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The Standard Involution of SQ_n

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Abstract

On the rational group algebra $\mathbb{Q}S_n$, the map given by $g^* = g^{-1}$, for $g \in S_n$, and extended linearly, is an involution. Since $\mathbb{Q}S_n$ is a semisimple algebra, it has a unique decomposition into simple two-sided ideals. We determine, for $3 \leq n \leq 5$, the restriction of * to these simple two-sided ideals. This is accomplished using a positive definite symmetric bilinear form. Also, we construct a counterexample to part of a theorem found in *The Representation Theory of the Symmetric Group* by James and Kerber.

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1 Introduction

Let F be a field and G be a finite group. On the group algebra FG, the map given by $g^* = g^{-1}$ for $g \in G$ and extended linearly to FG, is an involution. If the characteristic of F does not divide the order of G, then, by Maschke's Theorem, FG is a semisimple algebra. In other words, it has a decomposition into simple factors, $FG = I_1 \times \ldots \times I_n$, and this decomposition is unique (up to order of the factors). Applying the involution * to FG we have that $FG = I_1^* \times \ldots \times I_n^*$ is also a decomposition into simple factors. So $I_j^* = I_k$ for some k. Since I_k is a simple finite dimensional algebra, it is isomorphic to a matrix algebra $M_{n_k}(D_k)$, with entries in a finite dimensional division algebra D_k.

We are interested in the rational group algebra $\mathbb{Q}S_n$. If $\mathbb{Q}S_n = I_1 \times \ldots \times I_\ell$ is the decomposition into simple factors, then each such factor is split, that is, the division algebras are isomorphic to $\mathbb{Q}$. Moreover, it turns out that $I_j^* = I_j$, for $1 \leq j \leq \ell$. Through the use of a positive definite symmetric bilinear form, we determine $^*|_{I_j}$ for $1 \leq j \leq \ell$.

Chapter 2 is an introduction to the topics and results which will be needed in the chapters that follow. In Chapter 4, the positive definite symmetric bilinear form is introduced, and the restrictions of the map * to the simple factors of $\mathbb{Q}S_n$ are determined. To help in some computations, functions for computing in $\mathbb{Q}S_n$ were written in *Mathematica*. To realize the form mentioned above, a theorem in James and Kerber [J&K] giving an explicit form of the simple factors of $\mathbb{Q}S_n$, and special properties of certain elements of $\mathbb{Q}S_n$, was referred to. However, discrepancies in computations based on a part of this theorem suggested there was an error with the theorem. In Chapter 3 we construct a counterexample to part of this theorem in [J&K].

2 Preliminaries

In this chapter, we establish notation, give definitions, and state basic results, often without proof, that will be needed in the following chapters. As there is a lack of consistent notation, terminology, and definitions, see [Boe], [J&K], [Rut], and [Sag]; we will mostly follow those set out in [J&K].

2.1 S_n and Young Tableaux

For $n \in \mathbb{N}$, the positive integers, let $\mathbf{n}$ denote the set $\{1, 2, \ldots, n\}$. A *permutation* of $\mathbf{n}$ is a bijective map $\sigma : \mathbf{n} \to \mathbf{n}$, and the set S_n of all permutations of $\mathbf{n}$ forms a group under composition, called the *symmetric group*. Since the group operation is composition, the product $\tau\sigma$ is the bijection obtained by first applying σ and then τ to $\mathbf{n}$. It is clear that the order of S_n is $n!$. Similarly, if E is any finite set, then S_E is the set of permutations of E.

A permutation $\sigma \in S_n$ is usually denoted as a $2 \times n$ matrix

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}.$$

So if $\sigma \in S_3$ is given by

$$\sigma(1) = 3, \quad \sigma(2) = 2, \quad \sigma(3) = 1,$$

it would be denoted by the matrix

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

There is no reason for the first row to contain the elements of $\mathbf{n}$ in their natural order. In general, a permutation can be denoted as

$$\begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix},$$

where $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ are the elements of $\mathbf{n}$ in some order. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

A permutation of the form

$$\begin{pmatrix} i_1 & i_2 & \dots & i_{k-1} & i_k & i_{k+1} & \dots & i_n \\ i_2 & i_3 & \dots & i_k & i_1 & i_{k+1} & \dots & i_n \end{pmatrix}$$

is called a k-cycle, and is denoted

$$(i_1\ i_2\ \dots\ i_k)(i_{k+1})\cdots(i_n), \text{ or just } (i_1\ i_2\ \dots\ i_k).$$

It should be noted that

$$(i_1\ i_2\ \dots\ i_k) = (i_2\ i_3\ \dots\ i_k\ i_1) = \dots = (i_k\ i_1\ i_2\ \dots\ i_{k-1}),$$

and that when using the latter notation for a k-cycle it is not possible to determine in which S_n the k-cycle belongs. Continuing with our example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3)(2) = (1\ 3) = (3\ 1).$$

The *identity permutation* is the permutation which consists only of 1-cycles, and will be denoted 1_{S_n}. A 2-cycle is called a *transposition*, and any transposition σ satisfies $\sigma^2 = 1_{S_n}$.

A *partition of* n is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$$

which satisfies

$$\text{(i) } \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k,$$

$$\text{(ii) } \sum_{i=1}^{k} \alpha_i = n.$$

That α is a partition of n will be denoted $\alpha \vdash n$.

To any partition

$$\alpha = (\alpha_1, \dots, \alpha_k) \vdash n$$

we associate a *Young diagram* $[\alpha]$ (where we suppress the parentheses), which consists of n nodes $\times$ placed in k left justified rows and where each row contains α_i nodes. For example, to the partition

$$(2, 2, 1) \vdash 5$$

corresponds the Young diagram

$$[2,2,1] = \begin{array}{cc} \times & \times \\ \times & \times \\ \times & \end{array}.$$

The lengths α'_i of the columns of $[\alpha]$ form another partition α' of n, called the *partition associated with* α. More precisely

$$\alpha'_i = \sum_{\substack{j \\ \alpha_j \geq i}} 1.$$

Correspondingly, $[\alpha']$ is called the *Young diagram associated with* $[\alpha]$. So to $[2,2,1]$ is associated the diagram

$$[(2,2,1)'] = \begin{array}{ccc} \times & \times & \times \\ \times & \times & \end{array}.$$

A *Young tableau with diagram* $[\alpha]$, or α*-tableau* or just *tableau*, t^α is obtained by replacing the n nodes $\times$ in $[\alpha]$ bijectively with the elements of $\mathbf{n}$. For any partition $\alpha \vdash n$ there are $n!$ distinct α-tableaux. For example, here are three of the 5! Young tableaux with diagram $[2,2,1]$:

$$\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}, \quad \begin{array}{cc} 2 & 5 \\ 1 & 3 \\ 4 & \end{array}, \quad \begin{array}{cc} 5 & 1 \\ 4 & 2 \\ 3 & \end{array}.$$

Let t^α, $\alpha \vdash n$, be a tableau. Then the rows $R_1, R_2, \ldots, R_k$ of t^α and the columns $C_1, C_2, \ldots, C_l$ of t^α form two partitions of $\mathbf{n}$. These partitions of $\mathbf{n}$ are used to construct the *horizontal group of* t^α

$$H(t^\alpha) = \mathrm{S}_{R_1} \times \mathrm{S}_{R_2} \times \ldots \times \mathrm{S}_{R_k}$$

and the *vertical group of* t^α

$$V(t^\alpha) = \mathrm{S}_{C_1} \times \mathrm{S}_{C_2} \times \ldots \times \mathrm{S}_{C_k}.$$

Both $H(t^\alpha)$ and $V(t^\alpha)$ are subgroups of S_n. For example, consider the $(2,2,1)$-tableau

$$\begin{matrix} & 2 & 5 \\ t = & 1 & 3\,. \\ & 4 & \end{matrix}$$

Then

$$H(t) = \{1_{S_n}, (2\ 5), (1\ 3), (2\ 5)(1\ 3)\}$$

and

$$V(t) = \{1_{S_n}, (1\ 2), (1\ 4), (2\ 4), (1\ 2\ 4), (1\ 4\ 2), (3\ 5), (1\ 2)(3\ 5), (1\ 4)(3\ 5), (2\ 4)(3\ 5), (1\ 2\ 4)(3\ 5), (1\ 4\ 2)(3\ 5)\}$$

Since $H(t^\alpha)$ and $V(t^\alpha)$ are groups, for any $\rho \in H(t^\alpha)$

$$\rho H(t^\alpha) = H(t^\alpha)\rho = H(t^\alpha)\,, \tag{2.1}$$

and for any $\pi \in V(t^\alpha)$

$$\pi V(t^\alpha) = V(t^\alpha)\pi = V(t^\alpha)\,. \tag{2.2}$$

Any permutation in $H(t^\alpha)$ interchanges only the elements of each row amongst themselves, and any permutation in $V(t^\alpha)$ interchanges only the elements of each column amongst themselves, so

$$H(t^\alpha) \cap V(t^\alpha) = \{1_{S_n}\}.$$

For t^α and $\sigma \in S_n$, define σt^α to be the tableau obtained by applying σ to the elements of t^α. For example, if

$$\begin{matrix} & 2 & 5 \\ t = & 1 & 3 \\ & 4 & \end{matrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 1 & 4 & 2 & 5 \end{pmatrix}$$

then

$$\begin{matrix} & 5 & 1 \\ \sigma t^\alpha = & 4 & 3\,. \\ & 2 & \end{matrix}$$

We introduce an equivalence relation on the collection of α-tableaux by stating that tableaux t_1 and t_2 are *row equivalent* if their rows contain the same elements, and denote the equivalence class of a tableau t^α by $\{t^\alpha\}$. The action of σ on t^α induces an action by σ on $\{t^\alpha\}$ given by

$$\sigma\{t^\alpha\} = \{\sigma t^\alpha\}.$$

The horizontal groups and vertical groups of t^α and σt^α satisfy

$$H(\sigma t^\alpha) = \sigma H(t^\alpha)\sigma^{-1} \tag{2.3}$$

and

$$V(\sigma t^\alpha) = \sigma V(t^\alpha)\sigma^{-1}. \tag{2.4}$$

To see this, let $\tau \in H(\sigma t^\alpha)$. Then

$$\begin{aligned}
\tau \in H(\sigma t^\alpha) &\Leftrightarrow \tau\{\sigma t^\alpha\} = \{\sigma t^\alpha\} \\
&\Leftrightarrow \tau\sigma\{t^\alpha\} = \sigma\{t^\alpha\} \\
&\Leftrightarrow \sigma^{-1}\tau\sigma\{t^\alpha\} = \{t^\alpha\} \\
&\Leftrightarrow \sigma^{-1}\tau\sigma \in H(t^\alpha) \\
&\Leftrightarrow \tau \in \sigma H(t^\alpha)\sigma^{-1}.
\end{aligned}$$

The result for the vertical groups follows from the fact that $V(t^\alpha) = H((t^\alpha)')$.

