uOttawa

L'Université canadienne
Canada's university

**FACULTÉ DES ÉTUDES SUPÉRIEURES ET POSTOCTORALES**

uOttawa

L'Université canadienne
Canada's university

**FACULTY OF GRADUATE AND POSDOCTORAL STUDIES**


Christopher Dionne
AUTEUR DE LA THÈSE / AUTHOR OF THESIS

M.Sc. (Mathematics)
GRADE / DEGREE

Department of Science, Mathematics and Statistics
FACULTÉ, ÉCOLE, DÉPARTEMENT / FACULTY, SCHOOL, DEPARTMENT


Deligne-Lusztig Varieties

TITRE DE LA THÈSE / TITLE OF THESIS

M. Nevins
DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

E. Neher

P. Mezo

Gary W. Slater
Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

# Deligne-Lusztig Varieties

Christopher Dionne

Thesis Submitted to the Faculty of Graduate and Postdoctoral Studies
In partial fulfilment of the requirements for the degree of Master of Science in Mathematics [1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa



[1]The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

*Your file Votre référence*
*ISBN:* 978-0-494-61219-4
*Our file Notre référence*
*ISBN:* 978-0-494-61219-4

# Abstract

In this thesis we expose some of the theory behind Deligne-Lusztig varieties. To this end, we develop the general theory of flag varieties, the theory of $BN$-pairs, and the theory of varieties defined over finite ground fields. With these tools in hand, we define Deligne-Lusztig varieties. We then study the Deligne-Lusztig varieties associated with general linear groups using flags, and give a complete description in the case of $\mathrm{GL}_3$.

# Acknowledgements

Here is where the truth comes out; where I confess that this thesis, which I claimed to have authored, is in fact mostly due to other people. First, I thank my Thesis advisor, and mathematical mother: Monica Nevins. She has been looking after me for a long time now; I thank her for all of her support be it moral, mathematical or financial.

I am eternally grateful to Elizabeth Maltais who has for nearly two years imbued my life with an intense form of happiness theretofore unknown to me. I am glad that we could make this journey together. Writing this thesis, as well as every other task, has been more joyful with her by my side. She has my thanks as well as my heart. I must also thank her for her complete mastery of LaTeX, which she is always willing to share.

Next I thank my family: My parents Constanc Dionne and Beverly Boyd, my sister Katy and nephew Cole. I would especially like to thank my grandparents Jack and Mildred Boyd to whom I owe so much. I thank them all for their constant encouragement. I also thank the Maltais family who have been so welcoming.

Of course I would be nowhere without my friends: Jesse Collingwood, Andrew Staal, Matthew Mazowita, Marc Casey, Samantha Marion, Éric Serré, Hemza Yagoub, and Jerome Lefebvre. They have all, in their own way, made profound and indelible marks upon my life; I am who I am largely because of them.

I also would like to thank Erhard Neher for his encouragement and wisdom, and

Daniel Daigle for his work in the schemes seminar. I also thank my examiners, Paul Mezo and Erhard Neher for their helpful suggestions and kind comments.

Finally, I thank the University of Ottawa, particularly the Department of Mathematics and Statistics, for giving me a place to work and the means to do so.

# Dedication

For Jack.

# Contents

# Chapter 1

# Introduction

This thesis is mainly concerned with two things. Although it is the secondary goal, we mention first that we would like to better understand the structure of the general linear groups. We do this by using them as the running example in this thesis. The primary goal is writing down a definition for the Deligne-Lusztig varieties associated with certain algebraic groups. These varieties were introduced in the paper [Deligne-Lusztig], as a tool for studying the representation theory of finite groups of Lie type. We now state the definition found in that paper. Of course, they are not called Deligne-Lusztig varieties there, but are simply denoted $X(w)$.

**Definition 1.0.1** [Deligne-Lusztig, 1.4] Let $G$ be a connected, reductive algebraic group defined over $\mathbb{F}_q$ with Frobenius map $F$. Let $X$ be the flag variety and let $W$ be the Weyl group. Then, for $w \in W$, $X(w) \subset X$ is the locally closed subscheme of $X$ consisting of all Borel subgroups $B$ of $G$ such that $B$ and $F(B)$ are in relative position $w$.

It is clear that, especially for the beginning graduate student, there are many things to be explained before this can be made sense of: What does it mean for a variety to be *defined over* $\mathbb{F}_q$? What is a *flag variety*, *Weyl group*, *Borel subgroup*? Moreover, what is a *scheme*, and what does it mean for things to be *in relative position* $w$? So when we say that this thesis is about writing down a definition, we mean that we want to shed some light on the concepts on which the definition relies.

Before we delve too deeply into this, let's take a moment to establish what sort of background is assumed. Vaguely put, the reader is assumed to have a "good" grasp of "basic" algebraic geometry and algebraic groups; something like [Hartshorne, 1], and [Springer, 1-2,5]. We now make this more precise.

From algebraic geometry it is important to be comfortable with the notion of an algebraic variety. Of course the answer to the question: What is an algebraic variety? depends on who is asked. It is most desirable for us to use the following definition taken from [Springer, 1]:

**Definition 1.0.2** An *algebraic variety* is a compact separated space $X$ together with a sheaf of reduced $k$-algebras $\mathcal{O}_X$, such that $X$ can be covered by open sets each of which is isomorphic to a closed subset of an affine space.

If $X$ is isomorphic to a (Zariski) closed subset of an affine space then it is called an *affine variety.*

Such things are sometimes called, as in [Deligne-Lusztig], *reduced schemes of finite type over* $k$. These are not schemes in the usual sense, that is in the sense of [Hartshorne, 2]. Moreover, we do not need to be bothered too much about sheaves. Familiarity with the coordinate ring of an affine variety suffices most of the time, and in cases where it would not, it turns out we are mainly interested in the underlying topological space. Other key concepts we require are morphisms of varieties, see [Humphreys, 4] or [Springer, 5], and products of varieties presented in [Springer, 1] or [Humphreys, 2].

From the theory of algebraic groups we require the contents of [Springer, 2]; in particular all of our algebraic groups are linear. Any other prerequisite knowledge is listed at the start of the appropriate chapter.

Returning to Definition 1.0.1, note first that *scheme* is to be interpreted in the sense of Definition 1.0.2 and so, from here on, no actual scheme theory is needed in this thesis. We have stressed this point because, after reading Definition 1.0.1 a newcomer may be tempted to spend a great deal of time learning about schemes; this is a fine and noble cause, but in this case is ultimately not required.

Now, if we ask what it means for things to be *in relative position* $w$, and what is the *Weyl group*, we are led to the notion of the *Bruhat decomposition* of a reductive algebraic group. It is true that every reductive linear algebraic group has a Bruhat decomposition, and details of this very long chain of arguments can be found in [Springer, 7,8]. These notions can be captured using the idea of *Tits systems.* In fact it follows from the discussion mentioned in [Springer, 7,8], that a reductive linear algebraic group comes equipped with a Tits system. It is this idea that we study in Chapter 2. The advantage is that Tits systems have been axiomatized (see [Bourbaki]) and so can be studied abstractly without all the baggage of algebraic geometry.

Varieties defined over $\mathbb{F}_q$ are presented in Chapter 3. This has much the same flavour as an introduction to algebraic geometry over algebraically closed fields, but the extra structure we get from the Frobenius morphism does add some spice.

*Borel subgroups* and the *flag variety* are studied in Chapter 4; up to this point the discussion might be considered elementary, but from here on the work is much more serious. The proofs are typically much longer, and often require us to call upon high powered results from [Springer, 5]. The payoff is that we obtain very good results that hold for all linear algebraic groups.

Then, finally in Chapter 5, we face the Deligne-Lusztig varieties with some small hope of understanding them. The main goal in this chapter is to introduce and organize the Deligne-Lusztig varieties, as well as some interesting related varieties. At last, in Section 5.5, we investigate the Deligne-Lusztig varieties associated with the groups $\mathrm{GL}_n$ using flags. This approach allows us to completely settle the case of $\mathrm{GL}_3$.

The Deligne-Lusztig varieties are studied for their applications to representation theory; information gleaned from them, by means of zeta functions and cohomology, can be used to construct certain kinds of representations of finite groups of Lie type. See [Deligne-Lusztig] or [Geck].

# Chapter 2

# The Theory of $BN$-pairs

The first stop along the road to Deligne-Lusztig varieties is, as discussed in the introduction, the Weyl group and the Bruhat decomposition. Following [Bourbaki, 4.2] and [Geck, 1.6], we show that any abstract group satisfying the axioms of a Tits system has a Bruhat decomposition (Theorem 2.3.6), and then we show that general linear groups admit Tits systems. In Section 2.4 we examine some of the internal structure of a group $G$ that admits a Tits system. We then prove a technical result called the exchange condition in Section 2.5. We put this result to use in discussing the longest element of the Weyl group.

The background required for this chapter is minimal; in particular no knowledge of algebraic geometry is required. All that is needed is some group theory and linear algebra.

## 2.1 Tits systems

If $G$ is a group and $B$ is a subgroup of $G$ then $B \times B$ acts on $G$ by $(b_1, b_2)g = b_1 g b_2^{-1}$. The orbits of this action are the sets $BgB$. They are called **double cosets** and the set of double cosets is denoted $B\backslash G/B$. The double cosets of any subgroup partition

$G$.

If $C = BgB$ and $C' = Bg'B$ then $CC' = \bigcup_{b_1,b_2 \in B} B(gb_1b_2g')B$. Thus $CC'$ is a union of double cosets.

**Definition 2.1.1** [Bourbaki, 4.2] Let $G$ be a group and let $B$ and $N$ be subgroups of $G$ such that $T = B \cap N$ is a normal subgroup of $N$. Let $W = N/T$, and let $S$ be a subset of $W$. The quadruple $(G, B, N, S)$ is called a **Tits system** if the following four axioms are satisfied:

(T1) The subgroup generated by $B \cup N$ is $G$.

(T2) The set $S$ is a generating set for $W$, and consists of elements of order two.

(T3) If $s \in S$ and $w \in W$ then $\dot{s}B\dot{w} \subseteq B\dot{w}B \cup B\dot{s}\dot{w}B$, for any representatives $\dot{s}$ of $s$ in $G$, and $\dot{w}$ of $w$ in $G$.

(T4) If $s \in S$ then $\dot{s}B\dot{s}^{-1} \not\subseteq B$, for any representative $\dot{s}$ of $s$ in $G$.

Since $T \subseteq B$ it suffices to verify axioms $(T3)$ and $(T4)$ for a single choice of representatives.

**Example 2.1.2** Let $G = \mathfrak{S}_3$ be the symmetric group on 3 letters, and $B = \{e, (13)\}$, and $N = \{e, (12)\}$. It is easily checked that the subgroup generated by $B$ and $N$ is $G$. Since $B \cap N = \{e\}$ we have $W = N$. Let $S = \{(12)\}$. Then $S$ generates $W$, and axioms $(T3), (T4)$ are easily checked. Thus $(\mathfrak{S}_3, B, N, S)$ is a Tits system.

Groups that do not admit Tits systems are easy to find. For example if $G$ is abelian then axiom $(T4)$ can never be satisfied. The quaternions provide a non-abelian example of a group that does not admit a Tits system: Since every subgroup is normal there is once again no possible choice for $B$. If $G$ admits a Tits system we may also say that $G$ is a group with a **BN-pair**, and the group $W$ is called the **Weyl group**.

**Definition 2.1.3** [Geck, 1.6] If $(G, B, N, S)$ is a Tits system such that $\bigcap_{n \in N} nBn^{-1} \subseteq T$, then we say that $(G, B, N, S)$ satisfies the **saturation axiom**.

**Example 2.1.4** Consider the Tits system $(\mathfrak{S}_3, B, N, S)$, of Example 2.1.2. We have $\bigcap_{n\in N} nBn^{-1} = B \cap \{e, (23)\} = \{e\} = T$. Thus $(\mathfrak{S}_3, B, N, S)$ satisfies the saturation axiom.

**Example 2.1.5** [Kumar, 5.1] Suppose that $(G, B, N, S)$ is a Tits system and let $H$ be a non-trivial group. Then it is straightforward to verify that $(G\times H, B\times H, N\times 1, S')$ is a Tits system, where $S' = \{(s, 1) \mid s \in S\}$. But, even if $(G, B, N, S)$ satisfies the saturation axiom, we show that $(G \times H, B \times H, N \times 1, S')$ does not. Indeed,

$$\bigcap_{n\in N} (n, 1)(B \times H)(n^{-1}, 1) = \bigcap_{n\in N} nBn^{-1} \times H \nsubseteq T \times 1.$$

**Definition 2.1.6** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, and suppose

1. there is a normal subgroup $U$ of $B$ such that $B = UT$;
2. for all $n \in N$, $nUn^{-1} \cap B \subseteq U$.

Then we say that $G$ has a **split $BN$-pair**.

**Example 2.1.7** Consider the Tits system $(\mathfrak{S}_3, B, N, S)$, of Example 2.1.2. Let $B = U$ then $U$ is normal in $B$ and $B = UT$; recall $T = \{e\}$. It is clear that (2) is satisfied. Thus $\mathfrak{S}_3$ has a split $BN$-pair.

In the next section we produce a non-trivial example.

## 2.2 Tits Systems for General Linear Groups

Following [Geck, 1.6] we show that, for a field $k$, the group $G = \mathrm{GL}_n(k)$ admits a Tits system. This can also be found in [Bourbaki, 4.2]. Let $B$ be the subgroup of upper-triangular matrices, and let $N$ be the subgroup of monomial matrices. A monomial matrix is one with exactly one non-zero entry in each row and column.

Then $T = B \cap N$ is the subgroup of diagonal matrices. Also, let $U$ be the upper-triangular unipotent matrices: These are upper-triangular matrices in which each diagonal entry is 1; they are sometimes called upper-unitriangular.

**Proposition 2.2.1** The subgroup $B$ may be written as a semi-direct product of $U$ and $T$, that is $B = U \rtimes T$.

**Proof:** We first show that $B = UT$. Let $b = (b_{ij})$, and take $t$ to be the diagonal matrix given by $t_{ii} = b_{ii}$, for $1 \leq i \leq n$. Now let $u \in U$ be given by $u_{ij} = b_{ij}t_{jj}^{-1}$, for $j > i$. Then $b = ut$, and thus $B = UT$. We now show that $U$ is normal in $B$. Let $u \in U$ and $b \in B$. Then $bub^{-1} \in B$ since $B$ is a subgroup. Moreover, the eigenvalues of any such element are all equal to 1, Therefore $bub^{-1} \in U$, which gives that $U$ is normal in $B$. Finally, since $T \cap U = \{1\}$, we have $B = U \rtimes T$. ▲

Recall that a **permutation matrix** is one that is obtained by reordering the columns of an identity matrix. The set of $n \times n$-permutation matrices is a subgroup of $\mathrm{GL}_n$ that is isomorphic to $\mathfrak{S}_n$. This isomorphism can be realized by the map that, for $1 \leq i \leq n-1$, sends the transposition $(i\ i+1) \in \mathfrak{S}_n$ to the matrix $s_i$ which is obtained by switching the $i^{th}$ and $i+1^{th}$ columns of the identity. It can be shown that the set of permutation matrices form a complete set of coset representatives for $N/T$. It follows that $W = N/T$ is a group which is isomorphic to $\mathfrak{S}_n$. When speaking of the Weyl group $W$ of $\mathrm{GL}_n$ we freely use the language of symmetric groups; that is we identify $W$ with $\mathfrak{S}_n$. The following lemma is immediate.

**Lemma 2.2.2** The group $W$ is generated by the subset $S = \{s_i \mid 1 \leq i \leq n-1\}$, in which each element has order two.

Now, if $i \neq j$ and $\alpha \in k$ let $X_{ij}(\alpha)$ be the matrix whose $(k, l)$ entry is $\alpha$ if $k = i$ and $l = j$, and is $\delta_{kl}$ otherwise, where $\delta_{kl}$ is the Kroenecker delta.

**Lemma 2.2.3** Let $1 \leq i, j \leq n$ and $i \neq j$. Then $X_{ij} = \{X_{ij}(\alpha) \mid \alpha \in k\}$ is a subgroup of $G$.

**Proof:** For any $\alpha \in k$ the matrix $X_{ij}(\alpha)$ is either upper or lower unitriangular, and hence has determinant 1. Thus $X_{ij} \subseteq G$.

From the formula for matrix multiplication we see that $X_{ij}(\alpha)X_{ij}(\beta) = X_{ij}(\alpha + \beta)$. Thus $X_{ij}$ is closed under multiplication, and $X_{ij}(\alpha)^{-1} = X_{ij}(-\alpha)$. Thus $X_{ij}$ is a subgroup of $G$. ▲

**Definition 2.2.4** The subgroups $X_{ij}$ of $G$ are called **root subgroups**.

**Proposition 2.2.5** Let $w \in W$, and let $\dot{w}$ be a representative of $w$ in $G$. Then $\dot{w}X_{ij}\dot{w}^{-1} = X_{w(i)w(j)}$.

**Proof:** Let $E_{ij}(\alpha)$ be the matrix whose $(i,j)$ entry is $\alpha$, and is 0 otherwise. Then $X_{ij}(\alpha) = I + E_{ij}(\alpha)$. Thus, it suffices to show that $\dot{w}E_{ij}\dot{w}^{-1} = E_{w(i)w(j)}$. It also suffices to show that $s_l E_{ij} s_l = E_{s_l(i)s_l(j)}$, for all $s_l \in S$, since $S$ is a generating set for $W$. Multiplication on the left by $s_l$ switches rows $l$ and $l+1$, while multiplication on the right by $s_l$ switches columns $l$ and $l+1$. Thus the only non-zero entry of $s_l E_{ij} s_l$ is in position $(s_l(i), s_l(j))$. ▲

**Lemma 2.2.6** [Geck, 1.6] Let $V_i$ be the set of upper-unitriangular matrices whose $(i, i+1)$-entry is 0, and set $X_i = X_{i,i+1}$ and $X_{-i} = X_{i+1,i}$. Then $V_i$ is a subgroup of $U$ that satisfies

1. $U = X_i V_i = V_i X_i$, and $V_i \cap X_i = \{I\}$, for $1 \leq i \leq n-1$;
2. $s_i V_i s_i = V_i$, and $s_i X_i s_i = X_{-i}$.

**Proof:** That $V_i$ is a subgroup, and that $U = X_i V_i = V_i X_i$ is an exercise in matrix multiplication which we omit. It is clear that $V_i \cap X_i = \{I\}$. The proof of (2) is similar to that of Proposition 2.2.5. ▲

**Theorem 2.2.7** [Geck, 1.6] The quadruple $(G, B, N, S)$ is a Tits system.

**Proof:** We show that $(G, B, N, S)$ satisfies axioms $(T1) - (T4)$.

$(T1)$ We must show that $G$ is generated by $B \cup N$. Let $g \in G$, and choose $b \in B$ such that $bg$ has as many leading zeroes as possible; that is, the total number of zeroes that appear, in any row, before the first non-zero entry. In this case we show that each row has a different number of leading zeroes. Indeed, if this were not the case we could row reduce $bg$ by subtracting some row from an earlier row to increase the number of leading zeroes, but this row reduction amounts to multiplying $bg$ on the left by some $X_{ij}(\alpha)$ with $i < j$. That is, we have found some $b' \in B$ such that $b'g$ has more leading zeroes than $bg$, contradicting the choice of $b$. Thus $bg$ differs by a permutation matrix, which is of course monomial, from an element of $B$. So $nbg \in B$ for some $n \in N$.

