI_1 = \begin{bmatrix} EI_1^y & 0 \\ 0 & EI_1^z \end{bmatrix}$

with EI_1^y and EI_1^z denoting the flexural rigidity of the boom for deflections in the j_1 and k_1 directions respectively. We note that in the case of non uniform boom, EI_1 would be a function of x_1 .

The potential energy V_2 of the tower is obtained in a similar way. Denoting transverse deflections of the tower in the j_2 and k_2 directions by $y_2 = y_2(x_2, t)$ and $z_2 = z_2(x_2, t)$ respectively, we have

$$V_2 = \frac{1}{2} \int_0^{L_2} EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \cdot \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} dx_2, \quad (3.11)$$

where $EI_2 = \begin{bmatrix} EI_2^y & 0 \\ 0 & EI_2^z \end{bmatrix}$

In (3.11), EI_2 denotes the flexural rigidity and L_2 the length of the tower.

The dynamics of the spacecraft in the absence of any control is obtained from the variational formulation of Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} [T_b + T_1 + T_2 - V_1 - V_2] dt, \quad (3.12)$$

subject to the end conditions

$$\delta \theta_1 = 0, \quad \delta \theta_2 = 0, \quad \delta \theta_3 = 0,$$

and

$$\delta y_1 = 0, \quad \delta z_1 = 0,$$

$$\delta y_2 = 0, \quad \delta z_2 = 0,$$

at $t = t_1, t_2$, where θ_1, θ_2 and θ_3 are the Euler angles describing the orientation of the body frame relative to the reference frame.

The virtual displacement of the boom mass element dm_1 , which is defined by the vector \tilde{r}_1 in the inertial space (see equation 3.7), is given by (c.f. (2.27))

$$\delta \tilde{r}_1 = \delta R_0 + \delta \theta \times (R_1 + M_1 r_1) + M_1 \delta r_1, \quad (3.13)$$

where $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is the virtual rotation of the body frame relative to the inertial space. Then, using the rule of differentiation in a rotating reference frame i.e. (2.12), we have

$$\delta\left(\frac{d\tilde{r}_1}{dt}\right) = \delta v_0 + \delta\omega \times (R_1 + M_1 r_1) + \delta\theta \times \{\omega \times (R_1 + M_1 r_1) + M_1 \dot{r}_1\} + M_1 \frac{d}{dt} (\delta r_1), \quad (3.14)$$

where $v_0 = \frac{dR_0}{dt}$ denotes the velocity of the origin of the body frame in the inertial space.

The virtual displacement and the virtual velocity of an incremental mass element dm_2 of the tower are obtained in a similar way and are given by

$$\delta\tilde{r}_2 = \delta R_0 + \delta\theta \times (R_2 + M_2 r_2) + M_2 \delta r_2 \quad (3.15)$$

$$\delta\left(\frac{d\tilde{r}_2}{dt}\right) = \delta v_0 + \delta\omega \times (R_2 + M_2 r_2) + \delta\theta \times \{\omega \times (R_2 + M_2 r_2) + M_2 \dot{r}_2\} + M_2 \frac{d}{dt} (\delta r_2). \quad (3.16)$$

We recall that, by assumption, the satellite describes a fixed circular orbit, so that $\delta v_0 = 0$. Then using (3.5)-(3.11) in (3.12) and taking the first variation, one obtains:

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L dt = 0 &= \int_{t_1}^{t_2} \delta \omega \cdot (I_b + I_1 + I_2) \cdot \omega dt \\
&+ \sum_{i=1}^2 \int_{t_1}^{t_2} \delta \omega \cdot \int_0^{L_i} \rho_i (R_i + M_i r_i) \times M_i \dot{r}_i dx_i dt \\
&+ \sum_{i=1}^2 \int_{t_1}^{t_2} \int_0^{L_i} \rho_i \frac{d}{dt} (\delta r_i) \cdot M_i^* \{ \omega \times (R_i + M_i r_i) + M_i \dot{r}_i \} dx_i dt \\
&- \sum_{i=1}^2 \int_{t_1}^{t_2} \int_0^{L_i} \frac{\partial^2}{\partial x_i^2} \left[EI_i \frac{\partial^2}{\partial x_i^2} \begin{pmatrix} y_i \\ z_i \end{pmatrix} \right] \cdot \delta \begin{pmatrix} y_i \\ z_i \end{pmatrix} dx_i dt \\
&- \sum_{i=1}^2 \int_{t_1}^{t_2} EI_i \frac{\partial^2}{\partial x_i^2} \begin{pmatrix} y_i \\ z_i \end{pmatrix} \cdot \delta \left[\frac{\partial}{\partial x_i} \begin{pmatrix} y_i \\ z_i \end{pmatrix} \right] \Big|_0^{L_i} dt \\
&+ \sum_{i=1}^2 \int_{t_1}^{t_2} \frac{\partial}{\partial x_i} \left[EI_i \frac{\partial^2}{\partial x_i^2} \begin{pmatrix} y_i \\ z_i \end{pmatrix} \right] \cdot \begin{pmatrix} \delta y_i \\ \delta z_i \end{pmatrix} \Big|_0^{L_i} dt \quad (3.17)
\end{aligned}$$

By assumption, the boom and the tower are inextensible so that $\delta r_1 = (0, \delta y_1, \delta z_1)$ and $\delta r_2 = (0, \delta y_2, \delta z_2)$. Then, integrating the first five terms in (3.17) by parts using the fact that $\delta \omega = \frac{d}{dt} (\delta \theta)$ and noting that the variations $\delta \theta_1, \delta \theta_2, \delta \theta_3, \delta y_1, \delta z_1, \delta y_2, \delta z_2$ are arbitrary, one obtains the set of coupled differential equations for the dynamics of the flexible spacecraft as given below:

Bus Dynamics (without control)

$$\frac{d}{dt} [(I_b + I_1 + I_2) \cdot \omega] + \frac{d}{dt} \left[\int_0^{L_1} \rho_1 (R_1 + M_1 r_1) \times M_1 \dot{r}_1 dx_1 \right] \\ + \frac{d}{dt} \left[\int_0^{L_2} \rho_2 (R_2 + M_2 r_2) \times M_2 \dot{r}_2 dx_2 \right] = 0 ,$$

Boom Dynamics (without control)

$$\rho_1 \frac{d}{dt} \left[M_1^* \{ M_1 \dot{r}_1 + \omega \times (R_1 + M_1 r_1) \} \right]_{j_1, k_1} + \frac{\partial^2}{\partial x_1^2} \left[EI_1 \frac{\partial^2}{\partial x_1^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} \right] = 0 , \quad (3.18)$$

Tower Dynamics (without control)

$$\rho_2 \frac{d}{dt} \left[M_2^* \{ M_2 \dot{r}_2 + \omega \times (R_2 + M_2 r_2) \} \right]_{j_2, k_2} + \frac{\partial^2}{\partial x_2^2} \left[EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right] = 0 ,$$

where we have used the notation $[A]_{j,k} = [A_j, A_k]$ to denote the j -th and k -th component of the vector A .

In order to obtain the dynamics in a more convenient form, we perform the differentiations in (3.18) using (2.12) and use the facts that $\dot{r}_i = v_i = (0, \frac{\partial y_i}{\partial t}, \frac{\partial z_i}{\partial t})$, $i=1,2$. Then the complete dynamics of the flexible spacecraft along with the control torques and forces is given by:

Bus Dynamics (with control)

$$(I_b + I_1 + I_2) \cdot \dot{\omega} + \omega \times [(I_b + I_1 + I_2) \cdot \omega] + \int_0^{L_1} \rho_1 (R_1 + M_1 r_1) \times M_1 \dot{v}_1 dx_1 \\ + \int_0^{L_2} \rho_2 (R_2 + M_2 r_2) \times M_2 \dot{v}_2 dx_2 + 2 \int_0^{L_1} \rho_1 (R_1 + M_1 r_1) \times (\omega \times M_1 v_1) dx_1 \\ + 2 \int_0^{L_2} \rho_2 (R_2 + M_2 r_2) \times (\omega \times M_2 v_2) dx_2 = T , \quad (3.19)$$

Boom Dynamics (with control)

$$\rho_1 \frac{\partial^2}{\partial t^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \frac{\partial^2}{\partial x_1^2} \left[EI_1 \frac{\partial^2}{\partial x_1^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} \right] + \rho_1 Q M_1^* \alpha_1 = F_1 , \quad (3.20)$$

$$\alpha_1 = \dot{\omega} \times (R_1 + M_1 r_1) + \omega \times (\omega \times (R_1 + M_1 r_1)) + 2\omega \times M_1 v_1 ,$$

Tower Dynamics (with control)

$$\rho_2 \frac{\partial^2}{\partial t^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} + \frac{\partial^2}{\partial x_2^2} \left[EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right] + \rho_2 Q M_2^* \alpha_2 = F_2, \quad (3.21)$$

$$\alpha_2 = \omega \times (R_2 + M_2 r_2) + \omega \times (\omega \times (R_2 + M_2 r_2)) + 2 \omega \times M_2 v_2,$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where T is the torque applied to the bus and F_1 , F_2 are the forces applied to the boom and tower respectively. To obtain the boundary conditions for the boom and the tower dynamics (3.20)-(3.21), we make the following assumptions:

- A1) The boom is rigidly attached to the bus.
- A2) The remote end of the tower is free.
- A3) The joint between the boom and the tower is rigid.
- A4) The i_1 - j_1 plane of the boom coincides with the i_2 - j_2 plane of the tower.
- A5) The deflections of the boom and the tower are small.

On the basis of these assumptions and the equation (3.17), we obtain the boundary conditions for (3.20) and (3.21) as follows:

$$\left\{ \begin{array}{l} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} (0, t) = 0, \quad \frac{\partial}{\partial x_1} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} (0, t) = 0, \\ EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} (L_2, t) = 0, \quad \frac{\partial}{\partial x_2} \left[EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right] (L_2, t) = 0, \end{array} \right. \quad (3.22.a)$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x_1} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} (L_1, t) = \frac{\partial}{\partial x_2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} (0, t), \\ EI_1 \frac{\partial^2}{\partial x_1^2} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} (L_1, t) = EI_2 \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} (0, t), \end{array} \right. \quad (3.22.b)$$

$$y_1(L_1, t) \cos \theta = y_2(0, t), \quad z_1(L_1, t) = z_2(0, t),$$

$$\frac{\partial}{\partial x_1} \left(EI_1^y \frac{\partial^2 y_1}{\partial x_1^2} \right) (L_1, t) = \frac{\partial}{\partial x_2} \left(EI_2^y \frac{\partial^2 y_2}{\partial x_2^2} \right) (0, t) \cos \theta, \quad (3.22.c)$$

$$\frac{\partial}{\partial x_1} \left(EI_1^z \frac{\partial^2 z_1}{\partial x_1^2} \right) (L_1, t) = \frac{\partial}{\partial x_2} \left(EI_2^z \frac{\partial^2 z_2}{\partial x_2^2} \right) (0, t),$$

where θ denotes the angle between the boom and the tower measured in the usual positive direction, i.e. the angle between the i_1 and i_2 axes as shown in Fig. 3.2.

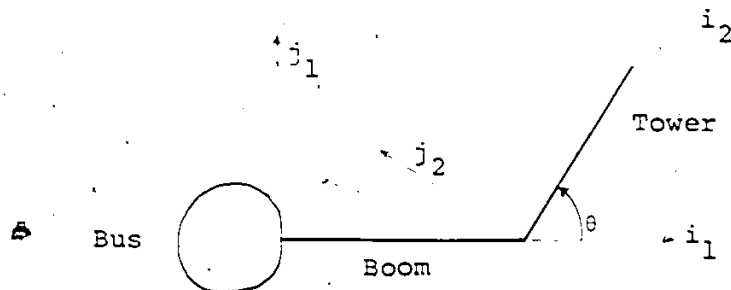


Fig. 3.2

We note here that the boundary conditions (3.22.a) follows easily from the assumptions (A1) and (A2). The assumption (A1) implies that the displacement and the slope of displacement at the fixed end of the boom must vanish. The free end assumption (A2) of the tower shows that the bending moment and the shear force must be zero. The rigidity assumption (A3) of the joint between the boom and the tower gives rise to the conditions (3.22.b). This is because of the fact that the slope of displacement and its

(spatial) derivative on either side of the joint must be equal in order that the joint be rigid. Finally, the conditions (3.22.c) are obtained from the assumptions (A3)-(A5) and simple geometrical arguments.

Remark 3.1

The assumption (A4) does not pose any restriction on the spacecraft configuration since the boom and the tower are only one dimensional elements. Selection of a reference frame is always at one's disposal and is usually done suitably to obtain mathematical simplicity.

3.2.3 Complete System Dynamics

The complete dynamics of the flexible spacecraft consisting of a rigid body, a boom and a tower is, therefore, given by the hybrid dynamics (3.19)-(3.21) along with the boundary conditions (3.22). From these equations, it is clear that the attitude dynamics of the bus and the transverse vibrations of the boom and the tower are strongly coupled. Any vibration of these elastic members would induce motions of the bus and vice versa. As in Chapter II, we note that in the case of infinitely rigid boom and tower, the bus attitude dynamics (3.19) reduces to the usual dynamics [36] of a rigid body satellite. Similarly, if the boom and the tower are considered to be isolated from the bus, equations (3.20) and (3.21) reduce to the usual Euler equations describing the transverse vibrations of a beam.

3.2.3.1 Spacecraft Dynamics with Multiple Beams

The dynamics of a spacecraft consisting of a rigid bus and several flexible beams located at arbitrary positions on the bus and placed at arbitrary angles with respect to the body frame can be obtained easily from (3.19)-(3.22), and is given by:

Bus Dynamics

$$I^T \ddot{\omega} + \omega \times (I^T \omega) + \sum_i \int_0^{L_i} \rho_i (R_i + M_i r_i) \times M_i \dot{v}_i dx_i + 2 \sum_i \int_0^{L_i} \rho_i (R_i + M_i r_i) \times (\omega \times M_i v_i) dx_i = T, \quad (3.23)$$

Beam Dynamics

$$\rho_i \frac{\partial^2 \phi_i}{\partial t^2} + \frac{\partial^2}{\partial x_i^2} \left[EI_i \frac{\partial^2 \phi_i}{\partial x_i^2} \right] + \rho_i Q M_i^* \phi_i = F_i,$$

$$\phi_i(0, t) = 0, \quad EI_i \frac{\partial^2 \phi_i}{\partial x_i^2}(L_i, t) = 0, \quad (3.24)$$

$$\frac{\partial \phi_i}{\partial x_i}(0, t) = 0, \quad \frac{\partial}{\partial x_i} \left[EI_i \frac{\partial^2 \phi_i}{\partial x_i^2} \right](L_i, t) = 0,$$

$$\text{where } I^T = I_b + \sum_i I_i,$$

$$\rho_i = \rho \times r_i + \omega \times (\omega \times r_i) + 2 \omega \times v_i.$$

The coordinate transformation matrices between the body frame and the individual beam frames are denoted by M_i . In case the spacecraft consists of a rigid bus and only one beam located on the i_b axis, then M_1 is an identity matrix, and the dynamics (3.23)-(3.24) reduce to the dynamics (2.13) and (2.19) as derived in Chapter II.

3.2.4 Stabilization

In this section we study the problem of stabilization of the flexible spacecraft described by the coupled system of ordinary and partial differential equations (3.19)-(3.21) along with the boundary conditions (3.22). We will consider the following rest state for the system.

$$\begin{aligned}
 & \text{and } \omega_1 = 0, \quad \omega_2 = \omega_0, \quad \omega_3 = 0, \\
 & \left. \begin{aligned}
 y_1 &= 0, & z_1 &= 0, \\
 \frac{\partial y_1}{\partial t} &= 0, & \frac{\partial z_1}{\partial t} &= 0,
 \end{aligned} \right\} \text{for } x_1 \in [0, L_1] \\
 & \left. \begin{aligned}
 y_2 &= 0, & z_2 &= 0, \\
 \frac{\partial y_2}{\partial t} &= 0, & \frac{\partial z_2}{\partial t} &= 0,
 \end{aligned} \right\} \text{for } x_2 \in [0, L_2].
 \end{aligned} \tag{3.25}$$

We show that the spacecraft could be stabilized in the asymptotic sense by application of simple feedback controls. For the proof of stability we shall follow Lyapunov's approach with the total energy of the system as the natural choice of Lyapunov functional. We present this result in the following theorem.

Theorem 3.1

Consider the system described by (3.19)-(3.21) with the boundary conditions (3.22). Suppose that the controls applied to the system are given by the feedback law:

$$T = (-k_1 \omega_1, -k_2(\omega_2 - \omega_0), -k_3 \omega_3), \quad k_1, k_2, k_3 > 0,$$

$$F_1 = (-d_1(x_1) \frac{\partial y_1}{\partial t}, -d_2(x_1) \frac{\partial z_1}{\partial t}), \quad d_1, d_2 > 0 \text{ a.e. on } \Omega_1, \quad (3.26)$$

$$F_2 = (-d_3(x_2) \frac{\partial y_2}{\partial t}, -d_4(x_2) \frac{\partial z_2}{\partial t}), \quad d_3, d_4 > 0 \text{ a.e. on } \Omega_2.$$

Then the system is asymptotically stable in the Lyapunov sense with respect to the rest state (3.25).

Proof

For brevity we shall denote $\phi_1 = (y_1, z_1)$, $\phi_2 = (y_2, z_2)$, $\hat{F}_1 \equiv R_1 + M_1 r_1$ and $\hat{F}_2 \equiv R_2 + M_2 r_2$. Scalar multiplying both sides of (3.19) by ω and noting that the boom and the tower inertia matrices vary with time because of vibrations, we have

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} ((I_B + I_1 + I_2) \cdot \omega) \cdot \omega \right] - \frac{1}{2} (\dot{I}_1 \cdot \omega) \cdot \omega - \frac{1}{2} (\dot{I}_2 \cdot \omega) \cdot \omega \\ + \int (\hat{F}_1 \times M_1 \dot{V}_1) \cdot \omega \, dm_1 + \int (\hat{F}_2 \times M_2 \dot{V}_2) \cdot \omega \, dm_2 \\ + 2 \int (\hat{F}_1 \times (\omega \times M_1 V_1)) \cdot \omega \, dm_1 + 2 \int (\hat{F}_2 \times (\omega \times M_2 V_2)) \cdot \omega \, dm_2 = T \cdot \omega. \end{aligned} \quad (3.27)$$

Using the rules of elementary vector calculus (see Appendix A), it can be easily verified that, for $i = 1, 2$,

$$\begin{aligned} I_i \cdot \omega &= \int |\omega \times \hat{F}_i|^2 \, dm_i, \\ (\dot{I}_i \cdot \omega) \cdot \omega &= -2 \int (\omega \times (\omega \times \hat{F}_i)) \cdot M_i V_i \, dm_i. \end{aligned} \quad (3.28)$$

Using these results in (3.27), one obtains

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} (I_b \cdot \omega) \cdot \omega + \frac{1}{2} \int |\omega \times \hat{r}_1|^2 dm_1 + \frac{1}{2} \int |\omega \times \hat{r}_2|^2 dm_2 \right] \\ & + \int (\hat{r}_1 \times M_1 \dot{v}_1) \cdot \omega dm_1 - \int (\omega \times (\omega \times \hat{r}_1)) \cdot M_1 v_1 dm_1 \\ & + \int (\hat{r}_2 \times M_2 \dot{v}_2) \cdot \omega dm_2 - \int (\omega \times (\omega \times \hat{r}_2)) \cdot M_2 v_2 dm_2 = T \cdot \omega \quad (3.29) \end{aligned}$$

Scalar multiplying both sides of (3.20) by $\frac{\partial \phi_1}{\partial t}$ and integrating by parts over $\Omega_1 = [0, L_1]$ and using the facts that $v_1 = (0, \frac{\partial y_1}{\partial t}, \frac{\partial z_1}{\partial t}) = (0, \frac{\partial \phi_1}{\partial t})$ and noting that M_1 is orthogonal, we have

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \int |M_1 v_1|^2 dm_1 + \frac{1}{2} \int_{\Omega_1} EI_1 \frac{\partial^2 \phi_1}{\partial x_1^2} \cdot \frac{\partial^2 \phi_1}{\partial x_1^2} dx_1 \right] + \frac{\partial}{\partial x_1} \left[EI_1 \frac{\partial^2 \phi_1}{\partial x_1^2} \right] \cdot \phi_1 \Big|_0^{L_1} \\ & - EI_1 \frac{\partial^2 \phi_1}{\partial x_1^2} \cdot \frac{\partial \phi_1}{\partial x_1} \Big|_0^{L_1} + \int (\omega \times \hat{r}_1) \cdot M_1 v_1 dm_1 \\ & + \int (\omega \times (\omega \times \hat{r}_1)) \cdot M_1 v_1 dm_1 = \int_{\Omega_1} F_1 \cdot \frac{\partial \phi_1}{\partial t} dx_1 \quad (3.30) \end{aligned}$$

Similarly, using (3.21), one obtains

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \int |M_2 v_2|^2 dm_2 + \frac{1}{2} \int_{\Omega_2} EI_2 \frac{\partial^2 \phi_2}{\partial x_2^2} \cdot \frac{\partial^2 \phi_2}{\partial x_2^2} dx_2 \right] + \frac{\partial}{\partial x_2} \left[EI_2 \frac{\partial^2 \phi_2}{\partial x_2^2} \right] \cdot \phi_2 \Big|_0^{L_2} \\ & - EI_2 \frac{\partial^2 \phi_2}{\partial x_2^2} \cdot \frac{\partial \phi_2}{\partial x_2} \Big|_0^{L_2} + \int (\omega \times \hat{r}_2) \cdot M_2 v_2 dm_2 \\ & + \int (\omega \times (\omega \times \hat{r}_2)) \cdot M_2 v_2 dm_2 = \int_{\Omega_2} F_2 \cdot \frac{\partial \phi_2}{\partial t} dx_2 \quad (3.31) \end{aligned}$$

We now define the functional $V(t) \equiv V(\omega, \frac{\partial \phi_1}{\partial t}, \frac{\partial \phi_2}{\partial t}; t)$ by

$$\begin{aligned} V(t) & \equiv \frac{1}{2} (I_b \cdot \omega) \cdot \omega + \frac{1}{2} \int |M_1 v_1 + \omega \times \hat{r}_1|^2 dm_1 + \frac{1}{2} \int |M_2 v_2 + \omega \times \hat{r}_2|^2 dm_2 \\ & + \frac{1}{2} \int_{\Omega_1} EI_1 \frac{\partial^2 \phi_1}{\partial x_1^2} \cdot \frac{\partial^2 \phi_1}{\partial x_1^2} dx_1 + \frac{1}{2} \int_{\Omega_2} EI_2 \frac{\partial^2 \phi_2}{\partial x_2^2} \cdot \frac{\partial^2 \phi_2}{\partial x_2^2} dx_2 \quad (3.32) \end{aligned}$$

Then adding (3.29), (3.30) and (3.31) and using the boundary conditions (3.22), we have

$$\frac{dV}{dt} = T \cdot \omega + \int_{\Omega_1} F_1 \cdot \frac{\partial \phi_1}{\partial t} dx_1 + \int_{\Omega_2} F_2 \cdot \frac{\partial \phi_2}{\partial t} dx_2, \quad (3.33)$$

which for the given set of controls, becomes

$$\begin{aligned} \frac{dV}{dt} = & -k_1 \omega_1^2 - k_2 (\omega_2 - \omega_0)^2 - k_3 \omega_3^2 - \int_{\Omega_1} d_1(x_1) \left(\frac{\partial y_1}{\partial t} \right)^2 dx_1 \\ & - \int_{\Omega_1} d_2(x_1) \left(\frac{\partial z_1}{\partial t} \right)^2 dx_1 - \int_{\Omega_2} d_3(x_2) \left(\frac{\partial y_2}{\partial t} \right)^2 dx_2 - \int_{\Omega_2} d_4(x_2) \left(\frac{\partial z_2}{\partial t} \right)^2 dx_2. \end{aligned} \quad (3.34)$$

One could easily verify that V is positive definite, and from (3.34) it is clear that $\frac{dV}{dt}$ is negative semidefinite. Therefore, the rest state (3.25) is stable in the sense of Lyapunov. In what follows, we show that the rest state is actually asymptotically stable.

