A SURVEY
OF THE
STONE-DANIELL INTEGRAL

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ABSTRACT

The thesis presents a survey of a theory of integration due to P. J. Daniell and M. H. Stone. The author follows in the main M. H. Stone's "Notes on Integration" I - IV. Proc. Nat. Acad. Sci. U. S. A. 34(1948); 35(1949). A special feature of the thesis is that it provides a detailed description of the principal parts of this theory; the outline and the results are of course due to Stone.

INTRODUCTION

This thesis presents a survey of a theory of integration which is due to P. J. Daniell and M. H. Stone. Our treatment follows in the main Stone's definitive work "Notes on Integration" I - IV. Proc. Nat. Acad. Sci. U. S. A. 34(1948); 35(1949). Our task of studying Stone's work on the subject was much helped by the reading of suitable portions of a typescript version of Dr. Klambauer's forthcoming book entitled "Elements of Real Analysis" (American Elsevier Publishing Company, New York).

The theory of integration at hand commences with a postulated "elementary integral" defined for a class of "elementary functions", and proceeds by defining in terms of the elementary integral, a "norm", or upper integral, for all (extended) real-valued functions. A pseudometric is introduced in the class of all functions whose norm is finite; upon proper identification, the resulting function space is shown to be complete. Defining integrable function and integral, it is seen that these concepts have many of their usual properties, in particular, as it relates to monotone and dominated convergence and the integrability of certain Baire functions of integrable functions.

A noteworthy aspect of the theory is the definition of measurability. A function f is called measurable if and only if med (f,g,h) is integrable whenever g and h are integrable, where med (a,b,c) is the intermediate of the three numbers a, b and c. The class of measurable functions is seen to have most of its customary closure properties. In general, however, it need not be true that the constant function 1 be measurable. In the special case in which 1 is measurable, the theory is shown to be essentially identical with the classical measure theoretic approach to integration. The role of the theory of measure in Stone's theory is persued far enough to obtain an analogue of Egorov's theorem. We also discuss the representation of positive linear functionals on the space of continuous functions vanishing in a neighborhood of infinity in a locally compact Hausdorff space. Moreover, the role of L_p spaces is discussed upon obtaining pertinent generalizations of the Hoelder and Minkowski inequalities and Stone's generalization of the Mazur theorem on the mutual homeomorphism of the L_p spaces.

Adjoining to the postulates an assumption which in measure-theoretic language amounts to the requirement of σ-finiteness, we consider analogues of the Lebesgue decomposition theorem and the Radon-Nikodym theorem. The proof of the latter is by Stone's modification of von Neumann's proof using linear functionals in the appropriate Hilbert space. And, finally, Stone's version of Fubini's theorem is described.

For completeness of presentation we begin the thesis by presenting some pertinent facts from the theory of uniform approximation of continuous functions going back to M. H. Stone's famous paper entitled "The Generalized Weierstrass Approximation Theorem" (Math. Mag. 21(1948)).

Section 1

Approximation of Continuous Functions

Let $C = C(X,R^1)$ be the set of all continuous real-valued functions defined on a compact Hansdorff space X; the real-valued functions under consideration throughout this section are assumed always to be finite-valued. We shall moreover suppose that C is endowed with the topology of uniform convergence; this topology is defined by the norm $\|f\| = \sup_{x \in X} |f(x)|$; with $\|f-g\|$ as distance C is seen to be a complete metric space.

If H is subset of C, we shall say that a continuous real-valued function f on X can be uniformly approximated by functions in H if f lies in the closure of H in the space C, i.e., if, for each $\varepsilon > o$, there exists a function $g \in H$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. To say that every continuous real-valued function on X can be uniformly approximated by functions of H therefore means that H is dense in C.

On the set C the relation $f \leq g$ (which means that $f(x) \leq g(x)$ for all $x \in X$) is an order relation, with respect to which C is a lattice.

<u>Definition:</u> If X is any non-empty set, a set H of mappings of X into a set Y is said to <u>separate the elements of a subset</u> A of X if, given any two distinct elements x, y of A, there is a function $f \in H$ such that $f(x) \neq f(y)$.

<u>Proposition 1.1.:</u> (M.H. Stone) Let H be a subset of $C(X,R^1)$ having the following properties: (1) H contains all

constant functions; (2) H is a vector subspace of C; (3) if $h \in H$ then $|h| = \max\{h, -h\} \in H$; (4) H separates the points of X. Then every continuous real-valued function on X can be uniformly approximated by functions of H. (For a proof see Bourbaki, General Topology, part 2, page 311, theorem 2).

In the real q - dimensional Euclidean space R^q with the points $\xi = (\xi_1, \dots, \xi_q)$ let H_o denote the system of all functions h which can be obtained from the projections $\pi_i(\xi) = \xi_i$, $i = 1, \dots, q$, by the formation of finite linear combinations with real coefficients and by the formation of the absolute value.

H_o is characterized as the smallest system of functions with the following properties:

(I) H_o contains the projections $\pi_i(\xi) = \xi_i$ for $i = 1, 2, \dots, q$.

(II) If a_1 and a_2 are real numbers and f_1 and f_2 are functions in H_o, then $a_1 f_1 + a_2 f_2$ also belongs to H_o.

(III) If f is in H_o, then so is $|f|$.

<u>Proposition 1.2.:</u> Every continuous function on $T = \{\xi \in R^q : |\xi_1| + \dots + |\xi_q| = 1\}$ can be uniformly approximated by functions of H_o', where H_o' denotes the system of functions $h' = h/T$ (here h/T denotes the restriction of h to T), $h \in H_o$.

(Proof) H_o' satisfies all conditions of proposition 1.1. Indeed, if c is any constant, then the function $c \cdot \sum_{i=1}^{q} |\pi_i(\xi)| = c$ is in H_o'; moreover, the linearity of H_o' as well as its closure

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relative to the formation of the absolute value is evident; finally, the functions $\pi_i(\xi) = \xi_i$, $(i = 1, \ldots, q)$ already form a set of functions which separate the points.

Definition: A function $p: R^q \longrightarrow R$ is called positive homogeneous iff $p(\alpha\xi) = \alpha p(\xi)$ for all $\alpha \geq o$ and all $\xi \in R^q$.

The functions of H_o are positive homogeneous.

A positive homogeneous function is already determined by its values on the set T introduced above since $p(o) = o$ and $p(\xi) = r(\xi)\, p(\frac{\xi}{r(\xi)})$ for $\xi \neq o$, where $r(\xi) = |\xi_1| + \cdots + |\xi_q|$.

Proposition 1.3.: Every continuous positive homogeneous function p on R^q can be approximated "relative uniformly in R^q", i.e., for $\varepsilon > o$ there exists an $h \in H_o$ such that, for all $\xi \in R^q$, $|p(\xi) - h(\xi)| < \varepsilon\, r(\xi)$.

(Proof) Let $\varepsilon > o$ be given. Then there exists $h \in H_o$ such that $\|h' - p'\| < \varepsilon$, where $h' = h/T$ and $p' = p/T$. Hence $|h(\xi) - p(\xi)|$ $= r(\xi)\, |h(\xi/r(\xi)) - p(\xi/r(\xi))| \leq r(\xi)\, \|h' - p'\| < r(\xi)\varepsilon$. For all $\xi \in R^q$.

In order to obtain similar approximation theorems for functions φ on R^q which are not positive homogeneous, we consider an extension of the function system H_o. Let H_1 denote the set of all functions h on R^q which are obtained upon repeated application of the following operations from the special functions $\pi_i(\xi) = \xi_i$ $(i = 1, 2, \ldots, q)$:

1: $h_1, h_2 \longrightarrow a_1h_1 + a_2h_2$ with real constants a_1, a_2.
2: $h \longrightarrow |h|$
3: $h \longrightarrow \min\{h,1\}$

It is clear that $h(o) = o$ for every function so obtained. We have the following approximation theorem.

Proposition 1.4.: If φ is any continuous finite real-valued function on R^q with $\varphi(o) = o$, then there exists a sequence $(h_n)_{n\in N}$ of functions in H_1 which converges "monotonely away from zero" to φ. This means: For $\varphi(o) = o$ we have $h_n(x) = o$, $n = 1, 2, \ldots$; for $\varphi(x) \neq o$, with $s = \operatorname{sgn} \varphi(x)$, we have $o \leq sh_1(x) \leq sh_2(x) \leq \ldots \leq sh_n(x) \leq \ldots$ with $\lim_n h_n(x) = \varphi(x)$. The convergence is uniform in every bounded closed subset of R^q.

(Proof) The proof will be done in 5 steps.

Step 1: We observe that $\min\{h,a\} = a \min\{\frac{h}{a}, 1\}$ belongs to H_1 if $h \in H_1$ and a is a positive constant.

Step 2: We next define several functions in H_1, where a, b denote positive constants. $\beta(\pi_1(\xi); b,a) = \min\{\frac{1}{2}(\xi_1+|\xi_1|), b+a\} - \min\{\frac{1}{2}(\xi_1+|\xi_1|), b\}$, $\tau_1(\pi_1(\xi)) = \gamma(\pi_1(\xi); b,a) = \frac{1}{a}[\beta(\pi_1(\xi); b,a) - \beta(\pi_1(\xi); b+2a, a)]$. In addition we also have the function $\overline{\tau}_1(\pi_1(\xi)) = \tau_1(-\pi_1(\xi))$. The functions τ_1 and $\overline{\tau}_1$ are "stump functions" for the case $q = 1$.

Step 3: For $q \geq 2$ we construct the stump functions as follows:

$$\alpha(\pi_1(\xi), \dots, \pi_q(\xi); b,a) = \min\{\gamma(\pi_1(\xi), b,a), \dots, \gamma(\pi_q(\xi); b,a)\}.$$

From these stump functions in turn we obtain the general "stump function" τ_q by a rotation $\xi_i \to \sum a_{ik}\xi_k$ around the origin. $\tau(\xi) = \tau_q(\xi) = \alpha(\sum_k a_{1k}\xi_k, \dots, \sum_k a_{qk}\xi_k; b,a)$ with $\xi = (\xi_1, \dots, \xi_q)$. We have $0 \leq \tau \leq 1$; the open, respectively closed non-empty cubes $\{\xi \in R^q; \tau(\xi) > 0\}$ and $\{\xi \in R^q; \tau(\xi) = 1\}$ we shall refer to as the kernel, respectively principal kernel of τ.

Step 4: In the sequel we also need the function $\omega(\pi_1(\xi), \pi_2(\xi)) = \xi_1 + \xi_2 - \max\{\min\{\xi_1, \xi_2\}, 0\} - \min\{\max\{\xi_1, \xi_2\}, 0\}$; if the signs of ξ_1 and ξ_2 coincide, then the value of this function is ξ_i with ξ_i being the one whose absolute value equals $\max\{|\xi_1|, |\xi_2|\}$(and otherwise, i.e., if the signs of ξ_1 and ξ_2 are different, the value of the function is $\xi_1 + \xi_2$).

Step 5: Now to the approximation:

The set $G = \{\xi \in R^q : \varphi(\xi) \neq 0\}$ is open ($G = \varphi^{-1}(-\infty, 0) \cup \varphi^{-1}(0, +\infty)$ and φ is continuous) and does not contain the origin $\xi = 0$. Hence, for every point η of G and for every natural number n, there is a stump function τ and a constant ζ ($\neq 0$) with the following properties:

1) η belongs to the principal kernel K' of τ;

2) φ has everywhere in the kernel K of τ the same sign as ζ;

3) for the function $\varphi(\mathfrak{z};\eta, n) = \mathfrak{z}\tau(\mathfrak{z})$ we have

(a) $(\operatorname{sgn}\mathfrak{z})\,\varphi(\mathfrak{z};\eta, n) \leq (\operatorname{sgn}\mathfrak{z})\tau(\mathfrak{z})$ for $\mathfrak{z} \in K$.

(b) $|\varphi(\mathfrak{z};\eta, n) - \varphi(\mathfrak{z})| < \frac{1}{n}$ for $\mathfrak{z} \in K'$.

For fixed n we can represent G as the union of countably many K', eg. $K'_{n1}, K'_{n2}, \ldots$, where every compact subset of G can be covered by a finite subfamily of these sets. We arrange these sets K'_{nm} as a simple sequence $K'_1, K'_2, \ldots$ by enumerating in the usual fashion the doubly indexed sequence

$$\begin{array}{llllll}
K'_{11}, & K'_{12}, & K'_{13}, & \ldots, & K'_{1j}, & \ldots \\
K'_{21}, & K'_{22}, & K'_{23}, & \ldots, & K'_{2j}, & \ldots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
K'_{n'1}, & K'_{n'2}, & K'_{n'3}, & \ldots, & K'_{n'j}, & \ldots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots ;
\end{array}$$

we let the corresponding $\varphi(\mathfrak{z};\eta;n)$ be $\varphi_1(\mathfrak{z}), \varphi_2(\mathfrak{z}), \ldots$ (the corresponding kernels are situated in G by (a)).

In the sequence $h_1 = \varphi_1$, $h_2 = \omega(h_1, \varphi_2), \ldots$ we have the desired approximating sequence. Indeed; if B is a closed and bounded subset of R^q, and $B_0 = B \cap \{\mathfrak{z} \in R^q : \varphi(\mathfrak{z}) = 0\}$ then for a given $\varepsilon > 0$, the set $\mathcal{V} = |\varphi|^{-1}[(-\varepsilon, +\varepsilon)]$ is open, contains B_0 and $|\varphi(\mathfrak{z})| < \varepsilon$ for $\mathfrak{z} \in \mathcal{V}$. Then by (a) we have $|\varphi_n(\mathfrak{z})| < \varepsilon$ for all $n \in N$ and hence $|h_n(\mathfrak{z})| \leq |\varphi_n(\mathfrak{z})| < \varepsilon$, $n \in N$, $\mathfrak{z} \in \mathcal{V}$. For the closed bounded set $B_1 = B - B \cap \mathcal{V}$ we pick an index n_0, after having chosen $n' > \frac{1}{\varepsilon}$, such that amongst $K'_1, \ldots\ldots, K'_{n_0}$ we have all those finitely many $K'_{n'1}, K'_{n'2}, \ldots$ which have points in common with

B_1 and cover B_1. Then $|\varphi(\xi) - h_{n_o}(\xi)| < \frac{1}{n'} < \varepsilon$ for $\xi \in B_1$ (because of (b)) and so in general $|\varphi(\xi) - h_n(\xi)| < \varepsilon$ for all $\xi \in B$ and $n \geqslant n_o$. Therefore the claimed monotonicity property of the sequence $(h_n)_{n \in N}$ is manifest by step (4) and (a).

Q.E.D.

Remark: For arbitrary continuous functions φ on R^q we obtain an approximation theorem if in place of H_1 we consider the larger system H_2 which is obtained by using the projection functions $\pi_i(\xi) = \xi_i$ $(i = 1, 2, \ldots, q)$ and the constant unit function 1 and then applying to these the operations of forming finite linear combinations with real coefficients and of forming the absolute value.

Proposition 1.5.: If $\varphi | R^q$ is a finite real-valued function, then there exists a sequence of functions in H_2 which converges "monotonely away from zero" to φ ; here the convergence is uniform on every bounded closed subset of R^q.

(Proof) Let $\varphi = \varphi_o + k$, where $\varphi_o(o) = o$ then there exists (h_n) with $h_n \in H_1$, $n \in N$ such that $h_n \rightarrow \varphi_o$ as $n \rightarrow +\infty$. Hence $h_n + k \rightarrow \varphi_o + k = \varphi$ as $n \rightarrow +\infty$. Since $c \cdot 1 \in H_2$ for all $c \in R$ we have that $k + h_n \in H_2$. Finally $(k + h_n)_{n \in N}$ is monotone since $(h_n)_{n \in N}$ is so.

Section 2

Elementary and Norm Integrals

Let $\mathcal{E}$ be a vector lattice of finite real-valued functions f defined on a non-empty set A, i.e., $\mathcal{E}$ is a vector space over R^1 with the property $|f| \in \mathcal{E}$ whenever $f \in \mathcal{E}$. We call the elements of $\mathcal{E}$ elementary functions.

Let E be a real-valued function defined on $\mathcal{E}$ with the following properties:

(1c) $E(af) = aE(f)$ with a real and $f \in \mathcal{E}$.

(2c) $E(f_1 + f_2) = E(f_1) + E(f_2)$ with f_1 and f_2 in $\mathcal{E}$.

(3c) $E(|f|) \geqslant 0$ with $f \in \mathcal{E}$.

(4c) If $(f_n)_{n \in N}$ is a sequence in $\mathcal{E}$ with $f_n \downarrow 0$ then $E(f_n) \longrightarrow 0$.

We call E an elementary integral.

Let $\mathcal{G}$ denote the set of all extended real-valued functions g defined on A. For $g \in \mathcal{G}$ we define the norm of g associated with the elementary integral E by:

$$\text{(a)} \quad \mathcal{N}(g) = \inf \left\{ \sum_n E(|f_n|) : |g| \leq \sum_n |f_n| \text{ with } f_n \in \mathcal{E} \right\}$$

where we adopt the convention that the infimum of the empty set is $+\infty$.

The following properties of $\mathcal{N}$ follow immediately from the foregoing definition:

If $g, g_1, \ldots, g_n \cdots \in \mathcal{G}$, then

(b) $0 \leq \mathcal{N}(g) \leq +\infty$.

(c) $\mathcal{N}(ag) = |a|\mathcal{N}(g)$ for $a \in R^1 - \{0\}$.

(d) $|g| \leq \sum_n |g_n|$ implies $\mathcal{N}(g) \leq \sum_n \mathcal{N}(g_n)$.

(e) $\mathcal{N}(|g|) = \mathcal{N}(g)$.

(f) $|g_1| \leq |g_2|$ implies $\mathcal{N}(g_1) \leq \mathcal{N}(g_2)$.

(g) $\mathcal{N}(f) = E(|f|)$ for all $f \in \mathcal{E}$.

Indeed, (b), (e) and (f) are immediate consequences of the definition of $\mathcal{N}$. To see that (c) holds, we note that, for $a \neq 0$, $|ag| \leq \sum_n |f_n|$ and $|g| \leq \sum_n \frac{1}{|a|} |f_n|$ are equivalent statements.

To verify (d): Given $\varepsilon > 0$, there exist $(f_{n,k}) \subseteq \mathcal{E}$, $n = 1, 2, \ldots$ such that $|g_n| \leq \sum_k |f_{n,k}|$ and $\mathcal{N}(g_n) > \sum_k E(|f_{n,k}|) - \varepsilon 2^{-n}$; thus we have $|g| \leq \sum_{n,k} |f_{n,k}|$ and $\mathcal{N}(g) \leq \sum_{n,k} E(|f_{n,k}|) < \sum_n \mathcal{N}(g_n) + \varepsilon$; with $\varepsilon > 0$ the claim made in (d) follows.

To verify (g), we observe that condition (4c) is equivalent with

(5): $|f| \leq \sum_n |f_n|$ implies $E(|f|) \leq \sum_n E(|f_n|)$ whenever $f, f_1, \ldots, f_n, \ldots$ are in $\mathcal{E}$.

But condition (5) is equivalent with the claim made in (g).

Remark: The formal definition in (a) lends itself to the following informal presentation. Confining ourselves to non-negative functions, we may regard (a) as the condensed description of a measuring process. The set $\mathcal{E}$ provides us with a stock of measuring rods, the non-negative elementary functions, by means of which the non-negative functions in $\mathcal{G}$ are to be gauged. Each measuring rod has a magnitude given by its elementary integral. The basic measuring process consists in choosing from stock such an infinite sequence of measuring rods $f_n = |f_n| \geqslant o$ that by addition they combine to surpass $g = |g| \geqslant o$ indicated in the inequality $g \leqslant \sum_n f_n$. The real number $\lambda = \sum_n E(f_n)$ obtained by adding together the magnitudes of the particular measuring rods thus employed is then accepted as an estimate, generally in excess, of the magnitude of g. Repetitions of this basic measuring process furnish successively better estimates, convergent to the quantity $\mathcal{N}(g)$, when suitable precautions are taken. It is, of course, conceivable that the basic measuring process will fail to produce any real numbers as estimates of the magnitude of a particular function g, either because the inequality $g \leqslant \sum_n f_n$ cannot be realized or because it implies the divergence of the series $\sum_n E(f_n)$. Under these circumstances the formal definition requires that we take $\mathcal{N}(g) = +\infty$.

<u>Definitions:</u> A function $g \in \mathcal{J}$ is said to be a <u>null-function</u> if $\mathcal{N}(g) = o$. A subset of A is called a <u>null-set</u> if its characteristic function is a null-function, moreover, the phrase <u>"almost everywhere"</u> signifies "with the exception of the points of a certain null set".

As direct consequences of the properties (f) and (d) we see that any subset of a null-set must be a null-set and that the union of a countable number of null-sets will again be a null-set.

<u>Proposition 2.1</u> Every function f of $\mathcal{J}$ with $\mathcal{N}(f) < +\infty$ is finite almost everywhere.

(Proof) We show that if $Y = \{x \in A: \ f(x) = \pm\infty\}$, then Y is a null-set. It suffices to show that χ_Y is a null-function. Since $Y \subseteq \{x \in A : |f(x)| > M\} = Y_M$ we have that $\mathcal{N}(\chi_Y) \leq \mathcal{N}(\chi_{Y_M})$. We show that $\mathcal{N}(\chi_{Y_M}) \to 0$ as $M \to +\infty$. Since $\mathcal{N}(f) < +\infty$ there exists $(f_n) \subseteq \mathcal{E}$ such that $|f| \leq \sum_n |f_n|$ and $\sum_n E(|f_n|) < +\infty$. Since $Y_M \subseteq Z_M = \{x \in A : M \leq \sum_n |f_n(x)|\}$ we get $M\chi_{Z_M} \leq \sum_n |f_n|$ and so by c) and the definition of the norm we also have $M\mathcal{N}(\chi_{Z_M}) \leq \sum_n E(|f_n|)$ so that $\mathcal{N}(\chi_{Z_M}) \to 0$ as $M \to +\infty$. Thus $\mathcal{N}(\chi_{Y_M}) \to 0$ as $M \to +\infty$ on account of $\mathcal{N}(\chi_{Y_M}) \leq \mathcal{N}(\chi_{Z_M})$.