2.2 The Rational Group Algebra $\mathbb{Q}S_n$

Although we are concerned only with the rational group algebra $\mathbb{Q}S_n$, we begin by recalling some results from a more general setting. For more details, see [Boe] or [C&R].

Let F be a field and G a finite group. Then the *group algebra*, or *group ring*, FG is the set of all F-linear combinations of elements of G, i.e.,

$$FG = \Big\{ \sum_{g\in G} \lambda_g g \mid \forall\, g \in G,\ \lambda_g \in F \Big\}.$$

For $a \in FG$ and $g \in G$, we denote by $[a]_g$ the coefficient of g in a.

A *right ideal* in FG is a linear subspace I of FG such that

$$x \in I \;\Rightarrow\; \forall\, a \in FG\; xa \in I.$$

Left ideals and *two-sided ideals* are defined analogously. A right (or left) ideal which contains no proper right ideal is called *minimal*; a two-sided ideal which contains no proper two-sided ideal is called *simple*.

The irreducible representations of G are the same as those of FG, and each minimal right ideal affords an irreducible representation. Two minimal right ideals are said to be *equivalent* if they afford the same irreducible representation. If the characteristic of F does not divide the order of G, then, by Maschke's Theorem, we can write

$$FG = R_1 \oplus R_2 \oplus \ldots \oplus R_k\,, \tag{2.5}$$

where each R_i is a minimal right ideal, and this decomposition is unique up to order and equivalence.

Let $[R_i]$ denote the collection of minimal right ideals in the decomposition 2.5 which are equivalent to R_i. Then

$$I = \bigoplus_{R \in [R_i]} R$$

is a simple two-sided ideal. This gives a decomposition

$$FG = I_1 \oplus I_2 \oplus \ldots \oplus I_l \tag{2.6}$$

of FG as a direct sum of simple two-sided ideals, each of which is a direct sum of equivalent minimal right ideals. Unlike the decomposition into minimal right ideals, the decomposition into simple two-sided ideals is uniquely determined, save for the order of the summands. Moreover, we can bijectively associate to each simple two-sided ideal an irreducible representation.

If I_i and I_j are distinct simple two-sided ideals in FG, then they annihilate each other. In other words

$$\forall\, a_i \in I_i,\; \forall\, a_j \in I_j,\; a_i a_j = a_j a_i = 0.$$

An element $e \in FG$ is an *idempotent* if $e^2 = e$, and is *essentially idempotent* if there exists a $\kappa \in F \setminus \{0\}$ such that κe is an idempotent. If R is a right ideal, then there exists an idempotent e in R such that

$$R = eFG.$$

A nonzero idempotent e is *primitive* if the decomposition

$$e = e_1 + e_2$$

with

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 \neq 0, \quad e_2 \neq 0$$

cannot occur. If e is a primitive idempotent, then the right ideal $R = eFG$ is minimal, and conversely if $R = eFG$ is a minimal right ideal with e an idempotent, then e is primitive.

For a non-zero primitive idempotent e,

$$D = eFGe$$

is a division algebra with centre F, and

$$eFG \cong D^n.$$

Consider now the rational group algebra $\mathbb{Q}S_n$. The horizontal and vertical groups of a tableau t^α are used to construct three elements of $\mathbb{Q}S_n$, namely

$$\mathcal{V}(t^\alpha) = \sum_{\pi \in V(t^\alpha)} (-1)^\pi \pi \quad \text{and} \quad \mathcal{H}(t^\alpha) = \sum_{\rho \in H(t^\alpha)} \rho,$$

and their product

$$e^\alpha = \mathcal{V}(t^\alpha)\mathcal{H}(t^\alpha) = \sum_{\pi \in V(t^\alpha)} \sum_{\rho \in H(t^\alpha)} (-1)^\pi \pi\rho\,.$$

By [3.1.4, J&K], we have that e^α is essentially idempotent and generates a minimal right ideal in $\mathbb{Q}S_n$. So to each tableau t^α is associated an irreducible representation. It can be shown that tableaux with the same diagram yield equivalent representations, while tableaux with different diagrams yield inequivalent representations. Since the number of distinct diagrams is the same as the number of distinct conjugacy classes in S_n, we have a complete set of irreducible representations. Abusing notation, we denote by $[\alpha]$ the irreducible representation obtained from a tableau with diagram $[\alpha]$.

Denote by f^α the dimension of the irreducible representation $[\alpha]$. Then the simple two-sided ideal in the decomposition 2.6 of $\mathbb{Q}S_n$ associated with $[\alpha]$ is a direct sum of f^α minimal right ideals. Also by [3.1.4, J&K], if

$$\kappa^\alpha = \frac{n!}{f^\alpha},$$

then

$$\hat{e}^\alpha = \frac{1}{\kappa^\alpha} e^\alpha$$

is a primitive idempotent and

$$\hat{e}^\alpha \mathbb{Q}S_n = e^\alpha \mathbb{Q}S_n.$$

2.3 Properties of $\mathcal{H}(t^\alpha)$, $\mathcal{V}(t^\alpha)$, and e^α

By 2.1, if $\rho \in H(t^\alpha)$ then

$$\rho\mathcal{H}(t^\alpha) = \mathcal{H}(t^\alpha)\rho = \mathcal{H}(t^\alpha), \tag{2.7}$$

and, by 2.2, if $\pi \in V(t^\alpha)$ then

$$(-1)^\pi \pi \mathcal{V}(t^\alpha) = \mathcal{V}(t^\alpha) \cdot (-1)^\pi \pi = \mathcal{V}(t^\alpha). \tag{2.8}$$

Also, by 2.3 and 2.4, for any $\sigma \in S_n$ we have

$$\mathcal{H}(\sigma t^\alpha) = \sigma \mathcal{H}(t^\alpha) \sigma^{-1} \tag{2.9}$$

and

$$\mathcal{V}(\sigma t^\alpha) = \sigma \mathcal{V}(t^\alpha) \sigma^{-1}. \tag{2.10}$$

A $\sigma \in S_n$ appears at most once in the defining sum of e^α, for if

$$\sigma = \pi_1\rho_1 = \pi_2\rho_2, \quad \pi_i \in \mathcal{V}(t^\alpha), \quad \rho_i \in \mathcal{H}(t^\alpha),$$

then

$$\pi_2^{-1}\pi_1 = \rho_2\rho_1^{-1}$$

is an element of $H(t^\alpha) \cap V(t^\alpha)$. Thus

$$\pi_1 = \pi_2 \quad \text{and} \quad \rho_1 = \rho_2 .$$

Since $1_{S_n} \in H(t^\alpha) \cap V(t^\alpha)$, the coefficient of 1_{S_n} in both $e^\alpha = \mathcal{V}(t^\alpha)\mathcal{H}(t^\alpha)$ and $\mathcal{H}(t^\alpha)\mathcal{V}(t^\alpha)$ is 1, so that these two elements of the group algebra are not identically zero. Also, by 2.7 and 2.8,

$$\pi e^\alpha \rho = (-1)^\pi e^\alpha ,$$

for any $\pi \in V(t^\alpha)$ and any $\rho \in H(t^\alpha)$.

Let $R_1, R_2, \ldots, R_k$ denote the rows of the tableau t^α and $C_1, C_2, \ldots, C_l$ denote the columns of t^α. The elements $\mathcal{H}(t^\alpha)$ and $\mathcal{V}(t^\alpha)$ can then be factored as

$$\mathcal{H}(t^\alpha) = \mathcal{H}(R_1)\mathcal{H}(R_2)\cdots\mathcal{H}(R_k) \tag{2.11}$$

and

$$\mathcal{V}(t^\alpha) = \mathcal{V}(C_1)\mathcal{V}(C_2)\cdots\mathcal{V}(C_l) , \tag{2.12}$$

where

$$\mathcal{H}(R_i) = \sum_{\sigma\in S_{R_i}} \sigma \quad \text{and} \quad \mathcal{V}(C_i) = \sum_{\sigma\in S_{C_i}} (-1)^\sigma \sigma .$$

The terms in these factorizations commute as both the rows R_i and the columns C_j form partitions of $\mathbf{n}$.

We now give two factorizations of $\mathcal{H}(R)$ and $\mathcal{V}(C)$ which will be used in Chapter 3. For the first, suppose $a, b \in C$, where C is some column of t^α with k elements. Since

$$H = \{1_{S_n}, (a\ b)\}$$

is a subgroup of S_C, let

$$\{1_{S_n}, \sigma_2, \ldots, \sigma_{k!/2}\}$$

be a set of coset generators for H in S_C. Then the product

$$\left(1_{S_n} + (-1)^{\sigma_2}\sigma_2 + \ldots + (-1)^{\sigma_{k!/2}}\sigma_{k!/2}\right)\left(1_{S_n} - (a\ b)\right)$$

is a sum of $k!$ terms, and it is easy to see that these $k!$ terms are distinct. So

$$\mathcal{V}(C) = \left(1_{S_n} + (-1)^{\sigma_2}\sigma_2 + \ldots + (-1)^{\sigma_{k!/2}}\sigma_{k!/2}\right)\left(1_{S_n} - (a\ b)\right). \tag{2.13}$$

Similarly for

$$\mathcal{H}(R) = \left(1_{S_n} + \sigma_2 + \ldots + \sigma_{k!/2}\right)\left(1_{S_n} + (a\ b)\right), \tag{2.14}$$

where a, b are two elements in some row R of t^α.

