Strategic situations involving imperfect information

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Introduction

Game theory is the study of strategic interaction. It has had remarkable successes and applications in a variety of fields, including economics, international relations, and even biology. One of the main goals in game theory is to discover and understand equilibria in games. An equilibrium in a game is a strategy that is optimal in the sense that it is impossible for an individual player to obtain a better payoff by deviating from the equilibrium strategy.

Strategic situations in which the parties involved have complete information are relatively simple to model and reason about. In strategic situations where one or all of the parties involved have incomplete information however, things become much more complicated. Incomplete information can come in a variety of forms in a strategic situation. For example, players could be uncertain of the possible moves available to themselves or their opponents, a player could be unclear of the order in which moves are to be played, a player could be unsure of how much information their opponent has regarding the game, or a player could be uncertain of some payoffs within the game. This research focussed on situations involving incomplete information regarding payoffs.

The goal of this research project was to explore systematic methods of reasoning about strategic situations in which one or more parties have partial or incomplete information about each other's payoffs, using game trees and ideas from epistemic modal logic. As incomplete information is a reality in most real life strategic situations, developing systematic methods for reasoning about them would be very beneficial indeed, and give us tools to better understand the nature of decision-making in such situations.

Background

Game Trees

Game trees can be used as visual aids to help model game theoretic situations. The nodes in the game trees represent the player who currently has the opportunity to make a decision while the branches indicate the player's decision making options. The numbers appearing at the end of some branches indicate the payoffs for Player 1 and Player 2, respectively, as a result of the decision made. The node marked N, for "nature", is used to indicate that Player 1 does not know which situation the two players are actually in and hence has decided upon probabilities for the likelihood of each situation being reality. Players can calculate their expected values (payoffs) for making certain moves.

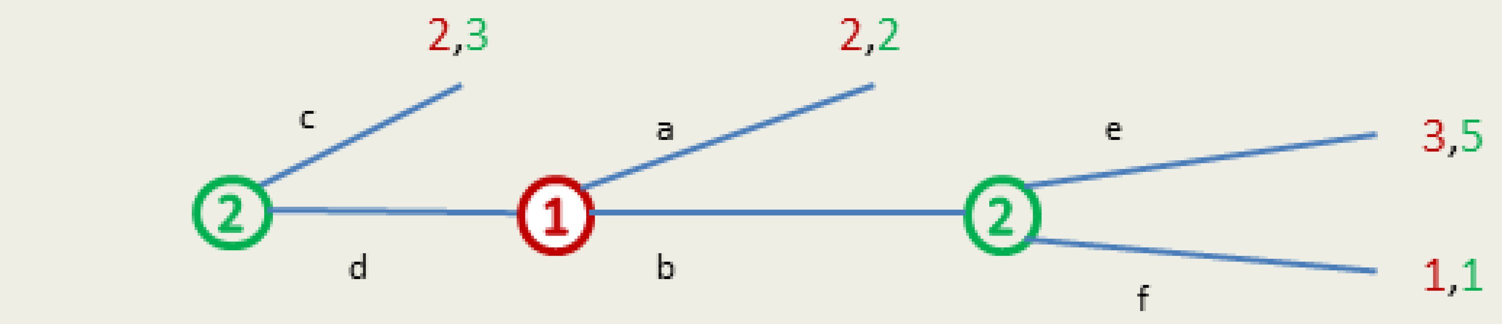
Epistemic Modal Logic

Epistemic logic is a type of logic that is used to reason about individuals' knowledge and beliefs. Probabilistic epistemic modal logic is used to reason about degrees of uncertainty and thus allows us to formally reason about situations in which individuals are uncertain about various pieces of knowledge. In our syntax, $□_i$ will be used to mean player 'i' knows a piece of information, while $P_i(p = 0.7)$ will mean that player 'i' believes p to be true with a probability of 0.7. Epistemic modal logic allows us to formalize our reasoning about situations involving incomplete information and various decision making possibilities. Furthermore, as with every logical system, there are reasoning principles that can be used to formally check correctness of reasoning. This is very important as it enables us to reason about the strategic situation in a systematic and rigorous way.

