

# The Elliptic Hall Algebra and the Quantum Heisenberg Category

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# Abstract

We define the affinization of an arbitrary monoidal category  $\mathcal{C}$ , corresponding to the category of  $\mathcal{C}$ -diagrams on the cylinder. We also give an alternative characterization in terms of adjoining dot generators to  $\mathcal{C}$ . The affinization formalizes and unifies many constructions appearing in the literature. In particular, we describe a large number of examples coming from Hecke-type algebras, braids, tangles, and knot invariants. When  $\mathcal{C}$  is rigid, its affinization is isomorphic to its horizontal trace, although the two definitions look quite different. In general, the affinization and the horizontal trace are not isomorphic.

We then use the affinization to show our main result, which is an explicit isomorphism between the central charge  $k$  reduction of the universal central extension of the elliptic Hall algebra and the trace, or zeroth Hochschild homology, of the quantum Heisenberg category of central charge  $k$ . We use this isomorphism to construct large families of representations of the universal extension of the elliptic Hall algebra.

# Résumé

Nous définissons l'affinisation d'une catégorie monoïdale  $\mathcal{C}$  arbitraire, correspondant à la catégorie des  $\mathcal{C}$ -diagrammes sur le cylindre. Nous donnons aussi une autre caractérisation en termes de l'adjonction à  $\mathcal{C}$  de générateurs pointés. L'affinisation formalise et unifie plusieurs constructions qui existent dans la littérature. En particulier, nous décrivons un grand nombre d'exemples provenant d'algèbres de type de Hecke, tresses, enchevêtrements, et invariants de nœuds. Lorsque  $\mathcal{C}$  est rigide, son affinisation est isomorphe à sa trace horizontale, bien que les deux définitions paraissent assez différentes. En général, l'affinisation et la trace horizontale ne sont pas isomorphes.

Nous utilisons ensuite l'affinisation pour montrer notre résultat principal, qui est un isomorphisme explicite entre la réduction  $k$  de la charge centrale de l'extension centrale universelle de l'algèbre de Hall elliptique et la trace, ou la homologie de Hochschild en degré zéro, de la catégorie quantique de Heisenberg avec charge centrale  $k$ . Nous utilisons cet isomorphisme pour construire une grande famille de représentations de l'extension centrale universelle de l'algèbre de Hall elliptique.

*To mom and dad, and my late grandma*

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# List of Symbols

Add	The additive envelope of a $\mathbb{k}$ -linear category . . . . .	20
Aff	The affinization functor . . . . .	23
$\mathit{End}(\mathcal{M})$	The category of endofunctors of $\mathcal{M}$ . . . . .	28
$\mathcal{I}$	A tensor ideal . . . . .	18
$\mathcal{P}$	The set of partitions . . . . .	102
$\mathcal{P}^+$	The set of partitions without a negative part . . . . .	104
$\mathit{EH}^\pm$	The positive and negative part of $\mathit{EH}_k$ . . . . .	73
$\mathit{EH}_k$	The elliptic Hall algebra of central charge $k$ . . . . .	72
$\mathfrak{E}\mathfrak{H}$	The underlying Lie algebra of the elliptic Hall algebra . . . . .	66
$\mathcal{FOT}(D)$	The category of framed oriented tangles over the disc . . . . .	36
$\mathcal{FT}(D)$	The category of framed tangles over the disc . . . . .	37
$\mathit{Heis}^\pm$	The positive and negative parts of $\mathit{Heis}_k$ . . . . .	87
$\mathit{Heis}_k$	The quantum Heisenberg category of central charge $k$ . . . . .	79
htr	The horizontal trace functor . . . . .	52
$\mathbb{k}$	A commutative ring . . . . .	17
$\mathbb{1}$	The unit object in a monoidal category . . . . .	7
$\mathcal{OS}$	The framed HOMFLYPT skein category . . . . .	44
$\mathcal{OT}(D)$	The category of oriented tangles over the disc . . . . .	37
Sym	The ring of symmetric functions over $\mathbb{k}$ . . . . .	83
$\mathcal{T}(D)$	The category of tangles over the disc . . . . .	37
$\mathcal{TL}$	The Temperley-Lieb category . . . . .	50
Tr	The trace of a $\mathbb{k}$ -linear category . . . . .	19
$\widehat{\mathfrak{E}\mathfrak{H}}$	The universal central extension of $\mathfrak{E}\mathfrak{H}$ . . . . .	66
$e_r$	The $r$ -th elementary symmetric function . . . . .	83
$h_r$	The $r$ -th complete homogeneous symmetric function . . . . .	83
$h_r$	The $r$ -th power symmetric function . . . . .	83
$k$	The central charge . . . . .	79
$P_\lambda$	A basis element of $\mathit{Sym} \otimes \mathit{Sym}$ . . . . .	102
$Q_{\lambda,\mu}$	A symmetric function in $\mathit{Sym} \otimes \mathit{Sym}$ . . . . .	104
$R_{\lambda,\mu}$	A symmetric function in $\mathit{Sym} \otimes \mathit{Sym}$ . . . . .	109
$z, t$	Two invertible elements in $\mathbb{k}$ with $z = q - q^{-1}$ . . . . .	79

# Chapter 1

## Introduction

This thesis is based on the papers with A. Savage [46] and [47]. Compared to the papers, the thesis contains more details and background and a few other minor results not in the papers.

The concept of *affinization/affine* appears in many areas of category theory and representation theory. We are particularly interested in the following uses of the concept:

- *Topological/diagrammatic*: The term *affine* is often used to refer to the topological or diagrammatic setting of a torus, annulus, or cylinder. For example, one often encounters the term *affine braids* to refer to braids on a cylinder. More generally, the term *affine* is often used to describe either monoidal categories drawn in terms of string diagrams on a cylinder or annulus, or where strings are allowed to carry *dots* with various properties. The description in terms of dots is often ad hoc. For example, the dots are sometimes equal to their own mates and sometimes they are not.

- *Algebraic*: In the context of Hecke-type algebras, the term *affine* refers to the introduction of a (Laurent) polynomial part of the algebra. (Of course, this can be explained in terms of the Hecke algebra of an affine Weyl group.) For example, the affine Hecke algebra  $H_r^{\text{aff}}$  of type  $A_{r-1}$  over a commutative ground ring  $\mathbb{k}$  is isomorphic, as a  $\mathbb{k}$ -module, to  $\mathbb{k}[x_1^{\pm 1}, \dots, x_r^{\pm 1}] \otimes_{\mathbb{k}} H_r$ , where  $H_r$  is the Iwahori–Hecke algebra of type  $A_{r-1}$ . Allowing  $r$  to vary, the affine Hecke algebras can be organized into a tower of algebras, which can be viewed as a monoidal category. In the string diagram calculus for monoidal categories, the elements  $x_i$  correspond to the *dots* mentioned above.

- *Representation theoretic*: The term *affine* often appears in the context of duality statements in representation theory. For example, quantum Schur–Weyl duality states that there is a surjective algebra homomorphism  $H_r \rightarrow \text{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes r})$ , where  $V$  is the quantum analogue of the natural representation of  $\mathfrak{gl}_n$ . This induces an algebra homomorphism from  $H_r$  to the endomorphism algebra of the functor

$V^{\otimes r} \otimes - : U_q(\mathfrak{gl}_n)\text{-mod} \rightarrow U_q(\mathfrak{gl}_n)\text{-mod}$ . This homomorphism is not surjective in general. Instead the braiding on  $U_q(\mathfrak{gl}_n)\text{-mod}$  coming from the  $R$ -matrix allows one to extend it to an algebra homomorphism from  $H_r^{\text{aff}}$  to the endomorphism algebra of the functor  $V^{\otimes r} \otimes -$ .

Preference for one of the languages listed above often depends on one's point of view. Topologists interested in skein theory and knot invariants will prefer the topological point of view, while representation theorists may prefer the algebraic or representation theoretic point of view. In some cases, the translation between the languages is well known. For instance, this is the case for the affine Hecke algebras mentioned above. Our aim is to completely unify the different languages with a general approach. The fact that the various viewpoints coincide gives a rich interplay between topology and representation theory.

Our starting point is the definition of the *affinization*  $\text{Aff}(\mathcal{C})$  of an arbitrary strict monoidal category  $\mathcal{C}$ . In terms of the usual string diagram calculus for monoidal categories,  $\text{Aff}(\mathcal{C})$  should be thought of as the category of  $\mathcal{C}$ -diagrams on the cylinder. Its definition (Definition 3.1.1) involves adjoining to  $\mathcal{C}$  invertible morphisms corresponding to strings wrapping around the cylinder, subject to natural relations. If  $\mathcal{C}$  is braided, then  $\text{Aff}(\mathcal{C})$  is a strict monoidal category, with the tensor product corresponding to nesting of cylinders. Furthermore, in this case, we can give an equivalent definition of the affinization involving the addition of *dot generators* on strands (Theorem 3.1.6). The equivalence of these two definitions is a very general and precise statement of the correspondence between the topological/diagrammatic and algebraic notions of *affine* described above. A significant advantage of the description of  $\text{Aff}(\mathcal{C})$  in terms of dot generators is that string diagrams become easier to draw, since we no longer need to draw them on a cylinder.

We next turn our attention to categorical actions. If  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor, then  $\mathcal{C}$  acts on  $\mathcal{M}$  via the action

$$X \cdot M = F(X) \otimes M, \quad f \cdot g = F(f) \otimes g,$$

for  $X \in \text{Ob}(\mathcal{C})$ ,  $M \in \text{Ob}(\mathcal{M})$ ,  $f \in \text{Mor}(\mathcal{C})$ , and  $g \in \text{Mor}(\mathcal{M})$ . If  $\mathcal{C}$  is *balanced* (i.e. it is braided and has a twist; see Section 3.2), then we show in Theorem 3.2.3 that this action can be extended in a natural way to the affinization  $\text{Aff}(\mathcal{C})$ . This yields the precise connection to the representation theoretic viewpoint mentioned above; see Corollary 3.2.4.

Our affinization procedure recovers a large number of examples appearing in the literature, unifying them into a single precise framework. Examples include the following:

- The affinization of the category of braids over the disc is the category of braids over the annulus (Proposition 4.2.1).

- The affinization of the tower of Iwahori–Hecke algebras of type  $A$ , naturally viewed as a monoidal category, is the tower of affine Hecke algebras of type  $A$  (Section 4.2.2).
- The affinization of the category of oriented tangles (respectively, framed oriented tangles) over the disc is the category of oriented tangles (respectively, framed oriented tangles) over the annulus (Propositions 4.3.2 and 4.3.4). Analogous results also hold for unoriented tangles (Propositions 4.4.1 and 4.4.3).
- The affinization of the framed HOMFLPYT skein category over the disc is the framed HOMFLPYT skein category over the annulus (Proposition 4.3.6).
- The affinization of the Kauffman skein category over the disc is the Kauffman skein category over the annulus (Proposition 4.4.7).
- The affinization of the Temperley–Lieb category is the affine Temperley–Lieb category (Section 4.4.4).

In all of the above examples, our general results give two descriptions of the affine categories, one in terms of string diagrams on the cylinder and the other in terms of string diagrams carrying dots. Some of these presentations have appeared previously in the literature, while others are new. (See Chapter 4 for references to the literature in each case.) The appeal of our approach is a completely uniform treatment.

Another construction that has appeared in the literature in the context of monoidal categories on the annulus is the *horizontal trace*. The horizontal trace  $\text{htr}(\mathcal{C})$  has the same objects as  $\mathcal{C}$ , and its morphisms are equivalence classes of certain morphisms in  $\mathcal{C}$ ; see Section 5.1. We show (Theorem 5.1.6) that, when  $\mathcal{C}$  is rigid (i.e. it has left and right duals), the affinization  $\text{Aff}(\mathcal{C})$  and the horizontal trace  $\text{htr}(\mathcal{C})$  are isomorphic. However, even in this case, the two definitions are quite different. The affinization involves adjoining additional morphisms subject to some natural relations, while the horizontal trace involves equivalence classes of morphisms. This difference makes the affinization easier to work with in many cases. In general (i.e. when  $\mathcal{C}$  is not rigid), the affinization and the horizontal trace are *not* isomorphic, and it is the affinization, and not the horizontal trace, that gives the correct notion of  $\mathcal{C}$ -diagrams on a cylinder for two reasons: First, the interpretation of morphisms in the horizontal trace as string diagrams on the cylinder (or annulus) involves cups and caps that have no precise meaning when  $\mathcal{C}$  is not rigid. String diagrams in the affinization avoid such cups and caps. Second, in specific cases, it is the affinization that gives the expected “affine” category. For example, if  $\mathcal{C}$  is the category of braids over the disc, then  $\text{Aff}(\mathcal{C})$  is the category of braids over the annulus, while  $\text{htr}(\mathcal{C})$  is quite different; see Example 5.1.8.

The *vertical trace*  $\text{Tr}_v(\mathcal{C})$  of a  $\mathbb{k}$ -linear category  $\mathcal{C}$  is the  $\mathbb{k}$ -module given by linear combinations of endomorphisms in  $\mathcal{C}$  modulo the relation  $f \circ g = g \circ f$  for morphisms  $f: X \rightarrow Y$ ,  $g: Y \rightarrow X$  in  $\mathcal{C}$ ; see (2.6.1). If  $\mathcal{C}$  is strict pivotal, elements of the trace are

often drawn as diagrams on the annulus. In Section 5.2 we discuss how the procedure of taking the vertical trace behaves with respect to the process of affinization. In particular, if  $\mathcal{C}$  is a balanced strict  $\mathbb{k}$ -linear monoidal category, then  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  can be viewed as the category of  $\mathcal{C}$ -diagrams on the torus, and it acts naturally on  $\mathrm{Tr}_v(\mathcal{C})$ . This action corresponds to placing an annular diagram representing a morphism in  $\mathrm{Tr}_v(\mathcal{C})$  inside the toroidal diagram representing a morphism in  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$ ; see Proposition 5.2.3 for a more general statement. We also show (Theorem 5.2.5) that if  $\mathcal{C}$  is a right-rigid or left-rigid braided strict monoidal category, then  $\mathrm{Tr}_v(\mathcal{C})$  is isomorphic, as a  $\mathbb{k}$ -algebra, to the center  $Z(\mathrm{Aff}(\mathcal{C})) := \mathrm{End}_{\mathrm{Aff}(\mathcal{C})}(\mathbb{1})$  of  $\mathrm{Aff}(\mathcal{C})$ .

We conclude the first part of this thesis with a brief discussion of how the concept of affinization can be extended to the setting of 2-categories (Section 3.4). For rigid 2-categories, the affinization is again isomorphic to the horizontal trace. However, the two concepts are different in general.

The *elliptic Hall algebra* associated to a smooth elliptic curve  $X$  over a finite field is the Drinfeld double of the Hall algebra of the category of coherent sheaves over  $X$ . In [15], Burban and Schiffmann gave an explicit realization of a *generic* elliptic Hall algebra  $\mathrm{EH}$ , depending on two formal parameters  $\sigma, \bar{\sigma}$ , which specializes to the elliptic Hall algebra for *any*  $X$ . The importance of the algebra  $\mathrm{EH}$  is underlined by the fact that versions of it (more precisely, its “positive half” or central extensions) have appeared in many different contexts under different names: a *generalized quantum affine algebra* [19], a  $(q, \gamma)$ -analogue of the  $W_{1+\infty}$  algebra [40], the *shuffle algebra* [23, 49], the *spherical  $\mathfrak{gl}_\infty$  double affine Hecke algebra* [22, 55], and the *quantum continuous  $\mathfrak{gl}_\infty$*  [22]. It is also intimately related to the equivariant  $K$ -theory of the Hilbert scheme of points on  $\mathbb{A}^2$  [22, 23, 48, 56]. We show in Chapter 7 that the elliptic Hall algebra is categorified by the *quantum Heisenberg category* defined in [13]. We then use this categorification to construct large families of representations of central extensions of the elliptic Hall algebra.

Let us explain our results in more detail. We first show that the elliptic Hall algebra has a universal central extension  $\widetilde{\mathrm{EH}}$  by a two-dimensional center (Proposition 6.1.2). Then, to any *central charge*  $k \in \widetilde{\mathbb{Z}}$ , one can define a natural central reduction  $\mathrm{EH}_k$ . In fact, *every* central reduction of  $\mathrm{EH}$  is isomorphic to  $\mathrm{EH}_k$  for some  $k$  (Proposition 6.2.1).

To this same central charge, one can associate a quantum Heisenberg category  $\mathcal{H}\mathrm{eis}_k$  as in [13]. This is a strict  $\mathbb{k}$ -linear pivotal monoidal category modelled on the affine Hecke algebras of type  $A$ . When  $k \neq 0$ , it acts naturally on the category of modules for cyclotomic Hecke algebras of level  $|k|$ . When  $k = -1$ , it extends the Heisenberg category introduced in [36]. The later is a  $q$ -deformation of Khovanov’s original Heisenberg category [35]. On the other hand, when  $k = 0$ , it is the framed HOMFLYPT skein category over the annulus and it acts naturally on the category of modules for  $U_q(\mathfrak{gl}_n)$ .

The *trace*, or *zeroth Hochschild homology*, of a small  $\mathbb{k}$ -linear category is the  $\mathbb{k}$ -

module

$$\mathrm{Tr}(\mathcal{C}) := \left( \bigoplus_{X \in \mathcal{C}} \mathrm{End}_{\mathcal{C}}(X) \right) / \mathrm{span}_{\mathbb{k}}\{f \circ g - g \circ f\},$$

where  $f$  and  $g$  run through all pairs of morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  in  $\mathcal{C}$ . The trace can be thought of as a categorical analogue of the cocenter of an algebra. If  $\mathcal{C}$  is monoidal, then  $\mathrm{Tr}(\mathcal{C})$  is naturally an associative  $\mathbb{k}$ -algebra. The main result of the current thesis (Theorem 7.4.1) is that there is an isomorphism of algebras

$$\mathrm{EH}_k \xrightarrow{\cong} \mathrm{Tr}(\mathcal{H}eis_k). \quad (1.0.1)$$

This isomorphism is given explicitly, by specifying the images of the elements of a natural basis for  $\mathrm{EH}_k$ .

When  $k = 0$ ,  $\mathrm{Tr}(\mathcal{H}eis_0)$  is isomorphic to the skein algebra of the torus. This skein algebra was identified with  $\mathrm{EH} = \mathrm{EH}_0$  by Morton and Samuelson [45]. On the other hand, when  $k = -1$ , the  $q$ -deformed Heisenberg category of [36] was identified with the positive half of  $\mathrm{EH}_{-1}$  by Cautis, Lauda, Licata, Samuelson, and Sussan [17]. This corresponds to the fact that the  $q$ -deformed Heisenberg category can be viewed as “half” of the quantum Heisenberg category  $\mathcal{H}eis_{-1}$ . (See Remark 7.4.9.) In some sense,  $\mathrm{Tr}(\mathcal{H}eis_k)$  can be thought of as a deformation of the skein algebra of the torus, depending on the central charge  $k$ , that breaks the symmetry between the two directions. This central charge deformation allows us to categorify arbitrary central reductions  $\mathrm{EH}_k$ .

The split Grothendieck ring  $K_0(\mathcal{H}eis_k)$  of the quantum Heisenberg category is conjecturally isomorphic to the central charge  $k$  reduction  $\mathrm{Heis}_k$  of the universal enveloping algebra of the infinite rank Heisenberg Lie algebra. (The corresponding statement for the *degenerate* Heisenberg category has been proved; see [11, Th. 1.1].) The Chern character map gives a homomorphism  $K_0(\mathcal{H}eis_k) \rightarrow \mathrm{Tr}(\mathcal{H}eis_k)$ . Assuming the aforementioned conjecture, this corresponds to a natural inclusion  $\mathrm{Heis}_k \hookrightarrow \mathrm{EH}_k$ . (See Remark 6.1.1.)

One immediate application of our categorification of  $\mathrm{EH}_k$  is that we obtain a large number of representations of this algebra. The first family of representations arises from the fact that the trace of a linear pivotal category acts naturally on its center, which is the endomorphism algebra of the unit object. For the quantum Heisenberg category, the center is isomorphic to  $\mathrm{Sym} \otimes \mathrm{Sym}$ , where  $\mathrm{Sym}$  is the algebra of symmetric functions. Thus, we obtain a natural family of actions of  $\mathrm{EH}_k$  on  $\mathrm{Sym} \otimes \mathrm{Sym}$  depending on a parameter  $t$  in the ground ring. This generalizes an action of  $\mathrm{EH} = \mathrm{EH}_0$  on  $\mathrm{Sym} \otimes \mathrm{Sym}$  described in [45, §4], corresponding to the action of the skein algebra of the torus acting on the skein of the annulus.

The second family of representations emerges from the natural action of  $\mathcal{H}eis_k$  on the category of modules for cyclotomic Hecke algebras. Passing to traces, this yields an action of  $\mathrm{EH}_k$  on the cocenters of cyclotomic Hecke algebras. We expect these actions to be related to the geometry of moduli spaces of framed torsion-free sheaves on  $\mathbb{P}^2$ , extending work of Schiffmann and Vasserot [56]. (See Remark 8.2.4.)

## So what's new in this thesis?

- A formal way (Definition 3.1.1) to express the idea of drawing diagrams of a monoidal category on a cylinder.
- If  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor, then  $\mathcal{C}$  acts on  $\mathcal{M}$  via the action

$$X \cdot M = F(X) \otimes M, \quad f \cdot g = F(f) \otimes g,$$

for  $X \in \text{Ob}(\mathcal{C})$ ,  $M \in \text{Ob}(\mathcal{M})$ ,  $f \in \text{Mor}(\mathcal{C})$ ,  $g \in \text{Mor}(\mathcal{M})$ .

In Theorem 3.2.3, we extend this action to the affinization  $\text{Aff}(\mathcal{C})$ . We also show that this action induces an action in the level of vertical traces, that is,  $\text{Tr}(\text{Aff}(\mathcal{C}))$  acts on  $\text{Tr}(\mathcal{M})$  (Proposition 5.2.3).

- We prove that if  $\mathcal{C}$  is a balanced monoidal category, then  $\mathcal{C}$  can be seen as a subcategory of its affinization  $\text{Aff}(\mathcal{C})$  (Corollary 3.2.5).
- The affinization of 2-categories is given (Section 3.4).
- We give the description of many examples of affine categories in terms of string diagrams carrying dots (see Chapter 4).
- A new monoidal structure on the horizontal trace is given (Proposition 5.1.2).
- The connection of the affinization to the horizontal trace is given (Theorem 5.1.6).
- The isomorphism in Theorem 5.2.5 connecting the vertical trace to the center of the affinization.
- The universal central extension of the elliptic Hall algebra is computed (Proposition 6.1.2), and a biangular presentation of it is given (Corollary 6.3.4).
- A categorification of the elliptic Hall algebra using the trace functor (Theorem 7.4.1). Our result is an extension of an earlier result of Cautis, Lauda, Licata, Samuelson and Sussan ([17, Th. 6.3]); see Remark 7.4.9 for the precise relation between the two results.
- A full description of an action of the elliptic Hall algebra  $\text{EH}_k$  on the tensor product  $\text{Sym}^{\otimes 2}$  of symmetric functions (Corollary 8.1.11 and Propositions 8.1.12 and 8.1.13). In the  $k = 0$  case, this action is a twist of an action by Morton and Samuelson (Proposition 8.1.7).
- A partial description of an action of  $\text{EH}_k$ , for  $k \neq -1$ , on the direct sum (over all ranks) of cocenters of cyclotomic Hecke algebras. When  $k = -1$ , this action extends the action in [17, §7] for “half” of  $\text{EH}_{-1}$ ; see Remark 8.2.4.

# Chapter 2

## Preliminaries

In this preliminary chapter we collect the definitions and facts about monoidal and linear categories that will be used throughout this thesis.

### 2.1 Monoidal categories and functors

**Definition 2.1.1** ([21, Def. 2.1.1]). A monoidal category is a quintuple  $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \iota)$  where  $\mathcal{C}$  is a category,  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is a bifunctor called the *tensor product bifunctor*,  $\alpha: (- \otimes -) \otimes - \xrightarrow{\cong} - \otimes (- \otimes -)$  is a natural isomorphism

$$\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z), \quad X, Y, Z \in \mathcal{C}, \quad (2.1.1)$$

called the *associativity constraint* (or *associativity isomorphism*),  $\mathbb{1} \in \mathcal{C}$  is an object of  $\mathcal{C}$ , called the *unit object*, and  $\iota: \mathbb{1} \otimes \mathbb{1} \rightarrow \mathbb{1}$  is an isomorphism, subject to the following two axioms.

- **The pentagon axiom.** The diagram

$$\begin{array}{ccc}
 & ((W \otimes X) \otimes Y) \otimes Z & \\
 \alpha_{W,X,Y} \otimes \text{id}_Z \swarrow & & \searrow \alpha_{W \otimes X, Y, Z} \\
 (W \otimes (X \otimes Y)) \otimes Z & & (W \otimes X) \otimes (Y \otimes Z) \\
 \downarrow \alpha_{W, X \otimes Y, Z} & & \downarrow \alpha_{W, X, Y \otimes Z} \\
 W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\text{id}_W \otimes \alpha_{X, Y, Z}} & W \otimes (X \otimes (Y \otimes Z))
 \end{array}$$

commutes for all objects  $X, Y, Z, W \in \mathcal{C}$ .

- **The unit axiom.** The functors

$$L_{\mathbb{1}}: X \rightarrow \mathbb{1} \otimes X \quad \text{and} \quad R_{\mathbb{1}}: X \rightarrow X \otimes \mathbb{1}, \quad (2.1.2)$$

of the left and right multiplication by  $\mathbb{1}$ , are autoequivalences of  $\mathcal{C}$ .

**Remark 2.1.2.** Definition 2.1.1 is not the traditional definition of a monoidal category. A more traditional definition can be found in [21, Def. 2.8.8]. The two definitions are equivalent as explained in [21, Rem. 2.2.9].

**Definition 2.1.3** ([21, Def. 2.1.4]). A *monoidal subcategory* of a monoidal category  $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$  is a quintuple  $(\mathcal{D}, \otimes, a, \mathbb{1}, \iota)$ , where  $\mathcal{D} \subseteq \mathcal{C}$  is a subcategory closed under the tensor product of objects and morphisms and containing  $\mathbb{1}$  and  $\iota$ .

Throughout this thesis, whenever we say subcategory of a monoidal category, we will always mean a monoidal subcategory.

Next, we record an important property of monoidal categories, the so-called *interchange law*.

**Interchange law.** Let  $X_1 \xrightarrow{f} X_2$ , and  $Y_1 \xrightarrow{g} Y_2$  be morphisms in a monoidal category  $\mathcal{C}$ . Then

$$(f \otimes 1_{Y_2}) \circ (1_{X_1} \otimes g) = f \otimes g = (1_{X_2} \otimes g) \circ (f \otimes 1_{Y_1}).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} X_1 \otimes Y_1 & \xrightarrow{1 \otimes g} & X_1 \otimes Y_2 \\ f \otimes 1 \downarrow & \searrow f \otimes g & \downarrow f \otimes 1 \\ X_2 \otimes Y_1 & \xrightarrow{1 \otimes g} & X_2 \otimes Y_2 \end{array} \quad (2.1.3)$$

This follows from the fact that  $\otimes$  is a bifunctor.

## The unit object and its endomorphisms

Here we give some basic properties of the unit object. The proofs of these properties can be found in the indicated references.

**Proposition 2.1.4** ([21, Prop. 2.2.6]). *The unit object in a monoidal category is unique up to a unique isomorphism.*

**Proposition 2.1.5** ([21, Prop. 2.2.10]). *Let  $\mathcal{C}$  be a monoidal category. Then  $\text{End}_{\mathcal{C}}(\mathbb{1})$  is a commutative monoid under composition.*

**Definition 2.1.6.** The monoid  $\text{End}_{\mathcal{C}}(\mathbb{1})$  is called the *center* of the monoidal category  $\mathcal{C}$ .

## Monoidal functors and their morphisms

**Definition 2.1.7** ([21, Def. 2.4.1]). Let  $(\mathcal{C}, \otimes, \mathbb{1}, a, \iota)$  and  $(\mathcal{C}', \otimes', \mathbb{1}', a', \iota')$  be two monoidal categories. A *monoidal functor*  $\mathcal{C} \rightarrow \mathcal{C}'$  is a pair  $(\mathcal{F}, \phi)$ , consisting of a functor

$$\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}',$$

and a natural isomorphism

$$\phi_{X,Y} : \mathcal{F}X \otimes' \mathcal{F}Y \rightarrow \mathcal{F}(X \otimes Y), \quad (2.1.4)$$

such that the diagram

$$\begin{array}{ccc} (\mathcal{F}X \otimes' \mathcal{F}Y) \otimes' \mathcal{F}Z & \xrightarrow{a'_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}}} & \mathcal{F}X \otimes' (\mathcal{F}Y \otimes' \mathcal{F}Z) \\ \downarrow \phi_{X,Y} \otimes' \text{id}_{\mathcal{F}(Z)} & & \text{id}_{\mathcal{F}(X)} \otimes' \phi_{Y,Z} \downarrow \\ \mathcal{F}(X \otimes Y) \otimes' \mathcal{F}(Z) & & \mathcal{F}(X) \otimes' \mathcal{F}(Y \otimes Z) \\ \downarrow \phi_{X \otimes Y, Z} & & \phi_{X, Y \otimes Z} \downarrow \\ \mathcal{F}((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & \mathcal{F}(X \otimes (Y \otimes Z)) \end{array}$$

commutes for all  $X, Y, Z \in \mathcal{C}$ . A monoidal functor  $\mathcal{F}$  is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

**Remark 2.1.8.** Note that what we have referred to as a monoidal functor is called a *strong monoidal functor* by some authors. For these authors a monoidal functor is a functor as in Definition 2.1.7 but without requiring the natural transformation (2.1.4) to be an isomorphism. In the terminology we chose, such a functor (without the isomorphism condition) is called a *lax monoidal functor*. However, we will not need this notion in this thesis.

**Definition 2.1.9** ([21, Def. 2.4.8]). Let  $(\mathcal{C}, \otimes, \mathbb{1}, a, \iota)$  and  $(\mathcal{C}', \otimes', \mathbb{1}', a', \iota')$  be two monoidal categories, and let  $(\mathcal{F}_1, \phi_1)$  and  $(\mathcal{F}_2, \phi_2)$  be two monoidal functors from  $\mathcal{C}$  to  $\mathcal{C}'$ . A *morphism* (or a *natural transformation*)  $\eta: (\mathcal{F}_1, \phi_1) \rightarrow (\mathcal{F}_2, \phi_2)$  of monoidal functors is a natural transformation  $\eta: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $\eta_{\mathbb{1}}$  is an isomorphism, and the diagram

$$\begin{array}{ccc} \mathcal{F}_1 X \otimes' \mathcal{F}_1 Y & \xrightarrow{\phi_{1X,Y}} & \mathcal{F}_1(X \otimes Y) \\ \downarrow \eta_X \otimes' \eta_Y & & \downarrow \eta_{X \otimes Y} \\ \mathcal{F}_2 X \otimes' \mathcal{F}_2 Y & \xrightarrow{\phi_{2X,Y}} & \mathcal{F}_2(X \otimes Y) \end{array}$$

is commutative for all  $X, Y \in \mathcal{C}$ .

## 2.2 Strict monoidal categories

**Definition 2.2.1** ([21, Def. 2.8.1]). A monoidal category  $\mathcal{C}$  is *strict* if for all objects  $X, Y, Z$  in  $\mathcal{C}$  one has equalities  $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$  and  $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$ , and the associativity (2.1.1) and unit constraints (2.1.2) are the identity maps.

**Definition 2.2.2.** A monoidal functor between two strict monoidal categories is *strict* if all the isomorphisms in (2.1.4) are the identity maps.

**Example 2.2.3.** The category  $\mathit{End}(\mathcal{C})$  of endofunctors of a category  $\mathcal{C}$  is an important example of a strict monoidal category. The objects in  $\mathit{End}(\mathcal{C})$  are functors from  $\mathcal{C}$  to itself, the bifunctor  $\otimes$  is composition of functors and morphisms are natural transformations.

**Theorem 2.2.4** (Mac Lane’s strictness theorem). *Any monoidal category is monoidally equivalent to a strict monoidal category.*

For a proof of this theorem, see [38, §VII.2]. From now on we will switch gears to strict monoidal categories.

## Graphical calculus

Graphical calculus (or string diagrams) is a powerful tool for doing computations in strict monoidal categories. We briefly illustrate how it works here. The interested reader is referred to [62, Ch. 2] for a detailed treatment of the subject, or [54, 57] for a brief treatment.

Let  $\mathcal{C}$  be a strict monoidal category. In the language of graphical calculus, we depict a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  by a strand with a *coupon* labelled  $f$ :



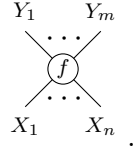
We use the convention that diagrams are read from bottom to top. The identity map  $1_X: X \rightarrow X$  is the strand with no coupon:



We sometimes omit the objects’ labels when it is clear from the context or unimportant. We do not draw the identity morphism of the unit object  $\mathbb{1}$ . Composition of morphisms is *vertical stacking* and the tensor product of morphisms is *horizontal juxtaposition*:

$$\begin{array}{c} \textcircled{f} \\ | \\ \textcircled{g} \\ | \end{array} = \textcircled{f \circ g} \quad , \quad \begin{array}{c} | \\ \textcircled{f} \\ | \end{array} \otimes \begin{array}{c} | \\ \textcircled{g} \\ | \end{array} = \begin{array}{c} | \\ \textcircled{f} \quad \textcircled{g} \\ | \end{array} .$$

A general morphism  $f: X_1 \otimes \cdots \otimes X_n \rightarrow Y_1 \otimes \cdots \otimes Y_m$  can be depicted as a coupon with  $n$  strands emanating from the bottom and  $m$  strands emanating from the top:



We draw endomorphisms  $f \in \text{End}_{\mathcal{C}}(\mathbb{1})$  as a *free-floating* coupon:

$$\textcircled{f}. \tag{2.2.1}$$

**Remark 2.2.5.** It follows from the axioms that in  $\text{End}_{\mathcal{C}}(\mathbb{1})$ , the vertical composition and horizontal juxtaposition coincide. So we can slide coupons (2.2.1) around each other at will. See [54, § 2.3] for an explanation of this fact.

**Interchange law revisited.** In the graphical calculus language, the interchange law (2.1.3) is depicted as:

$$\begin{array}{c} \textcircled{f} \\ | \\ \textcircled{g} \end{array} = \begin{array}{c} \textcircled{f} \\ | \\ \textcircled{g} \end{array} = \begin{array}{c} \textcircled{f} \\ | \\ \textcircled{g} \end{array},$$

which means we can slide coupons past each other.

## 2.3 Braided categories

**Definition 2.3.1.** A strict monoidal category  $\mathcal{C}$  is *braided* if it is equipped with a natural family of isomorphisms  $\beta_{X,Y}: X \otimes Y \rightarrow Y \otimes X$  satisfying

$$\beta_{X,Y \otimes Z} = (1_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes 1_Z), \quad \beta_{X \otimes Y,Z} = (\beta_{X,Z} \otimes 1_Y) \circ (1_X \otimes \beta_{Y,Z}), \tag{2.3.1}$$

for all  $X, Y, Z \in \text{Ob}(\mathcal{C})$ .

In the language of graphical calculus we draw the braiding as:

$$\beta_{X,Y} = \begin{array}{c} \diagdown \quad \diagup \\ X \quad Y \end{array}, \quad \beta_{X,Y}^{-1} = \begin{array}{c} \diagup \quad \diagdown \\ Y \quad X \end{array}.$$

Then the equations in (2.3.1) become

$$\begin{array}{c} \diagdown \quad \diagup \\ X \quad Y \otimes Z \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ X \quad Y \quad Z \end{array}, \quad \begin{array}{c} \diagdown \quad \diagup \\ X \otimes Y \quad Z \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ X \quad Y \quad Z \end{array}. \tag{2.3.2}$$

In what follows, we will use unlabeled strands to indicate that a relation holds for any labeling of the strands. So, for instance,

$$\textcircled{f} \text{ represents an arbitrary morphism } \begin{array}{c} Y \\ \textcircled{f} \\ X \end{array} \text{ in } \mathcal{C},$$

and we have

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \parallel \parallel = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}. \quad (2.3.3)$$

The naturality of the braiding means that

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \textcircled{f} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \textcircled{f}, \quad (2.3.4)$$

$$\textcircled{f} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \textcircled{f}, \quad \textcircled{f} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \textcircled{f}, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \textcircled{f} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \textcircled{f}. \quad (2.3.5)$$

In other words, the braid relations are satisfied, and coupons slide through crossings.

**Definition 2.3.2.** Let  $(\mathcal{C}, \otimes, \mathbb{1})$  and  $(\mathcal{C}', \otimes', \mathbb{1}')$  be braided strict monoidal categories. A *braided monoidal functor* between  $\mathcal{C}$  and  $\mathcal{C}'$  is a monoidal functor  $(\mathcal{F}, \phi): \mathcal{C} \rightarrow \mathcal{C}'$  that makes the following diagram commute

$$\begin{array}{ccc} \mathcal{F}X \otimes' \mathcal{F}Y & \xrightarrow{\beta_{\mathcal{F}(X), \mathcal{F}(Y)}}} & \mathcal{F}Y \otimes' \mathcal{F}X \\ \downarrow \phi_{X,Y} & & \downarrow \phi_{Y,X} \\ \mathcal{F}(X \otimes Y) & \xrightarrow{\mathcal{F}(\beta_{X,Y})} & \mathcal{F}(Y \otimes X) \end{array}$$

for all  $X, Y \in \mathcal{C}$ . Moreover,  $\mathcal{C}$  is called *strict braided monoidal functor* if it is a strict monoidal functor.

**Definition 2.3.3.** A braided strict monoidal category is called *symmetric* if  $\beta_{X,Y} \circ \beta_{Y,X} = \text{id}_{Y \otimes X}$ , for all  $X, Y \in \mathcal{C}$ .

A *symmetric monoidal functor* is a braided monoidal functor between symmetric monoidal categories.

**Remark 2.3.4.** When  $\mathcal{C}$  is symmetric, we can draw the braiding as

$$\beta_{X,Y} = \begin{array}{c} \diagup \diagdown \\ X \quad Y \end{array},$$

with no distinction between the over-crossing and under-crossing.

**Remark 2.3.5.** In Definition 2.3.1, we only considered the strict case as all the examples we treat in thesis are of this type. The definition of braiding involves the associator (2.1.1) if the category is not strict; see [21, §8.1] for details.

**Example 2.3.6.** Chapter 4 contains a lot of examples of braided strict monoidal categories. There is also an example of a symmetric strict monoidal category in Section 4.2.3. A typical example of a non-strict braided monoidal category is the category  $\mathcal{Vec}$  of vector spaces over a given field. The braiding in  $\mathcal{Vec}$  is given by the flip map  $V \otimes W \rightarrow W \otimes V$ ,  $v \otimes w \mapsto w \otimes v$ , for objects  $V, W \in \mathcal{Vec}$ . In fact,  $\mathcal{Vec}$  is symmetric. A less trivial example of a braided monoidal category is the category of finite dimensional modules over a quasi-triangular bialgebra; see [32, Ch. XIII] for details.

## 2.4 Rigid and pivotal categories

We start by defining rigidity in monoidal categories.

### Rigidity

**Definition 2.4.1** (right duals). Let  $\mathcal{C}$  be a strict monoidal category. An object  $X \in \text{Ob}(\mathcal{C})$  is said to have a right dual if there is another object  $X^\vee \in \text{Ob}(\mathcal{C})$ , called the *right dual* of  $X$ , together with *unit* and *counit* morphisms

$$\eta_X = \begin{array}{c} X \\ \curvearrowright \\ \mathbb{1} \end{array} : \mathbb{1} \rightarrow X^\vee \otimes X, \quad \epsilon_X = \begin{array}{c} \curvearrowleft \\ X \\ \mathbb{1} \end{array} : X \otimes X^\vee \rightarrow \mathbb{1}, \quad (2.4.1)$$

such that

$$\begin{array}{c} \downarrow \\ \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ X \end{array}, \quad \begin{array}{c} \uparrow \\ \uparrow \\ X \end{array} = \begin{array}{c} \uparrow \\ \uparrow \\ X \end{array}. \quad (2.4.2)$$

A strict monoidal category  $\mathcal{C}$  in which all objects have right duals is called *right rigid*.

Here we use an upward oriented string to denote the identity morphism of an object  $X$  and a downward oriented string to denote the identity morphism of its right dual  $X^\vee$ .

**Definition 2.4.2** (left duals). Let  $\mathcal{C}$  be a strict monoidal category  $\mathcal{C}$ . An object  $X \in \text{Ob}(\mathcal{C})$  is said to have a left dual if there is another object  ${}^\vee X \in \text{Ob}(\mathcal{C})$ , called the *left dual* of  $X$ , together with morphisms

$$\eta'_X = \begin{array}{c} X \\ \curvearrowleft \\ \mathbb{1} \end{array} : \mathbb{1} \rightarrow X \otimes {}^\vee X, \quad \epsilon'_X = \begin{array}{c} \curvearrowright \\ {}^\vee X \\ \mathbb{1} \end{array} : {}^\vee X \otimes X \rightarrow \mathbb{1}, \quad (2.4.3)$$

such that

$$\begin{array}{c} \uparrow \\ \text{hook} \\ \downarrow \\ X \end{array} = \begin{array}{c} \uparrow \\ | \\ \downarrow \\ X \end{array}, \quad \begin{array}{c} \downarrow \\ \text{hook} \\ \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ | \\ \downarrow \\ X \end{array}. \quad (2.4.4)$$

A strict monoidal category  $\mathcal{C}$  in which all objects have left duals is called *left rigid*.

Here we use an upward oriented string to denote the identity morphism of an object  $X$  and a downward oriented string to denote the identity morphism of its left dual  ${}^{\vee}X$ .

**Definition 2.4.3.** A strict monoidal category  $\mathcal{C}$  is *rigid* if it is both left rigid and right rigid.

The next result is well-known, and an algebraic proof of it can be found in [31, Prop. 7.2] for example. We want to give a demonstration of the usefulness of graphical calculus, so we will give a graphical proof here.

**Proposition 2.4.4** ([31, Prop. 7.2]). *If  $\mathcal{C}$  is a braided strict monoidal category, then it is left rigid if and only if it is right rigid (hence rigid). In this case, the left and right duals of  $X$  are isomorphic.*

**Proof:** Suppose  $\mathcal{C}$  is a braided strict monoidal category that is left rigid. Let  $X \in \text{Ob}(\mathcal{C})$ . There is  ${}^{\vee}X \in \text{Ob}(\mathcal{C})$  and morphisms  $\eta'_X$  and  $\epsilon'_X$  as in (2.4.3) satisfying (2.4.4). Define  $\eta_X: \mathbb{1} \rightarrow {}^{\vee}X \otimes X$  and  $\epsilon_X: X \otimes {}^{\vee}X \rightarrow \mathbb{1}$  by

$$\eta_X := \beta_{X, {}^{\vee}X} \circ \eta'_X = \begin{array}{c} \uparrow \\ \text{hook} \\ \downarrow \\ X \end{array} \quad \text{and} \quad \epsilon_X := \epsilon'_X \circ \beta_{X, {}^{\vee}X}^{-1} = \begin{array}{c} \downarrow \\ \text{hook} \\ \downarrow \\ X \end{array}.$$

We have

$$(1_{{}^{\vee}X} \otimes \epsilon_X) \circ (\eta_X \otimes 1_X) = \begin{array}{c} \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.3.5)}{=} \begin{array}{c} \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.3.3)}{=} \begin{array}{c} \downarrow \\ \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.4.4)}{=} \begin{array}{c} \downarrow \\ | \\ \downarrow \\ X \end{array},$$

which shows that  $\eta_X$  and  $\epsilon_X$  satisfy the first relation in (2.4.2). We also have

$$(\epsilon_X \otimes 1_X) \circ (1_X \otimes \eta_X) = \begin{array}{c} \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.3.5)}{=} \begin{array}{c} \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.3.3)}{=} \begin{array}{c} \uparrow \\ \text{hook} \\ \downarrow \\ X \end{array} \stackrel{(2.4.4)}{=} \begin{array}{c} \uparrow \\ | \\ \downarrow \\ X \end{array},$$

which shows that  $\eta_X$  and  $\epsilon_X$  also satisfy the second relation in (2.4.2). Thus  ${}^{\vee}X$  is a right dual of  $X$ . We can show in a similar way that a right dual of  $X$  is also a left dual of  $X$ .  $\blacksquare$

**Remark 2.4.5.** As a consequence of Proposition 2.4.4, if the category is braided, then there is no ambiguity in using a downward strand labeled  $X$  to denote the dual.

**Remark 2.4.6.** If  $X \in \text{Ob}(\mathcal{C})$  is *self-dual*, i.e.  $X = X^\vee = {}^\vee X$ , then the upward and downward oriented strands are equal, and it is natural to draw these strands without orientation. In particular, the left and right caps (resp. cups) above are equal.

**Definition 2.4.7.** Let  $\mathcal{C}$  be a strict monoidal category. If  $X$  and  $Y$  in  $\mathcal{C}$  have right duals  $X^\vee$  and  $Y^\vee$ , respectively, then a *right mate* of  $f: X \rightarrow Y$  is a morphism  $f^\vee: Y^\vee \rightarrow X^\vee$  given by

$$f^\vee := \begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowright (f) \text{---} \\ \downarrow \\ Y \end{array} .$$

If  $X$  and  $Y$  in  $\mathcal{C}$  have left duals  ${}^\vee X$  and  ${}^\vee Y$ , respectively, then a *left mate* of  $f: X \rightarrow Y$  is a morphism  ${}^\vee f: {}^\vee Y \rightarrow {}^\vee X$  given by

$${}^\vee f := \begin{array}{c} X \\ \uparrow \\ \text{---} \circlearrowleft (f) \text{---} \\ \downarrow \\ Y \end{array} .$$

If  $\mathcal{C}$  is right (resp. left) rigid, then we can define a functor  $\mathcal{R}: \mathcal{C} \rightarrow \mathcal{C}, X \rightarrow X^\vee$  (resp.  $\mathcal{L}: \mathcal{C} \rightarrow \mathcal{C}, X \rightarrow {}^\vee X$ ). This functor maps a morphism in  $\mathcal{C}$  to its right (resp. left) mate. The functor  $\mathcal{R}$  (resp.  $\mathcal{L}$ ) is a contravariant monoidal functor; see [63, §1.6.3].

**Remark 2.4.8.** We proved in Proposition 2.4.4 that

$$\begin{array}{c} \text{---} \circlearrowright \text{---} \\ \downarrow \end{array} = \begin{array}{c} | \end{array} .$$

This would be trivial if we were going to do computations modulo isotopy, as the equality would follow from the first Reidemeister move. However, for strict braided rigid categories, the equality of two diagrams holds only up to a regular isotopy. Regular isotopy is an equivalence on planar projections of 3-dimensional diagrams, and two diagrams are equivalent if and only if one can be obtained from the other through a sequence of moves involving only the second and third Reidemeister moves; see [57, §4.5] for more details. The right notion for computations modulo isotopy is the notion of pivotal categories, which we will define in the next subsection.

## Pivotal categories

**Definition 2.4.9.** A strict monoidal category  $\mathcal{C}$  is *pivotal* if it is a rigid strict monoidal category and we have the following:

(a) For all objects  $X$  and  $Y$  in  $\mathcal{C}$ ,

$$(X^\vee)^\vee = X, \quad (X \otimes Y)^\vee = Y^\vee \otimes X^\vee, \quad \mathbb{1}^\vee = \mathbb{1}.$$

(b) For all objects  $X$  and  $Y$  in  $\mathcal{C}$ , we have

$$\begin{array}{c} X \otimes Y \\ \curvearrowright \end{array} = \begin{array}{c} X Y \\ \curvearrowright \end{array} \quad \text{and} \quad \begin{array}{c} \curvearrowleft \\ X \otimes Y \end{array} = \begin{array}{c} \curvearrowleft \\ X Y \end{array}.$$

(c) For every morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ , its right and left mates are equal:

$$f^\vee := \begin{array}{c} X \\ \curvearrowright \\ f \\ \downarrow \\ Y \end{array} = \begin{array}{c} X \\ \downarrow \\ f \\ \curvearrowleft \\ Y \end{array}.$$

**Definition 2.4.10.** A strict monoidal category  $\mathcal{C}$  is said to be *balanced* if it is braided and has a *twist*, which is a natural transformation  $\theta: \text{id}_{\mathcal{C}} \rightarrow \text{id}_{\mathcal{C}}$  (where  $\text{id}_{\mathcal{C}}$  is the identity functor on  $\mathcal{C}$ ), whose components we will denote

$$\theta_X = \begin{array}{c} \theta \\ \downarrow \\ X \end{array} : X \rightarrow X, \quad X \in \text{Ob}(\mathcal{C}),$$

satisfying  $\theta_{\mathbb{1}} = 1_{\mathbb{1}}$  and

$$\begin{array}{c} \theta \\ \downarrow \\ X \otimes Y \end{array} = \begin{array}{c} \theta \quad \theta \\ \diagdown \quad \diagup \\ X \quad Y \\ \diagup \quad \diagdown \end{array}, \quad X, Y \in \text{Ob}(\mathcal{C}). \quad (2.4.5)$$

The fact that the family is natural means that the twists commute with morphisms:

$$\begin{array}{c} Y \\ \theta \\ \downarrow \\ f \\ \downarrow \\ X \end{array} = \begin{array}{c} Y \\ f \\ \downarrow \\ \theta \\ \downarrow \\ X \end{array}, \quad f: X \rightarrow Y \text{ in } \mathcal{C}. \quad (2.4.6)$$

The following result gives another characterization of braided pivotal categories.

