Webs, symmetrizers, and the oriented skein category

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Abstract

We define and study the *two-colour* $\mathfrak{gl}$*-web categories* $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, which generalize the $\mathfrak{gl}$-web categories that have previously appeared in the literature. After establishing some basic properties of these categories, we show that $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ is isomorphic to a partial idempotent completion of the oriented skein category $\mathcal{OS}$. This completion is obtained from $\mathcal{OS}$ by adjoining objects corresponding to the quantum *symmetrizer* and *antisymmetrizer* morphisms. We use this isomorphism to generalize results about web categories from the literature. In particular, we study the *web incarnation functors* from $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ to the category of type 1 weight modules for the quantum enveloping superalgebra $U_q(\mathfrak{gl}_{m|n})$. Our approach allows us to translate fullness and kernel results for $\mathcal{OS}$-incarnation functors into corresponding theorems for web incarnation functors, extending existing results to new cases where the base ring is not a field.

Dedications

For Brian, Kai-Lei, Zephyr, and Ari.

Acknowledgements

To my advisor, Alistair Savage: thank you so much. I simply could not have done this without your many years of mentorship and encouragement, starting all the way back in 2018 when you guided me through my first foray into mathematical research. I feel very lucky to have had the opportunity to work with you. It's been a pleasure.

To my family: thank you for always supporting me.

To my friends: thanks for all of the fun times and good memories.

Contents

Chapter 1

Introduction

1.1 Intended audience

This thesis is designed to be accessible to readers with some background in abstract algebra (algebras, modules, quotients, etc.) and familiarity with category theory (particularly monoidal categories; the key definitions and properties can be found in Appendix A). No previous knowledge of web categories is assumed. Some experience with super mathematics (super vector spaces, superalgebras, etc.) and quantum mathematics (specifically, quantum groups/enveloping algebras) would be helpful, but we recall the relevant definitions and results for these topics in §2.3 and §2.4, respectively.

1.2 History and overview

The modern study of web categories traces back to [Kup96], in which Kuperberg defined various *combinatorial spiders*, consisting of *web diagrams* subject to certain natural geometric/topological relations ("web category" is a more modern term). The spiders studied in [Kup96] correspond to the rank 2 simple Lie algebras (types A_2, B_2, and G_2) and their quantum enveloping algebras, with webs representing certain morphisms between representations of these Lie algebras or quantum groups. These sorts of diagrammatic approaches to representation theory have become increasingly popular in recent years; many identities that are opaque when written algebraically become much simpler and more intuitive when drawn diagrammatically. In addition, it is often possible to define a single diagrammatic category that captures the representation theory of a whole family of groups or algebras in one unified view.

In [CKM14], Cautis, Kamnitzer, and Morrison introduced the SL_n-*spiders*, extending Kuperberg's work to type A in general. In the years since, many new web categories have been defined and studied. In this thesis, we aim to unify and build upon previous approaches to type A web categories, giving economical presentations of the *two-colour* $\mathfrak{gl}$*-web category* $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and the *two-colour* $\mathfrak{gl}$*-web category with duals* $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, which generalize the $\mathfrak{gl}$-

web categories that have previously been studied. In the process, we fill in a few gaps in the literature by providing full proofs for some results that have been mentioned in previous papers. Our main result is Theorem 4.13, which establishes an isomorphism between $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ and $\mathcal{OS}^W$, the latter being a partial idempotent completion of the oriented skein category $\mathcal{OS}$. Specifically, $\mathcal{OS}^W$ is obtained from $\mathcal{OS}$ by formally adjoining objects that correspond to the idempotent quantum *symmetrizer* and *antisymmetrizer* morphisms, which are quantum analogues of Young (anti)symmetrizers. In terms of representation theory, these (anti)symmetrizer objects correspond to symmetric and exterior powers of the natural representation for $U_q(\mathfrak{gl}_{m|n})$, the quantum universal enveloping superalgebra of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$.

As applications of Theorem 4.13, we generalize some results about $\mathfrak{gl}$-web categories from the literature. For instance, we use the isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^W$ to show that our two-colour web categories $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ contain subcategories isomorphic to the monochromatic web categories that have previously appeared in the literature (see Proposition 5.1.6); this was previously known only for *specialized* $\mathfrak{gl}$-web categories, which correspond to particular dimension parameters $m, n \in \mathbb{N}$. Our main focus is on $\mathfrak{gl}$-web categories, though we also discuss $\mathfrak{sl}$-web categories in §5.3. In §5.4, we will discuss the precise connections between our web categories and those that have appeared previously in the literature. In Chapter 6, we give a new perspective on the *web incarnation functors* $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to U_q(\mathfrak{gl}_{m|n})$-mod, defining them in terms of the related $\mathcal{OS}$-incarnation functors $\mathcal{OS} \to U_q(\mathfrak{gl}_{m|n})$-mod. In general, Theorem 4.13 allows one to systematically translate definitions and results for $\mathcal{OS}$, which has been studied quite extensively, into corresponding definitions and results for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$.

At the same time, Theorem 4.13 provides a new perspective on the way that web categories are defined. The relations used to define $\mathfrak{sl}$-web categories in [CKM14] were derived using quantum skew Howe duality, which relates the actions of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{sl}_n)$ on various exterior power modules; similar dualities underlie the definitions of the $\mathfrak{gl}$-web categories in [TVW17] and [QS19]. The isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^W$ can be viewed as giving an alternate definition of the $\mathfrak{gl}$-web category in terms of the existing category $\mathcal{OS}$, independent of these dualities. As we will discuss further in §1.4, this sort of approach could be used to define new diagrammatic categories that are analogous to web categories, even in settings where suitable representation-theoretic tools (analogous to quantum skew Howe duality) have not yet been developed.

1.3 Originality of results

Chapter 2 is largely a review of standard definitions and results. In §2.4, we have included many remarks to help explain some details that are usually omitted in the literature. The facts we prove about (anti)symmetrizers in §2.5 would be unsurprising to those who have worked with such objects before, but the present author was unable to locate complete proofs in the literature for many of the properties we establish in that section.

Our two-colour upwards-oriented $\mathfrak{gl}$-web category $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ is isomorphic to the one studied in [TVW17], though our presentation is smaller; in §3.2, we show that our collection of defining relations implies the larger set used in [TVW17]. To the best knowledge of the present author, a two-colour $\mathfrak{gl}$-web category *with duals* has not previously appeared in the literature. Our definition of $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ extends the approach to $\mathfrak{gl}$-web categories with duals from [QS19] (which, in the language of this thesis, treated the monochromatic purple case) to the two-colour setting. The web category symmetry functors we describe in §3.1 are partially new. Many previous papers have discussed horizontal reflection symmetries of web categories, and a symmetry between the two colours of strands was noted in [TVW17] (though the colour-swapping symmetry we use in this thesis is slightly different; see Connection 5.4.3). Our functors corresponding to vertical reflection and 180° rotation appear to be novel, though similar symmetries have been described for other diagrammatic categories.

In §3.2, we prove Proposition 3.2.5, showing that the general "square-switch" identities follow from just the special case (MIRROR). This fact has been mentioned before, e.g. in [ST19, Rem. 2.5], but the present author is not aware of a complete proof in the literature.

The braided monoidal structure on $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ we define in §3.3 has appeared before in [TVW17]. Our extension of this structure to the $\mathfrak{gl}$-web category with duals is novel, but similar to the monochromatic case addressed in [QS19]. Our proof of Proposition 3.3.23 is new, though the idempotents appearing in that result have been studied before; see e.g. [RT16, Def. 2.11, Rem. 2.21].

The material in Chapter 4 is novel. The general idea of embedding one diagrammatic category into some suitable completion of another diagrammatic category has appeared before (for instance, this is one of the key techniques employed in [SSS24], a joint work between the present author and Alistair Savage), but this approach is relatively new, and has not previously been applied to web categories.

The first two sections in Chapter 5 are mostly concerned with extending existing material to our two-colour web categories. As noted in §1.2, Proposition 5.1.6 generalizes a result that had previously only been proved for specialized $\mathfrak{gl}$-web categories. Our presentation of the $\mathfrak{sl}_n$-web categories in §5.3 is new, taking inspiration from the presentation of the category q-**Schur** given in [Bru25]. The isomorphism of Theorem 5.3.19 (which is closely related to the material in Chapter 4) is also novel.

The incarnation functors we define in Chapter 6 have appeared previously in the literature in many forms. Our approach gives a new perspective on these functors, directly relating them to incarnation functors for $\mathcal{OS}$. We use these connections to extend existing fullness and kernel results for web incarnation functors to new cases where the base ring is not a field. Even in cases where such fullness and kernel results were previously known, our method of proof is new.

1.4 Further directions

- In Chapter 6, we prove the existence of web incarnation functors working over a fairly general base ring, and for arbitrary dimension parameters $m, n \in \mathbb{N}$. However, we only prove that these functors are full and describe their kernels in certain special cases, corresponding to what has been proved for related diagrammatic categories in the literature. We expect that existing methods can be extended to show that our fullness and kernel results hold more generally.

- In [DKMZ23], Davidson, Kujawa, Muth, and Zhu introduced a family of $\mathfrak{gl}$-web supercategories $\mathbf{Web}_I^{A,a}$ depending on a superalgebra A. There are incarnation functors from these A-enriched web categories to the category of modules for $\mathfrak{gl}_n(A)$, analogous to our incarnation functors in Chapter 6. In the case where A is a symmetric Frobenius superalgebra, it seems likely that the methods from Chapter 4 could be adapted to prove that the A-enriched $\mathfrak{gl}$-web category is isomorphic to an appropriate partial idempotent completion of the upwards-oriented part of the oriented Frobenius Brauer category $\mathcal{OB}(A)$, which first implicitly appeared as a subcategory of the Frobenius Heisenberg category $\mathcal{H}eis_{A,0}$ in [Sav19], and then was explicitly defined and studied in [MS23]. More generally, one could define a quantum version of $\mathbf{Web}_I^{A,a}$, which would be isomorphic to a partial idempotent completion of the upwards-oriented part of the quantum Frobenius Heisenberg category introduced in [BSW22].

- Given a braided monoidal category $\mathcal{C}$, one can define the *affinization of* $\mathcal{C}$, $\mathrm{Aff}(\mathcal{C})$, by adjoining invertible "dot" morphisms subject to appropriate relations; see [MS21] for details. Various affinizations of web categories have been studied to date, with close connections to the representation theory of their associated Lie algebras or quantum groups; see [SW26], [SW25a], [SSW25], [SW25b], and [DKM25]. Like in the previous bullet point, it seems likely that our methods from Chapter 4 could be adapted to the affine setting, yielding an isomorphism between affine web categories and a partial idempotent completion of the affinization of $\mathcal{OS}$ (or another appropriate diagrammatic category, depending on the specific web category under consideration).

- As briefly mentioned above, web categories have also been studied outside of type A. For instance, webs of type C were studied in [BERT21], types B, C, and D in [ST19], type P in [DKM24], and type Q in [BDK25] (the last two types appear only in the super setting). As above, we expect that the methods from this thesis could be adapted to these other types of web categories.

- Taking a broader view, Theorem 4.13 tells us that $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ can be obtained from $\mathcal{OS}$ by adjoining objects corresponding to symmetric and exterior powers of the natural $U_q(\mathfrak{gl}_{m|n})$-module. Here, Theorem 4.13 gives us a new perspective on a preexisting category. Taking the reverse approach could inspire definitions for new diagrammatic categories. Namely, given a diagrammatic category $\mathcal{C}$ with incarnation functor $\mathcal{C} \to A$-mod, one could investigate the completions of $\mathcal{C}$ at idempotent morphisms corresponding to certain families of A-modules, analogously to how (anti)symmetrizers correspond to symmetric and exterior powers in this thesis. One could then define a new diagrammatic category (analogous to web categories) specifically designed to be isomorphic to such a partial idempotent completion, but presented in a more convenient and compact way.

Chapter 2

Preliminaries

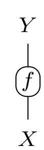
2.1 Conventions

Fix a commutative ground ring $\Bbbk$. We use the terms "$\Bbbk$-vector space" and "$\Bbbk$-module" interchangeably, regardless of whether or not $\Bbbk$ is a field. All of the modules we will consider are left modules. We set $\mathbb{N} = \mathbb{Z}_{\geq 0}$, so in particular $0 \in \mathbb{N}$.

2.2 String diagrams

We will make extensive use of string diagrams throughout this thesis. In this section, we give a quick rundown of the essential conventions and properties; a more detailed introduction to string diagrams can be found in [Sav21]. For a comprehensive treatment of the topic, see e.g. [TV17, §2]. The definitions of monoidal categories and related notions that we will mention below can be found in Appendix A.

Morphisms in a monoidal category can be represented by *string diagrams*. Given a morphism $f: X \to Y$, its corresponding string diagram is a line (called a *string* or *strand*) decorated with a *coupon* labelled by f:



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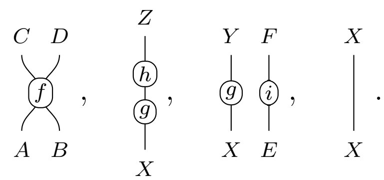
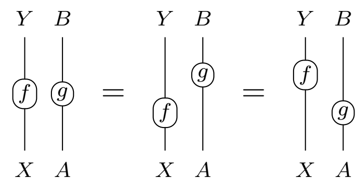
Note that the domain is indicated at the bottom and the codomain is indicated at the top, i.e. we read string diagrams bottom-to-top. Morphisms between tensor products of objects are represented by multiple strands going into/out of the coupon, one for each tensor factor. Composition of morphisms is represented by vertical stacking, and tensor products are represented by horizontal juxtaposition. Identity morphisms are represented by strings without coupons. For example, if $f: A \otimes B \to C \otimes D$, $g: X \to Y$, $h: Y \to Z$, and $i: E \to F$, are morphisms, then f, $h \circ g$, $g \otimes i$, and id_X would respectively be drawn as:

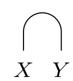


The fact that $\otimes$ is a bifunctor manifests as the *interchange law* for string diagrams, which says that one can freely move morphisms up and down along strings:



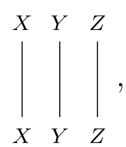
(2.2.1)

The unit object $\mathbb{1}$ is typically represented by the absence of a string. For instance,

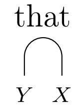
and


respectively represent morphisms $\mathbb{1} \to X$ and $X \otimes Y \to \mathbb{1}$. Here and throughout most of this thesis, we represent morphisms using certain shapes of string diagrams (sometimes with dots or other decorations) without giving those morphisms traditional names like f, g, h, etc.

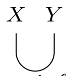
String diagrams are best suited for representing morphisms in strict monoidal categories. When working with non-strict monoidal categories, there is some mild ambiguity. For instance, $(\mathrm{id}_X \otimes \mathrm{id}_Y) \otimes \mathrm{id}_Z$ and $\mathrm{id}_X \otimes (\mathrm{id}_Y \otimes \mathrm{id}_Z)$ would both be drawn as



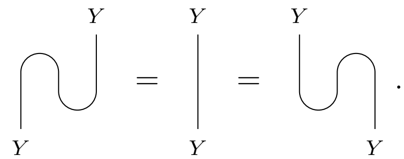
but those two morphisms need not be equal in a non-strict monoidal category. We mostly work with strict monoidal categories in this thesis, so these issues will not come up very much. When interpreting string diagrams in non-strict monoidal categories, one must insert appropriate associators and/or unitors to obtain well-defined morphisms and identities. Alternatively, one can reduce to the strict case by using the fact that every monoidal category is monoidally equivalent to a strict monoidal category (this follows from Mac Lane's coherence theorem; see [ML98, §2]).

Definition 2.2.1. Let $\mathcal{C}$ be a monoidal category and $X, Y \in \mathrm{ob}(\mathcal{C})$. We say that Y is *left dual to* X, or, equivalently, X is *right dual to* Y, if there exist morphisms 

and



(respectively called a *cap* and a *cup*, also known as an *evaluation* and a *coevaluation*) satisfying the *zigzag equations*:

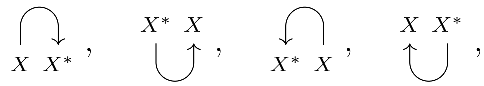


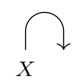
(ZIGZAG)

If Y is both left and right dual to X, we say it is *two-sided dual to* X, or simply *dual to* X.

Left and right duals (when they exist) are unique up to isomorphism, so we will often simply refer to "the" dual of an object X, which we typically denote X^*. We will usually leave the choice of cups and caps implicit when discussing duals.

An object X with a two-sided dual X^* is typically represented in string diagrams by an upwards-facing arrow, and its dual by a downwards-facing arrow. Thus, cups and caps for a dual pair (X, X^*) would be drawn as:



though some authors would omit the labels X^* and draw just e.g. 

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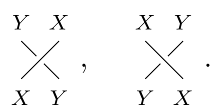
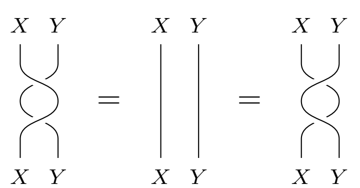
The braiding morphisms in braided monoidal categories are represented by crossings of strings. The diagrams corresponding to the braidings $B_{X,Y}$ and $B^{-1}_{X,Y}$ are, respectively:

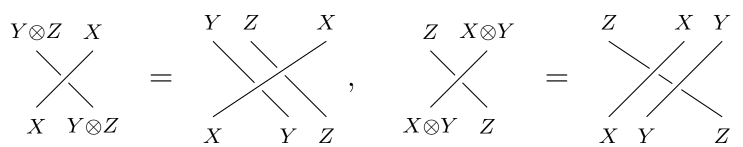


We call the first type of crossing an *over-crossing* or *positive crossing*, and the second type an *under-crossing* or *negative crossing*. The fact that positive and negative crossings are inverse to each other is represented diagrammatically by the *double-crossing relations*:



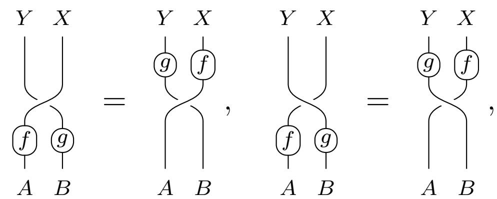
(2.2.2)

The braiding axioms (A.3) and (A.4) imply that crossings involving tensor products can be written in terms of crossings between the individual tensor factors:



(2.2.3)

It follows that the analogous identities hold for negative crossings as well. Hence, when defining crossings for a braided monoidal category generated (via tensor product) by certain objects, it suffices to specify the crossings between those generators, with general crossings implicitly recursively defined via (2.2.3). The requirement that the braiding is natural manifests in string diagrams as the *sliding relations*:



(SLIDE)

for any morphisms $f\colon A \to X$ and $g\colon B \to Y$. (Imagine the coupons sliding along their respective strings, past the crossing.) We will typically use the sliding identities in the special case where either f or g is an identity morphism.

2.3 Super-mathematics

In this section, we briefly recall the basic definitions related to super vector spaces and related structures.

Definition 2.3.1. A *super vector space* is a $\mathbb{Z}_2$-graded vector space. Explicitly, a super vector space is an ordinary vector space V equipped with a decomposition into subspaces labelled by $\mathbb{Z}_2 = \{0, 1\}$, i.e. $V = V_0 \oplus V_1$. The elements of V_0 are called *even*, and the elements of V_1 are called *odd*. Elements that are either even or odd are called *homogeneous*. If $v \in V_i$, then we write $\bar{v} = i$, and say that i is the *parity* of v. (Some authors use the term "grade" instead of "parity".) The *even dimension* of a super vector space V is $\dim(V_0)$. Similarly, the *odd dimension* is $\dim(V_1)$.

It is important to note that, unless $V_0 = \{0\}$ or $V_1 = \{0\}$, most elements of a super vector space V are not homogeneous, and hence $\bar{v}$ is not defined for all $v \in V$. Throughout this thesis, whenever we state definitions or theorems involving parity terms $\bar{v}$, we implicitly assume that all of the involved elements are homogeneous. The full definition or theorem is then obtained by linear extension to the whole super vector space.

Definition 2.3.2. A linear map $f\colon V \to W$ between super vector spaces is called *parity-preserving* or *even* if $\overline{f(v)} = \bar{v}$ for all (homogeneous) $v \in V$. We say f is *parity-reversing* or *odd* if we instead have $\overline{f(v)} = 1 + \bar{v}$.

It follows that $\mathrm{Hom}_{\Bbbk}(V, W) = \mathrm{Hom}_{\Bbbk}(V, W)_0 \oplus \mathrm{Hom}_{\Bbbk}(V, W)_1$, where $\mathrm{Hom}_{\Bbbk}(V, W)_0$ consists of even linear maps and $\mathrm{Hom}_{\Bbbk}(V, W)_1$ consists of odd linear maps. Hence, for any (homogeneous) linear map $f\colon V \to W$, we have $\overline{f(v)} = \bar{f} + \bar{v}$.

Definition 2.3.3. Let V and W be super vector spaces. The tensor product (of ordinary vector spaces) $V \otimes W$ becomes a super vector space when equipped with the $\mathbb{Z}_2$-grading given by

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

Thus, the tensor product is parity-preserving, i.e. $\overline{v \otimes w} = \bar{v} + \bar{w}$. If $f\colon V \to W$ and $g\colon X \to Y$ are linear maps between super vector spaces, $f \otimes g\colon V \otimes X \to W \otimes Y$ is defined on simple tensors by

$$(f \otimes g)(v \otimes x) = (-1)^{\bar{g}\bar{v}} f(v) \otimes g(x). \tag{2.3.1}$$

When considering individual spaces, there is no distinction between a "super vector space" and a "$\mathbb{Z}_2$-graded vector space". The difference only arises when considering tensor products and braidings. The tensor product of objects is the same for both types of vector space (as given in Definition 2.3.3). The tensor product of linear maps is different; for super vector spaces, one uses (2.3.1), but for $\mathbb{Z}_2$-graded vector spaces, one uses the ordinary definition of $f \otimes g$, without the sign term $(-1)^{\bar{g}\bar{v}}$. The standard braiding for the category of $\mathbb{Z}_2$-graded vector spaces is the usual symmetric one, given by $B_{X,Y}(x \otimes y) = y \otimes x$. On the other hand, the braiding for the category of super vector spaces is given by the *supersymmetric braiding* $B_{X,Y}(x \otimes y) = (-1)^{\bar{x}\bar{y}} y \otimes x$. That is, x and y commute past each other without a sign if at least one of them is even, but if x and y are both odd, then $B_{X,Y}(x \otimes y) = -y \otimes x$.

In definitions and theorems, appropriate sign terms should be included whenever two elements from super vector spaces swap places. For instance, this is why the sign term $(-1)^{\bar{g}\bar{v}}$ appears in (2.3.1). Such sign terms appear automatically if one phrases everything explicitly in terms of the supersymmetric braiding.

There is a small technical complication: the full category of super vector spaces, with arbitrary linear maps as morphisms, is not actually a monoidal category. Due to the sign term in (2.3.1), the tensor product does not satisfy the identity needed to be a bifunctor when acting on pairs of odd morphisms. These signs mean that the tensor product is instead a *superbifunctor*, and the full category of super vector spaces is a *monoidal supercategory*. However, the category of super vector spaces with only even linear maps as morphisms *is* monoidal in the ordinary sense. All of the functors we will define in this thesis involve only even maps, so we can largely ignore the distinction between monoidal categories and monoidal supercategories. For further details, see e.g. [SS22, §2, §3.1] or [BE17].

Definition 2.3.4. A *superalgebra* is a super vector space A equipped with a parity-preserving associative bilinear multiplication map $A^2 \to A$.

Let A be a superalgebra. An *A-module* M is defined just like a module for an ordinary algebra, with the additional condition that M is a super vector space and the action of A on M is parity-preserving, i.e. $\overline{a \cdot m} = \bar{a} + \bar{m}$ for all $a \in A$ and $m \in M$.

Suppose M and N are both A-modules. A $\Bbbk$-linear map $f\colon M \to N$ is called *A-equivariant* if the following identity holds for all $a \in A$ and $m \in M$:

$$f(a \cdot m) = (-1)^{\bar{f}\bar{a}} a \cdot f(m).$$

We write A-mod for the category of A-modules, with morphisms being A-equivariant $\Bbbk$-linear maps.

Definition 2.3.5. A *Lie superalgebra* is a super vector space $\mathfrak{g}$ equipped with a parity-preserving bilinear Lie bracket $[-,-]\colon \mathfrak{g}^2 \to \mathfrak{g}$, satisfying the skew symmetry and Jacobi identities:

$$[x,y] = -(-1)^{\bar{x}\bar{y}}[y,x], \quad [x,[y,z]] + (-1)^{\bar{x}\bar{z}+\bar{y}\bar{z}}[z,[x,y]] + (-1)^{\bar{x}\bar{y}+\bar{x}\bar{z}}[y,[z,x]] = 0.$$

If 2 is not invertible in $\Bbbk$, we additionally require $[x,x] = 0$ for all even $x \in \mathfrak{g}$. If 3 is not invertible in $\Bbbk$, we require that $[[x,x],x] = 0$ for all odd $x \in \mathfrak{g}$. These identities follow from skew symmetry and the Jacobi identity if 2 (resp. 3) is invertible.

Examples 2.3.6. Let $m, n \in \mathbb{N}$. Write $\Bbbk^{m|n}$ for the $(m+n)$-dimensional super vector space with even dimension m and odd dimension n. The *general linear Lie superalgebra* $\mathfrak{gl}_{m|n}$ is the super vector space $\mathrm{Hom}_{\Bbbk}(\Bbbk^{m|n}, \Bbbk^{m|n})$ (usually realized as the space of $(m+n) \times (m+n)$ matrices with entries from $\Bbbk$), equipped with the (super)commutator Lie bracket:

$$[X,Y] := XY - (-1)^{\bar{X}\bar{Y}}YX.$$

The *special linear Lie superalgebra* $\mathfrak{sl}_{m|n}$ is the subalgebra of $\mathfrak{gl}_{m|n}$ consisting of elements with supertrace equal to 0. (Suppose $X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix}$ is an $(m+n) \times (m+n)$ matrix written in block form, where X_{00} is $m \times m$, X_{01} is $m \times n$, X_{10} is $n \times m$, and X_{11} is $n \times n$. The *supertrace* of X is $\mathrm{tr}(X_{00}) - \mathrm{tr}(X_{11})$, where tr denotes the ordinary trace of a matrix.)

2.4 Quantum arithmetic, enveloping algebras, and modules

In this section, we first recall the basic definitions and properties related to *quantum arithmetic*. We then do the same for the quantum enveloping algebras $U_q(\mathfrak{gl}_{m|n})$ and some of their associated modules.

Definition 2.4.1. For each $n \in \mathbb{Z}$, we define the *quantum integer*

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}].$$

It follows that $[-n] = -[n]$, and if $n \in \mathbb{N}$,

$$[n] = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} = \sum_{i=1}^{n} q^{n+1-2i} = \sum_{i=1}^{n} q^{-n-1+2i}. \tag{2.4.1}$$

For $n \in \mathbb{N}$, we define the *quantum factorial*

$$[n]! = [n][n-1]\cdots[2][1].$$

(If $n = 0$, the above product is empty, so $[0]! = 1$.)

Setting $q = 1$ in Definition 2.4.1 recovers the usual integers and factorials. For instance, $[2] = q + q^{-1}$, which evaluates to 2 after setting $q = 1$.

We warn the reader that there is another convention for quantum integers commonly seen in the literature: $[n] := \sum_{i=0}^{n-1} q^i$. As such, one needs to be careful when comparing statements about quantum arithmetic between different sources. Definition 2.4.1 is standard in the context of web categories.

From this point forward (except where noted otherwise), we work over one of two base rings, denoted $\Bbbk_q$ and $\Bbbk_{q,t}$. There are two options:

- Take $\Bbbk_q = Q(\Bbbk[q])$, the total ring of fractions of $\Bbbk[q]$. (If $\Bbbk$ is an integral domain, then $Q(\Bbbk[q])$ is equal to $\Bbbk(q)$, the ring of rational functions in q.) Take $\Bbbk_{q,t} = Q(\Bbbk[q])[t, t^{-1}]$.
- Take $\Bbbk_q$ to be the smallest subring of $Q(\Bbbk[q])$ containing the elements q, q^{-1}, and, for all nonzero $n \in \mathbb{Z}$, $\frac{1}{[n]}$. For ease of later reference, we denote this subring $\widehat{Q(\Bbbk[q])}$. Similarly, take $\Bbbk_{q,t}$ to be the smallest subring of $Q(\Bbbk[q, t])$ containing $q, q^{-1}, t, t^{-1}, \frac{t-t^{-1}}{q-q^{-1}}$, and, for all nonzero $n \in \mathbb{Z}$, $\frac{1}{[n]}$. We denote this subring $\widehat{Q(\Bbbk[q, t])}$.

With rare exceptions that will be explicitly pointed out, all definitions and results in this thesis are valid with either choice of base rings. Note that, for either option, we may view $\Bbbk_q$ as the subring of $\Bbbk_{q,t}$ generated by $\Bbbk$, q, q^{-1}, and $\frac{1}{[n]}$ for $n \neq 0$. When working with objects that do not involve t, we will state definitions and results over $\Bbbk_q$. All such results in this thesis also hold over $\Bbbk_{q,t}$ (with the same proofs), and we will use them when working over the larger base ring without further comment. For instance, we will define and study *(anti)symmetrizer morphisms* over $\Bbbk_q$ in §2.5, then use (anti)symmetrizers in a $\Bbbk_{q,t}$-linear context later on.

Remark 2.4.2. In the literature, web categories are typically defined over $\Bbbk(q)$, $\Bbbk(q, t)$, or $\Bbbk(q)[t, t^{-1}]$, most often with $\Bbbk = \mathbb{C}$. The advantage of instead working over the smaller subrings $\widehat{Q(\Bbbk[q])}$ and $\widehat{Q(\Bbbk[q, t])}$ is that it allows one to take $q = 1$, which is ill-defined over $\Bbbk(q)$, $\Bbbk(q, t)$, and $\Bbbk(q)[t, t^{-1}]$. Setting $q = 1$ recovers the classical (that is, non-quantum) versions of the definitions and theorems that we will state in this thesis.

Note that, since $\Bbbk_q$ and $\Bbbk_{q,t}$ each contain $\frac{1}{[n]}$ for all nonzero $n \in \mathbb{Z}$, in order for setting $q = 1$ to be well-defined, one must assume that $\Bbbk$ contains $\frac{1}{n}$ for each such n. We implicitly assume this is the case whenever we mention setting $q = 1$.

Definition 2.4.3. Let $a \mapsto \bar{a}$ denote the involution of $\Bbbk_{q,t}$ that acts as the identity on $\Bbbk$ and maps $q \mapsto q^{-1}, t \mapsto t^{-1}$ (not to be confused with our notation $\bar{v} \in \{0, 1\}$ for the parity of an element in a super vector space). We use the same notation for the restriction of this involution to $\Bbbk_q$.

Let $a \mapsto a^*$ denote the involution of $\Bbbk_{q,t}$ that acts as the identity on $\Bbbk[t, t^{-1}]$ and maps $q \mapsto -q^{-1}$. We use the same notation for the restriction of this involution to $\Bbbk_q$.

We say a $\Bbbk$-linear map $f\colon V \to W$ between $\Bbbk_q$-vector spaces is *q-antilinear* if

$$f(av_1 + bv_2) = \bar{a}f(v_1) + \bar{b}f(v_2)$$

for all $a, b \in \Bbbk_q$ and $v_1, v_2 \in V$. If V and W are $\Bbbk_{q,t}$-vector spaces and f satisfies the same identity for all $a, b \in \Bbbk_{q,t}$, we say that f is *(q,t)-antilinear.* If we instead have

$$f(av_1 + bv_2) = a^* f(v_1) + b^* f(v_2),$$

we say that f is *q-skew antilinear*, regardless of whether V and W are $\Bbbk_{q,t}$-vector spaces or $\Bbbk_q$-vector spaces.

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between $\Bbbk_q$-linear categories is q-antilinear if the action on each morphism space is q-antilinear. We define (q,t)-antilinear and q-skew antilinear functors in the same way.

Definition 2.4.4. For all $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we define the *quantum binomial coefficients*

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1]\cdots[a-b+1]}{[b][b-1]\cdots[1]} \in \Bbbk_q. \tag{2.4.2}$$

In the special case $b = 0$, the products in the numerator and denominator are empty and should be interpreted as 1, i.e. $\begin{bmatrix} a \\ 0 \end{bmatrix} = \frac{1}{1} = 1$. For $b \in \mathbb{Z}_{<0}$, we define $\begin{bmatrix} a \\ b \end{bmatrix} = 0$.

Note that when $a, b \in \mathbb{N}$ and $b \le a$, we have $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}$.

Lemma 2.4.5. For all $m, n \in \mathbb{Z}$, we have

$$\overline{[n]} = [n], \qquad [n]^* = (-1)^{n+1}[n], \tag{2.4.3}$$

$$([n]!)^* = (-1)^{\binom{n}{2}}[n]! \quad \text{if } n \in \mathbb{N}, \tag{2.4.4}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}^* = (-1)^{mn+n}\begin{bmatrix} m \\ n \end{bmatrix}. \tag{2.4.5}$$

Proof. First,

$$\overline{[n]} = \overline{\frac{q^n - q^{-n}}{q - q^{-1}}} = \frac{q^{-n} - q^n}{q^{-1} - q} = \frac{q^n - q^{-n}}{q - q^{-1}} = [n].$$

If n is even, then $(-q)^n = q^n$, and so:

$$[n]^* = \left(\frac{q^n - q^{-n}}{q - q^{-1}}\right)^* = \frac{(-q)^{-n} - (-q)^n}{-q^{-1} + q} = \frac{q^{-n} - q^n}{q - q^{-1}} = -[n] = (-1)^{n+1}[n].$$

If n is odd, then $(-q)^n = -q^n$, and so:

$$[n]^* = \left(\frac{q^n - q^{-n}}{q - q^{-1}}\right)^* = \frac{(-q)^{-n} - (-q)^n}{-q^{-1} + q} = \frac{q^n - q^{-n}}{q - q^{-1}} = [n] = (-1)^{n+1}[n].$$

For (2.4.4), we compute:

$$([n]!)^* = \prod_{i=1}^{n}[i]^* \overset{(2.4.3)}{=} \prod_{i=1}^{n}(-1)^{i+1}[i] = (-1)^{\sum_{i=1}^{n}(i+1)}[n]! = (-1)^{\binom{n+1}{2}+n}[n]! = (-1)^{\binom{n}{2}}[n]!$$

Finally, if $n < 0$ then (2.4.5) holds trivially. If $n \geq 0$, then:

$$\begin{aligned}\begin{bmatrix} m \\ n \end{bmatrix}^* &= \frac{[m]^*[m-1]^*\cdots[m-n+1]^*}{[n]^*[n-1]^*\cdots[1]^*} \\ &\overset{(2.4.3)}{=} (-1)^{\left(\sum_{i=1}^{n} m+2-i\right)+\left(\sum_{i=1}^{n} n+2-i\right)}\frac{[m][m-1]\cdots[m-n+1]}{[n][n-1]\cdots[1]} \\ &= (-1)^{mn+2n-\binom{n+1}{2}+n^2+2n-\binom{n+1}{2}}\begin{bmatrix} m \\ n \end{bmatrix} = (-1)^{mn+n}\begin{bmatrix} m \\ n \end{bmatrix}. \qquad \square\end{aligned}$$

Quantum integers and binomial coefficients satisfy the following useful identities, for all $a, b \in \mathbb{Z}$ and $c \in \mathbb{N}$. These facts are standard, and follow from straightforward computations; we include the statements here as a reference for readers who are less familiar with quantum arithmetic.

$$[0] = 0, \quad [1] = 1, \quad [a][b+1] - [a+1][b] = [a-b],$$

$$[a][b] - [a-1][b-1] = [a+b-1], \quad [a+b] = q^b[a] + q^{-a}[b] = q^{-b}[a] + q^a[b], \quad \begin{bmatrix} a \\ 1 \end{bmatrix} = [a],$$

$$\begin{bmatrix} c \\ c \end{bmatrix} = 1, \quad \begin{bmatrix} a \\ c \end{bmatrix} \cdot \frac{[a+1]}{[c+1]} = \begin{bmatrix} a+1 \\ c+1 \end{bmatrix}, \quad \sum_{i=0}^{c}(-q)^i\begin{bmatrix} a-i \\ c \end{bmatrix}\begin{bmatrix} c \\ i \end{bmatrix} = q^{-ca+c^2},$$

$$\begin{bmatrix} a+1 \\ c \end{bmatrix} = q^{a+1-c}\begin{bmatrix} a \\ c-1 \end{bmatrix} + q^{-c}\begin{bmatrix} a \\ c \end{bmatrix} = q^c\begin{bmatrix} a \\ c \end{bmatrix} + q^{-a-1+c}\begin{bmatrix} a \\ c-1 \end{bmatrix} \text{ if } c \geq 1,$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = q^c\begin{bmatrix} a-1 \\ c \end{bmatrix} + q^{c-a}\begin{bmatrix} a-1 \\ c-1 \end{bmatrix} \text{ if } c \geq 1, \quad \begin{bmatrix} a \\ c \end{bmatrix} = 0 \text{ if } 0 \leq a < c.$$

Throughout the rest of this section, fix dimension parameters $m, n \geq 0$, not both equal to 0. For $i \in \{1-n, \ldots, m\}$, we define $\bar{i} = \begin{cases} 0 & \text{if } i \geq 1, \\ 1 & \text{if } i \leq 0, \end{cases}$ and $q_i = q^{(-1)^{\bar{i}}} = \begin{cases} q & \text{if } i \geq 1, \\ q^{-1} & \text{if } i \leq 0. \end{cases}$

In the following, we recall the definition of the quantum enveloping superalgebra $U_q(\mathfrak{gl}_{m|n})$, largely following the conventions in [SSS26, §5]. This definition is only applicable for the choice of base ring $\Bbbk_q = Q(\Bbbk[q])$, since it involves division by $q - q^{-1}$, which is not invertible in $\widehat{Q(\Bbbk[q])}$. In Definition 2.4.13 we will describe an alternate form of the quantum enveloping superalgebra that is well-defined over either choice of base ring.

Definition 2.4.6. The *quantum enveloping superalgebra of* $\mathfrak{gl}_{m|n}$ is denoted $U_q(\mathfrak{gl}_{m|n})$, and defined to be the unital associative $Q(\Bbbk[q])$-superalgebra generated by the elements

$$\{E_i, F_i, D_i, D_i^{-1} \mid i \in \{1-n, \ldots, m-1\}\} \cup \{D_m, D_m^{-1}\},$$

with parities given by

$$\overline{E_i} = \overline{F_i} = \begin{cases} 0 & \text{if } i \neq 0, \\ 1 & \text{if } i = 0, \end{cases} \qquad \overline{D_i} = \overline{D_i^{-1}} = 0.$$

These generators are subject to the following relations, for all $i, j \in \mathbb{Z}$ such that the expressions are well-defined:

$$D_iD_j = D_jD_i, \quad D_iD_i^{-1} = 1 = D_i^{-1}D_i, \tag{2.4.6}$$

$$D_iE_j = q_i^{\delta_{ij}-\delta_{i,j+1}}E_jD_i, \quad D_iF_j = q_i^{\delta_{i,j+1}-\delta_{ij}}F_jD_i, \tag{2.4.7}$$

$$E_iF_j - (-1)^{\overline{E_i}\,\overline{F_j}}F_jE_i = \delta_{ij}\frac{D_iD_{i+1}^{-1} - D_{i+1}D_i^{-1}}{q_i - q_i^{-1}}, \tag{2.4.8}$$

$$\text{for all } |i-j| \geq 2, \ E_iE_j = E_jE_i \text{ and } F_iF_j = F_jF_i, \tag{2.4.9}$$

$$E_0^2 = F_0^2 = 0, \tag{2.4.10}$$

$$E_0E_{-1}E_0E_1 + E_{-1}E_0E_1E_0 + E_0E_1E_0E_{-1} + E_1E_0E_{-1}E_0 - [2]E_0E_{-1}E_1E_0 = 0, \tag{2.4.11}$$

$$F_0F_{-1}F_0F_1 + F_{-1}F_0F_1F_0 + F_0F_1F_0F_{-1} + F_1F_0F_{-1}F_0 - [2]F_0F_{-1}F_1F_0 = 0. \tag{2.4.12}$$

For $i \neq 0$ we also impose:

$$E_i^2E_{i+1} - [2]E_iE_{i+1}E_i + E_{i+1}E_i^2 = 0, \quad E_i^2E_{i-1} - [2]E_iE_{i-1}E_i + E_{i-1}E_i^2 = 0, \tag{2.4.13}$$

$$F_i^2F_{i+1} - [2]F_iF_{i+1}F_i + F_{i+1}F_i^2 = 0, \quad F_i^2F_{i-1} - [2]F_iF_{i-1}F_i + F_{i-1}F_i^2 = 0. \tag{2.4.14}$$

There is a natural Hopf superalgebra structure on $U_q(\mathfrak{gl}_{m|n})$. We will not make explicit use of this structure, but we state the definitions of the structure morphisms here for completeness. (The Hopf structure is used to define tensor products of modules, dual modules, and the trivial module; we omit all of these details for the sake of brevity.) The coproduct Δ, counit ε, and antipode S are given by:

$$\Delta(E_i) = E_i \otimes 1 + D_iD_{i+1}^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes D_{i+1}D_i^{-1} + 1 \otimes F_i, \quad \Delta(D_i) = D_i \otimes D_i,$$

$$\Delta(D_i^{-1}) = D_i^{-1} \otimes D_i^{-1}, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(D_i) = 1,$$

$$S(E_i) = -D_{i+1}D_i^{-1}E_i, \quad S(F_i) = -F_iD_iD_{i+1}^{-1}, \quad S(D_i) = D_i^{-1}, \quad S(D_i^{-1}) = D_i.$$

The subalgebra of $U_q(\mathfrak{gl}_{m|n})$ generated by the E_i, F_i, $K_i := D_iD_{i+1}^{-1}$, and $K_i^{-1} := D_{i+1}D_i^{-1}$ is $U_q(\mathfrak{sl}_{m|n})$, the *quantum enveloping superalgebra of* $\mathfrak{sl}_{m|n}$.

We write V^+ for the natural module of $U_q(\mathfrak{gl}_{m|n})$, and V^- for its dual. These modules respectively have $\Bbbk_q$-bases $\{v_i^+ : 1-n \leq i \leq m\}$ and $\{v_i^- : 1-n \leq i \leq m\}$, with parities given by $\overline{v_i^+} = \overline{v_i^-} = \bar{i}$. The action of $U_q(\mathfrak{gl}_{m|n})$ on V^+ and on V^- is given by:

$$E_iv_j^+ = \delta_{i+1,j}v_{j-1}^+, \quad F_iv_j^+ = \delta_{ij}v_{j+1}^+, \quad D_iv_j^+ = q_i^{\delta_{ij}}v_j^+,$$

$$E_iv_j^- = \delta_{ij}v_{j+1}^-, \quad F_iv_j^- = \delta_{i+1,j}(-1)^{\delta_{i0}}v_{j-1}^-, \quad D_iv_j^- = q_i^{-\delta_{ij}}v_j^-.$$

Remark 2.4.7. The quantum enveloping superalgebra $U_q(\mathfrak{gl}_{m|n})$ is often defined with respect to a different choice of indices. Namely, the generators E_i and F_i usually have i ranging over $\{1,\dots,m+n-1\}$ rather than $\{1-n,\dots,m-1\}$, and the generators D_j and D_j^{-1} usually have j ranging over $\{1,\dots,m+n\}$ rather than $\{1-n,\dots,m\}$. Following the approach in [SSS26, §5], we use the latter indices since this yields nicer formulas. One can translate between the two conventions as follows. Let $\hat{U}_q(\mathfrak{gl}_{m|n})$ denote the version of $U_q(\mathfrak{gl}_{m|n})$ generated with respect to the indices $\{1,\dots,m+n-1\}$ and $\{1,\dots,m+n\}$. See e.g. [TVW17, Def. 3.1] for a full definition (their generators $L_i^{\pm1}$ correspond to our $D_i^{\pm1}$). We will use hats to denote the generators for $\hat{U}_q(\mathfrak{gl}_{m|n})$. An isomorphism $f\colon U_q(\mathfrak{gl}_{m|n}) \to \hat{U}_q(\mathfrak{gl}_{m|n})$ is given by:

$$f(E_i) = (-1)^{\overline{E_i}}\hat{F}_{m-i}, \quad f(F_i) = (-1)^{\overline{F_i}}\hat{E}_{m-i}, \quad f(D_i) = \hat{D}_{m+1-i}^{-1}, \quad f(D_i^{-1}) = \hat{D}_{m+1-i}.$$

Direct calculations show that f is indeed a well-defined isomorphism. To understand why f maps E_i to $\pm\hat{F}_{m-i}$ rather than $\pm\hat{E}_{m-i}$ (and similarly for F_i), one should look at the classical case. Our isomorphism f is a quantum analogue of the map $\mathfrak{gl}_{m|n} \to \mathfrak{gl}_{n|m}$ given by reversing the order of both the columns and rows of a given matrix; this action naturally exchanges the roles of the usual generating matrices E_i and F_i.

Remark 2.4.8. The module V^- is isomorphic (as a $U_q(\mathfrak{gl}_{m|n})$-module) to the usual Hopf dual $(V^+)^* = \mathrm{Hom}_{\Bbbk_q}(V^+, \Bbbk_q)$, but we warn the reader that the obvious map $v_i^* \mapsto v_i^-$ (where v_i^* denotes the functional given by $v_i^*(v_j^+) = \delta_{ij}$) is *not* an isomorphism of $U_q(\mathfrak{gl}_{m|n})$-modules. An isomorphism $\phi\colon (V^+)^* \to V^-$ is given by:

$$\phi(v_i^*) = (-1)^{i+\bar{i}+1} q^{|i-1|} v_i^-.$$

Using the basis $\{v_i^- : 1-n \le i \le m\}$ for V^- yields nicer formulas than those obtained when using the usual dual basis $\{v_i^* : 1-n \le i \le m\}$ for $(V^+)^*$; this is why we work with V^-.

Remark 2.4.9. As noted above, Definition 2.4.6 cannot be used with the base ring $\Bbbk_q = \widehat{Q(\Bbbk[q])}$ since $q - q^{-1}$ is not invertible in that ring, and hence the relation (2.4.8) is not well-defined. One can modify the definition of $U_q(\mathfrak{gl}_{m|n})$ by introducing a new family of even generators H_i (for $i \in \{1-n,\dots,m-1\}$), and replacing (2.4.8) with

$$E_iF_j - (-1)^{\overline{E_i}\,\overline{F_j}}F_jE_i = \delta_{ij}H_i, \quad (q_i - q_i^{-1})H_i = D_iD_{i+1}^{-1} - D_{i+1}D_i^{-1}, \tag{2.4.15}$$

which is well-defined over $\widehat{Q(\Bbbk[q])}$. It is straightforward to see that, working over $Q(\Bbbk[q])$, the resulting superalgebra is isomorphic to the original $U_q(\mathfrak{gl}_{m|n})$. After replacing (2.4.8) with (2.4.15), we obtain an *integral form* of $U_q(\mathfrak{gl}_{m|n})$, wherein all of the coefficients of the defining relations belong to $\mathbb{Z}[q,q^{-1}]$ (and hence the relations are well-defined over $\widehat{Q(\Bbbk[q])}$). This is the appropriate definition of $U_q(\mathfrak{gl}_{m|n})$ when working over $\widehat{Q(\Bbbk[q])}$. We obtain a corresponding integral form of $U_q(\mathfrak{sl}_{m|n})$ by taking the subalgebra generated by the E_i, F_i, and $K_i = D_iD_{i+1}^{-1}$.

For the case of $U_q(\mathfrak{sl}_{m|n})$, it is straightforward to see that, after evaluating at $q = 1$ and imposing the additional relations $K_i = 1$ for all $i \in \{1-n,\dots,m-1\}$, the resulting superalgebra is isomorphic to the ordinary universal enveloping superalgebra $U(\mathfrak{sl}_{m|n})$; the relations involving the E_i and F_i reduce to precisely the usual relations for $U(\mathfrak{sl}_{m|n})$.

Unfortunately, we cannot recover $U(\mathfrak{gl}_{m|n})$ in the same way. After setting $q = 1$ and imposing $K_i = 1$ for all $i \in \{1-n, \dots, m-1\}$, we get that $D_i = D_j$ for all $i, j \in \{1-n, \dots, m\}$, and that D_1 is central and invertible. If $\Bbbk$ is a field of characteristic 0 and $m \neq n$, one can obtain $U(\mathfrak{gl}_{m|n})$ from $U(\mathfrak{sl}_{m|n})$ by adjoining one additional central generator (corresponding to the one-dimensional centre of $\mathfrak{gl}_{m|n}$), but that generator is not invertible in $U(\mathfrak{gl}_{m|n})$, so the quotient of $U_{q=1}(\mathfrak{gl}_{m|n})$ we just specified is not isomorphic to $U(\mathfrak{gl}_{m|n})$. If $\Bbbk$ is not a field of characteristic 0 or $m = n$, it is a bit more complicated to extend $U(\mathfrak{sl}_{m|n})$ to $U(\mathfrak{gl}_{m|n})$, since generally $\mathfrak{gl}_{m|n} \ncong \mathfrak{sl}_{m|n} \oplus \Bbbk I$. Regardless of the choice of $\Bbbk$, one cannot straightforwardly recover $U(\mathfrak{gl}_{m|n})$ from $U_q(\mathfrak{gl}_{m|n})$ by setting $q = 1$. The idempotent form of $U_q(\mathfrak{gl}_{m|n})$ that we describe in Definition 2.4.13 remedies this issue.

The idempotent form of $U_q(\mathfrak{gl}_{m|n})$, to be defined shortly, is closely linked to "type 1" weight modules.