Proposition 2.2 A function f of $\mathcal{Y}$ is a null-function iff it is zero almost everywhere.

(Proof) "$\Longrightarrow$" Let $\mathcal{N}(f) = 0$. Then $Y = \{x \in A : \ f(x) \neq 0\}$ is a null-set as the countable union of null-sets. Indeed, $Y = \{x \in A : |f(x)| = +\infty\} \cup (\bigcup Y_\nu)$; $\nu = 0, \pm 1, \dots, Y_\nu = \{x \in A: \ 2^{\nu-1} < |f(x)| \leq 2^\nu\}$ and $2^{\nu-1}\chi_{Y_\nu} \leq |f|$, that is, $2^{\nu-1}\mathcal{N}(\chi_{Y_\nu}) \leq \mathcal{N}(|f|) = \mathcal{N}(f) = 0$.

"$\Longleftarrow$" Let $Y = \{x \in A : f(x) \neq 0\}$ be a null-set. For $g_n = n\chi_Y$, $n \in N$, we have $\mathcal{N}(g_n) = n\,\mathcal{N}(\chi_Y) = 0$, moreover $|f| \leq \sum_n |g_n|$ and so $\mathcal{N}(|f|) = \mathcal{N}(f) \leq \sum_n \mathcal{N}(|g_n|) = \sum_n \mathcal{N}(g_n) = 0$.

Corollary 2.3 If $f_1 = f_2$ a.e., then $\mathcal{N}(f_1) = \mathcal{N}(f_2)$.

(Proof) $f_1 = f_2$ a.e. implies $f_1 - f_2 = 0$ a.e. Hence $\mathcal{N}(f_1 - f_2) = 0$.

Since

$$|f_1| \leq |f_1 - f_2| + |f_2| \quad \text{and}$$
$$|f_2| \leq |f_1 - f_2| + |f_1| ,$$

we have by d), $\mathcal{N}(f_1) = \mathcal{N}(|f_1|) \leq o + \mathcal{N}(f_2)$, $\mathcal{N}(f_2) = \mathcal{N}(|f_2|) \leq o + \mathcal{N}(f_1)$ and hence $\mathcal{N}(f_1) = \mathcal{N}(f_2)$.

On the basis of Corollary 2.3 we can extend our concept of a function. We shall already speak of a function h on A whenever h is only defined almost everywhere on A. The set of all these functions we denote by $\mathcal{Z}$. Clearly, $\mathcal{J} \subseteq \mathcal{Z}$. Regardless of how we extend h to all of A, the resulting function $\bar{h} \in \mathcal{J}$ will give the same value for $\mathcal{N}(\bar{h})$ and we can therefore define in a unique manner $\mathcal{N}(h) = \mathcal{N}(\bar{h})$. At the same time we introduce in $\mathcal{Z}$ the equivalence relation:

(*) $h_1 = h_2$ iff $\bar{h}_1 = \bar{h}_2$ a.e.

Evidently, this definition is independent of the choice of the $\bar{h}_1$, $\bar{h}_2$ and moreover $h_1 = h_2$ implies $\mathcal{N}(h_1) = \mathcal{N}(h_2)$.

Let $\mathcal{F}$ be that part of $\mathcal{Z}$ which is characterized by the inequality $\mathcal{N}(h) < +\infty$. From proposition 2.1 we know that the functions in $\mathcal{F}$ are finite almost everywhere.

Clearly $\mathcal{E} \subsetneq \mathcal{F} \subseteq \mathcal{Z}$ and $\mathcal{F}$ together with the equality definition (*) is a normed vector space. Indeed, $a \in R^1$, f_1, $f_2 \in \mathcal{F}$ implies af_1 and $f_1 + f_2$ are defined a.e. with $\mathcal{N}(af_1) = |a| \mathcal{N}(f_1) < +\infty$, $\mathcal{N}(f_1 + f_2) \leq \mathcal{N}(f_1) + \mathcal{N}(f_2) < +\infty$ by d). Thus the correspondence $\mathcal{F} \times \mathcal{F} \ni (f_1, f_2) \longrightarrow \mathcal{N}(f_1 - f_2) \in [o, +\infty)$ is a metric when we agree to identify functions that are equal almost everywhere.

Definition A sequence $(f_n)_{n\in N}$ in $\mathcal{F}$ is said to converge in norm to f in $\mathcal{F}$ if $\lim\limits_n \mathcal{N}(f_n - f) = o$. We shall use the notation $f = \operatorname*{Lim}\limits_n f_n$ or $f_n \xrightarrow[\mathcal{N}]{} f$.

The following example shows that convergence in norm and convergence $\mathcal{N}$ - a.e. must be distinguished. Let A be the interval $(o, 1]$ and $\mathcal{E}$ be the set of all step functions f on $(o, 1]$ and E(f) be the Riemann integral of f. Let $f_n(x) = n$ for $o < x \leq \frac{1}{n}$ and $= o$ elsewhere; then $f_n \rightarrow o$, but $\mathcal{N}(f_n) = E(f_n) = 1$.

Proposition 2.4 If $f = \operatorname*{Lim}\limits_n f_n$, then every subsequence (f_{n_i}) contains a subsequence (f_{m_j}) such that $f = \lim\limits_j f_{mj}$ ($\mathcal{N}$- a.e.)

(Proof) We select (f_{m_j}) from (f_{n_i}) in such a way that $\mathcal{N}(f_{m_{j+1}} - f_{m_j}) < 2^{-j}$ which is possible since $\mathcal{N}(f_n - f) \Longrightarrow o$. We let $g = |f_{m_1}| + |f_{m_2} - f_{m_1}| + \dots$ and obtain $\mathcal{N}(g) \leq \mathcal{N}(f_{m_1}) + \mathcal{N}(f_{m_2} - f_{m_1}) + \dots \leq \mathcal{N}(f_{m_1}) + 2^{-1} + 2^{-2} + \dots < \mathcal{N}(f_{m_1}) + 1 < +\infty$. Since $\mathcal{N}(g) < +\infty$ it follows that g is finite $\mathcal{N}$- a.e. and so the function $\varphi = f_{m_1} + (f_{m_2} - f_{m_1}) + \dots$ is $\mathcal{N}$- a.e. absolutely and properly convergent, that is, $\varphi = \lim\limits_j f_{mj}$ ($\mathcal{N}$- a.e.).

From $\mathcal{N}(\varphi - f) \leq \mathcal{N}(f_{mj} - f) + \mathcal{N}(f_{m_{j+1}} - f_{m_j}) + \dots \leq \mathcal{N}(f_{mj} - f) + 2^{-j}$ we conclude, on letting $j \rightarrow +\infty$, that $\mathcal{N}(\varphi - f) = o$ and thus $\varphi = f$ ($\mathcal{N}$- a.e.)

Proposition 2.5 $\mathcal{F}$ is a Banach space.

(Proof) Let (f_n) be a Cauchy sequence. We pick a subsequence (f_{m_i}) from it such that $\mathcal{N}(f_{m_{i+1}} - f_{m_i}) < 2^{-i}$ and proceed as in the foregoing proof. In this manner we obtain in $\varphi = f_{m_i} + (f_{m_{i+1}} - f_{m_i}) + \dots$ a function of finite norm, namely, $\mathcal{N}(\varphi) \leq \mathcal{N}(f_{m_1}) + 1$ and such that $\mathcal{N}(\varphi - f_{m_j}) \longrightarrow 0$ as $j \rightarrow +\infty$.

Let $f \in \mathcal{F}$. We define the functions $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$; f^+ and f^- are non-negative a.e. and are in $\mathcal{F}$. We may therefore define the function F on $\mathcal{F}$ given by $F(f) = \mathcal{N}(f^+) - \mathcal{N}(f^-)$ with $f \in \mathcal{F}$; we observe that F is well-defined and finite.

<u>Proposition 2.6</u> $F|\mathcal{F}$ is continuous and $F(f) = E(f)$ whenever $f \in \mathcal{E}$.

(Proof) For the continuity we have

$$|F(f) - F(f_0)| = |\mathcal{N}(f^+) - \mathcal{N}(f^-) - \mathcal{N}(f_0^+) + \mathcal{N}(f_0^-)| \leq$$
$$|\mathcal{N}(f^+) - \mathcal{N}(f_0^+)| + |\mathcal{N}(f^-) - \mathcal{N}(f_0^-)| \leq \mathcal{N}(f^+ - f_0^+) +$$
$$\mathcal{N}(f^- - f_0^-) \leq 2\mathcal{N}(f - f_0) \text{ on account of } |f^+ - f_0^+| \leq |f - f_0|$$

and $|f^- - f_0^-| \leq |f - f_0|$. For the second statement of the proposition we have by (g) and the fact that $f = f^+ - f^-$, $F(f) = \mathcal{N}(f^+) - \mathcal{N}(f^-) = E(f^+) - E(f^-) = E(f^+ - f^-) = E(f)$.

We have already gained in $F|\mathcal{F}$ a continuous extension of $E|\mathcal{E}$. We are going now to gain a natural continuous extension of $E|\mathcal{E}$ to a positive linear functional and we confine ourselves to the set $\mathcal{L}$, which we get by closure of $\mathcal{E}$ in $\mathcal{F}$, i.e., $\mathcal{L} = \overline{\mathcal{E}}$ in $\mathcal{F}$.

Since the functions of $\mathcal{L}$ are "arbitrarily close" to $\mathcal{E}$, we call them "$\mathcal{N}$ - almost elementary functions".

We call the functions of $\mathcal{L}$ $\mathcal{N}$- integrable and $L(f)$ the $\mathcal{N}$ - integral (norm-integral) of f with $f \in \mathcal{L}$, where L is the restriction of F on $\mathcal{L}$.

Remark There is another process of obtaining the integrable functions without using the norm $\mathcal{N}$. In this process we start again with a given vector lattice $\mathcal{E}$ and an elementary integral E defined on $\mathcal{E}$ i.e., a finite real-valued function E which is defined on $\mathcal{E}$ and satisfies properties (1c) - (4c). After enlarging $\mathcal{E}$ by taking pointwise limits of increasing sequences of functions in $\mathcal{E}$ and denoting this family by $\mathcal{E}'$, we extend $E|\mathcal{E}$ to $E|\mathcal{E}'$ by defining $E(f') = \lim_n E(f_n)$, where $f_n \uparrow f'$, $(f_n) \subset \mathcal{E}$; we then introduce the upper and lower integrals of an arbitrary extended real-valued function $f: A \longrightarrow \hat{R}$ (with $\hat{R}$ the extended reals) by defining

$$\overline{E}(f) = \inf\{E(g) : g \geq f, g \in \mathcal{E}'\} \text{ and } \underline{E}(f) = -\overline{E}(-f).$$

Using the upper and lower integral of $f: A \longrightarrow \hat{R}$ we say that f is an "integrable function", i.e., $f \in \mathcal{L}$ if and only if $\overline{E}(f) = \underline{E}(f) = L(f) < +\infty$.

Proposition 2.7 (i) $\mathcal{L}$ is a complete normed vector lattice under the norm $\mathcal{N}(f) = \| f \|$.

(ii) $L|\mathcal{L}$ is a continuous elementary integral.

(iii) If $f_n \in \mathcal{L}$, $n \in N$ and $f_n \geqslant o$ for $n \in N$.
Then the function $f = \sum_n f_n$ belongs to $\mathcal{L}$ iff $\sum_n L(f_n) < +\infty$.
In that case the last sum equals L(f).

(Proof) (i) $\mathcal{L}$ is a subspace of $\mathcal{F}$ as the closure of the subspace $\mathcal{E}$ in $\mathcal{F}$. $\mathcal{L}$ is closed subspace of the Banach space $\mathcal{F}$ and so is a Banach space. Incidentally, $|f| \in \mathcal{L}$ follows from $f \in \mathcal{L}$. Indeed, $f \in \mathcal{L} \Longrightarrow \exists\, (f_n) \subset \mathcal{E}$ with $\mathcal{N}(f_n - f) \longrightarrow o$, but $\mathcal{N}(|f_n| - |f|) \leq \mathcal{N}(|f_n - f|) = \mathcal{N}(f_n - f)$ and $|f_n| \in \mathcal{E}$ for all $n \in N$. Hence $\mathcal{N}(|f_n| - |f|) \longrightarrow o$.

(ii) $L|\mathcal{L}$ is continuous since $F|\mathcal{F}$ is continuous.

To (1c):Let $a \in R^1$ and $f \in \mathcal{L}$. Then, there exists a sequence $(f_n) \subseteq \mathcal{E}$ such that $\mathcal{N}(f_n - f) \longrightarrow o$. Since $\mathcal{N}(af_n - af) = |a| \mathcal{N}(f_n - f)$ we have that $\mathcal{N}(af_n - af) \longrightarrow o$. Let $\varepsilon > o$ be given and let $n(\varepsilon) \in N$ be such that $\mathcal{N}(f_n - f) < \frac{\varepsilon}{2}$ for all $n > n(\varepsilon)$. Then $|E(f_n) - E(f_m)| \leqslant E(|f_n - f_m|) = \mathcal{N}(f_n - f_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for (n and m) $> n(\varepsilon)$. Thus $E(f_n) = L(f_n) \longrightarrow k \in R^1$. But $k = L(f)$ (since L is continuous). Thus $E(af_n) \longrightarrow L(af)$ and since $aE(f_n) \longrightarrow aL(f)$ we have that $L(af) = aL(f)$.

To (2c):Let $f_1, f_2 \in \mathcal{L}$ and $(f_{n1}), (f_{n2}) \subset \mathcal{E}$ such that $\mathcal{N}(f_{n1} - f_1) \longrightarrow o$, $\mathcal{N}(f_{n2} - f_2) \longrightarrow o$. Then $\mathcal{N}(f_{n1} + f_{n2} - f_1 - f_2) \leq \mathcal{N}(f_{n1} - f_1) + \mathcal{N}(f_{n2} - f_2)$ and hence $\mathcal{N}(f_{n1} + f_{n2} - (f_1 + f_2)) \longrightarrow 0$. Thus $L(f_{n1} + f_{n2}) \longrightarrow L(f_1 + f_2)$ and since $L(f_{n1} + f_{n2}) = E(f_{n1} + f_{n2}) = E(f_{n1}) + E(f_{n2}) = L(f_{n1}) + L(f_{n2}) \longrightarrow L(f_1) + L(f_2)$ we have that $L(f_1 + f_2) = L(f_1) + L(f_2)$.

To (3c): Let $f \in \mathcal{L}$ and $(f_n) \subseteq \mathcal{E}$ such that $\mathcal{N}(f_n - f) \longrightarrow o$. Then $\mathcal{N}(|f_n| - |f|) \leq \mathcal{N}(|f_n - f|) = \mathcal{N}(f_n - f) \longrightarrow o$. Thus $L(|f_n|) \longrightarrow L(|f|)$. But $L(|f_n|) = E(|f_n|) \geqslant o$ for all $n \in N$. Hence $L(|f|) \geqslant o$.

To (4c): It suffices to show (5), which is equivalent to (4c). Let $f, f_1, f_2, \ldots$

be in $\mathcal{L}$ and suppose that $|f| \leq \sum_n |f_n|$. Then by (d) we have $\mathcal{N}(|f|) \leq \sum_n \mathcal{N}(|f_n|)$, i.e., $\mathcal{N}(f) \leq \sum_n \mathcal{N}(f_n)$. But $\mathcal{N}(f) = L(|f|)$ for $f \in \mathcal{L}$. Hence $L(|f|) \leq \sum_n L(|f_n|)$.

(iii) "$\Rightarrow$" If $f \in \mathcal{L} \subseteq \mathcal{F}$ then $\mathcal{N}(f) < +\infty$ and $\sum_{n=1}^{k} L(f_n) = L(\sum_{n=1}^{k} f_n) \leq \mathcal{N}(f)$ thus $\sum_n L(f_n)$ converges.

"$\Leftarrow$" Suppose now that $\sum_n L(f_n) < +\infty$ then $\mathcal{N}(f) \leq \sum_n \mathcal{N}(f_n) = \sum_n L(f_n)$ $\Rightarrow$ $f \in \mathcal{F}$. But $\mathcal{N}(f - \sum_{n=1}^{k} f_n) \leq \sum_{n=k+1} \mathcal{N}(f_n) = \sum_{n=k+1}^{\infty} L(f_n) \longrightarrow 0$ as $k \to +\infty$. Thus $f \in \bar{\mathcal{L}} = \mathcal{L}$. Finally the continuity of L yields $L(f) = \lim_k L(\sum_{n=1}^{k} f_n) = \lim_k \sum_{n=1}^{k} L(f_n) = \sum_n L(f_n)$.

We are now going to consider a proposition concerning the relationship between two elementary integrals, where one of them is an extension of the other.

Proposition 2.8 Let $E|\mathcal{E}$ and $E'|\mathcal{E}'$ be elementary integrals for functions defined on the same basic set A and suppose that the second is an extension of the first, then we have the following relations amongst the corresponding norms $\mathcal{N}, \mathcal{N}'$, the domains of bounded norms $\mathcal{F}, \mathcal{F}'$ and the norm integrals L, L' and their domains $\mathcal{L}, \mathcal{L}'$:

1) $\mathcal{N}' \leq \mathcal{N}$; 2) $\mathcal{F}' \supseteq \mathcal{F}$; 3) $\mathcal{L}' \supseteq \mathcal{L}$; 4) $L'|\mathcal{L}'$ is an extension of $L|\mathcal{L}$.

5) In particular, if $\mathcal{E} \subseteq \mathcal{E}' \subseteq \mathcal{L}$ then we have $\mathcal{N}' = \mathcal{N}$, $\mathcal{F}' = \mathcal{F}$, $\mathcal{L}' = \mathcal{L}$ and $L' = L$.

(Proof) Concerning claims 1) and 2) we note that the set for which the infinum is formed in the determination of $\mathcal{N}'$ contains

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the set for which the infinum is taken in the determination of $\mathcal{N}$ and so $\mathcal{N}' \leq \mathcal{N}$. From this in turn it follows that $\mathcal{F}' \geq \mathcal{F}$. Regarding claims 3) and 4) we observe: If $f \in \mathcal{L}$, then there is a sequence of functions $(f_k) \subseteq \mathcal{E} \subseteq \mathcal{E}'$ converging in norm $\mathcal{N}$; by 1) this sequence also converges in norm $\mathcal{N}'$ and so $f \in \mathcal{L}'$. Here $L(f) = \lim_k E(f_k) = \lim_k E'(f_k) = L'(f)$. Finally we consider claim 5). Let $f \in \mathcal{H}$ and $\varepsilon > o$ be given. Then there exists a sequence $(f_n) \subset \mathcal{E}'$ such that $\mathcal{N}'(f) > \sum_n E'(|f_n|) - \varepsilon = \sum_n L'(|f_n|) - \varepsilon = \sum_n L(|f_n|) - \varepsilon = \sum_n \mathcal{N}(f_n) - \varepsilon$ and $|f| \leq \sum_n |f_n|$. Since $\mathcal{N}(f_n) < +\infty$ we have for suitable (f_{nk}) in $\mathcal{E}$, $|f_n| \leqslant \sum_k |f_{nk}|$ and $\mathcal{N}(f_n) > \sum_k E(|f_{nk}|) - \varepsilon 2^{-n}$. Hence $|f| \leqslant \sum_{n,k} |f_{nk}|$ and $\mathcal{N}'(f) \geqslant \sum_{k,n} E(|f_{nk}|) - \varepsilon - \varepsilon = \sum_{k,n} E(|f_{nk}|) - 2\varepsilon$. Thus $\mathcal{N}'(f) \geqslant \mathcal{N}(f) - 2\varepsilon$, that is, $\mathcal{N}'(f) \geqslant \mathcal{N}(f)$. This leads to $\mathcal{F}' \subseteq \mathcal{F}$ and thus $\overline{\mathcal{E}}' = \mathcal{L}' \subseteq \overline{\mathcal{L}} = \overline{\overline{\mathcal{E}}} = \mathcal{L}$. Finally we have by 4) that $L' = L$.

<u>Corollary 2.9</u> With $E'|\mathcal{E}' = L|\mathcal{L}$ the extension process from $E|\mathcal{E}$ to $L|\mathcal{L}$ yields again $L|\mathcal{L}$ when applied to $L|\mathcal{L}$.

<u>Proposition 2.10</u> (Monotone Convergence Theorem). If $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_n \leqslant \ldots$ is a non-decreasing sequence of functions f_n in $\mathcal{L}$ with $L(f_n) \leqslant M < +\infty$ for all $n \in N$, then $f = \lim_n f_n \in \mathcal{L}$ and $L(f) = \lim_n L(f_n)$.

(Proof) Since $o \leqslant f - f_n = (f_{n+1} - f_n) + (f_{n+2} - f_{n+1}) + \ldots$. we obtain $o \leqslant \mathcal{N}(f - f_n) \leqslant \mathcal{N}(f_{n+1} - f_n) + \mathcal{N}(f_{n+2} - f_{n+1}) + \ldots = L(f_{n+1} - f_n) + L(f_{n+2} - f_{n+1}) + \ldots = \lim_k L(f_k) - L(f_n) \longrightarrow o$ for $n \to +\infty$. Thus $f \in \mathcal{L}$. But $L|\mathcal{L}$ is continuous and the claimed equality follows.

A corollary of proposition 2.10 is the following Dominated Convergence Theorem:

Proposition 2.11 If $g, f_1, f_2, \ldots$ belong to $\mathcal{L}$ and if $f = \lim_n f_n \in \mathcal{F}$, and, moreover, if $|f_n| \leq |g|$ for $n \in N$, then $f \in \mathcal{L}$ and $L(f) = \lim_n L(f_n)$.

In place of a direct proof for proposition 2.11, we shall prove three lemmas; it will be apparent that Lemma 3 is equivalent to proposition 2.11.