$(T2)$ This is Lemma 2.2.2.

$(T3)$ We begin with the identity

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & -t^{-1} \end{pmatrix} \tag{2.2.1}$$

in $\mathrm{GL}_2$. Now, for $1 \le i \le n-1$ there is an embedding $\phi_i : \mathrm{GL}_2 \to \mathrm{GL}_n$ given by

$$A \mapsto \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

By Lemma 2.2.6 we have that $U = X_i V_i$ and $X_{-i} = s_i X_i s_i$. By Proposition 2.2.1 we have $B = UT$, thus $B = X_i V_i T$. So

$$s_i B s_i = s_i V_i X_i T s_i = s_i V_i s_i X_{-i} s_i T s_i = V_i X_{-i} T$$

Applying $\phi_i$ to Equation 2.2.1 gives $X_{-i} \subseteq \{I\} \cup X_i s_i X_i T$, so

$$V_i X_{-i} T \subseteq V_i(\{I\} \cup X_i s_i X_i T)T = V_i T \cup V_i X_i s_i X_i T \subseteq B \cup B s_i B.$$

Let $w \in W$, and let $1 \leq i \leq n-1$. There are two cases: $w^{-1}(i) < w^{-1}(i+1)$ and $w^{-1}(i) > w^{-1}(i+1)$. In the first case we have

$$\begin{aligned} s_i B \dot{w} &= s_i V_i X_i T \dot{w} \\ &= V_i s_i X_i T \dot{w} \\ &= V_i s_i \dot{w} \dot{w}^{-1} X_i \dot{w} \dot{w}^{-1} T \dot{w} \\ &= V_i s_i \dot{w} X_{w^{-1}(i) w^{-1}(i+1)} T \quad \text{(Proposition 2.2.5)} \\ &\subseteq V_i s_i \dot{w} U T \\ &\subseteq B s_i \dot{w} B. \end{aligned}$$

In the second case $w^{-1}(i) > w^{-1}(i+1)$, and let $y = s_i w$ so that $y^{-1}(i) = w^{-1}(i+1) < w^{-1}(i) = y^{-1}(i+1)$, and by the first case $s_i B \dot{y} \subseteq B s_i \dot{y} B = B \dot{w} B$. Thus,

$$\begin{aligned} s_i B \dot{w} &= s_i B s_i \dot{y} \\ &\subseteq (B \cup B s_i B) \dot{y} \\ &= B \dot{y} \cup B s_i B \dot{y} \\ &\subseteq B s_i \dot{w} \cup B(B \dot{w} B) \\ &\subseteq B s_i \dot{w} B \cup B \dot{w} B. \end{aligned}$$

So, for all $w \in W$ and $s_i \in S$, we have $s_i B \dot{w} \subseteq B s_i \dot{w} B \cup B \dot{w} B$, and so axiom $(T3)$ is satisfied.

($T4$) Conjugating $B$ by an element $s_i$ we obtain $s_iBs_i = s_iX_is_is_iV_is_is_iTs_i$, which by Lemma 2.2.6(2) is equal to $X_{-i}V_iT$, but $X_{-i}V_iT \not\subseteq B$. So, $s_iBs_i \not\subseteq B$ for any $s_i \in S$.

Thus $(G, B, N, S)$ is a Tits system. ▲

**Proposition 2.2.8** The Tits system $(G, B, N, S)$ for $G = \mathrm{GL}_n$ described above satisfies the saturation axiom.

**Proof:** Let $B^-$ be the subgroup of lower triangular matrices, and let $w_0 = (1\, n)(2\, n-1) \cdots \in \mathfrak{S}_n$. Then one verifies that $B^- = \dot{w}_0 B \dot{w}_0{}^{-1}$, and so $\bigcap_{n \in N} nBn^{-1} \subseteq B^- \cap B = T$. ▲

**Proposition 2.2.9** The group $G = \mathrm{GL}_n$ has a split $BN$-pair.

**Proof:** By Proposition 2.2.1 $B$ can be written as the semi-direct product $B = U \rtimes T$. Moreover, for all $n \in N$, $nUn^{-1} \cap B \subseteq U$, because the eigenvalues of any element of $nUn^{-1}$ are 1, and we are intersecting with $B$. ▲

## 2.3 The Bruhat Decomposition

Here $(G, B, N, S)$ is a Tits system with Weyl group $W$. For $w \in W$ let $\dot{w}$ be a representative of $w$ in $N$, and let $C(w) = B\dot{w}B$. If $\dot{w}$ and $\dot{w}'$ are representatives of $w$ then $\dot{w}^{-1}\dot{w}' \in T$. So $B\dot{w}B = B\dot{w}\dot{w}^{-1}\dot{w}'B = B\dot{w}'B$. This shows that $C(w)$ is well defined. It is clear that $C(1) = B$, $C(w_1w_2) \subseteq C(w_1)C(w_2)$ and $C(w^{-1}) = C(w)^{-1}$.

**Definition 2.3.1** [Geck, 1.6] The double cosets $C(w)$ for $w \in W$ are called **Bruhat cells.**

**Example 2.3.2** Let $G = \mathrm{GL}_2$, then $W = \{T, wT\} \cong \mathfrak{S}_2$, where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So $C(w) = BwB = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \neq 0 \right\}$, and $C(1) = B$.

**Definition 2.3.3** [Geck, 1.6] Let $w \in W$, and $w \neq 1$. Then, as $S$ generates $W$, we may write $w = s_1 \dots s_q$ with $s_1, \dots, s_q \in S$. If the number $q$ is as small as possible then the expression $w = s_1 \dots s_q$ is called **reduced**, and the number $q = l(w)$ is called the **length** of $w$. If $w = 1$ we set $l(w) = 0$.

Note that, for $s \in S$ and $w \in W$, we have $l(w) - 1 \leq l(sw) \leq l(w) + 1$. Indeed, let $w = t_1 \dots t_p$ be a reduced expression; if $l(sw) > l(w)$ then $l(sw) \leq l(w) + 1$, because $st_1 \dots t_p$ is an expression for $sw$. Conversely, suppose $l(sw) < l(w)$, and let $w = t_1 \dots t_p$ be a reduced expression. If $l(sw) < l(w) - 1$ then there is an expression $sw = r_1 \dots r_q$ with $q < p - 1$, but this yields an expression $w = sr_1 \dots r_q$ which is too short. Thus, $l(sw) \geq l(w) - 1$. Moreover, $l(sw) \neq l(w)$. This can be shown using the relationship between groups with a $BN$-par and Coxeter groups, as can be found in [Bourbaki, 4.1].

We have also that $l(wy) \geq |l(w) - l(y)|$.

**Example 2.3.4** Let $G = \mathrm{GL}_3$ then $W \cong \mathfrak{S}_3$, and $S = \{(12), (23)\}$. Let $w = (123)$ then $w = (12)(23)$, and this is a reduced expression for $w$ because $w \notin S$. Thus $l(w) = 2$.

**Lemma 2.3.5** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, and let $J \subseteq S$. Set $W_J \subseteq W$ to be the subgroup generated by $J$. Also set $N_J \subseteq N$ to be the subgroup generated by the representatives of $W_J$ in $N$. Then the set $P_J = BN_JB$ is a subgroup of $G$.

**Proof:** We have $P_J^{-1} = (BN_JB)^{-1} = BN_JB$ since $B$ and $N_J$ are subgroups, and it is clear that $P_J$ contains the identity. It remains to show that $P_J$ is closed under multiplication. Since $BP_J \subseteq P_J$ it is enough to show that $N_JP_J \subseteq P_J$. Moreover, it is enough to show that $\dot{s}P_J \subseteq P_J$ for $s \in J$. Now,

$$\begin{aligned}
\dot{s}P_J &= \dot{s}BN_JB \\
&= \dot{s} \bigcup_{n \in N_J} BnB \\
&= \bigcup_{n \in N_J} \dot{s}BnB \\
&\subseteq \bigcup_{n \in N_J} (B\dot{s}nB \cup BnB) \quad \text{(by axiom } (T3)) \\
&\subseteq P_J.
\end{aligned}$$

So $P_J$ is a subgroup of $G$. ▲

**Theorem 2.3.6 (The Bruhat Decomposition)** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system with Weyl group $W$. Then $G$ may be written as a disjoint union: $G = \coprod_{w \in W} C(w)$.

**Proof:** In Lemma 2.3.5 if we take $J = S$ we obtain $P_S = BNB$, a subgroup of $G$ which contains $B$ and $N$. Thus by axiom $(T1)$ we have $P_S = BNB = G$. Hence $G = \bigcup_{w \in W} C(w)$. It remains to show that this union is disjoint; it suffices, as double cosets are either equal or disjoint, to show that if $C(w) = C(y)$ then $w = y$. We proceed by induction on $\min\{l(y), l(w)\}$, and we may assume that $l(y) \leq l(w)$.

Suppose that $C(w) = C(y)$ for some $w, y \in W$. If $l(y) = 0$ then $y = 1$, and so $B = B\dot{y}B$. So $B\dot{w}B = B$ showing that $\dot{w} \in B$. So $\dot{w} \in B \cap N = T$, and $w = 1 = y$. Now suppose that $l(y) > 0$, in particular let $y = s_1 \dots s_r$ be a reduced expression for $y$. Let $x = s_2 \dots s_r$ so $y = s_1x$, and $l(y) = l(x) + 1$. Now, $\dot{s_1}\dot{x}B \subseteq B\dot{s_1}\dot{x}B = B\dot{y}B = B\dot{w}B$, so $\dot{x}B \subseteq \dot{s_1}B\dot{w}B$. Applying $(T3)$ to this relation we obtain $B\dot{x}B \subseteq B\dot{s}_1\dot{w}B \cup B\dot{w}B$. So $B\dot{x}B = B\dot{s}_1\dot{w}B$ or $B\dot{x}B = B\dot{w}B$, again because double cosets are either equal or disjoint. If $B\dot{x}B = B\dot{w}B$ then, by induction, $x = w$ which is absurd since $l(x) < l(w)$. So $B\dot{x}B = B\dot{s}_1\dot{w}B$ which shows, by induction, that $x = s_1 w$. Therefore, $y = s_1 x = w$. ▲

## 2.4 Some Multiplication Rules for Bruhat Cells

Here we use the Bruhat decomposition to prove some identities for the multiplication in $G$. We then relate these results to the length function which allows us to prove some key statements: that the subgroup $B$ is self normalizing, and that the set $S$ is unique.

**Proposition 2.4.1** [Bourbaki, 4.2] Let $(G, B, N, S)$ be a Tits system, $s \in S$ and $w \in W$. Then:

1. $C(s)C(w) = \begin{cases} C(sw) & \text{if } C(w) \not\subseteq C(s)C(w) \\ C(w) \cup C(sw) & \text{if } C(w) \subseteq C(s)C(w) \end{cases}$;
2. $C(s)C(s) = B \cup C(s)$;
3. $C(s)C(s)C(w) = C(w) \cup C(sw)$.

**Proof:**

1. By axiom $(T3)$, we have $C(s)C(w) \subseteq C(w) \cup C(sw)$. It is clear that $C(sw) \subseteq C(s)C(w)$, so if $C(w) \subseteq C(s)C(w)$ then $C(w) \cup C(sw) \subseteq C(s)C(w)$. Applying (T3) gives equality.

   If $C(w) \not\subseteq C(s)C(w)$ then $C(w) \cap C(s)C(w) = \emptyset$, since $C(s)C(w)$ is a disjoint union of double cosets. But, $C(s)C(w) \subseteq C(w) \cup C(sw)$, thus $C(s)C(w) \subseteq C(sw)$. We have equality since we have a single double coset on the right.

2. If $C(s) \not\subseteq C(s)C(s)$ then by part (1) $C(s)C(s) = C(s^2) = C(1) = B$, but this contradicts axiom $(T4)$. Thus, by part (1), $C(s)C(s) = C(s) \cup B$.

3. By part (2), $C(s)C(s) = B \cup C(s)$. Thus $C(s)C(s)C(w) = (B \cup C(s))C(w) = C(w) \cup C(s)C(w)$. By part (1) $C(s)C(w) = C(sw)$ or $C(s)C(w) = C(w) \cup C(sw)$. In either case $C(s)C(s)C(w) = C(w) \cup C(sw)$.

▲

By inverting the relation in axiom $(T3)$ we obtain, for $w \in W$ and $s \in S$, $\dot{w}^{-1}B\dot{s}^{-1} \subseteq C(sw)^{-1} \cup C(w)^{-1} = C(w^{-1}s) \cup C(w^{-1})$. So we obtain a right-handed version of axiom $(T3)$: $\dot{w}B\dot{s} \subseteq C(ws) \cup C(w)$. There is thus the following right-handed version of Proposition 2.4.1.

**Corollary 2.4.2** [Bourbaki, 4.2] Let $(G, B, N, S)$ be a Tits system, $s \in S$ and $w \in W$. Then:

1. $C(w)C(s) = \begin{cases} C(ws) & \text{if } C(w) \not\subseteq C(w)C(s) \\ C(w) \cup C(ws) & \text{if } C(w) \subseteq C(w)C(s) \end{cases}$;
2. $C(w)C(s)C(s) = C(w) \cup C(ws)$.

The following refined version of Proposition 2.4.1(1) is very useful, as it marries the notion of multiplying Bruhat cells to the length function on $W$.

**Proposition 2.4.3** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, $s \in S$, and $w \in W$. Then

$$C(s)C(w) = \begin{cases} C(sw) & \text{if } l(sw) > l(w) \\ C(w) \cup C(sw) & \text{if } l(sw) < l(w). \end{cases}$$

**Proof:** We proceed by induction on $l(w)$. If $l(w) = 0$ then, for all $s \in S$, $l(sw) = l(w) + 1$. Moreover, $C(s)C(w) = C(s) = C(sw)$.

Suppose that $l(sw) > l(w)$, and write $w = yt$ with $t \in S$ and $l(w) = l(y) + 1$. If $l(sy) < l(y)$ then $l(sw) = l(syt) \leq l(sy)+1 < l(y)+1 = l(w)$ which is a contradiction.

Hence, $l(sy) > l(y)$. Thus, by induction, we have $C(s)C(y) = C(sy)$. Now, by Proposition 2.4.1(1) we have that $C(s)C(w) = C(sw)$ or $C(s)C(w) \cap C(w) \neq \emptyset$. We show the second is impossible as follows. Since $w = yt$, it would yield $\dot{s}B\dot{y} \cap B\dot{w}B\dot{t}^{-1} \neq \emptyset$, and so $C(s)C(y) \cap C(w)C(t) \neq \emptyset$. Therefore, $C(sy) \cap C(w)C(t) \neq \emptyset$. Moreover, by the right handed version of Proposition 2.4.1, $C(w)C(t) \subseteq C(wt) \cup C(w)$. Thus, $C(sy) \cap C(wt) \neq \emptyset$, or $C(sy) \cap C(w) \neq \emptyset$. If $C(sy) \cap C(wt) \neq \emptyset$ then $C(sy) = C(wt)$, and so the Bruhat decomposition gives that $sy = wt = yt^2 = y$. Thus $s = 1$, which is impossible. Similarly, if $C(sy) \cap C(w) \neq \emptyset$ then $sy = w$, but this also yields a contradiction since $l(y) < l(w) < l(sw)$. Thus, $C(s)C(w) \cap C(w) = \emptyset$, and so $C(s)C(w) = C(sw)$.

Now suppose that $l(sw) < l(w)$ then $l(ssw) = l(w) > l(sw)$, and by the first case we have that $C(s)C(sw) = C(w)$. So $C(s)C(s)C(sw) = C(s)C(w)$, but by Proposition 2.4.1(3) we have that $C(s)C(s)C(sw) = C(sw) \cup C(w)$. Thus $C(s)C(w) = C(sw) \cup C(w)$. ▲

We now show that the subgroup $B$ is self-normalizing. This result will be very useful in Section 5.1, where we define the variety of Borel subgroups.

**Corollary 2.4.4** [Geck, 1.6] If $(G, B, N, S)$ is a Tits system then the normalizer of $B$ in $G$ is $B$; that is, $N_G(B) = B$.

**Proof:** Let $g \in N_G(B)$. Then $g \in G$ so, using the Bruhat decomposition, write $g = b_1\dot{w}b_2$ for some $w \in W$ and $b_1, b_2 \in B$. Then $B = gBg^{-1} = b_1\dot{w}b_2B(b_1\dot{w}b_2)^{-1}$, and so $\dot{w} \in N_G(B)$. We show that $w = 1$. If $w \neq 1$ we may choose some $s \in S$ such that $l(sw) < l(w)$, and so, by Proposition 2.4.3, we have $C(s)C(w) = C(w) \cup C(sw)$, in particular $\dot{s}B\dot{w} \cap B\dot{w}B \neq \emptyset$. This gives that $\dot{s} \in B\dot{w}B\dot{w}^{-1}B$, but $\dot{w} \in N_G(B)$ so $\dot{s} \in N_G(B)$. In particular $\dot{s}B\dot{s}^{-1} \subseteq B$ contradicting $(T4)$. So $w = 1$ and $g \in B$. ▲

**Corollary 2.4.5** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, and $w \in W$. Then $w \in S$ if and only if $w \neq 1$ and $B \cup C(w)$ is a subgroup.

**Proof:** If $s \in S$ then $B \cup C(s) = C(s)C(s)$ by Proposition 2.4.1(2). Thus $B \cup C(s)$ is closed under multiplication and inversion; it contains the identity since it contains $B$. Thus $B \cup C(s)$ is a subgroup.

Conversely, if $B \cup C(w)$ is a subgroup for some $w \in W$, and $w \neq 1$ then we may choose some $s \in S$ such that $l(sw) < l(w)$. In this case, Proposition 2.4.3 shows that $C(w) \subseteq C(s)C(w)$. Thus $\dot{s} \in C(w)C(w)^{-1} \subseteq B \cup C(w)$. So $\dot{s} \in B$ or $\dot{s} \in C(w)$, but $\dot{s} \notin B$. So $\dot{s} \in C(w)$, thus $C(s) = C(w)$. We deduce, using the Bruhat decomposition, that $w = s \in S$. ▲

Corollary 2.4.5 shows us that if $(G, B, N, S)$ is a Tits system, and $(G, B, N, S')$ is also a Tits system, then $S = S'$. So $S$ is intrinsic and we may say that $(G, B, N)$ is a Tits system. This also explains the terminology "$G$ has a $BN$-pair" used, for example, in [Geck].