Definition 2.4.10. We write $\mathbb{Z}^{m|n}$ for the additive group $\mathbb{Z}^{m+n}$, but with indices starting from $1 - n$ rather than 1 to match our conventions for $U_q(\mathfrak{gl}_{m|n})$. That is, if $\lambda \in \mathbb{Z}^{m|n}$, the first entry of λ is denoted λ_{1-n}, the second is $\lambda_{2-n}, \dots$, and the last entry is λ_m.

A *type 1 weight module* for $U_q(\mathfrak{gl}_{m|n})$ is a module that admits a decomposition into weight spaces,

$$W = \bigoplus_{\lambda \in \mathbb{Z}^{m|n}} W_\lambda,$$

with each generator D_i acting on W_λ as multiplication by $q_i^{\lambda_i}$.

Remark 2.4.11. While we will only work with type 1 weight modules in this thesis, we point out to the curious reader that the terminology "type 1" comes from the fact that there are 2^{m+n} types of weight modules in total, corresponding to choices of signs in the actions of the D_i on weight spaces. Precisely, if $s \in \{1, -1\}^{m+n}$ is a tuple encoding such a choice of signs (with indices starting at $1 - n$ like for $\mathbb{Z}^{m|n}$) and W is a type s weight module, then D_i acts on W_λ as multiplication by $s_i q_i^{\lambda_i}$. Thus, a type 1 weight module is equivalently a type $(1, 1, \dots, 1)$ weight module. For each type s, the category of type 1 weight modules is equivalent to the category of type s weight modules; an equivalence is given by tensoring with the one-dimensional representation upon which E_i and F_i act trivially, and D_i acts as multiplication by s_i. We work with type 1 weight modules in particular because they are the only type that form a monoidal subcategory of $U_q(\mathfrak{gl}_{m|n})$-mod; the tensor product of a type s and type r weight module is type sr, and the monoidal unit is type 1.

Examples 2.4.12. The natural module V^+ is a type 1 weight module for $U_q(\mathfrak{gl}_{m|n})$. We have $V^+ = \bigoplus_{i=1-n}^{m} V^+_{\epsilon_i}$, where $\epsilon_i \in \mathbb{Z}^{m|n}$ is the tuple with a 1 in position i and zeroes elsewhere. Each weight space $V^+_{\epsilon_i}$ is the span of v_i^+. Similarly, V^- is a type 1 weight module, with $V^- = \bigoplus_{i=1-n}^{m} V^-_{-\epsilon_i}$, and $V^-_{-\epsilon_i} = \mathrm{Span}(v_i^-)$.

If U and W are type 1 weight modules, then $U \otimes W$ is a type 1 weight module, with weight spaces given by $(U \otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} U_\mu \otimes W_\nu$, where μ and ν range over $\mathbb{Z}^{m|n}$.

Definition 2.4.13. *Lusztig's idempotent form* of the quantum enveloping superalgebra of $\mathfrak{gl}_{m|n}$ is the $\Bbbk_q$-linear category denoted $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$, whose objects are weights $\lambda \in \mathbb{Z}^{m|n}$, and whose morphisms are generated by

$$E_i \operatorname{id}_\lambda \colon \lambda \to \lambda + \alpha_i, \quad F_i \operatorname{id}_\lambda \colon \lambda \to \lambda - \alpha_i$$

for $i \in \{1-n, \dots, m-1\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1} \in \mathbb{Z}^{m|n}$ has a 1 in position i, a -1 in position $i+1$, and zeroes elsewhere. The parities of the generating morphisms are given by $\overline{E_i \operatorname{id}_\lambda} = \overline{F_i \operatorname{id}_\lambda} = \delta_{i0}$. These morphisms are subject to the following relations:

$$E_iF_j \operatorname{id}_\lambda - (-1)^{\overline{E_i}\,\overline{F_j}} F_jE_i \operatorname{id}_\lambda = \delta_{ij}[\lambda_i - (-1)^{\delta_{i0}}\lambda_{i+1}] \operatorname{id}_\lambda, \tag{2.4.16}$$

(For the avoidance of doubt, $[\lambda_i - (-1)^{\delta_{i0}}\lambda_{i+1}]$ denotes a quantum integer. In (2.4.16) and in what follows, we omit identity morphisms whose subscripts can be deduced from the rest of the expression. For instance, the first summand in (2.4.16) would be written in full as $E_i \operatorname{id}_{\lambda-\alpha_j} F_j \operatorname{id}_\lambda$; the omitted subscript $\lambda - \alpha_j$ is the codomain of $F_j \operatorname{id}_\lambda$.)

$$\text{for all } |i-j| \geq 2, \quad E_iE_j \operatorname{id}_\lambda = E_jE_i \operatorname{id}_\lambda \text{ and } F_iF_j \operatorname{id}_\lambda = F_jF_i \operatorname{id}_\lambda, \tag{2.4.17}$$

$$E_0^2 \operatorname{id}_\lambda = F_0^2 \operatorname{id}_\lambda = 0, \tag{2.4.18}$$

$$\begin{aligned} &E_0E_{-1}E_0E_1 \operatorname{id}_\lambda + E_{-1}E_0E_1E_0 \operatorname{id}_\lambda + E_0E_1E_0E_{-1} \operatorname{id}_\lambda \\ &\qquad + E_1E_0E_{-1}E_0 \operatorname{id}_\lambda - [2]E_0E_{-1}E_1E_0 \operatorname{id}_\lambda = 0, \end{aligned} \tag{2.4.19}$$

$$\begin{aligned} &F_0F_{-1}F_0F_1 \operatorname{id}_\lambda + F_{-1}F_0F_1F_0 \operatorname{id}_\lambda + F_0F_1F_0F_{-1} \operatorname{id}_\lambda \\ &\qquad + F_1F_0F_{-1}F_0 \operatorname{id}_\lambda - [2]F_0F_{-1}F_1F_0 \operatorname{id}_\lambda = 0. \end{aligned} \tag{2.4.20}$$

For $i \neq 0$ we also impose:

$$\begin{aligned} E_i^2E_{i+1} \operatorname{id}_\lambda - [2]E_iE_{i+1}E_i \operatorname{id}_\lambda + E_{i+1}E_i^2 \operatorname{id}_\lambda = 0, \\ E_i^2E_{i-1} \operatorname{id}_\lambda - [2]E_iE_{i-1}E_i \operatorname{id}_\lambda + E_{i-1}E_i^2 \operatorname{id}_\lambda = 0, \\ F_i^2F_{i+1} \operatorname{id}_\lambda - [2]F_iF_{i+1}F_i \operatorname{id}_\lambda + F_{i+1}F_i^2 \operatorname{id}_\lambda = 0, \\ F_i^2F_{i-1} \operatorname{id}_\lambda - [2]F_iF_{i-1}F_i \operatorname{id}_\lambda + F_{i-1}F_i^2 \operatorname{id}_\lambda = 0. \end{aligned} \tag{2.4.21}$$

A *$\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-module* is a $\Bbbk_q$-linear functor from $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ to the category of $\Bbbk_q$-super vector spaces. We write $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod for the category of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-modules, with natural transformations as morphisms.

Conceptually, $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ is a form of $U_q(\mathfrak{gl}_{m|n})$ partitioned according to its action on weight spaces. Given a $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-module R, the associated type 1 $U_q(\mathfrak{gl}_{m|n})$-weight module is $\bigoplus_{\lambda \in \mathbb{Z}^{m|n}} R(\lambda)$, with $R(\lambda)$ being the λ-weight space. The action of E_i on $R(\lambda)$ is given by $R(E_i \operatorname{id}_\lambda)$, and similarly for F_i. The category of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-modules is equivalent to the category of type 1 weight modules for $U_q(\mathfrak{gl}_{m|n})$; see [Lus10, §23] for details. All of the modules we will consider in this thesis are type 1 weight modules, so we are free to work with $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ rather than $U_q(\mathfrak{gl}_{m|n})$. The advantage of doing so is that we can take $q = 1$ in Definition 2.4.13 without any issues. Let W be any type 1 weight module for $U_q(\mathfrak{gl}_{m|n})$. When considering the action of $U_q(\mathfrak{gl}_{m|n})$ on the λ-weight space of W, the previously problematic relation (2.4.8) becomes

$$(E_iF_j - (-1)^{\overline{E_i}\,\overline{F_j}}F_jE_i)\,|_{W_\lambda} = \delta_{ij}\frac{q_i^{\lambda_i}q_{i+1}^{-\lambda_{i+1}} - q_{i+1}^{\lambda_{i+1}}q_i^{-\lambda_i}}{q_i - q_i^{-1}} = \delta_{ij}[\lambda_i - (-1)^{\bar{i}+\overline{i+1}}\lambda_{i+1}],$$

as in (2.4.16). This form of the relation is well-defined when taking $q = 1$. The category of modules for $\dot{\mathcal{U}}_{q=1}(\mathfrak{gl}_{m|n})$ is equivalent to the category of weight modules for $U(\mathfrak{gl}_{m|n})$ with integral weights. Hence, throughout the rest of the thesis, we can always recover the classical version of any construction involving $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ simply by setting $q = 1$ (with the choice of base ring $\Bbbk_q = \widehat{Q(\Bbbk[q])}$).

Remark 2.4.14. The definition of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ is sometimes given in terms of the *divided powers* $E_i^{(r)}\,\mathrm{id}_\lambda$ and $F_i^{(r)}\,\mathrm{id}_\lambda$, with $r \in \mathbb{N}$. In our setting, we can define $E_i^{(r)}\,\mathrm{id}_\lambda = \frac{E_i^r\,\mathrm{id}_\lambda}{[r]!}$ and $F_i^{(r)}\,\mathrm{id}_\lambda = \frac{F_i^r\,\mathrm{id}_\lambda}{[r]!}$ rather than introducing extra generators. One needs to explicitly include the $E_i^{(r)}$ and $F_i^{(r)}$ generators when working over $\mathbb{Z}[q, q^{-1}]$ or some other ring in which the quantum factorials $[r]!$ are not invertible.

The category $U_q(\mathfrak{gl}_{m|n})$-mod has a natural braided monoidal structure, depending on a choice of universal R-matrix $\mathcal{R} \in U_q(\mathfrak{gl}_{m|n})\hat{\otimes}U_q(\mathfrak{gl}_{m|n})$, where $\hat{\otimes}$ denotes a certain completion of the ordinary tensor product. We omit the details here for the sake of brevity, but emphasize that there are multiple possible choices, leading to different braided structures. We use the same choice of braiding/R-matrix as in [SSS26, §5]. Given $U_q(\mathfrak{gl}_{m|n})$-modules U and W, we write $B_{U,W}\colon U \otimes W \to W \otimes U$ for the associated braiding morphism. We will only make explicit use of the braiding for two copies of the natural module, as follows.

Definition 2.4.15. With respect to our chosen braided monoidal structure on $U_q(\mathfrak{gl}_{m|n})$-mod, the braiding morphism $B_{V^+,V^+}\colon V^+ \otimes V^+ \to V^+ \otimes V^+$ is given by

$$B_{V^+,V^+}(v_i^+ \otimes v_j^+) = \begin{cases} (-1)^{\bar{i}}q_iv_i^+ \otimes v_i^+ & \text{if } i = j, \\ (-1)^{\bar{i}\bar{j}}v_j^+ \otimes v_i^+ + (q - q^{-1})v_i^+ \otimes v_j^+ & \text{if } i < j, \\ (-1)^{\bar{i}\bar{j}}v_j^+ \otimes v_i^+ & \text{if } i > j. \end{cases} \tag{2.4.22}$$

Direct calculations show that $B_{V^+,V^+}^2 = \mathrm{id} + (q - q^{-1})B_{V^+,V^+}$, and the inverse of B_{V^+,V^+} acts as:

$$B_{V^+,V^+}^{-1}(v_i^+ \otimes v_j^+) = \begin{cases} (-1)^{\bar{i}}q_i^{-1}v_i^+ \otimes v_i^+ & \text{if } i = j, \\ (-1)^{\bar{i}\bar{j}}v_j^+ \otimes v_i^+ & \text{if } i < j, \\ (-1)^{\bar{i}\bar{j}}v_j^+ \otimes v_i^+ - (q - q^{-1})v_i^+ \otimes v_j^+ & \text{if } i > j. \end{cases} \tag{2.4.23}$$

Definition 2.4.16. Let W be a $U_q(\mathfrak{gl}_{m|n})$-module. We define the *(quantum-braided) symmetric superalgebra* $\mathrm{Sym}^\bullet W$ to be the quotient of the tensor superalgebra TW by the two-sided ideal generated by the following elements:

$$B_{W,W}(w) - qw \quad \text{for all } w \in W \otimes W. \tag{2.4.24}$$

The product in $\mathrm{Sym}^\bullet W$ is denoted by juxtaposition. Since $B_{W,W}$ is parity-preserving, $\mathrm{Sym}^\bullet W$ inherits a $\mathbb{Z}_2$-grading from TW, namely

$$\overline{w_1 w_2 \cdots w_d} = \overline{w_1} + \overline{w_2} + \cdots + \overline{w_d}.$$

For each $d \in \mathbb{N}$, we write $\mathrm{Sym}^d W$ for the subspace of $\mathrm{Sym}^\bullet W$ spanned by products of length d. We call $\mathrm{Sym}^d W$ the *dth symmetric power of* W. The $U_q(\mathfrak{gl}_{m|n})$-module structure on $\mathrm{Sym}^d W$ is the one induced by the $U_q(\mathfrak{gl}_{m|n})$-module structure on $W^{\otimes d}$.

Similarly, we define the *(quantum-braided) exterior superalgebra* $\Lambda^\bullet W$ to be the quotient of TW by the two-sided ideal generated by the following elements:

$$B_{W,W}(w) + q^{-1} w \quad \text{for all } w \in W \otimes W. \tag{2.4.25}$$

We will use the terms "exterior" and "alternating" interchangeably. The product in $\Lambda^\bullet W$ is denoted by $\wedge$. As in the symmetric case, we have

$$\overline{w_1 \wedge w_2 \wedge \cdots \wedge w_d} = \overline{w_1} + \overline{w_2} + \cdots + \overline{w_d}.$$

For each $d \in \mathbb{N}$, we write $\Lambda^d W$ for the subspace of $\Lambda^\bullet W$ spanned by products of length d. We call $\Lambda^d W$ the *dth exterior power of* W. These exterior powers inherit a $U_q(\mathfrak{gl}_{m|n})$-module structure from the one on $W^{\otimes d}$, as in the case of symmetric powers.

Remark 2.4.17. The definitions of $\mathrm{Sym}^d W$ and $\Lambda^d W$ differ by an application of the involution $q \mapsto -q^{-1}$. In Definition 3.1.8, we will introduce a functor that extends this involution to the $\mathfrak{gl}$-web category. This correspondence between symmetric and exterior/antisymmetric objects will be used extensively throughout this thesis.

2.5 Symmetrizers and antisymmetrizers

In this section, we define and study (quantum) *symmetrizer* and *antisymmetrizer* morphisms. Such morphisms have appeared many times in the literature, but often with different conventions, e.g. using the previously mentioned quantum integers $[n] := \sum_{i=0}^{n-1} q^i$ rather than the ones we specified in Definition 2.4.1. For the sake of completeness, we will provide proofs of some key identities related to (anti)symmetrizers here. Most of these identities have previously appeared in the literature in some form, but often without a full proof. We will use (anti)symmetrizers extensively in Chapter 4.

(Anti)symmetrizer morphisms in web categories are also referred to as "clasps"; this term was coined by Kuperberg in [Kup96]. We use the term "(anti)symmetrizer" to emphasize the connection to the Young (anti)symmetrizers that appear in the study of the representation theory of symmetric groups; our quantum (anti)symmetrizers are a somewhat more general, q-deformed version of those classical (anti)symmetrizers.

Definition 2.5.1. For $n \in \mathbb{N}$, the *Hecke algebra* H_n is the unital $\Bbbk_q$-algebra generated by the elements $t_1, \dots, t_{n-1}$ subject to the following relations:

$$t_i^2 = 1 + (q - q^{-1}) t_i \quad \text{for all } i \in \{1, \dots, n-1\}, \tag{2.5.1}$$

$$t_it_j = t_jt_i \quad \text{for all } i, j \in \{1, \ldots, n-1\} \text{ such that } |i-j| \geq 2, \tag{2.5.2}$$

$$t_it_{i+1}t_i = t_{i+1}t_it_{i+1} \quad \text{for all } i \in \{1, \ldots, n-2\}. \tag{2.5.3}$$

Note that in the special cases $n = 0$ and $n = 1$ there are no generators t_i, and so H_0 and H_1 are both the one-dimensional unital $\Bbbk_q$-algebra.

We extend the involution $-^*$ of $\Bbbk_q$ (see Definition 2.4.3) to a q-skew antilinear involution of H_n by setting $t_i^* = t_i$ for all $i \in \{1, \ldots, n-1\}$.

The following three lemmas state some basic properties of Hecke algebras. For proofs and a detailed treatment of these algebras, see e.g. [Mat99]. In the following and throughout the rest of this thesis, we write S_n for the symmetric group on n letters, $\tau_i \in S_n$ for the adjacent transposition that exchanges i and $i+1$, and $l(\sigma)$ for the minimal length among decompositions of a permutation σ into a product of adjacent transpositions, also called simply the *length* of σ.

Lemma 2.5.2. Let $n \geq 2$. Each generator $t_i \in H_n$ is invertible, with $t_i^{-1} = q^{-1} - q + t_i$.

Lemma 2.5.3. Let $n \geq 1$. If $\sigma = \tau_{i_1} \cdots \tau_{i_k}$ and $\sigma = \tau_{j_1} \cdots \tau_{j_k}$ are both minimal-length decompositions of a permutation $\sigma \in S_n$, then $t_{i_1} \cdots t_{i_k} = t_{j_1} \cdots t_{j_k}$ in H_n. Hence, for each $\sigma \in S_n$, there is a well-defined element $t_\sigma \in H_n$ given by $t_\sigma = t_{i_1} \cdots t_{i_k}$, where $\sigma = \tau_{i_1} \cdots \tau_{i_k}$ is any minimal-length decomposition of σ. In particular, $t_{\tau_i} = t_i$ for each i, and $t_{\mathrm{id}} = 1_{H_n}$. Moreover, the set $\{t_\sigma \mid \sigma \in S_n\}$ is a $\Bbbk_q$-basis for H_n.

Lemma 2.5.4. Let $n \geq 1$. If $\sigma, \kappa \in S_n$ satisfy $l(\sigma\kappa) = l(\sigma) + l(\kappa)$, then in H_n we have:

$$t_\sigma t_\kappa = t_{\sigma\kappa}. \tag{2.5.4}$$

Definition 2.5.5. For $n \geq 1$, we define the *(idempotent quantum) symmetrizer to be:*

$$e_n = \frac{q^{-\binom{n}{2}}}{[n]!} \sum_{\sigma \in S_n} q^{l(\sigma)} t_\sigma \in H_n, \tag{2.5.5}$$

and the *(idempotent quantum) antisymmetrizer* to be:

$$e'_n = \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} t_\sigma \in H_n. \tag{2.5.6}$$

Note that the exponent $\binom{n}{2}$ appearing in the definition of (anti)symmetrizers is the maximum length among elements of S_n.

It follows quickly from Lemma 2.4.5 that, for each $n \geq 1$, $e_n^* = e'_n$ and $(e'_n)^* = e_n$. In other words, mapping $q \mapsto -q^{-1}$ interchanges e_n and e'_n.

Lemma 2.5.6. Let $n \geq 1$. For all $i \in \{1, \ldots, n-1\}$ we have

$$t_ie_n = qe_n = e_nt_i, \qquad t_ie'_n = -q^{-1}e'_n = e'_nt_i. \tag{2.5.7}$$

More generally, for any $\sigma \in S_n$,

$$t_\sigma e_n = q^{l(\sigma)}e_n = e_nt_\sigma, \qquad t_\sigma e'_n = (-q)^{-l(\sigma)}e'_n = e'_nt_\sigma. \tag{2.5.8}$$

These identities are known as the *absorbing property* of quantum (anti)symmetrizers.

The following arguments are adapted from the proof of [Pag13, Thm. 4].

Proof. Fix $n \geq 1$ and let $i \in \{1, \ldots, n-1\}$. Define $D = \{\sigma \in S_n \mid l(\tau_i \sigma) = l(\tau_i) + l(\sigma)\}$ and $E = \{\sigma \in S_n \mid l(\tau_i \sigma) < l(\tau_i) + l(\sigma)\}$. We have that D and E are disjoint, and $D \cup E = S_n$. We claim that the map $\sigma \mapsto \tau_i \sigma$ is a bijection of S_n that interchanges D and E. Being a bijection is clear; this map is its own inverse. Noting that $l(\tau_i) = 1$, the definitions of D and E immediately imply that if $\sigma \in D$, $l(\tau_i \sigma) = l(\sigma) + 1$, and if $\sigma \in E$, $l(\tau_i \sigma) < l(\sigma)$. Since S_n is a Coxeter group, the inequality in the latter case implies that $l(\tau_i \sigma) = l(\sigma) - 1$.

If we had some $\sigma \in D$ such that $\tau_i \sigma \in D$, then we would get

$$l(\sigma) = l(\tau_i \tau_i \sigma) = l(\tau_i \sigma) + 1 = l(\sigma) + 2,$$

a contradiction. We get a similar contradiction if $\sigma, \tau_i \sigma \in E$. Thus the map $\sigma \mapsto \tau_i \sigma$ indeed interchanges D and E. Hence, S_n is the disjoint union of D and $E = \tau_i D$, and so:

$$\begin{aligned}
t_i e'_n = t_{\tau_i} \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in D} (-q)^{-l(\sigma)} t_\sigma + (-q)^{-l(\sigma)-1} t_{\tau_i \sigma} &\overset{(2.5.4)}{=} \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in D} (-q)^{-l(\sigma)} t_{\tau_i} t_\sigma + (-q)^{-l(\sigma)-1} t_{\tau_i} t_{\tau_i} t_\sigma \\
&\underset{(2.5.4)}{\overset{(2.5.1)}{=}} \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in D} (-q)^{-l(\sigma)} t_{\tau_i \sigma} + (-q)^{-l(\sigma)-1} (1 + (q - q^{-1}) t_{\tau_i}) t_\sigma \\
&\overset{(2.5.4)}{=} \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in D} (-q)^{-l(\sigma)-1} t_\sigma + ((-q)^{-l(\sigma)} - (-q)^{-l(\sigma)} + (-q)^{-l(\sigma)-2}) t_{\tau_i \sigma} \\
&= (-q)^{-1} \cdot \frac{q^{\binom{n}{2}}}{[n]!} \sum_{\sigma \in D} (-q)^{-l(\sigma)} t_\sigma + (-q)^{-l(\sigma)-1} t_{\tau_i \sigma} = -q^{-1} e'_n,
\end{aligned}$$

which establishes the first equality for antisymmetrizers in (2.5.7). The equality with multiplication by t_i on the right follows from essentially the same calculation, and the identities for symmetrizers follow after applying $-^*$.

For general $\sigma \in S_n$, let $\sigma = \tau_{i_1} \cdots \tau_{i_m}$ be a minimal-length decomposition. Then, by definition, $t_\sigma e'_n = t_1 \cdots t_m e'_n$, and similarly for e_n. The identity (2.5.8) then follows after using (2.5.7) $l(\sigma)$ times, and similarly for multiplication by t_σ on the right. $\square$

The following identities will be used later in this section to prove that e_n and e'_n are idempotent.

Lemma 2.5.7. For all $n \geq 1$, we have

$$\sum_{\sigma \in S_n} q^{2l(\sigma)} = \frac{[n]!}{q^{-\binom{n}{2}}}, \qquad \sum_{\sigma \in S_n} (-q)^{-2l(\sigma)} = \frac{[n]!}{q^{\binom{n}{2}}}. \tag{2.5.9}$$

Proof. It is well-known that the cardinality of $\{\sigma \in S_n : l(\sigma) = a\}$ is given by the coefficient of q^a in $\prod_{i=1}^{n} \sum_{j=0}^{i-1} q^j$. Hence $\sum_{\sigma \in S_n} q^{l(\sigma)} = \prod_{i=1}^{n} \sum_{j=0}^{i-1} q^j$. Replacing the variable q with q^2, we get

$$\sum_{\sigma\in S_n} q^{2l(\sigma)} = \prod_{i=1}^{n}\sum_{j=0}^{i-1} q^{2j} = \prod_{i=1}^{n} q^{i-1}\sum_{j=0}^{i-1} q^{-i+2j+1} \overset{(2.4.1)}{=} \prod_{i=1}^{n} q^{i-1}[i]$$
$$= [n]!q^{\sum_{i=1}^{n}(i-1)} = [n]!q^{\binom{n+1}{2}-n} = \frac{[n]!}{q^{-\binom{n}{2}}},$$

as desired. The second identity in (2.5.9) follows from the first after applying $-^*$. $\square$

Lemma 2.5.8. For all $n \in \mathbb{N}$, the following subspaces of H_n are one dimensional:

$$\{x \in H_n : t_i x = qx \text{ for all } i \in \{1,\dots,n-1\}\},$$
$$\{x \in H_n : t_i x = -q^{-1}x \text{ for all } i \in \{1,\dots,n-1\}\}.$$

Proof. We prove the claim about the first subspace; the claim about the second follows after applying $-^*$. Suppose $x = \sum_{\sigma\in S_n} a_\sigma t_\sigma \in H_n$ satisfies $t_i x = qx$ for all $i \in \{1,\dots,n-1\}$, or equivalently, $q^{-1}t_i x = x$. So we have

$$\sum_{\sigma\in S_n} q^{-1}a_\sigma t_i t_\sigma = \sum_{\sigma\in S_n} a_\sigma t_\sigma.$$

Using Lemma 2.5.4 and the exchange lemma for Coxeter groups, each product $t_i t_\sigma$ is either equal to $t_{\tau_i\sigma}$ (if $l(\tau_i\sigma) = 1 + l(\sigma)$) or to $t_{\tau_i\sigma} + (q+q^{-1})t_\sigma$ (if $l(\tau_i\sigma) < 1 + l(\sigma)$). Hence, for any $\kappa \in S_n$, if $l(\tau_i\kappa) = 1 + l(\kappa)$, then the coefficient of $t_{\tau_i\kappa}$ on the left hand side (relative to the basis $\{t_\sigma : \sigma \in S_n\}$) is $q^{-1}a_\kappa$, and so $q^{-1}a_\kappa = a_{\tau_i\kappa}$. It follows by induction on length that $a_\sigma = q^{l(\sigma)}a_{\mathrm{id}}$ for any $\sigma \in S_n$. This shows that any x satisfying our assumption is determined by the choice of scalar a_{id}, and hence the subspace in question is at most one dimensional. The symmetrizer e_n is nonzero and lies in this subspace by (2.5.7), so the dimension is 1. $\square$

Lemma 2.5.9. Fix some $n \geq 1$. Let $\rho_n \colon H_n \to H_n$ denote the $\Bbbk_q$-linear automorphism that acts on generating elements as $\rho_n(t_i) = -t_i^{-1}$. Then $\rho_n(e_n) = e_n'$ and $\rho_n(e_n') = e_n$.

Note that ρ_n is well-defined since it preserves all of the defining relations for H_n; this follows from straightforward calculations.

Proof. We will prove the first identity; the second follows from the fact that ρ_n is self-inverse. By (2.5.7) we have $t_i e_n = qe_n$ for all $i \in \{1,\dots,n-1\}$. Multiplying by $q^{-1}t_i^{-1}$ on both sides, we get $q^{-1}e_n = t_i^{-1}e_n$. With this in mind, we compute:

$$t_i\rho_n(e_n) = -\rho_n(t_i^{-1})\rho_n(e_n) = -\rho_n(t_i^{-1}e_n) = -q^{-1}\rho_n(e_n).$$

By Lemma 2.5.8, the subspace of elements $x \in H_n$ that satisfy the identity $t_i x = -q^{-1}x$ for all $i \in \{1,\dots,n-1\}$ is one dimensional, and that space contains e_n' by (2.5.7). Hence, $\rho_n(e_n) = ae_n'$ for some $a \in \Bbbk_q$. Using the fact that $\rho_n(t_\sigma) = (-1)^{l(\sigma)}t_{\sigma^{-1}}^{-1}$, which follows from a straightforward induction on the length of σ, we get:

$$\frac{q^{-\binom{n}{2}}}{[n]!}\sum_{\sigma\in S_n}(-q)^{l(\sigma)}t_{\sigma^{-1}}^{-1} = a\frac{q^{\binom{n}{2}}}{[n]!}\sum_{\sigma\in S_n}(-q)^{-l(\sigma)}t_\sigma.$$

Applying the sign character (given by $t_i \mapsto -q^{-1}$ for all i) to both sides then yields:

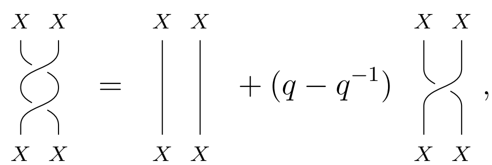
$$\frac{q^{-\binom{n}{2}}}{[n]!}\sum_{\sigma\in S_n}(-q)^{2l(\sigma)} = a\frac{q^{\binom{n}{2}}}{[n]!}\sum_{\sigma\in S_n}(-q)^{-2l(\sigma)}.$$

Using the second identity in (2.5.9), the right hand side simplifies to a. Applying $\bar{\cdot}$ and then $-^*$ to the first identity from (2.5.9), we get that the left hand side is equal to:

$$\frac{q^{-\binom{n}{2}}}{[n]!}\left(\frac{\overline{[n]!}}{\overline{q^{-\binom{n}{2}}}}\right)^{*} \overset{(2.4.3)}{=} \frac{q^{-\binom{n}{2}}}{[n]!}\left(\frac{[n]!}{q^{\binom{n}{2}}}\right)^{*} \overset{(2.4.4)}{=} \frac{q^{-\binom{n}{2}}}{[n]!}\cdot\frac{(-1)^{\binom{n}{2}}[n]!}{(-q)^{-\binom{n}{2}}} = 1.$$

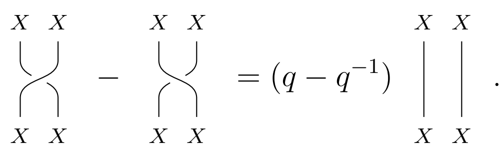
Hence $a = 1$, and so $\rho_n(e_n) = e_n'$, as desired. $\square$

Definition 2.5.10. We say that an object X in a $\Bbbk_q$-linear monoidal braided category is *Hecke-braided* if the following identity holds:



(2.5.10)

or, equivalently (after composing with a negative crossing and simplifying),

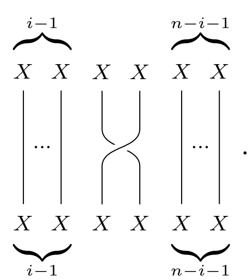


(2.5.11)

If X is Hecke-braided, then for each $n \geq 1$ there is a homomorphism of algebras

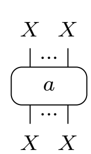
$$h_{X,n}\colon H_n \to \mathrm{End}(X^{\otimes n}), \tag{2.5.12}$$

which we refer to as the *Hecke-braid homomorphism*, given by sending each generator t_i to the positive crossing between the ith and $(i+1)$th strands, i.e.

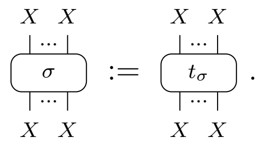
$h_{X,n}(t_i) =$ 

(2.5.13)

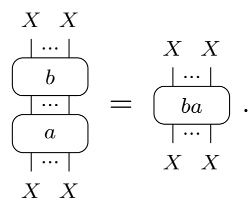
For each $a \in H_n$, we draw the $h_{X,n}$-image of a as:

$h_{X,n}(a) =$  $\in \mathrm{End}(X^{\otimes n}).$

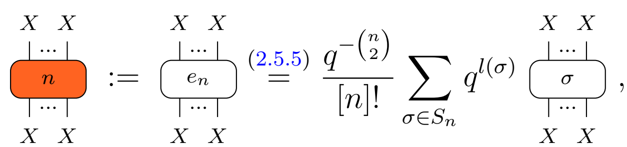
When using this notation, we will usually write just σ instead of t_σ (for $\sigma \in S_n$) for the sake of simplicity, i.e. we set

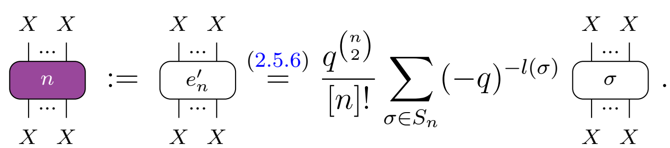


Note that, since $h_{X,n}$ is a homomorphism, for all $a, b \in H_n$ we have

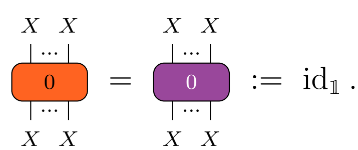


We use the following special notation for the images of the symmetrizer and antisymmetrizer:





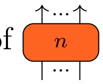
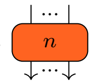
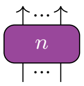
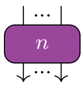
We will also refer to these image morphisms as (anti)symmetrizers. To allow for more uniform statements of identities, we also define:



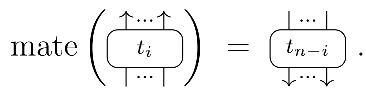
(Recall that $\mathbb{1}$ denotes the monoidal unit object.)

Examples 2.5.11. The natural $U_q(\mathfrak{gl}_{m|n})$-module V^+ is Hecke-braided with respect to the braiding specified in Definition 2.4.15. For each $d \geq 1$, the Hecke-braid homomorphism $h_{V^+,d}\colon H_d \to \operatorname{End}((V^+)^{\otimes d})$ is always surjective, and is in fact an isomorphism whenever $d \leq m+n$. These claims follow from [Bru17, Thm. 1.2].

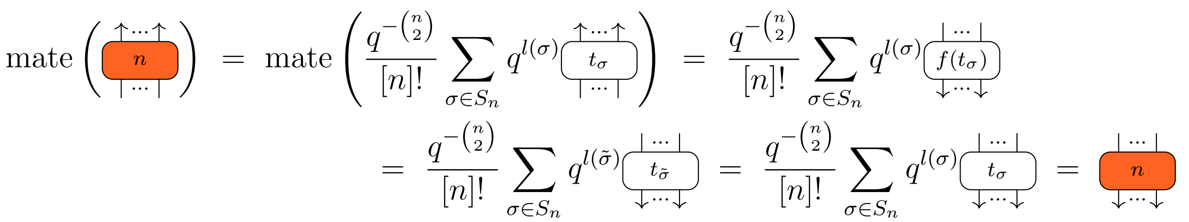
If $(\uparrow, \downarrow)$ is a dual pair of objects and $\uparrow$ is Hecke-braided, then $\downarrow$ is also Hecke-braided. This follows from the fact that the positive crossing for two copies of $\downarrow$ must be the mate of the positive crossing for two copies of $\uparrow$ (see Definition A.5 for the definition of "mate"), which is a consequence of the axioms of a braided monoidal category. In particular, we have that V^- is Hecke-braided.

Lemma 2.5.12. Suppose that $(\uparrow, \downarrow)$ is a dual pair of Hecke-braided objects in a strict pivotal category (see Definition A.7). For each $n \geq 1$, the mate of  is , and the mate of  is .

Proof. Fix some $n \geq 1$. As noted above, the positive crossing for two copies of $\downarrow$ is the mate of the positive crossing for two copies of $\uparrow$. Using the fact that taking mates reverses the order of tensor products, this implies that, for each $i \in \{1, \dots, n-1\}$,

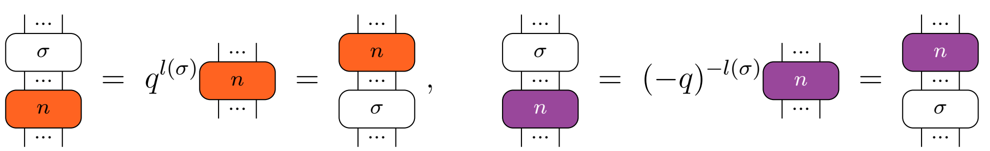


The action $t_i \mapsto t_{n-i}$ induces an automorphism f of H_n such that for all $\sigma \in S_n$, $f(t_\sigma) = t_{\tilde{\sigma}}$, where $\tilde{\sigma}$ denotes the conjugation of σ by the permutation that maps i to $n-i+1$ for each i. Note that we have $l(\tilde{\sigma}) = l(\sigma)$ since the map $\sigma \mapsto \tilde{\sigma}$ simply permutes the adjacent transpositions $\tau_1, \dots, \tau_{n-1}$. Thus:

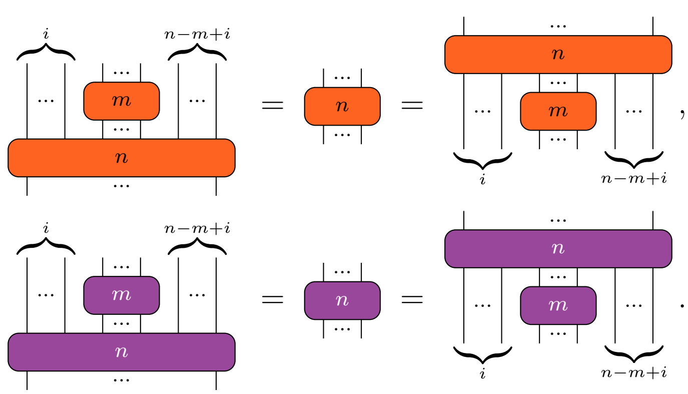


where in the second-to-last equality we use the fact that $\sigma \mapsto \tilde{\sigma}$ is an automorphism of S_n. The identity for antisymmetrizers follows after applying $-^*$. $\square$

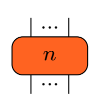
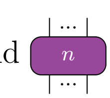
Lemma 2.5.13. Let $|$ be a Hecke-braided object. For all $1 \leq m \leq n$, all $i \in \mathbb{N}$ such that $i + m \leq n$, and all $\sigma \in S_n$ we have:



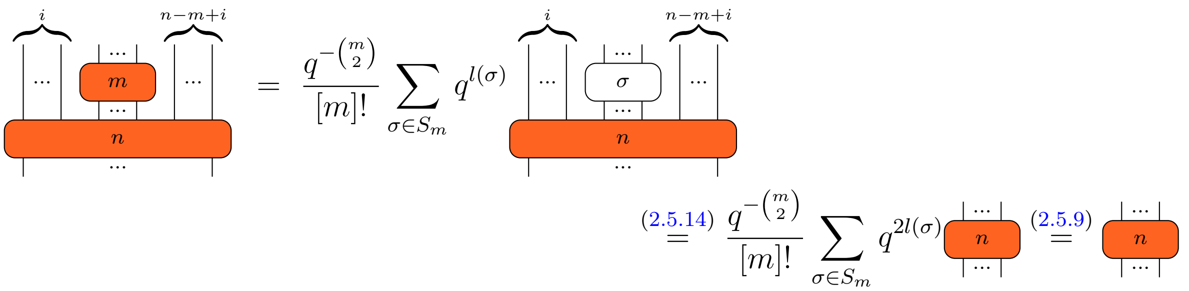
(2.5.14)



(2.5.15)

In particular, taking $m = n$ and $i = 0$ in (2.5.15), we get that  and  are idempotent.

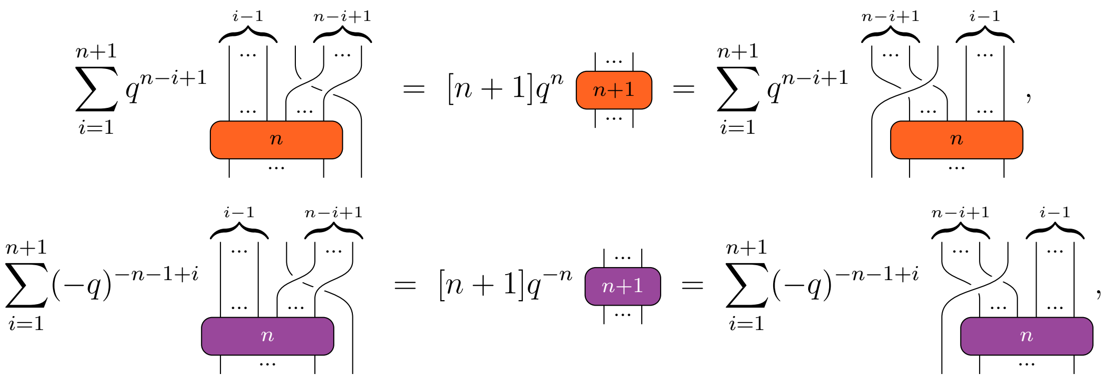
Proof. The first identity is simply the image of (2.5.8) via the Hecke-braid homomorphism $h_{|,n}$. For (2.5.15), we compute:



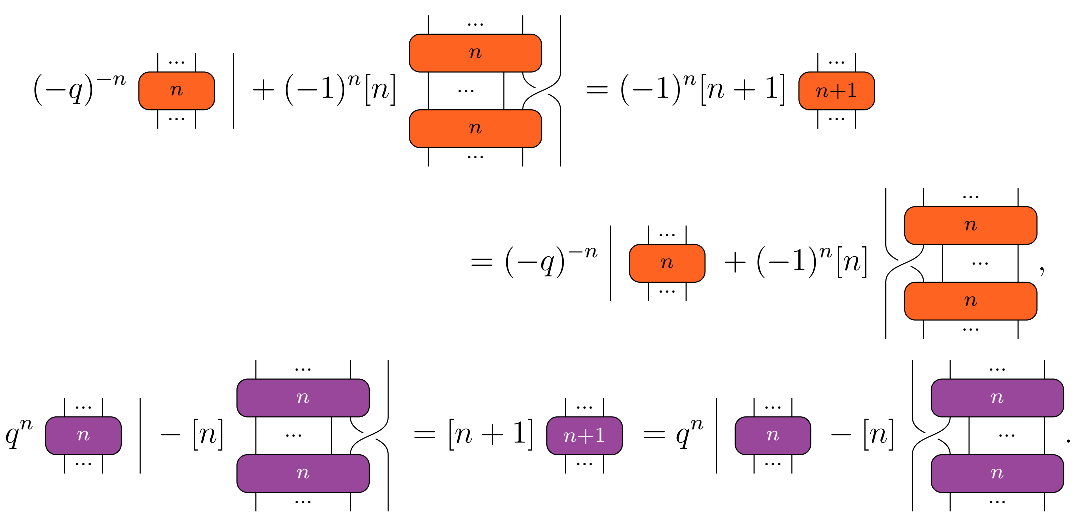
The second equality follows from essentially the same computation. The identities for anti-symmetrizers follow after applying $-^*$. $\square$

The following recursive formulas for (anti)symmetrizers will be used in the proof of Proposition 3.3.23, which forms the foundation for many results later in this thesis.

Lemma 2.5.14. Let $|$ be a Hecke-braided object. For each $n \in \mathbb{N}$ we have:



(2.5.16)

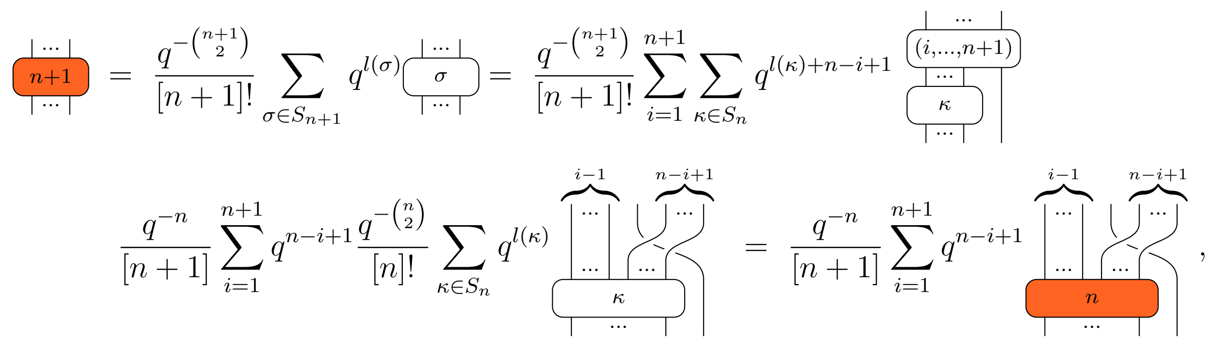


(2.5.17)

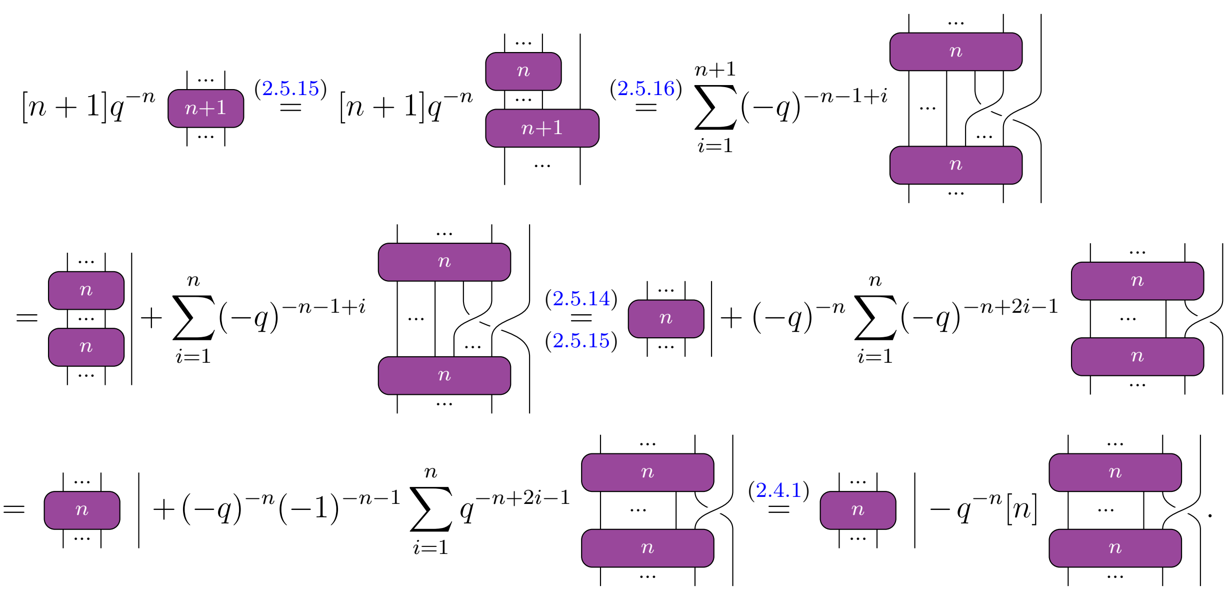
Proof. Every permutation $\sigma \in S_{n+1}$ has a unique factorization of the form

$$\sigma = (i, i+1, \ldots, n, n+1) \circ \kappa$$

for some $\kappa \in S_n \subseteq S_{n+1}$ and some $i \in \{1, \ldots, n+1\}$. Straightforward arguments then show that $l(\sigma) = l(i, \ldots, n+1) + l(\kappa) = l(\kappa) + n - i + 1$, and hence, by Lemma 2.5.4, $t_\sigma = t_{(i,\ldots,n+1)} t_\kappa$. Using these facts, we compute:



which proves one of the equalities in (2.5.16) after multiplying by $[n+1]q^n$. The second follows from a similar calculation, and the antisymmetrizer identities then follow after applying $-^*$. We will prove the antisymmetrizer part of (2.5.17); the symmetrizer part follows after applying $-^*$. We compute:



(In the fourth equality, we use (2.5.14) to absorb $n-i$ crossings from the middle of the diagram into the top antisymmetrizer.) This proves the first equality in (2.5.17) after dividing by q^{-n}. The second equality follows from totally analogous arguments. $\square$

Chapter 3

General linear web categories

In this chapter, we will define and establish some fundamental properties of the main objects of study in this thesis: the $\mathfrak{gl}$-web categories. Such web categories have appeared in many forms in the literature. Our definitions incorporate elements from various prior approaches, taking particular inspiration from the presentations in [TVW17] and [QS19].

3.1 Definitions and symmetries

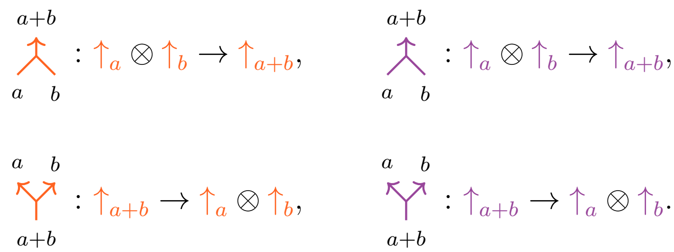
In this section, we introduce the two-colour $\mathfrak{gl}$-web categories $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. The former is isomorphic to the web category studied in [TVW17] (we will discuss this further in Connection 5.4.3), and the latter expands $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ by adjoining duals for each of the objects, extending the approach from [QS19] to the two-colour setting. We then define several functors that reflect different types of symmetry exhibited by these categories. Analogous symmetries have been studied for many diagrammatic categories, but to the best knowledge of the present author, our functors corresponding to vertical reflection, 180° rotation, and colour-swapping have not been discussed in the literature in the context of web categories (though a different colour-swap symmetry has been employed; see Connection 5.4.3).

Definition 3.1.1. The *(two-colour) $\mathfrak{gl}$-web category*, which we denote $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$, is a $\Bbbk_q$-linear strict monoidal category whose objects are generated (via tensor product) by:

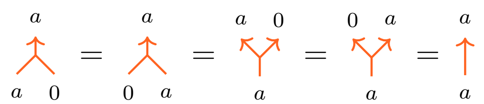
$$\{{\color{orange}\uparrow_a} : a \in \mathbb{N}_{>1}\} \cup \{{\color{purple}\uparrow_a} : a \in \mathbb{N}_{>1}\} \cup \{\uparrow_1\}.$$

We write $\uparrow_0$ for the monoidal unit of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. For the sake of convenience, we also set ${\color{orange}\uparrow_0} = {\color{purple}\uparrow_0} = \uparrow_0$ and ${\color{orange}\uparrow_1} = {\color{purple}\uparrow_1} = \uparrow_1$, i.e. strands labelled by 0 or 1 may be drawn in an arbitrary colour. We will sometimes refer to the label of a strand as its *thickness*. Strands of thickness 1 are called *thin*. We will discuss the intuitive meaning of these variously coloured strands just after the present definition.

