Spectral Asymptotics for Left-Definite Vector Sturm-Liouville Problems

By

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To the memory of
my father

Abstract

We study the spectrum of the vector boundary value problem

$$
\begin{aligned}
-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y &= \lambda\mathcal{R}(t)y \qquad\qquad a \le t \le b \\
y(a) = 0 &= y(b),
\end{aligned}
$$

where $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are $m \times m$ symmetric matrix functions, $\mathcal{P}(t)$ is positive definite, $\mathcal{R}(t)$ is nondefinite and the entries of $\mathcal{P}^{-1}(t)$,$\mathcal{Q}(t)$, $\mathcal{R}(t)$ belong to $L^\infty[a\ ,\ b]$, and in particular discuss the existence of two infinite sequences of real eigenvalues. We also present two asymptotic formulas for the distribution functions of the positive and negative eigenvalues when $\mathcal{R}(t)$ belongs to some special classes of matrix functions.

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Contents

Introduction

The problem of the existence of eigenvalues for the regular nondefinite scalar Sturm-Liouville problem

$$-(p(t)u')' + q(t)u = \lambda r(t)u \qquad a \leq t \leq b \tag{0.1}$$

$$u(a)cos\alpha - (pu')(a)sin\alpha = 0 \tag{0.2}$$

$$u(b)cos\beta - (pu')(b)sin\beta = 0 \tag{0.3}$$

where $0 \leq \alpha, \beta < \pi$ and $-\infty < a < b < \infty$, has been treated in many different respects. In fact, Richardson [35] and Atkinson-Mingarelli [3] have proved that :

Whenever

$$\int_a^b r_+(t)dt > 0 \quad and \int_a^b r_-(t)dt > 0 \tag{0.4}$$

the problem (0. 1-2-3) has two infinite sequences of real eigenvalues, one positive and one negative, and each one of which has $+\infty$ and $-\infty$ for its only points of accumulation, where $r_+ = 1/2[\mid r \mid +r\ ]$ and $r_- = 1/2[\mid r \mid -r\ ]$, ($r_+$(resp.$r_-$) are commonly described as the positive (resp. negative) part of r).

In [35] Richardson proves the above result for the Dirichlet problem i. e. , $\alpha = \beta = 0$ in (0. 2–3) when the coefficients p, q, r are continuous and $p(t) > 0$.The method he employs there depends on the behaviour of the zeros of solutions of (0. 1) for varying $|\lambda|$. On the other hand, Atkinson-Mingarelli prove the same result in [3] by using a modified Prüfer angle method when the coefficients have the mere properties $1/p, q, r \in L(a, b)$, (i. e. , are Lebesgue integrable on $[a, b]$) and $p(t) > 0$ a. e.

A vector boundary value problem corresponding to (0. 1–2–3) is

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \lambda\mathcal{R}(t)y \qquad a \leq t \leq b \tag{0.5}$$

$$(\mathcal{A}_{j1}y(a) + \mathcal{A}_{j2}\mathcal{P}(a)y'(a)) + (\mathcal{B}_{j1}y(b) + \mathcal{B}_{j2}\mathcal{P}(b)y'(b)) = 0 \tag{0.6}$$

$(j = 1, 2)$, where $\mathcal{P}(t)$, $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are $m \times m$ symmetric matrix functions, $\mathcal{P}(t)$ is positive definite for $t \in [a, b]$, while $\mathcal{A}_{ij}, \mathcal{B}_{ij}$ $(i, j = 1, 2)$ are $m \times m$ constant matrices satisfying

$$\mathcal{A}\mathcal{Y}_m\mathcal{A}^* = \mathcal{B}\mathcal{Y}_m\mathcal{B}^* \tag{0.7}$$

and the rank of the matrix $[\mathcal{A}, \mathcal{B}]$ equals $2m$.The matrices $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{Y}_m$ are defined by

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{pmatrix}$$

and

$$\mathcal{Y}_m = \begin{pmatrix} \mathcal{O} & \mathcal{I}_m \\ -\mathcal{I}_m & \mathcal{O} \end{pmatrix}.$$

Here we restrict our attention to the case when

$$\mathcal{A} = \begin{pmatrix} \mathcal{I}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{I}_m & \mathcal{O} \end{pmatrix}.$$

In this case the boundary conditions (0. 6) reduce to

$$y(a) = 0 = y(b) \tag{0. 8}$$

giving us the Dirichlet boundary conditions.

Similar to scalar boundary value problems, ([6], [9], [10]), the vector boundary value problem (0. 5–6) generates a *right-definite* problem (see Section 2. 1) when the matrix function $\mathcal{R}(t)$ is nonnegative definite for $t \in [a\ ,\ b]$ and the set $\{t : t \in [a\ ,\ b], \mathcal{R}(t)\ is\ positive\ definite\}$ is of positive measure.This problem has been studied systematically by Reid in [31].The existence of eigenvalues for the problem (0. 5–6) was treated there by means of variational methods.

The Sturm-Liouville boundary value problem (0. 5–6) is said to be *nondefinite* when the matrix function $\mathcal{R}(t)$ is nondefinite on $[a\ ,\ b]$. In this case we define nonnegative matrix functions $\mathcal{R}_+(t)$ and $\mathcal{R}_-(t)$ associated with $\mathcal{R}(t)$ (see Section 1. 1) similar to $r_+(t)$ and $r_-(t)$ in (0. 4) and consider the problem (0. 5–6) with the following assumptions on $\mathcal{R}_+(t)$ and $\mathcal{R}_-(t)$:

The sets

$$\{t : t \in [a\ ,\ b], \mathcal{R}_+(t)\ is\ positive\ definite\} \tag{0.9}$$

and

$$\{t : t \in [a\ ,\ b], \mathcal{R}_-(t)\ is\ positive\ definite\} \tag{0.10}$$

are of positive measure. These conditions can be interpreted in a way similar to (0. 4).

The existence theorem for eigenvalues of (0. 5–6) takes the form : (see Section 4. 2),

Whenever $\mathcal{R}_+(t)$ and $\mathcal{R}_-(t)$ have the properties (0. 9) and (0. 10) respectively the problem (0. 5–6) has two infinite sequences of real eigenvalues, one positive and one negative, and each one of which has $+\infty$ and $-\infty$ for its only points of accumulation.

Although this result is similar to the existence theorem we had for the scalar nondefinite problem none of the above cited methods for the scalar case seems to have a direct generalization to prove the result for the vector nondefinite problem.

We shall present a method to settle this result for the Dirichlet problem (0. 5–8) which seems possible to extend for the general nondefinite problem (0. 5–6). Our method will be based on a certain two parameter boundary value problem. For example, in order to show the existence of nonnegative eigenvalues of (0. 5–8) we consider

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y + \mu\mathcal{R}_-(t)y = \lambda(\mu)\mathcal{R}_+(t)y, \tag{0.11}$$

where $\mu \geq 0$, with boundary conditions (0. 8). This problem is a right-definite problem for each fixed μ. Thus we can use the results of Reid [31] to show the existence of eigenvalues $\lambda(\mu)$ of (0. 11). Such $\lambda(\mu)$ is a continuous nondecreasing function of μ (see Section 3. 2). Consequently, when $\lambda(\mu_0) = \mu_0$ for some μ_0 the equation (0. 11) is exactly equation (0. 5) with the same vector function y and $\lambda = \mu_0$ since $\mathcal{R}(t) = \mathcal{R}_+(t) - \mathcal{R}_-(t)$. Hence μ_0 is a nonnegative eigenvalue of (0. 5–8). Next interchanging the roles of $\mathcal{R}_+$ and $\mathcal{R}_-$ in (0. 11) we can generate the negative eigenvalues of (0. 5–8).

In the sequel we will establish some upper bounds for the number of negative eigenvalues of a right definite problem and also for the number of nonreal eigenvalues of a nondefinite problem of the type (0. 5–8) (see Sections 2. 2 and 4. 2).

Among the other results in our presentation are asymptotic formulas for the real eigenvalues of the nondefinite vector boundary value problem (0. 5–8). If $N_+(s)$ is the number of eigenvalues of (0. 1–2–3) which lie in the interval $[0\ ,\ s]$ then Atkinson-Mingarelli [3] showed that

$$\lim_{s\to\infty} \frac{N_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sqrt{(r(t)/p(t))_+}\,dt. \qquad (0.12)$$

This result has a natural extention to the vector problem (0. 5–8) if the matrix coefficients $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ are diagonal. In this case

$$\lim_{s\to\infty} \frac{n_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sum_{i=1}^{m} \sqrt{(r_i(t)/p_i(t))_+}\,dt \qquad (0.13)$$

where $n_+(s)$ is the number of eigenvalues of (0. 5–8) which lie in the interval $[0\ ,\ s]$,

$$\mathcal{P}(t) = diag(p_1(t), \ldots, p_m(t)) \qquad and \qquad \mathcal{R}(t) = diag(r_1(t), \ldots, r_m(t)).$$

If we denote the sum of the square roots of the moduli of all the eigenvalues of a square matix $\mathcal{A}$ by $\Lambda_{1/2}(\mathcal{A})$ then the previous equation becomes

$$\lim_{s\to\infty} \frac{n_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_+(t))\,dt \qquad (0.14)$$

since $\mathcal{P}(t)$ is positive definite. The asymptotic formula (0. 14) agrees with the result of Gohberg and Krein [15] where the same formula (0. 14) is obtained when

$\mathcal{P}$ and $\mathcal{R}$ are not necessarily diagonal but $\mathcal{R}(t)$ is nonnegative definite on $[a, b]$. Here we establish the result (0. 14) for the nondefinite problem (0. 5–8) when $\mathcal{R}(t)$ belongs to certain classes of matrix functions. In particular, the case when $\mathcal{R}$ is functionally commutative (see Section 1. 2) and $\mathcal{Q}$ is arbitrary will be considered thereby extending the results of [15] to this problem, (see Section 4. 3).

Another interesting result we present here is the dependence of the limit (0. 14) of $n_+(s)/\sqrt{s}$ for the nondefinite problem on the remainder in the asymptotic estimates of the eigenvalues associated with a certain right-definite problem, (see Section 4. 3).

We note that similar asymptotic results hold for the distribution function of the negative eigenvalues of (0. 5–8). i. e. , if $n_-(s)$ is the number of eigenvalues of (0. 5–8) which lie in $[-s, 0]$ then

$$\lim_{s\to\infty} \frac{n_-(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_-(t))dt. \tag{0.15}$$

This result has been established by Atkinson-Mingarelli [3] for the scalar boundary value problem (0. 1–2–3).

Chapter 1

Preliminary Results

In the study of differential equations we are frequently concerned with properties of the eigenvalues and eigenvectors of symmetric matrix functions. In first two sections we shall present a few such properties which we will use throughout this dissertation.

Most of the time we study differential equations associated with some given boundary conditions.Such problems are called boundary value problems. Problems of this kind can be studied more effectively through integral equations, in fact here we study boundary problems by means of compact integral operators on suitable Hilbert spaces. Section 3 will be devoted to the statement of some properties of compact operators.

1.1 Matrix Notation

We denote the linear vector space of ordered m−tuples of complex numbers, with complex scalars, by C_m. The linear vector space of ordered m−tuples of real numbers, with real scalars, is denoted by R_m. The m × m identity matrix is denoted by $\mathcal{I}_m$, or by merely $\mathcal{I}$ when there is no ambiguity. The symbol $\mathcal{O}$ is the zero matrix of any dimensions. The conjugate transpose of a matrix $\mathcal{A}$ is denoted by $\mathcal{A}^*$. When matrix $\mathcal{A}$ is real and $\mathcal{A} = \mathcal{A}^*$ we call $\mathcal{A}$ symmetric.

A vector is a matrix with one column. If ξ, η are vectors of C_m, then $[\xi, \eta]$ denotes the inner product $\sum_{\alpha=1}^{m} \overline{\eta}_\alpha \xi_\alpha = \eta^* \xi$. In particular $\sqrt{[\xi, \xi]} = | \xi |$ denotes the *norm* of a vector ξ. If $\mathcal{A}$ is an m × m matrix, then the *norm* of $\mathcal{A}$ is equal to the maximum of $| \mathcal{A}\xi |$ on the set $\{\xi; \xi \in C_m, | \xi | = 1\}$. Moreover, if $\mathcal{A}$ is a symmetric matrix then the norm of $\mathcal{A}$ is equal to the maximum of $| [\mathcal{A}\xi, \xi] |$ on the set $\{\xi; \xi \in C_m, | \xi | = 1\}$. The norm of the matrix $\mathcal{A}$ is denoted by $| \mathcal{A} |$.

If $\mathcal{A}$ is a symmetric matrix and $[\mathcal{A}\xi, \xi] > 0$ ($[\mathcal{A}\xi, \xi] \geq 0$) for every nonzero vector $\xi \in C_m$ then $\mathcal{A}$ is *positive (nonnegative) definite.* We denote this by $\mathcal{A} > \mathcal{O}$ ($\mathcal{A} \geq \mathcal{O}$). In general $\mathcal{A} > \mathcal{B}$ ($\mathcal{A} \geq \mathcal{B}$) means that $\mathcal{A}$ and $\mathcal{B}$ are symmetric matrices of the same dimension and the matrix $\mathcal{A} - \mathcal{B}$ is positive (nonnegative) definite.

When $\mathcal{A}$ and $\mathcal{B}$ are two symmetric matrices with $\mathcal{B} > \mathcal{O}$ there exist values λ satisfying

$$\mathcal{A}\xi = \lambda \mathcal{B}\xi \tag{1.1}$$

for $\xi \neq 0$. If such λ exists with corresponding vector ξ, then

$$[\mathcal{A}\xi, \xi] = \lambda[\mathcal{B}\xi, \xi] \quad \text{and} \quad [\xi, \mathcal{A}\xi] = \overline{\lambda}[\xi, \mathcal{B}\xi].$$

Therefore $(\lambda - \overline{\lambda})[\mathcal{B}\xi, \xi] = 0$ since $\mathcal{A}$ is symmetric. Consequently $\lambda = \overline{\lambda}$, as $\mathcal{B}$ is positive, and λ is real. We also note that if $\mathcal{A}$ is nonnegative, $\lambda[\mathcal{B}\xi, \xi] = [\mathcal{A}\xi, \xi]$ implies that λ is nonnegative. If λ_1 and λ_2 are two distinct real values satisfying (1. 1) with corresponding vectors ξ_1 and ξ_2 then $[\mathcal{A}\xi_1, \xi_2] = \lambda_1[\mathcal{B}\xi_1, \xi_2]$ and $[\xi_1, \mathcal{A}\xi_2] = \lambda_2[\xi_1, \mathcal{B}\xi_2]$. Therefore $(\lambda_1 - \lambda_2)$ $[\mathcal{B}\xi_1, \xi_2] = 0$ and

$$[\mathcal{B}\xi_1, \xi_2] = 0. \tag{1.2}$$

Also, if λ satisfies (1. 1) for two linearly independent vectors then we can choose two such vectors ξ_1 and ξ_2 with the property (1. 2).

In [38] H. F. Weinberger studies the vector equation (1. 1) systematically and gives a variational method to find those values λ.

Theorem 1.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be $m \times m$ symmetric matrices with $\mathcal{B} > \mathcal{O}$. Then there exists a sequence of real values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ satisfying (1. 1) with corresponding vectors $\xi = u_\beta$ such that*

(i) $[\mathcal{B}u_\alpha, u_\beta] = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, m,$

(ii) $\lambda_1 = [\mathcal{A}u_1, u_1]$ *is the maximum of the quadratic form* $\mathcal{A}[\xi] = [\mathcal{A}\xi, \xi]$ *on the class*

$$\Gamma_N = \{\xi;\ \xi \in \mathcal{C}_m, [\mathcal{B}\xi, \xi] = 1\},$$

(iii) for $\beta = 2, \ldots, m$, the class

$$\Gamma_{N\beta} = \{\xi;\ \xi \in \Gamma_N, [\mathcal{B}\xi, u_\delta] = 0\ \delta = 1, \ldots, \beta - 1\}$$

is nonempty and $\lambda_\beta = \mathcal{A}[u_\beta]$ is the maximum of $\mathcal{A}[\xi]$ on $\Gamma_{N\beta}$.

Proof. See ([38] page 39).

Corollary 1.2 *Let $V = \{v_1, \ldots, v_k\}$ where $1 \leq k \leq m$, be given vectors of $\mathcal{C}_m$, and $\lambda[V]$ denote the maximum of $\mathcal{A}[\xi]$ on the set*

$$\Gamma_N[V] = \{\xi;\ \xi \in \mathcal{C}_m,\ [\mathcal{B}\xi, \xi] = 1,\ [\mathcal{B}\xi, v_\delta] = 0,\ \delta = 1, \ldots, k\}$$

then λ_{k+1} is the minimum of $\lambda[V]$.

Proof. See ([38] page 56).

Corollary 1.3 *Let $\mathcal{A}_1, \mathcal{A}_2$ be symmetric and $\mathcal{B}$ be positive $m \times m$ matrices. Also let $\{\lambda_\alpha^1, u_\alpha^1\}$ and $\{\lambda_\alpha^2, u_\alpha^2\}$ satisfy*

$$\mathcal{A}_1 u_\alpha^1 = \lambda_\alpha^1 \mathcal{B} u_\alpha^1 \quad \text{and} \quad \mathcal{A}_2 u_\alpha^2 = \lambda_\alpha^2 \mathcal{B} u_\alpha^2$$

be given by Theorem 1. 1. Then $\mathcal{A}_1 > \mathcal{A}_2$ ($\mathcal{A}_1 \geq \mathcal{A}_2$) implies that $\lambda_\alpha^1 > \lambda_\alpha^2$ ($\lambda_\alpha^1 \geq \lambda_\alpha^2$), $\alpha = 1, \ldots, m$. In particular,

$$\mathcal{A}_1 - \epsilon\mathcal{B} < \mathcal{A}_2 < \mathcal{A}_1 + \epsilon\mathcal{B}$$

implies that $| \lambda_\alpha^1 - \lambda_\alpha^2 | < \epsilon, \alpha = 1, \ldots, m.$

Proof. The first part is proved in ([38] page 62). To prove the second part note that $| \mathcal{A}_1[\xi] - \mathcal{A}_2[\xi] | < \epsilon$ on $\Gamma_N[V]$ for every subset V. Therefore $| \lambda_\alpha^1 - \lambda_\alpha^2 | < \epsilon$ for every $\alpha = 1, \ldots, m$.

Corollary 1.4 *Let $\mathcal{A}$ be nonnegative and $\mathcal{B}_1, \mathcal{B}_2$ be positive $m \times m$ matrices. Also let $\{\mu_\alpha^1, v_\alpha^1\}$ and $\{\mu_\alpha^2, v_\alpha^2\}$ satisfy*

$$\mathcal{A} v_\alpha^1 = \mu_\alpha^1 \mathcal{B}_1 v_\alpha^1 \quad \text{and} \quad \mathcal{A} v_\alpha^2 = \mu_\alpha^2 \mathcal{B}_2 v_\alpha^2$$

be given by Theorem 1. 1. Then $\mathcal{B}_1 \geq \mathcal{B}_2$ implies that $\mu_\alpha^2 \geq \mu_\alpha^1$ for $\alpha = 1, \ldots, m$.