If $\frac{dV}{dt}$ were to vanish identically for all $t \in [t_1, t_2]$, then ω must be zero and $\frac{\partial \phi_1}{\partial t}, \frac{\partial \phi_2}{\partial t}$ must be zero almost everywhere on Ω_1 and Ω_2 respectively for all $t \in [t_1, t_2]$. This requires that

$$\omega = 0,$$

$$\frac{\partial^2 \phi_1}{\partial t^2} = 0 \quad \text{and} \quad \frac{\partial^2 \phi_2}{\partial t^2} = 0 \quad \text{a.e. on } \Omega_1 \text{ and } \Omega_2 \text{ respectively.}$$

Then, from (3.20) and (3.21), it follows that

$$\begin{aligned} \frac{\partial^2}{\partial x_1^2} \left[EI_1 \frac{\partial^2 \phi_1}{\partial x_1^2} \right] &= 0, \\ \frac{\partial^2}{\partial x_2^2} \left[EI_2 \frac{\partial^2 \phi_2}{\partial x_2^2} \right] &= 0. \end{aligned} \quad (3.35)$$

From the equations (3.35) along with the boundary conditions (3.22), one concludes that $\phi_1 \equiv 0$ and $\phi_2 \equiv 0$. Thus, $\frac{dV}{dt}$ is negative definite. Hence the functional V defined by (3.32) is a Lyapunov functional. Consequently, the rest state of the system is asymptotically stable. This completes the proof: \square

Remark 3.2

The Lyapunov function (3.32) represents the total energy of the system. We note that the first term in (3.32) gives the kinetic energy of the satellite bus; and the second and third terms give the kinetic energy of the boom and tower respectively due to vibration and rotation of the body axes. The elastic potential energy of the boom and the tower are represented by the last two terms of (3.32).

Remark 3.3

It is clear from (3.33), that in the absence of any controls, $V(t) = V(0)$ for all t , which implies that the system is conservative. That is the sum of the absolute bus energy, boom energy and tower energy remains constant. However, interchange of energy among the bus and the two elastic members (boom and tower) may take place during perturbations leading to changes in the amplitude of vibrations of the elastic members and the wobbling of the bus.

For practical applications, it is preferable to use localized controls rather than the distributed controls considered in Theorem 3.1. Since the elastic vibrations do not vanish identically over any finite subinterval of positive length of the boom or the tower, we have the following corollary:

Corollary 3.1

Consider the system(3.19)-(3.21) along with the boundary conditions (3.22) and consider the control scheme given by the feedback structure:

$$\begin{aligned} T &= (-k_1 \omega_1, -k_2(\omega_2 - \omega_0), -k_3 \omega_3), \quad k_1, k_2, k_3 > 0, \\ F_1 &= (-d_1(x_1) \frac{\partial y_1}{\partial t}, -d_2(x_1) \frac{\partial z_1}{\partial t}), \\ F_2 &= (-d_3(x_2) \frac{\partial y_2}{\partial t}, -d_4(x_2) \frac{\partial z_2}{\partial t}), \end{aligned} \quad (3.36)$$

with

$$\begin{aligned} d_i &= \sum_{j=1}^{N_i} a_j^i x_{A_j^i}(x_1), \quad a_j^i > 0, \quad i = 1, 2, \\ d_k &= \sum_{j=1}^{N_k} b_j^k x_{B_j^k}(x_2), \quad b_j^k > 0, \quad k = 3, 4, \end{aligned}$$

where x_E denotes the indicator function of the set E and $\{A_j^i, j=1, 2, \dots, N_i\}$, $i = 1, 2$ and $\{B_j^k, j=1, 2, \dots, N_k\}$, $k = 3, 4$ are any four families of Lebesgue measurable subsets of Ω_1 and Ω_2 respectively such that the velocities $\left\{ \frac{\partial y_i}{\partial t}, \frac{\partial z_i}{\partial t} \right\}$, $i=1, 2$ do not vanish identically on $\cup A_j^i$ and $\cup B_j^k$ respectively. Then the system is asymptotically stable with respect to the rest state (3.25).

Remark 3.4

The system (3.19)-(3.21) could also be asymptotically stabilized with respect to the origin using bang bang type of feedback controls (c.f. Corollary 2.1). In case stabilization of the system with respect to a small neighbourhood of the origin is permissible, it suffices to apply only deadzone type of feedback controls (c.f. Corollary 2.2).

3.2.5 Simulation Results

We now present numerical results on the stabilization of the flexible spacecraft governed by the coupled system of ordinary differential equations (3.19) and the partial differential equations (3.20)-(3.21) with the boundary conditions (3.22). For numerical simulation of the hybrid dynamics, we first convert the partial differential equations into a set of ordinary differential equations by using the semi-discretization technique, and then use Runge-Kutta method to solve these equations along with the bus equations (3.19). The algorithm is discussed in details in Appendix C.

We assume that the satellite bus inertia matrix is diagonal, and that the boom and the tower are uniform along their length. We use following data in the simulation:

Bus Data

$$I_1^b = 645 \text{ Slug ft.}$$

$$I_2^b = 100 \text{ "}$$

$$I_3^b = 669 \text{ "}$$

Boom and Tower Data

Flexural rigidity.

$$EI_1 = EI_2 = \text{Diag}(3550.8, 3550.8) \text{ lb ft}^2$$

$$\text{Mass density } \rho_1 = \rho_2 = 2.86 \times 10^{-2} \text{ slug/ft}$$

$$\text{Boom length } L_1 = 18.8 \text{ ft}$$

$$\text{Tower length } L_2 = 9.4 \text{ ft}$$

Boom located on the i_b axis

Distance from origin of F_b frame $R_1 = 3 \text{ ft.}$

Tower i_2 axis 95° with respect to the Boom i_1 axis.

We assume that the boom j_1 axis is parallel to the bus j_b axis and that the i_2-j_2 plane is coincident to the i_1-j_1 plane. Clearly, under these assumptions, the Euler transformation matrices M_1 and M_2 are given by

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.37)$$

$$M_2 = \begin{bmatrix} \cos 95 & -\sin 95 & 0 \\ \sin 95 & \cos 95 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.38)$$

For the initial conditions we take

$$\omega_1(0) = 0.03 \text{ rad/sec}$$

$$\omega_2(0) = 0.02 \quad "$$

$$\omega_3(0) = 0.01 \quad "$$

and $\frac{\partial y_1}{\partial t}(x_1, 0) = 0, \quad \frac{\partial z_1}{\partial t}(x_1, 0) = 0, \quad x_1 \in [0, L_1]$

$$\frac{\partial y_2}{\partial t}(x_2, 0) = 0, \quad \frac{\partial z_2}{\partial t}(x_2, 0) = 0, \quad x_2 \in [0, L_2]$$

$$\frac{\partial^2 y_1}{\partial x_1^2}(x_1, 0) = \frac{\partial^2 z_1}{\partial x_1^2}(x_1, 0) = \phi(x_1), \quad x_1 \in [0, L_1]$$

$$\frac{\partial^2 y_2}{\partial x_2^2}(x_2, 0) = \frac{\partial^2 z_2}{\partial x_2^2}(x_2, 0) = \phi(x_2 + L_1), \quad x_2 \in [0, L_2]$$

where

$$\phi(x) = \epsilon (\text{Cosh } \lambda L + \text{Cos } \lambda L) (\text{Sinh } \lambda x + \text{Sin } \lambda x) - (\text{Sinh } \lambda L + \text{Sin } \lambda L) (\text{Cosh } \lambda x + \text{Cos } \lambda x),$$

with $\text{Cosh } \lambda L \text{ Cos } \lambda L + 1 = 0$, and $L = L_1 + L_2$; $\epsilon = -0.00196$.

It has been observed in Chapter II that effective stabilization of the spacecraft requires a dual control scheme, i.e. a combination of torques applied to the bus and forces applied to the beam. Similar characteristics were observed also for the spacecraft considered in this chapter. This is also apparent from (3.34), since in case controls are applied only on the bus, or the boom, or the tower, the Lyapunov functional (3.32) has negative semidefinite derivative indicating mere stability of the system instead

of asymptotic stability. In this section, we present numerical results on stabilization of the flexible spacecraft by using proportional feedback controls (3.36) applied to the bus, the boom, and the tower. The feedback gains used in the simulation are as follows:

$$k_1 = 600.0, \quad k_2 = 200.0, \quad k_3 = 600.0,$$

and

$$d_1 = d_2 = 3 \times [0.95, 1.0],$$

$$d_3 = d_4 = 1 \times [0.95, 1.0],$$

where χ_E is the indicator function of the set E. The effects of the controls on stabilization of the spacecraft is presented in Figs. 3.3-3.16.

From Remark 3.3, it is clear that the uncontrolled system is conservative. That is the sum of the absolute energy of the bus, the boom and the tower remains constant for all t . But because of mechanical attachment of the bus and the two elastic members (boom and tower), interchange of energy among them may take place during perturbations leading to changes in the bus angular velocity trajectories and the vibrations of the elastic members. This could be clearly observed from the fluctuation of energy in Fig. 3.3, Case A. (Note that in case the bus and the two beams are considered decoupled from each other, then the energy of each element would remain constant.) With the application of the

suggested controls the bus kinetic energy and the vibrational energy of the boom and the tower decay to zero, thus stabilizing the system (Fig. 3.3, Case B). Stabilization of the various states of the system are shown separately in Figs. 3.4-3.16.

In Fig. 3.4 and 3.6, we show the deflections of the boom at the extreme end, i.e. $y_1(L_1, t)$ and $z_1(L_1, t)$. The corresponding velocities are shown in Fig. 3.5 and 3.7 respectively. The deflections $y_2(L_2, t)$, $z_2(L_2, t)$ and the corresponding velocities $\frac{\partial y_2}{\partial t}(L_2, t)$ and $\frac{\partial z_2}{\partial t}(L_2, t)$ of the tower at the free end are shown in Fig. 3.8-3.11. It is clear from these figures that in the absence of any control, the satellite would be unsuitable for any application because of continued vibrations of the elastic members (Fig. 3.4-3.11, Case A). With the application of the chosen feedback controls these vibrations eventually decay to zero as observed from Fig. 3.4-3.11, Case B.

In order to show that the vibrations of the boom and the tower are eliminated throughout their length, we consider their 'relative energy' defined by

$$\begin{aligned}
 E_1(t) &\equiv \int_0^{L_1} \left[\left(\frac{\partial y_1}{\partial t} \right)^2 + \left(\frac{\partial z_1}{\partial t} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 y_1}{\partial x_1^2} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 z_1}{\partial x_1^2} \right)^2 \right] dx_1 \\
 E_2(t) &\equiv \int_0^{L_2} \left[\left(\frac{\partial y_2}{\partial t} \right)^2 + \left(\frac{\partial z_2}{\partial t} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 y_2}{\partial x_2^2} \right)^2 + \frac{EI}{\rho} \left(\frac{\partial^2 z_2}{\partial x_2^2} \right)^2 \right] dx_2
 \end{aligned} \tag{3.39}$$

Clearly $E_1(t) = 0$ and $E_2(t) = 0$ imply that all the points of the boom and the tower are at the undeformed rest state. In Fig. 3.12-3.13, Case A, we observe that in the uncontrolled system these relative energies increase with time, indicating growing vibrations of the boom and the tower. Eventual decay of all the elastic vibrations by the action of the chosen feedback controls is clear from Fig. 3.12-3.13, Case B.

The effects of the controls in stabilizing the satellite bus angular velocities are shown in Fig. 3.14-3.16. As in Chapter II, we observe that in the uncontrolled system, Fig. 3.14-3.16, Case A, the angular velocities ω_2 and ω_3 experience larger fluctuations as compared to ω_1 . The underlying reasons are again location of the boom and the tower with respect to the body axes and relative magnitude of their inertia. Stabilization of the angular velocities by application of the feedback controls could be observed from Fig. 3.14-3.16, Case B.

Remark 3.5

Response of the system in the presence of bang bang or deadzone type of feedback controls have similar characteristics as those presented in Chapter II, and hence will be omitted.

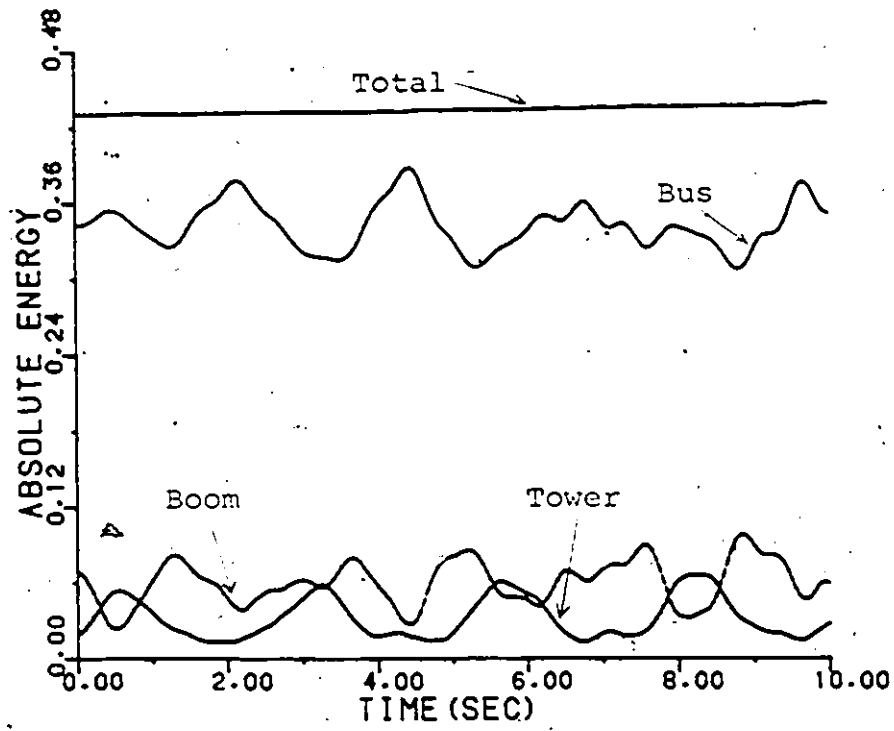


Fig. 3.3(A) Absolute Energy of the uncontrolled System.

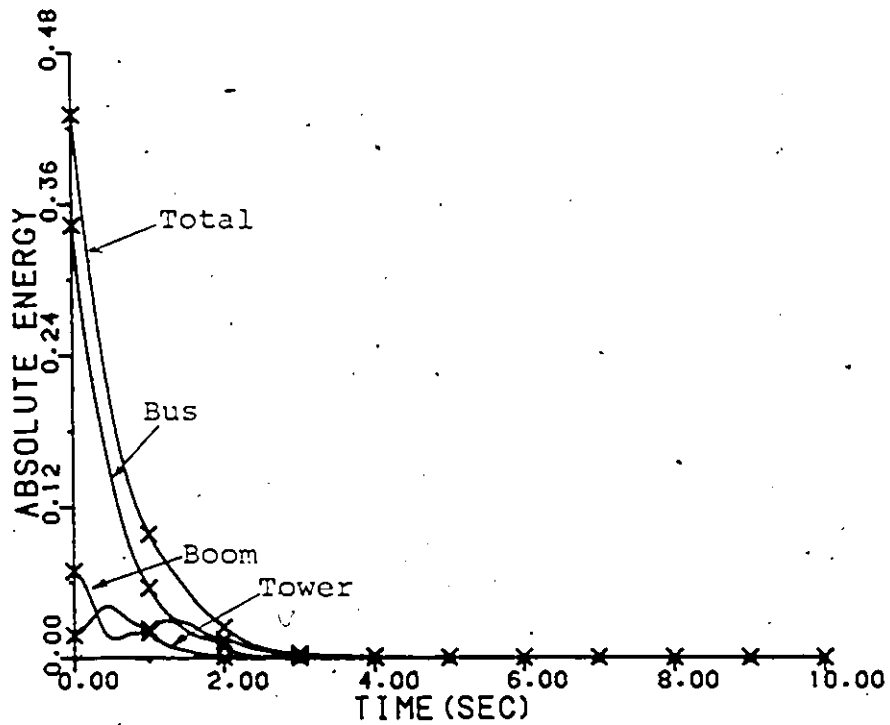
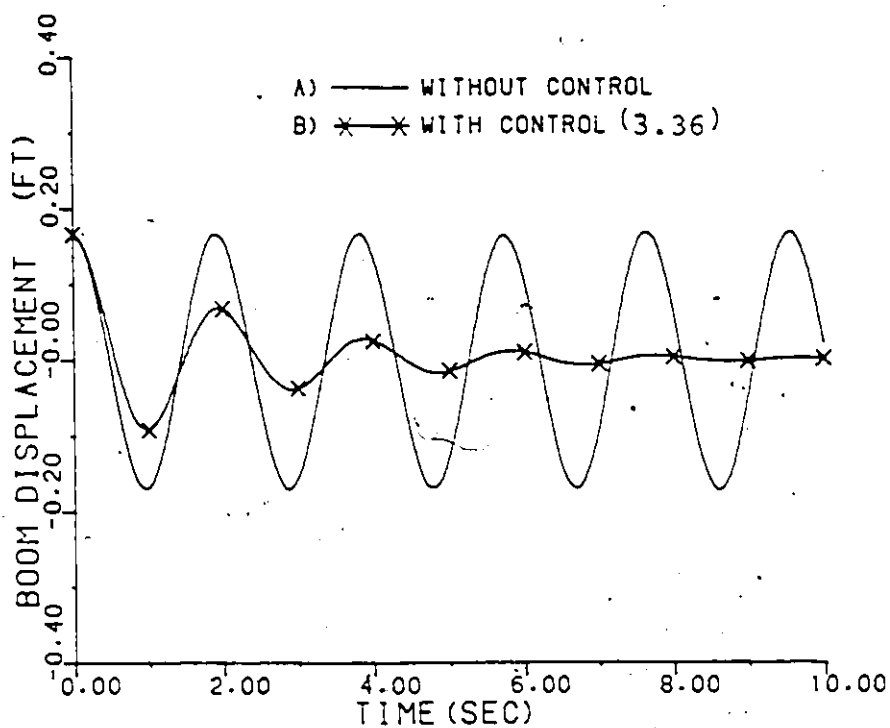
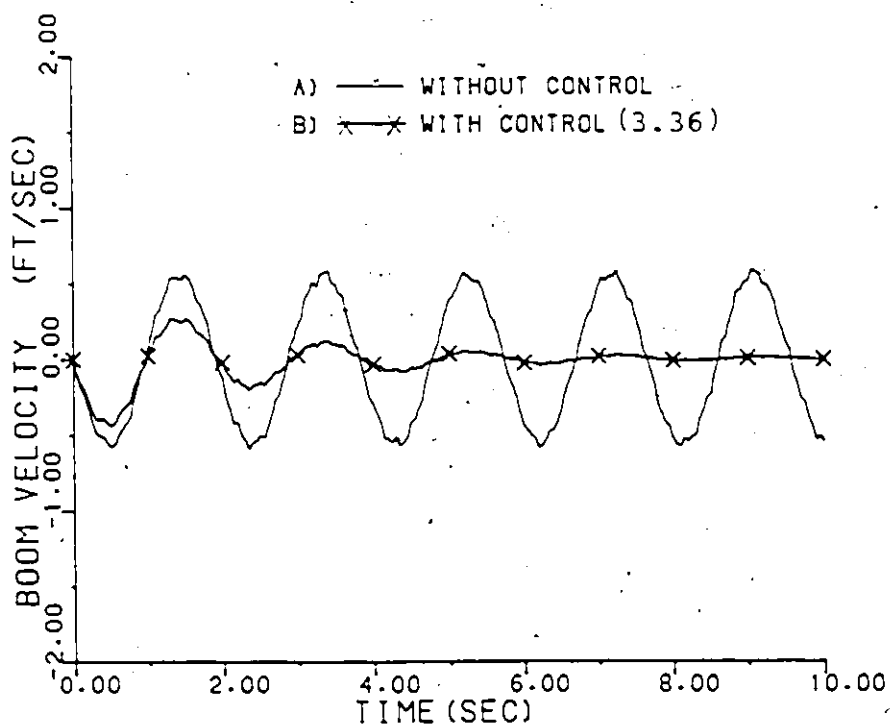


Fig. 3.3(B) Decay of (absolute) Energy with controls (3.36)

Fig. 3.4 Boom Displacement $y_1(L_1, t)$ Fig. 3.5 Boom Velocity $\frac{\partial y_1}{\partial t}(L_1, t)$