The morphisms in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ are generated (via tensor product, i.e. horizontal juxtaposition, composition, i.e. vertical stacking, and taking $\Bbbk_q$-linear combinations) by those of the following types, for $a, b \in \mathbb{N}$:

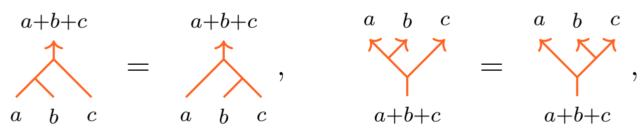


The first two are called *merge* morphisms, and the latter two are called *split* morphisms. These generators are subject to the following relations (for all $a, b, c \in \mathbb{N}$), together with the analogous relations obtained by changing the colour of all strands to be purple rather than orange:

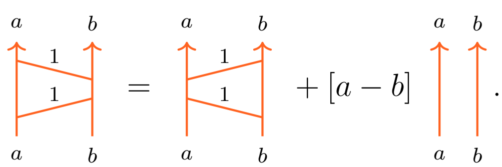


(TRIV)

("TRIV" is short for "trivial merge/split".)

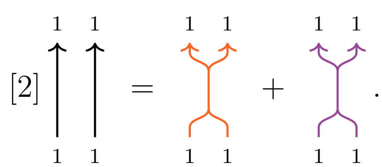


(ASSOC)

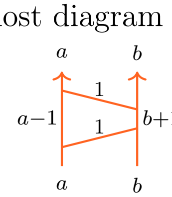


(MIRROR)

We also impose the following mixed-colour relation:



(MIX)

Note that, to avoid visual clutter, we often omit labels on strands that can be inferred from other labels. For instance, we left two labels implicit in the leftmost diagram of (MIRROR). Drawing all labels explicitly, that diagram would be depicted as  .

We adopt the following useful convention: any diagram involving a string labelled, either explicitly or implicitly, by a negative number is (by definition) equal to zero. This allows us to state and work with families of relations in a concise way, without needing to explicitly note constraints on the variables appearing in those relations. In fact, we already employed this convention when stating (MIRROR); some of the diagrams appearing in that relation would otherwise be undefined when $a = 0$ or $b = 0$ due to the presence of a strand labelled by -1. With this convention, it immediately follows that the identities in (TRIV), (ASSOC), and (MIRROR) hold more generally for all $a, b, c \in \mathbb{Z}$, i.e. where we allow a, b, and/or c to be negative.

The category $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ is closely related to the representation theory of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$, and it is often helpful to think about the properties of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and related categories through this lens. To aid readers who are not already familiar with web categories, we will now briefly outline some of the key connections.

As we will discuss in further detail in Chapter 6, for all $m, n \in \mathbb{N}$ there is an *incarnation functor* $I_{m|n} \colon \mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod which is known to be full under certain assumptions on $\Bbbk$, m, and n, and that we conjecture to be full more generally. For each $a \in \mathbb{N}$, $I_{m|n}$ sends $\uparrow_a$ to the symmetric power $\mathrm{Sym}^a V^+$ and $\uparrow_a$ to the exterior power $\Lambda^a V^+$ (recall that V^+ denotes the natural representation of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$). As such, $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ can be thought of as a diagrammatic version of the full subcategory of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-modules generated by such symmetric and alternating powers, for all choices of m and n at once. This also explains why we allow strands labelled by 0 or 1 to be drawn in an arbitrary colour; we have $\mathrm{Sym}^1 V^+ \cong V^+ \cong \Lambda^1 V^+$ and (by convention) $\mathrm{Sym}^0 V^+ \cong \mathbb{1} \cong \Lambda^0 V^+$, where $\mathbb{1}$ denotes the trivial representation.

The $I_{m|n}$-image of  is the morphism $\mathrm{Sym}^a V^+ \otimes \mathrm{Sym}^b V^+ \to \mathrm{Sym}^{a+b} V^+$ given by

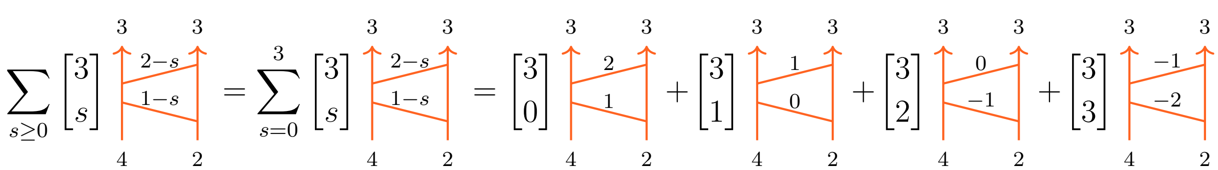
$$(v^+_{i_1} \cdots v^+_{i_a}) \otimes (v^+_{j_1} \cdots v^+_{j_b}) \mapsto v^+_{i_1} \cdots v^+_{i_a} v^+_{j_1} \cdots v^+_{j_b},$$

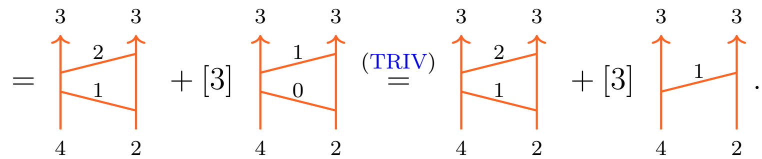
and similarly for purple merges. The images of split morphisms are more complicated to describe; we will fully specify such images in Chapter 6. In the purely even classical case, i.e. where $q = 1$ and $n = 0$, the $I_{m|0}$-image of  acts as:

$$v^+_{i_1} \dots v^+_{i_{a+b}} \mapsto \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} (v^+_{i_{\sigma(1)}} \cdots v^+_{i_{\sigma(a)}}) \otimes (v^+_{i_{\sigma(a+1)}} \cdots v^+_{i_{\sigma(a+b)}}).$$

The image of  has a similar action, with a factor of $\mathrm{sgn}(\sigma)$ in the sum. Many identities in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ become intuitively clear after taking $I_{m|n}$-images; for instance, the first relation in (ASSOC) corresponds to the fact that the symmetric superalgebra $\mathrm{Sym}^{\bullet} V^+$ is associative.

Example 3.1.2. Using our convention for negative labels in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$, we have:



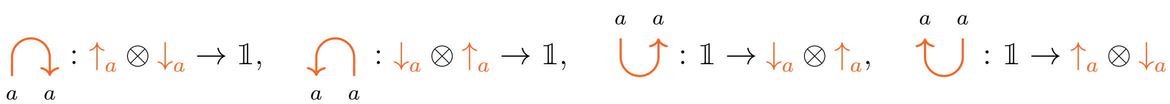


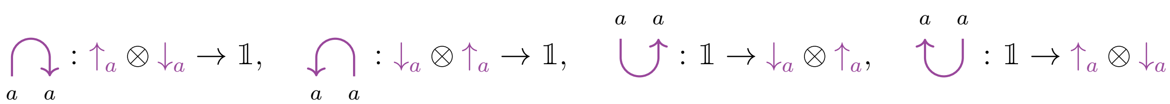
(This sum is the right hand side of the upcoming identity (3.2.5), with parameters $a = 4$, $b = 2$, $i = 2$, and $j = 1$.)

Definition 3.1.3. The *(two-colour)* $\mathfrak{gl}$*-web category with duals*, which we denote $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, is the category obtained from $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ by adjoining new generating objects

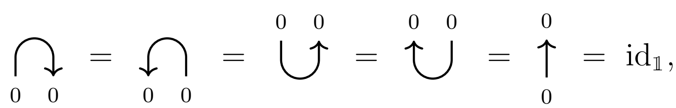
$$\{\downarrow_a : a \in \mathbb{N}_{>1}\} \cup \{\downarrow_a : a \in \mathbb{N}_{>1}\} \cup \{\downarrow_1\}$$

and generating morphisms

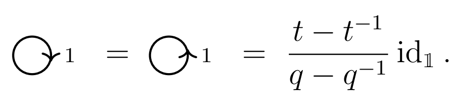




(for all $a \in \mathbb{N}$) subject to the following relations (for all $a, b \in \mathbb{N}$). We adopt the convention that a string labelled by 0 in either orientation (and in any colour) is equal to the monoidal unit, so $\uparrow_0 = \downarrow_0 = \mathbb{1}$. We also define $\downarrow_1 = \downarrow_1 = \downarrow_1$. The first two relations apply to black strands:



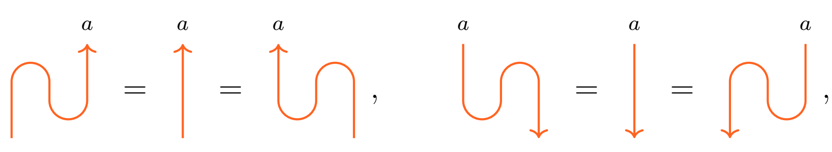
(TRIV2)



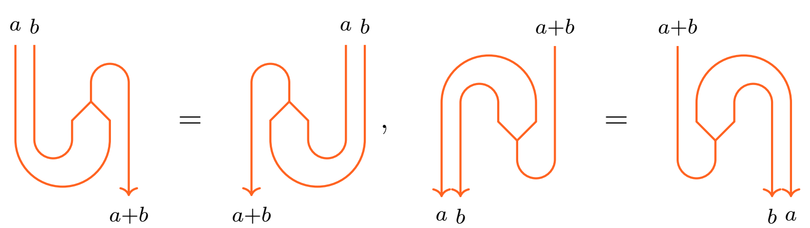
(SPEC)

("SPEC" is short for "specialization".)

The following two relations are imposed on both orange and purple strands; we draw only the orange versions for brevity.

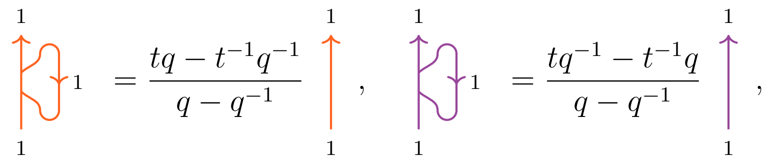


(ZIGZAG)

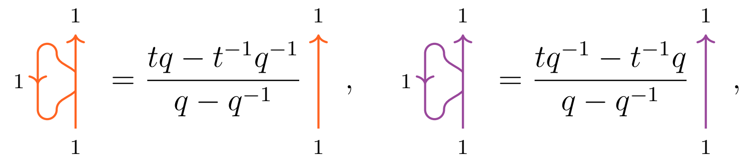


(MATE)

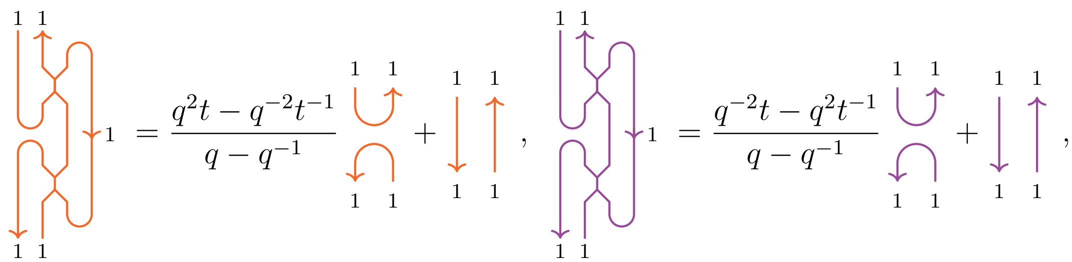
The final set of relations feature slight variations between the two colours:



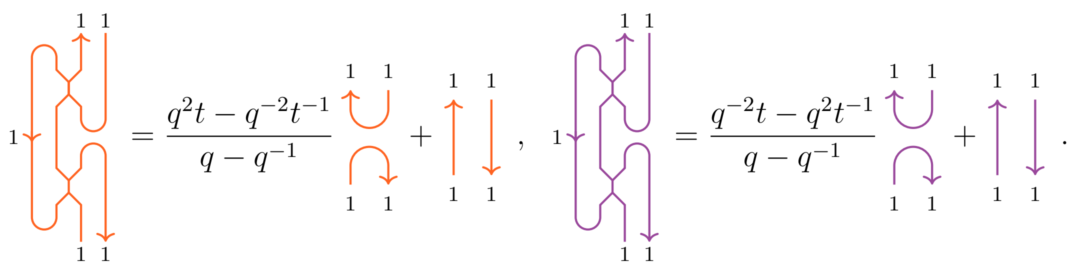
(RBLOB)



(LBLOB)



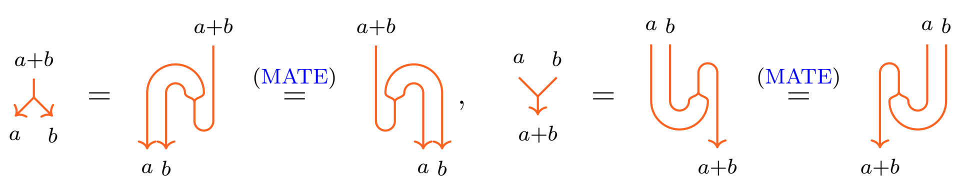
(RSQUIG)



(LSQUIG)

("SQUIG" is short for "squiggle".)

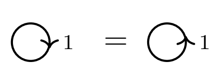
We define the *rotated split* and *rotated merge* morphisms by:



(3.1.1)

and the same for purple strands.

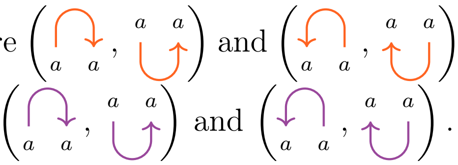
Remark 3.1.4. One should think of the parameter t appearing in the defining relations for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ as being equal to q^δ for some $\delta \in \mathbb{Z}$. After setting $t = q^\delta$ (as we will do in Chapter 6), the specialization relation (SPEC) simplifies to

 $= [\delta]\,\mathrm{id}_{\mathbb{1}}$,

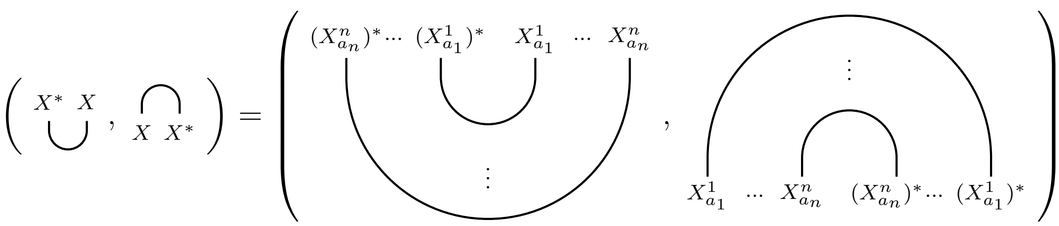
which, by definition, says that $\uparrow_1$ and $\downarrow_1$ each have *categorical dimension* equal to $[\delta]$.

Lemma 3.1.5. The category $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ is strict pivotal (see Definition A.7) with respect to the following duality data:

- For each $a \in \mathbb{N}$, we set $(\uparrow_a)^* = \downarrow_a$, $(\downarrow_a)^* = \uparrow_a$, $(\uparrow_a)^* = \downarrow_a$, and $(\downarrow_a)^* = \uparrow_a$.
- The chosen cap-cup pairs for the dual pair $(\uparrow_a, \downarrow_a)$ are … and … . Similarly, the chosen cup-cap pairs for $(\uparrow_a, \downarrow_a)$ are … and … .



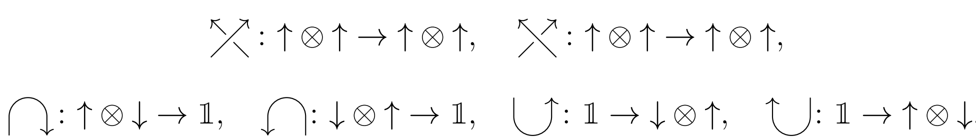
- The dual of a general object $X^1_{a_1} \otimes \cdots \otimes X^n_{a_n}$ (where we have $X^i \in \{\uparrow, \downarrow, \uparrow, \downarrow\}$ and $a_i \in \mathbb{N}$) is $(X^n_{a_n})^* \otimes \cdots \otimes (X^1_{a_1})^*$.
- For a general dual pair $(X, X^*) = (X^1_{a_1} \otimes \cdots \otimes X^n_{a_n}, (X^n_{a_n})^* \otimes \cdots \otimes (X^1_{a_1})^*)$, the chosen cup-cap pair is given by nesting, i.e.



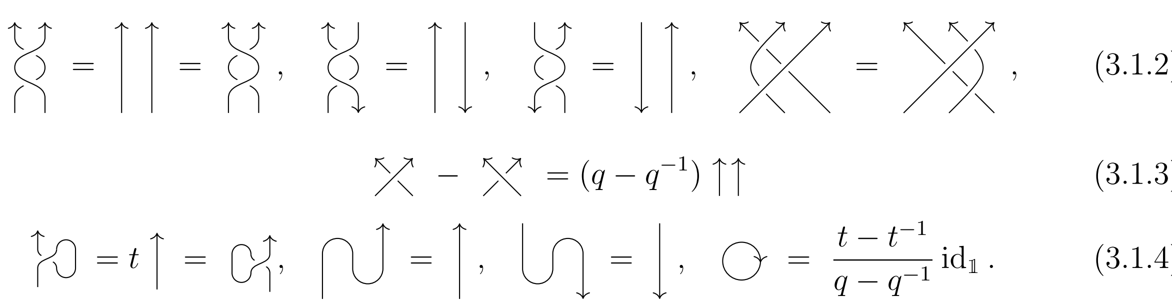
Proof. It is immediate that the first four properties in Definition A.7 are satisfied. For the final property regarding mates, recall that the morphisms in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ are generated by splits, merges, cups, and caps, and so it suffices to show that left and right mates coincide for each of these generators. For cups and caps this follows immediately from (ZIGZAG). For splits and merges, this is precisely given by the defining relation (MATE). □

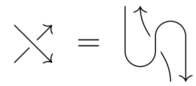
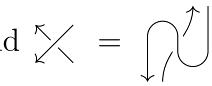
We now introduce another diagrammatic category that will play an important role in Chapter 4, and will also help motivate the upcoming Definition 3.1.8.

Definition 3.1.6. The *oriented skein category*, denoted $\mathcal{OS}$, is the strict $\Bbbk_{q,t}$-linear monoidal category with two generating objects, $\uparrow$ and $\downarrow$, and generating morphisms

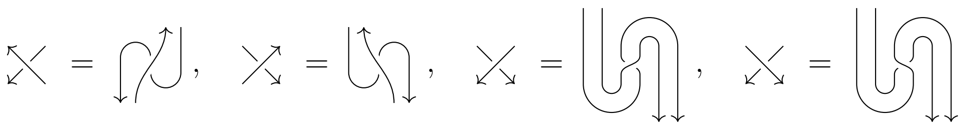


subject to the following relations:



The sideways crossings appearing in (3.1.2) are defined by  and .

We also define:



The variously oriented crossings (the generators and those defined in terms of the generators) equip $\mathcal{OS}$ with the structure of a braided monoidal category.

Note that (3.1.3) tells us that $\uparrow \in \mathrm{ob}(\mathcal{OS})$ is Hecke-braided, and hence $\downarrow \in \mathrm{ob}(\mathcal{OS})$ is also Hecke-braided (see Examples 2.5.11).

Recall that a "q-skew antilinear" map is $\Bbbk$-linear and sends q to $q^* = -q^{-1}$ (see Definition 2.4.3).

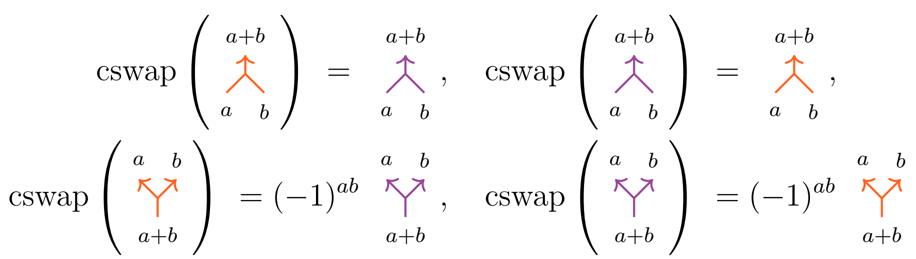
Definition 3.1.7. We write $Q\colon \mathcal{OS} \to \mathcal{OS}$ for the unique q-skew antilinear strict monoidal endofunctor that fixes each of the generating morphisms for $\mathcal{OS}$. All of the defining relations for $\mathcal{OS}$ are invariant under mapping $q \mapsto -q^{-1}$, and hence Q is well-defined.

Definition 3.1.8. We define a q-skew antilinear strict monoidal functor

$$\mathrm{cswap}\colon \mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow}(\mathfrak{gl}),$$

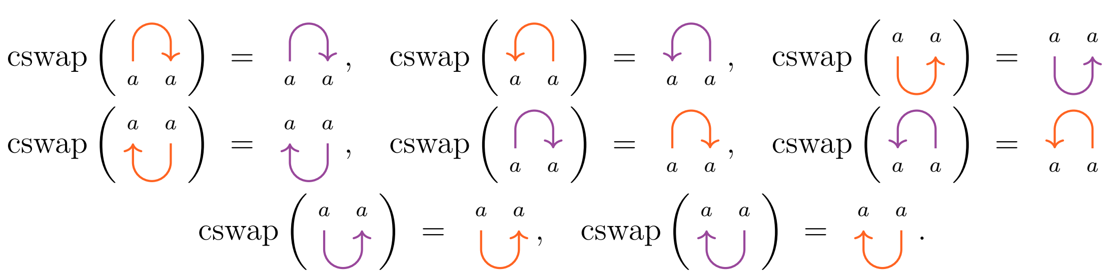
short for "colour swap", as follows. For all $a, b \in \mathbb{N}$ we set:

$$\mathrm{cswap}(\uparrow_a) = \uparrow_a, \quad \mathrm{cswap}(\uparrow_a) = \uparrow_a,$$

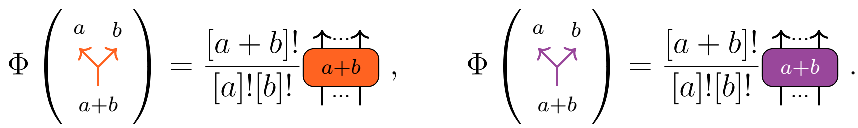


We also write cswap for the q-skew antilinear strict monoidal functor $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ that acts on the generators for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ as specified above, and acts on the new generators for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ by simply changing orange strands to purple and vice versa:

$$\mathrm{cswap}(\downarrow_a) = \downarrow_a, \quad \mathrm{cswap}(\downarrow_a) = \downarrow_a,$$



The sign term appearing in the action of cswap on split morphisms can be understood as follows. In Chapter 4, we will construct an isomorphism Φ between $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ and a partial idempotent completion of $\mathcal{OS}$. This isomorphism acts on splits as



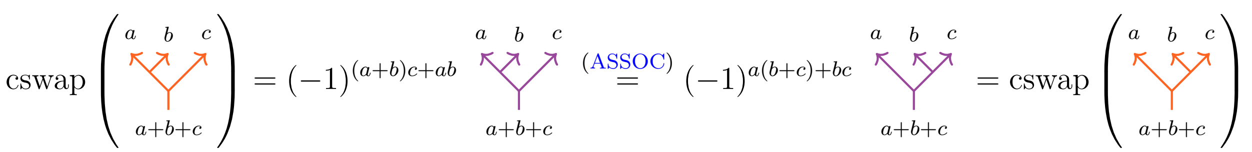
The functor $Q\colon \mathcal{OS} \to \mathcal{OS}$ from Definition 3.1.7 maps symmetrizers to antisymmetrizers and vice versa, and it is straightforward to show that $\left(\frac{[a+b]!}{[a]![b]!}\right)^* = (-1)^{ab}\frac{[a+b]!}{[a]![b]!}$ using (2.4.4). The functor cswap is the $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$-analogue of Q. More precisely, we have $\mathrm{cswap} = \Phi^{-1} \circ Q \circ \Phi$. This is why the factor of $(-1)^{ab}$ appears in our definition. (The isomorphism Φ sends merge morphisms to (anti)symmetrizers without a scalar factor, and hence there is no sign term when acting on those generators.)

Lemma 3.1.9. The functors

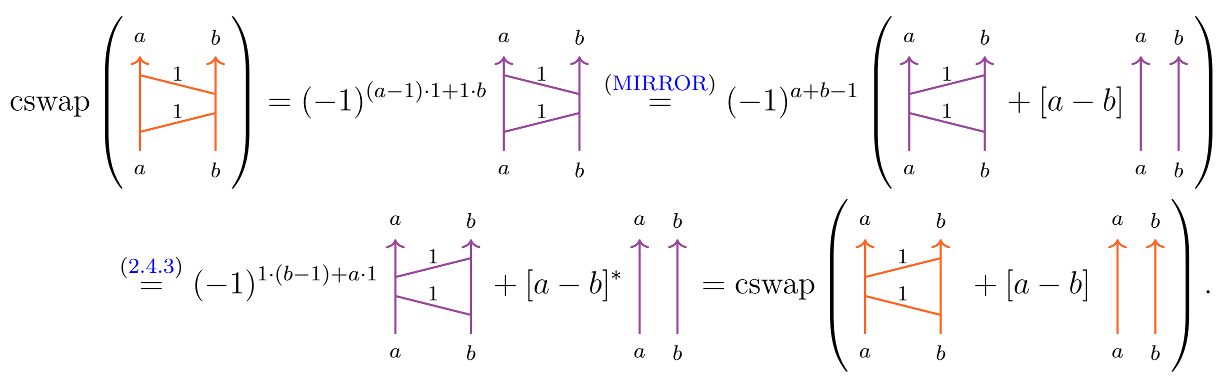
$$\mathrm{cswap}\colon \mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow}(\mathfrak{gl}), \qquad \mathrm{cswap}\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$$

are well-defined.

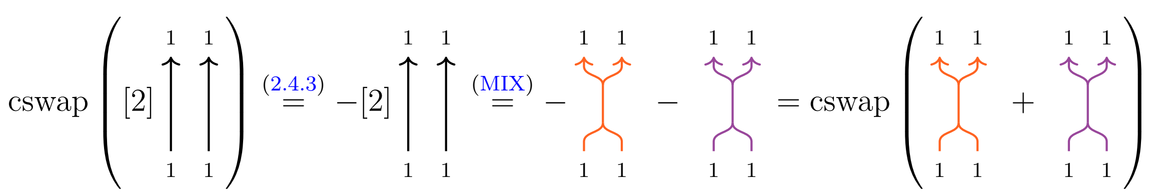
Proof. It suffices to show that the defining relations for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ are preserved by cswap. This is immediate for (TRIV) and the first identity in (ASSOC). For the second, we have:



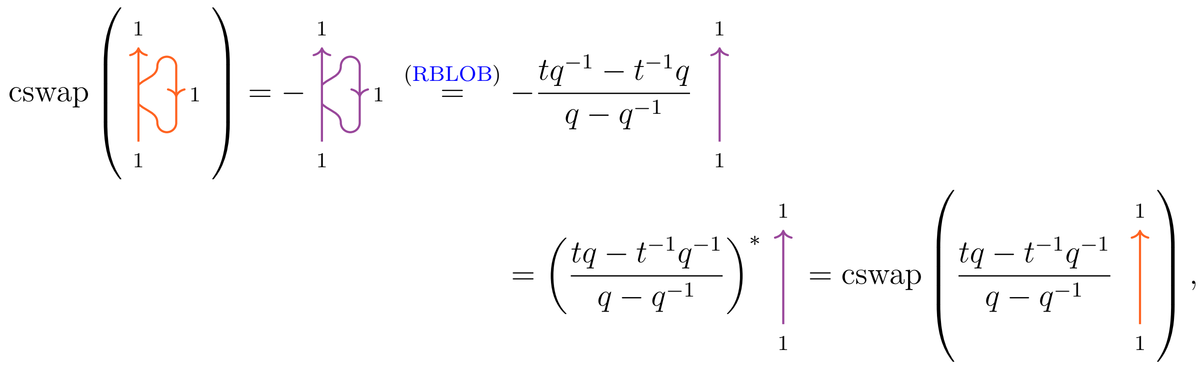
The argument for the purple version of the identity is the same. For (MIRROR), we compute:



Note that we use the fact that cswap is q-skew antilinear in the last equality. Again, the purple argument is the same. Next, we compute:



so (MIX) is preserved. This shows that cswap: $\mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow}(\mathfrak{gl})$ is well-defined. The fact that cswap preserves the additional defining relations for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ follows from similar direct calculations, e.g.



which shows that the first equality in (RBLOB) is preserved. The remaining calculations are analogous. □

Definition 3.1.10. Given a monoidal category $\mathcal{C}$, $\mathcal{C}^{\mathrm{rev}}$ denotes the monoidal category with the same objects, morphisms, composition, and monoidal unit as $\mathcal{C}$, but with the reversed tensor product on both objects and morphisms, given by $x \otimes_{\mathcal{C}^{\mathrm{rev}}} y = y \otimes_{\mathcal{C}} x$. Similarly, $\mathcal{C}^{\mathrm{op}}$ denotes the monoidal category with the same objects, morphisms, and monoidal structure as $\mathcal{C}$, but with the opposite composition, given by $x \circ_{\mathcal{C}^{\mathrm{op}}} y = y \circ x$.

It follows immediately from the definitions that, for any monoidal category $\mathcal{C}$, we have $(\mathcal{C}^{\mathrm{rev}})^{\mathrm{rev}} = \mathcal{C}$, $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$, and $(\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{rev}}$. We define the shorthand $(\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} = \mathcal{C}^{\mathrm{rot}}$, with rot being short for "rotated 180°", corresponding to the effect on string diagrams that reversing the order of tensor products and compositions has.

Definition 3.1.11. We define two (q,t)-antilinear strict monoidal symmetry functors associated to $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ as follows. The first is hflip: $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{rev}}$ (short for "horizontal flip"), which acts as the identity on generating objects, and on morphisms by reflecting diagrams horizontally (that is, left-to-right, across the vertical axis) and applying the involution $\bar{\cdot}$ (see Definition 2.4.3) to the coefficients on diagrams. The second is vflip: $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{op}}$ ("vertical flip"), given by vflip $=$ hflip $\circ$ mate, where mate: $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{rot}}$ is the $\Bbbk_{q,t}$-linear functor that sends each object to its dual and each morphism to its mate (using the duality data specified in Lemma 3.1.5).

Remark 3.1.12. Strictly speaking, the composition vflip $=$ hflip $\circ$ mate mentioned in Definition 3.1.11 is not well-defined, since the codomain of mate is $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{rot}}$, but the domain of hflip is $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. More precisely, we define vflip $=$ hflip$^{\mathrm{rot}} \circ$ mate, where

$$\mathrm{hflip}^{\mathrm{rot}}\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{rot}} \to (\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{rot}})^{\mathrm{rev}} = \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{op}}$$

has the same action on objects and morphisms as hflip. We identify functors in a similar way in Proposition 3.1.13 and throughout the rest of the thesis; for instance, the upcoming claim that mate is "self-inverse" really means that $\mathrm{mate}^{\mathrm{rot}} \circ \mathrm{mate} = \mathrm{id}_{\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})}$.

Proposition 3.1.13. The functor hflip is well-defined. The functor vflip acts on generating objects as

$$\mathrm{vflip}(\uparrow_a) = \downarrow_a, \quad \mathrm{vflip}(\downarrow_a) = \uparrow_a, \quad \mathrm{vflip}(\uparrow_a) = \downarrow_a, \quad \mathrm{vflip}(\downarrow_a) = \uparrow_a,$$

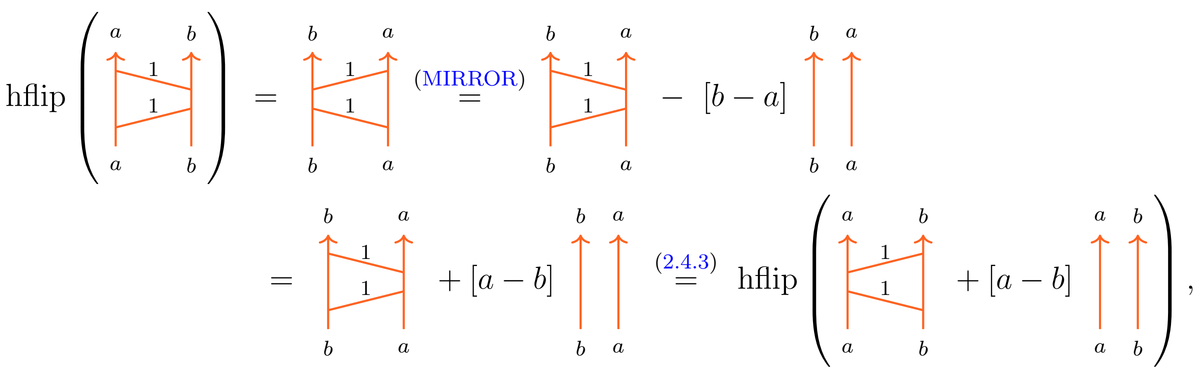
and on morphisms by reflecting diagrams vertically (that is, top-to-bottom, across the horizontal axis) and applying the involution $\bar{\cdot}$ to the coefficients on diagrams. The functor mate acts on generating objects as

$$\mathrm{mate}(\uparrow_a) = \downarrow_a, \quad \mathrm{mate}(\downarrow_a) = \uparrow_a, \quad \mathrm{mate}(\uparrow_a) = \downarrow_a, \quad \mathrm{mate}(\downarrow_a) = \uparrow_a,$$

and on morphisms by rotating diagrams by 180° (without changing the coefficients on said diagrams). All three of these symmetry functors are self-inverse, and we have

$$\mathrm{mate} = \mathrm{hflip} \circ \mathrm{vflip} = \mathrm{vflip} \circ \mathrm{hflip}\,.$$

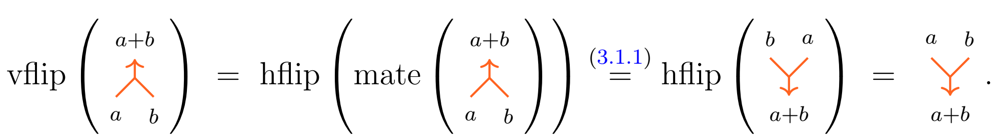
Proof. We need to show that hflip preserves the defining relations of $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. It is immediate that hflip preserves (TRIV), (ASSOC), (TRIV2), (ZIGZAG), and (MATE). Since quantum integers and $\frac{t-t^{-1}}{q-q^{-1}}$ are both invariant under $\bar{\cdot}$, (MIX) and (SPEC) are also preserved. For (MIRROR), we compute:



as desired. (The same calculation shows that the purple version of (MIRROR) is also preserved.) The identities in (LBLOB) and (RBLOB) are the hflip-images of each other, and the same is true for (LSQUIG) and (RSQUIG). Hence these four relations are preserved by hflip. This completes the proof that hflip is well-defined. Turning our attention to the claims for vflip, for any $a \in \mathbb{N}$ we have

$$\mathrm{vflip}(\uparrow_a) = \mathrm{hflip}(\mathrm{mate}(\uparrow_a)) = \mathrm{hflip}(\downarrow_a) = \downarrow_a, \quad \mathrm{vflip}(\downarrow_a) = \mathrm{hflip}(\mathrm{mate}(\downarrow_a)) = \mathrm{hflip}(\uparrow_a) = \uparrow_a,$$

and similarly for purple generating objects. All that remains is to show that vflip acts on generating morphisms by vertical reflection; the claim for general diagrams then follows from the fact that vflip is a functor from $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ to $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})^{\mathrm{op}}$. The desired equalities follow from direct calculations. For instance,



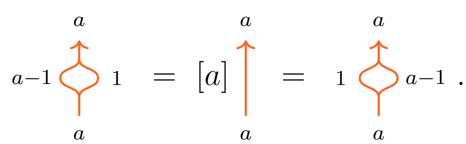
The calculations for the other generating morphisms are analogous.

The fact that hflip and vflip are self-inverse follows from the fact that horizontal and vertical reflection are each self-inverse transformations of the plane. For the claim about the action of the mate functor, note that $\mathrm{hflip} \circ \mathrm{vflip} = \mathrm{hflip} \circ \mathrm{hflip} \circ \mathrm{mate} = \mathrm{mate}$, and so mate acts on diagrams by flipping them vertically and then horizontally; this has the net effect of a 180° rotation. The same action can equivalently be obtained by first flipping diagrams horizontally and then vertically, and so we also have $\mathrm{vflip} \circ \mathrm{hflip} = \mathrm{mate}$. The fact that mate is self-inverse then follows from vflip and hflip being self-inverse. □

3.2 Key diagrammatic identities

In this section, we will establish several key identities in the $\mathfrak{gl}$-web categories. Several of these identities have previously been assumed as part of the definition of $\mathfrak{gl}$-web categories in the literature; for instance, the identity in Lemma 3.2.2 is typically assumed directly. The presentations of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ we gave in the previous section are smaller than most preexisting presentations (that is, our presentations involve fewer defining relations), and hence we need to provide proofs for these foundational identities.

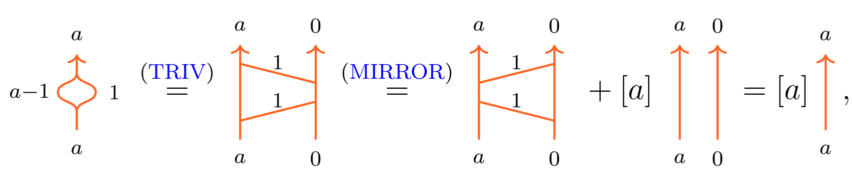
Lemma 3.2.1. The following identity holds in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ for all $a \in \mathbb{N}$, for both colours of strands:



(3.2.1)

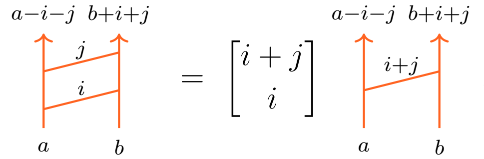
Proof. We prove the statement for orange strands; the purple version follows after applying cswap and dividing by $(-1)^{a-1}$.

If $a = 0$, then the first and third diagrams in (3.2.1) are zero due to the presence of a string labelled by -1, and the second diagram has a coefficient of $[a] = 0$, so the identity holds. Assuming $a \geq 1$, we compute:



where in the last equality we remove the first diagram since it has a string implicitly labelled by -1, and we remove the string labelled 0 from the second diagram since $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ is a strict monoidal category. Applying hflip then yields the second desired equality (recall that hflip is q-antilinear, and mapping $q \mapsto q^{-1}$ fixes quantum integers). □

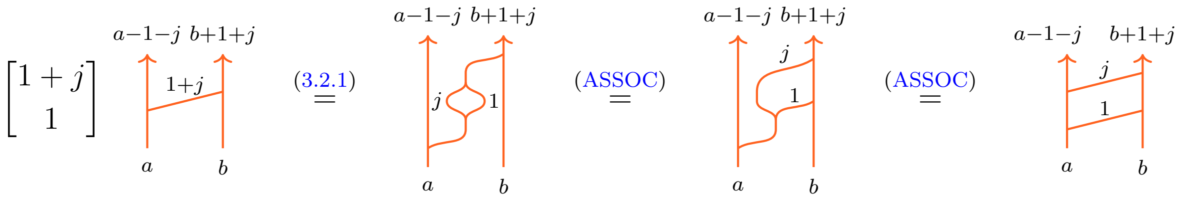
Lemma 3.2.2. The following identity holds in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ for all $a, b, i, j \in \mathbb{N}$, for both colours of strands:



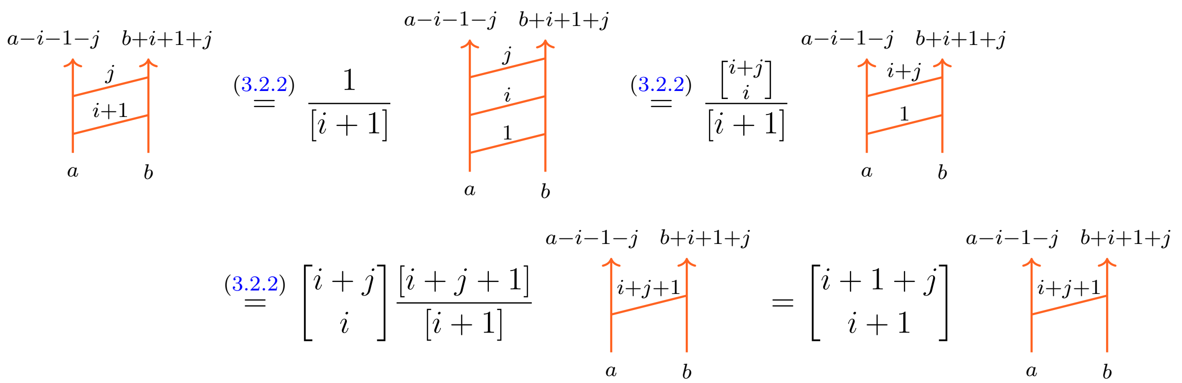
(3.2.2)

Proof. We prove the statement for orange strands; the purple version follows after applying cswap and dividing by $(-1)^{ai+aj+i+j+ij}$.

Let $a, b \in \mathbb{N}$. If $i = 0$ or $j = 0$, (3.2.2) follows immediately from (TRIV), noting that $\begin{bmatrix} i+j \\ i \end{bmatrix} = 1$ in such cases. For another special case, suppose $i = 1$ and $j \in \mathbb{N}$. Then

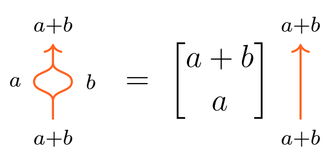


as desired. We now proceed by induction on i. Let $i \in \mathbb{N}_{>0}$, and suppose that (3.2.2) holds for all $j \in \mathbb{N}$ with that choice of i. Then, using the $i = 1$ case of (3.2.2) proved above,



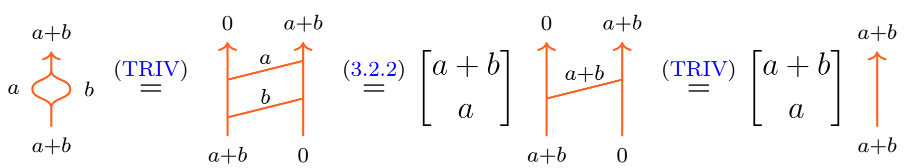
as desired. $\square$

Lemma 3.2.3. The following identity holds in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ for all $a, b \in \mathbb{N}$, for both colours of strands:



(3.2.3)

Proof. We compute:



The purple version of the identity follows after applying cswap and dividing by $(-1)^{ab}$. $\square$

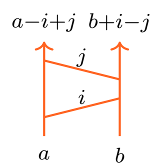
The following identity will be used in the proof of Proposition 3.2.5. It follows from a straightforward but tedious calculation, which we omit for the sake of brevity.

Lemma 3.2.4. For all $a, b, i, j, s \in \mathbb{N}$ we have

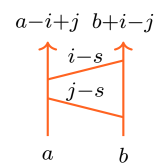
$$\frac{\begin{bmatrix} a-b-i+j \\ s \end{bmatrix}[i+1-s] + \begin{bmatrix} a-b-i+j-1 \\ s-1 \end{bmatrix}[a-b-2i-1+j]}{[i+1]} = \begin{bmatrix} a-b-i-1+j \\ s \end{bmatrix}. \tag{3.2.4}$$

The following result is mentioned in [ST19, Rem. 2.5], but a proof is only given for the case $i = 2, j = 1$. To the best knowledge of the present author, a full proof of Proposition 3.2.5 has not appeared in the literature to date, though the identity has been directly assumed in some presentations of web categories, e.g. [CKM14, (2.10)] and [TVW17, (2-8)].

Proposition 3.2.5. The following identity holds in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ for all $a, b, i, j \in \mathbb{N}$, for both colours of strands:

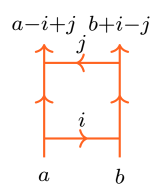


$$= \sum_{s \geq 0} \begin{bmatrix} a-b-i+j \\ s \end{bmatrix} \tag{3.2.5}$$

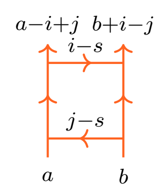


.

The above identities are sometimes referred to as the "square-switch" identities, corresponding to a slightly different way of drawing the diagrams:



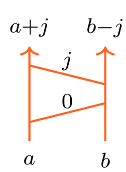
$$= \sum_{s \geq 0} \begin{bmatrix} a-b-i+j \\ s \end{bmatrix}$$



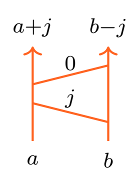
.

Proof. We prove the statement for orange strands; the purple version follows after applying cswap and dividing by $(-1)^{ai+bj+ij+i+j}$.

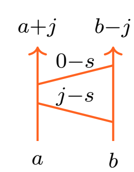
Let $a, b \in \mathbb{N}$. We begin by proving (3.2.5) in some special cases. If $i = 0$, then we have:



$\stackrel{\text{(TRIV)}}{=}$

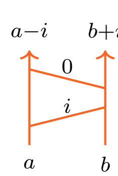


$$= \sum_{s \geq 0} \begin{bmatrix} a-b+j \\ s \end{bmatrix}$$

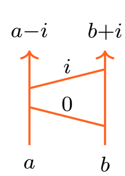


,

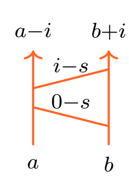
as desired. Note that in the second equality we use the fact that the summands corresponding to $s > 0$ are zero since they feature a string labelled by $-s < 0$. Similarly, if $j = 0$, then



$\stackrel{\text{(TRIV)}}{=}$

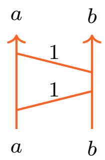


$$= \sum_{s \geq 0} \begin{bmatrix} a-b-i \\ s \end{bmatrix}$$

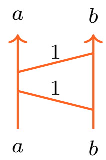


,

as desired. If $i = j = 1$, then

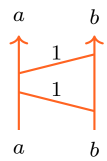


$\overset{\text{(MIRROR)}}{=}$

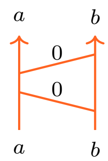


$+ [a-b]$

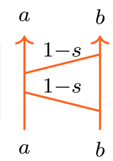
$\overset{\text{(TRIV)}}{=} \begin{bmatrix} a-b \\ 0 \end{bmatrix}$



$+ \begin{bmatrix} a-b \\ 1 \end{bmatrix}$

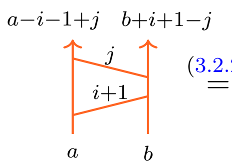


$= \sum_{s \geq 0} \begin{bmatrix} a-b \\ s \end{bmatrix}$

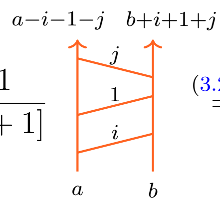


.

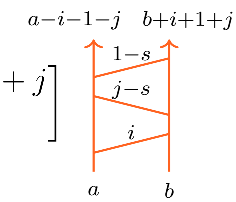
We have shown that (3.2.5) holds for all $i, j \in \mathbb{N}$ such that $i = 0$ or $j = 0$ or $i + j \leq 2$. We prove the remaining cases by induction on $i + j$. Let $k \in \mathbb{N}_{\geq 2}$, and suppose that (3.2.5) holds for all $i, j \in \mathbb{N}$ such that $i + j \leq k$. Let $i, j \in \mathbb{N}_{>0}$ such that $i + j = k$. Note that this implies $j + 1 \leq k$. Then:



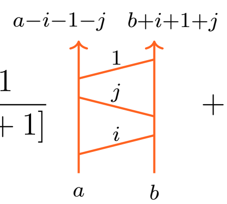
$\overset{(3.2.2)}{=} \frac{1}{[i+1]}$



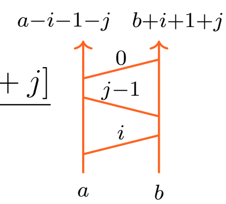
$\overset{(3.2.5)}{=} \frac{1}{[i+1]} \sum_{s \geq 0} \begin{bmatrix} a-i-b-i-1+j \\ s \end{bmatrix}$



$= \frac{1}{[i+1]}$

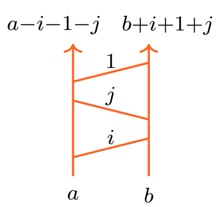


$+ \frac{[a-b-2i-1+j]}{[i+1]}$

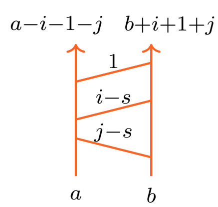


.