Proof. See ([38] Theorem 3. 8. 1).

If the matrix $\mathcal{B}$ is replaced by the identity matrix $\mathcal{I}$ in the preceding discussion then $\{\lambda_\alpha , u_\alpha\}$, $\alpha = 1, \ldots, m$, are the eigenvalues and eigenvectors of the matrix $\mathcal{A}$.

According to the spectral theorem for symmetric matrices, ([18] page 275), if $\lambda_{i_1} > \lambda_{i_2} > \ldots > \lambda_{i_k}$, $m \geq i_k$, are distinct eigenvalues of the symmetric matrix $\mathcal{A}$, then there exist symmetric matrices $\mathcal{E}_\alpha$, $\alpha = 1, \ldots, k$, such that

$$\begin{aligned} &\text{(i)} \quad \mathcal{A} = \lambda_{i_1}\mathcal{E}_1 + \ldots + \lambda_{i_k}\mathcal{E}_k, \\ &\text{(ii)} \quad \mathcal{I} = \mathcal{E}_1 + \ldots + \mathcal{E}_k \\ &\text{(iii)} \quad \mathcal{E}_\alpha\mathcal{E}_\beta = \delta_{\alpha\beta}\mathcal{E}_\alpha. \end{aligned}$$

Then note that since

$$[\mathcal{E}_\alpha\xi, \xi] = [\mathcal{E}_\alpha^2\xi, \xi] = [\mathcal{E}_\alpha\xi, \mathcal{E}_\alpha\xi] \geq \mathcal{O} \quad \text{for } \xi \in \mathcal{C}_{\mathfrak{H}},$$

every matrix $\mathcal{E}_\alpha$ is nonnegative definite.Consequently,

$$\lambda_{i_1}\mathcal{I} - \mathcal{A} = (\lambda_{i_1} - \lambda_{i_2})\mathcal{E}_2 + \ldots + (\lambda_{i_1} - \lambda_{i_k})\mathcal{E}_k$$

and

$$\mathcal{A} - \lambda_{i_k}\mathcal{I} = (\lambda_{i_1} - \lambda_{i_k})\mathcal{E}_1 + \ldots + (\lambda_{i_1} - \lambda_{i_{k-1}})\mathcal{E}_{k-1}$$

are nonnegative matrices. Hence we have the following

Theorem 1.5 *Let $\lambda_1(\mathcal{A})$ and $\lambda_m(\mathcal{A})$ be the maximum and minimum eigenvalues of the symmetric matrix $\mathcal{A}$ respectively. Then*

$$\lambda_m(\mathcal{A})\mathcal{I} \leq \mathcal{A} \leq \lambda_1(\mathcal{A})\mathcal{I}.$$

If $\mathcal{A}$ is a nonnegative matrix then there exists an unique nonnegative matrix $\mathcal{C}$ such that $\mathcal{A} = \mathcal{C}^2$ (see, e. g. , [13] page 223). We call such matrix $\mathcal{C}$ the *square root matrix* of $\mathcal{A}$. Now for given symmetric matrix $\mathcal{A}$, $\mathcal{A}^2$ is nonnegative and therefore $\mathcal{A}^2$ has a unique square root matrix. We denote this by $\mid \mathcal{A} \mid$ and define

$$\mathcal{A}_+ = \frac{1}{2}[\mid \mathcal{A} \mid + \mathcal{A}]$$

and

$$\mathcal{A}_- = \frac{1}{2}[\mid \mathcal{A} \mid - \mathcal{A}].$$

Then $\mathcal{A}_+$ and $\mathcal{A}_-$ are nonnegative matrices with

$$\mathcal{A}_+ \geq \mathcal{A} \quad \text{and} \quad \mathcal{A} \geq -\mathcal{A}_-.$$

More properties of these matrices can be found in ([37] sec. 108).

1.2 Some Properties of Matrix Functions

A matrix function is called *continuous, integrable,* etc. , when each element of that matrix possesses the specified property. For a given compact interval $[a , b]$ the symbols $C_{mr}[a , b]$, $AC_{mr}[a , b]$, $L_{mr}[a , b]$, $L^p_{mr}[a , b]$, $L^\infty_{mr}[a , b]$, will denote the class of matrix functions $\mathcal{A}(t) = [\mathcal{A}_{\alpha\beta}(t)]$, $(\alpha = 1, \ldots, m;\ \beta = 1, \ldots, r)$, which on $[a , b]$ are respectively continuous, absolutely continuous, Lebesgue integrable, Lebesgue measurable and $\mid \mathcal{A}_{\alpha\beta}(t) \mid^p$ integrable $(\alpha = 1, \ldots, m;\ \beta = 1, \ldots, r)$, measurable and essentially bounded, i. e. , there exists a finite constant $k(\mathcal{A})$ such that each of the sets $\{t;\ \mid \mathcal{A}_{\alpha\beta}(t) \mid > k(\mathcal{A})\}$ has measure zero. For convenience we will denote $C_{m1}[a , b]$, $AC_{m1}[a , b]$, $L_{m1}[a , b]$, $L^p_{m1}[a , b]$, $L^\infty_{m1}[a , b]$, by $C_m[a , b]$, $AC_m[a , b]$, $L_{m1}[a , b]$, $L^p_m[a , b]$, $L^\infty_m[a , b]$ respectively. Also when $m = 1$ and $r = 1$, $C[a , b]$, $AC[a , b]$, $L[a , b]$, $L^p[a , b]$, $L^\infty[a , b]$ denote the respective classes. When there is no ambiguity as to the interval $[a , b]$ under consideration we omit the symbol $[a , b]$.

If the elements of a matrix $\mathcal{A}(t)$ are of $AC[a , b]$, then $\mathcal{A}'(t)$ signifies the matrix of derivatives where these derivatives exist, and the zero matrix elsewhere; correspondingly, if the elements of $\mathcal{A}(t)$ are of $L[a , b]$ then $\int_a^b \mathcal{A}(t)dt$ denotes the matrix of integrals of respective elements of $\mathcal{A}(t)$. If $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are equal a. e. on there domain of definition, we write simply $\mathcal{A}(t) = \mathcal{B}(t)$.

The class of vector functions $L_m^2[a\ ,\ b]$ are characterized as the space consisting of all m dimensional vector functions

$$f(x) = (f_\alpha(x)) \qquad (\alpha = 1, \ldots, m)$$

$x \in [a\ ,\ b]$ with real valued measurable coordinates $f_\alpha(x)$ $(\alpha = 1, \ldots, m)$ such that

$$\int_a^b \sum_{\alpha=1}^m \mid f_\alpha(x) \mid^2 dx \ < \ \infty.$$

The inner product in $L_m^2[a\ ,\ b]$ is defined in a natural way by the formula

$$(f, g) \ = \ \int_a^b \sum_{\alpha=1}^m \overline{g_\alpha} f_\alpha$$

for $f \ = \ (f_\alpha)$ and $g \ = \ (g_\alpha)$ belonging to $L_m^2[a\ ,\ b]$.Then $L_m^2[a\ ,\ b]$ is a Hilbert space with respect to this inner product.

$L_{m\times m}^2([a\ ,\ b] \times [a\ ,\ b])$ is the Hilbert space consisting of matrix functions

$$\mathcal{A}(x,t) = \mathcal{A}_{\alpha\beta}(x,t), \qquad (\alpha, \beta = 1, \ldots, m)$$

with elements measurable on $[a\ ,\ b] \times [a\ ,\ b]$ such that

$$\int_a^b \int_a^b \sum_{\alpha,\beta=1}^m \mid \mathcal{A}_{\alpha\beta}(x,t) \mid^2 dxdt \ < \ \infty.$$

The inner product of two elements $\mathcal{A}(x,t)$ and $\mathcal{B}(x,t)$ is defined by

$$(\mathcal{A},\mathcal{B}) = \int_a^b \int_a^b \sum_{\alpha,\beta=1}^{m} \mathcal{A}_{\alpha\beta}(x,t)\mathcal{B}_{\alpha\beta}(x,t)dxdt.$$

Next we consider the vector equation

$$\mathcal{A}(t)u \ = \ \lambda\mathcal{B}(t)u \tag{1.3}$$

for $t \in [a\ ,\ b]$ when $\mathcal{A}(t)$ is symmetric and $\mathcal{B}(t)$ is positive definite on $[a\ ,\ b]$. Let $\lambda_\alpha(t)$, $\alpha = 1,\ldots,m$, be a set of real values satisfying (1. 3) with corresponding vectors $u_\alpha(t)$. Also let $\{\lambda(t)\ ;\ u_\alpha(t)\}$ satisfy Theorem 1. 1. for each $t \in [a\ ,\ b]$. Then the inequality

$$|\ \lambda_\delta(t)\ | = \frac{|\ [\mathcal{B}^{-1}\mathcal{A}u_\delta(t), u_\delta(t)]\ |}{[u_\delta(t), u_\delta(t)]} \leq |\ \mathcal{B}^{-1}\mathcal{A}(t)\ | \leq \sum_{\alpha,\beta,\delta=1}^{m} |\ \mathcal{B}^{-1}_{\alpha\delta}\mathcal{A}_{\delta\beta}\ |, \tag{1.4}$$

holds for $t \in [a\ ,\ b]$ and $\delta = 1,\ldots,m$ where $\mathcal{B}^{-1}(t) = [\mathcal{B}^{-1}_{\alpha\beta}(t)]$.

Theorem 1.6 *Let $\mathcal{A}(t)$ be symmetric, $\mathcal{B}(t) > \mathcal{O}$ on $[a\ ,\ b]$ and $\mathcal{A}(t)$, $\mathcal{B}^{-1}(t)$ be elements of class $L^\infty_{mm}[a\ ,\ b]$. Then the functions $\lambda_\delta(t)$ which satisfy (1. 3) are of class $L^\infty[a\ ,\ b]$.*

Proof. This result follows from the inequality (1. 4).

In order to study the properties of matrix functions $\mathcal{A}_+(t)$, $\mathcal{A}_-(t)$ of a given symmetric matrix function $\mathcal{A}(t)$ we need the following lemmas.

Lemma 1.7 *Let $\mathcal{U}(t)$ be an $m \times m$ matrix of class $L^{\infty}_{mm}[a\ ,\ b]$ and $\|\mathcal{U}\|_{\infty} \leq 1$ then the matrix*

$$\mathcal{V}(t) = \mathcal{I} - \sum_{k=1}^{\infty} c_k \mathcal{U}^k(t) \tag{1.5}$$

is defined on $[a\ ,\ b]$ with $c_1 = 1/2$, $c_k = (1.\ 2\ldots(2k-3))/(k!2^k)$, $(k = 2, 3, \ldots)$ and

(i) $\mathcal{V}(t)$ *is of class* L^{∞}_{mm}

(ii) $\mathcal{V}^2(t) = \mathcal{I} - \mathcal{U}(t)$

(iii) *If* $\mathcal{U}(t) \geq \mathcal{O}$ *then* $\mathcal{V}(t) \geq \mathcal{O}$.

Proof. This can be obtained by slightly changing Lemma 1. 4. of [30] and so is omitted.

Lemma 1.8 *Let $\mathcal{A}(t)$ be nonnegative definite and of class $L^{\infty}_{mm}[a\ ,\ b]$. Then*

$$\mathcal{A}(t) \leq \|\mathcal{A}\|_{\infty}\mathcal{I}$$

on $[a\ ,\ b]$ where

$$\|\mathcal{A}\|_{\infty} = ess \sup_{t\in[a\ ,\ b]} |\lambda_1(\mathcal{A}(t))|\,.$$

Proof. Since $\mathcal{A}(t) \leq \lambda_1(\mathcal{A}(t))\mathcal{I}$ for $t \in [a\ ,\ b]$ by Theorem 1. 5, the result follows from Theorem 1. 6.

Theorem 1.9 *Let $\mathcal{A}(t)$ be an $m \times m$ nonnegative matrix function of class L^{∞}_{mm}. Then there exists a nonnegative matrix $\mathcal{B}(t)$ such that* *(i)* $\mathcal{B}^2(t) = \mathcal{A}(t)$

(ii) $\mathcal{B}(t)$ *is of class* L^{∞}_{mm}

(iii) *If* $\mathcal{C}(t)$ *is another nonnegative matrix such that* $\mathcal{C}^2(t) = \mathcal{A}(t)$ *then* $\mathcal{B}(t) = \mathcal{C}(t)$.

Proof. Since $\mathcal{A}(t)$ is of class L^{∞}_{mm} there exists a positive constant k such that $\mathcal{O} \leq k^2\mathcal{A}(t) < \mathcal{I}$. The symmetric matrix $\mathcal{U}(t) = \mathcal{I} - k^2\mathcal{A}(t)$ satisfies $\mathcal{O} \leq \mathcal{U}(t) < \mathcal{I}$ and $\mathcal{U}(t)$ is of class L^{∞}_{mm}. If $\mathcal{V}(t)$ is defined by (1. 5) then $\mathcal{B}(t) = 1/k\mathcal{V}(t)$ is a nonnegative matrix satisfying (i) and (ii). Since $\mathcal{B}$ is the limit of polynomials in $\mathcal{A}$ clearly $\mathcal{A}$ commutes with $\mathcal{B}$. Then $\mathcal{CA} = \mathcal{C}^3 = \mathcal{AC}$ and hence $\mathcal{CB} = \mathcal{BC}$. Consequently $(\mathcal{C}+\mathcal{B})(\mathcal{C}-\mathcal{B}) = \mathcal{C}^2 - \mathcal{B}^2 = \mathcal{O}$, and also if $\mathcal{C}(t) \geq \mathcal{O}$ then individually $\mathcal{C}(\mathcal{C}-\mathcal{B}) = \mathcal{O}$ and $\mathcal{B}(\mathcal{C}-\mathcal{B}) = \mathcal{O}$. Thus $(\mathcal{C}-\mathcal{B})^2 = \mathcal{O}$ and $\mathcal{C} = \mathcal{B}$.

Corollary 1.10 *Let* $\mathcal{A}(t)$ *be a symmetric matrix function of class* L^{∞}_{mm}. *Then matrix functions* $\mathcal{A}_+(t)$ *and* $\mathcal{A}_\pm(t)$ *are also symmetric matrix functions of class* L^{∞}_{mm}.

Proof. Theorem 1. 9 shows that the nonnegative square root matrix $\mid \mathcal{A}(t) \mid$ of $\mathcal{A}^2(t)$ is of L^{∞}_{mm}. Thus $\mathcal{A}_+(t) = \frac{1}{2}[\mid \mathcal{A}(t) \mid + \mathcal{A}(t)]$ and $\mathcal{A}_-(t) = \frac{1}{2}[\mid \mathcal{A}(t) \mid - \mathcal{A}(t)]$ are symmetric and of L^{∞}_{mm}.

Theorem 1.11 *Let* $\mathcal{A}(t)$ *be symmetric and* $a_1(t) \geq a_2(t) \geq \ldots \geq a_m(t)$ *be its eigenvalues satisfying Theorem 1. 1 for* $t \in [a\ ,\ b]$. *Then*

$$a_1^+(t) \geq a_2^+(t) \geq \ldots \geq a_m^+(t) \geq 0$$

are the eigenvalues of $\mathcal{A}_+(t)$ *and*

$$a_m^-(t) \geq a_{m-1}^-(t) \geq \ldots \geq a_1^-(t) \geq 0$$

are the eigenvalues of $\mathcal{A}_-(t)$, *where* $a_i^+ = max\,(a_i\,,\,0)$ *and* $a_i^- = max\,(-a_i\,,\,0)$.

Proof. Since $\mathcal{A}(t)$ is symmetric there is an orthogonal matrix $\mathcal{S}(t)$ such that $\mathcal{S}^*\mathcal{A}\mathcal{S} = \Lambda = \text{diag}\,\{a_1, \ldots, a_m\}$. Then $\mathcal{A}^2 = \mathcal{S}\Lambda^2\mathcal{S}^*$ implies that $\mid \mathcal{A} \mid = \mathcal{S} \mid \Lambda \mid \mathcal{S}^*$ and hence $\mathcal{A}_+ = \frac{1}{2}\mathcal{S}\{\Lambda + \mid \Lambda \mid\}\mathcal{S}^*$. Therefore $\mathcal{S}^*\mathcal{A}_+\mathcal{S} = \text{diag}\{a_1^+, \ldots, a_m^+\}$ showing that $a_1^+, \ldots, a_m^+$ are the eigenvalues of $\mathcal{A}_+$. Similarly $a_1^-, \ldots, a_m^-$ are the eigenvalues of $\mathcal{A}_-$. The rest of the theorem follows from the fact that if $a(t) \geq b(t)$ then $a^+(t) \geq b^+(t)$ and $b^-(t) \geq a^-(t)$.

Corollary 1.12 *Let* $\mathcal{A}(t)$ *be as in Theorem 1. 11, then*

$$a_m^+(t)\mathcal{I} \leq \mathcal{A}_+(t) \leq a_1^+(t)\mathcal{I}$$

and

$$a_1^-(t)\mathcal{I} \leq \mathcal{A}_-(t) \leq a_m^-(t)\mathcal{I}.$$

Proof. This is immediate from Theorems 1. 5 and 1. 11.

Some other elementary properties of eigenvalues and eigenvectors of matrix functions can be found in [32].

Next we introduce a special class of matrix functions called *functionally commutative matrices.* An $m \times m$ matrix function $\mathcal{A}(t)$ for $t \in [a\,,\,b]$ is said to be functionally commutative in case

$$\mathcal{A}(t)\mathcal{A}(s) - \mathcal{A}(s)\mathcal{A}(t) = \mathcal{O}$$

for all $t,\ s,\ \in [a\ ,\ b]$.

If $\mathcal{A}(t)$ is symmetric and functionally commutative then there exists a constant nonsingular matrix $\mathcal{U}$ such that $\mathcal{U}^{-1}\mathcal{A}(t)\mathcal{U} = \mathcal{J}(t)$ where $\mathcal{J}(t) = diag(a_1(t), \ldots, a_m(t))$ and $a_i(t)$ is as in Theorem 1. 11, (see [12]). Then as in the proof of Theorem 1. 11 we have

$$\mathcal{U}^{-1}\mathcal{A}_+(t)\mathcal{U} = \mathcal{J}_+(t) \quad and \quad \mathcal{U}^{-1}\mathcal{A}_-(t)\mathcal{U} = \mathcal{J}_-(t)$$

where $\mathcal{J}_+(t) = diag(a_1^+(t), \ldots, a_m^+(t))$ and $\mathcal{J}_-(t) = diag(a_1^-(t), \ldots, a_m^-(t))$.