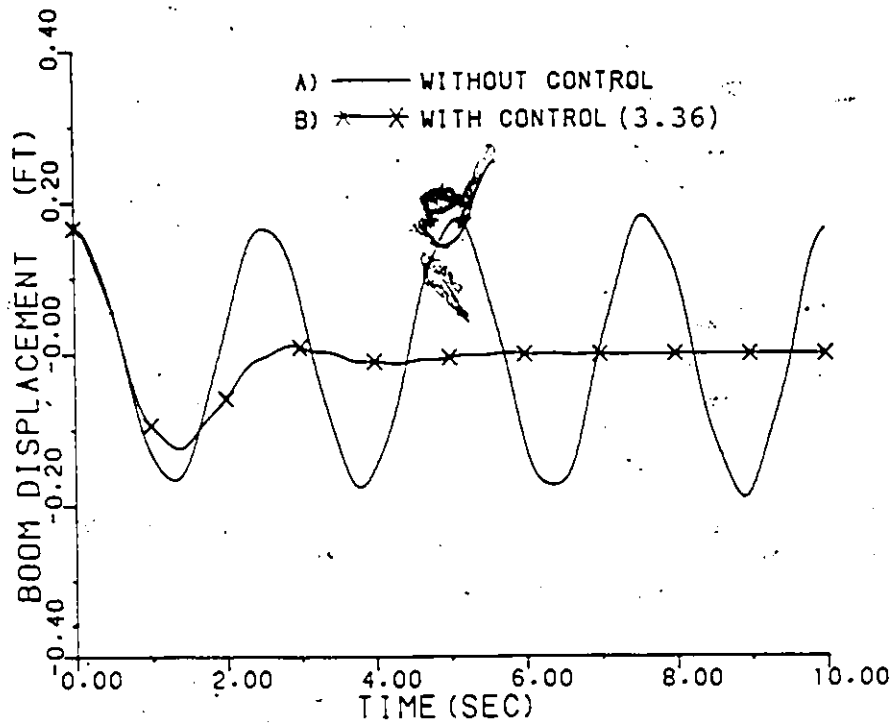


Fig. 3.6 Boom Displacement $z_1(L_1, t)$

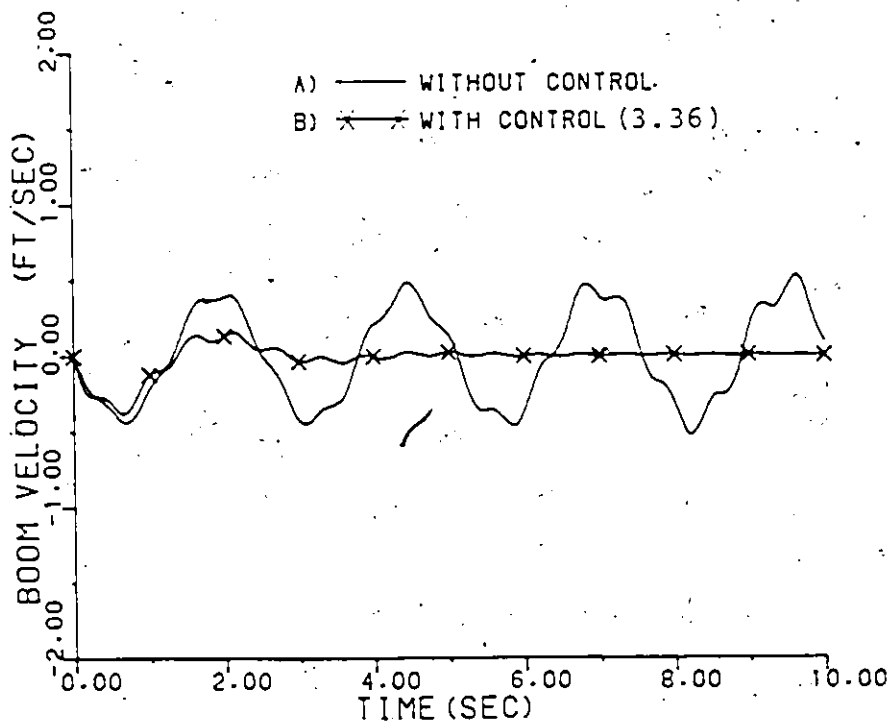


Fig. 3.7 Boom Velocity $\frac{\partial z_1}{\partial t}(L_1, t)$

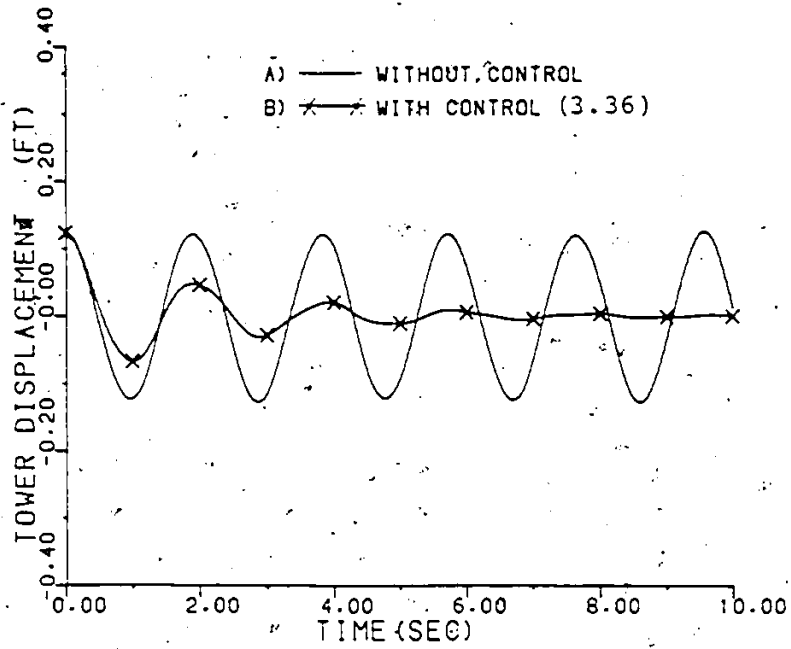


Fig. 3.8 Tower Displacement $y_2(L_2, t)$

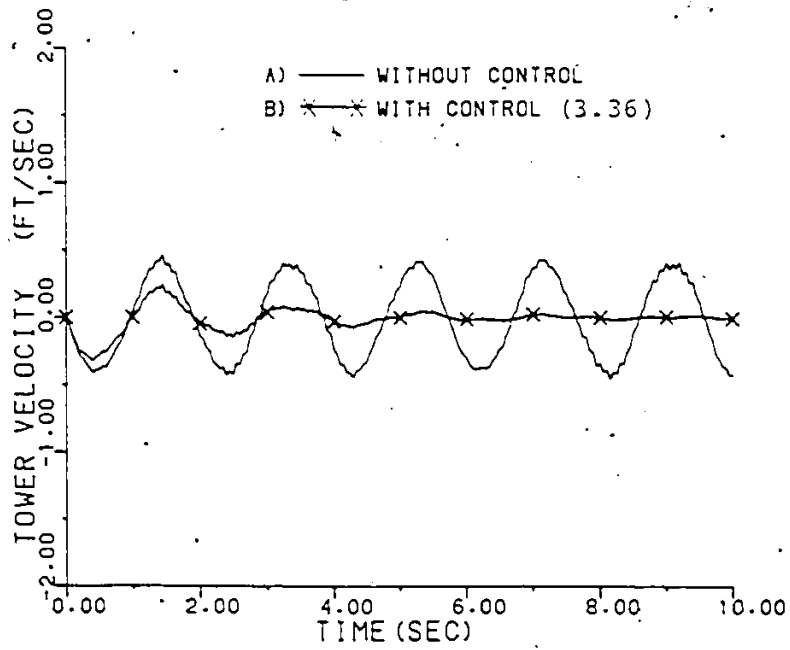
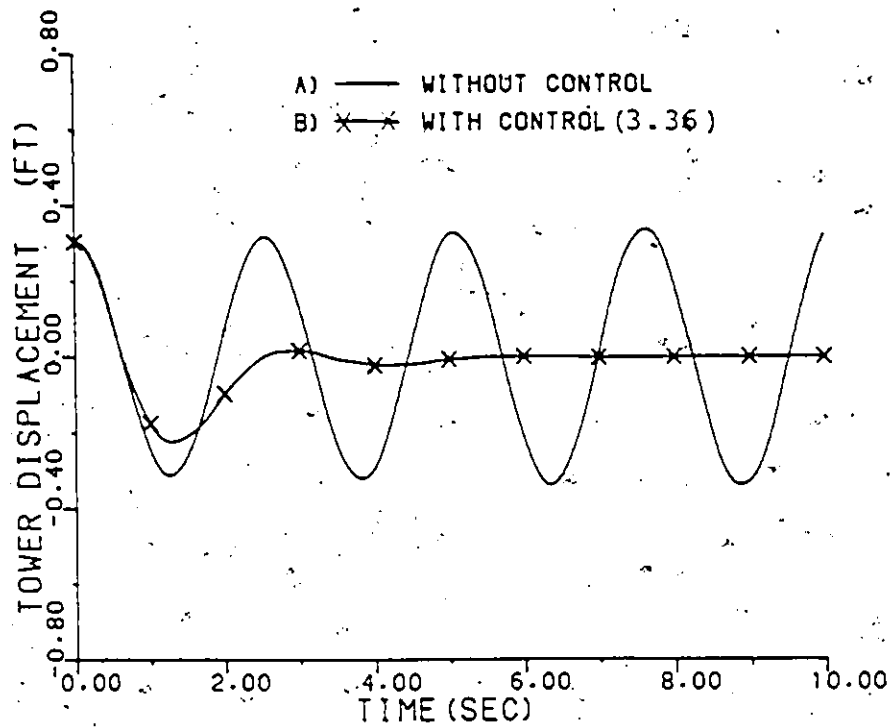
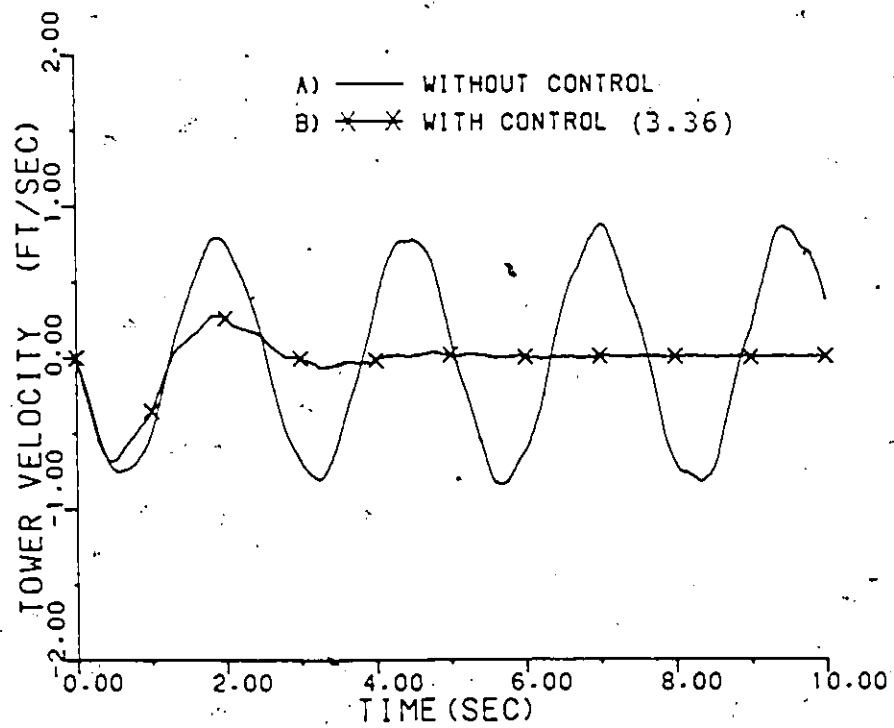


Fig. 3.9 Tower Velocity $\frac{\partial y_2}{\partial t}(L_2, t)$

Fig. 3.10 Tower Displacement $z_2(L_2, t)$ Fig. 3.11 Tower Velocity $\frac{\partial z_2}{\partial t}(L_2, t)$

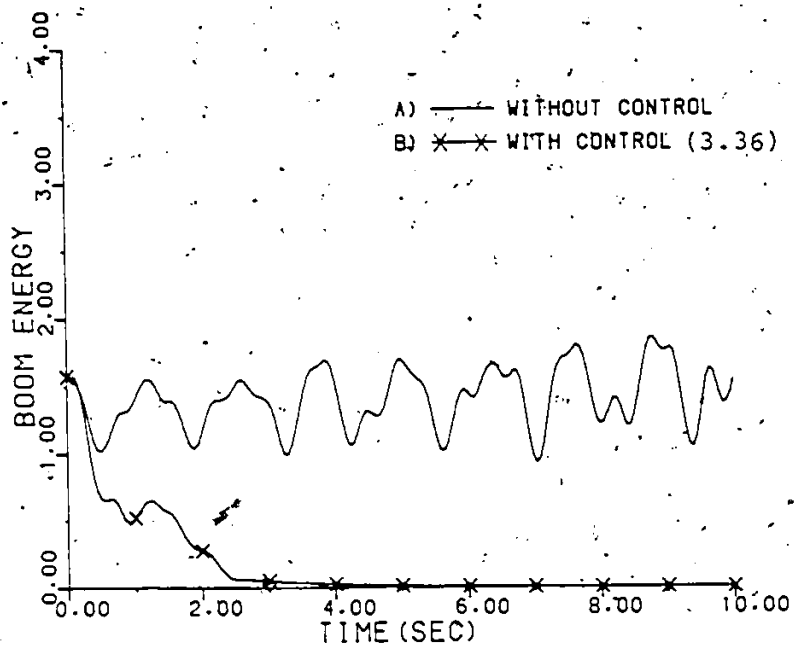


Fig. 3.12 Boom Relative Energy $E_1(t)$

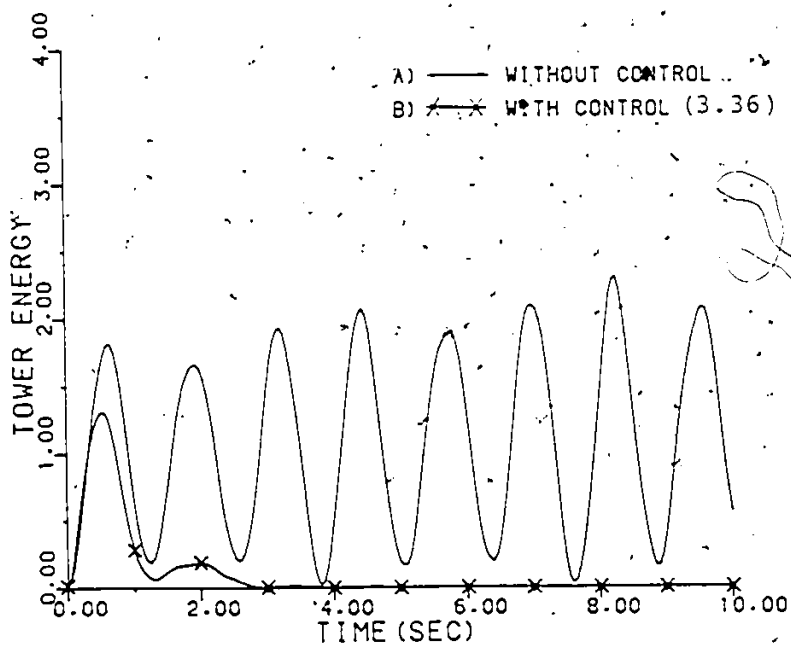
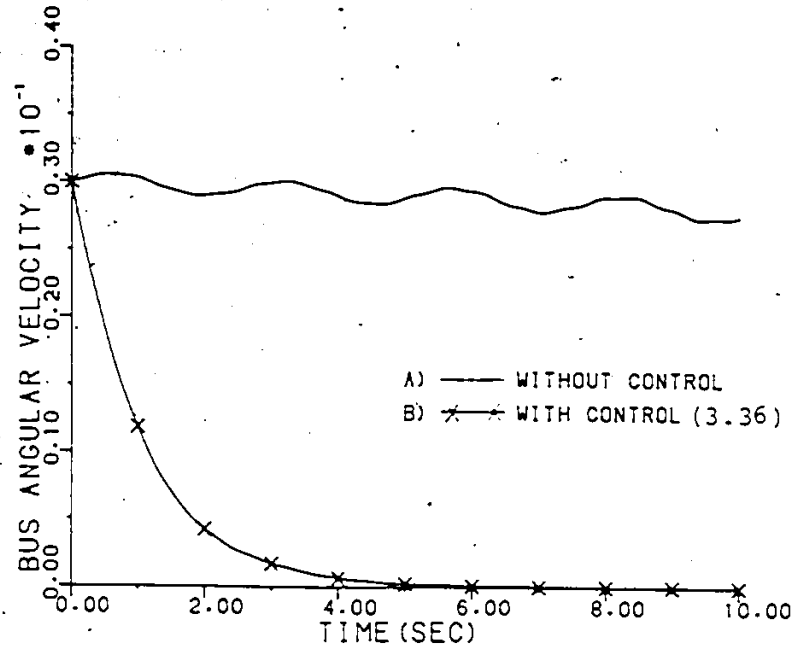
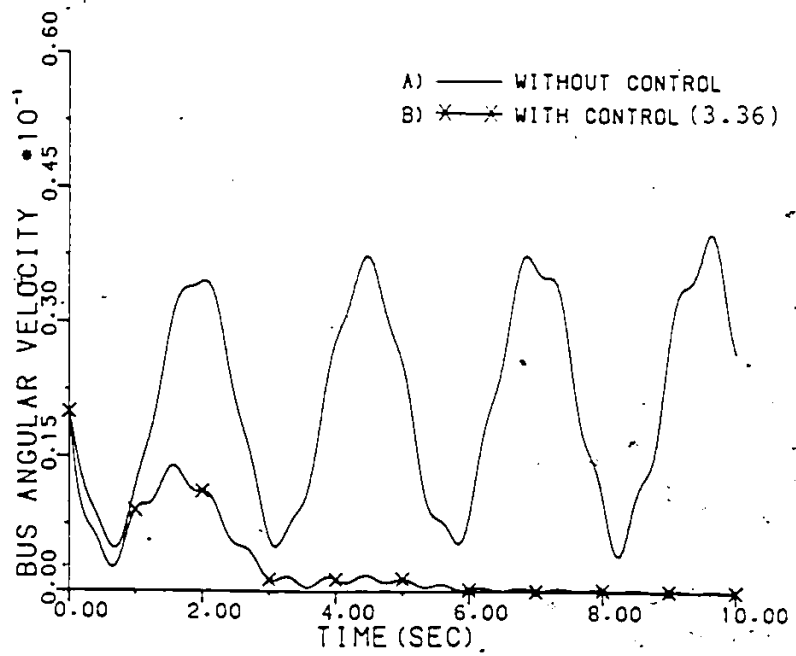


Fig. 3.13 Tower Relative Energy $E_2(t)$

Fig. 3.14 Bus Angular Velocity ω_1 (rad/sec)Fig. 3.15 Bus Angular Velocity ω_2 (rad/sec)

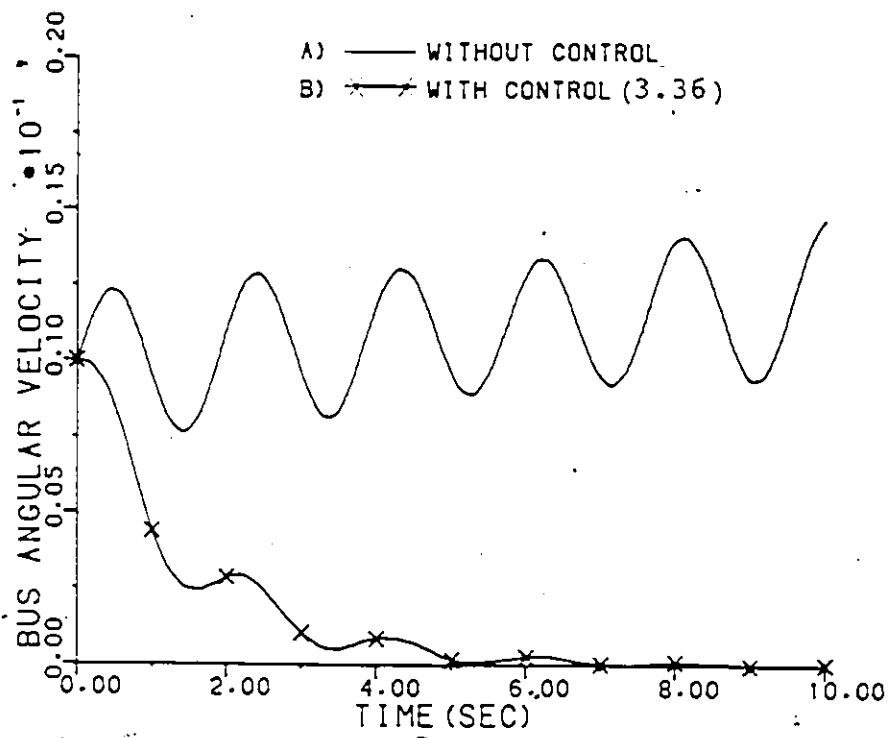


Fig. 3.16 Bus Angular Velocity ω_3 (rad/sec)

3.3 SPACECRAFT CONSISTING OF A BUS AND A SOLAR PANEL

In this section we consider the problem of modelling and stabilization of a spacecraft consisting of a rigid bus and a flexible solar panel as shown in Fig. 3.17. It is assumed that the solar panel is rigidly fixed to the bus, but its position and angle with respect to the bus is completely arbitrary. As in the previous cases, we shall assume that the satellite is on a fixed geo-synchronous orbit, and we develop the dynamics for the attitude motion of the bus and for the lateral vibrations of the solar panel.

3.3.1 Reference Frames and Inertia Tensors

Let $F_b \equiv (i_b, j_b, k_b)$ denote the body frame of reference fixed to the center of mass of the bus. Similarly we consider the orbital reference frame $F_r \equiv (i_r, j_r, k_r)$ as introduced in Section 2.2.1.

We shall use subscript 3 to denote any quantity associated with the solar panel. In order to describe the vibration of the solar panel, we define the coordinate system F_3 by $F_3 \equiv (i_3, j_3, k_3)$, with the axis i_3 along the major axis of the solar panel in the unperturbed position and the axis j_3 along its minor axis. The axis k_3 is, then, orthogonal to the i_3 - j_3 plane. It is further assumed that i_3 - j_3 plane coincides with the middle plane of the unperturbed solar panel. This is further illustrated in

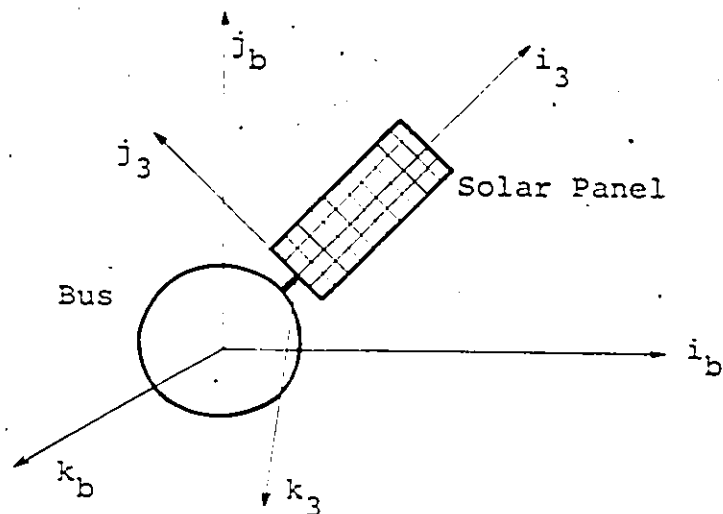


Fig. 3.17 Schematic Diagram of a Spacecraft with a Solar Panel.

Fig. 3.17. We shall assume that the F_3 frame is stationary with respect to the F_b frame.

Since the orientation of the solar panel with respect to the body frame is arbitrary, we need to introduce a Euler transformation matrix M_3 (c.f. equation(3.2)) to describe the orientation of the F_3 frame with respect to the F_b frame. Let a point P on the solar panel be denoted by the vector $r_3 = (x_3, y_3, z_3)$ in the F_3 frame of reference and by \tilde{r}_3 in the F_b frame, with the corresponding velocities denoted by v_3 and \tilde{v}_3 respectively. Then as discussed in Section 3.2.1, we have

$$\vec{r}_3 = R_3 + M_3 r_3. \quad (3.40)$$

$$\vec{v}_3 = M_3 v_3, \quad (3.41)$$

where R_3 is the vector denoting the origin of the F_3 frame with respect to the F_b frame. We note that the matrix M_3 is orthogonal and is obtained easily if the position of the unperturbed solar panel is given. Furthermore, since F_3 frame is stationary with respect to the F_b frame, M_3 is a constant matrix.

Let L_3^x be the length of the solar panel and L_3^y be its width. Then the inertia tensor I_3 of the solar panel is given by

$$I_3 = \int_{-L_3^y/2}^{L_3^y/2} \int_{-L_3^x/2}^{L_3^x/2} \rho_3 \{ (\vec{r}_3 \cdot \vec{r}_3) U_3 - \vec{r}_3 \vec{r}_3' \} dx_3 dy_3, \quad (3.42)$$

where $\rho_3 \equiv \rho_3(x_3, y_3)$ is the mass density per unit area and U_3 is a 3×3 identity matrix. We note here that because of vibrations, the vector r_3 and hence \vec{r}_3 would vary with time so that I_3 is also time varying. We shall assume that the solar panel will not elongate in the i_3 and j_3 directions. The deflections in the k_3 directions, denoted by $z_3 \equiv z_3(x_3, y_3, t)$, is governed by appropriate dynamic equations to be derived later in this section.

3.3.2 System Modelling

In order to derive the complete dynamics of the flexible spacecraft, we first introduce the total kinetic energy and the total potential energy of the system. The bus kinetic energy arising due to translational and rotational motions is given by (3.5). For the kinetic energy of the solar panel we have

$$T_3 = \frac{1}{2} \int \frac{d\tilde{r}_3}{dt} \cdot \frac{d\tilde{r}_3}{dt} dm_3 \quad (3.43)$$

with $\tilde{r}_3 = R_0 + R_3 + M_3 r_3$,

where \tilde{r}_3 denotes the absolute position of the incremental mass element dm_3 of the solar panel with respect to the inertial space, and $R_3 + M_3 r_3 = \hat{r}_3$ defines the position of dm_3 with respect to the body frame (see equation (3:40)). The position of the origin of the body frame in the inertial space is denoted by the vector R_0 .

The potential energy of the system arises due to elastic deformation of the solar panel. As mentioned earlier, we shall suppose that the solar panel will not elongate in the i_3 and j_3 direction, and the deflections in the k_3 direction will be denoted by $z_3 \equiv z_3(x_3, y_3, t)$. Assuming that the vibrations of the solar panel could be represented by that of a thin uniform elastic plate, the potential energy of the system could be given by [72,76]:

$$V_3 = \frac{1}{2} D \iint_{\Omega_3} \left\{ |\Delta z_3|^2 - 2(1-\eta) \left[\frac{\partial^2 z_3}{\partial x_3^2} \cdot \frac{\partial^2 z_3}{\partial y_3^2} - \left(\frac{\partial^2 z_3}{\partial x_3 \partial y_3} \right)^2 \right] \right\} dx_3 dy_3, \quad (3.44)$$

$$\text{with } \Delta \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2},$$

where D is the flexural rigidity of the plate and η is the Poisson's ratio. Letting h denote the thickness of the plate and E the Young's modulus, the flexural rigidity is given by $D = Eh^3/12(1-\eta^2)$.

With this introduction, we are now in a position to derive the dynamics of the flexible spacecraft. By Hamilton's principle, the dynamics of the system in the absence of any control is determined by the variational equation

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} (T_b + T_3 - V_3) dt, \quad (3.45)$$

subject to the end conditions

$$\delta \theta_1 = 0, \quad \delta \theta_2 = 0, \quad \delta \theta_3 = 0, \quad \delta z_3 = 0,$$

at $t = t_1$ and t_2 , where θ_i , $i = 1, 2, 3$, are the Euler angles describing the orientation of the body frame relative to the reference frame.

The virtual displacement and virtual velocity of an incremental mass element dm_3 of the solar panel are given by

$$\delta \tilde{r}_3 = \delta R_0 + \delta \theta_1 (R_3 + M_3 r_3) + M_3 \delta r_3, \quad (3.46)$$

$$\delta \left(\frac{d\tilde{r}_3}{dt} \right) = \delta v_0 + \delta \omega \times (R_3 + M_3 r_3) + \delta \theta \times \{ \omega \times (R_3 + M_3 r_3) + M_3 \dot{r}_3 \} + M_3 \frac{d}{dt} (\delta r_3). \quad (3.47)$$

By assumption the satellite describes a fixed circular orbit, so that $\delta R_0 = 0$, and $\delta v_0 = 0$. Then taking the first variation of the Lagrangian L as in (3.45) (with (3.5), (3.43), (3.44) substituted) and using the equations (3.46)-(3.47), and noting that $\delta \theta_1, \delta \theta_2, \delta \theta_3$, and δz_3 are arbitrary, we obtain the hybrid dynamics for the flexible spacecraft as given below:

Bus Dynamics (without control)

$$\frac{d}{dt} [(I_b + I_3) \cdot \omega] + \frac{d}{dt} \left[\int_{\Sigma_3} \rho_3 (R_3 + M_3 r_3) \times M_3 \dot{r}_3 \, dx_3 \, dy_3 \right] = 0,$$

Plate Dynamics (without control) (3.48)

$$\rho_3 \frac{d}{dt} \left[M_3^* \{ \omega \times (R_3 + M_3 r_3) + M_3 \dot{r}_3 \} \right]_k + \Delta (D \delta z_3) = 0,$$

where we have used the notation $(A)_k \equiv (A_1, A_j, A_k)_{k k} \equiv A_k$ to denote the k -th component of the vector A .