**Proposition 2.4.11** ([57, Cor. 4.21]). *A braided pivotal category is the same as a balanced rigid category.*

In particular, if  $\mathcal{C}$  is a braided strict pivotal category, then it has a twist given by

$$\theta_X := \begin{array}{c} \uparrow \\ \bigcirc \\ \downarrow \\ X \end{array}, \quad X \in \text{Ob}(\mathcal{C}). \quad (2.4.7)$$

**Definition 2.4.12.** A *ribbon category* (also called a *tortile category*) is a braided pivotal category satisfying

$$\begin{array}{c} \uparrow \\ \bigcirc \\ \downarrow \\ X \end{array} = \begin{array}{c} \uparrow \\ \bigcirc \\ \downarrow \\ X \end{array}, \quad X \in \text{Ob}(\mathcal{C}). \quad (2.4.8)$$

We say a category is a *strict ribbon category* if it is a ribbon category that is strict pivotal.

## 2.5 Linear and additive categories

In this section, we introduce some notions related to linear and additive categories. The reader interested in learning more about these can consult [21, §1] for example. Throughout this section we fix a commutative ring  $\mathbb{k}$ .

**Definition 2.5.1.** A  $\mathbb{k}$ -*linear category* is a category that is enriched over the category of  $\mathbb{k}$ -modules. That is, the morphism spaces  $\text{Hom}_{\mathcal{C}}(X, Y)$  are  $\mathbb{k}$ -modules such that the composition maps  $\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$ ,  $(f, g) \mapsto g \circ f$  are bilinear, for all  $X, Y, Z \in \mathcal{C}$ .

A (strict)  $\mathbb{k}$ -*linear monoidal category* is a category that is both (strict) monoidal and  $\mathbb{k}$ -linear, and such that the tensor product of morphisms is  $\mathbb{k}$ -bilinear.

The graphical calculus for strict  $\mathbb{k}$ -linear monoidal categories is the same as before, except that we allow  $\mathbb{k}$ -linear combinations of diagrams.

**Definition 2.5.2.** An *additive category* is a category in which

- every morphism space  $\text{Hom}_{\mathcal{C}}(X, Y)$  is an abelian group (written additively) such that composition of morphisms is biadditive,
- there exists a zero object  $0 \in \mathcal{C}$  such that:  $\text{Hom}_{\mathcal{C}}(0, 0) = 0$  and,
- for any objects  $X_1, X_2 \in \mathcal{C}$ , there exists a unique, up to isomorphism, object  $X_1 \oplus X_2 \in \mathcal{C}$ , and morphisms  $p_1: X_1 \oplus X_2 \rightarrow X_1, p_2: X_1 \oplus X_2 \rightarrow X_2, i_1: X_1 \rightarrow X_1 \oplus X_2, i_2: X_2 \rightarrow X_1 \oplus X_2$  such that  $p_1 i_1 = \text{id}_{X_1}, p_2 i_2 = \text{id}_{X_2}$ , and  $i_1 p_1 + i_2 p_2 = \text{id}_{X_1 \oplus X_2}$ .

Starting from a  $\mathbb{k}$ -linear category  $\mathcal{C}$ , one can embed it in an additive category, which can be defined as follows.

**Definition 2.5.3.** Let  $\mathcal{C}$  be a  $\mathbb{k}$ -linear category. The *additive envelope* or *completion* of  $\mathcal{C}$ , denoted  $\text{Add}(\mathcal{C})$ , is the category defined as follows:

- objects in  $\text{Add}(\mathcal{C})$  are finite direct sums

$$\bigoplus_{i \in I} X_i,$$

of objects in  $\mathcal{C}$  (including the empty sum),

- morphisms in  $f: \bigoplus_{i \in I} X_i \rightarrow \bigoplus_{j \in J} Y_j$  are  $|I| \times |J|$  matrices, where the  $(i, j)$ -entry is given by the morphism  $f_{i,j}: X_i \rightarrow Y_j$ . Composition of morphisms in  $\text{Add}(\mathcal{C})$  is given by matrix multiplication.

**Definition 2.5.4.** A  $\mathbb{k}$ -linear functor (monoidal)  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$  between two  $\mathbb{k}$ -linear (monoidal) categories  $\mathcal{C}$  and  $\mathcal{C}'$  is a functor such that the associated maps

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}'}(\mathcal{F}X, \mathcal{F}Y), \quad X, Y \in \mathcal{C},$$

are  $\mathbb{k}$ -linear.

**Definition 2.5.5.** Let  $\mathcal{C}$  be a  $\mathbb{k}$ -linear monoidal category. A *left tensor ideal*  $\mathcal{I}$  of  $\mathcal{C}$  is a collection of subspaces  $\mathcal{I}(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$  for all  $X, Y \in \mathcal{C}$  such that for all  $X, Y, Z, T \in \mathcal{C}$  the following holds:

- for all  $f \in \mathcal{I}(X, Y), g \in \text{Hom}_{\mathcal{C}}(Y, Z)$  and  $h \in \text{Hom}_{\mathcal{C}}(Z, X)$  we have  $g \circ f \in \mathcal{I}(X, Z)$  and  $f \circ h \in \mathcal{I}(Z, Y)$ ,
- for all  $f \in \mathcal{I}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{C}}(Z, T)$  we have  $f \otimes g \in \mathcal{I}(X \otimes Z, Y \otimes T)$ .

A *right tensor ideal* is defined in the obvious way. A *tensor ideal* is a left and right tensor ideal.

When  $\mathcal{I}$  is a left or right tensor ideal in a  $\mathbb{k}$ -linear monoidal category  $\mathcal{C}$ , one can define a  $\mathbb{k}$ -linear category  $\mathcal{C}/\mathcal{I}$  as the quotient of  $\mathcal{C}$  by  $\mathcal{I}$ , as follows: the objects of  $\mathcal{C}/\mathcal{I}$  are objects of  $\mathcal{C}$ , and  $\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y)$ . The composition of morphisms in  $\mathcal{C}/\mathcal{I}$  is the same as in  $\mathcal{C}$ . The category  $\mathcal{C}/\mathcal{I}$  is not monoidal in general if  $\mathcal{I}$  is only a left or right tensor ideal. However,  $\mathcal{C}/\mathcal{I}$  is monoidal if  $\mathcal{I}$  is a tensor ideal and, in this case, the tensor product in  $\mathcal{C}/\mathcal{I}$  is the same as in  $\mathcal{C}$ . We will see examples of tensor ideals and quotients in Section 8.2.

## 2.6 Trace decategorification

In this section we collect some important facts about traces of categories. We refer the reader to [3] for a more thorough treatment. Throughout this section  $\mathbb{k}$  denotes an arbitrary commutative ring.

**Definition 2.6.1.** The *trace* or *zeroth Hochschild homology* of a small  $\mathbb{k}$ -linear category  $\mathcal{C}$  is the  $\mathbb{k}$ -module

$$\mathrm{Tr}(\mathcal{C}) := \left( \bigoplus_{X \in \mathcal{C}} \mathrm{End}_{\mathcal{C}}(X) \right) / \mathrm{span}_{\mathbb{k}}\{f \circ g - g \circ f\}, \quad (2.6.1)$$

where  $f$  and  $g$  run through all pairs of morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  in  $\mathcal{C}$ . We let  $[f] \in \mathrm{Tr}(\mathcal{C})$  denote the class of an endomorphism  $f \in \mathrm{End}_{\mathcal{C}}(X)$ .

**Notation.** For  $f, g \in \bigoplus_{X \in \mathcal{C}} \mathrm{End}_{\mathcal{C}}(X)$ , we define

$$f \equiv g \iff [f] = [g]. \quad (2.6.2)$$

Thus, for example, we have

$$f \circ g \equiv g \circ f \quad \text{for all } f: X \rightarrow Y, g: Y \rightarrow X. \quad (2.6.3)$$

**Lemma 2.6.2.** *If  $\mathcal{C}$  is a  $\mathbb{k}$ -linear monoidal category, then  $\mathrm{Tr}(\mathcal{C})$  is an associative  $\mathbb{k}$ -algebra with multiplication given by*

$$[f][g] := [f \otimes g]. \quad (2.6.4)$$

A  $\mathbb{k}$ -linear functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  induces a linear map on traces:

$$\mathrm{Tr}(F): \mathrm{Tr}(\mathcal{C}) \rightarrow \mathrm{Tr}(\mathcal{D}), \quad [f] \mapsto [F(f)], \quad f \text{ an endomorphism in } \mathcal{C}. \quad (2.6.5)$$

*If  $\mathcal{C}, \mathcal{D}$  are monoidal categories and  $F$  is a monoidal functor, then (2.6.5) is a homomorphism of associative  $\mathbb{k}$ -algebras.*

**Proof:** This follows trivially from the definitions. ■

**Definition 2.6.3.** Let  $\{\mathcal{C}_i : i \in I\}$  be a family of small  $\mathbb{k}$ -linear categories. The disjoint union of categories  $\bigsqcup_{i \in I} \mathcal{C}_i$  is the category whose set of objects is  $\bigsqcup_{i \in I} \mathrm{Ob} \mathcal{C}_i$ , and whose spaces of morphisms are given by

$$\mathrm{Hom}_{\bigsqcup_{i \in I} \mathcal{C}_i}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}_i}(X, Y) & \text{if } X, Y \in \mathcal{C}_i, \\ \{0\} & \text{otherwise.} \end{cases}$$

**Lemma 2.6.4.** • If  $\mathcal{C}_i$ ,  $i \in I$ , are  $\mathbb{k}$ -linear categories, then we have a canonical isomorphism

$$\mathrm{Tr} \left( \bigsqcup_{i \in I} \mathcal{C}_i \right) \cong \bigoplus_{i \in I} \mathrm{Tr}(\mathcal{C}_i).$$

• If  $\mathcal{C}_1, \mathcal{C}_2$  are  $\mathbb{k}$ -linear subcategories of a  $\mathbb{k}$ -linear monoidal category  $\mathcal{C}$ , then the tensor product  $\otimes: \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}$  induces a linear map

$$\mathrm{Tr}(\mathcal{C}_1) \otimes \mathrm{Tr}(\mathcal{C}_2) \rightarrow \mathrm{Tr}(\mathcal{C}).$$

**Proof:** For the first statement, we have

$$\begin{aligned} \mathrm{Tr} \left( \bigsqcup_{i \in I} \mathcal{C}_i \right) &= \frac{\bigoplus_{i \in I} \bigoplus_{X \in \mathcal{C}_i} \mathrm{End}_{\mathcal{C}_i}(X)}{\bigoplus_{i \in I} \mathrm{span}_{\mathbb{k}} \{f \circ g - g \circ f \mid f, g \in \mathrm{Mor}(\mathcal{C}_i)\}} \\ &\cong \bigoplus_{i \in I} \mathrm{Tr}(\mathcal{C}_i), \end{aligned}$$

where the isomorphism is, explicitly, given by  $[\bigoplus_i f_i] \mapsto \bigoplus_i [f_i]$ , for  $f_i \in \mathrm{Mor}(\mathcal{C}_i)$ .

For the second statement, the induced linear map is given by  $[f] \otimes [g] \mapsto [f \otimes g]$ , where  $f \in \mathrm{Mor}(\mathcal{C}_1)$  and  $g \in \mathrm{Mor}(\mathcal{C}_2)$ . This is well defined as

$$\begin{aligned} (f_1 f_2 - f_2 f_1) \otimes g &= f_1 f_2 \otimes g - f_2 f_1 \otimes g \\ &\stackrel{(2.1.3)}{=} (f_1 \otimes 1) \circ (f_2 \otimes g) - (f_2 \otimes g) \circ (f_1 \otimes 1), \end{aligned}$$

and similarly

$$f \otimes (g_1 g_2 - g_2 g_1) = (f \otimes g_1) \circ (g_2 \otimes 1) - (g_2 \otimes 1) \circ (f \otimes g_1).$$

■

For a  $\mathbb{k}$ -linear category  $\mathcal{C}$ , let  $\mathrm{Add}(\mathcal{C})$  denote its additive envelope. If  $\mathcal{C}$  is monoidal, then  $\mathrm{Add}(\mathcal{C})$  inherits a natural monoidal structure.

**Lemma 2.6.5** ([3, Exercise 9]). *If  $\mathcal{C}$  is a  $\mathbb{k}$ -linear category, then the inclusion functor  $\mathcal{C} \rightarrow \mathrm{Add}(\mathcal{C})$  induces a linear isomorphism  $\mathrm{Tr}(\mathcal{C}) \cong \mathrm{Tr}(\mathrm{Add}(\mathcal{C}))$ . If  $\mathcal{C}$  is monoidal, then this is an isomorphism of associative  $\mathbb{k}$ -algebras.*

**Proof:** Objects of  $\mathrm{Add}(\mathcal{C})$  are formal direct sums  $\bigoplus_{i=1}^n X_i$ ,  $X_i \in \mathcal{C}$ . An endomorphism of such an object is a matrix  $(f_{ij})_{i,j=1}^n$  with  $f_{ij}: X_i \rightarrow X_j$ . For  $j = 1, \dots, n$ , we have canonical inclusion and projection maps

$$X_j \xrightarrow{\iota_j} \bigoplus_{i=1}^n X_i \xrightarrow{\pi_j} X_j.$$

Then we have

$$(f_{ij})_{i,j=1}^n = \sum_{i,j=1}^n \iota_j f_{ij} \pi_i \equiv \sum_{i,j=1}^n f_{ij} \pi_i \iota_j \equiv \sum_{i=1}^n f_{ii}.$$

Thus the map  $\mathrm{Tr}(\mathcal{C}) \rightarrow \mathrm{Tr}(\mathrm{Add}(\mathcal{C}))$  induced by the inclusion functor is surjective.

Similarly, for morphisms

$$\bigoplus_{i=1}^n X_i \xrightarrow{f=(f_{ij})} \bigoplus_{j=1}^m Y_j \xrightarrow{g=(g_{ji})} \bigoplus_{i=1}^n X_i$$

in  $\mathrm{Add}(\mathcal{C})$ , we have

$$fg - gf = \left( \sum_{i=1}^n f_{il} g_{ji} \right)_{j,l=1}^m - \left( \sum_{j=1}^m g_{jl} f_{ij} \right)_{i,l=1}^n \equiv \sum_{i=1}^n \sum_{j=1}^m (f_{ij} g_{ji} - g_{ji} f_{ij}).$$

Hence the map  $\mathrm{Tr}(\mathcal{C}) \rightarrow \mathrm{Tr}(\mathrm{Add}(\mathcal{C}))$  induced by the inclusion functor is also injective. As noted above, this map is a homomorphism of associative  $\mathbb{k}$ -algebras when  $\mathcal{C}$  is monoidal.  $\blacksquare$

If  $S \subseteq \mathrm{Ob}(\mathcal{C})$  is a subset of the set of objects of a small category  $\mathcal{C}$ , let  $\mathcal{C}|_S$  denote the full subcategory of  $\mathcal{C}$  with  $\mathrm{Ob}(\mathcal{C}|_S) = S$ .

**Lemma 2.6.6** ([5, Lem. 2.1]). *Suppose  $\mathcal{C}$  is a  $\mathbb{k}$ -linear additive category. Let  $S \subseteq \mathrm{Ob}(\mathcal{C})$  be a subset such that every object of  $\mathcal{C}$  is isomorphic to a direct sum of finitely many copies of objects in  $S$ . Then the inclusion functor  $\mathcal{C}|_S \rightarrow \mathcal{C}$  induces a linear isomorphism  $\mathrm{Tr}(\mathcal{C}|_S) \cong \mathrm{Tr}(\mathcal{C})$ .*

**Corollary 2.6.7.** *Suppose  $\mathcal{C}$  is a  $\mathbb{k}$ -linear (not necessarily additive) category. Let  $S \subseteq \mathrm{Ob}(\mathcal{C})$  be a subset such that every object of  $\mathcal{C}$  is isomorphic in  $\mathrm{Add}(\mathcal{C})$  to a direct sum of finitely many copies of objects in  $S$ . Then the inclusion functor  $\mathcal{C}|_S \rightarrow \mathcal{C}$  induces a linear isomorphism  $\mathrm{Tr}(\mathcal{C}|_S) \cong \mathrm{Tr}(\mathcal{C})$ .*

**Proof:** The inclusion functors  $\mathcal{C}|_S \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathrm{Add}(\mathcal{C})$  induce linear maps

$$\mathrm{Tr}(\mathcal{C}|_S) \xrightarrow{\mathrm{Tr}(F)} \mathrm{Tr}(\mathcal{C}) \xrightarrow{\mathrm{Tr}(G)} \mathrm{Tr}(\mathrm{Add}(\mathcal{C})).$$

The maps  $\mathrm{Tr}(G) \circ \mathrm{Tr}(F) = \mathrm{Tr}(G \circ F)$  and  $\mathrm{Tr}(G)$  are linear isomorphisms by Theorem 2.6.6 and Theorem 2.6.5, respectively. It follows that  $\mathrm{Tr}(F)$  is also a linear isomorphism.  $\blacksquare$

For the remainder of this section, we assume that  $\mathcal{C}$  is a small  $\mathbb{k}$ -linear additive category with

$$\text{Ob}(\mathcal{C}) = \{X_n : n \in \mathbb{N}\},$$

where  $\mathbb{N}$  denotes the set of nonnegative integers, and we assume that  $X_n \neq X_m$  for  $n \neq m$ . Furthermore, suppose that we have subsets  $D_{m,n} \subseteq \text{Hom}_{\mathcal{C}}(X_n, X_m)$ ,  $m, n \in \mathbb{N}$ , and  $D_n \subseteq \text{Hom}_{\mathcal{C}}(X_n, X_n)$ ,  $n \in \mathbb{N}$ , with the following properties:

(B1) We have  $D_{n,n} = \{1_{X_n}\}$  for all  $n \in \mathbb{N}$ , and  $\mathbf{B}_{m,n} := \bigsqcup_{l=0}^{\min(m,n)} D_{m,l} D_l D_{l,n}$  is a basis of  $\text{Hom}_{\mathcal{C}}(X_n, X_m)$  for each  $m, n \in \mathbb{N}$ . (Part of our assumption here is that the sets  $D_{m,l} D_l D_{l,n}$  are disjoint.)

(B2) For all  $n \in \mathbb{N}$ ,  $R_n := \text{span}_{\mathbb{k}} D_n$  is a subalgebra of  $\text{End}_{\mathcal{C}}(X_n)$ .

For  $n \in \mathbb{N}$ , let  $\mathcal{C}_n$  denote  $\mathbb{k}$ -linear subcategory of  $\mathcal{C}$  with one object  $X_n$  and  $\text{End}_{\mathcal{C}_n}(X_n) = R_n$ .

**Proposition 2.6.8.** *Under the above assumptions on  $\mathcal{C}$ , the inclusion  $\bigoplus_{n \in \mathbb{N}} \mathcal{C}_n \rightarrow \mathcal{C}$  induces a linear isomorphism*

$$\bigoplus_{n \in \mathbb{N}} \text{Tr}(\mathcal{C}_n) \xrightarrow{\cong} \text{Tr}(\mathcal{C}).$$

**Proof:** This is proved in [52, Prop. 2.11] using the equivalent language of locally unital algebras. ■

# Chapter 3

## Affinization of monoidal categories

Throughout this chapter, we will use the usual calculus of string diagrams for monoidal categories. We assume all categories are essentially small. For a category  $\mathcal{C}$ , we let  $\text{Ob}(\mathcal{C})$  denote its set of objects, and  $\text{Mor}(\mathcal{C})$  its set of morphisms. We let  $1_X$  denote the identity morphism of an object  $X$ . We use  $\mathbb{k}$  to denote a commutative ground ring. We let  $\mathbb{1}$  denote the unit object of a monoidal category. Throughout this chapter,  $\mathcal{C}$  denotes a strict monoidal category.

### 3.1 Affinization of monoidal categories

**Definition 3.1.1** (Affinization of a strict monoidal category). The *affinization* of a strict monoidal category  $\mathcal{C}$  is the category  $\text{Aff}(\mathcal{C})$  obtained from  $\mathcal{C}$  by adjoining invertible morphisms  $\xi_{X,Y}: X \otimes Y \rightarrow Y \otimes X$  for each pair of objects  $X, Y \in \text{Ob}(\mathcal{C})$ , subject to the relations

$$\xi_{X,Y \otimes Z} = \xi_{Z \otimes X, Y} \circ \xi_{X \otimes Y, Z}, \quad (3.1.1)$$

$$\xi_{X_2, Y_2} \circ (g \otimes f) = (f \otimes g) \circ \xi_{X_1, Y_1}, \quad (3.1.2)$$

for all  $X, X_1, X_2, Y, Y_1, Y_2, Z \in \text{Ob}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(Y_1, Y_2)$ , and  $g \in \text{Hom}_{\mathcal{C}}(X_1, X_2)$ . We call the  $\xi_{X,Y}$  *coils*. If  $\mathcal{C}$  is a  $\mathbb{k}$ -linear strict monoidal category, then  $\text{Aff}(\mathcal{C})$  is also naturally  $\mathbb{k}$ -linear.

It follows from (3.1.1) that  $\xi_{X, \mathbb{1}} = \xi_{X, \mathbb{1} \otimes \mathbb{1}} = \xi_{X, \mathbb{1}} \circ \xi_{X, \mathbb{1}}$  and so, since  $\xi_{X, \mathbb{1}}$  is invertible, we have

$$\xi_{X, \mathbb{1}} = 1_X \quad \text{for all } X \in \text{Ob}(\mathcal{C}). \quad (3.1.3)$$

We warn the reader that  $\xi_{\mathbb{1}, X}$  is different from  $\xi_{X, \mathbb{1}} = 1_X$ . We define  $\xi_X := \xi_{\mathbb{1}, X}$ . In terms of string diagrams, we should picture the morphisms of  $\text{Aff}(\mathcal{C})$  as string diagrams on a cylinder, where the coil  $\xi_{X,Y}$  corresponds to a strand labeled  $Y$  wrapping around

the cylinder:

$$\xi_{X,Y} = \text{cylinder with strands } X \rightarrow Y, \quad \xi_{X,Y}^{-1} = \text{cylinder with strands } Y \rightarrow X, \quad \xi_X = \text{cylinder with strand } X, \quad \xi_X^{-1} = \text{cylinder with strand } X. \quad (3.1.4)$$

**Remark 3.1.2.** Instead of considering diagrams on the cylinder, one can also consider a “pole” on the right-hand side of the diagrams. Then, rather than strands wrapping around the cylinder, they wrap around this pole. However, the monoidal structure to be discussed below is more intuitive from the cylindrical point of view.

**Remark 3.1.3.** The cylinder is topologically equivalent to the annulus, and, in the literature, one often sees “affine type” categories drawn in terms of string diagrams on the annulus. We choose here to use cylinders, since this allows us to draw the coils without drawing caps and cups (i.e. critical points with respect to the vertical coordinate). As we will recall below, cups and caps in string diagrams typically arise from dual objects. Thus, in categories that do not necessarily have duals, the cylindrical diagrammatics seem more natural. See Remark 5.1.5 for a similar situation.

In order to make cylindrical string diagrams easier to draw, we cut open the cylinder, drawing  $\xi_{X,Y}$  and  $\xi_{X,Y}^{-1}$  as the string diagrams

$$\xi_{X,Y} = \text{string diagram with strands } X \rightarrow Y, \quad \xi_{X,Y}^{-1} = \text{string diagram with strands } Y \rightarrow X.$$

where the dashed vertical edges are identified. Then the relations (3.1.1) and (3.1.2) become

$$\text{string diagram with } Y \otimes Z \text{ and } X \text{ strands} = \text{string diagram with } Y, Z \text{ and } X \text{ strands}, \quad \text{string diagram with } Y_2, X_2 \text{ and } X_1, Y_1 \text{ strands} = \text{string diagram with } Y_2, X_2 \text{ and } X_1, Y_1 \text{ strands}. \quad (3.1.5)$$

Analogous relations also hold for the inverses  $\xi_{X,Y}^{-1}$ . Intuitively, we can slide morphisms in  $\mathcal{C}$  around the cylinder.

**Remark 3.1.4** (Dehn twist). It follows from (3.1.2) that

$$\xi_Y \circ f = f \circ \xi_X, \quad f \in \text{Hom}_{\mathcal{C}}(X, Y). \quad (3.1.6)$$

In other words,  $(\xi_X)_{X \in \text{Ob}(\mathcal{C})}$  is a natural transformation of the identity functor  $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ . In terms of string diagrams, this corresponds to a *Dehn twist* of the cylinder.

We now wish to endow  $\text{Aff}(\mathcal{C})$  with the structure of strict monoidal category. Intuitively, viewing morphisms  $f, g$  in  $\text{Aff}(\mathcal{C})$  as diagrams on the cylinder, the tensor product  $f \otimes g$  is given by nesting the cylindrical diagram corresponding to  $g$  inside the cylindrical diagram corresponding to  $f$ . In order for this to make sense, we need  $\mathcal{C}$  to be a *braided* strict monoidal category, so that we can use the braiding to formalize what it means for strands in one diagram to pass over strands in another diagram.

**Proposition 3.1.5.** *If  $\mathcal{C}$  is a braided strict monoidal category, then there is a unique way to extend the tensor product of  $\mathcal{C}$  to  $\text{Aff}(\mathcal{C})$  such that  $\text{Aff}(\mathcal{C})$  is a strict monoidal category and*

$$\xi_{X,Y} \otimes 1_Z = \xi_{X \otimes Z, Y} \circ (1_X \otimes \beta_{Y,Z}), \quad 1_X \otimes \xi_{Y,Z} = (\beta_{X,Z}^{-1} \otimes 1_Y) \circ \xi_{X \otimes Y, Z}, \quad (3.1.7)$$

for all  $X, Y, Z \in \text{Ob}(\mathcal{C})$ .

**Proof:** Any tensor product on  $\text{Aff}(\mathcal{C})$  must satisfy

$$(g \circ f) \otimes 1_W = (g \otimes 1_W) \circ (f \otimes 1_W) \quad (3.1.8)$$

for all morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  in  $\text{Aff}(\mathcal{C})$  and  $W \in \text{Ob}(\mathcal{C})$ . Since

- $f \otimes 1_W$  must be the tensor product in  $\mathcal{C}$  if  $f$  is a morphism in  $\mathcal{C}$ , and is given by (3.1.7) for  $f$  a coil, and
- morphisms in  $\mathcal{C}$ , together with coils, generate the morphisms in  $\text{Aff}(\mathcal{C})$  under composition,

there is a unique way to define  $f \otimes 1_W$  for a morphism  $f$  in  $\text{Aff}(\mathcal{C})$ . Similarly, there is a unique way to define  $1_W \otimes f$ . Then, for morphisms  $f: X \rightarrow Y$  and  $g: Z \rightarrow W$  in  $\text{Aff}(\mathcal{C})$ , we must have

$$g \otimes f = (g \otimes 1_Y) \circ (1_Z \otimes f).$$

This proves the uniqueness statement in the proposition. It is then a straightforward verification to check that the tensor product, extended to  $\text{Aff}(\mathcal{C})$  as above, endows  $\text{Aff}(\mathcal{C})$  with the structure of a strict monoidal category.  $\blacksquare$

Diagrammatically, the equations in (3.1.7) become

$$\begin{array}{c} Y \quad X \\ \text{---} \text{---} \\ \text{---} \text{---} \\ X \quad Y \end{array} \otimes \begin{array}{c} | \\ | \\ | \\ Z \end{array} = \begin{array}{c} Y \quad X \\ \text{---} \text{---} \\ \text{---} \text{---} \\ X \quad Y \quad Z \end{array}, \quad \begin{array}{c} | \\ | \\ | \\ X \end{array} \otimes \begin{array}{c} Z \quad Y \\ \text{---} \text{---} \\ \text{---} \text{---} \\ Y \quad Z \end{array} = \begin{array}{c} Z \quad Y \\ \text{---} \text{---} \\ \text{---} \text{---} \\ X \quad Y \quad Z \end{array}. \quad (3.1.9)$$

Note that the strands of the left-hand diagram in the tensor product pass over those of the right-hand diagram.

For the remainder of this section, we assume  $\mathcal{C}$  is braided, and we view  $\text{Aff}(\mathcal{C})$  as a strict monoidal category with the tensor product of Proposition 3.1.5. Note that, in general,  $\text{Aff}(\mathcal{C})$  is no longer braided. (See, for example, (3.1.11) below.)

We now introduce some diagrammatic shorthand that allows us to dispose of cylindrical diagrams. We use a *positive dot* on a strand to denote the morphism  $\xi_X$  and a *negative dot* to denote  $\xi_X^{-1}$ :

$$\begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} := \xi_X = \begin{array}{c} X \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \ominus \\ \text{⋮} \end{array} := \xi_X^{-1} = \begin{array}{c} X \\ \text{⋮} \end{array}.$$

It then follows from our definition of the tensor product on  $\text{Aff}(\mathcal{C})$  that

$$\left| \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} \right| = \left| \begin{array}{c} \text{⋮} \\ \text{⋮} \end{array} \right|, \quad \left| \begin{array}{c} \text{⋮} \\ \ominus \\ \text{⋮} \end{array} \right| = \left| \begin{array}{c} \text{⋮} \\ \text{⋮} \end{array} \right|, \quad (3.1.10)$$

where there can be any number of strands to the left and right of the dots.

Using (2.3.4), (3.1.5), and (3.1.10), we have

$$\begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array}, \quad (3.1.11)$$

$$\begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \otimes \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array}, \quad \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array}, \quad (3.1.12)$$

for all  $X, Y \in \text{Ob}(\mathcal{C})$  and  $f \in \text{Mor}(\mathcal{C})$ . Note that the dots do indeed change the crossings in (3.1.11) and (3.1.12). Note also that the relations in (3.1.12) follow immediately from those in (3.1.11), together with the fact that the negative dot is inverse to the positive dot. The third and fourth relations in (3.1.11) are equivalent to the assertion that the collection  $(\xi_X)_{X \in \text{Ob}(\mathcal{C})}$  is a monoidal natural automorphism of the identity functor on  $\mathcal{C}$ ; see Remark 3.1.4.

The following result shows, in particular, that the positive and negative dots, together with the morphisms of  $\mathcal{C}$ , generate all morphisms of  $\text{Aff}(\mathcal{C})$  under composition and tensor product.

**Theorem 3.1.6.** *The affinization  $\text{Aff}(\mathcal{C})$  of a braided strict monoidal category  $\mathcal{C}$  is isomorphic to the strict monoidal category obtained from  $\mathcal{C}$  by adjoining invertible morphisms*

$$\xi_X = \begin{array}{c} \text{⋮} \\ \oplus \\ \text{⋮} \end{array} : X \rightarrow X$$

for all objects  $X \in \text{Ob}(\mathcal{C})$ , subject to the first, third, and fourth relations in (3.1.11).



**Proposition 3.1.7.** *Any strict monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between strict monoidal categories induces a functor  $\text{Aff}(F): \text{Aff}(\mathcal{C}) \rightarrow \text{Aff}(\mathcal{D})$  by defining*

$$\begin{aligned} \text{Aff}(F)(X) &= F(X), \quad X \in \text{Ob}(\mathcal{C}), & \text{Aff}(F)(f) &= F(f), \quad f \in \text{Mor}(\mathcal{C}), \\ \text{Aff}(F)(\xi_{X,Y}) &:= \xi_{F(X),F(Y)}, \quad X, Y \in \text{Ob}(\mathcal{C}). \end{aligned}$$

*If  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{E}$  are strict monoidal functors between strict monoidal categories, we have  $\text{Aff}(G \circ F) = \text{Aff}(G) \circ \text{Aff}(F)$ .*

**Proof:** To verify that  $\text{Aff}(F)$  is a well-defined functor, it suffices to verify that  $\text{Aff}(F)$  preserves the relations (3.1.1) and (3.1.2), which follows immediately from the fact that  $F$  is a strict monoidal functor. For example, for (3.1.1), we have

$$\begin{aligned} \text{Aff}(F)(\xi_{X,Y \otimes Z}) &= \xi_{F(X),F(Y) \otimes F(Z)} \\ &= \xi_{F(Z) \otimes F(X),F(Y)} \circ \xi_{F(X) \otimes F(Y),F(Z)} = \text{Aff}(F)(\xi_{Z \otimes X,Y}) \circ \text{Aff}(F)(\xi_{X \otimes Y,Z}). \end{aligned}$$

The final statement of the proposition also follows immediately from the definitions. ■

Recall that a braided monoidal functor between braided strict monoidal categories is a monoidal functor that respects the braiding (see Definition 2.3.2). The proof of the following result is a straightforward exercise.

**Proposition 3.1.8.** *If  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a braided strict monoidal functor between braided strict monoidal categories, then  $\text{Aff}(F)$  is a strict monoidal functor.*

**Remark 3.1.9.** We expect that the concept of the affinization of a monoidal category can be extended to the setting of monoidal categories which are not necessarily strict. We have chosen to focus on strict monoidal categories since it significantly simplifies the exposition and all of the examples and applications that we have in mind are strict monoidal categories. Furthermore, by the Mac Lane coherence theorem, every monoidal category is monoidally equivalent to a strict one.

## 3.2 Actions

Recall that, for any category  $\mathcal{M}$ , the category  $\mathcal{E}nd(\mathcal{M})$  of endofunctors and natural transformations is a strict monoidal category.

**Definition 3.2.1.** An *action* of a strict monoidal category  $\mathcal{C}$  on a monoidal category  $\mathcal{M}$  is a monoidal functor  $A: \mathcal{C} \rightarrow \mathcal{E}nd(\mathcal{M})$ . We adopt the notation  $X \cdot M = A(X)(M)$  for  $X \in \text{Ob}(\mathcal{C})$  and  $M \in \text{Ob}(\mathcal{M})$ .

If  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor, then  $\mathcal{C}$  acts on  $\mathcal{M}$  via the action

$$X \cdot M = F(X) \otimes M, \quad f \cdot g = F(f) \otimes g, \quad (3.2.1)$$

for  $X \in \text{Ob}(\mathcal{C})$ ,  $M \in \text{Ob}(\mathcal{M})$ ,  $f \in \text{Mor}(\mathcal{C})$ ,  $g \in \text{Mor}(\mathcal{M})$ . The goal of this section is to extend this action to the affinization  $\text{Aff}(\mathcal{C})$ . Intuitively, this action corresponds to placing string diagrams from  $\mathcal{M}$  inside the cylinder corresponding to diagrams from  $\text{Aff}(\mathcal{C})$ , then using the functor  $F$  to interpret this as a diagram in  $\mathcal{M}$ . In order for this action to be well defined, we need to make the additional assumption that  $\mathcal{C}$  is a balanced strict monoidal category (see Definition 2.4.10).

**Remark 3.2.2.** In most of the examples to be considered in this thesis, the objects of the category  $\mathcal{C}$  will be freely generated by some set of generating objects. In this case, any twist on  $\mathcal{C}$  is uniquely determined by the twists of the generating objects and (2.4.5). Furthermore, the category  $\mathcal{C}$  will often be a braided strict pivotal category, in which case we have a twist given by a ‘‘curl’’; see (2.4.7).

The following result extends the action of (3.2.1) to the affinization  $\text{Aff}(\mathcal{C})$ .

**Theorem 3.2.3.** *Suppose  $\mathcal{C}$  is a braided strict monoidal category,  $\mathcal{M}$  is a balanced strict monoidal category, and  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a braided monoidal functor. Then there is an action of  $\text{Aff}(\mathcal{C})$  on  $\mathcal{M}$  uniquely determined by*

$$X \cdot M := F(X) \otimes M, \quad f \cdot g := F(f) \otimes g, \quad (3.2.2)$$

$$\xi_X \cdot g := \beta_{N, F(X)} \circ (g \otimes \theta_{F(X)}) \circ \beta_{F(X), M}, \quad \xi_X^{-1} \cdot g := \beta_{F(X), N}^{-1} \circ (g \otimes \theta_{F(X)}^{-1}) \circ \beta_{M, F(X)}^{-1}, \quad (3.2.3)$$

for all  $X \in \text{Ob}(\mathcal{C}) = \text{Ob}(\text{Aff}(\mathcal{C}))$ ,  $f \in \text{Mor}(\mathcal{C})$ ,  $M, N \in \text{Ob}(\mathcal{M})$ ,  $g \in \text{Hom}_{\mathcal{M}}(M, N)$ .

In terms of string diagrams, (3.2.3) becomes

$$\xi_X \cdot g = \begin{array}{c} N \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \end{array} \theta, \quad \xi_X^{-1} \cdot g = \begin{array}{c} N \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \end{array} \theta^{-1}.$$

**Proof:** Let  $W, X, Y, Z$  be objects, and let  $f: Y \rightarrow Z$ ,  $f': Z \rightarrow W$ ,  $g$ , and  $g'$  be morphisms in  $\mathcal{C}$ . It is straightforward to verify that

$$\begin{aligned} (\xi_X \cdot g) \circ (\xi_X^{-1} \cdot g') &= 1_X \cdot (g \circ g') = (\xi_X^{-1} \cdot g) \circ (\xi_X \cdot g') \quad \text{and} \\ (f \cdot g) \circ (f' \cdot g') &= (f \circ f') \cdot (g \circ g') \end{aligned}$$

for all objects  $X$  in  $\mathcal{C}$ , morphisms  $f, f'$  in  $\mathcal{C}$ , and morphisms  $g, g'$  in  $\mathcal{M}$  such that the above compositions are defined.

It remains to show that (3.2.2) and (3.2.3) respect the first, third, and fourth relations in (3.1.11). Since  $f \cdot g = (f \cdot 1) \circ (1 \cdot g)$  for  $f \in \text{Mor}(\text{Aff}(\mathcal{C}))$  and  $g \in \text{Mor}(\mathcal{M})$ , it suffices to consider the action on identity morphisms. For the first relation in (3.1.11), we compute

$$\text{⊗} \cdot 1_M = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ M \end{array} = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ M \end{array} = \text{⊗} \cdot 1_M.$$

For the third relation, setting  $X' = F(X)$  and  $Y' = F(Y)$ , we have

$$\left( \begin{array}{c} \oplus \\ X \quad Y \end{array} \right) \cdot 1_Z = \begin{array}{c} Z \\ \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} = \begin{array}{c} Z \\ \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \quad Y' \end{array} \stackrel{(2.4.5)}{=} \begin{array}{c} Z \\ \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X' \otimes Y' \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ X \otimes Y \end{array} = \begin{array}{c} \oplus \\ X \otimes Y \end{array} \cdot 1_Z.$$

The fourth relation in (3.1.11) is straightforward to verify using (2.4.6). ■

The following result relates affinization of a monoidal category to the representation theoretic approach to affinization mentioned in the introduction.

**Corollary 3.2.4.** *Suppose  $\mathcal{C}$  is a braided strict monoidal category,  $\mathcal{M}$  is a balanced strict monoidal category, and  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a braided monoidal functor. Let  $A: \mathcal{C} \rightarrow \text{End}(\mathcal{M})$  be the action functor of (3.2.1).*

- (a) *For each  $X \in \text{Ob}(\mathcal{C})$ , the collection  $\Xi_X := (\xi_X \cdot 1_M)_{M \in \text{Ob}(\mathcal{M})}$  is an endomorphism of the functor  $A(X): \mathcal{M} \rightarrow \mathcal{M}$  (i.e. a natural transformation from  $A(X)$  to itself).*
- (b) *The collection  $(\Xi_X)_{X \in \text{Ob}(\mathcal{C})}$  is an endomorphism of  $A$ .*

**Proof:**

- (a) For a morphism  $g: M \rightarrow N$  in  $\mathcal{M}$ , we have

$$(A(X)(g)) \circ (\xi_X \cdot 1_M) = \begin{array}{c} N \\ \text{---} \\ \text{⊗} \\ \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} = \begin{array}{c} N \\ \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \\ F(X) \quad M \end{array} = (\xi_X \cdot 1_N) \circ (A(X)(g)).$$

Hence  $\Xi_X$  is a natural transformation from  $A(X)$  to itself.

(b) This follows from Theorem 3.2.3 and the last relation in (3.1.11). ■

For the remainder of this section, suppose that  $\mathcal{C}$  is a balanced strict monoidal category. Taking  $\mathcal{M} = \mathcal{C}$  and  $F = \text{id}_{\mathcal{C}}$  to be the identity functor, Theorem 3.2.3 implies that  $\text{Aff}(\mathcal{C})$  acts on  $\mathcal{C}$ . This extends the natural action of  $\mathcal{C}$  on itself given by the tensor product. Considering the action on the unit object and its identity morphism, it then follows that we have a functor

$$\text{Aff}(\mathcal{C}) \rightarrow \mathcal{C}, \quad X \mapsto X, \quad f \mapsto f \cdot 1_{\mathbb{1}}, \quad X \in \text{Ob}(\text{Aff}(\mathcal{C})), \quad f \in \text{Mor}(\text{Aff}(\mathcal{C})). \quad (3.2.4)$$

Note that, under the functor (3.2.4), we have

$$\xi_X \mapsto \theta_X, \quad \xi_{X,Y} \mapsto \beta_{X,Y} \circ (1_X \otimes \theta_Y),$$

where we use (3.1.13).

**Corollary 3.2.5.** *If  $\mathcal{C}$  is a balanced strict monoidal category, then the functor  $\mathcal{C} \rightarrow \text{Aff}(\mathcal{C})$  that is the identity on objects and sends  $f \in \text{Mor}(\mathcal{C})$  to the corresponding morphism in  $\text{Aff}(\mathcal{C})$  is faithful.*

**Proof:** It is straightforward to verify that the composite  $\mathcal{C} \rightarrow \text{Aff}(\mathcal{C}) \xrightarrow{(3.2.4)} \mathcal{C}$  is the identity functor, which implies the result. ■

In light of Corollary 3.2.5, we will view  $\mathcal{C}$  as a subcategory of  $\text{Aff}(\mathcal{C})$ . We expect that the natural functor  $\mathcal{C} \rightarrow \text{Aff}(\mathcal{C})$  is faithful even when  $\mathcal{C}$  is not balanced.

**Definition 3.2.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $\mathbb{k}$ -linear categories, then  $\mathcal{C} \boxtimes \mathcal{D}$  is the  $\mathbb{k}$ -linear category whose objects are pairs  $(X, Y)$  with  $X \in \text{Ob}(\mathcal{C})$  and  $Y \in \text{Ob}(\mathcal{D})$ . Morphisms are given by

$$\text{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}((X_1, Y_1), (X_2, Y_2)) = \text{Hom}_{\mathcal{C}}(X_1, X_2) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{D}}(Y_1, Y_2).$$

Composition is componentwise on simple tensors, and extended by linearity.

**Definition 3.2.7.** If  $\mathcal{C}$  is a strict  $\mathbb{k}$ -linear monoidal category, then a *module category* over  $\mathcal{C}$  is a  $\mathbb{k}$ -linear category  $\mathcal{M}$ , together with a  $\mathbb{k}$ -linear monoidal functor  $\mathcal{C} \rightarrow \mathcal{E}nd_{\mathbb{k}}(\mathcal{M})$ , where  $\mathcal{E}nd_{\mathbb{k}}(\mathcal{M})$  denotes the strict  $\mathbb{k}$ -linear monoidal category with objects that are the  $\mathbb{k}$ -linear endofunctors of  $\mathcal{M}$  and morphisms that are natural transformations.

**Remark 3.2.8.** A  $\mathbb{k}$ -module category  $\mathcal{M}$  over a strict  $\mathbb{k}$ -linear monoidal category  $\mathcal{C}$  is the same as a  $\mathbb{k}$ -linear functor  $-\otimes -: \mathcal{C} \boxtimes \mathcal{M} \rightarrow \mathcal{M}$  satisfying associativity and unity axioms. All of the results of the current section go through in this linear setting with the obvious modifications.

### 3.3 Pivotal structures

The following result is straightforward.

**Lemma 3.3.1.** *If  $X$  is right dual (resp. left dual) to  $Y$  in  $\mathcal{C}$ , then the same is true in  $\text{Aff}(\mathcal{C})$ , using the same units and counits. In particular, if  $\mathcal{C}$  is left rigid (resp. right rigid, rigid) then so is  $\text{Aff}(\mathcal{C})$ .*

**Theorem 3.3.2.** *If  $\mathcal{C}$  is a braided strict pivotal category, then the same units and counits endow  $\text{Aff}(\mathcal{C})$  with the structure of a strict pivotal category. Furthermore, we have*

$$\xi_X^\vee = \xi_{X^\vee}^{-1}, \quad X \in \text{Ob}(\mathcal{C}). \quad (3.3.1)$$

**Proof:** The left and right mates of the morphisms of  $\mathcal{C}$  are equal since  $\mathcal{C}$  is strict pivotal. Thus, to show that  $\text{Aff}(\mathcal{C})$  is strict pivotal, it suffices to show that the left and right mates of the positive dots are equal. (It then automatically follows that the left and right mates of the negative dots are equal, since they are inverses to the positive dots.) So it is enough to show that

$$\begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ \ominus \\ \downarrow \\ X \end{array}. \quad (3.3.2)$$

To illustrate the two viewpoints, topological and algebraic, we give two proofs of these identities.

For a topological proof, we compute

$$\begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ \ominus \\ \downarrow \\ X \end{array},$$

$$\begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} \stackrel{(3.1.5)}{=} \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ \ominus \\ \downarrow \\ X \end{array},$$

where, in the last equality, we used the fact that the two curls appearing in the penultimate diagram are inverses of each other (see (4.1.4)).