For the second, let $a_1, a_2, \ldots, a_k$ be the other elements in the column C which contains n in the tableau t^α. Since $S_{\{a_1,\ldots,a_k\}}$ is a subgroup of S_C of order $k!$, a set of coset representatives for $S_{\{a_1,\ldots,a_k\}}$ in S_C will consist of $k+1$ permutations. In fact, the $k+1$ permutations

$$1_{S_n},\ (a_1\ n),\ (a_2\ n),\ \ldots,\ (a_k\ n)$$

are a set of representatives. Since

$$\mathcal{V}(\{a_1,\ldots,a_k\})\left(1_{S_n} - (a_1\ n) - \ldots - (a_k\ n)\right)$$

is a sum of $(k+1)!$ terms and each term occurs in $\mathcal{V}(C)$, it suffices to show that the terms are distinct. Suppose to the contrary that

$$\sigma(a_i\ n) = \tau(a_j\ n),$$

where σ, τ are terms in $\mathcal{V}(\{a_1,\ldots,a_k\})$. Then

$$\tau^{-1}\sigma = (a_j\ n)(a_i\ n),$$

and since the left hand side fixes n, we must have that $i = k$. It then follows immediately that $\sigma = \tau$. Hence

$$\mathcal{V}(C) = \mathcal{V}(\{a_1,\ldots,a_k\})\left(1_{S_n} - (a_1\ n) - \ldots - (a_k\ n)\right) \tag{2.15}$$

and 1_{S_n}, $(a_1\ n)$, $(a_2\ n)$, $\ldots$, $(a_k\ n)$ are a set of representatives. Similarly

$$\mathcal{H}(R) = \left(1_{S_n} + (a_1\ n) + \ldots + (a_k\ n)\right)\mathcal{H}(\{a_1,\ldots,a_k\}), \tag{2.16}$$

where R is the row of t^α that contains n and $a_1, \ldots, a_k$ are the other elements of R.

2.4 Standard Tableaux

We have seen that the simple two-sided ideal corresponding to $[\alpha]$ is a direct sum of f^α minimal right ideals. However, there are $n!$ α-tableaux which give $n!$ primitive idempotents e^α, and so there are (at least) $n!$ minimal right ideals to choose from. But is there a subset from the collection of α-tableaux which will generate a suitable collection of minimal right ideals?

A tableau t^α is called a *standard tableau* if the elements occur in each row and each column in lexicographical order, where rows are read left to right and columns are read top to bottom. For example, there are 5 standard $(3, 2)$-tableaux:

$$\begin{matrix}1&3&5\\2&4&\end{matrix}\,,\quad\begin{matrix}1&2&5\\3&4&\end{matrix}\,,\quad\begin{matrix}1&3&4\\2&5&\end{matrix}\,,\quad\begin{matrix}1&2&4\\3&5&\end{matrix}\,,\quad\begin{matrix}1&2&3\\4&5&\end{matrix}\,.$$

By [3.1.13, J&K] the number of standard α-tableaux is equal to f^α. In Chapter 3 it is shown that the sum of the minimal right ideals generated by the standard tableaux is direct, which will show that the collection of standard α-tableaux is a suitable subset of α-tableaux.

Let $\mathrm{st}[\alpha]$ denote the collection of standard α-tableaux. We wish to introduce an ordering on $\mathrm{st}[\alpha]$. If $t^\alpha \in \mathrm{st}[\alpha]$ then the element n occurs at the end of a row and at the end of a column. Let $t^{\alpha *}$ denote the tableau obtained by removing the element n from t^α; then $t^{\alpha *}$ is also a standard tableau and the element $n-1$ occurs at the end of a row and at the end of a column. This allows the elements $t_1^\alpha, t_2^\alpha, \ldots, t_{f^\alpha}^\alpha$ to be placed in the *last letter sequence* using a recursive definition [3.1.16, J&K]: the numbers i and k of $t_i^\alpha, t_k^\alpha \in \mathrm{st}[\alpha]$ satisfy $i < k$ if and only if either

(i) the element n occurs in t_i^α in a higher row than in t_k^α, or

(ii) $t_i^{\alpha *} = t_j^\beta$ and $t_k^{\alpha *} = t_h^\beta$, $\beta \vdash n-1$, where $j < h$.

When $i < k$ we write $t_i^\alpha < t_k^\alpha$. For example

$$\begin{matrix}1&3&5\\2&4&\end{matrix} < \begin{matrix}1&2&5\\3&4&\end{matrix} < \begin{matrix}1&3&4\\2&5&\end{matrix} < \begin{matrix}1&2&4\\3&5&\end{matrix} < \begin{matrix}1&2&3\\4&5&\end{matrix}\,.$$

For $1 \leq i, k \leq f^\alpha$ let $\pi_{ik} \in \mathrm{S}_n$ be given by

$$t_i^\alpha = \pi_{ik} t_k^\alpha.$$

These permutations satisfy

$$\pi_{ik}^{-1} = \pi_{ki} \quad \text{and} \quad \pi_{ij}\pi_{jk} = \pi_{ik}.$$

Let

$$e_i^\alpha = \mathcal{V}(t_i^\alpha)\mathcal{H}(t_i^\alpha),$$

and let

$$e_{ik}^\alpha = e_i^\alpha \pi_{ik} \quad \text{with} \quad e_{ii}^\alpha = e_i^\alpha.$$

We conclude this section with the following lemma:

Lemma 2.1. [3.2.12, J&K] For each $a \in \mathbb{Q}S_n$ we have

$$e_{ij}^\alpha a e_{hk}^\alpha = \lambda \kappa^\alpha e_{ik}^\alpha\,,$$

where

$$\lambda = \left[e_{hj}^\alpha a\right]_{1_{S_n}}.$$

3 On a Theorem of James and Kerber

The following theorem is found in [J&K]:

Theorem 3.1. [3.1.24, J&K] The minimal left ideals

$$L_i^\alpha = \mathbb{Q}S_n \hat{e}_i^\alpha, \quad \alpha \vdash n, \quad 1 \le i \le f^\alpha,$$

yield the following decomposition of $\mathbb{Q}S_n$ into simple two-sided ideal I^α:

$$\mathbb{Q}S_n = \bigoplus_{\alpha \vdash n} I^\alpha,$$

where

$$I^\alpha = \bigoplus_{i=1}^{f^\alpha} L_i^\alpha.$$

The generating elements $\hat{e}_i^\alpha$ of the minimal left ideals L_i^α are pairwise orthogonal primitive idempotents, so that in particular

$$\hat{e}_i^\alpha \hat{e}_k^\beta = \delta_{\alpha\beta}\delta_{ik}\hat{e}_i^\alpha. \qquad \square$$

The main goal of this chapter is to give a counterexample, for $n \ge 5$, to the statement that the generating elements $\hat{e}_i^\alpha$ are pairwise orthogonal. We begin, however, by showing that the stated decomposition of $\mathbb{Q}S_n$ is valid.

3.1 The decomposition of $\mathbb{Q}S_n$

This section gives the chain of results, complete with proofs, used in [J&K] to prove the decomposition of $\mathbb{Q}S_n$ as stated in Theorem 3.1.

Theorem 3.2. [1.5.13, J&K] Let t and t' be α-tableaux. Then $\mathcal{H}(t)\mathcal{V}(t') = 0$ if and only if there exist two elements which occur in the same row in t and in the same column in t'.

The following lemma is needed in the proof:

Lemma 3.3. [1.5.7, J&K] If every two elements which occur in the same column of t, occur in different rows of $t' = \sigma t$, then σ is of the form $\pi\rho$ for suitable $\pi \in V(t)$, $\rho \in H(t)$.

Proof of the Lemma: By assumption, all the elements of the first row of t' occur in different columns of t, so that an element of $V(t)$ applied to t moves them into the first row. Leaving these elements fixed, another element of $V(t)$ moves the elements found in the second row of t' into the second row of t, and so on. Thus there exists $\pi \in V(t)$ such that $\pi t \in \{t'\}$. Hence, for a suitable $\rho' \in H(\pi t)$, we have $t' = \rho' \pi t = \pi \rho t$, where $\rho = \pi^{-1} \rho' \pi$. ■

Proof of the Theorem: Suppose a, b occur in row R in t and in column C in t'. Then, by 2.11, 2.12, 2.13, and 2.14,

$$\begin{aligned}\mathcal{H}(t)\mathcal{V}(t') &= \mathcal{H}(R_1)\cdots\mathcal{H}(R_k)\mathcal{V}(C_1)\cdots\mathcal{V}(C_l)\\ &= \left[\prod_{R_i \neq R} \mathcal{H}(R_i)\right] [\mathcal{H}(R)]\,[\mathcal{V}(C)] \left[\prod_{C_i \neq C} \mathcal{V}(C_i)\right]\\ &= [\cdots]\,[1_{S_n} + (a\ b)]\,[1_{S_n} - (a\ b)]\,[\cdots]\\ &= [\cdots]\,[1_{S_n} - (a\ b)^2]\,[\cdots]\\ &= 0.\end{aligned}$$