**Example 2.4.6** Let $G = \mathrm{GL}_3(\mathbb{F}_2)$. We have seen, in Section 2.2, that $G$ has a $BN$-pair, where $B$ is the upper-triangular subgroup, and $N$ is the subgroup of monomial matrices. Moreover,

$$S = \left\{ \sigma_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

but there is another element of order two in $W$, namely the matrix $\sigma_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Moreover, $S' = \{\sigma_{12}, \sigma_{13}\}$ is a generating set for $W$ consisting of elements of order two. But, even though axioms $(T1)$ and $(T2)$ are satisfied, we now show that $(G, B, N, S')$

is not a Tits system. Indeed, we have the equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The left hand side of this equation is an element of $B\sigma_{13}B\sigma_{13}B \subseteq C(\sigma_{13})C(\sigma_{13})$, and the right hand side is $\sigma_{123} \in C(\sigma_{123})$ which, by the Bruhat decomposition, is disjoint from $B$ and $C(\sigma_{13})$. Thus $B \cup C(\sigma_{13})$ is not a subgroup. So, by Corollary 2.4.5, $(G, B, N, S')$ is not a Tits system.

**Corollary 2.4.7** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system. If $w \in W$ has maximal length, then $G$ is generated by $B$ and $\dot{w}B\dot{w}^{-1}$.

**Proof:** If $w$ has maximal length then $l(sw) < l(w)$ for all $s \in S$. So, by Proposition 2.4.3, we have $C(s)C(w) = C(sw) \cup C(w)$. In particular $\dot{w} \in B\dot{s}B\dot{w}B$, which gives $\dot{s} \in B\dot{w}B\dot{w}^{-1}B$ which lies in the subgroup generated by $B$ and $\dot{w}B\dot{w}^{-1}$. So, since every $\dot{s} \in \langle B, \dot{w}B\dot{w}^{-1}\rangle$, we have that $N \subseteq \langle B, \dot{w}B\dot{w}^{-1}\rangle$. Thus $G = \langle B, \dot{w}B\dot{w}^{-1}\rangle$, as $G$ is generated by $B$ and $N$. ▲

## 2.5 The Exchange Condition and Longest Element

**Lemma 2.5.1** [Bourbaki, 4.2] Let $s_1, \ldots, s_q \in S$, and let $w \in W$. Then

$$C(s_1 \ldots s_q)C(w) \subseteq \bigcup_{\substack{i_1,\ldots,i_p \\ 0 \le p \le q \\ 1 \le i_1 < i_2 < \cdots < i_p \le q}} C(s_{i_1} \ldots s_{i_p} w).$$

**Proof:** We proceed by induction on $q$. If $q = 0$ there is nothing to prove, and if $q = 1$ then Proposition 2.4.1(1) says that $C(s)C(w) \subseteq C(sw) \cup C(w)$.

Now, suppose that $q > 1$, $s_1, \ldots, s_q \in S$, and $w \in W$. Since $C(s_1 \ldots s_q) \subseteq C(s_1)C(s_2 \ldots s_q)$ we have

$$\begin{aligned} C(s_1 \ldots s_q)C(w) &\subseteq C(s_1)C(s_2 \ldots s_q)C(w) \\ &\subseteq C(s_1) \bigcup_{\substack{j_1,\ldots,j_p \\ 0 \le p \le q-1 \\ 2 \le j_1 < j_2 < \cdots < j_p \le q}} C(s_{j_1} \ldots s_{j_p} w) \text{ by induction} \\ &= \bigcup_{\substack{j_1,\ldots,j_p \\ 0 \le p \le q-1 \\ 2 \le j_1 < j_2 < \cdots < j_p \le q}} C(s_1)C(s_{j_1} \ldots s_{j_p} w) \\ &\subseteq \bigcup_{\substack{j_1,\ldots,j_p \\ 0 \le p \le q-1 \\ 2 \le j_1 < j_2 < \cdots < j_p \le q}} \left(C(s_1 s_{j_1} \ldots s_{j_p} w) \cup C(s_{j_1} \ldots s_{j_p} w)\right). \end{aligned}$$

The last inclusion, which is obtained by an application Proposition 2.4.1(1), gives the desired result. ▲

**Proposition 2.5.2 (The Exchange Condition)** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, and $w \in W$. If $s_1 \ldots s_q$ is a reduced expression for $w$, and $s \in S$ is such that $l(sw) < l(w)$ then $sw = s_1 \ldots \hat{s}_i \ldots s_q$, for some $1 \le i \le q$; $\hat{s}_j$ indicates that $s_j$ is omitted from the expression.

**Proof:** If $l(sw) < l(w)$ then, by Proposition 2.4.3, $C(s)C(w) = C(sw) \cup C(w)$. In particular $\dot{s}B\dot{w} \cap B\dot{w}B \neq \emptyset$, and $\dot{s} \in B\dot{w}B\dot{w}^{-1}B = C(w)C(w^{-1})$. Let $w = s_1 \cdots s_q$ be a reduced expression. Then $\dot{s} \in C(w)C(w^{-1}) = C(s_1 \ldots s_q)C(w^{-1})$ which, by Lemma 2.5.1, is contained in $\bigcup_{i_1,\ldots,i_p} C(s_{i_1} \ldots s_{i_p} w^{-1})$ for $1 \le i_1 < i_2 < \cdots < i_p \le q$. Hence, by the Bruhat decomposition, there is some $x = s_{i_1} \cdots s_{i_p}$ such that $s = xw^{-1}$. Clearly $p < q$ else $s = ww^{-1} = 1$.

Let $t_1 \cdots t_r$ be a reduced expression for $x$. Then

$$1 = l(xw^{-1}) \ge l(w) - l(x) = q - r \ge q - p.$$

Thus $p = q - 1$ so $sw = x = s_1 \ldots \hat{s}_i \ldots s_p$ for some $1 \leq i \leq q$. ▲

**Proposition 2.5.3** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system, and let $J \subseteq S$. Let $W_J$ be the subgroup of $W$ generated by $J$. Then every $w \in W_J$ has a reduced expression consisting entirely of elements in $J$.

**Proof:** Let $w \in W_J$. Then we may write $w = s_1 \cdots s_q$ with each $s_i \in J$. If $l(w) < q$ then there is an $i \in \{1, \ldots, q\}$ such that $l(s_i \cdots s_q) < l(s_{i+1} \cdots s_q)$; otherwise we have a chain of inequalities:

$$1 = l(s_1) < l(s_1 s_2) < \cdots < l(s_1 \ldots s_q),$$

contradicting the fact that $l(w) < q$. Now we may apply the exchange condition to obtain $s_i \cdots s_q = s_{i+1} \cdots \hat{s}_j \cdots s_q$, for some $j$. Thus $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_q$, and we can continue in this way to obtain the desired reduced expression for $w$. ▲

If $(G, B, N, S)$ is a Tits system whose Weyl group $W$ is finite then there is some $w \in W$ that has maximal length. We will denote such an element by $w_0$, and refer to it as a **longest element**. More generally, if $J \subseteq S$ then the subgroup $W_J$ has a longest element, which we denote by $w_J$.

**Theorem 2.5.4** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system whose Weyl group $W$ is finite, and $J \subseteq S$. Let $w_J \in W_J$ be a longest element then, for any $w \in W_J$, $l(ww_J) = l(w_J) - l(w)$.

**Proof:** Proceed by induction on $l(w)$. If $l(w) = 0$ then $w = 1$, so $l(ww_J) = l(w_J) - 0$. Now suppose that $l(w) > 0$, and use Proposition 2.5.3 to take a reduced expression $w = s_1 \cdots s_p s$ for $w$, where $s \in J$ and each $s_i \in J$. Then, by induction, $l(s_1 \cdots s_p w_J) = l(w_J) - p$. Now, let $q = l(w_J) - p$ and take a reduced expression $t_1 \cdots t_q$ for $s_1 \cdots s_p w_J$, with each $t_i \in J$. Thus $w_J = s_p \cdots s_1 t_1 \cdots t_q$. Since $l(w_J)$ is maximal we have that $l(sw_J) < l(w_J)$, and so by Proposition 2.5.2 (the exchange condition) there are two cases. Either

$$(1)\ sw_J = s_p \cdots \hat{s}_i \cdots s_1 t_1 \cdots t_q \text{ for some } 1 \leq i \leq p,$$

or

$$(2)\ sw_J = s_p \cdots s_1 t_1 \cdots \hat{t}_j \cdots t_q \text{ for some } 1 \leq j \leq q.$$

If (1) is true then we can multiply on the right by $(s_1 \cdots s_p w_J)^{-1}$ to obtain $w^{-1} = ss_p \cdots s_1 = s_p \cdots \hat{s}_i \cdots s_1$, but this shows that $l(w^{-1}) < p$. Then $l(w) < p$, which is a contradiction.

Thus (2) holds. So by multiplying on the left by $s_1 \cdots s_p$ we obtain $ww_J = t_1 \cdots \hat{t}_j \cdots t_q$, and so $l(ww_J) \leq q-1$, but $l(ww_J) \geq q-1$ as $l(w) = p+1$. Thus, $l(ww_J) = q-1$ which is equal to $l(w_J) - l(w)$. ▲

**Corollary 2.5.5** [Geck, 1.6] Let $(G, B, N, S)$ be a Tits system whose Weyl group $W$ is finite, and let $J \subseteq S$. Then

1. there is a unique longest element $w_J \in W_J$. It has order two;
2. an element $w \in W_J$ is the longest element if and only if $l(sw) < l(w)$ for all $s \in J$.

**Proof:**

1. First note that $l(w_J w_J) = l(w_J) - l(w_J) = 0$, so $w_J w_J = 1$, and $w_J$ has order two. Now suppose that $w$ and $w_J$ are longest elements, then $l(ww_J) = l(w) - l(w_J) = 0$. So $ww_J = 1$ and $w^{-1} = w_J = w_J^{-1}$. Hence $w = w_J$.

2. If $w_J$ is a longest element then, for all $s \in J$, $l(sw_J) = l(w_J) - 1 < l(w_J)$. Now suppose $w \in W_J$ is such that $l(sw) < l(w)$ for all $s \in J$, and that $l(w) < l(w_J)$. Then $l(ww_J) > 0$, and so for some $s \in J$ we have $l(sww_J) < l(ww_J)$. But,

$l(sww_J) = l(w_J) - l(sw)$, and $l(ww_J) = l(w_J) - l(w)$. Thus $l(sw) > l(w)$, which contradicts our choice f $w$.

▲

Taking $J = S$ in Theorem 2.5.4 and Corollary 2.5.5 we obtain that $l(ww_0) = l(w_0) - l(w)$ for all $w \in W$, and that $W$ has a unique longest element. This longest element has order two, and is the only element that satisfies $l(sw_0) < l(w_0)$ for all $s \in S$.

Using these techniques we now determine the longest element in the Weyl group $W \cong \mathfrak{S}_n$ for the Tits system $(\mathrm{GL}_n, B, N, S)$ described in Section 2.2.

**Proposition 2.5.6** [Geck, 1.8] Let $G = \mathrm{GL}_n$ and $(G, B, N, S)$ the Tits system described in Section 2.2. We have, for $1 \leq i \leq n-1$ and $w \in W \cong \mathfrak{S}_n$ that $l(s_i w) > l(w)$ if and only if $w^{-1}(i) < w^{-1}(i+1)$.

**Proof:** We have seen, in the proof of Theorem 2.2.7, that if $w^{-1}(i) < w^{-1}(i+1)$ then $C(s_i)C(w) \subseteq C(s_i w)$. It follows from Proposition 2.4.3 that $l(s_i w) > l(w)$. Conversely, if $w^{-1}(i) > w^{-1}(i+1)$ put $y = s_i w$ so that

$$y^{-1}(i) = w^{-1}s_i(i) = w^{-1}(i+1) < w^{-1}(i) = w^{-1}s_i(i+1) = y^{-1}(i+1).$$

So as above $C(s_i)C(y) \subseteq C(s_i y)$, and so $l(s_i y) > l(y)$. That is, $l(w) > l(s_i w)$. ▲

It is now easy to see that the longest element, in the Weyl group of $\mathrm{GL}_n$, is that which corresponds to the permutation $(1\ n)(2\ n-1)\cdots \in \mathfrak{S}_n$; it is represented by the matrix

$$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

# Chapter 3

# Varieties Defined over $\mathbb{F}_q$

We now assume the background in algebraic geometry discussed in the introduction, although unless otherwise stated, varieties are always affine. From now on $k$ is an algebraic closure of a finite field $\mathbb{F}_p$, where $p$ is prime. We use $q$ for some power of $p$.

The goal of this chapter is to introduce linear algebraic groups that are defined over $\mathbb{F}_q$, and to prove Lang's theorem which is fundamental to the study of the Deligne-Lusztig varieties. We begin in Section 3.1 by developing the notion of a Frobenius morphism, which defines the affine varieties defined over $\mathbb{F}_q$. We mention now that varieties defined over $\mathbb{F}_q$ are not just varieties where we have replaced $k$ by $\mathbb{F}_q$, but are in fact $k$-varieties with extra structure. In Section 3.2 we give an algebraic interpretation of varieties defined over $\mathbb{F}_q$. A key idea is $\mathbb{F}_q$-structures on $k$-algebras. This can provide a framework for generalising the notion of being defined over a subfield to the case $\text{char}(k) = 0$, where there is no Frobenius morphism available. In Sections 3.3 and 3.4 we look at how general affine varieties defined over $\mathbb{F}_q$ relate to algebraic sets defined over $\mathbb{F}_q$, and put these objects into a category. Finally, in Section 3.5 we introduce linear algebraic groups defined over $\mathbb{F}_q$ and prove Lang's theorem.

## 3.1 Frobenius Morphisms

In this section we generalize the notion of the $q^{th}$-power map on $k$, which we denote by $F_q$. Consider the case $q = p$. Then, for $a, b \in k$, $F_p(a+b) = (a+b)^p$, and if we expand using the binomial theorem we see that every coefficient is divisible by $p$, except the coefficients of $a^p$ and $b^p$; both of which are 1. Thus, as $\text{char}(k) = p$, we have that $F_p(a + b) = (a + b)^p = a^p + b^p = F_p(a) + F_p(b)$. Clearly $F_p(ab) = F_p(a)F_p(b)$, and so $F_p$ is a ring homomorphism. It follows that $F_q$ is a ring homomorphism, because $F_q = F_p^r$, for some $r \geq 1$. Moreover, $F_q$ is injective as its domain is a field, and as $k$ is perfect, being algebraically closed, $F_q$ is surjective. We therefore have that $F_q$ is an automorphism of $k$.

Now suppose that $F_q(a) = a$. Then $a^q = a$, and so $a \in \mathbb{F}_q$. So the fixed points of $F_q$ is the subfield $\mathbb{F}_q$, and thus $F_q$ an an element of the Galois group of $k/\mathbb{F}_q$; it is called the **Frobenius map**.

It is clear that we can extend this map to a bijective morphism $\mathbb{A}^n \to \mathbb{A}^n$, that is just $F_q$ in each coordinate. We also denote this map by $F_q$.

If $A$ is a ring, we write $A^q$ for $\{a^q \mid a \in A\}$, and we denote the coordinate ring of an affine variety $X$ by $\Gamma(X)$.

**Definition 3.1.1** [Geck, 4.1] Let $X$ be an affine $k$-variety, and let $F : X \to X$ be a morphism such that the induced morphism $F^* : \Gamma(X) \to \Gamma(X)$ is injective . Suppose $q$ is a power of $p$ such that

1. the image of $F^*$ is $\Gamma(X)^q$;

2. for each $f \in \Gamma(X)$ there exists an $m > 0$ such that $(F^*)^m(f) = f^{q^m}$.

Then we say that $X$ is **defined over** $\mathbb{F}_q$, or $X$ admits an $\mathbb{F}_q$**-rational structure**, and $F$ is the corresponding **Frobenius morphism**.

**Example 3.1.2** [Geck, 4.1] Let $X = \mathbb{A}^n$, $F = F_q$. Then $F_q^*$ is injective as $F_q$ is surjective. Let $A = \Gamma(X) = k[T_1, \ldots T_n]$. We have that $F_q^*(A) = A^q$ since for each $1 \leq i \leq n$, $F_q^*(T_i) = T_i \circ F_q = T_i^q$. Finally, since $k$ is algebraic over $\mathbb{F}_p$, we have that each element of $k$ is contained in some finite subfield of $k$. So if $f \in A$ there is some $m > 0$ such that all the coefficients of $f$ are in $\mathbb{F}_{q^m}$. In this case $(F_q^*)^m(f) = f^{q^m}$. Therefore $F_q$ is a Frobenius morphism.

**Definition 3.1.3** [Geck, 4.1] Let $X$ be an affine $k$-variety defined over $\mathbb{F}_q$, and $F : X \to X$ a corresponding Frobenius morphism. The set $X^F = \{x \in X \mid F(x) = x\}$ is called the set of **$\mathbb{F}_q$-rational points** of $X$.

**Example 3.1.4** [Geck, 4.1] In Example 3.1.2 we have

$$X^{F_q} = \{(x_1, \ldots, x_n) \in k^n \mid (x_1, \ldots, x_n) = (x_1^q, \ldots, x_n^q)\} = \mathbb{F}_q^n.$$

## 3.2 Arithmetic Frobenius Morphisms

In this section we show that the geometric notion of being defined over $\mathbb{F}_q$ has a very nice algebraic meaning.

**Definition 3.2.1** [Geck, 4.1] Let $X$ be a $k$-variety defined over $\mathbb{F}_q$ with corresponding Frobenius morphism $F$. Then the map $\sigma : \Gamma(X) \to \Gamma(X)$ defined by

$$\sigma(f) = (F^*)^{-1}(f^q)$$

is called the **arithmetic Frobenius morphism**.

Note that $\sigma$ is well defined because $F^*$ is injective and $F^*(\Gamma(X)) = \Gamma(X)^q$.

**Example 3.2.2** Let $X = \mathbb{A}^n$, $F = F_q$ and $f = \sum \alpha_i T_1^{i_1} \dots T_n^{i_n} \in k[T_1, \dots, T_n] = \Gamma(X)$. Then $F^*(\sum \alpha_i T_1^{i_1} \dots T_n^{i_n}) = \sum \alpha_i T_1^{i_1} \dots T_n^{i_n} \circ F = \sum \alpha_i T_1^{qi_1} \dots T_n^{qi_n}$. So

$$\begin{aligned} \sigma(f) &= (F^*)^{-1}(f^q) \\ &= (F^*)^{-1} \sum \alpha_i^q T_1^{qi_1} \dots T_n^{qi_n} \\ &= \sum \alpha_i^q T_1^{i_1} \dots T_n^{i_n}. \end{aligned}$$

So in contrast with $F_q^*$, which acts on the variables, the arithmetic Frobenius morphism $\sigma$ raises the coefficients to the $q^{th}$ power.

**Proposition 3.2.3** [Geck, 4.1] Let $X$ be an affine $k$-variety defined over $\mathbb{F}_q$ with corresponding Frobenius morphism $F$. Then the arithmetic Frobenius morphism $\sigma$ satisfies the following properties:

1. It is an automorphism of $\Gamma(X)$;
2. For all $f \in \Gamma(X)$, and $\xi \in k$ we have $\sigma(\xi f) = \xi^q \sigma(f)$;
3. For all $f \in \Gamma(X)$ there is some $m > 0$ such that $\sigma^m(f) = f$.