1.3 s-Numbers of Compact Operators and their Properties

In this section all the operators are assumed to be defined on $L_m^2[a\ ,\ b]$.Also we assume definitions and basic properties of operators. Let $\mathcal{A}$ be a compact operator, then

$$\mathcal{H} = (\mathcal{A}^*\mathcal{A})^{1/2}$$

is also a compact operator [14]. The norm of an operator $\mathcal{A}$ is denoted simply by $|\mathcal{A}|$.

The eigenvalues of the operator $\mathcal{H}$ are called the $s-numbers$ of the operator $\mathcal{A}$.

We shall enumerate the nonzero s-numbers in decreasing order, taking acount of their multiplicities, so that

$$s_j(\mathcal{A}) = \lambda_j(\mathcal{H})$$

where $\lambda_j(\mathcal{H})$ is the j th eigenvalue of $\mathcal{H}$ obtained by the max-min principle (see [14] page 25).

We note that

$$s_1(\mathcal{A}) = \mid \mathcal{A} \mid .$$

If $\mathcal{A}$ is a compact self adjoint operator then

$$s_j(\mathcal{A}) = \mid \lambda_j(\mathcal{A}) \mid \qquad (j = 1, 2, \ldots\ldots).$$

It is also obvious that for any scalar c

$$s_j(c\mathcal{A}) = \mid c \mid s_j(\mathcal{A}) \qquad (j = 1, 2, \ldots\ldots).$$

Theorem 1.13 *Let $\mathcal{A}$ be a compact operator, then*

(i) $\quad s_j(\mathcal{A}) = s_j(\mathcal{A}^*) \qquad (j = 1, 2, \ldots\ldots)$

(ii) $\quad$ *For any bounded operator $\mathcal{B}$,*

$$s_j(\mathcal{B}\mathcal{A}) \leq \mid \mathcal{B} \mid s_j(\mathcal{A}) \qquad (j = 1, 2, \ldots\ldots),$$

$$s_j(\mathcal{AB}) \leq | \mathcal{B} | s_j(\mathcal{A}) \qquad (j = 1, 2, \ldots\ldots).$$

Proof. See ([14] section 2. 1).

Theorem 1.14 *Let $\mathcal{A}$ and $\mathcal{B}$ be compact operators, then*

$$s_{m+n-1}(\mathcal{A}+\mathcal{B}) \leq s_m(\mathcal{A}) + s_n(\mathcal{B}) \qquad (m, n = 1, 2, \ldots\ldots)$$

and

$$s_{m+n-1}(\mathcal{AB}) \leq s_m(\mathcal{A}) s_n(\mathcal{B}) \qquad (m, n = 1, 2, \ldots\ldots).$$

Proof. See [11].

Next we have an asymptotic theorem concerning s-numbers.The proof of this theorem is due to Fan [11].

Theorem 1.15 *Let $\mathcal{A}$ and $\mathcal{B}$ be compact operators and that for some $r > 0$*

$$\lim_{n\to\infty} n^r s_n(\mathcal{A}) = a \quad \text{and} \quad \lim_{n\to\infty} n^r s_n(\mathcal{B}) = 0.$$

Then

$$\lim_{n\to\infty} n^r s_n(\mathcal{A}+\mathcal{B}) = a.$$

Proof. From Theorem 1. 14 it follows that

$$s_{(k+1)m+j}(\mathcal{A}+\mathcal{B}) \le s_{km+j}(\mathcal{A}) + s_{m+1}(\mathcal{B})$$

$(m = 1, 2 \dots; j = 1, \dots, k)$.

Since any integer n admits the representation $n = (k+1)m + j$, it follows that

$$\overline{\lim_{n\to\infty}}\; n^r s_n(\mathcal{A}+\mathcal{B}) \le ((k+1)/k)^r a$$

and therefore, since k is arbitrary,

$$\overline{\lim_{n\to\infty}}\; n^r s_n(\mathcal{A}+\mathcal{B}) \le a. \tag{1.6}$$

On the other hand, $\mathcal{A} = (\mathcal{A}+\mathcal{B}) - \mathcal{B}$ and consequently

$$s_{(k+1)m+j}(\mathcal{A}) \le s_{km+j}(\mathcal{A}+\mathcal{B}) s_{m+1}(\mathcal{B}),$$

or

$$s_{km+j}(\mathcal{A}+\mathcal{B}) \ge s_{(k+1)m+j}(\mathcal{A}) - s_{m+1}(\mathcal{B}).$$

setting $n = km + j$ and letting n go to infinity, we obtain

$$\underline{\lim_{n\to\infty}}\; n^r s_n(\mathcal{A}+\mathcal{B}) \ge (k/(k+1))^r a,$$

or, since k is arbitrary,

$$\lim_{n\to\infty} n^r s_n(\mathcal{A}+\mathcal{B}) \ge a. \tag{1.7}$$

From (1. 6) and (1. 7) there follows the theorem.

Let $\mathcal{A}$, an operator acting on some suitable dense subspace of $L^2_m[a, b]$, be defined by

$$\mathcal{A}y = \int_a^b \mathcal{A}(t,s)y(s)ds \tag{1.8}$$

for $y \in L^2_m$, where $\mathcal{A}(t,s) = [\mathcal{A}_{ij}(t,s)]$. Then the adjoint $\mathcal{B} = \mathcal{A}^*$ of $\mathcal{A}$ is defined by

$$\mathcal{B}y = \int_a^b \mathcal{B}(t,s)y(s)ds$$

with $\mathcal{B}(t,s) = [\mathcal{B}_{ij}(t,s)]$ where

$$\mathcal{B}_{ij}(t,s) = \overline{\mathcal{A}_{ji}(s,t)} \qquad (i,j = 1,\ldots,m).$$

Theorem 1.16 *Let $\mathcal{A}$ be defined by (1.8), then $\mathcal{A}$ is a Hilbert- Schmidt operator if and only if $\mathcal{A} \in L^2_{mm}[a\,;\,b] \times [a\,,\,b]$. Moreover, Hilbert-Schmidt operators are compact operators.*

Proof. See ([14] Section III. 9).

Chapter 2

Right-Definite Boundary Problems

In this chapter we shall consider the vector differential equation

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \lambda\mathcal{R}(t)y \tag{2.1}$$

on the compact interval $[a, b]$. Here $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are real valued $m \times m$ matrix functions; $y(t) = (y_\alpha(t))$ is an m dimensional vector function and λ is a parameter. Throughout this chapter the matrix coefficients $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ satisfy the conditions

$\mathcal{H}_1$ *The matrix functions $\mathcal{P}(t), \mathcal{Q}(t)$ are symmetric, $\mathcal{P}(t)$ is positive definite and $\mathcal{P}^{-1}(t), \mathcal{Q}(t)$ are of class $L^\infty_{mm}[a, b]$.*

$\mathcal{H}_2$ *The matrix function $\mathcal{R}(t)$ is symmetric and of class $L^\infty_{mm}[a, b]$ with $\mathcal{R}(t) \geq \mathcal{O}$ on $[a, b]$. Also, the set $\{t;\ t \in [a, b],\ \mathcal{R}(t) > \mathcal{O}\}$ is of positive measure.*

Under these conditions, given any point $t_0 \in [a, b]$ and real m dimensional vectors ξ and η, there is a unique solution y of (2. 1) which exists on $[a\,,\,b]$ such that y and $\mathcal{P}y'$ are absolutely continuous on $[a\,,\,b]$, $y(t_0)=\xi$, $(\mathcal{P}y')(t_0)=\eta$ and (2. 1) is satisfied a. e. on $[a\,,\,b]$ (see, e. g. ,[29], section 15 and 16]). When the parameter λ is real this solution is always real valued.

In Section 1 we shall define the *right-definite* Sturm-Liouville boundary problems associated with the differential equation (2. 1) and the Dirichlet boundary conditions

$$y(a) \;=\; 0 \;=\; y(b). \tag{2.2}$$

and then discuss the properties of λ in such problems.

We devote Section 2 to extend some eigenvalue distribution results of I. C. Gohberg and M. G. Krein [15] to a general right-definite Sturm-Liouville problem.This will be done by introducing an integral operator corresponding to the problem (2. 1–2) and studying the s-numbers of that operator.

2.1 Existence of Eigenvalues

According to the terminology of Everitt [8] boundary problems of the type (2. 1–2) are called *right-definite* if $\mathcal{R}(t) \geq \mathcal{O}$. Moreover, we say the problem (2. 1–2) is *regular* if for no solution $y_0(t)$ ($\not\equiv 0$) of the equation $-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = 0$ the vector function $\mathcal{R}(t)y_0(t)$ is almost everywhere equal to zero.

The condition $\mathcal{H}_2$ will guarantee that problem (2. 1–2) is regular right-definite. We note that if $\mathcal{Q}(t) \geq \mathcal{O}$ in (2. 1) then $\mathcal{H}_2$ can be relaxed to

$\mathcal{H}_2'$ *The matrix function $\mathcal{R}(t)$ is symmetric and of class $L^{\infty}_{mm}[a\,,\,b]$ with $\mathcal{R}(t) \geq \mathcal{O}$ on $[a\,,\,b]$. Also, the set $\{t;\; t \in [a\,,\,b],\; \mathcal{R}(t) \neq \mathcal{O}\}$ is of positive measure,*

which will ensure the regularity of the right-definite problem (see [31] and [34]).

On the other hand if $\mathcal{Q}(t)$ is arbitrary then the condition $\mathcal{H}_2'$ is not sufficient to guarantee that problem (2. 1–2) is regular. For example, consider the scalar Sturm-Liouville problem

$$-u'' = \lambda r(t) u$$

$$u(a) = 0 = u(b),$$

where $r(t)$ is continuous and changes sign in $[a\,,\,b]$, and let λ_0 be an eigenvalue with eigenfunction $u_0(t)$. Then the vector problem (2. 1–2) with $\mathcal{P}(t) = \mathcal{I}_2$,

$$\mathcal{Q} = \begin{pmatrix} -\lambda_0 r & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

has a solution

$$y_0(t) = \begin{bmatrix} u_0(t) \\ 0 \end{bmatrix}$$

satisfying $-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = 0$ and $\mathcal{R}(t)y = 0$ showing (2. 1–2) is not regular.

The regular Sturm-Liouville problem with Dirichlet boundary conditions consists in finding the values of a parameter λ for which equation (2. 1) has a solution y (nonidentically zero vector) satisfying the boundary conditions (2. 2). If such value λ with corresponding vector function y exists we say λ is an *eigenvalue* and y is an *eigenfunction* of the problem (2. 1–2).

Theorem 2.1 *All the eigenvalues of the right definite Sturm-Liouville problem (2. 1–2) are real.*

Proof. Suppose λ is an eigenvalue of (2. 1–2) and $y = y_1 + \imath y_2$ is a corresponding eigenfunction with y_1, y_2 real valued vector functions. Then taking L^2_m inner product of y and (2. 1)

$$-\left((\mathcal{P}y')', y\right) + (\mathcal{Q}y, y) \;=\; \lambda(\mathcal{R}y, y). \tag{2.3}$$

Now integration by parts and use of (2. 2) yield the result

$$((\mathcal{P}y')', y) \;=\; -(\mathcal{P}y', y').$$

Also

$$\begin{aligned}(\mathcal{P}y', y') &= (\mathcal{P}y_1', y_1') + \imath(\mathcal{P}y_1', y_2') - \imath(\mathcal{P}y_2', y_1') + (\mathcal{P}y_2', y_2') \\ &= (\mathcal{P}y_1', y_1') + (\mathcal{P}y_2', y_2')\end{aligned}$$

since $\mathcal{P}(t)$ is real valued and symmetric. Therefore $(\mathcal{P}y', y')$ is real. Similarly $(\mathcal{Q}y, y)$ and $(\mathcal{R}y, y)$ are also real. Then the equation (2. 3) implies that $(\text{Im } \lambda)(\mathcal{R}y, y)$ must be equal to zero. But for no eigenfunction y can $(\mathcal{R}y, y)$ be zero. Therefore $\text{Im}\lambda = 0$ and λ is real.

Therefore we can find some λ_0, possibly complex, which is not an eigenvalue of (2. 1–2). For such λ_0 there exists a continuous matrix function $\mathcal{G}(t, s, \lambda_0)$ called *Green's matrix function* with the property that if λ is an eigenvalue of (2. 1–2) then

$$y(t) = (\lambda - \lambda_0) \int_a^b \mathcal{G}(t, s, \lambda_0)\mathcal{R}(s)y(s)ds \tag{2.4}$$

The fundamental properties of $\mathcal{G}(t, s, \lambda_0)$ and results related to integral equation (2. 4) can be found in [31], [34], [2].

Lemma 2.2 *The integral operator $G : L_m^2 \longrightarrow L_m^2$ defined by*

$$Gy = \int_a^b \mathcal{G}(t, s, \lambda_0)\mathcal{R}(s)y(s)ds \tag{2.5}$$

is compact.

Proof. Since $\mathcal{G}$ is continuous and $\mathcal{R} \in L_{mm}^\infty[a\,,\, b]$ the product matrix $\mathcal{G}(t, s, \lambda_0)\mathcal{R}(s) \in L_{mm}^2[a\,,\, b] \times [a\,,\, b]$. Now from Theorem 1. 16 operator G is compact.

Theorem 2.3 *The eigenvalues of right-definite Sturm-Liouville problem (2. 1–2) form a discrete set with no finite point of accumulation.*

Proof. Since G is compact its eigenvalues form a discrete set and zero is the only possible accumulation point. Now the theorem follows from (2. 4).

Next, we state the following well known result (see [7], [15], [31]).

Courant-Hilbert min-max principle

Let

$$J[y] = \{(\mathcal{P}y', y') + (\mathcal{Q}y, y)\},$$

$$\mathcal{D} = \left\{ y \in L^2_m[a\ ,\ b];\ y, \mathcal{P}y' \in AC_m[a\ ,\ b], (\mathcal{P}y')' \in L^2_m \text{ and } y(a) = 0 = y(b) \right\},$$

and

$$\mathcal{U} = \{y \in \mathcal{D};\ (\mathcal{R}y, y) = 1\}.$$

Also let $\lambda_1 \leq \lambda_2 \leq \ldots\ldots$ *be the complete sequence of eigenvalues of (2. 1–2). Then*

$$\lambda_n = \max_{\{p_1,\ldots,p_{n-1}\} \subset \mathcal{D}} \ \min_{\substack{y \in \mathcal{U} \\ (\mathcal{R}y, p_i) = 0 \\ i = 1,\ldots,n-1}} J[y].$$

if the following additional hypothesis is assumed.

$\mathcal{H}$. *There exists a real number* k_0 *such that* $J[y] - k_0(\mathcal{R}y, y)$ *is positive definite on* $\mathcal{D}$.

The following two corollaries are immediate consequences of the preceding principle.

Corollary 2.4 *Let $\{\lambda_i \ ; \ y_i\}$ be the sequence of eigenvalues and eigenfunctions corresponding to the functional $J[y]$. Then*

(i) $\{\alpha\lambda_i \ ; \ y_i\}$ is the sequence of eigenvalues and eigenfunctions corresponding to the functional $\alpha J[y]$ for a positive real number α.

(ii) $\{\lambda_i - \alpha \ ; \ y_i\}$ is the sequence of eigenvalues and eigenfunctions corresponding to the functional $J[y] - \alpha(\mathcal{R}y, y)$ for a real number α.

Corollary 2.5 *Let $\lambda_1^{(i)} \le \lambda_2^{(i)} \le \ldots\ldots$ be the complete sequence of eigenvalues of the problem*

$$-(\mathcal{P}_i(t)y')' + \mathcal{Q}_i(t)y = \lambda\mathcal{R}(t)y \qquad (2.6)$$

with boundary conditions (2. 2) for i=1,2. Also let

$$J_i[y] = \{(\mathcal{P}_i y', y') + (\mathcal{Q}_i y, y)\}.$$

If $J_1[y] \le J_2[y]$ for every vector function y in $\mathcal{U}$, then $\lambda_n^{(1)} \le \lambda_n^{(2)}$ for every n.

Theorem 2.6 *Let $\mu_1^{(i)} \le \mu_2^{(i)} \le \ldots\ldots$ be the complete sequence of eigenvalues of the problem*

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \mu\mathcal{R}_i(t)y$$

with boundary conditions (2. 2) for $i = 1, 2$ where both $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy $\mathcal{H}_2$. Also let $\mathcal{R}_1(t) \ge \mathcal{R}_2(t)$ on $[a\ ,\ b]$.

If $\mu_N^{(2)} < 0 \le \mu_{N+1}^{(2)}$ then $\mu_i^{(2)} \le \mu_i^{(1)} \le 0$ for $i = 1, \ldots, N$ and $0 \le \mu_i^{(1)} \le \mu_i^{(2)}$ for $i = N+1, N+2, \ldots\ldots$

Proof. This is a restatement of some Sturm separation and comparison theorems in [27] and [[28] Chapter IV] and so is omitted.

Thus in order to obtain the eigenvalues of (2. 1–2) from the min-max principle it remains to show that the eigenvalues of (2. 1–2) are bounded below.

Note that if $\mathcal{Q}(t) \geq \mathcal{O}$ on $[a\ ,\ b]$ then $J[y]$ is nonnegative on $\mathcal{D}$ and therefore $\lambda(\mathcal{R}y, y) = J[y] \geq 0$ for any eigenvalue λ of (2. 1–2). Hence λ is nonnegative. Next we consider the case that $\mathcal{Q}(t)$ is not necessarily nonnegative definite on $[a\ ,\ b]$. In this case we can assume without loss of genarality that the set $\{t : t \in [a\ ,\ b], \mathcal{Q}_-(t) \neq 0\}$ is of positive measure.

We now consider the boundary problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}_+(t)y + \lambda\mathcal{R}(t)y = \mu(\lambda)\mathcal{Q}_-(t)y \qquad (2.\ 7)_\lambda$$

with boundary conditions (2. 2). Then all the eigenvalues of $(2.\ 7)_\lambda$ are positive when λ is nonnegative and consequently the min-max principle can be used to obtain these eigenvalues $\mu(\lambda)$. Let us denote the eigenvalues of $(2.\ 7)_\lambda$-(2. 2) by

$$0 < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \ldots\ldots$$

for $\lambda \geq 0$.

When $\lambda = 0$ the equation $(2.\ 7)_\lambda$ is

$$-(\mathcal{P}(t)y')' + \mathcal{Q}_+(t)y = \mu\mathcal{Q}_-(t)y. \qquad (2.\ 7)_0$$

Then in view of Theorem 2. 3 the number of eigenvalues of the problem $(2.\ 7)_0-(2.\ 2)$ which lie in the interval [0 , 1] is finite.Let us denote this number by N.

Lemma 2.7 *Let $\{\mu_i\}_{i=1}^{\infty}$ be the sequence of eigenvalues of $(2.\ 7)_0-(2.\ 2)$ then*

$$\sum_{i=1}^{\infty}\frac{1}{\mu_i}=\int_a^b Tr\,\mathcal{H}(s,s)\mathcal{Q}_-(s)ds \tag{2.8}$$

where $\mathcal{H}(x,s)$ is the Green's matrix function of

$$-(\mathcal{P}(t)y')'+\mathcal{Q}_+(t)y=0$$

$$y(a)=0=y(b).$$

Proof. See [33].