Game Trees

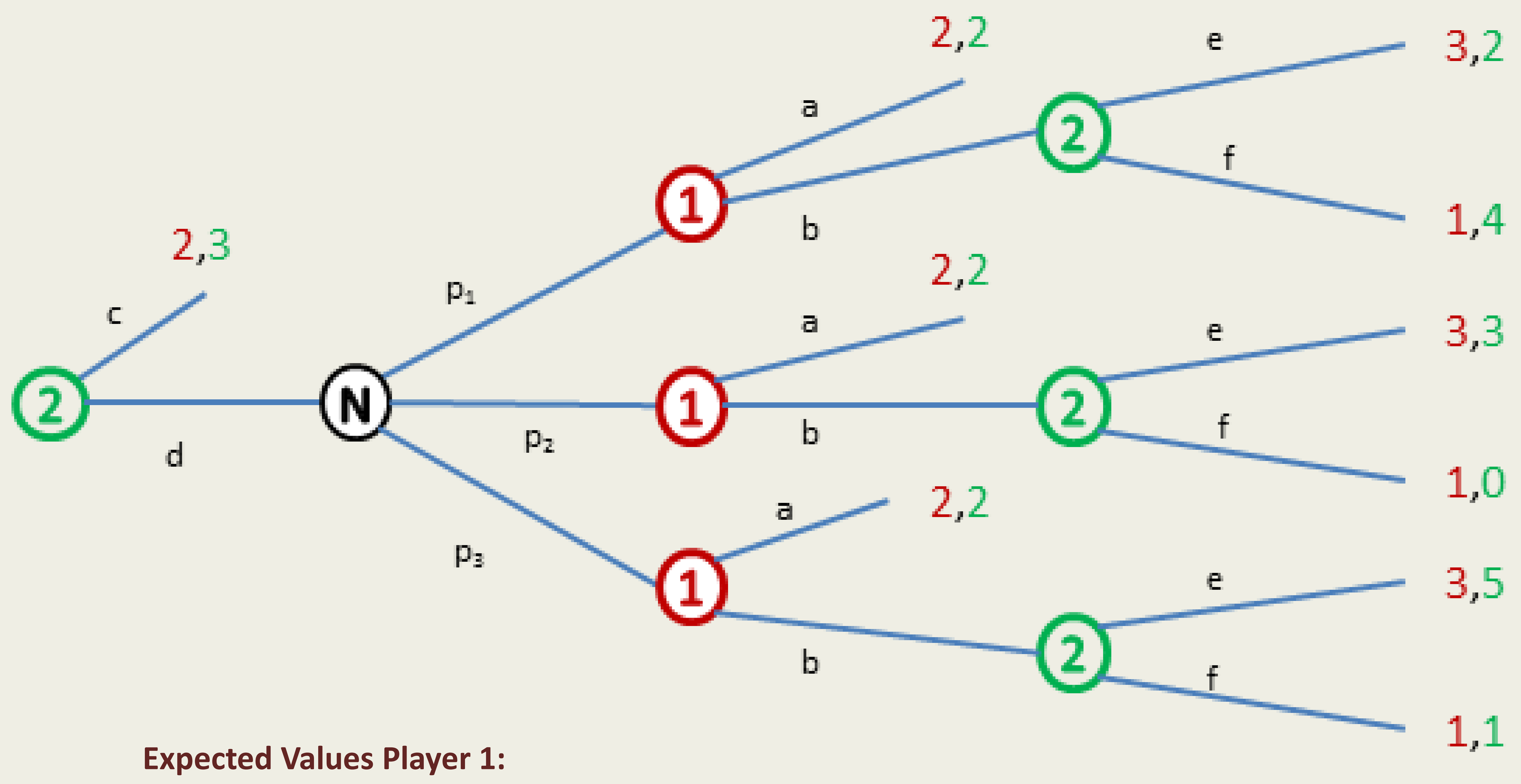
Complete Information:

In this game both Player 1 and Player 2 have complete information. This means that both players are aware of all the moves available to themselves and the other player and they are aware of all of the payoffs associated with these moves. In the first node, Player 2 can choose to end the game with move 'c' or play move 'd' which will allow Player 1 to move. Player 1 can then end the game with move 'a' or play move 'b' which will allow Player 2 to have the final move.