Using the rule (2.12) of differentiation in a rotating frame of reference in (3.48) and noting that $\dot{r}_3 = (0, 0, \frac{\partial z_3}{\partial t})$, the dynamics of the controlled spacecraft is thus given by

Bus Dynamics(with control)

$$\begin{aligned}
 (I_b + I_3) \cdot \ddot{\omega} + \omega \times [(I_b + I_3) \cdot \omega] + \int_{\Omega_3} \rho_3 (R_3 + M_3 r_3) \times M_3 \dot{v}_3 \, dx_3 \, dy_3 \\
 + 2 \int_{\Omega_3} \rho_3 (R_3 + M_3 r_3) \times (\omega \times M_3 v_3) \, dx_3 \, dy_3 = T, \quad (3.49)
 \end{aligned}$$

Plate Dynamics(with control)

$$\rho_3 \frac{\partial^2 z_3}{\partial t^2} + \Delta(D \Delta z_3) + \rho_3 Q_3 M_3^* \alpha_3 = F_3, \quad (3.50)$$

$$\text{with } \alpha_3 = \dot{\omega} \times (R_3 + M_3 r_3) + \omega \times (\omega \times (R_3 + M_3 r_3)) + 2 \omega \times M_3 v_3$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The boundary conditions for (3.50) are given below:

$$\text{At } x_3 = 0 : z_3 = 0 \quad \text{and} \quad \frac{\partial z_3}{\partial x_3} = 0,$$

$$\text{At } x_3 = L_3^x : \frac{\partial^2 z_3}{\partial x_3^2} + \eta \frac{\partial^2 z_3}{\partial y_3^2} = 0, \quad \frac{\partial^3 z_3}{\partial x_3^3} + (2-\eta) \frac{\partial^3 z_3}{\partial x_3 \partial y_3^2} = 0,$$

$$\text{At } y_3 = -L_3^y/2 \quad \text{and} \quad +L_3^y/2 : \quad (3.51)$$

$$\frac{\partial^2 z_3}{\partial y_3^2} + \eta \frac{\partial^2 z_3}{\partial x_3^2} = 0, \quad \frac{\partial^3 z_3}{\partial y_3^3} + (2-\eta) \frac{\partial^3 z_3}{\partial x_3^2 \partial y_3} = 0.$$

We note that the conditions at $x_3 = 0$ are the usual boundary conditions for the fixed edge of a plate, while rest of the conditions representing bending moment and shear force are the natural boundary conditions for free edges.

The complete dynamics of the flexible spacecraft consisting of a rigid bus and a solar panel, therefore, is given by the coupled system of ordinary differential equations (3.49) and the partial differential equations (3.50) along with the boundary conditions (3.51). Clearly, the two sets of equations are very strongly coupled. Any angular motion of the bus would induce vibrations of the solar panel and vice versa. As in the other cases, we observe that in the case of an infinitely rigid plate, the bus dynamics reduces to the usual dynamics of a rigid body satellite. Similarly, in the absence of any motions of the bus, the plate equation (3.50) reduces to the usual dynamics [72] for lateral vibration of a flexible plate.

3.3.3 Stabilization

We now study the question of stabilization of the flexible spacecraft described by the hybrid dynamics (3.49) - (3.51). We consider the following rest state for the system:

$$z_1 = 0, \quad z_2 = 0, \quad z_3 = 0,$$

and for $(x_3, y_3) \in [0, L_3^X] \times [-L_3^Y/2, +L_3^Y/2] \equiv \Omega_3$ (3.52)

$$z_3 = 0, \quad \frac{\partial z_3}{\partial t} = 0.$$

Following Lyapunov's approach, we show that the spacecraft could be stabilized in the asymptotic sense by application of simple feedback controls. This is stated in the following theorem.

Theorem 3.2

Consider the system described by the hybrid dynamics (3.49) - (3.50) with the boundary conditions (3.51) and consider the feedback controls given by

$$T = (-k_1 \omega_1, -k_2 (\omega_2 - \omega_0), -k_3 \omega_3), \quad k_1, k_2, k_3 > 0,$$

and

$$F = (-d(x_3, y_3) \frac{\partial z_3}{\partial t}), \quad d \geq 0 \text{ a.e. on } \Omega_3. \quad (3.53)$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (3.52).

Proof:

For brevity, we shall denote $\hat{r}_3 \equiv R_3 + M_3 r_3$. Scalar multiplying both sides of (3.49) by ω , we obtain

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} (I_b \cdot \omega) \cdot \omega + \frac{1}{2} \int |\omega \times \hat{r}_3|^2 dm_3 \right] + \omega \cdot \int \hat{r}_3 \times M_3 \dot{v}_3 dm_3 \\ - \int (\omega \times (\omega \times \hat{r}_3)) \cdot M_3 v_3 dm_3 = T \cdot \omega. \end{aligned} \quad (3.54)$$

Similarly, scalar multiplying both sides of (3.51) by $\frac{\partial z_3}{\partial t}$ and integrating by parts over Ω_3 , we have

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \int |M_3 v_3|^2 dm_3 + V_3 \right] + \int (\omega \times \hat{r}_3) \cdot M_3 v_3 dm_3 \\ + \int (\omega \times (\omega \times \hat{r}_3)) \cdot M_3 v_3 dm_3 = \int_{\Omega_3} F_3 \cdot \frac{\partial z_3}{\partial t} dx_3 dy_3, \end{aligned} \quad (3.55)$$

where V_3 is as given in (3.44).

We now introduce the total energy of the system denoted $V(t) \equiv V(\omega, z, \frac{\partial z_3}{\partial t})$, given by

$$V(t) = \frac{1}{2} (I_b \cdot \omega) \cdot \omega + \frac{1}{2} \int |M_3 v_3 + \omega \times r_3|^2 dm_3 + V_3. \quad (3.56)$$

Note that the first two terms in (3.56) represent the total kinetic energy, and V_3 is the potential energy of the plate. Then from (3.54) - (3.56) it follows that

$$\frac{dV}{dt} = T \cdot \omega + \int_{\sigma_3} F_3 \cdot \frac{\partial z_3}{\partial t} dx_3 dy_3. \quad (3.57)$$

The rest of the proof is similar to that of Theorem 3.1. \square

Remark 3.6

From (3.56) and (3.57), it is observed that in the absence of any control, the total system energy is conserved, although there may be interchange of energy during perturbations between the rigid and the elastic members of the spacecraft.

Remark 3.7

The dynamics of an antenna on-board a spacecraft could also be developed following similar procedure as discussed in Section 3.2 and Section 3.3.

3.4 SPACECRAFT CONSISTING OF A BUS, A BOOM, A TOWER, AND A SOLAR PANEL

The complete dynamics of a flexible spacecraft, consisting of a rigid bus, a boom, a tower and a solar panel could be developed following the procedure presented in Sections 3.2 and 3.3, and is given by:

Bus Dynamics

$$I^T \dot{\omega} + \omega \times (I^T \omega) + \sum_{i=1}^3 \int (R_i + M_i r_i) \times M_i \dot{v}_i dm_i \\ + 2 \sum_{i=1}^3 \int (R_i + M_i r_i) \times (\omega \times M_i v_i) dm_i = T$$

where $I^T = I_b + I_1 + I_2 + I_3$

Boom Dynamics

Given by equation (3.20)

Tower Dynamics

Given by equation (3.21)

Solar Panel Dynamics

Given by equation (3.50)

The boundary conditions for (3.20)-(3.21) are given by (3.22), and those for the equation (3.50) by (3.51).

The model could be extended to include any number of elastic elements. Also stabilization of the system using velocity feedback follows easily as shown in Theorem 3.1 and Theorem 3.2.

3.5 SUMMARY

In this chapter, we presented a methodology for developing the complete dynamics of a flexible spacecraft of general structure. Complete dynamic equations are developed for a spacecraft consisting of a rigid bus, a boom, a tower, and a solar panel. It is shown that the complete dynamics of the system could be described by a hybrid system of equations, i.e. a set of ordinary differential equations for the rigid members and a set of partial differential equations for the elastic members of the spacecraft. These equations indicate that the rigid and elastic members of the spacecraft interact very strongly during perturbations.

Stabilization of the system is proved following Lyapunov's approach. It is shown that flexible spacecraft could be stabilized by application of simple feedback controls proportional to the bus angular velocities and the vibrational velocity of the elastic members. These controls are practically implementable.

Although the system dynamics appears to be somewhat complicated, we have demonstrated that it could be simulated on a digital computer. A typical run for the results presented in this chapter took 90 seconds of CPU time on an AMDAHL 470/V 8 computer, and it could be further reduced by improving the efficiency of the program.

Chapter IV

STABILIZATION OF FLEXIBLE STRUCTURES IN THE PRESENCE OF RANDOM DISTURBANCES

Flexible structures in space are often subjected to random disturbances arising from various sources such as meteorite collisions, variations in solar pressure and magnetic fields due to disturbances occurring in the sun, as well as on-board disturbances such as sloshing of liquid fuel, disturbances due to motors and pumps etc. These disturbances may produce random torques as well as random spatially distributed forces on the flexible spacecraft; and may be very detrimental to the overall stability of the system. A stable spacecraft may gain some angular motions of the bus, and/or vibrations of the beam or other elastic members. If the disturbances persist, then in the absence of appropriate controls, these small motions may build up and lead to instability of the system.

In this chapter we consider the problem of stabilization of flexible spacecraft in the presence of random noise acting on the bus and on the beam. We shall assume that the structure of the spacecraft is the same as that considered in Chapter II. We shall further assume that the beam is

uniform along its length. First, we consider the question of stabilization of a flexible beam in the presence of distributed noise. This result would then be extended to the stabilization of a flexible spacecraft in a noisy environment.

4.1 NOTATIONS

Let Q be an open bounded set in R . We denote by $L_2(Q)$ the equivalence classes of Lebesgue measurable square integrable functions on Q . We shall use the standard notation $H^m(Q)$ to denote the Sobolev space [2] of order m on Q defined by

$$H^m(Q) = \left\{ y \in L_2(Q) : \frac{\partial^\alpha y}{\partial x^\alpha} \in L_2(Q), \alpha \leq m \right\}, \text{ integer } m \geq 1, \quad (4.1)$$

furnished with the norm

$$\|y\|_{H^m(Q)} = \left(\sum_{\alpha \leq m} \left\| \frac{\partial^\alpha y}{\partial x^\alpha} \right\|_{L_2(Q)}^2 \right)^{\frac{1}{2}}$$

The space $H^m(Q)$ equipped with the scalar product

$$(y, z)_{H^m(Q)} = \sum_{\alpha \leq m} \left(\frac{\partial^\alpha y}{\partial x^\alpha}, \frac{\partial^\alpha z}{\partial x^\alpha} \right)_{L_2(Q)}$$

is a Hilbert space.

The solutions of hyperbolic systems such as the beam equation are frequently defined in certain energy spaces. For the present problem we define the energy space \tilde{E} by

$$E \equiv Y \times Z ,$$

$$Y \equiv \{ \phi \in H^2(Q) : \phi(0) = 0, \frac{\partial \phi}{\partial x}(0) = 0 \} , \quad (4.2)$$

$$Z \equiv L_2(Q) .$$

The energy space E is given the norm

$$\| (y, z) \|_E \equiv \left(\int_Q \left(EI \left| \frac{\partial^2 y}{\partial x^2} \right|^2 + \rho |z|^2 \right) dx \right)^{\frac{1}{2}} , \quad (4.3)$$

where $EI > 0$ and $\rho > 0$ are constant.

Let (Ω, \mathcal{F}, P) denote a complete probability space, where Ω is the sample space, \mathcal{F} is the sigma algebra of events on Ω and P is the probability measure on \mathcal{F} . Let F be a Banach space. Then by $L_2(\Omega, F)$ we denote the equivalence classes of strongly P -measurable functions on Ω with values in F such that

$$\| f \|_{L_2(\Omega, F)} = \left(\int_{\Omega} \| f \|_F^2 dP \right)^{\frac{1}{2}} = \left(E(\| f \|_F^2) \right)^{\frac{1}{2}} < \infty , \quad (4.4)$$

where $E\{\cdot\}$ denotes the mathematical expectation of its argument. For any Banach space F , we shall use

$$B_F(r) \equiv \{ f \in F : \| f \|_F \leq r \} , \quad (4.5)$$

to denote the ball of radius r in F .

In addition to the notations stated above, we shall continue using the notations introduced in Chapter II.

4.2 STABILIZATION OF A FLEXIBLE BEAM

In this section we shall consider the problem of stabilization of a flexible beam in the presence of distributed noise. The dynamics of a uniform beam is given by the well-known Euler equation

$$\rho \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = u(t,x) + \xi(t,x), \quad t \geq 0, \quad x \in Q \equiv (0,L), \quad (4.6)$$

where $y \equiv y(t,x)$ is the transverse deflection of the beam and $\rho (>0)$ is the mass density per unit length and $EI (>0)$ is the flexural rigidity of the beam. We shall suppose that the beam is acted upon by a distributed white noise denoted by $\xi(t,x)$. We shall give further characterization of the noise process later in this section. We are interested to find a control $u(t,x)$ that stabilizes the system (4.6) which is otherwise unstable.

The boundary conditions for (4.6) depend on the physical conditions at the two ends of the beam, (such as rigidly fixed, free, or simply supported ends). In this study we consider a cantilever beam, motivated by the fact that the beam associated with a flexible spacecraft is cantilever in nature. Thus we have the following boundary conditions:

$$\begin{aligned} y(t,0) = 0, \quad \frac{\partial^2 y}{\partial x^2}(t,L) = 0, \\ \frac{\partial y}{\partial x}(t,0) = 0, \quad \frac{\partial^3 y}{\partial x^3}(t,L) = 0. \end{aligned} \quad (4.7)$$

As is wellknown, by applying a simple transformation of variables one can normalize the beam equation (4.6), which gives rise to certain mathematical simplifications. However, in this study we shall restrain ourselves from doing so, since we are interested to relate the vibration characteristics with the beam parameters such as ρ and EI.

The questions of control and stabilization of systems governed by the deterministic wave equation have been considered by several authors, [3,18,26-29,44,62,63] to name a few. Control schemes for exponential stabilization using distributed and boundary feedback have been developed in [26-29]. Stability of systems governed by the wave equation in the presence of distributed noise has been considered in [21]. For the parabolic systems stability questions under very general conditions has been treated in [10]. In this chapter, for the study of stability of the beam equation in the presence of distributed noise, we shall follow essentially the same method as developed in [10,21].

4.2.1 A Distributed Noise Process

In this section, we give a characterization of the noise process $\xi(t,x)$ perturbing the beam equation. Let $\{n_n(t), n=1,2,\dots,\infty\}$ denote a sequence of independent Gaussian white noise processes such that

$$\begin{aligned} E \{n_n(t)\} &= 0, \text{ for all } n, \\ E \{n_n(t) n_n(\tau)\} &= \delta(t-\tau), \text{ for all } n, \\ E \{n_n(t) n_m(\tau)\} &= 0, \text{ for all } n \neq m. \end{aligned} \tag{4.8}$$

We now define the distributed noise $\xi(t, x)$ by

$$\xi(t, x) \equiv \sum_{n=1}^{\infty} \sigma_n \eta_n(t) \phi_n(x), \quad (4.9)$$

$$\text{with } q \equiv \sum_{n=1}^{\infty} \sigma_n^2 < \infty,$$

where, for convenience, one can take the sequence $\{\phi_n, n=1, 2, \dots, \infty\}$ to be the eigenfunctions of the operator A defined by

$$A\psi \equiv \frac{\partial^4 \psi}{\partial x^4}, \quad x \in (0, L), \quad (4.10)$$

$$\text{with } D(A) \equiv \{\psi \in H^4(Q) : \psi(0) = 0, \frac{\partial \psi}{\partial x}(0) = 0, \frac{\partial^2 \psi}{\partial x^2}(L) = 0, \frac{\partial^3 \psi}{\partial x^3}(L) = 0\}.$$

However, any sequence $\{\phi_n\} \in D(A)$ which forms a basis for $L_2(Q)$ can be used.

It is known that the operator A has an infinite system of distinct positive eigenvalues $\{\lambda_n\}$ and the corresponding eigenfunctions $\{\phi_n\}$ form an orthonormal (after proper normalization if necessary) basis for $L_2(Q)$. Then, $\xi(t, x)$ represents a distributed white noise satisfying

$$\text{i) } E\{\xi\} = 0,$$

$$\text{ii) for } h, f \in L_2(Q),$$

$$\begin{aligned} (\text{Cov } \xi, h, f) &= E\{(\xi(t+\tau, \cdot), h) (\xi(\tau, \cdot), f)\} \\ &= \delta(t) \sum_{n=1}^{\infty} \sigma_n^2 (\phi_n, h) (\phi_n, f) \\ &= \delta(t) (\Gamma h, f) \end{aligned}$$

where the covariance operator Γ is defined by

$$\Gamma \equiv \sum_{n=1}^{\infty} \sigma_n^2 \phi_n \otimes \phi_n.$$

Then the trace of the covariance operator is given by

$$(\text{Trace Cov } \xi)(t) = \delta(t) \sum_{n=1}^{\infty} (\Gamma \phi_n, \phi_n) = \delta(t) \sum_{n=1}^{\infty} \sigma_n^2.$$

More precisely, letting $\{W_n, n = 1, 2, \dots, \infty\}$ denote a sequence of independent standard Wiener processes whose generalized derivatives are the white noise processes $\{\eta_n, n = 1, 2, \dots, \infty\}$, the L_2 -valued Wiener process $\{W(t), t \geq 0\}$ is given by

$$W(t) \equiv \sum_{n=1}^{\infty} \sigma_n W_n(t) \phi_n(x).$$

Then for $h, f \in L_2(Q)$, the covariance operator is given by

$$E\{(W(t)-W(s), h)(W(t)-W(s), f)\} = (t-s)(Th, f).$$

Note that, for each $n = 1, 2, \dots, \infty$, σ_n^2 represents the noise power associated with the n -th eigenmode of vibration and the sum $\sum_{n=1}^{\infty} \sigma_n^2 = q$ is the total power spectral density of the noise process. Finiteness of q implies that noise in very higher order modes are insignificant.

4.2.2 Mean Square Stabilization

With this development, we can now consider the question of stabilization of the system (4.6). It is clear that in the absence of any control, i.e. $u \equiv 0$, the noisy system (4.6) has no stable rest state; and in fact, the system is unstable. In what follows, we shall suppose that the control is provided by the velocity feedback, i.e. $u = -\alpha \frac{\partial y}{\partial t}$, $\alpha > 0$, and study the stability properties of the following controlled system:

$$\rho \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = u + \sum_{n=1}^{\infty} \sigma_n \eta_n(t) \phi_n(x), \quad x \in Q, \quad t \geq 0, \quad (4.11)$$

$$u = -\alpha \frac{\partial y}{\partial t}$$

satisfying the boundary conditions (4.7) and the initial conditions

$$y(0, x) = y_0(x), \quad \frac{\partial y}{\partial t}(0, x) = z_0(x).$$

Clearly, because of the perturbing noise, there is no stable rest state of the system (4.11). However, in the next theorem we show that the system, with $\alpha > 0$, is exponentially stable in the mean square sense, with respect to a ball in the energy space E where the energy space E is as defined by (4.2) and (4.3).

Definition 4.1 : Stability in the Mean

The system (4.11) is said to be stable in the mean if there exists a constant $k \geq 0$ such that for every $R \geq k$, there exists an $\tau > 0$ such that $E\{\|(y, \frac{\partial y}{\partial t})(t)\|_E^2\} < R$ for all $t \geq 0$ whenever $E\{\|(y_0, z_0)\|_E^2\} < \tau$.

Definition 4.2 : Exponential Stability in the Mean

The system (4.11) is said to be exponentially stable in the mean if for every $(y_0, z_0) \in L_2(Q, E)$, i.e. $E\{\|(y_0, z_0)\|_E^2\} < \infty$, there exists constants $\eta, \delta > 0$, $\beta \in R$ with $\eta + \beta \geq 0$, such that

$$E\{\|(y, \frac{\partial y}{\partial t})(t)\|_E^2\} \leq \eta + \beta e^{-\delta t}, \quad t \geq 0.$$

Theorem 4.1

Consider the beam equation (4.11) with the velocity feedback $u = -\alpha \frac{\partial v}{\partial t}$, $\alpha > 0$, and subjected to the distributed noise ε having the power spectral density $q \equiv \varepsilon \varepsilon_n^2 < \infty$. Let the initial state $\{y_0, z_0\} \in X \equiv L_2(\Omega, \mathcal{E})$ be arbitrary and independent of the noise process. Then the system is globally exponentially stable with respect to the ball $B_X(\varepsilon)$, $\varepsilon = \sqrt{(qk_2/\alpha k_1)}$, where k_1, k_2 , with $k_1 < k_2$, are two positive numbers depending on α .

Proof

Let the sequence $\{e_n\}$ and $\{f_n\}$ be as defined in Section 4.2.1. Since $\{e_n\}$ forms an orthonormal basis for $L_2(Q)$, we have the expansion

$$y(t, x) = \sum_{n=1}^{\infty} y_n(t) e_n(x) \quad (4.12)$$

where $\{y_n\}$ are random processes satisfying the stochastic differential equation

$$\dot{y}_n + \alpha \dot{z}_n + EI \lambda_n y_n = \varepsilon_n \eta_n(t), \quad n = 1, 2, \dots, \infty \quad (4.13)$$

Letting $z_n \equiv \dot{y}_n$, we can formally rewrite (4.13) as an Ito differential equation

$$\begin{bmatrix} dy_n \\ dz_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -EI \lambda_n & -\alpha \end{bmatrix} \begin{bmatrix} y_n \\ z_n \end{bmatrix} dt + \begin{bmatrix} 0 \\ \frac{\varepsilon_n}{\alpha} \end{bmatrix} dW_n \quad (4.14)_n$$

where $\{W_n, n=1, 2, \dots, \infty\}$ are independent standard Wiener processes whose generalized derivatives are the white noise processes $\{\eta_n, n=1, 2, \dots, \infty\}$.

Now for each integer n , we define the functions

$$V_n(a,b) = \left(\frac{\alpha}{2} + \frac{\rho EI \lambda_n}{\alpha} \right) a^2 + \rho ab + \frac{\rho^2}{\alpha} b^2, \quad (a,b) \in \mathbb{R}^2,$$

and introduce the processes $\{V_n(t), t \geq 0\}$ defined by

$$V_n(t) = V_n(y_n(t), z_n(t)), \quad (4.15)$$

where $\{y_n, z_n\}$ are solutions of equations (4.14) $_n$. Then, one verifies that the process $\{V_n\}$ has Ito differential given by

$$dV_n = \left(\frac{\alpha}{2} - EI \lambda_n y_n^2 - \rho z_n^2 \right) dt + c_n \left(y_n + \frac{2\rho}{\alpha} z_n \right) dW_n, \quad (4.16)$$

Integrating this last equation over $(0,t)$ and taking expectation and noting that $t \rightarrow E\{y_n^2, z_n^2\}$ are locally integrable, we have

$$E V_n(t) = E V_n(0) + E \int_0^t \left(\frac{\alpha}{2} - EI \lambda_n y_n^2 - \rho z_n^2 \right) ds. \quad (4.17)$$

We now define the functionals V and \tilde{V} by

$$V(t) = \int_Q \left(\frac{\alpha}{2} y^2 + \frac{\rho EI}{\alpha} \frac{\partial^2 y}{\partial x^2} + \frac{\rho^2}{\alpha} \left(\frac{\partial y}{\partial t} \right)^2 + \rho y \frac{\partial y}{\partial t} \right) dx, \quad (4.18)$$

$$\tilde{V}(t) = \int_Q \left(EI \frac{\partial^2 y}{\partial x^2} + \rho \left(\frac{\partial y}{\partial t} \right)^2 \right) dx = \left\| \left(y, \frac{\partial y}{\partial t} \right) \right\|_E^2, \quad (4.19)$$

where $y = y(t,x)$ is the solution of the system (4.11).

Using the elementary inequality

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2 \quad \text{for all } \epsilon > 0, \quad (4.20)$$

and the Poincaré's inequality

$$\int_Q y^2 dx \leq K \int_Q EI \frac{\partial^2 y}{\partial x^2} dx, \quad \text{for some } K > 0, \quad (4.21)$$

one can find two positive numbers k_1 and k_2 , with $k_1 < k_2$ and independent of y and $\frac{\partial y}{\partial t}$, such that

$$k_1 \tilde{V}(t) \leq V(t) \leq k_2 \tilde{V}(t). \quad (4.22)$$

Furthermore, substituting (4.12) into (4.18) and using Parseval's identity, we have

$$V(t) = \sum_{n=1}^{\infty} V_n(t),$$

where V_n is as defined in (4.15). Since each V_n is nonnegative, V is also nonnegative and it follows from monotone convergence theorem and Fubini's theorem that

$$\begin{aligned} E V(t) &= \sum_{n=1}^{\infty} E V_n(t) \\ &= E V(0) + \int_0^t \frac{1}{\alpha} \sum_{n=1}^{\infty} c_n^2 - E \left\{ \sum_{n=1}^{\infty} E I \lambda_n y_n^2 + \rho z_n^2 \right\} ds. \end{aligned} \quad (4.23)$$

Using Parseval's identity (for each $t \geq 0$)

$$\sum_{n=1}^{\infty} (E I \lambda_n y_n^2 + \rho z_n^2) = E I \left\| \frac{\partial^2 y}{\partial x^2} \right\|_{L_2(Q)}^2 + \rho \left\| \frac{\partial y}{\partial t} \right\|_{L_2(Q)}^2 = \tilde{V}, \quad (4.24)$$

and noting that $\sum_{n=1}^{\infty} c_n^2 = \alpha < \infty$, we obtain from (4.23) that

$$E V(t) = E V(0) + \int_0^t \left\{ \frac{\alpha}{\alpha} - E \tilde{V}(s) \right\} ds. \quad (4.25)$$

Combining the inequalities (4.22) with the equation (4.25), one obtains

$$E \tilde{V}(t) \leq \frac{\alpha}{\alpha} \frac{k_2}{k_1} + \frac{k_2}{k_1} \left(E \tilde{V}(0) - \frac{\alpha}{\alpha} \right) \exp\left(-\frac{1}{k_2} t\right), \quad (4.26)$$

which shows that the system is exponentially stable with respect to the ball $B_X(\varepsilon)$, $\varepsilon = \sqrt{qk_2/ak_1}$, where $X \equiv L_2(Q, E)$. In other words, as $t \rightarrow \infty$, the expected energy of vibration, i.e. $E\{ \|(y, \frac{\partial y}{\partial t})\|_E^2 \}$ would exponentially decay to some value ε^2 .