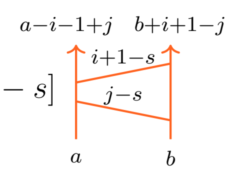
Next, we rewrite the last two diagrams in terms of simpler ones:



$\overset{(3.2.5)}{=} \sum_{s \geq 0} \begin{bmatrix} a-b-i+j \\ s \end{bmatrix}$

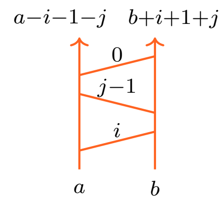


$\overset{(3.2.2)}{=} \sum_{s \geq 0} \begin{bmatrix} a-b-i+j \\ s \end{bmatrix} [i+1-s]$

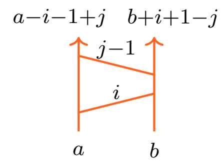


,

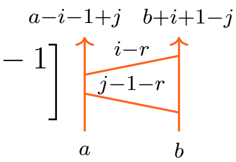
and



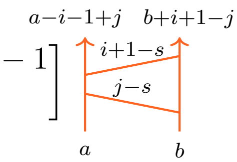
$\overset{(\text{TRIV})}{=}$



$\overset{(3.2.5)}{=} \sum_{r\geq 0} \begin{bmatrix} a-b-i+j-1 \\ r \end{bmatrix}$

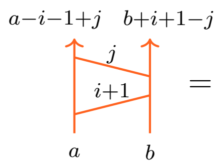


$= \sum_{s\geq 0} \begin{bmatrix} a-b-i+j-1 \\ s-1 \end{bmatrix}$

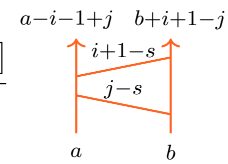


,

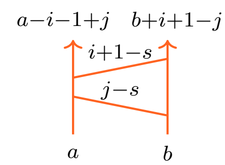
where in the last equality we switch the summation variable to $s = r+1$. Note that we have added one term to the sum after this switch of variables, namely the term where $s = 0$, but that summand is zero since $\begin{bmatrix} a-b-i+j-1 \\ -1 \end{bmatrix} = 0$ by definition. Putting our calculations together, we have



$$= \sum_{s\geq 0} \frac{\begin{bmatrix} a-b-i+j \\ s \end{bmatrix}[i+1-s] + \begin{bmatrix} a-b-i+j-1 \\ s-1 \end{bmatrix}[a-b-2i-1+j]}{[i+1]}$$



$$\overset{(3.2.4)}{=} \sum_{s\geq 0} \begin{bmatrix} a-b-i-1+j \\ s \end{bmatrix}$$

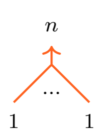


,

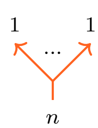
which shows that (3.2.5) holds with $i+1$ in place of i. An analogous calculation shows that (3.2.5) also holds with $j+1$ in place of j, and hence it holds for all $i + j \leq k + 1$. This completes the induction. ☐

We conclude this section by introducing some morphisms that will play an important role throughout the rest of the thesis.

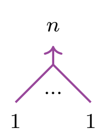
Definition 3.2.6. For $n \in \mathbb{N}$, we define the *multi-merge* and *multi-split* morphisms



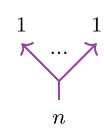
$: \uparrow_1^{\otimes n} \to \uparrow_n,$



$: \uparrow_n \to \uparrow_1^{\otimes n},$

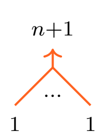


$: \uparrow_1^{\otimes n} \to \uparrow_n,$

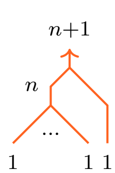


$: \uparrow_n \to \uparrow_1^{\otimes n}$

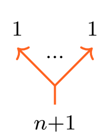
in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ as follows. For $n = 0$ and $n = 1$, we set the corresponding multi-merge and multi-split morphisms to be the identity on $\uparrow_n = \uparrow_n$. For $n \geq 1$, we inductively define:



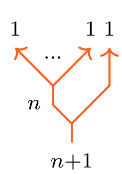
$=$



,

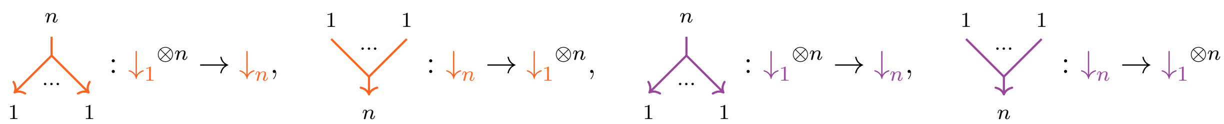


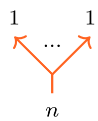
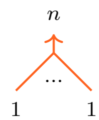
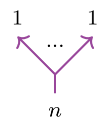
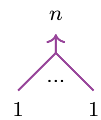
$=$



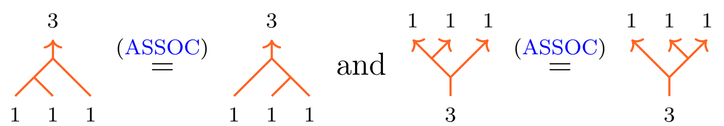
, (3.2.6)

and the same for purple strands. We also define the downwards-oriented multi-split and multi-merge morphisms

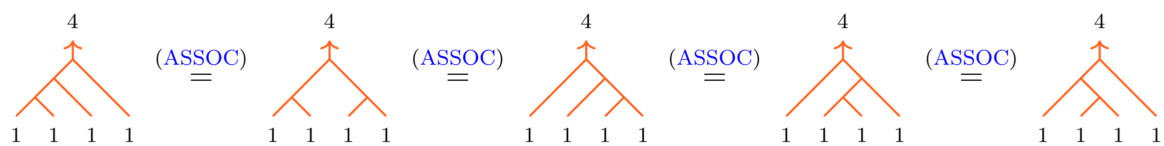


in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ to be, respectively, the mates of  ,  ,  , and  .

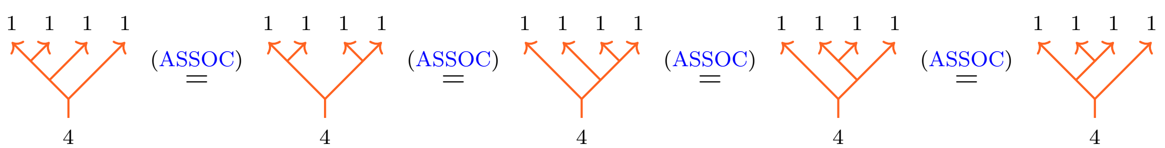
Example 3.2.7. The (orange) upwards-oriented multi-merge and multi-split morphisms for $n = 2$ are simply the generating morphisms  and  . For $n = 3$, the morphisms are



For $n = 4$, the morphisms are

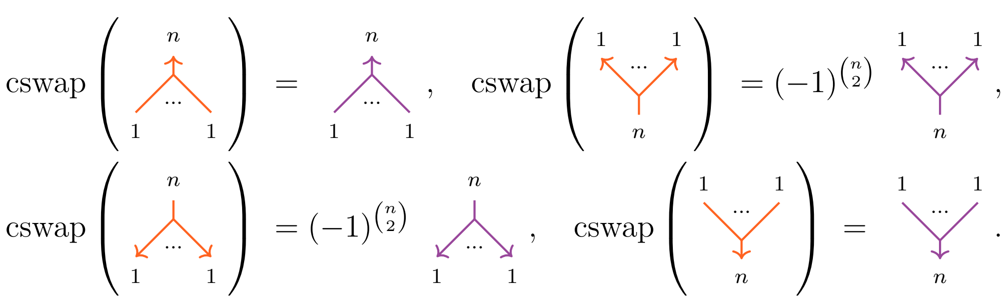


and



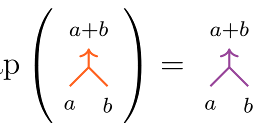
In general, (ASSOC) implies that the multi-merge and multi-split morphisms for any n are equal to the diagrams built by composing $n-1$ binary merge or split morphisms in an arbitrary order.

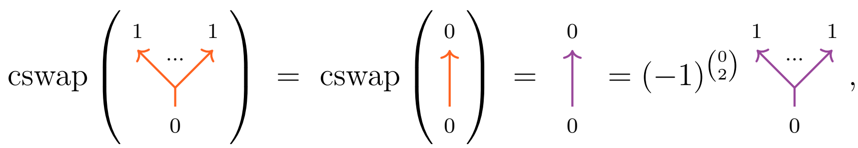
Lemma 3.2.8. The functor cswap acts on multi-merge and multi-split morphisms as follows:



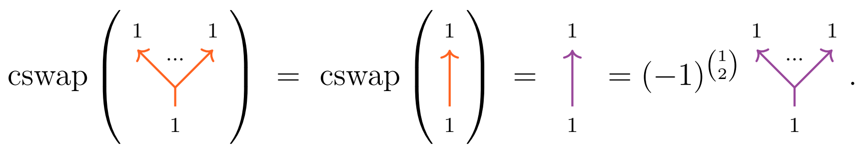
(3.2.7)

The same is true after exchanging the orange strands for purple ones and vice versa.

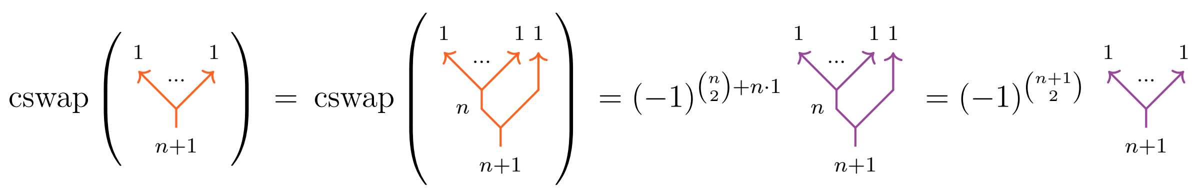
Proof. The first identity follows from the fact that . We prove the second identity by induction on n. For $n = 0$, we have



and similarly, for $n = 1$,

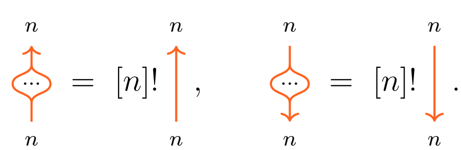


Assuming the identity holds for some $n \geq 1$, we compute:



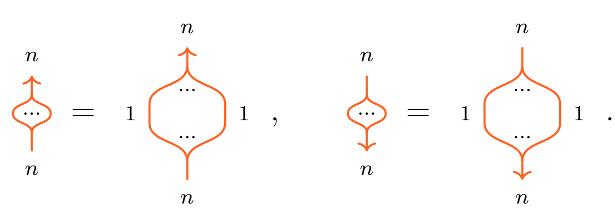
as desired. Taking mates yields the third and fourth identities in (3.2.7). The same arguments prove the corresponding identities with the roles of the colours swapped. $\square$

Lemma 3.2.9. For all $n \in \mathbb{N}$ and for both colours of strands, we have



(3.2.8)

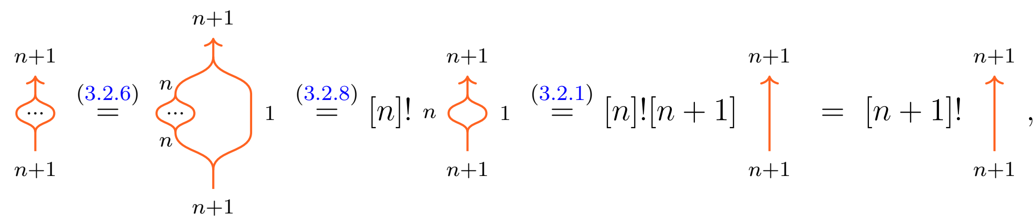
For the avoidance of doubt: the morphisms on the left hand sides of (3.2.8) are the composition of a multi-split and a multi-merge morphism. That is,



We draw such compositions with only one set of ellipses to reduce visual clutter.

Proof. We prove the identity for upwards-oriented orange strands; the purple version follows after applying cswap and dividing by $(-1)^{\binom{n}{2}}$, and the downwards-oriented identities then follow after taking mates.

The desired identity is trivial for $n = 0$ and $n = 1$. Suppose that (3.2.8) holds for some $n \geq 1$. Then:

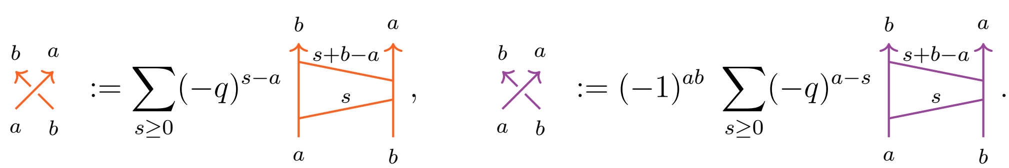


as desired. □

3.3 Braided monoidal structure

In this section, we will equip $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ each with the structure of a braided monoidal category, and then prove several results related to these structures. The braided structure on $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ has appeared previously in [TVW17]. The extension to $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ is new, though [QS19] previously treated the monochromatic purple case. As mentioned in the introduction, Proposition 3.3.23 appeared previously in [RT16, Rem. 2.21], but the approach outlined in that remark is quite different from the proof we will give in this section.

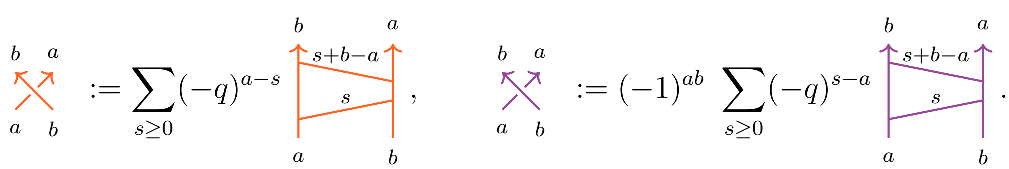
Definition 3.3.1. We define the *monochromatic positive crossings* in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ as follows, for $a, b \in \mathbb{N}$:



(3.3.1)

(Only finitely many of the summands appearing above are nonzero, since for $s > a$ the diagrams have a strand implicitly labelled by a negative integer.)

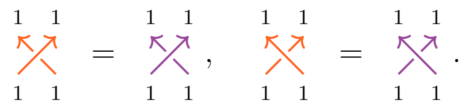
The *monochromatic negative crossings* are obtained by replacing q with q^{-1} in the above definition, i.e.



(3.3.2)

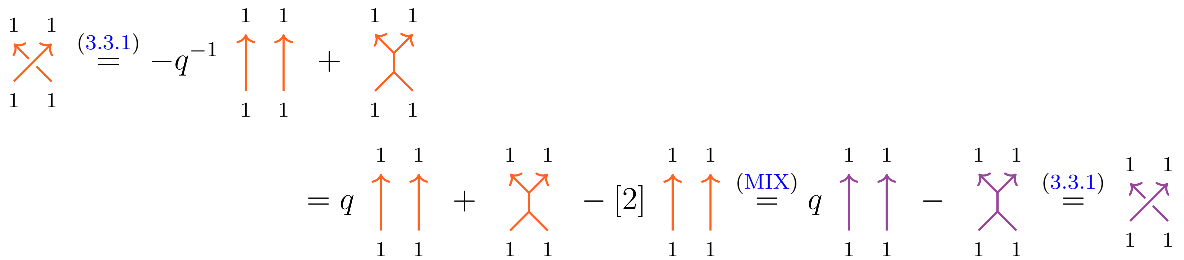
Remark 3.3.2. In the literature, $\mathfrak{gl}$-web categories featuring only one colour of strands, corresponding to our purple strands, have sometimes been equipped with crossings that differ from those specified above by a factor of $(-1)^{ab}$. These alternate crossings correspond to a different choice of braided monoidal structure on $U_q(\mathfrak{gl}_{m|n})$-mod, for which the braiding for two copies of the natural module V^+ is the negation of the one we specified in Definition 2.4.15. In order to define web category crossings that properly correspond to that alternate braided structure, one should multiply *both* the orange and purple crossings of Definition 3.3.1 by $(-1)^{ab}$.

Lemma 3.3.3. The thin orange and purple crossings coincide. That is,



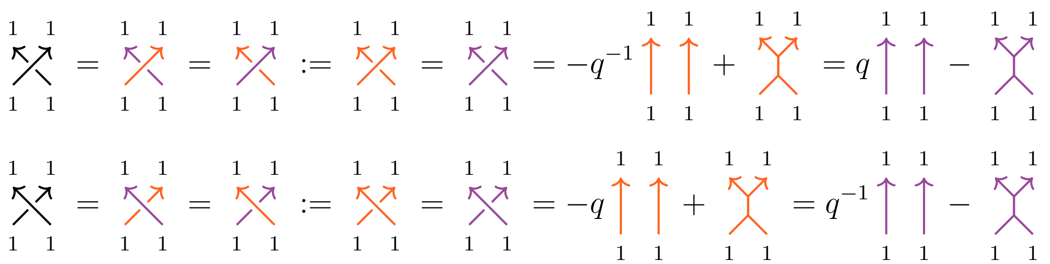
(3.3.3)

Proof. We will prove the identity for positive crossings; the calculation for negative crossings is totally analogous. Recalling that $\uparrow_1 = \uparrow_1$ by definition, we compute:



where in the second equality we use $[2] = q + q^{-1}$. $\square$

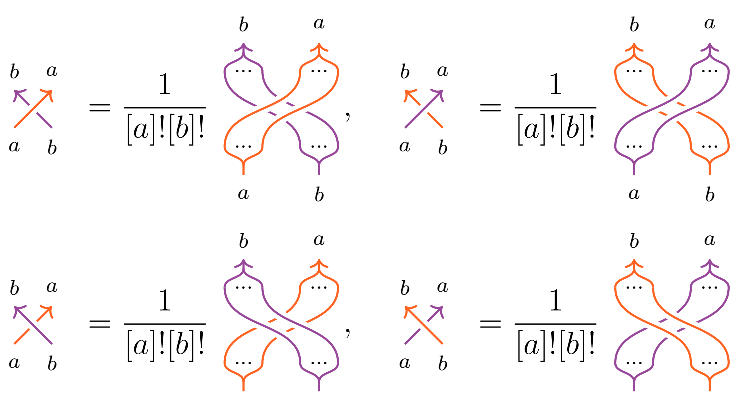
In light of (3.3.3), we can unambiguously draw thin crossings using strands of arbitrary (even mixed) colour:



(3.3.4)

These formulas for thin crossings immediately imply that $\uparrow_1$ is Hecke-braided. It follows that $\downarrow_1$ is also Hecke-braided (with respect to downwards-oriented crossings that we will define shortly).

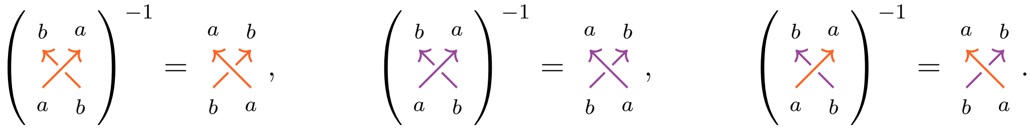
Definition 3.3.4. We define the *mixed-colour crossings* in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ as follows, for $a, b \in \mathbb{N}$:



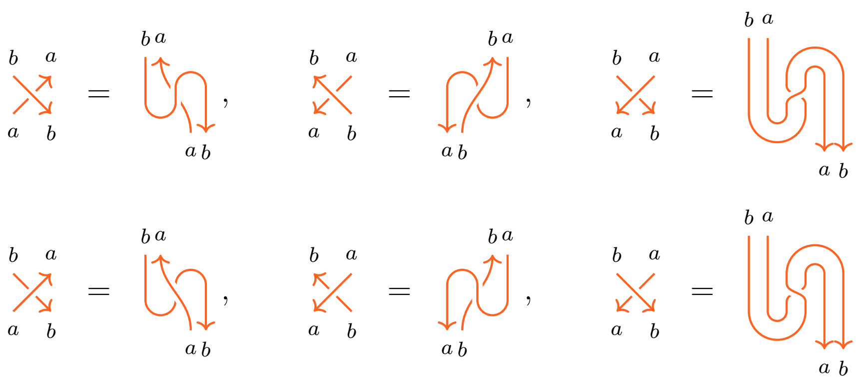
(3.3.5)

Note that the crossings on the right hand side of each definition are the thin ones from (3.3.4).

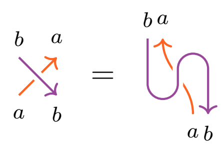
Proposition 3.3.5 ([TVW17, Prop. 2.22])**.** The crossings of Definitions 3.3.1 and 3.3.4 equip $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ with the structure of a braided monoidal category, with



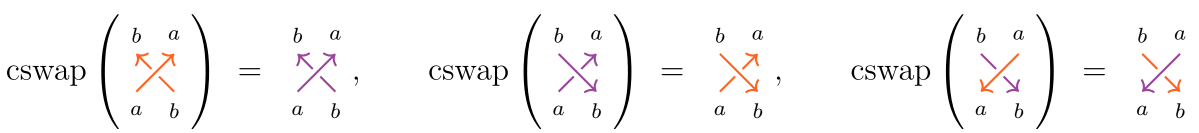
Definition 3.3.6. For all $a, b \in \mathbb{N}$ we define the following *rotated crossing* morphisms in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$:



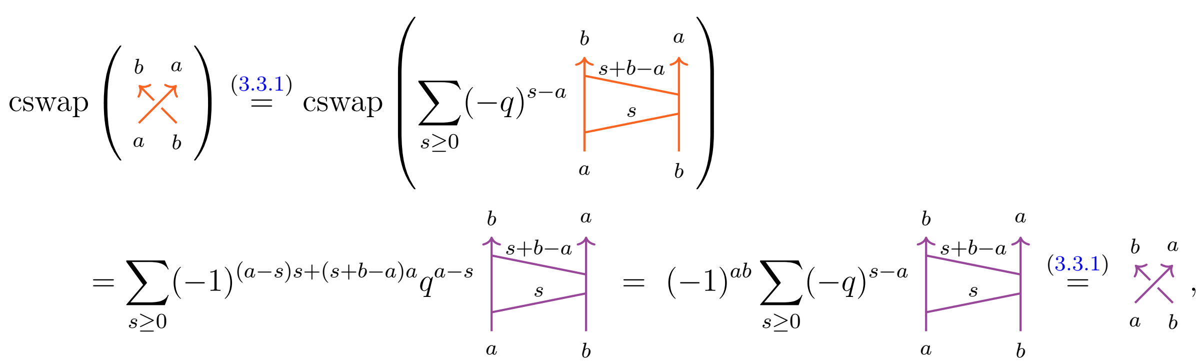
(3.3.6)

We adopt these definitions for arbitrary strand colours, so we also have e.g. 

Lemma 3.3.7. The functor cswap acts on crossings (with arbitrary strand labels, colours, and orientations) by simply swapping the colour of both strands. For instance,

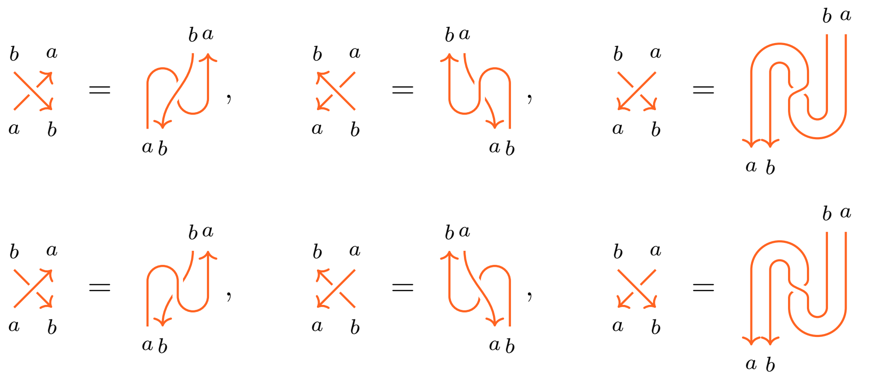


Proof. We have:

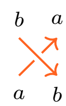


proving the first claimed identity. The other upwards-oriented monochromatic cases follow from totally analogous calculations. The upwards-oriented mixed-colour cases then follow from (2.4.4) and (3.2.7). The cases involving rotated crossings follow immediately from the upwards-oriented ones and the fact that cswap acts on caps and cups by simply swapping the colour of the strand. ☐

Lemma 3.3.8. For all $a, b \in \mathbb{N}$ and for arbitrary strand colours, we have:

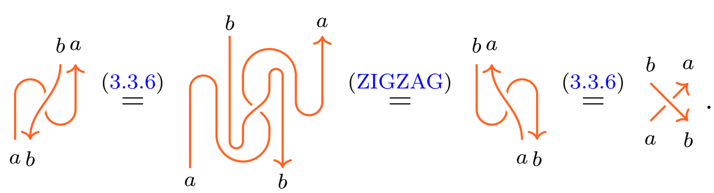


(3.3.7)

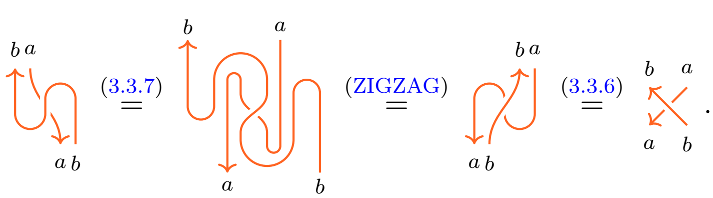
Note that the identities in (3.3.7) are slight variants of our original definitions of the rotated crossings in (3.3.6). For instance, the definition of  features a zigzag labelled by b, whereas the first identity in (3.3.7) features a zigzag labelled by a.

Proof. We prove the monochromatic orange identities; the arguments for other colours are identical.

The third and sixth identities follow from the fact that $\mathbf{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ is strict pivotal, and hence the left and right mates of the (positive or negative) upwards-oriented crossing coincide. For the first identity, we compute:

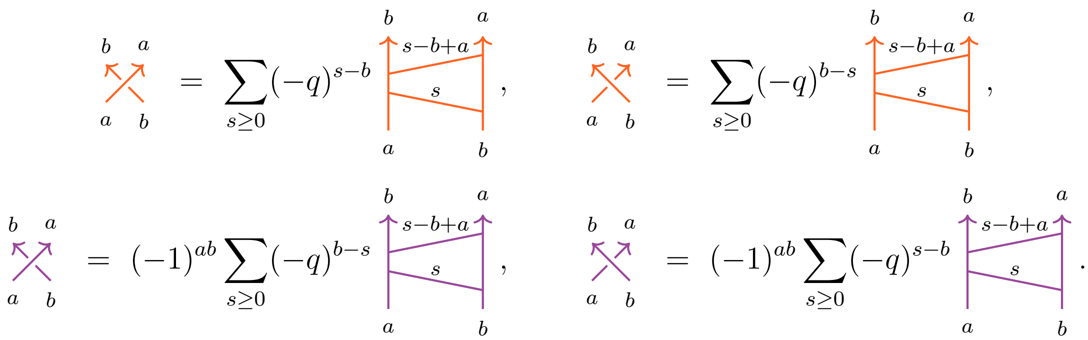


The second identity follows similarly:



(In the first equality above, we use the third identity from (3.3.7).) The fourth and fifth identities follow from totally analogous calculations. ☐

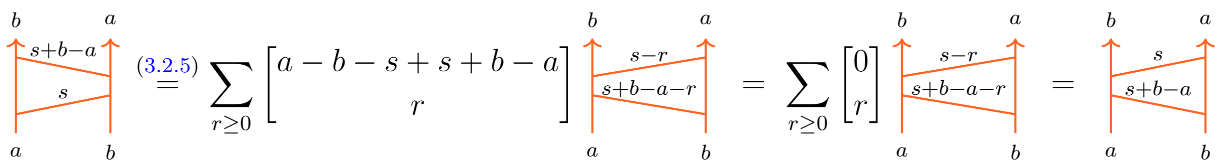
Lemma 3.3.9. For all $a, b \in \mathbb{N}$ we have:



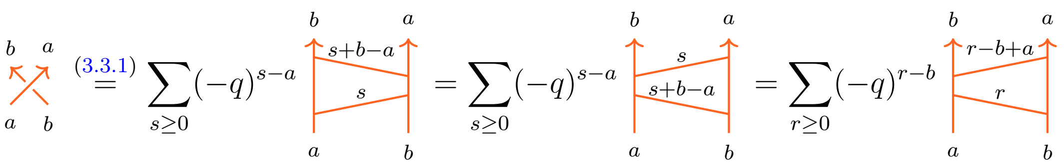
(3.3.8)

Note that the summand diagrams appearing on the right hand sides of (3.3.8) are mirrored versions of those from the original definition of the crossings, (3.3.1).

Proof. Recall that $\begin{bmatrix}0\\0\end{bmatrix} = 1$ and $\begin{bmatrix}0\\s\end{bmatrix} = 0$ for all $s \geq 1$. With this in mind, for any $s \geq 0$ we have:

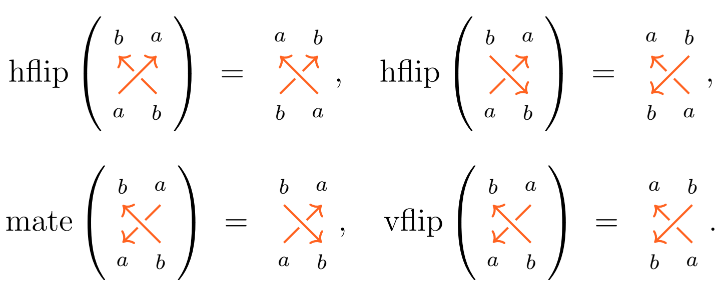


Hence:

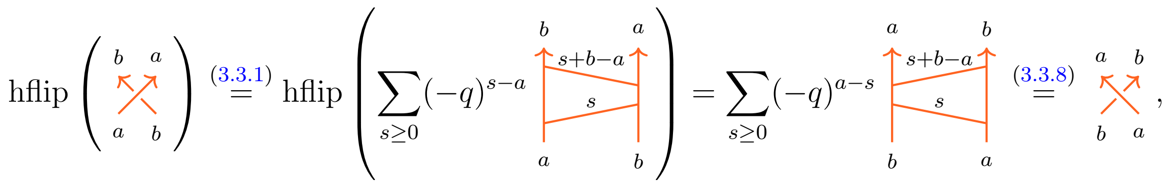


where we switch to a sum over $r = s + b - a$ in the final equality. The second identity follows after replacing q with q^{-1}. The purple identities then follow from the orange ones after applying cswap. $\square$

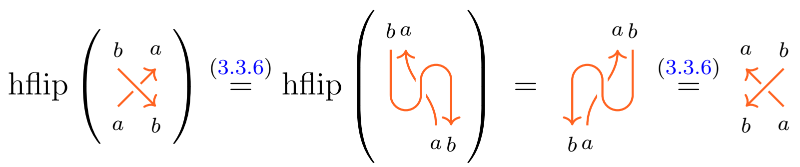
Corollary 3.3.10. The symmetry functors hflip, vflip, and mate act on crossings (either positive or negative, in any orientation, and with arbitrary strand colours) by, respectively, horizontally flipping (across the vertical axis), vertically flipping (across the horizontal axis), or rotating the crossing in question by 180°. For example, we have



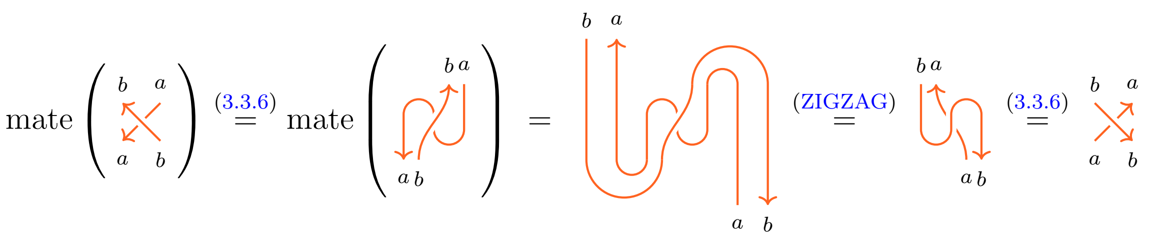
Proof. The claimed monochromatic identities follow from Definition 3.3.1, Definition 3.3.6, Lemma 3.3.8, Lemma 3.3.9, and the fact that mate = vflip ∘ hflip = hflip ∘ vflip, as shown in Proposition 3.1.13. We will provide proofs for the four examples given in the corollary statement; the other monochromatic cases follow from totally analogous calculations. First,



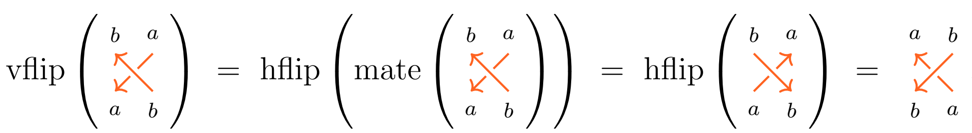
where in the second equality we use the fact that hflip is q-antilinear. We use this first identity in the proof of the second:



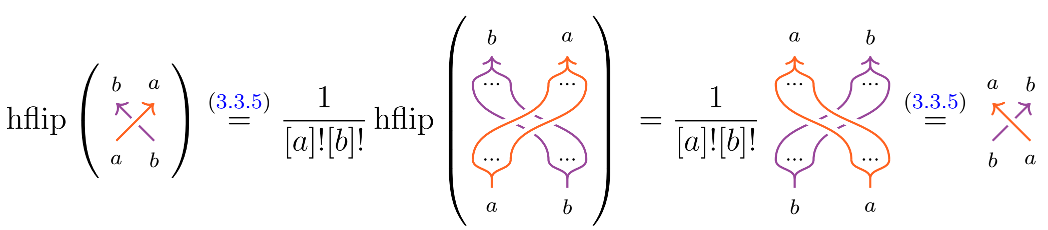
Next, we have



Finally,

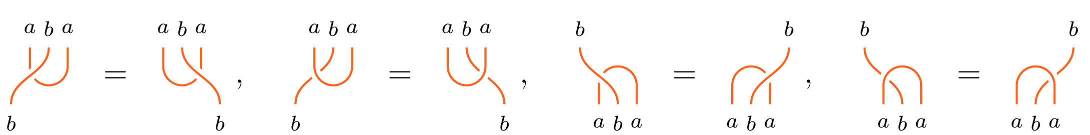


The identities for mixed-colour crossings follow from those for thin crossings. For instance,

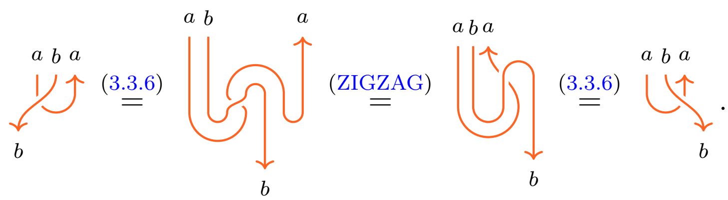


where in the second equality we use the fact that hflip fixes multi-merge and multi-split morphisms. □

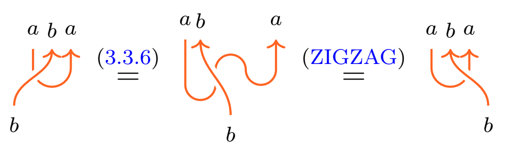
Lemma 3.3.11. The following *cup-* and *cap-sliding* relations hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, for all $a, b \in \mathbb{N}$, all possible orientations of the strands, and arbitrary strand colours:

 (3.3.9)

Proof. We will prove the monochromatic orange identities where the strand labelled b crosses over the strand labelled a; the identities with the other type of crossing and/or different strand colours follow from totally analogous arguments. For one orientation of the first identity, we compute:

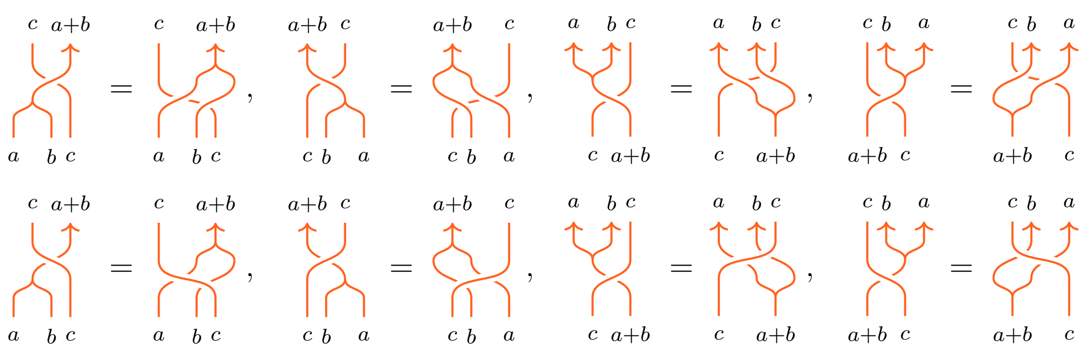


For another orientation, we compute:



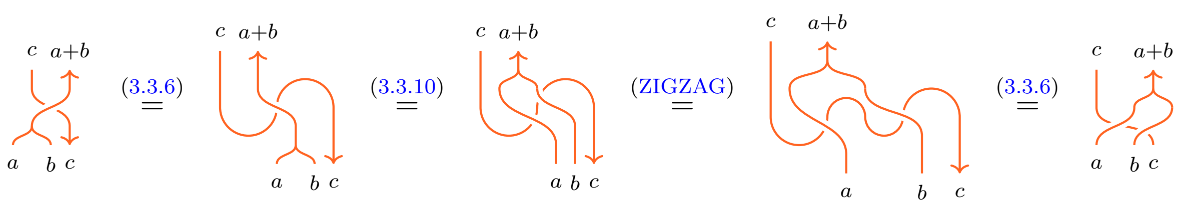
The remaining orientations of the desired identities follow from these two after applying hflip, vflip, and mate. ☐

Lemma 3.3.12. Upwards-oriented merge and split morphisms slide through all crossings. That is, for all $a, b, c \in \mathbb{N}$, for both possible orientations of the strand labelled c, and for arbitrary strand colours, the following identities hold:



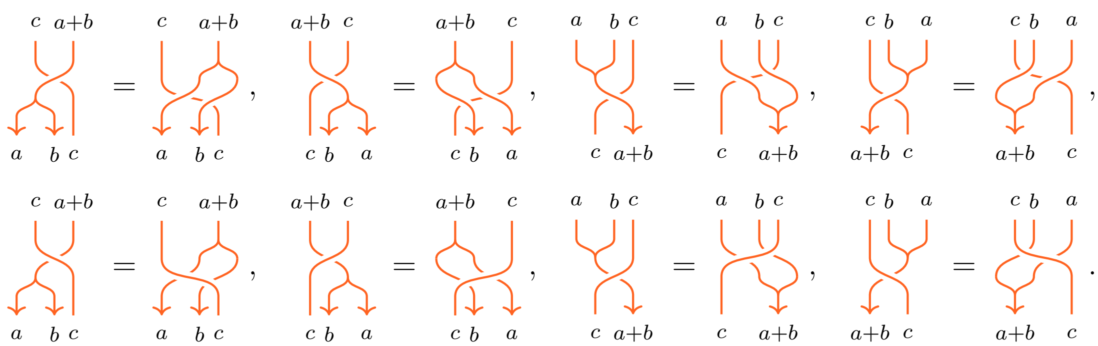
(3.3.10)

Proof. We will give a proof for the monochromatic orange case; the proofs for other colours of strands are identical. Proposition 3.3.5 tells us that the sliding identities hold when all strands are oriented upwards. For the case where the strand labelled c is oriented downwards, we compute:



which establishes the first identity from (3.3.10). Note that the use of (3.3.10) in the above calculation has all strands oriented upwards, so this argument is not circular. Applying hflip yields the second identity. The arguments for the third and fourth identities are analogous. The identities on the second line of (3.3.10) follow from essentially the same proofs as for those on the first line, with all positive crossings replaced by negative ones and vice versa. $\square$

Corollary 3.3.13. Downwards-oriented split and merge morphisms slide through braidings. That is, for all $a, b, c \in \mathbb{N}$, for both possible orientations of the strand labelled c, and for arbitrary strand colours, the following identities hold:

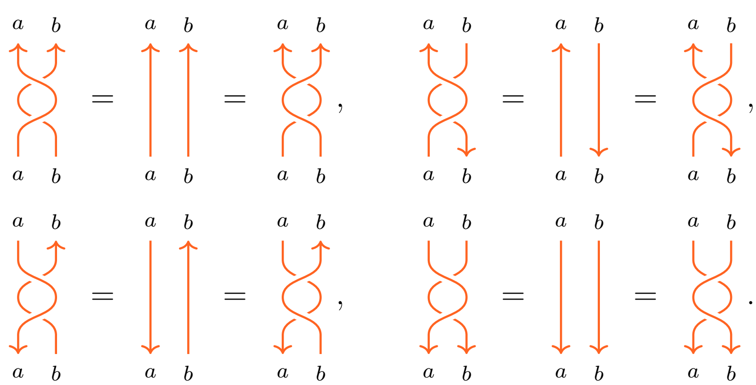


(3.3.11)

Proof. These identities are simply the vflip-images of those from Lemma 3.3.12. $\square$

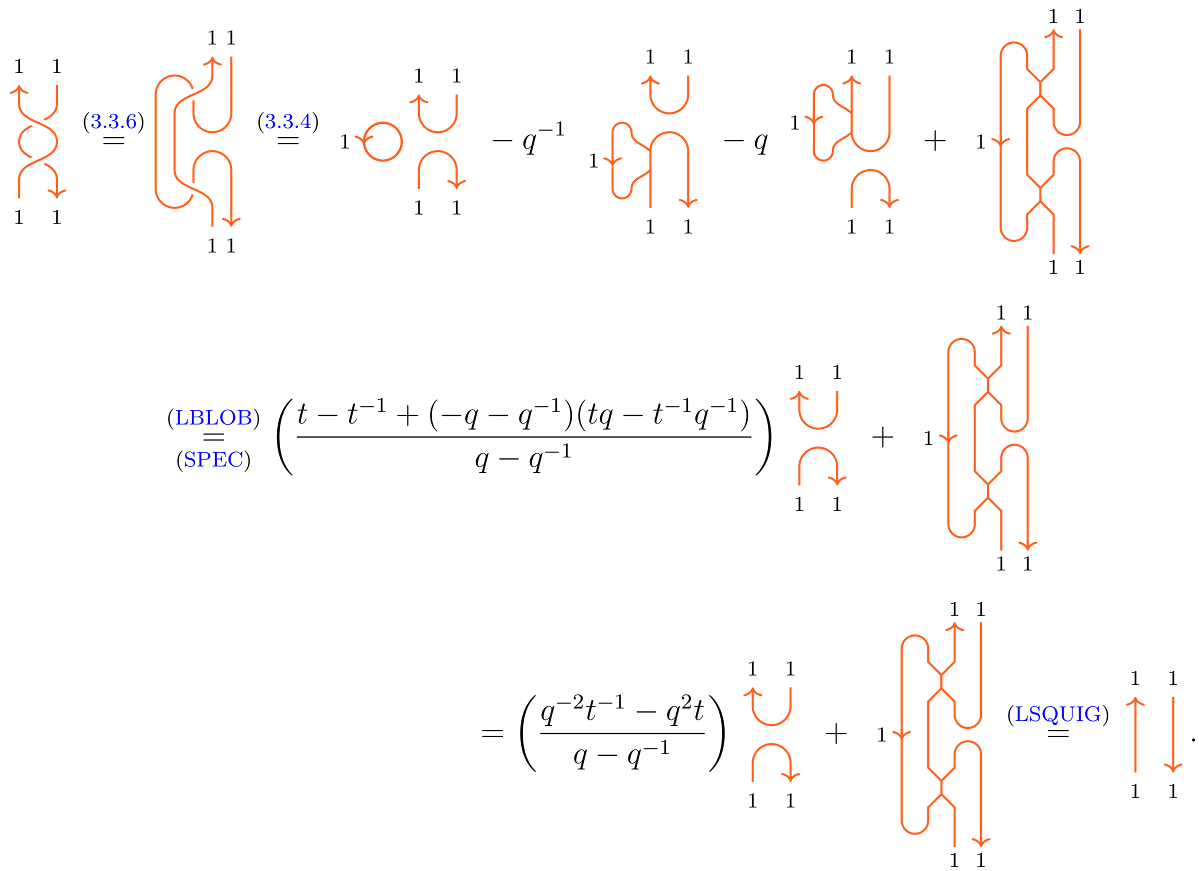
Note that Lemma 3.3.12 and Corollary 3.3.13 immediately imply that multi-split and multi-merge morphisms (in either orientation, and in both colours) slide through crossings.

Lemma 3.3.14. The *double-crossing relations* hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, for all $a, b \in \mathbb{N}$ and for arbitrary strand colours:

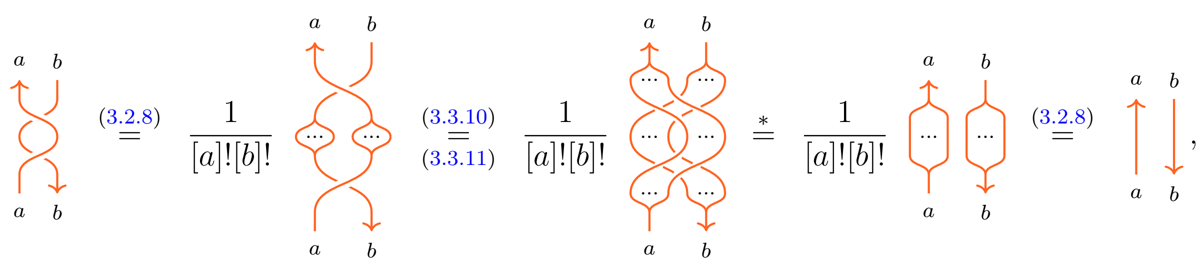


(DCROSS)

Proof. The upwards-oriented relation (for arbitrary strand colours) is part of Proposition 3.3.5. Taking mates yields the downwards-oriented relation. We will prove the first equality with mixed string orientations; the other equalities follow after applications of mate, hflip, and vflip. We first consider the thin case, and compute:

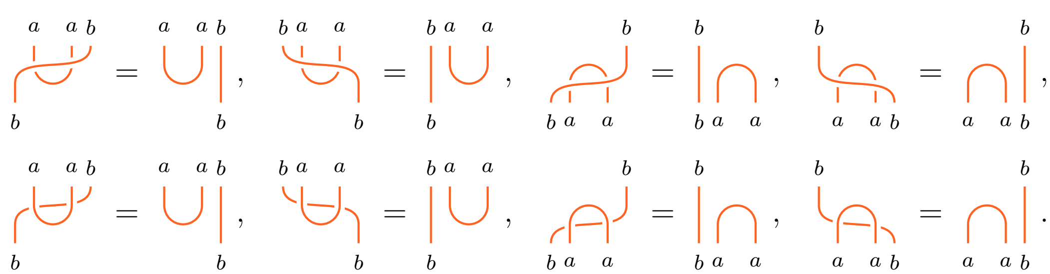


For general $a, b \in \mathbb{N}$ we then have:



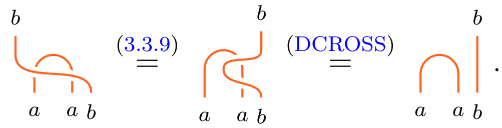
where in the equality labelled $*$ we use the thin case of the double-crossing identity ab times to disentangle the strands in the middle of the diagram. The same proof works for strands of arbitrary colour. $\square$

Lemma 3.3.15. Cups and caps in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ slide through crossings, i.e. for all $a, b \in \mathbb{N}$, for all possible orientations of the strands, and for arbitrary strand colours:



(3.3.12)

Proof. These identities follow immediately from (3.3.9) and (DCROSS). For instance, with strands of arbitrary orientation and colour,

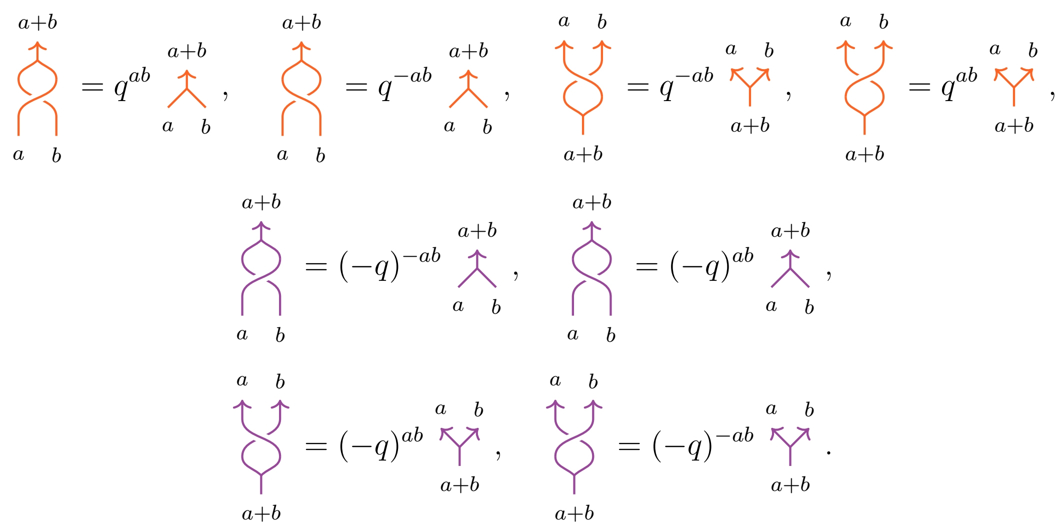


□

Proposition 3.3.16. The crossings of Definitions 3.3.1, 3.3.4, and 3.3.6 equip $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ with the structure of a braided monoidal category.