Incomplete Information:

This is the same game as before, except in this situation Player 1 has incomplete information about both players' payoffs from moves 'e' and 'f'. As such, Player 1 assigns probabilities p_1, p_2, p_3 to the likelihood of each pair of payoffs being the true set of payoffs. This is illustrated by adding the node marked N for "nature", as in Player 1's mind there is a chance that any one of these three situations could be the case. Player 2 is still aware that in reality the payoffs for move 'e' are (3, 5) and for move 'f' (1, 1), but he does not know the values of p_1, p_2, p_3 so he still faces some uncertainty.



Expected Values Player 2:

$EV_2(c) = 3$

$EV_2(d) = P(\text{player 1 choosing b})(5) + P(\text{player 1 choosing a})(2)$
$= P(EV_1(b) > EV_1(a))(5) + P(EV_1(b) < EV_1(a))(2)$
$= P(EV_1(b) > EV_1(a))(5) + (1 - P(EV_1(b) > EV_1(a)))(2)$

Expected Values Player 1:

$EV_1(a) = 2$

$EV_1(b) = p_1(1) + p_2(3) + p_3(3)$

Simple Example:

For a given set of probabilities: $\{p_1 = {}^1/_3,\ p_2 = {}^1/_3,\ p_3 = {}^1/_3\}$

we can calculate as follows: $EV_1(a) = 2$

$EV_1(b) = p_1(1) + p_2(3) + p_3(3)$
$= ({}^1/_3)(1) + ({}^1/_3)(3) + ({}^1/_3)(3)$
$= 7/3$

Hence, Player 1 would choose to play move 'b'. If player 2 is aware of these probabilities, he will play moves 'd' and 'e'. If Player 2 is unaware of the values of p_1, p_2, p_3 he will also face a choice characterized by uncertainty. If he plays move 'd' and Player 1 responds with 'a', his payoff will be 2 and he will be worse off than if he chose 'c' in the first place. Player 2 will therefore have to assign a probability to the likelihood of Player 1 choosing 'b' and calculate his expected values for moves 'c' and 'd' based on this probability.

Epistemic Modal Logic

Epistemic modal logic can be used to formalize our reasoning and also to deduce conclusions regarding player's strategic options. Reasoning with a formal logical system allows us to check the validity of our arguments using the principles of reasoning contained within the system.

Player 1 knows his payoff for move 'a':

$□_1(a_1 = 2)$

Player 1 knows that Player 2 knows Player 1's payoff for move 'a':

$□_1(□_2(a_1 = 2))$

Player 1 assigns a probability of ${}^1/_2$ that his payoff for choosing move b will be 1, a probability of ${}^1/_4$ that his payoff for choosing move 'b' will be 3, and a probability of ${}^1/_4$ that his payoff for choosing move 'b' will be 5. After performing expected value calculations, player 1 can deduce that choosing move 'b' is his optimal strategy:

$P_1((b_1 = 1) = {}^1/_2) \wedge P_1((b_1 = 3) = {}^1/_4) \wedge P_1((b_1 = 5) = {}^1/_4) \rightarrow □_1(EV_1(b) > EV_1(a))$

Applications

The use of game trees and epistemic modal logic allows for systematic and formal reasoning about strategic situations involving imperfect information. This could allow us to set up computer programs that could reason about these situations and deduce optimal strategies in arbitrarily complex scenarios.

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