For an algebraic proof of (3.3.2), we first note that by attaching appropriate cups and caps and using (2.4.2) and (2.4.4), the identities (3.3.2) are equivalent to

$$\begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array}, \quad \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array}. \quad (3.3.3)$$

We prove the first identity, since the proof of the second is analogous. Adding a positive dot to the bottom-right strand, we see that it suffices to prove

$$\begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \text{⌋} \\ \oplus \\ \downarrow \\ X \end{array}.$$

By the third relation in (3.1.11), the left-hand side is the composite of a single positive dot on a strand labelled  $X \otimes X^\vee$  and the counit (the right cap). Then, by the fourth relation in (3.1.11), we can slide this positive dot above the counit. Since  $\xi_1 = 1_1$ , this gives the right-hand side above.  $\blacksquare$

If  $\mathcal{C}$  is a braided strict pivotal category and  $X \in \text{Ob}(\mathcal{C})$ , then we define the invertible *dots*

$$\begin{array}{c} \uparrow \\ \circlearrowleft \\ X \end{array} := \begin{array}{c} \uparrow \\ \oplus \\ X \end{array}, \quad \begin{array}{c} \downarrow \\ \circlearrowright \\ X \end{array} := \begin{array}{c} \downarrow \\ \ominus \\ X \end{array}. \quad (3.3.4)$$

It then follows from (3.3.2) that dots slide over cups and caps:

$$\begin{array}{c} \uparrow \\ \circlearrowleft \\ \cap \end{array} = \begin{array}{c} \cap \\ \circlearrowleft \end{array}, \quad \begin{array}{c} \downarrow \\ \circlearrowright \\ \cup \end{array} = \begin{array}{c} \cup \\ \circlearrowright \end{array}, \quad \begin{array}{c} \downarrow \\ \circlearrowleft \\ \cap \end{array} = \begin{array}{c} \cap \\ \circlearrowleft \end{array}, \quad \begin{array}{c} \uparrow \\ \circlearrowright \\ \cup \end{array} = \begin{array}{c} \cup \\ \circlearrowright \end{array}. \quad (3.3.5)$$

If  $X$  is self-dual, so that  $X = X^\vee$ , the above convention for open dots is still valid, but not very useful in practice. As noted in Remark 2.4.6, one typically denotes the identity of a self-dual object by an unoriented strand, in which case we cannot introduce the open dot as above. Instead, we have

$$\begin{array}{c} \oplus \\ \cap \end{array} = \begin{array}{c} \cap \\ \oplus \end{array}, \quad \begin{array}{c} \oplus \\ \cup \end{array} = \begin{array}{c} \cup \\ \oplus \end{array}, \quad \begin{array}{c} \ominus \\ \cap \end{array} = \begin{array}{c} \cap \\ \ominus \end{array}, \quad \begin{array}{c} \ominus \\ \cup \end{array} = \begin{array}{c} \cup \\ \ominus \end{array}. \quad (3.3.6)$$

**Remark 3.3.3.** Many of our categories will have generating morphisms given by braidings for the generating objects and possibly cups and caps coming from dual objects. We will want to give a presentation for the affinization of such a category  $\mathcal{C}$ , following Theorem 3.1.6, by adjoining the  $\xi_X$  for  $X$  ranging over the set of generating objects. Then  $\xi_X$  is defined on all objects using the third relation in (3.1.11). We impose the first relation in (3.1.11) (and then the second relation follows) and then it remains to impose the fourth relation in (3.1.11) with the coupon there ranging over the generating morphisms of  $\mathcal{C}$ . When the coupon is a braiding on generating objects  $X$  and  $Y$ , we have, using the first two relations in (3.1.11)

$$\begin{array}{c} \diagdown \quad \diagup \\ \oplus \quad \oplus \\ X \quad Y \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \oplus \quad \oplus \\ X \quad Y \end{array} = \begin{array}{c} \oplus \quad \oplus \\ \diagdown \quad \diagup \\ X \quad Y \end{array}.$$

So the fourth relation in (3.1.11) is automatically satisfied when the coupon is a braiding. If the coupon is a right cap, then, as explained in the proof of Theorem 3.3.2, the fourth relation in (3.1.11) is equivalent to

$$\begin{array}{c} \oplus \\ \downarrow \\ X \end{array} = \begin{array}{c} \cap \\ \oplus \\ X \end{array},$$

and similarly for left caps. Thus, it suffices to impose the cap-slide relations in (3.3.5) (for non-self-dual objects) or (3.3.6) (for self-dual objects); the corresponding cup relations then follow using (2.4.2) and (2.4.4).

### 3.4 Affinization of 2-categories

In this final section, we briefly describe how the concept of affinization can be generalized to the setting of 2-categories. In a 2-category, we will use juxtaposition to denote horizontal composition of 1-morphisms and 2-morphisms, and we will use  $\circ$  to denote vertical composition.

**Definition 3.4.1** (Affinization of a 2-category). The *affinization*  $\text{Aff}(\mathfrak{C})$  of a 2-category  $\mathfrak{C}$  is the category defined as follows. Objects of  $\text{Aff}(\mathfrak{C})$  are 1-endomorphisms  $f: x \rightarrow x$  in  $\mathfrak{C}$ . The morphisms of  $\text{Aff}(\mathfrak{C})$  are obtained from the class of 2-morphisms  $\sigma: f \rightarrow g$  in  $\mathfrak{C}$  between 1-endomorphisms by adjoining invertible 2-morphisms

$$\xi_{f,g}: fg \rightarrow gf, \quad f: y \rightarrow x, \quad g: x \rightarrow y, \quad x, y \in \text{Ob}(\mathfrak{C}),$$

and imposing the relations

$$\xi_{f,gh} = \xi_{hf,g} \circ \xi_{f,g,h}, \tag{3.4.1}$$

$$\xi_{f_2,g_2} \circ (\tau\sigma) = (\sigma\tau) \circ \xi_{f_1,g_1}, \tag{3.4.2}$$

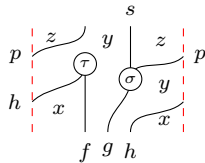
for all  $f: z \rightarrow x, g: y \rightarrow z, h: x \rightarrow y, x, y, z \in \text{Ob}(\mathfrak{C})$ , and all  $\sigma: g_1 \rightarrow g_2, \tau: f_1 \rightarrow f_2$  for  $g_1, g_2: x \rightarrow y, f_1, f_2: y \rightarrow x, x, y \in \text{Ob}(\mathfrak{C})$ .

The reader should compare Definition 3.4.1 to Definition 3.1.1. In particular, a strict monoidal category is the same as a 2-category with one object. In this case, Definitions 3.1.1 and 3.4.1 coincide.

As for the affinization of strict monoidal categories, morphisms in  $\text{Aff}(\mathfrak{C})$  can be naturally interpreted as diagrams on the cylinder. Cutting open the cylinder as in Section 3.1, we draw  $\xi_{f,g}$  and  $\xi_{f,g}^{-1}$  for  $f: y \rightarrow x, g: x \rightarrow y$  as the string diagrams

$$\xi_{f,g} = \begin{array}{c} \begin{array}{cc} g & f \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ f & g \end{array} \end{array}, \quad \xi_{f,g}^{-1} = \begin{array}{c} \begin{array}{cc} f & g \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ g & f \end{array} \end{array}.$$

Regions of the cylinder are labeled by objects of  $\mathfrak{C}$ , strings, which are allowed to wrap around the cylinder, are labeled by 1-morphisms in  $\mathfrak{C}$  and these strings can carry coupons labeled by 2-morphisms in  $\mathfrak{C}$ . For example,



is a 2-morphism in  $\text{Aff}(\mathfrak{C})$  for  $x, y, z \in \text{Ob}(\mathfrak{C})$ ,  $f: y \rightarrow x$ ,  $g: y \rightarrow y$ ,  $h: x \rightarrow y$ ,  $\sigma: g \rightarrow sp$ , and  $\tau: hf \rightarrow 1_y$ .

The horizontal trace of a 2-category is defined in [4, §2.4]. The discussion of Section 5.1 has a straightforward generalization to the setting of 2-categories. In particular, the horizontal trace of a 2-category  $\mathfrak{C}$  is isomorphic to  $\text{Aff}(\mathfrak{C})$  if  $\mathfrak{C}$  is rigid, but the two concepts are different in general. (Since a strict monoidal category can be considered as a 2-category with one object, Example 5.1.7 gives an example illustrating the difference.) For an arbitrary 2-category, it is the affinization that naturally corresponds to the category of  $\mathfrak{C}$ -diagrams on the cylinder.

# Chapter 4

## Affine presentation of certain categories

In this chapter, we apply our affinization procedure to a large number of diagrammatic categories appearing in the literature. In all the examples, we give two descriptions of the affine categories, one in terms of string diagrams on the cylinder and the other in terms of string diagrams carrying dots.

### 4.1 Tangles

Many of the examples to be discussed in the sequel will be constructed from categories of tangles. In this section, we fix our conventions and recall some concepts that will be common to many of these examples.

Let

$$D = (0, 1)^2 \quad \text{and} \quad A = S^1 \times (0, 1)$$

be the disc and the annulus, respectively. In order to make various categories of tangles strict, we need to fix a countable number of points in  $D$ , which will be the possible endpoints of our tangles. We choose the points

$$P_n = \left(1 - \frac{1}{2^n}, \frac{1}{2}\right), \quad n \in \mathbb{Z}_{>0}. \quad (4.1.1)$$

We make the identification

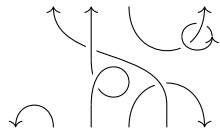
$$A = ([0, 1] \times (0, 1)) / \sim, \quad (4.1.2)$$

where  $\sim$  is the relation given by  $(0, b) \sim (1, b)$  for all  $b \in (0, 1)$ , and we also view the  $P_n$  as points in  $A$ . Up to isomorphism, our categories will not depend on the particular choice of points. We will typically draw them as equally spaced, or adjust the spacing to the particular tangle we draw.

We let  $\mathcal{FOT}(D)$  be the category of framed oriented tangles over  $D$ . Its objects are finite sequences  $(\varepsilon_1, \dots, \varepsilon_n)$  of elements of  $\{\uparrow, \downarrow\}$ . The unit object  $\mathbb{1}$  is the empty sequence. Morphisms in  $\mathcal{FOT}(D)$  from  $(\varepsilon_1, \dots, \varepsilon_m)$  to  $(\varepsilon'_1, \dots, \varepsilon'_n)$  are framed oriented tangles in  $D \times [0, 1]$ , up to ambient isotopy, with endpoints

$$(\{P_1, \dots, P_m\} \times \{0\}) \cup (\{P_1, \dots, P_n\} \times \{1\})$$

such that the orientation of the tangle at each  $P_i \times \{0\}$  agrees with  $\varepsilon_i$ , the orientation at each  $P'_i \times \{1\}$  agrees with  $\varepsilon'_i$ , and the framing at the point  $P_i \times \{0\}$  (respectively,  $P_i \times \{1\}$ ) points towards  $P_{i+1} \times \{0\}$  (respectively,  $P_{i+1} \times \{1\}$ ). We allow tangles to have closed components. For example,



$$\in \text{Hom}_{\mathcal{FOT}(D)}((\downarrow, \uparrow, \uparrow, \uparrow, \uparrow, \downarrow), (\uparrow, \uparrow, \downarrow, \uparrow)),$$

where we adopt the convention of blackboard framing (i.e. the framing is parallel to the page). The composite  $f \circ g$  is given by placing  $f$  above  $g$  and rescaling the vertical coordinate. The category  $\mathcal{FOT}(D)$  is a strict monoidal category, with tensor product  $f \otimes g$  given by placing the  $f$  to the left of  $g$  and rescaling. Objects can be written as (possibly empty) tensor products of the objects  $\uparrow$  and  $\downarrow$ .

We let  $\mathcal{OT}(D)$  denote the strict monoidal category of oriented tangles over  $D$ . This is defined as above, but without the framing. Similarly, forgetting orientations, we let  $\mathcal{FT}(D)$  and  $\mathcal{T}(D)$  denote the strict monoidal categories of framed tangles over  $D$  and tangles over  $D$ , respectively. Here the objects are natural numbers, since we have no orientations of the endpoints.

Replacing the disc  $D$  by the annulus  $A$ , we obtain the categories  $\mathcal{FOT}(A)$ ,  $\mathcal{OT}(A)$ ,  $\mathcal{FT}(A)$ , and  $\mathcal{T}(A)$  of framed oriented tangles, oriented tangles, framed tangles, and tangles over the annulus. We draw these by cutting along  $\{(0, b, c) : b \in (0, 1), c \in [0, 1]\}$  (see (4.1.2)) in order to draw, for example,

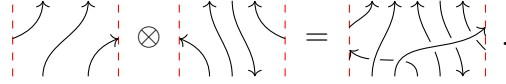


$$\in \text{Hom}_{\mathcal{FOT}(A)}(\uparrow \otimes \uparrow \otimes \downarrow, \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow), \quad (4.1.3)$$

where we identify the dashed vertical edges. We always isotope tangles so that they intersect the cut transversely.

The categories  $\mathcal{FOT}(A)$ ,  $\mathcal{OT}(A)$ ,  $\mathcal{FT}(A)$ , and  $\mathcal{T}(A)$  are also strict monoidal categories, although some care must be taken with the tensor product. Viewing  $A \times [0, 1]$  as the cylinder, the tensor product  $f \otimes g$  is given by placing the cylinder for  $g$  inside the cylinder for  $f$ , then rescaling and isotoping the endpoints of the tangles so that the endpoints of  $g$  are to the right of those of  $f$  (preserving the relative order of the

endpoints in  $f$  and the endpoints in  $g$ ). In terms of diagrams as in (4.1.3), this corresponds to placing the diagram of  $g$  to the right of the diagram of  $f$ , and then passing all strands of  $f$  exiting the right side of its diagram over the diagram for  $g$  and all strands of  $g$  exiting the left side of its diagram under the diagram for  $f$ . For example,



The category  $\mathbf{Braid}(D)$  of braids over  $D$  is the strict monoidal subcategory of  $\mathcal{T}(D)$  whose morphisms have no closed components. Equivalently,  $\mathbf{Braid}(D)$  is the strict monoidal subcategory of  $\mathcal{OT}(D)$  with objects generated by the object  $\uparrow$  and morphisms containing no closed components. We define  $\mathbf{Braid}(A)$  similarly.

For a commutative ring  $\mathbb{k}$ , we use a subscript  $\mathbb{k}$  to denote the  $\mathbb{k}$ -linearization of a strict monoidal category. For example,  $\mathcal{FOT}(D)_{\mathbb{k}}$  is the  $\mathbb{k}$ -linearization of  $\mathcal{FOT}(D)$ .

Many of the examples to be introduced in the sections to follow share some conventions and defining relations. In order to make our presentation efficient, we introduce these first here.

The “oriented” categories to follow will be generated by two objects,  $\uparrow$  and  $\downarrow$ , whose identity morphisms we denote by an upward and downward strand, respectively. Morphisms will be string diagrams with oriented strings. The domain and codomain of such diagrams can be read from the orientations of the strands at the bottom and top of the diagram in the usual way. The “unoriented” categories will be generated by a single object  $\downarrow$ , whose identity morphism we denote by an unoriented strand. String diagrams will then involve unoriented strings.

The following relations will be important (here  $z, t, \delta \in \mathbb{k}$ ):

$$\downarrow = \uparrow = \cup, \quad \downarrow = \downarrow', \quad \uparrow = \uparrow', \quad (\text{R0})$$

$$\downarrow = \downarrow = \downarrow, \quad (\text{R1})$$

$$\downarrow = \downarrow, \quad (\text{FR1})$$

$$\downarrow = \downarrow = \downarrow, \quad (\text{R2})$$

$$\downarrow = \downarrow, \quad (\text{R3})$$

$$\downarrow + \downarrow = z \downarrow + z \cup, \quad (\text{KS}_+)$$

$$\downarrow - \downarrow = z \downarrow - z \cup, \quad (\text{KS}_-)$$

$$\times := q \left| \right| + q^{-1} \cup, \quad (\text{KB})$$

$$\nearrow \nearrow - \nwarrow \nwarrow = z \uparrow \uparrow, \quad (\text{CS})$$

$$\left| \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} \right| = t, \quad (\text{T})$$

$$\bigcirc = \delta 1_{\mathbb{1}}. \quad (\text{D})$$

$$\begin{array}{c} \times \\ \oplus \end{array} = \begin{array}{c} \times \\ \otimes \end{array}, \quad \begin{array}{c} \cap \\ \oplus \end{array} = \begin{array}{c} \cap \\ \otimes \end{array}, \quad \begin{array}{c} \oplus \\ \oplus \end{array} = \begin{array}{c} \oplus \\ \oplus \end{array} = \left| \right|, \quad (\text{UA})$$

$$\begin{array}{c} \nearrow \\ \otimes \end{array} \begin{array}{c} \nearrow \\ \otimes \end{array} = \begin{array}{c} \nearrow \\ \otimes \end{array} \begin{array}{c} \nearrow \\ \otimes \end{array}, \quad \begin{array}{c} \curvearrowright \\ \otimes \end{array} = \begin{array}{c} \curvearrowright \\ \otimes \end{array}, \quad \begin{array}{c} \curvearrowleft \\ \otimes \end{array} = \begin{array}{c} \curvearrowleft \\ \otimes \end{array}, \quad \hat{\otimes} \text{ is invertible.} \quad (\text{OA})$$

We will also use oriented versions of (R0), (R1), (FR1), (R2), and (R3), referring to them by the same labels. We will interpret the above both as relations in strict monoidal categories described by generators and relations, and also as relations on ( $\mathbb{k}$ -linearizations of) tangle categories over the disc or annulus. When viewing them as relations on tangle categories, they are local relations drawn using the conventions outlined earlier in this section. The relations (KS<sub>+</sub>), (KS<sub>-</sub>), (KB), and (CS) are called the *Kauffman skein relation*, *Dubrovnik skein relation* (or *Kauffman skein relation in its Dubrovnik form*), *Kauffman bracket skein relation* and *Conway skein relation*, respectively.

We now recall some other relations that follow from various combinations of the above. First, note that (R0) implies the “windmill relation”:

$$(\text{R0}) \implies \times = \curvearrowright \cup, \quad \times = \curvearrowleft \cup. \quad (\text{W})$$

Also, it is a straightforward exercise to see that

$$(\text{R0}), (\text{R2}), (\text{R3}) \implies \begin{array}{c} \cap \\ \otimes \end{array} = \left( \begin{array}{c} \cap \\ \otimes \end{array} \right)^{-1}, \quad \begin{array}{c} \cup \\ \otimes \end{array} = \left( \begin{array}{c} \cup \\ \otimes \end{array} \right)^{-1}, \quad (4.1.4)$$

and this implication also holds for the strands oriented in either direction. Thus, if (R0), (R2), and (R3) hold, then relations (FR1) and (T) suffice to straighten all twists, in both the oriented and unoriented settings.

Next we consider bubbles. Relations (R0) and (R2) imply that the bubble is strictly central:

$$(\text{R0}) + (\text{R2}) \implies \bigcirc \left| \right| = \left| \right| \bigcirc$$

(The same implication holds in the oriented case.) Hence it is natural to impose the dimension relation (D). In fact, adjoining an indeterminate  $\delta$  to the coefficient ring and then imposing (D) simply corresponds to viewing the bubble as an element of the coefficient ring. In the oriented case, (FR1) and (T) imply that the clockwise and counterclockwise bubbles are equal:

$$(\text{FR1}) + (\text{T}) \implies \curvearrowright = t^{-1} \curvearrowleft = \curvearrowright. \quad (4.1.5)$$

Thus, in the presence of (FR1) and (T), imposing (D) for the clockwise bubble automatically means (D) is also satisfied for the counterclockwise bubble.

## 4.2 Towers of algebras

Many families of algebras appearing in representation theory can be combined into a *tower*. We discuss some of these families here, using the language of strict monoidal categories. We assume that all categories are linear over a commutative ground ring  $\mathbb{k}$ .

Suppose  $\mathcal{C}$  is a strict monoidal category such that

- the objects of  $\mathcal{C}$  are generated by a single object  $X$ , and
- we have  $\text{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m}) = 0$  when  $m \neq n$ .

For each  $n \in \mathbb{N}$ , we have the endomorphism algebra  $\mathcal{C}(n) := \text{End}_{\mathcal{C}}(X^{\otimes n})$ . The collection  $\mathcal{C}(n)$ ,  $n \in \mathbb{N}$ , is sometimes called a *tower of algebras* in the literature.

If  $\mathcal{C}$  is balanced, then, under the functor (3.2.4), the elements  $1_X^{\otimes(n-i)} \otimes \xi_X \otimes 1_X^{\otimes(i-1)}$ ,  $1 \leq i \leq n$ , are mapped to

$$J_{i,n} := 1_X^{\otimes(n-i)} \otimes \left( \beta_{X^{\otimes(i-1)}, X} \circ (1_X^{\otimes(i-1)} \otimes \theta_X) \circ \beta_{X, X^{\otimes(i-1)}} \right) \in \mathcal{C}(n). \quad (4.2.1)$$

For fixed  $n$ , the elements  $J_{1,n}, \dots, J_{n,n}$  are pairwise commuting elements of  $\mathcal{C}(n)$ . We call them the *Jucys–Murphy elements* of  $\mathcal{C}(n)$  since, in our examples below, they will correspond to classical Jucys–Murphy elements.

In fact, in many of the examples in this section, we have  $\theta_X = 1_X$ . Then the twist on arbitrary objects  $X^{\otimes n}$  is determined recursively by (2.4.5). In this case, we have

$$J_{i,n} = 1_X^{\otimes(n-i)} \otimes \left( \beta_{X^{\otimes(i-1)}, X} \circ \beta_{X, X^{\otimes(i-1)}} \right). \quad (4.2.2)$$

### 4.2.1 Category of braids

The category  $\mathbf{Braid}(D)$  of braids over the disc is isomorphic to the strict monoidal category generated by a single object  $\uparrow$ , and morphisms

$$\begin{array}{c} \nearrow \\ \searrow \end{array}, \begin{array}{c} \nwarrow \\ \searrow \end{array},$$

subject to the relations (R2) and (R3), with the strands oriented upwards. (The orientation of the strands does not play an important role here.) We will identify these two categories. It follows that  $\mathbf{Braid}(D)$  is the free braided monoidal category generated by a single object; see, for example, [64, Th. 5.6]. It is balanced, with twist determined by  $\theta_{\uparrow} = 1_{\uparrow}$ . The endomorphism algebra  $\text{End}_{\mathbf{Braid}(D)}(\uparrow^{\otimes n})$  is the group algebra of the braid group of type  $A_{n-1}$  and the elements  $J_{i,n}$  of (4.2.2) are the usual Jucys–Murphy elements.

**Proposition 4.2.1.** *The affinization  $\text{Aff}(\mathbf{Braid}(D))$  of the category of braids over the disc is isomorphic to the category  $\mathbf{Braid}(A)$  of braids over the annulus.*

**Proof:** Consider the functor  $F: \text{Aff}(\mathbf{Braid}(D)) \rightarrow \mathbf{Braid}(A)$  that is the identity on objects, and is given on morphisms as follows. Given a braid  $f \in \mathbf{Braid}(D)$ , we can naturally view it as a braid over  $A$  via (4.1.2). We define

$$F(\xi_{\uparrow}) = \begin{array}{c} \uparrow \\ \curvearrowright \\ \uparrow \end{array} \quad \text{and} \quad F(\xi_{\uparrow}^{-1}) = \begin{array}{c} \uparrow \\ \curvearrowleft \\ \uparrow \end{array},$$

where we view these as braids over  $A$  by identifying the vertical edges, as in (4.1.3). It is straightforward to verify that  $F$  respects the relations (3.1.1) and (3.1.2) and that it is a monoidal functor.

Now consider the functor  $G: \mathbf{Braid}(A) \rightarrow \text{Aff}(\mathbf{Braid}(D))$  that is the identity on objects, and is given on morphisms as follows. Given a braid  $f$  over the annulus, we cut the annulus as explained in Section 4.1 to obtain a tangle drawn in a rectangle with vertical edges identified. After isotoping if necessary, this diagram is a composite of diagrams not intersecting the vertical edges and diagrams of the form

$$\begin{array}{c} \uparrow \\ \diagup \\ \diagdown \\ \uparrow \end{array} \quad \text{and} \quad \begin{array}{c} \uparrow \\ \diagdown \\ \diagup \\ \uparrow \end{array}, \quad (4.2.3)$$

where there can be an arbitrary number of strands (including zero) in the middle (i.e. not intersecting the vertical edges of the rectangle). We then define  $G$  on such a morphism by declaring that it sends tangles not intersecting the vertical edges to the same tangles, naturally interpreted as tangles over the disc, and the tangles (4.2.3) to  $\xi_{\uparrow^{\otimes n}, \uparrow}$  and  $\xi_{\uparrow^{\otimes n}, \uparrow}^{-1}$ , respectively, where  $n$  is the number of strands in the middle of the diagrams (not intersecting the vertical edges). The relations (3.1.2) ensure that  $G$  is well defined, and it is straightforward to verify that  $F$  and  $G$  are mutually inverse.  $\blacksquare$

**Corollary 4.2.2.** *The category  $\mathbf{Braid}(A)$  of braids over the annulus is isomorphic to the strict monoidal category generated by a single object  $\uparrow$ , morphisms  $\curvearrowright$ ,  $\curvearrowleft$ , and an invertible morphism  $\hat{\phi}$ , subject to the relations (R2), (R3), and the first relation in (OA).*

The endomorphism algebra  $\text{End}_{\mathbf{Braid}(A)}(\uparrow^{\otimes n})$  is the extended affine braid group of type  $A_{n-1}$ , which is isomorphic to the braid group of type  $B_n$ . We refer the reader to [29, §2] for further discussion of viewing this braid group in terms of cylindrical braids.

## 4.2.2 Hecke algebras

Let  $\mathcal{H}(D)$  be the strict  $\mathbb{k}$ -linear monoidal category obtained from  $\mathbf{Braid}(D)_{\mathbb{k}}$  by imposing the Conway skein relation (CS). Then the endomorphism algebra  $\text{End}_{\mathcal{H}(D)}(\uparrow^{\otimes n})$  is

the Iwahori–Hecke algebra of type  $A_{n-1}$ . (One often sees the definition with  $z = q - q^{-1}$  for some  $q \in \mathbb{k}^\times$ .)

Applying our affinization procedure, we obtain the category  $\text{Aff}(\mathcal{H}(D))$ . By Theorem 3.1.6, this is the strict monoidal category generated by a single object  $\uparrow$ , and morphisms

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}, \quad \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array}, \quad \hat{\phi},$$

subject to the relations (R2) and (R2) with the strands oriented upwards, the relation (CS), and

$$\begin{array}{c} \nearrow \searrow \\ \swarrow \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array}, \quad \hat{\phi} \text{ is invertible.}$$

The endomorphism algebra  $\text{End}_{\text{Aff}(\mathcal{H}(D))}(\uparrow^{\otimes n})$  is the affine Hecke algebra of type  $A_{n-1}$ . The category  $\mathcal{H}(D)$  is balanced, with twist determined by  $\theta_{\uparrow} = 1_{\uparrow}$ . The elements  $J_{i,n}$  of (4.2.2) are the usual Jucys–Murphy elements. It is well known that symmetric functions in the Jucy–Murphy elements generate the center of the Iwahori–Hecke algebra of type  $A$ . This was conjectured by Dipper and James [20], and later proved by Francis and Graham in [24]. An interpretation of affine Hecke algebras in terms of the cylinder was given in [29].

The above discussion can be generalized to the setting of *quantum wreath product algebras*, which were introduced in [52, Def. 2.1] and depend on a Frobenius algebra  $A$ . They are the endomorphism algebras of the *quantum wreath product category* defined in [14, §2]. Affinization of this category yields the *quantum affine wreath product category* defined there, whose endomorphism algebras are the *quantum affine wreath product algebras* defined in [52, Def. 2.5]. When  $A = \mathbb{k}$ , this recovers the case of (affine) Hecke algebras described above. When  $A$  is the group algebra of a finite cyclic group, it corresponds to the (affine) Yokonuma–Hecke algebras.

### 4.2.3 Symmetric groups

Let  $\mathbf{Sym}$  be the free  $\mathbb{k}$ -linear symmetric monoidal category on a single object. Thus  $\mathbf{Sym}$  is the strict  $\mathbb{k}$ -linear monoidal category generated by a single object  $\uparrow$ , and morphism  $\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}$ , subject to the relations

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} = \uparrow \uparrow = \begin{array}{c} \nwarrow \nearrow \\ \swarrow \nearrow \end{array} \quad \text{and} \quad \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} = \begin{array}{c} \nwarrow \nearrow \\ \nwarrow \nearrow \end{array}.$$

Then  $\text{End}_{\mathbf{Sym}}(\uparrow^{\otimes n})$  is the group algebra of the symmetric group  $\mathfrak{S}_n$ .

Since  $\mathbf{Sym}$  is a braided monoidal category, we can apply our affinization procedure. It is easy to see, using Theorem 3.1.6, that  $\text{End}_{\text{Aff}(\mathbf{Sym})}(\uparrow^{\otimes n})$  is isomorphic to the wreath product algebra  $\mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rtimes \mathfrak{S}_n$ , where  $x_i$  corresponds to a positive dot on the  $i$ -th strand, and  $\mathfrak{S}_n$  acts on  $\mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  by permutation of the  $x_i$ .

Note that the endomorphism algebras of  $\text{Aff}(\mathcal{S}ym)$  are *not* degenerate affine Hecke algebras. To obtain the latter, one needs to instead consider a  $q \rightarrow 1$  degeneration of  $\text{Aff}(\mathcal{H}(D))$  from Section 4.2.2.

### 4.3 Oriented examples

In this section and the next, we give a number of examples of the affinization of monoidal categories coming from the theory of tangles and skein theory as introduced in Section 4.1. In each example, we see that affinization of such a category over the disc yields the corresponding category over the annulus. Furthermore, Theorem 3.1.6 gives us presentations of these annular categories involving dot generators.

#### 4.3.1 Oriented tangles

The category  $\mathcal{OT}(D)$  of oriented tangles over the disc is isomorphic to the strict monoidal category generated by objects  $\uparrow, \downarrow$ , and morphisms

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \nearrow, \nwarrow, \cup, \cup, \cap, \cap,$$

subject to the relations (R0), (R1), (R2), and (R3) for all orientations of the strands. This category is a strict ribbon category.

**Remark 4.3.1.** There exist more efficient presentations. For instance, using (W), it is enough to include the upward crossings as generators. One can then use (W) to *define* the other crossings. Also, for example, (R1) and (R2) for downward oriented strands follow from the upward oriented strand analogues, together with (R0). See [62, Th. 3.2], [25, Th. 3.5], or [32, Th. XII.2.2] for details. We choose to include all the crossings as generators to emphasize the structure as a braided monoidal category.

**Proposition 4.3.2.** *The affinization  $\text{Aff}(\mathcal{OT}(D))$  of the category of oriented tangles over the disc is isomorphic to the category  $\mathcal{OT}(A)$  of oriented tangles over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

**Corollary 4.3.3.** *The category  $\mathcal{OT}(A)$  of oriented tangles over the annulus is isomorphic to the strict monoidal category generated by objects  $\uparrow, \downarrow$ , and morphisms*

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \nearrow, \nwarrow, \cup, \cup, \cap, \cap, \dot{\uparrow}, \dot{\downarrow},$$

subject to relations (R0), (R1), (R2), (R3), and (OA).

### 4.3.2 Framed oriented tangles

The category  $\mathcal{FOT}(D)$  of framed oriented tangles over the disc is isomorphic to the strict monoidal category generated by objects  $\uparrow, \downarrow$ , and morphisms

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \cup, \cap, \cup, \cap,$$

subject to the relations (R0), (FR1), (R2), and (R3) for all orientations of the strands. As in Remark 4.3.1, there are actually more efficient presentations, i.e. presentations with fewer generators. See, for example, [25, Th. 3.5]. The category  $\mathcal{FOT}(D)$  is the ribbon category freely generated by a single object; see [59, Th. 6.4] and [64, Th. 9.1].

**Proposition 4.3.4.** *The affinization  $\text{Aff}(\mathcal{FOT}(D))$  of the category of framed oriented tangles over the disc is isomorphic to the category  $\mathcal{FOT}(A)$  of oriented tangles over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

**Corollary 4.3.5.** *The category  $\mathcal{FOT}(A)$  of oriented tangles over the annulus is isomorphic to the strict monoidal category generated by objects  $\uparrow, \downarrow$ , and morphisms*

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \cup, \cap, \cup, \cap, \uparrow, \downarrow,$$

subject to relations (R0), (FR1), (R2), (R3), and (OA).

### 4.3.3 HOMFLYPT skein category

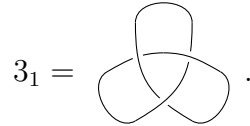
Fix  $z, \delta \in \mathbb{k}$  and  $t \in \mathbb{k}^\times$  satisfying  $z\delta = t - t^{-1}$ . Generically, we can work over the ring  $\mathbb{k} = \mathbb{Z}[z, t, t^{-1}, \delta]/(z\delta - t + t^{-1})$ . The *framed HOMFLYPT skein category*  $\mathcal{OS}(D; z, t, \delta)$  (resp.  $\mathcal{OS}(A; z, t, \delta)$ ) over the disc (resp. over the annulus) is the category obtained from  $\mathcal{FOT}(D)_{\mathbb{k}}$  (resp. from  $\mathcal{FOT}(A)_{\mathbb{k}}$ ) by imposing the Conway skein relation (CS), the twist relation (T) for the upward orientation of the strands, and the dimension relation (D) with either orientation of the bubble (see (4.1.5)). The reason for imposing the condition  $z\delta = t - t^{-1}$  in our coefficient ring is that we have

$$(t - t^{-1}) \left| \begin{array}{c} \uparrow \\ \text{(T)} \\ \uparrow \end{array} \right| - \left| \begin{array}{c} \uparrow \\ \text{(CS)} \\ \uparrow \end{array} \right| z \left| \begin{array}{c} \uparrow \\ \text{(D)} \\ \uparrow \end{array} \right| z\delta \left| \begin{array}{c} \uparrow \\ \text{(D)} \\ \uparrow \end{array} \right|. \quad (4.3.1)$$

When  $z \in \mathbb{k}^\times$ , we have  $\delta = (t - t^{-1})/z$ , and so we can omit  $\delta$  from the notation. In this case, we denote the category by  $\mathcal{OS}(D; z, t)$  (resp.  $\mathcal{OS}(A; z, t)$ ). The category  $\mathcal{OS}(D; z, t, \delta)$  was first introduced in [62, §5.2], where it was called the *Hecke category* (not to be confused with the more modern use of this term, which is related to the

category of Soergel bimodules). Our choice of the notation  $\mathcal{OS}$  comes from *oriented skein*.

The category  $\mathcal{OS}(D; z, t)$  underpins the HOMFLYPT polynomial in the following sense. Given an oriented link diagram  $L$ , define  $\text{writhe}(L)$  to be the number of positive crossings minus the number of negative crossings in  $L$ . Viewing  $L$  as an element of  $\text{End}_{\mathcal{OS}(D; z, t)}(\mathbb{1})$ , there is a unique scalar  $H_L(z, t) \in \mathbb{k}$  such that  $t^{-\text{writhe}(L)}L = H_L(z, t)\delta\mathbb{1}_{\mathbb{1}}$ . (The factor of  $\delta$  appears here to normalize the polynomial so that  $H_L(z, t) = 1$  when  $L$  is the unknot.) If  $\mathbb{k} = \mathbb{Z}[z, z^{-1}, t, t^{-1}]$ , then  $H_L(z, t)$  is precisely the HOMFLYPT polynomial of  $L$ . Since the writhe number of a knot is independent of its orientation, this gives an invariant of unoriented knots. The Alexander, Conway, and Jones polynomials are all specializations of the HOMFLYPT polynomial. We illustrate with the trefoil knot, denoted  $3_1$ , and depicted below



We add orientation to the trefoil knot, and we compute

$$\begin{aligned}
 & \text{Diagram} \stackrel{\text{(CS)}}{=} \text{Diagram} - z \text{Diagram} \\
 & \stackrel{\text{(R2), (CS)}}{=} \text{Diagram} - z \left( \text{Diagram} - z \text{Diagram} \right) \\
 & \stackrel{\text{(R2), (T)}}{\underset{\text{(D)}}{=}} \delta t^{-1} \mathbb{1}_{\mathbb{1}} - z (\delta^2 - z\delta t^{-1}) \mathbb{1}_{\mathbb{1}}.
 \end{aligned}$$

The writhe number of the trefoil knot is  $-3$ . Thus its HOMFLYPT polynomial is

$$H_{3_1}(z, t) = \delta^{-1}t^3 (\delta t^{-1} - z\delta^2 + z^2\delta t^{-1}) = -t^4 + 2t^2 + t^2z^2.$$

**Proposition 4.3.6.** *The affinization  $\text{Aff}(\mathcal{OS}(D; z, t, \delta))$  of the framed HOMFLYPT skein category over the disc is isomorphic to  $\mathcal{OS}(A; z, t, \delta)$ , the framed HOMFLYPT skein category over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

**Corollary 4.3.7.** *The framed HOMFLYPT skein category  $\mathcal{OS}(A; z, t, \delta)$  over the annulus is isomorphic to the strict monoidal category generated by objects  $\uparrow$ ,  $\downarrow$ , and morphisms*

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \nearrow, \nwarrow, \cup, \cap, \cup, \phi, \psi,$$

subject to relations (R0), (FR1), (R2), (R3), (CS), (T), (D), and (OA).

The presentation from Corollary 4.3.7 implies that  $\mathcal{OS}(A; z, t)$  is isomorphic to the affine oriented skein category  $\mathcal{AOS}(z, t)$  of [10, §4], which is also the quantum Heisenberg category of central charge zero (see [13]). The functor  $\mathcal{OS}(A; z, t) \rightarrow \mathcal{OS}(D; s, t)$  from (3.2.4) corresponds to the functor described in [10, Lem. 4.2] after rescaling the dots by a factor of  $t$ .

## 4.4 Unoriented examples

In this section we continue our study of examples of the affinization of monoidal categories, now focusing on unoriented categories of tangles and skein categories.

### 4.4.1 Tangles

The category  $\mathcal{T}(D)$  of tangles over the disc is isomorphic to the strict monoidal category generated by a single object  $\mathbb{1}$ , and morphisms

$$\times, \times, \cup, \cap,$$

subject to the relations (R0), (R1), (R2), and (R3); see [25, Th. 3.5]. The category  $\mathcal{T}(D)$  is a strict ribbon category.

**Proposition 4.4.1.** *The affinization  $\text{Aff}(\mathcal{T}(D))$  of the category of tangles over the disc is isomorphic to the category  $\mathcal{T}(A)$  of tangles over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

**Corollary 4.4.2.** *The category  $\mathcal{T}(A)$  of tangles over the annulus is isomorphic to the strict monoidal category generated by a single object  $\mathbb{1}$ , and morphisms*

$$\times, \times, \cup, \cap, \phi, \psi,$$

subject to the relations (R0), (R1), (R2), (R3), and (UA).

### 4.4.2 Framed tangles

The category  $\mathcal{FT}(D)$  of framed tangles over the disc is isomorphic to the strict monoidal category generated by a single object  $\mathbb{1}$ , and morphisms

$$\times, \times, \cup, \cap,$$

subject to the relations (R0), (FR1), (R2), and (R3); see [62, p. 436]. It is a strict ribbon category.

**Proposition 4.4.3.** *The affinization  $\text{Aff}(\mathcal{FT}(D))$  of the category of framed tangles over the disc is isomorphic to the category  $\mathcal{FT}(A)$  of tangles over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

**Corollary 4.4.4.** *The category  $\mathcal{FT}(A)$  of framed tangles over the annulus is isomorphic to the strict monoidal category generated by a single object  $\mathbb{1}$ , and morphisms*

$$\times, \times, \cup, \cap, \phi, \phi,$$

subject to the relations (R0), (FR1), (R2), (R3), and (UA).

### 4.4.3 Kauffman skein category

Let  $\varepsilon \in \{1, -1\}$ , and fix  $z, \delta \in \mathbb{k}$ ,  $t \in \mathbb{k}^\times$  satisfying  $z(\delta + \varepsilon) = t + \varepsilon t^{-1}$ . Generically, we work over the ring  $\mathbb{k} = \mathbb{Z}[z, t, t^{-1}, \delta]/(z(\delta + \varepsilon) - t - \varepsilon t^{-1})$ . The *Kauffman skein categories*  $\mathcal{KS}_\varepsilon(D; z, t, \delta)$  over the disc are the categories obtained from  $\mathcal{FT}(D)_\mathbb{k}$  by imposing the relation  $(\text{KS}_\varepsilon)$  (i.e.  $(\text{KS}_+)$  when  $\varepsilon = 1$  and  $(\text{KS}_-)$  when  $\varepsilon = -1$ ), the twist relation (T), and the dimension relation (D). Replacing the disc  $D$  by the annulus  $A$ , we get the analogous categories over the annulus. One might also use the term *Kauffman skein category* to refer to the choice  $\varepsilon = 1$  and *Dubrovnik skein category* to refer to the choice  $\varepsilon = -1$ .

The reason for imposing the condition  $z(\delta + \varepsilon) = t + \varepsilon t^{-1}$  in our coefficient ring is that we have

$$(t + \varepsilon t^{-1}) \left| \stackrel{\text{(T)}}{=} \right| \left| \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} \right| + \varepsilon \left| \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} \right| \stackrel{\text{(KS}_\varepsilon)}{=} z \left| \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} \right| + \varepsilon z \left| \stackrel{\text{(D)}}{=} z(\delta + \varepsilon) \right|. \quad (4.4.1)$$

When  $z \in \mathbb{k}^\times$ , we have  $\delta = 1 - \varepsilon(t + \varepsilon t^{-1})/z$ , and we denote the categories simply by  $\mathcal{KS}_\varepsilon(D; z, t)$ . These categories were introduced by Turaev in [62, §7.7].

The endomorphism algebras  $\text{End}_{\mathcal{KS}_\varepsilon(D; z, t)}(\mathbb{1}^{\otimes n})$  are the *Kauffman tangle algebras*, which are isomorphic to the *Birman–Murakami–Wenzl (BMW) algebras*; see [43].

Explicit bases for the morphism spaces of  $\mathcal{KS}_\pm(D; z, t)$  are given in [62, Th. 7.8]. In particular,  $\text{End}(\mathbb{1})$  is one-dimensional, and this gives rise to link invariants as for the HOMFLYPT skein category described in Section 4.3.3. Namely, suppose  $\mathbb{k} = \mathbb{Z}[z, t, t^{-1}, \delta]/(z(\delta + \varepsilon) - t - \varepsilon t^{-1})$ . Given a link diagram  $L$ , we can view  $L$  as an element of  $\text{End}_{\mathcal{KS}_\varepsilon(D; z, t)}(\mathbb{1})$ . Then there is a unique scalar  $F_{L, \varepsilon}(z, t) \in \mathbb{k}[z, t, t^{-1}]$  such that  $t^{-\text{writhe}(L)}L = F_{L, \varepsilon}(z, t)\delta\mathbb{1}_{\mathbb{1}}$ . The scalar  $F_{L, 1}(z, t)$  is the *Kauffman polynomial* of  $L$  and  $F_{L, -1}(z, t)$  is the *Dubrovnik polynomial* of  $L$ . For example in the case of the trefoil knot, we have

$$\begin{aligned} \varepsilon \text{ (trefoil)} &\stackrel{(\text{KS}_\varepsilon)}{=} z \text{ (two circles)} + z\varepsilon \text{ (trefoil)} - \text{ (trefoil)} \\ &\stackrel{(\text{KS}_\varepsilon), (\text{R2})}{=} z \left( z \text{ (trefoil)} + z\varepsilon \text{ (trefoil)} - \varepsilon \text{ (two circles)} \right) + z\varepsilon \text{ (trefoil)} - \text{ (trefoil)} \\ &\stackrel{(\text{R2}), (\text{T})}{=} z^2\delta t + z^2\varepsilon(z - t^{-1}\varepsilon)\delta - \varepsilon z\delta^2 + z\varepsilon\delta t^2 - (z - t^{-1}\varepsilon)\delta. \end{aligned}$$

Hence

$$F_{3_1, \varepsilon}(z, t) = zt^5 + \varepsilon z^2t^4 - t^4 + z^3t^3 - \varepsilon zt^3 - \varepsilon z^2t^2 + 2t^2.$$

It was noted by Lickorish (see [33, p. 466]) that the Kauffman and Dubrovnik polynomials are essentially equivalent when one extends scalars to a ring including a square root  $i = \sqrt{-1}$  of negative one, in the sense that

$$F_{L, 1}(z, t) = i^{-\text{writhe}(L)}(-1)^{c(L)+1}F_{L, -1}(-iz, it),$$

where  $c(L)$  is the number of components of  $L$ . More generally, one has the following.

**Lemma 4.4.5.** *Suppose  $\mathbb{k}$  contains a square root  $i$  of  $-1$ . Then, for each  $n \in \mathbb{N}$ , we have an isomorphism of algebras*

$$\text{End}_{\mathcal{KS}_+(D; z, t, \delta)}(\mathbb{1}^{\otimes n}) \rightarrow \text{End}_{\mathcal{KS}_-(D; -iz, it, -\delta)}(\mathbb{1}^{\otimes n}),$$

determined by

$$\begin{aligned} \mathbb{1}_{\mathbb{1}^{\otimes(j-1)}} \otimes \times \otimes \mathbb{1}_{\mathbb{1}^{\otimes(n-j-1)}} &\mapsto i\mathbb{1}_{\mathbb{1}^{\otimes(j-1)}} \otimes \times \otimes \mathbb{1}_{\mathbb{1}^{\otimes(n-j-1)}}, \\ \mathbb{1}_{\mathbb{1}^{\otimes(j-1)}} \otimes \cup \otimes \mathbb{1}_{\mathbb{1}^{\otimes(n-j-1)}} &\mapsto -\mathbb{1}_{\mathbb{1}^{\otimes(j-1)}} \otimes \cup \otimes \mathbb{1}_{\mathbb{1}^{\otimes(n-j-1)}}, \end{aligned} \tag{4.4.2}$$

for  $1 \leq j < n$ .

**Proof:** It is well-known that the elements appearing in (4.4.2) generate the endomorphism algebras of  $\mathcal{KS}_\pm(D; z, t, \delta)$ , i.e. the BMW algebras in their original and Dubrovnik forms. Then it is straightforward to verify, using the standard presentation of these algebras, that the given map is an isomorphism.  $\blacksquare$

Because of the above relationship between the two choices of  $\varepsilon$ , many authors make the choice that best suits their particular purpose. In the literature, the two choices are often referred to as “variants” of each other. However, if one is interested in the full monoidal categories, there is no such relationship, as the following result shows.

**Proposition 4.4.6.** *If  $\mathbb{k}$  is an integral domain of characteristic not equal to two, then there do not exist any choices of  $z_\pm, t_\pm, \delta_\pm$  for which  $\mathcal{KS}_+(D; z_+, t_+, \delta_+)$  and  $\mathcal{KS}_-(D; z_-, t_-, \delta_-)$  are isomorphic as monoidal categories.*

**Proof:** Suppose, towards a contradiction, that we have an isomorphism of monoidal categories

$$\Psi: \mathcal{KS}_+(D; z_+, t_+, \delta_+) \rightarrow \mathcal{KS}_-(D; z_-, t_-, \delta_-).$$

Since this induces an isomorphism after we extend the base ring, we may assume that  $\mathbb{k}$  is a field. By [62, Th. 7.8], the morphism space  $\text{Hom}(\mathbb{I}^{\otimes 2}, \mathbb{1})$  is one-dimensional (in both categories), spanned by  $\cap$ . Similarly,  $\text{Hom}(\mathbb{1}, \mathbb{I}^{\otimes 2})$  is one-dimensional, spanned by  $\cup$ . It follows that  $\Psi(\cap) = d\cap$  and  $\Psi(\cup) = c\cup$  for some  $c, d \in \mathbb{k}$ . Since  $\Psi$  must preserve the first equality in (R0), we conclude that  $cd = 1$ .

Let  $\text{Rot}$  be the linear operator on  $\text{End}(\mathbb{I}^{\otimes 2})$  giving by counterclockwise rotation by  $90^\circ$ . This operation is given by tensoring on the left and right by  $\mathbb{I}$ , then adding a cap to the top two strands, a cup to the bottom two strands, then using (R0). For any  $f \in \text{End}_{\mathcal{KS}_+(D; z_+, t_+, \delta_+)}(\mathbb{I}^{\otimes 2})$ , we have

$$\Psi(\text{Rot}(f)) = \Psi \left( \left[ \begin{array}{c} \cap \\ f \\ \cup \end{array} \right] \right) = \left[ \begin{array}{c} \cap \\ \Psi(f) \\ \cup \end{array} \right] = \text{Rot}(\Psi(f)).$$

In particular,  $\Psi$  preserves the eigenspaces of  $\text{Rot}$ . However, it follows from [62, Th. 7.8] that the endomorphism space  $\text{End}_{\mathcal{KS}_\varepsilon(D; z_\varepsilon, t_\varepsilon, \delta_\varepsilon)}(\mathbb{I}^{\otimes 2})$  is three-dimensional, with basis

$$\times + \times, \quad \times - \times, \quad | | - \varepsilon \cup.$$

In particular, the  $+1$  eigenspace is one-dimensional for  $\varepsilon = 1$  and it is two-dimensional for  $\varepsilon = -1$ . This contradicts the assumption that  $\Psi$  is an isomorphism.  $\blacksquare$

**Proposition 4.4.7.** *The affinization  $\text{Aff}(\mathcal{KS}_\varepsilon(D; z, t, \delta))$  of the Kauffman skein category over the disc is isomorphic to  $\mathcal{KS}_\varepsilon(A; z, t, \delta)$ , the Kauffman skein category over the annulus.*

**Proof:** The proof is analogous to that of Proposition 4.2.1. ■

By Theorem 3.1.6,  $\text{Aff}(\mathcal{KS}_\varepsilon(D; z, t, \delta))$  is the strict monoidal category generated by a single object  $\mathbb{1}$ , and morphisms

$$\times, \times, \cup, \cap, \phi, \phi,$$

subject to the relations (R0), (FR1), (R2), (R3), (KS $_\varepsilon$ ), (T), (D), and (UA). For the choice  $\varepsilon = -1$ , the category  $\text{Aff}(\mathcal{KS}_-(D; z, t, \delta))$  appeared in [26, Def. 1.3], where the authors called it the *affine Kauffman skein category* (although they have made the choice of the Dubrovnik skein relation), and gave a basis theorem for it and its cyclotomic quotients. The topological interpretation in terms of diagrams on the cylinder does not seem to appear there. For the choice  $\varepsilon = 1$ , this category does not seem to have been studied yet in the literature. The properties of the Jucys–Murphy elements do not seem to have been explored in either of the two cases.