Conversely, if two such points do not exist, then by Lemma 3.3 we can write $t = \pi' \rho' t'$, where $\pi' \in V(t')$, $\rho' \in H(t')$. Then

$$\begin{aligned}\mathcal{H}(t)\mathcal{V}(t') &= \mathcal{H}(\pi'\rho't')\mathcal{V}(t')\\ &= \pi'\rho'\mathcal{H}(t')\rho'^{-1}\pi'^{-1}\mathcal{V}(t')\\ &= \pi'\mathcal{H}(t')(-1)^{\pi'^{-1}}\mathcal{V}(t')\\ &= (-1)^{\pi'}\pi'\mathcal{H}(t')\mathcal{V}(t')\\ &\neq 0.\end{aligned}$$ ■

On the set of partitions of n,

$$P(n) = \{\alpha \mid \alpha \vdash n\},$$

we can define a partial order $\trianglelefteq$ by

$$\alpha \trianglelefteq \beta \ \Leftrightarrow\ \forall\, i,\ \sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j.$$

For example, when $n = 6$ we have

$$(2,2,1,1) \trianglelefteq (2,2,2) \quad \text{and} \quad (3,2,1) \trianglelefteq (4,1,1),$$

while

$$(3,1,1,1) \quad \text{and} \quad (2,2,2)$$

are incomparable. As shown in [J&K], this partial order can be characterized in terms of Young tableaux:

Lemma 3.4. [1.4.20, J&K] Let $\alpha, \beta \vdash n$ and let t^α be an α-tableau. Then $\alpha \trianglelefteq \beta$ if and only if there exists a β-tableau t^β such that any two points which occur in t^α in the same row occur in t^β in different columns. □

From the preceding lemma, and since the first part of the proof of Theorem 3.2 did not use the fact that the two tableaux had the same diagram, we have

Lemma 3.5. [1.5.17, J&K] Let $\alpha \ntrianglelefteq \beta$ and t^α, t^β be tableaux with diagrams $[\alpha], [\beta]$. Then there exists two points which occur in t^α in the same row while they occur in t^β in the same column. Also, $\mathcal{H}(t^\alpha)\mathcal{V}(t^\beta) = 0$, and hence $e^\alpha e^\beta = 0$. □

By the definition of the last letter sequence on st$[\alpha]$,

Lemma 3.6. [3.1.19, J&K] For $t_i^\alpha, t_k^\alpha \in$ st$[\alpha]$ we denote by $[\alpha^{(i,j)}]$ and $[\alpha^{(k,j)}]$ the diagrams of $t_i^{\alpha * \cdots *}$ and $t_k^{\alpha * \cdots *}$ (j asterisks), where $0 \leq j \leq n$. Then $t_i^\alpha < t_k^\alpha$ if and only if there exists an $h \in \mathbb{N}$ such that

$$\alpha^{(i,j)} = \alpha^{(k,j)}, \quad 0 \leq j \leq h-1,$$

while

$$\alpha^{(i,h)} \triangleleft \alpha^{(k,h)}.$$ □

Theorem 3.7. [3.1.20, J&K] If $t_i^\alpha < t_k^\alpha$ then $\hat{e}_k^\alpha \hat{e}_i^\alpha = 0$.

Proof: Denote by R the row which contains n and $a_1, \ldots, a_r$ the other elements of R; denote by C the column which contains n and $b_1, \ldots, b_s$ the other elements of C.

Then, by 2.11, 2.12, 2.15, and 2.16,

$$\begin{aligned}\mathcal{H}(t_k^\alpha) &= \prod_{i=1}^{p} \mathcal{H}(R_i) \\ &= \mathcal{H}(R) \left[\prod_{R_i \neq R} \mathcal{H}(R_i) \right] \\ &= \big(1_{S_n} + (a_1\ n) + \ldots + (a_k\ n)\big) \mathcal{H}(\{a_1, \ldots, a_k\}) \left[\prod_{R_i \neq R} \mathcal{H}(R_i) \right] \\ &= \big(1_{S_n} + (a_1\ n) + \ldots + (a_k\ n)\big) \mathcal{H}(t_k^{\alpha *}),\end{aligned}$$

and similarly

$$\mathcal{V}(t_i^\alpha) = \mathcal{V}(t_i^{\alpha *})\big(1_{S_n} - (a_1\ n) - \ldots - (a_k\ n)\big).$$

Since

$$\hat{e}_k^\alpha \hat{e}_i^\alpha = \frac{1}{(\kappa^\alpha)^2} \mathcal{V}(t_k^\alpha)\mathcal{H}(t_k^\alpha)\mathcal{V}(t_i^\alpha)\mathcal{H}(t_i^\alpha),$$

it suffices to show that

$$\mathcal{H}(t_k^\alpha)\mathcal{V}(t_i^\alpha) = 0.$$

We proceed by induction on n. If $[\alpha^{(i,1)}] = [\alpha^{(k,1)}]$, then $t_i < t_k$ forces $t_i^{\alpha *} < t_k^{\alpha *}$. By the induction hypothesis we have $\mathcal{H}(t_k^{\alpha *})\mathcal{V}(t_i^{\alpha *}) = 0$, and thus $\mathcal{H}(t_k^\alpha)\mathcal{V}(t_i^\alpha) = 0$. If $[\alpha^{(i,1)}] \neq [\alpha^{(k,1)}]$, then, by Lemma 3.6, we have $\alpha^{(i,1)} \lhd \alpha^{(k,1)}$, and so $\alpha^{(k,1)} \not\trianglelefteq \alpha^{(i,1)}$. Then, by Lemma 3.5, $\mathcal{H}(t_k^{\alpha *})\mathcal{V}(t_i^{\alpha *}) = 0$, and thus $\mathcal{H}(t_k^\alpha)\mathcal{V}(t_i^\alpha) = 0$. ■

Corollary 3.8. The sum

$$\mathbb{Q}S_n\hat{e}_1^\alpha + \mathbb{Q}S_n\hat{e}_2^\alpha + \ldots + \mathbb{Q}S_n\hat{e}_{f^\alpha}^\alpha$$

is direct.

Proof: Suppose

$$a_1\hat{e}_1^\alpha + a_2\hat{e}_2^\alpha + \ldots + a_{f^\alpha}\hat{e}_{f^\alpha}^\alpha = 0,$$

where $a_1, \ldots, a_{f^\alpha} \in \mathbb{Q}S_n$. Multiplying on the right with $\hat{e}_1^\alpha$ gives

$$\begin{aligned}0 &= (a_1\hat{e}_1^\alpha + a_2\hat{e}_2^\alpha + \ldots + a_{f^\alpha}\hat{e}_{f^\alpha}^\alpha)\hat{e}_1^\alpha \\ &= a_1(\hat{e}_1^\alpha)^2 + a_2\hat{e}_2^\alpha\hat{e}_1^\alpha + \ldots + a_{f^\alpha}\hat{e}_{f^\alpha}^\alpha\hat{e}_1^\alpha \\ &= a_1\hat{e}_1^\alpha.\end{aligned}$$

Multiplication with $\hat{e}_2^\alpha, \ldots, \hat{e}_{f^\alpha}^\alpha$ successively yields

$$a_2\hat{e}_2^\alpha = 0, \ldots, a_{f^\alpha}\hat{e}_{f^\alpha}^\alpha = 0.$$ ■

Hence the decomposition given in Theorem 3.1 has been proven. We now turn to the construction of a counterexample to the claimed orthogonality of the generating elements $\hat{e}_i^\alpha$.

3.2 A Counterexample to Theorem 3.1

If $\alpha \vdash n$ and $\beta \vdash n$ are distinct, then

$$\hat{e}_i^\alpha \hat{e}_k^\beta = \hat{e}_k^\beta \hat{e}_i^\alpha = 0\,.$$

For if $\alpha \neq \beta$, $\hat{e}_i^\alpha$ and $\hat{e}_k^\beta$ belong to distinct simple two-sided ideals in $\mathbb{Q}S_n$. So our counterexample must come from the product of two primitive idempotents belonging to the same simple two-sided ideal. Moreover, by Theorem 3.7, our counterexample

$$\hat{e}_i^\alpha \hat{e}_k^\alpha \neq 0$$

must satisfy

$$t_i^\alpha < t_k^\alpha\,.$$

Suppose t, t' are two α-tableaux that satisfy no two elements in the same row of t occur in the same column of t'. Let the tableau t'' have the element i in its x^{th} row and y^{th} column if i is found in the x^{th} row of t and in the y^{th} column of t'. So there exists an α-tableau t'' such that

$$H(t'') = H(t) \quad \text{and} \quad V(t'') = V(t').$$

For example, if

$$t = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{matrix} \quad \text{and} \quad t' = \begin{matrix} 2 & 4 & 3 \\ 6 & 1 & \\ 5 & & \end{matrix}\;,$$

then

$$t'' = \begin{matrix} 2 & 1 & 3 \\ 5 & 4 & \\ 6 & & \end{matrix}\;.$$

Lemma 3.9. [1.5.16, J&K] Suppose that t and t' are two tableaux with the same diagram, and $t' = \tau t$. Then $\mathcal{H}(t)\mathcal{V}(t') \neq 0$ if and only if $\tau = \pi' \rho$ for some $\pi' \in V(t')$, $\rho \in H(t)$.