Lemma 1 If $g \in \mathcal{L}$, $g_n \in \mathcal{L}$ with $g_n \leq g$, $n \in N$, then $s = \sup_n g_n \in \mathcal{L}$ and $L(s) \geqslant \sup_n L(g_n)$.

(Proof) Let $f_n \doteqdot \max\{g_1, \ldots, g_n\} \leq g$, then $f_n \in \mathcal{L}$, $s = \lim_n f_n$ and $L(f_n) \leq L(g) < +\infty$ so that we can use Proposition 2.10 and obtain: $s \in \mathcal{L}$ and $L(s) = \lim_n L(f_n) \geqslant \lim_n \max\{L(g_1), \ldots, L(g_n)\} = \sup_n L(g_n)$.

Lemma 2 If $g \in \mathcal{L}$, $g_n \in \mathcal{L}$ with $g_n \leq g$, $n \in N$ and letting $\lambda = \lim_n \sup L(g_n) > -\infty$, then $\varphi = \lim_n \sup g_n \in \mathcal{L}$ and $L(\varphi) \geqslant \lambda$.

(Proof) Let $\varphi = \lim_n \varphi_n$ with $\varphi_n = \sup\{g_n, g_{n+1}, \ldots\} \in \mathcal{L}$ (by Lemma 1) and $\varphi_n \geqslant \varphi_{n+1} \geqslant g_{n+1}$. For $m > n$ we therefore have $L(\varphi_n) \geqslant L(\varphi_m) \geqslant L(g_m)$ so that $L(\varphi_n) \geqslant \lambda > -\infty$. By proposition 2.10 we therefore get that $\varphi \in \mathcal{L}$ and $L(\varphi) = \lim_n L(\varphi_n) \geqslant \lambda$.

Lemma 3. If $g \in \mathcal{L}$, $g_n \in \mathcal{L}$ with $|g_n| \leq g$, $n \in N$, and if $\lim_n g_n = \varphi$ exist, then $\varphi \in \mathcal{L}$ and $L(\varphi) = \lim_n L(g_n)$.

(Proof) The assumption leads to $-g \leqq \pm g_n \leqq g$ (we shall keep both signs in the sequel). We get $L(-g) \leqq L(\pm g_n)$ and hence $\lim_n \sup L(\pm g_n) > -\infty$. By Lemma 2 we obtain $L(\pm \varphi) = \lim_n \sup(\pm L(g_n))$, or upon separation of the cases, $\lim_n \inf L(g_n) \geqq L(\varphi) \geqq \lim_n \sup L(g_n)$ which is only possible with the signs being equal.

Let Φ^q denote the family of all positive homogeneous continuous real-valued functions φ in the q variables $\zeta_1, \ldots, \zeta_q$; positive homogeneous means:
If $p \geqq 0$, then $\varphi(p\zeta_1, \ldots, p\zeta_q) = p\varphi(\zeta_1, \ldots, \zeta_q)$ for arbitrary $\zeta_1, \ldots, \zeta_q$.

In section 1 we have studied the system H_o of functions consisting of the functions $\pi_k(\zeta) = \zeta_k$ $(k = 1, \ldots, q)$ and all linear lattice combinations which can be formed from them. It is clear that Φ^q contains in particular the system H_o.

<u>Proposition 2.12</u> If $\varphi \in \Phi^q$ and $f_1, \ldots, f_q \in \mathcal{L}$ then $\varphi(f_1, \ldots, f_q) \in \mathcal{L}$.

(Proof) By proposition 1.2 every continuous function on $T = \{(\zeta_1, \ldots, \zeta_q) : |\zeta_1| + \ldots + |\zeta_q| = 1\}$ can be approximated uniformly by functions belonging to H_o. For $\varphi \in \Phi^q$ and $k > 0$ there is a Ψ_k in H_o such that $|\varphi - \Psi_k| < \frac{1}{k}$ on T. This means that $|\varphi - \Psi_k| < (|\zeta_1| + \cdots + |\zeta_k|)/k$. If $f_1, \ldots, f_q$ are $\mathcal{N}$- integrable, then by part (i) of proposition 2.7 $\Psi_k(f_1, \ldots, f_q)$ is $\mathcal{N}$- integrable and since $\mathcal{N}(\varphi(f_1, \ldots, f_q) - \Psi_k(f_1, \ldots, f_q)) \leqq (\mathcal{N}(f_1) + \ldots + \mathcal{N}(f_q))/k$ it follows, for $k \to +\infty$, that $\varphi(f_1, \ldots, f_q)$ is $\mathcal{N}$- integrable.

Let X be a topological space. A family $\tilde{\mathcal{R}}$ of real-valued functions on X is said to be <u>closed under pointwise limits</u> if $f \in \tilde{\mathcal{R}}$ whenever f is a real-valued function on X and, for some sequence $(f_n) \subseteq \tilde{\mathcal{R}}$, $f(x) = \lim_n f_n(x)$ for all $x \in X$.

Let $\mathcal{R}_0$ be the set of all real-valued functions continuous on X. If α is an ordinal number such that $0 < \alpha < \Omega$, with Ω the smallest non-denumerable ordinal, define $\mathcal{R}_\alpha$ to be the family of all functions f such that f is the pointwise limit of some sequence $(f_n) \subseteq \bigcup \{\mathcal{R}_\beta : \beta \text{ is an ordinal number}, \beta < \alpha\}$. The functions in $\mathcal{R}_\alpha$ are known as the <u>Baire functions of type</u> α. The family $\mathcal{R}(X)$ of all <u>Baire functions</u> on X is defined to be $\mathcal{R}(X) = \bigcup_{\alpha < \Omega} \mathcal{R}_\alpha$. The sets $\mathcal{R}_\alpha$ we shall call <u>Baire classes</u>.

We note that $\mathcal{R}(X)$ may also be described as the intersection of all families $\tilde{\mathcal{R}}$ of real-valued functions on X such that $\tilde{\mathcal{R}}$ contains all real-valued continuous functions on X and $\tilde{\mathcal{R}}$ is closed under pointwise limits. Observe that the set of all real-valued functions on X is such a class $\tilde{\mathcal{R}}$.

<u>Proposition 2.13</u> Let φ be a positive homogeneous Baire function of q real variables and suppose that $\varphi \mid T$ (with $T = \{(\xi_1, .., \xi_q): |\xi_1| + \cdots + |\xi_q| = 1\}$ is a bounded Baire function, then $\varphi(f_1, .., f_q)$ is $\mathcal{N}$- integrable whenever $f_1, \ldots, f_q$ are $\mathcal{N}$- integrable.

(Proof) (By induction): For the Baire class $\mathcal{R}_0$ this proposition coincides with proposition 2.12; suppose that the proposition is true for functions of every Baire class $<\alpha$ and let φ be a Baire function of type α with φ being positive homogeneous and $|\varphi| \leq M$ on T so that $|\varphi| \leq Mr$, where $r(x) = |\xi_1| + \dots + |\xi_q|$ with $x = (\xi_1, \dots, \xi_q)$. On T let $\varphi = \lim_n \varphi_n$, where φ_n are corresponding functions belonging to Baire classes $<\alpha$. We replace φ_n by $\psi_n(x) = \text{med}\{-Mr(x), \varphi_n(x), Mr(x)\}$, where $\text{med}\{f,g,h\} = \min\{\max\{f,g\}, \max\{g,h\}, \max\{h,f\}\}$. The functions ψ_n also belong to Baire classes $<\alpha$ with $\lim_n \psi_n = \varphi$, and $|\psi_n(x)| \leq Mr(x)$. By proposition 2.12 $M(|f_1| + \dots + |f_q|)$ is $\mathcal{N}$- integrable whenever $f_1, \dots, f_q$ are $\mathcal{N}$- integrable and, by the inductive assumption, the $\psi_n(f_1, \dots, f_q)$ are $\mathcal{N}$- integrable; thus proposition 2.11 applies and we get that $\varphi(f_1, \dots, f_q)$ is $\mathcal{N}$- integrable.

In order to be able to free ourselves of the restrictions that φ be positive homogeneous, as encountered in Proposition 2.13 we must impose in addition to the conditions 1c) to 4c) the following further requirement on $\mathcal{E}$:

(5c) $f \in \mathcal{E}$ implies $\min\{f,1\} \in \mathcal{E}$.

Proposition 2.14 Let the elementary integral E over $\mathcal{E}$ satisfy also the property 5c). Then for every finite Baire function $\varphi(\xi_1, \dots, \xi_q)$ in q real variables with $\varphi(0, \dots, 0) = 0$ we have that the function $g = \varphi(f_1, \dots, f_q)$ is $\mathcal{N}$- integrable whenever the functions $f_1, \dots, f_q$ are $\mathcal{N}$- integrable and if, moreover, there is an $\mathcal{N}$- integrable function h such that $|g| \leq h$.

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(Proof) By proposition 1.4 we may approximate every function of Baire class o with $\varphi(o) = o$ by functions ψ_n belonging to the system H_1 in such a manner that in this approximation $|\psi_n| \leq |\varphi|$ holds; thus $|\psi_n(f_1, \ldots, f_q)| \leq h$ and we may apply proposition 2.11 to see that $\varphi(f_1, \ldots, f_q)$ and $\psi_n(f_1, \ldots, f_q)$ are simultaneously $\mathcal{N}$ - integrable.

We next suppose that the proposition is true for the functions belonging to Baire class $< \alpha$, $\alpha > o$, and we assume that φ is of Baire type α with $\varphi(o) = o$ and $\varphi = \lim_n \varphi_n$, where the φ_n come from Baire classes $< \alpha$. We may assume here that $\varphi_n(o) = o$ and, moreover, that $|\varphi_n(x)| \leq nr(x)$ (otherwise we would take the functions $\varphi_n'(x) = \text{med}\{- nr(x), \varphi_n(x), nr(x)\}$ which are not of higher Baire classes). Since $|\varphi_n(f_1, \ldots, f_q)| \leq n(|f_1| + \ldots + |f_q|)$, the functions $\varphi_n(f_1, \ldots, f_q)$ are $\mathcal{N}$ - integrable, but so are also $\psi_n = \text{med}\{-h, \varphi_n(f_1, \ldots, f_q), h\}$ with $|\psi_n| \leq h$ and $\lim_n \psi_n = \varphi(f_1, \ldots, f_q)$; here again proposition 2.11 leads to the desired conclusion and the induction is complete.

<u>Remark:</u> We may drop the condition $\varphi(o) = o$ in the foregoing proposition, if instead of 5c) we impose the stronger requirement:

6c): The identically constant function 1 belongs to $\mathcal{E}$.

Indeed, referring to proposition 1.5, it is clear that we may adopt the proof of proposition 2.14 to the situation at hand. (Observe that $\min\{f, 1\} = \frac{1}{2}(1 + f - |f - 1|)$ is in $\mathcal{E}$ whenever $f \in \mathcal{E}$).

In the set $\mathfrak{Z}$ (introduced after Corollary 2.3) we consider the following two subsets:

$\mathfrak{M} = \{ f \in \mathfrak{Z} :$ (g and h) $\in \mathcal{E}$ implies med $\{g,f,h\} \in \mathcal{L} \}$

and

$\mathfrak{M}' = \{ f \in \mathfrak{Z} :$ (g and h) $\in \mathcal{L}$ implies med $\{g,f,h\} \in \mathcal{L} \}$

Since $\mathcal{E} \subseteq \mathcal{L}$ we have that $\mathfrak{M}' \subseteq \mathfrak{M}$.

<u>Proposition 2.15</u> $\mathfrak{M} = \mathfrak{M}'$

(Proof) If g and h are in $\mathcal{L}$, then there exist sequences (g_n) and (h_n) in $\mathcal{E}$ with $\operatorname{Lim}_n g_n = g$ and $\operatorname{Lim}_n h_n = h$. If $f \in \mathfrak{M}$ then med $\{g_n, f, h_n\} \in \mathcal{L}$; but Lim_n med$\{g_n, f, h_n\}$ = med $\{g,f,h\}$ and so med $\{g,f,h\} \in \mathcal{L}$ because $\mathcal{L}$ is closed. This shows that the inclusion $\mathfrak{M} \subseteq \mathfrak{M}'$ is also valid and the proof is complete.

<u>Definition</u> The functions in $\mathfrak{M}$ are called $\mathcal{N}$ - <u>measurable</u>.

<u>Proposition 2.16</u> If two functions f_1 and f_2 differ only by a null-function, then the $\mathcal{N}$ - measurability of one implies the $\mathcal{N}$ - measurability of the other.

(Proof) In the case at hand the functions Φ_1 = med $\{g, f_1, h\}$ and Φ_2 = med $\{g, f_2, h\}$ differ only by a null-function. This completes the proof.

We can see from the foregoing proposition that the concept of $\mathcal{N}$ - measurability is compatible with the notion of equality introduced in $\mathfrak{Z}$ by means of the relation (*).

Since $\mathcal{L}$ is a lattice,the following proposition is immediate:

<u>Proposition 2.17</u> Every $\mathcal{N}$- integrable function is $\mathcal{N}$- measurable.

<u>Proposition 2.18</u> If $f_1, f_2, \ldots \in \mathfrak{M}$ and if $\lim_n f_n = f$ in $\mathcal{Z}$ exists, then $f \in \mathfrak{M}$.

(Proof) By assumption $\mathrm{med}\{g, f_n, h\} \in \mathcal{L}$ for $g, h \in \mathcal{L}$ and for $n \in N$. Since $\min\{g, h\} \leq \mathrm{med}\{g, f_n, h\} \leq \max\{g, h\}$ and since the two ends of this inequality are elements of $\mathcal{L}$, proposition 2.11 applies with the effect that $\lim_n \mathrm{med}\{g, f_n, h\} = \mathrm{med}\{g, f, h\} \in \mathcal{L}$.

<u>Proposition 2.19</u> For $f_1, f_2 \in \mathfrak{M}$ and for all $a \neq 0$ the functions af_1, $\genfrac{}{}{0pt}{}{\min}{\max}\{f_1, f_2\}$ and $|f_1|$ belong to $\mathfrak{M}$; moreover $f_1 + f_2$ belongs to $\mathfrak{M}$ provided this sum is meaningful almost everywhere in A.

(Proof) The first claim follows from the equality $a\,\mathrm{med}\{\frac{g}{a}, f_1, \frac{h}{a}\} = \mathrm{med}\{g, af_1, h\}$ with $g, h \in \mathcal{L}$, the second claim results from the fact that $\mathcal{L}$ is a lattice and that $\mathrm{med}\{g, \min\{f_1, f_2\}, h\} = \min\{\mathrm{med}\{g, f_1, h\}, \mathrm{med}\{g, f_2, h\}\}$ holds, the third claim is verified analogously and the fourth claim is a consequence of the preceeding ones. The claim concerning the sum is established as follows: With $g, h \in \mathcal{L}$, let

$$\varphi_{i,n} = \mathrm{med}\{-n(|g|+|h|), f_i, n(|g|+|h|)\} \in \mathcal{L},$$

$i = 1,2$, $n \in N$ and let $\psi_n = \mathrm{med}\{g, \varphi_{1,n}+\varphi_{2,n}, h\} \in \mathcal{L}$ so that $\lim_n \psi_n = \mathrm{med}\{g, f_1+f_2, h\} \in \mathcal{L}$; this is so because $\lim_n \varphi_{i,n}(x) = f_i(x)$ in case $|g(x)| + |h(x)| > 0$, and for the rest proposition 2.11 again applies.

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Remark: $\mathfrak{M}$ is in fact a σ-complete lattice (i.e., every countable subset has a supremum and infimum). Indeed, if for example $s = \sup\{f_1, f_2, \dots\} = \lim\limits_n \varphi_n$ with $\varphi_n = \max\{f_1, \dots, f_n\}$, then by proposition 2.19, φ_n together with f_k are in $\mathfrak{M}$ and $\lim\limits_n \varphi_n$ converges on account of monotonicity, implying therefore that $s \in \mathfrak{M}$ by proposition 2.18.

Proposition 2.20 If $f \geq o$, then $f \in \mathfrak{M}$ if and only if $\min\{f,g\} \in \mathcal{L}$ for every non-negative function g in $\mathcal{L}$.

(Proof) Let $f \in \mathfrak{M}$, $g \geq o$ and $g \in \mathcal{L}$. Then $\text{med}\{o,f,g\} = \min\{f,g\}$. Suppose, conversely, the condition in the proposition is fulfilled. For arbitrary $g,h \in \mathcal{L}$ and setting $\underline{m} = \min\{g,h\}$, $\overline{m} = \max\{g,h\}$ and $m = \text{med}\{g,f,h\}$, we have $m = \text{med}\{\underline{m},f,\overline{m}\} =$
$= \max\{\min\{\underline{m},f\}, \min\{\overline{m},f\}, \min\{\underline{m},\overline{m}\}\} = \max\{\min\{\overline{m},f\},\underline{m}\}$.
But $\min\{\overline{m},f\} = \min\{\max\{\overline{m},o\},f\} + \min\{o,\overline{m}\}$ (as one may verify at once by considering the cases $\overline{m} \geq o$ and $< o$) and $\max\{\overline{m},o\} \geq o$ and so $\min\{\overline{m},f\} \in \mathcal{L}$ which in turn implies that $m \in \mathcal{L}$.

Corollary 2.21 $1 \in \mathfrak{M}$ if and only if $\min\{1,g\} \in \mathcal{L}$ for every non-negative function g in $\mathcal{L}$.

Remark: The condition in the foregoing proposition is certainly fulfilled in case property 5c) holds; this is so because 5c) implies:

(5'c): $f \in \mathcal{L}$ implies that $\min\{1,f\} \in \mathcal{L}$.

(Indeed, if $f \in \mathcal{L}$, there exists $(f_n) \subset \mathcal{E}$ such that $\lim\limits_n f_n = f$; then by 5c) $\min\{1,f_n\} \in \mathcal{E}$. Thus $\lim\limits_n \min\{1,f_n\} = \min\{1,f\} \in \mathcal{L}$.)

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We are now going to give an example where $1 \notin \mathcal{M}$. Let A be the closed unit internal $[o, 1]$, $\mathcal{E}$ the set of all functions of the form ax, where a is any finite real number, and $E(ax) = \frac{a}{2}$. It is easy to see that $\mathcal{E}$ is a vector lattice and E satisfies the properties 1c) to 4c). Moreover $\mathcal{L} = \mathcal{E}$, for if $f \in \mathcal{L}$ and $(f_n) \subseteq \mathcal{E}$ with $\lim_n f_n = f$, then $L(f) = \lim_n E(f_n) = \lim_n \frac{1}{2} a_n = \frac{1}{2} a$ (where $a = \lim_n a_n$). Hence $L(f) = L(f')$ where $f'(x) = ax$. Thus $f = f'$ a.e. which gives that $f \in \mathcal{E}$. Since now $\min\{1, 2x\} \notin \mathcal{L}$ we have by Corollary 2.21 that $1 \notin \mathcal{M}$.

For $o \leq x \leq +\infty$ let $s_o(x) = \min\{1, x\}$ and for $n \in N$ let

$$s_n(x) = \left\{ \begin{matrix} s_{n-1}(x) \text{ for } o \leq x \leq \frac{n}{2}, \\ s_{n-1}(\frac{n}{2}) + \frac{1}{2^n} s_o(x - \frac{n}{2}) \quad \text{for } x \geq \frac{n}{2} \end{matrix} \right\} \quad \text{and } s_n(-x) = -s_n(x).$$

We observe that $s(\zeta) = \lim_n s_n(\zeta)$ defines a homeomorphism of the extended real numbers $\hat{R}$ onto the interval $[-1, 1]$.

<u>Proposition 2.22</u> If the functions 1 and f are $\mathcal{N}$-measurable, then so is s(f).

(Proof) By definition $s_n(\zeta)$ can be obtained from the functions 1 and ζ upon a finite number of linear lattice combinations (i.e., formation of linear combinations and formation of absolute values). By proposition 2.19 we may therefore conclude that $s_n(f)$ is $\mathcal{N}$-measurable since 1 and f are. Invoking proposition 2.18, we may thus conclude that $\lim_n s_n(f) = s(f)$ is $\mathcal{N}$-measurable as well.

Following proposition 2.12 we defined the family of all Baire functions; to accommodate Baire functions which assume infinite values we permit that the operation of pointwise limits include improper convergence, i.e., infinite limits.

We shall use the mapping s whose definition preceeds proposition 2.22 in the following.

<u>Definition</u> By a Baire function in the domain $-\infty \leq \lambda_i \leq +\infty$, $i = 1, \ldots, q$, we mean a function which by means of the transformation $\mu_i = s(\lambda_i)$, $i = 1, \ldots, q$, becomes a Baire function $\bar{\varphi}(\mu_1, \ldots, \mu_q)$ on the cube $W^q = \{(\mu_1, \ldots, \mu_q): -1 \leq \mu_i \leq 1,\ i = 1, \ldots, q\}$ $(\bar{\varphi}(s(\lambda_1), \ldots, s(\lambda_q)) = \varphi(\lambda_1, \ldots, \lambda_q))$.

<u>Proposition 2.23</u> If 1, $f_1, \ldots, f_q$ are $\mathcal{N}$-measurable and if $\varphi(\lambda_1, \ldots, \lambda_q)$ is a Baire function in the domain $-\infty \leq \lambda_i \leq +\infty$, $i = 1, \ldots, q$, then $\varphi(f_1, \ldots, f_q)$ is also $\mathcal{N}$-measurable.

(Proof)

α) Let $\bar{\varphi} \mid W^q$ be a Baire function of order o, that is, a continuous function on W^q. By proposition 1.2, it is possible to approximate $\bar{\varphi}$ uniformly within arbitrarily small error by starting with the functions $1, \mu_1, \ldots, \mu_q$ and forming real linear combinations a finite number of times and forming absolute values. Putting $\mu_i = s(f_i)$, we obtain (by proposition 2.22) a sequence of functions $g_n \in \mathfrak{M}$ with $g = \lim_n g_n = \bar{\varphi}(s(f_1), \ldots, s(f_q)) \in \mathfrak{M}$ (on account of proposition 2.18).