Remark 4.1

Exponential stability of the deterministic beam equation corresponding to (4.11) (c.f. [26]) can be obtained directly from (4.26) by setting the noise power density $q = 0$, i.e.

$$\tilde{V}(t) = \frac{k_2}{k_1} \tilde{V}(0) \exp(-\frac{1}{k_2} t).$$

Remark 4.2

Using the energy function \tilde{V} as in equation (4.19) in conjunction with (4.14) one obtains

$$E \tilde{V}(t) = E \tilde{V}(0) + \int_0^t \left[\frac{q}{E} - 2 \cdot E \left\| \frac{\partial y}{\partial t} \right\|_{L_2(Q)}^2 \right] ds, \quad (4.27)$$

which indicates an asymptotic limit of $E \left\{ \left\| \frac{\partial y}{\partial t} \right\|_{L_2(Q)}^2 \right\}$. However, asymptotic stability of the system (4.11) cannot be ascertained from this relation since the invariance principle (commonly used in the stability analysis of finite dimensional problems) is not well established for the infinite dimensional systems. This problem has been avoided in this study by introducing the 'energy-like' functional V as in (4.18) and by establishing the inequalities (4.22). Note that because of the inequalities (4.22), the functional V is completely equivalent to the energy functional \tilde{V} .

Remark 4.3

It is also clear from (4.27) that in the absence of any damping, i.e. $\alpha = 0$, the (expected) energy grows in time, indicating that the uncontrolled system is unstable.

4.2.3 Almost Sure Stabilization

Theorem 4.1 gives mean square stability of the system which from practical point of view does not prevent certain trajectories from taking very large values with small but nonzero probability. Therefore, it is more appropriate to consider the question of stability in the almost sure sense which essentially characterizes the properties of the sample paths and is, in fact, a much more stronger result compared to the mean square stability. Almost sure stability of parabolic systems under very general conditions has been treated recently in [10]. For the beam equation, we present the following result, the proof of which is essentially similar to that of [10]. We first introduce the following definition for almost sure stability.

Definition 4.3 : Almost Sure Stability

The system (4.11) is said to be almost surely globally asymptotically stable if there exists an ϵ valued F measurable random variable κ such that for all $(y(0, \cdot), \frac{\partial y}{\partial t}(0, \cdot)) \in E$,

$$P\left\{ \lim_{t \rightarrow \infty} (y, \frac{\partial y}{\partial t})(t) = \kappa \right\} = 1 ;$$

and is said to be almost surely globally asymptotically stable with respect to a closed set $K \subset E$ if

$$P\left\{\lim_{t \rightarrow \infty} (y, \frac{\partial y}{\partial t})(t) \in K\right\} = 1.$$

Theorem 4.2

Consider the system (4.11) and suppose that the hypotheses of the Theorem 4.1 are satisfied. Then the system is almost surely asymptotically stable with respect to the ball $B_E(\delta)$, $\delta = \sqrt{\alpha k_2 / \alpha k_1}$, i.e. $P\left\{\lim_{t \rightarrow \infty} (y, \frac{\partial y}{\partial t})(t) \in B_E(\delta)\right\} = 1$.

Proof

Let F_s be an increasing family of completed subsigma algebras of the σ -algebra F generated by the random variables y_0, z_0 and the Wiener processes $\{W_n, n = 1, 2, \dots, \infty\}$. That is

$$F_s = \sigma(y_0, z_0, W_n(s), n=1, 2, \dots, \infty, 0 \leq s).$$

Integrating (4.16) from s to t , $0 \leq s \leq t$ and taking conditional expectation with respect to F_s , we obtain

$$E\{V_n(t) | F_s\} = V_n(s) + E\left[\int_s^t \left(\frac{\alpha}{\alpha} - E\{A_n^2 | F_s\} - \alpha z_n^2\right) ds | F_s\right].$$

Hence, by monotone convergence theorem and Parseval's identity it follows that

$$\begin{aligned} E\{V(t) | F_s\} &= V(s) + E\left[\int_s^t \left(\frac{\alpha}{\alpha} - \tilde{V}(s)\right) ds | F_s\right] \\ &\leq V(s) + \frac{1}{K_2} E\left[\int_s^t \left(\frac{\alpha}{\alpha} k_2 - V(s)\right) ds | F_s\right], \end{aligned} \quad (4.28)$$

where the last inequality follows from (4.22).

Let $s \geq 0$ be any arbitrary time and F_s is given. Suppose that

$$P \left\{ \left(y, \frac{\partial y}{\partial t} \right) (s) \notin B_{\varepsilon}(s) \right\} = 1.$$

That is $\| \left(y, \frac{\partial y}{\partial t} \right) (s) \|_{\varepsilon}^2 = \tilde{V}(s) > \varepsilon^2$. By the inequality (4.22), this also implies that $V(s) > \frac{\alpha}{2} k_2$. Let τ_s be first time the process $V(t)$ violates this inequality, i.e.

$$\tau_s = \inf \{ t \geq s : V(t) \leq \frac{\alpha}{2} k_2 \}.$$

Define the stopped process $V(t \wedge \tau_s)$ by

$$\begin{aligned} V(t \wedge \tau_s) &= V(t) & \text{if } t < \tau_s, \\ &= V(\tau_s) & \text{if } t \geq \tau_s. \end{aligned}$$

Then from (4.28), it follows that, for $s \leq \theta < t$,

$$\begin{aligned} E \{ V(t \wedge \tau_s) \mid F_{\theta \wedge \tau_s} \} &< V(\theta \wedge \tau_s) & \text{if } t < \tau_s \\ &\leq V(\theta \wedge \tau_s) & \text{if } t \geq \tau_s. \end{aligned}$$

Therefore, the process $\{ V(t \wedge \tau_s), F_{\theta \wedge \tau_s} \}$ is a supermartingale. Furthermore, $V(t \wedge \tau_s)$ is positive. Hence, by Doob's supermartingale convergence theorem ([55], Theorem T6, pp.-96),

$$\lim_{t \rightarrow \infty} V(t \wedge \tau_s) = V(\tau_s)$$

exists and is finite almost surely. In other words

$$\lim_{t \rightarrow \infty} V(t) \longrightarrow \gamma, \quad \gamma \leq \frac{\alpha}{2} k_2 \quad \text{with probability one.}$$

Hence, by the inequality (4.22)

$$\lim_{t \rightarrow \infty} \tilde{V}(t) \rightarrow \gamma', \gamma' \leq \frac{qk_2}{\alpha k_1} \quad \text{with probability one.}$$

Since $s \in [0, \infty)$ and $(y_0, z_0) \in E$ (P-a.s.) are arbitrary the conclusion of the theorem follows. \square

4.2.4 A Suboptimal Choice of Damping

It would be desirable to choose the feedback coefficient (damping) α so that one obtains maximum decay rate and minimum size for the attractor. From Theorem 4.1 and Theorem 4.2 we observe that the radius r of the attractor and the decay rate d are given by

$$r = \sqrt{qk_2 / \alpha k_1}, \quad d = 1/k_2,$$

where the constants k_1 and k_2 can be obtained easily from (4.18)-(4.22) as

$$\begin{aligned} k_1 &= \min \left\{ \frac{q}{\alpha}, \frac{q}{\alpha} - \frac{q}{2\varepsilon_1} \right\} \text{ defined for } \alpha > 0, \frac{q}{2} < \varepsilon_1 \leq \alpha \\ k_2 &= \max \left\{ K \left(\frac{\alpha}{2} + \frac{q\varepsilon_2}{2} \right) + \frac{q}{\alpha}, \frac{q}{\alpha} + \frac{1}{2\varepsilon_2} \right\} \text{ defined for } \alpha > 0, 0 < \varepsilon_2 < \infty. \end{aligned} \quad (4.29)$$

It is clear from the above that such an α exists and can be determined in the following way.

Since we are interested to find the smallest attractor and the largest decay rate, it is necessary to obtain the supremum of k_1 and the infimum of k_2 over the domain as defined in (4.29). First fixing α , we have

$$\begin{aligned} \text{Sup}_{\frac{\rho}{2} < \varepsilon_1 < \alpha} k_1 &= \frac{c}{2\alpha}, \\ \text{Inf}_{0 < \varepsilon_2 < \infty} k_2 &= \frac{\rho}{\alpha} + \frac{1}{-\frac{\rho}{\alpha} + \left(\frac{\rho^2}{c^2} + \frac{4}{\rho K} \right)^{\frac{1}{2}}}. \end{aligned}$$

and hence, for the given α , one obtains

$$\begin{aligned} r^2 &= \frac{2c}{\alpha \left[1 - \left(1 + \frac{4c}{\alpha^2 K} \right)^{-\frac{1}{2}} \right]}, \\ d &= \frac{\rho}{c} \left[1 - \left(1 + \frac{4c}{\alpha^2 K} \right)^{-\frac{1}{2}} \right]. \end{aligned} \tag{4.30}$$

Asymptotic behavior of the decay rate d in (4.30) with respect to α indicates that as $\alpha \rightarrow \infty$, $d \approx \frac{2}{K\alpha} \rightarrow 0$. That is the decay rate of energy does not increase arbitrarily for large values of α . This conforms with an observation made in [26], pp. -73 for the deterministic wave equation. Also one observes that $d \rightarrow 0$ as $\alpha \rightarrow 0$. The asymptotic properties of the size of the attractor with respect to α are just the opposite. This justifies existence of an optimal damping α^* for maximum decay rate and minimum size of the attractor; and from (4.30) one easily obtains

$$\alpha^* = \left[(\sqrt{5} - 1) \frac{2c}{K} \right]^{\frac{1}{2}}$$

At this stage we need an estimate of the constant K in the Poincare's inequality (4.21). It is easy to verify that an estimate of K could be given by $K = (L)^4 / (8EI)$. Thus,

we have the following estimates for the best damping coefficient and the corresponding size of the attractor.

$$\alpha^* = 4(\sqrt{5} - 1)^{\frac{3}{2}} \frac{\sqrt{EI} \sqrt{c}}{L^2}, \quad (\text{Damping Coefficient})$$

$$d^* = (\sqrt{5} - 1)^{\frac{5}{2}} \frac{1}{L^2} \sqrt{\frac{EI}{c}}, \quad (\text{Decay Rate})$$

$$(r^2)^* = \frac{2}{(\sqrt{5} - 1)^{\frac{5}{2}}} \frac{c L^2}{\sqrt{EI} \sqrt{c}} \quad (\text{Attractor Size}).$$

It is clear from the above relations that the more the beam is rigid, the faster is the decay of vibrations and the smaller is the size of the attractor. On the other hand a heavier beam would require longer time to settle down after a disturbance. Similarly, a longer beam would vibrate for a longer time with a larger amplitude.

The radius of the attractor increases with the increase of the noise strength g , implying that the amplitude of vibration will be larger if the noise is stronger; and shrinks to zero in the absence of any noise as in the deterministic case. It is interesting to note that the best damping coefficient α^* is independent of the noise strength and is determined solely by the properties of the beam material.

Remark 4.4

The damping coefficient α^* given above is not optimal in the strict sense, since it is obtained using the particular choice of the V function as in (4.18).

4.3 STABILIZATION OF A FLEXIBLE SPACECRAFT IN A NOISY ENVIRONMENT

As mentioned in the introduction of this chapter, satellites in space are often subjected to random disturbances arising from various sources. In this section, we consider the problem of stabilization of flexible spacecraft in the presence of these random disturbances. We show that stability of the system could be achieved by application of simple feedback controls. We shall follow the same notations as introduced in Section 2.1 and Section 4.1.

4.3.1 System Dynamics

We shall suppose that the structure of the spacecraft is the same as shown in Fig. 2.1 of Chapter II. For simplicity, we shall assume that the beam is uniform along its length. Then the complete dynamics of the system as developed in Section 2.2 is given by:

Bus Dynamics

$$I^T \ddot{\theta} + \omega \times (I^T \dot{\theta}) + 2\rho \int_0^L r \times (\omega \times v) dx + \rho \int_0^L (r \times \dot{v}) dx = T, \quad (4.31.a)$$

Beam Dynamics

$$\rho \frac{\partial^2}{\partial t^2} \begin{bmatrix} y \\ z \end{bmatrix} + EI \frac{\partial^4}{\partial x^4} \begin{bmatrix} y \\ z \end{bmatrix} + \rho M \alpha = F, \quad (4.31.b)$$

where $\alpha = \dot{\omega} \times r + \omega \times (\omega \times r) + 2 \omega \times v$,

and
$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

with the boundary conditions.

$$\begin{aligned} \begin{bmatrix} y \\ z \end{bmatrix} (t,0) &= 0, & \frac{\partial^2}{\partial x^2} \begin{bmatrix} y \\ z \end{bmatrix} (t,L) &= 0, \\ \frac{\partial}{\partial x} \begin{bmatrix} y \\ z \end{bmatrix} (t,0) &= 0, & \frac{\partial^3}{\partial x^3} \begin{bmatrix} y \\ z \end{bmatrix} (t,L) &= 0. \end{aligned} \quad (4.32)$$

and the initial conditions

$$\begin{aligned} \omega(0) &= \omega^0, \\ \begin{bmatrix} y \\ z \end{bmatrix} (0,x) &= \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} (x) = \phi^0, & \frac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} (0,x) &= \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} (x) = \psi^0, \end{aligned}$$

where ω denotes the angular velocity of the satellite bus, and y, z are the beam displacements. (Rest of the notations used in the above equations are given in page 18).

In the equation (4.31), the terms in the right hand side denote the total torque and the total force acting on the system and would consist of two components, viz. a) a control component, and b) a noise component, i.e.

$$\begin{aligned} T &= T_{\text{control}} + T_{\text{noise}}, \\ F &= F_{\text{control}} + F_{\text{noise}}. \end{aligned}$$

For the control component, we shall consider proportional feedback as in Chapter II, i.e.

$$\begin{aligned} T_{\text{control}} &= -K \omega, \text{ with } K \text{ positive definite,} \\ F_{\text{control}} &= -D \frac{\partial \phi}{\partial t}, \text{ with } D \text{ positive definite,} \end{aligned} \quad (4.33)$$

where $\phi = (y, z)$. We shall assume that the damping is applied uniformly on the beam so that D is a constant matrix.

For the noise component of torque perturbing the bus equation, we shall consider Gaussian White noise processes given by

$$T_{\text{noise}} = v \xi_t \quad (4.34.a)$$

where $\xi_t = (\xi_t^1, \xi_t^2, \xi_t^3)$ and $v = \text{diag}(v_1, v_2, v_3)$. The three mutually independent components of ξ_t affecting the bus angular motions about the three axes of the body frame satisfy

$$E \{ \xi^i(t) \} = 0, \quad i=1,2,3,$$

$$E \{ \xi^i(t) \xi^i(\tau) \} = \delta(t-\tau), \quad \text{for all } i=1,2,3,$$

$$E \{ \xi^i(t) \xi^j(\tau) \} = 0, \quad \text{for } i \neq j.$$

The noise perturbing the beam will be modelled as a distributed white noise as discussed in Section 4.2.1. Let $\{c_n\}$ denote the sequence of eigenfunctions of the operator $\hat{A} = \frac{\partial^4}{\partial x^4}$ defined in (4.10) and $\{\lambda_n\}$ the corresponding eigenvalues. Then the distributed white noise affecting the beam vibrations will be given by

$$F_{\text{noise}} = \sum_{n=1}^{\infty} c_n \eta_n(t) c_n(x) \quad (4.34.b)$$

$$\text{with } \text{Tr} \left\{ \sum_{n=1}^{\infty} (c_n' c_n) \right\} = q < \infty,$$

where $\eta_n(t) = (\eta_n^1(t), \eta_n^2(t))$, $n=1,2,\dots,\infty$ denote a sequence of independent Gaussian white noise processes with each component satisfying (4.8), and $c_n = \text{diag}(c_n^1, c_n^2)$. The power spectral density of the noise process is then given by $\text{Tr}_{n=1}^{\infty} (c_n' c_n)$.

We shall further assume that the noise processes $\xi_t, \eta_n^1, \eta_n^2, n=1,2,\dots,\infty$ are all mutually independent.

At this stage we introduce a first order approximation in the system dynamics, more precisely in the total inertia matrix I^T in the bus equation. We note that I^T is composed of two components, viz. a) the bus inertia I^b , which is a constant, and b) the beam inertia I^B , which varies with time during vibrations. In practice most of the control and instrumentation hardware would be contained in the satellite bus, so that the bus inertia I^b would be very large compared to the beam inertia I^B . Furthermore, in practical applications the beam vibrations would not be allowed to exceed certain reasonable limits. Hence for all practical purposes, it is reasonable to assume that the total inertia $I^T \equiv I^b + I^B \approx I^b + I_0^B \equiv I_0^T$, where I_0^B is the rest state inertia of the beam. Consequently, we replace I^T in (4.31.a) by I_0^T , and throughout the rest of this chapter we shall assume that the beam deflections are small. Note that I_0^T represents the total rest state inertia of the satellite. For brevity, we shall suppress the index T in the term I_0^T .

Thus the complete dynamics of the flexible spacecraft in the presence of random disturbances and the feedback controls is represented by the coupled system of stochastic differential equations (4.31)-(4.34). In the next section, we study the question of stability of this system. As in the previous section, we shall follow Ritz-Galerkin approach in order to prove the stability of the system.

4.3.2 Mean Square StabilizationProposition 4.1

Consider the system described by (4.31) with the boundary conditions (4.32), and subjected to the noise processes (4.34). Let the initial state (x^0, y^0, z^0) be arbitrary such that

$$E\left\{\int_0^T x^2 dt\right\} < \infty, E\left\{\int_0^T \frac{y^2}{3x^2} dt\right\} < \infty, E\left\{\int_0^T z^2 dt\right\} < \infty,$$

and independent of the noise processes. Then

- i) the uncontrolled system is unstable,
- ii) the controlled system with the velocity feedback (4.33) is finite time stable for all initial conditions satisfying

$$E\left\{(Kx^0 - y^0) + (Dy^0 - z^0)\right\}_{L_2(0,L)} \leq \frac{1}{2} \text{Tr}(D^* I_0^{-1} D) + \frac{1}{2\epsilon} \text{Tr} \sum_{n=1}^{\infty} (c_n^* c_n)$$

Proof

Let $\{\varphi_n\}$ be the sequence of eigenfunctions of the operator $A \equiv \frac{\partial^2}{\partial x^2}$ satisfying (4.10) and $\{\lambda_n\}$ be the corresponding eigenvalues. Since the sequence $\{\varphi_n\}$ forms an orthonormal basis for $L_2(0,L)$, we can construct an approximate solution $\{x^N, y^N, z^N\}$ of (4.31)-(4.34) by defining

$$\begin{aligned} x^N &= \sum_{n=1}^N y_n(t) \varphi_n(x) \\ z^N &= \sum_{n=1}^N z_n(t) \varphi_n(x) \end{aligned} \quad (4.35)$$

For brevity, we shall suppress the index N in $\{x^N, y^N, z^N\}$. We shall use the notation $r_n = (x_n, y_n, z_n)$, where $x_n = \int_0^t (x+R) r_n(x) dx$ and $v_n = (0, \dot{y}_n, \dot{z}_n)$. Then using the expression (4.35), we can verify that the approximate solution of the bus equation (4.31.a) satisfies

$$\dot{I}_0^N = \lambda(I_0^N) + 2\epsilon \sum_{n=1}^N (r_n \cdot \dot{v}_n) + \epsilon \sum_{n=1}^N (r_n \cdot \ddot{v}_n) + K_0 = \dot{I}_0 \quad (4.36)$$

where I_0^N is the N-th approximation of I_0 and $I_0^N \rightarrow I_0$ as $N \rightarrow \infty$.

Let w^D denote a standard Wiener process (vector) whose generalized derivative is the Gaussian white noise process \dot{w}^D . Then equation (4.36) can be formally written as the following stochastic differential equation:

$$\dot{I}_0^N dt = \lambda(I_0^N) dt + 2\epsilon \sum_{n=1}^N (r_n \cdot \dot{v}_n) dt + \epsilon \sum_{n=1}^N (r_n \cdot \ddot{v}_n) dt = \dot{I}_0 dt + \epsilon \sum_{n=1}^N (r_n \cdot \dot{v}_n) dt \quad (4.37)$$

Similarly, using (4.35) in (4.31.b), one can verify that, for each $n = 1, 2, \dots, N$, $\{y_n\}$ and $\{z_n\}$ are random processes satisfying:

$$\ddot{y}_n + D \dot{y}_n + EI y_n + \epsilon X y_n = \epsilon_n r_n(t),$$

$$\ddot{z}_n + D \dot{z}_n + EI z_n + \epsilon X z_n = \epsilon_n r_n(t),$$

with $r_n = \int_0^t r_n + \dots + (1 - r_n) + 2 \dots v_n$.

Letting $s_n = (y_n, z_n)$ and $v_n = (\dot{y}_n, \dot{z}_n)$, the above equation could be formally rewritten as an Ito differential equation

$$ds_n = v_n dt$$

$$dv_n = \left[-\frac{1}{\epsilon} Dv_n - \frac{1}{\epsilon} EI s_n - X(s_n \cdot r_n) - 2X(\dot{v}_n \cdot v_n) \right] dt$$

$$+ X(dw_n \cdot r_n) + \frac{1}{\epsilon} \epsilon_n dw_n^B, \quad n=1, 2, \dots, N, \quad (4.38)$$

where $\{W_n^B, n=1,2,\dots,\infty\}$ are independent standard Wiener processes whose generalized derivatives are the white noise processes $\{\dot{w}_n, n=1,2,\dots,\infty\}$.

We now introduce the process $\{V^N(t), t \geq 0\}$ defined by

$$V^N(t) = V^N(t), \dot{c}_n(t), \dot{v}_n(t), n=1,2,\dots,N$$

$$= \frac{1}{2} (I^B \cdot 1) \cdot 1 + \frac{1}{2} \sum_{n=1}^N \dot{v}_n + \alpha x_n^2 + \frac{1}{2} EI \sum_{n=1}^N \dot{c}_n^2, \quad (4.39)$$

where $v_n \equiv (0, \dot{v}_n, \dot{z}_n) \equiv (0, \dot{v}_n)$, $r_n \equiv (x_n, y_n, z_n) \equiv (x_n, \dot{c}_n)$ and $\{r_n, (\dot{c}_n, \dot{v}_n)\}$ satisfy (4.37) and (4.38) respectively. Then using the Ito calculus, it can be shown that the process $\{V^N(t), t \geq 0\}$ has the Ito differential given by

$$dV^N = \frac{\partial V^N}{\partial t} \cdot dt + \sum_{n=1}^N \frac{\partial V^N}{\partial \dot{c}_n} \cdot d\dot{c}_n + \sum_{n=1}^N \frac{\partial V^N}{\partial \dot{v}_n} \cdot d\dot{v}_n$$

$$+ \frac{1}{2} \text{Tr} \left(\frac{\partial^2 V^N}{\partial \dot{c}_n \partial \dot{c}_n} I_0^{-1} \right) dt + \frac{1}{2} \text{Tr} \left(\frac{\partial^2 V^N}{\partial \dot{v}_n \partial \dot{v}_n} I_0^{-1} \right) dt. \quad (4.40)$$

Using (4.39), one easily obtains

$$\frac{\partial V^N}{\partial t} = I_0^{-1} + \sum_{n=1}^N (r_n \cdot v_n),$$

$$\frac{\partial V^N}{\partial \dot{c}_n} = EI \dot{c}_n + \alpha M((\alpha x_n) \times \dot{c}_n) + \alpha M(v_n \times \dot{c}_n), \quad (4.41)$$

$$\frac{\partial V^N}{\partial \dot{v}_n} = \alpha \dot{v}_n + \alpha M(\alpha x_n),$$

$$\frac{\partial^2 V^N}{\partial \dot{c}_n \partial \dot{c}_n} = I_0^{-1}, \quad \frac{\partial^2 V^N}{\partial \dot{v}_n \partial \dot{v}_n} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

Substituting the equations (4.36), (4.38) and (4.41) into (4.40) and after some algebraic simplifications, we obtain

$$dV^N = \left[\frac{1}{2} \text{Tr}(v' I_0^{-1} v) - K_0 \cdot v \right] dt + \sum_{n=1}^N \left[\frac{1}{2\alpha} \text{Tr}(c_n' c_n) - D_0 \cdot c_n \right] dt + \dots + \sum_{n=1}^N c_n' c_n \cdot dW_n^B \quad (4.42)$$

Integrating this last equation over $(0, t)$ and taking expectation and assuming that $t \rightarrow E\{v^2, c_n^2\}$ are locally integrable, we have

$$E V^N(t) = E V^N(0) + E \int_0^t \left[\frac{1}{2} \text{Tr}(v' I_0^{-1} v) - K_0 \cdot v \right] ds + E \int_0^t \sum_{n=1}^N \left[\frac{1}{2\alpha} \text{Tr}(c_n' c_n) - D_0 \cdot c_n \right] ds \quad (4.43)$$

We now introduce the functional V defined by

$$V(t) \equiv \frac{1}{2} (I^b \cdot v) \cdot v + \frac{1}{2} \int_0^L v + \frac{1}{2} \int_0^L v^2 dx + \frac{1}{2} EI \int_0^L \frac{v^2}{3x^2} dx \quad (4.44)$$

Note that the functional V represents the total energy of the system and, as discussed in Chapter II (see equation (2.42) and Remark 2.1), is positive definite. Substituting (4.35) into (4.44) and using the Parseval's identity, one easily verifies that

$$V(t) = \lim_{N \rightarrow \infty} V^N(t),$$

where V^N is as defined in (4.39). Note that V^N is also positive definite. Then using Fubini's theorem, Parseval's identity and taking the limit $N \rightarrow \infty$, it follows that

$$E V(t) = E V(0) + \int_0^t \left[\left\{ \frac{1}{2} \text{Tr}(v' I_0^{-1} v) + \frac{1}{2\rho} \text{Tr} \sum_{n=1}^{\infty} (\sigma_n' c_n) \right\} - E \{ (K\omega \cdot \omega) + D \left(\frac{\partial \phi}{\partial t} \cdot \frac{\partial \phi}{\partial t} \right)_{L_2(0,L)} \} \right] ds.$$

It is clear from the above that in the absence of any damping, i.e. $K \equiv 0$ and $D \equiv 0$, the system (expected) energy increases with time indicating that the uncontrolled system is unstable. On the other hand, for

$$E \left\{ (K\omega \cdot \omega) + D \left(\frac{\partial \phi}{\partial t} \cdot \frac{\partial \phi}{\partial t} \right)_{L_2(0,L)} \right\} (0) > \left\{ \frac{1}{2} \text{Tr}(v' I_0^{-1} v) + \frac{1}{2\rho} \text{Tr} \sum_{n=1}^{\infty} (\sigma_n' c_n) \right\},$$

the system expected energy is nonincreasing. Thus the system vibrations will remain bounded once it starts with finite energy.