#### 4.4.4 Temperley–Lieb category

For  $\delta \in \mathbb{k}$ , the *Temperley–Lieb category*  $\mathcal{TL}(\delta)$  is the free  $\mathbb{k}$ -linear rigid monoidal category generated by a self-dual object of dimension  $\delta$ . It is generated by a single object  $\mathbb{1}$ , and morphisms  $\cup$  and  $\cap$ , subject to the relation (D) and the first two equalities in (R0).

For the remainder of this section, assume there exists  $q \in \mathbb{k}^\times$  such that

$$\delta = -q^2 - q^{-2}. \quad (4.4.3)$$

Then  $\mathcal{TL}(\delta)$  is braided with the braiding given by

$$\times := q \left| \begin{array}{c} | \\ | \end{array} \right| + q^{-1} \begin{array}{c} \cup \\ \cap \end{array}, \quad \times := q^{-1} \left| \begin{array}{c} | \\ | \end{array} \right| + q \begin{array}{c} \cup \\ \cap \end{array}, \quad (4.4.4)$$

and then extended to arbitrary objects using (2.3.2); see [6, Prop. 2.6] (note that our  $q$  is the  $q^{1/2}$  of [6]). The cups and caps make  $\mathcal{TL}(\delta)$  a strict pivotal category, and so we can define a twist as in (2.4.7):

$$\theta_{\mathbb{1}} := \left| \begin{array}{c} \circlearrowright \end{array} \right| = -q^3. \quad (4.4.5)$$

Extending the twist to a general object  $\mathbb{1}^{\otimes n}$  using (2.4.5) recovers the twist defined in [6, Prop. 2.10].

Note that the definition (4.4.4) corresponds precisely to the relation (KB) and its image under clockwise rotation by  $90^\circ$ . Thus  $\mathcal{TL}(-q^2 - q^{-2})$  is isomorphic to a quotient of either of the Kauffman skein categories  $\mathcal{KS}_\pm(D; q \pm q^{-1}, -q^3, -q^2 - q^{-2})$  by (KB).

The category  $\mathcal{TL}(-q^2 - q^{-2})$  underpins the Jones polynomial just like the oriented skein category underpins the HOMFLYPT polynomial (see Section 4.3.3). Given an unoriented link  $L$ , let  $\vec{L}$  be an oriented link obtained by choosing some orientation of the components of  $L$ . Then there is a unique scalar  $V_{\vec{L}}(q) \in \mathbb{k}$  such that  $(-q^3)^{-\text{writhe}(\vec{L})} L = V_{\vec{L}}(q) \delta 1_{\mathbb{1}}$  in  $\mathcal{TL}(-q^2 - q^{-2})$ . If  $\mathbb{k} = \mathbb{Z}[q, q^{-1}]$  and  $L$  is a knot (in which case the writhe number is independent of the chosen orientation), then  $V_L(q)$  is the Jones polynomial of  $L$  (a Laurent polynomial in an indeterminate  $t$ ), specialized at  $t = q^4$ .

We can now apply our general affinization procedure to obtain the *affine Temperley–Lieb category*  $\text{Aff}(\mathcal{TL}(\delta))$ . It is the strict monoidal category generated by the object  $\mathbb{1}$  and morphisms

$$\cup, \cap, \times, \times, \phi, \phi,$$

subject to the relations (R0), (FR1), (R2), (R3), (KB), and (UA). Alternatively, if we want a presentation without crossings,  $\text{Aff}(\mathcal{TL}(\delta))$  is the strict monoidal category generated by an object  $\mathbb{1}$  and morphisms

$$\cup, \cap, \phi, \phi,$$

subject to the relations from (R0) and (UA) that do not involve crossings, together with the relation

$$q \left| \begin{array}{c} | \\ \phi \\ | \end{array} \right. - q^{-1} \left| \begin{array}{c} | \\ \phi \\ | \end{array} \right. = q \begin{array}{c} \cup \\ \phi \cap \end{array} - q^{-1} \begin{array}{c} \cup \phi \\ \cap \end{array}. \quad (4.4.6)$$

The affine Temperley–Lieb category was introduced in [28, Def. 2.5], where the morphism spaces were defined to be linear combinations of Temperley–Lieb diagrams on the cylinder. (The  $q$  of [28] is our  $q^2$ .) We were not able to find the above presentation of the affine Temperley–Lieb category, with the relation (4.4.6), in the literature.

One can also form the affine Temperley–Lieb category  $\text{Aff}(\mathcal{TL}(\delta))$  for general  $\delta$ , not necessarily of the form (4.4.3). However, one then loses the braided monoidal structure on  $\mathcal{TL}(\delta)$ . Thus we cannot use the presentation coming from Theorem 3.1.6, and we lose the monoidal structure on  $\text{Aff}(\mathcal{TL}(\delta))$ .

# Chapter 5

## Affinization, traces of categories and factorization homology

In this chapter we make connections between affinization of monoidal categories and some other related constructions from the literature. The constructions we consider are the horizontal trace, the vertical trace and factorization homology.

### 5.1 Relation to the horizontal trace

In several places in the literature, the *horizontal trace* is used as the formal concept embodying the idea of monoidal categories on annuli or cylinders. For example, the horizontal trace is defined in [4, §2.4], where it is stated that the horizontal trace of  $\mathcal{C}$  can be naturally regarded as the category of  $\mathcal{C}$ -diagrams on the annulus *when  $\mathcal{C}$  admits biadjoints*. A particular instance of this can be found in [16, Prop. 4.4], where the authors identify the annular spider category with the horizontal trace of the usual spider category.

In this section, we give a precise relationship between the horizontal trace and the affinization of a monoidal category. The two are different in general, but are isomorphic *when  $\mathcal{C}$  is rigid*. This explains why the horizontal trace corresponds to diagrams on the annulus for rigid categories. For example, the spider category considered in [16] is rigid. However, when  $\mathcal{C}$  is *not* rigid, it is the affinization, and not the horizontal trace, that naturally corresponds to  $\mathcal{C}$ -diagrams on the annulus. Below we give examples to illustrate this distinction.

Suppose  $\mathcal{C}$  is an essentially small monoidal category. Fix two objects  $X, Y$  in  $\mathcal{C}$ , and consider pairs  $(Z, f)$ , where  $Z$  is an object of  $\mathcal{C}$  and  $f: X \otimes Z \rightarrow Z \otimes Y$  is a morphism in  $\mathcal{C}$ . We define an equivalence relation on such pairs generated by the

relations

$$(Z, f \circ (1_X \otimes g)) \sim (Z', (g \otimes 1_Y) \circ f), \quad g \in \text{Hom}_{\mathcal{C}}(Z, Z'), \quad f \in \text{Hom}_{\mathcal{C}}(X \otimes Z', Z \otimes Y). \quad (5.1.1)$$

**Definition 5.1.1.** The *horizontal trace*  $\text{htr}(\mathcal{C})$  of  $\mathcal{C}$  is the category with the same objects as  $\mathcal{C}$  and with  $\text{Hom}_{\text{htr}(\mathcal{C})}(X, Y)$  the set of equivalence classes  $[Z, f]$  of such pairs  $(Z, f)$ . Composition of morphisms is given by

$$[Z', g: X \otimes Z' \rightarrow Z' \otimes Y] \circ [Z, f: W \otimes Z \rightarrow Z \otimes X] := [Z \otimes Z', (1_Z \otimes g) \circ (f \otimes 1_{Z'})]. \quad (5.1.2)$$

**Proposition 5.1.2.** *If  $\mathcal{C}$  is a braided strict monoidal category, then  $\text{htr}(\mathcal{C})$  is a strict monoidal category, where the tensor product of objects is the same as in  $\mathcal{C}$ , and the tensor product of morphisms is given by*

$$\begin{aligned} & [Z_1, f_1: X_1 \otimes Z_1 \rightarrow Z_1 \otimes Y_1] \otimes [Z_2, f_2: X_2 \otimes Z_2 \rightarrow Z_2 \otimes Y_2] \\ & := [Z_1 \otimes Z_2, (1_{Z_1} \otimes \beta_{Y_1, Z_2} \otimes 1_{Y_2}) \circ (f_1 \otimes f_2) \circ (1_{X_1} \otimes \beta_{Z_1, X_2}^{-1} \otimes 1_{Z_2})]. \end{aligned} \quad (5.1.3)$$

**Proof:** Let  $X_i, Y_i, Z_i \in \mathcal{C}$ , for  $i \in \{1, 2, 3\}$ . To shorten notation, write  $[f_i] := [Z_i, f_i: X_i \otimes Z_i \rightarrow Z_i \otimes Y_i]$ . We also write  $XY$  instead of  $X \otimes Y$ , for  $X, Y \in \mathcal{C}$ , and  $fg$  instead of  $f \otimes g$ , for  $f, g \in \text{Mor}(\mathcal{C})$ . (The composition is denoted  $\circ$ .)

We compute

$$\begin{aligned} ([f_1] \otimes [f_2]) \otimes [f_3] &= (1_{Z_1 Z_2} \beta_{Y_1, Z_3} 1_{Y_2 Y_3}) \circ (1_{Z_1 Z_2 Y_1} \beta_{Y_2, Z_3} 1_{Y_3}) \circ (1_{Z_1} \beta_{Y_1, Z_2} 1_{Y_2 Z_3 Y_3}) \\ &\circ (f_1 f_2 f_3) \circ (1_{X_1} \beta_{Z_1, X_2}^{-1} 1_{Z_2 X_3 Z_3}) \circ (1_{X_1 X_2 Z_1} \beta_{Z_2, X_3}^{-1} 1_{Z_3}) \circ (1_{X_1 X_2} \beta_{Z_1, X_3}^{-1} 1_{Z_2 Z_3}), \end{aligned}$$

and

$$\begin{aligned} [f_1] \otimes ([f_2] \otimes [f_3]) &= (1_{Z_1 Z_2} \beta_{Y_1, Z_3} 1_{Y_2 Y_3}) \circ (1_{Z_1} \beta_{Y_1, Z_2} 1_{Z_3 Y_2 Y_3}) \circ (1_{Z_1 Y_1 Z_2} \beta_{Y_2, Z_3} 1_{Y_3}) \\ &\circ (f_1 f_2 f_3) \circ (1_{X_1 Z_1 X_2} \beta_{Z_2, X_3}^{-1} 1_{Z_3}) \circ (1_{X_1} \beta_{Z_1, X_2}^{-1} 1_{X_3 Z_2 Z_3}) \circ (1_{X_1 X_2} \beta_{Z_1, X_3}^{-1} 1_{Z_2 Z_3}). \end{aligned}$$

Since

$$(1_{Z_1 Z_2 Y_1} \beta_{Y_2, Z_3} 1_{Y_3}) \circ (1_{Z_1} \beta_{Y_1, Z_2} 1_{Y_2 Z_3 Y_3}) \stackrel{(2.1.3)}{=} (1_{Z_1} \beta_{Y_1, Z_2} 1_{Z_3 Y_2 Y_3}) \circ (1_{Z_1 Y_1 Z_2} \beta_{Y_2, Z_3} 1_{Y_3}),$$

and

$$(1_{X_1} \beta_{Z_1, X_2}^{-1} 1_{Z_2 X_3 Z_3}) \circ (1_{X_1 X_2 Z_1} \beta_{Z_2, X_3}^{-1} 1_{Z_3}) \stackrel{(2.1.3)}{=} (1_{X_1 Z_1 X_2} \beta_{Z_2, X_3}^{-1} 1_{Z_3}) \circ (1_{X_1} \beta_{Z_1, X_2}^{-1} 1_{X_3 Z_2 Z_3}),$$

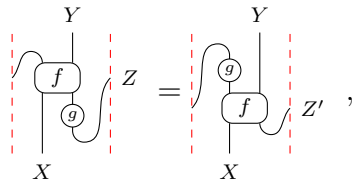
the claim follows. ■

**Remark 5.1.3.** In the case that  $\mathcal{C}$  is a braided monoidal category, a formula for a tensor product on  $\text{htr}(\mathcal{C})$  is asserted in [16, §2.4.2]. However, the tensor product given there does *not* satisfy the properties required of a tensor product in a braided monoidal category. For example, when  $\mathcal{C}$  is the category of braids over the disc (see Section 4.2.1) with generating object  $\mathbb{1}$  and  $f = \eta_1 \circ \epsilon_1$ , then  $(f \otimes 1_1) \circ (1_1 \otimes f) \neq f \otimes f \neq (1_1 \otimes f) \circ (f \otimes 1_1)$  with the tensor product of [16, §2.4.2].

**Remark 5.1.4.** It can be helpful to visualize the morphism  $[Z, f], f: X \otimes Z \rightarrow Z \otimes Y$ , as the following diagram on the cylinder:

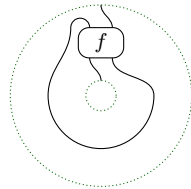

(5.1.4)

where, as usual, we identify the vertical edges. The relation (5.1.1) then corresponds to the equality


(5.1.5)

where we think of sliding the coupon  $g$  around the cylinder. However, in general, *one should not view this as a precise string diagram*. The reason is that the cups and caps drawn above do not have any precise meaning, since our category may not be rigid! (This is why we do not orient the strands in (5.1.4).) This observation is essential, since it explains the difference between the affinization and horizontal trace constructions. As we will see below, for non-rigid categories the affinization and horizontal trace differ in general. We will see in Theorem 5.1.6 that if our category is rigid then the affinization and horizontal trace agree, essentially because we can view (5.1.4) as a string diagram, with the cup and cap being a unit and counit morphism as in (2.4.1).

**Remark 5.1.5.** One also sometimes sees the morphism  $[Z, f], f: X \otimes Z \rightarrow Z \otimes Y$ , drawn as a diagram on the annulus:



However, the same pitfalls are present here as they are with the cylindrical depiction of Remark 5.1.4, since the cups and caps appearing in this diagram do not have a precise meaning in general. See also Remark 3.1.3.



Taking inverses shows that  $\Theta$  preserves (3.1.1). Next, for  $f \in \text{Hom}_{\mathcal{C}}(Y_1, Y_2)$  and  $g \in \text{Hom}_{\mathcal{C}}(X_1, X_2)$ , we compute

$$\begin{aligned} \Theta(\xi_{X_2, Y_2}^{-1}) \circ \Theta(f \otimes g) &= [Y_2, 1_{Y_2} \otimes 1_{X_2} \otimes 1_{Y_2}] \circ [\mathbb{1}, f \otimes g] = [Y_2, f \otimes g \otimes 1_{Y_2}] \\ &\stackrel{(5.1.1)}{=} [Y_1, 1_{Y_1} \otimes g \otimes f] = [\mathbb{1}, g \otimes f] \circ [Y_1, 1_{Y_1} \otimes 1_{X_1} \otimes 1_{Y_1}] = \Theta(g \otimes f) \circ \Theta(\xi_{X_1, Y_1}^{-1}), \end{aligned}$$

and so  $\Theta$  preserves (3.1.2).

(b) Suppose  $\mathcal{C}$  is left rigid. To verify that the functor  $\Theta'$  is well-defined, suppose  $g \in \text{Hom}_{\mathcal{C}}(Z, Z')$  and  $f \in \text{Hom}_{\mathcal{C}}(X \otimes Z', Z \otimes Y)$ . Then

$$\begin{aligned} \Theta'([Z, f \circ (1_X \otimes g)]) &= \begin{array}{c} \text{Diagram 1: } \\ \text{A box labeled } f \text{ with input } X \text{ and output } Y. \text{ A wire from } Z \text{ goes into } f \text{ from the bottom.} \\ \text{A wire from } Z \text{ goes into } g \text{ from the bottom, then } g \text{ outputs to } f \text{ from the bottom.} \end{array} \\ &= \begin{array}{c} \text{Diagram 2: } \\ \text{A box labeled } f \text{ with input } X \text{ and output } Y. \text{ A wire from } Z \text{ goes into } f \text{ from the bottom.} \\ \text{A wire from } Z \text{ goes into } g \text{ from the bottom, then } g \text{ outputs to } f \text{ from the bottom.} \end{array} \stackrel{(3.1.5)}{=} \\ &= \begin{array}{c} \text{Diagram 3: } \\ \text{A box labeled } f \text{ with input } X \text{ and output } Y. \text{ A wire from } Z \text{ goes into } f \text{ from the bottom.} \\ \text{A wire from } Z \text{ goes into } g \text{ from the bottom, then } g \text{ outputs to } f \text{ from the bottom.} \end{array} \\ &= \begin{array}{c} \text{Diagram 4: } \\ \text{A box labeled } f \text{ with input } X \text{ and output } Y. \text{ A wire from } Z \text{ goes into } f \text{ from the bottom.} \\ \text{A wire from } Z \text{ goes into } g \text{ from the bottom, then } g \text{ outputs to } f \text{ from the bottom.} \end{array} \\ &= \Theta'([Z, (g \otimes 1_Y) \circ f]). \end{aligned}$$

(c) Now suppose  $\mathcal{C}$  is rigid. We check that the functors  $\Theta$  and  $\Theta'$  are mutually inverse. For  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ , we have

$$\Theta' \circ \Theta(f) = \Theta'([\mathbb{1}, f]) = f \circ \xi_{X, \mathbb{1}} \stackrel{(3.1.3)}{=} f.$$

For objects  $X, Y$  in  $\mathcal{C}$ , we have

$$\Theta' \circ \Theta(\xi_{X, Y}^{-1}) = \Theta'([Y, 1_Y \otimes 1_X \otimes 1_Y]) = \begin{array}{c} \text{Diagram 1: } \\ \text{A box labeled } 1_Y \text{ with input } Y \text{ and output } Y. \text{ A wire from } X \text{ goes into } 1_Y \text{ from the bottom.} \\ \text{A wire from } Y \text{ goes into } 1_Y \text{ from the bottom.} \end{array} \stackrel{(3.1.5)}{=} \begin{array}{c} \text{Diagram 2: } \\ \text{A box labeled } 1_Y \text{ with input } Y \text{ and output } Y. \text{ A wire from } X \text{ goes into } 1_Y \text{ from the bottom.} \\ \text{A wire from } Y \text{ goes into } 1_Y \text{ from the bottom.} \end{array} = \begin{array}{c} \text{Diagram 3: } \\ \text{A box labeled } 1_Y \text{ with input } Y \text{ and output } Y. \text{ A wire from } X \text{ goes into } 1_Y \text{ from the bottom.} \\ \text{A wire from } Y \text{ goes into } 1_Y \text{ from the bottom.} \end{array} = \xi_{X, Y}^{-1}.$$

Thus  $\Theta' \circ \Theta$  is the identity functor. It is straightforward to verify that  $\Theta \circ \Theta'$  is also the identity.

(d) Suppose  $\mathcal{C}$  is braided and rigid. By part (c), it suffices to show that the functor  $\Theta$  is a monoidal functor. For morphisms  $f, g$  in any monoidal category, we have

$$f \otimes g = (f \otimes 1) \circ (1 \otimes g).$$

Thus, it suffices to show that  $\Theta(f) \otimes \Theta(1_Z) = \Theta(f \otimes 1_Z)$  and  $\Theta(1_Z) \otimes \Theta(f) = \Theta(1_Z \otimes f)$  for all objects  $Z$  and morphisms  $f$  in  $\text{Aff}(\mathcal{C})$ . For  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  and  $Z \in \text{Ob}(\mathcal{C})$ , we have

$$\Theta(f) \otimes \Theta(1_Z) = [\mathbb{1}, f] \otimes [\mathbb{1}, 1_Z] = [\mathbb{1}, \beta_{Y, \mathbb{1}} \circ f \circ (1_X \otimes \beta_{\mathbb{1}, \mathbb{1}}^{-1})] = [\mathbb{1}, f] = \Theta(f \otimes 1_Z).$$

Similarly,  $\Theta(1_Z) \otimes \Theta(f) = \Theta(1_Z \otimes f)$ . For  $X, Y, Z \in \text{Ob}(\mathcal{C})$ , we have

$$\begin{aligned}
 \Theta(\xi_{X,Y}) \otimes \Theta(1_Z) &\stackrel{(5.1.6)}{=} [Y^\vee, \eta_Y \otimes 1_X \otimes \epsilon_Y] \otimes [\mathbb{1}, 1_Z] \\
 &\stackrel{(5.1.3)}{=} [Y^\vee, (1_{Y^\vee} \otimes \beta_{Y \otimes X, \mathbb{1}} \otimes 1_Z) \circ (\eta_Y \otimes 1_X \otimes \epsilon_Y \otimes 1_Z) \circ (1_X \otimes 1_Y \otimes \beta_{Y^\vee, Z}^{-1})] \\
 &= [Y^\vee, (\eta_Y \otimes 1_X \otimes \epsilon_Y \otimes 1_Z) \circ (1_X \otimes 1_Y \otimes \beta_{Y^\vee, Z}^{-1})] \\
 &\stackrel{(R0)}{=} [Y^\vee, (\eta_Y \otimes 1_X \otimes 1_Z \otimes \epsilon_Y) \circ (1_X \otimes \beta_{Y,Z} \otimes 1_{Y^\vee})] \\
 &\stackrel{(5.1.2)}{=} [Y^\vee, \eta_Y \otimes 1_X \otimes 1_Z \otimes \epsilon_Y] \circ [\mathbb{1}, 1_X \otimes \beta_{Y,Z}] \\
 &\stackrel{(5.1.6)}{=} \Theta(\xi_{X \otimes Z, Y} \circ (1_X \otimes \beta_{Y,Z})) \\
 &\stackrel{(3.1.7)}{=} \Theta(\xi_{X,Y} \otimes 1_Z)
 \end{aligned}$$

and

$$\begin{aligned}
 \Theta(1_X) \otimes \Theta(\xi_{Y,Z}) &\stackrel{(5.1.6)}{=} [\mathbb{1}, 1_X] \otimes [Z^\vee, \eta_Z \otimes 1_Y \otimes \epsilon_Z] \\
 &\stackrel{(5.1.3)}{=} [Z^\vee, (\beta_{X, Z^\vee} \otimes 1_Z \otimes 1_Y) \circ (1_X \otimes \eta_Z \otimes 1_Y \otimes \epsilon_Z) \circ (1_X \otimes \beta_{1, Y \otimes Z}^{-1} \otimes 1_{Z^\vee})] \\
 &= [Z^\vee, (\beta_{X, Z^\vee} \otimes 1_Z \otimes 1_Y) \circ (1_X \otimes \eta_Z \otimes 1_Y \otimes \epsilon_Z)] \\
 &\stackrel{(R0)}{=} [Z^\vee, (1_{Z^\vee} \otimes \beta_{X,Z}^{-1} \otimes 1_Y) \circ (\eta_Z \otimes 1_X \otimes 1_Y \otimes \epsilon_Z)] \\
 &\stackrel{(5.1.2)}{=} [\mathbb{1}, \beta_{X,Z}^{-1} \otimes 1_Y] \circ [Z^\vee, \eta_Z \otimes 1_X \otimes 1_Y \otimes \epsilon_Z] \\
 &\stackrel{(5.1.6)}{=} \Theta((\beta_{X,Z}^{-1} \otimes 1_Y) \circ \xi_{X \otimes Y, Z}) \\
 &\stackrel{(3.1.7)}{=} \Theta(1_X \otimes \xi_{Y,Z}).
 \end{aligned}$$

■

We now give an example to show that the affinization and horizontal trace are *not* isomorphic (or even equivalent) in general.

**Example 5.1.7.** Let  $\mathcal{C}$  be the free monoidal category on one generating object  $\mathbb{1}$ . Thus  $\text{Hom}_{\mathcal{C}}(\mathbb{1}^{\otimes m}, \mathbb{1}^{\otimes n})$  is empty when  $m \neq n$  and consists of only  $1_{\mathbb{1}^{\otimes n}}$  when  $m = n$ . Then  $\text{Aff}(\mathcal{C})$  is generated by the morphisms  $\xi_{\mathbb{1}^{\otimes m}, \mathbb{1}^{\otimes n}} : \mathbb{1}^{\otimes(m+n)} \rightarrow \mathbb{1}^{\otimes(m+n)}$ ,  $m, n \in \mathbb{N}$ , and their inverses. In particular, since  $\xi_{\mathbb{1}, \mathbb{1}} = 1_{\mathbb{1}}$  by (3.1.3), we have  $\text{End}_{\text{Aff}(\mathcal{C})}(\mathbb{1}) = \{1_{\mathbb{1}}\}$ . However,  $\text{End}_{\text{htr}(\mathcal{C})}(\mathbb{1}) = \{\mathbb{1}^{\otimes n}, 1_{\mathbb{1}^{\otimes n}} : n \in \mathbb{N}\}$  has countably many elements. In terms of cylindrical diagrams,  $\text{htr}(\mathcal{C})$  contains closed string diagrams wrapping around the cylinder, while  $\text{Aff}(\mathcal{C})$  does not.

On the other hand,  $\text{End}_{\text{Aff}(\mathcal{C})}(\mathbb{1}) = \{\xi_{\mathbb{1}}^n : n \in \mathbb{Z}\}$  is an infinite cyclic group, while  $\text{End}_{\text{htr}(\mathcal{C})}(\mathbb{1}) = \{\mathbb{1}^{\otimes n}, 1_{\mathbb{1}^{\otimes(n+1)}} : n \in \mathbb{N}\}$  is an infinite cyclic monoid (whose generator

is not invertible). In terms of cylindrical diagrams,  $\text{End}_{\text{Aff}(\mathcal{C})}(\mathbb{I})$  contains strands that enter at the bottom of the cylinder, wrap around the cylinder in either direction, and exit out the top, while in  $\text{End}_{\text{htr}(\mathcal{C})}(\mathbb{I})$  such strands can only wrap in one direction. This illustrates the asymmetry inherent in the definition of the horizontal trace.

The next example illustrates how, in general, it is the affinization of  $\mathcal{C}$ , and not its horizontal trace, that naturally corresponds to the category of  $\mathcal{C}$ -diagrams on the cylinder or annulus.

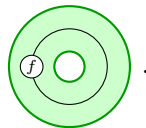
**Example 5.1.8.** As we saw in Proposition 4.2.1 the affinization  $\text{Aff}(\mathbf{Braid}(D))$  of the category of braids over the disc is the category  $\mathbf{Braid}(A)$  of braids over the annulus. On the other hand,  $\text{htr}(\mathbf{Braid}(D))$  is quite different. Similar to the situation in Example 5.1.7, the “braids” in  $\text{htr}(\mathbf{Braid}(D))$  can only wrap in one direction around the cylinder, and one also has closed components wrapping around the cylinder.

## 5.2 The vertical trace

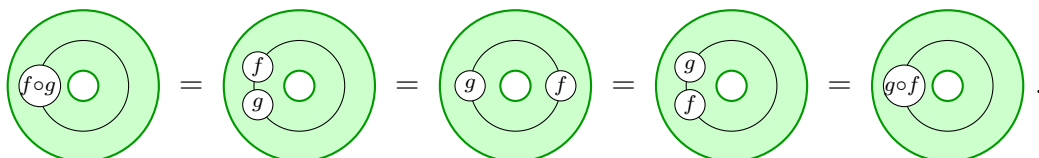
We already gave the definition of the trace of a  $\mathbb{k}$ -linear category in Section 2.6. We just note that this trace is also commonly called *vertical trace*, this is to differentiate it from the other trace in Section 5.1.

**Remark 5.2.1.** One can also consider the vertical trace of categories that are not necessarily linear. In this case, one replaces (2.6.1) by the set  $\text{Tr}_v(\mathcal{C}) = \left( \bigsqcup_{X \in \text{Ob}(\mathcal{C})} \text{End}_{\mathcal{C}}(X) \right) / \sim$ , where  $\sim$  is the equivalence relation generated by  $f \circ g \sim g \circ f$  for all pairs of morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  in  $\mathcal{C}$ . Since most of our examples of interest are linear categories, we work in the linear setting in this section. However, all of the results go through in the not-necessarily-linear setting with the obvious modifications.

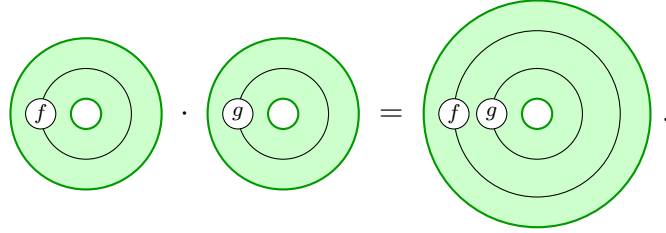
If  $\mathcal{C}$  is strict pivotal (in particular, this means that it is strict monoidal), we can naturally think of the trace as consisting of diagrams on the annulus. In particular, if  $f$  is an endomorphism in  $\mathcal{C}$ , then we picture  $[f]$  as



The fact that  $[f \circ g] = [g \circ f]$  in  $\text{Tr}_v(\mathcal{C})$  then corresponds to the fact we can slide diagrams around the annulus:

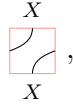


We also have that  $\mathrm{Tr}_v(\mathcal{C})$  is a  $\mathbb{k}$ -algebra, with multiplication given by  $[f] \cdot [g] = [f \otimes g]$ . This corresponds to nesting of annuli:



We refer the reader to [3] for further details and properties of the trace.

Now suppose  $\mathcal{C}$  is a strict  $\mathbb{k}$ -linear monoidal category. Then  $\mathrm{Aff}(\mathcal{C})$  is a  $\mathbb{k}$ -linear category, and we can pass to its vertical trace  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$ . If  $\mathcal{C}$  is strict pivotal, this naturally corresponds to string diagrams drawn on the torus. For example, if  $X \in \mathrm{Ob}(\mathcal{C})$ , then  $[\xi_X]$  corresponds to the diagram



where we identify the vertical edges and the horizontal edges, thereby obtaining a diagram on the torus.

If  $\mathcal{C}$  is a braided strict  $\mathbb{k}$ -linear monoidal category, then  $\mathrm{Aff}(\mathcal{C})$  is a strict monoidal category, and hence  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  is a  $\mathbb{k}$ -algebra. If  $\mathcal{C}$  is also strict pivotal, the product on  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  corresponds to the nesting of tori. This can be visualized as in (3.1.9), except that we only consider classes of endomorphisms and we identify the top and bottom of the diagrams (as well as the dashed vertical edges). For example, for  $X, Y \in \mathrm{Ob}(\mathcal{C})$ , we have

$$[\xi_X] \cdot [1_Y] = \begin{array}{c} X \\ \square \\ X \end{array} \cdot \begin{array}{c} \square \\ Y \end{array} = \begin{array}{c} X \\ \square \\ X \quad Y \end{array} = [\xi_{Y,X} \circ \beta_{X,Y}].$$

The action described in Section 3.2 induces an action of the trace, as we now explain.

**Lemma 5.2.2.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are  $\mathbb{k}$ -linear categories, we have an isomorphism of  $\mathbb{k}$ -modules*

$$\mathrm{Tr}_v(\mathcal{C} \boxtimes \mathcal{D}) \xrightarrow{\cong} \mathrm{Tr}_v(\mathcal{C}) \otimes \mathrm{Tr}_v(\mathcal{D}), \quad [f \otimes g] \mapsto [f] \otimes [g], \quad (5.2.1)$$

for  $f$  an endomorphism in  $\mathcal{C}$  and  $g$  an endomorphism in  $\mathcal{D}$ , extended by  $\mathbb{k}$ -linearity. If  $\mathcal{C}$  and  $\mathcal{D}$  are also monoidal, then (5.2.1) is an isomorphism of  $\mathbb{k}$ -algebras.

**Proof:** For  $f_1, f_2$  endomorphisms in  $\mathcal{C}$  and  $g_1, g_2$  endomorphisms in  $\mathcal{D}$ , we have

$$([f_1] \otimes [g_1]) \circ ([f_2] \otimes [g_2]) = [f_1 \circ f_2] \otimes [g_1 \circ g_2] = [f_2 \circ f_1] \otimes [g_2 \circ g_1] = ([f_2] \otimes [g_2]) \circ ([f_1] \otimes [g_1]).$$

Thus the map (5.2.1) is well defined. The map

$$\mathrm{Tr}_v(\mathcal{C}) \otimes \mathrm{Tr}_v(\mathcal{D}) \rightarrow \mathrm{Tr}_v(\mathcal{C} \boxtimes \mathcal{D}), \quad [f] \otimes [g] \mapsto [f \otimes g],$$

provides an inverse to (5.2.1). To see that this map is well-defined, we compute

$$[(f_1 \circ f_2) \otimes g] = [(f_1 \otimes g) \circ (f_2 \otimes 1)] = [(f_2 \otimes 1) \circ (f_1 \otimes g)] = [(f_2 \circ f_1) \otimes g]$$

and

$$[f \otimes (g_1 \circ g_2)] = [(f \otimes g_1) \circ (1 \otimes g_2)] = [(1 \otimes g_2) \circ (f \otimes g_1)] = [f \otimes (g_2 \circ g_1)].$$

■

**Proposition 5.2.3.** *Suppose  $\mathcal{C}$  is a strict  $\mathbb{k}$ -linear monoidal category,  $\mathcal{M}$  is a balanced strict  $\mathbb{k}$ -linear monoidal category, and  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor. Then  $\mathrm{Tr}_v(\mathcal{M})$  is a  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$ -module, with action given by*

$$[f] \cdot [g] = [f \cdot g], \quad f \in \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(X), \quad g \in \bigoplus_{X \in \mathrm{Ob}(\mathcal{M})} \mathrm{End}_{\mathcal{M}}(X), \quad (5.2.2)$$

where, on the right-hand side,  $f \cdot g$  denotes the action from Theorem 3.2.3.

**Proof:** By (the  $\mathbb{k}$ -linear version of) Theorem 3.2.3, we have a  $\mathbb{k}$ -linear functor  $\mathrm{Aff}(\mathcal{C}) \boxtimes \mathcal{M} \rightarrow \mathcal{M}$ . This induces a  $\mathbb{k}$ -linear map  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}) \boxtimes \mathcal{M}) \rightarrow \mathrm{Tr}_v(\mathcal{M})$ . By Lemma 5.2.2, this induces a  $\mathbb{k}$ -linear map  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C})) \otimes \mathrm{Tr}_v(\mathcal{M}) \rightarrow \mathrm{Tr}_v(\mathcal{M})$ . The associativity and unity axioms follow from the corresponding properties of the action of  $\mathrm{Aff}(\mathcal{C})$  on  $\mathcal{M}$ . ■

If  $\mathcal{C}$  is a balanced strict  $\mathbb{k}$ -linear monoidal category, then we can take  $\mathcal{M} = \mathcal{C}$  and  $F$  the identity functor in Proposition 5.2.3 to see that  $\mathrm{Tr}_v(\mathcal{C})$  is a  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$ -module. If  $\mathcal{C}$  is strict pivotal, this action can be interpreted diagrammatically as follows. As described above, the element  $[f] \in \mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  can be viewed as a  $\mathcal{C}$ -diagram on the torus, while  $[g] \in \mathrm{Tr}_v(\mathcal{C})$  can be viewed as a  $\mathcal{C}$ -diagram in the annulus. Thickening the annulus, we view  $[g]$  as a diagram in the solid torus. Then  $[f] \cdot [g]$  is obtained by placing this solid torus inside the torus carrying the diagram of  $f$ , viewing the result as a diagram in the solid torus, which we then project onto the annulus, using the braiding to formalize what it means for one strand to pass over another.



We define an inverse to (5.2.3) by sending this element to  $[f]$ . The argument that this inverse is well defined is analogous to the argument given above that (5.2.3) is well defined. ■

### 5.3 Relation to factorization homology

The connection of affinization to factorization homology was brought to our attention by Adrien Brochier, and the goal of this section is to give the precise relationship between the two concepts. The definition of factorization homology is very technical and we will only give a very brief glimpse of what it is, without diving into details; the interested reader may consult [2, 27]. Note that none of the results of this thesis rely on this section; moreover, the reader can skip the definitions of this section and jump directly to Proposition 5.3.9, which gives an alternate definition of the factorization homology of the annulus.

**Definition 5.3.1.** An  $(\infty, 1)$ -category is an  $\infty$ -category for which all  $n$ -morphisms for  $n > 1$  are invertible.

**Definition 5.3.2.** Let  $\mathcal{Mfld}_n$  be the symmetric monoidal  $(\infty, 1)$ -category whose objects are (oriented or framed)  $n$ -dimensional manifolds, whose Hom-spaces  $\text{Hom}(X, Y)$  are the  $\infty$ -groupoids of (orientation-preserving or framed) embeddings  $X \hookrightarrow Y$  (that is to say, objects of  $\text{Hom}(X, Y)$  are smooth (oriented or framed) embeddings, the 1-morphisms are isotopies, the 2-morphisms are homotopies between the 1-morphisms and so on). The symmetric monoidal product in  $\mathcal{Mfld}_n$  is disjoint union.

**Definition 5.3.3.** Let  $\mathcal{Disc}_n$  be the full subcategory of  $\mathcal{Mfld}_n$  whose objects are finite disjoint unions of  $\mathbb{R}^n$ .

**Definition 5.3.4.** Let  $\mathcal{V}$  be a symmetric monoidal  $(\infty, 1)$ -category. An  $E_n$ -algebra is a symmetric monoidal functor  $F: \mathcal{Disc}_n \rightarrow \mathcal{V}$ .

**Remark 5.3.5.** Note that the  $E_n$ -algebra is determined by the image of  $\mathbb{R}^n$ , and for this reason it is common to call the image of  $\mathbb{R}^n$  an  $E_n$ -algebra in  $\mathcal{V}$ , where  $\mathcal{V}$  is the target category.

**Example 5.3.6.** This example is the content of [60, Th. 1.5]. Let  $\mathbb{k}$  be a field. Let  $F: \mathcal{Disc}_1^{or} \rightarrow \mathcal{Vect}_{\mathbb{k}}$  be a symmetric monoidal functor from the category of oriented 1-dimensional discs to the category of  $\mathbb{k}$ -vector spaces. Objects of  $\mathcal{Disc}_1^{or}$  are  $\emptyset, \mathbb{R}, \mathbb{R} \sqcup \mathbb{R}, \dots$  and morphisms are orientation-preserving smooth embeddings  $X \hookrightarrow Y$  up to isotopy. Then  $F(\mathbb{R})$  is a unital associative  $\mathbb{k}$ -algebra. Conversely, every unital associative  $\mathbb{k}$ -algebra can be obtained from an  $E_1$ -algebra in  $\mathcal{Vect}_{\mathbb{k}}$ . The unit in  $F(\mathbb{R})$  is  $F(\eta): \mathbb{k} \rightarrow$

$F(\mathbb{R})$ , where  $\eta$  is the unique embedding  $\emptyset \hookrightarrow \mathbb{R}$ . Now, up to isotopy, there are exactly two embeddings  $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ . This was explained in the proof of [60, Th. 1.5], but we will reproduce the explanation here for the reader's convenience. Let  $\iota: \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$  be a smooth orientation-preserving embedding. Order the two connected components of the domain  $\mathbb{R} \sqcup \mathbb{R}$ , and consider the image of each component under  $\iota$ . The orientation of the codomain either respects this order or not. This gives the two isotopy classes to which  $\iota$  can belong. Then one can define a multiplication on  $F(\mathbb{R})$  as  $F(\iota): F(\mathbb{R}) \otimes F(\mathbb{R}) \rightarrow F(\mathbb{R})$ , where  $\iota$  is one of the two embeddings  $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ .

**Definition 5.3.7.** Let  $\mathcal{V}$  be a symmetric monoidal  $(\infty, 1)$ -category and let  $F: \mathbf{Disc}_n \rightarrow \mathcal{V}$  be a framed  $E_n$ -algebra with  $\mathcal{E} = F(\mathbb{R}^n)$ . If it exists, the left Kan extension of the diagram

$$\begin{array}{ccc} \mathcal{Mfld}_n & & \\ \uparrow & \searrow \int_- \mathcal{E} & \\ \mathbf{Disc}_n & \xrightarrow{F} & \mathcal{V} \end{array}$$

is called the factorization homology with coefficients in  $\mathcal{E}$ , its image on the manifold  $M$  is called the factorization homology of  $M$  with coefficients in  $\mathcal{E}$  and is denoted  $\int_M \mathcal{E}$ .

**Remark 5.3.8.** The left Kan extension exists at least when  $\mathcal{V}$  is  $\otimes$ -presentable; see [2, Def. 3.4, Rem. 3.8] for details. We are essentially interested in the 2-dimensional case ( $n = 2$ ), and where the target category,  $\mathcal{V}$ , is a certain symmetric  $(2, 1)$ -category  $\mathcal{Pr}$ . This is the  $(2, 1)$  category of compactly generated presentable categories with compact and cocontinuous functors and their natural isomorphisms; we will not explain what this means, the interested reader can consult [7, §4] for details.

### 5.3.1 Factorization homology of the annulus

As mentioned in Remark 5.3.8, we consider the case  $n = 2$  with the target category  $\mathcal{Pr}$ . The  $E_2$ -algebra, or rather the image  $\mathcal{E} = F(\mathbb{R}^2)$ , is a braided monoidal category  $\mathcal{C}$ . This is the data needed to define the factorization homology of the annulus  $\int_{S^1 \times [0,1]} \mathcal{C}$ , with coefficients in  $\mathcal{C}$ . We can actually define the factorization homology of the annulus via the following universal property.

**Proposition 5.3.9** ([7, Prop. 3.10]). *Let  $\mathcal{C}$  be a braided monoidal category, and let  $\mathcal{M}$  be a monoidal category. Then monoidal functors  $\int_{S^1 \times [0,1]} \mathcal{C} \rightarrow \mathcal{M}$  are naturally identified with pairs  $(F, \eta)$ , where  $F$  is a monoidal functor from  $\mathcal{C}$  to  $\mathcal{M}$ , and  $\eta$  is a natural automorphism of  $F$  satisfying*

$$\eta_X \otimes 1_Y = F(\beta_{Y,X})(1_Y \otimes \eta_X) F(\beta_{X,Y}) \tag{5.3.1}$$

and

$$\eta_{X \otimes Y} = \eta_X \otimes \eta_Y \beta_{Y,X} \beta_{X,Y}, \quad (5.3.2)$$

for all  $X, Y \in \mathcal{C}$ .

We have a similar characterization in the case of the affinization.

**Proposition 5.3.10.** *Let  $\mathcal{C}$  be a braided strict monoidal category, and let  $\mathcal{M}$  be a strict monoidal category. Let  $F: \text{Aff}(\mathcal{C}) \rightarrow \mathcal{M}$  be a monoidal functor, and let  $F|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{M}$  be its restriction to  $\mathcal{C}$ . Then the assignment  $F \mapsto (F|_{\mathcal{C}}, \nu)$ , with  $\nu_X = F(\xi_X)$ , gives a one-to-one correspondence between monoidal functors from  $\text{Aff}(\mathcal{C})$  to  $\mathcal{M}$  and pairs  $(F, \nu)$ , where  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor and  $\nu$  is a natural automorphism of  $F$  satisfying*

$$\nu_X \otimes 1_Y = F(\beta_{Y,X})(1_Y \otimes \nu_X) F(\beta_{X,Y}) \quad (5.3.3)$$

and

$$\nu_{X \otimes Y} = \nu_X \otimes \nu_Y, \quad (5.3.4)$$

for all  $X, Y \in \mathcal{C}$ .

**Proof:** Let  $F: \text{Aff}(\mathcal{C}) \rightarrow \mathcal{M}$  be a monoidal functor. We let  $F': \mathcal{C} \rightarrow \mathcal{M}$  be the restriction of  $F$  to  $\mathcal{C}$ , and we define the natural automorphism  $\nu \in \text{Aut}(F')$  by  $\nu_X = F(\xi_X)$  for  $X \in \mathcal{C}$ . (Recall that  $\xi_X$  is invertible, so  $\nu$  is indeed an automorphism.) Then  $\nu$  satisfies (5.3.3) and (5.3.4) as these relations are just the image, under  $F$ , of the relations (3.1.11) satisfied by  $\xi_X$ .

Conversely, suppose we are given  $F': \mathcal{C} \rightarrow \mathcal{C}$  and  $\nu \in \text{Aut}(F')$  satisfying relations (5.3.3) and (5.3.4). Then we can extend  $F'$  to  $F: \text{Aff}(\mathcal{C}) \rightarrow \mathcal{M}$  by setting  $F(\xi_X) = \nu_X$ . ■

The two equations (5.3.2) and (5.3.4) differ by the double-braiding which appears in the first. However, if  $\mathcal{C}$  is balanced with twist  $\theta$  and if  $\eta$  is a natural automorphism, as in Proposition 5.3.9, that satisfies (5.3.1) and (5.3.2), then  $\eta^{-1}\theta^{-1}$  satisfies (5.3.3) and (5.3.4) (this was already mentioned in [7, Rem. 3.6], and it can easily be shown graphically). Thus, we get the following result.

**Corollary 5.3.11.** *Suppose  $\mathcal{C}$  is balanced with twist  $\theta$ . Then there is an equivalence of monoidal categories from  $\text{Aff}(\mathcal{C})$  to  $\int_{S^1 \times [0,1]} \mathcal{C}$  given by  $f \mapsto f, f \in \text{Mor}(\mathcal{C})$ , and  $\xi_X \mapsto \eta_X^{-1}\theta_X^{-1}$ , where  $\eta$  is the natural automorphism associated to the identity functor of  $\int_{S^1 \times [0,1]} \mathcal{C}$ , and obtained via Proposition 5.3.9.*

### 5.3.2 Braided module categories

We saw in Theorem 3.2.3 that under certain conditions a module category over  $\mathcal{C}$  can be made into a category over  $\text{Aff}(\mathcal{C})$ . Using the connection of the affinization to the factorization homology together with the results of [7], we give a closely related result.

**Definition 5.3.12** ([8, §5.1]). Let  $\mathcal{C}$  be a braided tensor category, with braiding  $\beta$ . A *strict braided module category* over  $\mathcal{C}$  is a strict right  $\mathcal{C}$ -module category  $\mathcal{M}$  equipped with a natural automorphism,  $E$ , of the action bifunctor  $\otimes: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$  satisfying the following axioms for all  $M \in \mathcal{M}, X, Y \in \mathcal{C}$ :

$$E_{M \otimes X, Y} = (1_M \otimes \beta_{X, Y}^{-1}) (E_{M, Y} \otimes 1_X) (1_M \otimes \beta_{Y, X}^{-1}), \quad (5.3.5)$$

and

$$E_{M, X \otimes Y} = (E_{M, X} \otimes 1_Y) E_{M \otimes X, Y}. \quad (5.3.6)$$

A category  $\mathcal{M}$  having a right module structure over  $\int_{S^1 \times [0, 1]} \mathcal{C}$  means there is a monoidal functor  $\int_{S^1 \times [0, 1]} \mathcal{C} \rightarrow \text{End}(\mathcal{M})$ ,  $X \mapsto - \otimes X$ . By Proposition 5.3.9, this is the same as a functor  $\mathcal{C} \rightarrow \text{End}(\mathcal{M})$ ,  $X \mapsto - \otimes X$  equipped with a natural automorphism  $\eta_X^M: M \otimes X \rightarrow M \otimes X$  ( $M \in \mathcal{M}$  and  $X \in \mathcal{C}$ ), satisfying (5.3.1) and (5.3.2). Defining  $E_{M, X} := \eta_X^M$ , we see that  $E$  satisfies relations (5.3.5) and (5.3.6). This makes  $\mathcal{M}$  into a braided  $\mathcal{C}$ -module category. This gives the following identification.

**Proposition 5.3.13** ([7, Theorem 3.11]). *Let  $\mathcal{C}$  be a braided tensor category. Then right modules over  $\int_{S^1 \times [0, 1]} \mathcal{C}$  are naturally identified with  $\mathcal{C}$ -braided module categories.*

Hence we get the following characterization of right module categories over  $\text{Aff}(\mathcal{C})$ .

**Corollary 5.3.14.** *If  $\mathcal{C}$  is balanced then strict right  $\text{Aff}(\mathcal{C})$ -module categories are identified with strict right  $\mathcal{C}$ -module categories  $\mathcal{M}$  together with a natural automorphism of the action bifunctor  $\mathcal{M} \otimes \mathcal{C} \rightarrow \mathcal{M}$  satisfying (5.3.5) and (5.3.6).*

**Example 5.3.15.** Let  $\mathcal{C}$  be a balanced monoidal category with braiding  $\beta$  and twist  $\theta$ . The natural automorphism of the bifunctor  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  given by  $E_{X, Y} := \beta_{X, Y} \circ (1_X \otimes \theta_Y)$  makes  $\mathcal{C}$  into a braided module category over itself. This action of  $\mathcal{C}$  over itself extends to an action of  $\text{Aff}(\mathcal{C})$  over  $\mathcal{C}$  where a morphism  $g$  in  $\text{Aff}(\mathcal{C})$  acts on a morphism  $f$  in  $\mathcal{C}$  by  $f \cdot g := f \otimes F(g)$ , where  $F$  is the functor (3.2.4). More examples of braided module categories can be found in [7, §1.1].