β) For functions of class $< \alpha$, $\alpha >$ o, let the statement be true and let $\bar{\varphi}$ be of class α. Then $\lim_n \bar{\varphi}_n = \bar{\varphi}$, where the $\bar{\varphi}_n$'s are of classes $< \alpha$. Since $\bar{\varphi}_n(s(f_1), \ldots, s(f_q)) \in \mathfrak{M}$ for $n \in N$, we have by proposition 2.18 that $\bar{\varphi}(s(f_1), \ldots, s(f_q)) \in \mathfrak{M}$. Hence the proposition is true for Baire functions of all classes.

As an application of the foregoing proposition we mention:

Proposition 2.24 (i) If $1, t \in \mathfrak{M}$ and $t(x) \geq 0$ for $x \in A$, then we also have $(t)^{\frac{1}{2}} \in \mathfrak{M}$. (ii) If $1, f, g \in \mathfrak{M}$, $g \geq 0$ and $\mathcal{N}$-a.e. $g > 0$, then we have $f/g \in \mathfrak{M}$.

(Proof) $\sqrt{t}$ is a Baire function and hence by proposition 2.23, $\sqrt{t} \in \mathfrak{M}$. ii) For every $n \in N$, let $f_n = \frac{nf}{ng+1}$. By proposition 2.23 we have that $f_n \in \mathfrak{M}$; since $f/g = \lim_n f_n$ $\mathcal{N}$-a.e., we get by proposition 2.18 that $f/g \in \mathfrak{M}$.

Proposition 2.25 Each of the following conditions is necessary and sufficient for the $\mathcal{N}$-integrability of a function f:

(i) $f \in \mathfrak{M}$ and $\mathcal{N}(f) < +\infty$.

(ii) $f \in \mathfrak{M}$ and there exist $\mathcal{N}$-integrable functions g and h with $g \leq f \leq h$.

(Proof) If f is $\mathcal{N}$-integrable, then by definition $\mathcal{N}(f) < +\infty$, and (ii) is trivially satisfied with $g = h = f$. If, on the other hand, $\mathcal{N}(f) < +\infty$, then there exist $(f_n) \subseteq \mathcal{E}$, $n \in N$, with $|f| \leq \sum_n |f_n|$ and $\sum_n E(|f_n|) < +\infty$; here $k = \sum_n |f_n| \in \mathcal{L}$ by part (iii) of proposition 2.7, $g \leq f \leq h$ with $g = -k$ and $h = k$ in $\mathcal{L}$. However, as soon as there are g and h, we have on account of the $\mathcal{N}$-measurability of f immediately that $\mathcal{L} \ni \text{med}\{g, f, h\} = f$ and the proposition is proved.

Remark: The foregoing proposition may be expressed in abbreviated form by the equation

$$\mathcal{L} = \mathfrak{M} \cap \mathcal{F}.$$

Proposition 2.26 (i) If $\mathcal{N}(f) < +\infty$, then there is a $g \in \mathcal{L}$ with $|f| \leq g$ and $\mathcal{N}(f) = L(g)$.

(ii) If $1 \in \mathfrak{M}$ and f is a characteristic function, then we may take in (i) for g a characteristic function as well.

(Proof) We first prove part (i). By assumption there exist elementary functions $f_{m,k}$, (m and k in N) with $|f| \leq \sum_k |f_{m,k}| = g_m$ and $\mathcal{N}(f) \leq \sum_k E(|f_{m,k}|) \leq \mathcal{N}(f) + \frac{1}{m}$, where also $\mathcal{N}(f) \leq \mathcal{N}(g_m) = L(g_m) \leq \mathcal{N}(f) + \frac{1}{m}$; thus $g_m \in \mathcal{L}$. For $g = \lim_m \min\{g_1, \dots, g_m\}$ we have by proposition 2.10 $g \in \mathcal{L}$ with $L(g) = \mathcal{N}(f)$ and $g \geq |f|$.

We next prove part (ii). Let f be a characteristic function with $\mathcal{N}(f) < +\infty$. The $g \in \mathcal{L}$, which exists by part (i), we can replace by $g_1 = \min\{1, g\} \in \mathcal{L}$ according to corollary 2.21 and g_1 in turn we can replace by the $\mathcal{N}$-measurable $g_n = (g_1)^n$ for arbitrary n = 2, 3, ... according to proposition 2.23. We have $|f| \leq g_n \leq g_1$ and $g_n \in \mathcal{L}$ by proposition 2.25. Letting $h = \lim_n g_n$, we get $|f| \leq h \in \mathcal{L}$ by proposition 2.10 and $\mathcal{N}(f) = L(h)$, where evidently h can only assume the values o and 1, that is, h is a characteristic function.

Section 3

Connections to measure theory

We commence with the function $\mathcal{N}|\mathcal{Y}$, where $\mathcal{Y}$ is the set of all extended real-valued functions on A.

Definition For every $X \in \mathcal{P}(A)$ we define $m^*(X) = \mathcal{N}(\chi_X)$, where χ_X is the characteristic function of X and $\mathcal{P}(A)$ is the set of all subsets of A.

It is easy to see that m^* is an outer measure on $\mathcal{P}(A)$:

(I) $m^*(X)$ is unambiguously defined for $X \in \mathcal{P}(A)$ with $0 \leq m^*(X) \leq +\infty$ and $m^*(\emptyset) = 0$, where $\emptyset$ is the empty set.

(II) for each sequence $(X_n)_{n \in N}$ in $\mathcal{P}(A)$ with $\mathcal{P}(A) \ni X \subseteq \bigcup_n X_n$ we have $m^*(X) \leq \sum_n m^*(X_n)$.

Indeed, the unambiguity of m^* is clear, the rest of (I) follows from property (b) of the norm $\mathcal{N}$ and the fact that $E(0) = 0$.

The statement (II) follows from

$\chi_X \leq \chi_{\bigcup_n X_n} \leq \sum_n \chi_{X_n}$ by property (d) of the norm $\mathcal{N}$.

We set $\mathfrak{m} = \{ X \in \mathcal{P}(A) : \chi_X \in \mathfrak{M} \}$ and call every X in $\mathfrak{m}$ $\mathcal{N}$-measurable. It is clear that $\mathcal{R} = \{ Y \in \mathcal{P}(A) : \chi_Y \in \mathcal{L} \}$ is a subset of $\mathfrak{m}$ because $\mathcal{L} \subseteq \mathfrak{M}$; the sets in $\mathcal{R}$ we call properly $\mathcal{N}$-measurable.

We observe that $\mathcal{R}$ is a <u>field of sets</u>, that is,
if $B, C \in \mathcal{R}$, then $B \cup C \in \mathcal{R}$;
if $B, C \in \mathcal{R}$ and $C \subseteq B$, then $B - C \in \mathcal{R}$

For a fixed $T \in \mathcal{R}$, let

$\mathcal{R}_T = \{ X \in \mathcal{R} : X \subseteq T \}$. It can be seen at once that $\mathcal{R}_T$ is a σ - algebra of sets. Indeed; if X_1, X_2 are in $\mathcal{R}_T$ we have

(1) $\chi_{X_1 \cup X_2} = \chi_{X_1} \vee \chi_{X_2} \in \mathcal{L}$

(2) $\chi_{X_1 \cap X_2} = \chi_{X_1} \wedge \chi_{X_2} \in \mathcal{L}$

and moreover if $(X_n)_{n \in N}$ is a sequence in $\mathcal{R}_T$, then $\chi_{\bigcup_n X_n} = \lim_n (\chi_{X_1} \vee \dots \vee \chi_{X_n})$, with $\chi_{X_1 \cup \dots \cup X_n} \in \mathcal{L}$, $L(T) < +\infty$ and the Monotone Convergence Theorem is applicable and hence $\chi_{\bigcup_n X_n} \in \mathcal{L}$.

<u>Definition</u>: We define m on $\mathcal{R}$ by setting $m(X) = L(\chi_X)$.

If $Y \subseteq A$, we say that Y is m-<u>enclosable</u> if for each $\varepsilon > 0$ there exist two sets A_1, A_2 in $\mathcal{R}$ with $A_1 \subseteq Y \subseteq A_2$ and $m(A_2 - A_1) < \varepsilon$.

The set function m on $\mathcal{R}$ is said to be <u>complete</u> with respect to $\mathcal{P}(A)$ if every m-enclosable set $Y \subseteq A$ belongs to $\mathcal{R}$.

<u>Proposition 3.1</u> The function $m | \mathcal{R}$, defined by $m(X) = L(\chi_X)$ for $X \in \mathcal{R}$ is countably additive, non-negative and finite-valued (for brevity we say that m is a <u>content</u>) on the field of sets $\mathcal{R}$. In particular, $m / \mathcal{R}_T$ (for fixed $T \in \mathcal{R}$) is a bounded measure on the σ -algebra $\mathcal{R}_T = \{ X \in \mathcal{R} : X \subseteq T \}$. Moreover, $m | \mathcal{R}$ is a complete content with respect to the power set of A and $m | \mathcal{R}_T$ is a complete measure with respect to the power set of T (i.e., every m-enclosable set $X \subseteq T$ belongs to $\mathcal{R}_T$).

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(Proof) We first prove that $m|\mathcal{R}$ is countably additive. Let $(X_n)_{n \in N}$ be a sequence of pairwise disjoint sets in $\mathcal{R}$ and suppose that $\bigcup_n X_n \in \mathcal{R}$. Then

$$\chi_{\bigcup_n X_n} = \lim_n (\chi_{X_1} + \cdots + \chi_{X_n}),$$

$$\chi_{X_1} \leq \chi_{X_1} + \chi_{X_2} \leq \cdots \leq \chi_{X_1} + \cdots + \chi_{X_n} \leq \cdots,$$

$\chi_{X_1} + \cdots + \chi_{X_n} \in \mathcal{L}$ and $L(\chi_{X_1} + \cdots + \chi_{X_n}) \leq L(\bigcup_n X_n) < +\infty$.

Hence the Monotone Convergence Theorem is applicable and

$m(\bigcup_n X_n) = L(\chi_{\bigcup_n X_n}) = \lim_n L(\chi_{X_1} + \cdots + \chi_{X_n}) = \sum_n L(\chi_{X_n}) = \sum_n m(X_n)$. Now that $m|\mathcal{R}_T$ is bounded measure is obvious from what we have said above and the fact that $m(X) \leq m(T) < +\infty$ for all $X \in \mathcal{R}_T$.

To the completeness: If $Y \subseteq A$ is m-enclosable, i.e., if there exist for every $n \in N$ sets X_n' and X_n'' in $\mathcal{R}$ with $X_n' \subseteq Y \subseteq X_n''$ and $m(X_n'' - X_n') < \frac{1}{n}$, then we obtain, taking

$$\underline{X} = \bigcup_n X_n' \text{ and } \overline{X} = \bigcap_n X_n'',$$

$\underline{X} \subseteq Y \subseteq \overline{X}$; using proposition 2.11, we see that $\underline{X}$, $\overline{X} \in \mathcal{R}$ and $m(\underline{X}) = m(\overline{X})$. Thus the functions $\chi_{\underline{X}}$ and $\chi_{\overline{X}}$ differ only by a null-function and so $\chi_{\underline{X}}$ and χ_Y also differ only by a null-function; hence $Y \in \mathcal{R}$ and $m(Y) = m(\underline{X})$.

<u>Remark</u>: One can easily see that the foregoing notion of completeness is equivalent with the usual notion of completeness, according to which every subset of a measurable set of measure zero is again measurable and has measure zero.

The connection between $\mathcal{N}$-measurable sets and the <u>additive decomposers</u> of m^*, i.e., those sets Z with $m^*(Y) = m^*(Y \cap Z) + m^*(Y \cap Z^c)$ for all $Y \subseteq A$ with $m^*(Y) < +\infty$, is given by the following:

<u>Proposition 3.2</u> Every $\mathcal{N}$-measurable set is an additive decomposer of m^*; the converse holds if and only if $1 \in \mathcal{M}$.

(Proof) Let X be $\mathcal{N}$-measurable. For an outer measure m^* we always have $m^*(Y) \leq m^*(Y \cap X) + m^*(Y \cap X^c)$. In this inequality we have to prove equality under the assumption that $m^*(Y) = N(\chi_Y) < +\infty$. By proposition 2.26 there is a function $g \in \mathcal{L}$ with $\chi_Y \leq g$ and $m^*(Y) = L(g)$. Since $\chi_{Y \cap X} = \chi_X \chi_Y \leq g\chi_X \leq g$ and $g\chi_X \in \mathcal{M}$ (Observe that $g\chi_X = \lim_n \min\{g, n\chi_X\}$) we have by proposition 2.25 that $g\chi_X \in \mathcal{L}$ and $m^*(Y \cap X) \leq L(g\chi_X)$. Moreover, $\chi_{Y - (Y \cap X)} = \chi_Y(1 - \chi_X) \leq g - g\chi_X \in \mathcal{L}$ and so $m^*(Y \cap X^c) \leq L(g - g\chi_X)$. Addition of the foregoing inequalities yields $m^*(Y \cap X) + m^*(Y \cap X^c) \leq L(g\chi_X) + L(g - g\chi_X) = m^*(Y)$, proving the first part of the proposition.

Since the basic set A is always an additive decomposer of m^*, for the converse to be valid we must have that $1 \in \mathcal{M}$. On the other hand, we assume that $1 \in \mathcal{M}$ and that Z is an additive decomposer of m^* and we show that Z is $\mathcal{N}$-measurable by using the measurability criterion established in proposition 2.20. To this end we let g be an arbitrary non-negative $\mathcal{N}$-integrable function; in addition, we let $\varphi_n(x) = 0$

for $o \leq x < \frac{1}{n}$ and $\varphi_n(x) = 1$ for $\frac{1}{n} \leq x \leq +\infty$ and we let $c_n = \varphi_n(g)$, $g_n = gc_n$, $n \in N$. By the definition of φ_n we have that $c_n \leq ng$ and $g_n \leq g$; thus c_n and g_n are $\mathcal{N}$-integrable by propositions 2.23, 2.25. Incidentally, c_n is the characteristic function of a set Y_n with $m^*(Y_n) = L(c_n)$. Since $\lim_n g_n = g$, we have $\min\{\chi_Z, g\} = \lim_n \min\{\chi_Z, g_n\} = \lim_n \min\{\chi_Z c_n, g\} \leq g$, from which it follows that if $\chi_Z c_n$, $n \in N$ is in $\mathcal{L}$, then so is also $\min\{\chi_Z, g\}$. It is therefore sufficient to show that $\chi_Z c_n \in \mathcal{L}$. But $\chi_Z c_n = \chi_{Z \cap Y_n}$ and $m^*(Z \cap Y_n) \leq m^*(Y_n) < +\infty$. Part (ii) of proposition 2.26 yields a characteristic function χ_{W_n} with $\chi_{Z \cap Y_n} \leq \chi_{W_n} \in \mathcal{L}$ and $\mathcal{N}(\chi_{Z \cap Y_n}) = L(\chi_{W_n})$, i.e., $Z \cap Y_n \subseteq W_n$ and $m^*(Z \cap Y_n) = m^*(W_n)$. Here we may assume that $W_n \subseteq Y_n$ and hence $m^*(W_n) < +\infty$ and $Z \cap Y_n = Z \cap W_n$; otherwise we replace W_n by $W_n \cap Y_n$. (Observe that $Z \cap Y_n \subseteq W_n \cap Y_n \subseteq W_n$ and hence also $m^*(Z \cap Y_n) = m^*(W_n \cap Y_n)$; furthermore, since $o \leq \chi_{W_n \cap Y_n} \leq c_n \chi_{W_n} \leq c_n \in \mathcal{L}$ and $\chi_{W_n} \in \mathcal{L}$, we have that $\chi_{W_n \cap Y_n} \in \mathcal{L}$). Now we obtain $m^*(W_n) = m^*(W_n \cap Z) + m^*(W_n \cap Z^c) \geq m^*(W_n \cap Z) = m^*(Z \cap Y_n) = m^*(W_n)$, hence $m^*(W_n \cap Z^c) = o$ and so $W_n - (W_n \cap Z) = W_n - (Z \cap Y_n)$ is a null-set. Thus χ_{W_n} and $\chi_Z c_n$ differ only by a null-function and so $\chi_Z c_n$ is $\mathcal{N}$-integrable as well.

Remark In view of the foregoing proposition we have:
If $1 \in \mathfrak{M}$, then the system of all $\mathcal{N}$-measurable sets is a σ-algebra.

Proposition 3.3 If $1 \in \mathfrak{M}$, then we have:

(a) $\mathcal{R} = \{ Y : Y \in \mathcal{M} \text{ and } m^*(Y) < +\infty \}$

(b) $m^*(Y) = \min\limits_{\substack{Y \subseteq Z \\ Z \in \mathcal{R}}} m(Z)$ provided that , the set over which we take the min is not empty; otherwise $m^*(Y) = +\infty$.

(Proof) To part (a): If $m^*(Y) = \mathcal{N}(\chi_Y) < +\infty$, then proposition 2.26 yields a χ_Z with $Y \subseteq Z \in \mathcal{R}$ and $m^*(Y) = m(Z)$. If moreover $Y \in \mathcal{M}$, then by proposition 3.2 Y is an additive decomposer of m^*, hence $m^*(Z) = m^*(Y) + m^*(Z - Y)$, but $m^*(Z) = m(Z)$ and so $m^*(Z - Y) = 0$; χ_Z and χ_Y differ only by a null-function and so $Y \in \mathcal{R}$ if $Z \in \mathcal{R}$. Hence $\{Y: Y \in \mathcal{M} \text{ and } m^*(Y) < +\infty\} \subseteq \mathcal{R}$. By proposition 3.1 the inclusion in the other direction is evident.

To part (b): If $\{Y \subseteq Z \in \mathcal{R}\} \neq \phi$, then clearly $m^*(Y) < +\infty$ and so by the proof of part (a) the claim in (b) also holds; if however the set in question is empty, then $m^*(Y)$ cannot be finite.

<u>Definition</u> The set Z with $Y \subseteq Z \in \mathcal{R}$ and $m^*(Y) = m(Z)$ introduced in the proof of part (a) is called the <u>equimeasurable hull</u> of Y.

<u>Proposition 3.4</u> Let $1 \in \mathcal{M}$ and f be finite and $A(\alpha, \beta) = \{x \in A: \alpha \leq f(x) < \beta\}$. If $A(\alpha, \beta)$ is $\mathcal{N}$-measurable for all rationals α, β then f is $\mathcal{N}$-measurable; if f is $\mathcal{N}$-measurable, then $A(\alpha, \beta)$ is $\mathcal{N}$-measurable for all reals α, β.

(Proof) Let $\chi_{\alpha\beta}$ denote the characteristic function of the interval $[\alpha, \beta)$. If f is $\mathcal{N}$-measurable, then by proposition 2.23 the function $\chi_{\alpha\beta}$ is $\mathcal{N}$-measurable, i.e., the set $A(\alpha, \beta)$ is $\mathcal{N}$-measurable for arbitrary real numbers α and β. Conversely, if these sets are $\mathcal{N}$-measurable for all rationals α, β with $\alpha < \beta$, then we consider for every natural number p a monotone sequence ..., $a_{-2}, a_{-1}, a_o, a_1, a_2, \ldots$ of rational numbers with $a_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$, $a_n \longrightarrow -\infty$ as $n \longrightarrow -\infty$, $a_{n+1} - a_n < \frac{1}{p}$, and numbers δ_n with $a_n \leq \delta_n \leq a_{n+1}$ for $n = o, \pm 1, \pm 2, \ldots$

Letting $\chi_n = \chi_{a_n a_{n+1}}$ and $f_p = \sum_{-\infty}^{+\infty} \delta_n \cdot \chi_n(f)$, we have $|f_p(x) - f(x)| < \frac{1}{p}$ for $x \in A$ and f_p is $\mathcal{N}$-measurable by propositions 2.18, 2.19; but $f(x) = \lim_p f_p(x)$ for $x \in A$ and so f is $\mathcal{N}$-measurable also.

<u>Proposition 3.5</u> If $1 \in \mathfrak{M}$ and $f \in \mathfrak{M}$, then for every real or extended real number q the following sets are $\mathcal{N}$-measurable: $\{x \in A: f(x) = q\}$, $\{x \in A: f(x) \neq q\}$, $\{x \in A: f(x) > q\}$ and $\{x \in A: f(x) \geqslant q\}$.

(Proof) For a real number q (and denoting by Q the set of all rational numbers),

$$\chi_{\{x \in A:\ f(x) = q\}} = \lim_n \chi_{\{x \in A:\ \alpha_n \leq f(x) \leq \beta_n\}}$$

α_n, $\beta_n \in Q$, $\alpha_n \uparrow q$, $\beta_n \downarrow q$ is measurable by proposition 3.4 and by proposition 2.18. Since $1 \in \mathfrak{M}$, we have by proposition 2.19 that $\chi_{\{x \in A:\ f(x) \neq q\}} = 1 - \chi_{\{x \in A:\ f(x) = q\}} \in \mathfrak{M}$.

Again by proposition 2.19 we have $\max\{q, f\} \in \mathfrak{M}$, hence $\{x \in A: \max\{q, f\}(x) \neq q\}$, that is, $\{x \in A: f(x) > q\}$, is measurable and with it is also $\chi_{\{x \in A:\ f(x) = +\infty\}} = \lim_n \chi_{\{x \in A:\ f(x) > n\}}$ and so forth.

<u>Definition</u> A real-valued function f on A is said to be <u>continuous with respect to $\mathcal{N}$-decomposition</u> if for each $\varepsilon > 0$ there exists a decomposition $A = \bigcup_n X_n$ into countably many disjoint $\mathcal{N}$-measurable sets X_n such that on each X_n the function f is bounded and its oscillation is less than ε (i.e., $\sup f(X_n) - \inf f(X_n) < \varepsilon$).

<u>Proposition 3.6</u> If $1 \in \mathfrak{M}$, then for a finite function continuity with respect to $\mathcal{N}$-decomposition and $\mathcal{N}$-measurability are equivalent.