Proof. In light of Lemma 3.3.14, all that remains is to show that each of the generating morphisms for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ slide through crossings in all possible orientations. Lemma 3.3.12 tells us that this is true for merge and split morphisms, and Lemma 3.3.15 shows that this is also true for cups and caps. □

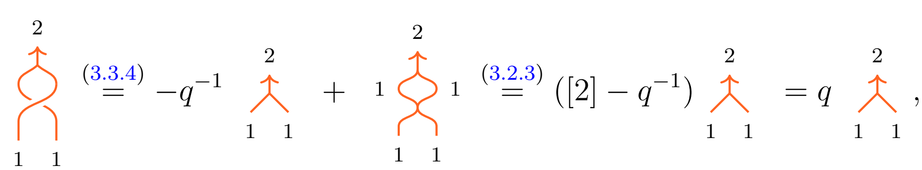
Lemma 3.3.17. For all $a, b \in \mathbb{N}$, the following identities hold in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$:



(3.3.13)

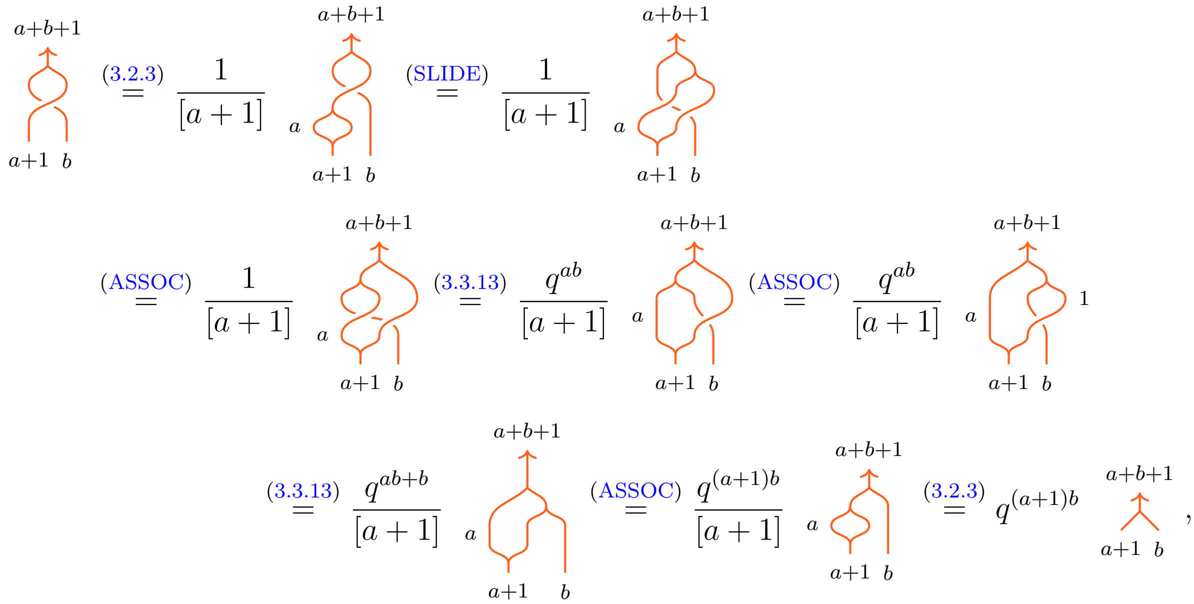
The following argument takes inspiration from the proof of [LT26, Lem. 2B.17].

Proof. It suffices to prove the first and third orange identities; the others follow after applications of hflip and/or cswap. The arguments for merges and splits are essentially the same, so we will only prove the first identity. For the thin case,



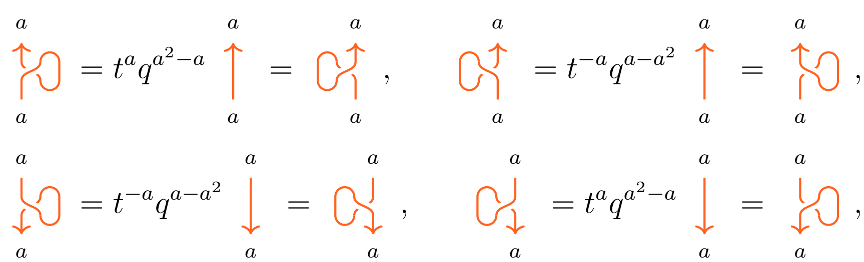
as desired. Assume that the first identity in (3.3.13) holds for all $a, b \in \mathbb{N}$ such that $a+b \leq k$ for some fixed $k \in \mathbb{N}$ (the thin base case establishes this for $k = 2$; the cases where $a = 0$ or

$b = 0$ follow immediately from (TRIV)). Let $a, b \in \mathbb{N}_{>0}$ such that $a + b = k$. Note that this implies $a + 1, b + 1 \leq k$, and hence our induction hypothesis tells us that the first identity in (3.3.13) holds when the bottom strands are labelled (a, b), $(b, 1)$, or $(1, a)$. We then have:

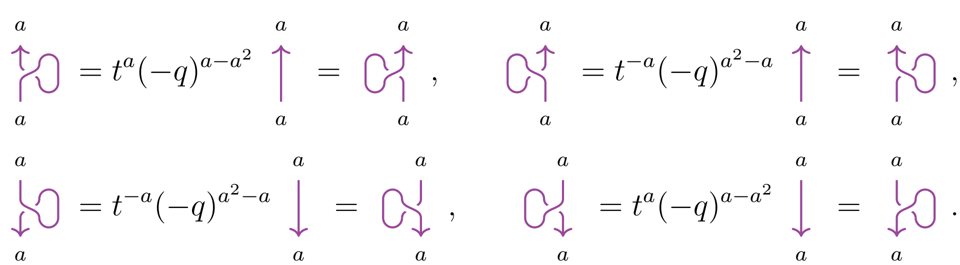


and so the first identity in (3.3.13) holds when the bottom strands are labelled $(a+1, b)$. An analogous calculation shows that the same is true for strands labelled $(a, b + 1)$, completing the induction. $\square$

Lemma 3.3.18. The following *curl relations* hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, for all $a \in \mathbb{N}$:

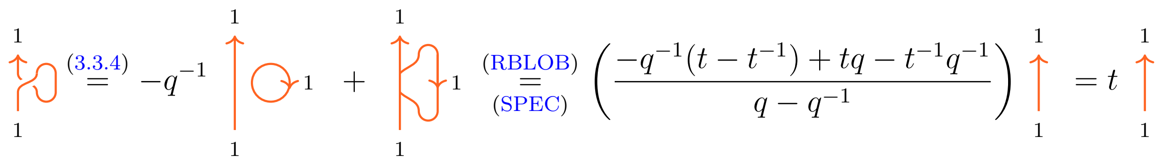


(3.3.14)

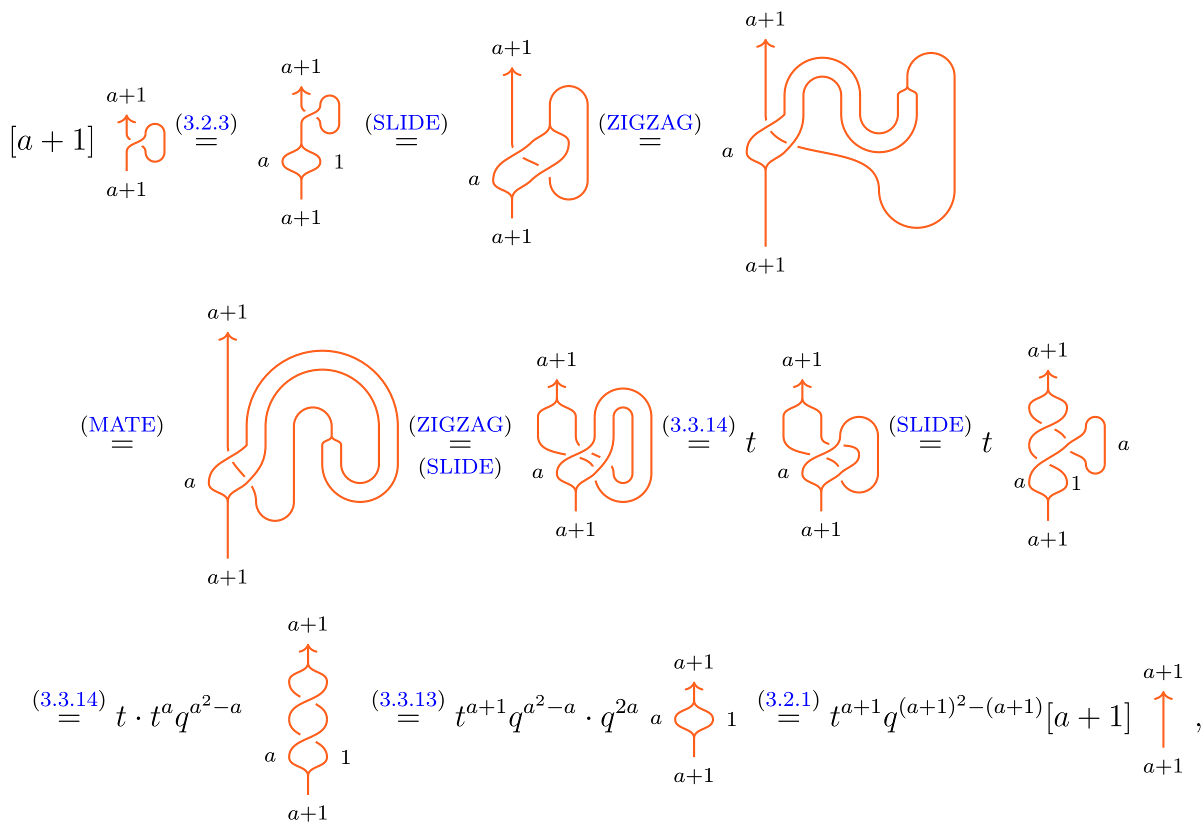


(3.3.15)

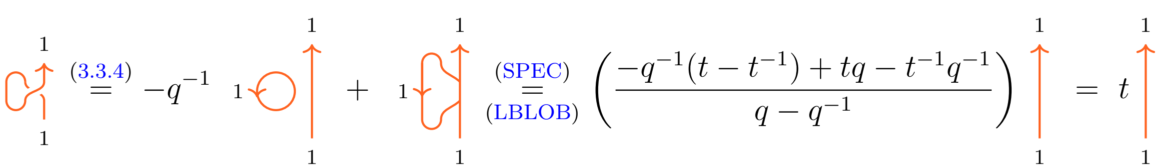
Proof. We will prove the first pair of equalities; the others follow after applications of hflip, vflip, mate, and cswap. In the thin case we have:



Assuming that the first equality in (3.3.14) holds for some $a \in \mathbb{N}$, we then have:



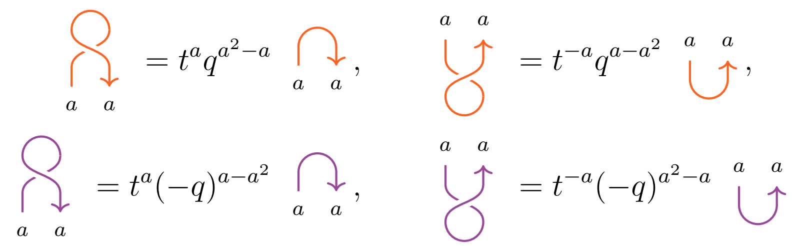
which establishes the first equality in (3.3.14) by induction. Similarly,



establishing the thin case of the second equality in (3.3.14). For general $a \in \mathbb{N}$, that equality then follows from an inductive argument that is totally analogous to the one given above. $\square$

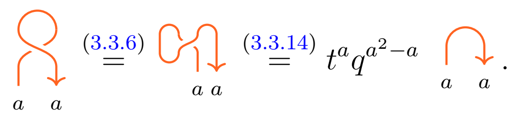
Lemma 3.3.19. The following *cap-* and *cup-delooping* relations hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, for all

$a \in \mathbb{N}$:



(3.3.16)

Proof. We compute:



The remaining identities follow after applications of vflip and/or cswap. □

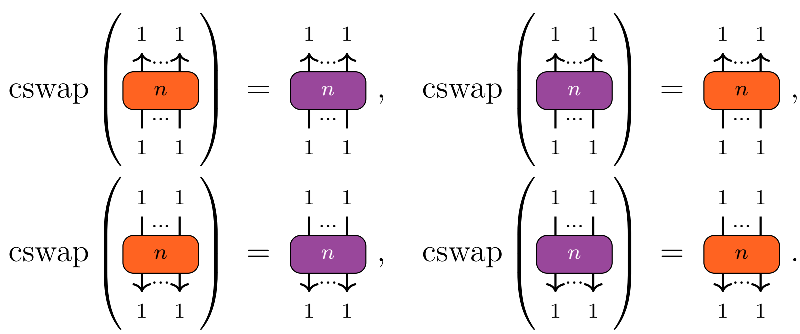
Corollary 3.3.20. In $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ the following *bubble-flip* identities hold for all $a \in \mathbb{N}$:

(3.3.17)

Proof. We have:

□

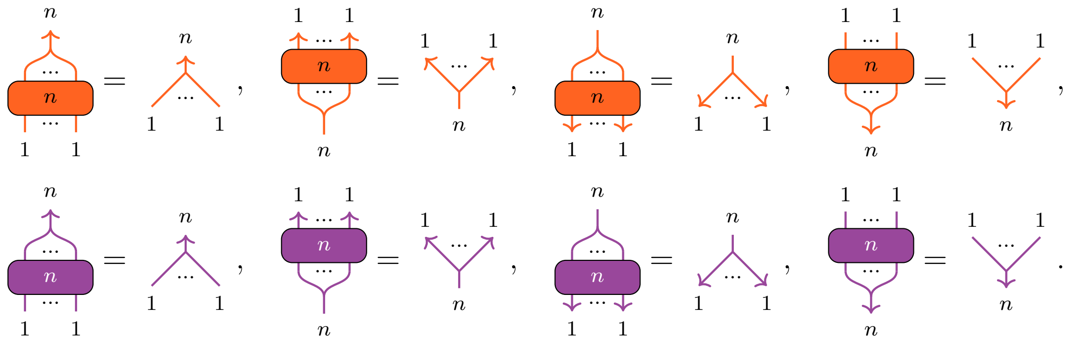
Lemma 3.3.21. For all $n \in \mathbb{N}$, the following identities hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$:



(3.3.18)

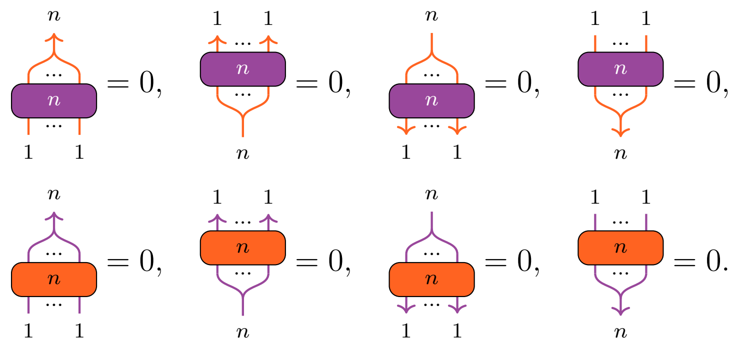
Proof. The upwards-oriented identities follow immediately from the fact that cswap fixes thin crossings (see Lemma 3.3.7) and that, as noted previously, mapping $q \mapsto -q^{-1}$ sends symmetrizers to antisymmetrizers and vice versa (this follows directly from Definition 2.5.5 and Lemma 2.4.5). The downwards-oriented identities then follow after taking mates and using Lemma 2.5.12. □

Lemma 3.3.22. For all $n \in \mathbb{N}$, the following identities hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$:



(3.3.19)

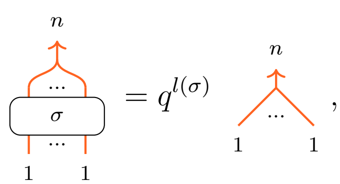
For all $n \geq 2$, the following identities also hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$:



(3.3.20)

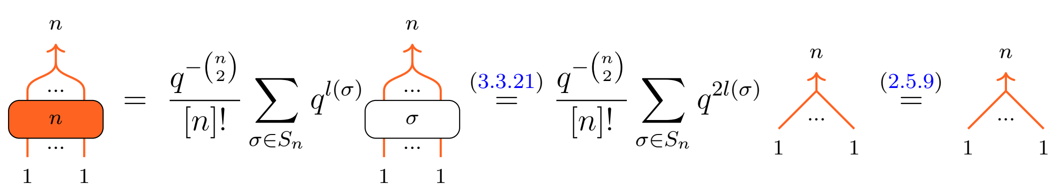
Note that the colours of the strands and (anti)symmetrizers match in (3.3.19), whereas in (3.3.20) the colours are opposite.

Proof. In light of (ASSOC) and (3.3.13), for any adjacent transposition $\sigma \in S_n$ we have

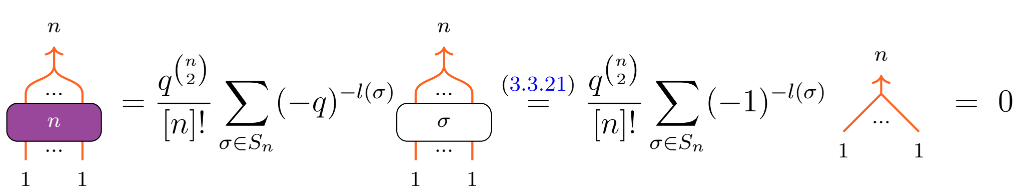


(3.3.21)

which immediately implies that (3.3.21) holds for arbitrary permutations $\sigma \in S_n$. Hence:



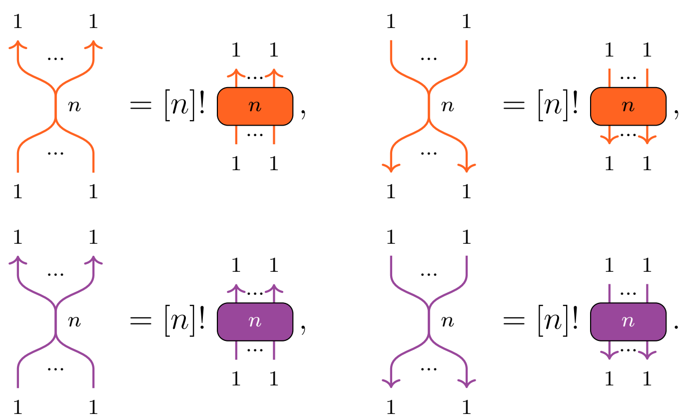
and so the first identity in (3.3.19) holds. For (3.3.20), we compute:



where the last equality follows from the fact that S_n contains the same number of even and odd permutations. For both (3.3.19) and (3.3.20), the second identity follows from essentially the same argument used to prove the first one. The third and fourth identities follow after taking mates and using Lemma 2.5.12. Applying cswap yields the remaining identities. □

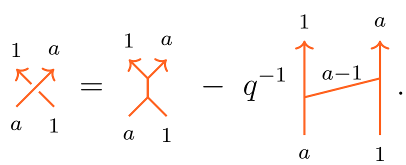
The following result will play a crucial role in Chapter 4, in which we establish a precise correspondence between $\uparrow_n$ and the n-strand upwards-oriented symmetrizer, and similarly for the other types of strands.

Proposition 3.3.23. For each $n \in \mathbb{N}$, the following identities hold in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$:



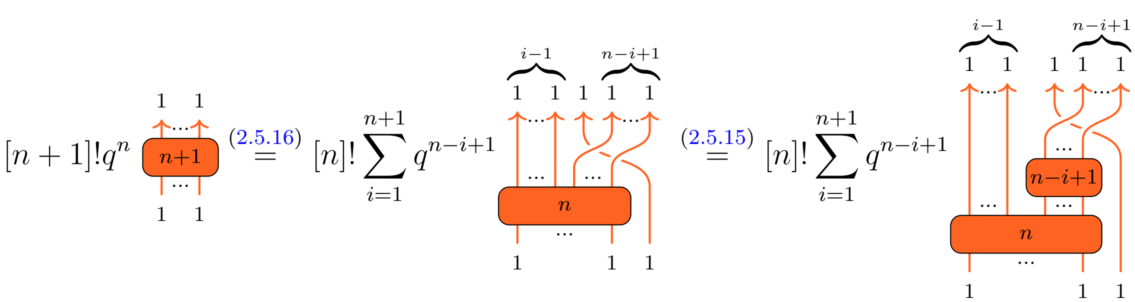
(3.3.22)

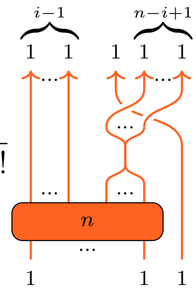
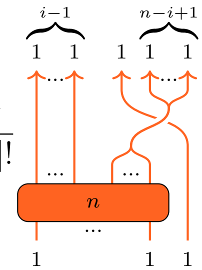
Proof. For use in the calculation below, note that taking $b = 1$ in Definition 3.3.1 yields

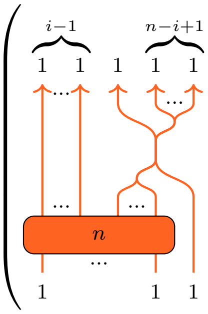
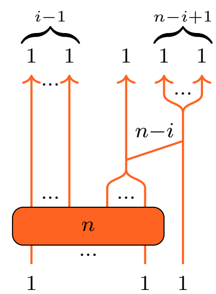


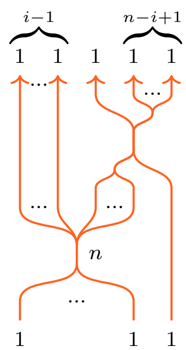
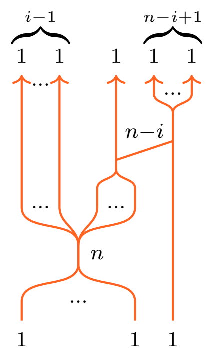
(3.3.23)

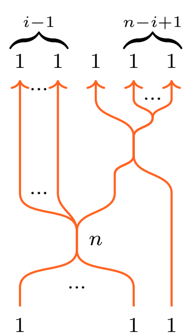
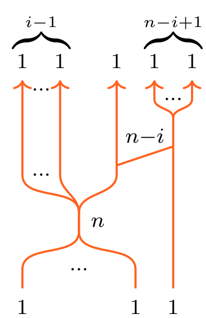
We will prove the first identity by induction on n; the second follows after taking mates and using Lemma 2.5.12, and the purple identities follow after applying cswap and dividing by $(-1)^{\binom{n}{2}}$. The equalities for $n = 0$ and $n = 1$ hold trivially. Assuming the first equality in (3.3.22) holds for some $n \geq 1$, we compute:

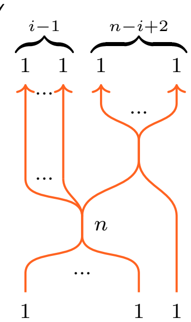
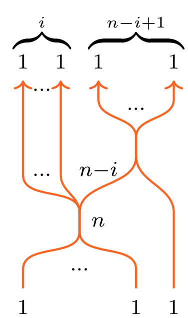


$$\overset{(3.3.22)}{=} \sum_{i=1}^{n+1} \frac{[n]!q^{n-i+1}}{[n-i+1]!}$$  $$\overset{(\mathrm{SLIDE})}{=} \sum_{i=1}^{n+1} \frac{[n]!q^{n-i+1}}{[n-i+1]!}$$ 

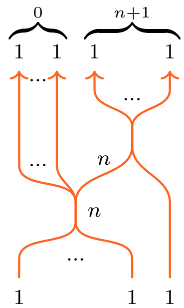
$$\overset{(3.3.23)}{=} \sum_{i=1}^{n+1} \frac{[n]!q^{n-i+1}}{[n-i+1]!} \Bigg($$  $$-\, q^{-1}$$  $$\Bigg)$$

$$\overset{(3.3.22)}{=} \sum_{i=1}^{n+1} \frac{q^{n-i+1}}{[n-i+1]!} \Bigg($$  $$-\, q^{-1}$$  $$\Bigg)$$

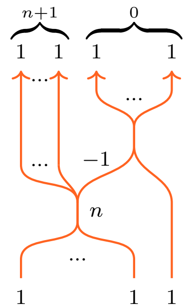
$$\overset{(3.2.8)}{=} \sum_{i=1}^{n+1} q^{n-i+1} \Bigg($$  $$-\, q^{-1}$$  $$\Bigg)$$

$$= q^{n+1} \sum_{i=1}^{n+1} q^{-i} \Bigg($$  $$-\, q^{-1}$$  $$\Bigg)$$

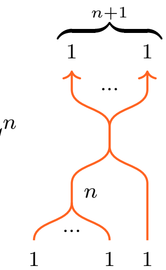
$\overset{*}{=} q^{n+1} \Big(q^{-1}$



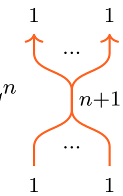
$- q^{-n-2}$



$\Big) \overset{**}{=} q^n$



$\overset{(3.3.22)}{=} q^n$



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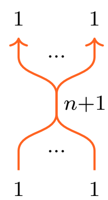
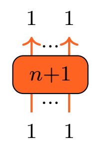
where in the step marked $*$ we collapse the telescoping sum, and in the step marked $**$ we use the fact that any diagram with a strand labelled by -1 is equal to zero. Dividing through by q^n yields



$= [n+1]!$



,

as desired. ∎

Chapter 4

Isomorphism of categories

In this chapter, we construct an isomorphism between $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ and a partial idempotent completion of $\mathcal{OS}$, denoted $\mathcal{OS}^W$. As discussed in the introduction, some similar functors have been defined involving other (completions of) diagrammatic categories, but this approach is relatively new, and has not yet been applied to web categories. Our isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^W$ allows one to systematically translate theorems for $\mathcal{OS}$ into corresponding theorems for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. In the remaining chapters, we use this approach to further study the $\mathfrak{gl}$-web categories.

Definition 4.1. Given a category $\mathcal{C}$, the *Karoubi envelope of* $\mathcal{C}$ (also called the *idempotent completion* of $\mathcal{C}$) is a category denoted $\mathrm{Kar}(\mathcal{C})$. The objects of this category are pairs (X, e), where $X \in \mathrm{ob}(\mathcal{C})$ and $e\colon X \to X$ is an idempotent morphism in $\mathcal{C}$, i.e. $e \circ e = e$. A morphism $\alpha\colon (X, e) \to (Y, f)$ in $\mathrm{Kar}(\mathcal{C})$ is a $\mathcal{C}$-morphism $\alpha\colon X \to Y$ that satisfies $f \circ \alpha \circ e = \alpha$. Composition of morphisms in $\mathrm{Kar}(\mathcal{C})$ is inherited from $\mathcal{C}$. The identity morphism for (X, e) is e.

If $\mathcal{C}$ is monoidal, then there is a natural monoidal structure on $\mathrm{Kar}(\mathcal{C})$ given on objects by $(X, e) \otimes (Y, f) = (X \otimes_{\mathcal{C}} Y, e \otimes_{\mathcal{C}} f)$, with the tensor product on morphisms being inherited from $\mathcal{C}$. The monoidal unit in $\mathrm{Kar}(\mathcal{C})$ is $(\mathbb{1}_{\mathcal{C}}, \mathrm{id}_{\mathbb{1}_{\mathcal{C}}})$.

There is a full and faithful embedding $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ that acts as $X \mapsto (X, \mathrm{id}_X)$ on objects and as the identity on morphisms. We will bring up other properties of Karoubi envelopes as they become relevant.

Note that some authors instead write $\mathrm{Kar}(\mathcal{C})$ for the *additive Karoubi envelope* of $\mathcal{C}$, i.e. the additive envelope of the category $\mathrm{Kar}(\mathcal{C})$ we defined above.

Definition 4.2. We write $\mathcal{OS}^W$ (short for "$\mathcal{OS}$-Web") for the full subcategory of $\mathrm{Kar}(\mathcal{OS})$ generated (via tensor product) by the objects

$$\mathrm{Sym}^n := \left(\uparrow^{\otimes n}, \boxed{n}\right), \quad \Lambda^n := \left(\uparrow^{\otimes n}, \boxed{n}\right),$$

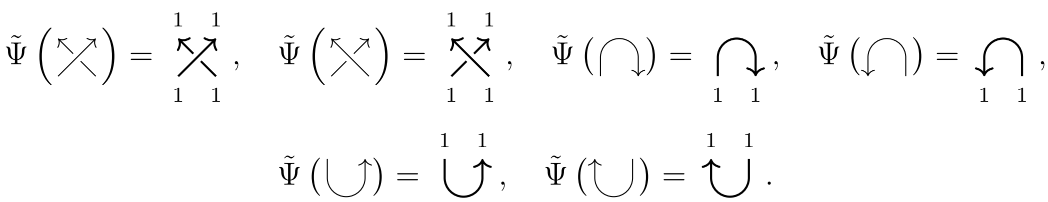
$$\mathrm{Sym}^{-n} := \left(\downarrow^{\otimes n}, \boxed{n}\right), \quad \Lambda^{-n} := \left(\downarrow^{\otimes n}, \boxed{n}\right),$$

for all $n \in \mathbb{N}$. This category is the *partial idempotent completion* of $\mathcal{OS}$ that we mentioned at the start of the chapter; we are adjoining objects to $\mathcal{OS}$ corresponding only to the (anti)symmetrizers, not all idempotent morphisms.

Remark 4.3. We have $\mathrm{Sym}^1 = \Lambda^1 = (\uparrow, \mathrm{id}_\uparrow)$ and $\mathrm{Sym}^{-1} = \Lambda^{-1} = (\downarrow, \mathrm{id}_\downarrow)$. Hence, $\mathcal{OS}^W$ contains a full subcategory isomorphic to $\mathcal{OS}$, namely the image of the canonical embedding $\mathcal{OS} \to \mathrm{Kar}(\mathcal{OS})$.

We now work to define a functor $\Psi \colon \mathcal{OS}^W \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, which we will eventually prove is an isomorphism.

Definition 4.4. We define a $\Bbbk_{q,t}$-linear strict monoidal functor $\tilde{\Psi} \colon \mathcal{OS} \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ as follows. On objects, we set $\tilde{\Psi}(\uparrow) = \uparrow_1$ and $\tilde{\Psi}(\downarrow) = \downarrow_1$. On morphisms, we set:

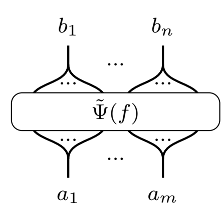


(4.1)

Lemma 4.5. The functor $\tilde{\Psi}$ specified above is well-defined.

Proof. We need to show that $\tilde{\Psi}$ respects the defining relations for $\mathcal{OS}$. The fact that the images of the relations in (3.1.2) hold follows from Proposition 3.3.16. The image of (3.1.3) follows immediately from the definition of the thin crossings, (3.3.4). The images of the curl relations in (3.1.4) are two of the identities from Lemma 3.3.18 (with $a = 1$). The images of the $\mathcal{OS}$-zigzag relations are simply the defining relations (ZIGZAG) (with $a = 1$). Finally, the image of the last relation from (3.1.4) is the defining relation (SPEC). ☐

Definition 4.6. We define a $\Bbbk_{q,t}$-linear strict monoidal functor $\Psi \colon \mathcal{OS}^W \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ as follows. On objects, for all $a \in \mathbb{Z}$ we set $\Psi(\mathrm{Sym}^a) = |_a$ and $\Psi(\Lambda^a) = |_a$. Here and throughout the rest of this chapter, we adopt the convention that an unoriented orange strand labelled by $a \geq 0$ represents $\uparrow_a$, an unoriented orange strand labelled by $a < 0$ represents $\downarrow_{|a|}$, and similarly for purple strands. (This convention overrides the previous one that said any diagram featuring a negative label was zero.) For the action on morphisms, suppose that $f \colon X \to Y$ is a morphism in $\mathcal{OS}^W$, where $X = X_1^{a_1} \otimes \cdots \otimes X_m^{a_m}$ and $Y = Y_1^{b_1} \otimes \cdots \otimes Y_n^{b_n}$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Z}$, and $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \{\mathrm{Sym}, \Lambda\}$. Then we set:

$$\Psi(f) = \frac{1}{s^\uparrow(a_1, \ldots, a_m) s^\downarrow(b_1, \ldots, b_n)}$$  ,

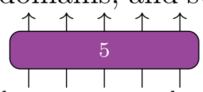
where $\tilde{\Psi}$ is the functor specified in Definition 4.4, and $s^\uparrow$ and $s^\downarrow$ are given by:

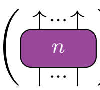
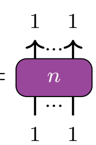
$$s^\uparrow(a) = \begin{cases} [a]! & \text{if } a \geq 0, \\ 1 & \text{otherwise,} \end{cases} \qquad s^\uparrow(a_1, \ldots, a_m) = \prod_{i=1}^m s^\uparrow(a_i),$$

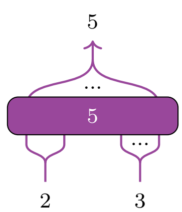
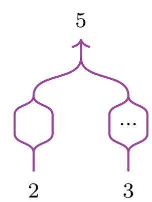
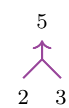
$$s^{\downarrow}(b) = \begin{cases} [|b|]! & \text{if } b \leq 0, \\ 1 & \text{otherwise,} \end{cases} \qquad s^{\downarrow}(b_1, \dots, b_n) = \prod_{i=1}^{n} s^{\downarrow}(b_i).$$

The strands appearing in the definition of $\Psi(f)$ should be coloured according to the values of the X_i and Y_i. Explicitly, the multi-split morphism with bottom label a_i appearing in the above definition is coloured orange if $X_i = \mathrm{Sym}$, and purple if $X_i = \Lambda$. Similarly, the multi-merge morphism with top label b_i is coloured orange if $Y_i = \mathrm{Sym}$, and purple if $Y_i = \Lambda$. (All of the strands appearing in $\tilde{\Psi}(f)$ are thin, so they can be drawn in an arbitrary colour.)

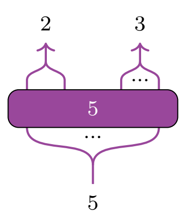
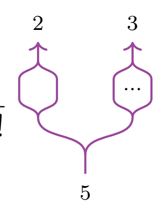
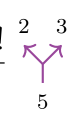
Note that for any $a_1, \dots, a_m \in \mathbb{Z}$, we have $s^{\uparrow}(a_1, \dots, a_m)s^{\downarrow}(a_1, \dots, a_m) = [|a_1|]! \cdots [|a_m|]!$.

Examples 4.7. It is important to note that a given morphism in $\mathcal{OS}$ may correspond to several morphisms in $\mathcal{OS}^W$ with different domains and codomains, and such morphisms have different Ψ-images. For instance, in light of (2.5.15), $f =$  is a morphism from $\Lambda^2 \otimes \Lambda^3$ to Λ^5. (Indeed, f is a morphism from $\Lambda^a \otimes \Lambda^b$ to $\Lambda^c \otimes \Lambda^d$ for any $a, b, c, d \in \mathbb{N}$ satisfying $a + b = 5 = c + d$. We will illustrate the case $a = 2, b = 3, c = 5, d = 0$ here; other cases are analogous.) While we will typically leave the domain and codomain of morphisms in $\mathcal{OS}^W$ implicit to avoid visual clutter, for the following examples we will explicitly indicate them by writing the domain as a subscript and the codomain as a superscript, e.g. $f^{\Lambda^5}_{\Lambda^2 \otimes \Lambda^3}$.

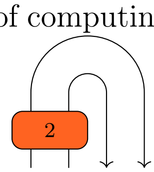
It follows immediately from Definition 4.4 that $\tilde{\Psi}\Bigg($  $\Bigg) =$  for each $n \in \mathbb{N}$ (and similarly for symmetrizers), so:

$$\Psi\left(f^{\Lambda^5}_{\Lambda^2 \otimes \Lambda^3}\right) = \frac{1}{s^{\uparrow}(2,3)s^{\downarrow}(5)}$$  $$\overset{(3.3.19)}{=} \frac{1}{[2]![3]!}$$  $$\overset{(3.2.8)}{=}$$  .

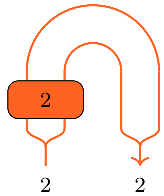
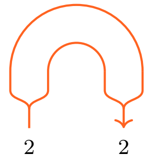
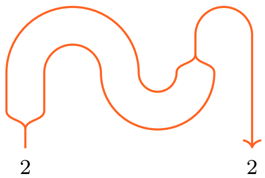
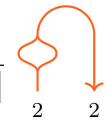
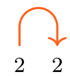
On the other hand, we may view f as a morphism from Λ^5 to $\Lambda^2 \otimes \Lambda^3$, and then:

$$\Psi\left(f^{\Lambda^2 \otimes \Lambda^3}_{\Lambda^5}\right) = \frac{1}{s^{\uparrow}(5)s^{\downarrow}(2,3)}$$  $$\overset{(3.3.19)}{=} \frac{1}{[5]!}$$  $$\overset{(3.2.8)}{=} \frac{[2]![3]!}{[5]!}$$  .

Using (3.2.3), we have $\Psi(f^{\Lambda^5}_{\Lambda^2 \otimes \Lambda^3}) \circ \Psi(f^{\Lambda^2 \otimes \Lambda^3}_{\Lambda^5}) = \mathrm{id}_{\uparrow_5}$, which is the Ψ-image of the 5-string antisymmetrizer viewed as the identity morphism for Λ^5, i.e. $\Psi(f^{\Lambda^5}_{\Lambda^5})$.

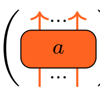
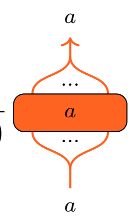
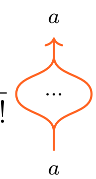
For an example of computing the Ψ-image of a morphism involving downwards-oriented strings, consider $g =$ . It is straightforward to check that g is a morphism from

$\mathrm{Sym}^2 \otimes \mathrm{Sym}^{-2}$ to $\mathbb{1} = \mathrm{Sym}^0$. Then:

$$\Psi\left(g^{\mathrm{Sym}^0}_{\mathrm{Sym}^2\otimes\mathrm{Sym}^{-2}}\right) = \frac{1}{s^{\uparrow}(2,-2)s^{\downarrow}(0)}$$

$$\overset{(3.3.19)}{=} \frac{1}{[2]!}$$

$$\overset{(3.1.1)}{=} \frac{1}{[2]}$$

$$\overset{(\text{ZIGZAG})}{=} \frac{1}{[2]}$$

$$\overset{(3.2.1)}{=}$$

.

Proposition 4.8. The functor Ψ specified above is well-defined.

Proof. It is an immediate consequence of Definition 4.6 that $\Psi(f \otimes g) = \Psi(f) \otimes \Psi(g)$ for all morphisms f and g. For each $a \geq 0$, we have:

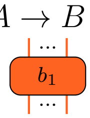
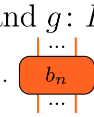
$$\Psi\left(\begin{array}{c}\mathrm{Sym}^a\\ | \\ \mathrm{Sym}^a\end{array}\right) = \Psi\Bigg($$

$$\Bigg) = \frac{1}{s^{\uparrow}(a)s^{\downarrow}(a)}$$

$$\overset{(3.3.19)}{=} \frac{1}{[a]!}$$

$$\overset{(3.2.8)}{=}$$

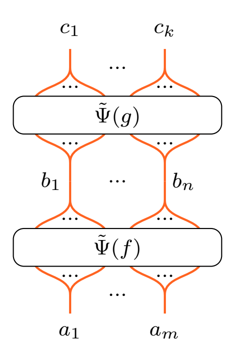
.

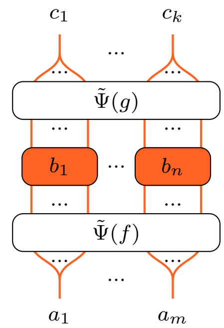
The calculations for $a < 0$ and for the identity on Λ^a are essentially the same. Since the objects of $\mathcal{OS}^W$ are monoidally generated by Sym^a and Λ^a for $a \in \mathbb{Z}$, we conclude that Ψ preserves all identity morphisms.

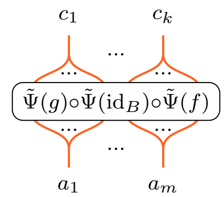
Since we already know that $\tilde{\Psi}$ is well-defined, all that remains is to show that Ψ respects composition. We will prove this in the special case where the domains and codomains of the composed morphisms are tensor products of objects of the form Sym^a. The general case is analogous. Let $a_1, \dots, a_m, b_1, \dots, b_n, c_1, \dots, c_k \in \mathbb{Z}$, set

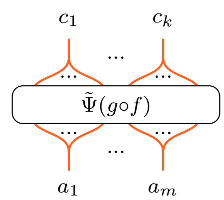
$$A = \mathrm{Sym}^{a_1} \otimes \cdots \otimes \mathrm{Sym}^{a_m}, B = \mathrm{Sym}^{b_1} \otimes \cdots \otimes \mathrm{Sym}^{b_n}, C = \mathrm{Sym}^{c_1} \otimes \cdots \otimes \mathrm{Sym}^{c_k},$$

and suppose that $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathcal{OS}^W$. For use in the following calculation, note that  ...  $\in \mathrm{Hom}_{\mathcal{OS}^W}(B, B)$ is the identity morphism for B. We compute:

$$\Psi(g) \circ \Psi(f) = \frac{1}{s^{\uparrow}(a_1,\dots,a_m)s^{\downarrow}(b_1,\dots,b_n)s^{\uparrow}(b_1,\dots,b_n)s^{\downarrow}(c_1,\dots,c_k)}$$


$$\overset{(3.3.22)}{=} \frac{[|b_1|]! \cdots [|b_n|]!}{s^{\uparrow}(a_1, \dots, a_m)[|b_1|]! \cdots [|b_n|]! s^{\downarrow}(c_1, \dots, c_k)}$$ 

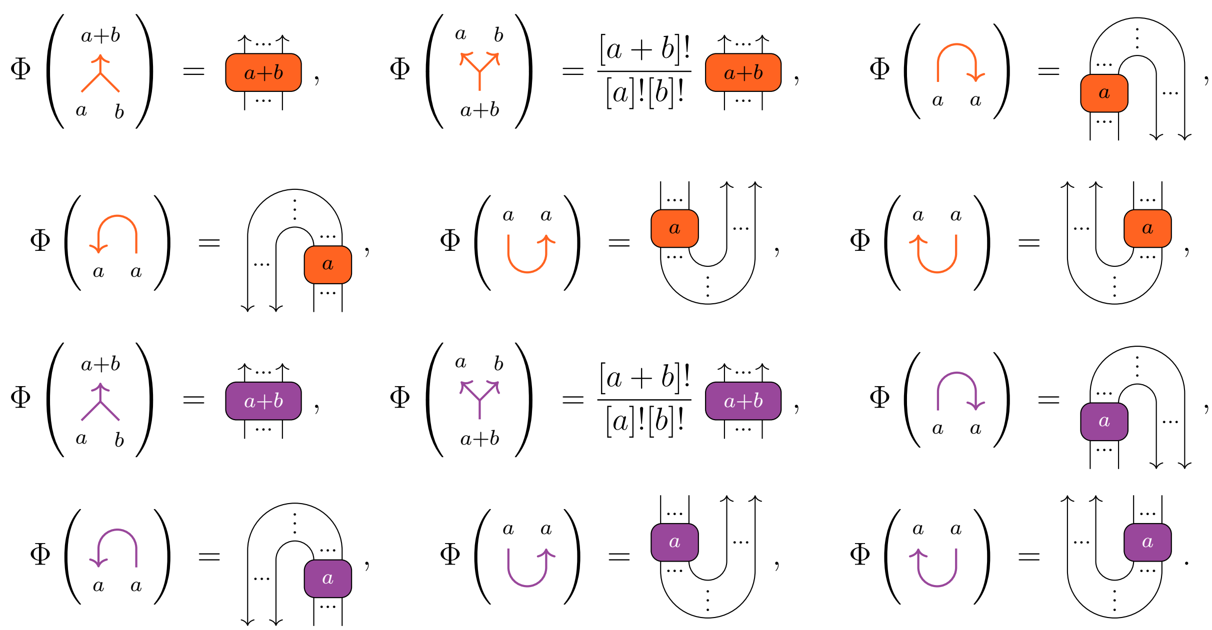
$$= \frac{1}{s^{\uparrow}(a_1, \dots, a_m) s^{\downarrow}(c_1, \dots, c_k)}$$ 

$$= \frac{1}{s^{\uparrow}(a_1, \dots, a_m) s^{\downarrow}(c_1, \dots, c_k)}$$  $= \Psi(g \circ f),$

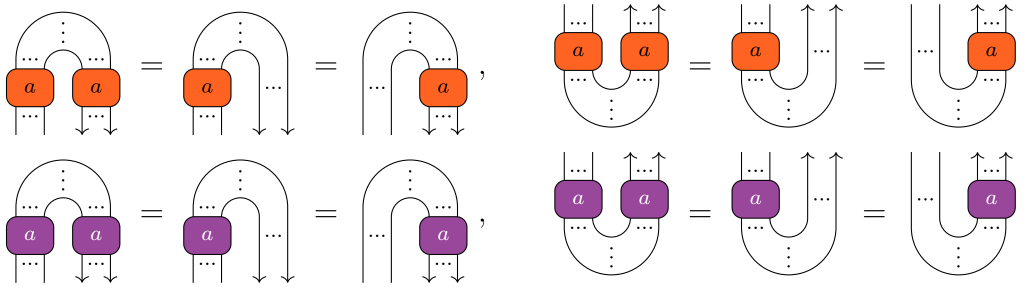
as desired. □

We now introduce a new functor Φ, which we will show is the inverse of Ψ.

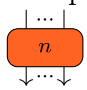
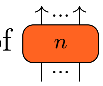
Definition 4.9. We define a $\Bbbk_{q,t}$-linear functor $\Phi\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathcal{OS}^W$ as follows. On objects, for each $a \in \mathbb{Z}$ we set $\Phi(|_a) = \mathrm{Sym}^a$ and $\Phi(|_a) = \Lambda^a$. The action on generating morphisms is given by:

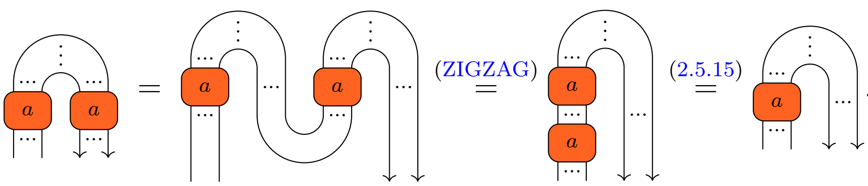


Lemma 4.10. For all $a \in \mathbb{N}$, the following identities hold in $\mathcal{OS}$:



(4.2)

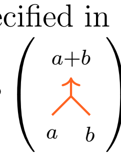
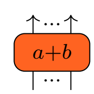
Proof. We will prove the first equality; the others follow from totally analogous calculations. Lemma 2.5.12 tells us that  is the mate of . Hence:



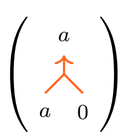
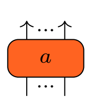
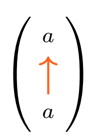
$\square$

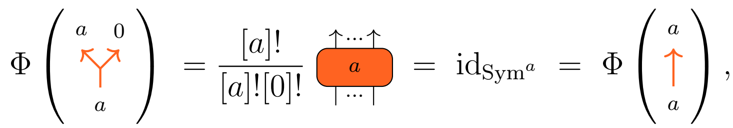
It follows immediately from Definition 4.9 that $\Phi \circ \mathrm{cswap} = Q \circ \Phi$; we will use this property in the following proof to easily translate between orange and purple identities. (Here and in what follows, we slightly abuse notation and write Q for the q-skew antilinear endofunctor of $\mathcal{OS}^W$ extended from the endofunctor $Q\colon \mathcal{OS} \to \mathcal{OS}$ introduced in Definition 3.1.7.)

Proposition 4.11. The functor Φ specified in Definition 4.9 is well-defined.

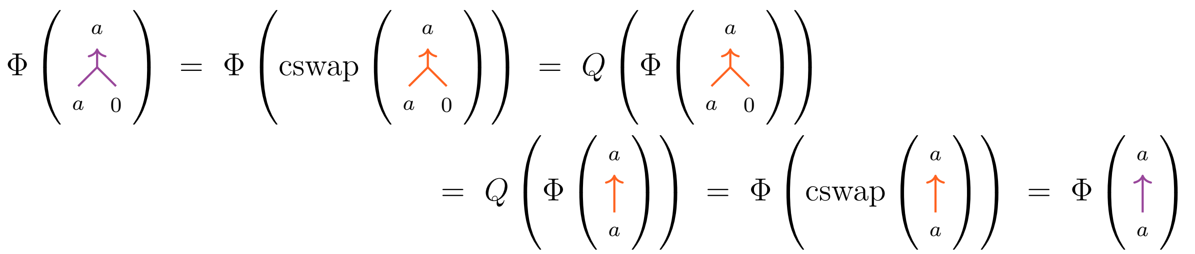
Proof. It follows quickly from (2.5.15) and (4.2) that the images specified in Definition 4.9 are indeed morphisms with the desired domain and codomain, e.g. Φ  $=$  is a morphism from $\mathrm{Sym}^a \otimes \mathrm{Sym}^b$ to Sym^{a+b}. We need to show that the defining relations for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, namely (TRIV), (ASSOC), (MIRROR), (MIX), (TRIV2), (SPEC), (ZIGZAG), (MATE), (RBLOB), (LBLOB), (RSQUIG), and (LSQUIG), are preserved by Φ. For the monochromatic identities, we will prove only the orange case. Using the previously mentioned fact that $\Phi \circ \mathrm{cswap} = Q \circ \Phi$, the orange version of each needed identity quickly implies the purple version. We will illustrate this below for the first identity; the process is essentially the same for all of the others.