Remark 4.5

The above result shows that the system is merely stable. However, from the numerical results presented below, it appears that the system is indeed asymptotically stable with respect to some neighbourhood of the zero state. It would be interesting to prove asymptotic stability of the flexible spacecraft; and in case it is not globally asymptotically stable, one would be interested to determine the domain of asymptotic stability.

4.3.2.1 Simulation Results

As a numerical experiment, we investigate the stability properties of the flexible spacecraft described by (4.31)-(4.33) and under the control law (2.46). We shall use the same system parameters as in Section 2.3.1 and the gains in the control structure (2.46) as in Section 2.3.1.1. For the noise, we take $v = \text{diag}(0.2, 0.2, 0.2)$ and $\sigma_1 = \text{diag}(0.02, 0.02)$ and $\sigma_n = 0$ for all $n = 2, 3, \dots, \infty$. In the simulation, Gaussian noise processes were generated using the IBM subroutine GAUSS and numerical integration was carried out using a Runge-Kutta-like algorithm as discussed in [61]. The response of the noisy system under the action of the feedback controls are shown in Fig. 4.1-4.10.

Fig. 4.1 shows the absolute energy of the uncontrolled system. Comparing Fig. 4.1 with Fig 2.2 (for the corresponding deterministic system), we observe an increasing trend in the total energy of the system which indicates that the uncontrolled system is unstable. The feedback control (2.46) stabilizes the system as shown in Fig. 4.2 with respect to a small neighbourhood of the zero state.

The displacements and the velocities of the beam at the free end are shown in Fig. 4.3-4.6. The small amplitude vibrations of the beam (at larger values of time) could be clearly observed from these figures. This is more visible in Fig. 4.7 where the nonvanishing beam relative energy is a definite indication of sustained beam vibrations.

The decay of bus angular motions due to the effect of the feedback controls is shown in Fig. 4.8-4.10. We observe that, unlike in the deterministic case (shown in Fig. 2.10-2.12) the satellite keeps on wobbling at small angular velocities. This is because of the presence of the noise perturbing the system. We further observe that the angular velocity ω_2 experiences larger fluctuations as compared to ω_1 and ω_3 . The main reason for this is the smaller bus inertia I_2 .

Remark 4.7

In Proposition 4.1, we require uniform damping on the beam. Although in the control law (2.46) we use localized damping, numerical studies shows that the effects of this control law is very similar (qualitatively) to that of uniform damping.

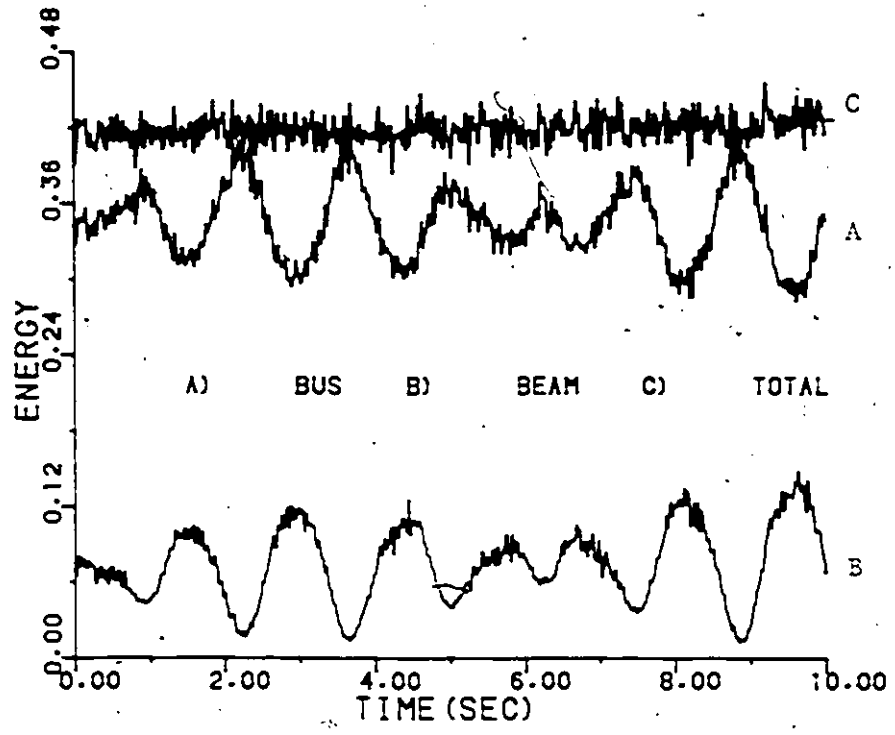


Fig. 4.1 Uncontrolled System Absolute Energy .

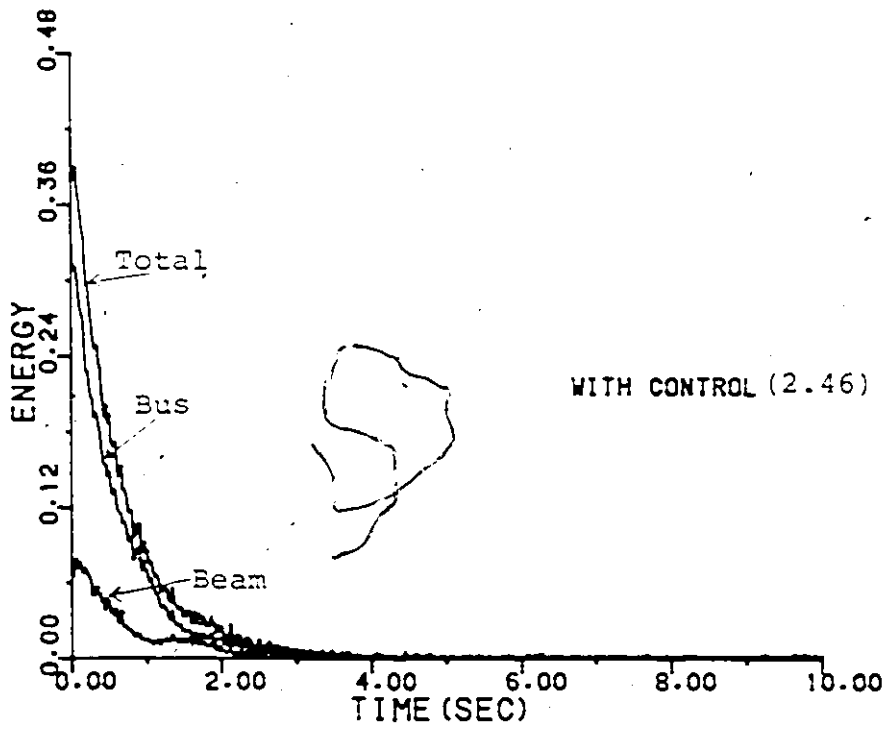
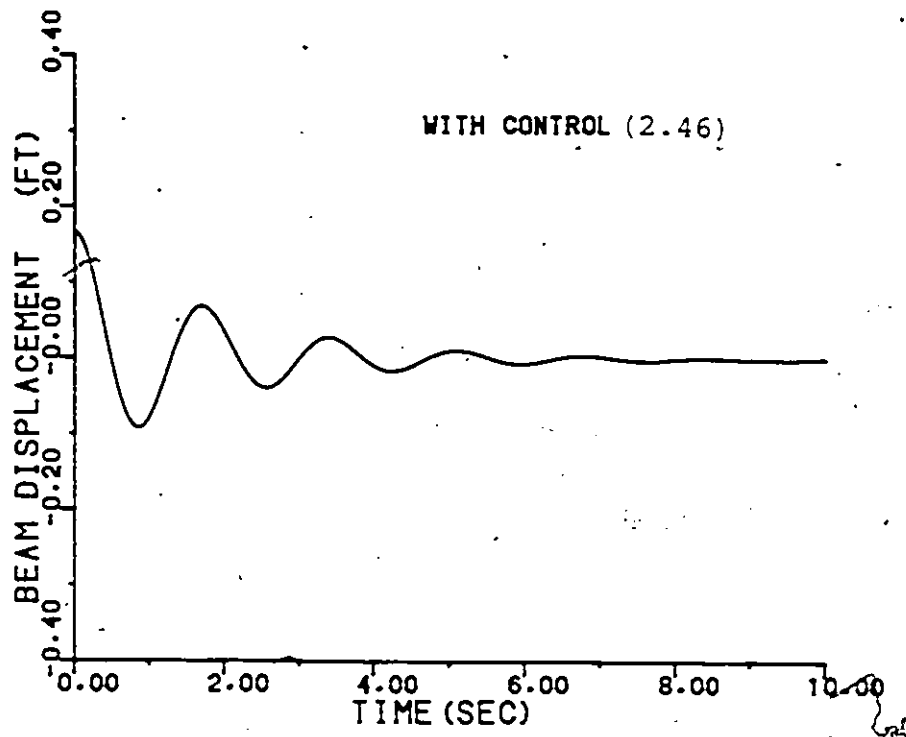
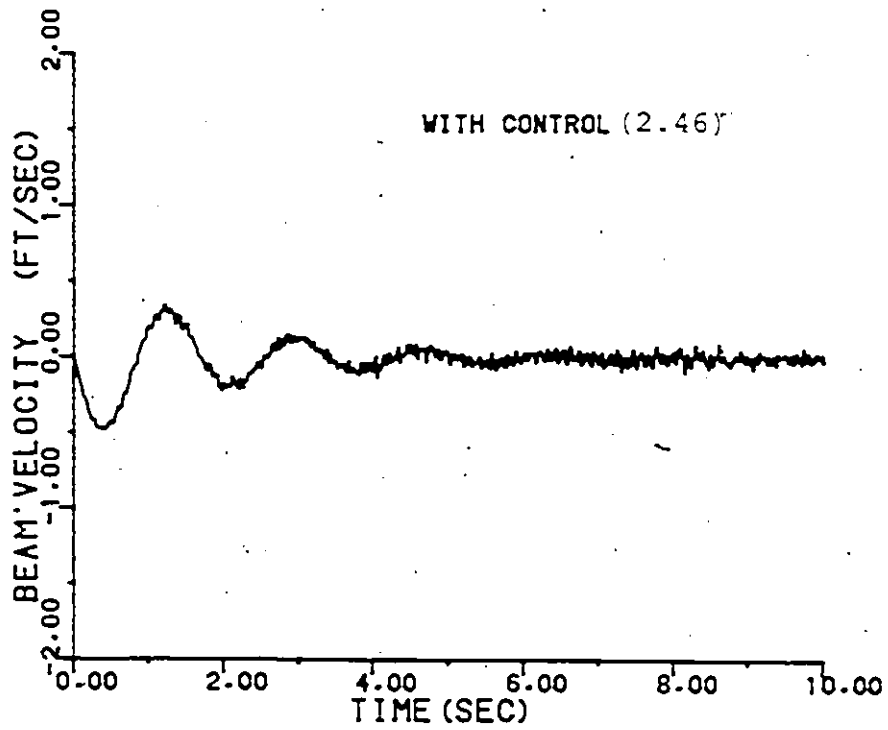
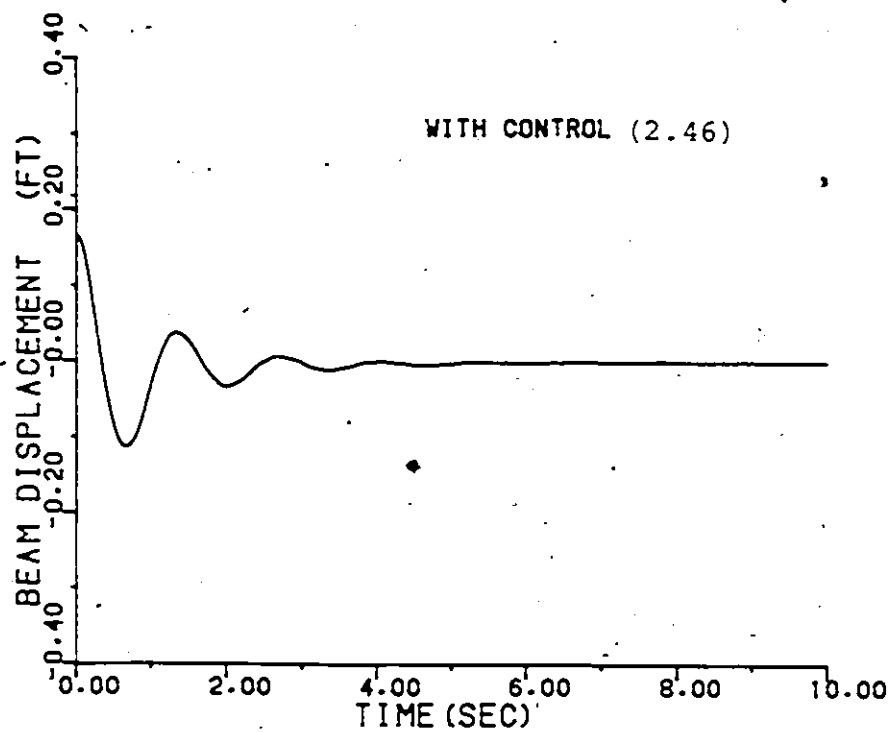
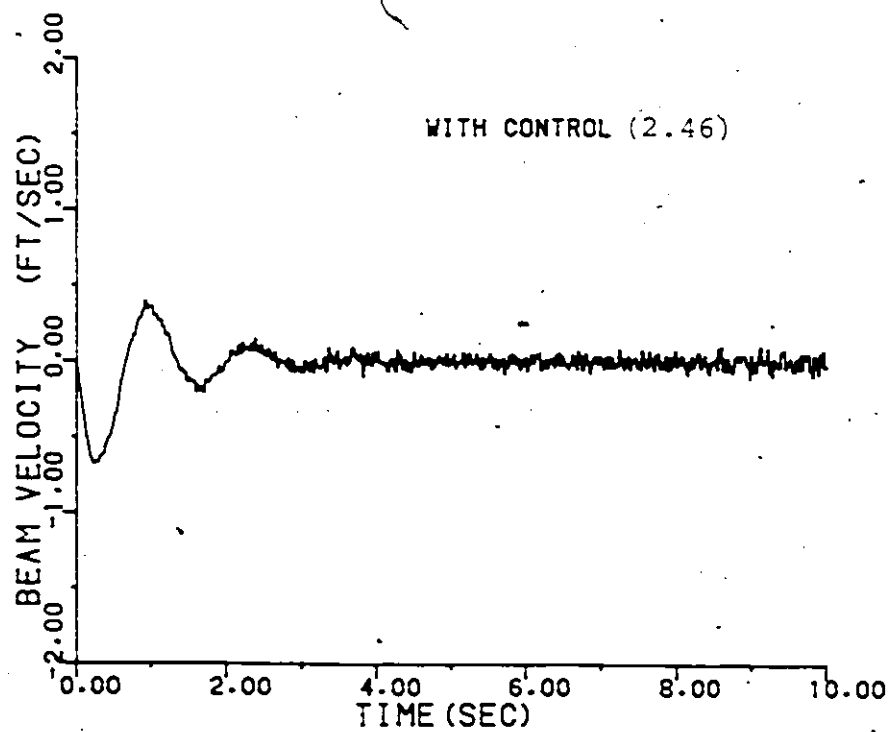


Fig. 4.2 Decay of Absolute Energy with Control.

Fig. 4.3 Beam Displacement $y(t,L)$ Fig. 4.4 Beam Velocity $\frac{\partial y}{\partial t}(t,L)$

Fig. 4.5 Beam Displacement $z(t,L)$ Fig. 4.6 Beam Velocity $\frac{\partial z}{\partial t}(t,L)$

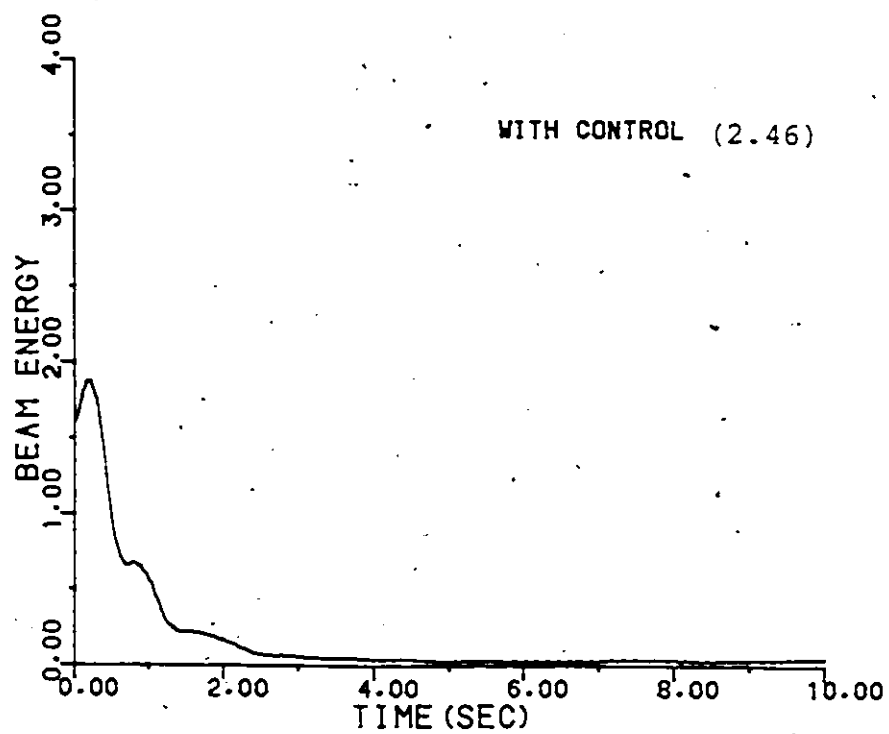
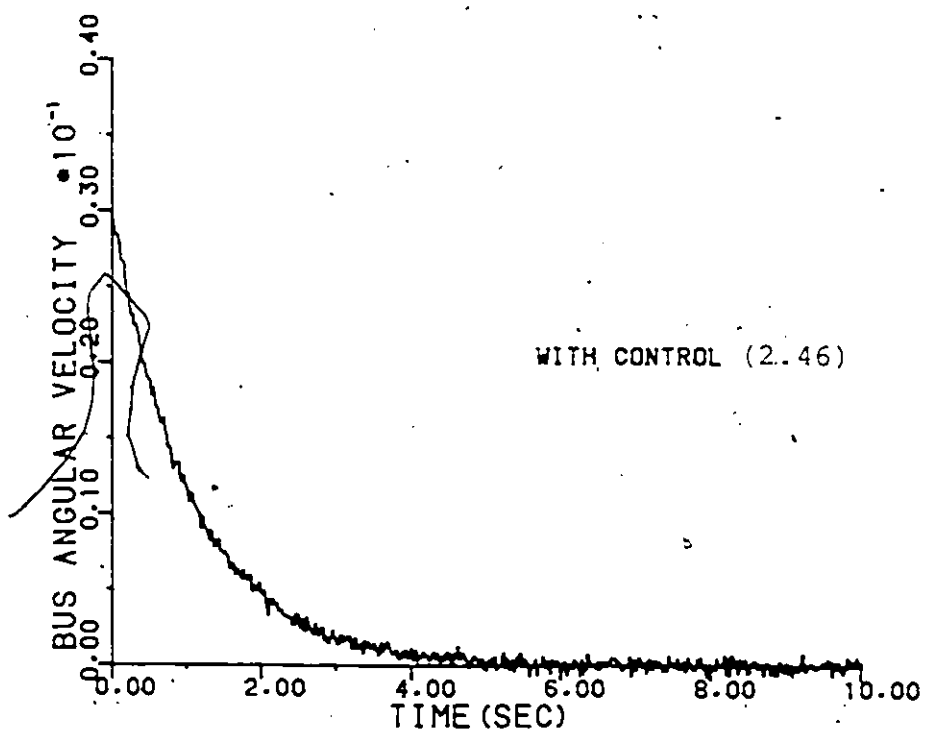


Fig. 4.7 Beam Relative Energy

Fig. 4.8 Bus Angular Velocity ω_1 (rad/sec)

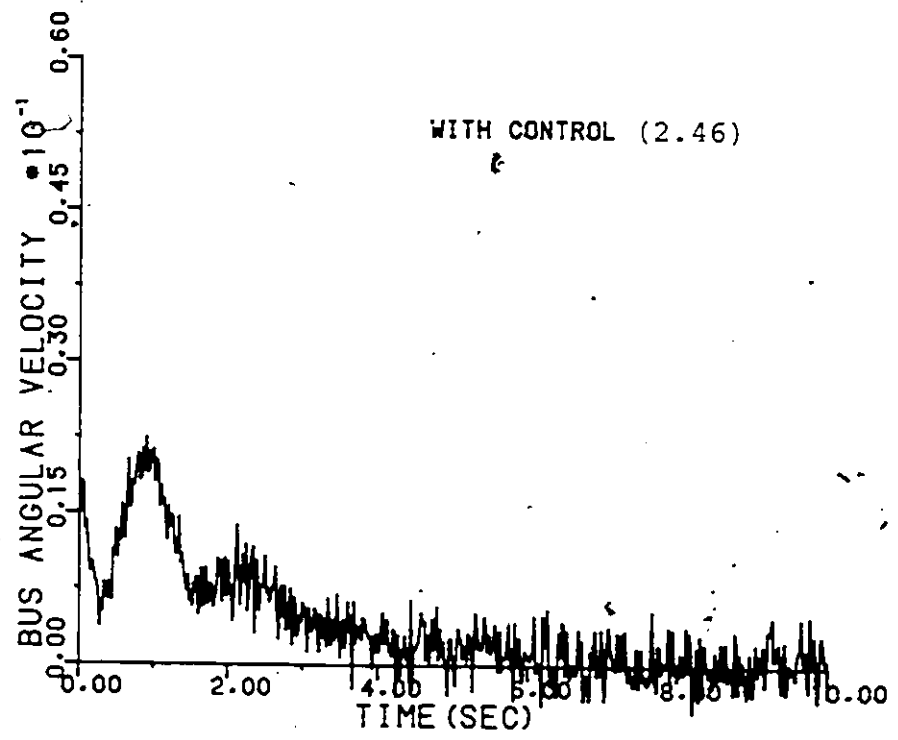


Fig. 4.9 Bus Angular Velocity ω_2 (rad/sec)

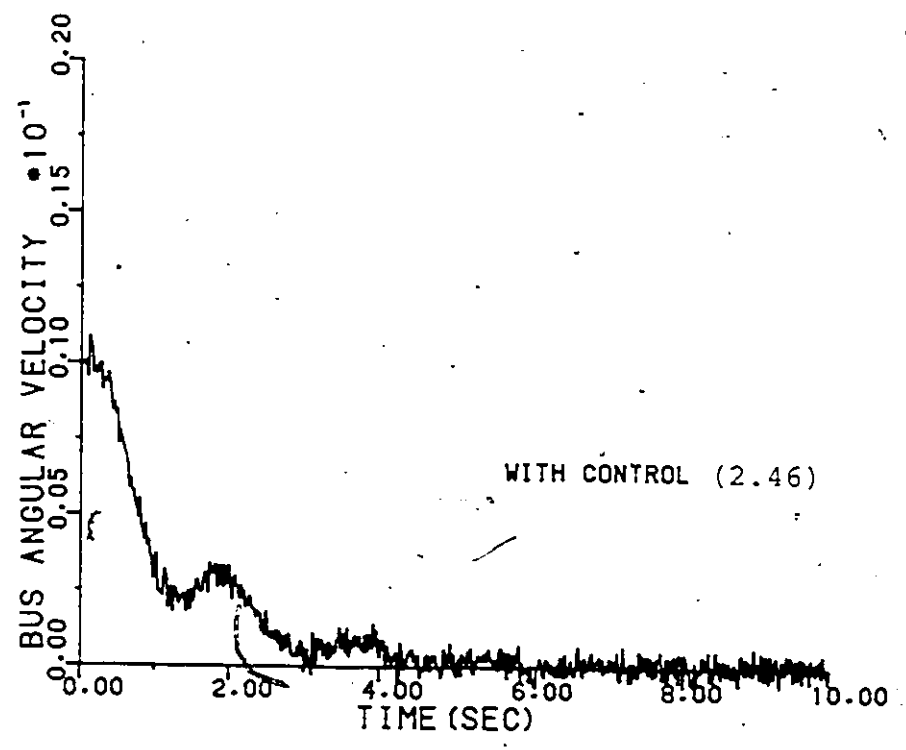


Fig. 4.10 Bus Angular Velocity ω_3 (rad/sec)

4.4 SUMMARY

In this chapter we study the stability properties of flexible structures in the presence of distributed noise. It is shown that a flexible beam perturbed by these disturbances becomes unstable in the absence of any control. By application of velocity feedback the system can be stabilized in the mean square sense and almost sure sense with respect to a ball in the energy space. The radius of this attractor increases with the increase of the noise strength and shrinks to the origin in the absence of any noise as in the deterministic case. The radius of the attractor and the decay rate of energy depend also on the damping. It has been found that too much of damping deteriorates the stability properties of the system. An optimal damping coefficient has been derived in order to maximize the size of the attractor and minimize the decay rate.