(Proof) If f is $\mathcal{N}$-measurable, then we have $A = \bigcup \{x \in A: \frac{m}{n} \leq f(x) < \frac{m+1}{n}$ with $m = 0, \pm 1, \pm 2, \ldots\}$; by proposition 3.4 this is a decomposition of the required kind with $\varepsilon = \frac{1}{n}$, $n \in N$. If conversely, $A = \bigcup_n X_n$ is a decomposition in the sense of the above definition with $\varepsilon = \frac{1}{n}$, then X_n is measurable and so are χ_{X_n} and $\sup f(X_n) \chi_{X_n}$; by proposition 2.18 we therefore see that $f_n = \sum_n \sup f(X_n) \chi_{X_n}$ is measurable. Since $|f_n - f| < \frac{1}{n}$, we have $\lim_n f_n = f$ and hence also $f \in \mathfrak{M}$.

A direct consequence of the foregoing proposition is:

<u>Proposition 3.7</u> If $1 \in \mathfrak{M}$, f is non-negative, finite and $\mathcal{N}$-integrable, then $m^*(\{x \in A: f(x) > 0\}) > 0$ if and only if $L(f) > 0$.

(Proof) If $m^*(\{x \in A: f(x) > 0\}) > 0$, then by proposition 3.6 $0 < m^*(\{x \in A: f(x) > 0\}) \leq \sum_{n=1}^{+\infty} m^*(A(n,n+1)) + \sum_{k=1}^{+\infty} m^*(A(\frac{1}{k+1},\frac{1}{k}))$, so that at least one of the summands on the right must be positive, say $A(n,n+1)$. Taking into account the fact: $\chi_{\alpha\beta}(f) \in \mathcal{L}$ whenever $f \in \mathcal{L}$, $\alpha < \beta$ but not $\alpha < 0 < \beta$ (observe that if for example $0 < \alpha < \beta$, then $0 \leq \chi_{\alpha\beta}(f) \leq \frac{1}{\alpha}|f|$ and proposition 2.25 is applicable), we see that $0 < n\, m^*(A(n,n+1)) = n\, \mathcal{N}(\chi_{n,n+1}(f)) = n\, L(\chi_{n,n+1}(f)) =$ $= L(n\chi_{n,n+1}(f)) \leq L(f)$. Conversely, from $L(f) > 0$ it follows that $m^*(\{x \in A: f(x) > 0\}) \geq m^*(A(\frac{1}{n},n)) = L(\chi_{\frac{1}{n},n}(f)) = \frac{1}{n}L(n\chi_{\frac{1}{n},n}(f)) \geq$ $\geq \frac{1}{n}L(f\chi_{\frac{1}{n},n}(f)) > 0$ for sufficiently large n. (Observe that for $n \to +\infty$ we have the Dominated Convergence Theorem, $f\chi_{\frac{1}{n},n}(f) \to f$ and so $L(f\chi_{\frac{1}{n},n}(f)) \to L(f) > 0$, that is, $L(f\chi_{\frac{1}{n},n}(f))$ is eventually positive.)

<u>Proposition 3.8</u> (EGOROV): If X is an $\mathcal{N}$-measurable subset of A with $m(X) < +\infty$ and $(f_n)_{n \in N}$ is a sequence of finite $\mathcal{N}$-measurable functions on X which tends to the finite function f on X, then there is a decomposition $X = X_0 \cup (\bigcup_n X_n)$ into $\mathcal{N}$-measurable parts such that X_0 is an $\mathcal{N}$-null set and for $k \in N$ the sequence $(f_n)_{n \in N}$ converges uniformly on X_k.

(Proof) For $n \in N$, let $g_n(x) = \sup\{|f_{n'}(x) - f_{n''}(x)|: n' \geq n, n'' \geq n\}$, then g_n is an $\mathcal{N}$-measurable function (see the remark after proposition 2.19) with $\lim_n g_n(x) = 0$ for all $x \in X$. For $j \in N$ we form the

$\mathcal{N}$-measurable sets $K_{n,j} = \left\{ x: g_n(x) < \frac{1}{j} \right\} \subseteq X$. For fixed j we have $K_{1,j} \subseteq K_{2,j} \subseteq \dots\,$ and $\bigcup_k K_{k,j} = X$. Since m is σ-additive, we have $\lim_k m(K_{k,j}) = m(X)$. For $j = 1$ we determine a k_1 with $m(X - K_{k_1,1}) \leq 2^{-2} m(X)$, then a $k_2 > k_1$ with $m(K_{k_1,1} \cap (X - K_{k_2,2})) \leq 2^{-3}\, m(X)$ and so forth. We let $X_1 = \bigcap_j K_{k_j,j}$ and $X' = (X - K_{k_1,1}) \cup (K_{k_1,1} \cap (X - K_{k_2,2})) \cup \dots$; we have the decomposition $X \doteq X_1 \cup X'$, where $m(X') \leq m(X) \sum_n \frac{1}{2^{n+1}} = \frac{1}{2}\, m(X)$ and on X_1 we have that g_n tends to zero uniformly.

We repeat the process, described above for the set X, on the set X' and obtain a decomposition $X' \doteq X_2 \cup X''$ with $m(X'') \leq \frac{1}{2}\, m(X') \leq \frac{1}{4}\, m(X)$ and with the sequence g_n tending uniformly to zero on X_2. Continuation of this process leads to a decomposition $X = (\bigcup_n X_n) \cup X_o$, where $X_o = X - (\bigcup_n X_n)$ and, since $m(X) \geqslant m(\bigcup_n X_n) = \sum_n m(X_n) \geqslant m(X)$, we see that X_o is an $\mathcal{N}$-null-set; moreover, on each individual X_j the g_n, and hence also the f_n, converge uniformly.

Using the notation introduced in the proof of proposition 3.4 we assign to each $\mathcal{N}$-measurable function f formally the "Lebesgue sums".

$$\Lambda_p(f) = \sum_n{}' \sigma_n \mathcal{N}(\chi_n(f)), \quad p \in N$$

where the a_n and χ_n satisfy the same conditions as in the proof of proposition 3.4 and the σ_n are selected according to the following rule:

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If $a_n \geqslant \frac{1}{p}$ or $a_{n+1} \leq -\frac{1}{p}$, then let $a_n \leq \sigma_n \leq a_{n+1}$; otherwise let $\sigma_n = 0$. The symbol $\sum'$ here denotes summation over all integers n with $\sigma_n \neq 0$.

<u>Proposition 3.9</u> If $1 \in \mathfrak{M}$ and f is finite and $\mathcal{N}$-integrable, then the Lebesque sums are absolutely and properly convergent and $\lim_p \Lambda_p(f) = L(f)$.

(Proof) Let $f_p = \sum' \sigma_n \chi_n(f)$ and $\gamma_n = \min \{|a_n|, |a_{n+1}|\}$. Then for $\gamma_n \geqslant \frac{1}{p}$ (i.e., $a_n \geqslant \frac{1}{p}$ or $a_{n+1} \leq -\frac{1}{p}$) we see that $|\sigma_n| \leq \min\{|a_n|, |a_{n+1}|\} + \frac{1}{p} \leq 2\gamma_n$ and $|f| \geqslant \gamma_n \chi_n(f)$ and hence $|f_p| \leq 2 \sum' \gamma_n \chi_n(f) \leq 2|f|$. Just as in the proof of proposition 3.4 follows the $\mathcal{N}$-measurability of $\chi_n(f)$ and f_p; by part (i) of proposition 2.25 now follows the $\mathcal{N}$-integrability of these functions. The foregoing inequality gives the dominated convergence of the series for f_p and so by proposition 2.11 we have $L(f_p) = \sum' L(\sigma_n \chi_n(f)) = \Lambda_p(f)$; we also have dominated convergence in $f_p \longrightarrow f$ and so $L(f) = \lim_p \Lambda_p(f)$.

<u>Definition</u> We call the limit of $\Lambda_p(f)$ the <u>Lebesgue integral of</u> f <u>over</u> A <u>with respect to</u> m and write $\int_A f\, dm$.

<u>Definition</u> If X is a topological space, we call a real-valued function on X a <u>K-function</u> if it is finite, continuous and vanishes outside a compact subset of X. In other words, K-functions are functions of compact support.

Is is easy to see that every K-function is bounded and the set of all K-functions of a topological space forms a linear lattice.

Proposition 3.10 If X is a locally compact Hausdorff space, $\mathcal{E}$ the set of all K-functions on X and E is a positive linear functional on $\mathcal{E}$, then $E|\mathcal{E}$ satisfies also the properties 4c) and 5c).

(Proof) Property 5c) reads: if $f \in \mathcal{E}$, then $\min\{f, 1\} \in \mathcal{E}$; it is evident that this property is satisfied. Since 4c) holds if and only if property 5) holds, it suffices to show that property 5) is satisfied.

Let f, f_1, f_2, ... be in $\mathcal{E}$ such that $|f| \leq \sum_n |f_n|$, and let $K = \overline{\{x \in X: f(x) \neq o\}}$. Since f is a K-function, $\{x \in X: f(x) \neq o\}$ is compact subset of X and hence closed (observe that X is Hausdorff). Thus $K = \overline{\{x \in X: f(x) \neq o\}}$ is compact. Now each $x \in K$ has an open neighborhood $\mathcal{V}$ with compact closure $\overline{\mathcal{V}}$. We can cover K already with finitely many of these $\mathcal{V}$. The union $\mathcal{V}^*$ of these finitely many $\mathcal{V}$ is open and has a compact closure $\overline{\mathcal{V}}^*$. Let $X^* = X \cup \{p\}$ be the one-point compactification of X. Then X^* is a compact Hausdorff space and hence normal, so by Urysohn's lemma there exists a continuous real-valued function g such that $o \leq g(x) \leq 1$ for all $x \in X^*$, $g(X^* - \mathcal{V}^*) = o$ and $g(K) = 1$. This function g is a K-function.

We consider the sequence of K-functions $g_m = \max\left\{o, |f| - \sum_{n=1}^{m} |f_n|\right\}$ which tends to zero non-increasingly. Here $g_m(x) = o$ for $x \notin K$ and $m \in N$; moreover we have uniform convergence by Dini's theorem. Thus for $\varepsilon > o$ $\exists$ $m_o = m_o(\varepsilon)$ such that $o \leq g_m(x) < \varepsilon g(x)$ for $x \in X$ and $m > m_o$. But $|f| - \sum_{n=1}^{m} |f_n| \leq g_m < \varepsilon g$, i.e., $|f| \leq \sum_{n=1}^{m} |f_n| + \varepsilon g$ implies

$$E(|f|) \leq \sum_{n=1}^{m} E(|f_n|) + \varepsilon E(g) \leq \sum_{n} E(|f_n|) + \varepsilon E(g)$$ and from this we get 5) by letting $\varepsilon \to o$.

We have the following representation theorem.

Proposition 3.11 If X is a locally compact Hausdorff space and $E|\mathcal{E}$ is a positive linear functional on the set $\mathcal{E}$ of all K-functions on X, then there is an outer measure m^* on the family of all subsets of X such that, for each $f \in \mathcal{E}$, E(f) is representable as Lebesque integral of f with respect to m, in symbols: $E(f) = \int_X fdm$.

(Proof) By proposition 3.10 above and by corollary 2.21 we have that $1 \in \mathfrak{M}$. Hence the various propositions concerning m^* established further above in this section apply; in particular we have that proposition 3.9 is applicable since $f \in \mathcal{E}$ implies $f \in \mathcal{L}$ and f is bounded.

On occasion, one requires of the elementary integrals under consideration not only 5c) or $1 \in \mathfrak{M}$ be satisfied, but also that the additional property, called σ-finiteness, be satisfied:

(7c): There is a sequence $(g_n)_{n \in N}$ of $\mathcal{N}$-integrable functions such that $\bigcap_n \{x : g_n(x) = o\}$ is an $\mathcal{N}$-null set.

Proposition 3.12 Under the assumption that $1 \in \mathfrak{M}$, each of the following statements is equivalent with (7c).

(a) There are bounded everywhere positive functions $f_o \in \mathcal{L}$.

(b) There is a sequence $(f_n)_{n \in N}$ of non-negative elementary functions for which $\bigcap_n \{x: f_n(x) = o\}$ is an $\mathcal{N}$-null-set.

(c) Every $\mathcal{N}$-measurable set can be represented as a countable union of $\mathcal{N}$-measurable sets of finite measure.

(d) There is a sequence $(g_n)_{n \in \mathbb{N}}$ of functions in $\mathcal{E}$ with $\sum_n |g_n(x)| = +\infty$ for all $x \in A$.

(Proof) 7c) $\Longrightarrow$ (a)

The statement is trivial if the basic set A itself is a null-set. Otherwise we can leave out in the sequence $(g_n)_{n \in N}$ of 7c) all those functions for which $L(|g_n|) = o$, since for such the set $\{x: g_n(x) = o\}$ differs from A only by a null-set. Thus, $L(|g_n|) \neq o$ for all $n \in N$. Then, setting $f^* = \sum_n \frac{g_n}{2^n L(|g_n|)}$, we have that $f^* \geqslant o$ and $f^* \in \mathcal{L}$. Moreover, $\{x: f^*(x) = o\} = \bigcap_n \{x: g_n(x) = o\}$ is a null-set. If we therefore set:

$$f_o(x) = \begin{cases} \min\{f^*(x), 1\} \text{ for } f^*(x) > o \\ 1 \text{ for } f^*(x) = o \end{cases}$$

we then have the desired function f_o with $o < f_o(x) \leq 1$ for all $x \in A$.

a) $\Longrightarrow$ 7c) Setting $g_n = f_o$ for all $n \in N$ we have the desired sequence.

Concerning part (b): Observe that (b) is stronger than 7c). On the other hand, 7c) implies by (a) the existence of the function f_o to which there corresponds a sequence $(\varphi_n)_{n \in N}$ of elementary functions with $f_o = \operatorname{Lim}_n \varphi_n$. But then we also have $\operatorname{Lim}_n |\varphi_n| = f_o$ and in $f_n = \max\{|\varphi_1|, \ldots, |\varphi_n|\}$ we have the desired functions. Indeed, $o \leq f_n \in \mathcal{E}$ and $f_n(x) = o$ for all $n \in N$ means that $\varphi_n(x) = o$ for all $n \in N$. But by proposition 2.4 there is a subsequence $|\varphi_{n_1}|, |\varphi_{n_2}|, \ldots$ with $\lim_j |\varphi_{n_j}(x)| = f_o(x) > o$ $\mathcal{N}$- a.e. and so $f_n(x) = o$ for all $n \in N$ can only occur at the points x of a null-set.

7c) $\Longrightarrow$ c) By proposition 3.3 it follows that for a $Y \in \mathcal{M}$. either $m^*(Y) = m(Y) < +\infty$ or $m^*(Y) = +\infty$. In the first case we set $T_n = Y$ and are finished. If now $m^*(Y) = +\infty$, with χ_Y we also have that $\varphi_n = \min\{\chi_Y, nf_o\}$ is measurable and since $o \leq \varphi_n \leq nf_o$ we have $\varphi_n \in \mathcal{L}$. Letting $B_n = \{x: \varphi_n(x) = 1\}$, we have that $\chi_{B_n} \in \mathcal{L}$, i.e., B_n is measurable and has finite measure. (Note: If $f \in \mathcal{L}$, then for every real $r \neq o$ the function $\chi_{\{x \in A: f(x) = r\}}$ belongs to $\mathcal{L}$. This is so because $\chi_{\{x \in A: f(x) = r\}} =$ $\lim_n \chi_{\{x \in A: \alpha_n \leq f(x) < \beta_n\}}$ with rational α_n, β_n, $\alpha_n \uparrow r \downarrow \beta_n$ and $\chi_{\{x \in A: \alpha_n \leq f(x) < \beta_n\}}$ is in $\mathfrak{M}$ for all $n \in N$, so that $\chi_{\{x \in A: f(x) = r\}}$ is in $\mathfrak{M}$. Moreover we have

$$-\left|\frac{1}{r} f(x)\right| \leq \chi_{\{x \in A: f(x) = r\}} \leq \left|\frac{1}{r} \cdot f(x)\right|$$

and $f \in \mathcal{L}$.)

Since φ_n converges monotonely to χ_Y, we have that $Y = \bigcup_n B_n$.

c) $\Longrightarrow$ 7c) Since $1 \in \mathfrak{M}$, A is measurable; hence there exists a representation $A = \bigcup_n B_n$ with $B_n \in \mathcal{M}$ and $m(B_n) < +\infty$, where we may assume that the B_n are pairwise disjoint and $m(B_n) > o$ (provided we do not have the trivial case $m(A) = o$).

We then put $f_o(x) = (2^n m(B_n))^{-1}$ for $x \in B_n$, $n \in N$; we get that $f_o > o$, and by proposition 2.7 we see that $L(f_o) \leq 1$; thus (a) holds and so property 7c) is satisfied.

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d) $\Longrightarrow$ 7c). If d) holds, we put $f_n = |g_1| + \cdots + |g_n|$. Then $f_n \in \mathcal{E}$, $\sup_n f_n = \sum_k |g_k|$ and $\{x : \sup_n f_n(x) = 0\} = \phi$, hence condition (b) holds and thus 7c) is satisfied.

7c) $\Longrightarrow$ (d). For the function f_0 in (a) we have, since $f_0 \in \mathcal{L}$, the representation $\operatorname{Lim}_k g_k = f_0$ with $g_k \in \mathcal{E}$, or, if we envisage that a suitable subsequence has been chosen already, with $\lim_k g_k = f_0$, hence also $\lim_k |g_k| = |f_0| = f_0$ $\mathcal{N}$- a.e. Since $f_0 > 0$, we therefore have $|g_1| + |g_2| + \cdots = +\infty$ $\mathcal{N}$- a.e. with $|g_n| \in \mathcal{E}$. Since for every measurable set of finite measure and in particular for every null-set Y there are sequences $(f_n)_{n \in N}$ of elementary functions with $\sum_n |f_n| > 0$ for $x \in Y$ (because on account of $\mathcal{N}(\chi_Y) < +\infty$ there exists a sequence $(f_n)_{n \in N}$ in $\mathcal{E}$ with $\chi_Y \leq \sum_n |f_n|$), we obtain upon suitable choice of the f_n in $\mathcal{E}$ in the functions $h_n = |g_n| + |f_1| + \cdots + |f_n|$ a sequence in $\mathcal{E}$ with $\sum_n h_n(x) = +\infty$ for all x and so (d) is established.

We now consider a generalization of the Dominated Convergence Theorem.

Proposition 3.13 Assume that 7c) holds and that $1 \in \mathcal{M}$. Suppose that for all $n \in N$ the functions f_n, g_n and h_n belong to $\mathcal{L}$ and that $f_n \leq g_n \leq h_n$ holds $\mathcal{N}$- a.e. Moreover, let $\operatorname{Lim}_n f_n = f \in \mathcal{L}$, $\operatorname{Lim}_n h_n = h \in \mathcal{L}$ and suppose that $\lim_n g_n = g$ exists $\mathcal{N}$- a.e. Then $g \in \mathcal{L}$ and $L(g) = \lim_n L(g_n)$.

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(Proof) We have that $f \leq g \leq h$ holds $\mathcal{N}$-a.e.; we may assume without loss of generality that $f_n \leq g \leq h_n$ holds $\mathcal{N}$-a.e. as well (we may replace f_n by $f - |f_n - f|$ and h_n by $h + |h_n - h|$).

Taking into account propositions 3.12 (c), 3.8, we can decompose the basic set A into $A = A_o \cup (\bigcup_m A_m)$ where A_o is an $\mathcal{N}$-null-set and for m = 1, 2, ... every A_m is $\mathcal{N}$-measurable and of finite measure and such that the sequence $(g_n)_{n \in N}$ converges uniformly on A_m. Letting $B_m = \bigcup_{k=1}^{m} A_k$ and $C_m = A - B_m$ and noting that g is $\mathcal{N}$- a.e. finite (because $f \leq g \leq h$), we obtain $\mathcal{N}(g - g_n) \leq \mathcal{N}((g - g_n)\chi_{B_m}) + \mathcal{N}((g - g_n)\chi_{C_m})$, where $\mathcal{N}((g - g_n)\chi_{C_m}) = \mathcal{N}(|g - g_n|\chi_{C_m}) \leq \mathcal{N}((h_n - f_n)\chi_{C_m}) \leq \mathcal{N}(f - f_n) + \mathcal{N}((f - h)\chi_{C_m}) + \mathcal{N}(h - h_n)$. On account of the proper convergence of the series $L((h - f)\chi_{A_1}) + L((h - f)\chi_{A_2}) + \dots = L(h - f) = \mathcal{N}(h - f)$ with the series remainders $\mathcal{N}((h - f)\chi_{C_m})$ we can, for every prescribed $\varepsilon > o$, find an m' such that $\mathcal{N}((f - h)\chi_{C_m}) < \varepsilon/4$. Since $g_n \longrightarrow g$ uniformly on $B_{m'}$, we can find an n' such that $\mathcal{N}(|g - g_n|\chi_{B_{m'}}) < \varepsilon/4$ for $n > n'$, one may of course pick n' so large that $\mathcal{N}(f - f_n) < \varepsilon/4$ and $\mathcal{N}(h - h_n) < \varepsilon/4$ hold as well for $n > n'$. Thus $\mathcal{N}(g - g_n) < \varepsilon$ for $n > n'$ and so $g \in \mathcal{L}$ and $L(g) = \lim_n L(g_n)$ follow.

<u>Proposition 3.14</u> Let property 7c) hold and $1 \in \mathfrak{M}$. Suppose also that an $\mathcal{N}$-measurable function φ has the property that $\varphi f \in \mathcal{L}$ whenever $f \in \mathcal{L}$, where, moreover, $L(\varphi f) \geq o$ whenever $f \geq o$. Then $\varphi \geq o$ holds $\mathcal{N}$-a.e.