Lemma 2.8 *Let $0<\mu_1\leq\mu_2\leq\ldots\leq\mu_N\leq 1<\mu_{N+1}\leq\ldots\ldots$ be the sequence of eigenvalues of $(2.\ 7)_0-(2.\ 2)$ then*

$$N\leq\int_a^b Tr\,\mathcal{H}(s,s)\mathcal{Q}_-(s)ds. \tag{2.9}$$

Proof. From (2. 8)

$$\sum_{i=1}^{N}\frac{1}{\mu_i}+\sum_{i=N+1}^{\infty}\frac{1}{\mu_i}=\int_a^b Tr\,\mathcal{H}(s,s)\mathcal{Q}_-(s)ds. \tag{2.10}$$

Note that

$$N\leq\sum_{i=1}^{N}\frac{1}{\mu_i} \tag{2.11}$$

since $0<\mu_i\leq 1$ for $i=1,\ldots,N$.Then the lemma follows from (2. 10) since the left side of (2. 10) is certainly greater than $\sum_{i=1}^{N}1/\mu_i$.

Theorem 2.9 *The right-definite Sturm-Liouville problem (2. 1–2) can have at most N negative eigenvalues, where N is as in Lemma 2. 8.*

Proof. Suppose $-\lambda_1 \geq -\lambda_2 \geq \ldots\ldots$ are negative eigenvalues of (2. 1–2) with $\lambda_i > 0$, $i = 1, 2, \ldots\ldots$. Then rewriting (2. 1) we have

$$-(\mathcal{P}(t)y')' + \mathcal{Q}_+(t)y + \lambda_i \mathcal{R}(t)y \ = \ \mathcal{Q}_-(t)y.$$

Therefore, for evey i, $\mu_{m_i}(\lambda_i) = 1$ for some m_i, $i = 1, 2, \ldots\ldots$, with the convention that $m_i > m_{i+1}$ if $\lambda_i = \lambda_{i+1}$.

If $m_1 > N$ then using Corollary 2. 5 we have $\mu_{m_1}(0) \leq \mu_{m_1}(\lambda_1) = 1$ for $m_1 > N$. This is a contradiction as $(2.\ 7)_0$–(2. 2) has only N eigenvalues in the interval $[0\ ,\ 1]$.Therefore $m_1 \leq N$. Next we will prove that $m_2 < m_1$. If $\lambda_2 = \lambda_1$ then by the convetion $m_2 < m_1$. So we suppose $\lambda_2 > \lambda_1$. Applying Corollary 2. 5 to problem $(2.\ 7)_{\lambda_1}$ and $(2.\ 7)_{\lambda_2}$ we have $\mu_{m_2}(\lambda_1) \leq \mu_{m_2}(\lambda_2)$. On the other hand $\mu_{m_2}(\lambda_2) = \mu_{m_1}(\lambda_1) = 1$ .Thus $\mu_{m_2}(\lambda_1) \leq \mu_{m_1}(\lambda_1)$ and hence $m_2 \leq m_1$. If $m_1 = m_2$ then $\mu_{m_1}(\lambda_1) = \mu_{m_1}(\lambda_2) = 1$ and consequently, by Corollary 2. 5, for every λ in the interval $[\lambda_1\ ,\ \lambda_2]$

$$1 = \mu_{m_1}(\lambda_1) \leq \mu_{m_1}(\lambda) \leq \mu_{m_1}(\lambda_2) = 1.$$

i. e. , $\mu_{m_1}(\lambda) = 1$ for every λ in $[\lambda_1\ ,\ \lambda_2]$. Now rewriting $(2.\ 7)_\lambda$ with $\mu_{m_1}(\lambda) = 1$ we have every $-\lambda$ in the interval $[-\lambda_2\ ,\ -\lambda_1]$ is an eigenvalue of (2. 1–2). This contradicts Theorem 2. 3 and therefore $m_2 < m_1$.

Now, if (2. 1–2) has $N+1$ negative eigenvalues, after repeating above argument at most N times we have $0 < m_{N+1} < 1$. This is a contradiction again as m_{N+1} must be an integer. Therefore (2. 1–2) has at most N negative eigenvalues.

Corollary 2.10 *If the number of negative eigenvalues of the right definite Sturm-Liouville problem (2. 1-2) is N then*

$$N \leq \int_a^b Tr\,\mathcal{H}(s,s)\mathcal{Q}_-(s)ds.$$

In view of Theorem 2. 9 we have the following

Theorem 2.11 *The eigenvalues of the right definite Sturm-Liouville problem (2. 1–2) are bounded below.*

2.2 Asymptotic Distribution of the Eigenvalues

In this section we shall prove the following

Theorem 2.12 *Let $n(r)$ be the number of eigenvalues of the right-definite problem (2. 1-2) which lie in the interval $[0\ ,\ r]$. Then $n(r)$ satisfies the asymptotic formula*

$$\lim_{r\to\infty} \frac{n(r)}{\sqrt{r}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t))dt \qquad (2.12)$$

where

$$\Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t)) = \sum_{j=1}^{m}\sqrt{\omega_j(t)}$$

($\sqrt{\omega}_j \geq 0$; $j = 1, 2, \ldots, m$) and $\omega_1(t) \geq \ldots \geq \omega_m(t)\ (\geq 0)$ are the roots of the equation $det(\mathcal{R}(t) - \omega\mathcal{P}(t)) = 0$.

This theorem is an extention of the following result of I. C. Gohberg and M. G. Krein.

Theorem 2.13 *Let $n_0(r)$ be the number of eigenvalues of the right-definite problem (2. 1-2) with $Q(t) \equiv 0$ which lie in the interval $[0, r]$. Then $n_0(r)$ satisfies the asymptotic formula*

$$\lim_{r \to \infty} \frac{n_0(r)}{\sqrt{r}} = \frac{1}{\pi} \int_a^b \Lambda_{1/2}(P^{-1}(t)\mathcal{R}(t))dt. \tag{2.13}$$

Proof. See ([15] sec. VI. 8. 3).

Let

$$0 < \lambda_1' \leq \lambda_2' \leq \ldots\ldots$$

be the sequence of eigenvalues of (2. 1–2) with $Q(t) \equiv 0$ and

$$-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{i_0} < 0 \leq \lambda_{i_0+1} \leq \ldots\ldots$$

be the sequence of eigenvalues of (2. 1–2). The existence of such an integer i_0 follows from Theorem 2. 9.

If $\{x_m\}$ is a sequence such that $x_m \nearrow \infty$ as $m \longrightarrow \infty$, then for every $m \geq M$ (sufficiently large) we find some integer $k(m)$ with the property that

$$0 < \lambda_{k(m)-1} \leq x_m \leq \lambda_{k(m)}. \tag{2.14}$$

However,

$$n(\lambda_{k(m)-1}) \leq n(x_m) \leq n(\lambda_{k(m)}).$$

i. e. ,

$$(k(m) - 1 - i_0) \leq n(x_m) \leq (k(m) - i_0). \tag{2.15}$$

Now combining (2. 14) and (2. 15) we have

$$\frac{k(m) - (i_0 + 1)}{\lambda_{k(m)}^{1/2}} \leq \frac{n(x_m)}{x_m^{1/2}} \leq \frac{k(m) - i_0}{\lambda_{k(m)-1}^{1/2}}$$

for $m \geq M$. Consequently, the proof of Theorem 2. 12 follows from the next theorem which will be proved later.

Theorem 2.14 *The eigenvalues λ_n of the right-definite problem (2. 1-2) have the property*

$$\lim_{n\to\infty}\frac{n}{\lambda_n^{1/2}}=\frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t))dt. \tag{2.16}$$

In order to prove Theorem 2. 14 we first assume that $\lambda = 0$ is not an eigenvalue of the problem (2. 1–2). This assumption is not restrictive as the translation $\lambda \to \lambda + c$ where $c \in R$, $c \neq 0$ is sufficiently small, transform (2. 1) into a problem of the same form wherein $\lambda = 0$ is not an eigenvalue. We then observe that $k = 1$ is not an eigenvalue of the boundary problem

$$-(\mathcal{P}(t)y')' = -k\mathcal{Q}(t)y \tag{2.17}$$

with boundary conditions (2. 2).Moreover, when $-k$ is not an eigenvalue of this problem there exist an $m \times m$ Green's matrix function $\mathcal{G}(t,s;k)$ such that the unique solution of nonhomogeneous problem

$$-(\mathcal{P}(t)y')' = -k\mathcal{Q}(t)y + f \tag{2.18}$$

with boundary conditions (2. 2) is given by

$$y(t)=\int_a^b \mathcal{G}(t,s;k)f(s)ds$$

for any vector function $f \in L_m^\infty[a\ ,\ b]$, where the matrix function $\mathcal{G}(t,s;k)$ has the property that

$$\mathcal{G}(t,s;k)=[\mathcal{G}(s,t;k)]^*$$

(see [2], [31] Chapter VI). Therefore every solution of the problem (2. 1–2. 2) is given by

$$y(t) = \lambda \int_a^b \mathcal{G}(t,s;1)\mathcal{R}(s)y(s)ds. \tag{2.19}$$

Since for each $t \in [a\ ,\ b]$ the matrix $\mathcal{R}(t)$ is nonnegative definite,there exists for each $t \in [a\ ,\ b]$ a unique matrix of the same type whose square is $\mathcal{R}(t)$. We denote this matrix by $\mathcal{R}^{1/2}(t)$. Then by Theorem 1. 8, $\mathcal{R}^{1/2}(t) \in L^{\infty}_{mm}$.

If in (2. 19) we make the substitution

$$z(t) = \mathcal{R}^{1/2}(t)y(t) \tag{2.20}$$

and multiply the resulting equation from the left by $\mathcal{R}^{1/2}(t)$, we obtain an integral equation in $L^2_m[a\ ,\ b]$,

$$z(t) = \lambda \int_a^b \mathcal{R}^{1/2}(t)\mathcal{G}(t,s;1)\mathcal{R}^{1/2}(s)z(s)ds, \tag{2.21}$$

with the symmetric Hilbert-Schmidt matrix kernel

$$\mathcal{R}^{1/2}(t)\mathcal{G}(t,s;1)\mathcal{R}^{1/2}(s). \tag{2.22}$$

Conversely, if $z(t)$ ($\in L^2_m[a\ ,\ b]$) is a solution of (2. 21), then it follows that $z(t)$ must have the form (2. 20), where

$$y(t) = \lambda \int_a^b \mathcal{G}(t,s;1)\mathcal{R}^{1/2}(s)z(s)ds. \tag{2.23}$$

Thus two equations (2. 19) and (2. 21) are equivalent (see [15] page 257).

We shall next define the integral operator $\mathcal{A}_k$ on $L^2_m[a, b]$ by

$$\mathcal{A}_k y(t) = \int_a^b \mathcal{R}^{1/2}(t)\mathcal{G}(t,s;k)\mathcal{R}^{1/2}(s)y(s)ds. \tag{2.24}$$

Then $\mathcal{A}_k$ is a self adjoint compact operator as its kernel is symmetric and Hilbert-Schmidt.

Since $k = 0$ is not an eigenvalue of the problem (2. 17–2. 2) the preceding discussion is also valid when $k = 0$. Thus, in particular, we have the self adjoint compact operator $\mathcal{A}_0$ on $L^2_m[a , b]$ defined by

$$\mathcal{A}_0 y(t) = \int_a^b \mathcal{R}^{1/2}(t)\mathcal{G}(t,s;0)\mathcal{R}^{1/2}(s)y(s)ds. \tag{2.25}$$

Then the eigenvalues of $\mathcal{A}_0$ are the reciprocals of the eigenvalues of problem (2. 1–2) with $\mathcal{Q}(t) \equiv \mathcal{O}$. Therefore

$$s_n(\mathcal{A}_0) = \frac{1}{\lambda'_n} \tag{2.26}$$

Lemma 2.15 *The s-numbers of the operator $\mathcal{A}_0$ have the property*

$$\lim_{n\to\infty} n^2 s_n(\mathcal{A}_0) = \left(\frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t))dt\right)^2. \tag{2.27}$$

Proof. This follows from Theorem 2. 13 by taking the limit in (2. 13) through the sequence $\{\lambda_n'\}$.

Lemma 2.16 *Green's function $\mathcal{G}(t,s;\cdot)$ satisfies the resolvent equation*

$$\mathcal{G}(t,s;k) - \mathcal{G}(t,s;l) = (l-k)\int_a^b \mathcal{G}(t,\xi;l)\mathcal{Q}(\xi)\mathcal{G}(\xi,s;k)d\xi \tag{2.28}$$

if k and l are not eigenvalues of boundary problem (2. 17)-(2. 2).

Proof. For $f \in L_m^2[a\ ,\ b]$, the solution of problem (2. 18)-(2. 2) is

$$y(t) = \int_a^b \mathcal{G}(t,s;k)f(s)ds.$$

On the other hand since $k = l$ is not an eigenvalue of (2. 17)-(2. 2) this y may be also written as

$$\begin{aligned} y(t) &= \int_a^b \mathcal{G}(t,s;l)\left\{(l-k)\mathcal{Q}(s)y(s) + f(s)\right\}ds \\ &= \int_a^b \mathcal{G}(t,s;l)f(s)ds + (l-k)\int_a^b \mathcal{G}(t,\xi;l)\mathcal{Q}(\xi)\int_a^b \mathcal{G}(\xi,s;k)f(s)ds. \end{aligned}$$

Combining these two results we have

$$\int_a^b \left\{\mathcal{G}(t,s;k) - \mathcal{G}(t,s;l) - (l-k)\int_a^b \mathcal{G}(t,\xi;l)\mathcal{Q}(\xi)\mathcal{G}(\xi,s;k)d\xi\right\} f(s)ds = 0.$$

Since this is true for every $f \in L_m^2[a\ ,\ b]$ there must hold the identity (2. 28).

Since the two integral operators $\mathcal{A}_1$ and $\mathcal{A}_0$ have the same domain $L_m^2[a\ ,\ b]$ we shall consider the integral operator $\mathcal{A}_1 - \mathcal{A}_0$ which is given by

$$(\mathcal{A}_1 - \mathcal{A}_0)y(t) = \int_a^b \mathcal{R}^{1/2}(t)[\mathcal{G}(t,s;1) - \mathcal{G}(t,s;0)]\mathcal{R}^{1/2}(s)y(s)ds \tag{2.29}$$

for $y \in L^2_m[a\ ,\ b]$.Now making use of the identity (2. 28) the equation (2. 29) becomes

$$(\mathcal{A}_1 - \mathcal{A}_0)y(t) = -\int_a^b \mathcal{R}^{1/2}(t)\left\{\int_a^b \mathcal{G}(t,\xi;0)\mathcal{Q}(\xi)\mathcal{G}(\xi,s;1)d\xi\right\}\mathcal{R}^{1/2}(s)y(s)ds$$

for every $y \in L^2_m[a\ ,\ b]$.Therefore

$$\mathcal{A}_1 - \mathcal{A}_0 = -\mathcal{B}_0\mathcal{B}_1$$

where $\mathcal{B}_1$ and $\mathcal{B}_0$ are the operators

$$\mathcal{B}_1 y(t) = \int_a^b \mathcal{G}(t,\xi;1)\mathcal{R}^{1/2}(\xi)y(\xi)d\xi$$

and

$$\mathcal{B}_0 y(t) = \int_a^b \mathcal{R}^{1/2}(t)\mathcal{G}(t,\xi;0)\mathcal{Q}(\xi)y(\xi)d\xi$$

defined on $L^2_m[a\ ,\ b]$.

It is easy to verify that both matrix kernels

$$\mathcal{G}(t,\xi;1)\mathcal{R}^{1/2}(\xi) \qquad \textit{and} \qquad \mathcal{R}^{1/2}(t)\mathcal{G}(t,\xi;0)\mathcal{Q}(\xi)$$

are elements of $L^2_{mm}[a\ ,\ b] \times [a\ ,\ b]$.Thus $\mathcal{B}_1$ and $\mathcal{B}_0$ are compact operators.Therefore by Theorem 1. 13

$$s_j(\mathcal{B}_0) \leq |\mathcal{R}^{1/2}|| \mathcal{Q} | s_j(\mathcal{C}_0)$$

and

$$s_j(\mathcal{B}_1) \leq |\mathcal{R}^{1/2}| s_j(\mathcal{C}_1)$$

for $(j = 1, 2, \ldots\ldots)$, where $\mathcal{C}_0$ and $\mathcal{C}_1$ are the respective operators

$$\mathcal{C}_0 y(t) = \int_a^b \mathcal{G}(t, \xi; 0) y(\xi) d\xi$$

$$\mathcal{C}_1 y(t) = \int_a^b \mathcal{G}(t, \xi; 1) y(\xi) d\xi$$

defined on $L^2_m[a\ ,\ b]$.Note that the s-numbers of $\mathcal{C}_0$ are the reciprocals of the eigenvalues of the boundary problem

$$-(\mathcal{P}(t)y')' = \lambda y$$

$$y(a) = 0 = y(b)$$

while s-numbers of $\mathcal{C}_1$ are the reciprocals of the absolute value of eigenvalues of the problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \lambda y$$

$$y(a) = 0 = y(b).$$

Therefore

$$\lim_{n\to\infty} n^2 s_n(\mathcal{C}_0) = \left(\frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t))dt\right)^2 \tag{2.30}$$

from Theorem 2. 13 with $\mathcal{R}(t) = \mathcal{I}$ and

$$\lim_{n\to\infty} s_n(\mathcal{C}_1) = 0 \tag{2.31}$$

since the eigenvalues of corresponding boundary problem have no finite accumulation points.

From Theorem 1. 14

$$s_{2n}(\mathcal{B}_0\mathcal{B}_1) \le s_{2n-1}(\mathcal{B}_0\mathcal{B}_1) \le s_n(\mathcal{B}_0)s_n(\mathcal{B}_1)$$

and therefore

$$(2n)^2 s_{2n}(\mathcal{B}_0\mathcal{B}_1) \le 4n^2 s_n(\mathcal{B}_0)s_n(\mathcal{B}_1) \tag{2.32}$$

and

$$(2n-1)^2 s_{2n-1}(\mathcal{B}_0\mathcal{B}_1) \le 4(n-1/2)^2 s_n(\mathcal{B}_0)s_n(\mathcal{B}_1). \tag{2.33}$$

Now in view of (2. 30) and (2. 31) the right sides of both (2. 32) and (2. 33) tend to zero as $n \longrightarrow \infty$. Consequently,

$$\lim_{n\to\infty} n^2 s_n(\mathcal{B}_0\mathcal{B}_1) = 0. \tag{2.34}$$

Lemma 2.17 *The s-numbers of the operator $\mathcal{A}_1$ have the property*

$$\lim_{n\to\infty} n^2 s_n(\mathcal{A}_1) = \left(\frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t))dt\right)^2 . \tag{2.35}$$

Proof. Since $\mathcal{A}_1 = \mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_1$ the proof follows from Theorem 1. 15 and equation (2. 34).