Let $a \in \mathbb{N}$. Then:

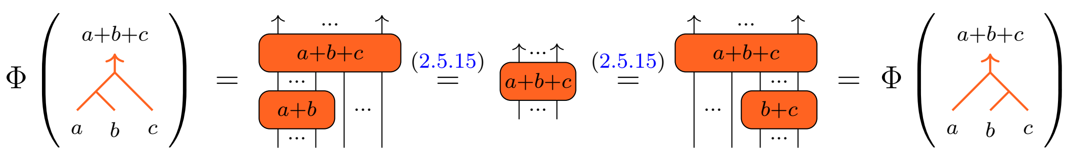
Φ  $=$  $= \mathrm{id}_{\mathrm{Sym}^a} = \Phi$  , so the orange version of the first relation in (TRIV) is preserved. Similarly,



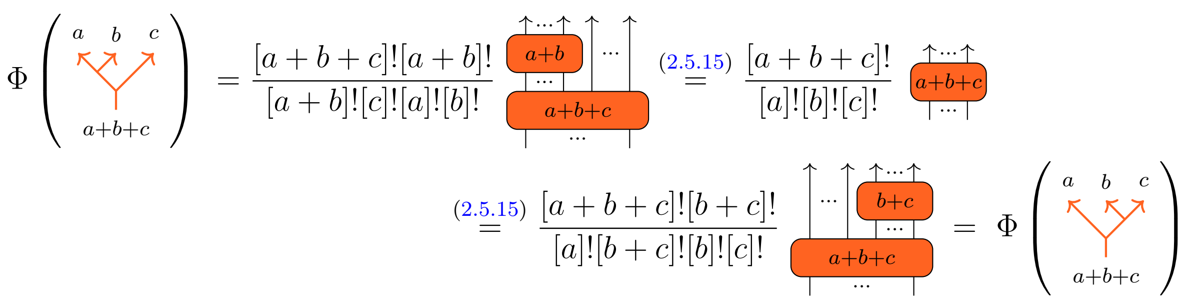
so the third relation in (TRIV) is preserved. The second and fourth relations follow from essentially the same calculations. For the purple version of the first identity, we have:



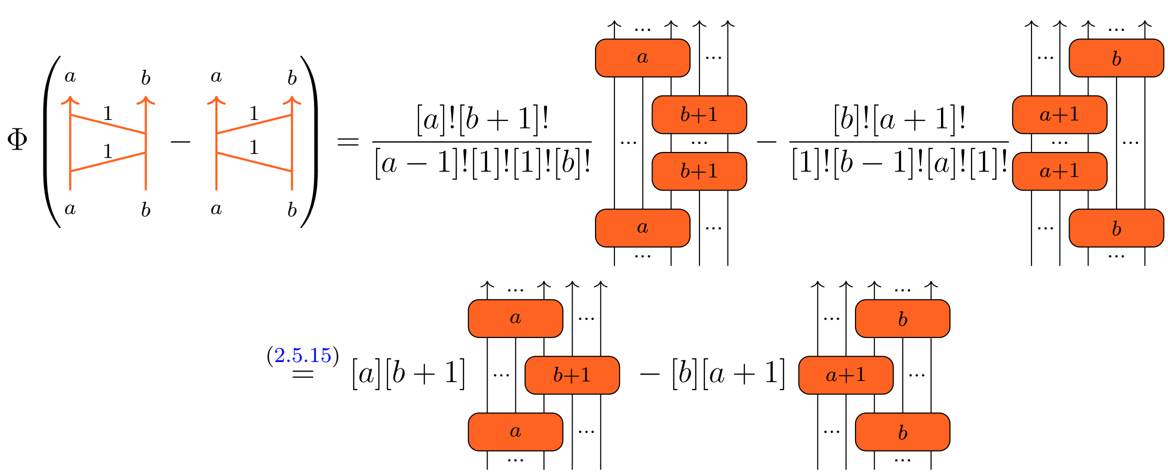
Next, for any $a, b, c \in \mathbb{N}$ we have:

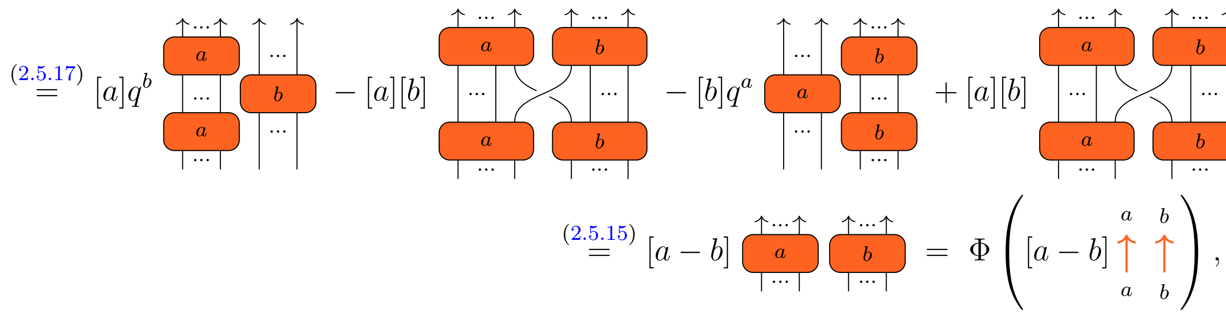


and similarly,

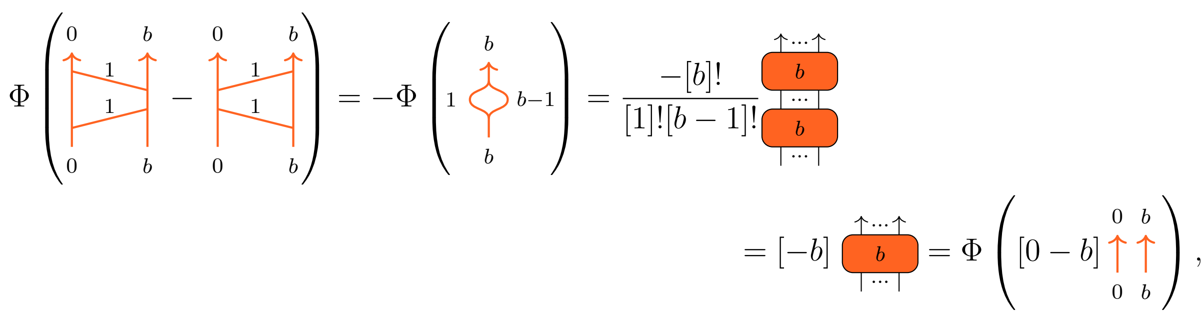


so (ASSOC) is preserved. Next, for $a, b \in \mathbb{N}_{\geq 1}$,

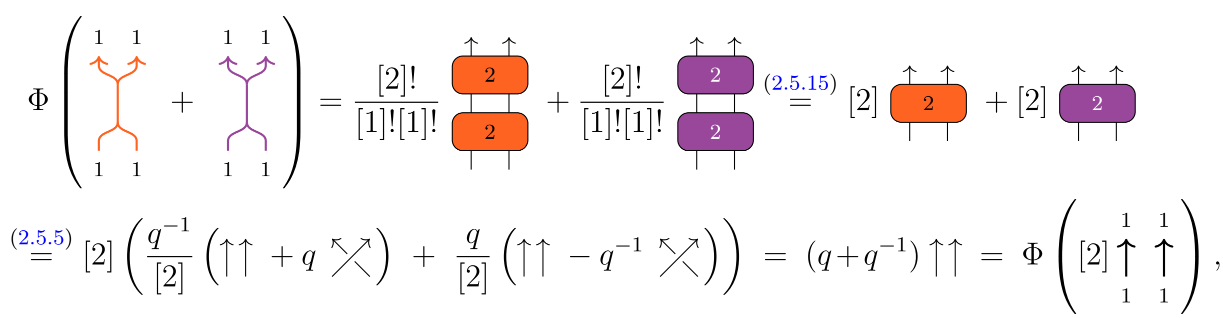




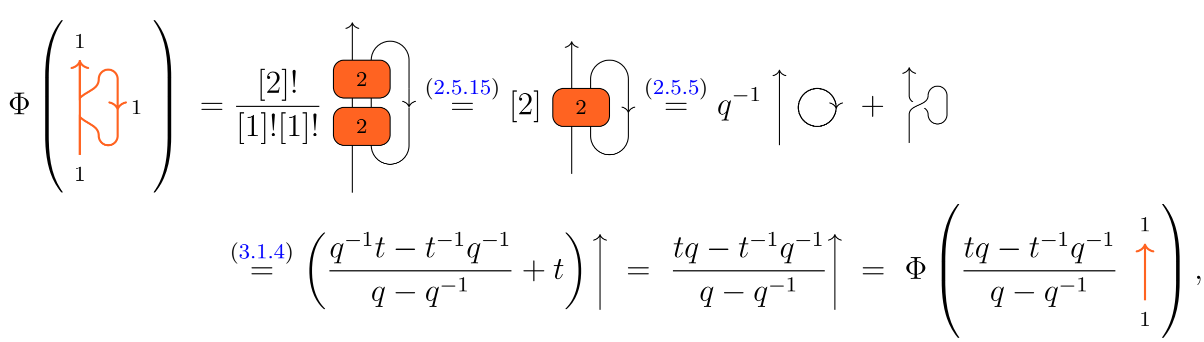
so (MIRROR) is preserved for all such a and b. In the special case where $a = b = 0$, (MIRROR) simply states that $0 = 0$, which is trivially preserved by Φ. If $a = 0$ and $b > 0$, we have:



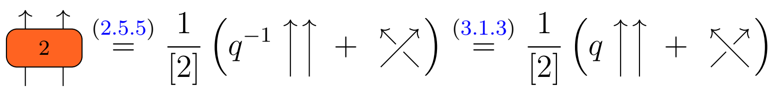
so (MIRROR) is preserved in this case as well. The case $a > 0, b = 0$ is analogous. Next,



which shows that (MIX) is preserved. The fact that (TRIV2), (ZIGZAG), and (MATE) are preserved by Φ follows from Lemma 4.10 and $\mathcal{OS}$ being strict pivotal. The Φ-image of (SPEC) is simply the last relation in (3.1.4) together with the analogous relation for a bubble with the opposite orientation. (That relation also holds in $\mathcal{OS}$; it follows from a calculation analogous to the proof of Corollary 3.3.20, with $a = 1$.) Next, we have:

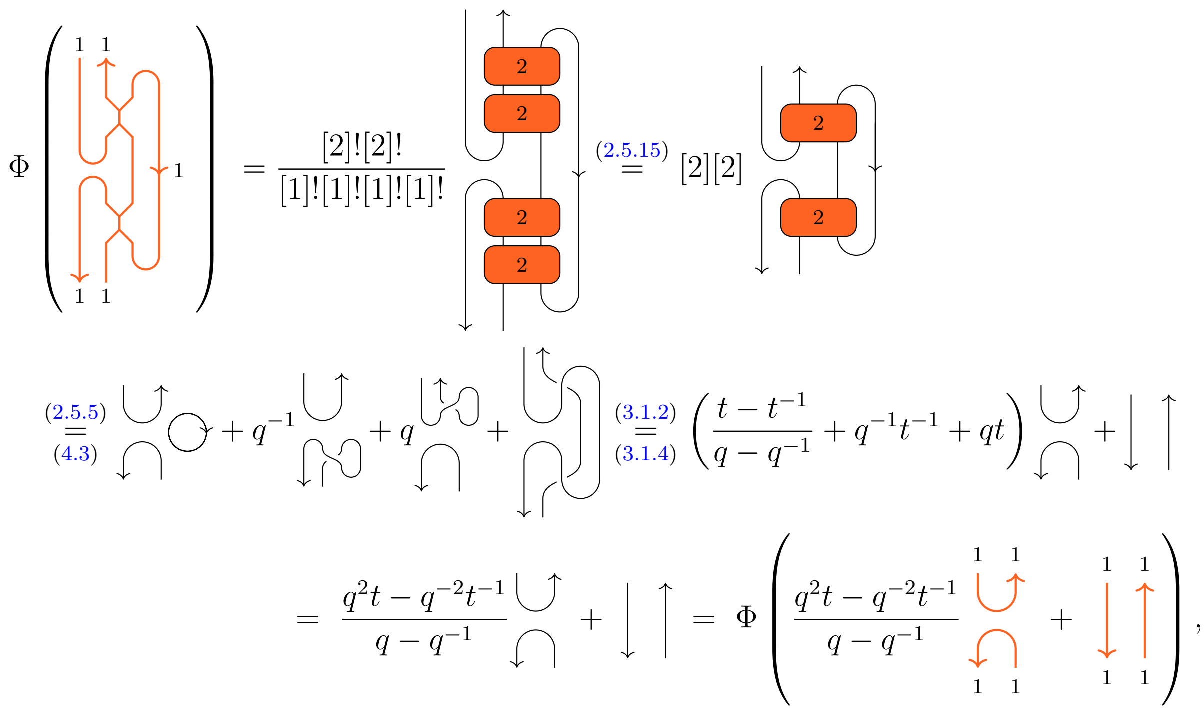


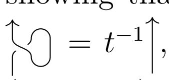
which shows that (RBLOB) is preserved. The calculation for (LBLOB) is essentially the same. For the final relations, we will use the fact that (3.1.3) allows us to express symmetrizers in terms of negative crossings:



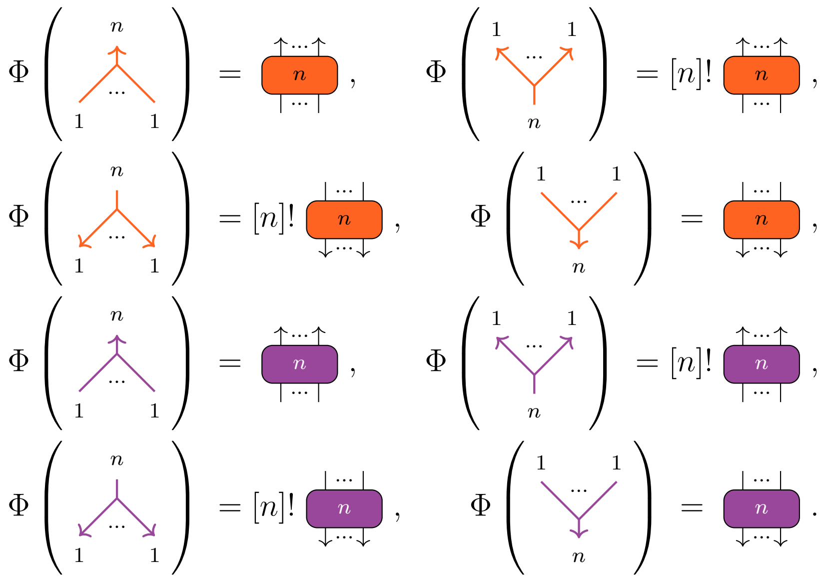
(4.3)

With this in mind, we compute:



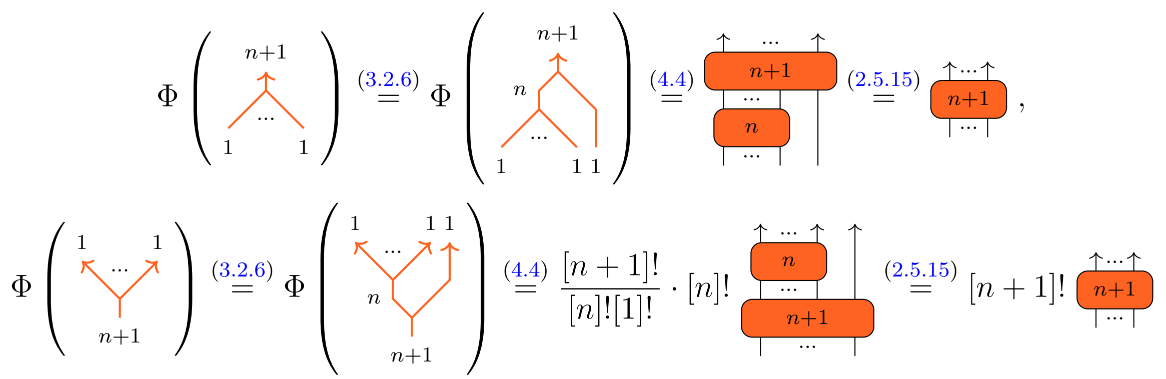
showing that the Φ-image of (RSQUIG) holds. (In the fifth equality, we use the identity , which follows straightforwardly from (3.1.3) and (3.1.4).) The calculation for (LSQUIG) is analogous. □

Lemma 4.12. For all $n \in \mathbb{N}$, we have:



(4.4)

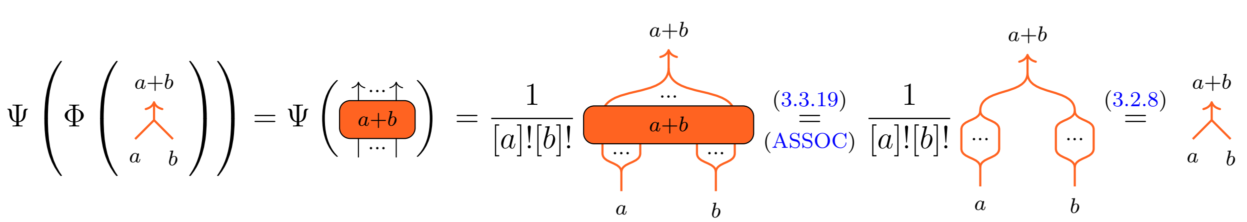
Proof. We will prove the first two equalities; the next two follow after taking mates, using the fact that Φ sends caps to caps and cups to cups, and hence mates to mates. The purple identities follow easily from the orange ones using the fact that $\Phi \circ \mathrm{cswap} = Q \circ \Phi$, as in the proof of Proposition 4.11. The desired identities hold trivially for $n = 0$ and $n = 1$. Assuming the first equality in (4.4) holds for some $n \geq 1$, we compute:

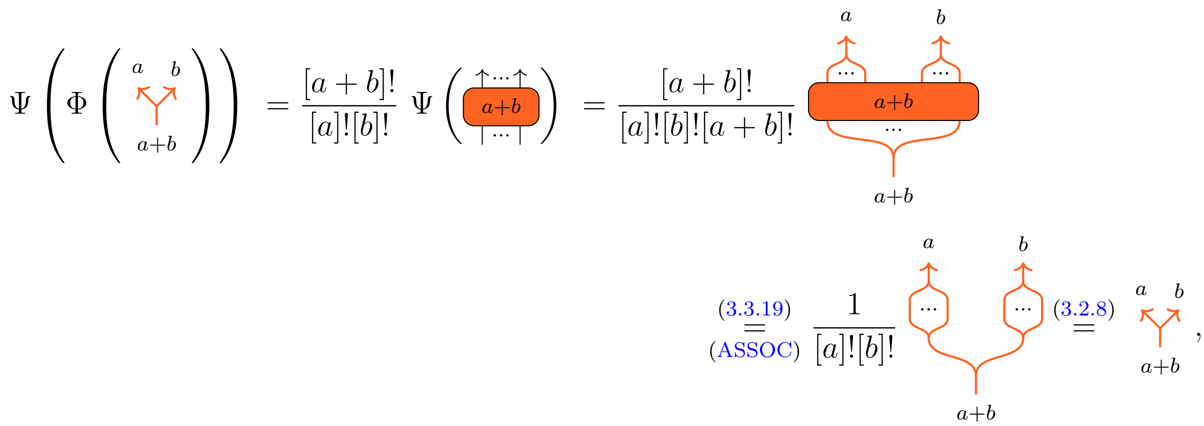


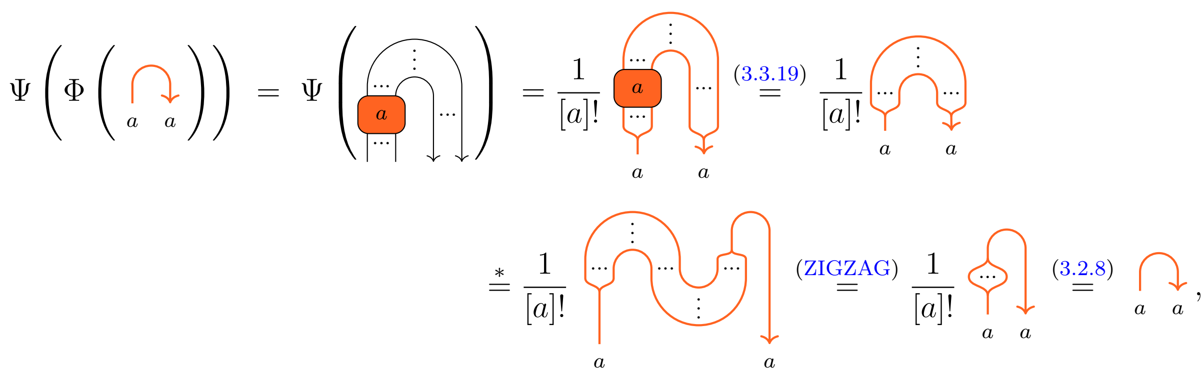
as desired. $\square$

Theorem 4.13. The functors $\Psi \colon \mathcal{OS}^W \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ and $\Phi \colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \mathcal{OS}^W$ are mutually inverse, and hence $\mathcal{OS}^W$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ are isomorphic as $\Bbbk_{q,t}$-linear strict monoidal categories.

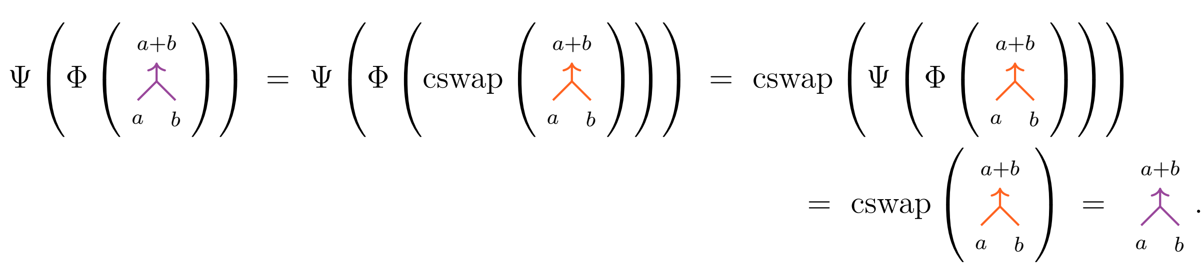
Proof. The composite functors $\Psi \circ \Phi$ and $\Phi \circ \Psi$ clearly act as the identity on objects, so it suffices to show that they each act as the identity on morphisms. Let $a, b \in \mathbb{N}$. We compute:



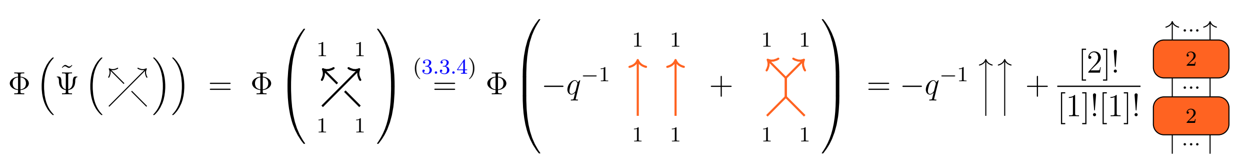


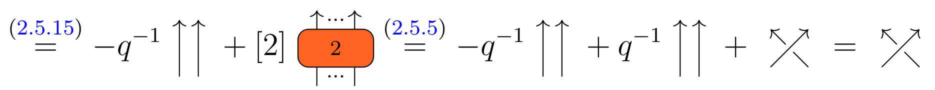


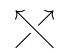
where in the equality marked $*$ we use the definition of downwards-oriented multi-split morphisms; see Definition 3.2.6. The calculations for the other orientation of cap and for the two orientations of cup are analogous. This shows that $\Psi \circ \Phi$ acts as the identity on orange generating morphisms. The same being true for purple generators follows quickly using $\Phi \circ \mathrm{cswap} = Q \circ \Phi$ and $\Psi \circ Q = \mathrm{cswap} \circ \Psi$ (the former equality was noted previously, and the latter is immediate from the definitions of Ψ, Q, and cswap). For instance, we have:

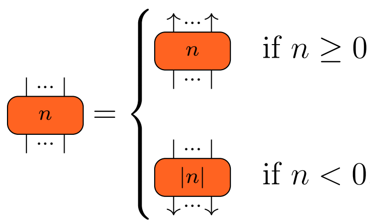


This completes the proof that $\Psi \circ \Phi = \mathrm{id}$. Next, we will show that $\Phi \circ \tilde{\Psi} = \mathrm{id}$, where we identify $\mathcal{OS}$ with the evident full subcategory of $\mathcal{OS}^W$ it is isomorphic to, as mentioned in Remark 4.3. (We remind the reader that $\tilde{\Psi} \colon \mathcal{OS} \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ was introduced in Definition 4.4). We compute:

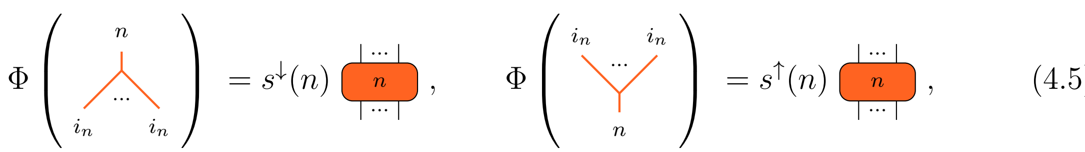




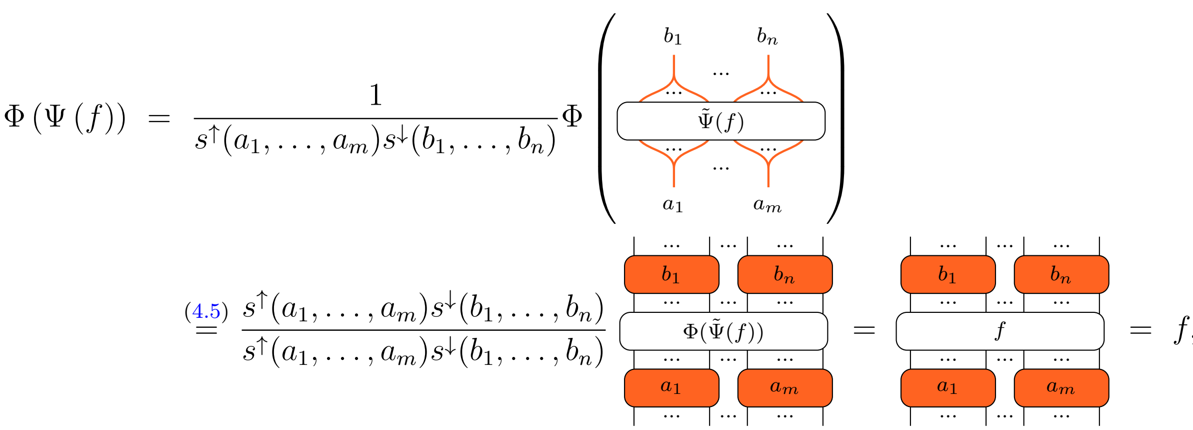
A totally analogous calculation shows that $\Phi\left(\tilde{\Psi}\left(\right.\right.$  $\left.\left.\right)\right) =$ . The fact that $\Phi \circ \tilde{\Psi}$ acts as the identity on caps and cups follows immediately from the definition of Φ and $\tilde{\Psi}$. Thus we indeed have $\Phi \circ \tilde{\Psi} = \mathrm{id}$. Finally, towards showing that $\Phi \circ \Psi = \mathrm{id}$, let $f\colon X \to Y$ be an arbitrary morphism in $\mathcal{OS}^W$, where $X = \mathrm{Sym}^{a_1} \otimes \cdots \otimes \mathrm{Sym}^{a_m}$ and $Y = \mathrm{Sym}^{b_1} \otimes \cdots \otimes \mathrm{Sym}^{b_n}$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Z}$. As in the proof of Proposition 4.8, we will only consider the all-symmetric case; the general case is essentially the same. For use in the following calculation, we temporarily adopt the notation



Using this notation, the orange half of (4.4) can be restated as follows, for $n \in \mathbb{Z}$:



where $i_n = \begin{cases} 1 & \text{if } n \geq 0, \\ -1 & \text{if } n < 0. \end{cases}$ With this in mind, we compute:



where the last equality holds by the definition of f being a morphism from X to Y (see Definition 4.1). $\square$

Chapter 5

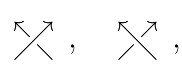
Variant $\mathfrak{gl}$-web categories

In this chapter, we will define and study several variants of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. We use the isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^W$ of Theorem 4.13 (and several slight variants thereof) to establish connections between these various web categories. This approach allows us to prove Proposition 5.1.6, extending a result previously only known for specialized web categories (to be defined in §5.2). In §5.3 we give a new presentation of the *special linear web categories* $\mathrm{Web}(\mathfrak{sl}_n)$ which emphasizes their connection to the $\mathfrak{gl}$-web categories, and prove an $\mathfrak{sl}$-analogue of Theorem 4.13. In the final section, we describe precisely how our web categories relate to some of those that have previously appeared in the literature.

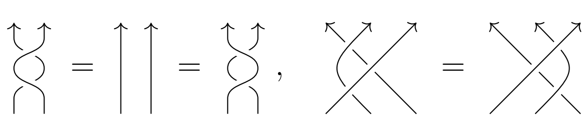
5.1 Upwards-oriented and monochromatic subcategories

Our first goal in this section is to establish that $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ is (isomorphic to) the full subcategory of $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ generated by upwards-oriented objects. In principle, the generators and relations for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ involving downwards-oriented strands could imply new identities for purely upwards-oriented diagrams, i.e. beyond what can be proved in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. Proposition 5.1.5 shows that this is not actually the case. A similar result was proved for the monochromatic purple case in [QS19, Prop. 6.8], but our proof is different, relying on Theorem 4.13.

Definition 5.1.1. The *braid category* $\mathcal{BR}$ is the strict $\Bbbk_q$-linear monoidal category generated by a single object $\uparrow$, and morphisms

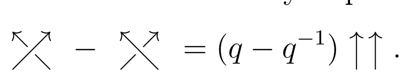


subject to the following relations:



(5.1.1)

The *Hecke braid category* $\mathcal{H}$ is obtained from $\mathcal{BR}$ by imposing the additional relation:



(5.1.2)

In other words, $\mathcal{BR}$ is the free braided strict monoidal category generated by a single object, and $\mathcal{H}$ is the free strict monoidal category generated by a single Hecke-braided object.

Lemma 5.1.2. Let $\iota\colon \mathcal{H} \to \mathcal{OS}$ denote the functor that sends each generator of $\mathcal{H}$ to the generator in $\mathcal{OS}$ drawn in the same way. Then ι is full and faithful. Hence, the full subcategory of $\mathcal{OS}$ generated by $\uparrow$ is isomorphic to $\mathcal{H}$.

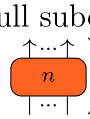
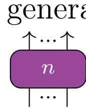
Proof. It is immediate from the definition of $\mathcal{H}$ that $\mathrm{Hom}_{\mathcal{H}}(\uparrow^{\otimes a}, \uparrow^{\otimes b})$ is trivial whenever $a \neq b$, and for all $a \in \mathbb{N}$, the Hecke-braid homomorphism $h_{\uparrow,a}\colon H_a \to \mathrm{Hom}_{\mathcal{H}}(\uparrow^{\otimes a}, \uparrow^{\otimes a})$ (see (2.5.12)) is an isomorphism. The basis theorem [Bru17, Thm. 1.2] for $\mathcal{OS}$ implies that the Hecke-braid homomorphism $\tilde{h}_{\uparrow,a}\colon H_a \to \mathrm{Hom}_{\mathcal{OS}}(\uparrow^{\otimes a}, \uparrow^{\otimes a})$ is also an isomorphism, and that $\mathrm{Hom}_{\mathcal{OS}}(\uparrow^{\otimes a}, \uparrow^{\otimes b})$ is trivial whenever $a \neq b$. For each $a \in \mathbb{N}$, the composition

$$H_a \xrightarrow{h_{\uparrow,a}} \mathrm{Hom}_{\mathcal{H}}(\uparrow^{\otimes a}, \uparrow^{\otimes a}) \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{OS}}(\uparrow^{\otimes a}, \uparrow^{\otimes a})$$

is equal to $\tilde{h}_{\uparrow,a}$; this follows immediately from the definition of the maps in question. Hence ι is full and faithful. $\square$

Definition 5.1.3. The *symmetric* $\mathfrak{gl}$*-web category*, denoted $\mathrm{Web}^{\uparrow}_{\mathrm{Sym}}(\mathfrak{gl})$, is the full subcategory of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ monoidally generated by orange objects. The *symmetric* $\mathfrak{gl}$*-web category with duals*, denoted $\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})$, is the full subcategory of $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ monoidally generated by orange objects. Similarly, the *alternating* $\mathfrak{gl}$*-web category* $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})$ and *alternating* $\mathfrak{gl}$*-web category with duals* $\mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl})$, are, respectively, the full subcategories of $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ generated by purple objects.

Definition 5.1.4. For use in the proofs of Propositions 5.1.5 and 5.1.6, we define:

- $\mathcal{H}^W$ to be the full subcategory of $\mathrm{Kar}(\mathcal{H})$ generated via tensor product by the objects $\mathrm{Sym}^n := \Big(\uparrow^{\otimes n},$  $\Big)$ and $\Lambda^n := \Big(\uparrow^{\otimes n},$  $\Big)$, $n \in \mathbb{N}$;
- $\mathcal{H}^{\mathrm{Sym}}$ to be the full subcategory of $\mathrm{Kar}(\mathcal{H})$ generated via tensor product by the objects Sym^n, $n \in \mathbb{N}$;
- $\mathcal{H}^{\Lambda}$ to be the full subcategory of $\mathrm{Kar}(\mathcal{H})$ generated via tensor product by the objects Λ^n, $n \in \mathbb{N}$;
- $\mathcal{OS}^{\mathrm{Sym}}$ to be the full subcategory of $\mathrm{Kar}(\mathcal{OS})$ generated, via tensor product and taking duals, by the objects Sym^n, $n \in \mathbb{N}$;
- $\mathcal{OS}^{\Lambda}$ to be the full subcategory of $\mathrm{Kar}(\mathcal{OS})$ generated, via tensor product and taking duals, by the objects Λ^n, $n \in \mathbb{N}$;
- $(\mathcal{OS}^W)^{\uparrow}$ to be the full subcategory of $\mathcal{OS}^W$ (see Definition 4.2) generated by upwards-oriented objects.

Proposition 5.1.5. The evident inclusion $\mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ (sending each generating object and morphism to the generator that is drawn the same way in the target category) is full and faithful. The same is true for the analogous functors $\mathrm{Web}^{\uparrow}_{\mathrm{Sym}}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl})$.

Proof. The definitions and arguments from Chapter 4 are still valid when restricted to purely upwards-oriented diagrams. In this setting, instead of working with the partial idempotent completion $\mathcal{OS}^W$, we use the category $\mathcal{H}^W$. The upwards-oriented version of Theorem 4.13 then yields an isomorphism $p\colon \mathcal{H}^W \cong \mathrm{Web}^{\uparrow}(\mathfrak{gl})$.

By Lemma 5.1.2, there is a full and faithful functor $\iota\colon \mathcal{H} \to \mathcal{OS}$. Composing with the inclusion $\mathcal{OS} \to \mathrm{Kar}(\mathcal{OS})$ yields a full and faithful functor $\iota'\colon \mathcal{H} \to \mathrm{Kar}(\mathcal{OS})$. By the universal property of Karoubi envelopes, there exists another full and faithful functor $\mathrm{Kar}(\iota')\colon \mathrm{Kar}(\mathcal{H}) \to \mathrm{Kar}(\mathcal{OS})$ extending ι'. Restricting to $\mathcal{H}^W$ then yields a full and faithful functor $\iota^W\colon \mathcal{H}^W \to (\mathcal{OS}^W)^{\uparrow}$.

The original version of Theorem 4.13 yields an isomorphism $\mathcal{OS}^W \cong \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. Restricting to $(\mathcal{OS}^W)^{\uparrow}$ gives a full and faithful functor $u\colon (\mathcal{OS}^W)^{\uparrow} \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. Consider the following composition:

$$\mathrm{Web}^{\uparrow}(\mathfrak{gl}) \xrightarrow{p} \mathcal{H}^W \xrightarrow{\iota^W} (\mathcal{OS}^W)^{\uparrow} \xrightarrow{u} \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}).$$

Each arrow is full and faithful, so the composition is too. This composite functor is the desired one from the proposition statement; the fact that it has the claimed action follows directly from the definitions of the composed functors. The claims for $\mathrm{Web}_{\mathrm{Sym}}$ and Web_{Λ} follow after restricting to the monochromatic subcategories. $\square$

We now shift our focus to the monochromatic web subcategories introduced in Definition 5.1.3. Analogously to what we discussed at the start of this section, a priori the two-colour relation (MIX) could imply new monochromatic identities in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$, e.g. an equality involving purely orange diagrams that does not follow from just the defining monochromatic orange relations for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. The following proposition rules out this possibility. Hence, the two-colour web categories we study in this thesis contain subcategories (the monochromatic ones) isomorphic to the $\mathfrak{gl}$-web categories from the literature that are defined using just one colour of strand; we will discuss this further in §5.4.

Proposition 5.1.6. Let $\overline{\mathrm{Web}^{\uparrow}_{\mathrm{Sym}}(\mathfrak{gl})}$ (resp. $\overline{\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})}$) denote the $\Bbbk_q$-linear (resp. $\Bbbk_{q,t}$-linear) strict monoidal category generated by the orange generating objects and morphisms for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ (resp. $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$), subject to the monochromatic orange defining relations for that category (see Definitions 3.1.1 and 3.1.3). Similarly, we let $\overline{\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})}$ and $\overline{\mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl})}$ respectively denote the categories generated by the purple generating objects, morphisms, and relations for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$.

For each category $\mathcal{C} \in \{\mathrm{Web}^{\uparrow}_{\mathrm{Sym}}(\mathfrak{gl}), \mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl}), \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}), \mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl})\}$, we have $\mathcal{C} \cong \overline{\mathcal{C}}$, with an isomorphism given by the map sending each generating object and morphism to the generator that is drawn in the same way in the target category.

In light of this proposition, we will identify the pairs of categories $\mathcal{C} \cong \overline{\mathcal{C}}$ after the following proof.

The purple case of Proposition 5.1.6 is a generalization of [TVW17, Cor. 2.16]. The argument in [TVW17] uses incarnation functors, and hence it only applies to specialized $\mathfrak{gl}_{m|n}$-web categories (which we will define and discuss shortly), not the general $\mathfrak{gl}$-web category.

Proof. We will give the proof for $\mathcal{C} = \mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})$ for the sake of concreteness; the arguments for the other three categories are analogous. First, note that all of the monochromatic orange results proved for $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ earlier in this thesis also hold in $\overline{\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})}$ since all of the proofs for such results used only other monochromatic orange results. Hence, all of the definitions and arguments of Chapter 4 are still valid when restricted to the monochromatically generated orange case. In this setting, instead of working with the category $\mathcal{OS}^W$, we use $\mathcal{OS}^{\mathrm{Sym}}$. (In the purple case we would instead work with $\mathcal{OS}^{\Lambda}$. For the categories with only upwards-oriented strands, we would replace $\mathcal{OS}$ with $\mathcal{H}$.)

The monochromatic orange version of Theorem 4.13 yields $\overline{\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})} \cong \mathcal{OS}^{\mathrm{Sym}}$. If we instead restrict the original two-colour version of Theorem 4.13 to $\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})$, we get that $\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl}) \cong \mathcal{OS}^{\mathrm{Sym}}$. Composing these two isomorphisms gives the desired result. $\square$

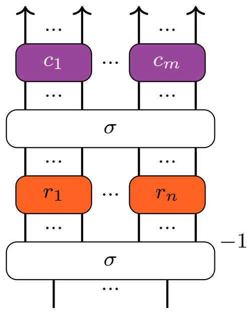
5.2 Idempotent quotients

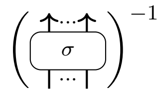
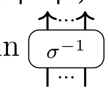
In this section, we define quotients of our diagrammatic categories by certain idempotent morphisms. As we will see in Chapter 6, these idempotents generate the kernels of the previously mentioned incarnation functors $I_{m|n} \colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod. Hence, one can think of these quotients as specializations of our categories to particular dimension parameters m and n.

The following definition adapts [TVW17, Def. 2.26] to the notation of this thesis. That definition is in turn based on [Gyo86] and [AM98]. As in those papers, we use the English notation for Young diagrams.

Definition 5.2.1. Let $\uparrow$ be a Hecke-braided object in some category $\mathcal{C}$. Let λ be a Young diagram with a boxes, m columns, and n rows. Write c_i for the number of boxes appearing in column i of λ, and r_i for the number of boxes in row i. Let $\lambda_{\rightarrow}$ denote the associated Young tableau obtained by filling the boxes of λ with $1, \ldots, a$ in increasing order from left to right, starting with the top row and then moving downwards. Similarly, let $\lambda_{\downarrow}$ denote the Young tableau obtained by filling the boxes of λ with $1, \ldots, a$ in increasing order from top to bottom, starting with the leftmost column and then moving to the right. Let $\sigma \in S_a$ be the unique permutation that maps $\lambda_{\rightarrow}$ to $\lambda_{\downarrow}$. We define the *quasi-idempotent associated to*

λ to be the following morphism:

$$\tilde{i}_\lambda :=$$  $$\in \mathrm{Hom}_{\mathcal{C}}(\uparrow^{\otimes a}, \uparrow^{\otimes a}).$$

Note that the morphism appearing at the bottom of the diagram is . This inverse is equal to the morphism obtained by replacing each positive crossing in  with a negative crossing.

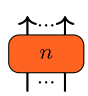
As noted in [TVW17, Def. 2.26], [AM98, Thm. 12] implies that $\tilde{i}_\lambda^2 = \alpha_\lambda \tilde{i}_\lambda$ for some invertible scalar $\alpha_\lambda \in \Bbbk_q$. We define i_λ to be the idempotent morphism $\frac{\tilde{i}_\lambda}{\alpha_\lambda}$.

Remark 5.2.2. An explicit formula for α_λ is given after the proof of [AM98, Thm. 12], but note that the (anti)symmetrizers used in [AM98] are unscaled, i.e. defined without the factors $\frac{q^{\pm\binom{n}{2}}}{[n]!}$ found in our Definition 2.5.5. Hence, one must multiply the scalar computed via the formula in [AM98] by these missing factors in order to obtain the correct α_λ for the quasi-idempotents we defined above.

Examples 5.2.3. Let $n \in \mathbb{N}$, and consider the Young diagram $\lambda = \underbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \end{array}}_{n \text{ boxes}}$. We have

$$\lambda_\rightarrow = \lambda_\downarrow = \begin{array}{|c|c|c|}\hline 1 & \cdots & n \\ \hline \end{array},$$

and hence the associated permutation is $\sigma = \mathrm{id}$. Recalling that (anti)symmetrizers of size 1 are simply the identity, we thus have:

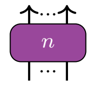
$$\tilde{i}_\lambda =$$  .

In this case, $\tilde{i}_\lambda$ is already idempotent, so $\alpha_\lambda = 1$ and $i_\lambda = \tilde{i}_\lambda$.

Next, take $\lambda = n \text{ boxes} \left\{ \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array} \right.$. We have

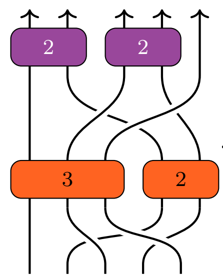
$$\lambda_\rightarrow = \lambda_\downarrow = \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline \end{array},$$

so again the associated permutation is $\sigma = \mathrm{id}$. Then:

$\tilde{i}_\lambda =$  ,

which is again already idempotent.

For an example where the permutation σ is nontrivial, take $\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$. Then we have $\lambda_\rightarrow = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$ and $\lambda_\downarrow = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$, so the associated permutation is $\sigma = (2354)$. Then:

$\tilde{i}_\lambda =$  .

We have $\alpha_\lambda = \frac{[4]}{[2]^3}$, and so $i_\lambda = \frac{[2]^3}{[4]}\tilde{i}_\lambda$. (This scalar was computed using the previously mentioned formula from [AM98].)

The idempotents i_λ are quantum analogues of the classical Young idempotents. Fix a Young diagram λ with a boxes, and some $m, n \in \mathbb{N}$. Write $V^+_{m|n}$ for the natural $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-module, and $I_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod for the corresponding incarnation functor, sending $\uparrow_1$ to $V^+_{m|n}$ (as mentioned previously, we will precisely define these functors in Chapter 6). Then $I_{m|n}(i_\lambda)\colon (V^+_{m|n})^{\otimes a} \to (V^+_{m|n})^{\otimes a}$ projects onto a copy of the quantum Schur module $(V^+_{m|n})^\lambda$ appearing in the irreducible decomposition of $(V^+_{m|n})^{\otimes a}$, if such a copy exists, and is the zero map otherwise. For instance, if $\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$, then $I_{3|0}(i_\lambda)$ is the quantum analogue of the Young antisymmetrizer, projecting onto the unique copy of $\Lambda^3(V^+_{3|0})$ contained in $(V^+_{3|0})^{\otimes 3}$. On the other hand, $I_{2|0}(i_\lambda) = 0$, corresponding to the fact that $\Lambda^3(V^+_{2|0})$ is the zero module. In general, the kernel of $I_{m|n}$ is generated by i_λ for a certain Young diagram λ depending on m and n. The following definition introduces notation for the quotients of our diagrammatic categories by these kernels. These quotients are known as *specialized* categories.

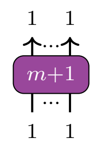
Definition 5.2.4. For $d \in \mathbb{Z}$, we define $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}, t = q^d)$ to be the $\Bbbk_q$-linear category obtained from the $\Bbbk_{q,t}$-linear category $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ by setting $t = q^d$.

The $\mathfrak{gl}_{m|n}$*-web category*, denoted $\mathrm{Web}^{\uparrow}(\mathfrak{gl}_{m|n})$, is the $\Bbbk_q$-linear category obtained from $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ by imposing the following relation:

$$i_{\lambda(m+1,n+1)} = 0, \qquad \text{(IDEM0)}$$

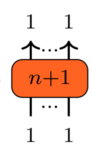
where $\lambda(m+1,n+1)$ denotes the rectangular Young diagram with $m+1$ rows and $n+1$ columns. Similarly, the $\mathfrak{gl}_{m|n}$-*web category with duals*, denoted $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}_{m|n})$, is the $\Bbbk_q$-linear category obtained from $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}, t = q^{m-n})$ by imposing (IDEM0). We adopt analogous definitions for all of the other varieties of $\mathfrak{gl}$-web categories we have previously defined, as well as for $\mathcal{OS}$ and $\mathcal{H}$. For instance, $\mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl}_{m|n})$ is obtained from $\mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl})$ by setting $t = q^{m-n}$ and then imposing (IDEM0), and $\mathcal{H}_{m|n}$ is obtained from $\mathcal{H}$ by imposing (IDEM0). (For the specializations of web categories, the idempotent $i_{\lambda(m+1,n+1)}$ appearing in (IDEM0) is an endomorphism of $\uparrow_1^{\otimes(m+1)(n+1)}$, whereas for the specializations of $\mathcal{OS}$ and $\mathcal{H}$, it is an endomorphism of $\uparrow^{\otimes(m+1)(n+1)}$.)

Recall that, as mentioned in Remark 3.1.4, after setting $t = q^{m-n}$, (SPEC) tells us that the categorical dimension of $\uparrow_1$ is $[m-n]$. This is equal to the categorical dimension of the natural representation of $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$.

Remark 5.2.5. In the case $n = 0$, (IDEM0) says that  $= 0$ (see Examples 5.2.3). In light of (3.2.8), (3.3.19), and (3.3.22), this relation is equivalent to

$$\uparrow_{m+1} = 0. \tag{5.2.1}$$

This latter form of the relation is more common in the literature. Using (3.2.3), we get that $\uparrow_k$ is a zero object in $\mathrm{Web}^{\uparrow}(\mathfrak{gl}_{m|0})$ for all $k > m$.

Similarly, if $m = 0$, then (IDEM0) says that  $= 0$, which is equivalent to $\uparrow_{n+1} = 0$, and this implies that $\uparrow_k$ is a zero object in $\mathrm{Web}^{\uparrow}(\mathfrak{gl}_{0|n})$ for all $k > n$.

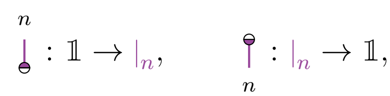
5.3 Special linear web categories

Special linear web categories (for general dimension parameters) first appeared in the literature in [CKM14]. In this section, we give a novel presentation of such categories, defining them in terms of $\mathfrak{gl}$-web categories. We then prove some key results about our $\mathfrak{sl}$-web categories, establishing that they are equivalent to those found in the literature; this will be discussed further in Connection 5.4.1. Finally, we prove Theorem 5.3.19, which is the $\mathfrak{sl}$-analogue of Theorem 4.13.

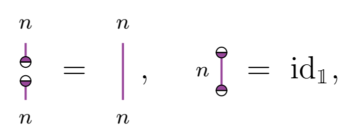
Throughout this section, we work over the base ring $\Bbbk_{q^{1/n}}$, defined analogously to $\Bbbk_q$. Explicitly, $\Bbbk_{q^{1/n}}$ can be taken to be either the total ring of fractions $Q(\Bbbk[q^{1/n}])$ (which is equal to the ring of rational functions $\Bbbk(q^{1/n})$ if $\Bbbk$ is an integral domain), or the smallest subring of $Q(\Bbbk[q^{1/n}])$ that contains $q^{1/n}, q^{-1/n}$, and $\frac{1}{[a]}$ for all nonzero $a \in \mathbb{Z}$. All of the definitions and identities for $\mathfrak{gl}$-web categories established earlier in this thesis are also valid over this extended base ring. These nth roots of q will be used to define a braided monoidal structure on the $\mathfrak{sl}_n$-web category, corresponding to the natural braided structure on the category of $U_q(\mathfrak{sl}_n)$-modules.

Our approach in the following definition, specifically the use of new generators encoding an isomorphism $|_n \cong \mathbb{1}$, was inspired by the presentations of the category q-**Schur** given in [Bru25].

