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A STUDY OF HYSTERESIS

by

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Submitted to the Department of Electrical Engineering, The University of Ottawa, Ottawa, July, 1971, in partial fulfillment of the requirements for the degree of Master of Science

ACKNOWLEDGEMENT

The author would like to express his sincere thanks to his supervisor, Dr. G. S. Glinski, for his generous guidance and understanding.

Thanks are also due to Drs. W.K.Chung and Nicholas Georganas, for their encouragement, helpful discussion and correcting.

ABSTRACT

The phenomenon of hysteresis is observed in many branches of science. Several attempts have been made in the past for the mathematical modelling of this phenomenon.

In this thesis, hysteresis is considered from a general point of view. Basic features are described and the most known models are reviewed. Finally, two types of models are proposed.

CONTENTS

CHAPTER I

INTRODUCTION

Hysteresis has been widely studied in the past years. The earliest document available now is J. A. Ewing's book " Magnetic Induction in Iron and Other Related Metals " (1) published in 1900. In that book, Ewing gave the name 'hysteresis' to the closed loop relation between the magnetism B and the magnetic force H from the Greek word 'ὑστερέω' which means 'lag behind'. *

After Ewing's discovery, many attemps have been tried to use conventional mathematical techniques to describe that mysterious phenomenon.

However, to take a fast review, those works can be classified into two types:

1. From the microscopic point of view, the Physicists and chemists paid their attention to the

internal structure of materials which present hysteresis and studied the possible factors which might contribute to the forming of that phenomenon. Those include the interactive forces among molecules in solid materials (like magnets and iron) and the different phases among nonhomogeneous thermal states. In the former case, Weiss (2) proposed a ' domain ' model for the explanation of the hysteresis phenomenon in ferromagnetic materials. This model has been repeatedly used in various books. ** (3) (4) (5)

In the latter case, Everett and his co-authors studied the hysteresis phenomenon observed in chemical processes, and a series of papers were published. (6) (7) (8) (9) (10) (11)

2. From macro-scopic point of view, the engineers - electrical and mechanical - studied the input-output relations of systems with hysteresis and tried to use various conventional mathematical formulae to express or to approximate them. Some useful methods like ' describing function ' (12), ' harmonic analysis' (13), and 'complex equivalent gain' (14) have been applied in some problems in electrical and mechanical

systems containing nonlinear elements with hysteresis.

This thesis will study the hysteresis phenomenon from engineers' stand. However, it should be emphasized here that, Ernst Weber, in his celebrated paper "Nonlinear Physical Phenomena"(15) stated that 'No mathematical formulation of this double valued function ever exists which can be used for computational purposes'.

The purpose of this thesis is:

1. To review the most known models of hysteresis.

2. To describe the general features of hysteresis.

3. To set up some assumptions on making models for hysteresis. and

4. To propose reasonable approaches to the modelling of this phenomenon.

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* Ewing's naming of hysteresis, from author's point of view, is not quite correct. In the latter part of this thesis, it turns out that besides the "lagging behind" magnetic hysteresis, there does exist a "leading ahead" hysteresis.. For example, the hysteresis formed on the stress-strain plane by elasto-plastic materials is of such type. (16)

** Everett claimed that the idea of ' domains ' was introduced by Ewing. (6)

CHAPTER II

HYSTERESIS PHENOMENA

2.1 Introdution

In this chapter, the general nature of hysteresis phenomena is outlined and some basic features are described.

2.2 The Nature of Hysteresis

Generally speaking, hysteresis is a kind of physical phenomenon occuring in a system which presents a nonlinear, non-single valued relation between its input and output.(6) Different types of systems present different forms of hysteresis. However, they still have some common behavior. A list of those hysteresis types is in the following Table 1.

TABLE I

SUMMARY OF HYSTERESIS PHENOMENA

PROCESS	INPUT (indep. variable)	OUTPUT (dep. variable)	CAUSE
magnetization of ferromagnetics	magnetic field strength	intensity of magnetization	domain wall and domain rotation
polarization of ferroelectrics	electric field strength	electric polarization	domain wall and domain rotation
solid transition in crystals and alloys	temperature	density or molar volume	nonhomogeneous mixture
absorption of H_2 in Pd	partial pressure of gases	amount absorbed	nonhomogeneous phases
application of stress to solid	stress	strain	molecular attraction and repulsion forces (17)
gear transmission	rotation	rotation	motion and friction
thermocouple	temperature	deflection	nonhomgeneous (18)

2.3 Features of Hysteresis

The following features are common for all types of hysteresis.

2.3.1 The amount of energy loss in hysteresis is finite. (4) The output of a system with hysteresis is usually bounded. A saturative nonlinearity is usually present. This is obvious for magnetic hysteresis. (Whereas in elasto-plastic hysteresis, the specimen under test breaks at certain stress.)

2.3.2 The primary trajectory of hysteresis is shown in Fig. 2-1

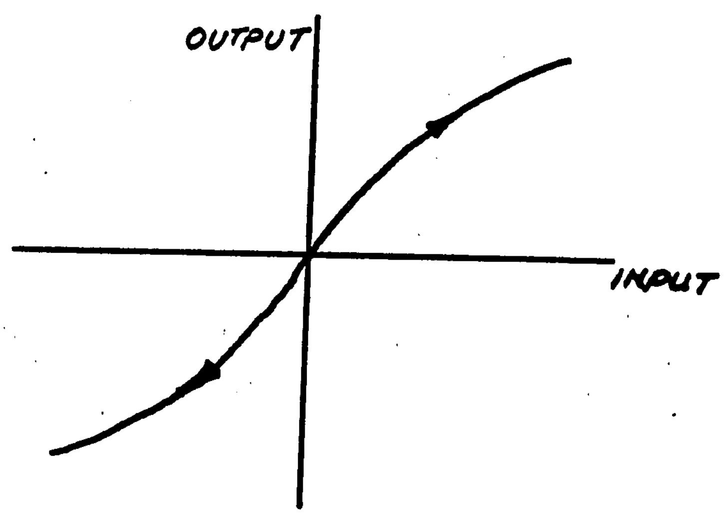

Fig. 2-1 Primary magnetization curve

It is the input-output characterestic of an untreated specimen. This characterestic curve is symmetric to the origin.

It is known that for any nonhysteretic nonlinearity of this type, (Fig 2-2) with sinusoidal input, the corresponding output contains a large portion of odd harmonics, especially the third harmonic.

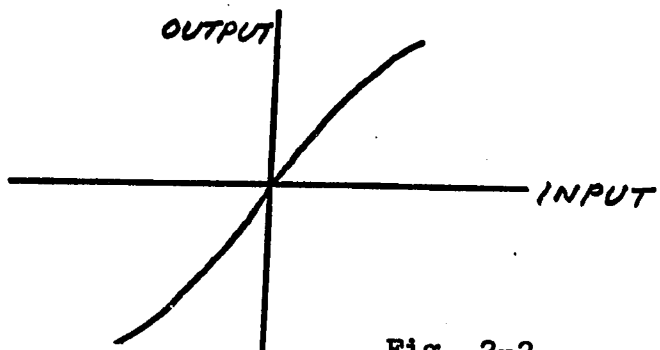

Fig. 2-2

For a system with hysteresis, the input-output relation is usually of the form in Fig.2-3

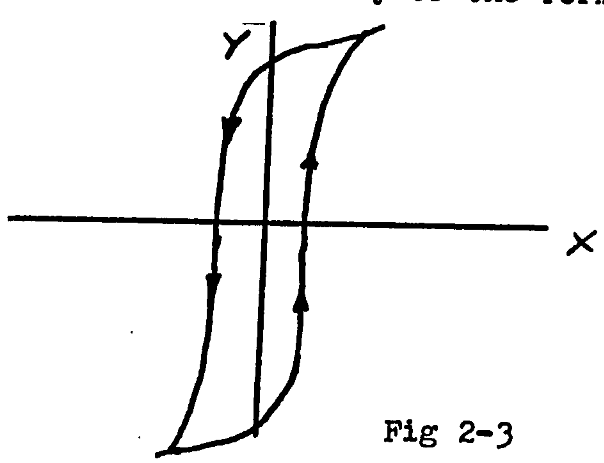

Fig 2-3

An experimental result (19) shows that with sinusoidal input, the output also contains a large portion of third harmonics. The difference between nonhysteretic and hysteretic nonlinearities is that, the former one shows no phase shift and the latter one indicates phase shift between input and output signals. This type of hysteretic nonlinearity is also called complex nonlinearity.*(20)

2.3.3 In magnetic hysteresis, the primary hysteresis curve does not change smoothly with increasing input, but by jumps, which are called Barkhaussen jumps. (3) Analogously, in elasto-plastic hysteresis, there is also a vibrating region between lower and upper yielding-points in elongation test.(16) Ewing (1) explained the Barkhaussen jumps as a result of the "changing role" between "domain rotation " and "domain wall motion". The vibration can be considered as the result of the interaction between molecular attraction and repulsing forces in the solids. (16) From this, a term called <u>nonhomogeneous structure</u> is defined as: " A system having more than one factors which react to the input. "**

For example, the domain wall motion and domain rotation in the magnetization process *** are nonhomogeneous. (10)

2.3.4 All hysteresis characterestics corresponding to periodic inputs are <u>closed curves</u> (6), which are also called <u>loops</u>. In all hysteresis phenomena, there is a largest closed curve (it is finite), which encloses all other trajectories. This is called the <u>boundary curve</u> (<u>or major loop)</u>. The other closed curves are called <u>minor loops</u>. They are shown in Fig. 2-4.