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(Proof) For $r < o$ we have that $\{x: \varphi(x) < r\}$ is measurable; by part (c) of proposition 3.12 this set is either an $\mathcal{N}$-null set or contains a measurable subset X of finite positive measure. If the latter were the case, we would have $\chi_X \in \mathcal{L}$ and $L(\varphi \chi_X) \leq rm(X) < o$ in contradiction to the hypothesis of the proposition. Thus $\{x: \varphi(x) < r\}$ is a null-set for all $r < o$ and so the set $\{x: \varphi(x) < o\}$ will also be a null-set.

Proposition 3.15 If φ is $\mathcal{N}$-measurable and bounded and if $\mathcal{E} \ni g \geq o$ implies $L(\varphi g) \geq o$ then $\varphi \geq o$ holds $\mathcal{N}$-a.e.

(Proof) Let $\mathcal{L} \ni f \geq o$, then there exists $(g_n)_{n \in N} \subset \mathcal{E}$ with $\mathrm{Lim}_n\, g_n = f$. Without loss of generality we may suppose that $g_n \geq o$ (or else we would use $\max\{g_n, o\}$ in place of g_n in the following). Taking $M = \sup_{x \in A} |\varphi(x)| < +\infty$, we obtain that $|\varphi g_n| \leq M g_n \in \mathcal{L}$ and hence by proposition 3.13 $\lim_n L(\varphi g_n) = L(\varphi f)$; but $L(\varphi g_n) \geq o$ and so $L(\varphi f) \geq o$. Now proposition 3.14 is applicable and the proof is finished.

Section 4

The Function Spaces $\mathcal{F}_p$ and $\mathcal{L}_p$

For $p \geq 1$ we define

$$\mathcal{F}_p = \{f: f|f|^{p-1} \in \mathcal{F}\}, \mathcal{L}_p = \{f: f|f|^{p-1} \in \mathcal{L}\} \text{and}$$
$\mathcal{N}_p(f) = (\mathcal{N}(|f|^p))^{1/p}$.

It is clear that $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{N}_1 = \mathcal{N}$.

Proposition 4.1 $\mathcal{N}_p$ satisfies the inequalities of Hoelder and Minkowski:

(1) (Hoelder) For $p > 1$, $p + q = pq$, $f \in \mathcal{F}_p$, $g \in \mathcal{F}_q$ we have $\mathcal{N}(fg) \leq \mathcal{N}_p(f)\mathcal{N}_q(g)$

(2) (Minkowski) For $p \geq 1$, f, $g \in \mathcal{F}_p$ we have $\mathcal{N}_p(f+g) \leq \mathcal{N}_p(f) + \mathcal{N}_p(g)$.

(Proof) First we consider (1). We may of course ignore the trivial case when $\mathcal{N}(|f|^p)$ or $\mathcal{N}(|g|^q)$ is zero. For $\alpha, \beta > 0$ and taking $\gamma = \left((\frac{p}{q})^{1/p} + (\frac{q}{p})^{1/q}\right)^{-1}$, the function $\gamma(\alpha\zeta^p + \beta\zeta^{-q})$ over the interval $0 < \zeta < \infty$ has only at the point $\zeta_0 = (\frac{\beta q}{\alpha p})^{1/pq}$ an absolute minimum and because $p + q = pq$; thus $\alpha^{1/p} \cdot \beta^{1/q} \leq \gamma(\alpha\zeta^p + \beta\zeta^{-q})$. We therefore obtain $|fg| \leq \gamma(|f|^p \cdot \zeta^p + |g|^q \cdot \zeta^{-q})$ ($\alpha = |f|^p$, $\beta = |g|^q$) from which we get that $\mathcal{N}(fg) \leq \gamma(\alpha'\zeta^p + \beta'\zeta^{-q})$ for all $\zeta \in (0, +\infty)$, ($\alpha' = \mathcal{N}(|f|^p)$, $\beta' = \mathcal{N}(|g|^q)$).

For the special value $\zeta = \zeta_1 = (\frac{\beta' q}{\alpha' p})^{1/pq}$ we get Hoelder's inequality.

Next we consider (2). The case $p = 1$ we have proved earlier on (see property (d) of the norm $\mathcal{N}$, discussed at the beginning of Section 2). The case $p > 1$ we reduce to (1). Since for reals x and y we have in general that $|x+y|^p \le 2^{p-1}(|x|^p + |y|^p)$, it follows that:

$\mathcal{N}(|f+g|^p) \le 2^{p-1}(\mathcal{N}(|f|^p) + \mathcal{N}(|g|^p))$; this means that with f and g the function f+g is also in $\mathcal{F}_p$. From $|f+g|^p \le$ $\le (|f|+|g|)|f+g|^{p-1}$ we obtain (by using Hoelder's inequality): $\mathcal{N}(|f+g|^p) \le \mathcal{N}(|f|\,|f+g|^{p-1}) + \mathcal{N}(|g|\,|f+g|^{p-1}) \le \mathcal{N}_p(f)$ $\mathcal{N}_q(|f+g|^{p-1}) + \mathcal{N}_p(g)\mathcal{N}_q(|f+g|^{p-1})$ which yields Minkowski's inequality on account of $\mathcal{N}_q(|f+g|^{p-1}) = (\mathcal{N}(|f+g|^p))^{1-\frac{1}{p}}$.

We define in $\mathcal{F}_p$ equality by f=g iff $\mathcal{N}_p(f-g) = o$. Since $\mathcal{N}_p(f) = o$ if and only if $\mathcal{N}(f) = o$, we see that this definition of equality is identical with the one introduced in $\mathcal{L}$. Minkowski's inequality shows that $\mathcal{F}_p$ is a normed vector space under the norm $\mathcal{N}_p$.

<u>Proposition 4.2</u> $\mathcal{F}_p$ is a Banach space.

(Proof) The proof will be done for $p > 1$. Let V be the mapping which associates with every $f \in \mathcal{F}_p$ a function $V(f) = f|f|^{p-1}$ from $\mathcal{F}$. V is a homeomorphism. Indeed, if $g = V(f) = |f|^p$ sign f, then $f = |g|^{1/p}$ sign $g = V^{-1}(g)$, the inverse of V, defined for all $g \in \mathcal{F}$. We use the inequalities, due to S. Mazur ("Une remarque sur l'homéomorphie des champs fonctionnels" Studia Math., 1, 83-85 (1929)).

(1) $2^{1-p}|x-y|^p \le \big|x|x|^{p-1} - y|y|^{p-1}\big| \le p\,|x-y|\,(|x|^{p-1} + |y|^{p-1})$

which hold even for complex x and y. From the first half of inequality (1) follows for $g_1, g_2 \in \mathcal{F}$.

$$|V^{-1}(g_1) - V^{-1}(g_2)|^p \le 2^{p-1}|g_1 - g_2|$$

Thus $\mathcal{N}_p(V^{-1}(g_1) - V^{-1}(g_2)) \leq 2^{1/q}(\mathcal{N}(g_1 - g_2))^{1/p}$; this shows the uniform continuity of V^{-1}. From the second half of inequality (1) we get for $f_1, f_2 \in \mathcal{F}_p$:

$$|V(f_1) - V(f_2)| \leq p\,|f_1 - f_2|\;|f_1|^{p-1} + p\,|f_1 - f_2|\;|f_2|^{p-1}$$

upon using Hoelder's inequality, we obtain $\mathcal{N}(V(f_1) - V(f_2)) \leq$

$$p\mathcal{N}_p(f_1 - f_2)(\mathcal{N}_q(|f_1|^{p-1}) + \mathcal{N}_q(|f_2|^{p-1})) = p\mathcal{N}_p(f_1 - f_2)$$

$(|\mathcal{N}_p(f_1)|^{p/q} + |\mathcal{N}_q(f_2)|^{p/q})$. This shows that $V(f)$ is uniformly continuous on all bounded parts of $\mathcal{F}_p$. Finally, we take a Cauchy sequence $(f_n)_{n \in N}$ in $\mathcal{F}_p$. Since $(f_n)_{n \in N}$ is bounded in $\mathcal{F}_p$ we have that $(V(f_n))_{n \in N}$ is a Cauchy sequence in $\mathcal{F}$. But $\mathcal{F}$ is known to be complete by proposition 2.5 and so $V(f_n) \longrightarrow g \in \mathcal{F}$. Since V^{-1} is continuous, $\lim_n f_n$ exists in $\mathcal{F}_p$ and is equal to $V^{-1}(g)$. And so the completeness of $\mathcal{F}_p$ is established.

Remark From the foregoing proof we see that the spaces $\mathcal{F}_p$, $p \geq 1$, are all homeomorphic to each other.

Turning now to the consideration of $\mathcal{L}_p$, we proceed to specialize and sharpen the results of the previous two propositions.

Proposition 4.3

(1') (Hoelder) If $p > 1$, $p + q = pq$, $f \in \mathcal{L}_p$, $g \in \mathcal{L}_q$, then $fg \in \mathcal{L}$ and $|L(fg)| \leq (L(|f|^p))^{1/p}\,(L(|g|^q))^{1/q}$. Here equality holds if and only if $f|f|^{p-1}$ and $g|g|^{q-1}$ are linearly dependent. In case $p = q = 2$, we have the inequality of Schwarz; in it equality holds only if f and g are proportional.

(2') (Minkowski's inequality) If $p \geq 1$, f and g belongs to $\mathcal{L}_p$, then $f + g \in \mathcal{L}_p$ and $(L(|f+g|^p))^{1/p} \leq (L(|f|^p))^{1/p} + (L(|g|^p))^{1/p}$ Here equality holds in case $p = 1$ if and only if f and g have the same sign almost everywhere, in case $p > 1$ if and only if f and g are linearly dependent.

(3') $\mathcal{L}_p$ is a closed linear subspace of $\mathcal{F}_p$ (and hence is a Banach space).

(Proof) We first consider (1'). The function $\varphi(x_1, x_2) = |x_1|^{1/p} |x_2|^{1/q} \operatorname{sign} x_1x_2$ is a positive homogeneous and continuous (hence Baire) function and so, by proposition 2.12, $fg = \varphi(f', g')$ with $f' = f|f|^{p-1} \in \mathcal{L}$ and $g' = g|g|^{q-1} \in \mathcal{L}$ is also a member of $\mathcal{L}$. From the inequality $|fg| \leq \gamma(|f|^p \xi_1^{\,p} + |g|^q \xi_1^{\,-q})$ derived in the proof of proposition 4.1 it follows by L-integration that $|L(fg)| \leq L(|fg|) \leq \gamma(L(|f|^p)\xi_1^{\,p} + L(|g|^q)\xi_1^{\,-q}) = |L(|f|^p)|^{1/p} |L(|g|^q)|^{1/q}$. We have equality between the end terms of these inequalities only if on the one hand $|L(fg)| = L(|fg|)$, i.e., $\pm L(fg) = L(|fg|)$, or $L(\pm fg - |fg|) = o$, i.e., for a fixed sign $\pm fg - |fg|$ is a null-function, and, on the other hand, if the function $h = |fg| - \gamma(|f|^p \xi_1^{\,p} + |g|^q \xi_1^{\,-q})$ is a null-function as well. In case the values of f and $g \neq o$, then h will only be zero if $\xi_1 = (\frac{|g|^q q}{|f|^p p})^{1/pq}$, or $|g|^q = \frac{p}{q} \xi_1^{\,pq} |f|^p$ according to the minimum considerations carried out in the proof of proposition 4.1. It is evident that the same conclusion can be reached in case f and g is zero. Together with the above condition we obtain that for fixed sign almost everywhere $g|g|^{q-1} = \pm\frac{p}{q}\xi_1^{pq} f|f|^{p-1}$ and so the necessity of the condition is verified. That the condition is also sufficient follows immediately from its derivation.

In case $p = q = 2$ the last equation reads: $g|g| = \pm\xi_1^4 f|f|$; from it we get that $g = \pm\xi_1^2 f$.

Next we consider (2'). With the Baire function $\varphi(\lambda,\theta) = (|\lambda|^{1/p}\operatorname{sign}\lambda + |\theta|^{1/p}\operatorname{sign}\theta)\,\big||\lambda|^{1/p}\operatorname{sign}\lambda + |\theta|^{1/p}\operatorname{sign}\theta\big|^{p-1}$ and with $f' = f|f|^{p-1}\in\mathcal{L}$, $g' = g|g|^{p-1}\in\mathcal{L}$, that is, f and g $\in\mathcal{L}_p$, we have $(f+g)|f+g|^{p-1} = \varphi(f', g')$ and so $f+g\in\mathcal{L}_p$. The proof of the stated inequality proceeds as in proposition 4.1. An inspection of the proof of proposition 4.1 shows that equality can only occur if on the one hand almost everywhere $|f+g| = |f|+|g|$, which means that f and g have almost everywhere the same sign. In case $p = 1$ this is already sufficient.

On the other hand, if $p>1$ and equality is to hold, we need in addition that $\mathcal{N}(|f|\,|f+g|^{p-1}) = \mathcal{N}_p(|f|)\mathcal{N}_q(|f+g|^{p-1})$ be satisfied. By (1') we then get that $|f|^p$ and $|f+g|^{(p-1)q} = |f+g|^p$ are linearly dependent, and similarly for g that $|g|^p$ and $|f+g|^p$ are linearly dependent. Since the trivial that $f+g$ be a null-function requires that f and g be null-functions, we obtain linear dependence of $|f|^p$ and $|g|^p$ hence also linear dependence between $|f|$ and $|g|$; but the signs are equal and so there must be linear dependence between f and g themselves.

Finally, we consider (3'). If $f \in \mathcal{L}_p$, then αf and $|f|$ also belong to $\mathcal{L}_p$ and so $\mathcal{L}_p$ is a linear sublattice of $\mathcal{F}_p$. Moreover $V^{-1}(\mathcal{L}) = \mathcal{L}_p$ (V^{-1} is the homeomorphism studied in the proof of proposition 4.2) and since $\mathcal{L}$ is closed in $\mathcal{F}$, so is $\mathcal{L}_p$ in $\mathcal{F}_p$.

In the remainder of this section we assemble certain important properties of the space $\mathcal{L}_2$. $\mathcal{L}_2$ consists of all real-valued functions f, defined on some basic set A, such that $f\,|f| \in \mathcal{L}$; L is the $\mathcal{N}$-integral which belongs to an elementary integral $E|\mathcal{E}$ over A. For $f \in \mathcal{L}_2$, $L(f^2) = L(|f\,|f||)$ is finite with the norm $\|f\| = (L(f^2))^{1/2}$ and the metric $\|f - g\|$ for any two elements f and g of $\mathcal{L}_2$. Here again we have equality almost everywhere.

The mapping $\langle , \rangle : \mathcal{L}_2 \times \mathcal{L}_2 \to \hat{R}$ defined by $\langle f, g \rangle = L(fg)$ is finite by proposition 4.3, part (1); it is called the *inner product* of f and g; the inner product is a continuous bilinear, symmetric function, that is, $\langle f, g \rangle = \langle g, f \rangle$, $\langle (f_1 + f_2), g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$, $\langle af, g \rangle = a \langle f, g \rangle$, $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$ for arbitrary f, f_1, f_2, g belonging to $\mathcal{L}_2$ and real a. From the last inequality follows the continuity of $\langle , \rangle$.

The functions f and g in $\mathcal{L}_2$ are said to be <u>orthogonal</u> if $\langle f, g\rangle = o$. It can be seen that $\langle f, f\rangle = o$ if and only if $f = o$ a.e.

<u>Proposition 4.4</u> Let $E|\mathcal{E}$ satisfy (5c) (i.e., $f \in \mathcal{E}$ implies $\min\{1, f\} \in \mathcal{E}$), then, for $f \in \mathcal{L}_2$ and $n \in N$, we have that $f_n = \text{med}\{-n, f, n\} \in \mathcal{L}_2$ and $f_n \xrightarrow[\mathcal{N}]{} f$ in $\mathcal{L}_2$ for $n \to \infty$.

(Proof) Since $f|f| \in \mathcal{L}$ and $f_n|f_n| = \text{med}\{-n^2, f|f|, n^2\} = n^2 \min\{\frac{1}{n^2}(-\min\{n^2, -f|f|\}), 1\}$, we have that $f_n|f_n| \in \mathcal{L}$, i.e., $f_n \in \mathcal{L}_2$.

Moreover, $\varphi_n = (f_n - f)^2 = |(f_n - f)|f_n - f|| \in \mathcal{L}$ since $f_n - f \in \mathcal{L}_2$ and $o \leq \varphi_{n+1} \leq \varphi_n$ and the monotone convergence theorem (see proposition 2.10) is applicable. But $\lim_n \varphi_n(x)$ for all x with finite f(x) is zero $\mathcal{N}$- a.e. and hence is an $\mathcal{N}$-null-function; it follows that $\|f_n - f\|^2 = L((f_n - f)^2) \to o$.

<u>Definition</u> A subset $\mathfrak{A}$ of $\mathcal{L}_2$ is said to be a <u>total set in</u> $\mathcal{L}_2$ if the linear subspace $[\mathfrak{A}]$ generated by $\mathfrak{A}$ is dense in $\mathcal{L}_2$.

<u>Proposition 4.5</u> A subset $\mathfrak{A}$ of $\mathcal{L}_2$ is total if an only if for any two functions f_1, f_2 in $\mathcal{L}_2$ the following condition holds:

(α) $\langle f_1, g\rangle = \langle f_2, g\rangle$ for all g in $\mathfrak{A}$ implies $f_1 = f_2$ a.e.

(Proof) If (α) is satified, then $\langle (f_1 - f_2), g\rangle = o$ for all $g \in \mathfrak{A}$. Claim: $\langle (f_1 - f_2), h\rangle = o$ for all $h \in \mathcal{L}_2$.

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Indeed, if $h \in \mathcal{L}_2 = \overline{[\mathfrak{A}]}$ then there exist $(\varphi_n)_{n \in N} \subseteq [\mathfrak{A}]$ such that $\| \varphi_n - h \| \longrightarrow o$ as $n \to +\infty$, and since $\langle , \rangle$ is continuous we have that

$$\langle f_1 - f_2, \varphi_n \rangle = \langle f_1 - f_2, \sum_{k=1}^{m} a_{nk}\, g_{nk} \rangle =$$

$$\sum_{k=1}^{m} a_{nk} \langle f_1 - f_2, g_{nk} \rangle = o \longrightarrow \langle f_1 - f_2, h \rangle \text{ as } n \to \infty.$$ Thus $\langle f_1 - f_2, h \rangle = o$. In particular for $h = f_1 - f_2$ we obtain $\langle f_1 - f_2, f_1 - f_2 \rangle = o$ implying that $f_1 - f_2 = o$ a.e.

" $\Leftarrow$ " Now suppose that $\mathfrak{A}$ is not a total set in $\mathcal{L}_2$; then $\mathcal{L}_2 - \overline{[\mathfrak{A}]} \neq \emptyset$ and hence there exist an $h \neq o$ with $\langle h, g \rangle = o$ for all $g \in [\mathfrak{A}]$, (For a proof see G.F. SIMMONS, Introduction to Topology and Modern Analysis, page 249, Th.B) and hence in particular for all $g \in \mathfrak{A}$. Thus $\langle 2h, g \rangle = o = \langle h, g \rangle$ for all $g \in \mathfrak{A}$, that is, condition (α) does not hold in general.

<u>Corollary 4.6</u> $\mathcal{L}_2$ is itself a total set.

<u>Proposition 4.7</u> (Riesz Representation theorem)

To each continuous linear functional F in $\mathcal{L}_2$ corresponds a unique element g_F in $\mathcal{L}_2$ such that for all $f \in \mathcal{L}_2$, we have $F(f) = L(fg_F)$, where $\| g_F \| \leq M_F$. If moreover, F is positive, that is, $F(f) \geqslant o$ for $f \geqslant o$, then g_F is $\mathcal{N}$- a.e. non-negative.

(Proof) The uniqueness of the representation is easy to see; by Corollary 4.6.

Next we show the existence of such a g_F. Let $\mathfrak{A} = \operatorname{Ker} F$ (Ker F = kernel of F); since F is continuous and linear, $\mathfrak{A}$ is a closed linear subspace in $\mathcal{L}_2$ and so is either identical with $\mathcal{L}_2$, in which case we may set $g_F = o$, or is a proper subspace of $\mathcal{L}_2$, in which case is not dense and hence there is an element $g \neq o$ orthogonal to $\mathfrak{A}$, i.e., $\langle g, h \rangle = o$ for all $h \in \mathfrak{A}$. For $g_F = \frac{F(g)}{\| g \|^2} g$ we therefore have $F(h) = o = \langle h, g_F \rangle$ for all $h \in \mathfrak{A}$ and moreover, $F(g) = \langle g, g_F \rangle$; for arbitrary $f \in \mathcal{L}_2$ with $c = \frac{F(f)}{F(g)}$ the function $f^* = f - cg$ is in $\mathfrak{A}$ (since $F(f^*) = F(f) - cF(g) = o$) and so $F(f) = F(f^*) + cF(g) = \langle f^*, g_F \rangle + c \langle g, g_F \rangle = \langle f, g_F \rangle$. For $f = g_F$ we get $F(g_F) = \langle g_F, g_F \rangle$ and hence (since F is bounded linear functional), $M_F \| g_F \| \geqslant | F(g_F) | = | \langle g_F, g_F \rangle | = \| g_F \|^2$, that is, $M_F \geqslant \| g_F \|$.

Finally, we assume that F is positive; if g_F is in $\mathcal{L}_2$, then $| g_F |$ and $| g_F | - g_F = f_F \in \mathcal{L}_2$ and $F(f_F) = L(f_F g_F) \geqslant o$ But $f_F \cdot g_F \leq o$ and so $L(f_F g_F) \leq o$ by the positivity of L; hence $o = L(f_F \cdot g_F) = -L(-f_F g_F) = -\mathcal{N}(f_F g_F)$. This implies that $(| g_F | - g_F) g_F = o$ a.e. i.e., $| g_F | \cdot g_F = g_F^2$ a.e. and hence that $g_F \geqslant o$ a.e.