We also consider the problem of stabilization of a spacecraft which is partly rigid and partly flexible and subjected to random disturbances. We show that the uncontrolled system is unstable and that the system vibrations can be stabilized by application of simple feedback controls. This is also clarified by simulation results.



Chapter V-

IDENTIFICATION OF BEAM PARAMETERS .

An important and essential aspect of modelling any physical system is the identification of parameters in the model equation. These model equations are usually inferred on the basis of fundamental physical laws and some idealizing assumptions; but contains certain parameters which are completely unknown because of lack of precise understanding of the system, or only partly known because of poor measurement data. The analyst must determine these parameters on the basis of the available field data.

This problem of identification of parameters arises naturally in the modelling of flexible spacecraft. The system model developed in Chapter II and Chapter III contain certain unknown parameters such as a) moment of inertia of the rigid body, and b) mass density, flexural rigidity, and structural damping of the elastic body. Determination of mass moment of inertia of the rigid body is relatively simple and could be done experimentally through dynamometer test or retardation test. On the other hand, the elastic members of a flexible spacecraft may consist of trusses rather than a solid continuum of metal. While

conceptually, it is possible to develop a detailed dynamic model of a truss beam taking into account each and every element of the truss, numerical simulation of such a model would be extremely difficult and costly. An inexpensive solution to this problem is to represent [68,69] the truss structure by an equivalent continuum beam or plate. Then, the problem that remains to solve is to determine the parameters of the equivalent beam or plate.

The problem of identification of parameters and, in general, operators in systems governed by first and second order evolution equations have been discussed recently in [11]. Following the method developed in [11], in this chapter, we present certain necessary conditions for optimal identification of parameters in a class of linear second order hyperbolic systems. These results are then used to determine the parameters of a flexible beam. We shall use the following notations in the sequel.

5.1 NOTATIONS.

Let H be a real separable Hilbert space and V a linear subspace of H carrying the structure of a reflexive Banach space with V dense in H and the injection $V \hookrightarrow H$ continuous. We identify H with its dual, so that we have $V \hookrightarrow H \hookrightarrow V'$, where V' is the topological dual of V . Let $\|\cdot\|_H$, $\|\cdot\|_V$ and $\|\cdot\|_{V'}$ denote the norms in H , V and V' respectively. We shall

denote the scalar product in H by $(\cdot, \cdot)_H$ or (\cdot, \cdot) , and the duality pairings for the pair $\{V, V'\}$ by $\langle \cdot, \cdot \rangle_{V, V'}$ or $\langle \cdot, \cdot \rangle_{V', V}$.

For any Banach space E and a bounded interval $I \subset \mathbb{R}$, let $L_2(I, E)$ denote the equivalence classes of strongly measurable functions f on I with values in E such that $\int_I \|f(t)\|_E^2 dt < \infty$. The space $L_2(I, E)$ furnished with the norm topology $\|f\|_{L_2(I, E)} = \left(\int_I \|f(t)\|_E^2 dt \right)^{1/2}$ is a Banach space. In case E is Hilbert, $L_2(I, E)$ is also a Hilbert space. Similarly we use $L_\infty(I, E)$ to denote the Banach space of strongly measurable E valued functions on I with the norm $\|f\|_{L_\infty(I, E)} = \text{ess sup}\{\|f(t)\|_E, t \in I\}$. Let $C(I, E)$ denote the space of strongly continuous E valued functions on I . Furnished with the uniform topology $\|f\|_{C(I, E)} = \sup\{\|f(t)\|_E, t \in I\}$, $C(I, E)$ is a Banach space. We use $L(E, F)$ to denote the space of bounded linear operators from a Banach space E to a Banach space F . Further notations will be introduced in the sequel as required.

5.2 PROBLEM FORMULATION

The dynamics of flexible structures such as beam or plate could be described by second order linear evolution equations of the form:

$$\ddot{x} + A(t, q) x = f, \quad (5.1)$$

$$\ddot{x} + \gamma A(t, q) \dot{x} + A(t, q) x = f, \quad (5.2)$$

where A denotes the appropriate spatial differential operator. The second equation (5.2) is known as the structurally damped hyperbolic system with γ being a unknown parameter. Also unknown is the parameter q (q may be a vector as well) in the spatial differential operator A . Appropriate assumptions on A will be introduced shortly.

Let H denote the Hilbert space of observations and $\tilde{y} \in C(\bar{I}, H)$ the observed data or the response of the natural system. We shall assume that the observation equation for the model system is given by

$$y = Cx,$$

where $C \in L(H, H)$ and x is the response of the model system. We define a cost index or identification error as the mean square difference between the model output y and the observed data \tilde{y} , i.e.

$$J(q, \gamma) = \frac{1}{2} \int_0^T \|Cx(q, \gamma) - \tilde{y}\|_H^2 dt, \quad (5.3)$$

Then the problem of identification of parameters γ and q could be considered as the problem of minimizing the cost (5.3) subject to the dynamic constraint (5.1) or (5.2). We shall follow the techniques of optimal control of distributed systems [7] and develop the necessary conditions characterizing the optimal elements for the unknown.

5.3 NECESSARY CONDITIONS FOR OPTIMAL IDENTIFICATION

5.3.1 Identification of Structural Damping γ

We first assume that the operator A is completely known (i.e. q is known) and let $I \equiv (0, T)$, $T < \infty$ and $\{A(t, q) \equiv A(t), t \in \bar{I}\}$ denote a family of linear operators with values $A(t) \in L(V, V')$. Consider the structurally damped system

$$\begin{aligned} \ddot{x} + \gamma A(t) \dot{x} + A(t) x &= f, \\ x(0) &= x_0, \quad \dot{x}(0) = x_1. \end{aligned} \quad (5.4)$$

We shall suppose that the operator A satisfies the following conditions:

A1) $t \rightarrow \langle A(t)\phi, \phi \rangle$ is measurable for each $\phi, \psi \in V$ and that $A \in C^1(I, L(V, V'))$ such that

$$|\langle \dot{A}(t)\phi, \phi \rangle| \leq c \|\phi\|_V \|\phi\|_V \text{ for all } t \in \bar{I},$$

A2) A satisfies the Garding's inequality, i.e. there exists an $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\langle A(t)\phi, \phi \rangle_{V', V} + \beta \|\phi\|_H^2 \geq \alpha \|\phi\|_V^2 \text{ for all } t \in \bar{I}.$$

Then we have the following results:

Theorem 5.1 [46]

Suppose that the operator A is self adjoint and satisfies the basic assumptions (A1) and (A2), and let A be symmetric and $\gamma > 0$. Then for every $f \in L_2(I, V')$, $x_0 \in V$ and $x_1 \in H$, the system (5.4) has a unique solution $x \equiv x(\gamma)$ satisfying

$$i) x \in L_\infty(I, V) \subset L_2(I, V),$$

$$ii) \dot{x} \in L_\infty(I, H) \subset L_2(I, V),$$

$$iii) \ddot{x} \in L_2(I, V'),$$

and except for a null set

$$iv) x \in C(\bar{I}, V) \text{ and } \dot{x} \in C(\bar{I}, H).$$

Let $\mathcal{P} \equiv [0, \beta]$, $\beta < \infty$, denote the class of admissible parameters for the unknown γ . Then, we consider the problem of identification of an element $\gamma \in \mathcal{P}$ so that the identification error (5.3) is minimum subject to the dynamic constraint (5.4). For the proof of necessary conditions of optimality in identification, we shall make use of the Gâteaux differential of $x(\gamma)$ with respect to the parameter γ . Indeed we show that the Gâteaux differential of x at γ^0 in the direction γ , defined as

$$\hat{x}(\gamma^0, \gamma) = \lim_{\varepsilon \rightarrow 0} (x(\gamma^0 + \varepsilon\gamma) - x(\gamma^0)) / \varepsilon,$$

exists and that it is the solution of a related differential equation. In this regard we have the following result:

Lemma 5.1

Let $x(\gamma)$ denote the weak solution of the Cauchy problem (5.4) corresponding to $\gamma \in \Gamma$. Then at every point $\gamma^0 \in \Gamma$, the function $\gamma \rightarrow x(\gamma)$ has a weak Gâteaux differential in the direction $\gamma - \gamma^0$, denoted $\hat{x}(\gamma^0, \gamma - \gamma^0)$, and it is the solution of the Cauchy Problem

$$\begin{aligned} \ddot{\hat{x}} + \gamma^0 A \dot{\hat{x}} + A \hat{x} &= (\gamma^0 - \gamma) A \dot{x}^0, \\ \hat{x}(0) &= 0, \text{ and } \dot{\hat{x}}(0) = 0, \end{aligned} \quad (5.5)$$

where $\dot{x}^0 = \dot{x}(\gamma^0)$ is the response of the system (5.4) corresponding to $\gamma = \gamma^0$. Further, $\hat{x} \in L_2(I, V) \cap C(\bar{I}, V)$ and $\dot{\hat{x}} \in L_2(I, V) \cap C(\bar{I}, H)$.

Proof

Since Γ is convex, we have, for all $\gamma^0, \gamma \in \Gamma$, $\gamma^\varepsilon = \gamma^0 + \varepsilon(\gamma - \gamma^0) \in \Gamma$ for all $0 \leq \varepsilon \leq 1$. Define

$$\phi^\varepsilon = (x(\gamma^\varepsilon) - x(\gamma^0)) / \varepsilon.$$

Then using the differential equation (5.4) one easily obtains

$$\ddot{\phi}^\varepsilon + \gamma^\varepsilon A \dot{\phi}^\varepsilon + A \phi^\varepsilon = (\gamma^0 - \gamma) A \dot{x}^0, \quad (5.6)$$

$$\phi^\varepsilon(0) = 0, \quad \dot{\phi}^\varepsilon(0) = 0.$$

We show that the weak limit of ϕ^ε exists and indeed, is the weak solution of (5.5).

Scalar multiplying the first equation of (5.6) by $\dot{\phi}^\varepsilon$, we have

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\dot{\phi}^\varepsilon\|_H^2 + \gamma^\varepsilon \langle A \dot{\phi}^\varepsilon, \dot{\phi}^\varepsilon \rangle + \frac{d}{dt} \frac{1}{2} \langle A \phi^\varepsilon, \phi^\varepsilon \rangle \\ = \frac{1}{2} \langle \dot{A} \phi^\varepsilon, \phi^\varepsilon \rangle + \langle (\gamma^0 - \gamma) A \dot{x}^0, \dot{\phi}^\varepsilon \rangle \end{aligned}$$

Integrating the above expression over $(0, t)$ and using the assumptions (A1) and (A2), Schwarz inequality and the elementary inequality

$$2ab \leq \frac{1}{m} a^2 + m b^2 \text{ for all } m > 0,$$

one obtains

$$\begin{aligned} \|\phi^\varepsilon\|_H^2 + 2\gamma^\varepsilon \alpha \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta + \alpha \|\phi^\varepsilon\|_V^2 &\leq |\beta| \|\phi^\varepsilon\|_H^2 + c \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta \\ &+ 2\gamma^\varepsilon |\beta| \int_0^t \|\phi^\varepsilon\|_H^2 \, d\theta + \frac{1}{n} \int_0^t \|(\gamma^0 - \gamma) A x^0\|_V^2 \, d\theta + n \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta \end{aligned} \quad (5.7)$$

It can be easily verified that,

$$\|\phi^\varepsilon(t)\|_H^2 \leq T \int_0^t \|\phi^\varepsilon\|_H^2 \, d\theta,$$

so that, from (5.7) we have, for $n = \gamma^\varepsilon \alpha$,

$$\begin{aligned} \|\phi^\varepsilon\|_H^2 + \gamma^\varepsilon \alpha \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta + \alpha \|\phi^\varepsilon\|_V^2 &\leq (|\beta| T + 2\gamma^\varepsilon |\beta|) \int_0^t \|\phi^\varepsilon\|_H^2 \, d\theta \\ &+ c \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta + \frac{1}{\gamma^\varepsilon \alpha} \int_0^t \|(\gamma^0 - \gamma) A x^0\|_V^2 \, d\theta \end{aligned} \quad (5.8)$$

Clearly

$$\begin{aligned} \|\phi^\varepsilon\|_H^2 + \alpha \|\phi^\varepsilon\|_V^2 &\leq (2\gamma^\varepsilon |\beta| + |\beta| T) \int_0^t \|\phi^\varepsilon\|_H^2 \, d\theta \\ &+ c \int_0^t \|\phi^\varepsilon\|_V^2 \, d\theta + \frac{1}{\gamma^\varepsilon \alpha} \int_0^t \|(\gamma^0 - \gamma) A x^0\|_V^2 \, d\theta, \end{aligned}$$

and hence, by Gronwall's lemma, one concludes that

$$\|\phi^\varepsilon\|_H^2 + \alpha \|\phi^\varepsilon\|_V^2 \leq \left(\frac{1}{\gamma^\varepsilon \alpha} \int_0^T \|(\gamma^0 - \gamma) A x^0\|_V^2 \, d\theta \right) \exp(Kt), \quad (5.9)$$

where $K = \max \{ 2\gamma^\varepsilon |\beta| + |\beta| T, \frac{c}{\alpha} \}$.

Energy estimate for ϕ^ε now follows easily from (5.8) and (5.9) and is given by, for almost all $t \in I$,

$$\|\dot{\phi}^\varepsilon\|_H^2 + \gamma^\varepsilon \alpha \int_0^t \|\dot{\phi}^\varepsilon\|_V^2 ds + \alpha \|\phi^\varepsilon\|_V^2 = \left(\frac{1}{\gamma^\varepsilon \alpha} \int_0^T (\gamma^0 - \gamma) \|Ax^0\|_V^2 ds \right) \exp(KT), \quad (5.10)$$

which holds uniformly for all $0 \leq \varepsilon \leq 1$. It is clear from the above inequality that the sequence $\{\phi^\varepsilon\}$ is contained in a bounded subset of $L_\infty(I, V) \subset L_2(I, V)$ and $\{\dot{\phi}^\varepsilon\}$ in a bounded subset of $L_\infty(I, H) \cap L_2(I, V)$. Since $L_2(I, V)$ is a reflexive Banach space, we can extract subsequences $\{\phi^{\varepsilon_n}\} \subset \{\phi^\varepsilon\}$ and $\{\dot{\phi}^{\varepsilon_n}\} \subset \{\dot{\phi}^\varepsilon\}$, with $0 \leq \varepsilon_n \leq 1$, relabelled as $\{\phi^n\}$ and $\{\dot{\phi}^n\}$, and two elements ϕ^* and $\dot{\phi}^*$ such that

$$\begin{aligned} \{\phi^n\} &\longrightarrow \phi^* \text{ weakly in } L_2(I, V), \\ \{\dot{\phi}^n\} &\longrightarrow \dot{\phi}^* \text{ weakly in } L_2(I, V). \end{aligned} \quad (5.11)$$

Hence the Gâteaux differential of $\hat{x}(\gamma)$ exists and is given by $\hat{x}(\gamma^0, \gamma - \gamma^0) = \dot{\phi}^*$. It remains to show that ϕ^* is a solution of (5.5).

Indeed, since $A\phi^n \longrightarrow A\phi^*$ in $L_2(I, V')$ weakly, and

$$\gamma^\varepsilon A\dot{\phi}^n = (\gamma^0 + \varepsilon_n(\gamma - \gamma^0))A\dot{\phi}^n \xrightarrow{w} \gamma^0 A\dot{\phi}^* \text{ in } L_2(I, V'),$$

it follows from (5.6) that $\ddot{\phi}^n \in L_2(I, V')$ and $\ddot{\phi}^n \longrightarrow \varepsilon$ in $L_2(I, V')$ for a suitable $\varepsilon \in L_2(I, V')$ and that ε is the distributional derivative $\ddot{\phi}^*$. Hence ϕ^* satisfies the equality

$$\ddot{\phi}^* + \gamma^0 A\dot{\phi}^* + A\phi^* \varepsilon = (\gamma^0 - \gamma)Ax^0$$

Since $\phi^* \in L_2(I, V)$ and $\ddot{\phi}^* \in L_2(I, V')$, it follows from intermediate derivative theorem [46] that $\phi^* \in C(\bar{I}, H)$. Similarly, from (5.11) it is clear that $\phi^* \in C(\bar{I}, V)$. Then $\phi^*(0)$ and $\dot{\phi}^*(0)$ are well defined elements and equal $\phi^n(0)$ and $\dot{\phi}^n(0)$ respectively for all n . Hence ϕ^* satisfies the differential equation (5.5). This completes the proof. \square

With the help of the above results, we now prove the following theorem on the necessary conditions of optimality characterizing the optimal parameter $\gamma^0 \in \Gamma$.

Theorem 5.2

Consider the hyperbolic system (5.4) with the structural damping $\gamma > 0$ unknown. Let $f \in L_2(I, V')$, $x_0 \in V$, and $x_1 \in H$. Then the best approximation $\gamma^0 \in \Gamma$ for the unknown parameter γ is determined by the simultaneous solution of the system equation

$$\begin{cases} \ddot{x}^0 + \gamma^0 A \dot{x}^0 + A x^0 = f, \\ x^0(0) = x_0 \text{ and } \dot{x}^0(0) = x_1, \end{cases} \quad (5.12)$$

the adjoint system

$$\begin{cases} \ddot{z}^0 - \gamma^0 A^* \dot{z}^0 + A^* z^0 = C^* A_H (C \dot{x}^0 - \tilde{y}) \\ z^0(T) = 0 \text{ and } \dot{z}^0(T) = 0, \end{cases} \quad (5.13)$$

and the inequality

$$\gamma^0 \int_0^T \langle A \dot{x}^0, z^0 \rangle dt \geq \gamma \int_0^T \langle A \dot{x}^0, z^0 \rangle dt \quad (5.14)$$

for all $\gamma \in \Gamma$.

Proof

Since $\gamma \longrightarrow x(\gamma)$ has weak Gâteaux differential in \mathcal{T} , it follows that the identification error J , defined in (5.3), also has a Gâteaux differential. Then in order that J attains its minimum at $\gamma^0 \in \mathcal{T}$, it is necessary that

$$J'_{\gamma^0}(\gamma - \gamma^0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [J(\gamma^0 + \varepsilon(\gamma - \gamma^0)) - J(\gamma^0)] \geq 0. \quad (5.15)$$

Using the results of Theorem 5.1 and Lemma 5.1, it follows from the above that

$$J'_{\gamma^0}(\gamma - \gamma^0) = \int_0^T \langle C\hat{x}(\gamma^0, \gamma - \gamma^0), Cx(\gamma^0) - \tilde{y} \rangle_H dt \geq 0, \quad (5.16)$$

where $\hat{x}(\gamma^0, \gamma - \gamma^0)$ is the weak Gâteaux differential of x at the point $\gamma^0 \in \mathcal{T}$ in the direction $\gamma - \gamma^0$.

Using the canonical isomorphism A_H defined by

$$(A_H h, h)_{H', H} = \|h\|_H^2, \quad \|A_H h\|_{H'} = \|h\|_H,$$

for all $h \in H$, the inequality (5.16) could be rewritten as

$$J'_{\gamma^0}(\gamma - \gamma^0) = \int_0^T \langle \hat{x}(\gamma^0, \gamma - \gamma^0), C^* A_H (Cx(\gamma^0) - \tilde{y}) \rangle_{V, V} dt \geq 0, \quad (5.17)$$

for all $\gamma \in \mathcal{T}$, where $C^* \in L(H', H)$ denotes the adjoint of the operator C .

The inequality (5.17) could be further simplified by introducing the so called adjoint variable z which is the solution of the following equation:

$$\begin{aligned} \dot{z} - \gamma^0 A^* z + A^* z &= C^* A_H (Cx(\gamma^0) - \tilde{y}), \\ z(T) &= 0 \text{ and } \dot{z}(T) = 0. \end{aligned} \quad (5.18)$$

Since $C^* A_H (Cx(\gamma^0) - \tilde{y}) \in L_2(I, H) \subset L_2(I, V')$, reversing the flow of time $t \rightarrow T-t$, it follows from Theorem 5.1 that the adjoint system (5.18) also has a weak solution $z \in L_2(I, V) \cap C(\bar{I}, V)$ and $\dot{z} \in L_2(I, V) \cap C(\bar{I}, H)$.

Utilizing (5.18) into the inequality (5.17) and integrating by parts, one obtains

$$\int_0^T \langle \ddot{x} + \gamma^0 \dot{A}x + Ax, z \rangle_{V', V} dt \geq 0 \quad (5.19)$$

The necessary inequality (5.14) now follows from (5.5) and (5.19). This completes the proof. \square

Equations (5.12)-(5.14) represent a complete set of necessary conditions using which the unknown parameter γ^0 could be iteratively computed.

5.3.2 Identification of the Parameter q

The necessary conditions of optimality for identification of the parameter q in the undamped system (5.1) or the structurally damped system (5.2) could be obtained following the procedure discussed in [11]. In what follows, we shall present the main results, while the proof could be found in [11].

Let Q be a compact metric space and τ_m denote the metric topology on Q and denote the corresponding topological space

by Q_m . We consider the undamped system (5.1) and seek an element $q^0 \in Q_m$ so as to minimize the identification error (5.3). We will need the following assumptions in the sequel.

A3) The family of operators $\{A(\cdot, q), q \in Q\}$ satisfy the assumptions (A1) and (A2) with certain constants c, α, β independent of $q \in Q_m$.

A4) The mapping $q \rightarrow A(t, q)$ from Q to $L(V, V')$ is continuous in the strong operator topology, in the sense that whenever

$$q^n \rightarrow q^0 \text{ in } Q, \quad A(t, q^n)v \rightarrow A(t, q^0)v$$

strongly for all $t \in I =]0, T)$ and $v \in V$.

Lemma 5.2

Suppose that the operator A satisfy the assumptions (A3) and (A4) and $q \rightarrow A(t, q)$ is once Gâteaux differentiable in the weak operator topology of $L(V, V')$ and the Gâteaux differential is weakly measurable on I . Then the solution x of the system (5.1) has a weak Gâteaux differential at each point in the direction $(q - q^0)$, denoted $\hat{x}(q^0, q - q^0)$, and is given by the weak solution of

$$\begin{cases} \ddot{\hat{x}} + A(t, q) \hat{x} = A'(q^0, q - q^0) x^0, \\ \hat{x}(0) = 0, \quad \dot{\hat{x}}(0) = 0, \end{cases} \quad (5.20)$$

where $A'(q^0, q - q^0)$ is the Gateaux differential of A at $q^0 \in Q_m$ in the direction $q^0 - q$ and $x^0 \equiv x(q^0)$ is the response of the system (5.1) corresponding to $q = q^0$. Further, $\tilde{x} \in L_2(I, V) \cap C(\bar{I}, V)$ and $\tilde{x} \in L_2(I, H) \cap C(\bar{I}, H)$.