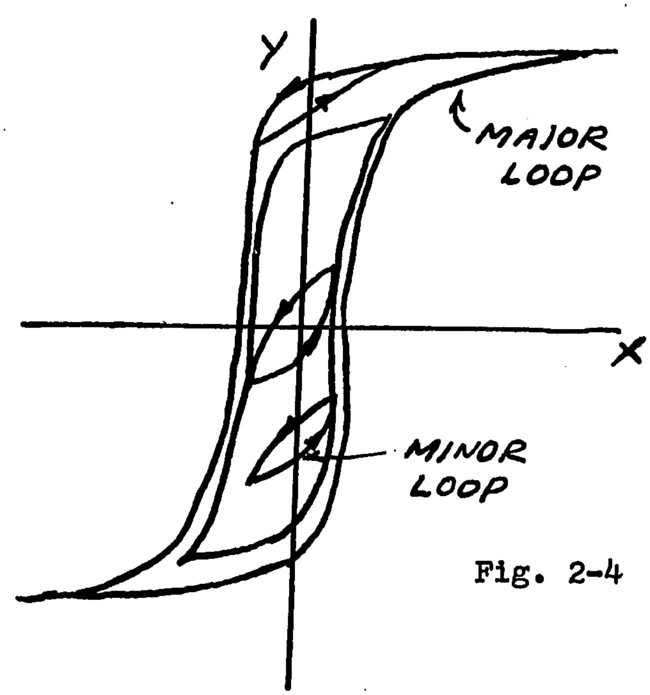

Fig. 2-4

2.3.5 At any specific point within a hysteresis loop, there are two possible directions of next move which depend on the past history of the trajectory and the present input. This will be shown in Fig. 2-5 .

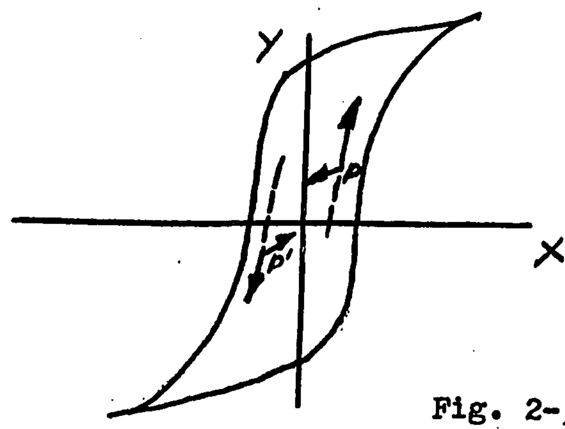

Fig. 2-5

2.3.6 The impulse responses of a system with hysteresis are points on the boundary curve. Those are called the "remanence points" of hysteresis. They are shown in Fig. 2-6.

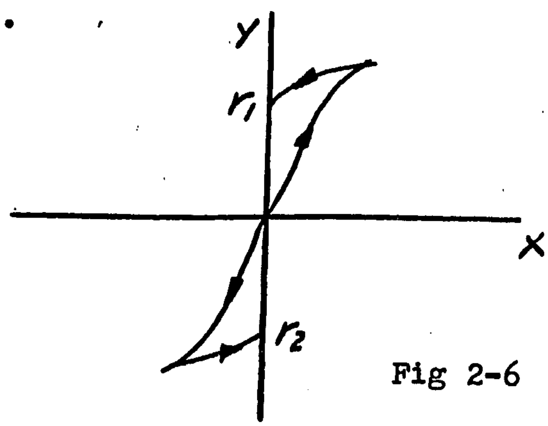

Fig 2-6

2.3.7 In a system with hysteresis, if a process is taken throuth a series of oscillations of the input x(t), after the n-th. reversal, the process moves toward the point at which the n-1 th. reversal occurred. If the process is carried through the n-1 th. point, it moves toward the n-3 th. reversal point, etc. (8) A sketch of the process is in Fig 2-7.

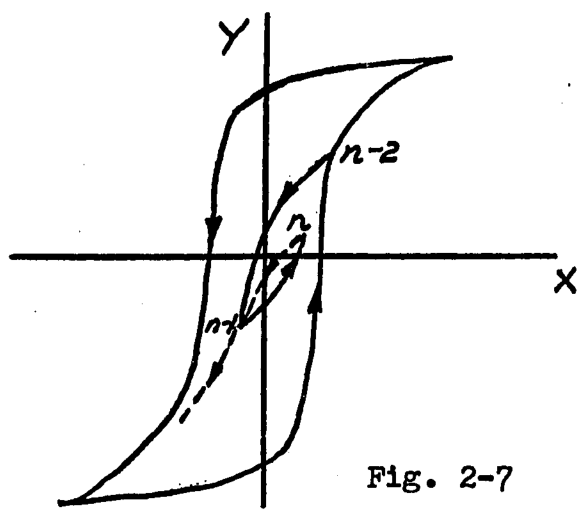

Fig. 2-7

2.3.8 If two periodic inputs to a system with hysteresis have responses of equal magnititudes, the areas of the corresponding minor loops are equal. It has been shown by Everett and Smith in an experiment. (7) See Fig. 2-8a.

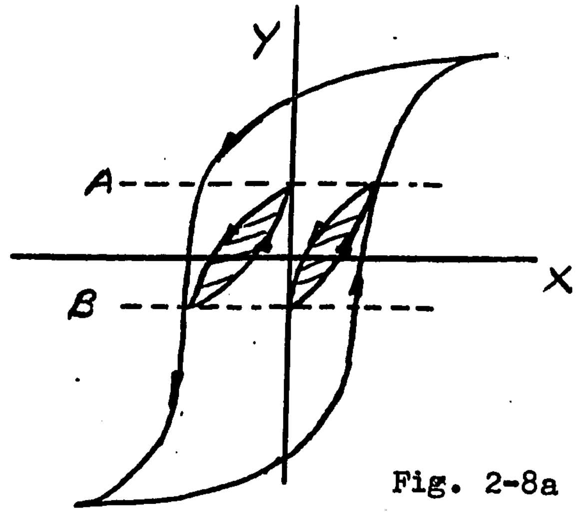

Fig. 2-8a

2.3.9 If two periodic inputs to a system with hysteresis vary between the same magnititude as in Fig. 2-8b ,then the areas of the corresponding minor loops are equal. (8)

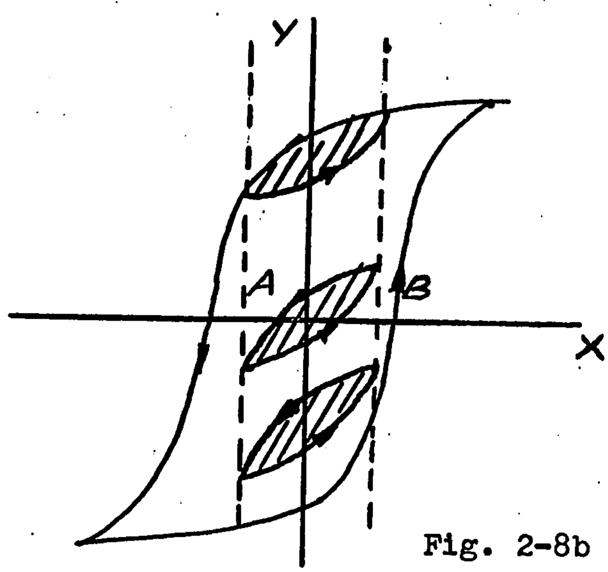

Fig. 2-8b

2.3.10 A system with hysteresis generates continuous output signals with continuous input signals. Whenever the input signal reverses, the corresponding output signal reverses at the same instant. This is called <u>the principle of synchroneity</u>.

2.3.11 The slope of any ascending trajectory inside the boundary closed curve is always less than the slope of the ascending boundary curve corresponding to the same interval of input values. The slope of any descending trajectory inside the boundary curve is always less than the slope of the descending boundary curve corresponding to the same interval of input values. It is shown in Fig. 2-9 and 2-10.

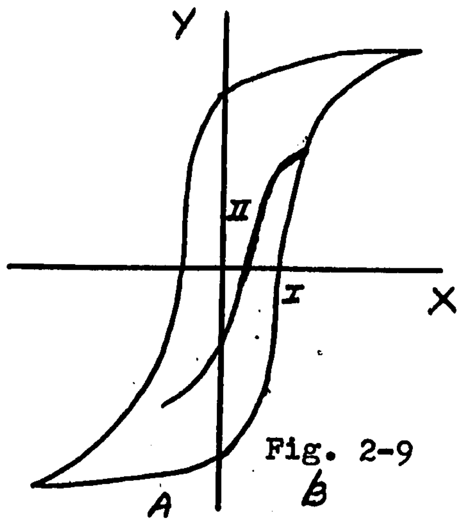

Fig. 2-9

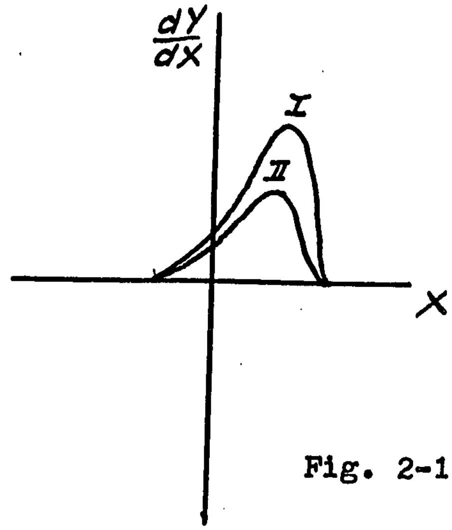

Fig. 2-10

2.3.12 All the ascending trajectories converge to the ascending boundary curve, and all the descending trajectories converge to the descending boundary curve. (6) They are shown in Fig. 2-11 and Fig. 2-12.