Proof of Theorem 2. 14. For sufficiently large n we have λ_n is positive and therefore $\lambda_n = s_n(\mathcal{A}_1)$. Now the result follows from Lemma 2. 17.

The value $\sqrt{r}\left(1/\pi\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}(t))dt\right)$ is called the *first term in the asymptotic equation of the eigenvalue distribution function $n(r)$ of the right-definite problem (2. 1-2).* Then in view of Theorem 2. 12 we have that the first term of $n(r)$ is independent of the coefficient $\mathcal{Q}(t)$.We will make use of this phenomenon later to obtain asymptotics for eigenvalue distribution functions when the matrix $\mathcal{R}(t)$ is nondefinite.

Chapter 3

Two Parameter Boundary Problems

As we have remarked at the end of last chapter the asymptotic formula for the distribution function of eigenvalues $n(r)$ of (2. 1–2) does not depend on the coefficient matrix function $\mathcal{Q}(t)$.This motivated us to study the right definite Sturm-Liouville problem with a varying coefficient matrix function $\mathcal{Q}(t)$. In section 3. 1 we consider the boundary value problem

$$-(\mathcal{P}(t)y')' + (\mathcal{Q}(t) + \mu\mathcal{R}_-(t))y = \lambda(\mu)\mathcal{R}_+(t)y \qquad (3.\ 1)_\mu$$

for each real μ with boundary conditions (2. 2).

The boundary value problem $(3.\ 1)_\mu - (2.\ 2)$ can be treated as a two parameter problem. The history of two parameter differential equations goes back to the begining of this century. According to Richardson [36] there were many people who studied two parameter problems before him. But Richardson himself investigated the exact problem that we have here in 1912 for scalar differntial equations.Since then, various aspects of two parameter eigenvalue problems have been investigated by several people. Notable researchers include P.Binding and P.Browne.In [4] and [5] they study the eigenvalue problem for an operator equation in two parameters.

An application of these latter results to Sturm- Liouville boundary problems can be found in [20]. Very recently Allegretto and Mingarelli [1] used the two parameter problem in connection with nondefinite scalar boundary problems.

In section 3. 2 we discuss some properties of $\lambda(\mu)$ and $y(x,\mu)$ as functions of μ and will make use of these results in Chapter 4.

Section 3. 3 will extend the results in Chapter 2. 2 about integral equations corresponding to right-definite problems to the case of two parameter boundary problems $(3.\ 1)_\mu - (2.\ 2)$.

3.1 Existence of Eigenvalues

Here we are concerned with two parameter boundary value problem $(3.\ 1)_\mu - (2.\ 2)$ when the coefficient matrix functions $\mathcal{P}, \mathcal{Q}$ satisfy condition $\mathcal{H}_1$ and both $\mathcal{R}_+$ and $\mathcal{R}_-$ satisfy conditions $\mathcal{H}_2$ and $\mathcal{H}_*$ with $\mathcal{R}$ replaced by $\mathcal{R}_+, \mathcal{R}_-$.

We note that when matrix functions $\mathcal{P}, \mathcal{Q}, \mathcal{R}_+, \mathcal{R}_-$ satisfy those conditions boundary value problem $(3.\ 1)_\mu - (2.\ 2)$ is right-definite for each fixed μ. Consequently, the results in Chapter 2 are valid for $(3.\ 1)_\mu - (2.\ 2)$ and in particular for each real number μ the eigenvalues of the problem $(3.\ 1)_\mu - (2.\ 2)$ are discrete, bounded below and can be obtained by the Courant-Hilbert min-max principle.

We recall the space

$$\mathcal{D} = \left\{ y \in L^2_m[a\ ,\ b];\ y, \mathcal{P}y' \in AC_m[a\ ,\ b], (\mathcal{P}y')' \in L^2_m \text{ and } y(a) = 0 = y(b) \right\}$$

and the set

$$\mathcal{U}\ =\ \{y \in \mathcal{D};\ (\mathcal{R}_+ y, y)\ =\ 1\}.$$

Let the functional $J[y;\mu]$ be defined by

$$J[y;\mu]\ =\ \{(\mathcal{P}y', y') + ((\mathcal{Q} + \mu\mathcal{R}_-)y, y)\}.$$

Then by the Courant-Hilbert min-max principle,

$$\lambda_n(\mu) = \max_{\{p_1,\ldots,p_{n-1}\}\subset\mathcal{D}} \min_{\substack{y\in\mathcal{U} \\ (\mathcal{R}y,p_i)=0 \\ i=1,\ldots,n-1}} J[y;\mu]$$

where

$$\lambda_1(\mu) \le \lambda_2(\mu) \le \lambda_3(\mu) \le \ldots\ldots$$

is the complete sequence of eigenvalues of $(3.\ 1)_\mu - (2.\ 2)$.

If μ is chosen to be equal to -1, then the smallest eigenvalue $\lambda_1(-1)$ of $(3.\ 1)_{-1} - (2.\ 2)$ is the minimum of $J[y;-1]$ over the set $\mathcal{U}$. Therefore

$$J[y\ ;\ -1] = \{(\mathcal{P}y',y') + ((\mathcal{Q}-\mathcal{R}_-)y,y)\} \ge \lambda_1(-1)(\mathcal{R}_+y,y) \qquad (3.2)$$

for $y\in\mathcal{U}$. *Now we denote* $\lambda_1(-1)$ *by* λ_0 *for convenience and fix this.*

Next, we introduce the quadratic functional, which is again denoted by $J[y\ ;\ \mu]$,

$$J[y\ ;\ \mu] = \{(\mathcal{P}y',y') + ((\mathcal{Q}+\mu\mathcal{R}_-)y,y) - \lambda_0(\mathcal{R}_+y,y)\} \qquad (3.3)$$

for $y\in\mathcal{D}$. Then the inequality (3. 2) implies that $J[y\ ;\ -1]$ is nonnegative definite on $\mathcal{U}$. Consequently $J[y\ ;\ \mu]$ is also nonnegative definite on $\mathcal{U}$ for every $\mu \ge -1$ since $J[y\ ;\ \mu] \ge J[y\ ;\ -1]$ for every such μ. Moreover, by Corollary 2. 4(ii), the successive minmax ratios of $J[y\ ;\ \mu]$ on $\mathcal{U}$ yield the values $\lambda_n(\mu) - \lambda_0$ for every positive integer n.

3.2 Continuous dependence of eigenvalues and eigenfunctions

We begin this section with the follwing

Theorem 3.1 *For every positive integer n, $\lambda_n(\mu) : [0\ ,\ \infty) \longrightarrow [\lambda_1(0)\ ,\ \infty)$ is a nondecreasing continuous function of μ.*

Proof. Since

$$J[y\ ;\ \mu_2]\ \geq\ J[y\ ;\ \mu_1]$$

for $y \in \mathcal{U}$ and $\mu_2 \geq \mu_1 \geq 0$, by Corollary 2. 5

$$\lambda_n(\mu_2)\ -\ \lambda_0\ \geq\ \lambda_n(\mu_1)\ -\ \lambda_0$$

for every positive integer n.Thus

$$\lambda_n(\mu_2)\ \geq\ \lambda_n(\mu_1)$$

and hence $\lambda_n(\mu)$ is a nondecreasing function.

To show $\lambda_n(\mu)$ is continuous consider $\mu_0 \in [0\ ,\ \infty)$.Then, for $y \in \mathcal{U}$,

$$\begin{aligned} \mid J[y\ ;\ \mu]\ -\ J[y\ ;\ \mu_0] \mid\ &=\ \mid (\mu\ -\ \mu_0)(\mathcal{R}_- y, y) \mid \\ &=\ \mid \mu\ -\ \mu_0 \mid (\mathcal{R}_- y, y) \\ &\leq\ \mid \mu\ -\ \mu_0 \mid ((\mathcal{R}_- y, y) + J[y\ ;\ \mu_0 - 1]) \end{aligned}$$

since $J[y\ ;\ \mu_0 - 1]\ \geq\ 0$.Now

$$(\mathcal{R}_- y, y) + J[y\ ;\ \mu_0 - 1]\ =\ J[y\ ;\ \mu_0].$$

Therefore we can rewrite the preceding inequality

$$\mid J[y\ ;\ \mu]\ -\ J[y\ ;\ \mu_0] \mid \leq \mid \mu\ -\ \mu_0 \mid J[y\ ;\ \mu_0]$$

and hence

$$(1 - \mid \mu - \mu_0 \mid) J[y\ ;\ \mu_0]\ \leq\ J[y\ ;\ \mu] \leq (1 + \mid \mu - \mu_0 \mid) J[y\ ;\ \mu_0]$$

for $y \in \mathcal{U}$. Then a combination of Corollaries 2. 4 and 2. 5 now yields that

$$(1- \mid \mu - \mu_0 \mid)(\lambda_n(\mu_0) - \lambda_0) \leq \lambda_n(\mu) - \lambda_0 \leq (1+ \mid \mu - \mu_0 \mid)(\lambda_n(\mu_0) - \lambda_0)$$

and therefore

$$- \mid \mu - \mu_0 \mid (\lambda_n(\mu_0) - \lambda_0) \leq \lambda_n(\mu) - \lambda_n(\mu_0) \leq \mid \mu - \mu_0 \mid (\lambda_n(\mu_0) - \lambda_0).$$

The monotonicity results in Corollary 2. 5 show that for $\mu \geq 0$,

$$\lambda_n(\mu_0) \geq \lambda_n(-1) \geq \lambda_1(-1) = \lambda_0$$

and hence

$$\lambda_n(\mu_0) - \lambda_0 \geq 0.$$

Thus

$$\mid \lambda_n(\mu) - \lambda_n(\mu_0) \mid \leq \mid \mu - \mu_0 \mid (\lambda_n(\mu_0) - \lambda_0)$$

and hence $\lambda_n(\mu)$ is continuous at $\mu = \mu_0$.

The second order differential equation $(3.\ 1)_\mu$ can be written as a first order system

$$u' = \mathcal{P}^{-1}(x)v, \quad v' = \mathcal{G}(x, \mu, \lambda(\mu))u \tag{3.4}$$

with

$$u(x) = y(x),$$

$$v(x) = \mathcal{P}(x)y'(x)$$

and

$$\mathcal{G}(x, \mu, \lambda(\mu)) = \mathcal{Q}(x) + \mu\mathcal{R}_-(x) - \lambda(\mu)\mathcal{R}_+(x).$$

Then the corresponding first order matrix system is

$$\mathcal{U}' = \mathcal{P}^{-1}(x)\mathcal{V}, \quad \mathcal{V}' = \mathcal{G}(x, \mu, \lambda(\mu))\mathcal{U} \tag{3.5}$$

where $\mathcal{U}$ and $\mathcal{V}$ are $m \times m$ matrix functions.

We now consider the initial value problem associated with (3. 5) and the initial conditions

$$\mathcal{U}(a, \mu, \lambda(\mu)) = \mathcal{O}, \quad \mathcal{V}(a, \mu, \lambda(\mu)) = \mathcal{I}. \tag{3.6}$$

This initial value problem has a unique solution $(\mathcal{U}(x, \mu, \lambda(\mu)), \mathcal{V}(x, \mu, \lambda(\mu)))$ in the interval $[a , b]$ for every fixed $\mu \in [0 , \infty)$ when $\mathcal{P}, \mathcal{Q}, \mathcal{R}_+, \mathcal{R}_-$ satisfy the conditions $\mathcal{H}_1$ and $\mathcal{H}_2$.

It is not difficult to see that

$$u(x, \mu, \lambda(\mu)) = \mathcal{U}(x, \mu, \lambda(\mu))\xi$$

is a solution of $(3.\ 1)_\mu$ satisfying $u(a) = 0$ for any arbitrary vector $\xi \in R_m$. Therefore $y = \mathcal{U}\xi$ is a solution of the second order differential equation $(3.\ 1)_\mu$ satisfying $y(a) = 0$. Thus $\lambda(\mu)$ will be an eigenvalue of $(3.\ 1)_\mu - (2.\ 2)$ if and only if

$$y(b) = \mathcal{U}(b, \mu, \lambda(\mu))\xi = O$$

for some nonzero vector ξ. i. e. if and only if

$$det\, \mathcal{U}(b, \mu, \lambda(\mu)) = 0.$$

Therefore the eigenvalues of the problem $(3.\ 1)_\mu - (2.\ 2)$ are given by the zeros of the function $det\,\mathcal{U}(b,\mu,\lambda(\mu)) = 0$. If the problem $(3.\ 1)_\mu - (2.\ 2)$ has nonsimple eigenvalues we have the following result.

Theorem 3.2 *The multiplicity of any eigenvalue of* $(3.\ 1)_\mu - (2.\ 2)$ *coincides with its multiplicity as a root of the function*

$$det\,\mathcal{U}(b,\mu,\lambda(\mu)) = 0.$$

Proof. See [[15] Theorem VI. 1. 2].

Since the preceding results are valid for any $\mu \in [0\ ,\ \infty)$ we can study the matrix function $\mathcal{U}_n(x,\mu)$ defined by

$$\mathcal{U}_n(x,\mu) = \mathcal{U}(x,\mu,\lambda_n(\mu))$$

as a function of μ.

Lemma 3.3 *The matrix function* $\mathcal{U}_n(x,\mu)$ *is continuous as a function of* μ *for each fixed* $x \in [a\ ,\ b]$.

Proof. Integrating the two equations in (3. 5) and using (3. 6) we have

$$\mathcal{U}(x,\mu) = \int_a^x \mathcal{P}^{-1}(t)dt - \int_a^x \mathcal{P}^{-1}(t)\int_a^t \mathcal{G}(s,\mu)\mathcal{U}(s,\mu)dsdt$$

where the subscript n of $\mathcal{U}_n(x,\mu)$ and $\mathcal{G}_n(s,\mu)$ is suppressed for convenience. Therefore

$$\mathcal{U}(x,\mu) - \mathcal{U}(x,\mu_0) = \int_a^x \mathcal{P}^{-1}(t)\int_a^t \mathcal{G}(s,\mu)\mathcal{U}(s,\mu)dsdt - \int_a^x \mathcal{P}^{-1}(t)\int_a^t \mathcal{G}(s,\mu_0)\mathcal{U}(s,\mu_0)dsdt$$

$$= \int_a^x \int_a^t \mathcal{P}^{-1}(t)(\mathcal{G}(s,\mu) - \mathcal{G}(s,\mu_0))\mathcal{U}(s,\mu_0)dsdt$$
$$+ \int_a^x \int_a^t \mathcal{P}^{-1}(t)\mathcal{G}(s,\mu)(\mathcal{U}(s,\mu) - \mathcal{U}(s,\mu_0))dsdt.$$

Then

$$\begin{aligned} | \mathcal{U}(x,\mu) - \mathcal{U}(x,\mu_0) | \;\leq\; & \int_a^x \int_a^t | \mathcal{P}^{-1}(t) || \mathcal{G}(s,\mu) - \mathcal{G}(s,\mu_0) || \mathcal{U}(s,\mu_0) | dsdt \\ & + \int_a^x \int_s^x | \mathcal{P}^{-1}(t) || \mathcal{G}(s,\mu) | \mathcal{U}(s,\mu) - \mathcal{U}(s,\mu_0) | dtds. \end{aligned}$$

Now

$$| \mathcal{P}^{-1}(t) | \leq \|\mathcal{P}^{-1}\|_\infty,$$
$$| \mathcal{U}(s,\mu) | \leq \|\mathcal{U}_{\mu_0}\|_\infty,$$
$$| \mathcal{G}(s,\mu) | \leq | \mu | \; \|\mathcal{R}_-\|_\infty + | \lambda(\mu) | \; \|\mathcal{R}_+\|_\infty + \|\mathcal{Q}\|_\infty$$

and

$$\begin{aligned} | \mathcal{G}(s,\mu) - \mathcal{G}(s,\mu_0) | \;\leq\; & | \mu - \mu_0 | \, | \mathcal{R}_-(s) | + | \lambda(\mu) - \lambda(\mu_0) | \, | \mathcal{R}_+(s) | \\ \leq\; & | \mu - \mu_0 | \; \|\mathcal{R}_-\|_\infty + | \lambda(\mu) - \lambda(\mu_0) | \; \|\mathcal{R}_+\|_\infty \end{aligned}$$

yield the result

$$\begin{aligned} | \mathcal{U}(x,\mu) \; - \; & \mathcal{U}(x,\mu_0) | \\ \leq\; & \|\mathcal{P}^{-1}\|_\infty \, \|\mathcal{U}_{\mu_0}\|_\infty \, (| \mu - \mu_0 | \; \|\mathcal{R}_-\|_\infty + | \lambda(\mu) - \lambda(\mu_0) | \; \|\mathcal{R}_+\|_\infty)\,(b-a)^2 + \\ & \|\mathcal{P}^{-1}\|_\infty \, (| \mu | \; \|\mathcal{R}_-\|_\infty + | \lambda(\mu) | \; \|\mathcal{R}_+\|_\infty + \|\mathcal{Q}\|_\infty) \int_a^x | \mathcal{U}(s,\mu) - \mathcal{U}(s,\mu_0) | \, ds \end{aligned}$$

This can be written as

$$\begin{aligned} | \mathcal{U}(x,\mu) - \mathcal{U}(x,\mu_0) | \;\leq\; & c_1 | \mu - \mu_0 | + c_2 | \lambda(\mu) - \lambda(\mu_0) | \\ & + c_3(\mu) \int_a^x | \mathcal{U}(s,\mu) - \mathcal{U}(s,\mu_0) | \, ds \end{aligned} \qquad (3.7)$$

with

$$c_1 = \|\mathcal{P}^{-1}\|_\infty \, \|\mathcal{U}_{\mu_0}\|_\infty \, \|\mathcal{R}_-\|_\infty \, (b-a)^2, c_2 = \|\mathcal{P}^{-1}\|_\infty \, \|\mathcal{U}_{\mu_0}\|_\infty \, \|\mathcal{R}_+\|_\infty \, (b-a)^2$$

and

$$c_3(\mu) = \|\mathcal{P}^{-1}\|_\infty \, (| \mu | \; \|\mathcal{R}_-\|_\infty + | \lambda(\mu) | \; \|\mathcal{R}_+\|_\infty + \|\mathcal{Q}\|_\infty)\,.$$

An application of Gronwall's inequality to (3. 7) gives

$$| \mathcal{U}(x,\mu) - \mathcal{U}(x,\mu_0) | \leq (c_1 | \mu - \mu_0 | + c_2 | \lambda(\mu) - \lambda(\mu_0) |) \, exp\{c_3(\mu)(b-a)\}$$

Since $\lambda(\mu)$ and $c_3(\mu)$ are continuous at $\mu = \mu_0$ the continuity of $\mathcal{U}(x,\mu)$ at $\mu = \mu_0$ now follows.