<u>Proposition 4.8</u> If $F(f,g)$ is a bounded bilinear functional in $\mathcal{L}_2$ (i.e., if $F(f,g)$, $f, g \in \mathcal{L}_2$ is a linear functional in f for each fixed g and vice versa and moreover there exists a constant M_F such that $| F(f,g) | \leq M_F \| f \| \cdot \| g \|$ for all $f, g \in \mathcal{L}_2$) then there exists a bounded linear operator T mapping $\mathcal{L}_2$ into itself such that $F(f, g) = L(fT(g))$ and $M_T \leq M_F$ (where $\| T(g) \| \leq M_T \| g \|$ for all $g \in \mathcal{L}_2$).

(Proof) With fixed $g \in \mathcal{L}_2$ we write $F(f, g) = F^*(f)$. Then $F^*(f)$ is a bounded linear functional, hence by proposition 4.7 $F^*(f) = \langle f, g_{F^*} \rangle$ with $g_{F^*} \in \mathcal{L}_2$ and $\| g_{F^*} \| \leq M_{F^*} = M_F \| g \|$.

If we put $T(g) = g_F{}^*$ for $g \in \mathcal{L}_2$, then T is uniquely defined by proposition 4.7 and is a bounded operator by the inequality just proved with T mapping $\mathcal{L}_2$ into $\mathcal{L}_2$ and $M_T \leq M_F$; here $F(f, g) = \langle f, T(g) \rangle$ for all f and g belonging to $\mathcal{L}_2$. From the last equality it follows however that $\langle f, T(\alpha g_1, \beta g_2) \rangle = \alpha \langle f, T(g_1) \rangle + \beta \langle f, T(g_2) \rangle = \langle f, (\alpha T(g_1) + \beta T(g_2) \rangle$ for fixed f and using the bilinearity of F; this being so for an arbitrary $f \in \mathcal{L}_2$, we may conclude, using corollary 4.6, that $T(\alpha g_1 + \beta g_2) = \alpha T(g_1) + \beta T(g_2)$. Thus T is seen to be linear and the proof is complete.

Definition A bilinear functional $F(f,g)$, respectively, a linear operator $T(f)$ in $\mathcal{L}_2$ is said to be positive if for non-negative f and g we have $F(f, g) \geq 0$, respectively if $T(f) \geq 0$ $\mathcal{N}$- a.e.

Corollary 4.9 Let F be as in proposition 4.8 and in addition assume that F is positive. Then the operator T introduced in propositon 4.8 will also be positive.

(Proof) Indeed, for fixed non-negative $g \in \mathcal{L}_2$ we have that $F(f, g) = F^*(f)$ is positive; by propositon 4.7 $g_F{}^* = T(g)$ is $\mathcal{N}$-a.e. non-negative.

Section 5

Fubini's Theorem in Stone's Version

For $i = 1,2$ let $\mathcal{E}_i$ be a linear lattice of elementary functions f_i with domain of definition X_i and let $E_i | \mathcal{E}_i$ be an elementary integral, i.e., we assume that both elementary integrals satisfy conditions 1c) to 4c) introduced at the beginning of Section 2. For brevity we shall write $E_i f_i$ in place of $E_i(f_i)$ provided that this does not lead to misunderstanding and we shall use similar notation for other functionals. We shall consider functions f whose domain of definition X_3 is the Cartesian product set $X_3 = X_1 \times X_2$ and in the following we shall make use of the simplifying idea that one may view f with $x_1 \in X_1$ held fixed as a function on X_2 (in other words, we identify the partial function f on $\{x_1\} \times X_2$ with a function on X_2). Let $\mathcal{E}_1 * \mathcal{E}_2$ denote the set of all functions f on X_3 for which on the one hand f with fixed but arbitrary $x_1 \in X_1$ as a function on X_2 belongs to $\mathcal{E}_2$ so that the function $E_2 f$ with X_1 as domain of definition exists and for which on the other hand $E_2 f \in \mathcal{E}_1$ so that the number $E_1 E_2 f = E_1(E_2 f)$ is defined.

We note that the set $\mathcal{E}_1 * \mathcal{E}_2$ does not satisfy the condition: $|f| \in \mathcal{E}_1 * \mathcal{E}_2$ whenever $f \in \mathcal{E}_1 * \mathcal{E}_2$ and hence is not a linear lattice. For example, if $E_i | \mathcal{E}_i$ is the Riemann integral on the interval $[0,1]$ for $i = 1,2$ and
$$f(x_1,x_2) = \begin{cases} 0 & \text{for } x_1 = 0 \\ \frac{1}{x_1} \operatorname{sign}(\frac{1}{2} - x_2) & \text{for } 0 < x_1 \leq 1, \end{cases}$$
then $f \in \mathcal{E}_1 * \mathcal{E}_2$ but not $|f| \in \mathcal{E}_1 * \mathcal{E}_2$.

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On account of the apparent shortcomings of $\mathcal{E}_1 * \mathcal{E}_2$ we shall make the following restrictive assumptions:

(1) $X_3 = X_1 \times X_2$;

(2) for i = 1,2,3, on X_i there is given a linear lattice of elementary functions $\mathcal{E}_i$ and with it is given an elementary integral $E_i | \mathcal{E}_i$ such that

(3) $\mathcal{E}_3 \subseteq \mathcal{E}_1 * \mathcal{E}_2$ and $E_3 | \mathcal{E}_3 = E_1 E_2 | \mathcal{E}_3$ holds.

We stipulate that to $E_i | \mathcal{E}_i$ belong the norm $\mathcal{N}_i$, the function systems $\mathcal{G}_i$, $\mathcal{Z}_i$ and $\mathcal{F}_i$, and, finally, the $\mathcal{N}_i$-integral L_i with the corresponding function space $\mathcal{L}_i$, for i = 1, 2, 3.

By $\mathcal{L}_1 * \mathcal{L}_2$ we denote the set of all functions f with domain of definition $X_3 = X_1 \times X_2$ satisfying the following requirements:

(I) For f there is an $\mathcal{N}_1$-null-set $X_1^* \subseteq X_1$ such that for every fixed $x_1 \in X_1 - X_1^*$ the function f as function on X_2 is equal to a function in $\mathcal{L}_2$;

(II) The real-valued function $L_2 f$ with domain of definition $X_1 - X_1^*$ equals a function in $\mathcal{L}_1$.

<u>Proposition 5.1</u> If (3) holds, then, for arbitrary $f \in \mathcal{G}_3$; $\mathcal{N}_1 \mathcal{N}_2 f \leq \mathcal{N}_3 f$.

(Proof) We need only to consider the case where $\mathcal{N}_3 f < +\infty$. For $\varepsilon > o$, there exists a sequence $(f_n) \subseteq \mathcal{E}_3$, $n \in N$ such that $|f| \leq \sum_n |f_n|$ and $\sum_n E_1 E_2 |f_n| = \sum_n E_3 |f_n| < \mathcal{N}_3 f + \varepsilon$ hold.

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On the other hand, using properties (d) and (g) of the norm discussed early in Section 2 we see that

$\mathcal{N}_2 f \le \sum_n \mathcal{N}_2 f_n = \sum_n E_2 |f_n|$ and so

$\mathcal{N}_1 \mathcal{N}_2 f \le \sum_n \mathcal{N}_1 \mathcal{N}_2 |f_n| = \sum_n E_1 E_2 |f_n| < \mathcal{N}_3 f + \varepsilon$ follows;

that is, $\mathcal{N}_1 \mathcal{N}_2 f \le \mathcal{N}_3 f$.

Corollary 5.2 $\mathcal{N}_1 \mathcal{N}_2 f$ in $\mathfrak{Z}_3$ and $L_1 L_2 f$ in $\mathcal{L}_1 * \mathcal{L}_2$ are uniquely defined.

(Proof) Indeed, $f = f'$ in the sense of equality in $\mathfrak{Z}_3$ means that

$$\mathcal{N}_3 \mathcal{X} = 0$$

for

$$\mathcal{X} = \mathcal{X}\left\{(x_1, x_2) \in X_3 : f(x_1, x_2) \neq f\ (x_1, x_2)\right\}$$

However, by the foregoing result this means that the number $\mathcal{N}_1 \mathcal{N}_2 \mathcal{X} = 0$, or the function $\mathcal{N}_2 \mathcal{X} = 0$ on $X_1 - X_1^{*}$, where X_1^{*} is an $\mathcal{N}_1$-null-set such that $\left\{x_2 \in X_2 : f(x_1, x_2) \neq f\ (x_1, x_2)\right\}$ for $x_1 \in X_1 - X_1^{*}$, is an $\mathcal{N}_2$-null-set. We therefore have that $\mathcal{N}_2 f = \mathcal{N}_2 f'$ for $x_1 \in X_1 - X_1^{*}$; this in turn means that $\mathcal{N}_1 \mathcal{N}_2 f = \mathcal{N}_1 \mathcal{N}_2 f'$. If, moreover, $f, f' \in \mathcal{L}_1 * \mathcal{L}_2$ (with corresponding $\mathcal{N}_1$-null-sets X_1^{**} and X_1^{***}) then we obtain that $L_2 f(x_1) = L_2 f'(x_1)$ for $x_1 \in X_1 - (X_1^{*} \cup X_1^{**} \cup X_1^{***})$ and so $L_1 L_2 f = L_1 L_2 f'$.

Proposition 5.3 (Fubini-Stone): If the elementary integrals $E_i \,|\, \mathcal{E}_i$, i = 1, 2, 3, satisfy the condition (3), then the corresponding

norm integrals $L_i \,|\, \mathcal{L}_i$ satisfy the related conditions
$\mathcal{L}_3 \subseteq \mathcal{L}_1 * \mathcal{L}_2$ and $L_3 f = L_1 L_2 f$ for $f \in \mathcal{L}_3$.

(Proof) Let $f \in \mathcal{L}_3$. Then there is an $f_n \in \mathcal{E}_3$, $n \in N$, such that $\mathcal{N}_3(f - f_n) < 2^{-n}$. Since $g = \sum_n \mathcal{N}_2(f - f_n) \in \mathcal{G}_1$, we get by proposition 5.1 that $\mathcal{N}_1 g \leq \sum_n \mathcal{N}_1 \mathcal{N}_2(f - f_n) \leq \sum_n \mathcal{N}_3(f - f_n) < \sum_n 2^{-n} = 1$; thus $g \in \mathcal{F}_1$, i.e., $o \leq g(x_1) < +\infty$ for $x_1 \in X_1 - X_1^*$, where X_1^* denotes an $\mathcal{N}_1$-null-set. For fixed $x_1 \in X_1 - X_1^*$ we therefore have $\lim_n \mathcal{N}_2(f - f_n) = o$ and so f as function on X_2 belongs to $\mathcal{L}_2$; for f_n with x_1 fixed belongs to $\mathcal{E}_2$. We consider $L_2 f = f^*$ as function belonging to $\mathcal{G}_1$ (the values in X_1^* do not matter) and we obtain $|f^* - E_2 f_n| = |L_2(f - f_n)| \leq \mathcal{N}_2(f - f_n)$ so that
$\mathcal{N}_1(f^* - E_2 f_n) \leq \mathcal{N}_1 \mathcal{N}_2 (f - f_n) \leq \mathcal{N}_3(f - f_n) < 2^{-n}$; thus, since $E_2 f_n \in \mathcal{E}_1$, by the definition of $\mathcal{L}_1$ we have that $f^* \in \mathcal{L}_1$. Finally, we have

$$L_1 L_2 f = L_1 f^* = \lim_n E_1 E_2 f_n = \lim_n E_3 f_n = L_3 f.$$

There is a partial converse to the foregoing proposition.

Proposition 5.4 Let the elementary integrals $E_i \,|\, \mathcal{E}_i$, $i = 1, 2, 3$, satisfy the condition (3). If the function f with domain of definition X_3 is $\mathcal{N}_3$-measurable and if for $n \in N$ there are functions $f_n \in \mathcal{F}_3$, i.e., with $\mathcal{N}_3 f_n < +\infty$ and with $|f| \leq \sum_n |f_n|$, then $|f| \in \mathcal{L}_1 * \mathcal{L}_2$ already implies $f \in \mathcal{L}_3$.

(Proof) By part (i) of proposition 2.26 we may without restriction assume that $f_n \in \mathcal{L}_3$. Since $|f_n| \in \mathfrak{M}_3$ and $s_n = |f_1| + \dots + |f_n| \in \mathcal{L}_3$, we have by proposition 2.20 that $g_n = \min\{|f|, s_n\} \in \mathcal{L}_3$. Since $0 \leq g_n \leq |f|$, we have that $\mathcal{N}_2 g_n \leq \mathcal{N}_2 f$ and $\mathcal{N}_1 \mathcal{N}_2 g_n \leq \mathcal{N}_1 \mathcal{N}_2 f$. From $|f| \in \mathcal{L}_1 * \mathcal{L}_2$ follows the existence of an $\mathcal{N}_1$-null-set X_1^* such that $L_2 |f| = \mathcal{N}_2 f$ for fixed $x_1 \in X_1 - X_1^*$ and in addition that $\mathcal{N}_2 f \in \mathcal{L}_1$. Thus $\mathcal{N}_1 \mathcal{N}_2 f = L_1 \mathcal{N}_2 f < +\infty$. On the other hand, $\mathcal{N}_3 f \geq L_3 g_n \geq L_3 g_{n-1}$ holds. From this we may conclude that $\mathcal{N}_3 f = \lim_n L_3 g_n$ in case $\lim_n L_3 g_n = +\infty$; if, on the other hand, this limit is finite, the Monotone Convergence Theorem can be used (as g_n tends monotonely to $|f|$) to establish the same equality. Applying proposition 5.3, we get $L_3 g_n = \mathcal{N}_1 \mathcal{N}_2 g_n \leq \mathcal{N}_1 \mathcal{N}_2 f$ which leads to $\mathcal{N}_3 f \leq \mathcal{N}_1 \mathcal{N}_2 f < +\infty$ upon letting $n \to +\infty$. Finally we obtain $f \in \mathcal{F}_3 \cap \mathfrak{M}_3 = \mathcal{L}_3$ (by proposition 2.25) and the proof is complete.

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Section 6

The Radon-Nikodym Theorem in Stone's Version

In this section we shall study the relations between different elementary integrals $E|\mathcal{E}$, $E'|\mathcal{E}$, ... which are defined on the same set $\mathcal{E}$ of elementary functions f over a fixed basic set A and we consider the corresponding $\mathcal{N}$-integrals.

The set $\mathfrak{M}$ of all elementary integrals $E|\mathcal{E}$ over the same fixed domain $\mathcal{E}$ of elementary functions f over A forms a partially ordered commutative semigroup with respect to addition admitting multiplication by positive real scalars:

The functionals $E + E'$ and αE are defined by $(E + E')(f) = E(f) + E'(f)$ and $(\alpha E)(f) = \alpha E(f)$ for $f \in \mathcal{E}$. From $E = E' + E''$ follows $E' \leq E$ and $E'' \leq E$; if $E' \leq E$, then there is a unique E'' given by $E''(f) = E(f) - E'(f)$, with $f \in \mathcal{E}$, such that $E = E' + E''$.

The functionals and function spaces associated with the elementary integrals E, E', ..., E_1, E_1', ... will be distinguished by the use of the corresponding primes and subscripts. Here we have in particular that $\mathcal{E} = \mathcal{E}' = \ldots$ and $\mathcal{J} = \mathcal{J}' = \ldots$. With this convention in mind, we prove

<u>Proposition 6.1</u> If $E' \leq E$, then (i) $\mathcal{N}'(f) \leq \mathcal{N}(f)$ for $f \in \mathcal{J}$; (ii) $\mathfrak{Z}' \supseteq \mathfrak{Z}$; (iii) $\mathfrak{F}' \supseteq \mathfrak{F}$; (iv) $\mathcal{L}' \supseteq \mathcal{L}$; (v) $L'(f) \leq L(f)$ for non-negative f in $\mathcal{L}$; (vi) if the set $\{x \in A : f_1(x) \neq f_2(x)\}$ is an $\mathcal{N}$-null set, then it is also an $\mathcal{N}'$-null set; (vii) $\mathfrak{M}' \supset \mathfrak{M}$.

(Proof) To (i): By definition we can find for each value which serves for the formation of $\mathcal{N}(f)$ as infimum a non-greater one for the formation of $\mathcal{N}'(f)$.

To (ii): By part (i), $\mathcal{N}(f) = o$ implies that $\mathcal{N}'(f) = o$ and so every $\mathcal{N}$-null-set will also be an $\mathcal{N}'$-null-set; hence each member of $\mathcal{Z}$ is also a member of $\mathcal{Z}'$.

To (iii): By part (i), we have that $\mathcal{N}(f) < +\infty$ implies $\mathcal{N}'(f) < +\infty$, that is, $f \in \mathcal{F}$ implies $f \in \mathcal{F}'$.

To (iv): $f \in \mathcal{L}$ means that $f = \mathcal{N}\text{-}\operatorname{Lim}_n f_n$ with $(f_n) \subseteq \mathcal{E}$ by the definition of $\mathcal{L}$; by part (i) we then also have that $f = \mathcal{N}'\text{-}\operatorname{Lim}_n f_n$ and so $f \in \mathcal{L}'$.

To (v): The claim is once again a consequence of part (i) since for a non-negative function the integral and the norm coincide.

To (vi): The claim is a consequence of part (ii).

To (vii): If $f \in \mathcal{M}$, then $\operatorname{med}\{g,f,h\} \in \mathcal{L} \subseteq \mathcal{L}'$ for $g,h \in \mathcal{E}$ and thus $f \in \mathcal{M}'$. The proof is finished.

<u>Definition</u> An elementary integral $E|\mathcal{E}$ satisfying (5c) and

(8c): There is a sequence of non-negative functions f_n in $\mathcal{E}$ with $\sum_n f_n(x) = +\infty$ for all $x \in A$,

we call a <u>normal integral</u>.

We assume from now on that properties 5c) and 8c) are satisfied. Observe that a normal integral is σ-finite in the sense of section 3 and note that propositions 3.12 and 3.13 are satisfied. Moreover, 5c) implies that $1 \in \mathfrak{M}$ (see corollary 2.21).

<u>Proposition 6.2</u> If $E|\mathcal{E}$ is a normal integral, $o \leq g \in \mathfrak{M}$ and $s = \sup\{L(f): f \in \mathcal{L}$ and $o \leq f \leq g\} < +\infty$, then $g \in \mathcal{L}$ with $L(g) = s$.

(Proof) Let $(f_n)_{n \in N}$ be a sequence satisfying 8c) and let $g_n = \min\{f_1 + f_2 \ \ldots. + f_n, g\}$. Then $(g_n)_{n \in N}$ is a monotone non-decreasing sequence which converges to g with $L(g_n) \leq s$. By the Monotone Converges Theorem we get $g \in \mathcal{L}$ and clearly $L(g) = s$. This completes the proof.

In order to fix uniquely the product of two extended real numbers, we agree to let the product be zero in case one of the factors is zero.

<u>Proposition 6.3</u> (Radon-Nikodym) If E and E' are normal integrals over $\mathcal{E}$ and $E' \leq E$, then there exists one, and up to an $\mathcal{N}$-null-set only one, real function Φ^* defined on A such that

(1) $o \leq \Phi^* \leq 1$ and $\Phi^* \in \mathfrak{M}$;
(2) $f \in \mathcal{E}$ implies $E'(f) = L(\Phi^* f)$;
(3) $f' \in \mathcal{L}' \Leftrightarrow \Phi^* f' \in \mathcal{L}$;
(4) $f' \in \mathcal{L}'$ implies $L'(f') = L(\Phi^* f')$.

(Proof) If h, $h^* \in \mathcal{L}_2$, then $h h^* \in \mathcal{L} \subset \mathcal{L}'$. Since $|L'(h\ h^*)| \leq L'(|h\ h^*|) \leq L(|h\ h^*|) \leq [L(h^2)]^{\frac{1}{2}}\ [L(h^{*2})]^{\frac{1}{2}}$ (see proposition 4.3), we have that $L'(h\ h^*)$ is a positive bounded bilinear functional in $\mathcal{L}_2$ and so by proposition 4.8 there is a positive bounded linear operator T mapping $\mathcal{L}_2$ into itself such that $L'(h\ h^*) = L(hT(h^*))$. If g is a bounded function belonging to $\mathfrak{M}$, then gh and gh^* will also belong to $\mathcal{L}_2$; indeed, for example $gh\ |g|\,|h|$ is $\mathcal{N}$-a.e. finite and being a product of $\mathcal{N}$-measurable functions, is $\mathcal{N}$-measurable; moreover, $gh\ |g|\,|h|$ is wedged in between $-sh^2$ and $+sh^2$ with $s = \sup_{x \in A} [g(x)]^2$ and so must be $\mathcal{N}$-integrable by proposition 2.25. The equations $L(hT(gh^*)) = L'(hgh^*) = L(hgT(h^*))$ hold for all $h \in \mathcal{L}_2$ and so, by proposition 4.6 we have $T(gh^*) = gT(h^*)$ $\mathcal{N}$-a.e.

To exploit the symmetry of g and h^* on the left hand side of the last equation, we substitute for g a function $g_o = f_o^{\frac{1}{2}} \in \mathcal{L}_2$ according to part (a) of proposition 3.12 and we substitute for h^* the functions $h_n^* = \text{med}\{-n, h^*, n\}$, $n \in N$, all of which are bounded for $h^* \in \mathcal{L}_2$ and belong to $\mathcal{L}_2$. We obtain: $g_oT(h_n^*) = T(g_oh_n^*) = T(h_n^*g_o) = h_n^*T(g_o)$. Since $h_n^* \longrightarrow h^*$ in $\mathcal{L}_2$ by proposition 4.4, we get by continuity considerations that $g_oT(h^*) = h^*T(g_o)$ for $h^* \in \mathcal{L}_2$. Setting $\Phi^* = \frac{T(g_o)}{g_o}$ we obtain an $\mathcal{N}$-measurable function; indeed, since $T(g_o) \in \mathcal{L}_2$, we have $|T(g_o)|^2 \in \mathcal{L} \subset \mathfrak{M}$ and, since $T(g_o) \geqslant 0$ by proposition 4.9, it follows by part (i) of proposition 2.24 that $T(g_o) \in \mathfrak{M}$ and in like manner that $g_o \in \mathfrak{M}$ so that part (ii) of proposition 2.24 is applicable. Thus we have $T(h^*) = \Phi^*h^*$. Now, if $f \in \mathcal{L}$, then, letting $h^* = |f|^{\frac{1}{2}} \in \mathcal{L}_2$

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and $h = |f|^{\frac{1}{2}} \text{sign } f$ in $\mathcal{L}_2$, we get $f = h \cdot h^*$ and $\mathcal{L} \ni hT(h^*) = h\Phi^* h^* = \Phi^* f$, that is,

(q): $L'(f) = L'(h\, h^*) = L(hT(h^*)) = L(\Phi^* f)$.