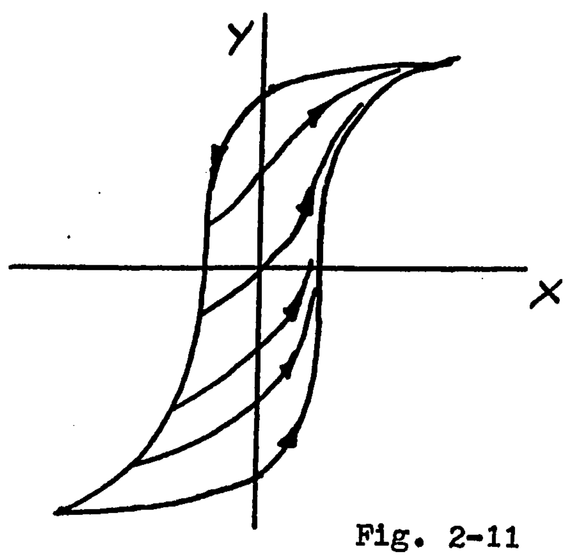

Fig. 2-11

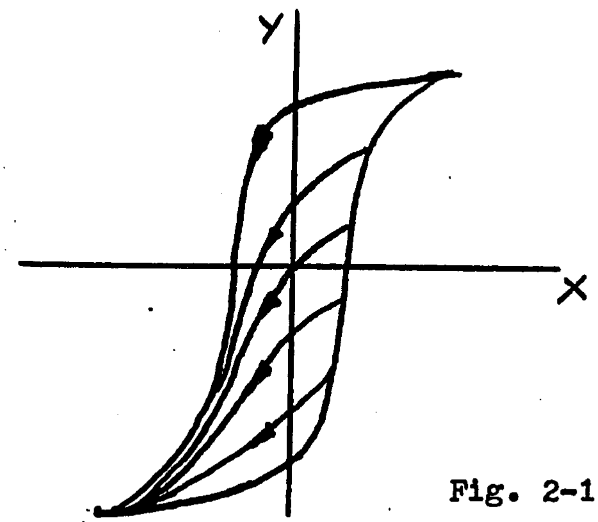

Fig. 2-12

2.3.13 If the input signal is in the form as is shown in Fig. 2-13,

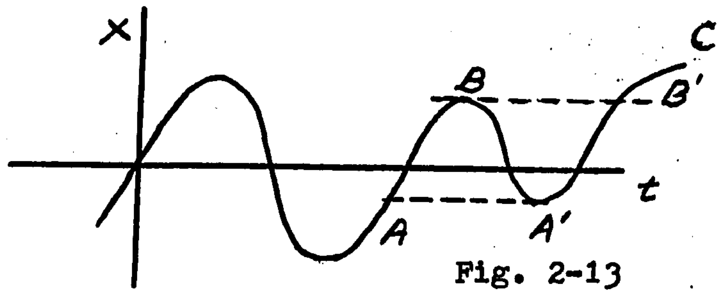

Fig. 2-13

then a minor loop is formed with the trajectory shown in Fig. 2-14 .

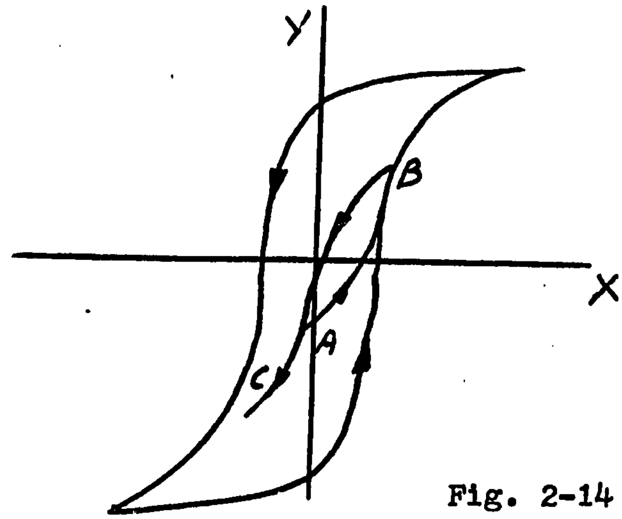

Fig. 2-14

It is the same trajectory as if the part (a,b) is deleted. (III) This is called <u>the principle of retrieval.</u> (Fig. 2-15)

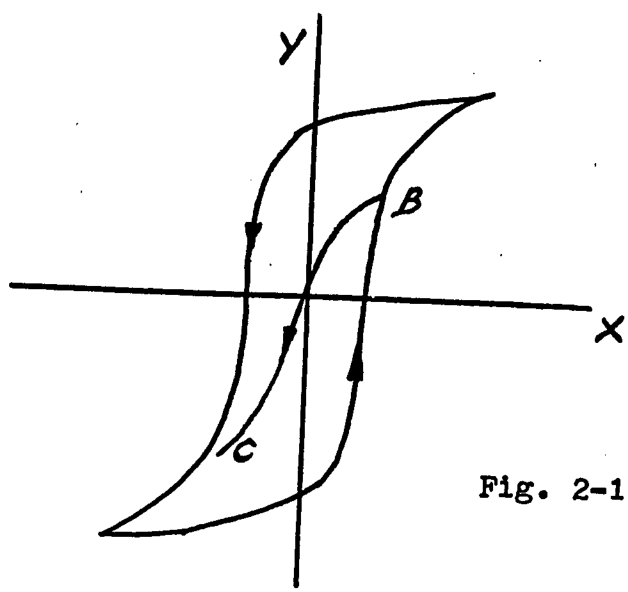

Fig. 2-15

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* Complex nonlinearities, in general, are characterized by describing functions which have at least nonzero phase angles, and sometimes both amplitude ratios and associated phase angles are dependent on both input amplitude and frequency. (14)

** There is no evidence indicating whether those factors are independent or not.

*** Curie Point : If a magnet is heated to a specified temperature (ie, for iron, T= 500 -700 C), it becomes nonhysteretic. (3) The authour suggests that this phenomenon is the evidence of the homogenization of nonhomgeneous structures under special circumstances.

CHAPTER III

MODELS FOR HYSTERESIS

3.1 Introduction

In this chapter, a brief review of previous models will be presented. Some examples are given for illustration. Several assumptions are abstracted. Finally, two types of models are developed.

3.2 A Historical Sketch

A short review of the most known models of hysteresis is provided in this section. Comments on those models are also listed.

3.2.1 Middleton's Model (21)

Middleton's model is a pair of parametrized equations for input h and output b ,

$$h = \sin (t + \delta)$$
$$b = \tan^{-1} (k \sin t) / \tan^{-1} k \qquad (3\text{-}1)$$

where k is the amplitude and δ is the delay angle.

If h is applied to the horizontal deflection plates and b is applied to the vertical plates of a cathode ray oscilloscope, then the resulting display on the screen of CRO is a closed loop similar to a hysteresis loop. This simulation, however, didn't provide a satisfactory representation to hysteresis, because:

1. In this model, the nonlinear part is played by an arctangent function , while in practical cases, the saturative nonlinearity is not that simple.

2. This model is subject to steady state sinusoidal signals only. The shape and the size of the loop is a function of specified parameters k and δ .

3. As it has been mentioned before, besides the ' lag behind ' hysteresis, there does exist a kind of hysteresis which is actually look like ' leading ahead ', * for example, the elasto-plastic hysteresis. (3) A comparison

of these two types of hysteresis is given in Fig. 3-1 and Fig. 3-2.

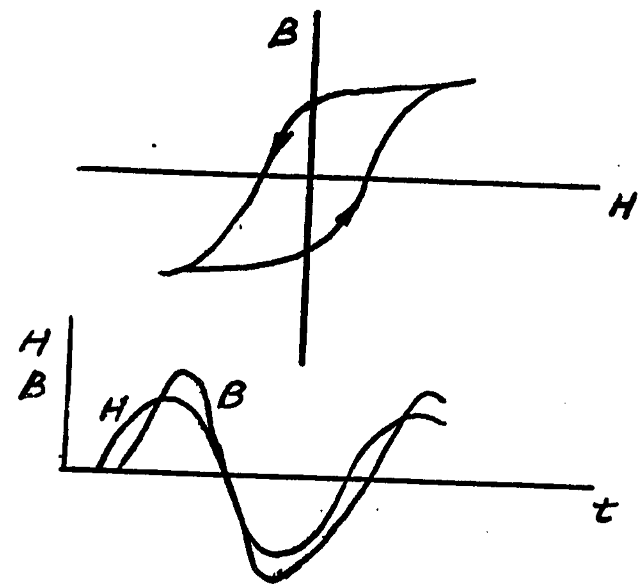

Fig. 3-1 ' Lagging behind ' hysteresis

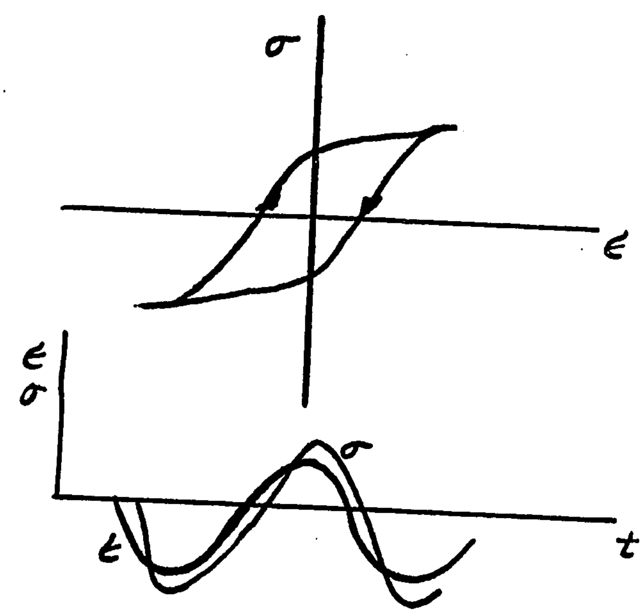

Fig. 3-2 ' Leading ahead ' hysteresis

In the lagging hysteresis, the loop is a counter clockwise oriented closed curve, whereas in the leading hysteresis, the loop is a clockwise oriented closed curve. The work which Middleton had done was only a nice curve-fitting work.