Proof: First suppose $\mathcal{H}(t)\mathcal{V}(t') \neq 0$. Then there exists a tableau t'' such that $H(t'') = H(t)$ and $V(t'') = V(t')$. Let σ, σ' be such that $t'' = \sigma' t' = \sigma t$. Then $\sigma' \in V(t')$, $\sigma \in H(t)$, and $t' = \sigma'^{-1}\sigma t$. Suppose now that $\tau = \pi' \rho$ for some $\pi' \in V(t')$, $\rho \in H(t)$. Two elements which occur in the same row of t occur in the same row of ρt, and so occur in the same row of ${\pi'}^{-1} t'$. Thus they occur in different columns of ${\pi'}^{-1} t'$, and since ${\pi'}^{-1} \in V(t')$, they must also occur in different columns of t'. The result now follows from Theorem 3.2. ■

For α-tableaux s and t, let $\pi_{s,t}$ be given by

$$s = \pi_{s,t} t$$

and define

$$e_{s,t} = \mathcal{V}(s)\mathcal{H}(s)\pi_{s,t} .$$

This extends the definitions of π_{ik} and e^{α}_{ik} (Section 2.4), which were given for the standard α-tableaux, to the collection of α-tableaux. The $\pi_{s,t}$ satisfy

$$\pi_{s,t}^{-1} = \pi_{t,s} \quad \text{and} \quad \pi_{s,t}\pi_{t,u} = \pi_{s,u}.$$

To any permutation $\tau \in S_n$ we can associate an α-tableau t' such that $\tau = \pi_{s,t'}$; t' can be obtained by applying τ^{-1} to s. By 2.10, $\mathcal{V}(t') = \tau^{-1}\mathcal{V}(s)\tau$, and we have the following theorem:

Theorem 3.10. Let τ $(= \pi_{s,t'})$ be a permutation. If $\mathcal{H}(t)\mathcal{V}(t') = 0$ then τ has coefficient 0 in $e_{s,t}$. If $\mathcal{H}(t)\mathcal{V}(t')$ $\big(= \mathcal{H}(t'')\mathcal{V}(t'')\big) \neq 0$, then the coefficient of τ in $e_{s,t}$ is ± 1 according as $\pi_{t',t''}$ is even or odd.

Proof: First suppose τ has nonzero coefficient in $e_{s,t}$. Since, by 2.10,

$$e_{s,t} = \mathcal{V}(s)\mathcal{H}(s)\pi_{s,t} = \mathcal{V}(s)\pi_{s,t}\mathcal{H}(t) ,$$

τ is of the form $\pi_s\pi_{s,t}\rho_t$, where $\pi_s \in V(s)$ and $\rho_t \in H(t)$. Then, by 2.8 and 2.10,

$$\begin{aligned}\mathcal{H}(t)\mathcal{V}(t') &= \mathcal{H}(t)\tau^{-1}\mathcal{V}(s)\tau \\ &= \mathcal{H}(t)\rho_t^{-1}\pi_{t,s}\pi_s^{-1}\mathcal{V}(s)\pi_s\pi_{s,t}\rho_t \\ &= \mathcal{H}(t)\pi_{t,s}\mathcal{V}(s)\pi_{s,t}\rho_t \\ &= \mathcal{H}(t)\mathcal{V}(t)\rho_t \\ &\neq 0.\end{aligned}$$

Hence $\mathcal{H}(t)\mathcal{V}(t') = 0$ implies that τ has coefficient 0 in $e_{s,t}$.

Suppose now that $\mathcal{H}(t)\mathcal{V}(t') \neq 0$. By Lemma 3.9

$$\pi_{t',t} = \pi_{t'}\rho_t\,,$$

where $\pi_{t'} \in V(t')$ and $\rho_t \in H(t)$. Then, by 2.4, we have

$$\begin{aligned}\pi_{s,t} &= \pi_{s,t'}\pi_{t',t} \\ &= \pi_{s,t'}\pi_{t'}\rho_t \\ &= \pi_s\pi_{s,t'}\rho_t \\ &= \pi_s\tau\rho_t\,,\end{aligned}$$

and so

$$\tau = \pi_s^{-1}\pi_{s,t}\rho_t^{-1}.$$

Thus τ occurs in $e_{s,t}$ with coefficient $(-1)^{\pi_s} = (-1)^{\pi_{t'}}$. ■

Consider first the partition $\alpha = (3,2) \vdash 5$. By applying Lemma 2.1 to $e_1^\alpha e_5^\alpha = e_1^\alpha 1_{S_n} e_5^\alpha$, we have that $e_1^\alpha e_5^\alpha = 24\lambda e_{15}^\alpha$, where λ is the coefficient of 1_{S_n} in e_{51}^α. Theorem 3.10, with $\tau = 1_{S_n}$ and $e_{s,t}^\alpha = e_{51}^\alpha$, can be used to evaluate λ as follows. Since $\tau = 1_{S_n}$ we have that

$$t' \ (= \tau^{-1}t_5) = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array}.$$

With the rows of

$$t \ (= t_1) = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

and the columns of t' we build the tableau

$$t'' = \begin{matrix} 1 & 5 & 3 \\ 4 & 2 & \end{matrix}.$$

Thus $\pi_{t',t''} = (2\ 5)$, and so, by Theorem 3.10, 1_{S_n} has coefficient -1 in e^{α}_{51}. Hence $e^{\alpha}_1 e^{\alpha}_5 \neq 0$.

By adjoining $6, 7, \ldots, n$ to the first row of t^{α}_1 and t^{α}_5, we obtain the standard tableaux with diagram $[\beta] = [n-2, 2]$

$$t_{i_o} = \begin{matrix} 1 & 3 & 5 & 6 & 7 & \ldots & n \\ 2 & 4 & & & & & \end{matrix}$$

and

$$t_{k_o} = \begin{matrix} 1 & 2 & 3 & 6 & 7 & \ldots & n \\ 4 & 5 & & & & & \end{matrix},$$

which satisfy $t_{i_o} < t_{k_o}$. Applying Lemma 2.1 and Theorem 3.10 to $e^{\beta}_{i_o}$ and $e^{\beta}_{k_o}$ as in the previous paragraph, we find that $\pi_{t',t''} = (2\ 5)$. Thus, for all $n > 5$, $e^{\beta}_{i_o} e^{\beta}_{k_o} \neq 0$. Hence we have a counterexample, for $n \geq 5$, to the orthogonality of the generating elements $\hat{e}^{\alpha}_i$ as claimed in Theorem 3.1.

4 Involutions on $\mathbb{Q}S_n$

Let R be a ring. An *involution* on R is a map

$$^* : R \to R$$

which satisfies

$$(a+b)^* = a^* + b^*\,, \quad (ab)^* = b^*a^*\,, \quad (a^*)^* = a$$

for all $a, b \in R$. An element $a \in R$ is said to be *symmetric* if $a^* = a$.

On the group algebra FG, the map given by $g^* = g^{-1}$ for $g \in G$ and extended linearly to FG is an involution. In particular,

$$\Big(\sum_{\sigma \in S_n} \lambda_\sigma \sigma\Big)^* = \sum_{\sigma \in S_n} \lambda_\sigma \sigma^{-1}$$

for $\mathbb{Q}S_n$.

Since

$$(\mathbb{Q}S_n)^* = \mathbb{Q}S_n$$

and

$$\mathbb{Q}S_n = \bigoplus_{\alpha \vdash n} I^\alpha\,,$$

where the I^α are simple two-sided ideals, we have that

$$\mathbb{Q}S_n = \bigoplus_{\alpha \vdash n} (I^\alpha)^*$$

is another decompositition into simple two-sided ideals. Since this decompostition is unique, for each $\alpha \vdash n$ there is a $\beta \vdash n$ such that $(I^\alpha)^* = I^\beta$. Actually, $(I^\alpha)^* = I^\alpha$ for all $\alpha \vdash n$ by Lemma 4.1 below.

We wish to determine the restriction $^*|_{I^\alpha}$, for $\alpha \vdash n$. This will be accomplished through the use of a positive definite symmetric bilinear form b defined on the minimal right ideal $e_i^\alpha \mathbb{Q}S_n$. Since for $1 \le i \le f^\alpha$ these minimal left ideals are equivalent, we are free to choose which minimal left ideal we use.

Given a primitive and essentially idempotent element e_i^α, we first need to find a primitive symmetric and essentially idempotent element E_i^α such that $E_i^\alpha \mathbb{Q}S_n = e_i^\alpha \mathbb{Q}S_n$. A candidate is $e_i^\alpha (e_i^\alpha)^*$, since by construction this element is symmetric.

Lemma 4.1. Let $0_{\mathbb{Q}S_n} \neq a \in \mathbb{Q}S_n$. Then $[aa^*]_{1_{S_n}} > 0$.

Proof: Let

$$a = \sum_{\sigma \in S_n} a_\sigma \sigma.$$

Then

$$a^* = \sum_{\sigma \in S_n} a_\sigma \sigma^{-1},$$

and hence

$$[aa^*]_{1_{S_n}} = \sum_{\sigma \in S_n} a_\sigma^2 .$$

∎

Putting

$$\xi^\alpha = [e_i^\alpha (e_i^\alpha)^*]_{1_{S_n}},$$

we have, by Lemma 2.1,

$$\begin{aligned}\left(\frac{1}{\xi^\alpha \kappa^\alpha} e_i^\alpha (e_i^\alpha)^*\right)^2 &= \frac{1}{(\xi^\alpha)^2 (\kappa^\alpha)^2}\left(e_i^\alpha (e_i^\alpha)^* e_i^\alpha\right)(e_i^\alpha)^* \\ &= \frac{\xi^\alpha \kappa^\alpha}{(\xi^\alpha)^2 (\kappa^\alpha)^2} e_i^\alpha (e_i^\alpha)^* \\ &= \frac{1}{\xi^\alpha \kappa^\alpha} e_i^\alpha (e_i^\alpha)^* .\end{aligned}$$

Thus $e_i(e_i^\alpha)^*$ is essentially idempotent.

Clearly $e_i^\alpha(e_i^\alpha)^* \in e_i^\alpha \mathbb{Q}S_n$. Also, by 2.1, $e_i^\alpha (e_i^\alpha)^* e_i^\alpha = \xi^\alpha \kappa^\alpha e_i^\alpha \neq 0$ gives

$$\{0\} \subsetneqq \left(e_i^\alpha (e_i^\alpha)^*\right)\mathbb{Q}S_n \subseteq e_i^\alpha \mathbb{Q}S_n.$$

By minimality of $e_i^\alpha \mathbb{Q}S_n$ we have that $\left(e_i^\alpha(e_i^\alpha)^*\right)\mathbb{Q}S_n = e_i^\alpha \mathbb{Q}S_n$. Thus $e_i^\alpha(e_i^\alpha)^*$ is primitive.