**Proof:**

1. We have that $\sigma$ is a ring homomorphism as it is the composition of the ring homomorphisms $f \mapsto f^q$ and $(F^*)^{-1}$. We show that $\sigma$ is injective. Let $f \in \ker(\sigma)$ then $0 = \sigma(f) = (F^*)^{-1}(f^q)$. So $f^q = F^*(0) = 0$, and so $f = 0$. Finally, if $g \in \Gamma(X)$ then, as $F^*(\Gamma(X)) = \Gamma(X)^q$, we have $F^*(g) = f^q$ for some $f \in \Gamma(X)$. So, as $F^*$ is injective, $g = (F^*)^{-1}(f^q) = \sigma(f)$. Thus $\sigma$ is surjective.

2. Let $\xi \in k$, and $f \in \Gamma(X)$. Then $\sigma(\xi f) = (F^*)^{-1}((\xi f)^q) = (F^*)^{-1}(\xi^q f^q) = \xi^q (F^*)^{-1}(f^q) = \xi^q \sigma(f)$.

3. We first show that $F^*$ and $\sigma$ commute. Let $f \in \Gamma(X)$ then $F^*(\sigma(f)) = F^*((F^*)^{-1}(f^q)) = f^q$, and $\sigma(F^*(f)) = (F^*)^{-1}(F^*(f)^q) = (F^*)^{-1}(F^*(f^q)) = f^q$. So $F^*$ and $\sigma$ commute.

Now, let $f \in \Gamma(X)$. Since $F$ is a Frobenius morphism there is an $m > 0$ such that $(F^*)^m(f) = f^{q^m}$. So for any $n$, $f^n = (F^*)^{-m}((f^{q^m})^n)$. We show that $(F^*)^m(\sigma^m(f)) = (F^*)^m(f)$ then, as $F^*$ is injective it follows that $\sigma^m(f) = f$. So,

$$\begin{aligned}(F^*)^m(\sigma^m(f)) &= \sigma^m((F^*)^m(f)) \quad \text{(since } F^* \text{ and } \sigma \text{ commute)} \\ &= \sigma^m(f^{q^m}) \quad \text{(by choice of } m\text{)} \\ &= (F^*)^{-m}((f^{q^m})^{q^m}) \quad \text{(by definition of } \sigma\text{)} \\ &= f^{q^m} \quad \text{(by choice of } m\text{)} \\ &= (F^*)^m(f).\end{aligned}$$

Thus $\sigma^m(f) = f$.

▲

Before we continue with our study of the arithmetic Frobenius map we pause to prove a nice result from linear algebra. First note that $\{w_1, \ldots, w_n\} \subset k$ is linearly independent over $\mathbb{F}_q$ if and only if, for every $r \geq 0$, the same is true of $\{F_q^r(w_1), \ldots, F_q^r(w_n)\}$.

**Lemma 3.2.4** [Goss, 1] Let $\{w_1, \ldots, w_n\} \subset k$, and let $A$ be the matrix

$$A = \begin{pmatrix} w_1 & \cdots & w_n \\ \vdots & & \vdots \\ F_q^{n-1}(w_1) & \cdots & F_q^{n-1}(w_n) \end{pmatrix} = \begin{pmatrix} w_1 & \cdots & w_n \\ \vdots & & \vdots \\ w_1^{q^{n-1}} & \cdots & w_n^{q^{n-1}} \end{pmatrix}.$$

Then, $\{w_1, \ldots, w_n\}$ is linearly independent over $\mathbb{F}_q$ if and only if $\det(A) \neq 0$.

**Proof:** To show the forwards implication, proceed by induction on $n$. If $n = 1$ then there is nothing to show. Suppose that, for any linearly independent set with cardinality $t$, that the corresponding matrix has non-zero determinant. Let $\{w_1, \ldots, w_{t+1}\}$ be linearly independent over $\mathbb{F}_q$. Suppose, towards a contradiction, that $\det(A) = 0$. Then the columns of $A$ are linearly dependent over $k$. That is, there exist $\alpha_1, \ldots \alpha_{t+1} \in k$, (not all 0) that satisfy the following linear system:

$$\begin{array}{ccccccc} \alpha_1 w_1 & + & \cdots & + & \alpha_{t+1} w_{t+1} & = & 0 \\ \alpha_1 w_1^q & + & \cdots & + & \alpha_{t+1} w_{t+1}^q & = & 0 \\ & \vdots & & \vdots & & & \\ \alpha_1 w_1^{q^t} & + & \cdots & + & \alpha_{t+1} w_{t+1}^{q^t} & = & 0. \end{array}$$

We assume, without loss of generality, that $\alpha_1 = 1$. Since $\mathrm{char}(k) = p$, and $q$ is a power of $p$, raising any one of these equations to the $q^{th}$ power is the same as raising each of the terms to the $q^{th}$ power. Thus when we subtract the $q^{th}$ power of the $i^{th}$ equation from the $i+1^{st}$ equation we obtain

$$\begin{array}{ccccccc} (\alpha_2 - \alpha_2^q) w_2^q & + & \cdots & + & (\alpha_{t+1} - \alpha_{t+1}^q) w_{t+1}^q & = & 0 \\ & \vdots & & \vdots & & & \\ (\alpha_2 - \alpha_2^q) w_2^{q^t} & + & \cdots & + & (\alpha_{t+1} - \alpha_{t+1}^q) w_{t+1}^{q^t} & = & 0. \end{array}$$

Since $\{w_2, \ldots, w_{t+1}\}$ is linearly independent so is $\{w_2^q, \ldots, w_{t+1}^q\}$, and we have by induction that $\det(A') \neq 0$, where $A'$ is the matrix:

$$A' = \begin{pmatrix} w_2^q & \cdots & w_{t+1}^q \\ \vdots & & \vdots \\ w_2^{q^t} & \cdots & w_{t+1}^{q^t} \end{pmatrix}$$

It follows, for each $2 \leq i \leq t+1$, that $\alpha_i - \alpha_i^q = 0$; that is each $\alpha_i \in \mathbb{F}_q$. Thus the original equation

$$w_1 + \alpha_2 w_2 + \cdots + \alpha_{t+1} w_{t+1} = 0,$$

has coefficients in $\mathbb{F}_q$ contradicting the fact that $\{w_1, \ldots, w_{t+1}\}$ is linearly independent over $\mathbb{F}_q$.

Conversely, suppose that $\det(A) \neq 0$, and that $\sum_{i=1}^n \alpha_i w_i = 0$, with each $\alpha_i \in \mathbb{F}_q$. Then,

$$\alpha_1 \begin{pmatrix} w_1 \\ \vdots \\ w_1^{q^{n-1}} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} w_n \\ \vdots \\ w_n^{q^{n-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, $(\alpha_1, \ldots, \alpha_n)^T \in \ker(A)$, which is 0. It follows that each $\alpha_i = 0$, and so $\{w_1, \ldots, w_n\}$ is linearly independent over $\mathbb{F}_q$. ▲

**Lemma 3.2.5** [Geck, 4.1] 4.1 Let $X$ be an affine $k$-variety defined over $\mathbb{F}_q$, $F$ the corresponding Frobenius morphism, and $\sigma$ the arithmetic Frobenius morphism. Then, for $f \in \Gamma(X)$, the subspace $V = \mathrm{span}\{\sigma^j(f) \mid j \geq 0\}$ is finite dimensional, and has a basis consisting of elements fixed by $\sigma$.

**Proof:** We may assume that $f \neq 0$. Choose, by Proposition 3.2.3(3), an $m$ such that $\sigma^m(f) = f$, so that $\dim(V) \leq m$, and let $\{\xi_0, \ldots, \xi_{m-1}\}$ be an $\mathbb{F}_q$-basis for the subfield $\mathbb{F}_{q^m}$ of $k$. Define, for $0 \leq i \leq m-1$,

$$\tilde{f}_i = \sum_{j=0}^{m-1} \sigma^j(\xi_i f).$$

We show, for each $0 \leq i \leq m-1$, that $\sigma(\tilde{f}_i) = \tilde{f}_i$.

$$\begin{aligned}
\sigma(\tilde{f}_i) &= \sum_{j=0}^{m-1} \sigma^{j+1}(\xi_i f) \\
&= \sigma^m(\xi_i f) + \sum_{j=1}^{m-1} \sigma^j(\xi_i f) \\
&= \xi_i f + \sum_{j=1}^{m-1} \sigma^j(\xi_i f) \quad (\xi_i^{q^m} = \xi_i \text{ and } \sigma^m(f) = f) \\
&= \sum_{j=0}^{m-1} \sigma^j(\xi_i f) \\
&= \tilde{f}_i.
\end{aligned}$$

Consider the matrix

$$M = \begin{pmatrix} \xi_0 & \cdots & \xi_{m-1} \\ \vdots & & \vdots \\ \xi_0^{q^{m-1}} & \cdots & \xi_{m-1}^{q^{m-1}} \end{pmatrix}.$$

The set $\{\xi_0, \ldots, \xi_{m-1}\}$ is linearly independent over $\mathbb{F}_q$, and so, by Lemma 3.2.4, the matrix $M$ is invertible. Thus, by definition of the $\tilde{f}_i$,

$$M^{-1} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_{n-1} \end{pmatrix} = \begin{pmatrix} \sigma^0(f) \\ \vdots \\ \sigma^{m-1}(f) \end{pmatrix},$$

which shows that the $\tilde{f}_i$ span $V$. ▲

**Definition 3.2.6** [Springer, 1] Let $A$ be a $k$-algebra. Then an **$\mathbb{F}_q$-structure** on $A$ is an $\mathbb{F}_q$-algebra $A_0$ contained in $A$ such that the map $m : k \otimes_{\mathbb{F}_q} A_0 \to A$, given by $m(\sum \alpha_i \otimes x_i) = \sum \alpha_i x_i$, is an isomorphism. When $a \in A_0$ we say that $a$ is **rational over $\mathbb{F}_q$**, or just **$\mathbb{F}_q$-rational**.

**Theorem 3.2.7** [Geck, 4.1] Let $X$ be an affine $k$-variety defined over $\mathbb{F}_q$, $F$ the corresponding Frobenius morphism, and $\sigma$ the arithmetic Frobenius morphism. Let

$$A_0 = \{f \in \Gamma(X) \mid \sigma(f) = f\} = \{f \in \Gamma(X) \mid F^*(f) = f^q\}.$$

Then $A_0$ is an $\mathbb{F}_q$-structure on $\Gamma(X)$, which we call the **$\mathbb{F}_q$-structure induced by** $F$.

**Proof:** Let $\{f_1, \ldots, f_m\}$ be a set of algebra generators for $\Gamma(X)$. Then, by Lemma 3.2.5, the subspaces $V_i = \text{span}\{\sigma^j(f_i) \mid j \geq 0\}$ are finite dimensional and have bases consisting of elements that are fixed by $\sigma$. Thus there is a set of algebra generators for $\Gamma(X)$ consisting of elements which are fixed by $\sigma$: Let $\{g_1, \ldots, g_l\}$ be linearly independent set with this property.

Let $B$ be the $\mathbb{F}_q$-algebra generated by $\{g_1, \ldots, g_l\}$. It is clear that $B$ is a finitely generated $\mathbb{F}_q$-subalgebra, and that the map $k \otimes_{\mathbb{F}_q} B \to \Gamma(X)$, given by $\sum \alpha_i \otimes h_i \mapsto \sum \alpha_i h_i$, is an isomorphism. We claim that $B = A_0$. Let $\xi \in \mathbb{F}_q$, then

$$\sigma(\xi g_i) = \xi^q \sigma(g_i) = \xi g_i.$$

It is clear that $\sigma(g_i + g_j) = g_i + g_j$, and $\sigma(g_i g_j) = g_i g_j$, so $B \subseteq A_0$. Since $\{g_1, \ldots, g_l\}$ generates $\Gamma(X)$ we have that $\{g_1^{i_1} \ldots g_l^{i_l} \mid i_j \geq 0\}$ spans $\Gamma(X)$; choose a basis $\Xi$ from this set. Now, let $f \in A_0$ and write $f = \sum_{\Xi} \alpha_i g_1^{i_1} \ldots g_l^{i_l}$, $\alpha_i \in k$. Then

$$\begin{aligned} f &= \sigma(f) \\ &= \sum \sigma(\alpha_i g_1^{i_1} \ldots g_l^{i_l}) \\ &= \sum \alpha_i^q g_1^{i_1} \ldots g_l^{i_l}. \end{aligned}$$

Thus $\alpha_i^q = \alpha_i$, and so $\alpha_i \in \mathbb{F}_q$. This shows that $A_0 \subseteq B$, and so $A_0 = B$. ▲

**Example 3.2.8** We saw in Example 3.2.2, that for the Frobenius morphism $F_q$ on $\mathbb{A}^n$, that the map $\sigma$ takes a polynomial and raises its coefficients to the $q^{th}$ power. Thus, the condition that $f \in A_0$, namely $\sigma(f) = f$, is equivalent to the coefficients of $f$ being in $\mathbb{F}_q$. Thus $A_0 = \mathbb{F}_q[T_1, \ldots T_n]$.

## 3.3 The Category of Affine Varieties Defined over $\mathbb{F}_q$

Here we define morphisms between varieties defined over $\mathbb{F}_q$, and so we construct a category.

**Definition 3.3.1** [Geck, 4.1] Let $X$, $Y$ be varieties defined over $\mathbb{F}_q$, and let $F_1 : X \to X$, $F_2 : Y \to Y$ be the corresponding Frobenius morphisms. Let $A_0 \subseteq \Gamma(X)$ and $B_0 \subseteq \Gamma(Y)$ be the $\mathbb{F}_q$-structures induced by $F_1$ and $F_2$ respectively. A morphism $\phi : X \to Y$ is called **defined over** $\mathbb{F}_q$ if $\phi^*(B_0) \subseteq A_0$.

It is clear that identity maps are defined over $\mathbb{F}_q$ and that the composite of two morphisms defined over $\mathbb{F}_q$ is defined over $\mathbb{F}_q$. Hence, there is indeed a category of affine varieties defined over $\mathbb{F}_q$.

**Example 3.3.2** Suppose $-1$ is not a square in $\mathbb{F}_q$. Let $X = Y = \mathbb{A}^1$, $i = \sqrt{-1}$ and let $\varphi : X \to Y$ be given by $\varphi(x) = ix$. Then $\varphi$ is a morphism, but $\varphi^*(T) = iT \notin \mathbb{F}_q[T]$. Thus $\varphi$ is not defined over $\mathbb{F}_q$.

We will give, in Proposition 3.4.6 below, a characterisation of morphisms defined over $\mathbb{F}_q$ in the case that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are closed subsets.

**Proposition 3.3.3** [Geck, 4.1] Let $X$, $Y$ be varieties defined over $\mathbb{F}_q$, $F_1 : X \to X$, $F_2 : Y \to Y$ be the corresponding Frobenius morphisms. Let $\phi : X \to Y$ be a morphism. Then $\phi$ is defined over $\mathbb{F}_q$ if and only if $F_2 \circ \phi = \phi \circ F_1$.

**Proof:** Let $A_0 \subseteq \Gamma(X)$, $B_0 \subseteq \Gamma(Y)$ be the induced $\mathbb{F}_q$-structures. Suppose that $\phi$ is defined over $\mathbb{F}_q$, and let $f \in B_0$. Then

$$\begin{aligned}\phi^*(F_2^*(f)) &= \phi^*(f^q) \quad \text{(since } f \in B_0)\\ &= \phi^*(f)^q\\ &= F_1(\phi^*(f)) \quad \text{(since } \phi^*(f) \in A_0).\end{aligned}$$

Thus, for all $f \in B_0$, $(F_2 \circ \phi)f = (\phi \circ F_1)f$. So, as $B_0$ generates $\Gamma(Y)$ as a $k$-algebra, $F_2 \circ \phi = \phi \circ F_1$.

Now, suppose that $F_2 \circ \phi = \phi \circ F_1$, and let $f \in B_0$. Then

$$\begin{aligned}F_1^*(\phi^*(f)) &= \phi^*(F_2^*(f))\\ &= \phi^*(f^q) \quad \text{(since } f \in B_0)\\ &= \phi^*(f)^q.\end{aligned}$$

Thus $\phi^*(f) \in A_0$. ▲

## 3.4 Embeddings in Affine Space

In this section we show that any affine $k$-variety defined over $\mathbb{F}_q$ is essentially a closed subset of some $\mathbb{A}^n$ that is the vanishing set of some polynomials with coefficients in $\mathbb{F}_q$.

**Proposition 3.4.1** [Geck, 4.1] Let $X$ be an affine variety defined over $\mathbb{F}_q$, and let $F$ be the corresponding Frobenius morphism. Let $F_q$ be the standard Frobenius morphism, $F_q(\alpha_1, \ldots, \alpha_n) = (\alpha_1^q, \ldots, \alpha_n^q)$, on $\mathbb{A}^n$. Then, for some $n$, there is a closed embedding $\iota : X \to \mathbb{A}^n$ that is defined over $\mathbb{F}_q$.

By Proposition 3.3.3 this says that when $X$ is an affine variety defined over $\mathbb{F}_q$, there is a closed embedding $\iota : X \to \mathbb{A}^n$ such that

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathbb{A}^n \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_q} \\ X & \xrightarrow[\iota]{} & \mathbb{A}^n \end{array}$$

is commutative.

**Proof:** Let $A_0 \subseteq \Gamma(X)$ be the $\mathbb{F}_q$-structure on $\Gamma(X)$ induced by $F$: See Theorem 3.2.7. Since $A_0$ is a finitely generated $\mathbb{F}_q$-algebra we may write $A_0 \cong \mathbb{F}_q[T_1, \ldots, T_n]/I_0$, for some ideal $I_0 \subseteq \mathbb{F}_q[T_1, \ldots, T_n]$. Let $I$ be the ideal of $k[T_1, \ldots, T_n]$ generated by $I_0$. Then because $A_0$ is an $\mathbb{F}_q$-structure on $\Gamma(X)$, $\Gamma(X) \cong A_0 \otimes k \cong (\mathbb{F}_q[T_1, \ldots, T_n]/I_0) \otimes k \cong k[T_1, \ldots, T_n]/I$. Let $\pi' : k[T_1, \ldots, T_n] \to k[T_1, \ldots, T_n]/I$ be the canonical epimorphism, and let $\pi$ be the composite

$$k[T_1, \ldots, T_n] \xrightarrow{\pi'} k[T_1, \ldots, T_n]/I \xrightarrow{\cong} \Gamma(X).$$

Then $\pi$ is a $k$-algebra homomorphism so there is a morphism $\iota : X \to \mathbb{A}^n$ such that $\pi = \iota^*$. Moreover, $\iota$ is injective since $\pi$ is surjective, and $\iota(X)$ is a closed set in $\mathbb{A}^n$ being the vanishing set of $I$. Thus $\iota$ is a closed embedding. Finally, as $\pi$ takes $\mathbb{F}_q[T_1, \ldots, T_n]$ to $A_0$, we have that $\iota$ is defined over $\mathbb{F}_q$. ▲

**Corollary 3.4.2** In the setup of Proposition 3.4.1 we have: For each $x \in X$ there is an $m > 0$ such that $F^m(x) = x$.