3.2.2 Functional Model

A mathematical approach to the representation of hysteresis has been provided by V. Volterra (23) and Kostitzin (24), by the use of integral operators.

$$y(t)=MF(x(t)) + \int_{t_0}^{t} k(t,u)F(x(u))du \qquad (3\text{-}2)$$

(Kostitzin) - nonlinear heredity

$$y(t)=ax(t) + \int_{t_0}^{t} k(t,u)x(u)du \qquad (3\text{-}3)$$

(Volterra) - linear heredity

The integral operator models have the following characteristics:

1. They form a closed loop in the x-y plane when the input signal is periodic.

2. If input is periodic with period T, then the output will also be periodic with the same period. (after sufficiently many periods have elapsed)

3. Due to the integration sign, there is a time lag between input and output.

A linear integral model can be easily realized by an R-L circuit:

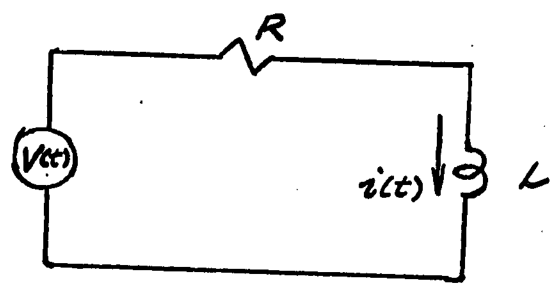

Fig. 3-3 An R-L circuit

$$v(t) = Ri(t) + (1/L)\int_{t_0}^{t} i(t)\,dt$$

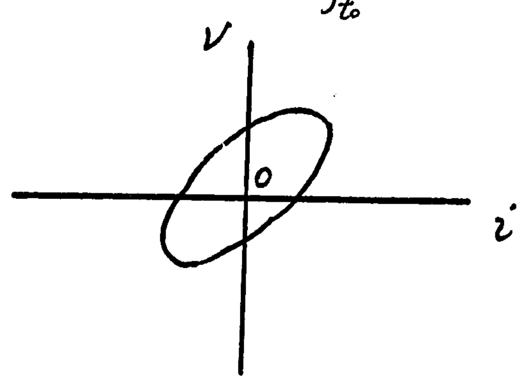

Fig. 3-4 Display of v-i on CRO

which looks like a hysteresis loop. It is obvious that this circuit has a time (decay) constant $T = L/R$ which means (according to Volterra) that the memory built in the hysteresis will fade away gradually when time passes. But, in general cases, the memory in hysteresis is supposed to be permanent (although it is impossible to say how permanent it is). (23) According to Artly (28) and Distler (22), a good approximation can be achieved when $L/R \gg 1/f$, where f is the frequency of the input signal.

Another obvious difference between a hysteresis and a R-L circuit i-v display is that: in the hysteresis loop, all slopes are positive, but in a R-L loop, in some region, the slope is negative. ** (10) (25) They are shown in Fig.3-5.

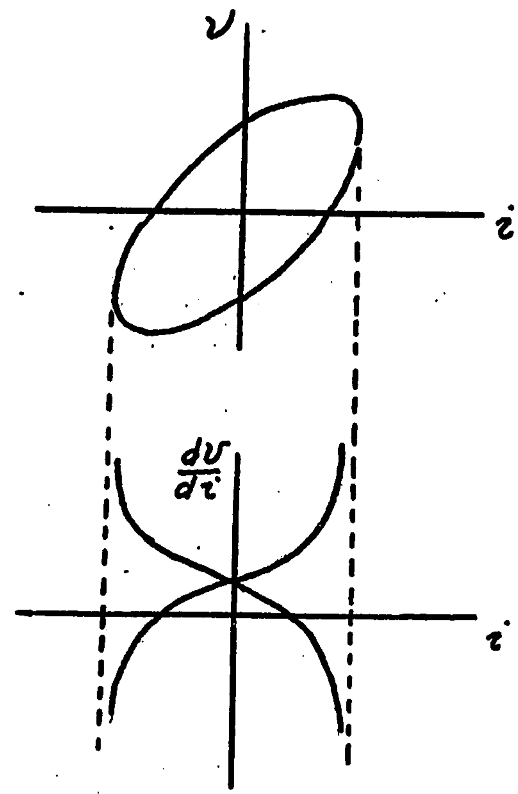

Fig. 3-5 Density function of v-i curve

3.2.3 Density Function Model

A density function method in representing magnetic hysteresis was suggested by Weygandt and Charp.*** (25) They used a pair of "moving " Gaussian functions in the dB/dH vs H axis, ie,

$$dB/dH = P(H \pm H_c)$$

$$B = B_s \, erf[(H \pm H_c)/H_c] \qquad (3-4)$$

Where

B_s= saturation value of flux density.

H_c= coercive force.

P( ) is the Gaussian function, ie.

$$P(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-x^2/2\sigma^2)$$

erf( ) is the error function.

The sign of H_c in equation (3-4) is plus when H is increasing, and is minus when H is decreasing.

The density functions are shown in Fig. 3-6.

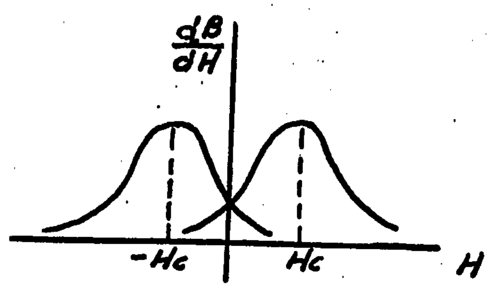

Fig. 3-6 Density functions

A loop is formed as shown in Fig. 3-7.

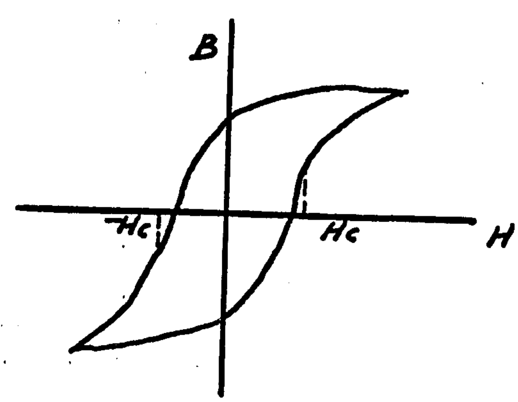

Fig. 3-7 A hysteresis loop

Everett (6) and Otheru (18) proposed similar approaches in later years.

Some comments on the density model are listed in the following:

1. For a general hysteresis , there might exist more than one kind of density functions, and those density functions might be strongly correlated. (26) (27)
2. It was claimed that the area of hysteresis changes with the frequency of the input signal. Then the mean value and the variance of the density function in that model should be a function of dx/dt and d^2x/dt^2 etc., but it is not easy to set up such relations.
3. It is very difficult to represent minor loops by using density function model.
4. It should be noted here that the density function mentioned above has nothing to do with probability.

Duhem (11) and Takagi (28) made models for hysteresis by using a family of monotonically increasing and decreasing curves to represent the branches of the hysteresis loops, with the following formula:

$$dy/dx = aF(x,y) + bG(x,y) \qquad (3\text{-}5)$$

where

a=1 when x(t) increasing

a=0 when x(t) decreasing

b=1 when x(t) decreasing

b=0 when x(t) increasing

x(t) is the input signal.

The curves are shown in Fig. 3-8.

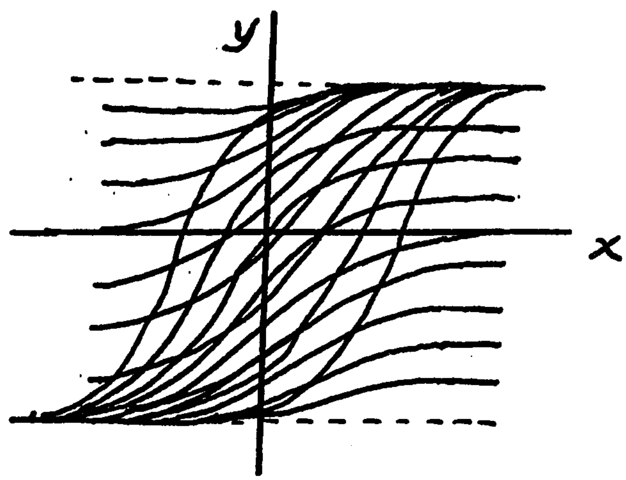

Fig. 3-8 Duhem and Takagi's model

Actually, this model is only an alternative expression of the density model.

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* The Causality has been discussed by Bridgman. (5)

** The authour accepts that there is not enough theoretical / experimental evidence to support the statement that the slopes (the ratio between input and output) are always positive. However, the density function model follows from that fact.

3.3 Examples of Hysteresis

Basic features of hysteresis have given in section 2-3. In this section, some examples of hysteresis will be discussed.

3.3.1 Hysteresis in Gear transmission (backlash)

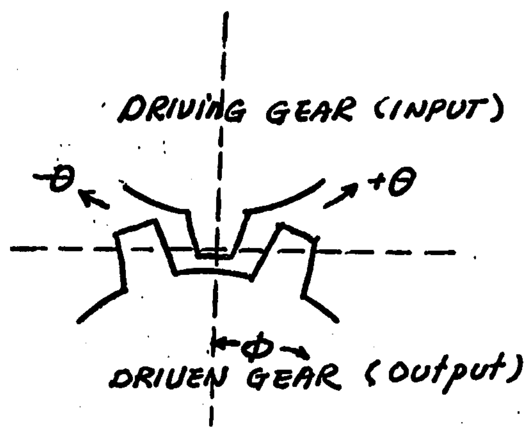

Fig. 3-9 Gear transmission

In Fig. 3-9, the driving gear rotates between $+\theta_m$ and $-\theta_m$. When it rotates towards $+\theta_m$, at the beginning of the motion, there is no contact (hence no friction) between the two gears. Therefore, in the region before the tooth contact, the driven gear does not move (or moves very slowly due to cohesion).
In this region, the ratio of two angular velocties is very small, let it be denoted by δ .