Let $\overset{*}{e}{}_i^\alpha$ denote $e_i^\alpha(e_i^\alpha)^*$. Hence $\overset{*}{e}{}_i^\alpha$ is a primitive symmetric and essentially idempotent element satisfying $\overset{*}{e}{}_i^\alpha \mathbb{Q}S_n = e_i^\alpha \mathbb{Q}S_n$.

Using this primitive symmetric and essentially idempotent element, we now define the form b:

$$b : \overset{*}{e}{}_i^\alpha \mathbb{Q}S_n \times \overset{*}{e}{}_i^\alpha \mathbb{Q}S_n \to \mathbb{Q}$$

given by

$$(\overset{*}{e}{}_i^\alpha x, \overset{*}{e}{}_i^\alpha y) \mapsto \left[(\overset{*}{e}{}_i^\alpha x)(\overset{*}{e}{}_i^\alpha y)^*\right]_{1_{S_n}}.$$

It is clear that for $x, y \in \mathbb{Q}\mathrm{S}_n$, $[xy]_{1_{\mathrm{S}_n}} = [yx]_{1_{\mathrm{S}_n}}$.

Lemma 4.2. For all $a \in \mathbb{Q}\mathrm{S}_n$, $\overset{*}{e}{}^{\alpha}_{i} a \overset{*}{e}{}^{\alpha}_{i} = \lambda \kappa^\alpha \overset{*}{e}{}^{\alpha}_{i}$, where $\lambda = [\overset{*}{e}{}^{\alpha}_{i} a]_{1_{\mathrm{S}_n}}$.

Proof: Since $\mathbb{Q} \cong \mathbb{Q}\overset{*}{e}{}^{\alpha}_{i} \subseteq \overset{*}{e}{}^{\alpha}_{i}\mathbb{Q}\mathrm{S}_n\overset{*}{e}{}^{\alpha}_{i} \cong \mathbb{Q}$, $\overset{*}{e}{}^{\alpha}_{i} a \overset{*}{e}{}^{\alpha}_{i} = \mu \overset{*}{e}{}^{\alpha}_{i}$ for some $\mu \in \mathbb{Q}$. Comparing the coefficients of 1_{S_n} on both sides gives $\mu\xi^\alpha = \left[\overset{*}{e}{}^{\alpha}_{i} a \overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} = \left[\overset{*}{e}{}^{\alpha}_{i}\overset{*}{e}{}^{\alpha}_{i} a\right]_{1_{\mathrm{S}_n}} = \left[\xi^\alpha\kappa^\alpha\overset{*}{e}{}^{\alpha}_{i} a\right]_{1_{\mathrm{S}_n}} = \xi^\alpha\kappa^\alpha\left[\overset{*}{e}{}^{\alpha}_{i} a\right]_{1_{\mathrm{S}_n}}$. From $\xi^\alpha \neq 0$ we have $\mu = \kappa^\alpha\left[\overset{*}{e}{}^{\alpha}_{i} a\right]_{1_{\mathrm{S}_n}}$. ■

Let h be a non-degenerate bilinear form on a vector space V, and let $\phi \in \mathrm{End}(V)$. Then the adjoint $\bar{\phi}$ of ϕ, which satisfies

$$h\big(u\phi, v\big) = h\big(u, v\bar{\phi}\big), \quad \forall\, u, v \in V,$$

induces an involution on $\mathrm{End}(V)$.

Theorem 4.3. The mapping b is a positive definite symmetric bilinear form. Moreover, the involution induced by b on $\overset{*}{e}{}^{\alpha}_{i}\mathbb{Q}\mathrm{S}_n$ is the involution $*$.

Proof: Let $\overset{*}{e}{}^{\alpha}_{i}x, \overset{*}{e}{}^{\alpha}_{i}x', \overset{*}{e}{}^{\alpha}_{i}y, \in \overset{*}{e}{}^{\alpha}_{i}\mathbb{Q}\mathrm{S}_n$ and let $\lambda \in \mathbb{Q}$. By Lemma 4.2, the map b is symmetric, as

$$\begin{aligned}
b(\overset{*}{e}{}^{\alpha}_{i}x, \overset{*}{e}{}^{\alpha}_{i}y) &= \left[(\overset{*}{e}{}^{\alpha}_{i}x)(\overset{*}{e}{}^{\alpha}_{i}y)^*\right]_{1_{\mathrm{S}_n}} \\
&= \left[\overset{*}{e}{}^{\alpha}_{i}xy^*\overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} \\
&= \left[\left[\overset{*}{e}{}^{\alpha}_{i}xy^*\right]_{1_{\mathrm{S}_n}}\kappa^\alpha\overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} \\
&= \left[\left[\big((yx^*)\overset{*}{e}{}^{\alpha}_{i}\big)^*\right]_{1_{\mathrm{S}_n}}\kappa^\alpha\overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} \\
&= \left[\left[\overset{*}{e}{}^{\alpha}_{i}yx^*\right]_{1_{\mathrm{S}_n}}\kappa^\alpha\overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} \\
&= \left[\overset{*}{e}{}^{\alpha}_{i}yx^*\overset{*}{e}{}^{\alpha}_{i}\right]_{1_{\mathrm{S}_n}} \\
&= \left[(\overset{*}{e}{}^{\alpha}_{i}y)(\overset{*}{e}{}^{\alpha}_{i}x)^*\right]_{1_{\mathrm{S}_n}} \\
&= b(\overset{*}{e}{}^{\alpha}_{i}y, \overset{*}{e}{}^{\alpha}_{i}x).
\end{aligned}$$

To show that b is bilinear, it now suffices to show linearity in only one coordinate. Then

$$\begin{aligned} b(\overset{*}{e_i^\alpha}x + \overset{*}{e_i^\alpha}x', \overset{*}{e_i^\alpha}y) &= \left[(\overset{*}{e_i^\alpha}x + \overset{*}{e_i^\alpha}x')(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= \left[(\overset{*}{e_i^\alpha}x + \overset{*}{e_i^\alpha}x')(y^*\overset{*}{e_i^\alpha})\right]_{1_{S_n}} \\ &= \left[\overset{*}{e_i^\alpha}xy^*\overset{*}{e_i^\alpha} + \overset{*}{e_i^\alpha}x'y^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} \\ &= \left[\overset{*}{e_i^\alpha}xy^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} + \left[\overset{*}{e_i^\alpha}x'y^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} \\ &= \left[(\overset{*}{e_i^\alpha}x)(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} + \left[(\overset{*}{e_i^\alpha}x')(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= b(\overset{*}{e_i^\alpha}x, \overset{*}{e_i^\alpha}y) + b(\overset{*}{e_i^\alpha}x', \overset{*}{e_i^\alpha}y) \end{aligned}$$

and

$$\begin{aligned} b(\lambda\overset{*}{e_i^\alpha}x, \overset{*}{e_i^\alpha}y) &= \left[(\lambda\overset{*}{e_i^\alpha}x)(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= \left[\lambda(\overset{*}{e_i^\alpha}x)(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= \lambda\left[(\overset{*}{e_i^\alpha}x)(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= \lambda b(\overset{*}{e_i^\alpha}x, \overset{*}{e_i^\alpha}y) . \end{aligned}$$

So b is a bilinear form. That b is positive definite follows immediately from Lemma 4.1. Lastly, let $\sigma \in S_n$. Then

$$\begin{aligned} b(\overset{*}{e_i^\alpha}x\sigma, \overset{*}{e_i^\alpha}y) &= \left[(\overset{*}{e_i^\alpha}x\sigma)(\overset{*}{e_i^\alpha}y)^*\right]_{1_{S_n}} \\ &= \left[\overset{*}{e_i^\alpha}x\sigma y^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} \\ &= \left[\overset{*}{e_i^\alpha}x(y\sigma^*)^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} \\ &= \left[\overset{*}{e_i^\alpha}x(y\sigma^{-1})^*\overset{*}{e_i^\alpha}\right]_{1_{S_n}} \\ &= \left[(\overset{*}{e_i^\alpha}x)(\overset{*}{e_i^\alpha}y\sigma^{-1})^*\right]_{1_{S_n}} \\ &= b(\overset{*}{e_i^\alpha}x, \overset{*}{e_i^\alpha}y\sigma^{-1}). \end{aligned}$$

■

The basis e_{ik}^α, $1 \le k \le f^\alpha$, of $\overset{*}{e_i^\alpha}\mathbb{Q}S_n$ is used to construct the matrix

$$B = \left(b(\overset{*}{e_i^\alpha}e_{ik}^\alpha, \overset{*}{e_i^\alpha}e_{i\ell}^\alpha)\right)_{1\le k,\ell\le f^\alpha} .$$

Since B is symmetric, it is *congruent* to some diagonal matrix D, i.e., there exists a non-singular matrix P such that $D = P^TAP$, and B, D represent the form b over different bases. Also, D induces the same involution as B. As the determinant $\det(P^TAP) = \det(A)\det^2(P)$, the *discriminant* of a non-degenerate symmetric bilinear form b on a vector space V over the field K is defined to be the class in $K^\times/K^{\times 2}$ of the determinant of the matrix of b with respect to some basis of V, and this is independent of the choice of basis. The discriminant is an invariant of the quadratic form, or of the associated bilinear form.

Since the involution is given by

$$X \mapsto DX^TD^{-1},$$

we can also multiply D by any nonzero scalar $\lambda \in \mathbb{Q}$ without changing the involution. However, the discriminant remains unchanged if f^α is even, but is multiplied by λ if f^α is odd.

A function called `InvolutionMatrices` (see Appendix B) was written in *Mathematica* which calculates the matrix B and then finds a square-free diagonal matrix D which is congruent to B. Using pen and paper, the diagonal matrix D is further simplified, using both congruent transformations (isometries, denoted $\sim$) and multiplications by a nonzero scalar a (homotheties, denoted $\overset{a}{\simeq}$).