# Chapter 6

## The elliptic Hall algebra

In this chapter we introduce the main object of study for this second half of the thesis, the elliptic Hall algebra. We define this algebra, compute its universal central extension and then explain its connection to the algebra of Burban and Schiffmann [15] (which is not needed for the results of the thesis).

Unless otherwise specified, we work over an arbitrary ring  $\mathbb{k}$  of characteristic zero, and we fix  $q \in \mathbb{k}^\times$  such that

$$\{d\} := q^d - q^{-d} \in \mathbb{k}^\times \text{ for all } d \neq 0. \quad (6.0.1)$$

Thus the most generic choice is  $\mathbb{k} = \mathbb{Z}[q^{\pm 1}, \{d\}^{-1} : d \geq 1]$ . Note that  $\{d\}$  is defined for  $d = 0$ , but we only require it to be invertible when  $d \neq 0$ . All algebras and tensor products are over  $\mathbb{k}$  unless otherwise indicated.

### 6.1 The elliptic Hall algebra and its universal central extension

Let

$$\begin{aligned} \mathbf{Z} &:= \mathbb{Z}^2, & \mathbf{Z}^* &:= \mathbf{Z} \setminus \{(0, 0)\}, \\ \mathbf{Z}^+ &:= \{(r, n) \in \mathbf{Z} : n > 0 \text{ or } n = 0, r > 0\}, & \mathbf{Z}^- &:= -\mathbf{Z}^+. \end{aligned}$$

Let  $\mathfrak{EH}$  be the Lie algebra over  $\mathbb{k}$  with basis  $w_{\mathbf{x}}$ ,  $\mathbf{x} \in \mathbf{Z}^*$ , and Lie bracket given by

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\}w_{\mathbf{x}+\mathbf{y}}, \quad \text{where } d = \det(\mathbf{x} \ \mathbf{y}). \quad (6.1.1)$$

Here  $(\mathbf{x} \ \mathbf{y})$  denotes the  $2 \times 2$  matrix with columns  $\mathbf{x}$  and  $\mathbf{y}$ . We will write  $w_{r,n}$  for  $w_{(r,n)}$ , and we adopt the convention that  $w_{0,0} = 0$ . It is a straightforward computation to verify that (6.1.1) satisfies the axioms of a Lie bracket. It is also not hard to see that

$\mathfrak{E}\mathfrak{H}$  is perfect, that is,  $[\mathfrak{E}\mathfrak{H}, \mathfrak{E}\mathfrak{H}] = \mathfrak{E}\mathfrak{H}$ . Thus,  $\mathfrak{E}\mathfrak{H}$  has a universal central extension, which we now describe.

Let  $\mathbf{Z}_{\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}} \mathbf{Z} \cong \mathbb{k}^2$ . It is straightforward to check that the  $\mathbb{k}$ -bilinear map

$$\mathfrak{E}\mathfrak{H} \times \mathfrak{E}\mathfrak{H} \rightarrow \mathbf{Z}_{\mathbb{k}}, \quad (w_{\mathbf{x}}, w_{\mathbf{y}}) \mapsto \delta_{\mathbf{x}, -\mathbf{y}} \mathbf{x},$$

is a 2-cocycle, where we view  $\mathbf{Z}_{\mathbb{k}}$  as a trivial  $\mathfrak{E}\mathfrak{H}$ -module. Let  $\widetilde{\mathfrak{E}\mathfrak{H}}$  be the corresponding central extension. Thus,  $\widetilde{\mathfrak{E}\mathfrak{H}} = \mathfrak{E}\mathfrak{H} \oplus \mathbf{Z}_{\mathbb{k}}$  as  $\mathbb{k}$ -modules, with Lie bracket given by the fact that the elements of  $\mathbf{Z}_{\mathbb{k}}$  are central and

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}} + \delta_{\mathbf{x}, -\mathbf{y}} \mathbf{x}, \quad \text{where } d = \det \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix}. \quad (6.1.2)$$

**Remark 6.1.1.** The Lie algebra  $\widetilde{\mathfrak{E}\mathfrak{H}}$  contains a copy of the infinite rank Heisenberg algebra for every rank one sublattice of  $\mathbf{Z}$ . More precisely, for  $\mathbf{x} \in \mathbf{Z}^*$ , we have

$$[w_{i\mathbf{x}}, w_{j\mathbf{x}}] = i\delta_{i,-j} \mathbf{x}, \quad i, j \in \mathbb{Z} \setminus \{0\}, \quad (6.1.3)$$

and so  $\text{span}_{\mathbb{k}}\{w_{i\mathbf{x}}, \mathbf{x} : i \in \mathbb{Z} \setminus \{0\}\}$  is an infinite rank Heisenberg algebra with central element  $\mathbf{x}$ .

The main result of this section is the following proposition.

**Proposition 6.1.2.** *The Lie algebra  $\widetilde{\mathfrak{E}\mathfrak{H}}$  is the universal central extension of  $\mathfrak{E}\mathfrak{H}$ .*

The proof of Proposition 6.1.2 will occupy the rest of the section. The proof uses an argument inspired by the one in [37]. Let

$$0 \rightarrow \mathfrak{Z} \rightarrow \widehat{\mathfrak{E}\mathfrak{H}} \xrightarrow{\hat{\pi}} \mathfrak{E}\mathfrak{H} \rightarrow 0 \quad (6.1.4)$$

be an arbitrary central extension of  $\mathfrak{E}\mathfrak{H}$ . We must show that there exists a unique homomorphism of Lie algebras  $\zeta: \widetilde{\mathfrak{E}\mathfrak{H}} \rightarrow \widehat{\mathfrak{E}\mathfrak{H}}$  such that  $\hat{\pi}\zeta = \tilde{\pi}$ , where

$$\tilde{\pi}: \widetilde{\mathfrak{E}\mathfrak{H}} \rightarrow \mathfrak{E}\mathfrak{H}, \quad w_{\mathbf{x}} \mapsto w_{\mathbf{x}}, \quad z \mapsto 0, \quad \mathbf{x} \in \mathbf{Z}^*, \quad z \in \mathbf{Z}_{\mathbb{k}}.$$

Fix a linear map (not necessarily a homomorphism of Lie algebras)  $\zeta_1: \mathfrak{E}\mathfrak{H} \rightarrow \widehat{\mathfrak{E}\mathfrak{H}}$  such that  $\hat{\pi}\zeta_1 = \text{id}_{\mathfrak{E}\mathfrak{H}}$ . Since

$$\hat{\pi}([\zeta_1(x), \zeta_1(y)]) - \hat{\pi}\zeta_1([x, y]) = [x, y] - [x, y] = 0, \quad x, y \in \mathfrak{E}\mathfrak{H},$$

it follows that  $[\zeta_1(x), \zeta_1(y)] - \zeta_1([x, y])$  is contained in  $\mathfrak{Z}$ , for all  $x, y \in \mathfrak{E}\mathfrak{H}$ , and so we can define

$$\vartheta: \mathfrak{E}\mathfrak{H} \times \mathfrak{E}\mathfrak{H} \rightarrow \mathfrak{Z}, \quad \vartheta(x, y) = [\zeta_1(x), \zeta_1(y)] - \zeta_1([x, y]), \quad x, y \in \mathfrak{E}\mathfrak{H}. \quad (6.1.5)$$

It follows immediately from the fact that the Lie bracket on  $\mathfrak{E}\mathfrak{H}$  is alternating and satisfies the Jacobi identity that

$$\vartheta(x, y) = -\vartheta(y, x), \quad \vartheta(x, x) = 0, \quad (6.1.6)$$

$$\vartheta([x, y], z) + \vartheta([y, z], x) + \vartheta([z, x], y) = 0, \quad (6.1.7)$$

for all  $x, y, z \in \mathfrak{E}\mathfrak{H}$ . (The two equations in (6.1.6) are equivalent when  $\text{char}(\mathbb{k}) \neq 2$ .)

**Lemma 6.1.3.** *The function*

$$\mathbf{Z} \rightarrow \mathfrak{Z}, \quad \mathbf{x} \mapsto \begin{cases} \vartheta(w_{\mathbf{x}}, w_{-\mathbf{x}}) & \text{if } \mathbf{x} \neq 0, \\ 0 & \text{if } \mathbf{x} = 0, \end{cases}$$

is a homomorphism of additive groups.

**Proof:** Suppose  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  satisfy  $\det(\mathbf{x} \ \mathbf{y}) \neq 0$ , that is,  $\mathbf{x}$  and  $\mathbf{y}$  are not collinear. Taking  $x = w_{\mathbf{x}}, y = w_{\mathbf{y}}, z = w_{-\mathbf{x}-\mathbf{y}}$  in (6.1.7), and letting  $\mathbf{z} = -\mathbf{x} - \mathbf{y}$ , we have

$$\{\det(\mathbf{x} \ \mathbf{y})\}\vartheta(w_{\mathbf{x}+\mathbf{y}}, w_{-\mathbf{x}-\mathbf{y}}) + \{\det(\mathbf{y} \ \mathbf{z})\}\vartheta(w_{-\mathbf{x}}, w_{\mathbf{x}}) + \{\det(\mathbf{z} \ \mathbf{x})\}\vartheta(w_{-\mathbf{y}}, w_{\mathbf{y}}). \quad (6.1.8)$$

The determinants appearing in (6.1.8) are all equal, and so, also using (6.1.6), we have

$$\vartheta(w_{\mathbf{x}+\mathbf{y}}, w_{-\mathbf{x}-\mathbf{y}}) = \vartheta(w_{\mathbf{x}}, w_{-\mathbf{x}}) + \vartheta(w_{\mathbf{y}}, w_{-\mathbf{y}}).$$

On the other hand, if  $\mathbf{x}$  and  $\mathbf{y}$  are collinear, choose  $\mathbf{z}$  that is not collinear with  $\mathbf{x}$  (equivalently, not collinear with  $\mathbf{y}$ ). Then, using the non-collinear case proved above, we have

$$\begin{aligned} \vartheta(w_{\mathbf{x}+\mathbf{y}}, w_{-\mathbf{x}-\mathbf{y}}) &= \vartheta(w_{\mathbf{x}+\mathbf{z}}, w_{-\mathbf{x}-\mathbf{z}}) + \vartheta(w_{\mathbf{y}-\mathbf{z}}, w_{-\mathbf{y}+\mathbf{z}}) \\ &= \vartheta(w_{\mathbf{x}}, w_{-\mathbf{x}}) + \vartheta(w_{\mathbf{z}}, w_{-\mathbf{z}}) + \vartheta(w_{\mathbf{y}}, w_{-\mathbf{y}}) + \vartheta(w_{-\mathbf{z}}, w_{\mathbf{z}}) \\ &\stackrel{(6.1.6)}{=} \vartheta(w_{\mathbf{x}}, w_{-\mathbf{x}}) + \vartheta(w_{\mathbf{y}}, w_{-\mathbf{y}}), \end{aligned}$$

completing the proof. ■

**Lemma 6.1.4.** *We have*

$$\vartheta(w_{r,0}, w_{s,0}) = 0 = \vartheta(w_{0,m}, w_{0,n}), \quad r, s, m, n \in \mathbb{Z} \setminus \{0\}, \quad r \neq -s, \quad m \neq -n.$$

**Proof:** We prove the first equality, since the proof of the second is similar. Since the case  $r = s$  follows immediately from (6.1.6), suppose  $r \neq s$ . Taking  $x = w_{0,1}, y = w_{0,-1}, z = w_{r+s,0}$  in (6.1.7), and then dividing by  $\{r+s\}$ , we have

$$\vartheta(w_{r+s,-1}, w_{0,1}) + \vartheta(w_{r+s,1}, w_{0,-1}) = 0. \quad (6.1.9)$$

Next, take  $x = w_{s,-1}, y = w_{0,1}, z = w_{r,0}$  in (6.1.7) to get

$$\{s\}\vartheta(w_{s,0}, w_{r,0}) - \{r\}\vartheta(w_{r,1}, w_{s,-1}) - \{r\}\vartheta(w_{r+s,-1}, w_{0,1}) = 0. \quad (6.1.10)$$

Then take  $x = w_{r,1}, y = w_{0,-1}, z = w_{s,0}$  in (6.1.7) to get

$$-\{r\}\vartheta(w_{r,0}, w_{s,0}) + \{s\}\vartheta(w_{s,-1}, w_{r,1}) + \{s\}\vartheta(w_{r+s,1}, w_{0,-1}) = 0. \quad (6.1.11)$$

Subtracting  $\{s\}$  times (6.1.10) from  $\{r\}$  times (6.1.11), then using (6.1.6) and (6.1.9), we get

$$(\{s\}^2 - \{r\}^2)\vartheta(w_{r,0}, w_{s,0}) = \{s+r\}\{s-r\}\vartheta(w_{r,0}, w_{s,0}) = 0.$$

Since  $r \neq \pm s$ , the result follows.  $\blacksquare$

Define

$$\zeta_2: \mathfrak{E}\mathfrak{H} \rightarrow \mathfrak{Z}, \quad \zeta_2(w_{r,n}) = \begin{cases} \frac{1}{\{rn\}}\vartheta(w_{r,0}, w_{0,n}) & \text{if } r, n \neq 0, \\ \frac{1}{\{n\}}\vartheta(w_{1,n}, w_{-1,0}) & \text{if } r = 0, n \neq 0, \\ \frac{1}{\{r\}}\vartheta(w_{0,-1}, w_{r,1}) & \text{if } r \neq 0, n = 0. \end{cases} \quad (6.1.12)$$

**Lemma 6.1.5.** *We have*

$$\vartheta(w_{\mathbf{x}}, w_{\mathbf{y}}) = \zeta_2([w_{\mathbf{x}}, w_{\mathbf{y}}]), \quad \mathbf{x}, \mathbf{y} \in \mathbf{Z}^*, \quad \mathbf{x} + \mathbf{y} \neq 0. \quad (6.1.13)$$

**Proof:** Let  $\mathbf{x} = (r, n)$ ,  $\mathbf{y} = (s, m)$ . If  $n = m = 0$  or  $r = s = 0$ , then the result holds by Lemma 6.1.4. If  $s = n = 0$ , then we have

$$\vartheta(w_{r,0}, w_{0,m}) \stackrel{(6.1.12)}{=} \{rm\}\zeta_2(w_{r,m}) \stackrel{(6.1.1)}{=} \zeta_2([w_{r,0}, w_{0,m}]).$$

The case  $r = m = 0$  then follows by using the fact that both sides of (6.1.13) are antisymmetric in the arguments  $w_{\mathbf{x}}$  and  $w_{\mathbf{y}}$ . We have now proved that (6.1.13) holds when at least two of  $r, s, m, n$  are zero. Therefore, for the remainder of the proof, we assume that at most one of these is zero.

Suppose that  $m = 0$  and  $r + s \neq 0$ . Taking  $x = w_{-s,n}$ ,  $y = w_{s,0}$ ,  $z = w_{r+s,0}$  in (6.1.7), we have

$$-\{sn\}\vartheta(w_{0,n}, w_{r+s,0}) + \{(r+s)n\}\vartheta(w_{r,n}, w_{s,0}) = 0.$$

Thus, we have

$$\vartheta(w_{r,n}, w_{s,0}) = \frac{\{sn\}}{\{(r+s)n\}}\vartheta(w_{0,n}, w_{r+s,0}) \stackrel{(6.1.12)}{=} -\{sn\}\zeta_2(w_{r+s,n}) \stackrel{(6.1.1)}{=} \zeta_2(w_{r,n}, w_{s,0}).$$

On the other hand, if  $m = 0 = r + s$ , then, taking  $x = w_{-1,0}$ ,  $y = w_{r+1,n}$ ,  $z = w_{-r,0}$  in (6.1.7), we have

$$-\{n\}\vartheta(w_{r,n}, w_{-r,0}) + \{rn\}\vartheta(w_{1,n}, w_{-1,0}) = 0.$$

Therefore

$$\vartheta(w_{r,n}, w_{-r,0}) = \frac{\{rn\}}{\{n\}}\vartheta(w_{1,n}, w_{-1,0}) \stackrel{(6.1.12)}{=} \{rn\}\zeta_2(w_{0,n}) \stackrel{(6.1.1)}{=} \zeta_2([w_{r,n}, w_{-r,0}]).$$

This completes the proof of (6.1.13) when  $m = 0$ . The case  $n = 0$  then follows by using the fact that both sides of (6.1.13) are antisymmetric in the arguments  $w_{\mathbf{x}}$  and  $w_{\mathbf{y}}$ .

Next, we prove the cases  $r = 0$  and  $s = 0$ . Suppose that  $s = 0$  and  $m + n \neq 0$ . Taking  $x = w_{r,-m}$ ,  $y = w_{0,m}$ ,  $z = w_{0,m+n}$  in (6.1.7), we have

$$\{rm\}\vartheta(w_{r,0}, w_{0,m+n}) - \{r(m+n)\}\vartheta(w_{r,n}, w_{0,m}) = 0.$$

Thus,

$$\vartheta(w_{r,n}, w_{0,m}) = \frac{\{rm\}}{\{r(m+n)\}}\vartheta(w_{r,0}, w_{0,m+n}) \stackrel{(6.1.12)}{=} \{rm\}\zeta_2(w_{r,n}) \stackrel{(6.1.1)}{=} \zeta_2([w_{r,n}, w_{0,m}]).$$

On the other hand, if  $s = 0 = m + n$ , then, taking  $x = w_{0,-1}$ ,  $y = w_{r,n+1}$ ,  $z = w_{0,-n}$  in (6.1.7), we have

$$\{r\}\vartheta(w_{r,n}, w_{0,-n}) - \{rn\}\vartheta(w_{r,1}, w_{0,-1}) = 0.$$

Thus,

$$\vartheta(w_{r,n}, w_{0,-n}) = \frac{\{rn\}}{\{r\}}\vartheta(w_{r,1}, w_{0,-1}) \stackrel{(6.1.12)}{=} -\{rn\}\zeta_2(w_{r,n}) \stackrel{(6.1.1)}{=} \zeta_2([w_{r,n}, w_{0,-n}]).$$

The case  $r = 0$  now follows by using the fact that both sides of (6.1.13) are antisymmetric in the arguments  $w_{\mathbf{x}}$  and  $w_{\mathbf{y}}$ .

It remains to consider the case where  $r, s, m, n$  are all nonzero. In this case, taking  $x = w_{s,m}$ ,  $y = w_{r,0}$ ,  $z = w_{0,n}$  in (6.1.7) gives

$$-\{rm\}\vartheta(w_{r+s,m}, w_{0,n}) + \{rn\}\vartheta(w_{r,n}, w_{s,m}) - \{sn\}\vartheta(w_{s,m+n}, w_{r,0}) = 0.$$

Thus

$$\begin{aligned} \vartheta(w_{r,n}, w_{s,m}) &= \frac{\{rm\}}{\{rn\}}\vartheta(w_{r+s,m}, w_{0,n}) + \frac{\{sn\}}{\{rn\}}\vartheta(w_{s,m+n}, w_{r,0}) \\ &= \frac{\{rm\}}{\{rn\}}\zeta_2([w_{r+s,m}, w_{0,n}]) + \frac{\{sn\}}{\{rn\}}\zeta_2([w_{s,m+n}, w_{r,0}]) \\ &= \left( \frac{\{rm\}}{\{rn\}}\{(r+s)n\} - \frac{\{sn\}}{\{rn\}}\{r(m+n)\} \right) \zeta_2(w_{r+s,m+n}) \\ &= \{rm - sn\}\zeta_2(w_{r+s,m+n}) \\ &\stackrel{(6.1.1)}{=} \zeta_2([w_{r,n}, w_{s,m}]), \end{aligned}$$

where, in the second equality, we used the previous cases. ■

Define the linear map  $\zeta: \widetilde{\mathfrak{E}\mathfrak{H}} \rightarrow \widehat{\mathfrak{E}\mathfrak{H}}$  by

$$\begin{aligned} \zeta(x) &= \zeta_1(x) + \zeta_2(x), & x \in \mathfrak{E}\mathfrak{H}, \\ \zeta((a, b)) &= a\vartheta(w_{1,0}, w_{-1,0}) + b\vartheta(w_{0,1}, w_{0,-1}), & (a, b) \in \mathbf{Z}_{\mathbf{k}}. \end{aligned} \tag{6.1.14}$$

**Lemma 6.1.6.** *The map  $\zeta$  is a homomorphism of Lie algebras.*

**Proof:** To avoid potential confusion with the Lie bracket on  $\mathfrak{E}\mathfrak{H}$ , we denote the Lie bracket on  $\widetilde{\mathfrak{E}\mathfrak{H}}$  by  $[\cdot, \cdot]'$  in this proof. If  $x, y \in \widetilde{\mathfrak{E}\mathfrak{H}}$  with  $x \in \mathbf{Z}_k$  or  $y \in \mathbf{Z}_k$ , then

$$\zeta([x, y]') = 0 = [\zeta(x), \zeta(y)].$$

It remains to show that  $\zeta([w_{\mathbf{x}}, w_{\mathbf{y}}]) = [\zeta(w_{\mathbf{x}}), \zeta(w_{\mathbf{y}})]$  for  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ . We have

$$\zeta([w_{\mathbf{x}}, w_{\mathbf{y}}]') \stackrel{(6.1.2)}{=} \zeta([w_{\mathbf{x}}, w_{\mathbf{y}}] + \delta_{\mathbf{x}, -\mathbf{y}}\mathbf{x}) = \zeta_1([w_{\mathbf{x}}, w_{\mathbf{y}}]) + \zeta_2([w_{\mathbf{x}}, w_{\mathbf{y}}]) + \delta_{\mathbf{x}, -\mathbf{y}}\zeta(\mathbf{x}).$$

If  $\mathbf{x} + \mathbf{y} \neq 0$ , then, by Lemma 6.1.5, we have

$$\zeta([w_{\mathbf{x}}, w_{\mathbf{y}}]') = \zeta_1([w_{\mathbf{x}}, w_{\mathbf{y}}]) + \vartheta(w_{\mathbf{x}}, w_{\mathbf{y}}) \stackrel{(6.1.5)}{=} [\zeta_1(w_{\mathbf{x}}), \zeta_1(w_{\mathbf{y}})] = [\zeta(w_{\mathbf{x}}), \zeta(w_{\mathbf{y}})],$$

where the last equality follows from the fact that the image of  $\zeta_2$  is contained in  $\mathfrak{Z}$ , and hence in the center of  $\widetilde{\mathfrak{E}\mathfrak{H}}$ . On the other hand, if  $\mathbf{x} + \mathbf{y} = 0$ , with  $\mathbf{x} = (r, n)$ , then

$$\begin{aligned} \zeta([w_{\mathbf{x}}, w_{-\mathbf{x}}]') &= \zeta(\mathbf{x}) \\ &= r\vartheta(w_{1,0}, w_{-1,0}) + n\vartheta(w_{0,1}, w_{0,-1}) \\ &= \vartheta(w_{\mathbf{x}}, w_{-\mathbf{x}}) && \text{(by Lemma 6.1.3)} \\ &\stackrel{(6.1.5)}{=} [\zeta_1(w_{\mathbf{x}}), \zeta_1(w_{-\mathbf{x}})] \\ &= [\zeta(w_{\mathbf{x}}), \zeta(w_{-\mathbf{x}})], \end{aligned}$$

where the last equality again follows from the fact that the image of  $\zeta_2$  is contained in  $\mathfrak{Z}$ . ■

We are now ready to prove Proposition 6.1.2.

**Proof: (Proof of Proposition 6.1.2)**

Suppose we have a central extension as in (6.1.4). By Lemma 6.1.6, the map  $\zeta$  defined by (6.1.14) is a homomorphism of Lie algebras. For  $\mathbf{x} \in \mathbf{Z}^*$  and  $z \in \mathbf{Z}_k$ , we have

$$(\hat{\pi}\zeta)(w_{\mathbf{x}}) = \hat{\pi}(\zeta_1(w_{\mathbf{x}}) + \zeta_2(w_{\mathbf{x}})) = (\hat{\pi}\zeta_1)(w_{\mathbf{x}}) = w_{\mathbf{x}} = \tilde{\pi}(w_{\mathbf{x}}), \quad (\hat{\pi}\zeta)(z) = 0 = \tilde{\pi}(z).$$

Hence  $\hat{\pi}\zeta = \tilde{\pi}$ , and so  $\zeta$  is a morphism of extensions, as desired.

It remains to show uniqueness of  $\zeta$ . Suppose we have another homomorphism of Lie algebras  $\zeta': \widetilde{\mathfrak{E}\mathfrak{H}} \rightarrow \widetilde{\mathfrak{E}\mathfrak{H}}$  such that  $\hat{\pi}\zeta' = \tilde{\pi}$ . Let  $z \in \widetilde{\mathfrak{E}\mathfrak{H}}$ . Since  $\mathfrak{E}\mathfrak{H}$  is easily seen to be perfect (that is,  $[\widetilde{\mathfrak{E}\mathfrak{H}}, \widetilde{\mathfrak{E}\mathfrak{H}}] = \widetilde{\mathfrak{E}\mathfrak{H}}$ ), there exist  $x, y \in \widetilde{\mathfrak{E}\mathfrak{H}}$  such that  $z = [x, y]$ . Then we have

$$\zeta'(x) - \zeta(x), \zeta'(y) - \zeta(y) \in \ker(\hat{\pi}) = \mathfrak{Z}.$$

Since  $\mathfrak{Z}$  is contained in the center of  $\widehat{\mathfrak{EH}}$ , this implies that

$$\zeta'(z) = [\zeta'(x), \zeta'(y)] = [\zeta(x), \zeta(y)] = \zeta(z).$$

Thus  $\zeta' = \zeta$ , as desired. ■

**Corollary 6.1.7.** *The second cohomology module  $H^2(\mathfrak{EH}; \mathbb{k})$  has rank two, with basis given by the classes of the two cocycles  $\mathfrak{EH} \times \mathfrak{EH} \rightarrow \mathbb{k}$  defined by*

$$(w_{\mathbf{x}}, w_{\mathbf{y}}) \mapsto \delta_{\mathbf{x}, -\mathbf{y}} r, \quad \text{and} \quad (w_{\mathbf{x}}, w_{\mathbf{y}}) \mapsto \delta_{\mathbf{x}, -\mathbf{y}} n,$$

for  $\mathbf{x} = (r, n) \in \mathbf{Z}^*$ ,  $\mathbf{y} \in \mathbf{Z}^*$ .

## 6.2 Central reductions

Let  $\widetilde{\text{EH}}$  be the universal enveloping algebra of  $\widetilde{\mathfrak{EH}}$ . For a  $\mathbb{Z}$ -linear map  $\lambda: \mathbf{Z} \rightarrow \mathbb{Z}$ , define the corresponding central reduction

$$\text{EH}_\lambda = \widetilde{\text{EH}} / \langle \mathbf{x} - \lambda(\mathbf{x}) : \mathbf{x} \in \mathbf{Z} \rangle. \quad (6.2.1)$$

For  $k \in \mathbb{Z}$ , let

$$\lambda_k: \mathbf{Z} \rightarrow \mathbb{Z}, \quad (r, n) \mapsto kn,$$

and define  $\text{EH}_k := \text{EH}_{\lambda_k}$ . Thus,  $\text{EH}_k$  is the associative  $\mathbb{k}$ -algebra generated by  $w_{\mathbf{x}}$ ,  $\mathbf{x} \in \mathbf{Z}^*$ , and relations

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}} + kn \delta_{\mathbf{x}, -\mathbf{y}}, \quad \text{where } d = \det(\mathbf{x} \ \mathbf{y}), \ \mathbf{x} = (r, n). \quad (6.2.2)$$

We will denote the image of  $w_{\mathbf{x}}$  in  $\text{EH}_k$  again by  $w_{\mathbf{x}}$ .

The integral general linear group  $\text{GL}_2(\mathbb{Z})$  acts on  $\widetilde{\text{EH}}$  by  $\mathbb{k}$ -algebra automorphisms via

$$\widetilde{\text{EH}} \xrightarrow{\cong} \widetilde{\text{EH}}, \quad w_{\mathbf{x}} \mapsto \det(\gamma) w_{\gamma \mathbf{x}}, \quad \mathbf{x} \mapsto \gamma \mathbf{x}, \quad \mathbf{x} \in \mathbf{Z}, \ \gamma \in \text{GL}_2(\mathbb{Z}). \quad (6.2.3)$$

For a  $\mathbb{Z}$ -linear map  $\lambda: \mathbf{Z} \rightarrow \mathbb{Z}$ , this induces isomorphisms

$$\text{EH}_\lambda \xrightarrow{\cong} \text{EH}_{\lambda \gamma^{-1}}, \quad \gamma \in \text{GL}_2(\mathbb{Z}). \quad (6.2.4)$$

The importance of the  $\text{EH}_k$  is given by the following result, which says that every central reduction is isomorphic to some  $\text{EH}_k$ .

**Proposition 6.2.1.** *For every  $\mathbb{Z}$ -linear map  $\lambda: \mathbf{Z} \rightarrow \mathbb{Z}$ , there exists  $k \in \mathbb{Z}$  such that  $\text{EH}_\lambda \cong \text{EH}_k$  as algebras.*

**Proof:** Let  $\lambda: \mathbf{Z} \rightarrow \mathbb{Z}$  be a  $\mathbb{Z}$ -linear map. Thus, there exist  $a, b \in \mathbb{Z}$  such that  $\lambda(r, n) = ar + bn$ . Let  $k = \gcd(a, b)$ , and choose  $c, d \in \mathbb{Z}$  such that  $ac + bd = k$ . Define

$$\gamma = \begin{pmatrix} d & -c \\ a/k & b/k \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad \text{so that} \quad \gamma^{-1} = \begin{pmatrix} b/k & c \\ -a/k & d \end{pmatrix}.$$

Then we have  $\lambda\gamma^{-1} = \lambda_k$ , and the result follows from (6.2.4).  $\blacksquare$

**Remark 6.2.2.** Note that  $\mathrm{EH}_0 \cong U(\mathfrak{EH})$ . Furthermore, by [45, Th. 2, Th. 3],  $\mathrm{EH}_0$  is isomorphic to the elliptic Hall algebra of [15], specialized at  $\bar{\sigma} = q^2 = \sigma^{-1}$ .

**Remark 6.2.3.** Being the universal enveloping algebra of a Lie algebra,  $\widetilde{\mathrm{EH}}$  has a natural Hopf algebra structure. For  $\mathbb{Z}$ -linear maps  $\lambda_1, \lambda_2: \mathbf{Z} \rightarrow \mathbb{Z}$ , the coproduct on  $\widetilde{\mathrm{EH}}$  induces an algebra homomorphism  $E_{\lambda_1 + \lambda_2} \rightarrow E_{\lambda_1} \otimes E_{\lambda_2}$ . In particular, if  $M$  is an  $\mathrm{EH}_k$ -module and  $N$  is an  $\mathrm{EH}_l$ -module, then  $M \otimes N$  is naturally an  $\mathrm{EH}_{k+l}$ -module.

Let  $\mathrm{EH}^\pm$  be the subalgebra of  $\widetilde{\mathrm{EH}}$  generated by  $w_{\mathbf{x}}$ ,  $\mathbf{x} \in \mathbf{Z}^\pm$ . Note that, for any  $k \in \mathbb{Z}$ ,  $\mathrm{EH}^\pm$  is also isomorphic to the subalgebra of  $\mathrm{EH}_k$  generated by the  $w_{\mathbf{x}}$ ,  $\mathbf{x} \in \mathbf{Z}^\pm$ . It follows from the PBW theorem that multiplication induces a linear isomorphism

$$\mathrm{EH}^+ \otimes \mathrm{EH}^- \xrightarrow{\cong} \mathrm{EH}_k. \quad (6.2.5)$$

We have automorphisms

$$\psi: \widetilde{\mathrm{EH}} \xrightarrow{\cong} \widetilde{\mathrm{EH}}, \quad w_{\mathbf{x}} \mapsto w_{-\mathbf{x}}, \quad \mathbf{x} \mapsto -\mathbf{x}, \quad (6.2.6)$$

$$\omega: \widetilde{\mathrm{EH}} \xrightarrow{\cong} \widetilde{\mathrm{EH}}, \quad w_{r,n} \mapsto (-1)^{n+1} w_{r,-n}, \quad (r, n) \mapsto (r, -n), \quad (6.2.7)$$

and  $\psi(\mathrm{EH}^\pm) = \mathrm{EH}^\mp$ . (Note that  $\psi$  is the automorphism (6.2.3) for  $\gamma$  equal to negative the identity matrix.) For  $k \in \mathbb{Z}$ ,  $\psi$  and  $\omega$  induce algebra isomorphisms

$$\psi_k: \mathrm{EH}_k \xrightarrow{\cong} \mathrm{EH}_{-k}, \quad w_{\mathbf{x}} \mapsto w_{-\mathbf{x}}, \quad (6.2.8)$$

$$\omega_k: \mathrm{EH}_k \xrightarrow{\cong} \mathrm{EH}_{-k}, \quad w_{r,n} \mapsto (-1)^{n+1} w_{r,-n}. \quad (6.2.9)$$

### 6.3 Biangular presentations

**Lemma 6.3.1.** *The subalgebra  $\mathrm{EH}^\pm$  is generated by the elements  $w_{r,\pm 1}$ ,  $r \in \mathbb{Z}$ , and  $w_{\pm r,0}$ ,  $r \geq 1$ .*

**Proof:** It suffices to consider  $\mathrm{EH}^+$ , since the result for  $\mathrm{EH}^-$  then follows by applying the involution  $\psi$  from (6.2.6). Let  $\mathrm{EH}'$  be the subalgebra of  $\widetilde{\mathrm{EH}}$  generated by the elements  $w_{r,1}$ ,  $r \in \mathbb{Z}$ , and  $w_{r,0}$ ,  $r \geq 1$ . We show by induction on

$n \geq 1$  that  $w_{r,n} \in \text{EH}'$  for all  $r \in \mathbb{Z}$ , from which the lemma follows. The base case  $n = 1$  holds by definition. Let  $n \geq 1$ , and assume that  $w_{r,n} \in \text{EH}'$  for all  $r \in \mathbb{Z}$ . If  $r \neq 0$ , then  $w_{r,n+1} = \{r\}^{-1}[w_{r,n}, w_{0,1}] \in \text{EH}'$ . Otherwise, if  $r = 0$ , we have  $w_{0,n+1} = \{n+1\}^{-1}[w_{1,n}, w_{-1,1}] \in \text{EH}'$ .  $\blacksquare$

**Lemma 6.3.2.** *Suppose  $\mathfrak{g}$  is a Lie algebra with  $\mathbb{k}$ -module decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , where  $\mathfrak{g}_1, \mathfrak{g}_2$  are Lie subalgebras of  $\mathfrak{g}$ . Furthermore, suppose that  $S_i$  is a set of generators of  $\mathfrak{g}_i$ , as a Lie algebra, for  $i = 1, 2$ , and that  $[x, y] \in \text{span}_{\mathbb{k}}(S_1 \cup S_2)$  for all  $x \in S_1, y \in S_2$ . Then*

$$U(\mathfrak{g}) \cong (U(\mathfrak{g}_1) \star U(\mathfrak{g}_2)) / \langle xy - yx - [x, y] : x \in S_1, y \in S_2 \rangle,$$

where  $\star$  denotes the free product of associative algebras.

**Proof:** Let  $I$  be the ideal  $\langle xy - yx - [x, y] : x \in S_1, y \in S_2 \rangle$ . Consider the sequence

$$U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \xrightarrow{f} (U(\mathfrak{g}_1) \star U(\mathfrak{g}_2)) / I \xrightarrow{g} U(\mathfrak{g}),$$

where  $f$  is the  $\mathbb{k}$ -linear map given by multiplication, and  $g$  is the algebra homomorphism arising from the universal property of the free product and the fact that the generators of  $I$  are zero in  $U(\mathfrak{g})$ . The assumption  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  implies that  $g$  is surjective. Now, elements of  $U(\mathfrak{g}_1) \star U(\mathfrak{g}_2)$  can be written as linear combinations of words in  $S_1 \cup S_2$ . In the quotient  $(U(\mathfrak{g}_1) \star U(\mathfrak{g}_2)) / I$ , one can use the generators of  $I$  to move elements of  $U(\mathfrak{g}_1)$  to the left of elements of  $U(\mathfrak{g}_2)$  modulo shorter words. This implies that the map  $f$  is surjective. Finally, the composition  $gf$  is a linear isomorphism by the PBW theorem. It follows that  $g$  is surjective, and hence an isomorphism.  $\blacksquare$

**Proposition 6.3.3.** *The algebra  $\widetilde{\text{EH}}$  is isomorphic to the free product of the algebras  $\text{EH}^+ \otimes U(\mathbf{Z}_{\mathbb{k}})$  and  $\text{EH}^-$  modulo the following relations:*

$$[w_{s,-1}, w_{1,1}] = \{s+1\}w_{s+1,0} - \delta_{s,-1}(1,1), \quad s \in \mathbb{Z}, \quad (6.3.1)$$

$$[w_{s,\pm 1}, w_{\mp r,0}] = \{r\}w_{s\mp r,\pm 1}, \quad r \geq 1, s \in \mathbb{Z}, \quad (6.3.2)$$

$$[w_{r,0}, w_{-s,0}] = \delta_{r,s}(r,0), \quad r, s \geq 1, \quad (6.3.3)$$

$$[\mathbf{x}, w_{r,-1}] = [\mathbf{x}, w_{-s,0}] = 0, \quad r \in \mathbb{Z}, s \geq 1, \mathbf{x} \in \mathbf{Z}. \quad (6.3.4)$$

**Proof:** Let  $\mathfrak{g} = \mathfrak{E}\mathfrak{H}$ ,  $\mathfrak{g}_1 = \text{span}_{\mathbb{k}}\{w_{\mathbf{x}, \mathbf{y}} : \mathbf{x} \in \mathbf{Z}^+, \mathbf{y} \in \mathbf{Z}\}$ ,  $\mathfrak{g}_2 = \text{span}_{\mathbb{k}}\{w_{\mathbf{x}} : \mathbf{x} \in \mathbf{Z}^-\}$ ,  $S_1 = \{\mathbf{x}, w_{r,1}, w_{s,0} : \mathbf{x} \in \mathbf{Z}, r \in \mathbb{Z}, s \geq 1\}$ ,  $S_2 = \{w_{r,-1}, w_{-s,0} : r \in \mathbb{Z}, s \geq 1\}$ . Then it follows from Lemmas 6.3.1 and 6.3.2 that  $\widetilde{\text{EH}}$  is isomorphic to the free product of the algebras  $\text{EH}^+ \otimes U(\mathbf{Z}_{\mathbb{k}})$  and  $\text{EH}^-$  modulo the relations (6.3.2) to (6.3.4) and

$$[w_{s,-1}, w_{r,1}] = \{r+s\}w_{r+s,0} + \delta_{r,-s}(s,-1), \quad r, s \in \mathbb{Z}, \quad (6.3.5)$$

which specializes to (6.3.1) when  $r = 1$ .

It remains to show that the relations (6.3.1) to (6.3.4) imply (6.3.5). We first prove that they imply the  $r \geq 1$  cases of (6.3.5) by induction on  $r$ . Fix  $r \geq 1$  and suppose that (6.3.5) holds for all  $s \in \mathbb{Z}$ . Then, for  $s \in \mathbb{Z}$ , we have

$$\begin{aligned}
\{1\}[w_{s,-1}, w_{r+1,1}] &\stackrel{(6.3.2)}{=} [w_{s,-1}, [w_{1,0}, w_{r,1}]] \\
&= [[w_{s,-1}, w_{1,0}], w_{r,1}] + [w_{1,0}, [w_{s,-1}, w_{r,1}]] \\
&\stackrel{(6.3.2)}{=} \{1\}[w_{s+1,-1}, w_{r,1}] + \{r+s\}[w_{1,0}, w_{r+s,0}] + \delta_{r,-s}[w_{1,0}, (s, -1)] \\
&\stackrel{(6.3.3)}{=} \{1\}\{r+s+1\}w_{r+s+1,0} + \{1\}\delta_{r+1,-s}(s+1, -1) - \{1\}\delta_{r+1,-s}(1, 0) \\
&\stackrel{(6.3.4)}{=} \{1\}\{r+s+1\}w_{r+s+1,0} + \{1\}\delta_{r+1,-s}(s, -1),
\end{aligned}$$

where we used the Jacobi identity in the second equality and the induction hypothesis in the third and fourth equalities. Dividing both sides by  $\{1\}$ , this completes the proof of the induction step.

Finally, we prove that (6.3.1) to (6.3.4) imply (6.3.5) for  $r \leq 1$  by induction on  $r$ . Fix  $r \leq 1$  and suppose that (6.3.5) holds for all  $s \in \mathbb{Z}$ . Then, for  $s \in \mathbb{Z}$ , we have

$$\begin{aligned}
\{1\}[w_{s,-1}, w_{r-1,1}] &\stackrel{(6.3.2)}{=} [w_{s,-1}, [w_{r,1}, w_{-1,0}]] \\
&= [[w_{s,-1}, w_{r,1}], w_{-1,0}] + [w_{r,1}, [w_{s,-1}, w_{-1,0}]] \\
&\stackrel{(6.3.2)}{=} \{r+s\}[w_{r+s,0}, w_{-1,0}] + \delta_{r,-s}[(s, -1), w_{-1,0}] - \{1\}[w_{r,1}, w_{s-1,-1}] \\
&\stackrel{(6.3.3)}{=} \{1\}\delta_{r-1,-s}(1, 0) + \{1\}\{r+s-1\}w_{r+s-1,0} - \{1\}\delta_{r-1,-s}(r, 1) \\
&\stackrel{(6.3.4)}{=} \{1\}\{r+s-1\}w_{r+s-1,0} + \{1\}\delta_{r-1,-s}(s, -1),
\end{aligned}$$

where we used the Jacobi identity in the second equality, the induction hypothesis in the third and fourth equalities, and the relation  $[w_{s,-1}, w_{-1,0}] = -\{1\}w_{s-1,-1}$  in  $\text{EH}^-$  in the third equality.  $\blacksquare$

**Corollary 6.3.4.** *For  $k \in \mathbb{Z}$ , the algebra  $\text{EH}_k$  is isomorphic to the free product of the algebras  $\text{EH}^+$  and  $\text{EH}^-$  modulo the relations*

$$[w_{s,-1}, w_{1,1}] = \{s+1\}w_{s+1,0} - \delta_{s,-1}k, \quad s \in \mathbb{Z}, \quad (6.3.6)$$

$$[w_{s,\pm 1}, w_{\mp r,0}] = \{r\}w_{s \mp r, \pm 1}, \quad r \geq 1, \quad s \in \mathbb{Z}, \quad (6.3.7)$$

$$[w_{r,0}, w_{-s,0}] = 0, \quad r, s \geq 1. \quad (6.3.8)$$

**Lemma 6.3.5.** *For  $k \in \mathbb{Z}$ ,  $\text{EH}_k$  is generated, as an algebra, by  $w_{r,\pm 1}$ ,  $r \in \mathbb{Z}$ .*

**Proof:** Fix  $k \in \mathbb{Z}$ . Let  $A$  denote the subalgebra of  $\text{EH}_k$  generated by  $w_{r,\pm 1}$ ,  $r \in \mathbb{Z}$ . By Lemma 6.3.1, it suffices to show that  $w_{r,0} \in A$  for all nonzero  $r \in \mathbb{Z}$ . But this follows easily from the fact that, for  $r \in \mathbb{Z}$ ,  $r \neq 0$ , we have  $[w_{0,-1}, w_{r,1}] = \{r\}w_{r,0}$ . ■

**Corollary 6.3.6.** *The isomorphism  $\omega_k$  from (6.2.9) is the unique isomorphism  $\text{EH}_k \xrightarrow{\cong} \text{EH}_{-k}$  such that  $w_{r,\pm 1} \mapsto w_{r,\mp 1}$  for  $r \in \mathbb{Z}$ .*

## 6.4 Relation to the algebra of Burban and Schiffmann

In this section we show (in Proposition 6.4.1) how the central reductions defined in Section 6.2 are specializations of a central extension of the elliptic Hall algebra of Burban and Schiffmann [15]. This relationship is not used elsewhere in the thesis. The fundamental ingredient here is the work of Morton and Samuelson [45], who described an isomorphism between the HOMFLYPT skein algebra of the torus and the elliptic Hall algebra. This corresponds to the case of central charge  $k = 0$ . The case  $k = -1$  was treated in [17]; see Remark 6.4.2.

We work here over the field  $\mathbb{C}(v, q)$  of rational functions in two indeterminates. We first recall from [15, Def. 6.4] the definition of the central extension  $\widetilde{\text{BS}}$  of the elliptic Hall algebra. This central extension is denoted  $\widetilde{\mathcal{E}}_{\mathbf{K}}$  in [15], where  $\mathbf{K} = \mathbb{C}(v, q)$ . Our  $v$  and  $q$  are denoted  $\sigma^{1/2}$  and  $\bar{\sigma}^{1/2}$  in [15]. For  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ , we define

$$\epsilon_{\mathbf{x}} = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathbf{Z}^+, \\ -1 & \text{if } \mathbf{x} \in \mathbf{Z}^-, \end{cases} \quad \text{and} \quad \epsilon_{\mathbf{x}, \mathbf{y}} = \text{sign}(\det(\mathbf{x} \ \mathbf{y})) \in \{\pm 1\}, \quad \text{if } \mathbf{x}, \mathbf{y} \text{ are not collinear.}$$

For  $a \in \mathbb{C}(v, q)^\times$  and  $d \in \mathbb{Z}$ , define

$$\{d\}_a := a^d - a^{-d} \quad \text{and} \quad [d]_a := \frac{\{d\}_a}{\{1\}_a} = \frac{a^d - a^{-d}}{a - a^{-1}}.$$

These are both elements of  $\mathbb{Z}[a^{\pm 1}] \subseteq \mathbb{C}(v, q)$ . Note also that  $[d]_1 = d$ . For  $d \geq 1$ , we define

$$\alpha_d := \frac{1}{d}(1 - v^{2d})(1 - q^{2d})(1 - (vq)^{-2d}) = \frac{1}{d}\{d\}_v\{d\}_q\{d\}_{vq}.$$

Define  $\widetilde{\text{BS}}$  to be the  $\mathbb{C}(v, q)$ -algebra with generators

$$\kappa_{\mathbf{x}}, \ \mathbf{x} \in \mathbf{Z}, \quad u_{\mathbf{x}}, \ \mathbf{x} \in \mathbf{Z}^*,$$

modulo the following relations:

(a) The  $\kappa_{\mathbf{x}}$  are central, and we have

$$\kappa_{(0,0)} = 1, \quad \kappa_{\mathbf{x}}\kappa_{\mathbf{y}} = \kappa_{\mathbf{x}+\mathbf{y}}.$$

(b) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are collinear, then

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \delta_{\mathbf{x}, -\mathbf{y}} \frac{\kappa_{\mathbf{x}} - \kappa_{\mathbf{x}}^{-1}}{\alpha_{\gcd(\mathbf{x})}},$$

where  $\gcd(\mathbf{x})$  denotes the greatest common denominator of the components of  $\mathbf{x}$ .

(c) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are not collinear,  $\gcd(\mathbf{x}) = 1$ , and the triangle in  $\mathbf{Z}$  with vertices  $\{(0, 0), \mathbf{x}, \mathbf{x} + \mathbf{y}\}$  has no element of  $\mathbf{Z}$  in its interior, then

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x}, \mathbf{y}} \kappa_{\alpha(\mathbf{x}, \mathbf{y})} \frac{\theta_{\mathbf{x} + \mathbf{y}}}{\alpha_1},$$

where

$$\alpha(\mathbf{x}, \mathbf{y}) = \begin{cases} \epsilon_{\mathbf{x}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x} + \mathbf{y}}(\mathbf{x} + \mathbf{y}))/2 & \text{if } \epsilon_{\mathbf{x}, \mathbf{y}} = 1, \\ \epsilon_{\mathbf{y}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x} + \mathbf{y}}(\mathbf{x} + \mathbf{y}))/2 & \text{if } \epsilon_{\mathbf{x}, \mathbf{y}} = -1, \end{cases}$$

and where the elements  $\theta_{\mathbf{z}}, \mathbf{z} \in \mathbf{Z}^*$ , are determined by

$$\sum_{i \geq 1} \theta_{i\mathbf{x}_0} w^i = \exp \left( \sum_{r \geq 1} \alpha_r u_{r\mathbf{x}_0} w^r \right)$$

for any  $\mathbf{x}_0 \in \mathbf{Z}^*$  such that  $\gcd(\mathbf{x}_0) = 1$ . Here  $w$  is a formal variable.