When the driving gear rotates and its tooth contacts the tooth of the driven gear, the ratio of angular velocities will be

$$d\phi/d\theta = \omega_2/\omega_1 = 1.$$

Since this is independent of whether the driving gear rotates clockwisely or counter-clockwisely, a symmetrical relation exists between $d\phi/d\theta$ and θ, as in Fig. 3-10.

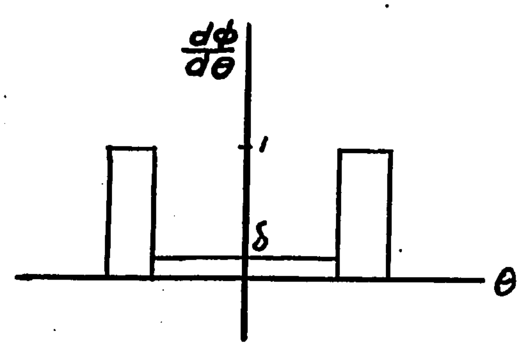

Fig. 3-10 Angular velocity ratio in gear transmission

The corresponding relation between ϕ and θ is in Fig. 3-11.

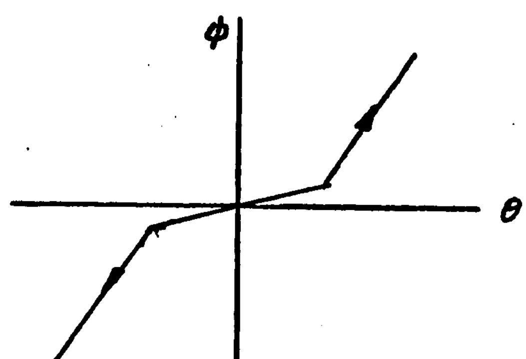

Fig. 3-11 ϕ -θ curve

When the driving gear arrives at $+\theta_m$, if it changes it direction towarding $-\theta_m$, then the driven gear at beginning rotates with low $d\phi/d\theta = \delta$ and follows with $d\phi/d\theta = 1$, as is shown in Fig . 3-12.

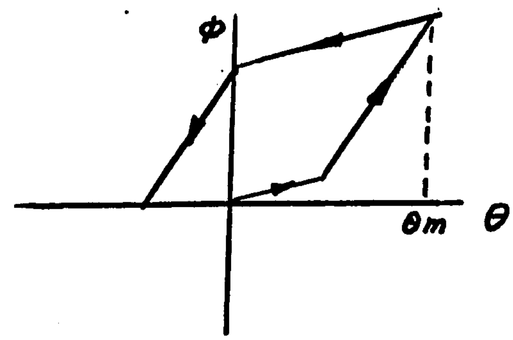

Fig. 3-12 Forming of a hysteresis loop

Recycling the same process, a loop can be formed and is shown in Fig. 3-13.

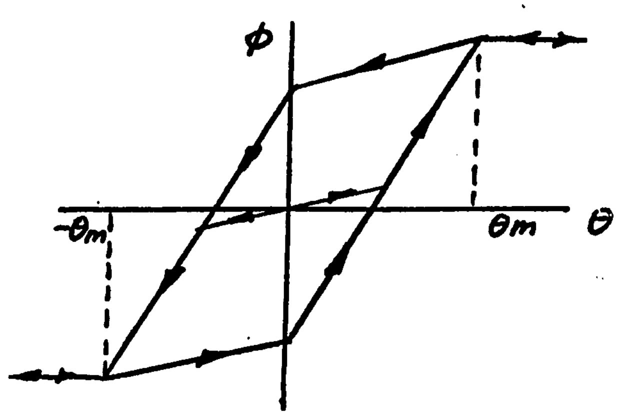

Fig. 3-13 Hysteresis loop in ' backlash '

Discussion:

1. The angular velocity ratios (or, clearly, the density functions), contain two parts. One part is very ' sensitive ' to the variation of the input, and the other part is 'less sensitive! From another point of view, there is a priority between the two factors. In a special case, if $\delta = 0$ (see Fig. 3-14), then one can simply say that there is a ' moving ' density function, which moves according to the variation of the input (as Weygant and Charp have proposed).

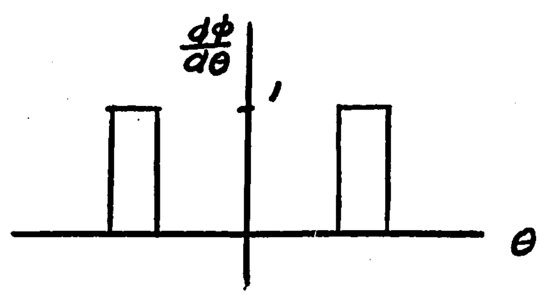

Fig. 3-14 ' Moving ' density in gear transmission

2. Within the region of $d\phi/d\theta = \delta$, the process is reversible (in ideal case).

3. The minor loops will occur when the input

is with maximun value less than $|\theta_m|$. It is shown in Fig. 3-15.

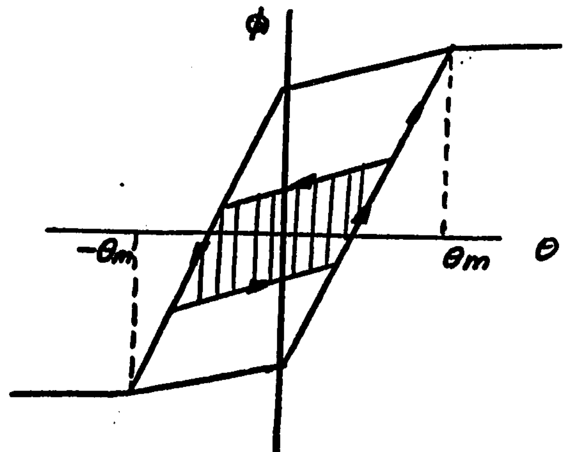

Fig. 3-15 Minor loop in backlash

4. The hysteresis is of counter-clockwise type.

3.3.2. .Elasto-plastic Hysteresis.

This kind of hysteresis is caused by the nonhomogeneous factors in molecular attraction forces and repulsion forces. The relation among those forces is in Fig. 3-16.

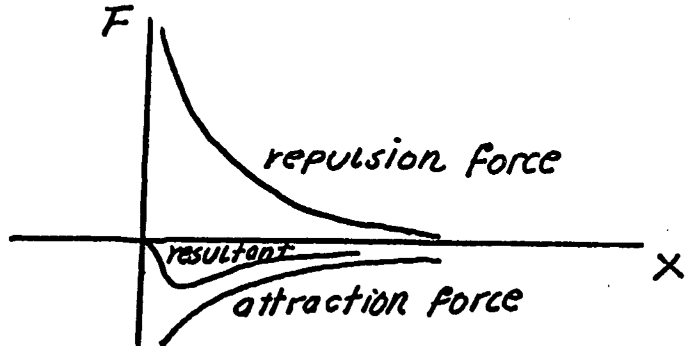

Fig. 3-16 Forces among molecules

The attraction forces are strongly against the applied stress on the material, and repulsion forces play a negative role and with small effect to the stress. (elongation)

The density functions corresponding to a unstressed material are shown in Fig. 3-17.

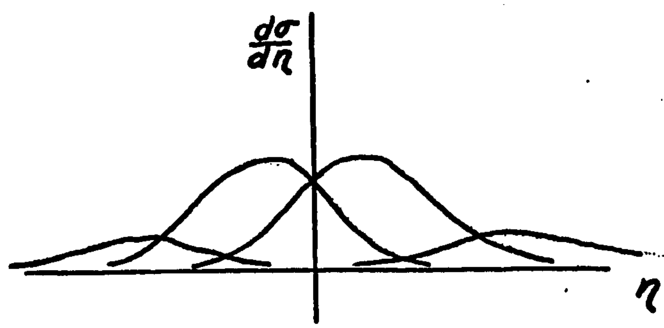

Fig. 3-17 Density functions

It can be idealized as is shown in Fig. 3-18.

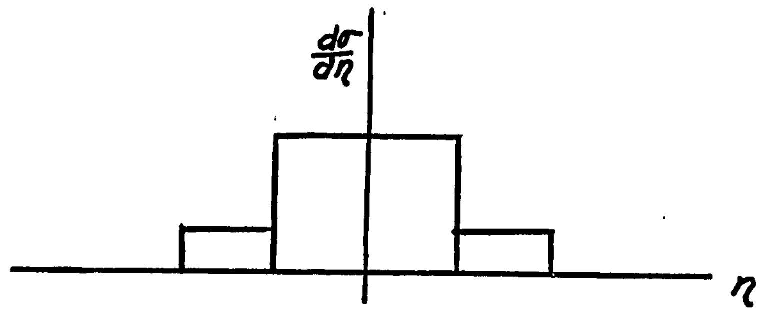

Fig. 3-18 Idealized density function

And with corresponding stress-strain is shown in Fig. 3-19

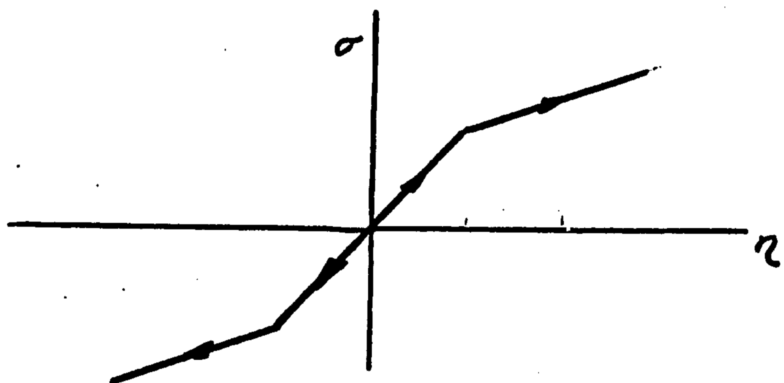

Fig. 3-19 σ -η curve

After the material has been stressed, the density function becomes:

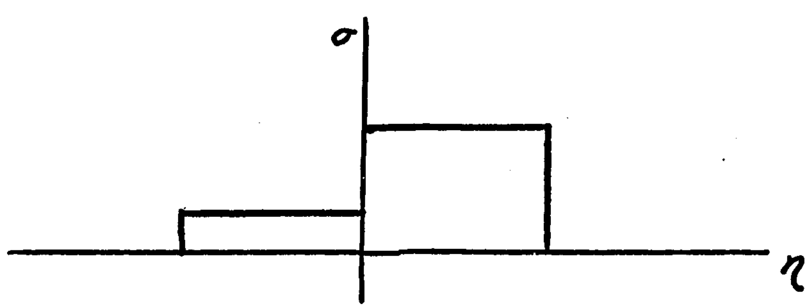

Fig. 3-20 Density function corresponding to stressed material

Following the procedure described in 3.3.1. , a loop can be formed as in Fig. 3-21.