**Proof:** It follows from Proposition 3.4.1 that, for all $m$, $\iota(F^m(x)) = F_q^m(\iota(x))$. So, let $x \in X$, and $\iota(x) = (\alpha_1, \ldots, \alpha_n)$. Then, there is an $m > 0$ such that each $\alpha_i \in \mathbb{F}_{q^m}$. So, $F_{q^m}(\alpha_1, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_n)$, but $F_{q^m} = F_q^m$. Thus,

$$\iota(x) = F_{q^m}(\iota(x)) = \iota(F^m(x))$$

and so $F^m(x) = x$, because $\iota$ is injective. ▲

Let $\mathcal{V}$, $\mathcal{I}$ be the vanishing and ideal operators in $\mathbb{A}^n$. We have, from the proof of 3.4.1, that $\iota(X) = \mathcal{V}(I)$ since $\Gamma(X) \cong k[T_1, \ldots, T_n]/I$.

**Corollary 3.4.3** [Geck, 4.1] Let $X$ be an affine variety defined over $\mathbb{F}_q$ by a Frobenius map $F$. Then,

1. $F$ is bijective;
2. the set $X^F$ of $\mathbb{F}_q$-rational points is finite.

**Proof:**

1. We first show that $F$ is injective. Suppose that $x, y \in X$, and $F(x) = F(y)$. Choose an $\iota$ as in Proposition 3.4.1. Then $\iota F(x) = \iota F(y)$, and so $F_q(\iota(x)) = F_q(\iota(y))$, but $F_q \circ \iota$ is injective so $x = y$. Thus $F$ is injective.

   Now let $x \in X$. By Corollary 3.4.2, there is some $m > 0$ such that $F^m(x) = x$. Thus $F(F^{m-1}(x)) = x$ and so $F$ is surjective.

2. First, for all $x \in X^F$, $F_q(\iota(x)) = \iota(F(x)) = \iota(x)$. Thus, $\iota(X^F) \subseteq (\mathbb{A}^n)^{F_q} = \mathbb{F}_q^n$, which is finite. Then, as $\iota$ is injective, $X^F$ is finite.

▲

In light of the discussion above we see that we can restrict our attention to closed subsets of $\mathbb{A}^n$ that are defined over $\mathbb{F}_q$.

**Proposition 3.4.4** [Geck, 4.1] Let $X$ be an algebraic set defined over $\mathbb{F}_q$. Let $A_0$ be the $\mathbb{F}_q$-structure on $\Gamma(X)$ induced by $F = F_q$. Then, for a closed subset $Y$ of $X$, the following are equivalent:

1. $F(Y) \subseteq Y$;
2. $F(Y) = Y$;
3. $\mathcal{I}(Y)$ may be generated by a subset of $A_0$;
4. $Y = \mathcal{V}(S)$ for some $S \subseteq A_0$.

**Proof:**

($1 \implies 2$) Let $y \in Y$ then, as $F$ is bijective, there is a unique $x \in X$ such that $F(x) = y$. We show that $x \in Y$. Choose an $m > 0$ such that $F^m(x) = x$, then $F^{m-1}(y) = x$. Thus $x$ is in the image of $F$, which by hypothesis is contained in $Y$. So $F(Y) = Y$.

($2 \implies 3$) We first show that $I = \mathcal{I}(Y)$ is invariant under $\sigma$. Let $f \in I$, let $y \in Y$, and let $g \in \Gamma(X)$ be such that $g = \sigma(f)$; that is $F^*(g) = f^q$. Then $g(F(y)) = F^*(g)(y) = f^q(y) = 0$. So $g$ vanishes on $F(Y)$, which is equal to $Y$, and so $g \in \mathcal{I}(Y)$.

Now, let $S$ be a finite set of generators for $I$. Then, from the proof of Proposition 3.2.7, we may assume that $S$ consists of elements fixed by $\sigma$, equivalently $S \subseteq A_0$.

($3 \implies 4$) Let $S$ be a generating set for $\mathcal{I}(Y)$ that is contained in $A_0$. Then $Y = \mathcal{V}(S)$.

($4 \implies 1$) Let $f \in S$. Then, by Theorem 3.2.7, $S \subseteq A_0$ means that $F^*(f) = f^q$. So, for $y \in Y$, we have $f(F(y)) = F^*(f)(y) = f^q(y) = f(y)^q = 0$. Thus, $F(y) \in Y$ and $F(Y) \subseteq Y$.

▲

Notice that the proof ($1 \implies 2$) did not require that $X$ be an algebraic set with $F = F_q$. Thus it is true for any affine variety $X$ defined over $\mathbb{F}_q$ by a Frobenius map $F$.

**Proposition 3.4.5** [Geck, 4.1] Let $X$ be an algebraic set defined over $\mathbb{F}_q$ by the Frobenius map $F = F_q$. If $Y \subseteq X$ is closed and the conditions in Proposition 3.4.4 are satisfied then $F|_Y$ is a Frobenius morphism for an $\mathbb{F}_q$-structure on $Y$.

**Proof:** Let $\phi = (F|_Y)^*$, and let $I = \mathcal{I}(Y)$. We have that $I$ is $F^*$-stable, because $F(Y) = Y$. Then $\phi$ is injective, as $F|_Y$ is surjective. Identifying $\Gamma(Y)$ with $\Gamma(X)/I$ we obtain

$$\phi(\Gamma(Y)) = \phi(\Gamma(X)/I) = F^*(\Gamma(X))/I = \Gamma(X)^q/I = \Gamma(Y)^q.$$

Thus $\phi(\Gamma(Y)) = \Gamma(Y)^q$. Finally, if $f + I \in \Gamma(Y)$, then there is some $m > 0$ such that $(F^*)^m(f) = f^{q^m}$, and so $\phi^m(f + I) = (F^*)^m(f) + I = f^{q^m} + I = (f + I)^{q^m}$. Thus $F|_Y$ is a Frobenius morphism. ▲

We now give the promised characterisation of morphisms defined over $\mathbb{F}_q$ between algebraic sets that are defined over $\mathbb{F}_q$. Suppose that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are algebraic sets with $\Gamma(X) = k[T_1, \ldots, T_n]/I$ and $\Gamma(Y) = k[S_1, \ldots, S_m]/J$. Recall, that given a $k$-algebra homomorphism $\phi : \Gamma(Y) \to \Gamma(X)$ we can construct a morphism $\varphi : X \to Y$ such that $\varphi^* = \phi$ as follows: Put $f_i + I = \phi(S_i + J)$, and then let $\varphi = (f_1, \ldots, f_m)$. The restriction of $\varphi$ to $X$ is the desired map. See [Hartshorne, 1.3].

**Proposition 3.4.6** [Geck, 4.1] Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be varieties defined over $\mathbb{F}_q$, and let $\varphi : X \to Y$ be a morphism. Then, $\varphi$ is defined over $\mathbb{F}_q$ if and only if $\varphi = (f_1, \dots, f_m)$, for some $f_1, \dots, f_m \in \mathbb{F}_q[T_1, \dots, T_n]$.

**Proof:** Let $A_0$ and $B_0$ the the $\mathbb{F}_q$-structures on $\Gamma(X)$ and $\Gamma(Y)$ respectively.

It is clear that if $\varphi = (f_1, \dots, f_m)$, for some $f_1, \dots, f_m \in \mathbb{F}_q[T_1, \dots, T_n]$ then $\varphi$ commutes with $F_q$, and thus $\varphi$ is defined over $\mathbb{F}_q$.

Conversely, if $\varphi$ is defined over $\mathbb{F}_q$, that is $\varphi^*(B_0) \subseteq A_0$ then the $f_1, \dots f_m$ discussed above are elements of $\mathbb{F}_q[T_1, \dots, T_n]$, as the $S_i + J \in B_0$. ▲

## 3.5 Lang's Theorem

From now on knowledge of linear algebraic groups is assumed. In this section we give a brief overview of what it means for a linear algebraic group to be defined over $\mathbb{F}_q$, and we prove Lang's theorem which, as we shall see, is important in the study of Deligne-Lusztig varieties.

**Definition 3.5.1** [Geck, 4.1] Let $G$ be a linear algebraic group. If $G$ is defined over $\mathbb{F}_q$ as a variety, and the multiplication and inversion maps are defined over $\mathbb{F}_q$, then $G$ is **defined over** $\mathbb{F}_q$.

**Example 3.5.2** Let $G = \mathrm{GL}_n(k)$. Then $G$ is defined over $\mathbb{F}_q$. Indeed, as a principal open set of $\mathbb{A}^{n^2}$, we have that $G$ is a closed set in $\mathbb{A}^{(n+1)^2}$; a principle open set is the non-vanishing set of a single polynomial. It is given by $\mathcal{V}(T\det - 1)$. The polynomial det has coefficients in $\mathbb{F}_q$, and so $G$ is defined over $\mathbb{F}_q$. It is clear that the multiplication is defined over $\mathbb{F}_q$, and the inversion map is seen to be defined over $\mathbb{F}_q$ using the adjoint formula for the inverse of a matrix.

As should be expected, it is the groups $\mathrm{GL}_n$ that are the special objects in the category of linear algebraic groups defined over $\mathbb{F}_q$. The strategy is to combine Proposition

3.4.1 with the linearisation of affine algebraic groups to obtain, for any affine algebraic group $G$ defined over $\mathbb{F}_q$, a closed embedding $\varphi : G \to \mathrm{GL}_n$ that is defined over $\mathbb{F}_q$. See [Geck, 4.1] for further details. Diagrammatically we have, for all $G$ there is an $n$ and a group homomorphism $\varphi$ such that

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \mathrm{GL}_n \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_q} \\ G & \xrightarrow[\varphi]{} & \mathrm{GL}_n \end{array}$$

commutes.

**Definition 3.5.3** [Geck, 4.1] Let $G$ be a linear algebraic group defined over $\mathbb{F}_q$, and let $F : X \to X$ be its Frobenius morphism. Then, the map $L : G \to G$, given by $L(g) = g^{-1}F(g)$, is called the **Lang map** induced by $F$.

**Definition 3.5.4** Let $X, Y$ be varieties. A morphism $\phi : X \to Y$ is called **dominant** if $\phi(X)$ is dense in $Y$.

**Theorem 3.5.5 (Lang's Theorem)** [Müller] The Lang map is surjective.

**Proof:** The group $G$ acts on itself by $g \cdot x = g^{-1}xF(g)$. By [Springer, 2] there is a closed orbit $O$ for this action. Thus, if $x \in O$, the map $\varphi : G \to O$ given by $\varphi(g) = g \cdot x = g^{-1}xF(g)$ is a dominant morphism. We show that $\varphi$ has finite fibres. It then follows from [Humphreys, 4] that $\dim G = \dim O$.

Let $\psi : G \to G$ be given by $\psi(g) = xF(g)x^{-1}$. Then the fibre over $x$ of $\varphi$ is the set of fixed points of $\psi$. Since $F$ is a Frobenius map there is some $m > 0$ such that $F^m(x) = x$. Set $h = xF(x) \cdots F^{m-1}(x)$. Then $F^m(h) = h$, but $F^m$ is a Frobenius morphism that defines $G$ over $\mathbb{F}_{q^m}$; see [Geck, 4.1]. Thus, $h$ is an element of the group $G^{F^m}$ which is finite by Corollary 3.4.3. So $h$ has finite order; let $r$ be the order of $h$. Now, for $g \in G$, it is easy to see that $\psi^{mr}(g) = F^{mr}(g)$. Thus the fixed points of $\psi^{mr}$ lie in $G^{F^{mr}}$ which is finite. So $\psi^{mr}$ has finitely many fixed points, but every fixed point of $\psi$ is a fixed point of $\psi^{mr}$. Thus, $\psi$ has finitely many fixed points, and so the fibres of $\varphi$ are finite.

So $O \subseteq G$ have the same dimension, but $G$ is connected and hence irreducible; $O$ is also irreducible being the image of $G$ by $\varphi$. It follows, since $O$ is closed in $G$, that $G = O$. Thus $1 \in O$, and so $G = \{g \cdot 1 \mid g \in G\} = L(G)$. So $L$ is surjective. ▲

# Chapter 4

# Flag Varieties of Linear Algebraic Groups

In this section $G$ denotes a linear algebraic group, and $k$ is an algebraically closed field. Our task in this section is to define a Borel subgroup $B$ of $G$, and study the coset space $G/B$. This space is called the flag variety. It is the last major piece of the Deligne-Lusztig varieties puzzle.

Understanding the variety structure of $G/H$, when $H$ is a closed subgroup, is in general difficult; for our purposes it suffices to know that it has the quotient topology. The details are in [Springer, 5.5].

The material found in this chapter is in most books about linear algebraic groups, but we recommend [Springer] as a reference for the presentation given here. This is because there are two equivalent, yet very different, approaches to collecting the results found in this chapter: The main difference is in the definition of parabolic subgroups. We say that a subgroup $P$ is parabolic if $G/P$ is complete (Definition 4.2.1), and then prove that a subgroup is parabolic if and only if it contains a Borel subgroup (Theorem 4.3.4). In [Borel] and [Humphreys] these roles are reversed.

## 4.1 Complete varieties

**Definition 4.1.1** [Springer, 6.1] An algebraic variety $X$ over $k$ is **complete** if for any variety $Y$ over $k$ the projection morphism $X \times Y \to Y$ is closed.

It is typical to compare completeness for varieties with compactness for locally compact Hausdorff spaces; this comparison is made in both [Springer] and [Humphreys]. The reason for this comparison is that a locally compact Hausdorff space $X$ is compact if and only if it satisfies the condition in Definition 4.1.1 with $Y$ replaced by any locally compact space. The following result strengthens the analogy.

**Proposition 4.1.2** The image of a complete variety under a morphism is complete.

**Proof:** Let $X$ be a complete variety and let $\varphi : X \to Y$ be a surjective morphism. Let $Z$ be any variety and let $C$ be a closed subset of $Y \times Z$. Consider the following commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{(\varphi, \mathrm{id}_Z)} & Y \times Z \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle p} & \\ Z & & \end{array}$$

where $p$ and $q$ are the second coordinate projections. Then $p(C) = q((\varphi, \mathrm{id}_Z)^{-1}(C))$ which is closed by completeness of $X$. Thus $Y$ is complete. ▲

It is obvious from Proposition 4.1.2 that completeness is preserved by homeomorphisms.

**Example 4.1.3** Let $X$ be a complete variety, and $\phi : X \to Y$ a morphism with graph $\mathcal{G} = \{(x, \phi(x)) \mid x \in X\}$. Then $\mathcal{G}$ is complete, being homeomorphic to $X$.

**Example 4.1.4** If $X$ is a singleton then $X$ is complete. Indeed, for any variety $Y$ the projection $X \times Y \to Y$ is a homeomorphism.

**Example 4.1.5** Let $X = Y = \mathbb{A}^1$. Then $X \times Y = \mathbb{A}^2$. The image of the closed set $\{(x, y) \mid xy = 1\} \subseteq X \times Y$ under the second coordinate projection is the non-closed set $\mathbb{A}^1 \setminus \{0\}$. Thus $\mathbb{A}^1$ is not complete.

**Proposition 4.1.6** [Springer, 6.1] Let $X$ be a complete variety.

1. A closed subvariety of $X$ is complete.
2. If $Y$ is a complete variety then so is $X \times Y$.
3. If $\phi : X \to Y$ is a morphism of varieties then $\phi(X)$ is complete, and closed in $Y$.
4. If $X$ is a subvariety of a variety $Y$ then $X$ is closed in $Y$.
5. A variety $Y$ is complete if and only if the irreducible components of $Y$ are complete.
6. If $X$ is irreducible then any regular function on $X$ is constant.
7. If $X$ is affine then $X$ is finite.

**Proof:** Suppose $X$ is a complete variety.

1. Let $Y$ be a closed subvariety of $X$, and let $Z$ be any variety. Then the projection map $Y \times Z \to Z$ is the restriction, to the closed subset $Y \times Z$, of the closed map $X \times Z \to Z$, and hence is closed. So $Y$ is complete.

2. Suppose that $Y$ is a complete variety. Then for any variety $Z$ the projection $(X \times Y) \times Z \to Z$ is the composite of the closed maps $X \times (Y \times Z) \to Y \times Z$ and $Y \times Z \to Z$. Thus it is closed.

3. Suppose $\phi : X \to Y$ is a morphism. Then its graph $\mathcal{G}$ is a closed set in $X \times Y$ [Springer, 1.6]. Thus, by completeness of $X$ the image of $\mathcal{G}$ under the projection $X \times Y \to Y$ is closed. This image is $\phi(X)$. We have seen in Proposition 4.1.2 that $\phi(X)$ is complete.

4. If $X$ is a subvariety of $Y$ then $X$ is the image of the inclusion morphism, and so is closed in $Y$ by (3).

5. If $X$ is complete then each irreducible component is complete, by (1).

   Conversely, if each $X_i$ is complete, and $Y$ is any variety then a closed set $C$ in $X \times Y$ can be written $C = \bigcup C_i$ where $C_i = C \cap (X_i \times Y)$. So the image of $C$ is closed in $Y$ being a finite union of closed sets.

   Note that this fact together with Example 4.1.4 shows that a finite variety is complete.

6. Suppose that $X$ is irreducible, and that $f : X \to k$ is a regular function. Then $f$ defines a morphism $f : X \to \mathbb{A}^1$. $X$ is irreducible so $f(X)$ is also irreducible, and by (3) is closed in $\mathbb{A}^1$. So if $f$ is non-constant then $f(X) = \mathbb{A}^1$ because $\dim(\mathbb{A}^1) = 1$. This would imply that $\mathbb{A}^1$ is a complete variety, which by Example 4.1.5 is not the case. Thus any regular function on $X$ is constant.

7. Suppose first that $X$ is affine and irreducible. Then by (6) any regular function on $X$ is constant so $\mathcal{O}_X \cong k$. But, $X$ is affine so $\mathcal{O}_X \cong \Gamma(X)$. Thus $\Gamma(X) \cong k$. This shows that $\Gamma(X)$ is a field, and thus $X$ is a point.

   So, given that an affine variety $X$ has finitely many irreducible components, and that each such component is a point, we have that $X$ is finite.

▲

So far we have seen an example of a non-complete variety, but the complete varieties that we have seen have not been very interesting. Fortunately, there is a large class of interesting varieties that are complete. First we state a lemma from commutative algebra.

**Lemma 4.1.7 (Nakayama's lemma)** [Atiyah-MacDonald, 2] Let $R$ be a ring, and

$M \subset R$ a maximal ideal. If $A$ is an $R$-module such that $A = MA$ then there is an $x \in A \setminus M$ such that $xA = \{0\}$.

**Theorem 4.1.8** [Humphreys, 6.2] Projective varieties are complete.

**Proof:** Since a projective variety is a closed subvariety of $\mathbb{P}^n$ for some $n$, and a closed subvariety of a complete variety is complete by Proposition 4.1.6, it suffices to show, for each $n$, that $\mathbb{P}^n$ is complete. So what we have to show is that, for any variety $Y$, the projection map $\mathbb{P}^n \times Y \to Y$ is closed.