Consider the $(2,1,1,1)$ partition of 5. The matrices outputted by `InvolutionMatrices` are

$$B = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ -1 & 1 & -1 & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

Consider first the submatrix

$$S = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$$

of D. Suppose this S corresponds to some basis $\{u, v\}$, and let q_S denote the quadratic form determined by S. Then

$$(u+v)^\perp = 2u - v\,, \quad q_S(u+v) = 9\,, \quad q_S(2u-v) = 18\,.$$

Thus the matrix of q_S with respect to the basis $\{\frac{1}{3}(u+v), \frac{1}{3}(2u-v)\}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and so

$$D \sim \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&10 \end{pmatrix} \overset{2}{\simeq} \begin{pmatrix} 2&0&0&0 \\ 0&2&0&0 \\ 0&0&4&0 \\ 0&0&0&20 \end{pmatrix} \sim \begin{pmatrix} 2&0&0&0 \\ 0&2&0&0 \\ 0&0&1&0 \\ 0&0&0&5 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&5 \end{pmatrix}.$$

To show the last isometry, consider the submatrix

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Again, suppose this S corresponds to some basis $\{u, v\}$, and let q_S denote the quadratic form determined by S. Then

$$(u+v)^\perp = u - v, \quad q_S(u+v) = 4, \quad q_S(u-v) = 4.$$

This shows that S is congruent to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $f^{(2,1,1,1)} = 4$ is even, multiplication by a scalar does not change the discriminant. Since the discriminant of B is 5, we can do no better than the matrix D_3.

Consider now the partition associated with $(2,1,1,1)$, namely $(4,1)$. It has as initial diagonal matrix

$$D = \begin{pmatrix} 2&0&0&0 \\ 0&30&0&0 \\ 0&0&15&0 \\ 0&0&0&5 \end{pmatrix} \overset{5}{\simeq} \begin{pmatrix} 10&0&0&0 \\ 0&6&0&0 \\ 0&0&3&0 \\ 0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0 \\ 0&3&0&0 \\ 0&0&6&0 \\ 0&0&0&10 \end{pmatrix},$$

and, by the work done for the $(2,1,1,1)$ partition, D induces the same involution as

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&5 \end{pmatrix}.$$

For each n the two trivial partitions (n) and $(1,1,\ldots,1)$ correspond to one-dimensional two-sided ideals, and so $*|_{I^{(n)}}$ and $*|_{I^{(1,1,\ldots,1)}}$ are trivial. What follows is a list of the diagonal matrices which induce the involutions $*|_{I^\alpha}$, obtained for the nontrivial partitions $\alpha \vdash n$ where $3 \le n \le 5$. See Appendix A for additional calculations and comments.

Partition	Diagonal Matrix	Discriminant
$(2,1)$	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	2
$(2,1,1)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1
$(2,2)$	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	2
$(3,1)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1
$(2,1,1,1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$	5
$(2,2,1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$	1
$(3,1,1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$	5
$(3,2)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$	1
$(4,1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$	5

We see that if β is the partition associated with α, then the involutions $*|_{I^\alpha}$ and $*|_{I^\beta}$ can be induced by the same diagonal matrix.

A Determining the Matrices

Much of the simplifications for the diagonal matrices D outputted by the function `InvolutionMatrices` (see Appendix B) were achieved by considering 2×2 submatrices of the diagonal of D. In Section 3 it was shown that

$$\left(\begin{smallmatrix} 3 & 0 \\ 0 & 6 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \quad \text{and} \quad \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) .$$

For

$$S = \left(\begin{smallmatrix} 6 & 0 \\ 0 & 10 \end{smallmatrix}\right)$$

corresponding to some basis $\{u, v\}$ and q_S denoting the quadratic form determined by S,

$$(u+v)^{\perp} = 5u - 3v\,, \quad q_S(u+v) = 16\,, \quad q_S(5u-3v) = 240\,,$$

and so

$$\left(\begin{smallmatrix} 6 & 0 \\ 0 & 10 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 15 \end{smallmatrix}\right) .$$

For

$$S = \left(\begin{smallmatrix} 15 & 0 \\ 0 & 15 \end{smallmatrix}\right)$$

corresponding to some basis $\{u, v\}$ and q_S denoting the quadratic form determined by S,

$$(2u+v)^{\perp} = u - 2v\,, \quad q_S(2u+v) = 75\,, \quad q_S(u-2v) = 75\,,$$

and so

$$\left(\begin{smallmatrix} 15 & 0 \\ 0 & 15 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right) .$$

For

$$S = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right)$$

corresponding to some basis $\{u, v\}$ and q_S denoting the quadratic form determined by S,

$$(2u+v)^{\perp} = u - 2v\,, \quad q_S(2u+v) = 25\,, \quad q_S(u-2v) = 25\,,$$

and so

$$\left(\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) .$$

For

$$S = \left(\begin{smallmatrix} 35 & 0 \\ 0 & 14 \end{smallmatrix}\right)$$

corresponding to some basis $\{u, v\}$ and q_S denoting the quadratic form determined by S,

$$(u+v)^{\perp} = 2u - 5v\,, \quad q_S(u+v) = 49\,, \quad q_S(2u-5v) = 490\,,$$

and so

$$\left(\begin{smallmatrix} 35 & 0 \\ 0 & 14 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 10 \end{smallmatrix}\right).$$

Lastly, consider

$$S = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$$

corresponding to some basis $\{u, v, w\}$ and q_S denoting the quadratic form determined by S. Then

$$u - v \in (u+v+w)^{\perp}\,, \quad u + v - 2w \in \{u+v+w, u-v\}^{\perp}$$
$$q_S(u+v+w) = 9\,, \quad q_S(u-v) = 6\,, \quad q_S(u+v-2w) = 18\,,$$

and so

$$\left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{smallmatrix}\right).$$

In what follows, the matrix D_α, $\alpha \vdash n$, is the diagonal matrix associated to α outputted by `InvolutionMatrices`, $A \sim B$ denotes that the matrix A is congruent to the matrix B, $A \overset{a}{\simeq} B$ denotes that the matrix B is obtained from the matrix A by multiplication with the scalar a, and squares are removed from the diagonal without special mention.

No additional calculations were required for the $(2,1)$ and $(2,2)$ partitions.

Partitions $(2,1,1)$ and $(3,1)$

$$D_{(3,1)} = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right),$$

and

$$D_{(2,1,1)} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{smallmatrix}\right) \overset{2}{\simeq} \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right) \sim D_{(3,1)}.$$

Partitions $(2,2,1)$ and $(3,2)$

$$D_{(2,2,1)} = \begin{pmatrix} 2&0&0&0&0\\0&30&0&0&0\\0&0&30&0&0\\0&0&0&2&0\\0&0&0&0&1 \end{pmatrix} \overset{3}{\simeq} \begin{pmatrix} 6&0&0&0&0\\0&10&0&0&0\\0&0&10&0&0\\0&0&0&6&0\\0&0&0&0&3 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0\\0&15&0&0&0\\0&0&15&0&0\\0&0&0&1&0\\0&0&0&0&3 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0\\0&3&0&0&0\\0&0&3&0&0\\0&0&0&1&0\\0&0&0&0&3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&3&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix} \overset{3}{\simeq} \begin{pmatrix} 3&0&0&0&0\\0&3&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix},$$

and

$$D_{(3,2)} = \begin{pmatrix} 3&0&0&0&0\\0&6&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&6 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0\\0&2&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&6 \end{pmatrix} \sim \begin{pmatrix} 3&0&0&0&0\\0&3&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&3 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&3&0&0\\0&0&0&3&0\\0&0&0&0&3 \end{pmatrix}.$$

Over fields of characteristic not 2, we can identify symmetric bilinear forms with quadratic forms and vice-versa, and the classification of non-degenerate quadratic forms over the rationals is a classical problem; for example, see [Sch] or [Jon]. In particular,

(i) two forms over $\mathbb{Q}$ are isometric if and only if they are isometric over every p-adic field $\mathbb{Q}_p$, including $p = \infty$, and

(ii) two forms over $\mathbb{Q}_p$, including $p = \infty$, are isometric if and only if they have the same dimension, discriminant and Hasse symbol c_p.

If A is the matrix of a quadratic form q with respect to some basis, then the Hasse symbol is defined by

$$c_p(q) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p \,,$$

where D_i is the $i \times i$ determinant in the upper left corner of the matrix A and $(\alpha, \beta)_p$, $\alpha, \beta \in \mathbb{Q}_p$, is the Hilbert Symbol, i.e., $(\alpha, \beta)_p = +1$ or -1 according as $\alpha x_1^2 + \beta x_2^2 = 1$ has a solution in $\mathbb{Q}_p$ or not. It can be shown that

(i) $(\alpha, -\alpha)_p = 1$,

(ii) $(\alpha\delta^2, \beta\gamma^2)_p = (\alpha, \beta)$, and

(iii) if p is an odd prime, $\alpha = p^a \alpha_1$, $\beta = p^b \beta_1$ with α_1, β_1 units (in $\mathbb{Q}_p$), then $(\alpha, \beta)_p = (-1|p)^{ab} (\alpha_1|p)^b (\beta_1|p)^a$, where $(\alpha|p)$ is the value of the Legendre Symbol $(a_0|p)$ with a_0 the leading term in the p-adic expansion of α.

Using these properties, $c_3(< 1,1,1,1,1 >) = 1$ while $c_3(< 1,1,1,3,3 >) = -1$. So over the 3-adics, these two forms are not isometric, and thus they are not isometric over the rationals. Hence, the form $< 1,1,1,3,3 >$ cannot be further simplified.