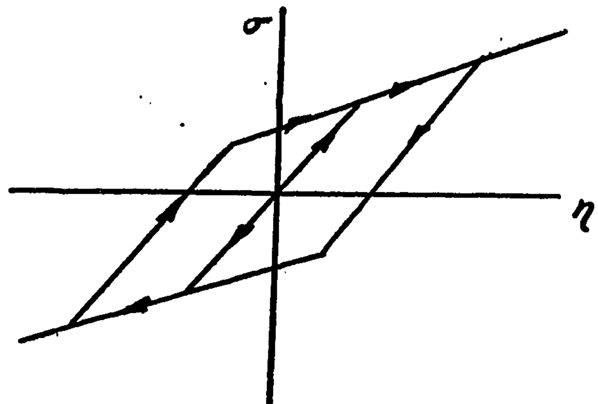

Fig. 3-21 Hysteresis loop on $\sigma - \eta$ plane

It should noticed that this hysteresis loop is quite different from the loop in 3.3.1 . A comparison between those two is in the following:

1. The density function with larger value is sensitive to the variation of the input, while in 3.3.1, the density function with smaller value is sensitive to this variation.

2. The reversible part is on the lines with larger slope. (which is also in an ideal case)

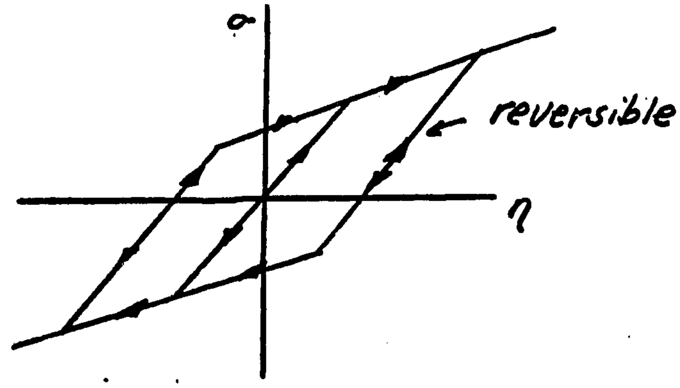

Fig. 3-22 Reversible part in hysteresis

3. The minor loops are in the following form : (see Fig,3-23)

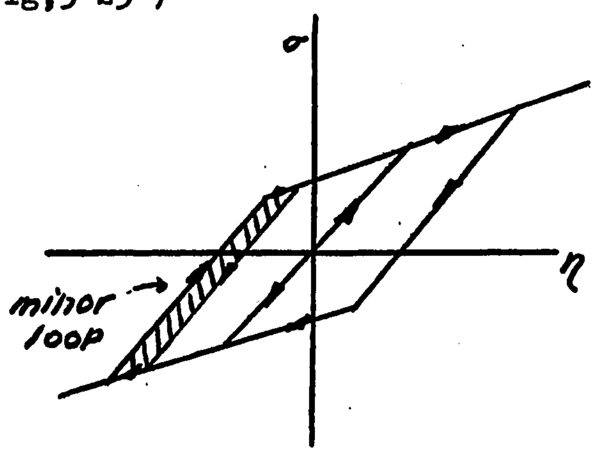

Fig. 3-23 Minor loops in $\sigma - \eta$ hysteresis

4. The hysteresis is of clockwise oriented type.

3.3.3 Magnetic Hysteresis:

The magnetic hysteresis is more complicate in making models as compare to the cases in 3.3.1 and 3.3.2 . The reasons are:

1. Instead of two kinds of nonhomogeneous factors in the other types of hysteresis, there are (possibly) three kinds of nonhomogeneous factors, ie,

   a. reversible wall motion
   b. irreversible wall motion
   c. domain rotation

2. Each of the factors corresponds to a density function and those density functions are correlated. (26)

3. The parameters of those density functions vary with frequency of the input signals.

A typical density function and magnetization curve are in the following:

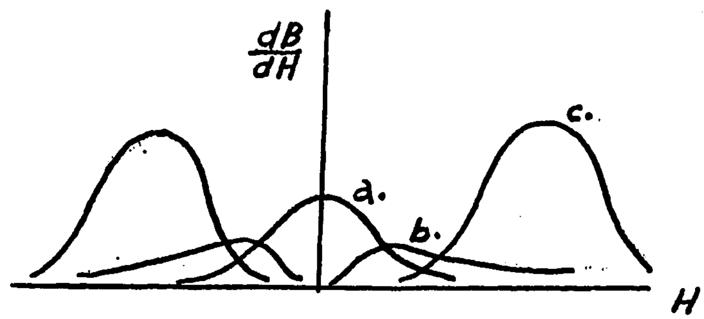

Fig. 3-24 Density functions of magnetic hysteresis

with hysteresis loop:

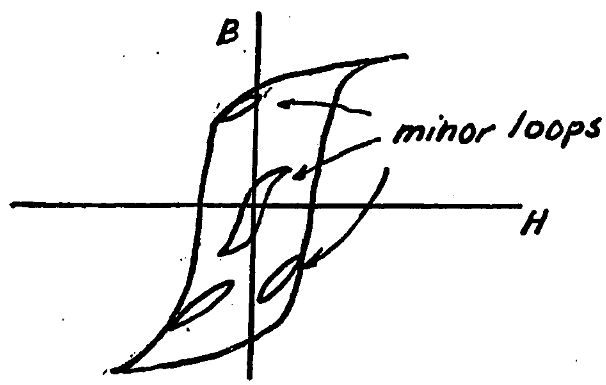

Fig. 3-25 Magnetic hysteresis

It should be noted that:

1. There are still priorities among those density functions, but the orders are not easy to be described.

2. The density function is defined over $(-\infty, \infty)$ and its normalized range lies in $[0, 1]$.

3. There is no reversible region in any part of the loop. (according to the condition that the density function for irreversible wall motion has its domain on $(0, \infty)$).

4. The minor loops may be formed anywhere in the major hysteresis loop.

3.4 Assumptions for Mathematical Modelling

Thomas K. Caughey, in a series of papers (29) (30), considered 'heredity' in a system with the following differential equation:

$$\ddot{x}+\beta\dot{x}+\omega_0 x+\eta\, g(x, \dot{x}, t)=f(t) \qquad (3\text{-}6)$$

Where f(t) is the forcing function . The function g( ) contains both velocity and displacement terms [illegible]

Instead of trying to define the so called 'heredity' and building a model for $g(\ddot{x}, \dot{x}, t)$, Caughey derived an approximation method (equivalent linearization technique) to treat the whole dynamic system. (29)

Henry D. Block, in his paper " periodic solutions of forced systems having hysteresis" (31), dealt with the following system :

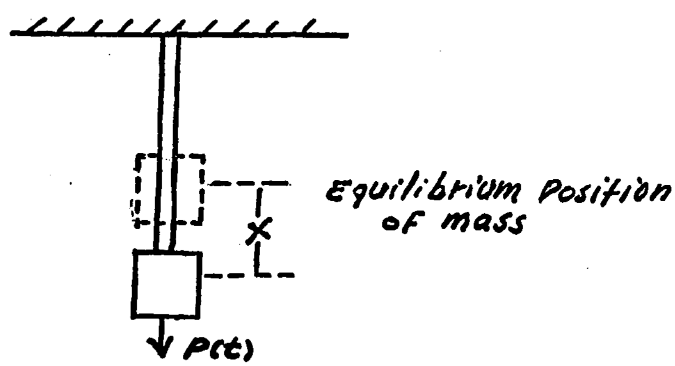

Fig. 3-26 A dynamic system with hysteresis

The system contains a unit mass supported by a rod and acted on by a force $p(t)$.

Letting x(t) denote the displacement of the mass from the initial equilibrium and F denote the restoring force exercised by the rod on the mass, the equation of motion is:

$$\ddot{x} + F = p(t) \qquad (3\text{-}7)$$

Where F is the restoring force, which relates the stress and strain, and has a hysteresis character.

To discuss the property of F, an experiment was performed in Block's paper, as is shown in Fig. 3-27.

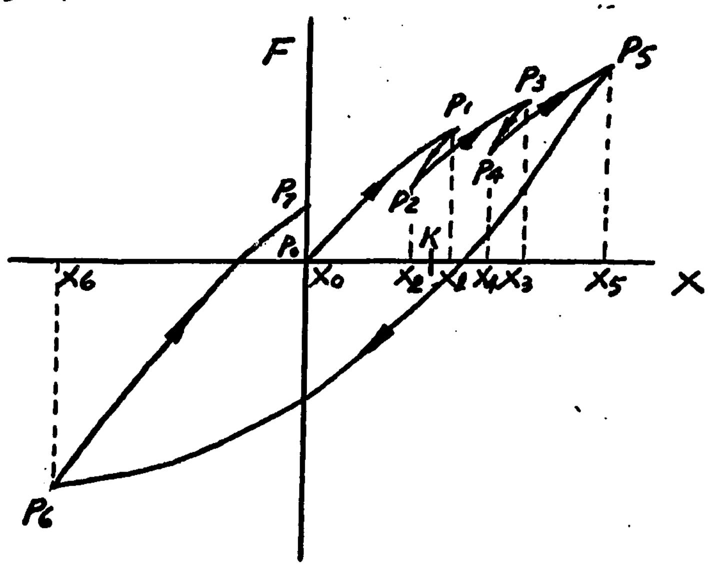

Fig. 3-27 The function F(x)

With x increased from x_0 to x_1 and then reduced x to x_2, increased it to x_3, decreased it to x_4, increased it to x_5, decreased it to x_6, and finally increased x back to x_0. The corresponding point in (x , F) plane runs along the path shown from P_0 to P_1, P_1 to P_2, ... P_6 to P_7. Then Block had the following conclusions on hysteresis:

1. F is not a 'function' of the value of x. (Recall the mathematical definition that F is a function of x if, for each specified value assigned to x, a corresponding value for F is uniquely determined.)