We reduce to the case that $Y$ is affine and irreducible. First, if the irreducible components of $Y$ are $\{Y_i\}_{i \leq m}$, and $C \subseteq \mathbb{P}^n \times Y$ is a closed set then $C \cap \mathbb{P}^n \times Y_i$ is closed for each $i$. So, if the projection $\mathbb{P}^n \times Y_i \to Y_i$ is closed then so is the projection $\mathbb{P}^n \times Y \to Y$. Now, a subset $C$ is closed if and only if its intersection with each element of an open cover is closed. We can cover $Y$ by finitely many affine open subsets; say $Y = \cup Y_i$. Then the sets $\mathbb{P}^n \times Y_i$ form a finite open cover of $\mathbb{P}^n \times Y$. So the projection $\mathbb{P}^n \times Y \to Y$ is closed if and only if each of the projections $\mathbb{P}^n \times Y_i \to Y_i$ are closed. Thus, we may assume that $Y$ is affine and irreducible.

Let $Y$ be an irreducible affine with $R = \Gamma[Y]$, and let $U_i = \{[x_0, \ldots, x_n] \mid x_i \neq 0\}$ so that $\{U_i\}_{i=0}^n$ is the standard affine open cover of $\mathbb{P}^n$. Let $V_i = U_i \times Y$, then $\{V_i\}_{0=1}^n$ is an affine open cover of $\mathbb{P}^n \times Y$. Moreover,

$$\begin{aligned}\Gamma[V_i] &= \Gamma[U_i \times Y] \\ &= \Gamma[U_i] \otimes \Gamma[Y] \\ &= k[X_0/X_i, \ldots, X_n/X_i] \otimes R \\ &\cong R[X_0/X_i, \ldots, X_n/X_i].\end{aligned}$$

Let $C \subseteq \mathbb{P}^n \times Y$ be closed, and let $y \in Y \setminus p(C)$. If there is an $f \in R$ such that $f \notin I(\{y\})$ and $f \in \mathcal{I}(p(C))$ then $D(f)$ is an open neighbourhood of $y$ that does not meet $p(C)$. Doing this for all $y \in Y \setminus p(C)$ shows that $p(C)$ is closed.

Let $S = R[X_0, \ldots, X_n]$, with the usual grading by degree $S = \bigoplus S_m$. Now define $I_m = \{f \in S_m \mid f(X_0/X_i, \ldots, X_n/X_i) \in \mathcal{I}(C_i) \text{ for all } i\}$ where $C_i = C \cap V_i$. Then let $I$ be the homogeneous ideal $I = \bigoplus I_m$. Note that, as elements of the fraction field, if $f \in S_m$ then $f(X_0/X_i, \ldots, X_n/X_i) = (\frac{1}{X_i^m})f(X_0, \ldots, X_n)$.

We first show that for all $g \in \mathcal{I}(C_i)$ there is some $m$ such that $X_i^m g \in I_m$. Fix an $i \in \{0, \ldots, n\}$, and let $g \in \mathcal{I}(C_i)$. Then $g$ is in particular an element of $R_i = R[X_0/X_i, \ldots, X_n/X_i]$. Let $m = \deg(g) + 1$. Then $X_i^{m-1}g \in S_{m-1}$, thus $(X_i^{m-1}/X_j^{m-1})g \in R_j$. It is clear (since $g$ vanishes on $C_i$) that $(X_i^{m-1}/X_j^{m-1})g$ is defined and vanishes on $C_i \cap V_j$, but $C_i \cap V_j = C \cap V_i \cap V_j = C_j \cap V_i$ thus $(X_i^{m-1}/X_j^{m-1})g$ vanishes on $C_j \cap V_i$. Now, $X_i/X_j$ vanishes on $V_j \setminus V_i$, and so $(X_i^m/X_j^m)g$ vanishes on $(C_j \setminus V_i) \cup (C_j \cap V_i) = C_j$. This is true for any $j$ thus $X_i^m g \in I_m \subseteq I$.

The sets $C_i$ and $U_i \times \{y\}$ are disjoint closed subsets of the affine variety $V_i$, so $\mathcal{I}(C_i)$ and $\mathcal{I}(U_i \times \{y\})$ generate the unit ideal in $\Gamma(V_i) = R_i$. Let $M = \mathcal{I}(\{y\})$, so $\mathcal{I}(U_i \times \{y\}) = R_i \otimes M = MR_i$. Then there is an equation $1 = f_i + \sum_j m_{ij}g_{ij}$ where $f_i \in \mathcal{I}(C_i), m_{ij} \in M$ and $g_{ij} \in R_i$. We have just shown that multiplication by some power of $X_i$ takes $f_i$ into $I$. We may choose an integer $m'$ large enough such that this is true for all $i$ , and that for all $i, j$, $X_i^{m'}g_{ij} \in S_{m'}$. So $X_i^{m'} = X_i^{m'}(1) = X_i^{m'}(f_i + \sum_j m_{ij}g_{ij}) = X_i^{m'}f_i + \sum_j m_{ij}X_i^{m'}g_{ij} \in I_{m'} + MS_{m'}$ for all $i$. So there is an $m \geq m'$ such that all monomials of degree $m$ are in $I_m + MS_m$. Then $S_m \subseteq I_m + MS_m$, and hence these sets are equal. Taking the quotient by $I_m$ we obtain $S_m/I_m = M(S_m/I_m)$, thus by Lemma 4.1.7 there is an $h \in R \setminus M$ (so $h(y) \neq 0$) such that $hS_m \subseteq I_m$. In particular we have for each $i$ that $X_i^m h \in I_m$, and so $h \in \mathcal{I}(C_i)$ for all $i$, so $h \in \mathcal{I}(C)$. Now, $h \in R$ so $p^*(h) = h$, and so $h$ vanishes on $p(C)$. Therefore, $D(h)$ is a neighbourhood of $y$ that is disjoint from $p(C)$ as required.
▲

## 4.2 Parabolic Subgroups

**Definition 4.2.1** [Springer, 6.2] A closed subgroup $P$ of an algebraic group $G$ is called **parabolic** if the quotient $G/P$ is a complete variety.

Since a parabolic subgroup is closed $G/P$ is a quasi-projective variety, as can be found in [Springer, 5.5]. But $G/P$ is complete and hence closed by Proposition 4.1.6(4). Thus $G/P$ is projective. Conversely if $H$ is a subgroup such that $G/H$ is projective then $G/H$ is also complete by Theorem 4.1.8. Thus we may have taken the definition of a parabolic subgroup to one such that taking the quotient yields a projective variety.

**Example 4.2.2** Let $G^0$ be the identity component of a linear algebraic group $G$. Then $G^0$ is of finite index. So $G/G^0$ is finite, and hence complete by Proposition 4.1.6(5). Thus $G^0$ is parabolic in $G$.

Let $H$ be a closed subgroup of $G$. Since the canonical map $\pi : G \to G/H$ is an equivariant homomorphism of $G$-spaces, it follows from [Springer, 5.3] that for any variety $Z$ the map $(\pi, \mathrm{id}_Z) : G \times Z \to G/H \times Z$ is an open map. Thus, as it is also a surjective morphism it is topologically a quotient map.

**Lemma 4.2.3 (Transitivity)** [Springer, 6.2] If $Q$ is parabolic in $P$, and $P$ is parabolic in $G$ then $Q$ is parabolic in $G$.

**Proof:** Let $Z$ be any variety, and consider the following diagram:

$$\begin{array}{ccccc}
P \times G \times Z & \xrightarrow{(\mu', \mathrm{id}_Z)} & G \times Z & \xrightarrow{(\pi_1, \mathrm{id}_Z)} & G/Q \times Z \\
\downarrow{\scriptstyle (\pi_2, \mathrm{id}_{G\times Z})} & & & & \\
P/Q \times G \times Z & & & & \downarrow{\scriptstyle p_1} \\
\downarrow{\scriptstyle p_2} & & & & \\
G \times Z & & & & \\
\downarrow{\scriptstyle (\pi_3, \mathrm{id}_Z)} & & & & \\
G/P \times Z & & \xrightarrow[p_3]{\hspace{6em}} & & Z
\end{array}$$

where $\mu'$ is multiplication restricted to $P \times G$, $\pi_1, \pi_2, \pi_3$ are the canonical quotient maps, and $p_1, p_2, p_3$ are coordinate projections. Note that every map in the above diagram is the identity in the $Z$-component, and so the diagram commutes.

We need to show that the map $p_1 : G/Q \times Z \rightarrow Z$ is a closed map, whence $Q$ is parabolic in $G$. Let $A \subseteq G/Q \times Z$ be closed, let $B = (\pi_1, \mathrm{id}_Z)^{-1}(A)$, and $C = (\mu', \mathrm{id}_{G\times Z})^{-1}(B)$. Then $C$ is a closed set in $P \times G \times Z$, which maps to $A$. So by the commutativity of the diagram it suffices to show that the image of $C$ in $Z$, when sent around the diagram in the counterclockwise direction, is closed in $Z$.

Let $D = (\pi_2, \mathrm{id}_{G\times Z})(C)$. We show that $(\pi_2, \mathrm{id}_{G\times Z})^{-1}(D) = C$; that $C \subseteq (\pi_2, \mathrm{id}_{G\times Z})^{-1}(D)$ is clear. Let $(pQ, g, z) \in D$, with $(p, g, z) \in C$, and let $(p', g', z') \in (\pi_2, \mathrm{id}_{G\times Z})^{-1}(pQ, g, z)$. Then $(pQ, g, z) = (p'Q, g', z')$, and thus $z = z'$, $g = g'$ and $pQ = p'Q$. Since $pQ = p'Q$ we have $p^{-1}p' \in Q$, and so there is some $q \in Q$ such that $p' = pq$. Now, $(p, g, z) \in C$ so $(gp, z) \in B$, but $B$, being the full pre-image of $A$, has the property that if $(h, x) \in B$ then $(hq, x) \in B$ for all $q \in Q$. Thus, for all $q \in Q$, $(gpq, z) \in B$. In particular $(gp', z) \in B$, and so $(p', g, z) \in C$. Hence $(\pi_2, \mathrm{id}_{G\times Z})^{-1}(D) \subseteq C$ and so $(\pi_2, \mathrm{id}_{G\times Z})^{-1}(D) = C$ which is closed. It follows that $D$ is closed because $(\pi_2, \mathrm{id}_{G\times Z})$ is a quotient map.

Let $E = p_2(D)$. Since $Q$ is parabolic in $P$ the variety $P/Q$ is complete. Thus, $p_2$ is a closed map, and so $E$ is closed.

Let $F = (\pi_3, \mathrm{id}_Z)(E)$. We show that $(\pi_3, \mathrm{id}_Z)^{-1}(F) = E$. Let $(gP, z) \in F$ with $(g, z) \in E$, and let $(g', z') \in (\pi_3, \mathrm{id}_Z)^{-1}(gP, z)$. Then $(gP, z) = (g'P, z')$, so $z = z'$ and $gP = g'P$. Now, $g^{-1}g' \in P$ so there is some $p \in P$ such that $g = g'p$. Since $(g, z) \in E$ there is some $p' \in P$ such that $(p', g, z) \in C$, and thus $(gp', z) \in B$, and so $(g'pp', z) \in B$. Let $p'' = pp'$ so that $(g'p'', z) \in B$ which yields that $(p'', g', z) \in C$, and so is mapped into $E$ by $p_2 \circ (\pi_2, \mathrm{id}_Z)$. But $p_2((\pi_2, \mathrm{id}_Z)(p'', g', z)) = (g', z)$, thus $(g', z) \in E$. Thus, $(\pi_3, \mathrm{id}_Z)^{-1}(F) = E$ which is closed in $G \times Z$. It follows that $F$ is closed because $(\pi_3, \mathrm{id}_Z)$ is a quotient map.

Finally, we show that $p_3(F)$ is closed in $Z$. Indeed, $P$ is parabolic in $G$ so $G/P$ is complete. Thus $p_3(F) = p_1(A)$ is closed in $Z$. So $G/Q$ is complete, and $Q$ is parabolic in $G$. ▲

**Lemma 4.2.4** [Springer, 6.2]

1. If $Q$ is a closed subgroup of $G$ containing a parabolic subgroup $P$ then $Q$ is parabolic.
2. $P$ is parabolic in $G$ if and only if $P^0$ is parabolic in $G^0$.

**Proof:**

1. Let $Q$ be a closed subgroup of $G$ containing a parabolic subgroup $P$. Since $G/Q$ is the image of the complete variety $G/P$, under the morphism $gP \mapsto gQ$, it is complete by Proposition 4.1.6(3). Hence $Q$ is parabolic in $G$.

2. For both implications we will use that $G^0$ is parabolic in $G$; see Example 4.2.2. Suppose that $P$ is parabolic in $G$. Then $P^0$ is parabolic in $P$, and so by transitivity (Lemma 4.2.3) we have that $P^0$ is parabolic in $G$. Then, as $G^0/P^0$ is closed in $G/P^0$, it is complete. Therefore $P^0$ is parabolic in $G^0$. Conversely suppose that $P^0$ is parabolic in $G^0$. Then since $G^0$ is parabolic in $G$, we have

$P^0$ is parabolic in $G$ by transitivity. Thus, by (1), $P$ is parabolic in $G$ since it contains the parabolic subgroup $P^0$.

▲

**Lemma 4.2.5** [Springer, 6.2] Let $X$ and $Y$ be homogeneous spaces for $G$, and let $\phi : X \to Y$ be a bijective $G$-morphism. Then $X$ is complete if and only if $Y$ is complete.

**Proof:** Recall, [Springer, 5.3], that under these hypotheses the map $(\phi, \mathrm{id}_Z) : X \times Z \to Y \times Z$ is an open map. So $(\phi, \mathrm{id}_Z)$ is bijective, continuous and open. In other words it is a homeomorphism. In particular it and its inverse are closed maps, thus $X$ is complete if and only if $Y$ is complete. ▲

The following theorem is really the heart of the section. It is the main tool that we use to study Borel subgroups in the next section.

**Theorem 4.2.6** [Springer, 6.2] A connected group $G$ has no proper parabolic subgroups if and only if $G$ is solvable.

**Proof:** By assumption $G$ is linear so we may take $G$ to be a closed subgroup of $\mathrm{GL}(V)$ for some (finite dimensional) vector space $V$.

Suppose that $G$ has no proper parabolic subgroups. Since $G \subseteq \mathrm{GL}(V)$ we have that $G$ acts on $\mathbb{P}(V)$, and this action has a closed orbit $X$ [Springer, 2]. So $X$ is closed in $\mathbb{P}(V)$, and hence, by Theorem 4.1.8, $X$ is complete. Let $x \in X$ and let $P$ be the isotropy group of $x$. Define a map $\phi : G/P \to X$ by $gP \mapsto g \cdot x$. Then $\phi$ is a bijective morphism of homogeneous $G$-spaces, and so $G/P$ is complete by Lemma 4.2.5. Thus $P = G$ since $G$ has no proper parabolic subgroups, and so $X = \{x\}$.

Let $x_1$ be a representative of $x$ in $V$. Then $\mathrm{span}\{x_1\}$ is a $G$-stable subspace.

Let $m < \dim(V)$, and suppose that $\{x_1, \ldots, x_m\}$ is linearly independent such that for each $i \in \{1, \ldots, m\}$ the subspace spanned by $\{x_1, \ldots, x_i\}$ is $G$-stable. Let $W = \operatorname{span}\{x_1, \ldots, x_m\}$ and $V' = V/W$. Then $G$ acts on $\mathbb{P}(V')$, and as above this action has a closed orbit $X$, consisting of a single point $\{x\}$. Let $x_{m+1}$ be a representative of $x$ in $V$. Then $\{x_1, \ldots, x_{m+1}\}$ is such that for each $i \in \{1, \ldots, m+1\}$ the subspace spanned by $\{x_1, \ldots, x_i\}$ is $G$-stable.

Thus, there is a basis $\{x_1, \ldots, x_{\dim(V)}\}$ such that for each $i \in \{1, \ldots, \dim(V)\}$ the subspace spanned by $\{x_1, \ldots, x_i\}$ is $G$-stable; that is there is a basis $\Xi$ of $V$ such that, for all $g \in G$ the matrix of $g$ with respect to $\Xi$ is upper-triangular. Thus, $G$ is isomorphic to a subgroup of the upper-triangular subgroup in $\mathrm{GL}_n$, which is solvable. Hence $G$ is solvable.

Conversely suppose that $G$ is solvable. Recall $G$ is connected by hypothesis. We proceed by induction on $\dim(G)$. Suppose that $\dim(G) = 0$ then $G$, being the trivial group, has no proper parabolic subgroups. Now suppose that $\dim(G) > 0$, and let $P \subsetneq G$ be a proper parabolic subgroup of maximal dimension. By Lemma 4.2.4(2) the identity component $P^0$ of $P$ is parabolic, moreover $\dim(P^0) = \dim(P)$ so we may assume that $P$ is connected. Let $(G, G)$ be the commutator subgroup, and let $Q = \langle P, (G, G) \rangle$. Since $(G, G)$ is normal $Q = (G, G)P$. Moreover, $(G, G)$ is a closed connected subgroup [Springer, 2] whence $Q$ is a closed connected subgroup containing $P$. By Lemma 4.2.4(1) $Q$ is parabolic, so by maximality of $P$, $Q = G$ or $Q = P$.

Suppose that $Q = G$. The set $(G, G)/(G, G) \cap P$ is a homogeneous $(G, G)$-space with respect to left multiplication, as is the set $G/P$. Let

$$\phi : (G, G)/(G, G) \cap P \to G/P$$

be given by $x(G, G) \cap P \mapsto xP$. Note that $\phi$ is well defined since $(G, G) \cap P \subseteq P$. Then $\phi$ is a morphism of $(G, G)$-spaces which we claim is bijective. Indeed, if

$$\phi(x(G, G) \cap P) = \phi(y(G, G) \cap P)$$

then $y^{-1}x \in P$, but $x, y \in (G,G)$ so $y^{-1}x \in (G,G) \cap P$. Thus $\phi$ is injective. Now let $gP \in G/P$ then, since $Q = G$, we may write $g = g'p$ for some $g' \in (G,G)$ and $p \in P$. Then $gP = g'pP = g'P = \phi(g'(G,G) \cap P)$. Thus $\phi$ is surjective. Thus, by Lemma 4.2.5, $(G,G) \cap P$ is parabolic in $(G,G)$ since $P$ is parabolic in $G$. Since $G$ is solvable $\dim((G,G)) < \dim(G)$ so, by induction, $(G,G)$ has no proper parabolic subgroups. Thus $(G,G) \cap P = (G,G)$, and so $(G,G) \subseteq P$, but this implies that $Q = P$, which is a contradiction.