Partition $(3,1,1)$

$$D_{(3,1,1)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&7&0&0&0\\ 0&0&0&70&0&0\\ 0&0&0&0&10&0\\ 0&0&0&0&0&5 \end{pmatrix} \overset{5}{\cong} \begin{pmatrix} 5&0&0&0&0&0\\ 0&5&0&0&0&0\\ 0&0&35&0&0&0\\ 0&0&0&14&0&0\\ 0&0&0&0&2&0\\ 0&0&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&10&0&0\\ 0&0&0&0&2&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$\overset{2}{\cong} \begin{pmatrix} 2&0&0&0&0&0\\ 0&2&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&5&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&2 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&5&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&5 \end{pmatrix}.$$

Since $f^{(3,1,1)}$ is even, multiplication by a scalar does not change the discriminant, and so we can do no better than the above matrix.

B *Mathematica* Functions

The following functions were written in *Mathematica*, and assume that the packages `DiscreteMath'Combinatorica'` and `LinearAlgebra'MatrixManipulation'` have been loaded. The file "S#.dat" contains the members of the symmetric group on # elements, ordered by conjugacy class. The file "M#.dat" contains the multiplication table for the symmetric group on # elements.

`InvolutionMatrices`: Given a partition, this function calculates the matrix B, as defined in Chapter 4, and then finds a diagonal matrix D which is congruent to B. As a means of error detection, the function does this for each standard tableaux associated to the partition.

`TableauxBasis`: Given a standard tableaux t, this function determines the i and α such that $t = t_i^\alpha$, and calculates a basis for the minimal right ideal $e_i^\alpha \mathbb{Q}\mathrm{S}_n$, namely $\{e_{ik}^\alpha \mid 1 \leq k \leq f^\alpha\}$. This function returns a list of length two: the first element is a list which contains the basis elements; the second element is the value i.

`TableauStabilizer`: Given a tableau t, this function returns $\mathcal{V}(t)\mathcal{H}(t)$.

`TableauPermutation`: Given two tableaux t_1, t_2, this function calculates the permutation π such that $t_1 = \pi t_2$. It returns this permutation as an element of $\mathbb{Q}\mathrm{S}_n$.

`QSnInvolution`: Given an element $a \in \mathbb{Q}\mathrm{S}_n$, this function returns the element a^*.

`QSnMultiplication`: Given two elements $a, b \in \mathbb{Q}\mathrm{S}_n$, this function returns the element ab.

`Diagonalize`: Given a symmetric matrix M, this function computes a diagonal matrix which is congruent to M. It then removes the greatest common factor from the diagonal, and makes the diagonal entries square-free. It returns this modified diagonal matrix.

```
InvolutionMatrices[p_List?PartitionQ] :=
 Module[
  {n1, n2, i, listtab, temp, basis, e, einv, psi, m1, m2, t, d, j, k},
  listtab = Tableaux[p];
  n1 = Length[listtab];
  n2 = Sum[p[[i]], {i, 1, Length[p]}];
  Do[
   Print[""]; Print[""];
   Print["Working with tableau:  ", TableForm[ listtab[[i]] ]];
   temp = TableauBasis[listtab[[i]]];
   basis = temp[[1]];
   e = basis[[ temp[[2]] ]];
   einv = QSnInvolution[e, n2];
   psi = QSnMultiplication[e, einv, n2];
   m1 = Table[0, {j, 1, n1}, {k, 1, n1}];
   Do[
    t = QSnMultiplication[ QSnMultiplication[QSnMultiplication[psi,
         basis[[j]], n2], QSnInvolution[basis[[k]], n2], n2 ], psi, n2];
    m1[[j, k]] = t[[1]],
    {j, 1, n1}, {k, 1, n1}
   ];
   m2 = m1 / Apply[GCD, Flatten[m1]];
   Print["Matrix w.r.t. basis is:  ", MatrixForm[m2]];
   d = Diagonalize[m2];
   Print["Diagonal matrix w.r.t basis:  ", MatrixForm[d]],
   {i, 1, n1}
  ];
 ];
```

```
TableauBasis[t_List?TableauQ] :=
  Module[
   {n, id, shape, i, listtab, numtab, e, tableaubasis, perm, k, prod},
   n = Max[t];
   id = Table[0, {n!}]; id[[1]] = 1;
   shape = {};
   Do[
    shape = Append[shape, Length[t[[i]]]],
    {i, 1, Length[t]}
   ];
   listtab = Tableaux[shape];
   numtab = Length[listtab];
   e = TableauStabilizer[t];
   tableaubasis = {};
  Do[
   perm = TableauPermutation[t, listtab[[i]]];
   If[perm == id, k = i];
   prod = QSnMultiplication[e, perm, n];
   tableaubasis = Append[tableaubasis, prod],
   {i, 1, numtab}
  ];
 {tableaubasis, k}
];
```

```
TableauStabilizer[t_List?TableauQ] :=
  Module[
   {rows, cols, n, p, id, rowstab, curRow, nRow, rowPerm, nPerm, curPerm,
    ind, curRowstab, colstab, curCol, nCol, colPerm, curColstab},
   rows = Length[t]; cols = Length[ t[[1]] ];
   n = Length[Flatten[t] ];
   p = Which[n == 3, Import["S3.dat"], n == 4, Import["S4.dat"],
         n == 5, Import["S5.dat"], n == 6, Import["S6.dat"] ];
   id = Table[i, {i, 1, n}];

   rowstab = Table[0, {n!}]; rowstab[[1]] = 1;
   Do[
    curRowstab = Table[0, {n!}]; curRow = t[[i]];
    nRow = Length[curRow]; rowPerm = Permutations[curRow];
    nPerm = Length[rowPerm];
    Do[
     curPerm = id;
     Do[
      curPerm[[ curRow[[k]] ]] = rowPerm[[j, k]],
          {k, 1, nRow} ];
     ind = Position[p, curPerm]; ind = ind[[1, 1]];
     curRowstab[[ind]] += 1,
     {j, 1, nPerm} ];
   rowstab = QSnMultiplication[rowstab, curRowstab, n],
   {i, 1, rows} ];

  tt = TransposeTableau[t];
  colstab = Table[0, {n!}]; colstab[[1]] = 1;
  Do[
   curColstab = Table[0, {n!}]; curCol = tt[[i]];
   nCol = Length[curCol]; colPerm = Permutations[curCol];
   nPerm = Length[colPerm];
   Do[
    curPerm = id;
    Do[
     curPerm[[ curCol[[k]] ]] = colPerm[[j, k]],
         {k, 1, nCol} ];
    ind = Position[p, curPerm]; ind = ind[[1, 1]];
    curColstab[[ind]] += SignaturePermutation[curPerm],
    {j, 1, nPerm} ];
  colstab = QSnMultiplication[colstab, curColstab, n],
          {i, 1, cols} ];
 tableaustab = QSnMultiplication[colstab, rowstab, n];
 tableaustab ];
```

```
TableauPermutation[t1_List?TableauQ, t2_List?TableauQ] :=
  Module[
   {n, q, perm, answer, p, i, j},
   n = Max[t1];
   q = Which[n == 3, Import["S3.dat"],
         n == 4, Import["S4.dat"],
         n == 5, Import["S5.dat"],
         n == 6, Import["S6.dat"]
    ];
   perm = Table[0, {n}];
   answer = Table[0, {n!}];
   Do[
    p = Position[t2, i][[1]];
    perm[[i]] = t1[[p[[1]], p[[2]]]],
    {i, 1, n}
   ];
   j = Position[q, perm][[1, 1]];
   answer[[j]] = 1;
   answer
  ];
```

```
QSnInvolution[a_List, n_Integer?Positive] :=
  Module[
   {i, j, l = Length[a], p, answer = a},
   p = Which[n == 3, Import["S3.dat"],
         n == 4, Import["S4.dat"],
         n == 5, Import["S5.dat"],
         n == 6, Import["S6.dat"]
    ];
   Do[
    If[Not[InvolutionQ[ p[[i]] ]],
     j = Position[p, InversePermutation[p[[i]]]];
     answer[[ j[[1, 1]] ]] = a[[i]];
    ],
    {i, l}
   ];
  answer
 ];
```

```
QSnMultiplication[a_List, b_List, n_Integer?Positive] :=
  Module[
   {i, j, l = Length[a], answer, m, p},
   m = Which[n == 3, Import["M3.dat"],
         n == 4, Import["M4.dat"],
         n == 5, Import["M5.dat"],
         n == 6, Import["M6.dat"]
    ];
   answer = Table[0, {l}];
   Do[
    p = m[[i, j]];
    answer[[p]] += (a[[i]] * b[[j]]),
    {i, 1, l}, {j, 1, l}
   ];
   answer
  ]; /; Length[a] == Length[b]
```

```
Diagonalize[M_List?MatrixQ] :=
  Module[
   {n, i, j, k, l, vectors, linfunc, diag, a, p, d, diagM, f},
   n = Length[M]; i = 1; j = 1;
   While[i ≤ n && M[[i, i]] == 0,
        i++ ];
   j = i;
   If[i > n,
    i = 0; j = 0;
    Do[
     If[M[[k, l]] ≠ 0,
      i = k; j = l; Break[];],
     {k, 1, n - 1}, {l, k + 1, n} ]; ];

  vectors = {}; linfunc = {}; diag = {};
  a = Table[0, {n}];
  a[[i]] = 1; a[[j]] = 1;
  p = a.M; d = a.p;
  vectors = Append[vectors, a]; linfunc = Append[linfunc, p];
  diag = Append[diag, d];
  Print["Step 1: ", a, " ", p, " ", d];
  Do[
   a = NullSpace[linfunc][[1]];
   p = a.M; d = a.p;
   vectors = Append[vectors, a]; linfunc = Append[linfunc, p];
   diag = Append[diag, d];
   Print["Step ", k, ": ", a, " ", p, " ", d],
   {k, 2, n} ];

 diag = diag / Apply[GCD, diag];
 Do[
  f = FactorInteger[diag[[k]]];
  Do[
   f[[l, 2]] = Mod[f[[l, 2]], 2],
   {l, 1, Length[f]} ];
  diag[[k]] = Product[f[[l, 1]]^f[[l, 2]], {l, 1, Length[f]}],
  {k, 1, n} ];
diagM = DiagonalMatrix[diag];
diagM ];
```

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