The relations imply that the  $\mathbb{C}(v, q)$ -subalgebra  $K$  generated by the  $\kappa_{\mathbf{x}}, \mathbf{x} \in \mathbf{Z}$ , is isomorphic to the group algebra, over  $\mathbb{C}(v, q)$ , of the abelian group  $\mathbf{Z}$ , and that  $\widetilde{\text{BS}}$  is naturally a  $K$ -algebra.

Fix a  $\mathbb{Z}$ -linear map  $\lambda: \mathbf{Z} \rightarrow \mathbb{Z}$  and define  $\widetilde{\text{BS}}_{\lambda}$  to be the  $\mathbb{C}(v, q)$ -algebra obtained from  $\widetilde{\text{BS}}$  by imposing the additional relations

$$\kappa_{\mathbf{x}} = (vq)^{\lambda(\mathbf{x})}, \quad \mathbf{x} \in \mathbf{Z}.$$

Now define the following elements of  $\widetilde{\text{BS}}_{\lambda}$ :

$$w_{\mathbf{x}} := \{\gcd(\mathbf{x})\}_v u_{\mathbf{x}}, \quad \mathbf{x} \in \mathbf{Z}^*.$$

Let  $\widetilde{\text{BS}}'_{\lambda}$  be the  $\mathbb{C}[v, q, \{d\}_v^{-1}, \{d\}_q^{-1} : d \geq 1]$ -subalgebra of  $\widetilde{\text{BS}}_{\lambda}$  generated by the  $w_{\mathbf{x}}, \mathbf{x} \in \mathbf{Z}^*$ . Thus  $\widetilde{\text{BS}}'_{\lambda}$  is the  $\mathbb{C}[v, q, \{d\}_v^{-1}, \{d\}_q^{-1} : d \geq 1]$ -algebra generated by  $w_{\mathbf{x}}, \mathbf{x} \in \mathbf{Z}^*$ , subject to the following relations:

(a) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are collinear, then

$$[w_{\mathbf{y}}, w_{\mathbf{x}}] = \delta_{\mathbf{x}, -\mathbf{y}} d \frac{\{d\}_v}{\{d\}_q} \left[ \frac{\lambda(\mathbf{x})}{d} \right]_{(vq)^d}, \quad \text{where } d = \gcd(\mathbf{x}).$$

- (b) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are such that  $\gcd(\mathbf{x}) = 1$  and the triangle in  $\mathbf{Z}$  with vertices  $\{(0, 0), \mathbf{x}, \mathbf{x} + \mathbf{y}\}$  has no interior lattice point, then

$$[w_{\mathbf{y}}, w_{\mathbf{x}}] = \epsilon_{\mathbf{x}, \mathbf{y}} \kappa_{\alpha(\mathbf{x}, \mathbf{y})} \{1\}_v \{\gcd(\mathbf{y})\}_v \frac{\theta_{\mathbf{x} + \mathbf{y}}}{\alpha_1}.$$

Now, by [45, Lem. 5.4], we have, for  $\mathbf{x} \in \mathbf{Z}^*$ ,

$$\frac{\theta_{\mathbf{x}}}{\alpha_1} = ([\gcd(\mathbf{x})]_v)^2 u_{\mathbf{x}} = -\frac{[\gcd(\mathbf{x})]_q}{\{1\}_q} w_{\mathbf{x}} \quad \text{when } q = v^{-1}.$$

(Note that our  $v$  and  $q$  are the  $q^{1/2}$  and  $t^{-1/2}$  of [45], respectively.) Let  $\text{BS}_{\lambda} := \widetilde{\text{BS}}_{\lambda}/(vq-1)$ . Thus, setting  $\mathbb{k} = \mathbb{C}[q^{\pm 1}, \{d\}^{-1} : d \geq 1]$ , we see that  $\text{BS}_{\lambda}$  is the  $\mathbb{k}$ -algebra generated by  $w_{\mathbf{x}}, \mathbf{x} \in \mathbf{Z}^*$ , subject to the following relations:

- (a) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are collinear, then

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \delta_{\mathbf{x}, -\mathbf{y}} \lambda(\mathbf{x}). \quad (6.4.1)$$

- (b) If  $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$  are such that  $\gcd(\mathbf{x}) = 1$  and the triangle in  $\mathbf{Z}$  with vertices  $\{(0, 0), \mathbf{x}, \mathbf{x} + \mathbf{y}\}$  has no element of  $\mathbf{Z}$  in its interior, then

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \epsilon_{\mathbf{x}, \mathbf{y}} \{\gcd(\mathbf{y})\}_q [\gcd(\mathbf{x} + \mathbf{y})]_q w_{\mathbf{x} + \mathbf{y}}. \quad (6.4.2)$$

**Proposition 6.4.1.** *We have an isomorphism of  $\mathbb{k}$ -algebras*

$$\text{EH}_{\lambda} \xrightarrow{\cong} \text{BS}_{\lambda}, \quad w_{\mathbf{x}} \mapsto w_{\mathbf{x}}, \quad \mathbf{x} \in \mathbf{Z}^*.$$

**Proof:** When  $\lambda = 0$ , this is precisely [45, Th. 5.6] after recalling that the  $s, v, q, t$  of [45] are  $q, t^{-1}, v^2, q^{-2}$  in our notation. To prove the result for general  $\lambda$ , we make the dependence on  $\lambda$  explicit by letting  $[\cdot, \cdot]_{\lambda}$  denote the bracket on  $\text{BS}_{\lambda}$  given by (6.4.1) and (6.4.2). Then we have

$$[w_{\mathbf{x}}, w_{\mathbf{y}}]_{\lambda} = [w_{\mathbf{x}}, w_{\mathbf{y}}]_0 + \delta_{\mathbf{x}, -\mathbf{y}} \lambda(\mathbf{x}). \quad (6.4.3)$$

Comparing to (6.1.2) and (6.2.1), we see that this is precisely the relationship between the bracket in  $\text{EH}_0$  and the one in  $\text{EH}_{\lambda}$ . ■

**Remark 6.4.2.** When  $\lambda = 0$ ,  $\text{EH}_0$  is the elliptic Hall algebra (no central extension) denoted  $\mathcal{E}_{\sigma, \bar{\sigma}}$  in [15], specialized at  $\sigma^{-1/2} = q = \bar{\sigma}^{1/2}$ . When  $\lambda = \lambda_{-1}$ ,  $\text{EH}_{-1}$  is the algebra denoted  $\mathbb{E}$  in [17, Def. 4.4].

# Chapter 7

## The elliptic Hall algebra and the quantum Heisenberg category

In the first sections of this chapter, we recall the definition of the quantum Heisenberg category introduced in [13] and state some important relations that will be used in our computations to follow. In the final section, we prove the main result of the chapter Theorem 7.4.1.

Throughout this chapter, unless otherwise stated, we fix an integral domain  $\mathbb{k}$  containing  $\mathbb{Q}$ , and  $q \in \mathbb{k}^\times$  satisfying (6.0.1). Let  $z = q - q^{-1} = \{1\}$  and choose  $t \in \mathbb{k}^\times$ . The most generic choice of ground ring is thus  $\mathbb{k} = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}, \{d\}^{-1} : d \geq 1]$ . Another valid choice is a field  $\mathbb{k}$  of characteristic zero, with  $q, t \in \mathbb{k}^\times$  such that  $q$  is not a root of unity. We also fix a *central charge*  $k \in \mathbb{Z}$ .

### 7.1 Quantum Heisenberg category

#### 7.1.1 Definition

**Definition 7.1.1** ([13, Def. 4.1]). The *quantum Heisenberg category*  $\mathcal{H}eis_k$  is the strict  $\mathbb{k}$ -linear monoidal category generated by objects  $\uparrow, \downarrow$  and morphisms

$$\begin{aligned} \nearrow, \searrow: \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow, \quad \hat{\phi}: \uparrow \rightarrow \uparrow, \\ \cup: \mathbb{1} \rightarrow \downarrow \otimes \uparrow, \quad \cap: \uparrow \otimes \downarrow \rightarrow \mathbb{1}, \quad \cup: \mathbb{1} \rightarrow \uparrow \otimes \downarrow, \quad \cap: \downarrow \otimes \uparrow \rightarrow \mathbb{1}, \end{aligned}$$

subject to relations that we now describe. First, we require  $\hat{\phi}$ , which we call a *dot*, to be invertible. For  $r \in \mathbb{Z}$ , we let  $\hat{\phi}_r$  denote the composition of  $r$  dots if  $r \geq 0$  and the composition of  $|r|$  inverse dots if  $r < 0$ . We then impose the following additional relations:

$$\begin{aligned} \begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \searrow \\ \nearrow \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array}, \end{aligned} \tag{7.1.1}$$

$$\begin{array}{c} \nearrow \searrow - \searrow \nearrow = z \uparrow \uparrow, \end{array} \quad (7.1.2)$$

$$\begin{array}{c} \nearrow \searrow \circlearrowleft = \searrow \nearrow, \quad \searrow \nearrow \circlearrowright = \nearrow \searrow, \end{array} \quad (7.1.3)$$

$$\begin{array}{c} \cap = \uparrow, \quad \cup = \downarrow, \end{array} \quad (7.1.4)$$

$$\begin{array}{c} \text{crossing} = \uparrow \downarrow - t^{-1} z \text{cup} + z^2 \sum_{r,s>0} \text{bubble}, \end{array} \quad (7.1.5)$$

$$\begin{array}{c} \text{crossing} = \downarrow \uparrow + tz \text{cup} + z^2 \sum_{r,s>0} \text{bubble}, \end{array} \quad (7.1.6)$$

$$\begin{array}{c} \text{loop} = \delta_{k,0} t^{-1} \uparrow \quad \text{if } k \geq 0, \quad r \text{ loop} = \frac{\delta_{r,0} t - \delta_{r,k} t^{-1}}{z} 1_{\mathbb{1}} \quad \text{if } 0 \leq r \leq k, \end{array} \quad (7.1.7)$$

$$\begin{array}{c} \text{loop} = \delta_{k,0} t \uparrow \quad \text{if } k \leq 0, \quad r \text{ loop} = \frac{\delta_{r,-k} t - \delta_{r,0} t^{-1}}{z} 1_{\mathbb{1}} \quad \text{if } 0 \leq r \leq -k. \end{array} \quad (7.1.8)$$

(In fact, the second relation in (7.1.3) is redundant, since it follows from the first relation in (7.1.3) and the first two equalities in (7.1.1).) In the above relations we have used right and left crossings defined by

$$\begin{array}{c} \nearrow \searrow := \text{right crossing}, \quad \searrow \nearrow := \text{left crossing}, \quad \searrow \nearrow := \text{right crossing}, \quad \nearrow \searrow := \text{left crossing}, \end{array} \quad (7.1.9)$$

and (+)-bubbles defined by

$$\begin{array}{c} (+) \text{ bubble } r := \text{loop } r, \quad r \text{ bubble } (+) := r \text{ loop}, \quad r > 0, \end{array} \quad (7.1.10)$$

and

$$\begin{array}{c} (+) \text{ bubble } r-k := t^{r+1} z^{r-1} \det (k+i-j+1 \text{ loop}),_{i,j=1,\dots,r}, \quad r \leq k, \end{array} \quad (7.1.11)$$

$$\begin{array}{c} r+k \text{ bubble } (+) := -t^{-r-1} z^{r-1} \det (-\text{loop } -k+i-j+1),_{i,j=1,\dots,r}, \quad r \leq -k, \end{array} \quad (7.1.12)$$

where we interpret the determinants as  $\delta_{r,0}$  when  $r \leq 0$ . In particular, note that the sums appearing in (7.1.5) and (7.1.6) are finite. When we wish to make the parameters  $z$  and  $t$  explicit, we will write  $\mathcal{H}eis_k(z, t)$  for  $\mathcal{H}eis_k$ . This completes the definition of  $\mathcal{H}eis_k$ .

As explained in the proof of [13, Th. 4.2], the defining relations of  $\mathcal{H}eis_k$  imply that we have the isomorphisms

$$\begin{array}{c} (\nearrow \cap \circlearrowleft \cdots \circlearrowleft_{k-1})^T: \uparrow \otimes \downarrow \rightarrow \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k} \quad \text{if } k \geq 0, \\ (\nearrow \cup \circlearrowright \cdots \circlearrowright_{-k-1}): \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus (-k)} \rightarrow \downarrow \otimes \uparrow \quad \text{if } k \leq 0, \end{array} \quad (7.1.13)$$

in  $\text{Add}(\mathcal{H}eis_k)$ , where  $\text{Add}$  denotes the additive envelope.

### 7.1.2 Additional relations

We now recall some additional relations that hold in  $\mathcal{H}eis_k$ . It follows from the defining relations that

$$\begin{aligned} \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} &= \begin{array}{c} \uparrow \\ \downarrow \end{array} \text{ if } k < 0, & \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} &= \begin{array}{c} \downarrow \\ \uparrow \end{array} \text{ if } k > 0, & \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array} &= \begin{array}{c} \downarrow \\ \uparrow \end{array} \text{ if } k = 0, & \begin{array}{c} \searrow \\ \nearrow \\ \nearrow \\ \searrow \end{array} &= \begin{array}{c} \uparrow \\ \downarrow \end{array} \text{ if } k = 0. \end{aligned} \quad (7.1.14)$$

(See [13, (4.14)–(4.16)].)

Note that (7.1.4) implies that  $\uparrow$  is right dual to  $\downarrow$ . We also have (see [13, Lem. 3.7])

$$\begin{array}{c} \curvearrowright \\ \downarrow \end{array} = \downarrow, \quad \begin{array}{c} \uparrow \\ \curvearrowleft \end{array} = \uparrow, \quad (7.1.15)$$

so that  $\uparrow$  is also left dual to  $\downarrow$ . In fact  $\mathcal{H}eis_k$  is strictly pivotal, with duality functor defined on morphisms by rotating diagrams through  $180^\circ$ ; see [13, (3.2.1)]. Thus, for example, we can define downward crossings and dots by

$$\begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} := \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} := \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array}, \quad \begin{array}{c} \downarrow \\ \circ \\ \downarrow \end{array} := \begin{array}{c} \downarrow \\ \circ \\ \downarrow \end{array},$$

and we have right, left, and downwards skein relations,

$$\begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = z \begin{array}{c} \downarrow \\ \downarrow \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} - \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} = z \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = z \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array}. \quad (7.1.16)$$

as well as right, left, and downward versions of (7.1.3). In what follows, we will freely use the pivotal structure, referring to a relation by equation number even when we use a rotated version of it. In addition, it follows from the pivotal structure on  $\mathcal{H}eis_k$  that dots slide over cups and caps. Therefore, we will sometimes draw dots at the critical points of cups or caps, since this causes no ambiguity.

It follows from repeated use of (7.1.2) and (7.1.3) that the following relations hold for  $r \in \mathbb{Z}$ :

$$\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{cases} \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} - z \sum_{\substack{a+b=r \\ a,b>0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r > 0, \\ \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} + z \sum_{\substack{a+b=r \\ a,b \leq 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r \leq 0; \end{cases} \quad \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} = \begin{cases} \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} + z \sum_{\substack{a+b=r \\ a,b \geq 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r \geq 0, \\ \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} - z \sum_{\substack{a+b=r \\ a,b < 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r < 0; \end{cases} \quad (7.1.17)$$

$$\begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} = \begin{cases} \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} + z \sum_{\substack{a+b=r \\ a,b>0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r > 0, \\ \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} - z \sum_{\substack{a+b=r \\ a,b \leq 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r \leq 0; \end{cases} \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{cases} \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} - z \sum_{\substack{a+b=r \\ a,b \geq 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r \geq 0, \\ \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} + z \sum_{\substack{a+b=r \\ a,b < 0}} a \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} b & \text{if } r < 0. \end{cases} \quad (7.1.18)$$

We define  $(-)$ -bubbles (see [13, (2.18)]) by

$$\curvearrowright_r := \circlearrowleft_r - \circlearrowright_r, \quad r \circlearrowleft := r \circlearrowleft - r \circlearrowright, \quad r \in \mathbb{Z}. \quad (7.1.19)$$

We then have the *infinite grassmannian relations* ([13, Lem. 3.4])

$$\sum_{r+s=n} \circlearrowright_r \circlearrowright_s = \sum_{r+s=n} \curvearrowright_r \curvearrowright_s = -\delta_{n,0} z^{-2} 1_{\mathbb{1}}, \quad n \in \mathbb{Z}, \quad (7.1.20)$$

and the relations

$$\circlearrowright_r = \delta_{r,-k} t z^{-1} 1_{\mathbb{1}}, \quad r \leq -k, \quad \circlearrowright_r = -\delta_{r,k} t^{-1} z^{-1} 1_{\mathbb{1}}, \quad r \leq k, \quad (7.1.21)$$

$$\curvearrowright_r = \delta_{r,0} t z^{-1} 1_{\mathbb{1}}, \quad r \geq 0, \quad \curvearrowright_r = -\delta_{r,0} t^{-1} z^{-1} 1_{\mathbb{1}}, \quad r \geq 0. \quad (7.1.22)$$

It will be useful to express some of our relations in terms of generating functions in an indeterminate  $u$ . Define

$$\circlearrowright(u) := t^{-1} z \sum_{r \in \mathbb{Z}} \circlearrowright_r u^{-r} \in u^k 1_{\mathbb{1}} + u^{k-1} \text{End}_{\mathcal{H}eis_k}(\mathbb{1})[[u^{-1}]], \quad (7.1.23)$$

$$\circlearrowleft(u) := -t z \sum_{r \in \mathbb{Z}} \circlearrowleft_r u^{-r} \in u^{-k} 1_{\mathbb{1}} + u^{-k-1} \text{End}_{\mathcal{H}eis_k}(\mathbb{1})[[u^{-1}]], \quad (7.1.24)$$

$$\curvearrowright(u) := -t z \sum_{r \in \mathbb{Z}} \curvearrowright_r u^{-r} \in 1_{\mathbb{1}} + u \text{End}_{\mathcal{H}eis_k}(\mathbb{1})[[u]], \quad (7.1.25)$$

$$\curvearrowleft(u) := t^{-1} z \sum_{r \in \mathbb{Z}} \curvearrowleft_r u^{-r} \in 1_{\mathbb{1}} + u \text{End}_{\mathcal{H}eis_k}(\mathbb{1})[[u]]. \quad (7.1.26)$$

Then (7.1.20) can be restated as

$$\circlearrowright(u) \circlearrowleft(u) = \curvearrowright(u) \curvearrowleft(u) = 1_{\mathbb{1}}. \quad (7.1.27)$$

The following *curl relations* hold for all  $r \in \mathbb{Z}$  ([13, Lem. 4.4]):

$$r \circlearrowleft \uparrow = z \sum_{a \geq 0} \circlearrowright_{r-a} \uparrow_a - z \sum_{a > 0} \curvearrowright_{r+a} \uparrow_{-a}, \quad r \circlearrowright \uparrow = z \sum_{a > 0} \circlearrowleft_{r-a} \uparrow_a - z \sum_{a \geq 0} \curvearrowleft_{r+a} \uparrow_{-a}, \quad (7.1.28)$$

$$\uparrow \circlearrowright_r = z \sum_{a \geq 0} -a \uparrow_{r+a} \curvearrowright - z \sum_{a > 0} a \uparrow_{r-a} \circlearrowleft, \quad \uparrow \circlearrowleft_r = z \sum_{a > 0} -a \uparrow_{r+a} \curvearrowleft - z \sum_{a \geq 0} a \uparrow_{r-a} \circlearrowright. \quad (7.1.29)$$

By [13, Lem. 4.5] (see [14, Lem. 5.7] for a proof, taking  $A = \mathbb{k}$  there) we have the following braid relation for alternating crossings:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \sum_{\substack{r,s \geq 0 \\ a > 0}} -r-s-a \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad \text{if } k \geq 0, \quad (7.1.30)$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \sum_{\substack{r,s \geq 0 \\ a > 0}} a \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \quad \text{if } k \leq 0. \quad (7.1.31)$$

For all other orientations of the strands, the usual braid relation holds.

### 7.1.3 The center

Recall that the *center* of a monoidal category  $\mathcal{C}$  is the endomorphism algebra  $\text{End}_{\mathcal{C}}(\mathbb{1})$  of the unit object. In this subsection, we describe the center of the quantum Heisenberg category and how elements of the center slide past strands.

Let  $\text{Sym}$  denote the ring of symmetric functions with coefficients in  $\mathbb{k}$ . For  $r \in \mathbb{N}$ , let  $h_r$ ,  $e_r$ , and  $p_r$  denote the  $r$ -th complete homogeneous, elementary, and power sum symmetric functions, respectively. For  $f \in \text{Sym}$ , define the following elements of  $\text{Sym} \otimes \text{Sym}$ :

$$f^+ := f \otimes 1, \quad f^- := 1 \otimes f. \quad (7.1.32)$$

**Proposition 7.1.2.** *We have an isomorphism*

$$\beta: \text{Sym} \otimes \text{Sym} \rightarrow \text{End}_{\mathcal{H}\text{eis}_k}(\mathbb{1}), \quad (7.1.33)$$

given by, for  $r \geq 1$ ,

$$h_r^+ \mapsto -tz \begin{array}{c} \oplus \\ r+k \end{array}, \quad h_r^- \mapsto t^{-1}z \begin{array}{c} \ominus \\ -r \end{array}, \quad (7.1.34)$$

$$e_r^+ \mapsto (-1)^r t^{-1}z \begin{array}{c} \oplus \\ r-k \end{array}, \quad e_r^- \mapsto (-1)^{r-1} tz \begin{array}{c} \ominus \\ -r \end{array}, \quad (7.1.35)$$

$$p_r^+ \mapsto z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{c} \oplus \\ s \end{array} \begin{array}{c} \oplus \\ r-s \end{array}, \quad p_r^- \mapsto z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{c} \ominus \\ -s \end{array} \begin{array}{c} \ominus \\ s-r \end{array}. \quad (7.1.36)$$

**Proof:** The fact that we have an isomorphism  $\beta$  given by (7.1.34) and (7.1.35) was first shown in [13, Cor. 10.2], although we use here the sign conventions of [14, Cor. 9.3] (where the Frobenius algebra  $A$  there is  $\mathbb{k}$ ). For the power sums, recall that  $p_r = \sum_{s=0}^r (-1)^{s-1} s e_s h_{r-s}$ ; see [39, p. 33]. Also note that the maps (7.1.34) to (7.1.36) are valid for  $r = 0$  when we adopt the usual conventions that  $h_0 = e_0 = 1$  and  $p_0 = 0$ . The image of  $p_r^-$  given in (7.1.36) follows immediately. For the image of  $p_r^+$ , we have

$$p_r^+ \mapsto z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{c} \oplus \\ s-k \end{array} \begin{array}{c} \oplus \\ r-s+k \end{array} = z^2 \sum_{s \in \mathbb{Z}} (s+k) \begin{array}{c} \oplus \\ s \end{array} \begin{array}{c} \oplus \\ r-s \end{array} \stackrel{(7.1.20)}{=} z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{c} \oplus \\ s \end{array} \begin{array}{c} \oplus \\ r-s \end{array}.$$



Next we recall how bubbles slide past strings. The precise relation is easiest to state using the generating functions (7.1.23) to (7.1.26) and dots labelled by formal power series. We define

$$x^r \uparrow \circlearrowleft := r \uparrow \circlearrowleft, \quad r \in \mathbb{Z}.$$

Then, expanding linearly, we can also label dots by polynomials  $a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{k}[x]$ , or even by Laurent series in  $\mathbb{k}[x]((u^{-1}))$  or  $\mathbb{k}[x]((u))$ . For example, expanding in  $\mathbb{k}[x]((u^{-1}))$ , we have

$$xu(u-x)^{-2} \uparrow \circlearrowleft = u^{-1} \uparrow \circlearrowleft + 2u^{-2} \uparrow \circlearrowleft_2 + 3u^{-3} \uparrow \circlearrowleft_3 + 4u^{-4} \uparrow \circlearrowleft_4 + \cdots .$$

We adopt the convention that, in any equation involving the generating functions (7.1.25) and (7.1.26), we expand all rational functions as Laurent series in  $\mathbb{k}[x]((u))$ . In all other equations, we expand rational functions as Laurent series in  $\mathbb{k}[x]((u^{-1}))$ . With these conventions, we have the following *bubble slides*:

$$\left( \uparrow \circlearrowleft(u) \right) \uparrow = \uparrow \circlearrowleft_{1-z^2xu(u-x)^{-2}} \left( \uparrow \circlearrowleft(u) \right), \quad \uparrow \left( \uparrow \circlearrowleft(u) \right) = \uparrow \circlearrowleft_{1-z^2xu(u-x)^2} \left( \uparrow \circlearrowleft(u) \right). \quad (7.1.37)$$

(See [13, Lem. 4.6] for the statement and [14, Lem. 5.6] for a proof, taking  $A = \mathbb{k}$  there.) In fact, as we see in the next result, the bubble slides are simpler when using the images under  $\beta$  of the power sums. For  $r \in \mathbb{Z}$ , we define the following element of  $\text{End}_{\mathcal{H}eis_k}(\mathbb{1})$ :

$$\circlearrowleft(r) := \begin{cases} -\{r\}^{-1} \beta(p_r^+) & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -\{r\}^{-1} \beta(p_{-r}^-) & \text{if } r < 0. \end{cases} \quad (7.1.38)$$

It follows that the center of  $\mathcal{H}eis_k$  is a polynomial algebra in the  $\circlearrowleft(r)$ ,  $r \neq 0$ :

$$\text{End}_{\mathcal{H}eis_k}(\mathbb{1}) = \mathbb{k}[\circlearrowleft(r) : r \in \mathbb{Z}, r \neq 0], \quad (7.1.39)$$

and we have

$$\beta(\text{Sym} \otimes \mathbb{1}) = \mathbb{k}[\circlearrowleft(r) : r > 0], \quad \beta(\mathbb{1} \otimes \text{Sym}) = \mathbb{k}[\circlearrowleft(r) : r < 0]. \quad (7.1.40)$$

**Proposition 7.1.3.** *For  $r \in \mathbb{Z}$ , we have*

$$\circlearrowleft(r) \uparrow = \uparrow \circlearrowleft(r) + \{r\} \uparrow \circlearrowleft_r. \quad (7.1.41)$$

**Proof:** The statement is trivial for  $r = 0$ . Now suppose  $r > 0$  and consider the generating functions

$$H_+(u) := \sum_{r \geq 0} h_r^+ u^{-r}, \quad E_+(u) := \sum_{r \geq 0} e_r^+ u^{-r}, \quad P_+(u) := \sum_{r \geq 1} p_r^+ u^{1-r}. \quad (7.1.42)$$

Then we have

$$H_+(u)E_+(-u) = 1, \quad P_+(u) = -u^2 H'_+(u)/H_+(u) = -u^2 H'_+(u)E_+(-u). \quad (7.1.43)$$

(See, for example, [39, (I.2.6) and (I.2.10)] setting the  $t$  there equal to  $u^{-1}$ .) Furthermore,

$$\beta(H_+(u)) = u^k \oplus (u), \quad \beta(E_+(-u)) = u^{-k} \ominus (u), \quad (7.1.44)$$

Let  $f(u) = 1 - z^2 x u(u-x)^{-2}$  be the rational function appearing in (7.1.37). Then we have

$$\beta(H_+(u)) \uparrow \stackrel{(7.1.37)}{=} f(u) \uparrow \circ \beta(H_+(u)).$$

Differentiating with respect to  $u$  gives

$$\beta(H'_+(u)) \uparrow = f(u) \uparrow \circ \beta(H'_+(u)) + f'(u) \uparrow \circ \beta(H_+(u)).$$

Multiplying on the left by  $-u^2 \beta(E_+(-u) \otimes 1)$  and using (7.1.37) again, we have

$$\beta(P_+(u)) \uparrow = \uparrow \circ \beta(P_+(u)) + -u^2 f'(u)/f(u) \uparrow \circ \uparrow.$$

Now, noting that  $f(u) = \frac{(u-q^2x)(u-q^{-2}x)}{(u-x)^2}$ , we have

$$\begin{aligned} \frac{f'(u)}{f(u)} &= \frac{\partial}{\partial u} \ln(f(u)) = \left( \frac{1}{u-q^2x} + \frac{1}{u-q^{-2}x} - \frac{2}{u-x} \right) \\ &= u^{-1} \sum_{r \geq 0} \left( \left( \frac{q^2x}{u} \right)^r + \left( \frac{q^{-2}x}{u} \right)^r - 2 \left( \frac{x}{u} \right)^r \right) = \sum_{r \geq 1} \{r\}^2 x^r u^{-r-1}. \end{aligned}$$

Thus we have

$$\beta(P_+(u)) \uparrow = \uparrow \circ \beta(P_+(u)) - \sum_{r \geq 1} \{r\}^2 u^{1-r} \uparrow \circ_r,$$

and (7.1.41) follows after dividing both sides by  $-\{r\}$  and equating coefficients of  $u$ . To prove the case  $r < 0$ , we work with power series in  $u$ , as opposed to  $u^{-1}$ . We use the generating functions

$$H_-(u) := \sum_{r \geq 0} h_r^- u^r, \quad E_-(u) := \sum_{r \geq 0} e_r^- u^r, \quad P_-(u) := \sum_{r \geq 1} p_r^- u^{r-1}.$$

Then we have  $H_-(u)E_-(-u) = 1$  and  $P_-(u) = H'_-(u)/H_-(u) = H'_-(u)E_-(-u)$ . (See, for example, [39, (I.2.6) and (I.2.10)] setting the  $t$  there equal to  $u$ .) Furthermore,

$$\beta(H_-(u)) = \ominus(u), \quad \beta(E_-(-u)) = \curvearrowright(u).$$

As above, we have

$$\beta(H'_-(u)) \uparrow = f(u) \circ \uparrow \beta(H'_-(u)) + f'(u) \circ \uparrow \beta(H_-(u)).$$

Multiplying on the left by  $\beta(E_-(-u))$  and using (7.1.37) again, we have

$$\beta(P_-(u)) \uparrow = \uparrow \beta(P_-(u)) + f'(u)/f(u) \circ \uparrow.$$

Now, expanding as a Laurent series in  $u$ , we have

$$\begin{aligned} \frac{f'(u)}{f(u)} &= \frac{\partial}{\partial u} \ln(f(u)) = \left( \frac{1}{u - q^2x} + \frac{1}{u - q^{-2}x} - \frac{2}{u - x} \right) \\ &= - \sum_{r \geq 0} (q^{-2r-2}x^{-r-1}u^r + q^{2r+2}x^{-r-1}u^r - 2x^{-r-1}u^r) = - \sum_{r \geq 1} \{r\}^2 x^{-r} u^{r-1}. \end{aligned}$$

Dividing by  $-\{r\}$  and equating coefficients of  $u^r$  then yields (7.1.41). ■

### 7.1.4 Basis theorem

We now recall the important basis theorem for the morphism spaces of  $\mathcal{H}eis_k$ . Let  $X = X_r \otimes \cdots \otimes X_1$  and  $Y = Y_s \otimes \cdots \otimes Y_1$  be objects of  $\mathcal{H}eis_k$  for  $X_i, Y_j \in \{\uparrow, \downarrow\}$ . An  $(X, Y)$ -*matching* is a bijection between  $\{i : X_i = \uparrow\} \sqcup \{j : Y_j = \downarrow\}$  and  $\{i : X_i = \downarrow\} \sqcup \{j : Y_j = \uparrow\}$ . A *reduced lift* of an  $(X, Y)$ -matching is a diagram representing a morphism  $X \rightarrow Y$  such that

- the endpoints of each string are points corresponding under the given matching;
- there are no floating bubbles and no dots on any string;
- there are no self-intersections of strings and no two strings cross each other more than once.

Fix a set  $B(X, Y)$  consisting of a choice of reduced lift for each of the  $(X, Y)$ -matchings. Let  $B_\circ(X, Y)$  be the set of all morphisms that can be obtained from the elements of  $B(X, Y)$  by adding dots labelled with integer multiplicities near to the terminus of each string. Using the morphism  $\beta$  of Proposition 7.1.2, we can make the morphism space  $\text{Hom}_{\mathcal{H}eis_k}(X, Y)$  into a right  $\text{Sym} \otimes \text{Sym}$ -module:

$$\phi\theta := \phi \otimes \beta(\theta), \quad \phi \in \text{Hom}_{\mathcal{H}eis_k}(X, Y), \quad \theta \in \text{Sym} \otimes \text{Sym}.$$

**Theorem 7.1.4** ([13, Th. 10.1]). *For any objects  $X, Y \in \mathcal{H}eis_k$ , the morphism space  $\text{Hom}_{\mathcal{H}eis_k}(X, Y)$  is a free right  $\text{Sym} \otimes \text{Sym}$ -module with basis  $B_o(X, Y)$ .*

By [13, Th. 3.2], there is a unique isomorphism of  $\mathbb{k}$ -linear monoidal categories

$$\Omega_k: \mathcal{H}eis_k(z, t) \xrightarrow{\cong} \mathcal{H}eis_{-k}(z, t^{-1})^{\text{op}} \quad (7.1.45)$$

given on the generating morphisms by

$$\begin{array}{c} \nearrow \searrow \mapsto - \searrow \nearrow, \quad \nwarrow \nearrow \mapsto - \nwarrow \searrow, \quad \hat{\phi} \mapsto \hat{\psi}, \quad \cup \mapsto \cap, \quad \cap \mapsto \cup, \quad \cup \mapsto -\cap, \quad \cap \mapsto -\cup. \end{array} \quad (7.1.46)$$

The isomorphism  $\Omega_k$  acts on bubbles as

$$\begin{array}{c} \textcircled{\pm} \mapsto -\textcircled{\pm}, \quad \textcircled{\pm} \mapsto -\textcircled{\pm}, \quad \textcircled{s} \mapsto -\textcircled{s}, \end{array} \quad (7.1.47)$$

for  $r, s \in \mathbb{Z}$ ,  $s \neq 0$ . Since the image of the power sums is not stated in [13, Th. 3.2], we include the details here. For  $s \geq 1$ , we have

$$\begin{aligned} \textcircled{s} &\stackrel{(7.1.36)}{=} z^2 \sum_{a \in \mathbb{Z}} a \textcircled{\pm}_a \textcircled{\pm}_{s-a} \mapsto z^2 \sum_{a \in \mathbb{Z}} a \textcircled{\pm}_{s-a} \textcircled{\pm}_a = z^2 \sum_{b \in \mathbb{Z}} (s-b) \textcircled{\pm}_b \textcircled{\pm}_{s-b} \\ &\stackrel{(7.1.20)}{=} -z^2 \sum_{b \in \mathbb{Z}} b \textcircled{\pm}_b \textcircled{\pm}_{s-b} = -\textcircled{s} \end{aligned}$$

and for  $s \leq -1$ , we have

$$\begin{aligned} \textcircled{s} &\stackrel{(7.1.36)}{=} z^2 \sum_{a \in \mathbb{Z}} a \textcircled{\ominus}_{-a} \textcircled{\ominus}_{a+s} \mapsto z^2 \sum_{a \in \mathbb{Z}} a \textcircled{\ominus}_{a+s} \textcircled{\ominus}_{-a} = z^2 \sum_{b \in \mathbb{Z}} (-s-b) \textcircled{\ominus}_{-b} \textcircled{\ominus}_{b+s} \\ &\stackrel{(7.1.20)}{=} -z^2 \sum_{b \in \mathbb{Z}} b \textcircled{\ominus}_{-b} \textcircled{\ominus}_{b+s} = -\textcircled{s}. \end{aligned}$$

## 7.2 Partial quantum Heisenberg categories

In this short section we define certain subcategories of  $\mathcal{H}eis_k$ , which we call *partial quantum Heisenberg categories*. Roughly speaking, our aim is to split  $\mathcal{H}eis_k$  in half in such a way that, when we later identify its trace with the elliptic Hall algebra  $\text{EH}_k$ , the two halves correspond to  $\text{EH}^+$  and  $\text{EH}^-$ . We remind the reader that we continue with the assumptions on  $\mathbb{k}$ ,  $q$ ,  $z$ ,  $t$ , and  $k$  made at the beginning of the present chapter.

**Definition 7.2.1.** Define  $\mathcal{H}eis^+$  to be the strict  $\mathbb{k}$ -linear monoidal category generated by the object  $\uparrow$  and morphisms

$$\begin{array}{c} \nearrow \searrow, \quad \nwarrow \nearrow: \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow, \quad \hat{\phi}: \uparrow \rightarrow \uparrow, \quad \textcircled{r}: \mathbb{1} \rightarrow \mathbb{1}, \quad r > 0, \end{array}$$

subject to the relations (7.1.1) to (7.1.3), relation (7.1.41) for  $r > 0$ , and the relation that the dot is invertible. Define  $\mathcal{H}eis^-$  to be the strict  $\mathbb{k}$ -linear monoidal category generated by the object  $\downarrow$  and morphisms

$$\begin{array}{c} \searrow \\ \downarrow \\ \swarrow \end{array}, \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array}: \downarrow \otimes \downarrow \rightarrow \downarrow \otimes \downarrow, \quad \dot{\downarrow}: \downarrow \rightarrow \downarrow, \quad \textcircled{r}: \mathbb{1} \rightarrow \mathbb{1}, \quad r < 0,$$

subject to the 180° rotation of relations (7.1.1) to (7.1.3) and (7.1.41) for  $r < 0$ , and the relation that the dot is invertible.

Note that the definition of  $\mathcal{H}eis^\pm$  does not involve  $k$ .

**Proposition 7.2.2.** *For  $k \in \mathbb{Z}$ , we have faithful  $\mathbb{k}$ -linear monoidal functors*

$$\Psi_k^\pm: \mathcal{H}eis^\pm \rightarrow \mathcal{H}eis_k,$$

mapping the generating objects and morphisms of  $\mathcal{H}eis^\pm$  to the objects and morphisms in  $\mathcal{H}eis_k$  denoted by the same symbols.

**Proof:** We give the proof for  $\Psi_k^+$ , since the proof for  $\Psi_k^-$  is analogous. Because all generating morphisms of  $\mathcal{H}eis^+$  are endomorphisms, we have  $\text{Hom}_{\mathcal{H}eis^+}(\uparrow^{\otimes n}, \uparrow^{\otimes m}) = 0$  for  $n \neq m$ . Since the defining relations of  $\mathcal{H}eis^+$  hold in  $\mathcal{H}eis_k$ , the functor is well defined.

Fix  $n \in \mathbb{N}$  define  $B_0(\uparrow^{\otimes n}, \uparrow^{\otimes n})$  as before Theorem 7.1.4. Let  $\mathbf{B}_{\text{Sym}}$  be the basis of  $\text{Sym}$  consisting of the power sum symmetric functions  $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$ , where  $\lambda = (\lambda_1, \dots, \lambda_l)$  is a partition. A standard straightening argument shows that the morphisms

$$\phi \otimes \beta(f \otimes 1), \quad \phi \in B_0(\uparrow^{\otimes n}, \uparrow^{\otimes n}), \quad f \in \mathbf{B}_{\text{Sym}},$$

span  $\text{Hom}_{\mathcal{H}eis^+}(\uparrow^{\otimes n}, \uparrow^{\otimes n})$ . (See for example, the proof of [13, Th. 10.1].) By Theorem 7.1.4, the images of these morphisms under  $\Psi_k^+$  are linearly independent in  $\mathcal{H}eis_k$ . ■

For  $k \in \mathbb{Z}$ , define

$$\mathcal{H}eis_k^\pm := \Psi_k^\pm(\mathcal{H}eis^\pm).$$

It follows that elements of  $\beta(\text{Sym} \otimes 1)$  are morphisms in  $\mathcal{H}eis_k^+$ , while elements of  $\beta(1 \otimes \text{Sym})$  are morphisms in  $\mathcal{H}eis_k^-$ . It also follows from Proposition 7.2.2 that we have isomorphisms of  $\mathbb{k}$ -linear monoidal categories

$$\mathcal{H}eis_k^\pm \cong \mathcal{H}eis_l^\pm, \quad k, l \in \mathbb{Z}. \tag{7.2.1}$$

### 7.3 Skein algebra of the torus

In this section we recall the definition of the skein algebra of the torus and identify it with the trace of the quantum Heisenberg category of central charge zero. Throughout this section we work over an arbitrary commutative ground ring  $\mathbb{k}$  and  $z, t \in \mathbb{k}^\times$ . (Although we introduced  $\mathcal{Heis}_k$ , at the beginning of the chapter, under more restrictive assumptions on  $\mathbb{k}$ , all the results used in the current section hold more generally, as shown in [13].)

Consider the annulus

$$A = [0, 1]^2 / \sim, \tag{7.3.1}$$

where  $\sim$  is the relation given by  $(0, b) \sim (1, b)$  for all  $b \in [0, 1]$ . We will denote points in  $A$  by representatives of the equivalence classes under  $\sim$ . In order to make the categories we are about to describe strict, we fix a countable number of points in  $A$ , which will be the possible endpoints of tangles. We choose the points

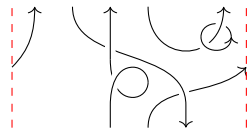
$$P_n = \left(1 - \frac{1}{2^n}, \frac{1}{2}\right) \in A, \quad n \in \mathbb{Z}_{>0}.$$

Up to isomorphism, our categories will not depend on the particular choice of points. We will typically draw them as equally spaced, or adjust the spacing to the particular tangle we draw.

We let  $\mathcal{FOT}(A)$  be the category of framed oriented tangles over  $A$ . Its objects are finite sequences  $(\varepsilon_1, \dots, \varepsilon_n)$  of elements of  $\{\uparrow, \downarrow\}$ . The unit object  $\mathbb{1}$  is the empty sequence. Morphisms in  $\mathcal{FOT}(A)$  from  $(\varepsilon_1, \dots, \varepsilon_m)$  to  $(\varepsilon'_1, \dots, \varepsilon'_n)$  are framed oriented tangles in  $A \times [0, 1]$ , up to ambient isotopy, with endpoints

$$(\{P_1, \dots, P_m\} \times \{0\}) \cup (\{P_1, \dots, P_n\} \times \{1\})$$

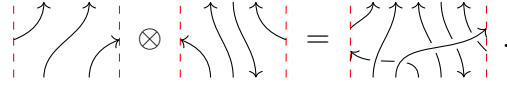
such that the orientation of the tangle at each  $P_i \times \{0\}$  agrees with  $\varepsilon_i$ , the orientation at each  $P'_i \times \{1\}$  agrees with  $\varepsilon'_i$ , and the framing at the point  $P_i \times \{0\}$  (respectively,  $P_i \times \{1\}$ ) points towards  $P_{i+1} \times \{0\}$  (respectively,  $P_{i+1} \times \{1\}$ ). We allow tangles to have closed components. For example,



$$\in \text{Hom}_{\mathcal{FOT}(A)}(\uparrow \otimes \uparrow \otimes \downarrow, \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow), \tag{7.3.2}$$

where we adopt the convention of blackboard framing (i.e. the framing is parallel to the page) and we identify the dashed vertical edges. We always isotope tangles so that they intersect the cut transversely. The composite  $f \circ g$  is given by placing  $f$  above  $g$  and rescaling the vertical coordinate. The category  $\mathcal{FOT}(A)$  is a strict monoidal category. Viewing  $A \times [0, 1]$  as the cylinder, the tensor product  $f \otimes g$  is given by placing the cylinder for  $g$  inside the cylinder for  $f$ , then rescaling and isotoping the endpoints of

the tangles so that the endpoints of  $g$  are to the right of those of  $f$  (preserving the relative order of the endpoints in  $f$  and the endpoints in  $g$ ). In terms of diagrams as in (7.3.2), this corresponds to placing the diagram of  $g$  to the right of the diagram of  $f$ , and then passing all strands of  $f$  exiting the right side of its diagram over the diagram for  $g$  and all strands of  $g$  exiting the left side of its diagram under the diagram for  $f$ . For example,



Let  $\mathcal{FOT}(A)_{\mathbb{k}}$  denote the  $\mathbb{k}$ -linearization of  $\mathcal{FOT}(A)$ . Thus, the morphisms in  $\mathcal{FOT}(A)_{\mathbb{k}}$  are formal  $\mathbb{k}$ -linear combinations of morphisms in  $\mathcal{FOT}(A)$ , with composition and tensor product extended by linearity. The framed HOMFLYPT skein category  $OS(A; z, t)$  over the annulus is the category obtained from  $\mathcal{FOT}(A)_{\mathbb{k}}$  by imposing the relations

$$\nearrow \searrow - \searrow \nearrow = z \uparrow \uparrow, \quad \bigcirc = t^{-1} \uparrow, \quad \bigcirc = \frac{t - t^{-1}}{z} 1_{\mathbb{1}}. \quad (7.3.3)$$

Note that these are precisely the relations (7.1.2) and (7.1.7) with  $k = 0$ . In fact, we have the following result, which states that the framed HOMFLYPT skein category over the annulus is the quantum Heisenberg category at central charge zero.

**Proposition 7.3.1** (Cor.4.3.7). *We have an isomorphism of monoidal categories*

$$\mathcal{Heis}_0 \xrightarrow{\cong} OS(A; z, t). \quad (7.3.4)$$

This isomorphism sends the generators  $\nearrow \searrow$ ,  $\searrow \nearrow$ ,  $\cup$ ,  $\cap$ ,  $\cup$ , and  $\cap$  to the tangles with the same diagrams, and the image of the dots are

$$\dot{\nearrow} \mapsto \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array}, \quad \dot{\searrow} \mapsto \begin{array}{c} \searrow \\ \vdots \\ \searrow \end{array}.$$

Now consider the torus

$$T^2 = [0, 1]^2 / \approx \quad (7.3.5)$$

where  $\approx$  is the relation given by  $(0, b) \approx (1, b)$  for all  $b \in [0, 1]$ , and  $(a, 0) \approx (a, 1)$  for all  $a \in [0, 1]$ . We let  $\text{Sk}(T^2; z, t)$  be the skein algebra of the torus. As a  $\mathbb{k}$ -module, this is space of  $\mathbb{k}$ -linear combinations of framed oriented links in  $T^2 \times [0, 1]$ , up to isotopy, modulo the relations (7.3.3). The product in  $\text{Sk}(T^2; z, t)$  is defined as follows. Consider the two embeddings

$$\begin{aligned} \iota_1: T^2 \times [0, 1] &\hookrightarrow T^2 \times [0, 1], & (a, b) &\mapsto (a, (b + 2)/3), \\ \iota_2: T^2 \times [0, 1] &\hookrightarrow T^2 \times [0, 1], & (a, b) &\mapsto (a, b/3). \end{aligned}$$

Then, for  $x, y \in \text{Sk}(T^2; z, t)$ , we define  $xy := \iota_1(x) \sqcup \iota_2(y)$ . Intuitively, the product  $xy$  is given by stacking  $x$  above  $y$ .

**Proposition 7.3.2.** *We have an isomorphism of algebras*

$$\mathrm{Tr}(\mathcal{OS}(A; z, t)) \cong \mathrm{Sk}(T^2; z, t).$$

**Proof:** Consider the natural surjection  $A \twoheadrightarrow T^2$  sending the equivalence class of  $(a, b)$  under  $\sim$  to the equivalence class of  $(a, b)$  under  $\approx$ . Under this surjection, any endomorphism in  $\mathcal{OS}(A; z, t)$  can be viewed as an element of  $\mathrm{Sk}(T^2; z, t)$  by identifying the top and bottom of the string diagrams in  $\mathcal{OS}(A; z, t)$ . This clearly descends to an algebra homomorphism  $f: \mathrm{Tr}(\mathcal{OS}(A; z, t)) \rightarrow \mathrm{Sk}(T^2; z, t)$ . Conversely, we can isotope framed oriented tangles in  $T^2$  so that they intersect the circle  $\{(a, 0) : a \in [0, 1]\} = \{(a, 1) : a \in [0, 1]\} \subseteq T^2$  transversely. Then, cutting along this circle gives a map  $g: \mathrm{Sk}(T^2; z, t) \rightarrow \mathrm{Tr}(\mathcal{OS}(A; z, t))$ ; the trace condition (2.6.3) ensures that this map is well-defined. It is straightforward to verify that  $f$  and  $g$  are mutually inverse.  $\blacksquare$

**Corollary 7.3.3.** *We have an isomorphism of algebras*

$$\mathrm{Sk}(T^2; z, t) \xrightarrow{\cong} \mathrm{Tr}(\mathcal{Heis}_0).$$

**Proof:** This follows immediately from Propositions 7.3.1 and 7.3.2.  $\blacksquare$

## 7.4 Trace of the quantum Heisenberg category

In this section, we prove our main result (Theorem 7.4.1). Namely, we describe an algebra isomorphism from  $\mathrm{EH}_k$  to the trace of the quantum Heisenberg category. We continue with the assumptions on  $\mathbb{k}, q, z, t$ , and  $k$  made at the beginning of the chapter.