2. When x is at the value k, for example, there are four possibles for F ,(as shown in Fig. 3-27) and there would be also many other possible values for F if x arrived at k by a different sequence of events. The specification of the value of $\dot{x}$, $\ddot{x}$, etc, will not help to determine F. Therefore, it is clear that F is not determined as a function of t, valid for arbitory loading histories. Thus F is not a function of one or more real variables at all, and so the

equation $\ddot{x} + F = p(t)$
is not the usual type of differential equation.

3. If the function $x(t)$ is specified, for $0 \leq t < T$, then F is determined as a function of t for $0 \leq t < T$. In particular, if $x(t)$ is given for all $t > 0$, then F is determined for all $t > 0$. Thus F is a mapping of the function $x(t)$ into another function $F(x(t))$. (Thus it is deterministic.)

4. In a very special case, F would be written in the form $f(x,\dot{x},\ddot{x},t)$. When the whole function $x(\ )$ is specified, then, at each t, $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are determined and $f(x,\dot{x},\ddot{x},t)$ becomes a function of t, hence it is of the type $F(x)$. For the purpose of distinguishing between the two types, the former $f(x,\dot{x},\ddot{x},t)$ should be referred as a point-function restoring force and the latter $F(x)$ as a hysteresis-type restoring force.

A modern approach to the modelling of hysteresis has been suggested by M. Brilliant. He adapted system theory terminology to define hysteresis and created a ' hysteretic system '. See Fig. 3-28.

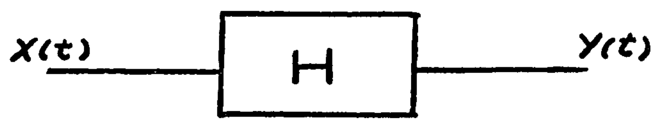

Fig. 3-28 The hysteretic system

In his report, (32) he defined the following as basic properties of a 'hysteretic system' :

1. It is possible to specify two inputs different for $t < 0$ but equal for $t \geq 0$, for which the difference between the corresponding outputs does not converge to zero as t approaches to infinite.

2. It is possible to specify an input for $t > 0$ in such a way that for any two inputs, arbitrarily different for $t < 0$ but with the specified form for $t \geq 0$, the difference between the corresponding outputs always converges to zero as $t \to \infty$.

The first property shows that a hysteretic system conserves some past history of input, and this conservation is ' permanent '. The second property shows that the permanent conservation can be discharged by a special type of input.

3.5 An Approximation Model

This model considers hysteresis as a linear combination of a nonlinear memoryless part and a linear part with memory.

It was proposed by Chihiro Hayashi in the study of the influence on nonlinear resonance. (34)

A description of this model is in the following:

A nonlinear circuit, as shown in Fig. 3-29,

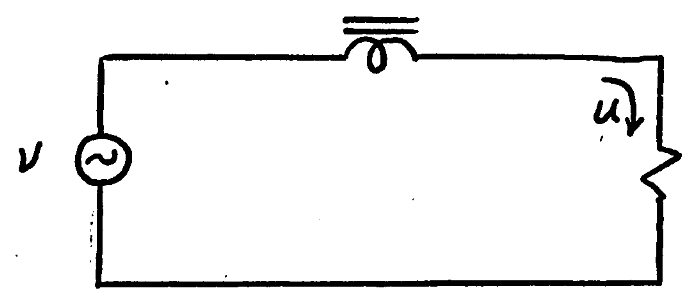

Fig. 3-29 A circuit containing nonlinear element

under the assumption that the magnetic flux v varies sinusoidally with a fixed amptitude, a form of the magnetic hysteresis can be expressed as:

$$u(t) = c_1 v(t) + c_3 v^3(t) + \frac{1}{\omega} \frac{a}{1+b^2 r^2} \frac{dv(t)}{dt} \qquad (3\text{-}8)$$

where c_1, c_3 and a, b are constants which characterize the hysteresis loop , and r is the amptitude of the flux v, ie. ,

$$v(t) = r \sin(\omega t + \theta)$$

The first two terms

$$u_0 = c_1 v(t) + c_3 v^3(t)$$

represent the saturation curve of the magnetic core, (which is an odd function) the third term

$$u_h(t) = \frac{1}{\omega} \frac{a}{1+b^2 r^2} \frac{dv(t)}{dt}$$

is associated with the memory part of hysteresis.

The hysteresis is represented by an ellipse in the u_h-v plane and oriented along the saturation curve. (see Fig. 3-30)

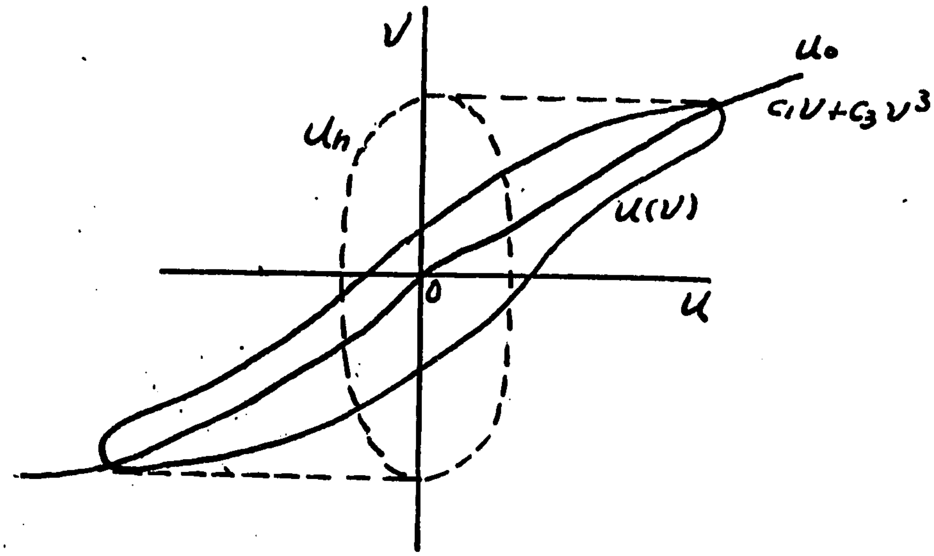

Fig. 3-30 An approximation model for hysteresis

Since this model is in a conventional mathematical form, it can be inserted directly into any differential equation other than the above one.

Some comments on this model are in the following:

1. The memoryless part, which represents the saturative nonlinearity, can be expressed in a more general form,

$$u_0(t) = c_1 v(t) + c_3 v^3(t) + \ldots\ldots + c_{2n+1}\, v^{2n+1}(t) + \ldots \qquad (3\text{-}9)$$

which should be bounded for all value of v( ).

An analytical expression provided by Langevin is in the following: (24)

$$u_0(v,t) = \frac{e^{av} + e^{-av}}{e^{av} - e^{-av}} - \frac{1}{av} \qquad (3\text{-}10)$$

which satisfies the condition that

$|u_0(t)| \leq 1$ for all v.

2. If $c_3 = 0$, then

$$u(t) = c_1\, v(t) + \frac{1}{\omega}\, \frac{a}{1+b^2 r^2}\, \frac{dv(t)}{dt}, \qquad (3\text{-}11)$$

which is a linear differential equation, and is very similar to Volterra's integral equation model.

3. This model does not represent permanent memory storage.

4. The model can be extended in the following way:

Let $x(t)$ be the input, $y(t)$ be the output.

Let $R(x)$ be a nonlinear memoryless function.

Let $F(x)$ be an integral type functional.

Let $G(x)$ be a differential type functional.

then there are various cases of combinations which representing the model for hysteresis. *

y(t) =		
$R(x) + F(x)$ $-R(x) + F(x)$	$R(x) - F(x)$ $-R(x) - F(x)$	(inductive)
$R(x) - G(x)$ $-R(x) - G(x)$	$R(x) + G(x)$ $-R(x) + G(x)$	(capacitive)
$R_1(x) \oplus R_2(x)$ $-R_1(x) \oplus -R_2(x)$	$-R_1(x) \oplus R_2(x)$ $-R_2(x) \oplus R_1(x)$	(resistive) **
(All loops in the above are counter-clockwise)	(All loops in the above are clockwise)	

TABLE 2 VARIOUS TYPES OF HYSTERESIS

---

* The inductive type hysteresis agrees with the Kostitzin's and Volterra's models.

** 1. Some of them might not be phycically realizable.

2. It should be noticed that in the resistive case, there is no memory type subsystem, but there is a memory dependence between each memoryless subsystem, ie., R_1 and R_2 interchange when x( ) change its sign . ($\oplus$ means that R_1 and R_2 are not necessarilly operating simultaneously.)