Following the same procedure as in the preceding section, one proves the following necessary conditions for optimality for identification of the parameter q .

Theorem 5.3

Suppose the assumptions of Lemma 5.2 hold, and $x_0 \in V$, $x_1 \in H$, $f \in L_2(I, H)$. Then in order that $q^0 \in Q_m$ be the best approximation to the unknown parameter q , it is necessary that there exists a pair $\{x^0, z^0\} \in C(\bar{I}, V) \times C(\bar{I}, V)$ satisfying

$$\begin{cases} \ddot{x}^0 + A(t, q^0) x^0 = f, \\ x^0(0) = x_0, \quad \dot{x}^0(0) = x_1, \end{cases} \quad (5.21)$$

$$\begin{cases} \ddot{z}^0 + A^*(t, q^0) z^0 = C^*_{H} (Cx(q^0) - \tilde{y}), \\ z^0(T) = 0, \quad \dot{z}^0(T) = 0, \end{cases} \quad (5.22)$$

along with the inequality

$$\int_0^T \langle A'(q^0, q^0 - q) x^0, z^0 \rangle dt \geq 0, \quad (5.23)$$

for all $q \in Q_m$.

Remark 5.1

Necessary conditions of optimality for the parameter q , or for q and γ together for the system (5.2) could be developed following the same procedure as discussed above.

5.4 COMPUTATIONAL ALGORITHM AND EXAMPLES

Based on the necessary conditions presented in the preceding section, an iterative procedure can be developed [11] for determining the optimal parameter to approximate the unknown. For simplicity of presentation, we discuss the contents of the algorithm with reference to Theorem 5.2; but the same algorithm holds for Theorem 5.3 if references are made to the appropriate equations.

Rewriting the inequality (5.14) as

$$J'_{\gamma^0}(\gamma - \gamma^0) = \left(- \int_0^T \langle A\dot{x}(\gamma^0), z^0 \rangle dt \right) (\gamma - \gamma^0) \geq 0,$$

we can identify, for each $\gamma \in \Gamma$,

$$J'_{\gamma} = - \int_0^T \langle A\dot{x}(\gamma), z \rangle_{V', V} dt \quad (5.24)$$

as the gradient of the cost function (5.3), where x and z are the solutions corresponding to $\gamma \in \Gamma$, of the system equation (5.12) and the adjoint equation (5.13) respectively. Using this gradient, we can use the following algorithm to compute the unknown parameter. This algorithm is a special case of the general algorithm presented in [11].

Algorithm

1. Guess $\gamma^1 \in \Gamma$ for γ^0 and set $i = 1$.
2. Solve the system equation (5.12) with $\gamma^0 = \gamma^i$.
3. Solve the adjoint equation (5.13) with $\gamma^0 = \gamma^i$.
4. Compute the Gradient J'_{γ^i} as given in (5.24).
5. Compute the search direction.

a) Gradient method: $S^i = -J'_{\gamma^i}$,

or b) Conjugate Gradient method:

$$S^i = -J'_{\gamma^i} + \beta^i S^{i-1},$$

where $\beta^i = \|J'_{\gamma^i}\|^2 / \|J'_{\gamma^{i-1}}\|^2$

6. Update the parameter by setting

$$\gamma^{i+1} = \gamma^i + \epsilon S^i$$

where $\epsilon > 0$ is chosen sufficiently small so that

$$J(\gamma^{i+1}) \leq J(\gamma^i) \quad \text{and} \quad \gamma^{i+1} \in \Gamma.$$

7. Set $i = i+1$ and repeat from step 2 until a convergence criterion is satisfied, for example

$$|J(\gamma^{i+1}) - J(\gamma^i)| \leq \epsilon$$

where $\epsilon > 0$ is chosen suitably small.

Remark 5.2

As in any gradient method, the iterated parameter would tend to seek a local minimum. This could be avoided by repeating the procedure for different initial guesses for the unknown.

The cost index or the identification error considered in the examples is given by

$$J(y, q) = \frac{\lambda}{2} \int_0^T \|y(y, q; t) - \tilde{y}\|_{L_2(\Omega)}^2 dt, \quad (5.25)$$

where \tilde{y} is the "given" observation data, y is the solution of the model equation as a function of the unknown parameter $y \in P$, $q \in Q_m$ and $\lambda > 0$ is a weighting factor. With reference to the equation (5.3), we note that the observation space $H \equiv L_2(\Omega)$ and C is the identity map.

For numerical solution of the model equation and the corresponding adjoint equation, we have used finite difference / semi-discretization methods with the spatial grid size of 0.05. The observation data for the actual system i.e. $\hat{y}(\cdot, t)$ were generated with known values of the parameter in the equations which, in the sequel, will be referred to as the "true parameter".

Example 5.1

We first consider the structurally damped beam equation given by

$$\frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial}{\partial t} \frac{\partial^4 y}{\partial x^4} + \frac{\partial^4 y}{\partial x^4} = 0, \quad x \in (0, 1) \equiv \Omega, \quad t \in [0, 1],$$

$$y(0, t) = 0, \quad y(1, t) = 0,$$

$$\frac{\partial y}{\partial x}(0, t) = 0, \quad \frac{\partial y}{\partial x}(1, t) = 0, \quad (5.26)$$

$$y(x, 0) = y_0(x), \quad \frac{\partial y}{\partial t}(x, 0) = 0,$$

$$y_0(x) = c[(\text{Cosh } \lambda - \text{Cos } \lambda)(\text{Sinh } \lambda x - \text{Sin } \lambda x) - (\text{Sinh } \lambda - \text{Sin } \lambda)(\text{Cosh } \lambda x - \text{Cos } \lambda x)], \quad c = -0.00012,$$

and λ satisfies $\text{Cosh } \lambda \text{ Cos } \lambda = 1$, $\lambda \neq 0$,

where the damping parameter γ is unknown. For this example, we can consider $v = H_0^2(x)$, $H \equiv L_2(\Omega)$, and $v' = H^{-2}(\Omega)$. From the inequality (5.14), one can obtain the necessary gradient as

$$J'_\gamma = - \int_0^T \int_0^1 \frac{\partial^3 v}{\partial t \partial x^2} \cdot \frac{\partial^2 z}{\partial x^2} dx dt \quad (5.27)$$

The "true" parameter is taken as $\gamma^* = 0.001$ and the weighting factor in the cost index (5.25) is assumed to be $\lambda = 10^{12}$. In Table I, we summarize the convergence of the parameter γ corresponding to the initial guess $\gamma^1 = 0$. CPU time taken in this example was 75 seconds.

Table I

Iter.	γ	$J(\gamma)$
0	0.0	0.2146177D 06
1	0.3377381D-03	0.8834628D 05
2	0.5820642D-03	0.3362766D 05
3	0.7550348D-03	0.1119205D 05
4	0.8710657D-03	0.3035701D 04
5	0.9420695D-03	0.6049853D 03
6	0.9796204D-03	0.7436543D 02
7	0.9953119D-03	0.3924051D 01
8	0.9996251D-03	0.2507304D-01
9	0.1000039D-02	0.2646957D-03
10	0.9999876D-03	0.2764182D-04
11	0.1000007D-02	0.9550624D-05
True	0.001	-

Example 5.2

In this example we consider the problem of identification of the parameter q in the undamped beam equation:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} + q \frac{\partial^4 y}{\partial x^4} &= 0, \quad x \in (0,1) \equiv \Omega, \quad t \in [0,1], \\ y(0,t) &= 0, \quad \frac{\partial^2 y}{\partial x^2}(1,t) = 0, \\ \frac{\partial y}{\partial x}(0,t) &= 0, \quad \frac{\partial^3 y}{\partial x^3}(1,t) = 0, \\ y(x,0) &= y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = 0, \end{aligned} \quad (5.28)$$

where $y_0(x) = \frac{1}{2}[(\text{Cosh } \lambda + \text{Cos } \lambda)(\text{Sinh } \lambda x - \text{Sin } \lambda x) - (\text{Sinh } \lambda + \text{Sin } \lambda)(\text{Cosh } \lambda x - \text{Cos } \lambda x)]$
with $\text{Cosh } \lambda \text{Cos } \lambda + 1 = 0$.

For this example, we can take

$$V \equiv \{ \phi \in H^2(\Omega) : \phi(0) = 0, \frac{\partial \phi}{\partial x}(0) = 0 \},$$

and $H = L_2(\Omega)$. The space V' is the space of all continuous linear functionals on V . Using the necessary condition (5.23) for optimal identification, the gradient J'_q is obtained as

$$J'_q = - \int_0^1 \int_0^1 \frac{\partial^2 y}{\partial x^2} \cdot \frac{\partial^2 z}{\partial x^2} dx dt. \quad (5.29)$$

For the "true" parameter, we take $q^* = 1.0$, and the weighting factor λ in (5.25) is taken as $\lambda = 10^{11}$. The iteration was initiated with $q^1 = 0.5$ and convergence was obtained in 12 iterations. The results are shown in Table II. The CPU time required for this example was 15 seconds.

Table II

Iter.	q	J(q)
0	0.5000000D 00	0.1843745D 06
1	0.7656415D 00	0.2632193D 05
2	0.8435816D 00	0.1056968D 05
3	0.8966337D 00	0.4330934D 04
4	0.9347122D 00	0.1656497D 04
5	0.9617468D 00	0.5529317D 03
6	0.9799527D 00	0.1491455D 03
7	0.9910922D 00	0.2913371D 02
8	0.9969367D 00	0.3426452D 01
9	0.9993286D 00	0.1642089D 00
10	0.9999556D 00	0.7175859D-03
11	0.1000005D 01	0.1027614D-04
12	0.9999982D 00	0.1212595D-05
True	1.0	-

Remark 5.2

In case the parameters (mass density and flexural rigidity) of the beam equation (2.16) are functions of x , one can parametrize them by using appropriate polynomials. Then the identification problem reduces to that of finding a set of constant parameters.

5.5 SUMMARY

In this chapter, we consider the problem of identification of parameters in vibrating systems described by second order evolution equations. Typical examples of this class of systems are vibrating strings, transverse and torsional vibration of beams, transverse vibration of plates etc. For identification of parameters in a particular system, it suffices to consider the appropriate spatial differential operator A in the system dynamics.

Following the techniques of optimal control theory, we have developed the necessary conditions of optimality for identification of the unknown parameters. It is shown that the optimal parameter minimizing the mean square difference between the 'observed' data and the response of the model equation, is determined by simultaneous solution of the system equation, the corresponding adjoint equation, and an associated maximality condition.

For numerical computation of the unknown parameter, a gradient algorithm is presented which utilizes the maximality condition characterizing the optimal parameter. Illustrative numerical examples are presented for identification of parameters in the beam equation. In developing the computer software, no special attention was given to minimize the computer time. Yet the CPU time required in these examples was found to be quite small. It is expected that the CPU time could be further reduced by improving the efficiency of the algorithm.

Chapter VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 CONCLUSIONS

Development in control theory have taken place mainly on systems which can be adequately described either by ordinary differential equations or by partial differential equations. But there are some practical systems which can be accurately described only by a coupled system of ordinary and partial differential equations. An important and prominent example in this category is the flexible spacecraft which is the subject of this thesis; and this has been a topic of continued research at NASA and other space organizations. Some other interesting examples [78] in this class of systems are: power systems, mass transport system, aerodynamic reentry vehicle with ablative surface etc.

In general, large flexible spacecraft would be partly rigid and partly flexible. In this thesis, a methodology has been developed for rigorous modelling of such spacecraft. It has been shown that the complete dynamics of the system could be described by a hybrid system of equations, i.e. a set of ordinary differential equations for the rigid parts, and a set of partial differential equations

for the flexible parts. These equations indicate a very strong and intricate nature of interaction between the rigid and the flexible parts of the spacecraft. Complete dynamic equations have been derived for spacecraft consisting of i) a rigid bus and a flexible beam, ii) a rigid bus with a bend beam and a solar panel, and iii) a rigid bus and multiple beams.

An important aspect of this model is that the system is described in terms of variables which can be physically measured. This is extremely important in the analysis of system behavior and also in the design of stabilizing controls. Furthermore, the deformations of the elastic members are described with respect to coordinate systems which are fixed at their unperturbed positions. This makes the system variables physically more meaningful and reduces computational complexities.

Stabilization of the flexible spacecraft has been proved using Lyapunov's approach. It has been shown that asymptotic stabilization of the system could be obtained by application of simple feedback controls. A combination of bus velocity feedback and beam damping have been found to be most effective. These controls are practically implementable. It has also been shown that these control laws are equally effective irrespective of the order of accuracy in modelling the beam dynamics.

Although the system dynamics appears to be somewhat complicated, it can be simulated on a digital computer for numerical studies. An algorithm has been developed for simultaneous solution of the hybrid system. The CPU time required for a typical run for the examples presented in this thesis is quite small and could be further reduced by improving the efficiency of the algorithm.

It is to be noted here that the equations developed in this thesis gives an exact dynamics of the spacecraft under the given assumptions. One might be tempted to obtain a simplified dynamics by approximating some of the coupling terms between the ordinary and the partial differential equations. Such an attempt has been made by the author. It has been found that a first order approximation of displacement ≈ 0 and velocity ≈ 0 in the coupling terms gives results which are more or less close to the solutions obtained from the exact dynamics. But approximation of acceleration gives rise to completely erroneous results. Thus it is essential to study the system behavior on the basis of coupled system of ordinary and partial differential equations.

Stability of flexible structures in the presence of distributed white noise have also been investigated. It has been found that by application of velocity feedback a flexible beam perturbed by distributed noise could be

stabilized in the mean square and almost sure sense with respect to ball in the energy space. The amount of damping could be suitably chosen in order to maximize the decay rate and minimize the size of the attractor. It is also shown that the uncontrolled flexible spacecraft would become unstable if it is perturbed by random disturbances. Stability of system vibrations could be achieved by application of simple feedback controls.

Techniques of optimal control theory have been utilized in identifying the parameters of vibrating systems. It has been shown that the optimal parameter minimizing the mean square difference between the model equation output and the actual observed data is determined by simultaneous solution of the system equation, the corresponding adjoint equation, and an associated maximality condition. An algorithm which effectively utilizes this maximality condition, has been presented in order to compute the unknown parameter.

In brief, control of attitude motion and suppression of elastic vibrations of flexible spacecraft are important problems in space technology. To ensure satisfactory performance of large flexible spacecraft, it is essential to consider the distributed nature of the elastic members in the system modelling and the controller design. A method has been presented in this thesis to obtain such a rigorous model of flexible spacecraft. Also several simple practically implementable feedback controllers have been suggested for stabilization of the system.

6.2 SUGGESTIONS FOR FURTHER RESEARCH

As a continuation of this thesis, further research could be conducted along several directions. A model of flexible spacecraft without the assumption of a fixed orbit, and including the Euler angles would be more elaborate and comprehensive. In addition, one would be interested to include the gravity gradient torques in the system model.

The control structure developed in this study uses distributed control supported on a small section of the beam. Design of stabilizing controllers using boundary feedback would be of interest. Experience with rigid body satellites show that passive controllers can be used for stabilization of the system in the case of small disturbances. Similar passive devices for stabilization of the flexible spacecraft would be of practical significance for the purpose of saving fuel. Design of optimum controllers should also be carried out to stabilize the spacecraft in minimum time and with minimum expenditure of fuel.

Robustness of a stabilizing controller requires stabilization in the presence of random disturbances. In this respect, it is required to prove asymptotic stability and to estimate the size of the attractor for the spacecraft governed by the stochastic hybrid system.

For the identification part, it is important to obtain the unknown parameter from experimental data which may, actually, be available at certain spatial discrete points. A computer software package for parameter identification of distributed systems would be very useful to industries.

And finally, to the knowledge of the author, not much work has been reported on systems which are governed by a coupled system of ordinary differential equations and partial differential equations. It would be of significant interest to give an abstract formulation of the problem and to pursue further research on control, stabilization, optimization and related areas for this class of systems and their stochastic versions.

Appendix A

SOME IDENTITIES FROM VECTOR ALGEBRA

Some vector relations frequently used in Chapters II-IV are given below. Let A, B, C be three vectors and T be a tensor in the rectangular coordinate system $(\hat{x}, \hat{y}, \hat{z})$, defined by

$$A = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z$$

$$B = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z$$

$$C = \hat{x} C_x + \hat{y} C_y + \hat{z} C_z$$

$$T = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Vector Relations

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

$$A \times (B \times C) + (B \times A) \times C = B \times (A \times C)$$

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

Vector and Tensor Operations

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z$$

$$A \times B = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)$$

$$T \cdot A = \hat{x} (T_{xx} A_x + T_{xy} A_y + T_{xz} A_z) \\ + \hat{y} (T_{yx} A_x + T_{yy} A_y + T_{yz} A_z) \\ + \hat{z} (T_{zx} A_x + T_{zy} A_y + T_{zz} A_z)$$

Appendix B

NUMERICAL SOLUTION OF HYBRID DYNAMICS: ALGORITHM-A

For numerical solution of the hybrid dynamics (2.32), a combination of Runge-Kutta method (for the ODE system) and finite difference method (for the PDE system) was used. The Runge-Kutta algorithm is available in many standard texts. The beam equation in (2.32) could be written in two first order equations as:

$$\begin{aligned} \frac{\partial \phi_1}{\partial t} &= -\hat{B}_1 \phi_1 - \frac{\partial^2 \phi_2}{\partial x^2} - \hat{B}_3 \phi + \hat{F}, \\ \frac{\partial \phi_1}{\partial t} &= \frac{\partial^2 \phi_1}{\partial x^2}, \end{aligned} \quad (\text{B-1})$$

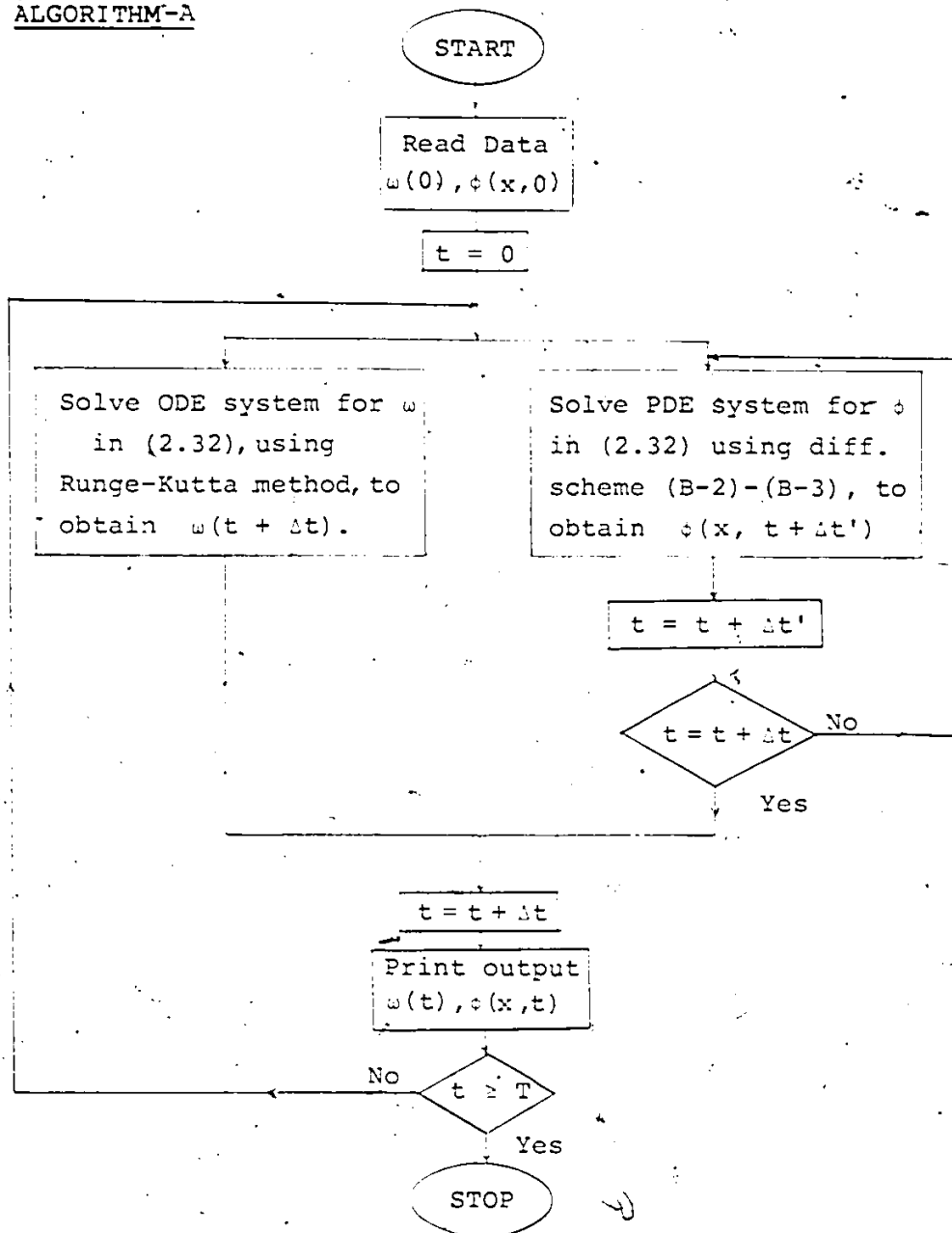
where $\phi_1 \equiv \frac{\partial \phi}{\partial t}$, and $\phi_2 \equiv \frac{\partial^2 \phi}{\partial x^2}$. (Equation (B-1) is shown in the normalized form assuming a uniform beam). Then a finite difference approximation for (B-1) is given by [56]:

$$\begin{aligned} \phi_1(i, j+1) &= \phi_1(i, j) - \frac{\Delta t}{(\Delta x)^2} [\phi_2(i+1, j) - 2\phi_2(i, j) + \phi_2(i-1, j)] \\ &\quad + [-\hat{B}_1 \phi_1(i, j) - \hat{B}_3 \phi(i, j) + \hat{F}(i, j)] \Delta t, \end{aligned} \quad (\text{B-2})$$

$$\phi_2(i, j+1) = \phi_2(i, j) + \frac{\Delta t}{(\Delta x)^2} [\phi_1(i+1, j+1) - 2\phi_1(i, j+1) + \phi_1(i-1, j+1)],$$

where i denotes the spatial and j the time discretization points. The displacement ϕ in (B-2) is obtained easily from ϕ_2 and using the boundary conditions as:

$$\phi(i+1, j+1) = 2\phi(i, j+1) - \phi(i-1, j+1) + \phi_2(i, j+1) (\Delta x)^2. \quad (\text{B-3})$$

ALGORITHM-A

For accuracy and numerical stability, it is required to take the PDE-solution-time step $\Delta t'$ smaller than the ODE-solution-time-step Δt .

Appendix C

NUMERICAL SOLUTION OF HYBRID DYNAMICS: ALGORITHM-B

The semidiscretization scheme involves approximating the partial differential equation by a set of ordinary differential equations. Defining

$$\begin{aligned} \xi_1 &\equiv \phi_1 \\ \xi_2 &\equiv \frac{\partial \phi_1}{\partial t} \\ \xi_3 &\equiv \frac{\partial^2 \phi_1}{\partial x_1^2} \end{aligned} \quad (C-1)$$

the boom dynamics (3.20) could be written as three first order equations as in (B-1). Note that, in terms of the notations of Chapter III, $\phi_1 \equiv (y_1, z_1)$ is the displacement vector. Then the approximate ODE system for the boom dynamics (3.20) is given by:

$$\begin{aligned} \dot{\xi}_1(i) &= \xi_2(i) \\ \dot{\xi}_2(i) &= -\frac{1}{(\Delta x)^2} [\xi_3(i+1) - 2\xi_3(i) + \xi_3(i-1)] \\ &\quad + [-\hat{B}_1 \xi_2(i) - \hat{B}_3 \xi_1(i) + \hat{F}(i)] \\ \dot{\xi}_3(i) &= \frac{1}{(\Delta x)^2} [\xi_2(i+1) - 2\xi_2(i) + \xi_2(i-1)] \end{aligned} \quad (C-2)$$

where i denotes the spatial discretization points and $\dot{\xi}$ denotes the time derivative.

Similarly, the tower dynamics is also approximated by a system of ordinary differential equations in terms of the variables η_1 , η_2 , and η_3 defined as:

$$\begin{aligned}\eta_1 &\equiv \phi_2 \\ \eta_2 &\equiv \frac{\partial \phi_2}{\partial t} \\ \eta_3 &\equiv \frac{\partial^2 \phi_2}{\partial x_2^2}\end{aligned}\tag{C-3}$$

Thus the hybrid dynamics (3.19)-(3.21) is written as a complete system of ordinary differential equations of the form

$$\begin{aligned}\dot{X} &= F(X), \\ X(0) &= X_0,\end{aligned}\tag{C-4}$$

where

$$X \equiv \{(\omega, (\xi_1(i), \xi_2(i), \xi_3(i)), i = 1, 2, \dots, N), (\eta_1(i), \eta_2(i), \eta_3(i)), i = 1, 2, \dots, M)\}.$$

Runge-Kutta subroutine is then used for solving the equations (C-4).

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