If the eigenfunction corresponding to $\lambda_n(\mu)$ is denoted by $y_n(x,\mu)$, then

$$y_n(x,\mu) \;=\; \mathcal{U}_n(x,\mu)\xi,$$

for some $\xi \in R_m$, which is a solution of the system of linear equations

$$\mathcal{U}_n(b,\mu)\xi \;=\; O.$$

Note that $y_n'(x,\mu) = \mathcal{U}_n'(x,\mu)\xi = \mathcal{P}^{-1}(x)\mathcal{V}_n(x,\mu)\xi$. Thus $y_n'(a,\mu) = \mathcal{P}^{-1}(a)\mathcal{V}_n(a,\mu)\xi$ or $\xi = \mathcal{P}(a)y_n'(a,\mu)$ since $\mathcal{V}_n(a,\mu) = \mathcal{I}$ from (3. 6). Hereafter, without loss of generality, we assume that every eigenfunction y_n has the property $| \mathcal{P}(a)y_n'(a,\mu) | = 1$.

Lemma 3.4 *If $\{\mu_m\}$ is a sequence such that $\mu_m \longrightarrow \mu_0$ then the sequence of eigenfunctions $\{y_n(x,\mu_m)\}_{m=1}^{\infty}$ has a subsequence converging to $y_n(x,\mu_0)$ for every positive integer n.*

Proof. Suppose $y_n(x,\mu_m) = \mathcal{U}_n(x,\mu_m)\xi_m$. Then $\{\xi_m\}$ is a sequence of vectors in R_m with the property that $\mathcal{U}_n(b,\mu_m)\xi_m = O$ and $| \xi_m | = 1$ for every m. Using Bolzano-Weiérstrass theorem we can extract a convergent subsequence of

$\{\xi_m\}$,which is again denoted by $\{\xi_m\}$ for convenience.Let $\lim_{m\to\infty}\xi_m = \xi_0$.Then $|\xi_0| = 1$. Therefore,

$$\begin{aligned} |\mathcal{U}_n(b,\mu_0)\xi_0| &= |\mathcal{U}_n(b,\mu_0)\xi_0 - \mathcal{U}_n(b,\mu_m)\xi_m| \\ &\le |\mathcal{U}_n(b,\mu_0) - \mathcal{U}_n(b,\mu_m)| + |\mathcal{U}_n(b,\mu_0)||\xi_0 - \xi_m| \end{aligned}$$

and the right side goes to zero as $m \to \infty$. Hence $\mathcal{U}_n(b,\mu_0)\xi_0 = O$ and $|\xi_0| = 1$.But then

$$y_n(x,\mu_0) = \mathcal{U}_n(x,\mu_0)\xi_0.$$

Now

$$\begin{aligned} |y_n(x,\mu_m) - y_n(x,\mu_0)| &= |\mathcal{U}_n(x,\mu_0)\xi_0 - \mathcal{U}_n(x,\mu_m)\xi_m| \\ &\le |\mathcal{U}_n(x,\mu_0) - \mathcal{U}_n(x,\mu_m)| + |\mathcal{U}_n(x,\mu_0)||\xi_0 - \xi_m| \end{aligned}$$

and this goes to zero as $m \to \infty$ at every $x \in [a\,,\,b]$. Hence there follows the lemma.

Theorem 3.5 *For every positive integer n, $\lambda_n(\mu)$ is*

(i) differentiable almost everywhere on the interval $[0\,,\,\infty)$ and

$$\frac{d\lambda_n}{d\mu} = \frac{(\mathcal{R}_- y_n(x,\mu), y_n(x,\mu))}{(\mathcal{R}_+ y_n(x,\mu), y_n(x,\mu))} \tag{3.8}$$

whenever differentiable.

(ii) Lipschitz continuous and strictly increasing on the interval $[0,\infty)$.

Proof. From Theorem 3. 1 $\lambda_n(\mu)$ is nondecreasing and hence differentiable almost everywhere on $[0\,,\,\infty)$.

Left-multiplying the equation $(3.\ 1)_\mu$ by $y_n(x,\mu_0)$ when $\lambda_n(\mu)$ is an eigenvalue and integrating from a to b yields

$$\begin{aligned} \lambda(\mu)(\mathcal{R}_+ y(x,\mu), y(x,\mu_0)) = (\mathcal{P}y'(x,\mu), y'(x,\mu_0)) &+ (\mathcal{Q}y(x,\mu), y(x,\mu_0)) \\ &+ \mu(\mathcal{R}_- y(x,\mu), y(x,\mu_0)) \end{aligned}$$

where the subscript n of $\lambda_n(\mu)$ and $y_n(x,\mu)$ is suppressed for convenience.Similarly

$$\lambda(\mu_0)(\mathcal{R}_+ y(x,\mu_0), y(x,\mu)) = (\mathcal{P}y'(x,\mu_0), y'(x,\mu)) + (\mathcal{Q}y(x,\mu_0), y(x,\mu)) + \mu_0(\mathcal{R}_- y(x,\mu_0), y(x,\mu)).$$

Then

$$\frac{\lambda(\mu)-\lambda(\mu_0)}{\mu-\mu_0} = \frac{(\mathcal{R}_- y(x,\mu), y(x,\mu_0))}{(\mathcal{R}_+ y(x,\mu), y(x,\mu_0))}. \tag{3.9}$$

Let $\{\mu_m\}$ be a sequence with $\mu_0 < \mu_{m+1} < \mu_m$ and $\lim_{m\to\infty}\mu_m = \mu_0$. Then Lemma 3. 4 implies the existence of a subsequence of $\{y(x,\mu_m)\}$, which is again denoted by the same notation for convenience, such that

$$y(x,\mu_m) \longrightarrow y(x,\mu_0)$$

as $m \to \infty$. Therefore

$$\begin{aligned}\lim_{m\to\infty}\frac{\lambda(\mu_m)-\lambda(\mu_0)}{\mu_m-\mu_0} &= \lim_{m\to\infty}\frac{(\mathcal{R}_- y(x,\mu_m), y(x,\mu_0))}{(\mathcal{R}_+ y(x,\mu_m), y(x,\mu_0))}\\ &= \frac{(\mathcal{R}_- y(x,\mu_0), y(x,\mu_0))}{(\mathcal{R}_+ y(x,\mu_0), y(x,\mu_0))},\end{aligned}$$

by using the continuity of L^2_m inner product, showing the right derivative of $\lambda(\mu)$ at $\mu=\mu_0$ is equal to the right side of (3. 8) and hence follows (i).

To prove (ii) we first note that

$$\lambda_n(\mu_1) \ge (\mathcal{P}y'(x,\mu_2), y'(x,\mu_2)) + (\mathcal{Q}y(x,\mu_2), y(x,\mu_2)) + \mu_1(\mathcal{R}_- y(x,\mu_2), y(x,\mu_2))$$

because of the min-max principle. This can be written as

$$\lambda_n(\mu_1) \ge \lambda_n(\mu_2) + (\mu_1-\mu_2)(\mathcal{R}_- y(x,\mu_2), y(x,\mu_2)).$$

A similar consideration will give us

$$\lambda_n(\mu_2) \ge \lambda_n(\mu_1) + (\mu_2-\mu_1)(\mathcal{R}_- y(x,\mu_1), y(x,\mu_1)).$$

Therefore

$$\begin{aligned}\frac{|\lambda_n(\mu_1)-\lambda_n(\mu_2)|}{|\mu_1-\mu_2|} &\le \max\{(\mathcal{R}_- y(x,\mu_1), y(x,\mu_1)), (\mathcal{R}_- y(x,\mu_2), y(x,\mu_2))\}\\ &\le \|\mathcal{R}_-\|_\infty \max\{(y(x,\mu_1), y(x,\mu_1)), (y(x,\mu_2), y(x,\mu_2))\}.\end{aligned}$$

We next make use of the fact

$$|y(x,\mu_i)| = \mathcal{O}\left(e^{c_i\sqrt{\mu_i}}\right)$$

for $x \in [a\ ,\ b]$ (see [2]) and hence obtain that

$$\max\{(y(x,\mu_1),y(x,\mu_1)),(y(x,\mu_2),y(x,\mu_2))\} \leq Ce^{c_1\sqrt{\mu_1}}e^{c_2\sqrt{\mu_2}}$$

where c_i and C are constants. Consequently, on compact sets this is uniformly bounded and hence $\lambda_n(\mu)$ is Lipschitz continuous.

Since Lipschitz continuous functions are absolutely continuous we have

$$\lambda_n(\mu) = \lambda_n(0) + \int_0^{\mu} \lambda_n'(s)ds$$

for $\mu \in [0\ ,\ \infty)$. Now the result follows from this since $\lambda_n'(\mu)$ is positive a. e. by (i).

3.3 Related Integral Equations

If for some $\mu_0 > 0$, all the eigenvalues of $(3.\ 1)_{\mu}$-(2. 2) are positive then $\mu = -\mu_0$ cannot be an eigenvalue of the problem

$$-\left(\mathcal{P}(t)y'\right)' + \mathcal{Q}(t)y = \mu\mathcal{R}_-(t)y \tag{3.10}$$

with boundary conditions (2. 2).Hence the nonhomogeneous problem

$$-\left(\mathcal{P}(t)y'\right)' + \mathcal{Q}(t)y + \mu_0\mathcal{R}_-(t)y = f \tag{3.11}$$

with boundary conditions (2. 2) has a unique solution and it is given by

$$y(t) = \int_a^b \mathcal{G}(t,s;\mu_0)f(s)ds \tag{3.12}$$

for any $f \in L_m^\infty[a\ ,\ b]$, where the Green's matrix function $\mathcal{G}(t, s; \mu_0)$ has similar properties to those stated in Chapter 2. 2. In particular $\mathcal{G}(t, s; \mu_0) = [\mathcal{G}(s, t; \mu_0)]^*$ and if $\mu > \mu_0$

$$\mathcal{G}(t, s; \mu) - \mathcal{G}(t, s; \mu_0) = (\mu - \mu_0) \int_a^b \mathcal{G}(t, \xi; \mu_0)\mathcal{R}_-(\xi)\mathcal{G}(\xi, s; \mu)d\xi. \tag{3.13}$$

The condition $\mu > \mu_0$ guarantees the existence of $\mathcal{G}(t, s; \mu)$ as all the eigenvalues of (3. 1)$_\mu$-(2. 2) are then positive. Thus any solution of (3. 1)$_\mu$-(2. 2) is given by

$$y(t) = \lambda(\mu) \int_a^b \mathcal{G}(t, s; \mu)\mathcal{R}_+(s)y(s)ds \tag{3.14}$$

for $\mu \geq \mu_0$. A consideration similar to that we had in Chapter 2. 2 will show the equation (3. 14) is equivalent to

$$z(t) = \lambda(\mu) \int_a^b \mathcal{R}_+^{1/2}(t)\mathcal{G}(t, s; \mu)\mathcal{R}_+^{1/2}(s)z(s)ds \tag{3.15}$$

where $z(t) = \mathcal{R}_+^{1/2}(t)y(t)$. Consequently, we have

$$\frac{1}{\lambda_n(\mu)} = s_n(\mathcal{A}_\mu) \quad (n = 1, 2, \ldots\ldots) \tag{3.16}$$

for $\mu \geq \mu_0$, if we define the integral operator $\mathcal{A}_\mu$ on $L_m^2[a\ ,\ b]$ by

$$\mathcal{A}_\mu y(t) = \int_a^b \mathcal{R}_+^{1/2}(t)\mathcal{G}(t, s; \mu)\mathcal{R}_+^{1/2}(s)y(s)ds \tag{3.17}$$

Then

$$(\mathcal{A}_\mu - \mathcal{A}_{\mu_0})y(t) = \int_a^b \mathcal{R}_+^{1/2}(t)(\mathcal{G}(t,s;\mu) - \mathcal{G}(t,s;\mu_0))\mathcal{R}_+^{1/2}(s)y(s)ds \tag{3.18}$$

and making use of (3. 13) yields

$$(\mathcal{A}_\mu - \mathcal{A}_{\mu_0})y(t) = (\mu_0 - \mu)\int_a^b \int_a^b \mathcal{R}_+^{1/2}(t)\mathcal{G}(t,\xi;\mu_0)\mathcal{R}_-(\xi)\mathcal{G}(\xi,s;\mu)\mathcal{R}_+^{1/2}(s)y(s)d\xi ds. \tag{3.19}$$

Thus

$$(\mathcal{A}_\mu - \mathcal{A}_{\mu_0})y = (\mu_0 - \mu)\mathcal{B}_{\mu_0}^*\mathcal{B}_\mu y \tag{3.20}$$

for $y \in L_m^2[a\,,\,b]$, if we define

$$\mathcal{B}_\mu y(t) = \int_a^b \mathcal{R}_-^{1/2}(t)\mathcal{G}(t,s;\mu)\mathcal{R}_+^{1/2}(s)y(s)ds. \tag{3.21}$$

Although the next result is a corollary to Theorem 2. 12 we state here as a separate theorem since its applications are far reaching, (see Chapter 4. 3). This theorem can also be viewed as a consequence of equation (3. 20) and Theorem 2. 13.

Theorem 3.6 *Let $n_\mu(r)$ be the number of eigenvalues of the right-definite problem $(3.\,1)_\mu - (2.\,2)$ which lie in the interval $[0\,,\,r]$. Then $n_\mu(r)$ satisfies the asymptotic formula*

$$\lim_{r\to\infty} \frac{n_\mu(r)}{\sqrt{r}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_+(t))dt, \tag{3.22}$$

for every $\mu > \mu_0$, where μ_0 is defined as in (3. 11) and $\Lambda_{1/2}$ is defined as in Theorem 2. 12.

Chapter 4

Non-Definite Boundary Problems

We once again consider the vector Sturm-Liouville boundary problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \lambda\mathcal{R}(t)y \tag{4.1}$$

on the compact interval $[a, b]$ with Dirichlet boundary conditions (2. 2). In this occasion the coefficient matrix functions $\mathcal{P}, \mathcal{Q}$ satisfy $\mathcal{H}_1$ while the matrix function $\mathcal{R}$ satisfies the condition

$\mathcal{H}_3$ *Matrix function $\mathcal{R}(t)$ is symmetric and of class $L^{\infty}_{mm}[a, b]$. Also, the sets $\{t : t \in [a, b], \mathcal{R}_+(t) > \mathcal{O}\}$ and $\{t : t \in [a, b], \mathcal{R}_-(t) > \mathcal{O}\}$ are of positive measure.*

In Section 1 we shall discuss some properties of the eigenvalues and eigenfunctions of (4. 1)–(2. 2).Most of these results are natural extensions from the scalar nondefinite problems to the vector case herein.

In Section 2 we will settle the existence of positive and negative eigenvalues of (4. 1)–(2. 2) under the hypothesis $\mathcal{H}_3$. In fact the method we use here will show that these eigenvalues are generated by certain two parameter boundary problems.

Asymptoptics for distribution functions of positive and negative eigenvalues of (4. 1)-(2. 2) will be obtained in Section 3 when $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ belong to certain classes of matrix functions.

4.1 Non-Real Eigenvalues

The *regular nondefinite case* arises when the coefficient matrix function $\mathcal{R}(t)$ of (4. 1) satisfies the condition $\mathcal{H}_3$. One of the features of the nondefinite case is the possible existence of nonreal eigenvalues. The existence of possibly nonreal eigenvalues for the scalar Sturm-Liouville problem was first hinted by O. Haupt [16] in 1915 and then by R.G.D.Richardson [35] in 1918. However, as A.B.Mingarelli [26] points out neither author gave an instance of such an occurence. More properties of the eigenvalues and eigenfunctions of nondefinite scalar Sturm-Liouville problem have been discussed in [21], [22] and [24].

In [17] E. Hilb showed that the scalar boundary problem

$$-u'' + q(t)u = \lambda u \tag{4.2}$$

with Dirichlet boundary conditions admits a countably infinity of complex eigenvalues with no finite point of accumulation when $q(t)$ is a complex valued continuous function. So if we choose $q(t) = q_1(t) + \imath q_2(t)$ so that $\lambda = \mu + \imath\nu,\ \nu \neq 0$ is a nonreal eigenvalue of the scalar boundary problem with the corresponding eigenfunction u, then

$$-u'' - (\mu - q_1(t))u = \imath(\nu - q_2(t))u$$

and

$$u(a) = 0 = u(b).$$

Thus $\lambda = \imath$ is an eigenvalue of a problem of the form (4. 1–2. 2) where $\mathcal{P}(t) = \mathcal{I}$, $\mathcal{Q}(t) = (\mu - q_1)\mathcal{I}$ and $\mathcal{R}(t) = (\nu - q_2)\mathcal{I}$ with the corresponding eigenfunction $y = (u)$. This example shows the possible existence of nonreal eigenvalues for nondefinite vector boundary problems.

Lemma 4.1 *Let λ be a nonreal eigenvalue of (4. 1–2. 2) with the corresponding eigenfunction y. Then*

$$(\mathcal{R}y, y) \;=\; 0.$$

Proof. Taking L^2_m inner product of y and (4. 1) we have

$$-((\mathcal{P}y')', y) + (\mathcal{Q}y, y) \;=\; \lambda(\mathcal{R}y, y).$$

Now integration by parts and use of (2. 2) yield the result that

$$(\mathcal{P}y', y') + (\mathcal{Q}y, y) \;=\; \lambda(\mathcal{R}y, y) \tag{4.3}$$

Since $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are real valued and symmetric, the left side of (4. 3) is real. Because of the same reasons $(\mathcal{R}y, y)$ is real. Therefore

$$(Im\lambda)(\mathcal{R}y, y) = 0$$

and since $Im\lambda \neq 0$ we have

$$(\mathcal{R}y, y) = 0.$$

Therefore, if λ is a nonreal eigenvalue of (4. 1–2. 2) with the corresponding eigenfunction y, then from (4. 3)

$$(\mathcal{P}y', y') + (\mathcal{Q}_+y, y) \;=\; (\mathcal{Q}_-y, y)$$

and hence $(\mathcal{Q}_{-}y, y)$ is positive. Thus the coefficient matrix function $\mathcal{Q}(t)$ of (4. 1) must have the property that $\mathcal{Q}_{-}(t) \geq \mathcal{O}$ on $[a\ ,\ b]$ and the set $\{t : t \in [a\ ,\ b], \mathcal{Q}_{-}(t) \neq \mathcal{O}\}$ is of positive measure.

Lemma 4.2 *Let λ be a nonreal eigenvalue of (4. 1–2. 2) with corresponding eigenfunction y. Then $\overline{\lambda}$ is also an eigenvalue of (4. 1–2. 2) with corresponding eigenfunction $\overline{y}$.*

Proof. Taking the complex conjugate of both sides of (4. 1) and observing that the coefficient matrix functions are real valued there follows the lemma.