Suppose that $Q = P$. Then $(G,G) \subseteq P$ which gives that $P$ is a normal subgroup of $G$. In this case $G/P$ is affine [Springer, 5.5.10], but $G/P$ is is complete since $P$ is parabolic. Thus $G/P$ is finite being both affine and complete; Proposition 4.1.6(7). Moreover, $G/P$ is connected since $G$ is, so $G/P$ is the trivial group showing that $P = G$ which is a contradiction. ▲

Notice that in the proof of Theorem 4.2.6, particularly in the forwards direction, we were interested in how $G$ acted on complete varieties. This connection is further explained in the next section by Borel's fixed point theorem.

## 4.3 Borel Subgroups

**Definition 4.3.1** [Springer, 6.2] A subgroup $B$ of $G$ is called a **Borel subgroup** if $B$ is maximal among closed, connected, solvable subgroups.

**Example 4.3.2** Let $G = \mathrm{GL}_n$ and let $B$ be the subgroup of upper-triangular matrices. Then $B$ is a Borel subgroup.

**Corollary 4.3.3 (Borel's fixed point theorem)** [Springer, 6.2] Let $G$ be a connected, solvable, linear algebraic group, and let $X$ be a complete $G$-variety. Then there is a point of $X$ fixed by all elements of $G$.

**Proof:** There is a closed orbit for the action of $G$ on $X$ [Springer, 2.3]. Then, as in the proof of Theorem 4.2.6, the stabilizer of a point $x \in X$ is a parabolic subgroup, which by Theorem 4.2.6 is the whole of $G$. ▲

The following characterisation of parabolic subgroups is taken as the definition in [Humphreys]. It is a powerful theorem that equates the technical condition that $G/P$ be complete with a simple containment relation.

**Theorem 4.3.4** [Springer, 6.2] A closed subgroup of $G$ is parabolic if and only if it contains a Borel subgroup. Furthermore, if $B$ is a Borel subgroup and $P$ is a parabolic subgroup then $P$ contains a conjugate of $B$.

**Proof:** We may assume by Lemma 4.2.4(2) that $G$ is connected. Let $P$ be a parabolic subgroup of $G$, and let $B$ be a Borel subgroup of $G$. We show that $P$ contains a conjugate of $B$. Since $B$ acts on the complete variety $G/P$ by left multiplication, by Corollary 4.3.3 there is a fixed point for this action. So, there is a $gP \in G/P$ such that for all $b \in B$, $bgP = gP$. In other words $g^{-1}bg \in P$ for all $b \in B$. This shows that $P$ contains a conjugate of $B$. This conjugate is also a Borel subgroup. Indeed it is isomorphic to $B$. Thus, $P$ contains a Borel subgroup.

Conversely, by Lemma 4.2.4(1), it suffices to show that a Borel subgroup is parabolic. Let $B$ be a Borel subgroup of $G$. Note that if $G$ is solvable then $B = G$ so $B$ is parabolic. We proceed by induction. Suppose $\dim(G) = 0$. Then $G = \{1\}$, being connected, and hence is solvable. We now suppose that $\dim(G) > 0$, and that $G$ is non-solvable. Then we can choose, by Theorem 4.2.6, a proper parabolic subgroup $P$, which by the first part of this proof contains a conjugate $B'$ of $B$. Now $G$ is connected, and hence irreducible, and $P$ is a proper closed subvariety. Thus $\dim(P) < \dim(G)$. So $B'$ is parabolic in $P$ by induction. Thus by transitivity (Lemma 4.2.3) $B'$ is parabolic in $G$, and so $G/B'$ is complete. Finally, $G/B' \cong G/B$ since $B$ and $B'$ are conjugates, and so $B$ is parabolic in $G$. ▲

**Corollary 4.3.5** [Springer, 6.2]

1. Any two Borel subgroups are conjugate.

2. If $\phi : G \to G'$ is a surjective homomorphism of linear algebraic groups then the image of a parabolic subgroup is parabolic.

**Proof:**

1. Suppose that $B$, and $B'$ are Borel subgroups. Then by Theorem 4.3.4 both are parabolic, and we have that $B$ contains a conjugate of $B'$ and $B'$ contains a conjugate of $B$. Thus, $B$ and $B'$ are conjugate.

2. Let $\phi : G \to G'$ be a surjective homomorphism of linear algebraic groups, and let $P$ be parabolic in $G$. Let $P' = \phi(P)$. The morphism of varieties $G/P \to G'/P'$ induced by $\phi$ is surjective so $G'/P'$ is complete by Proposition 4.1.2. Hence $P'$ is parabolic in $G'$.

▲

Consider the case of $G = \mathrm{GL}_n$. It can be shown, see [Alperin-Bell, 2.5], that any subgroup that contains the upper-triangular subgroup $B$ is either $G$ or is one of the the so called **staircase groups**: the block upper-triangular subgroups. So any proper parabolic subgroup in $\mathrm{GL}_n$ is a conjugate of a staircase group. For the sake of having a picture, let $n = 3$. Besides $B$ and $G$ there are two staircase groups: one that consists of matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},$$

and the other consists of matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

**Corollary 4.3.6** Let $B$ be a Borel subgroup of $G$. Then $G/B$ is a projective variety.

**Proof:** By Theorem 4.3.4 $B$ is parabolic whence $G/B$ is projective. ▲

We have now shown that if $B$ is a Borel subgroup of $G$ then the quotient $G/B$ is a projective variety. Moreover, by the conjugacy of Borel subgroups, this variety is independent up to isomorphism of the choice of $B$.

**Definition 4.3.7** For a Borel subgroup $B$ of an algebraic group $G$, the quotient $G/B$ is called the **flag variety** of $G$.

The flag variety is so named because, in the case of $G = \mathrm{GL}_n$, there is a strong connection between the variety $G/B$ and the set of *flags* in $V = k^n$ on which $G$ acts; see Section 5.5. We preview this by computing the flag variety for $\mathrm{GL}_2$.

**Example 4.3.8** Let $G = \mathrm{GL}_2$. Then $G$ acts transitively on $\mathbb{P}^1$ by multiplication. So we have a morphism $G \to \mathbb{P}^1$ defined by $g \mapsto g \cdot [1:0]$. The fibre of this map over $[1:0]$ is the Borel subgroup $B$. Thus $G/B \cong \mathbb{P}^1$.

# Chapter 5

# Deligne-Lusztig Varieties

In this final chapter we define the Deligne-Lusztig varieties, but not until Section 5.4. We are still missing two pieces of the puzzle: Frobenius stable Borel subgroups, which we prove exist in Section 5.3, and the final mystery of "relative position" which is discussed in Section 5.2. Sections 5.2 and 5.3 are concerned with collecting some key related varieties that are mentioned in the literature, and clarifying some of the terminology surrounding them. Then, in Section 5.5 we examine some Deligne-Lusztig varieties for general linear groups in terms of flags, and this brings our journey to an end.

## 5.1 The Variety of Borel Subgroups

Let $G$ be a linear algebraic group equipped with a $BN$-pair in which $B$ is a Borel subgroup. Then, as shown in Chapter 4, $X = G/B$ is a projective variety called the flag variety for $G$. Sometimes in the literature, see for example [Deligne-Lusztig], the flag variety is referred to as the variety of Borel subgroups of $G$. In this section we explain this phenomenon.

Let $\mathfrak{B}$ denote the set of Borel subgroups of $G$, and define a map $\omega : G/B \to \mathfrak{B}$

by $\omega(gB) = gBg^{-1}$.

**Theorem 5.1.1** The map $\omega$ is bijective.

**Proof:** We first confirm that $\omega$ is well-defined. If $gB = hB$ then $g^{-1}h \in B$, and so $g^{-1}hBh^{-1}g = B$. Thus, $hBh^{-1} = gBg^{-1}$, and $\omega$ is well-defined.

If $hBh^{-1} = gBg^{-1}$ then $B = g^{-1}hBh^{-1}g$, and so $g^{-1}h \in N_G(B)$ which is equal to $B$, by Corollary 2.4.4. So, $g^{-1}h \in B$ whence $gB = hB$, and $\omega$ is injective.

Now, let $B'$ be a Borel subgroup. Then, by Corollary 4.3.5, $B' = gBg^{-1}$ for some $g \in G$. Thus, $B' = \omega(gB)$ and $\omega$ is surjective. ▲

Using this bijection, we can transport the projective variety structure of $G/B$ to $\mathfrak{B}$ and speak of the variety of Borel subgroups. This language is traditional, as it is used in the original definition of Deligne-Lusztig varieties found in [Deligne-Lusztig]. We, however, typically prefer to work directly with the flag variety $X = G/B$.

## 5.2 Relative Position of Borel Subgroups

In this section we discuss the notion of relative position of Borel subgroups and define some new varieties using it. We are mainly concerned here with introducing these varieties and sorting out the terminology as the names and notation tend to varyt, and so this area can be confusing to the newcomer.

**Definition 5.2.1** Let $G$ be a linear algebraic group with a $BN$-pair, where $B$ is a Borel subgroup. Let $B_1, B_2 \in \mathcal{B}$, say $B_1 = gBg^{-1}$ and $B_2 = hBh^{-1}$. We say that $B_1$ and $B_2$ are in **relative position** $w$, for $w \in W$, if $g^{-1}h \in C(w)$.

The terminology of being in relative position can be misleading as it suggests that $B_1$ and $B_2$ are in relative position $w$ if and only if $B_2$ and $B_1$ are in relative position $w$. This is generally not the case as $g^{-1}h \in C(w)$ implies that $h^{-1}g \in C(w)^{-1} = C(w^{-1})$.

**Definition 5.2.2** [Deligne-Lusztig, 0.2] Let $w \in W$ and set

$$O(w) = \{(B_1, B_2) \in \mathfrak{B} \times \mathfrak{B} \mid B_1 \text{ and } B_2 \text{ are in relative position } w\}.$$

The sets $O(w)$ are sometimes called **Schubert cells.**

We find it more convenient to interpret these sets in terms of the flag variety:

$$O(w) = \{(gB, hB) \in X \times X \mid g^{-1}h \in C(w)\}.$$

**Example 5.2.3** Let $w = 1$. Then $O(w) = \{(gB, hB) \mid g^{-1}h \in B\}$. Thus $O(1)$ is the diagonal in $X \times X$; in particular $O(1) \cong X$.

**Example 5.2.4** Let $G = \mathrm{GL}_2$. Then $W = \{1, w_0\}$ where $w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As in the previous example, $O(1) \cong X$ (which in this case is isomorphic to $\mathbb{P}^1$). If $(gB, hB) \in X \times X$, and $g^{-1}h \notin B$, then by the Bruhat Decomposition $g^{-1}h \in C(w_0)$. Thus, we may write $X \times X = O(1) \cup O(w_0)$, and this union is disjoint.

Example 5.2.4 portrays a general trend which we now prove.

**Proposition 5.2.5** We may write $X \times X$ as a disjoint union $X \times X = \coprod_{w \in W} O(w)$.

**Proof:** Let $(gB, hB) \in X \times X$. Then, by the Bruhat decomposition, $g^{-1}h \in C(w)$ for some $w \in W$. Thus, $(gB, hB) \in O(w)$. Moreover, if $(gB, hB) \in O(w) \cap O(y)$ then $g^{-1}h \in C(w) \cap C(y)$. Hence, $w = y$, and $X \times X = \coprod_{w \in W} O(w)$. ▲

Note that $X \times X$ is a $G$-space, where $G$ acts by $g \cdot (xB, yB) = (gxB, gyB)$.

**Proposition 5.2.6** Let $w \in W$. Then $G$ acts transitively on $O(w)$.

**Proof:** Let $O_w$ denote the orbit of $(B, \dot{w}B)$. So for $(gB, g\dot{w}B) \in O_w$, we have $g^{-1}g\dot{w} = \dot{w} \in B\dot{w}B$ and so $O_w \subseteq O(w)$.

Conversely, if $(gB, hB) \in O(w)$ then $g^{-1}h \in B\dot{w}B$, say $g^{-1}h = x\dot{w}y$, with $x, y \in B$. Then:

$$\begin{aligned} gx \cdot (B, \dot{w}B) &= (gxB, gx\dot{w}B) \\ &= (gB, gx\dot{w}yB) \\ &= (gB, gg^{-1}hB) \\ &= (gB, hB). \end{aligned}$$

Thus $O(w) \subseteq O_w$. So $O(w) = O_w$, and $G$ acts transitvely on $O(w)$. ▲

It follows that $O(w)$ is a locally closed subset of $X \times X$, and hence has the structure of a variety; see [Springer, 2.2.3]. We refrain however from dubbing the $O(w)$ Schubert varieties. As we shall see, this terminology is already overworked. In fact the closure $\overline{O(w)}$ of a Schubert cell is sometimes called a **Schubert variety**; see [Hansen, 1].

Now that we have shown the $O(w)$ to be homogeneous spaces for $G$ we can define a surjective morphism $\varphi : G \to O(w)$ given by $\varphi(g) = g \cdot (B, \dot{w}B)$. So, the $O(w)$ are irreducible. We now examine the fibres over $\varphi$. As these fibres are all isomorphic it suffices, for our purposes, to consider the fibre over $(B, \dot{w}B)$.

If $g$ is in the fibre over $(B, \dot{w}B)$, then $(gB, g\dot{w}B) = (B, \dot{w}B)$. It follows that $g \in B \cap \dot{w}^{-1}B\dot{w}$. So what we have is that $O(w) \cong G/(B \cap \dot{w}^{-1}B\dot{w})$. Using this we immediately recover the result of Example 5.2.3; that $O(1) \cong G/B = X$. We also remark that, in light of Theorem 4.3.4, a Schubert cell $O(w)$ is projective if and only if $w = 1$.

**Example 5.2.7** Let $G = \mathrm{GL}_n$. We consider the longest element $w_0 \in W$; it is represented by the matrix $\dot{w}_0 = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. A straightforward computation shows that $B \cap \dot{w}_0 B \dot{w}_0 = T$, the diagonal subgroup. Thus $O(w_0) \cong G/T$.

Going back to the general case, as $T$ is normal in $N$, we have that $T \subseteq \dot{w}^{-1}B\dot{w}$ for all $w \in W$. Thus we always have the following bounds on the dimension of $O(w)$:

$$\dim(X) \leq \dim(O(w)) \leq \dim(G/T).$$

We now construct another collection of objects that take the name Schubert varieties, see [Springer, 8.5]. Using the Bruhat decomposition it is clear that we can write $X$ as a disjoint union $X = \coprod_{w \in W} C(w)/B$. The $C(w)/B$ are, like their pre-images under the quotient map, called **Bruhat cells**, see [Springer, 8.5]. In [Springer] the Bruhat cells just defined are denoted $X(w)$. We will not use this notation as $X(w)$ is normally, that is in [Deligne-Lusztig], used to denote the Deligne-Lusztig varieties of Definition 5.4.1.

**Definition 5.2.8** [Springer, 8.5] Let $w \in W$. The closure of a Bruhat cell $C(w)/B$ is called a **Schubert variety**.

It would be nice if the Schubert varieties $\overline{O(w)}$ and $\overline{C(w)/B}$ were isomorphic, but as we now show, this is not the case.

Let $p_1 : O(w) \to X$ be the first coordinate projection. It is clear that $p_1$ is surjective. We show that the fibres of $p_1$ are isomorphic to $C(w)/B$. Let $gB \in X$, and put $Y = p_1^{-1}(gB)$. Define $\varphi : Y \to C(w)/B$ by $\varphi(gB, hB) = g^{-1}hB$. We show that $\varphi$ is an isomorphism. First note that the codomain of $\varphi$ is indeed $C(w)/B$, as the domain is a subset of $O(w)$. Now, if $g^{-1}hB = g^{-1}kB$ then $hB = kB$ so $\varphi$ is injective. If $xB \in C(w)/B$ then $xB = \varphi(gB, gxB)$, and $(gB, gxB) \in Y$ so $\varphi$ is surjective.

It can be shown, as it is in [Springer, 8.5], that a Bruhat cell $C(w)/B$ is isomorphic to $\mathbb{A}^{l(w)}$. In particular $\dim(\overline{C(w)/B}) = \dim(C(w)/B) = l(w)$. Now, by the preceding paragraph, $O(w)$ is an affine bundle over $X$ with fibres isomorphic to $C(w)/B$. Thus, $\dim(\overline{O(w)}) = \dim(O(w)) = \dim(X) + l(w)$; see [Geck, 2.2]. So, as they do not have the same dimension, the $\overline{O(w)}$ and $\overline{C(w)/B}$ are not isomorphic. But, it is clear that they share a connection. As we have shown, both the $O(w)$ and the $C(w)/B$

give covers of their ambient spaces that are indexed by the Weyl group, and their dimensions are related by a simple formula. So even though it can be confusing the abuse of language is understandable.

## 5.3 Frobenius Stable Borel Subgroups

It is clear that when we take the $\mathbb{F}_q$-structure of Example 3.1.2 that the upper-triangular subgroup $B$ of $\mathrm{GL}_n$ is Frobenius stable. In this section we show that any linear algebraic group defined over $\mathbb{F}_q$ has Frobenius stable Borel subgroups. It then follows that if $G$ has a $BN$-pair in which $B$ is a Borel subgroup then we can choose that Borel subgroup to be Frobenius stable.

**Lemma 5.3.1** [Geck, 4.3] Let $G$ be a connected linear algebraic group defined over $\mathbb{F}_q$ by a Frobenius morphism $F$. Let $Y$ be a non-empty set on which $G$ acts transitively. Suppose there is a map $F' : Y \to Y$ satisfying

$$F'(g \cdot y) = F(g) \cdot F'(y),$$

for all $g \in G$ and $y \in Y$. Then $Y^{F'}$ is non-empty. Moreover, if the stabilizer of any $y \in Y$ is closed in $G$ and there is a $y_0 \in Y^{F'}$ whose stabilizer is connected then $G^F$ acts transitively on $Y^{F'}$.

**Proof:** We first show that $Y^{F'}$ is non-empty. Let $y \in Y$. Then, because $G$ acts transitively on $Y$, $F'(y) = g^{-1} \cdot y$ for some $g \in G$. Since $G$ is connected we may apply Lang's theorem and write $g = h^{-1}F(h)$ for some $h \in G$. So,

$$F'(y) = g^{-1} \cdot y = F(h^{-1})h \cdot y$$

which implies that $F(h) \cdot F'(y) = h \cdot y$. Hence $F'(h \cdot y) = h \cdot y$, and $h \cdot y \in Y^{F'}$.

Now, suppose that the stabilizer of any $y \in Y$ is closed. Let $y_0 \in Y^{F'}$ be such that the stabilizer of $y_0$ is connected, and let $H$ be that stabilizer. We first show that

$F(H) \subseteq H$. Let $s \in H$. Then

$$F(s) \cdot y_0 = F(s) \cdot F'(y_0) = F'(s \cdot y_0) = F'(y_0) = y_0.$$

Thus $F(s) \in H$, and so $F(H) \subseteq H$. Then, as $H$ is closed, we have that $F|_H$ is a Frobenius morphism for $H$ (Proposition 3.4.5). Now, let $y \in Y^{F'}$. We show that $y$ and $y_0$ are in the same $G^F$-orbit. Write $y = g \cdot y_0$ for some $g \in G$. Then $y = F'(y) = F'(g \cdot y_0) = F(g) \cdot F'(y_0) = F(g) \cdot y_0$, and so $g^{-1}F(g) \in H$. By Lang's theorem, applied to $H$, we may write $g^{-1}F(g) = h^{-1}F(h)$ for some $h \in H$. Then $hg^{-1} \in G^F$ and $hg^{-1} \cdot y = h \cdot y_0 = y_0$. Thus $y$ and $y_0$ are in the same $G^F$-orbit, and $G^F$ acts transitively on $Y^{F'}$. ▲

**Corollary 5.3.2** [Geck, 4.3] Let $G$ be a connected linear algebraic group defined over $\mathbb{F}_q$ with Frobenius morphism $F$. Then $G$ has an $F$-stable Borel subgroup.