For $o \leq f \in \mathcal{L}$, we have by part (v) of proposition 6.1 that $o \leq L'(f) \leq L(f)$, that is, $o \leq L(\Phi^* f) \leq L(f)$ for all such f, and so, by proposition 3.14, $o \leq \Phi^* \leq 1$ $\mathcal{N}$-a.e. (because the right-hand side of the second last inequality may also be written as $o \leq L((1 - \Phi^*)f)$.

It is now easily seen that up to an $\mathcal{N}$-null-set there can only be one such function Φ^*. Indeed, if Φ_1 and Φ_2 are $\mathcal{N}$-measurable functions with values in the interval $[o, 1]$ (so that $\Phi_1 f$ and $\Phi_2 f$ belong to $\mathcal{L}$ whenever f belong to $\mathcal{L}$) and Φ_1 and Φ_2 enjoy the property that $L(\Phi_1 f) = L'(f) = L(\Phi_2 f)$ for all $f \in \mathcal{L}$, then it follows that $L((\Phi_1 - \Phi_2)f) = o$ for all $f \in \mathcal{L}$ and so by proposition 3.14 we have $\mathcal{N}$-a.e. both $\Phi_1 - \Phi_2 \leq o$ and $\geq o$, that is, $= o$. Without restriction we can choose Φ^* such that $o \leq \Phi^*(x) \leq 1$ for all $x \in A$. Thus parts (1) and (2) of the proposition are proven; for, in particular, the statement (q) further up in this proof holds for $f \in \mathcal{E}$.

Next, let $f' \in \mathcal{L}'$. This means that for $\varepsilon > o$ there are functions $g, f_1, f_2, \ldots$ belonging to $\mathcal{E}$ with $|f' - g| \leq \sum_n |f_n|$ and $\sum_n E(|f_n|) \leq \varepsilon$. Then $|L'(f') - L'(g)| \leq L'(|f' - g|) \leq \sum_n E(|f_n|) \leq \varepsilon$ holds and thus $\Phi^* f \in \mathcal{L}$ because $\Phi^* g \in \mathcal{L}$. Furthermore it follows that $| L(\Phi^{*}f') - L'(g) | = | L(\Phi^{*}f' - \Phi^{*}g) | \leq L(| \Phi^{*}f' - \Phi^{*}g |) = \mathcal{N}(\Phi^{*}f' - \Phi^{*}g) \leq \varepsilon$, hence $| L(\Phi^{*}f') - L'(f') | \leq | L(\Phi^{*}f') - L'(g) | + | L'(g) - L'(f') | \leq 2\varepsilon$, and with $\varepsilon \to 0$ finally that $L(\Phi^{*}f') = L'(f')$. This proves claim (4) and the "only if" part in claim (3).

We now take up the "if" part in claim (3).

(I): We show first that $P = \{ x \in A: \Phi^{*}(x) = 0 \}$ is an $\mathcal{N}'$-null-set. Indeed, since $\Phi^{*} \in \mathfrak{M}$, $\Phi^{*} \in \mathfrak{M}'$ by part (vii) of proposition 6.1 and so by proposition 3.5 we obtain that $\chi_{P} \in \mathfrak{M}'$. Since by our general assumptions the basic set A is $\mathcal{N}'$-measurable, we have by part (c) of proposition 3.12 that $A = \bigcup_{n=1}^{\infty} A_{n}$, where each A_{n} is measurable and of finite measure. The latter means that $\chi_{A_{n}} \in \mathcal{L}'$. By proposition 2.25 we therefore see that $\chi_{P} \chi_{A_{n}} \in \mathcal{L}'$ and so by part (4), for all $n \in N$, $L'(\chi_{P} \chi_{A_{n}}) = L(\Phi^{*} \chi_{P} \chi_{A_{n}}) = 0$ because $\Phi^{*} \chi_{P} = 0$. From $P = \bigcup_{n=1}^{\infty} (P \cap A_{n})$ it now follows that $\mathcal{N}'(\chi_{P}) \leq \sum_{n} L'(\chi_{P} \chi_{A_{n}}) = 0$, verifying the claim. (II): Since $\Phi^{*}f'$ is $\mathcal{N}$- a.e. finite, the same holds $\mathcal{N}'$-a.e. on account of (I); we therefore see that f' is $\mathcal{N}'$-a.e. finite.

Since $f' = |f'| - (|f'| - f')$, it will suffice to consider the case $f' \geq 0$ only. Then $h_{n} = \frac{n \Phi^{*}f'}{n \Phi^{*} + 1}$, $n \in N$, is a sequence which converges non-decreasingly to an $\mathcal{N}'$-a.e. finite function h, where

$$h(x) = \begin{cases} 0 \text{ for } x \in P \\ f'(x) \text{ for } x \in A - P \end{cases} \text{ and } \Phi^{*}h = \Phi^{*}f'.$$

Since h_{n} as an $\mathcal{N}$-measurable function with $0 \leq h_{n} \leq n \Phi^{*}f' \in \mathcal{L}$ also belongs to $\mathcal{L}$, we have in addition that $h_{n} \in \mathcal{L}'$; thus

$L'(h_n) = L(\bar{\Phi}^* h_n) \leq L(\bar{\Phi}^* f')$ because $h_n \leq f'$. By the Monotone Convergence Theorem we get that $h \in \mathcal{L}'$. By (I) the functions h and f' differ only on an $\mathcal{N}'$-null-set and so with h belonging to $\mathcal{L}'$ so does f'.

The foregoing proposition has the following converse.

<u>Proposition 6.4</u> If the normal integral $E|\mathcal{E}$ and $\Phi \in \mathfrak{M}$ with $0 \leq \Phi \leq 1$ are given, then $E'|\mathcal{E}$, defined by

$$E'(f) = L(\Phi f),\ f \in \mathcal{E}$$

is a normal integral with $E' \leq E$ and the function Φ^* belonging to E' by virtue of proposition 6.3 differs from Φ by an $\mathcal{N}$-null-function only.

(Proof) Properties (1c) to 5c) for $E'|\mathcal{E}$ follow immediately from the analogous properties of $L|\mathcal{L}$; the normality of the integral E' is trivial. By (2) of proposition 6.3 we have for Φ^* belonging to E':

$$L(\Phi^* f) = E'(f) = L(\Phi f) \text{ for } f \in \mathcal{E}.$$

By proposition 3.14 we may conclude that $\Phi^* = \Phi$ $\mathcal{N}$- a.e.

<u>Definitions</u> Let $x \vee y$ and $x \wedge y$ denote $\sup\{x, y\}$ and $\inf\{x, y\}$, respectively. A lattice V is said to be <u>distributive</u> if and only if, for every $x, y, z \in V$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A lattice is said to be σ-<u>complete</u> if every countable subset of it has a supremum and an infimum and is called <u>conditionally</u> σ-<u>complete</u> if every countable subset of it which has an upper(lower) bound has a supremum (infimum).

From propositions 6.3 and 6.4 follows:

Proposition 6.5 The set $\mathfrak{N}^*$ of all normal integrals E over a fixed domain $\mathcal{E}$ is a conditionally σ-complete distributive lattice.

(Proof)

The mapping :

$$\varphi : \mathfrak{N}^*_E = \{ E' : E \geq E', E' \in \mathfrak{N}^* \} \to \mathfrak{M}_1 = \{ \Phi^* : \Phi^* \in \mathfrak{M} \text{ and } 0 \leq \Phi^* \leq 1 \}$$

defined in proposition 6.3 is an isomorphic mapping of the partially ordered set $(\mathfrak{N}^*_E, \leq)$ onto the partially ordered set $(\mathfrak{M}_1, \leq)$.

Since $\mathfrak{M}_1$ is σ-complete, so is $\mathfrak{N}^*_E$. If we do not restrict ourselves to subsets $\mathfrak{N}^*_E$ and consider $\mathfrak{N}^*$, only conditional σ-completeness remains. For example, $\mathfrak{N}^*$ - sup $\{ E_1, E_2, \ldots \}$ exists if there is an $E \in \mathfrak{N}^*$ such that $E_i \leq E$, $i \in N$; then one forms the functions $\Phi^*_i = \varphi(E_i)$ and obtains $\mathfrak{N}^*$ - sup $\{E_1, E_2, \ldots \}(f) = L(f \sup \{\Phi^*_1, \Phi^*_2, \ldots \})$.

For a given E_i, $i = 1, 2, 3$, and $E_4 = E_1 + E_2 + E_3$ we have $E_i \leq E_4$ $i = 1, 2. 3$ and hence $\varphi(E_1 \wedge (E_2 \vee E_3)) = \varphi(E_1) \wedge \varphi(E_2 \vee E_3)$ $= \varphi(E_1) \wedge (\varphi(E_2) \vee \varphi(E_3)) = \Phi^*_1 \wedge (\Phi^*_2 \vee \Phi^*_3)$ and $\varphi((E_1 \wedge E_2) \vee (E_1 \wedge E_3)) = (\varphi(E_1) \wedge \varphi(E_2)) \vee (\varphi(E_1) \wedge \varphi(E_3)) =$ $(\Phi^*_1 \wedge \Phi^*_2) \vee (\Phi^*_1 \wedge \Phi^*_3)$ and since $\mathfrak{M}_1$ is distributive, and φ is one-to-one we have $E_1 \wedge (E_2 \vee E_3) = (E_1 \wedge E_2) \vee (E_1 \wedge E_3)$. (If we are given the lattices, S and S', then every isomorphic mapping of S onto S', understood in the sense of the partial ordering, will be an isomorphic mapping of the lattice S onto the lattice S'. (See A.G. Kurosh, "Lectures in General Algebra" page 165.)).

<u>Definition</u> For $E^{\prime}, E^{\prime\prime}$ in $\mathfrak{n}^*$ we define an order relation by $E^{\prime} \lhd E^{\prime\prime}$ iff every $\mathcal{N}^{\prime\prime}$-null-set is an $\mathcal{N}^{\prime}$-null-set.

By proposition 2.2 it is evident that the foregoing condition is equivalent with: Every $\mathcal{N}^{\prime\prime}$-null-function is an $\mathcal{N}^{\prime}$-null-function. We observe that $E^{\prime} \leqq E^{\prime\prime}$ implies $E^{\prime} \lhd E^{\prime\prime}$ (by part (i) of proposition 6.1) and that the relation $\lhd$ defined on $\mathfrak{n}^*$ is reflexive and transitive (but not necessarily anti-symmetric).

<u>Proposition 6.6</u> If E_1, E_2 and E are normal integrals over $\mathcal{E}$ with $E_i \leqq E$ and if Φ^*_i are Φ^* functions corresponding to E_i, i = 1, 2 by virtue of proposition 6.3, then the following are equivalent.

(i) $E_1 \lhd E_2$;

(ii) $K = \left\{ x \in A: \Phi^*_2(x) = o \right\} \cap \left\{ x \in A: \Phi^*_1(x) \neq o \right\}$ is an $\mathcal{N}$-null-set.

(Proof) We have the representation $L_i(f) = L(\Phi^*_i f)$ for $f \in \mathcal{L}_i$ and we put $K_i = \left\{ x \in A: \Phi^*_i(x) = o \right\}$ (i = 1, 2)

Suppose that statement (ii) holds. If Z_2 is any $\mathcal{N}_2$-null-set, then $L_2(\chi_{Z_2}) = o$, or $L(\Phi^*_2 \chi_{Z_2}) = o$. By proposition 2.2, $Z_2 \cap (A - K_2)$ is an $\mathcal{N}$-null-set. Consider

$$Z_2 \cap K_2 = (Z_2 \cap K_2 \cap K_1) \cup (Z_2 \cap K_2) \cap (A - K_1).$$

The set $Z_2 \cap K_2 \cap K_1$, being a subset of K_1, is an $\mathcal{N}_1$-null-set; this can be seen from the proof of proposition 6.3 (see claim (I)).

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The set $(Z_2 \cap K_2) \cap (A - K_1)$, being a subset of K, is an $\mathcal{N}_1$-null-set also because K is by assumption an $\mathcal{N}$-null-set. Therefore Z_2 is an $\mathcal{N}_1$-null-set and we have $E_1 \triangleleft E_2$.

Suppose, finally, that the set K is not an $\mathcal{N}$-null-set. Since Φ_1^* and Φ_2^* belong to $\mathfrak{M}$, we have that K is $\mathcal{N}$-measurable (by proposition 3.4); by part (c) of proposition 3.12 there exists an $\mathcal{N}$-measurable subset K_o of K with $o < m(K_o) < +\infty$. Using part (a) of proposition 3.12 we obtain $\chi_{K_o} f_o \in \mathcal{L}_1$ and $L_1(\chi_{K_o} f_o) = L(\Phi_1^* \chi_{K_o} f_o)$. But $\{x \in A: \Phi_1^*(x) \chi_{K_o}(x) f_o(x) > o\} = K_o$ and hence by proposition 2.2 $L_1(\chi_{K_o} f_o) > o$ and so $m_1^*(K_o) > o$, showing that K_o cannot be an $\mathcal{N}_1$-null-set. Since K_o is a subset of K and K is an $\mathcal{N}_2$-null-set, we conclude that the relation $E_1 \triangleleft E_2$ fails to be fulfilled.

We now take up the generalized Radon-Nikodym Theorem, that is a generalization of propositions 6.3 and 6.4.

<u>Proposition 6.7</u> (I): Let E' and E'' be normal integrals over $\mathcal{E}$ with $E' \triangleleft E''$. Then there exists up to $\mathcal{N}''$-null-sets a unique non-negative function $\psi \in \mathfrak{M}''$ with the following properties:

(α): $f \in \mathcal{L}'$ if and only if $\psi f \in \mathcal{L}''$;
(β): $f \in \mathcal{L}'$ implies $L'(f) = L''(\psi f)$;
in particular $E'(f) = L''(\psi f)$ for $f \in \mathcal{E}$.

(II) If $E''|\mathcal{E}$ is a normal integral and ψ a non-negative function belonging to $\mathfrak{M}''$ with the property that $\psi f \in \mathcal{L}''$ for all $f \in \mathcal{E}$, then $E'(f) = L''(\psi f)$ defines a normal integral with $E' \triangleleft E''$.

(Proof) We consider (I) first and put $E = E' + E''$. We then have $E' \leq E$ and $E'' \leq E$; by proposition 6.3 there are representations $E'(f) = L(\Phi' f)$ and $E''(f) = L(\Phi'' f)$ for $f \in \mathcal{E}$, where Φ', Φ'' are in $\mathfrak{M} \subset \mathfrak{M}' \cap \mathfrak{M}''$ and $0 \leq \Phi' \leq 1$ and $0 \leq \Phi'' \leq 1$. Here $\Phi' + \Phi'' = 1$, $\{x \in A : \Phi''(x) = 0\} \cap \{x \in A : \Phi''(x) \neq 0\} = \{x \in A : \Phi''(x) = 0\}$ is an $\mathcal{N}$-null-set. Thus

$$\psi = \Phi' / \Phi''$$

is finite $\mathcal{N}$-a.e., non-negative and $\psi \in \mathfrak{M} \subset \mathfrak{M}''$. By (3) of proposition 6.3, $f \in \mathcal{L}'$ if and only if $\Phi' f \in \mathcal{L}$, that is, $\Phi''(\psi f) \in \mathcal{L}$, which in turn is equivalent with $\psi f \in \mathcal{L}''$. Further, by (4) of proposition 6.3, we have $L'(f) = L(\Phi' f) = L(\Phi''(\psi f)) = L''(\psi f)$.

Next, we consider (II). Let E'' and ψ have the mentioned properties. We first observe that $E'(f) = L''(\psi f)$ defines a normal integral. Indeed, 1c) to 3c) are clear on account of the corresponding properties of L'', 5)(which is equivalent to 4c)) follows from property (d) for $\mathcal{N}''$, the normality is trivial. We put $E = E' + E''$ and have the representations $E'(f) = L(\Phi' f)$ and $E''(f) = L(\Phi'' f)$ for $f \in \mathcal{E}$. Moreover,

$$L(\Phi' f) = E'(f) = L''(\psi f) = L(\Phi''(\psi f)) \text{ for all } f \text{ in } \mathcal{E}.$$

The last gives $L(\Phi' f) = L(\Phi'' \psi f)$ for $f \in \mathcal{L}$.

Indeed; For such f let $(f_n)_{n \in N}$ be a sequence from $\mathcal{E}$ with $\mathrm{Lim}_n f_n = f$ and (upon selection of a subsequence if necessary) $\lim_n f_n = f$ $\mathcal{N}$-a.e. Thus we also have $\lim_n \Phi''\psi f_n = \Phi''\psi f$ and $\lim_n \Phi' f_n = \Phi' f$ $\mathcal{N}$-a.e. Since moreover $L(|\Phi''\psi f_m - \Phi''\psi f_n|) = L(\Phi''\psi |f_m - f_n|) = L(\Phi' |f_m - f_n|) \leq L(|f_m - f_n|)$, $\mathrm{Lim}_n \Phi''\psi f_n$ and $\mathrm{Lim}_n \Phi' f_n$ exist and are equal to $\Phi''\psi f$ and $\Phi' f$, respectively. Here we have $L''(\Phi''\psi f) = \lim_n L(\Phi''\psi f_n) = \lim_n L(\Phi' f_n) = L(\Phi' f)$; but this is what we have claimed. Going on with the proof, we assume that $f_o \in \mathcal{L}$ is chosen according to (a) of proposition 3.12; then by the above we have $\Phi''\psi f_o \in \mathcal{L}$, hence

$$\Phi''\psi = \frac{\Phi''\psi f_o}{f_o} \in \mathcal{M} \text{ (by proposition 2.24)}$$

and so $\Phi' - \Phi'' \cdot \psi \in \mathcal{M}$ as well.

From $L((\Phi' - \Phi''\psi)f) = o$ for all $f \in \mathcal{L}$ follows by proposition 3.14 that $\Phi' - \Phi'\psi = o$, that is, $\Phi' = \Phi''\psi$ $\mathcal{N}$-a.e. Thus $\{x \in A: \Phi''(x) = o\} \cap \{x \in A: \Phi'(x) \neq o\}$ is an $\mathcal{N}$-null-set and hence $E' \triangleleft E''$.

The essential uniqueness of ψ in (I) can be seen by the following argument: If ψ_1 is a function which satisfied (α) and (β), then $L'(f') = L(\Phi' f') = L''(\psi_1 f') = L(\Phi''\psi_1 f')$ for every $f' \in \mathcal{L}'$. and a fortiori $L(\Phi' f) = L(\Phi''\psi_1 f)$ for $f \in \mathcal{L} \subseteq \mathcal{L}'$.

Invoking proposition 3.14, we see that $\Phi' - \Phi''\psi_1$ is an $\mathcal{N}$-null-function and so $\psi_1 = \frac{\Phi'}{\Phi''} = \psi$ $\mathcal{N}$-a.e.

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We now consider the Lebesque Decomposition Theorem.

Proposition 6.8 If E' and E'' are normal integrals over $\mathcal{E}$, then E' has a unique decomposition

$$E' = E_1' + E_2'$$

with $E_1' \triangleleft E''$ and $\inf \{E_2', E''\}(|f|) = o$ for all $f \in \mathcal{E}$.

(Proof) We put $E' + E'' = E$ and get by proposition 6.3 the representations $E'(f) = L(\Phi' f)$ and $E''(f) = L(\Phi'' f)$ and we set

$$\Phi_1'(x) = \begin{cases} 0 \text{ for } & \Phi''(x) = o \\ \Phi'(x) \text{ for } & \Phi''(x) \neq o \end{cases}$$

and $$\Phi_2'(x) = \begin{cases} \Phi'(x) \text{ for } \Phi''(x) = o \\ o \text{ for } \Phi''(x) \neq o \end{cases}$$

Then $\Phi_1' + \Phi_2' = \Phi'$,

$\{x \in A: \Phi''(x) = o\} \cap \{x \in A: \Phi_1'(x) \neq o\}$ is empty and $\inf \{\Phi_2', \Phi''\} = o$; with $E_i'(f) = L(\Phi_i' f)$, $i = 1, 2$, we therefore have a representation of the desired kind. We still have to show that there is essentially only one such decomposition of Φ'. For a decomposition of E' of the desired kind there are in any case representations $E_i'(f) = L(\Psi_i f)$ and for the Ψ_i we must have:

$$\Psi_1 + \Psi_2 = \Phi' \text{ and } \inf \{\Psi_2, \Phi''\} = o \ \mathcal{N}\text{- a.e.}$$

and $\{x \in A: \Phi''(x) = o\} \cap \{x \in A: \Psi_1(x) \neq o\}$ is an $\mathcal{N}$-null-set. The last leads to $\Psi_1(x) = \Phi_1'(x)$ $\mathcal{N}$- a.e. for $\Phi''(x) = o$ and the second last to $\Psi_2(x) = \Phi_2'(x)$ $\mathcal{N}$-a.e. for $\Phi''(x) \neq o$, that is, $\Psi_1(x) = \Phi_1'(x)$ $\mathcal{N}$-a.e. for $\Phi''(x) \neq o$. Thus $\Psi_1 = \Phi_1'$ and $\Psi_2 = \Phi_2'$ $\mathcal{N}$- a.e.

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