Lemma 4.3 *Let λ_i , λ_j be nonreal eigenvalues of (4. 1–2. 2) with $\lambda_i \neq \overline{\lambda}_j$. Let ϕ_i , ϕ_j be two corresponding eigenfunctions. Then $\phi_i \neq \phi_j$ and*

$$(\mathcal{P}\phi_i', \phi_j') + (\mathcal{Q}\phi_i, \phi_j) = 0 \tag{4.4}$$

Proof. The result $\phi_i \neq \phi_j$ is trivially true since the problem (4. 1–2. 2) is regular.To establish (4. 4) first take the L_m^2 inner product of (4. 1) when λ_i is an eigenvalue with ϕ_j.Now use of (2. 2) yields

$$(\mathcal{P}\phi_i', \phi_j') + (\mathcal{Q}\phi_i, \phi_j) = \lambda_i(\mathcal{R}\phi_i, \phi_j).$$

If $(\mathcal{R}\phi_i, \phi_j) = 0$ then we have (4. 4).Otherwise, interchanging the roles of i and j in the preceding argument we have

$$(\mathcal{P}\phi_j', \phi_i') + (\mathcal{Q}\phi_j, \phi_i) = \overline{\lambda}_j(\mathcal{R}\phi_j, \phi_i).$$

Using the properties of the inner product this equation is the same as

$$\overline{(\mathcal{P}\phi_i', \phi_j')} + \overline{(\mathcal{Q}\phi_i, \phi_j)} = \lambda_j\overline{(\mathcal{R}\phi_i, \phi_j)}.$$

Therefore

$$(\overline{\lambda}_i - \lambda_j)\overline{(\mathcal{R}\phi_i, \phi_j)} = 0 \qquad or \qquad (\lambda_i - \overline{\lambda}_j)(\mathcal{R}\phi_i, \phi_j) = 0$$

showing that $\lambda_i = \overline{\lambda}_j$. This contradicts the fact that $\lambda_i \neq \overline{\lambda}_j$. Thus $(\mathcal{R}\phi_i, \phi_j) = 0$ and hence follows (4. 4).

Note that if λ is a nonreal eigenvalue of (4. 1)-(2. 2) then taking the complex conjugate on both sides of (4. 1) we have

$$-(\mathcal{P}(t)\overline{y}')' + \mathcal{Q}(t)\overline{y} = \overline{\lambda}\mathcal{R}(t)\overline{y}$$

since the coefficient matrices are real valued. Therefore, $\overline{\lambda}$ is also an eigenvalue of (4. 1)-(2. 2). Hence nonreal eigenvalues of (4. 1)-(2. 2) occur in pairs. Let the number of pairs of nonreal eigenvalues of (4. 1-2. 2) be M. Also, let the eigenvalues μ_i with corresponding eigenfunctions ψ_i of the problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}_+(t)y = \mu\mathcal{Q}_-(t)y \tag{4.5}$$

and (2. 2) satisfy

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N \leq 1 < \mu_{N+1} \leq \mu_{N+2} \leq \ldots\ldots$$

for some integer N. Then N is finite in view of the results in Theorem 2. 9.

Theorem 4.4 *Nonreal eigenvalues of the nondefinite Sturm-Liouville problem (4. 1-2. 2) have the property that*

$$M \leq N.$$

Proof. If $M = 0$ then the theorem is trivially true. So we may suppose that $M \geq 1$. If (4. 1–2. 2) has a nonreal eigenvalue then for a vector function $y \in \mathcal{D}$ we have

$$(\mathcal{P}y', y') + (\mathcal{Q}y, y) = 0.$$

Rewriting this as

$$(\mathcal{P}y', y') + (\mathcal{Q}_{+}y, y) = (\mathcal{Q}_{-}y, y)$$

we see that $\mu_1 \leq 1$ since μ_1 is the minimum of the ratio

$$\{(\mathcal{P}y', y') + (\mathcal{Q}_{+}y, y)\}/(\mathcal{Q}_{-}y, y) \tag{4.6}$$

over the set $\mathcal{D}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ denote a collection of nonreal eigenvalues labelled in such a way that $\lambda_i \neq \overline{\lambda}_j$ for all i, j, $1 \leq i, j \leq M$. Let $\phi_1, \ldots, \phi_M$ denote M corresponding eigenfunctions.

Assume the contrary that $M > N$. Then construct $f(t) = \sum_{i=1}^{M} c_i\phi_i$ where the generally complex c_i, $1 \leq i \leq M$ are chosen so that

$$(\mathcal{Q}_{-}f, \psi_i) \;=\; 0 \quad i = 1, 2, \ldots, N. \tag{4.7}$$

Now (4. 7) describes a system of N linear equations in M unknowns. Since $M > N$ this system has a nontrivial solution $c_1, c_2, \ldots, c_M$ which we now fix. Note that then $f \not\equiv 0$ and $f \in \mathcal{D}$. Furthermore use of Lemma 4. 3 shows that f satisfies

$$(\mathcal{P}f', f') + (\mathcal{Q}f, f) \;=\; 0$$

or

$$(\mathcal{P}f', f') + (\mathcal{Q}_{+}f, f) \;=\; (\mathcal{Q}_{-}f, f).$$

Once again by variational principles μ_{N+1} is the minimum of (4. 6) over vector functions $y \in \mathcal{D}$ satisfying $(\mathcal{Q}_{-}y, \psi_i) = 0$ for $i = 1, \ldots, N$. Since f is such a function we have that μ_{N+1} cannot exceed 1. i. e. , $\mu_{N+1} \leq 1$, which is a contradiction. Thus $M \leq N$ and so the nonreal eigenvalues of (4. 1–2. 2) form a finite set.

We note that the preceding theorem also follows from a more general result of Mingarelli (see [25]).

Corollary 4.5 *If the number of nonreal eigenvalues of the nondefinite Sturm-Liouville problem is 2M then*

$$M \leq \int_a^b Tr\,\mathcal{H}(s,s)\mathcal{Q}_-(s)ds$$

where $\mathcal{H}(t,s)$ is defined as in Lemma 2. 7.

Proof. This follows from Theorem 4. 4 and Lemma 2. 8.

Thus not every $\lambda \in C$ is an eigenvalue of (4. 1–2. 2). Consequently for some λ_0 which is not an eigenvalue of (4. 1–2. 2) there exists a continuous Green's matrix function $\mathcal{G}(t,s,\lambda_0)$ with the property that if λ is an eigenvalue of (4. 1–2. 2) then

$$y(t) = (\lambda - \lambda_0)\int_a^b \mathcal{G}(t,s,\lambda_0)\mathcal{R}(s)y(s)ds$$

for some $y \in \mathcal{D}$. Lemma 2. 2 shows that the integral operator $G : L^2_m \longrightarrow L^2_m$ defined by

$$Gy = \int_a^b \mathcal{G}(t,s,\lambda_0)\mathcal{R}(s)y(s)ds$$

is compact. Now invoking the properties of compact operators we have

Theorem 4.6 *The eigenvalues of the nondefinite Sturm-Liouville problem (4. 1–2. 2) form a discrete set in C with no finite accumulation point except possibly $\pm\infty$.*

Lemma 4.7 *Let λ_i and λ_j be nonzero distinct real eigenvalues of (4. 1–2. 2) with corresponding eigenfunctions ϕ_i and ϕ_j. Then*

$$(\mathcal{R}\phi_i, \phi_j) = 0.$$

Proof. A method similar to the proof of Lemma 4. 3 can be used to show this and so is omitted.

Another interesting property of eigenfunctions corresponding to nonzero real eigenvalues of (4. 1–2. 2) is given by the following

Theorem 4.8 *The nondefinite Sturm-Liouville boundary problem (4. 1–2. 2) has at most finite (though possibly empty) set of positive (negative) eigenvalues with the property that corresponding eigenfunctions y satisfy*

$$(\mathcal{R}y, y) \leq 0 \quad (resp. \geq 0). \tag{4.8}$$

Proof. It suffices to prove the theorem for positive eigenvalues as the other case follows from similar considerations.If (4. 8) is satisfied by some eigenfunction y of (4. 1–2. 2), then $(\mathcal{P}y', y') + (\mathcal{Q}y, y)$ is nonpositive and so $\mathcal{Q}_-(t) \geq \mathcal{O}$ on $[a\,,\,b]$ and the set $\{t : t \in [a\,,\,b], \mathcal{Q}_-(t) \neq \mathcal{O}\}$ must be of positive measure.Therefore we can now consider the boundary problem (4. 5–2. 2).We recall that the eigenvalues μ_i of (4. 5–2. 2) with correspoding eigenfunctions ψ_i have the property that $\mu_1 \not\leq \mu_2 \leq \ldots \leq \mu_N \leq 1 < \mu_{N+1}$ for some integer N.

Assume the contrary that there exists $M > N$ distinct positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of (4. 1–2. 2) and each of which has an eigenfunction satisfying (4. 8).Let us fix such a set of eigenfunctions and label them $\phi_1, \phi_2, \ldots, \phi_M$, the order being of no
importance here.Then each ϕ_i satisfies (4. 1) and by Lemma 4. 7 $(\mathcal{R}\phi_i, \phi_j) = 0$, for $i \neq j$,$1 \leq i, j \leq M$.Now let $f = \sum_{j=1}^{M} c_j\phi_j$ be chosen so that

$$(\mathcal{Q}_- f, \psi_i) = 0$$

for $i = 1, 2, \ldots, N$.Since $M > N$ a nontrivial collection of $c_1, c_2, \ldots, c_M$ may always be found.Fixing such a set of c_j's we have f satisfies (2. 2) and

$$\begin{aligned}(\mathcal{R}f, f) &= (\sum_{i=1}^{M} c_i \mathcal{R}\phi_i, \sum_{j=1}^{M} c_j \phi_j) \\ &= \sum_{i=1}^{M} |\, c_i \,|^2 (\mathcal{R}\phi_i, \phi_i) \\ &\leq 0.\end{aligned}$$

Moreover,

$$\begin{aligned}(\mathcal{P}f', f') + (\mathcal{Q}f, f) &= (\sum_{i=1}^{M} c_i \mathcal{P}\phi_i', \sum_{j=1}^{M} c_j \phi_j') + (\sum_{i=1}^{M} c_i \mathcal{Q}\phi_i, \sum_{i=1}^{M} c_j \phi_j) \\ &= \sum_{i=1}^{M} |\, c_i \,|^2 [(\mathcal{P}\phi_i', \phi_i') + (\mathcal{Q}\phi_i, \phi_i)]\end{aligned}$$

since

$$(\mathcal{P}\phi_i', \phi_j') + (\mathcal{Q}\phi_i, \phi_j) = 0 \;\; for \;\; i \neq j.$$

Therefore

$$\begin{aligned}(\mathcal{P}f', f') + (\mathcal{Q}f, f) &= \sum_{i=1}^{M} |\, c_i \,|^2 (\mathcal{R}\phi_i, \phi_i) \\ &\leq 0\end{aligned}$$

or

$$(\mathcal{P}f', f') + (\mathcal{Q}_+ f, f) \leq (\mathcal{Q}_- f, f) \tag{4.9}$$

Now μ_{N+1} is the minimum of the ratio (4. 6) over those $y \in \mathcal{D}$ satisfying $(\mathcal{Q}_- y, \psi_i) = 0$ for $i = 1, \ldots, N$. Since f is such a vector function μ_{N+1} does not exceed one by (4. 9) and this is a contradiction. Thus $M \leq N$ and hence follows the theorem.

4.2 Existence and Distribution of the Real Eigenvalues

Hereafter we are concerned with only the real eigenvalues of (4. 1–2. 2). We will use those results found in Chapter 3 about two parameter problems to show the existence of nonnegative eigenvalues of (4. 1–2. 2).

If γ (≥ 0) is an eigenvalue of (4. 1–2. 2) with corresponding eigenfunction y, then

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \gamma(\mathcal{R}_+(t) - \mathcal{R}_-(t))y$$

and

$$y(a) = 0 = y(b).$$

But this is equivalent to

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y + \gamma\mathcal{R}_-(t)y = \gamma\mathcal{R}_+(t)y$$

and

$$y(a) = 0 = y(b).$$

Therefore γ is an eigenvalue of $(3.\ 1)_\gamma$–(2. 2) with eigenfunction y.
i. e. , $\gamma = \lambda_i(\gamma)$ for some i.

Conversely, if the curve $\lambda_i(\mu)$ in $\mu\lambda$-plane intersects the line $\lambda = \mu$ at the point $\mu = \mu_0$ then $\lambda_i(\mu_0) = \mu_0$ and therefore there is some $y \in \mathcal{D}$ satisfying

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y + \mu_0\mathcal{R}_-(t)y = \mu_0\mathcal{R}_+(t)y.$$

or

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \mu_0\mathcal{R}(t)y.$$

Thus μ_0 is an eigenvalue of (4. 1–2. 2). Hence we have the following

Theorem 4.9 *A nonnegative real number μ is an eigenvalue of (4. 1–2. 2) if and only if μ is an eigenvalue of $(3.\ 1)_\mu$–(2. 2).*

In this section we shall also need the conditions

$\mathcal{H}^+$ *Hypothesis $\mathcal{H}_*$ is satisfied with $\mathcal{R}$ replaced by $\mathcal{R}_+$.*

and

$\mathcal{H}^-$ *Hypothesis $\mathcal{H}_*$ is satisfied with $\mathcal{R}$ replaced by $\mathcal{R}_-$.*

We next consider the curve $\lambda_n(\mu)$ for every n such that $\lambda_n(0) > 0$ and show that $\lambda_n(\mu) \leq \mu$ for sufficiently large μ, which in turn shows the curve $\lambda_n(\mu)$ intersects the line $\lambda = \mu$ in $\mu\lambda$-plane.In fact it suffices to show above result for $\tilde{\lambda}_n(\mu)$, eigenvalues of the problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}_+(t)y + \mu\mathcal{R}_-(t)y = \tilde{\lambda}(\mu)\mathcal{R}_+(t)y$$

with boundary conditions (2. 2), since $\tilde{\lambda}_n(\mu) \geq \lambda_n(\mu)$.If $\mu > 0$, then dividing through above equation by μ we get

$$-\frac{1}{\mu}(\mathcal{P}(t)y')' + \frac{1}{\mu}\mathcal{Q}_+(t)y + \mathcal{R}_-(t)y = \frac{\tilde{\lambda}(\mu)}{\mu}\mathcal{R}_+(t)y.$$

So if $\mu_2 > \mu_1 > 0$, then $1/\mu_1 > 1/\mu_2$ and consequently from Corollary 2. 5

$$\tilde{\lambda}_n(\mu_1)/\mu_1 \geq \tilde{\lambda}_n(\mu_2)/\mu_2 \qquad \text{for every } n$$

showing that $\tilde{\lambda}_n(\mu)/\mu$ is a nonincreasing function of μ. Therefore $\tilde{\lambda}_n(\mu) \leq \mu$ for sufficiently large μ.

From the preceding discussion we have that for every n such that $\lambda_n(0) > 0$, the curve $\lambda_n(\mu)$ intersects the line $\lambda = \mu$ in $\mu\lambda$-plane. These curves then generate infinite number of positive eigenvalues of (4. 1)-(2. 2).

A similar consideration with interchanging the roles of $\mathcal{R}_+$ and $\mathcal{R}_-$ we will get the sequence of negative eigenvalues of (4. 1)-(2. 2). We have proved the following

Theorem 4.10 *Let $\mathcal{P}, \mathcal{Q}$ satisfy the conditions $\mathcal{H}_1, \mathcal{H}^+, \mathcal{H}^-$ and $\mathcal{R}$ satisfy the condition $\mathcal{H}_3$, then the nondefinite Sturm-Liouville boundary value problem (4. 1)-(2. 2) has two infinite sequences of real eigenvalues, one positive and one negative, and each one of which has $+\infty$ and $-\infty$ for its only points of accumulation.*

In view of Theorem 4. 10 all the nonnegative eigenvalues of (4. 1–2. 2) are generated by the curves $\lambda_i(\mu)$ and can be obtained by studying the points of intersection of $\lambda_i(\mu)$ and the line $\lambda = \mu$ in $\mu\lambda$-plane.

Theorem 4.11 *There exists an integer i_1 such that $\lambda_i(\mu)$ generates exactly one eigenvalue of (4. 1–2. 2) whenever $i \geq i_1$.*

Proof. Since $\lambda_i(0)$ is positive for sufficiently large i we can certainly choose an integer i_1 such that $\lambda_i(\mu) > 0$ for every $i \geq i_1$ and μ. Then from the discussion prior to Theorem 4. 10 we have such $\lambda_i(\mu)$ must intersect the line $\lambda = \mu$ at least once. On the other hand from Theorem 3. 5 (ii) $\lambda_i(\mu)$ is strictly increasing and hence it can cross the line $\lambda = \mu$ only once. Therefore $\lambda_i(\mu)$ will generate exactly one eigenvalue when $i \geq i_1$.

From Theorem 2. 10 the eigenvalues of the boundary problem

$$-(\mathcal{P}(t)y')' + \mathcal{Q}(t)y = \delta\mathcal{R}_-(t)y \tag{4.10}$$

with boundary conditions (2. 2) are bounded below. Thus there exists a smallest eigenvalue δ_1 of (4. 10–2. 2) such that $\delta_1 > -\infty$. Note that if $\delta_1 \leq 0$ then $(-\delta_1, 0)$ is the point of intersection of $\lambda_1(\mu)$ and the μ axis in $\mu\lambda$-plane.

Let

$$0 \le \gamma_0 \le \gamma_1 \le \gamma_2 \le \ldots\ldots\ldots$$

be the complete sequence of nonnegative eigenvalues of (4. 1–2. 2). The existence of such a sequence follows from Theorem 4. 10. For convenience we denote the curve which generates γ_0 by $\lambda_{T_0}(\mu)$ and the eigenvalue generated by $\lambda_{i_1}(\mu)$ by γ_{T_1}, where i_1 is as in Theorem 4. 11. If for every i, $\lambda_i(\mu)$ generates n_i eigenvalues of (4. 1–2. 2) then the quantity T_2 defined by $\sum_{i=T_0}^{i_1}(n_i - 1)$ is finite from Theorem 4. 8. We now fix T_0, T_1 and T_2 together with $s_0 = \max\{\gamma_{T_1}, -\delta_1\}$.

Theorem 4.12 *For every $n \ge T_1$ the n^{th} eigenvalue γ_n of (4. 1)-(2. 2) is generated by the curve $\lambda_{n+i_1-T_1}(\mu)$. Moreover,*

$$\lambda_{n+i_1-T_1}(\gamma_n) = \gamma_n \tag{4.11}$$

Proof. Since $\lambda_i(\mu)$ generates exactly one eigenvalue of (4. 1)-(2. 2) for $i > i_1$, the index of the curve which generates γ_n must be $(n - T_1) + i_1$. The equation (4. 11) is now obvious.