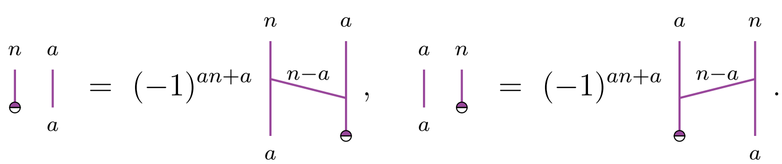
Definition 5.3.1. Let $n \geq 1$. The $\mathfrak{sl}_n$*-web category*, which we denote $\mathrm{Web}(\mathfrak{sl}_n)$, is the $\Bbbk_{q^{1/n}}$-linear strict monoidal category obtained from $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$ by adjoining two new generating morphisms and imposing some new relations, as detailed below. In the following, we draw diagrams in $\mathrm{Web}(\mathfrak{sl}_n)$ using unoriented strands rather than upwards-oriented ones; the reason for this switch of notation will be discussed after the present definition. The new generating morphisms we add to obtain $\mathrm{Web}(\mathfrak{sl}_n)$ are



subject to the following relations, for all $a \in \{0, \dots, n\}$:



(DOTS)



(SLSLIDE)

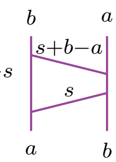
As we will prove in Lemma 5.3.7, the objects $|_0, \dots, |_n$ in $\mathrm{Web}(\mathfrak{sl}_n)$ form dual pairs $(|_a, |_{n-a})$. Hence, there is no need to add a new family of dual generating objects to $\mathrm{Web}(\mathfrak{sl}_n)$, as was done when moving from $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ to $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. Accordingly, we will typically draw the generating objects and morphisms for $\mathrm{Web}(\mathfrak{sl}_n)$ using unoriented strands rather than upwards-oriented ones, i.e. $|_a$ rather than $\uparrow_a$. We will still occasionally draw morphisms in $\mathrm{Web}(\mathfrak{sl}_n)$ using upwards-oriented strands; the precise meaning of such oriented morphisms will be discussed after Definition 5.3.4.

Remark 5.3.2. Recall that purple strands in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ correspond (via an incarnation functor) to exterior powers of the natural representation V^+ of $U_q(\mathfrak{gl}_{n|m})$. In the purely even case, i.e. $m = 0$, we have that $\Lambda^n V^+$ is the quantum analogue of the one-dimensional determinant representation of $\mathfrak{gl}_n \cong \mathfrak{gl}_{n|0}$. Viewed as a $U_q(\mathfrak{sl}_n)$-representation (using the evident inclusion $U_q(\mathfrak{sl}_n) \to U_q(\mathfrak{gl}_n)$), we correspondingly have that $\Lambda^n V^+$ is isomorphic to the trivial representation. The new generators  and , together with the relation (DOTS), translate this isomorphism of representations into the language of web categories. The isomorphism $|_n \cong \mathbb{1}$ will allow us to define cups and caps for the dual pairs $(|_a, |_{n-a})$.

Remark 5.3.3. We impose (SLSLIDE) so that $\mathrm{Web}(\mathfrak{sl}_n)$ becomes braided monoidal when equipped with the crossings specified in Definition 5.3.4. As we will see in the proof of Lemma 5.3.5, (SLSLIDE) precisely says that  slides through those crossings.

Definition 5.3.4. We define the *positive crossings* in $\mathrm{Web}(\mathfrak{sl}_n)$ as follows, for $a, b \in \mathbb{N}$:

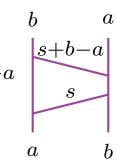
$$:= (-1)^{ab} q^{-\frac{ab}{n}} \sum_{s \geq 0} (-q)^{a-s}$$



$$. \qquad (5.3.1)$$

The *negative crossings* are obtained by replacing q with q^{-1} in the above definition, i.e.

$$:= (-1)^{ab} q^{\frac{ab}{n}} \sum_{s \geq 0} (-q)^{s-a}$$



$$. \qquad (5.3.2)$$

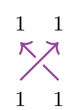
Note that these crossings differ from those defined for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ in Definition 3.3.1 by a factor of $q^{\pm \frac{ab}{n}}$.

The $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$-crossings from Definition 3.3.1 still play an important role in $\mathrm{Web}(\mathfrak{sl}_n)$. In particular, they will be used in the proof of the following lemma about the new crossings from Definition 5.3.4. Since we will need to work with both types of crossings, we adopt the following diagrammatic convention: crossings drawn with unoriented strands represent the crossings defined for $\mathrm{Web}(\mathfrak{sl}_n)$ in Definition 5.3.4, and those drawn with oriented strands represent the crossings defined for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ in Definition 3.3.1. For instance,

$$\overset{(5.3.1)}{=} q^{1-\frac{1}{n}}$$

$$- q^{-\frac{1}{n}}$$

$$\overset{(3.3.4)}{=} q^{-\frac{1}{n}}$$

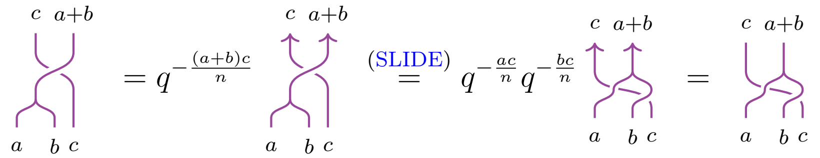


$$.$$

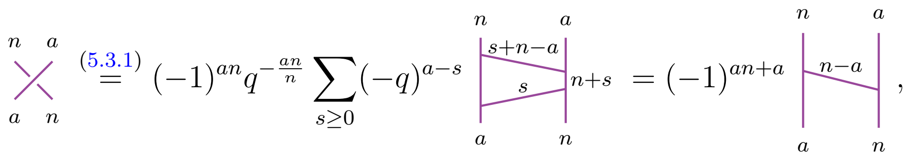
Note that $|_1$ is simply an alternate notation for $\uparrow_1$ that we use when working in $\mathrm{Web}(\mathfrak{sl}_n)$, so both the $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$- and $\mathrm{Web}(\mathfrak{sl}_n)$-crossings are endomorphisms of $|_1 \otimes |_1 = \uparrow_1 \otimes \uparrow_1$.

Lemma 5.3.5. The crossings of Definition 5.3.4 equip $\mathrm{Web}(\mathfrak{sl}_n)$ with the structure of a braided monoidal category.

Proof. The fact that the proposed crossings satisfy the double-crossing relations, and that splits and merges slide through them, follows quickly from the corresponding identities for the crossings defined for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. We will prove one such identity; the others follow from totally analogous calculations. We compute:

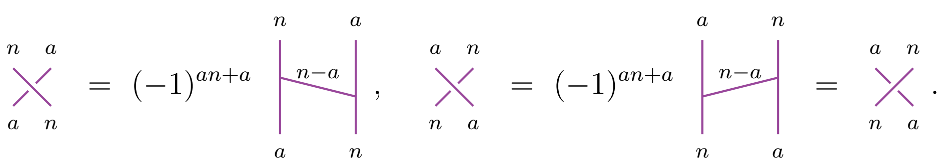


All that remains to show is that the new generators  and  slide through crossings. Recall that $|_k$ is a zero object for all $k > n$ (see Remark 5.2.5). Hence, any diagram in $\mathrm{Web}(\mathfrak{sl}_n)$ featuring a strand labelled by some $k > n$ is zero. With this in mind, for all $a \in \{0, \ldots, n\}$ we have:



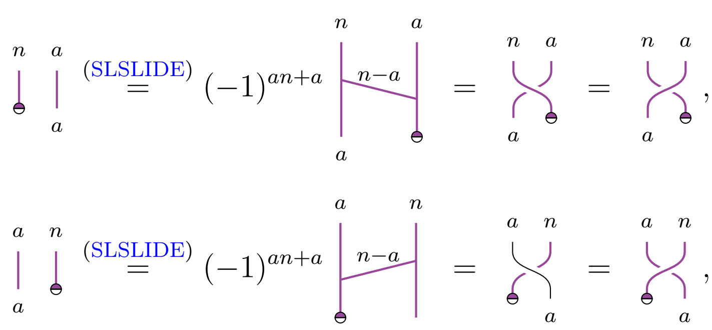
(5.3.3)

and similarly,

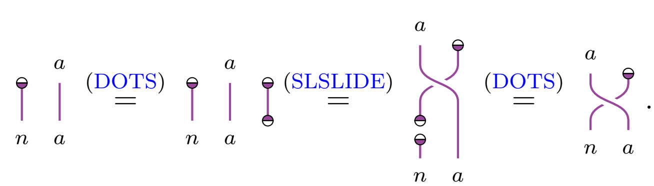


(5.3.4)

Thus:

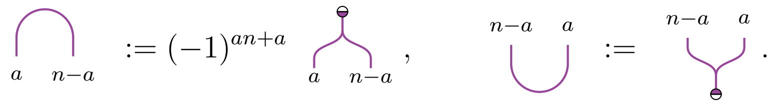


so  slides through all crossings. This implies that  also slides through crossings, e.g.



The other sliding identities follow from similar calculations. □

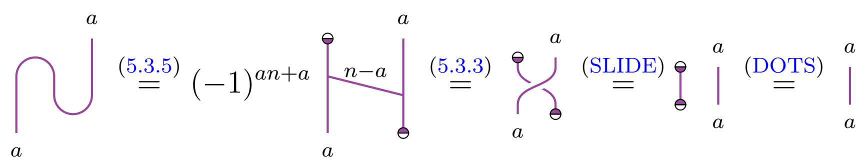
Definition 5.3.6. We define the following cap and cup morphisms in $\mathrm{Web}(\mathfrak{sl}_n)$, for all $a \in \{0, \dots, n\}$:



(5.3.5)

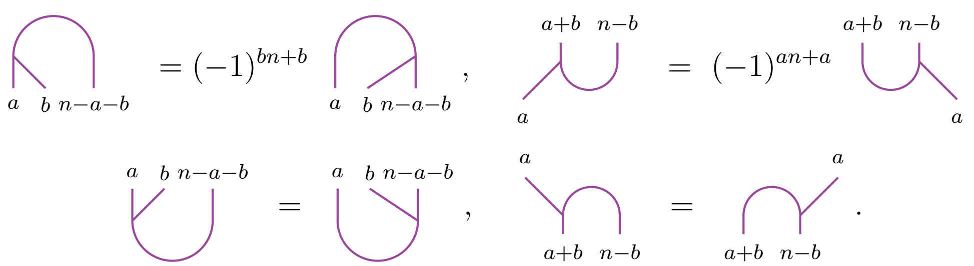
Lemma 5.3.7. The caps and cups of Definition 5.3.6 satisfy the zigzag equations, (ZIGZAG). Hence, $(|_a, |_{n-a})$ is a dual pair in $\mathrm{Web}(\mathfrak{sl}_n)$ for each $a \in \{0, \dots, n\}$.

Proof. Let $a \in \{0, \dots, n\}$. We compute:



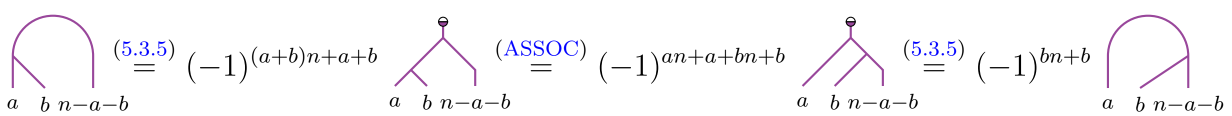
The other zigzag equation follows from a similar calculation, using the analogue of Lemma 3.3.9 for the $\mathrm{Web}(\mathfrak{sl}_n)$-crossings. $\square$

Lemma 5.3.8. The following identities hold in $\mathrm{Web}(\mathfrak{sl}_n)$ for all $a, b \in \{0, \dots, n\}$:

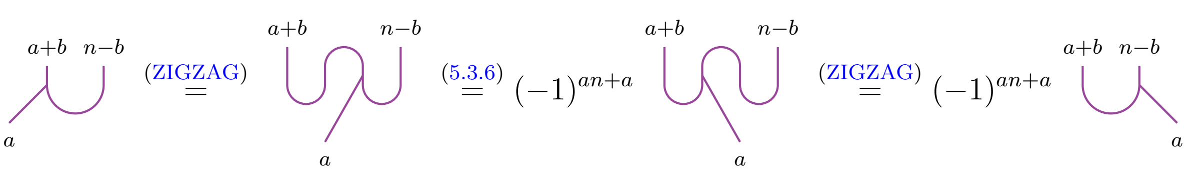


(5.3.6)

Proof. We have:

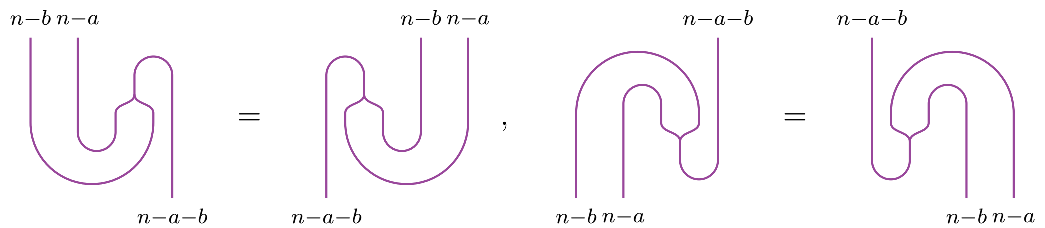


proving the first identity, which quickly implies the second:



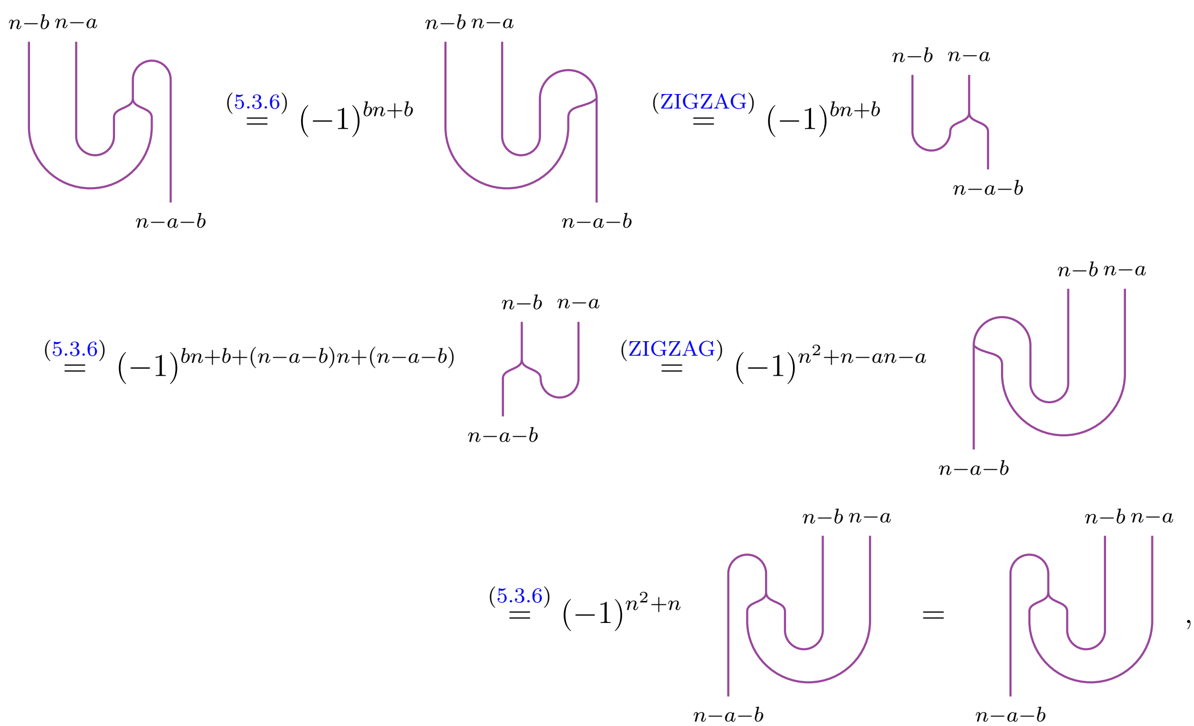
The final two identities follow from analogous computations. $\square$

Lemma 5.3.9. The left and right mates of merge and split morphisms in $\mathrm{Web}(\mathfrak{sl}_n)$ agree. Explicitly, for all $a, b \in \{0, \dots, n\}$, we have:

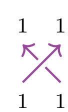


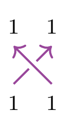
(5.3.7)

Proof. We will prove the first identity; the second follows from an analogous computation.



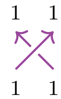
where in the last equality we use the fact that $n^2 + n = n(n+1)$ is even. □

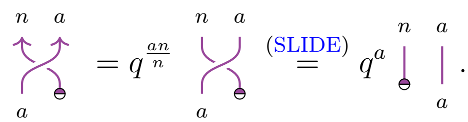
We will soon show that a category constructed from $\mathcal{H}_{n|0}$ (see Definition 5.2.4) is isomorphic to $\mathrm{Web}(\mathfrak{sl}_n)$, with $\uparrow \in \mathrm{ob}(\mathcal{H}_{n|0})$ corresponding to $|_1 \in \mathrm{ob}(\mathrm{Web}(\mathfrak{sl}_n))$. However, $|_1$ is *not* Hecke-braided with respect to the crossings from Definition 5.3.4. In the following, all antisymmetrizers in $\mathrm{Web}(\mathfrak{sl}_n)$ are defined using the $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$-crossings 

and 

, with respect to which $|_1$ is Hecke-braided.[1] However, it is important to note that

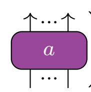
[1]Strictly speaking, our definition of "Hecke-braided" states that these crossings must come from a braided monoidal structure on the whole category, but all that is needed for our purposes is a *local braiding*, i.e. a braiding for the full subcategory monoidally generated by $|_1$.

these crossings cannot be extended to a braided monoidal structure on all of $\mathrm{Web}(\mathfrak{sl}_n)$. A straightforward argument shows that any braided monoidal structure on $\mathrm{Web}(\mathfrak{sl}_n)$ such that the positive $(|_1, |_1)$-crossing is  must fully coincide with the braided monoidal structure from $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$; see e.g. [QS19, Lem. 5.11]. However, the generators  and  do not slide through the $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$-crossings. For instance, since these generators *do* slide through the $\mathrm{Web}(\mathfrak{sl}_n)$-crossings, we have

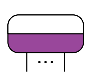
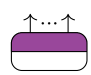


Thus, even though we use the $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$-crossings for the purpose of defining antisymmetrizers, the rescaled crossings from Definition 5.3.4 are needed in order to equip the whole category $\mathrm{Web}(\mathfrak{sl}_n)$ with a braided monoidal structure.

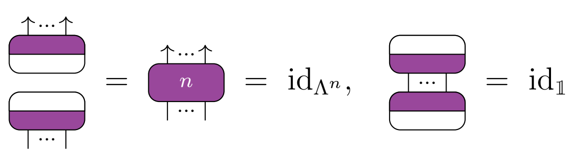
Definition 5.3.10. Let $\mathcal{H}^{\Lambda}_{n|0}$ denote the full monoidal subcategory of $\mathrm{Kar}(\mathcal{H}_{n|0})$ generated by the objects

$$\Lambda^a := \left(\uparrow^{\otimes a},\ \right.$$  $$\left.\right), \quad a \in \mathbb{N}.$$

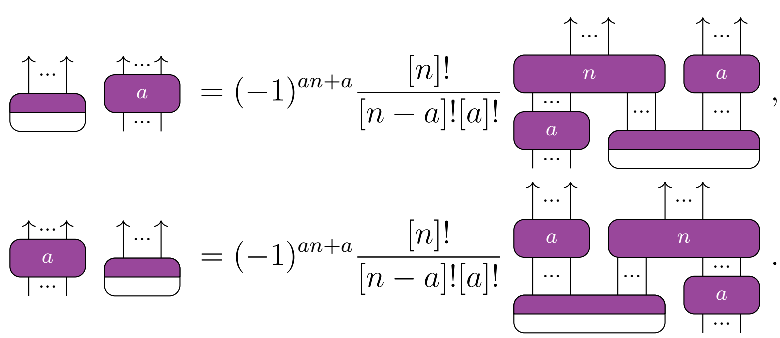
We then define the category $\mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0}$ by adjoining two new generating morphisms to $\mathcal{H}^{\Lambda}_{n|0}$, which we call "billboards",

 $: \Lambda^n \to \mathbb{1}$,  $: \mathbb{1} \to \Lambda^n$,

subject to the following relations, for all $a \in \{0, \dots, n\}$:



(5.3.8)

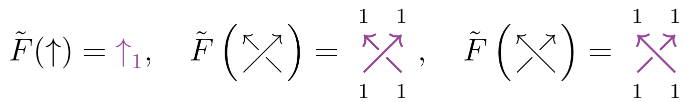


(5.3.9)

For the avoidance of doubt, the morphisms on the first line of (5.3.9) have domain Λ^a and codomain $\Lambda^n \otimes \Lambda^a$, while the morphisms on the second line have domain Λ^a and codomain $\Lambda^a \otimes \Lambda^n$.

The relations (5.3.9) are the $\mathcal{H}_{n|0}^{\Lambda_{\mathfrak{sl}}}$-analogues of the relations (SLSLIDE) for $\mathrm{Web}(\mathfrak{sl}_n)$. More precisely, (5.3.9) is the image of (SLSLIDE) via an isomorphism $\mathrm{Web}(\mathfrak{sl}_n) \cong \mathcal{H}_{n|0}^{\Lambda_{\mathfrak{sl}}}$ (analogous to the isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^{\Lambda}$ from Theorem 4.13) that we will construct in the remainder of this section.

Definition 5.3.11. Let $\tilde{F}\colon \mathcal{H} \to \mathrm{Web}^{\uparrow}(\mathfrak{gl})$ be the $\Bbbk_{q^{1/n}}$-linear strict monoidal functor given on generators by

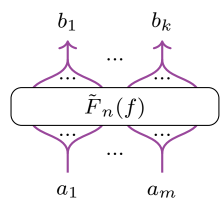


(The $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$-crossings are those from Definition 3.3.1.) We write $\tilde{F}_n\colon \mathcal{H}_{n|0} \to \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$ for the functor with the same action on generators as $\tilde{F}$.

Lemma 5.3.12. The functors $\tilde{F}$ and $\tilde{F}_n$ are well-defined.

Proof. Proposition 3.3.5 tells us that $\tilde{F}$-images of the relations (5.1.1) hold in $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. We previously noted that $\uparrow_1$ is Hecke-braided, which precisely says that the $\tilde{F}$-image of (5.1.2) holds. Hence, $\tilde{F}$ is well-defined. The $\tilde{F}_n$-image of the defining relation (IDEM0) for $\mathcal{H}_{n|0}$ holds in $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$ by definition, so $\tilde{F}_n$ is also well-defined. ☐

Definition 5.3.13. Define a $\Bbbk_{q^{1/n}}$-linear strict monoidal functor $\overline{F_n}\colon \mathcal{H}^{\Lambda}_{n|0} \to \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$ as follows. The action on generating objects is given by $\overline{F_n}(\Lambda^a) = \uparrow_a$. Suppose that $f\colon X \to Y$ is a morphism in $\mathcal{H}^{\Lambda}_{n|0}$, where $X = \Lambda^{a_1} \otimes \cdots \otimes \Lambda^{a_m}$ and $Y = \Lambda^{b_1} \otimes \cdots \otimes \Lambda^{b_k}$ for some $a_1, \ldots, a_m, b_1, \ldots, b_k \in \mathbb{N}$. Then we set:

$$\overline{F_n}(f) = \frac{1}{[a_1]! \cdots [a_m]!}$$  .

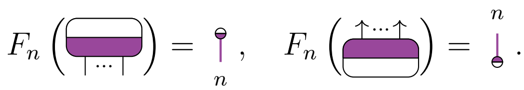
Lemma 5.3.14. The functor $\overline{F_n}$ is well-defined.

Proof. This follows from essentially the same arguments that were used to show that the functor $\Psi\colon \mathcal{OS}^W \to \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$ is well-defined (see Proposition 4.8). The main difference is that all strands are oriented upwards in the codomain of $\overline{F_n}$, whereas the codomain of Ψ features both upwards- and downwards-oriented strands. To make the parallel between these two functors more obvious, note that the scalar factor appearing in the definition of $\overline{F_n}$ above can be rewritten as

$$\frac{1}{[a_1]! \cdots [a_m]!} = \frac{1}{s^{\uparrow}(a_1, \ldots, a_m) s^{\downarrow}(b_1, \ldots, b_k)}$$

since all of the a_i and b_j are nonnegative, where $s^{\uparrow}$ and $s^{\downarrow}$ are the functions introduced in Definition 4.6. ☐

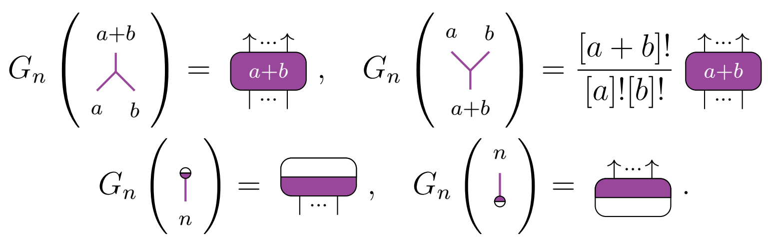
Definition 5.3.15. We define a $\Bbbk_{q^{1/n}}$-linear strict monoidal functor $F_n \colon \mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0} \to \mathrm{Web}(\mathfrak{sl}_n)$ as follows. On objects and non-billboard generating morphisms, F_n has the same action as $\overline{F_n}$. On billboards, we set



Lemma 5.3.16. The functor F_n is well-defined.

Proof. Since we already know $\overline{F_n}$ is well-defined, all that remains is to show that the F_n-images of (5.3.8) and (5.3.9) hold in $\mathrm{Web}(\mathfrak{sl}_n)$. These images are simply the relations (DOTS) and (SLSLIDE), which hold in $\mathrm{Web}(\mathfrak{sl}_n)$ by definition. ☐

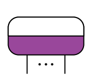
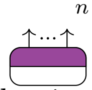
Definition 5.3.17. We define a $\Bbbk_{q^{1/n}}$-linear strict monoidal functor $G_n \colon \mathrm{Web}(\mathfrak{sl}_n) \to \mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0}$ as follows. On objects, for each $a \in \mathbb{N}$ we set $G_n(|_a) = \Lambda^a$. The action on generating morphisms is given by:



Lemma 5.3.18. The functor G_n is well-defined.

Proof. We need to show that the defining relations for $\mathrm{Web}(\mathfrak{sl}_n)$ are preserved by G_n. The arguments from the proof of Proposition 4.11 show that the defining relations for $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})$ are preserved. The G_n-images of (IDEM0), (DOTS), and (SLSLIDE), are, respectively, (IDEM0) (this relation is part of the definition of both the source and target category), (5.3.8), and (5.3.9), which hold in $\mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0}$ by definition. ☐

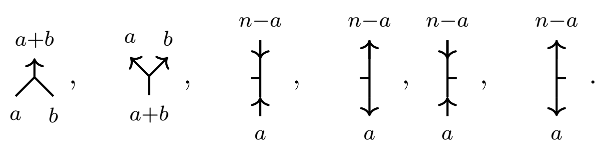
Theorem 5.3.19. The functors $F_n \colon \mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0} \to \mathrm{Web}(\mathfrak{sl}_n)$ and $G_n \colon \mathrm{Web}(\mathfrak{sl}_n) \to \mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0}$ are mutually inverse, and hence $\mathcal{H}^{\Lambda_{\mathfrak{sl}}}_{n|0}$ and $\mathrm{Web}(\mathfrak{sl}_n)$ are isomorphic as strict monoidal categories.

Proof. The composite functors $F_n \circ G_n$ and $G_n \circ F_n$ clearly act as the identity on objects. The calculations from the proof of Theorem 4.13 show that $F_n \circ G_n$ acts as the identity on merge and split morphisms, and it follows immediately from the definitions of F_n and G_n that the same is true for  and . Similarly, it is immediate that $G_n \circ F_n$ acts as the identity on  and , and the arguments from the proof of Theorem 4.13 show that $G_n \circ F_n$ acts as the identity on diagrams without billboards. Thus, F_n and G_n are indeed mutually inverse. ☐

5.4 Connections to web categories in the literature

In this section, we describe how the web categories studied in this thesis relate to several of those that have appeared previously in the literature. Some of the web categories we will discuss were defined over rings other than the ones used in the present thesis. The difference in rings does not impact any of the diagrammatics, so we ignore this detail in what follows.

Connection 5.4.1. The definition of a general $\mathfrak{sl}_n$-web category first appeared in the literature in [CKM14], building upon [Mor07], the PhD thesis of one of that paper's authors. There was a brief discussion of $\mathfrak{gl}_n$-web categories in [CKM14, §5] (referred to as "ladder categories"), but the main focus was on $\mathfrak{sl}_n$-webs. The spider category $\mathcal{S}p(\mathrm{SL}_n)$ described in [CKM14, §2.2] is essentially the same as our web category $\mathrm{Web}(\mathfrak{sl}_n)$, with some technical differences. The objects of $\mathcal{S}p(\mathrm{SL}_n)$ are generated by the singleton sequences (a^+) and (a^-), for $a \in \{1, \ldots, n-1\}$, respectively corresponding to our objects $\uparrow_a$ and $\downarrow_a$ in $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$. The generating morphisms for $\mathcal{S}p(\mathrm{SL}_n)$ are of the following types:

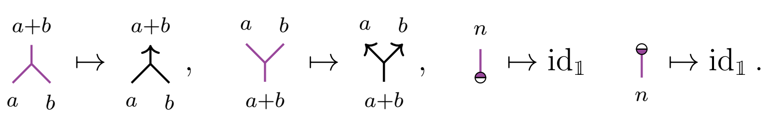


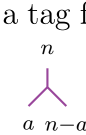
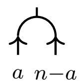
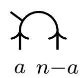
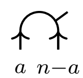
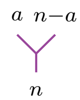
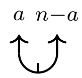
There are implicitly also cup and cap generating morphisms for the dual pairs $((a^+), (a^-))$. The last four types of generators above are called "tags". These tags are diagrammatic analogues of the isomorphisms $\Lambda^a V^+ \cong (\Lambda^{n-a} V^+)^*$. Due to the way that we defined $\mathrm{Web}(\mathfrak{sl}_n)$ by adjoining generators to $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$ and imposing new relations, our $\mathfrak{sl}_n$-web category contains an infinite sequence of zero objects, $|_{n+1}, |_{n+2}, \ldots$ (see Remark 5.2.5). There are no analogous zero objects in $\mathcal{S}p(\mathrm{SL}_n)$.

There is an equivalence of monoidal categories from the full monoidal subcategory of $\mathrm{Web}(\mathfrak{sl}_n)$ generated by $|_1, \ldots, |_n$ (that is, our $\mathfrak{sl}_n$-web category after discarding all of the zero objects) to $\mathcal{S}p(\mathrm{SL}_n)$ acting on objects as

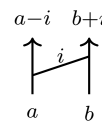
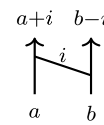
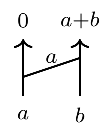
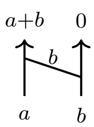
$$|_a \mapsto (a^+) \text{ for all } 1 \le a \le n-1, \quad |_n \mapsto \mathbb{1}, \quad |_0 \mapsto \mathbb{1},$$

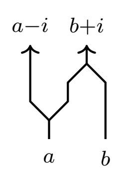
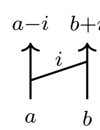
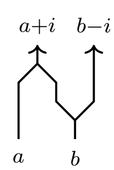
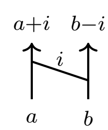
and on generating morphisms as:



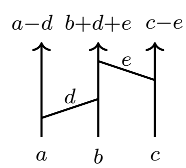
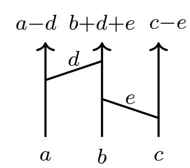
In the images for merge and split morphisms specified above, we employ the following diagrammatic convention from [CKM14]: any strand labelled by 0 or n should be deleted. When a strand labelled by 0 is deleted, the result is an identity morphism, as in our relation (TRIV). When a strand labelled by n is deleted, the result is a cup or cap, with a tag facing in the same direction as the orientation of the deleted strand. Thus, for instance,  gets mapped to  $:=$  $=$ , where the second equality follows from the defining relations for $\mathcal{S}p(\mathrm{SL}_n)$. Similarly,  gets mapped to .

Connection 5.4.2. Various $\mathfrak{gl}$-web categories were defined and studied in [QS19], all corresponding to the monochromatic purple case in this thesis. The web categories in [QS19] are defined by taking the additive envelope of diagrammatic categories, and thus correspond to the additive envelopes of our web categories. Almost all of the arguments in [QS19] are phrased in terms of just the diagrammatic categories themselves, not the additive envelopes, so we ignore this detail.

The category Sp^+ described in [QS19, Def. 5.1] is isomorphic to our category $\mathrm{Web}_\Lambda^{\uparrow}(\mathfrak{gl})$, though the presentation is somewhat different. Instead of using merges and splits as generators, Sp^+ is defined in terms of the ladder morphisms  and . One can recover merge and split morphisms from these ladders by taking $i = a$ or $i = b$. Namely, one could either define  to be equal to  or , and similarly for splits. The fact that these two definitions coincide follows from the defining relations of Sp^+. It then follows that ladders in Sp^+ are equal to diagrams built up from merges and splits in the natural way, namely:

 $=$ ,  $=$ .

With this correspondence between merges, splits, and ladders in mind, one can show that the defining relations for Sp^+ are equivalent to those for $\mathrm{Web}_\Lambda^{\uparrow}(\mathfrak{gl})$. In other words, $\mathsf{Sp}^+ \cong \mathrm{Web}_\Lambda^{\uparrow}(\mathfrak{gl})$ via the map that simply changes the colour of strands from black to purple. For instance, the first relation from (ASSOC) is the special case $d = a, e = c$ of the defining relation [QS19, (5.5c)]:

 $=$ ,

and conversely, [QS19, (5.5c)] (for general strand labels) follows from a single application of (ASSOC). There are many relations to check; we leave the details to the interested reader.

The category $\mathsf{Sp}(\beta)$, introduced in [QS19, Def. 6.5], is isomorphic to $\mathrm{Web}_\Lambda^{\uparrow\downarrow}(\mathfrak{gl})$, again via the map that changes the colour of strands from black to purple. Their parameter q^β corresponds to our variable t. With this in mind, the new defining relations for

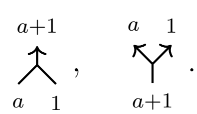
$\mathsf{Sp}(\beta)$ (beyond those for Sp^{+}) are almost the same as those for $\mathrm{Web}_{\Lambda}^{\uparrow\downarrow}(\mathfrak{gl})$. The relations [QS19, (6.15a), (6.15b)] are simply (RBLOB), (LBLOB), and (SPEC). Relations [QS19, (6.15c), (6.15d)] are equivalent to (RSQUIG) and (LSQUIG).

Connection 5.4.3. The green-red web category ∞-$\mathbf{Web}_{\mathrm{gr}}$ defined and studied in [TVW17] is isomorphic to our category $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. Their green and red strands correspond to our purple and orange ones, respectively. Up to this change of colour, the generating objects and morphisms for the two categories are the same. The defining relations for ∞-$\mathbf{Web}_{\mathrm{gr}}$ are a strict superset of those for $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$. The relations [TVW17, (2-6), (2-7), (2-9)] are our (ASSOC), (3.2.3) (which we proved as a consequence of our defining relations, rather than assuming it), and (MIX). Finally, the defining relation [TVW17, (2-8)] is our (3.2.5), which we only assume in the special case where both rungs are labelled by 1, i.e. (MIRROR). The quotients $N|M$-$\mathbf{Web}_{\mathrm{gr}}$ specified in [TVW17, Def. 5.1] are then straightforwardly isomorphic to our categories $\mathrm{Web}^{\uparrow}(\mathfrak{gl}_{N|M})$. (Indeed, as we noted just before Definition 5.2.1, our definition of $\mathrm{Web}^{\uparrow}(\mathfrak{gl}_{N|M})$ is an adaptation of the definitions from [TVW17].)

In [TVW17, Rem. 2.4], the authors point out that ∞-$\mathbf{Web}_{\mathrm{gr}}$ is symmetric under exchanging green and red. Phrased another way, this says there is a $\Bbbk_q$-linear monoidal endofunctor of ∞-$\mathbf{Web}_{\mathrm{gr}}$ sending each generator to the corresponding generator in the other colour. Note that this endofunctor is *not* the same as our colour-swapping endofunctor $\mathrm{cswap}\colon \mathrm{Web}^{\uparrow}(\mathfrak{gl}) \to \mathrm{Web}^{\uparrow}(\mathfrak{gl})$ from Definition 3.1.8, since cswap is q-skew antilinear rather than $\Bbbk_q$-linear, and its action on split morphisms involves a sign term. Simply swapping the colours of strands does not yield a valid $\Bbbk_q$-linear endofunctor of the category with duals, $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl})$, since the defining relations (RBLOB), (LBLOB), (RSQUIG), (LSQUIG) are not preserved by such a colour change.

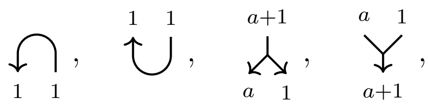
Connection 5.4.4. The categories $\mathbf{SWeb}_{\uparrow}(\mathfrak{gl}_{\mathbf{n}})$ and $\mathbf{SWeb}_{\uparrow,\downarrow}(\mathfrak{gl}_{\mathbf{n}})$ introduced in [LT26, Def. 2A.7, Def. 2A.23] are, respectively, isomorphic as monoidal categories to our categories $\mathrm{Web}_{\mathrm{Sym}}^{\uparrow}(\mathfrak{gl})$ and $\mathrm{Web}_{\mathrm{Sym}}^{\uparrow\downarrow}(\mathfrak{gl}, t = q^{n})$. In both cases, an isomorphism is given by simply changing the colour of strands from black to orange. The crossings for $\mathbf{SWeb}_{\uparrow}(\mathfrak{gl}_{\mathbf{n}})$ and $\mathbf{SWeb}_{\uparrow,\downarrow}(\mathfrak{gl}_{\mathbf{n}})$, specified in [LT26, Defs. 2A.2, 2A.3, 2A.19, 2A.21], are the orange strand analogues of the $\mathrm{Web}(\mathfrak{sl}_n)$-crossings from Definition 5.3.4. (**SWeb** stands for "symmetric web", corresponding to our orange strands.)

The presentations of the web categories in [LT26] are broadly similar to, but more minimal than those given in this thesis; see [LT26, Thm. 4.7] for a precise statement about the minimality of their presentation. Whereas we defined $\mathrm{Web}^{\uparrow}(\mathfrak{gl})$ using merge and split generators with arbitrary strand labels, the generators for $\mathbf{SWeb}_{\uparrow}(\mathfrak{gl}_{\mathbf{n}})$ consist of merges and splits of the following form:



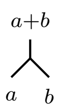
General merges and splits are then defined using multi-merge and multi-split morphisms; see [LT26, Def. 2A.5]. In $\mathrm{Web}_{\mathrm{Sym}}^{\uparrow}(\mathfrak{gl})$, those definitions hold as identities. One can show that the defining relations for $\mathrm{Web}_{\mathrm{Sym}}^{\uparrow}(\mathfrak{gl})$ are equivalent to those for $\mathbf{SWeb}_{\uparrow}(\mathfrak{gl}_{\mathbf{n}})$. For instance, in $\mathbf{SWeb}_{\uparrow}(\mathfrak{gl}_{\mathbf{n}})$, the relations (ASSOC) are assumed only in special cases, but as shown in [LT26, Lem. 2A.12], the general form of (ASSOC) follows from these special cases. We leave the remaining details (for both the upwards-oriented categories and the categories with duals) to the interested reader.

The category with duals, $\mathbf{SWeb}_{\uparrow,\downarrow}(\mathfrak{gl}_\mathbf{n})$, adds only two new cap and cup morphisms, together with an infinite family of downwards-oriented merges and splits, as generators:

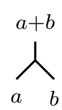
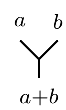


with morphisms of these shapes with general labels being defined in terms of the above generators; see [LT26, Def. 2A.20, 2A.22]. Those definitions hold as identities in $\mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl})$.

Connection 5.4.5. The categories q-**Schur** and q-**Schur**$_n$ studied in [Bru25] are, respectively, equivalent to our categories $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})$ and $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0})$. The objects of q-**Schur** are monoidally generated by the length-one compositions (a), with $a \geq 0$. These generators correspond to the objects $\uparrow_a \in \mathrm{ob}(\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}))$. In $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})$, the object $\uparrow_0$ is defined to be strictly equal to the monoidal unit $\mathbb{1}$, whereas in q-**Schur** (0) and $\mathbb{1}$ are distinct but isomorphic. To this end, q-**Schur** features generating morphisms $\underset{\bullet}{\mid}\colon \mathbb{1} \to (0)$ and $\overset{\bullet}{\mid}\colon (0) \to \mathbb{1}$, together with relations stating that these generators are mutually inverse. As noted previously, this approach inspired our treatment of the $\mathfrak{sl}_n$-web category in Definition 5.3.1.

In the following, we refer to the presentation of q-**Schur** given in [Bru25, Thm. 6.3]. The generating morphisms are the aforementioned $\underset{\bullet}{\mid}$ and $\overset{\bullet}{\mid}$, together with merge and split morphisms  and , for all $a, b \geq 0$. The defining relation [Bru25, (6.4)] is the q-**Schur**-analogue of our relation (TRIV). Their relation (6.5) is our (ASSOC), and their square-switch relation (1.3) is our (3.2.5) (together with an equivalent mirrored version of the relation).

Explicitly, an equivalence $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}) \to q$-**Schur** is given by:

$\uparrow_0 \mapsto \mathbb{1}$, $\quad \uparrow_a \mapsto (a)$ for $a \geq 1$,  $\mapsto$  for $a, b \geq 1$,  $\mapsto$  for $a, b \geq 1$,

and mapping merge/split generators with one (or both) strands labelled by 0 to the corresponding identity morphism in q-**Schur**, as in (TRIV). The category q-**Schur**$_n$ is defined to be the quotient of q-**Schur** by the tensor ideal generated by the identity morphisms for strands with a label strictly larger than n, and hence, by Remark 5.2.5, the functor specified above descends to an equivalence $\mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{n|0}) \to q$-**Schur**$_n$.

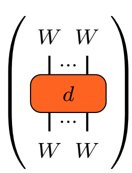
Chapter 6

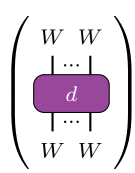
Incarnation functors

In this chapter, we will define and study the *web incarnation functors* that we have referred to several times earlier in this thesis. Such functors have previously been studied in various forms. Our approach, using the isomorphism $\mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}) \cong \mathcal{OS}^W$ from Theorem 4.13, allows us to precisely link web incarnation functors to incarnation functors for $\mathcal{OS}$. This in turn lets us extend existing results to some new cases; in Theorem 6.10, we will show that certain web incarnation functors are full and have kernels generated by the idempotents introduced in Definition 5.2.4. In [Eli15, Thm. 2.58], Elias established an analogue of Theorem 6.10 for exterior $\mathfrak{sl}_n$-webs over an arbitrary $\mathbb{Z}[q, q^{-1}]$-algebra, but similar results for $\mathfrak{gl}$-webs (and for symmetric or two-colour web categories) have only appeared in the literature for the case where $\Bbbk$ is a field. (Recall that, in this thesis, we assume only that $\Bbbk$ is a commutative ring.) Even for the cases where such results were already known, our method of proof is novel.

From this point forward, fix dimension parameters $m, n \geq 0$, not both zero. (Our main results in this chapter, Proposition 6.8 and Theorem 6.10, hold trivially when $m = n = 0$, but many of the upcoming arguments only make sense when $m + n \geq 1$. As such, we ignore the case $m = n = 0$ to simplify the exposition.) We remind the reader that V^+ denotes the natural $U_q(\mathfrak{gl}_{m|n})$-module, and V^- denotes its dual. Recall that we introduced the $U_q(\mathfrak{gl}_{m|n})$-modules $\mathrm{Sym}^d W$ and $\Lambda^d W$ in Definition 2.4.16, and that $B_{W,W} : W \otimes W \to W \otimes W$ denotes the braiding morphism for two copies of a module W.

Definition 6.1. Let W be a Hecke-braided $U_q(\mathfrak{gl}_{m|n})$-module. For each $d \in \mathbb{N}$, we define

$\Gamma^d W = \mathrm{im}$ 

, $\quad \Gamma^d_\Lambda W = \mathrm{im}$ 

.

We call these $U_q(\mathfrak{gl}_{m|n})$-modules the *dth (symmetric) divided power of* W and the *dth alternating divided power of* W, respectively.

Symmetric and exterior divided powers were previously defined and studied in a more general context in [BZ08]. The following lemma establishes variants of the canonical isomorphisms [BZ08, (2.3)], adapted to the setting of this thesis.

Lemma 6.2. Let W be a Hecke-braided $U_q(\mathfrak{gl}_{m|n})$-module. For each $d \geq 1$, the following maps define isomorphisms of $U_q(\mathfrak{gl}_{m|n})$-modules:

$$\mathrm{Sym}^d W \to \Gamma^d W, \qquad w_1 \cdots w_d \mapsto \underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}} (w_1 \otimes \cdots \otimes w_d)\,,$$

$$\Lambda^d W \to \Gamma^d_\Lambda W, \qquad w_1 \wedge \cdots \wedge w_d \mapsto \underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}} (w_1 \otimes \cdots \otimes w_d)\,.$$

Proof. The claims follow immediately when $d = 1$, so we assume $d \geq 2$. We will show that the first map is an isomorphism; the claim for alternating powers follows after mapping $q \mapsto -q^{-1}$.

Recall that $\mathrm{Sym}^\bullet W = TW/K$, where K is the two-sided ideal of the tensor superalgebra TW generated by $\{B_{W,W}(w) - qw : w \in W\}$. Hence, $\mathrm{Sym}^d W$ is the quotient of $W^{\otimes d}$ by the submodule M generated by

$$\{B_i(w) - qw : w \in W^{\otimes d}, 1 \leq i \leq d-1\},$$

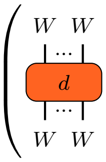
where $B_i \colon W^{\otimes d} \to W^{\otimes d}$ denotes the map that acts via $B_{W,W}$ on the ith and $(i+1)$th tensor factors. Given $\sigma \in S_d$, we will write $B_\sigma \colon W^{\otimes d} \to W^{\otimes d}$ for the image of $t_\sigma \in H_d$ via the Hecke-braid homomorphism $h_{W,d} \colon H_d \to \mathrm{End}(W^{\otimes d})$ (see (2.5.13)). It follows that M contains the elements $B_\sigma(w) - q^{l(\sigma)} w$ for all $\sigma \in S_d$ and $w \in W^{\otimes d}$.

It suffices to show that $M = \ker\left(\underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}}\right)$; the first isomorphism theorem then implies that the map from the lemma statement is an isomorphism. Recall the absorbing property of symmetrizers: (2.5.7). In our present notation, this says $\underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}} \circ B_i = q \underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}}$ for each $1 \leq i \leq d-1$. Hence $M \subseteq \ker\left(\underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}}\right)$. For the other inclusion, let $w \in \ker\left(\underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}}\right)$.

Then:

$$0 = \underset{W \cdots W}{\overset{W \cdots W}{\boxed{d}}} (w) = \frac{q^{-\binom{d}{2}}}{[d]!} \sum_{\sigma \in S_d} q^{l(\sigma)} B_\sigma(w) = \frac{q^{-\binom{d}{2}}}{[d]!} \sum_{\sigma \in S_d} q^{2l(\sigma)} w + q^{l(\sigma)}(B_\sigma(w) - q^{l(\sigma)} w)$$

$$\overset{(2.5.9)}{=} w + \frac{q^{-\binom{d}{2}}}{[d]!} \sum_{\sigma \in S_d} q^{l(\sigma)}(B_\sigma(w) - q^{l(\sigma)} w),$$

and hence $w = -\frac{q^{-\binom{d}{2}}}{[d]!} \sum_{\sigma \in S_d} q^{l(\sigma)}(B_\sigma(w) - q^{l(\sigma)}w)$. Each term $B_\sigma(w) - q^{l(\sigma)}w$ lies in M, as noted above, and so $w \in M$. Thus $M = \ker\Bigg($ 

$\Bigg)$, as desired. $\square$

It follows from Lemma 2.5.12 that, for each Hecke-braided $U_q(\mathfrak{gl}_{m|n})$-module W with dual W^* and each $d \geq 1$, $\Gamma^d W^*$ is dual to $\Gamma^d W$, and $\Gamma^d_\Lambda W^*$ is dual to $\Gamma^d_\Lambda W$. Using Lemma 6.2, we get $(\mathrm{Sym}^d W)^* \cong \mathrm{Sym}^d W^*$ and $(\Lambda^d W)^* \cong \Lambda^d W^*$.

Remark 6.3. When working in a more general setting than we consider in this thesis, the isomorphisms $\mathrm{Sym}^d W \cong \Gamma^d W$ and $\Lambda^d W \cong \Gamma^d_\Lambda W$ from Lemma 6.2 may no longer hold. It is for this reason that we choose not to fully identify $\mathrm{Sym}^d W$ with $\Gamma^d W$ and $\Lambda^d W$ with $\Gamma^d_\Lambda W$. In particular, we will define our web category incarnation functors in terms of $\Gamma^d W$ and $\Gamma^d_\Lambda W$, since this is what naturally arises from the proof of Theorem 6.10.