Let  $\mathcal{Heis}_k$  be the *quantum Heisenberg category* introduced in Section 7.1. Before stating our main result, we introduce some notation. For  $i, j \in \mathbb{N}$ , define

$$\sigma_{i,j} := \begin{array}{c} \begin{array}{c} \nearrow \cdots \nearrow \nearrow \nearrow \\ \curvearrowright \cdots \curvearrowright \curvearrowright \curvearrowright \\ \searrow \cdots \searrow \searrow \searrow \end{array} \\ \in \mathrm{End}_{\mathcal{Heis}_k}(\uparrow^{\otimes(i+j+1)}), \end{array} \quad (7.4.1)$$

$$\sigma_{-i,-j} := \begin{array}{c} \begin{array}{c} \searrow \cdots \searrow \searrow \searrow \\ \curvearrowleft \cdots \curvearrowleft \curvearrowleft \curvearrowleft \\ \nearrow \cdots \nearrow \nearrow \nearrow \end{array} \\ \in \mathrm{End}_{\mathcal{Heis}_k}(\downarrow^{\otimes(i+j+1)}), \end{array} \quad (7.4.2)$$

where the strand crossing all the others passes over  $i$  strands and under  $j$  strands. Then define

$$\sigma_n := \frac{z}{\{n\}} \sum_{i=0}^{n-1} \sigma_{i,n-i-1}, \quad \sigma_{-n} := \frac{z}{\{n\}} \sum_{i=0}^{n-1} \sigma_{-i,i-n+1}, \quad n \geq 1. \quad (7.4.3)$$

Note that  $\sigma_1 = 1_\uparrow$  and  $\sigma_{-1} = 1_\downarrow$ .

For  $r \in \mathbb{Z}$ ,  $n \in \mathbb{Z}_{>0}$ , define

$$\chi_{r,n} := \frac{\{r\}}{\{rn\}} \left( r \hat{\uparrow} \uparrow \cdots \uparrow + \uparrow r \hat{\uparrow} \cdots \uparrow + \cdots + \uparrow \uparrow \cdots \hat{\uparrow} r \right) \in \text{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n}), \quad (7.4.4)$$

$$\chi_{r,-n} := \frac{\{r\}}{\{rn\}} \left( r \hat{\downarrow} \downarrow \cdots \downarrow + \downarrow r \hat{\downarrow} \cdots \downarrow + \cdots + \downarrow \downarrow \cdots \hat{\downarrow} r \right) \in \text{End}_{\mathcal{H}eis_k}(\downarrow^{\otimes n}), \quad (7.4.5)$$

where we adopt the convention that  $\frac{\{r\}}{\{rn\}} = \frac{1}{n}$  when  $r = 0$ , so that  $\chi_{0,n} = 1_\uparrow^{\otimes n}$  and  $\chi_{0,-n} = 1_\downarrow^{\otimes n}$

**Theorem 7.4.1.** *For  $k \in \mathbb{Z}$ , there is a unique isomorphism of algebras*

$$\varphi_k: \text{EH}_k \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k)$$

such that

$$w_{r,1} \mapsto [\hat{\uparrow} r], \quad w_{r,-1} \mapsto [\hat{\downarrow} r], \quad r \in \mathbb{Z}. \quad (7.4.6)$$

Under  $\varphi_k$ , we also have

$$w_{r,0} \mapsto [\textcircled{r}], \quad r \in \mathbb{Z}, \quad r \neq 0, \quad (7.4.7)$$

$$w_{r,n} \mapsto [\chi_{r,n} \sigma_n], \quad r, n \in \mathbb{Z}, \quad n \neq 0. \quad (7.4.8)$$

The proof of Theorem 7.4.1 is given at the end of this section, after some preparatory results. Note that the definition of  $\text{EH}_k$  and the isomorphism of Theorem 7.4.1 are independent of  $t$ , even though the definition of quantum Heisenberg category  $\mathcal{H}eis_k$  involves  $t$ .

Recall the notation  $\equiv$  from (2.6.2), which we will use frequently in this section.

**Proposition 7.4.2.** *The  $k = 0$  case of Theorem 7.4.1 holds.*

**Proof:** It follows from Corollary 7.3.3 and [45, Th. 2] that we have an isomorphism

$$\varphi_0: \text{EH}_0 \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_0) \quad (7.4.9)$$

satisfying (7.4.6). (Recall that our  $q$  and  $t$  are the  $s$  and  $v^{-1}$  of [45], respectively. While [45] works over the ground ring  $\mathbb{k} = \mathbb{C}[q^{\pm 1}, t^{\pm 1}, \{d\}^{-1} : d \geq 1]$ , the result we use here holds more generally; see Remark 7.4.4.) More precisely, under the isomorphism of Corollary 7.3.3, we have

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto [\hat{\uparrow} r], \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto [\hat{\downarrow} r].$$

where we identify the vertical dashed edges with each other and the horizontal dashed edges with each other, and the curves wrap  $r$  times in the horizontal direction (we

have drawn the case  $r = 3$ ). Since  $\pm 1$  is coprime to  $r$  (see [45, p. 810]), under the isomorphism of [45, Th. 2] we have

$$w_{r,1} \mapsto \begin{array}{|c} \hline \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array}, \quad w_{r,-1} \mapsto \begin{array}{|c} \hline \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array},$$

where again the curves wrap  $r$  times in the horizontal direction. (Note that the skein of the torus  $\text{Sk}(T^2; z, t)$  is denoted  $H(T^2)$  in [45].) On the other hand, by Lemma 6.3.5, the isomorphism  $\varphi_0$  is uniquely determined by where it maps  $w_{r,\pm 1}$ ,  $r \in \mathbb{Z}$ .

To show (7.4.7), it suffices to prove that, for  $r \in \mathbb{Z}$ ,  $r \neq 0$ ,

$$-\{r\} [\textcircled{r}] = [r \uparrow \downarrow] - [\downarrow \uparrow r], \tag{7.4.10}$$

since then the result follows after applying  $\varphi_0$  and using (6.1.1) and (7.4.6). We start by proving the equality in (7.4.10) for  $r > 0$ . For  $r > 0$ , we have

$$r \uparrow \downarrow \stackrel{(7.1.14)}{=} r \uparrow \downarrow \stackrel{(7.1.17)}{\equiv} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} - z \sum_{\substack{a+b=r \\ a,b>0}} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \stackrel{(7.1.2)}{\equiv} \downarrow \uparrow r \stackrel{(7.1.14)}{=} \downarrow \uparrow r - z \sum_{\substack{a+b=r \\ a \geq 0, b > 0}} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array}.$$

Hence

$$\begin{aligned} r \uparrow \downarrow - \downarrow \uparrow r &\stackrel{(7.1.29)}{\equiv} -z^2 \sum_{\substack{a+b=r \\ a \geq 0, b > 0}} \left( \sum_{c>0} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} - \sum_{c \geq 0} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \right) \stackrel{(7.1.10)}{\equiv} z^2 \sum_{\substack{a+b=r \\ a \geq 0, b > 0}} \sum_{c=0}^a \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \\ &= z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \stackrel{(7.1.36)}{\equiv} \beta(p_r^+) = -\{r\} \textcircled{r}, \end{aligned}$$

as desired. To prove the remaining case of (7.4.10), we will still assume  $r > 0$  but replace the  $r$  with  $-r$  in the strand. We have

$$-r \uparrow \downarrow \stackrel{(7.1.14)}{=} -r \uparrow \downarrow \stackrel{(7.1.17)}{\equiv} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} + z \sum_{\substack{a+b=r \\ a,b \geq 0}} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \stackrel{(7.1.2)}{\equiv} \downarrow \uparrow r + z \sum_{\substack{a+b=r \\ a > 0, b \geq 0}} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array}.$$

Hence

$$\begin{aligned} -r \uparrow \downarrow - \downarrow \uparrow r &\stackrel{(7.1.29)}{\equiv} z^2 \sum_{\substack{a+b=r \\ a > 0, b \geq 0}} \left( \sum_{c>0} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} - \sum_{c \geq 0} \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \right) \\ &\stackrel{(7.1.10)}{\equiv} z^2 \sum_{\substack{a+b=r \\ a > 0, b \geq 0}} \sum_{c=1}^a \begin{array}{|c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \hline \end{array} \end{aligned}$$

$$= z^2 \sum_{s \in \mathbb{Z}} s \begin{array}{c} \ominus \\ -s \end{array} \begin{array}{c} \ominus \\ s-r \end{array} \stackrel{(7.1.36)}{=} \beta(p_r^-) = \{r\} \ominus r,$$

as desired.

It remains to prove (7.4.8). By [45, Def. 2.5], we have

$$\varphi_0(w_{0,n}) = [\sigma_n], \quad n \neq 0.$$

Thus, for  $r, n \in \mathbb{Z} \setminus \{0\}$ , we have

$$\varphi_0(w_{r,n}) = \{rn\}^{-1} \varphi_0([w_{r,0}, w_{0,n}]) \stackrel{(7.4.7)}{=} \{rn\}^{-1} [\begin{array}{c} \ominus \\ r \end{array} \otimes \sigma_n - \sigma_n \otimes \begin{array}{c} \ominus \\ r \end{array}] \stackrel{(7.1.41)}{=} [\chi_{r,n} \sigma_n].$$

■

**Remark 7.4.3.** As we see from the proof of Proposition 7.4.2, we use the results of [45] to prove the  $k = 0$  case of Theorem 7.4.1. The proof of [45, Th. 1] involves induction on  $\det(\mathbf{x} \ \mathbf{y})$ , starting with the base cases

$$[[\begin{array}{c} \ominus \\ r \end{array}], [\uparrow]] = \{r\} [\hat{\phi}_r], \quad [[\downarrow], \hat{\phi}_r] = \{r\} [\begin{array}{c} \ominus \\ r \end{array}], \quad r \in \mathbb{Z} \setminus \{0\}. \quad (7.4.11)$$

In [45], the proof of the first equation in (7.4.11) relies on [42, Th. 4.2], while the proof of the second involves direct skein manipulation. Note that the first equation in (7.4.11) is precisely the bubble slide relation (7.1.41), while the second equation in (7.4.11) is (7.4.10). Thus, in order to make the arguments of the current chapter independent of the results of [45], one would only need to include the inductive argument of [45, §3.2]; this is a purely algebraic argument, involving no skein theory.

**Remark 7.4.4.** Remark 7.4.3 allows us to see that [45, Th. 1 & 2] hold over the more arbitrary ground ring  $\mathbb{k}$  considered in this section. The assumption that  $\mathbb{k}$  contains  $\mathbb{Q}$  is needed in the proof of [45, Lem. 3.1], since this proof uses [61, Th. 1], which involves the  $\sigma_{n,0}$  (denoted  $A_{n,0}$  in [45]), whereas [45, Lem. 3.1] involves the  $\sigma_n$  (denoted  $P_n$  in [45]); solving for the  $\sigma_{n,0}$  in terms of the  $\sigma_n$  requires division by  $n$  (see [45, Rem. 2.4]). Essentially, at issue is the fact that the power sums generate  $\text{Sym}$  over  $\mathbb{Q}$ , but not over  $\mathbb{Z}$ ; see (7.1.38). The inductive argument of [45, §3.2] only requires division by  $\{d\}$ ,  $d \geq 1$ .

**Remark 7.4.5.** In the case  $k = 0$ , our explicit description (7.4.7) and (7.4.8) of the image of the isomorphism  $\varphi_k$  differs from that given in [45, Def. 2.5], which involves decorated framed oriented curves. There is no contradiction here, as the presence of the skein relations (7.3.3) means that different linear combinations of classes of framed oriented tangles can be equal in the framed HOMFLYPT skein algebra of the torus.

**Proposition 7.4.6.** *If Theorem 7.4.1 holds for central charge  $k$ , then it holds for central charge  $-k$ .*

**Proof:** Suppose Theorem 7.4.1 holds for some  $k \in \mathbb{Z}$ . Recall the isomorphisms  $\Omega_k$  and  $\omega_k$  from (7.1.45) and (6.2.9), respectively. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{EH}_k & \xrightarrow{\varphi_k} & \mathrm{Tr}(\mathcal{H}\mathrm{eis}_k(z, t^{-1})) \\ \omega_{-k} \uparrow & & \downarrow \mathrm{Tr}(\Omega_k) \\ \mathrm{EH}_{-k} & \xrightarrow{\varphi_{-k}} & \mathrm{Tr}(\mathcal{H}\mathrm{eis}_{-k}(z, t)) \end{array}$$

We use here the fact that, for any small  $\mathbb{k}$ -linear category, there is a canonical isomorphism  $\mathrm{Tr}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\cong} \mathrm{Tr}(\mathcal{C})$ ,  $[f] \mapsto [f]$ . By assumption,  $\varphi_k$  is an isomorphism of algebras. Since  $\mathrm{Tr}(\Omega_k)$  and  $\omega_{-k}$  are also isomorphisms, there is an algebra isomorphism  $\varphi_{-k}: \mathrm{EH}_{-k} \xrightarrow{\cong} \mathrm{Tr}(\mathcal{H}\mathrm{eis}_{-k}(z, t))$  making the above diagram commute. For  $r \in \mathbb{Z}$ , we have

$$\begin{aligned} \varphi_{-k}(w_{r,1}) &= \mathrm{Tr}(\Omega_k) \circ \varphi_k \circ \omega_{-k}(w_{r,1}) = [\hat{\phi}^r] \quad \text{and} \\ \varphi_{-k}(w_{r,-1}) &= \mathrm{Tr}(\Omega_k) \circ \varphi_k \circ \omega_{-k}(w_{r,-1}) = [\check{\phi}^r]. \end{aligned}$$

Thus the first statement in Theorem 7.4.1 holds for central charge  $-k$ .

To prove that (7.4.7) also holds for central charge  $-k$ , we compute, for  $r \neq 0$ ,

$$\varphi_{-k}(w_{r,0}) = \mathrm{Tr}(\Omega_k) \circ \varphi_k \circ \omega_{-k}(w_{r,0}) = [\textcircled{r}].$$

Finally, to prove that (7.4.8) also holds for central charge  $-k$ , we compute, for  $n \neq 0$ ,

$$\varphi_{-k}(w_{r,n}) = \mathrm{Tr}(\Omega_k) \circ \varphi_k \circ \omega_{-k}(w_{r,n}) = (-1)^{n+1} \mathrm{Tr}(\Omega_k)([\chi_{r,-n}\sigma_{-n}]) = [\sigma_n\chi_{r,n}] = [\chi_{r,n}\sigma_n],$$

where we use the fact that  $\Omega_k(\sigma_{-n}) = (-1)^n\sigma_n$  and  $\Omega_k(\chi_{r,-n}) = -\chi_{r,n}$ . ■

In light of Propositions 7.4.2 and 7.4.6, it suffices to prove Theorem 7.4.1 for central charge  $k < 0$ . Thus,

*for the remainder of this section we assume  $k < 0$ .*

The proof of the following proposition is inspired by that of [51, Prop. 6.2].

**Proposition 7.4.7.** *The tensor product*

$$\mathcal{H}\mathrm{eis}_k^+ \times \mathcal{H}\mathrm{eis}_k^- \xrightarrow{\otimes} \mathcal{H}\mathrm{eis}_k$$

*induces a linear isomorphism*

$$\mathrm{Tr}(\mathcal{H}\mathrm{eis}_k^+) \otimes \mathrm{Tr}(\mathcal{H}\mathrm{eis}_k^-) \xrightarrow{\cong} \mathrm{Tr}(\mathcal{H}\mathrm{eis}_k).$$

**Proof:** Since  $k < 0$ , it follows from (7.1.13) that every object of  $\text{Add}(\mathcal{H}eis_k)$  is isomorphic to a direct sum of objects of the form  $\uparrow^{\otimes m} \otimes \downarrow^{\otimes n}$ ,  $m, n \in \mathbb{N}$ . Let  $\mathcal{C}$  be the full subcategory of  $\mathcal{H}eis_k$  whose objects are  $\uparrow^{\otimes m} \otimes \downarrow^{\otimes n}$ ,  $m, n \in \mathbb{N}$ . Then, by Corollary 2.6.7, the inclusion functor  $\mathcal{C} \rightarrow \mathcal{H}eis_k$  induces a linear isomorphism  $\text{Tr}(\mathcal{C}) \cong \text{Tr}(\mathcal{H}eis_k)$ .

For  $m \in \mathbb{Z}$ , let  $\mathcal{C}^{(m)}$  be the full subcategory of  $\mathcal{H}eis_k$  whose objects are  $\uparrow^{\oplus(m+n)} \otimes \downarrow^{\otimes n}$  for  $n \geq \max(0, -m)$ . Then there are no morphisms between objects of  $\mathcal{C}^{(m)}$  and  $\mathcal{C}^{(n)}$  for  $m \neq n$ . Thus  $\mathcal{C} = \bigsqcup_{m \in \mathbb{Z}} \mathcal{C}^{(m)}$ , and so  $\text{Tr}(\mathcal{C}) = \bigoplus_{m \in \mathbb{Z}} \text{Tr}(\mathcal{C}^{(m)})$ .

Fix  $m \in \mathbb{Z}$  for the remainder of the proof. For  $n \in \mathbb{Z}$  with  $n \geq \max(0, -m)$ , let

$$X_n := \uparrow^{\otimes(m+n)} \otimes \downarrow^{\otimes n}.$$

Recall the definition of a *reduced lift* given above Theorem 7.1.4, and let  $\mathbf{B}_{\text{Sym}}$  be a basis of  $\text{Sym}$  (e.g. we can take  $\mathbf{B}_{\text{Sym}}$  to be the set of Schur functions). Fix a set  $D(n_2, n_1)$  consisting of a choice of reduced lift of each  $(X_{n_1}, X_{n_2})$ -matching. Then, for  $n, n_1, n_2 \geq \max(0, -m)$ , define the following:

- If  $n_1 > n_2$ , let  $D_{n_2, n_1}$  denote the set of all morphisms obtained from elements of  $D(n_2, n_1)$  containing no cups by adding to each string involved in a cap an integer number of dots near the terminus of the string.
- If  $n_1 < n_2$ , let  $D_{n_2, n_1}$  denote the set of all morphisms obtained from elements of  $D(n_2, n_1)$  containing no caps by adding to each string involved in a cup an integer number of dots near the terminus of the string.
- Let  $D_{n, n} = \{1_{X_n}\}$ .
- Let  $D_n$  denote the set of all morphisms that can be obtained from the elements of  $D(n, n)$  containing no cups or caps by adding to each string an integer number of dots near the terminus of the string, and then placing an element of  $\beta(\mathbf{B}_{\text{Sym}} \otimes 1)$  in between the downward and upward strings (i.e. to the right of all downward strings and to the left of all upward strings), and then placing an element of  $\beta(1 \otimes \mathbf{B}_{\text{Sym}})$  to the right of all strings.

We claim that the sets  $D_{n_2, n_1}$ ,  $D_n$  satisfy the conditions (B1) and (B2) of Section 2.6, where

$$R_n = \text{End}_{\mathcal{H}eis_k^+}(\uparrow^{\otimes(m+n)}) \otimes \text{End}_{\mathcal{H}eis_k^-}(\downarrow^{\otimes n}).$$

Given the claim, the current proposition then follows from Proposition 2.6.8.

It remains to prove the claim. Condition (B2) is clear. To see that (B1) is satisfied, we need to verify that, for each  $m, n \in \mathbb{N}$ , the set  $\mathbf{B}_{n_2, n_1} = \bigsqcup_{l=0}^{\min(n_1, n_2)} D_{n_2, l} D_l D_{l, n_1}$  is a basis of  $\text{Hom}_{\mathcal{C}}(X_{n_1}, X_{n_2})$ . The difference between the elements of  $\mathbf{B}_{n_2, n_1}$  and the elements described in Theorem 7.1.4 is that



Figure 7.1: Typical basis elements for  $m = -1$ ,  $n_1 = 3$ ,  $n_2 = 4$ .

- for strings connecting the top and bottom of the diagram, the basis elements in Theorem 7.1.4 have dots near the termini of the strands, whereas the dots on such strands in the elements of  $\mathbf{B}_{n_2, n_1}$  are in the middle of the diagram;
- the basis elements in Theorem 7.1.4 have all bubbles on the right side of the diagram, whereas the (+)-bubbles appearing in elements of  $\mathbf{B}_{n_2, n_1}$  are in the middle of the diagram.

Using (7.1.2) and (7.1.3), dots slide through crossings modulo diagrams with fewer total crossings. Similarly, by (7.1.37), bubbles slide through strands modulo diagrams with fewer total dots. For example, in Figure 7.1, the left-hand diagram is a typical element of  $D_{4,2}D_2D_{2,3}$ , while the right-hand diagram is a typical element of the basis from Theorem 7.1.4. The left-hand diagram is equal to the right-hand diagram modulo diagrams with fewer total crossings. Thus the claim follows by a standard triangularity argument. ■

**Proposition 7.4.8.** *We have a linear isomorphism*

$$\varphi_k: \text{EH}_k \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k)$$

satisfying (7.4.6) to (7.4.8). Furthermore, the restriction of  $\varphi_k$  to  $\text{EH}_k^\pm$  yields isomorphism of algebras

$$\text{EH}_k^\pm \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k^\pm).$$

**Proof:** The map  $\varphi_k$  is the composite of linear isomorphisms

$$\begin{aligned} \text{EH}_k &\xrightarrow{\cong} \text{EH}^+ \otimes \text{EH}^- && \text{(see (6.2.5))} \\ &\xrightarrow{\cong} \text{EH}_0 && \text{(see (6.2.5))} \\ &\xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_0) && \text{(Theorem 7.4.2)} \\ &\xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_0^+) \otimes \text{Tr}(\mathcal{H}eis_0^-) && \text{(Theorem 7.4.7)} \\ &\xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k^+) \otimes \text{Tr}(\mathcal{H}eis_k^-) && \text{(see (7.2.1))} \end{aligned}$$

$$\xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k).$$

It follows from Theorems 7.4.2 and 7.4.7 that restriction of  $\varphi_0$  to  $\text{EH}_0^\pm$  yields isomorphisms of algebras

$$\text{EH}_0^\pm \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_0^\pm).$$

Thus, restriction of  $\varphi_k$  to  $\text{EH}_k^\pm$  yields the following composite of algebra isomorphisms:

$$\text{EH}_k^\pm \xrightarrow{\cong} \text{EH}_0^\pm \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_0^\pm) \xrightarrow{\cong} \text{Tr}(\mathcal{H}eis_k^\pm).$$

■

We are now ready to prove Theorem 7.4.1.

**Proof: (Proof of Theorem 7.4.1)**

By Propositions 7.4.2 and 7.4.6, it suffices to give the proof for the case  $k < 0$ , which we assume for the remainder of the proof. By Proposition 7.4.8, we have a homomorphism of algebras

$$\text{EH}_k^+ \star \text{EH}_k^- \rightarrow \text{Tr}(\mathcal{H}eis_k)$$

(recall that  $\star$  denotes the free product of algebras) satisfying

$$w_{r,1} \mapsto [\hat{\phi} r], \quad w_{r,-1} \mapsto [\check{\phi} r], \quad w_{s,0} \mapsto [\textcircled{s}], \quad r, s \in \mathbb{Z}, \quad s \neq 0.$$

By Corollary 6.3.4, if we verify that the images of the relations (6.3.6) to (6.3.8) hold in  $\text{Tr}(\mathcal{H}eis_k)$ , it follows that we have an induced algebra homomorphism  $\text{EH}_k \rightarrow \text{Tr}(\mathcal{H}eis_k)$ , which is equal to the linear isomorphism  $\varphi_k$  of Proposition 7.4.8.

It is clear the image of (6.3.8) holds in  $\text{Tr}(\mathcal{H}eis_k)$  since  $\text{Tr}(\beta(\text{Sym} \otimes \text{Sym}))$  is a commutative subalgebra of  $\text{Tr}(\mathcal{H}eis_k)$ . To verify that the image of (6.3.7) holds, we compose all morphisms in (7.1.41) with  $\hat{\phi} s$  to see that

$$\textcircled{r} \uparrow \circlearrowleft s = s \uparrow \circlearrowleft r + \{r\} \uparrow \circlearrowleft s+r.$$

Passing to  $\text{Tr}(\mathcal{H}eis_k)$  gives the desired relations.

It remains to verify that the image of (6.3.6) is satisfied. More precisely, we must show that

$$[\hat{\phi} \check{\phi} s] = [s \check{\phi} \hat{\phi}] - \{s+1\} \left[ \textcircled{s+1} \right] + \delta_{s,-1} k. \tag{7.4.12}$$

(Recall our convention that  $\textcircled{0} = 0$ ; see (7.1.38).)

We first prove (7.4.12) for  $s \geq 0$ . In this case we have

$$\begin{aligned} \uparrow \circlearrowleft s &\stackrel{(7.1.14)}{=} \uparrow \circlearrowleft s \\ &\stackrel{(7.1.3)}{=} \uparrow \circlearrowleft s \\ &\stackrel{(7.1.17)}{=} s \uparrow \circlearrowleft - z \sum_{\substack{a+b=s \\ a,b \geq 0}} \uparrow \circlearrowleft a \\ &\equiv s \uparrow \circlearrowleft - z \sum_{\substack{a+b=s \\ a,b \geq 0}} \uparrow \circlearrowleft a \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(7.1.6)}{=} s \downarrow \uparrow + tz \circlearrowleft_{s+1} + z^2 \sum_{a>0} a \circlearrowleft_{s+a+1} \oplus_{-a} - z^2 \sum_{\substack{a+b=s \\ a,b \geq 0}} \left( \sum_{c \geq 0} \circlearrowleft_{b-c+1} \oplus_{a+c} - \sum_{c>0} \circlearrowleft_{b+c+1} \oplus_{a-c} \right) \\
 & \stackrel{(7.1.19)}{=} s \downarrow \uparrow + z^2 \sum_{a>0} a \oplus_{s+a+1} \oplus_{-a} + z^2 \sum_{\substack{a+b=s \\ a,b \geq 0}} \sum_{c>0} \circlearrowleft_{b+c+1} \oplus_{a-c} \\
 & \stackrel{(7.1.21)}{=} s \downarrow \uparrow + z^2 \sum_{r>s} (r-s) \oplus_{r+1} \oplus_{s-r} + z^2 \sum_{r=1}^s r \oplus_{r+1} \oplus_{s-r} + z^2 \sum_{r>s} s \oplus_{r+1} \oplus_{s-r} \\
 & \stackrel{(7.1.21)}{=} s \downarrow \uparrow + z^2 \sum_{r \geq 1} r \oplus_{r+1} \oplus_{s-r} \stackrel{(7.1.20)}{=} s \downarrow \uparrow + z^2 \sum_{r \in \mathbb{Z}} r \oplus_r \oplus_{s+1-r} \stackrel{(7.1.36)}{=} s \downarrow \uparrow - \{s+1\} \oplus_{s+1}.
 \end{aligned}$$

Thus (7.4.12) holds.

Finally, we prove (7.4.12) for  $s < 0$ . In this case we have

$$\begin{aligned}
 & \stackrel{(7.1.14)}{=} s \downarrow \uparrow \stackrel{(7.1.3)}{=} \circlearrowleft_s \stackrel{(7.1.17)}{=} s \downarrow \uparrow + z \sum_{\substack{a+b=s \\ a,b < 0}} \circlearrowleft_{b+1}^a \stackrel{(7.1.17)}{=} s \downarrow \uparrow + z \sum_{\substack{a+b=s \\ a,b < 0}} \circlearrowleft_{b+1}^a \\
 & \stackrel{(7.1.6)}{=} s \downarrow \uparrow + tz \circlearrowleft_{s+1} + z^2 \sum_{a,b>0} \circlearrowleft_{s+a+b+1} \oplus_{-a-b} + z^2 \sum_{\substack{a+b=s \\ a,b < 0}} \left( \sum_{c \geq 0} \circlearrowleft_{b-c+1} \oplus_{a+c} - \sum_{c>0} \circlearrowleft_{b+c+1} \oplus_{a-c} \right) \\
 & \stackrel{(7.1.19)}{=} s \downarrow \uparrow + tz \circlearrowleft_{s+1} + z^2 \sum_{r>0} (r-1) \circlearrowleft_{s+r+1} \oplus_{-r} + z^2 \sum_{\substack{a+b=s \\ a,b < 0}} \left( \sum_{c \geq 0} \circlearrowleft_{b-c+1} \oplus_{a+c} - \sum_{c>0} \circlearrowleft_{b+c+1} \oplus_{a-c} \right),
 \end{aligned}$$

where, in the final sum above, we used the fact that  $\oplus_m \oplus_n = 0$  whenever  $m+n < 0$ ,

by (7.1.21). Now,

$$\begin{aligned}
 & \sum_{r>0} (r-1) \circlearrowleft_{s+r+1} \oplus_{-r} \stackrel{(7.1.19)}{=} \sum_{r>0} (r-1) \oplus_{s+r+1} \oplus_{-r} + \sum_{r>0} (r-1) \circlearrowleft_{s+r+1} \oplus_{-r} \\
 & \stackrel{(7.1.20)}{=} \delta_{s,-1}(k+1)z^{-2}1_{\mathbb{1}} + \sum_{r>0} (r-1) \circlearrowleft_{s+r+1} \oplus_{-r} \\
 & \stackrel{(7.1.21)}{=}
 \end{aligned}$$

and

$$\sum_{\substack{a+b=s \\ a,b < 0}} \sum_{c>0} \circlearrowleft_{b+c+1} \oplus_{a-c} = \sum_{a=1}^{-s-1} \sum_{c>0} \circlearrowleft_{s+a+c+1} \oplus_{-a-c} = \sum_{a,c>0} \circlearrowleft_{s+a+c+1} \oplus_{-a-c} = \sum_{r>0} (r-1) \circlearrowleft_{s+r+1} \oplus_{-r},$$

where the second equality follows from the fact that, when  $a \geq -s$  and  $c > 0$ , we have  $s + a + c + 1 > 0$ . Thus

$$\begin{aligned}
 \begin{array}{c} \uparrow \\ \circ \\ \downarrow \end{array} \begin{array}{c} | \\ \circ \\ | \end{array} s - \begin{array}{c} | \\ \circ \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \circ \\ | \end{array} &= \delta_{s,-1}(k+1)1_{\mathbb{1}} + tz \begin{array}{c} \circ \\ \circ \\ \circ \\ s+1 \end{array} + z^2 \sum_{\substack{a+b=s \\ a,b < 0}} \sum_{c \geq 0} \begin{array}{c} \circ \\ \circ \\ \circ \\ b-c+1 \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ a+c \end{array} \\
 &\stackrel{(7.1.19)}{=} \delta_{s,-1}(k+1)1_{\mathbb{1}} + z^2 \begin{array}{c} \circ \\ \circ \\ \circ \\ s+1 \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ 0 \end{array} + z^2 \sum_{b=s+2}^0 \sum_{c \leq 0} \begin{array}{c} \circ \\ \circ \\ \circ \\ b+c \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ s+1-b-c \end{array} \\
 &\stackrel{(7.1.21)}{=} \delta_{s,-1}(k+1)1_{\mathbb{1}} + z^2 \sum_{b=s+1}^0 \sum_{c \leq 0} \begin{array}{c} \circ \\ \circ \\ \circ \\ b+c \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ s+1-b-c \end{array} \\
 &\stackrel{(7.1.22)}{=} \delta_{s,-1}(k+1)1_{\mathbb{1}} + z^2 \sum_{r \in \mathbb{Z}} (r+1) \begin{array}{c} \circ \\ \circ \\ \circ \\ -r \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ r+s+1 \end{array} \\
 &\stackrel{(7.1.20)}{=} \delta_{s,-1}k1_{\mathbb{1}} + z^2 \sum_{r \in \mathbb{Z}} r \begin{array}{c} \circ \\ \circ \\ \circ \\ -r \end{array} \begin{array}{c} \circ \\ \circ \\ \circ \\ r+s+1 \end{array} \stackrel{(7.1.36)}{=} \delta_{s,-1}k - \{s+1\} \begin{array}{c} \circ \\ \circ \\ \circ \\ s+1 \end{array} .
 \end{aligned}$$

This completes the proof of (7.4.12). ■

**Remark 7.4.9.** When  $k = -1$ , Theorem 7.4.1 can be seen as an extension of [17, Th. 6.3], which gives an isomorphism between “half” of  $\text{EH}_{-1}$  and the trace of the  $q$ -deformed Heisenberg category of [36], which is isomorphic to the monoidal subcategory of  $\mathcal{H}\text{eis}_{-1}(z, -z^{-1})$  consisting of all objects and all morphisms *not involving negative dots* (which thus can be viewed as “half” of  $\mathcal{H}\text{eis}_{-1}$ ). Note that, even in the case  $k = -1$ , our approach has significant advantages. In particular, the extension of the isomorphism to the full  $\text{EH}_k$  allows one to work with the simpler presentation given in Corollary 6.3.4. This allows one, for example, to avoid many of the lengthy and technical arguments of [17, §4], such as [17, Prop. 4.10].

# Chapter 8

## Representations of the elliptic Hall algebra

### 8.1 Action on symmetric functions

There is a natural action of the trace  $\text{Tr}(\mathcal{H}eis_k)$  on the center  $\text{End}_{\mathcal{H}eis_k}(\mathbb{1})$ , which we now explore. Throughout this section we continue with the assumptions on  $\mathbb{k}, q, z, t$ , and  $k$  made at the beginning of Chapter 7. Let us depict an endomorphism  $f \in \text{End}_{\mathcal{H}eis_k}(X)$  by

$$\begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \text{---} \end{array},$$

where the thick vertical strand is a horizontal juxtaposition of upward and downward strands corresponding to  $1_X$ . Then we define the action of  $\text{Tr}(\mathcal{H}eis_k)$  on  $\text{End}_{\mathcal{H}eis_k}(\mathbb{1})$  by

$$\left[ \begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \text{---} \end{array} \right] \cdot g = \boxed{f} \circ g, \quad f \in \text{End}_{\mathcal{H}eis_k}(X), X \in \mathcal{H}eis_k, g \in \text{End}_{\mathcal{H}eis_k}(\mathbb{1}), \quad (8.1.1)$$

and extend by linearity.

There is a unique map  $\rho: \text{EH}_k \otimes \text{Sym}^{\otimes 2} \rightarrow \text{Sym}^{\otimes 2}$  making the diagram

$$\begin{array}{ccc} \text{EH}_k \otimes \text{Sym}^{\otimes 2} & \xrightarrow{\rho} & \text{Sym}^{\otimes 2} \\ \cong \downarrow \varphi_k \otimes \beta & & \cong \downarrow \beta \\ \text{Tr}(\mathcal{H}eis_k) \otimes \text{End}_{\mathcal{H}eis_k}(\mathbb{1}) & \longrightarrow & \text{End}_{\mathcal{H}eis_k}(\mathbb{1}) \end{array} \quad (8.1.2)$$

commute, where the bottom horizontal map is given by the action (8.1.1). The map  $\rho$  gives an action of  $\text{EH}_k$  on  $\text{Sym}^{\otimes 2}$  and our goal is to give an explicit description of this action. We will use the notation  $a \cdot \theta$  for  $\rho(a \otimes \theta)$ ,  $a \in \text{EH}_k$ ,  $\theta \in \text{Sym}^{\otimes 2}$ .

Recall that, for  $r \geq 1$ ,  $h_r$ ,  $e_r$ , and  $p_r$  denote the degree  $r$  complete homogeneous symmetric function, elementary symmetric function, and power sum, respectively. We also adopt the conventions

$$h_0 = e_0 = 1, \quad p_0 = 0 \quad \text{and} \quad h_r = e_r = p_r = 0 \quad \text{for } r < 0.$$

Recall also the notation  $f^\pm$  for  $f \in \text{Sym}$  given in (7.1.32). Then (7.1.34) to (7.1.36) are valid for all  $r \in \mathbb{Z}$ .

**Lemma 8.1.1.** *We have*

$$w_{\pm r,0} \cdot \theta = \mp \{r\}^{-1} p_r^\pm \theta, \quad r \geq 1, \quad \theta \in \text{Sym}^{\otimes 2}, \quad (8.1.3)$$

$$w_{r,1} \cdot 1 = -t^{-1} z^{-1} h_{r-k}^+ + t z^{-1} h_{-r}^-, \quad r \in \mathbb{Z}, \quad (8.1.4)$$

$$w_{r,-1} \cdot 1 = (-1)^{r+k} t z^{-1} e_{r+k}^+ + (-1)^{r-1} t^{-1} z^{-1} e_{-r}^-, \quad r \in \mathbb{Z}. \quad (8.1.5)$$

**Proof:** Equation (8.1.3) follows from (7.1.38) and (7.4.7). To see (8.1.4) and (8.1.5), we compute

$$w_{r,1} \cdot 1 = \beta^{-1} (r \circlearrowleft) \stackrel{(7.1.19)}{=} \beta^{-1} (r \circlearrowright + r \circlearrowleft) \stackrel{(7.1.34)}{=} -t^{-1} z^{-1} h_{r-k}^+ + t z^{-1} h_{-r}^-,$$

and

$$\begin{aligned} w_{r,-1} \cdot 1 &= \beta^{-1} (r \circlearrowright) \stackrel{(7.1.19)}{=} \beta^{-1} (r \circlearrowright + r \circlearrowleft) \\ &\stackrel{(7.1.35)}{=} (-1)^{r+k} t z^{-1} e_{r+k}^+ + (-1)^{r-1} t^{-1} z^{-1} e_{-r}^-. \end{aligned}$$

■

Let  $\mathcal{P}$  be the set of all  $(\lambda_{-\ell_-}, \dots, \lambda_{-2}, \lambda_{-1}, \lambda_1, \lambda_2, \dots, \lambda_{\ell_+}) \in \mathbb{Z}^{\ell_- + \ell_+}$ ,  $\ell_-, \ell_+ \in \mathbb{N}$ , satisfying

$$\lambda_{-\ell_-} \leq \dots \leq \lambda_{-1} < 0 < \lambda_1 \leq \dots \leq \lambda_{\ell_+}. \quad (8.1.6)$$

For such an element  $\lambda \in \mathcal{P}$ , we define

$$\ell_\pm(\lambda) := \ell_\pm, \quad \ell(\lambda) := \ell_+ + \ell_-, \quad I_\lambda := \{-\ell_-, \dots, -1, 1, \dots, \ell_+\}, \quad (8.1.7)$$

$$|\lambda| := \sum_{i \in I_\lambda} \lambda_i, \quad \{\lambda\} := \prod_{i \in I_\lambda} \{\lambda_i\}, \quad \circlearrowleft := \prod_{i \in I_\lambda} \circlearrowleft_{\lambda_i}, \quad P_\lambda := \frac{1}{\{\lambda\}^2} \left( \prod_{i=1}^{\ell_+} p_{\lambda_i}^+ \right) \left( \prod_{i=1}^{\ell_-} p_{-\lambda_i}^- \right). \quad (8.1.8)$$

We think of elements of  $\mathcal{P}$  as partitions whose parts can be either positive or negative. We allow  $\ell_-$  or  $\ell_+$  (or both) to be zero. In particular, the empty partition  $\emptyset$  is an element of  $\mathcal{P}$ . It follows from Proposition 7.1.2 and (7.1.38) that

$\{P_\lambda : \lambda \in \mathcal{P}\}$  is a basis for  $\text{Sym} \otimes \text{Sym}$  and  $\{\circlearrowleft : \lambda \in \mathcal{P}\}$  is a basis for  $\text{End}_{\mathcal{H}eis_k}(\mathbb{1})$ .

Furthermore, it follows from (7.1.38) that

$$\beta(P_\lambda) = \frac{(-1)^{\ell(\lambda)}}{\{\lambda\}} \circlearrowleft \lambda, \quad \lambda \in \mathcal{P}. \quad (8.1.9)$$

For  $\lambda, \mu \in \mathcal{P}$  we define

$$\begin{aligned} \mu \trianglelefteq \lambda &\iff \mu = (\lambda_{i_a}, \dots, \lambda_{i_1}, \lambda_{j_1}, \dots, \lambda_{j_b}) \\ &\text{for some } -\ell_-(\lambda) \leq i_a < \dots < i_1 < 0 < j_1 < \dots < j_b \leq \ell_+(\lambda). \end{aligned} \quad (8.1.10)$$

In other words,  $\mu \trianglelefteq \lambda$  if  $\mu$  is obtained from  $\lambda$  by deleting some of its parts (allowing also  $\mu = \lambda$ ).

**Lemma 8.1.2.** *For  $\lambda \in \mathcal{P}$ , we have*

$$\circlearrowleft \lambda \uparrow = \{\lambda\} \sum_{\mu \trianglelefteq \lambda} \frac{1}{\{\mu\}} |\lambda - |\mu|| \circlearrowleft \mu, \quad \circlearrowleft \lambda \downarrow = \{\lambda\} \sum_{\mu \trianglelefteq \lambda} \frac{(-1)^{\ell(\lambda) - \ell(\mu)}}{\{\mu\}} |\lambda - |\mu|| \circlearrowleft \mu. \quad (8.1.11)$$

**Proof:** This follows from Proposition 7.1.3 by induction on  $\ell(\lambda)$ . ■

**Theorem 8.1.3.** *The action  $\rho$  in (8.1.2) is the unique action of  $\text{EH}_k$  on  $\text{Sym} \otimes \text{Sym}$  given by*

$$w_{r,1} \cdot P_\lambda = z^{-1} \sum_{\mu \trianglelefteq \lambda} \left( -t^{-1} h_{r+|\lambda|-|\mu|-k}^+ + t h_{-r-|\lambda|+|\mu|}^- \right) P_\mu, \quad (8.1.12)$$

$$w_{r,-1} \cdot P_\lambda = z^{-1} \sum_{\mu \trianglelefteq \lambda} (-1)^{r+\ell(\lambda)-\ell(\mu)+|\lambda|+|\mu|} \left( (-1)^k t e_{r+|\lambda|-|\mu|+k}^+ - t^{-1} e_{-r-|\lambda|+|\mu|}^- \right) P_\mu, \quad (8.1.13)$$

for  $r \in \mathbb{Z}$ ,  $\lambda \in \mathcal{P}$ . Furthermore,  $\text{Sym} \otimes \text{Sym}$  is a cyclic  $\text{EH}_k$ -module generated by  $1 \in \text{Sym} \otimes \text{Sym}$ .

**Proof:** The given expressions for  $w_{r,\pm 1} \cdot P_\lambda$  follow from (8.1.4), (8.1.5), and (8.1.11). The uniqueness follows from the fact that  $\text{EH}_k$  is generated by  $w_{r,\pm 1}$ ,  $r \in \mathbb{Z}$ ; see Lemma 6.3.5. The fact that the module is generated by 1 follows from (8.1.3). ■

Note that, even though the definition of  $\text{EH}_k$  does not depend on  $t$ , its action on  $\text{Sym} \otimes \text{Sym}$  does. So we obtain a family of modules depending on the parameter  $t$ . In addition, the following result shows that the cyclic vector  $1 \in \text{Sym} \otimes \text{Sym}$  is an eigenvector for many of the  $w_{0,n}$ , with the eigenvalues depending on  $t$ .

**Proposition 8.1.4.** *We have*

$$w_{0,n} \cdot 1 = \frac{t^n - t^{-n}}{\{n\}} \quad \text{if } k = 0, n \neq 0, \quad w_{0,n} \cdot 1 = \frac{t^n}{\{n\}} \quad \text{if } k, n > 0 \text{ or } k, n < 0. \quad (8.1.14)$$

**Proof:** First suppose  $k = 0$ . Then we have

$$\begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} = t \begin{array}{c} \uparrow \\ | \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} = t^{-1} \begin{array}{c} \uparrow \\ | \\ \uparrow \end{array}, \quad \begin{array}{c} \circ \\ \downarrow \end{array} = t \begin{array}{c} | \\ \downarrow \end{array}, \quad \begin{array}{c} \circ \\ \downarrow \end{array} = t^{-1} \begin{array}{c} | \\ \downarrow \end{array}. \quad (8.1.15)$$

This can be seen by using the fact that  $\mathcal{H}eis_0$  is the framed HOMFLYPT skein category (Proposition 7.3.1) or by using (7.1.21), (7.1.22), (7.1.28), and (7.1.29).

If  $n > 0$ , we have

$$w_{0,n} \cdot 1 \stackrel{(7.4.8)}{=} \frac{z}{\{n\}} \sum_{i=0}^{n-1} [\sigma_{i,n-i-1}] \cdot 1 \stackrel{(8.1.15)}{\stackrel{(7.1.7)}{=}} \frac{z}{\{n\}} \sum_{i=0}^{n-1} t^{n-2i-1} \frac{t - t^{-1}}{z} = \frac{t^n - t^{-n}}{\{n\}}.$$

For  $n = -m < 0$ , we have

$$w_{0,-m} \cdot 1 \stackrel{(7.4.8)}{=} \frac{z}{\{m\}} \sum_{i=0}^{m-1} [\sigma_{-i,i-m-i+1}] \cdot 1 \stackrel{(8.1.15)}{\stackrel{(7.1.8)}{=}} \frac{z}{\{m\}} \sum_{i=0}^{m-1} t^{m-2i-1} \frac{t - t^{-1}}{z} = \frac{t^n - t^{-m}}{\{m\}}.$$

Now suppose  $n, k > 0$ . Then, by (7.1.21), (7.1.22), (7.1.28), and (7.1.29), we have

$$\begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} = t \begin{array}{c} \uparrow \\ | \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \circ \\ \uparrow \end{array} = 0. \quad (8.1.16)$$

Thus

$$w_{0,n} \cdot 1 \stackrel{(7.4.8)}{=} \frac{z}{\{n\}} \sum_{i=0}^{n-1} [\sigma_{i,n-i-1}] \cdot 1 \stackrel{(8.1.16)}{\stackrel{(7.1.7)}{=}} \frac{z}{\{n\}} t^{n-1} \frac{t}{z} = \frac{t^n}{\{n\}}.$$

The case  $n, k < 0$  can be proved directly in an analogous manner, or obtained from the  $n, k > 0$  case by applying the isomorphism  $\Omega_k$  from (7.1.45).  $\blacksquare$

Next, we relate the action described in Theorem 8.1.3 in the case  $k = 0$  to the natural action of the skein algebra of the torus on the skein module of the annulus as described in Example 5.2.4 and explicitly computed in [45, Th. 4.1]. Unless stated otherwise, assume for the rest of this section that  $k = 0$ . For convenience, we will write out the action in [45, Th. 4.1] here. It is shown in [30] that the skein module of the annulus, which is isomorphic, as an algebra, to  $\text{Sym}^{\otimes 2}$ , has a  $\mathbb{k}$ -basis given by elements  $Q_{\lambda,\mu}$ , where  $\lambda, \mu$  range over all partitions (without a negative part). The elements have

a skein theoretic interpretation and enjoy some nice properties [45, §4], for instance they satisfy  $Q_{\lambda, \emptyset} = s_\lambda \otimes 1$  and  $Q_{\emptyset, \mu} = 1 \otimes s_\mu$ , where  $s_\lambda$  is the Schur function of the partition  $\lambda$ . The action of [45, §4] is given in the basis  $\{Q_{\lambda, \mu}\}_{\lambda, \mu}$ .

Let  $\mathcal{P}^+ \subset \mathcal{P}$  denote the set of usual partitions (without negative parts). We will identify partitions with their Young diagrams and use the English convention for Young diagrams. For  $\lambda, \mu \in \mathcal{P}^+$  with  $\mu \subseteq \lambda$ , write  $\lambda/\mu$  for the skew partition consisting of the cells contained in  $\lambda$  but not in  $\mu$ . Write  $\lambda + n$  for the set of partitions  $\alpha \supseteq \lambda$  such that  $\alpha/\lambda$  is a border strip with  $n$  boxes (a border strip is a skew Young diagram that is connected and contains no  $2 \times 2$  squares). Write  $\lambda - n$  for the set of partitions  $\beta \subseteq \lambda$  such that  $\lambda/\beta$  is a border strip with  $n$  boxes. Finally write  $\text{ht}(\theta)$  for the height of a border strip, defined as one less than the number of rows in  $\theta$ .