**Proof:** Define $F' : \mathfrak{B} \to \mathfrak{B}$ by $F'(B) = F(B)$. Recall, by Corollary 4.3.5 that $G$ acts transitively on $\mathfrak{B}$ by conjugation. Now, for $g \in G$, $F'(g \cdot B) = F'(gBg^{-1}) = F(g)F(B)F(g)^{-1} = F(g) \cdot F'(B)$. Thus $\mathfrak{B}^{F'}$ is non-empty; that is $G$ has an $F$-stable Borel subgroup. ▲

Since the stabilizer of any $B' \in \mathfrak{B}$ is $B'$ (Corollary 2.4.4) it is closed and connected, being a Borel subgroup. Thus, we have that $G^F$ acts transitively on $\mathfrak{B}^{F'}$. In particular $\mathfrak{B}^{F'}$ is finite since $G^F$ is finite. So $G$ has finitely many $F$-stable Borel subgroups.

## 5.4 Deligne-Lusztig Varieties

Here, $G$ is a linear algebraic group defined over $\mathbb{F}_q$ by a Frobenius map $F$. Assume $G$ has a $BN$-pair in which $B$ is an $F$-stable Borel subgroup. Let $W$ be the Weyl group, and $X = G/B$ the flag variety. At long last we can define the Deligne-Lusztig varieties.

**Definition 5.4.1** Let $w \in W$. The **Deligne-Lusztig variety** corresponding to $w$ is given by $X(w) = \{gB \in X \mid g^{-1}F(g) \in C(w)\}$.

We may also regard $X(w)$ as the subset of $\mathfrak{B}$ consisting of Borel subgroups that are in relative position $w$ with their image under Frobenius. This is the original approach taken in [Deligne-Lusztig, 1.3].

Alternatively, if $\mathcal{G}$ is the graph of the Frobenius map $F$, then the first coordinate projection $O(w) \cap \mathcal{G} \to X(w)$ is an isomorphism. In fact, $O(w) \cap \mathcal{G}$ is taken, in [Hansen, 1.3], to be the definition of $X(w)$. We have that $O(w) \cap \mathcal{G}$ is locally closed in $X \times X$ and so has a variety structure.

Note that the $X(w)$ are denoted $X_w$ in [Geck, 4.3] and $Z(w)$ in [Springer, 8.3].

We have that $X(1) = \{gB \mid g^{-1}F(g) \in B\}$. This set has other interesting interpretations. We now translate this into the language of Borel subgroups and relative position. If $B' = gBg^{-1}$ is an $F$-stable Borel subgroup then $gBg^{-1} = F(gBg^{-1}) = F(g)BF(g)^{-1}$. Thus $g^{-1}F(g) \in N_G(B) = B$, that is $B' \in X(1)$. Conversely, suppose $B' \in X(1)$. If $B' = gBg^{-1}$ and $g^{-1}F(g) \in B$, that is if $gB \in X(1)$, then $B = g^{-1}F(g)BF(g)^{-1}g$. So $B' = F(g)BF(g)^{-1} = F(gBg^{-1})$ and so $B'$ is $F$-stable. Thus $X(1)$ is the set of $F$-stable Borel subgroups. Next, we give a very concrete interpretation of $X(1)$.

**Lemma 5.4.2** [Geck, 4.7] Let $H \subseteq G$ be a closed, connected and Frobenius stable subgroup. If $F(gH) = gH$ then there is an $h \in H$ such that $F(gh) = gh$.

**Proof:** We have that $H$ acts transitively on $gH$ by $h \cdot gh' = gh'h^{-1}$. Moreover, as $F(gH) = gH$ and $H$ is $F$-stable, we can define a map $F' : gH \to gH$ by $F'(gh) = F(gh)$. Then $F'(h \cdot gh') = F(gh'h^{-1}) = F(gh')F(h)^{-1} = F(h) \cdot F'(gh')$.

Now, the stabilizer of an element of $gH$ is the trivial group which is closed and connected. Thus the conditions of Lemma 5.3.1 are satisfied, and so $(gH)^{F'} \neq \emptyset$. This gives that $F(gh) = gh$ for some $h \in H$. ▲

Now, with the setup of Lemma 5.4.2, we show that $G^F/H^F \cong (G/H)^F$. Indeed, if $g \in G^F$ then $gH = F(g)H = F(gH)$, that is $gH \in (G/H)^F$. So there is a map $\pi' : G^F \longrightarrow (G/H)^F$ given by $g \mapsto gH$, and if $\pi'(g) = \pi'(g')$ then $g^{-1}g' \in H$, but $g^{-1}g' \in G^F$ so the fibres of $\pi'$ are the cosets of $H^F$. Moreover, $\pi'$ is surjective by Lemma 5.4.2. Thus, $G^F/H^F \cong (G/H)^F$. In particular $G^F/B^F \cong (G/B)^F$.

**Proposition 5.4.3** We have $X(1) \cong G^F/B^F$.

**Proof:** We have $X(1) = \{gB \mid g^{-1}F(g) \in B\} = \{gB \mid gB = F(gB)\} = (G/B)^F \cong G^F/B^F$. ▲

Notice that, as $X(1) = G^F/B^F$ which is finite but not a point, the Deligne-Lusztig varieties are not generally irreducible in contrast with the Schubert cells $O(w)$. We do, however, still have a decomposition of $X$ as the disjoint union of the $X(w)$.

**Example 5.4.4** Let $G = \mathrm{GL}_n$. Then $G^F$ is just the general linear group over $\mathbb{F}_q$, and $B^F$ is its upper triangular-subgroup. Thus $X(1)$ is the flag variety of the finite algebraic group $G^F$.

The next and final section is devoted to the Deligne-Lusztig varieties associated with general linear groups.

## 5.5 General Linear Groups

Here $G = GL_n(k)$ is defined over $\mathbb{F}_q$, as in Example 3.1.2, and $(G, B, N, S)$ our usual Tits system from Section 2.2. Let $V = k^n$ and $\mathcal{F}(V)$ the set of full flags in $V$; see

[Alperin-Bell, 2]. We denote an element of $\mathcal{F}(V)$ by $(V_i)$. We now close the thesis by studying some Deligne-Lusztig varieties associated with $G$; we give a complete description in the case that $n = 3$, and we are also able to characterise the $X(w)$ for certain $w \in W$. To do this we exploit a correspondence between Borel subgroups and flags.

**Definition 5.5.1** Let $(V_i) \in \mathcal{F}(V)$. A **marking** of $(V_i)$ is an ordered basis $(v_1, \dots, v_n)$ of $V$ such that, for each $1 \le i \le n$, $V_i = \text{span}\{v_1, \dots, v_i\}$.

**Example 5.5.2** Let $(e_1, \dots, e_n)$ be the standard basis of $k^n$, and let $E_i = \text{span}\{e_1, \dots, e_i\}$. The flag $(E_i)$, marked by $e = (e_1, \dots, e_n)$ is called the **standard flag**.

**Proposition 5.5.3** [Alperin-Bell, 2.5] The action of $G$ on $\mathcal{F}(V)$ given by $g \cdot (V_i) = (gV_i)$ is transitive.

**Proof:** Let $(V_i) \in \mathcal{F}(V)$, and let $(v_1, \dots, v_n)$ be a marking of $(V_i)$. Let $g$ be the column matrix $[v_1 \dots v_n]$. Then $g \in GL_n$ since $\{v_1, \dots, v_n\}$ is linearly independent. Moreover, for each $i \in \{1, \dots, n\}$ we have $ge_i = v_i$. Thus $g \cdot (E_i) = (V_i)$, and so $G$ acts transitively on $\mathcal{F}(V)$. ▲

We have already used, in the proof Theorem 4.2.6, the easy fact that the stabilizer of the standard flag is $B$. It follows that the stabilizer of a full flag in $V$ is a conjugate of $B$. Then, by the conjugacy of Borel subgroups, the map $\psi : \mathfrak{B} \to \mathcal{F}(V)$, sending $gBg^{-1}$ to $g \cdot (E_i)$ is bijective.

**Definition 5.5.4** Let $(V_i), (V_i') \in \mathcal{F}(V)$. Then $(V_i)$ and $(V_i')$ are in **relative position** $w$ their corresponding Borel subgroups are in relative position $w$.

More precisely, $(V_i) = g \cdot (E_i)$ and $(V_i') = h \cdot (E_i)$ are in relative position $w$ if $gBg^{-1}$ and $hBh^{-1}$ are in relative position $w$, that is if $g^{-1}h \in C(w)$.

**Theorem 5.5.5** Let $(V_i), (V_i') \in \mathcal{F}(V)$. Then $(V_i)$ and $(V_i')$ are in relative position $w$ if and only if there is a marking $v = (v_1, \ldots, v_n)$ of $(V_i)$ and a marking $v' = (v_1', \ldots, v_n')$ of $(V_i')$ such that, for each $1 \leq i \leq n$, $v_i' = v_{w(i)}$.

**Proof:** Suppose that $(V_i)$ and $(V_i')$ are in relative position $w$, and that $(V_i) = g{\cdot}(E_i)$ and $(V_i') = h{\cdot}(E_i)$. Then $g^{-1}h \in C(w)$. Writing $g^{-1}h = b_1 w b_2$ yields $b_1^{-1}g^{-1}hb_2^{-1} = w$, and so letting $g' = gb_1$ and $h' = hb_2^{-1}$ we see that, as $b_1, b_2$ stabilize $(E_i)$, we may, without loss of generality, choose $g, h$ such that $g^{-1}h = w$. So, for each $1 \leq i \leq n$, we have $g^{-1}he_i = we_i$, which gives $he_{w^{-1}(i)} = ge_i$. Let $v_i = ge_i$ and $v_i' = he_i$. Then $v_i = ge_i = he_{w^{-1}(i)}$, which implies $v_{w(i)} = ge_{w(i)} = he_{w^{-1}w(i)} = he_i = v_i'$.

Conversely, suppose that there is a marking $v = (v_1, \ldots, v_n)$ of $(V_i)$ and a marking $v' = (v_1', \ldots, v_n')$ of $(V_i')$ such that, for each $1 \leq i \leq n$, we have $v_i' = v_{w(i)}$. Then define $g, h \in G$ such that $v_i = ge_i$ and $v_i' = he_i$. Then $he_i = v_i' = v_{w(i)}$, so letting $j = w^{-1}(i)$ we obtain, for all $1 \leq j \leq n$, $he_{w^{-1}(j)} = ge_j$. Thus $g^{-1}hw^{-1}e_j = e_j$. So, $g^{-1}h = w \in C(w)$, and the flags $(V_i)$, $(V_i')$ are in relative position $w$. ▲

We are now in position to study the Deligne-Lusztig varieties associated with general linear groups in terms of flags. We adopt the following shorthand. We say that a flag $g \cdot (E_i) \in X(w)$ if $g^{-1}F(g) \in C(w)$; that is we consider $X(w)$ to be the set of flags whose stabilizer is in $X(w)$. We also abuse notation and write $F$ for the Frobenius morphism on $V$. Theorem 5.5.5 tells us that $X(w)$ is the set of flags $(V_i)$ that have a marking $(v_i)$, such that $(v_{w(i)})$ is a marking of $(FV_i)$.

**Example 5.5.6** When $w = 1$, $X(w)$ is the set of flags $(V_i)$ that have a marking $(v_i)$, such that $(v_i)$ is a marking of $(FV_i)$. Thus, for each $1 \leq i \leq n$, $FV_i = V_i$, and so $X(w)$ consists of flags $(V_i)$ in which each $V_i$ is $F$-stable.

We now give a complete description of the Deligne-Lusztig varieties associated with $G = \mathrm{GL}_3$ in terms of flags.

**Example 5.5.7** Let $G = \mathrm{GL}_3$, $w \in W$. We begin by deriving some necessary conditions that a flag $(V_i)$ be in $X(w)$.

Let $w = 1$. As in Example 5.5.6, if $(V_i) \in X(w)$ then $FV_i = V_i$, for each $1 \leq i \leq 3$.

Let $w = (12)$. If $(V_i) \in X(w)$ then there is a marking $(v_1, v_2, v_3)$ of $(V_i)$ such that $(v_2, v_1, v_3)$ is marking of $(FV_i)$. So, $FV_1 = \mathrm{span}\{v_2\} \neq V_1$, but $FV_2 = \mathrm{span}\{v_2, v_1\} = V_2$.

Let $w = (23)$. If $(V_i) \in X(w)$ then there is a marking $(v_1, v_2, v_3)$ of $(V_i)$ such that $(v_1, v_3, v_2)$ is marking of $(FV_i)$. So, $FV_1 = V_1$, but $FV_2 = \mathrm{span}\{v_1, v_3\} \neq V_2$.

Let $w = (13)$. If $(V_i) \in X(w)$ then there is a marking $(v_1, v_2, v_3)$ of $(V_i)$ such that $(v_3, v_2, v_1)$ is marking of $(FV_i)$. So, $FV_1 = \mathrm{span}\{v_3\} \not\subset V_2$, and $FV_2 \cap V_2 = \mathrm{span}\{v_2\} \neq V_1$.

Let $w = (123)$. If $(V_i) \in X(w)$ then there is a marking $(v_1, v_2, v_3)$ of $(V_i)$ such that $(v_2, v_3, v_1)$ is marking of $(FV_i)$. So, $FV_1 = \mathrm{span}\{v_2\} \neq V_1$, but $FV_1 \subseteq V_2$. Finally, $V_2 \cap FV_2 = \mathrm{span}\{v_2\} \neq V_1$.

Let $w = (132)$. If $(V_i) \in X(w)$ then there is a marking $(v_1, v_2, v_3)$ of $(V_i)$ such that $(v_3, v_1, v_2)$ is marking of $(FV_i)$. So, $FV_1 = \mathrm{span}\{v_3\} \neq V_1$, $FV_1 \not\subseteq V_2$ and $FV_2 = \mathrm{span}\{v_3, v_1\} \neq V_2$. Finally, $V_2 \cap FV_2 = V_1$.

Let us take stock of what we have learned in the following table:

| $w =$ | 1 | (12) | (23) | (13) | (123) | (132) |
|---|---|---|---|---|---|---|
| $FV_1 = V_1$ | ✓ | × | ✓ | × | × | × |
| $FV_1 \subset V_2$ | ✓ | ✓ | ✓ | × | ✓ | × |
| $FV_2 = V_2$ | ✓ | ✓ | × | × | × | × |
| $FV_2 \cap V_2 = V_1$ | × | × | ✓ | × | × | ✓ |

We can now see, by looking at the table, that the necessary conditions derived above actually characterise the $X(w)$.

We finish the thesis by giving a characterisation of the Deligne-Lusztig varieties associated with general linear groups when $w$ is a transposition of the form $w = (i, i+1)$.

**Theorem 5.5.8** Let $w \in W$ be the transposition $w = (i, i+1)$. Then $X(w)$ consists of flags $(V_i)$ that satisfy the following conditions:

1. If $j \neq i$ then $FV_j = V_j$;

2. If $j = i$ then $FV_j \neq V_j$.

**Proof:** Suppose that $(V_i) \in X(w)$. Then, by Theorem 5.5.5, there is a marking $(v_1, \dots, v_n)$ of $(V_i)$ such that $(v_i, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2} \dots v_n)$ is a marking of $(FV_i)$. So, conditions 1 and 2 are satisfied.

Conversely, suppose that $(V_i)$ is a flag satisfying 1 and 2. If $j < i$ or $j > i+1$ then, by 1, $FV_j = V_j$, and $FV_{j-1} = V_{j-1}$ ($V_0 = \{0\}$). So $F(V_j \setminus V_{j-1}) = V_j \setminus V_{j-1}$. If $j = i$ then, by 2, $FV_j \neq V_j$, but $FV_j \subseteq V_{j+1}$. So we have a marking of $(V_i)$ of the form $(v_1, \dots, v_i, Fv_i, v_{i+2}, \dots, v_n)$. To see that $(v_1, \dots, Fv_i, v_i, v_{i+2}, \dots, v_n)$ is a marking of $(FV_i)$ we only need to show that $v_i \in FV_{i+1} \backslash FV_i$, but this is true as $\{v_1, \dots, v_i, Fv_i\}$ is linearly independent and $\{v_1, \dots v_{i-1}, Fv_i\} \subset FV_i$. Thus, $(v_1, \dots, Fv_i, v_i, v_{i+2}, \dots, v_n)$ is a marking of $(FV_i)$, and so $(V_i) \in X(w)$ by Theorem 5.5.5. ▲

As we have seen Theorem 5.5.5 is a useful tool. We have used it to turn the question of being in some $X(w)$ into questions about $F$-stable subspaces.

Here we are at the end. At times the task has been arduous, but it seems hopeful that we can now claim a clearer understanding of the Deligne-Lusztig varieties and their accoutrements; as well as a good sense of the structure of general linear groups.

# Bibliography

[Alperin-Bell] J.L. Alperin, Rowen B. Bell, **Groups and representations**. Graduate Texts in Mathematics, 162. Springer-Verlag, New York, 1995.

[Atiyah-MacDonald] M. F. Atiyah, I. G. Macdonald, I, **Introduction to commutative algebra**. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[Borel] Armand Borel. **Linear algebraic groups**. Second enlarged edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.

[Bourbaki] Nicolas Bourbaki. **Lie groups and Lie algebras**. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[Deligne-Lusztig] Pierre Deligne, George Lusztig, "Representations of reductive groups over finite fields", Ann. of Math. 103 1976, 103-161.

[Geck] Meinolf Geck, **An introduction to algebraic geometry and algebraic groups**. Oxford Graduate Texts in Mathematics, 10. Oxford University Press, Oxford, 2003.

[Goss] David Goss, **Basic Structures of Function Field Arithmetic**, A Series of Modern Surveys in Mathematics Volume 35, Springer 1991.

[Hansen] Søren Have Hansen, The geometry of Deligne-Lusztig varieties; Higher-Dimensional AG codes, PhD thesis University of Aarhus, 1999.

[Hartshorne] Robin Hartshorne **Algebraic geometry**. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Humphreys] James E Humphreys, **Linear algebraic groups**. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg 1975.

[Kumar] Shrawan Kumar, **Kac-Moody groups, their flag varieties and representation theory**. Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[Müller] Peter Müller, "Algebraic groups over finite fields, a quick proof of Lang's theorem", Proc. Amer. Math. Soc. 131, 2 (2003), 369–370.

[Springer] T.A. Springer, **Linear algebraic groups**. Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, 1998.