Note that the isomorphisms from Lemma 6.2 involve division by $[d]!$, since the definition of our idempotent (anti)symmetrizers features a factor of $\frac{q^{\pm\binom{d}{2}}}{[d]!}$. Our base rings $\Bbbk_q$ and $\Bbbk_{q,t}$ contain the inverses of all nonzero quantum integers by definition, and hence this division is well-defined. One can study symmetric and alternating powers more generally, e.g. over $\mathbb{Z}[q, q^{-1}]$, in which most quantum integers are not invertible, and hence the isomorphisms from Lemma 6.2 are ill-defined. In such situations one can still define the *unscaled symmetrizer* and *unscaled antisymmetrizer*, given by

$$\overline{e_d} = \sum_{\sigma \in S_d} q^{l(\sigma)} t_\sigma \in H_d, \quad \overline{e'_d} = \sum_{\sigma \in S_d} (-q)^{-l(\sigma)} t_\sigma \in H_d,$$

but these elements are not idempotent for $d \geq 2$, and hence the associated maps $W^{\otimes d} \to W^{\otimes d}$ are no longer projections onto copies of $\mathrm{Sym}^d W$ or $\Lambda^d W$ contained in $W^{\otimes d}$. In the general setting, one should define

$$\Gamma^d W = \{w \in W^{\otimes d} : B_\sigma(w) = q^{l(\sigma)} w \text{ for all } \sigma \in S_d\},$$

$$\Gamma^d_\Lambda W = \{w \in W^{\otimes d} : B_\sigma(w) = (-q)^{-l(\sigma)} w \text{ for all } \sigma \in S_d\},$$

which are quantum analogues of the subspaces of symmetric and alternating tensors. With these definitions, over any commutative ring we have

$$(\mathrm{Sym}^d W)^* \cong \Gamma^d W^*, \quad (\Lambda^d W)^* \cong \Gamma^d_\Lambda W^*,$$

but in general, when $[d]!$ is not invertible,

$$\mathrm{Sym}^d W \not\cong \Gamma^d W, \quad \Lambda^d W \not\cong \Gamma^d_\Lambda W.$$

While the notation $\Gamma^d W$ for symmetric divided powers is standard, the author of this thesis is not aware of any commonly used notation for the alternating divided powers. In the purely even setting, i.e. working with $U_q(\mathfrak{gl}_n)$-modules, we in fact *do* have $\Lambda^d W \cong \Gamma^d_\Lambda W$ over any commutative ring, and hence there is no need to introduce $\Gamma^d_\Lambda W$ as a separate object.

The morphisms in the following definition will be used to specify our incarnation functors.

Definition 6.4. In all of the diagrams appearing below, upwards-oriented strands represent V^+, and downwards-oriented strands represent V^-.

Fix $a, b \in \mathbb{N}$. Let $\alpha_{a,b} \colon (V^+)^{\otimes a} \otimes (V^+)^{\otimes b} \to (V^+)^{\otimes(a+b)}$ denote the natural associator isomorphism, and let $\iota_a^{\mathrm{Sym},\pm} \colon \Gamma^a V^\pm \to (V^\pm)^{\otimes a}$ and $\iota_a^{\Lambda,\pm} \colon \Gamma_\Lambda^a V^\pm \to (V^\pm)^{\otimes a}$ denote the inclusion maps. Define maps

$$m_{a,b}^{\mathrm{Sym}} \colon \Gamma^a V^+ \otimes \Gamma^b V^+ \to \Gamma^{a+b} V^+, \qquad m_{a,b}^{\Lambda} \colon \Gamma_\Lambda^a V^+ \otimes \Gamma_\Lambda^b V^+ \to \Gamma_\Lambda^{a+b} V^+$$

(m being short for "merge") by

$$m_{a,b}^{\mathrm{Sym}} = \boxed{a+b} \circ \alpha_{a,b} \circ \left(\iota_a^{\mathrm{Sym},+} \otimes \iota_b^{\mathrm{Sym},+} \right), \qquad m_{a,b}^{\Lambda} = \boxed{a+b} \circ \alpha_{a,b} \circ \left(\iota_a^{\Lambda,+} \otimes \iota_b^{\Lambda,+} \right).$$

Similarly, define maps

$$s_{a,b}^{\mathrm{Sym}} \colon \Gamma^{a+b} V^+ \to \Gamma^a V^+ \otimes \Gamma^b V^+, \qquad s_{a,b}^{\Lambda} \colon \Gamma_\Lambda^{a+b} V^+ \to \Gamma_\Lambda^a V^+ \otimes \Gamma_\Lambda^b V^+$$

(s being short for "split") by

$$s_{a,b}^{\mathrm{Sym}} = \alpha_{a,b}^{-1} \circ \iota_{a+b}^{\mathrm{Sym},+}, \qquad s_{a,b}^{\Lambda} = \alpha_{a,b}^{-1} \circ \iota_{a+b}^{\Lambda,+}.$$

It follows from (2.5.15) that the image of $\alpha_{a,b}^{-1} \circ \iota_{a+b}^{\mathrm{Sym},+}$ lies in $\Gamma^a V^+ \otimes \Gamma^b V^+$, and similarly for the antisymmetric/alternating case, so these maps are well-defined with the claimed codomains.

Next, we recall the (co)evaluation morphisms specified in [SSS26, §5]. Define evaluation morphisms $C \colon V^+ \otimes V^- \to \mathbb{1}$ and $C' \colon V^- \otimes V^+ \to \mathbb{1}$ by

$$C(v_i^+ \otimes v_j^-) = \delta_{ij}(-q)^{|i|}, \quad C'(v_i^- \otimes v_j^+) = \delta_{ij}(-1)^{i+\bar{i}} q^{m+n-|i-1|}.$$

The corresponding coevaluation morphisms are $U \colon \mathbb{1} \to V^- \otimes V^+$ and $U' \colon \mathbb{1} \to V^+ \otimes V^-$, given by

$$U(1) = \sum_{i=1-n}^{m} (-q)^{-|i|} v_i^- \otimes v_i^+, \quad U'(1) = \sum_{i=1-n}^{m} (-1)^{i+\bar{i}} q^{|i-1|-m-n} v_i^+ \otimes v_i^-.$$

For each $a \in \mathbb{N}$, we define $C_a \colon (V^+)^{\otimes a} \otimes (V^-)^{\otimes a} \to \mathbb{1}$ and $C'_a \colon (V^-)^{\otimes a} \otimes (V^+)^{\otimes a} \to \mathbb{1}$ to be the maps obtained by nesting a copies of C or C', like in Lemma 3.1.5. Explicitly,

$$C_a((v_{i_1}^+ \otimes \cdots \otimes v_{i_a}^+) \otimes (v_{j_1}^- \otimes \cdots \otimes v_{j_a}^-)) = C(v_{i_1}^+ \otimes v_{j_a}^-) C(v_{i_2}^+ \otimes v_{j_{a-1}}^-) \cdots C(v_{i_a}^+ \otimes v_{j_1}^-),$$

and similarly for C'_a. We define the coevaluation maps $U_a \colon \mathbb{1} \to (V^-)^{\otimes a} \otimes (V^+)^{\otimes a}$ and $U'_a \colon \mathbb{1} \to (V^+)^{\otimes a} \otimes (V^-)^{\otimes a}$ by nesting copies of U or U' in the same way.

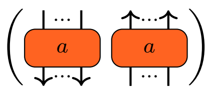
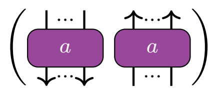
Finally, we define (co)evaluation morphisms

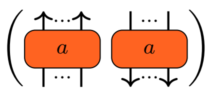
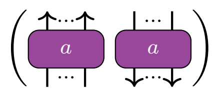
$$C_a^{\mathrm{Sym}} \colon \Gamma^a V^+ \otimes \Gamma^a V^- \to \mathbb{1}, \quad C_a^{\Lambda} \colon \Gamma_\Lambda^a V^+ \otimes \Gamma_\Lambda^a V^- \to \mathbb{1},$$

$$(C'_a)^{\mathrm{Sym}}\colon \Gamma^a V^- \otimes \Gamma^a V^+ \to \mathbb{1}, \quad (C'_a)^{\Lambda}\colon \Gamma^a_\Lambda V^- \otimes \Gamma^a_\Lambda V^+ \to \mathbb{1},$$
$$U_a^{\mathrm{Sym}}\colon \mathbb{1} \to \Gamma^a V^- \otimes \Gamma^a V^+, \quad U_a^{\Lambda}\colon \mathbb{1} \to \Gamma^a_\Lambda V^- \otimes \Gamma^a_\Lambda V^+,$$
$$(U'_a)^{\mathrm{Sym}}\colon \mathbb{1} \to \Gamma^a V^+ \otimes \Gamma^a V^-, \quad (U'_a)^{\Lambda}\colon \mathbb{1} \to \Gamma^a_\Lambda V^+ \otimes \Gamma^a_\Lambda V^-$$

as follows:

$$C_a^{\mathrm{Sym}} = C_a \circ (\iota_a^{\mathrm{Sym},-} \otimes \iota_a^{\mathrm{Sym},+}), \quad C_a^{\Lambda} = C_a \circ (\iota_a^{\Lambda,-} \otimes \iota_a^{\Lambda,+}),$$
$$(C'_a)^{\mathrm{Sym}} = C'_a \circ (\iota_a^{\mathrm{Sym},+} \otimes \iota_a^{\mathrm{Sym},-}), \quad (C'_a)^{\Lambda} = C'_a \circ (\iota_a^{\Lambda,+} \otimes \iota_a^{\Lambda,-}),$$

$U_a^{\mathrm{Sym}} =$  $\circ\, U_a$, $\quad U_a^{\Lambda} =$  $\circ\, U_a$,

$(U'_a)^{\mathrm{Sym}} =$  $\circ\, U'_a$, $\quad (U'_a)^{\Lambda} =$  $\circ\, U'_a$.

Conjugating by the isomorphisms from Lemma 6.2, the merge morphisms $m_{a,b}^{\mathrm{Sym}}$ and $m_{a,b}^{\Lambda}$ simply become the multiplication maps from $\mathrm{Sym}^\bullet V^+$ and $\Lambda^\bullet V^+$, i.e.

$$\mathrm{Sym}^a V^+ \otimes \mathrm{Sym}^b V^+ \to \mathrm{Sym}^{a+b} V^+, \quad (v_{i_1}^+ \cdots v_{i_a}^+) \otimes (v_{j_1}^+ \cdots v_{j_b}^+) \to v_{i_1}^+ \cdots v_{i_a}^+ v_{j_1}^+ \cdots v_{j_b}^+,$$
$$\Lambda^a V^+ \otimes \Lambda^b V^+ \to \Lambda^{a+b} V^+, \quad (v_{i_1}^+ \wedge \cdots \wedge v_{i_a}^+) \otimes (v_{j_1}^+ \wedge \cdots \wedge v_{j_b}^+) \to v_{i_1}^+ \wedge \cdots \wedge v_{i_a}^+ \wedge v_{j_1}^+ \wedge \cdots \wedge v_{j_b}^+.$$

Working in full generality, the split morphisms $s_{a,b}^{\mathrm{Sym}}$ and $s_{a,b}^{\Lambda}$ are less straightforward to describe after conjugation. As we mentioned much earlier in this thesis, in the purely even classical case these maps are given by

$$v_{i_1}^+ \cdots v_{i_{a+b}}^+ \mapsto \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} (v_{i_{\sigma(1)}}^+ \cdots v_{i_{\sigma(a)}}^+) \otimes (v_{i_{\sigma(a+1)}}^+ \cdots v_{i_{\sigma(a+b)}}^+),$$
$$v_{i_1}^+ \wedge \cdots \wedge v_{i_{a+b}}^+ \mapsto \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} \mathrm{sgn}(\sigma)(v_{i_{\sigma(1)}}^+ \wedge \cdots \wedge v_{i_{\sigma(a)}}^+) \otimes (v_{i_{\sigma(a+1)}}^+ \wedge \cdots \wedge v_{i_{\sigma(a+b)}}^+).$$

In the super case, one needs to introduce a term in the sums accounting for the signs picked up when permuting the v_{i_j} by σ. In the purely even quantum case, one can succinctly define the action of these split morphisms on the spanning elements $v_{i_1}^+ \cdots v_{i_{a+b}}^+$ and $v_{i_1}^+ \wedge \cdots \wedge v_{i_{a+b}}^+$, where $i_1 \geq \cdots \geq i_{a+b}$. For details, see e.g. [CKM14, §3.1], which treats the alternating case; their morphism $M'_{k,l}$ corresponds to our $s_{k,l}^{\Lambda}$. One can extend this approach to the general super quantum case, but the expressions involved become somewhat unwieldy, so we omit the details here.

We now introduce some maps and definitions that will be used to prove Proposition 6.8.

Lemma 6.5. There exists a $\Bbbk_q$-linear isomorphism of superalgebras

$$P = P_{m|n}\colon U_q(\mathfrak{gl}_{m|n}) \to U_q(\mathfrak{gl}_{n|m})$$

acting on generators as:

$$P(E_i) = F'_{-i}, \quad P(F_i) = E'_{-i}, \quad P(D_i) = (D'_{1-i})^{-1}, \quad P(D_i^{-1}) = D'_{1-i},$$

where we use primes to denote the generators in the target superalgebra to avoid confusion. Using the integral form of the quantized enveloping superalgebra discussed in Remark 2.4.9, this map acts on the additional generators as $P(H_i) = -(-1)^{\overline{E_i}} H'_{-i} = (-1)^{\delta_{i0}+1} H'_{-i}$.

Proof. The specified action of P on generators is evidently parity-preserving. The fact that P is well-defined, i.e. that it preserves the defining relations for $U_q(\mathfrak{gl}_{m|n})$, follows from direct calculations. For instance,

$$\begin{aligned}
P(E_iF_j - (-1)^{\overline{E_i}\,\overline{F_j}}F_jE_i) &= F'_{-i}E'_{-j} - (-1)^{\overline{E_i}\,\overline{F_j}}E'_{-j}F'_{-i} \\
&= -(-1)^{\overline{E'_{-j}}\,\overline{F'_{-i}}}(E'_{-j}F'_{-i} - (-1)^{\overline{E'_{-j}}\,\overline{F'_{-i}}}F'_{-i}E'_{-j}) \\
&\overset{(2.4.8)}{=} -(-1)^{\overline{E'_{-j}}\,\overline{F'_{-i}}}\delta_{-i,-j}\frac{D'_{-i}(D'_{-i+1})^{-1} - D'_{-i+1}(D'_{-i})^{-1}}{q_{-i} - q_{-i}^{-1}} \\
&\overset{*}{=} -(-1)^{\overline{E'_{-j}}\,\overline{F'_{-i}}+\overline{-i}}\delta_{ij}\frac{D'_{-i}(D'_{-i+1})^{-1} - D'_{-i+1}(D'_{-i})^{-1}}{q - q^{-1}} \\
&\overset{**}{=} (-1)^{\bar{i}}\delta_{ij}\frac{D'_{-i}(D'_{-i+1})^{-1} - D'_{-i+1}(D'_{-i})^{-1}}{q - q^{-1}} \overset{*}{=} \delta_{ij}\frac{D'_{-i}(D'_{-i+1})^{-1} - D'_{-i+1}(D'_{-i})^{-1}}{q_i - q_i^{-1}} \\
&= P\left(\delta_{ij}\frac{D_{i+1}^{-1}D_i - D_i^{-1}D_{i+1}}{q_i - q_i^{-1}}\right) \overset{(2.4.6)}{=} P\left(\delta_{ij}\frac{D_iD_{i+1}^{-1} - D_{i+1}D_i^{-1}}{q_i - q_i^{-1}}\right),
\end{aligned}$$

where in the equalities marked $*$ we use the fact that

$$q_i - q_i^{-1} = (-1)^{\bar{i}}(q - q^{-1}),$$

and in the equality marked $**$ we use

$$\overline{E'_{-i}}\,\overline{F'_{-i}} + \overline{-i} = \delta_{i0} + \overline{-i} = \bar{i} + 1.$$

This shows that P preserves the defining relation (2.4.8). The calculations for the other relations are similar. The inverse of $P_{m|n}$ is $P_{n|m}$. $\square$

The isomorphism P discussed in Lemma 6.5 induces an isomorphism of categories

$$\tilde{P}\colon U_q(\mathfrak{gl}_{m|n})\text{-mod} \to U_q(\mathfrak{gl}_{n|m})\text{-mod}.$$

Explicitly, $\tilde{P}$ acts as the identity on morphisms, and sends each $U_q(\mathfrak{gl}_{m|n})$-module M to the $U_q(\mathfrak{gl}_{n|m})$-module with the same underlying abelian group, and $U_q(\mathfrak{gl}_{n|m})$-action given by

$$X \cdot m := P^{-1}(X) \cdot m,$$

where the action on the right is the original $U_q(\mathfrak{gl}_{m|n})$-action. It is clear that $\tilde{P}$ sends type 1 weight modules to type 1 weight modules, and so it restricts to an isomorphism

$$\dot{P}\colon \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod} \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{n|m})\text{-mod}.$$

(Recall that $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$ is equivalent to the category of type 1 weight modules for $U_q(\mathfrak{gl}_{m|n})$.)

Definition 6.6. Let A be a superalgebra and M an A-module. We define the *parity-shifted module*

$$\Pi M := \{\pi m : m \in M\},$$

where π is a formal symbol. The parities of elements in ΠM are given by $\overline{\pi m} = \overline{m} + 1$. The A-module structure on ΠM is given by $a \cdot \pi m = (-1)^{\bar{a}}\pi(a \cdot m)$. It follows that the map $\pi\colon M \to \Pi M$ (given by $m \mapsto \pi m$) is an odd isomorphism of A-modules.

Lemma 6.7. There is a $\Bbbk_q$-linear isomorphism $\rho\colon \mathcal{H} \to \mathcal{H}$ acting on generators as:

$$\rho(\uparrow) = \uparrow, \quad \rho\left(\nwarrow\!\!\!\!\nearrow\right) = -\nwarrow\!\!\!\!\nearrow, \quad \rho\left(\nwarrow\!\!\!\!\nearrow\right) = -\nwarrow\!\!\!\!\nearrow.$$

Note that ρ restricts on endomorphism spaces to the automorphisms $\rho_n\colon H_n \to H_n$ discussed in Lemma 2.5.9.

Proof. The fact that ρ preserves the defining relations for $\mathcal{H}$ follows from direct calculations. For instance,

$$\rho\left(\nwarrow\!\!\!\!\nearrow - \nwarrow\!\!\!\!\nearrow\right) = -\nwarrow\!\!\!\!\nearrow + \nwarrow\!\!\!\!\nearrow \overset{(5.1.2)}{=} (q-q^{-1})\uparrow\uparrow = \rho\left((q-q^{-1})\uparrow\uparrow\right),$$

which shows that (5.1.2) is preserved. □

Proposition 6.8. For all $m,n \in \mathbb{N}$, there is a $\Bbbk_q$-linear monoidal *incarnation functor*

$$I^{\mathcal{OS}}_{m|n}\colon \mathcal{OS}(t = q^{m-n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}$$

that acts as follows:

$$I^{\mathcal{OS}}_{m|n}(\uparrow) = V^+, \qquad I^{\mathcal{OS}}_{m|n}(\downarrow) = V^-,$$

$$I^{\mathcal{OS}}_{m|n}\left(\nwarrow\!\!\!\!\nearrow\right) = B_{V^+,V^+}, \quad I^{\mathcal{OS}}_{m|n}\left(\nwarrow\!\!\!\!\nearrow\right) = B^{-1}_{V^+,V^+},$$

$$I^{\mathcal{OS}}_{m|n}\left(\curvearrowright\right) = C, \quad I^{\mathcal{OS}}_{m|n}\left(\curvearrowleft\right) = C', \quad I^{\mathcal{OS}}_{m|n}\left(\smile\!\uparrow\right) = U, \quad I^{\mathcal{OS}}_{m|n}\left(\uparrow\!\smile\right) = U'.$$

If any of the following conditions hold:

- $\Bbbk_q = \mathbb{C}(q)$, or
- We are in the purely even case (that is, $n = 0$), or
- We are in the purely odd case (that is, $m = 0$),

then $I^{\mathcal{OS}}_{m|n}$ is full, and its kernel is the tensor ideal generated by the idempotent $i_{\lambda(m+1,n+1)}$ (see Definition 5.2.4). Hence, in any of these three cases, the induced functor

$$\tilde{I}^{\mathcal{OS}}_{m|n}\colon\ \mathcal{OS}_{m|n} \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}$$

is full and faithful.

Proof. The existence of $I^{\mathcal{OS}}_{m|n}$ was proved over $\mathbb{C}(q)$ in [SSS26, Prop. 6.1]. More precisely, that proposition deals with modules over the superalgebra U_σ, which contains $U_q(\mathfrak{gl}_{m|n})$ as a subalgebra. Our incarnation functor $I^{\mathcal{OS}}_{m|n}$ is the composition of the incarnation functor from [SSS26, Prop. 6.1] and the restriction map U_σ-mod $\to U_q(\mathfrak{gl}_{m|n})$-mod. Almost all of the arguments from the proof of that proposition are valid over our base ring $\Bbbk_q$. The only step which needs to be adjusted is the verification that the curl identities from (3.1.4) are preserved. The proof in [SSS26] uses a form of Schur's lemma (namely: every endomorphism of a simple module is a scalar multiple of the identity) which does not necessarily hold over $\Bbbk_q$. We sidestep this issue as follows: a direct computation in the proof of [SSS26, Prop. 6.1] shows that $I^{\mathcal{OS}}_{m|n}\big(\ \big)$ acts as the scalar q^{m-n} on one of the basis vectors for V^+, and that calculation is still valid over $\Bbbk_q$. Since the action of $U_q(\mathfrak{gl}_{m|n})$ on V^+ is transitive, this implies that the image morphism is simply $q^{m-n} \cdot \mathrm{id}_{V^+}$, as desired. The argument for the other curl relation is essentially the same.

If $\Bbbk_q = \mathbb{C}(q)$, the claim about fullness was proved in [QS19, Prop. 6.4], and a proof for the claim about the kernel was outlined in [QS19, Rem. 6.19]. That remark addresses the kernel of the incarnation functor out of $\mathsf{Sp}(m-n) \cong \mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl}, t = q^{m-n})$, but the argument applies equally to $\mathcal{OS}(t = q^{m-n})$. Alternatively, one could obtain the kernel result for $\mathcal{OS}(t = q^{m-n})$ from the corresponding result for $\mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl}, t = q^{m-n})$ by restricting to the subcategory monoidally generated by $\uparrow_1$ and $\downarrow_1$, which is isomorphic to $\mathcal{OS}(t = q^{m-n})$.

Next, suppose that $n = 0$, with no assumptions on $\Bbbk_q$ (beyond those present in our definition of $\Bbbk_q$). We will deduce fullness and the kernel claim from [Bru25, Thm. 4]. That theorem establishes the existence of a full and faithful functor $\Sigma_m \colon q\text{-}\mathbf{Schur}_m \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|0})$-mod (note that the theorem is stated in [Bru25] with the codomain being the subcategory of tilting modules, but this change of codomain does not affect fullness or faithfulness). This result is first proved in [Bru25] over $\mathbb{Z}[q, q^{-1}]$, and then extended by way of a base change argument to any field $\mathbb{F}$, with an arbitrary choice of quantum parameter $q \in \mathbb{F}^\times$. In fact, the same base change argument is valid when $\mathbb{F}$ is any commutative ring. In particular, [Bru25, Thm. 4] holds over our base ring $\Bbbk_q$.

Recall that there is an equivalence $e\colon \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{m|0}) \to q\text{-}\mathbf{Schur}_m$, as discussed in Connection 5.4.5. The purple upwards-oriented version of Theorem 4.13 gives an isomorphism $\mathcal{H}^{\Lambda} \to \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl})$, which descends to an isomorphism $\mathcal{H}^{\Lambda}_{m|0} \to \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{m|0})$; we remind the reader that $\mathcal{H}^{\Lambda}_{m|0}$ was introduced in Definition 5.3.10 (see also Definition 5.2.4). Hence, the composite functor

$$\mathcal{H}_{m|0} \xrightarrow{\iota} \mathcal{H}^{\Lambda}_{m|0} \xrightarrow{\cong} \mathrm{Web}^{\uparrow}_{\Lambda}(\mathfrak{gl}_{m|0}) \xrightarrow{e} q\text{-}\mathbf{Schur}_m \xrightarrow{\Sigma_m} \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|0})\text{-mod}$$

is full and faithful; call this composite functor J. Here, ι denotes the indicated inclusion map, identifying $\mathcal{H}_{m|0}$ with the full subcategory of $\mathcal{H}^{\Lambda}_{m|0}$ generated by $\uparrow$ as in Remark 4.3. Tracing the definitions of the functors involved in the definition of J yields $J(\uparrow) = V^+$, $J\big(\ \big) = B_{V^+,V^+}$, and $J\big(\ \big) = B^{-1}_{V^+,V^+}$. In other words, J agrees with $\tilde{I}^{\mathcal{OS}}_{m|0}\colon \mathcal{OS}_{m|0} \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|0})$-mod on $\mathcal{H}_{m|0}$ (viewed as a subcategory of $\mathcal{OS}_{m|0}$, like in Lemma 5.1.2). Since $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|0})$-mod is a ribbon category and the objects of $\mathcal{OS}_{m|0}$ are generated by $\uparrow$ and $\downarrow$, it follows from abstract arguments that J being full and faithful implies that $\tilde{I}^{\mathcal{OS}}_{m|0}$ is also full and faithful; see [QS19, Prop. 6.2] for details (they only discuss fullness, but essentially the same arguments work for faithfulness). Fullness of $\tilde{I}^{\mathcal{OS}}_{m|0}$ implies that $I^{\mathcal{OS}}_{m|0}$ is also full. Faithfulness of $\tilde{I}^{\mathcal{OS}}_{m|0}$ implies that the kernel of $I^{\mathcal{OS}}_{m|0}$ is the tensor ideal generated by the antisymmetrizer $i_{\lambda(m+1,1)}$.

Finally, suppose that $m = 0$. We will deduce the desired properties of $I^{\mathcal{OS}}_{0|n}$ from those we just established in the purely even case. Let K denote the following composition:

$$\mathcal{H} \xrightarrow{\rho} \mathcal{H} \xrightarrow{\iota} \mathcal{OS}(t = q^n) \xrightarrow{I^{\mathcal{OS}}_{n|0}} \dot{\mathcal{U}}_q(\mathfrak{gl}_{n|0})\text{-mod} \xrightarrow{\dot{P}} \dot{\mathcal{U}}_q(\mathfrak{gl}_{0|n})\text{-mod}\,.$$

Here, ρ is the map from Lemma 6.7, ι is inclusion from Lemma 5.1.2 (that lemma holds with $\mathcal{OS}(t = q^n)$ in place of $\mathcal{OS}$, with the same proof), and $\dot{P}$ is the isomorphism discussed after the proof of Lemma 6.5. We showed above that $I^{\mathcal{OS}}_{n|0}$ is full, and that its kernel is generated by the antisymmetrizer $i_{\lambda(n+1,1)}$. Since ρ, ι, and $\dot{P}$ are each full and faithful, we get that K is full, and that its kernel is generated by the preimage of $i_{\lambda(n+1,1)}$ along the composition

$$\mathcal{H} \xrightarrow{\rho} \mathcal{H} \xrightarrow{\iota} \mathcal{OS}(t = q^n).$$

Using Lemma 2.5.9, this preimage is the symmetrizer $i_{\lambda(1,n+1)}$.

Direct calculations show that $K(\uparrow)$ is isomorphic (as a $U_q(\mathfrak{gl}_{0|n})$-module) to ΠV^+ (here, V^+ denotes the natural module for $U_q(\mathfrak{gl}_{0|n})$). Identifying $K(\uparrow)$ and ΠV^+, further direct calculations show that

$K\Big($  $\Big) = \pi^{\otimes 2} \circ B_{V^+,V^+} \circ (\pi^{\otimes 2})^{-1}, \quad K\Big($  $\Big) = \pi^{\otimes 2} \circ B^{-1}_{V^+,V^+} \circ (\pi^{\otimes 2})^{-1}.$

In other words, K agrees with $I^{\mathcal{OS}}_{0|n}$ on $\mathcal{H} \subseteq \mathcal{OS}(t = q^{-n})$ up to parity shifts and the presence of the symbols π. It follows that the restriction of $I^{\mathcal{OS}}_{0|n}$ to $\mathcal{H}$ is full and has kernel generated by $i_{\lambda(1,n+1)}$. As in the final steps of the argument for the purely even case above, this implies that $I^{\mathcal{OS}}_{0|n}$ is full and has kernel generated by $i_{\lambda(1,n+1)}$, as desired. $\square$

Remark 6.9. We expect the fullness and kernel results from Proposition 6.8 to hold more generally, i.e. for any dimension parameters m and n, and any commutative ring $\Bbbk$. The proof of [QS19, Prop. 6.4] (which we cited above) relies on a quantized version of Schur-Weyl duality. Quantized super Schur-Weyl duality was established over $\mathbb{Q}(q)$ in [Mit06], and we expect this duality to hold over more general base rings too. As such, it should be possible to extend the arguments in [QS19] to obtain the fullness and kernel results from Proposition 6.8 in the case $\Bbbk_q = \mathbb{Q}(q)$ (without any assumptions on m or n), and possibly over other base rings in the future. It also seems likely that the material in [Bru25] can be extended to the super setting, which would imply our fullness and kernel results without any assumptions on m, n, or $\Bbbk_q$ (beyond those present in our definition of $\Bbbk_q$).

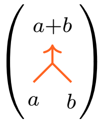
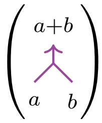
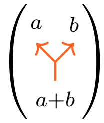
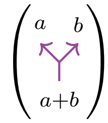
Since our web category incarnation functors (see below) are defined in terms of those for $\mathcal{OS}$, any new fullness and kernel results for $\mathcal{OS}$-incarnation functors would immediately imply corresponding results for web categories.

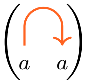
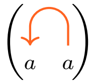
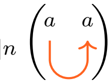
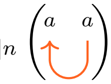
Theorem 6.10. For all $m, n \in \mathbb{N}$, there is a $\Bbbk_q$-linear *incarnation functor*

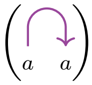
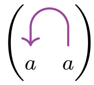
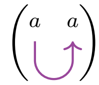
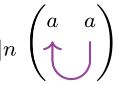
$$I_{m|n} \colon \ \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}, t = q^{m-n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}$$

acting on generators as follows:

$$I_{m|n}(\uparrow_a) = \Gamma^a V^+, \quad I_{m|n}(\uparrow_a) = \Gamma^a_\Lambda V^+, \quad I_{m|n}(\downarrow_a) = \Gamma^a V^-, \quad I_{m|n}(\downarrow_a) = \Gamma^a_\Lambda V^-,$$

$I_{m|n}$  $= m^{\mathrm{Sym}}_{a,b}$, $I_{m|n}$  $= m^{\Lambda}_{a,b}$, $I_{m|n}$  $= s^{\mathrm{Sym}}_{a,b}$, $I_{m|n}$  $= s^{\Lambda}_{a,b}$,

$I_{m|n}$  $= C^{\mathrm{Sym}}_a$, $I_{m|n}$  $= (C'_a)^{\mathrm{Sym}}$, $I_{m|n}$  $= U^{\mathrm{Sym}}_a$, $I_{m|n}$  $= (U'_a)^{\mathrm{Sym}}$,

$I_{m|n}$  $= C^{\Lambda}_a$, $I_{m|n}$  $= (C'_a)^{\Lambda}$, $I_{m|n}$  $= U^{\Lambda}_a$, $I_{m|n}$  $= (U'_a)^{\Lambda}$.

We write

$$I^{\mathrm{Sym}}_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl}, t = q^{m-n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod},$$

$$I^{\Lambda}_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl}, t = q^{m-n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}$$

for the corresponding monochromatic incarnation functors, obtained by restricting $I_{m|n}$ to the indicated subcategories. If any of the following conditions hold:

- $\Bbbk_q = \mathbb{C}(q)$, or
- We are in the purely even case (that is, $n = 0$), or
- We are in the purely odd case (that is, $m = 0$),

then each functor $I_{m|n}$, $I^{\mathrm{Sym}}_{m|n}$, and $I^{\Lambda}_{m|n}$ is full, and its kernel is the tensor ideal generated by the idempotent $i_{\lambda(m+1,n+1)}$. Hence, in any of these three cases, the induced functors

$$\tilde{I}_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}_{m|n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod},$$

$$\tilde{I}^{\mathrm{Sym}}_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}_{\mathrm{Sym}}(\mathfrak{gl}_{m|n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}, \quad \tilde{I}^{\Lambda}_{m|n}\colon \mathrm{Web}^{\uparrow\downarrow}_{\Lambda}(\mathfrak{gl}_{m|n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}$$

are full and faithful.

Proof. Since $\dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod is idempotent complete, the universal property of Karoubi envelopes yields a functor $\mathrm{Kar}(I^{\mathcal{OS}}_{m|n})\colon \mathrm{Kar}(\mathcal{OS}(t = q^{m-n})) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})$-mod extending $I^{\mathcal{OS}}_{m|n}$. Restricting to the full subcategory generated by the objects Λ^a and Sym^a yields a functor

$$\mathcal{OS}^W(t = q^{m-n}) \to \dot{\mathcal{U}}_q(\mathfrak{gl}_{m|n})\text{-mod}.$$

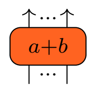
Precomposing with the isomorphism $\Phi\colon \mathrm{Web}^{\uparrow\downarrow}(\mathfrak{gl}, t = q^{m-n}) \cong \mathcal{OS}^W(t = q^{m-n})$ obtained from Theorem 4.13 yields the desired functor $I_{m|n}$. In the cases mentioned in the bullet points from the theorem statement, the claimed properties for $I_{m|n}$ follow straightforwardly from the corresponding ones for $I^{\mathcal{OS}}_{m|n}$ using the universal property of Karoubi envelopes. In turn, the claims for $I^{\mathrm{Sym}}_{m|n}$ and $I^{\Lambda}_{m|n}$ follow from those for $I_{m|n}$.

The fact that $I_{m|n}$ acts on generators as specified above follows directly from the way $\mathrm{Kar}(I^{\mathcal{OS}}_{m|n})$ is defined. The relevant property is as follows. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a strict monoidal functor, $\mathcal{D}$ is idempotent complete, and $\psi\colon (X, e) \to (Y, f)$ is a morphism in

Kar($\mathcal{C}$). Let $\widehat{F(X)}$ and $\widehat{F(Y)}$ respectively denote the objects in $\mathcal{D}$ that $F(e)$ and $F(f)$ project onto. Then Kar$(F)(\psi)$ is given by the following composition:

$$\widehat{F(X)} \xrightarrow{\iota_X} F(X) \xrightarrow{F(\psi)} F(Y) \xrightarrow{\pi_Y} \widehat{F(Y)},$$

where ι_X and π_Y are, respectively, the natural inclusion and projection maps. In the non-strict case, one must insert appropriate associators and/or unitors to make this composition well-defined. For instance, we have

$\Phi\Bigg($  $\Bigg) =$  $\colon \mathrm{Sym}^a \otimes \mathrm{Sym}^b \to \mathrm{Sym}^{a+b}.$

Using the notation from Definition 6.4, this morphism is mapped by $\mathrm{Kar}(I^{\mathcal{OS}}_{m|n})$ to the composition

$$\Gamma^a V^+ \otimes \Gamma^b V^+ \xrightarrow{\iota_a^{\mathrm{Sym},+} \otimes \iota_b^{\mathrm{Sym},+}} (V^+)^{\otimes a} \otimes (V^+)^{\otimes b} \xrightarrow{\alpha_{a,b}} (V^+)^{\otimes (a+b)} \xrightarrow{I^{\mathcal{OS}}_{m|n}\left(\boxed{a+b}\right)} (V^+)^{\otimes (a+b)} \xrightarrow{\pi} \Gamma^{a+b} V^+ ,$$

where π is the indicated projection map. Noting that $I^{\mathcal{OS}}_{m|n}$ preserves braided structure and hence sends symmetrizers to symmetrizers, this coincides with the definition of $m^{\mathrm{Sym}}_{a,b}$ from Definition 6.4. The only difference is that the projection via π was left implicit when defining $m^{\mathrm{Sym}}_{a,b}$, i.e. we considered $I_{m|n}\left(\boxed{a+b}\right)$ as a map from $(V^+)^{\otimes (a+b)}$ to $\Gamma^{a+b} V^+$ for simplicity (since its image tautologically lies in $\Gamma^{a+b} V^+$). $\square$

Appendix A

Categorical structures

In this appendix, we recall the key definitions and properties of the various types of categories studied in this thesis.

Definition A.1. A $\Bbbk$-*linear category* is a category $\mathcal{C}$ such that, for all $X, Y \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\Bbbk$-vector space, and composition of morphisms is $\Bbbk$-bilinear. That is, if $a, b \in \Bbbk$, $g, h \colon X \to Y$, and $e, f \colon Y \to Z$, then

$$(ae + bf) \circ g = a(e \circ g) + b(f \circ g), \quad f \circ (ag + bh) = a(f \circ g) + b(f \circ h).$$

Definition A.2. A *monoidal category* is a category $\mathcal{C}$ equipped with the following structure:

- A bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product* for $\mathcal{C}$;
- A *unit object*, denoted $\mathbb{1}_{\mathcal{C}} \in \mathrm{ob}(\mathcal{C})$ (or just $\mathbb{1}$ if the category is clear);
- A natural isomorphism α called the *associator*, with components

$$\alpha_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z);$$

- A natural isomorphism λ called the *left unitor*, with components $\lambda_X \colon \mathbb{1} \otimes X \to X$;
- A natural isomorphism ρ called the *right unitor*, with components $\rho_X \colon X \otimes \mathbb{1} \to X$,

such that the following two diagrams, respectively called the *triangle* and *pentagon diagrams*, commute for any objects W, X, Y, Z in $\mathcal{C}$:

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\rho_X \otimes \mathrm{id}_Y"] \arrow[dr, "\alpha_{X,\mathbb{1},Y}"'] & & X \otimes Y \\
& X \otimes (\mathbb{1} \otimes Y) \arrow[ur, "\mathrm{id}_X \otimes \lambda_Y"'] &
\end{tikzcd},$$

$$\begin{tikzcd}[column sep=tiny]
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y}\otimes \mathrm{id}_Z"'] & & (W \otimes (X \otimes (Y \otimes Z))) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}.$$

A monoidal category is called *strict* if the associator and both unitors are identity maps. Thus, in a strict monoidal category, the following identities hold for all $X, Y, Z \in \mathrm{ob}(\mathcal{C})$:

- $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$;
- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.

Example A.3. The prototypical $\Bbbk$-linear monoidal category is $\mathrm{Vect}_\Bbbk$, the category of $\Bbbk$-vector spaces, equipped with the usual tensor product, associator, and unitor morphisms, with the unit object being $\Bbbk$ (considered as a one-dimensional $\Bbbk$-vector space).

Definition A.4. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Write $\mathbb{1}_\mathcal{C}$, $\otimes_\mathcal{C}$, $\alpha^\mathcal{C}, \rho^\mathcal{C}$ and $\lambda^\mathcal{C}$ for the monoidal unit, tensor product, associator, and unitors for $\mathcal{C}$, and similarly for $\mathcal{D}$. A *weak monoidal functor from $\mathcal{C}$ to $\mathcal{D}$* (sometimes called a *lax monoidal functor*) is a functor $F\colon \mathcal{C} \to \mathcal{D}$ together with the following *structure morphisms*:

- A $\mathcal{D}$-morphism $\varepsilon\colon \mathbb{1}_\mathcal{D} \to F(\mathbb{1}_\mathcal{C})$;
- A natural transformation with components $\phi_{X,Y}\colon F(X) \otimes_\mathcal{D} F(Y) \to F(X \otimes_\mathcal{C} Y)$,

such that the following associativity diagram commutes for all $X, Y, Z \in \mathrm{ob}(\mathcal{C})$:

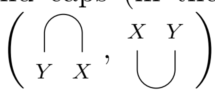
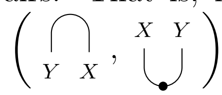
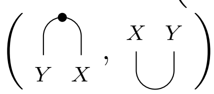
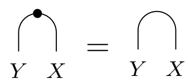
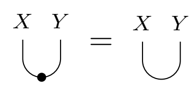
$$\begin{tikzcd}
(F(X) \otimes_\mathcal{D} F(Y)) \otimes_\mathcal{D} F(Z) \arrow[r, "\alpha^\mathcal{D}_{F(X),F(Y),F(Z)}"] \arrow[d, "\phi_{X,Y}\otimes_\mathcal{D} \mathrm{id}_{F(Z)}"'] & F(X) \otimes_\mathcal{D} (F(Y) \otimes_\mathcal{D} F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes_\mathcal{D} \phi_{Y,Z}"] \\
F(X \otimes_\mathcal{C} Y) \otimes_\mathcal{D} F(Z) \arrow[d, "\phi_{X\otimes_\mathcal{C} Y,Z}"'] & F(X) \otimes_\mathcal{D} F(Y \otimes_\mathcal{C} Z) \arrow[d, "\phi_{X,Y\otimes_\mathcal{C} Z}"] \\
F((X \otimes_\mathcal{C} Y) \otimes_\mathcal{C} Z) \arrow[r, "F(\alpha^\mathcal{C}_{X,Y,Z})"'] & F(X \otimes_\mathcal{C} (Y \otimes_\mathcal{C} Z))
\end{tikzcd}, \tag{A.1}$$

and the following unitality diagrams commute for all $X \in \mathrm{ob}(\mathcal{C})$:

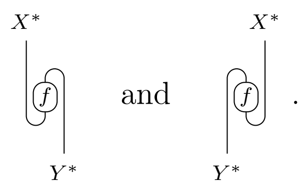
$$\begin{tikzcd}
\mathbb{1}_\mathcal{D} \otimes_\mathcal{D} F(X) \arrow[r, "\varepsilon\otimes_\mathcal{D} \mathrm{id}_{F(X)}"] \arrow[d, "\lambda^\mathcal{D}_{F(X)}"'] & F(\mathbb{1}_\mathcal{C}) \otimes_\mathcal{D} F(X) \arrow[d, "\phi_{\mathbb{1}_\mathcal{C},X}"] \\
F(X) & F(\mathbb{1}_\mathcal{C} \otimes_\mathcal{C} X) \arrow[l, "F(\lambda^\mathcal{C}_X)"]
\end{tikzcd}, \quad
\begin{tikzcd}
F(X) \otimes_\mathcal{D} \mathbb{1}_\mathcal{D} \arrow[r, "\mathrm{id}_{F(X)} \otimes_\mathcal{D} \varepsilon"] \arrow[d, "\rho^\mathcal{D}_{F(X)}"'] & F(X) \otimes_\mathcal{D} F(\mathbb{1}_\mathcal{C}) \arrow[d, "\phi_{X,\mathbb{1}_\mathcal{C}}"] \\
F(X) & F(X \otimes_\mathcal{C} \mathbb{1}_\mathcal{C}) \arrow[l, "F(\rho^\mathcal{C}_X)"]
\end{tikzcd}. \tag{A.2}$$

A *(strong) monoidal functor* is a weak monoidal functor such that ε is an isomorphism and $\phi_{-,-}$ is a natural isomorphism. A *strict monoidal functor* is a monoidal functor such that ε and $\phi_{-,-}$ are identity maps.

From this point forward, we represent morphisms using string diagrams, as outlined in §2.2.

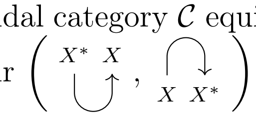
Cups and caps (in the sense of Definition 2.2.1) form unique pairs. That is, if the morphisms  satisfy the zigzag equations and so do  and , then we must have  and . This follows from a straightforward calculation using (ZIGZAG).

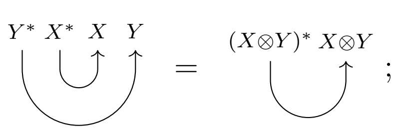
Definition A.5. Let X and Y be objects with two-sided duals. Let $f\colon X \to Y$ be a morphism. The *right mate* and *left mate* of f are defined to be, respectively,

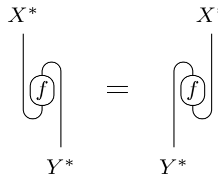


Note that these mates depend on the choice of cups and caps for the dual pairs (X, X^*) and (Y, Y^*), which we have left implicit. If the left and right mates of f are equal, we refer to that common morphism as just the *mate* of f, without specifying a side.

Example A.6. Each finite-dimensional vector space $V \in \mathrm{ob}(\mathrm{Vect}_{\Bbbk})$ has two-sided dual $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. The canonical cap for left duality is given by $f \otimes v \mapsto f(v)$, and the corresponding cup is given by $1 \mapsto \sum_{i=1}^{n} v_i \otimes v_i^*$, where $v_1, \dots, v_n$ is a basis of V and $v_1^*, \dots, v_n^*$ denotes the associated dual basis. The cup and cap for right duality are essentially the same, with the order of the tensor factors reversed. The mate of a linear map $f\colon V \to W$ between finite-dimensional vector spaces is the linear map $f^*\colon W^* \to V^*$ given by $f^*(\phi) = \phi \circ f$.

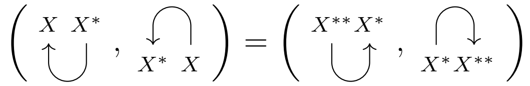
Definition A.7. A *strict pivotal category* is a strict monoidal category $\mathcal{C}$ equipped with, for each $X \in \mathrm{ob}(\mathcal{C})$, a choice of right dual X^* and cup-cap pair , such that the following conditions hold for all $X, Y \in \mathrm{ob}(\mathcal{C})$ and all morphisms $f\colon X \to Y$:

- $\mathbb{1}^* = \mathbb{1}$;
- $X^{**} = X$;
- $(X \otimes Y)^* = Y^* \otimes X^*$;
- 

- 

, i.e. the right and left mates of f are equal.

Note that the last identity involves left mates, which require left duality data to define. The condition $X^{**} = X$ implies that X^* (the chosen right dual for an object X) is also left dual to X via the cup-cap pair

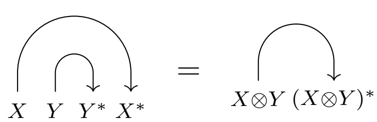


i.e. the chosen right duality cup-cap pair for X^*. These are the cups and caps used to define the left mates appearing above.

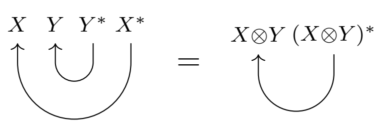
More generally, a *pivotal category* is a monoidal category $\mathcal{C}$ with chosen right duals for each object, equipped with a monoidal natural isomorphism $\pi\colon \mathrm{id} \to (-^*)^*$, where $-^*$ denotes the functor sending each object to its right dual, and each morphism to its right mate. A strict pivotal category is then a pivotal category for which π is the identity.

Lemma A.8. The following properties hold in any strict pivotal category $\mathcal{C}$, for all objects X, Y and morphisms $f\colon X \to Y$:

(a) The chosen cap and cup for $\mathbb{1}$ are both $\mathrm{id}_{\mathbb{1}}$;

(b) 

;

(c) 

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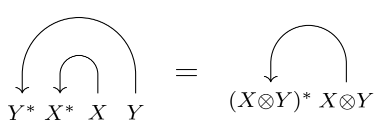
(d) 

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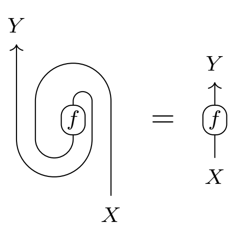
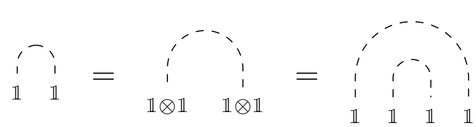
(e) 

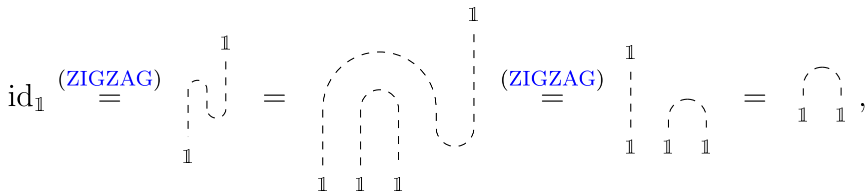
, i.e. the mate of the mate of f is equal to f.

Proof. To prove (a), we compute:



.

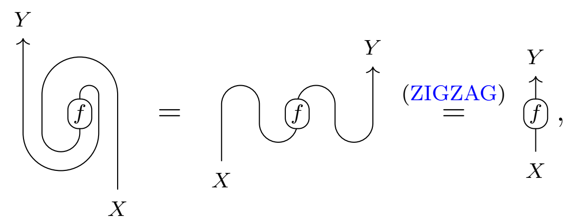
(Here, we employ the convention that $\mathbb{1}$ should be represented by a dashed string whenever it explicitly appears in string diagrams.) Attaching a cup to the bottom right of the first and last diagrams then yields:



which implies  $= \mathrm{id}_{\mathbb{1}}$ by the uniqueness of cup-cap pairs.

Identity (b) follows from the fourth identity in Definition A.7 using the uniqueness of cup-cap pairs. Identities (c) and (d) are, respectively, simply identity (b) from this lemma and the fourth identity from Definition A.7, with X^* in place of Y and Y^* in place of X (recall that $X^{**} = X$ and $Y^{**} = Y$).

Identity (e) follows from the fact that (by the definition of a strict pivotal category) left and right mates coincide in $\mathcal{C}$:



as desired. ☐

Definition A.9. A *braided monoidal category* is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism B, called the braiding, with components $B_{X,Y} \colon X \otimes Y \to Y \otimes X$, such that the following diagrams commute:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "B_{X,Y} \otimes \mathrm{id}_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "B_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "\mathrm{id}_Y \otimes B_{X,Z}"'] & Y \otimes (Z \otimes X)
\end{tikzcd}, \tag{A.3}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X \otimes B_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "B_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[r, "B_{X,Z} \otimes \mathrm{id}_Y"'] & (Z \otimes X) \otimes Y
\end{tikzcd}. \tag{A.4}$$

A *symmetric monoidal category* is a braided monoidal category whose braiding is self-inverse, i.e. satisfying $B_{Y,X} \circ B_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all objects X and Y.

Example A.10. The monoidal category $\mathrm{Vect}_{\Bbbk}$ has a natural symmetric braiding given on simple tensors by $B_{X,Y}(x \otimes y) = y \otimes x$.

The monoidal category of $\mathbb{Z}$-graded vector spaces can be equipped with a family of braidings, depending on a choice of parameter $a \in \Bbbk^{\times}$. Fixing such an a, the corresponding braiding acts on simple tensors by $B_{X,Y}(x \otimes y) = a^{\bar{x}\bar{y}} y \otimes x$, where $\bar{x}$ and $\bar{y}$ respectively denote the grades of homogeneous elements $x \in X$ and $y \in Y$. This braiding is symmetric if and only if $a^2 = 1$.

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