Following [45], the content of a cell  $x$  of  $\lambda \in \mathcal{P}^+$ , denoted  $x \in \lambda$ , is the integer  $c(x) = j - i$  if the cell  $x$  is in row  $i$  and column  $j$ . The content of  $\lambda$  is the Laurent polynomial

$$C_\lambda(u) = \sum_{x \in \lambda} u^{c(x)} \in \mathbb{k}[u^{\pm 1}].$$

If  $\mu \subseteq \lambda$ , then the content of  $\lambda/\mu$  is defined as

$$C_{\lambda/\mu} = C_\lambda - C_\mu.$$

For  $r \in \mathbb{Z}$  and  $\gamma$  a skew partition, define  $b \in \mathbb{k}$  with

$$b(r, \gamma) = (-1)^{\text{ht}(\gamma)} t^r C_\gamma(q^{2r}).$$

Finally for  $\lambda, \mu \in \mathcal{P}^+$  and  $r \in \mathbb{Z} \setminus \{0\}$ , define scalars  $a_{\lambda, \mu}(r) \in \mathbb{k}$  by

$$a_{\lambda, \mu}(r) = \frac{t^r - t^{-r}}{q^r - q^{-r}} + \{r\} (t^r C_\lambda(q^{2r}) - t^{-r} C_\mu(q^{-2r})).$$

To avoid confusion let us denote the action of [45] with  $\star$ . Then this action is as follows.

**Definition 8.1.5** ([45, Th. 4.6]). *For  $r, n \in \mathbb{Z} \setminus \{0\}$ , we have the following equalities.*

$$w_{r,0} \star Q_{\lambda, \mu} = a_{\lambda, \mu}(r) Q_{\lambda, \mu} \tag{8.1.17}$$

$$w_{0,n} \star Q_{\lambda, \mu} = \sum_{\alpha \in \lambda + n} (-1)^{\text{ht}(\alpha/\lambda)} Q_{\alpha, \mu} + \sum_{\beta \in \mu - n} (-1)^{\text{ht}(\mu/\beta)} Q_{\lambda, \beta} \tag{8.1.18}$$

$$w_{r,n} \star Q_{\lambda, \mu} = \frac{\{r\}}{\{rn\}} \left( \sum_{\alpha \in \lambda + n} b(r, \alpha/\lambda) Q_{\alpha, \mu} + \sum_{\beta \in \mu - n} b(-r, \mu/\beta) Q_{\lambda, \beta} \right). \tag{8.1.19}$$

**Remark 8.1.6.** The formula for  $w_{0, \pm n} \star Q_{\lambda, \mu}$ , with  $n > 0$ , is exactly the expression of the product  $p_n^\pm Q_{\lambda, \mu}$  in the basis  $\{Q_{\lambda, \mu}\}_{\lambda, \mu}$ , see [45, §4.1]. That is

$$w_{0, \pm n} \star f = p_n^\pm f, \quad n > 0, f \in \text{Sym}^{\otimes 2}. \tag{8.1.20}$$

For instance  $w_{0,n} \star Q_{\lambda, \emptyset}$ ,  $n > 0$ , recovers the Murnaghan–Nakayama rule for multiplying a power sum function with a Schur function.

To state the relation between the action in this proposition and the one in Theorem 8.1.3, we need some more notation. So let

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad \xi: \mathrm{Sym}^{\otimes 2} \rightarrow \mathrm{Sym}^{\otimes 2}, \quad \xi(P_\lambda) = (-1)^{\ell(\lambda)} \{\lambda\} P_\lambda, \quad \lambda \in \mathcal{P}.$$

**Proposition 8.1.7.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{EH}_0 \otimes \mathrm{Sym}^{\otimes 2} & \xrightarrow{\rho} & \mathrm{Sym}^{\otimes 2} \\ \downarrow \gamma \otimes \xi & & \downarrow \xi \\ \mathrm{EH}_0 \otimes \mathrm{Sym}^{\otimes 2} & \longrightarrow & \mathrm{Sym}^{\otimes 2} \end{array},$$

where  $\gamma: \mathrm{EH}_0 \rightarrow \mathrm{EH}_0$  is the isomorphism (6.2.4) and the bottom horizontal arrow is the action of Proposition 8.1.5.

We need some preparatory results to prove the proposition. The ring of symmetric functions  $\mathrm{Sym}$  has  $\mathbb{k}$ -basis  $\{A_\lambda\}_{\lambda \in \mathcal{P}^+}$  called Turaev's basis [61]. This basis has a skein theoretic interpretation as the closure of certain braids in the annulus. We will not need this interpretation here; we will rather use the following characterization from [44, Th. 2] as a definition.

$$1 + z \sum_{r=1}^{\infty} A_r u^r = \exp \left( \sum_{r=1}^{\infty} \frac{\{r\}}{r} p_r u^r \right). \quad (8.1.21)$$

Here the equality holds in  $\mathrm{Sym} \otimes_{\mathbb{k}} \mathbb{k}[[u]]$  and  $A_\lambda = A_{\lambda_1} \cdots A_{\lambda_{\ell(\lambda)}}$ . Similarly define monomials  $\bar{A}_r$  via

$$1 - z \sum_{r=1}^{\infty} \bar{A}_r u^r = \exp \left( - \sum_{r=1}^{\infty} \frac{\{r\}}{r} p_r u^r \right). \quad (8.1.22)$$

In the skein theoretic interpretation, the element  $\bar{A}_r$  is the mirror image of  $A_r$  in the annulus; see [41, 44] for details. In the following lemma we compute the image of the elementary and complete homogeneous functions under the algebra homomorphism  $\xi$ .

**Lemma 8.1.8.** *Let  $r > 0$ . We have*

$$\xi(e_r) = (-1)^r z A_r \quad \text{and} \quad \xi(h_r) = -z \bar{A}_r. \quad (8.1.23)$$

**Proof:** For  $\lambda \in \mathcal{P}^+$ , let  $m_i(\lambda)$  denote the multiplicity of  $i$  in  $\lambda$ . Define

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!.$$

Then  $e_r$  can be written in the basis  $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$  as

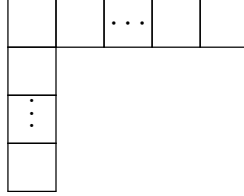
$$e_r = (-1)^r \sum_{\lambda \vdash r} \frac{(-1)^{\ell(\lambda)}}{z_\lambda} p_\lambda.$$

Thus

$$\xi(e_r) = (-1)^r \sum_{\lambda \vdash r} \frac{\{\lambda\}}{z_\lambda} p_\lambda \stackrel{(8.1.21)}{=} (-1)^r z A_r.$$

The last equality follows by comparing the coefficients of  $u^r$  on the LHS and RHS of (8.1.21) and using the definition of  $z_\lambda$ . The second equality in the lemma can be proved in a similar way using (8.1.22) and the expression of  $h_r$  in the basis  $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ .  $\blacksquare$

Let  $r > 0$ . A *hook partition*  $\lambda$  of  $r$  is a partition with  $r$  boxes whose Young diagram is of the form



We write  $\lambda = (i|j)$  if the (horizontal) arm has  $i$  boxes and the (vertical) leg has  $j$  boxes, so that  $r = |\lambda| = i + j + 1$ . The following result from [44] gives the expression of Turaev's monomials  $A_r$  in terms of Schur functions  $s_\lambda$ .

**Lemma 8.1.9** ([44, Th. 15]). *We have*

$$A_r = \sum_{i+j=r-1} (-1)^j q^{i-j} s_{(i|j)} \quad \text{and} \quad \bar{A}_r = \sum_{i+j=r-1} (-1)^j q^{j-i} s_{(i|j)}. \quad (8.1.24)$$

Let  $\rho'$  denote the bottom horizontal arrow in Proposition 8.1.7.

**Lemma 8.1.10.** *For  $r \in \mathbb{Z}$ , we have*

$$\xi \circ \rho(w_{r,\pm 1} \otimes 1) = \rho'(\gamma(w_{r,\pm 1}) \otimes 1). \quad (8.1.25)$$

**Proof:** Assume  $r > 0$ . A straightforward computation of (8.1.19), with  $\lambda = \mu = \emptyset$ , using (8.1.24), the definition of the coefficients  $b(\pm 1, \theta)$ , and the fact that  $Q_{\lambda, \emptyset} = s_\lambda \otimes 1$  and  $Q_{\emptyset, \mu} = 1 \otimes s_\mu$  gives  $\rho'(\gamma(w_{\pm r, \pm 1}) \otimes 1)$  in terms of Turaev's monomials  $A_r$  and  $\bar{A}_r$ . We get

$$\rho'(\gamma(w_{-r,-1}) \otimes 1) = -t^{-1} 1 \otimes A_r \stackrel{(8.1.23)}{=} (-1)^{r-1} t^{-1} z^{-1} \xi(e_r^-) \stackrel{(8.1.5)}{=} \xi \circ \rho(w_{-r,-1} \otimes 1).$$

Similarly

$$\rho'(\gamma(w_{-r,1}) \otimes 1) = -t 1 \otimes \bar{A}_r \stackrel{(8.1.23)}{=} t z^{-1} \xi(h_r^-) \stackrel{(8.1.4)}{=} \xi \circ \rho(w_{-r,1} \otimes 1).$$

On the other hand, we compute

$$\rho'(\gamma(w_{r,-1}) \otimes 1) = tA_r \otimes 1 \stackrel{(8.1.23)}{=} (-1)^r tz^{-1} \xi(e_r^+) \stackrel{(8.1.5)}{=} \xi \circ \rho(w_{r,-1} \otimes 1),$$

and

$$\rho'(\gamma(w_{r,1}) \otimes 1) = t^{-1}\overline{A}_r \otimes 1 \stackrel{(8.1.23)}{=} -t^{-1}z^{-1} \xi(h_r^+) \stackrel{(8.1.4)}{=} \xi \circ \rho(w_{r,1} \otimes 1).$$

Finally the case  $r = 0$  follows by comparing the first equation in (8.1.14) to (8.1.17) (by substituting  $r = \pm 1$  and  $\lambda = \mu = \emptyset$  in (8.1.17)).  $\blacksquare$

**Proof: (Proof of Proposition 8.1.7)**

It follows from (8.1.3) and (8.1.20) that

$$\xi \circ \rho(w_{r,0} \otimes f) = \rho'(\gamma(w_{r,0}) \otimes f), \quad (8.1.26)$$

for all  $r \in \mathbb{Z}$  and  $f \in \text{Sym}^{\otimes 2}$ . By Lemma 6.3.5, it is enough to show that

$$\xi \circ \rho(w_{r,\pm 1} \otimes P_\lambda) = \rho'(\gamma(w_{r,\pm 1}) \otimes P_\lambda), \quad (8.1.27)$$

for all  $r \in \mathbb{Z}$  and  $\lambda \in \mathcal{P}$ . We prove this by induction on  $\ell(\lambda)$ . For the base case  $\ell(\lambda) = 1$ , we need to verify that

$$\xi \circ \rho(w_{r,1} \otimes p_s^\pm) = \rho'(\gamma(w_{r,1}) \otimes p_s^\pm) \quad \text{and} \quad \xi \circ \rho(w_{r,-1} \otimes p_s^\pm) = \rho'(\gamma(w_{r,-1}) \otimes p_s^\pm), \quad (8.1.28)$$

where  $s > 0$ . We will illustrate the computations with one case  $\xi \circ \rho(w_{r,1} \otimes p_s^+) = \rho'(\gamma(w_{r,1}) \otimes p_s^+)$ , the other cases being similar. For this case we compute

$$\begin{aligned} w_{r,1} \cdot p_s^+ &\stackrel{(8.1.3)}{=} -\{s\}w_{r,1} \cdot (w_{s,0} \cdot 1) = -\{s\}w_{r,1}w_{s,0} \cdot 1 \stackrel{(6.1.1)}{=} \{s\}^2 w_{r+s,1} \cdot 1 - \{s\}w_{s,0}w_{r,1} \cdot 1 \\ &= \{s\}^2 w_{r+s,1} \cdot 1 - \{s\}w_{s,0} \cdot (w_{r,1} \cdot 1). \end{aligned}$$

So

$$\begin{aligned} \xi(w_{r,1} \cdot p_s^+) &= \{s\}^2 \xi(w_{r+s,1} \cdot 1) - \{s\} \xi(w_{s,0} \cdot (w_{r,1} \cdot 1)) \\ &\stackrel{(8.1.25)}{=} \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes 1) - \{s\} \rho'(\gamma(w_{s,0}) \otimes \xi(w_{r,1} \cdot 1)) \\ &\stackrel{(8.1.26)}{=} \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes 1) - \{s\} \rho'(\gamma(w_{s,0}) \otimes \xi(w_{r,1} \cdot 1)) \\ &\stackrel{(8.1.25)}{=} \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes 1) - \{s\} \rho'(\gamma(w_{s,0}) \otimes \rho'(\gamma(w_{r,1}) \otimes 1)) \\ &= \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes 1) - \{s\} \rho'(\gamma(w_{s,0}) \gamma(w_{r,1}) \otimes 1) \\ &= \{s\} \rho'((\{s\} \gamma(w_{r+s,1}) - \gamma(w_{s,0}) \gamma(w_{r,1})) \otimes 1) \\ &\stackrel{(6.1.1)}{=} -\{s\} \rho'(\gamma(w_{r,1}) \gamma(w_{s,0}) \otimes 1) \\ &\stackrel{(8.1.26)}{=} -\{s\} \rho'(\gamma(w_{r,1}) \otimes p_s^+) \\ &= \rho'(\gamma(w_{r,1}) \otimes \xi(p_s^+)), \end{aligned}$$

as wanted.

Now suppose that (8.1.27) holds for any  $\lambda \in \mathcal{P}$  with a fixed  $\ell(\lambda) \geq 1$ . We need to verify that

$$\xi \circ \rho(w_{r,1} \otimes p_s^\pm P_\lambda) = \rho'(\gamma(w_{r,1}) \otimes p_s^\pm P_\lambda), \quad (8.1.29)$$

for  $s > 0$ . Again we illustrate with one case, namely  $p_s^+$ , the other case is similar. As before, using (6.1.1) and (8.1.3), we compute

$$w_{r,1} \cdot (p_s^+ P_\lambda) = \{s\}^2 w_{r+s,1} \cdot P_\lambda - \{s\} w_{s,0} \cdot (w_{r,1} \cdot P_\lambda).$$

Then

$$\begin{aligned} \xi(w_{r,1} \cdot (p_s^+ P_\lambda)) &= \{s\}^2 \xi(w_{r+s,1} \cdot P_\lambda) - \{s\} \xi(w_{s,0} \cdot (w_{r,1} \cdot P_\lambda)) \\ &\stackrel{(8.1.26)}{=} \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes \xi(P_\lambda)) - \{s\} \rho'(\gamma(w_{s,0}) \otimes \xi(w_{r,1} \cdot P_\lambda)) \\ &= \{s\}^2 \rho'(\gamma(w_{r+s,1}) \otimes \xi(P_\lambda)) - \{s\} \rho'(\gamma(w_{s,0}) \otimes \rho'(\gamma(w_{r,1}) \otimes \xi(P_\lambda))) \\ &\stackrel{(6.1.1)}{=} -\{s\} \rho'((\gamma(w_{r,1}) \gamma(w_{s,0})) \otimes \xi(P_\lambda)) \\ &\stackrel{(8.1.26)}{=} -\{s\} \rho'((\gamma(w_{r,1})) \otimes p_s^+ \xi(P_\lambda)) \\ &= \rho'(\gamma(w_{r,1}) \otimes \xi(p_s^+ P_\lambda)). \end{aligned}$$

Note that we used the induction hypothesis in the second and third equalities. This completes the proof of the proposition.  $\blacksquare$

Define a basis  $\{R_{\lambda,\mu}\}_{\lambda,\mu \in \mathcal{P}^+}$  of  $\text{Sym}^{\otimes 2}$  with  $R_{\lambda,\mu} = \xi^{-1}(Q_{\lambda,\mu})$ . The action Theorem 8.1.3 in the basis  $\{R_{\lambda,\mu}\}_{\lambda,\mu \in \mathcal{P}^+}$  is as follows.

**Corollary 8.1.11.** *For  $k = 0$ , the action (8.1.2) of  $\text{EH}_0$  on  $\text{Sym}^{\otimes 2}$  is given by*

$$w_{0,n} \cdot R_{\lambda,\mu} = a_{\lambda,\mu}(n) R_{\lambda,\mu} \quad (8.1.30)$$

$$w_{r,0} \cdot R_{\lambda,\mu} = \sum_{\alpha \in \lambda+r} (-1)^{\text{ht}(\alpha/\lambda)} R_{\alpha,\mu} + \sum_{\beta \in \mu-r} (-1)^{\text{ht}(\mu/\beta)} R_{\lambda,\beta} \quad (8.1.31)$$

$$w_{r,n} \cdot R_{\lambda,\mu} = \frac{\{n\}}{\{rn\}} \left( \sum_{\alpha \in \lambda+r} b(-n, \alpha/\lambda) R_{\alpha,\mu} + \sum_{\beta \in \mu-r} b(n, \mu/\beta) R_{\lambda,\beta} \right), \quad (8.1.32)$$

for  $r, n \in \mathbb{Z} \setminus \{0\}$ .

**Proof:** This follows from Definition 8.1.5 and Proposition 8.1.7 together with the fact that  $a_{\lambda,\mu}(-n) = a_{\lambda,\mu}(n)$ .  $\blacksquare$

**Proposition 8.1.12.** *For  $k > 0$ , the action (8.1.2) is the unique action of  $\text{EH}_k$  on  $\text{Sym}^{\otimes 2}$  given by*

$$\begin{aligned} w_{r,0} \cdot R_{\lambda,\mu} &= \sum_{\alpha \in \lambda+r} (-1)^{\text{ht}(\alpha/\lambda)} R_{\alpha,\mu} + \sum_{\beta \in \mu-r} (-1)^{\text{ht}(\mu/\beta)} R_{\lambda,\beta}, \quad r \in \mathbb{Z} \setminus \{0\}, \\ w_{0,1} \cdot R_{\lambda,\mu} &= -\{k\}^{-1} \left( \sum_{\alpha \in \lambda-k} (-1)^{\text{ht}(\alpha/\lambda)} (d_{\alpha,\mu} - d_{\lambda,\mu}) R_{\alpha,\mu} + \sum_{\beta \in \mu+k} (-1)^{\text{ht}(\mu/\beta)} (d_{\lambda,\beta} - d_{\lambda,\mu}) R_{\lambda,\beta} \right), \\ w_{0,-1} \cdot R_{\lambda,\mu} &= -\frac{\{1\}}{\{k\}} \left( \sum_{\alpha \in \lambda-k} b(1, \alpha/\lambda) R_{\alpha,\mu} + \sum_{\beta \in \mu+k} b(-1, \mu/\beta) R_{\lambda,\beta} \right) - t^{-1} z^{-1} R_{\lambda,\mu}, \end{aligned}$$

where  $d_{\lambda,\mu} = a_{\lambda,\mu}(1) - tz^{-1}$ .

**Proof:** The action of  $w_{r,0}$  is the same as in the  $k = 0$  case in Corollary 8.1.11.

To avoid any possible confusion, we write  $\odot$  for the action of Theorem 8.1.3 in the  $k = 0$  case while we use  $\cdot$  for the action in the proposition. We have

$$w_{k,1} \cdot P_\lambda \stackrel{(8.1.12)}{=} -t^{-1} z^{-1} \sum_{\mu \leq \lambda} h_{|\lambda|-|\mu|}^+ P_\mu = w_{0,1} \odot P_\lambda - tz^{-1} P_\lambda, \quad \lambda \in \mathcal{P}.$$

(Here  $w_{k,1} \in \text{EH}_k$  while  $w_{0,1} \in \text{EH}_0$ . This should not cause confusion, since the two actions are denoted differently.) Then, for  $\lambda, \mu \in \mathcal{P}^+$ , we have

$$w_{k,1} \cdot R_{\lambda,\mu} = w_{0,1} \odot R_{\lambda,\mu} - tz^{-1} R_{\lambda,\mu} \stackrel{(8.1.30)}{=} (a_{\lambda,\mu}(1) - tz^{-1}) R_{\lambda,\mu} = d_{\lambda,\mu} R_{\lambda,\mu}. \quad (8.1.33)$$

Next, we compute

$$\begin{aligned} \{k\} w_{0,1} \cdot R_{\lambda,\mu} &\stackrel{(6.1.1)}{=} w_{k,1} w_{-k,0} \cdot R_{\lambda,\mu} - w_{-k,0} w_{k,1} \cdot R_{\lambda,\mu} \stackrel{(8.1.3)}{=} \{k\}^{-1} (w_{k,1} - d_{\lambda,\mu}) \cdot p_k^- R_{\lambda,\mu} \\ &\stackrel{(8.1.18)}{=} \stackrel{(8.1.20)}{=} - (w_{k,1} - d_{\lambda,\mu}) \cdot \left( \sum_{\alpha \in \lambda-k} (-1)^{\text{ht}(\alpha/\lambda)} R_{\alpha,\mu} + \sum_{\beta \in \mu+k} (-1)^{\text{ht}(\mu/\beta)} R_{\lambda,\beta} \right) \\ &\stackrel{(8.1.33)}{=} - \left( \sum_{\alpha \in \lambda-k} (-1)^{\text{ht}(\alpha/\lambda)} (d_{\alpha,\mu} - d_{\lambda,\mu}) R_{\alpha,\mu} + \sum_{\beta \in \mu+k} (-1)^{\text{ht}(\mu/\beta)} (d_{\lambda,\beta} - d_{\lambda,\mu}) R_{\lambda,\beta} \right). \end{aligned}$$

For the action of  $w_{0,-1}$ , we compute

$$\begin{aligned} w_{0,-1} \cdot P_\lambda &\stackrel{(8.1.13)}{=} (-1)^k t z^{-1} \sum (-1)^{\ell(\lambda) - \ell(\mu) + |\lambda| + |\mu|} e_{|\lambda|-|\mu|+k}^+ P_\mu - t^{-1} z^{-1} P_\lambda \\ &\stackrel{(8.1.13)}{=} w_{k,-1} \odot P_\lambda - t^{-1} z^{-1} P_\lambda. \end{aligned}$$

Then, by Corollary 8.1.11, we have

$$\begin{aligned}
w_{0,-1} \cdot R_{\lambda,\mu} &= w_{k,-1} \odot R_{\lambda,\mu} - t^{-1}z^{-1}R_{\lambda,\mu} \\
&= -\frac{\{1\}}{\{k\}} \left( \sum_{\alpha \in \lambda - k} b(1, \alpha/\lambda) R_{\alpha,\mu} + \sum_{\beta \in \mu + k} b(-1, \mu/\beta) R_{\lambda,\beta} \right) - t^{-1}z^{-1}R_{\lambda,\mu}.
\end{aligned}$$

Since the elements  $w_{0,\pm 1}$  and  $w_{r,0}$ , for  $r \in \mathbb{Z} \setminus \{0\}$ , generate  $\text{EH}_k$ , the proposition follows.  $\blacksquare$

**Proposition 8.1.13.** *For  $k < 0$ , the action (8.1.2) is the unique action of  $\text{EH}_k$  on  $\text{Sym}^{\otimes 2}$  given by*

$$\begin{aligned}
w_{r,0} \cdot R_{\lambda,\mu} &= \sum_{\alpha \in \lambda + r} (-1)^{\text{ht}(\alpha/\lambda)} R_{\alpha,\mu} + \sum_{\beta \in \mu - r} (-1)^{\text{ht}(\mu/\beta)} R_{\lambda,\beta}, \quad r \in \mathbb{Z} \setminus \{0\}, \\
w_{0,1} \cdot R_{\lambda,\mu} &= \frac{\{1\}}{\{k\}} \left( \sum_{\alpha \in \lambda + k} b(1, \alpha/\lambda) R_{\alpha,\mu} + \sum_{\beta \in \mu - k} b(-1, \mu/\beta) R_{\lambda,\beta} \right) - t^{-1}z^{-1}R_{\lambda,\mu}, \\
w_{0,-1} \cdot R_{\lambda,\mu} &= \{k\}^{-1} \left( \sum_{\alpha \in \lambda + k} (-1)^{\text{ht}(\alpha/\lambda)} (d_{\alpha,\mu} - d_{\lambda,\mu}) R_{\alpha,\mu} + \sum_{\beta \in \mu - k} (-1)^{\text{ht}(\mu/\beta)} (d_{\lambda,\beta} - d_{\lambda,\mu}) R_{\lambda,\beta} \right),
\end{aligned}$$

where  $d_{\lambda,\mu} = a_{\lambda,\mu}(1) - tz^{-1}$ .

**Proof:** This follows from the previous proposition by using the isomorphism  $\psi_k: \text{EH}_k \rightarrow \text{EH}_{-k}$ ,  $w_{r,n} \mapsto w_{-r,-n}$ .  $\blacksquare$

**Remark 8.1.14.** Using the formula for  $Q_{\lambda,\mu}$  in [30, §3.1], we can actually give an explicit definition for  $R_{\lambda,\mu}$ . Define  $\beta_r^\pm \in \text{Sym}^{\otimes 2}$  via

$$1 + \sum_{r \geq 1} \beta_r^\pm u^r = \exp \left( - \sum_{r \geq 1} \frac{p_r^\pm}{r \{r\}} u^r \right).$$

It is easy to check, using an argument similar to the proof of (8.1.23), that  $\beta_r^\pm$  satisfy  $\xi^{-1}(h_r^\pm) = \beta_r^\pm$ . We illustrate the definition of  $R_{\lambda,\mu}$  with the example  $\lambda = (3, 1)$  and  $\mu = (2, 1, 1)$ , the general construction can hopefully be readily guessed from this example; otherwise see [30, §3.1] or [45, §4.2], from which the example is taken. To define  $R_{\lambda,\mu}$ , we start by filling the diagonal of a with  $\ell(\lambda) + \ell(\mu)$  entries as shown below

$$\begin{pmatrix} \beta_1^- & & & & \\ & \beta_1^- & & & \\ & & \beta_2^- & & \\ & & & \beta_3^+ & \\ & & & & \beta_1^+ \end{pmatrix}.$$

We complete the rows by shifting the indices, from left to right, in an increasing order for the  $\beta_i^+$ 's, and a decreasing order for the  $\beta_i^-$ 's as shown below

$$M_{\lambda,\mu} = \begin{pmatrix} \beta_1^- & 1 & 0 & 0 & 0 \\ \beta_2^- & \beta_1^- & 1 & 0 & 0 \\ \beta_4^- & \beta_3^- & \beta_2^- & \beta_1^- & 1 \\ 1 & \beta_1^+ & \beta_2^+ & \beta_3^+ & \beta_4^+ \\ 0 & 0 & 0 & 1 & \beta_1^+ \end{pmatrix}.$$

Define  $R_{\lambda,\mu} = \det M_{\lambda,\mu}$ . This definition is such that  $\xi(R_{\lambda,\mu}) = Q_{\lambda,\mu}$ .

## 8.2 Action on cocenters of cyclotomic Hecke algebras

In this section, we describe the action of  $\mathrm{Tr}(\mathcal{H}eis_k)$  on traces of cyclotomic quotients of  $\mathcal{H}eis_k$  or, equivalently, on cocenters of cyclotomic Hecke algebras. Throughout this section we assume that  $\mathbb{k}$  is a field of characteristic zero and  $q, t \in \mathbb{k}^\times$ , with  $q$  not a root of unity.

Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $\mathbb{k}$ -linear categories. Recall the definition of the  $\mathbb{k}$ -linear category  $\mathcal{C} \boxtimes \mathcal{D}$  in Definition 3.2.6. It is straightforward to verify that we have a linear isomorphism

$$\mathrm{Tr}(\mathcal{C}) \otimes \mathrm{Tr}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Tr}(\mathcal{C} \boxtimes \mathcal{D}), \quad [f] \otimes [g] \mapsto [f \otimes g].$$

Recall the definition of a module category in Definition 3.2.7. If  $\mathcal{M}$  is a module category over  $\mathcal{C}$  with action functor  $F: \mathcal{C} \boxtimes \mathcal{M} \rightarrow \mathcal{M}$ , then we have an induced action of  $\mathrm{Tr}(\mathcal{C})$  on  $\mathrm{Tr}(\mathcal{M})$  given by

$$\mathrm{Tr}(\mathcal{C}) \otimes \mathrm{Tr}(\mathcal{M}) \xrightarrow{\cong} \mathrm{Tr}(\mathcal{C} \boxtimes \mathcal{M}) \xrightarrow{\mathrm{Tr}[F]} \mathrm{Tr}(\mathcal{M}). \quad (8.2.1)$$

The goal of this section is to use this fact to construct  $\mathrm{EH}_k$ -modules from certain module categories over the quantum Heisenberg category.

For  $n \in \mathbb{Z}_{\geq 1}$ , let  $\mathrm{H}_n$  denote the Iwahori–Hecke algebra of type  $A_{n-1}$ . This is the associative  $\mathbb{k}$ -algebra with generators  $\tau_1, \dots, \tau_{n-1}$  and relations

$$\begin{aligned} \tau_i \tau_j &= \tau_j \tau_i, & 1 \leq i, j \leq n-1, \quad |i-j| > 1, \\ \tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1}, & 1 \leq i \leq n-2, \\ \tau_i^2 &= z \tau_i + 1, & 1 \leq i \leq n-1. \end{aligned}$$

Let  $\mathrm{AH}_n$  denote the affine Hecke algebra of rank  $n$ . Thus  $\mathrm{AH}_n = \mathrm{H}_n \otimes_{\mathbb{k}} \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm n}]$  as  $\mathbb{k}$ -modules, with the two factors being subalgebras, and

$$\tau_i x_i = x_{i+1} \tau_i^{-1}, \quad 1 \leq i \leq n-1.$$

We adopt the convention that  $\mathrm{H}_0 = \mathrm{AH}_0 = \mathbb{k}$ .

It follows from (7.1.1) to (7.1.3) that we have an algebra homomorphism

$$\mathrm{AH}_n \rightarrow \mathrm{End}_{\mathcal{H}\mathrm{eis}_k}(\uparrow^{\otimes n}), \quad (8.2.2)$$

sending  $x_i$  to a dot on the  $i$ -th string and  $\tau_i$  to a positive crossing  $\nearrow \searrow$  of the  $i$ -th and  $(i+1)$ -st strings, where we number strings  $1, 2, \dots, n$  from right to left. In fact, it follows from Theorem 7.1.4 that this map is injective.

Fix a nonnegative integer  $l$  and a polynomial

$$f(u) = f_0 u^l + f_1 u^{l-1} + \dots + f_l \in \mathbb{k}[u], \quad f_0 = 1, \quad f_l = t^2. \quad (8.2.3)$$

(Note that this forces  $t = \pm 1$  if  $l = 0$ .) The *cyclotomic Hecke algebra*  $\mathrm{H}_n^f$  of level  $l$  associated to the polynomial  $f(u)$  is the quotient of  $\mathrm{AH}_n$  by the two-sided ideal generated by  $f(x_1)$ . By convention  $\mathrm{H}_0^f = \mathbb{k}$ . The basis theorem proved in [1, Th. 3.10] states that

$$\{x_1^{r_1} \cdots x_n^{r_n} \tau_g : 0 \leq r_1, \dots, r_n < l, \quad g \in \mathfrak{S}_n\}$$

is a basis for  $\mathrm{H}_n^f$  as a free  $\mathbb{k}$ -module, where  $\tau_g$  denotes the element of the finite Hecke algebra defined from a reduced expression for the element  $g$  of the symmetric group  $\mathfrak{S}_n$ .

Define the *tower of cyclotomic Hecke algebras* associated to  $f$  to be the  $\mathbb{k}$ -linear category  $\mathcal{H}^f$  with objects  $X_n$ ,  $n \in \mathbb{N}$ , and

$$\mathrm{Hom}_{\mathcal{H}^f}(X_n, X_m) = \begin{cases} \mathrm{H}_n^f & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Note that  $\mathcal{H}^f$  is *not* naturally a monoidal category. As we will now explain the tower  $\mathcal{H}^f$  can also be realized as a cyclotomic quotient of the quantum Heisenberg category.

Let  $\mathcal{I}(f)$  be the left tensor ideal generated by the morphism  $f(\dagger)$ . The *cyclotomic quantum Heisenberg category* associated to the polynomial  $f(u)$  is the quotient category

$$\mathcal{H}(f) := \mathcal{H}\mathrm{eis}_{-l} / \mathcal{I}(f). \quad (8.2.4)$$

Note that  $\mathcal{H}(f)$  is a  $\mathbb{k}$ -linear category, but it does not inherit the monoidal structure from  $\mathcal{H}\mathrm{eis}_{-l}$ . However, it is a left module category over  $\mathcal{H}\mathrm{eis}_{-l}$ .

**Proposition 8.2.1.** *The map (8.2.2) induces algebra isomorphisms*

$$\mathrm{H}_n^f \xrightarrow{\cong} \mathrm{End}_{\mathcal{H}(f)}(\uparrow^{\otimes n}), \quad n \in \mathbb{N}. \quad (8.2.5)$$

Furthermore, the functor  $\mathcal{H}^f \rightarrow \mathcal{H}(f)$  given on objects by  $X_n \mapsto \uparrow^{\otimes n}$  and on morphisms by (8.2.5) is an equivalence of  $\mathbb{k}$ -linear categories.

**Proof:** This is shown in [12, Lem. 5.13] and [13, Th. 9.5]. ■

**Remark 8.2.2.** As shown in [13, Th. 9.5], the equivalence of Proposition 8.2.1 induces an equivalence of  $\mathbb{k}$ -linear categories

$$\mathrm{Kar}(\mathcal{H}(f)) \rightarrow \bigoplus_{n \geq 0} \mathrm{H}_n^f\text{-pmod},$$

where  $\mathrm{Kar}(\mathcal{H}(f))$  denotes the additive Karoubi envelope of  $\mathcal{H}(f)$  and  $\mathrm{H}_n^f\text{-pmod}$  denotes the category of finitely-generated projective left  $\mathrm{H}_n^f$ -modules. Under this isomorphism, the natural action of  $\mathcal{H}eis_{-l}$  on  $\mathcal{H}(f)$  corresponds to an action of  $\mathcal{H}eis_{-l}$  on  $\bigoplus_{n \geq 0} \mathrm{H}_n^f\text{-pmod}$ , with the objects  $\uparrow$  and  $\downarrow$  acting by induction and restriction, respectively. We refer the reader to [13, §6] for details.

For an associative  $\mathbb{k}$ -algebra  $A$ , its *cocenter* is

$$C(A) := A / \mathrm{span}_{\mathbb{k}}\{ab - ba : ab \in A\}.$$

Note that this is the same as the trace of  $A$  considered as a monoidal category with one object. By Proposition 8.2.1 and Theorem 7.4.1, we have an action of  $\mathrm{EH}_{-l} \cong \mathrm{Tr}(\mathcal{H}eis_{-l})$  on

$$V_f := \mathrm{Tr}(\mathcal{H}(f)) \cong \bigoplus_{n \geq 0} C(\mathrm{H}_n^f). \quad (8.2.6)$$

Denote this action by  $\cdot$ , and let  $v_f$  denote the unit element in  $C(\mathrm{H}_0^f) \cong \mathbb{k}$ .

**Proposition 8.2.3.** *The  $\mathrm{EH}_{-l}$ -module  $V_f$  is cyclic, generated by  $v_f$ . Furthermore, we have*

$$w_{r,n} \cdot v_f = 0, \quad r \in \mathbb{Z}, \quad n < 0, \quad (8.2.7)$$

$$w_{r,1} \cdot v_f = - \sum_{i=1}^l f_i w_{r-i,1} \cdot v_f, \quad r \in \mathbb{Z}, \quad (8.2.8)$$

$$\sum_{r \geq 1} u^{1-r} \{r\} w_{r,0} \cdot v_f = (u^2 f'(u) f(u)^{-1} - lu) v_f, \quad (8.2.9)$$

$$\sum_{r \geq 1} u^{r-1} \{r\} w_{-r,0} \cdot v_f = (f'(u) f(u)^{-1}) v_f, \quad (8.2.10)$$

where (8.2.9) and (8.2.10) are equalities of Laurent series in  $\mathbb{k}((u^{-1}))$  and  $\mathbb{k}((u))$ , respectively.

**Proof:** The fact that  $V_f$  is cyclic, generated by  $v_f$ , follows from the fact that this module is a quotient of  $\mathrm{Tr}(\mathcal{H}eis_{-l})$ . The equalities (8.2.7) and (8.2.8) also follow immediately from the definition of  $\mathcal{I}(f)$ .

It is shown in [13, Lem. 9.2] that the ideal  $\mathcal{I}(f)$  contains the morphism  $\downarrow$  and the coefficients of the series

$$\oplus (u) - f(u)^{-1}, \quad \ominus (u) - t^2 f(u)^{-1}, \quad \oplus (u) - f(u), \quad \ominus (u) - t^{-2} f(u), \quad (8.2.11)$$

where the first and second occurrences of  $f(u)^{-1}$  are interpreted as Laurent series in  $u^{-1}$  and  $u$ , respectively; cf. (7.1.23) and (7.1.25). Thus, recalling the series defined in (7.1.42), we have

$$\sum_{r \geq 1} u^{1-r} \{r\} w_{r,0} \cdot v_f \stackrel{(7.4.7)}{=} \stackrel{(7.1.38)}{-\beta(P_+(u)) \cdot v_f} \stackrel{(7.1.43)}{=} u^2 \beta(H'_+(u)/H_+(u)) \cdot v_f$$

$$\stackrel{(7.1.44)}{=} \stackrel{(8.2.11)}{(u^2 f'(u) f(u)^{-1} - lu) \cdot v_f}.$$

This proves (8.2.9). To prove (8.2.10), we use the notation in the proof of Proposition 7.1.3. We have

$$\sum_{r \geq 1} u^{r-1} \{r\} w_{-r,0} \cdot v_f \stackrel{(7.4.7)}{=} \stackrel{(7.1.38)}{\beta(P_-(u)) \cdot v_f} = \beta(H'_-(u)/H_-(u)) \cdot v_f \stackrel{(8.2.11)}{=} (f'(u) f(u)^{-1}) v_f,$$

as wanted. ■

It follows from Proposition 8.2.3 that the cyclic vector  $v_f$  generates a one-dimensional subspace under the action of the commutative subalgebra of  $\text{EH}_{-l}$  generated by the  $w_{r,0}$ ,  $r \in \mathbb{Z} \setminus \{0\}$ , and that  $v_f$  is annihilated by the elements  $w_{r,n}$ ,  $r \in \mathbb{Z}$ ,  $n < 0$ . In this way,  $V_f$  is somewhat like a lowest weight module.

As for the action on the center described in Section 8.1, the action of  $\text{EH}_{-l}$  on  $V_f$  depends on  $t$  (since  $f$  can involve  $t$ ) even though  $\text{EH}_{-l}$  does not. Proposition 8.2.3 is not a complete algebraic description of the action of  $\text{EH}_{-l}$  on  $V_f$  since it only describes the action of certain elements on the cyclic vector  $v_f$ . To give a complete algebraic description of the action, one would need to give an explicit description of the images of the elements  $x_i^r \in \text{AH}_n$  in the cyclotomic quotients  $\text{H}_n^f$ .

**Remark 8.2.4.** When  $l = 1$ , we have  $\text{H}_n^f \cong \text{H}_n$ , with the  $x_i$  being sent to the Jucys–Murphy elements. In this case, explicit formulas are known, and the action on  $V_f \cong \text{Sym}$  (for  $t = -z^{-1}$  and  $f(u) = u + z^{-2}$ ) was computed in [17, §7] for “half” of  $\text{EH}_{-1}$ ; see Remark 7.4.9. The action computed in [17, §7] is a twist of the polynomial representation defined in [56, §1], where it is also realized in terms of the  $K$ -theory of the Hilbert scheme of  $\mathbb{A}^2$ . It is natural to expect that, for higher level  $l$ , the modules  $V_f$  are related to the  $K$ -theory of the moduli space of framed torsion-free sheaves on  $\mathbb{P}^2$ , which can be viewed as higher rank analogues of the Hilbert scheme; see [56, §8].

For the remainder of this section, we assume that  $\mathbb{k}$  is an algebraically closed field of characteristic zero. Let  $I$  be the union of the orbits of the roots of  $f(u)$  under the maps  $i \mapsto q^{\pm 2}i$ ,  $i \in \mathbb{k}$ . It follows from our assumption (6.0.1) that the map  $i \mapsto q^2i$  defines oriented edges making the set  $I$  into a quiver with connected components of type  $A_\infty$ . Let  $\mathfrak{g}$  denote the Kac–Moody Lie algebra associated to this quiver, and let

$\mathcal{U}(\mathfrak{g})$  be the corresponding Kac–Moody 2-category, as defined in [34, 53]. (See also [9], which unified the two approaches.)

For  $i \in I$ , let  $\mu_i$  be the multiplicity of  $i$  as a root of  $f(u)$ . Then let  $\mu := \sum_{i \in I} \mu_i \Lambda_i$  be the corresponding dominant integral weight of  $\mathfrak{g}$ , where  $\Lambda_i$  denotes the fundamental weight corresponding to  $i \in I$ . By [12, Th. B],  $\mathcal{H}(f)$  is isomorphic (as a locally unital algebra) to the cyclotomic quotient  $\mathcal{H}(\mu)$  of  $\mathcal{U}(\mathfrak{g})$  corresponding to  $\mu$ . As for the case of the quantum Heisenberg category discussed above, this implies that we have an action of  $\mathrm{Tr}(\mathcal{U}(\mathfrak{g}))$  on  $\mathrm{Tr}(\mathcal{H}(\mu)) \cong \mathrm{Tr}(\mathcal{H}(f)) \cong V_f$ .

To any symmetrizable Kac–Moody algebra  $\mathfrak{g}$ , one can associate a Lie algebra  $C\mathfrak{g}$  (denoted  $L\mathfrak{g}$  in [58, Def. 3.24] and, when  $\mathfrak{g}$  is simply laced,  $\mathcal{C}\mathfrak{g}$  in [5, §3.2]). The Lie algebra  $C\mathfrak{g}$  is isomorphic to the current algebra  $\mathfrak{g} \otimes \mathbb{k}[t]$  when  $\mathfrak{g}$  is of finite type  $ADE$ , but is larger in general; see [58, Rem. 3.26]. If  $\mathfrak{g}$  is of *finite* type  $A$ , then it follows from [58, Th. 1] and [5, Th. A, B], that  $\mathrm{Tr}(\mathcal{U}(\mathfrak{g}))$  is isomorphic to an idempotented form of the universal enveloping algebra  $U(C\mathfrak{g})$  and that the induced action on cyclotomic quotients realizes its Weyl modules. While [5, 58] do not treat type  $A_\infty$ , we expect that one should be able to take an appropriate limit to handle this case. In this way one would identify  $V_f$  as both a Weyl module for the current algebra and a module for the elliptic Hall algebra.

**Remark 8.2.5.** In general, one associates a generalized cyclotomic quotient  $\mathcal{H}(f|g)$  to a pair  $(f, g)$  of monic polynomials. If  $f$  and  $g$  are of degrees  $l$  and  $m$ , respectively, then  $\mathcal{H}(f|g)$  is a module category over  $\mathcal{H}eis_{m-l}$ . We have restricted our attention here to the case where  $g = 1$ , and hence to central charge  $k = -l < 0$ . As we saw in Proposition 8.2.3, this gives rise to a negative-central-charge module  $V_f$  generated by an eigenvector  $v_f$  for  $\mathrm{EH}^-$ . If we instead considered the case where  $f = 1$ , we would obtain positive-central-charge modules generated by an eigenvector for  $\mathrm{EH}^+$ . Alternatively these positive-central-charge modules can be obtained from the  $V_f$  by twisting by the automorphism  $\Omega_k$  from (7.1.45). The general case, of arbitrary  $f$  and  $g$ , would yield a tensor product of these positive- and negative-central-charge modules; see Remark 6.2.3. We refer the reader to [13, §9] for further details on the more general  $\mathcal{H}(f|g)$ .

### 8.3 Action on the trace of the category of representations of quantum $\mathrm{GL}_n$

In this final section, we give a partial description of an action of the elliptic Hall algebra  $\mathrm{EH}_0$  on the trace of the category of finite dimensional left  $U_q(\mathfrak{gl}_n)$ -modules. We follow the exposition of [13]. We work in the generic case, setting

$$\mathbb{k} := \mathbb{Q}(q), \quad z := q - q^{-1}, \quad t := q^n$$

for an indeterminate  $q$ . Let  $U_q(\mathfrak{gl}_n)$  be the quantized enveloping algebra of  $\mathfrak{gl}_n$  over  $\mathbb{k}$ . Its standard generators are  $\{e_i, f_i, d_j^{\pm 1} \mid i = 1, \dots, n-1, j = 1, \dots, n\}$ .

The natural module  $V^+$  and dual natural module  $V^-$  are the left  $U_q(\mathfrak{gl}_n)$ -modules with bases

$$\{v_i^+ \mid 1 \leq i \leq n\} \quad \text{and} \quad \{v_i^- \mid 1 \leq i \leq n\},$$

respectively, on which the generators act by

$$\begin{aligned} f_i v_j^+ &= \delta_{i,j} v_{i+1}^+, & e_i v_j^+ &= \delta_{i+1,j} v_i^+, & d_i v_j^+ &= q^{\delta_{i,j}} v_j^+, \\ f_i v_j^- &= \delta_{i+1,j} v_i^-, & e_i v_j^- &= \delta_{i,j} v_{i+1}^-, & d_i v_j^- &= q^{-\delta_{i,j}} v_j^-. \end{aligned}$$

Let  $U_q(\mathfrak{gl}_n)\text{-mod}$  be the category of finite dimensional left  $U_q(\mathfrak{gl}_n)$ -modules. This a balanced monoidal category with braiding  $R$  given by the  $R$ -matrix  $R_{M,N}: M \otimes N \xrightarrow{\sim} N \otimes M$  and twist  $\theta_M: M \rightarrow M$ . The definition of  $R$  as well as explicit formulae for  $R_{V^\pm, M}$  and  $R_{M, V^\pm}$ , for any finite dimensional  $U_q(\mathfrak{gl}_n)$ -module  $M$ , are given in [13]. (Actually the formulae in [13, Lem. 5.2] do not require  $M$  to be finite dimensional.)

Let  $\mathcal{OS}(z, t)$  be the HOMFLYPT skein category, as in Section 4.3.3, with  $q = t^n$ . There is a monoidal functor (see [13, §5])

$$\Psi: \mathcal{OS}(z, t) \rightarrow U_q(\mathfrak{gl}_n)\text{-mod} \tag{8.3.1}$$

to the category of left  $U_q(\mathfrak{gl}_n)$ -modules. The functor  $\Psi$  sends the generating objects  $\uparrow$  and  $\downarrow$  to  $V^+$  and  $V^-$ , respectively. It maps the various generating morphisms to the following  $U_q(\mathfrak{gl}_n)$ -module homomorphisms:

$$\begin{aligned} \nearrow : v_i^+ \otimes v_j^+ &\mapsto \begin{cases} v_j^+ \otimes v_i^+ & \text{if } i < j, \\ qv_j^+ \otimes v_i^+ & \text{if } i = j, \\ v_j^+ \otimes v_i^+ + zv_i^+ \otimes v_j^+ & \text{if } i > j; \end{cases} \\ \searrow : v_i^+ \otimes v_j^- &\mapsto \begin{cases} v_j^- \otimes v_i^+ & \text{if } i \neq j, \\ q^{-1}v_j^- \otimes v_i^+ - z \sum_{r=1}^{i-1} (-q)^{-r} v_{j-r}^- \otimes v_{i-r}^+ & \text{if } i = j; \end{cases} \\ \swarrow : v_i^- \otimes v_j^- &= \begin{cases} v_j^- \otimes v_i^- & \text{if } i > j, \\ qv_j^- \otimes v_i^- & \text{if } i = j, \\ v_j^- \otimes v_i^- + zv_i^- \otimes v_j^- & \text{if } i < j; \end{cases} \\ \nwarrow : v_i^- \otimes v_j^+ &\mapsto \begin{cases} v_j^+ \otimes v_i^- & \text{if } i \neq j, \\ q^{-1}v_j^+ \otimes v_i^- - z \sum_{r=1}^{n-i} (-q)^{-r} v_{j+r}^+ \otimes v_{i+r}^- & \text{if } i = j; \end{cases} \\ \nearrow : v_i^+ \otimes v_j^+ &\mapsto \begin{cases} v_j^+ \otimes v_i^+ & \text{if } i > j, \\ q^{-1}v_j^+ \otimes v_i^+ & \text{if } i = j, \\ v_j^+ \otimes v_i^+ - zv_i^+ \otimes v_j^+ & \text{if } i < j; \end{cases} \end{aligned}$$



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