Theorem 4.13 *Let $n_+(s)$ and $n_\mu(s)$ denote the number of eigenvalues of respective boundary problems (4. 1–2. 2) and (3. 1)$_\mu$–(2. 2) which lie in the interval $[0\ ,\ s]$. Then the following identity holds for $s > s_0$*

$$n_+(s) \;=\; n_s(s) + (T_2 - T_0 - 1). \tag{4.12}$$

Proof. Because of Theorem 3. 1 the largest eigenvalue of problem (4. 1–2. 2) and the largest eigenvalue of (3. 1)$_s$–(2 . 2) which lie in the interval $[0\ ,\ s]$ are generated by the same curve. Let us denote this curve by $\lambda_{T_s}(\mu)$. Then

$$n_+(s) = \sum_{i=T_0}^{T_s} n_i$$

and

$$n_s(s) = T_s$$

as $s > s_0$. Also, since $\lambda_i(\mu)$ generates only one eigenvalue of (4. 1-2. 2), for $i \geq i_1$ we have

$$\begin{aligned} T_2 &= \sum_{i=T_0}^{i_1} (n_i - 1) \\ &= \sum_{i=T_0}^{i_1} (n_i - 1) + \sum_{i=i_1+1}^{T_s} (n_i - 1) \\ &= \sum_{i=T_0}^{T_s} (n_i - 1) \end{aligned}$$

Then the expansion of this sum yields

$$\begin{aligned} T_2 &= \sum_{i=T_0}^{T_s} n_i - (T_s - T_0 - 1) \\ &= n_+(s) - n_s(s) + (T_0 + 1). \end{aligned}$$

From this there follows (4. 12).

The following two corollaries are immediate consequences of this theorem.

Corollary 4.14 *The* $\lim_{s\to\infty} n_+(s)/\sqrt{s}$ *exists if and only if the* $\lim_{s\to\infty} n_s(s)/\sqrt{s}$ *exists and when they exist the two limits are equal.*

Corollary 4.15 *The distribution function* $n_\mu(s)$ *satisfies the inequality*

$$n_s(s) \leq n_{s_0}(s) \qquad \textit{for } s > s_0.$$

4.3 Asymptotic Distribution of the Eigenvalues

The purpose of this section is to formulate a conjecture which we answer positively for some special classes of matrices $\mathcal{P}, \mathcal{Q}, \mathcal{R}$.

Conjecture 4.16 *Let $\mathcal{P}, \mathcal{Q}$ satisfy the condition $\mathcal{H}_1$ and $\mathcal{R}$ satisfy the condition $\mathcal{H}_3$, then the following asymptotic formulas hold for the eigenvalue distribution functions $n_+(s)$ and $n_-(s)$ of (4. 1)-(2. 2).*

$$\lim_{s\to\infty} \frac{n_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_+(t))dt \tag{4.13}$$

and

$$\lim_{s\to\infty} \frac{n_-(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_-(t))dt \tag{4.14}$$

where $\Lambda_{1/2}$ is defined as in Theorem 2. 12.

In the one dimensional case, i. e. , when $m = 1$, this is simply a special form of Jörgens conjecture [19]. In this case Atkinson-Mingarelli [3] have answered the conjecture positively and showed that the scalar boundary value problem

$$-(p(t)u')' + q(t)u = \lambda r(t)u \tag{4.15}$$

$$u(a) = 0 = u(b) \tag{4.16}$$

admits the following asymptotics :

$$\lim_{s\to\infty} \frac{N_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sqrt{\left(\frac{r(t)}{p(t)}\right)_+}\,dt \tag{4.17}$$

and

$$\lim_{s\to\infty} \frac{N_-(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sqrt{\left(\frac{r(t)}{p(t)}\right)_-}\,dt \tag{4.18}$$

where $N_+(s)$ and $N_-(s)$ are the respective positive and negative eigenvalue distributions of (4. 15–16) p, q, r satisfy $1/p, q, r \in L(a\ ,\ b)$ and $\int_a^b r_+(t)dt > 0$ *and* $\int_a^b r_-(t)dt > 0$.

Note that if we consider the problem

$$-(p(t)u')' + q(t)u = \lambda r_+(t)u \tag{4.19}$$

with boundary conditions (4. 16), then the distribution of the positive eigenvalues of this problem will have the same asymptotic formula (4. 17). Also the asymptotic formula for the distribution of positive eigenvalues of the problem

$$-(p(t)u')' + q(t)u = \lambda r_-(t)u \tag{4.20}$$

with boundary conditions (4. 16) will coincide with (4. 18). Therefore the asymptotic distribution of the positive eigenvalues of the nondefinite problem depends only on the positive part of the function $r(t)$ and the negative eigenvalues depends only on the negative part of $r(t)$.

An extension of the preceding remark to the vector boundary value problems and the asymptotic formula obtained by Gohberg and Krein (see Chapter 2) for such right-definite problems is the motivation for Conjecture 4. 16.

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ appear in (4. 1) be diagonal and denoted by $\mathcal{P}(t) = diag(p_1(t), \ldots, p_m(t))$ $\mathcal{Q}(t) = diag(q_1(t), \ldots, q_m(t))$ and $\mathcal{R}(t) = diag(r_1(t), \ldots, r_m(t))$. If λ is an eigenvalue of (4. 1)-(2. 2) with eigenfunction $y = (y_i)$, then

$$-(p_i(t)y_i')' + q_i(t)y_i = \lambda r_i(t)y_i \tag{4.21}$$

and

$$y_i(a) = 0 = y_i(b)$$

for at least one i since $y \neq 0$.

Conversely if λ is an eigenvalue of (4. 21) for some i then it is also an eigenvalue of (4. 1)-(2. 2) with eigenfunction

$$y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, λ is an eigenvalue of (4. 1)-(2. 2) if and only if it is an eigenvalue of (4. 21) for some i. Hence we have

$$n_+(s) = \sum_{i=1}^{m} N_+^{(i)}(s)$$

where $N_+^{(i)}(s)$ is the number of eigenvalues of (4. 21) which lie in the interval $[0\ ,\ s]$. From (4. 17) $\lim_{s\to\infty} N_+^{(i)}(s)/\sqrt{s} = 1/\pi \int_a^b \sqrt{(r_i(t)/p_i(t))_+}dt$ and therefore

$$\begin{aligned} \lim_{s\to\infty} \frac{n_+(s)}{\sqrt{s}} &= \sum_{i=1}^{m} \frac{1}{\pi} \int_a^b \sqrt{\left(\frac{r_i(t)}{p_i(t)}\right)_+} dt \\ &= \frac{1}{\pi} \int_a^b \sum_{i=1}^{m} \sqrt{\left(\frac{r_i(t)}{p_i(t)}\right)_+} dt \\ &= \frac{1}{\pi} \int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_+(t))dt. \end{aligned}$$

Similarly we can establish (4. 14) for the negative eigenvalues. Hence we have proved

Theorem 4.17 *Let $\mathcal{P}, \mathcal{Q}$ satisfy $\mathcal{H}_1$ and $\mathcal{R}$ satisfy $\mathcal{H}_3$ then Conjecture 4. 16 is true if the marices $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ in (4. 1) are diagonal.*

In order to answer Conjecture 4. 16 when the matrices $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are not necessarily diagonal, we will consider the right-definite problem (3. 1)$_\mu$–(2. 2). A combination of Corollary 4. 15 and Theorem 3. 6 will give us

$$\overline{\lim_{s\to\infty}} \frac{n_s(s)}{\sqrt{s}} \leq l \tag{4.22}$$

where $l = \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{P}^{-1}(t)\mathcal{R}_+(t))dt$. Then

$$\overline{\lim_{s\to\infty}} \frac{\nu_+(s)}{\sqrt{s}} \leq l$$

because of Theorem 4. 13.

If the matrix $\mathcal{Q}(t)$ in (3. 1)$_\mu$ is replaced by $\tilde{\mathcal{Q}}_+(t) = q_1^+(t)\mathcal{I}$ we get

$$-(\mathcal{P}(t)y')' + \tilde{\mathcal{Q}}_+(t)y + \mu\mathcal{R}_-(t)y = \lambda(\mu)\mathcal{R}_+(t)y \tag{4. 23$_\mu$}$$

where $q_1^+(t)$ is as in Theorem 1. 11. From Corollary 2. 5 the nth eigenvalue of the problem (3. 1)$_\mu$–(2. 2) does not exceed the nth eigenvalue of problem (4. 23)$_\mu$–(2. 2) since $\mathcal{Q}(t) \leq \mathcal{Q}_+(t) \leq \tilde{\mathcal{Q}}_+(t)$. Hence $n_\mu^0(s) \leq n_\mu(s)$ for every s and sufficiently large μ, where $n_\mu^0(s)$ is the number of eigenvalues of the problem (4. 23)$_\mu$–(2. 2) which lie in the interval $[0\ ,\ s]$. This in particular implies $n_s^0(s) \leq n_s(s)$ for sufficiently large s and

$$\underline{\lim_{s\to\infty}} \frac{n_s(s)}{\sqrt{s}} \geq \underline{\lim_{s\to\infty}} \frac{n_s^0(s)}{\sqrt{s}}. \tag{4.24}$$

We recall that if $\mathcal{R}(t)$ is functionally commutative then there exists a constant nonsingular matrix $\mathcal{U}$ such that $\mathcal{U}^{-1}\mathcal{R}_+(t)\mathcal{U} = \mathcal{J}_+(t)$ and $\mathcal{U}^{-1}\mathcal{R}_-(t)\mathcal{U} = \mathcal{J}_-(t)$ where $\mathcal{J}_+$ and $\mathcal{J}_-$ are diagonal matrices and their diagonal entries consist of the eigenvalues of $\mathcal{R}_+$ and $\mathcal{R}_-$ respectively (see Section 1. 2).

Theorem 4.18 *Let $\mathcal{Q}$ satisfy $\mathcal{H}_1, \mathcal{H}^+, \mathcal{H}^-$ and $\mathcal{R}$ satisfy $\mathcal{H}_3$ then Conjecture 4. 16 is true if $\mathcal{P}(t) \equiv \mathcal{I}$ and the matrix function $\mathcal{R}(t)$ is functionally commutative.*

Proof. When $\mathcal{P}(t) \equiv \mathcal{I}$ the equation (4. 23) becomes

$$-y'' + \tilde{\mathcal{Q}}_+(t)y + \mu\mathcal{R}_-(t)y = \lambda(\mu)\mathcal{R}_+(t)y.$$

Now substituting $y = \mathcal{U}z$ in this equation and then left-multiplying by $\mathcal{U}^{-1}$ we get

$$-z'' + \tilde{\mathcal{Q}}_+(t)z + \mu\mathcal{J}_-(t)z = \lambda(\mu)\mathcal{J}_+(t)z$$

since $\mathcal{U}^{-1}\tilde{\mathcal{Q}}_+(t)\mathcal{U} = \tilde{\mathcal{Q}}_+(t)$. Now considering the corresponding nondefinite problem with diagonal matrices

$$-z'' + \tilde{\mathcal{Q}}_+(t)z = \lambda\mathcal{J}(t)z$$

with boundary conditions (2. 2), where $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_-$, and making use of Corollary 4. 14 and Theorem 4. 17 we deduce

$$\begin{aligned} \lim_{s\to\infty} \frac{n_s^0(s)}{\sqrt{s}} &= \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{J}_+(t))dt \\ &= \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{R}_+(t))dt. \end{aligned}$$

Then

$$\lim_{s\to\infty} \frac{n_s(s)}{\sqrt{s}} \geq \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{R}_+(t))dt$$

from (4. 24) and formula (4. 13) follows from the inequality (4. 22) and Corollary 4. 14.

A similar consideration with interchanging the roles of $\mathcal{R}_+$ and $\mathcal{R}_-$ will give us (4. 14).

Corollary 4.19 *Let $\mathcal{Q}(t)$ satisfy the conditions in Theorem 4. 18 and $\mathcal{R}(t)$ be a constant symmetric matrix, then Conjecture 4. 16 is true if the nondefinite problem (4. 1)-(2. 2) is regular.*

Proof. The regularity of the problem (4. 1)-(2. 2) will allow us to use the results in Chapter 3 and Section 4. 2. Then the corollary follows from Theorem 4. 18 since every constant matrix is functionally commutative.

Theorem 4.20 *Let $\mathcal{Q}(t)$ satisfy $\mathcal{H}_1$ and $\mathcal{H}^+$. Then Conjecture 4. 16 is true for $n_+(s)$ if $\mathcal{P}(t) \equiv \mathcal{I}$ and the matrix function $\mathcal{R}(t)$ satisfies the conditions $\mathcal{H}_4$ and $\mathcal{H}_5$ where*

$\mathcal{H}_4$ *The matrix function $\mathcal{R}(t)$ satisfies $\mathcal{H}_3$ and has the property that $\mathcal{R}_+(t)$ is diagonal.*

$\mathcal{H}_5$. *The matrix function $\mathcal{R}(t)$ satisfies $\mathcal{H}_3$ and has the property that $\mathcal{R}(t)$ is either nonnegative definite or nonpositive definite at each $t \in [a\,,\,b]$.*

Proof. The eigenvalues $\tilde{\lambda}_n(\mu)$ of the problem

$$-y'' + \tilde{\mathcal{Q}}_+(t)y + \mu\tilde{\mathcal{R}}_-(t)y = \tilde{\lambda}(\mu)\mathcal{R}_+(t)y \qquad (4.\ 25)_\mu$$

with boundary conditions (2. 2) where $\tilde{\mathcal{Q}}_+(t) = q_1^+(t)\mathcal{I}$, $\tilde{\mathcal{R}}_-(t) = r_\mu^-(t)\mathcal{I}$, are comparable with $\lambda_n(\mu)$, the eigenvalues of problem $(4.\ 23)_\mu$-(2. 2), since

$$\mathcal{Q}_+(t) \leq \tilde{\mathcal{Q}}_+(t) \quad and \quad \mathcal{R}_-(t) \leq \tilde{\mathcal{R}}_-(t)$$

from Corollary 1. 12. In fact from Corollary 2. 5

$$\tilde{\lambda}_n(\mu) \geq \lambda_n(\mu) > 0 \qquad for \qquad n = 1, 2, \ldots$$

and hence

$$\tilde{n}_\mu(s) \leq n_\mu^0(s)$$

for every μ and s where $\tilde{n}_\mu(s)$ is the distribution function for the eigenvalues of (4. 25)$_\mu$-(2. 2).Thus, in particular,

$$\tilde{n}_s(s) \leq n_s^0(s). \tag{4.26}$$

Next, we construct a nondefinite boundary value problem corresponding to the two parameter problem (4. 25)$_\mu$-(2. 2), viz.,

$$-y'' + \tilde{\mathcal{Q}}_+(t)y = \gamma\tilde{\mathcal{R}}(t)y \tag{4.27}$$

with boundary conditions (2. 2) where $\tilde{\mathcal{R}}(t) = \mathcal{R}_+(t) - \tilde{\mathcal{R}}_-(t)$.Then $\tilde{\mathcal{R}}(t)$ is a diagonal matrix satisfying the condition $\mathcal{H}_3$ with $\tilde{\mathcal{R}}_+(t) = \mathcal{R}_+(t)$ since $\mathcal{R}_+(t) \geq \mathcal{O}$, $\tilde{\mathcal{R}}_-(t) \geq \mathcal{O}$ and

$$\mathcal{R}_+(t)\tilde{\mathcal{R}}_-(t) = \mathcal{R}_+(t)r_m^-(t)\mathcal{I} = r_m^-(t)\mathcal{R}_+(t) = \tilde{\mathcal{R}}_-(t)\mathcal{R}_+(t) = \mathcal{O}$$

because of condition $\mathcal{H}_5$.

Since the matrices $\tilde{\mathcal{Q}}_+$ and $\tilde{\mathcal{R}}$ in (4. 27) are diagonal we have from Theorem 4. 17,

$$\begin{aligned}\lim_{s\to\infty} \frac{\tilde{n}_s(s)}{\sqrt{s}} &= \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\tilde{\mathcal{R}}_+(t))dt \\ &= \frac{1}{\pi}\int_a^b \Lambda_{1/2}(\mathcal{R}_+(t))dt = l.\end{aligned}$$

Therefore, from (4. 26)

$$\underline{\lim}_{s\to\infty} \frac{n_s^0(s)}{\sqrt{s}} \geq l. \tag{4.28}$$

Combining this with (4. 24) we have the theorem from (4. 22) and Corollary 4. 14.

The following corollary can be proved by interchanging the roles of $\mathcal{R}_+$ and $\mathcal{R}_-$ in the above theorem.

Corollary 4.21 *Let $\mathcal{Q}(t)$ satisfy $\mathcal{H}_1$ and $\mathcal{H}^-$. Then Conjecture 4. 16 is true for $n_-(s)$ if $\mathcal{P}(t) \equiv \mathcal{I}$ and the matrix function $\mathcal{R}(t)$ satisfies the conditions $\mathcal{H}_4'$ and $\mathcal{H}_5$ where*

$\mathcal{H}_4'$. *The matrix function $\mathcal{R}(t)$ satisfies $\mathcal{H}_3$ and has the property that $\mathcal{R}_-(t)$ is diagonal.*

and $\mathcal{H}_5$ is as in Theorem 4. 20.

In view of the Theorem 3. 6 the eigenvalue distribution function $n_\mu(s)$ of (3. 1 $)_\mu$-(2. 2) satisfies the formula

$$\lim_{s\to\infty}\frac{n_\mu(s)}{\sqrt{s}} = l \tag{4.29}$$

Therefore, we can write an asymptotic equation for $n_\mu(s)$ as

$$n_\mu(s) = \sqrt{s}l + R(\mu, s) \tag{4.30}$$

with the remainder $R(\mu, s)$. Then

$$\lim_{s\to\infty}\frac{R(\mu,s)}{\sqrt{s}} = 0.$$

from (4. 29) for each fixed μ. On the other hand from Corollary 4. 14 the distribution function $n_+(s)$ of (4. 1)-(2. 2) will have the asymptotic formula (4. 13) if and only if

$$\lim_{s\to\infty}\frac{R(s,s)}{\sqrt{s}} = 0 \tag{4.31}$$

Thus in order to attain the formula (4. 13) it is sufficient that

$$R(\mu, s) = \mathcal{O}\left(\frac{1}{\mu^\delta}\right) \tag{4.32}$$

for every fixed s and some $\delta \geq 0$.Therefore, the asymptotics for the distribution function $n_+(s)$ of the nondefinite problem (4. 1)-(2. 2) depends on the remainder in the asymptotic equation (4. 30) of the distribution function $n_\mu(s)$ of the right definite problem (3. 1 $)_\mu$-(2. 2).

We state these results in the following

Theorem 4.22 *Conjecture 4. 16 is true for $n_+(s)$ if and only if the remainder $R(\mu, s)$ in the asymptotic equation (4. 30) of $n_\mu(s)$ has the property (4. 31). Moreover, if $R(\mu, s)$ has the property (4. 32) then $n_+(s)$ satisfies the formula (4. 13).*

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