3.6 A Special Function Model

Consider a squre loop hysteresis with input -output relation shown in Fig, 3-31. (35)

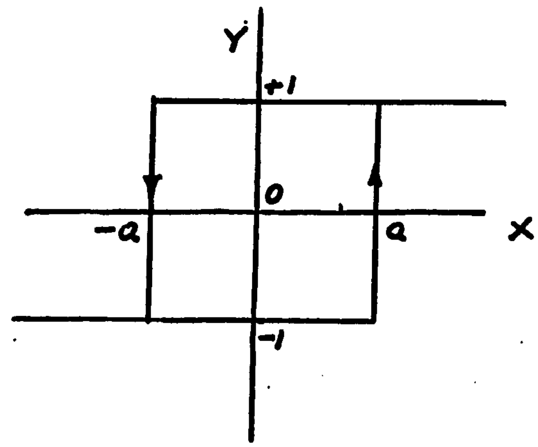

Fig. 3-31 Square loop hysteresis

Where +a and -a are coercive points and +1 and -1 are normalized ranges of output.(which are assumed to be on the same lever of remanent points)

Let the input be a sequence of discrete real numbers, $x(n)$, $n = 1,2, \ldots\ldots$. Let $y(n) = +1$, 0 or -1. Then the relation between input and output can be expressed as :

$$y(n) = \delta\,(y(n-1)-1)\ \mathrm{sgn}(x(n)+a) + \delta\,(y(n-1)+1)\ \mathrm{sgn}(x(n)-a) \qquad (3\text{-}12)$$

Where $\delta(\ )$ is the Kronecker delta function and sgn( ) will be defined in the supplement of this section. . The above equation is valid only when the magnet is in one of its remanent states, ie., when y(n-1)=+1 or -1. If y(n-1)=0, the equation should be:

$$y(n) = \frac{1}{2}(sgn(x(n)+a)+sgn(x(n)-a)) \qquad (3\text{-}13)$$

It is the primary magnetization curve. (Fig.3-32)

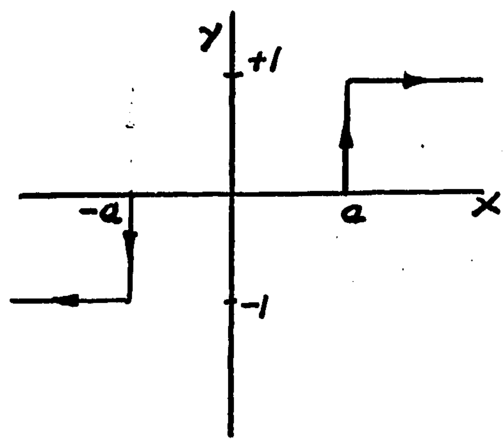

Fig. 3-32 Idealized primary magnetization curve

This model can be derived from the density function model. For a general type hysteresis curve shown in Fig. 3-33

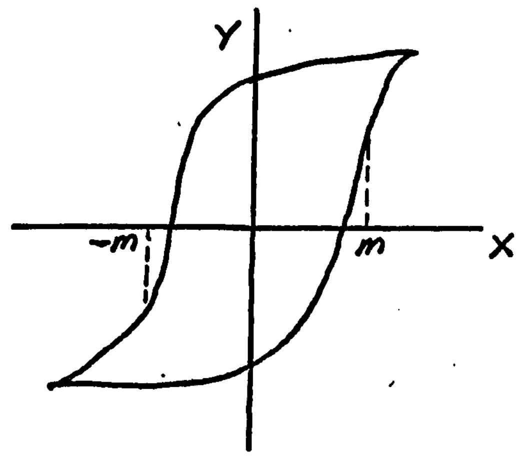

Fig. 3-33 A general type hysteresis

with density functions N(+m,σ) and N(-m,σ),where N(,) is the Gaussian function.

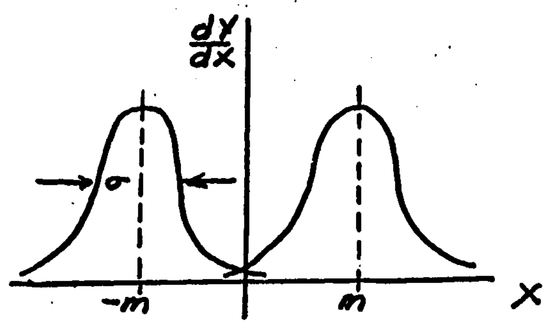

Fig. 3-34 Density function for hysteresis in 3-33

The functions are defined over $(-\infty,\infty)$ and are parametrized by mean $\pm m$ and variance σ. (28)

Let $\sigma \to 0$. The two density functions become two impulses.

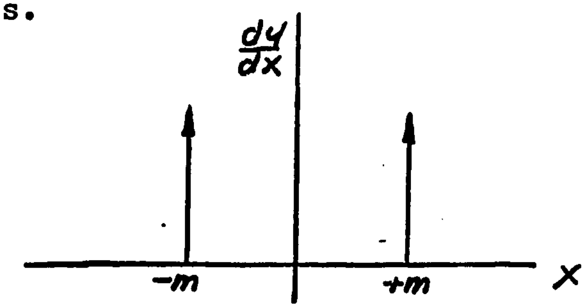

Fig. 3-35 The limiting case of density functions

Hence a loop is formed by integrating the density functions over x, ie, when $y(n-1) = -1$, the integration over x gives $y(n) = sgn(x(n)-a)$, when $y(n-1) = +1$, it gives $y(n) = sgn(x(n)+a)$.

In equation (3-12), it is obvious that y(n) depends on y(n-1), which means an <u>one-stage memory dependence</u>.

For a higher stage loop, see Fig. 3-36. The density functions are shown in Fig. 3-37.

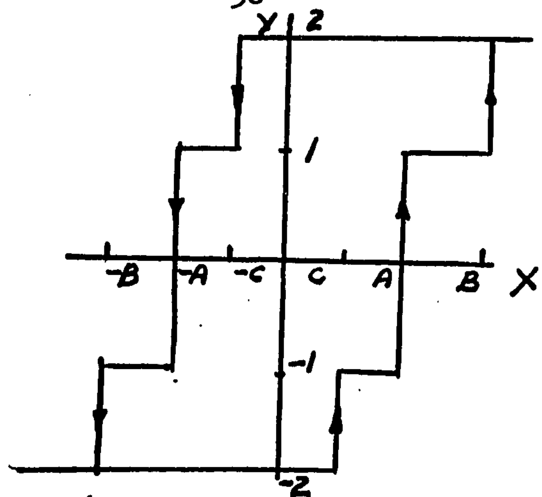

Fig. 3-36 A higher stage loop.

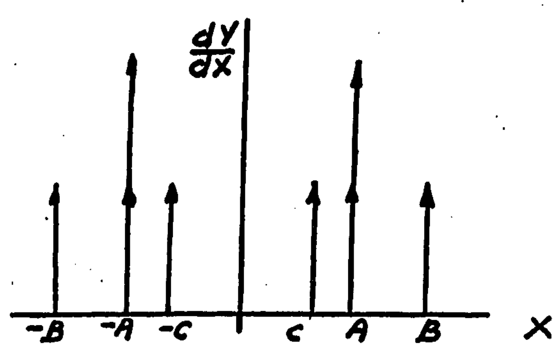

Fig. 3-37 Density function for the higher stage loop.

An analytic expression can also be derived and it should have a two-stage memory dependence, ie, y(n) is a function of y(n-1) and y(n-2).

Supplement to 3-6

A list of special functions

Def. 1. $u(x-x_o) = 1$ if $x \geqslant x_o$

$= 0$ otherwise

Def. 2. $\delta(x-x_o) = 1$ if $x = x_o$

$= 0$ otherwise

Def. 3. $\mathrm{sgn}(x-A) = +1$ if $x \geqslant A$

$= -1$ if $x < A$

Therefore, the following relations can be represented by the above special functions:

1. The piecewised saturative nonlinearity:

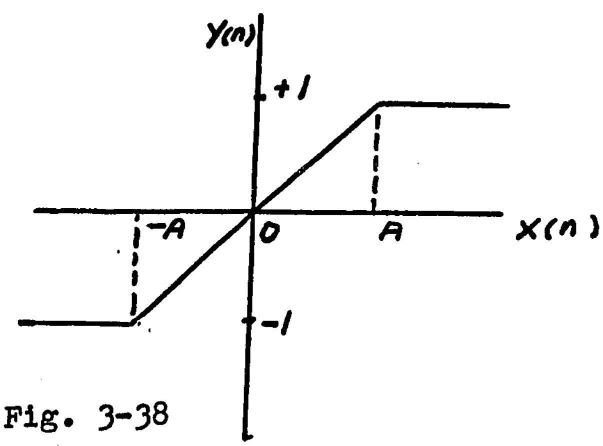

Fig. 3-38

$$y = +1 \quad \text{if} \quad x \geqslant A$$

$$y = -1 \quad \text{if} \quad x < -A$$

$$y = x \quad \text{if} \quad -A \leqslant x < A$$

with representation equation

$$y(n) = x(n) + \tfrac{1}{2}(\mathrm{sgn}(x(n)-A) + \mathrm{sgn}(x(n)+A))$$

2. The Coulomb friction:

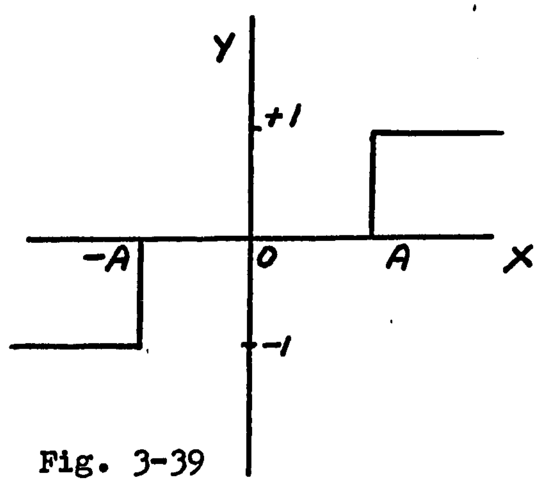

Fig. 3-39

where $y = 0$ for $-A \leqslant x < A$

has representation equation

$$y(n) = \tfrac{1}{2}(\mathrm{sgn}(x(n) - A) + \mathrm{sgn}\,(x(n) + A))$$

3. A single square loop:

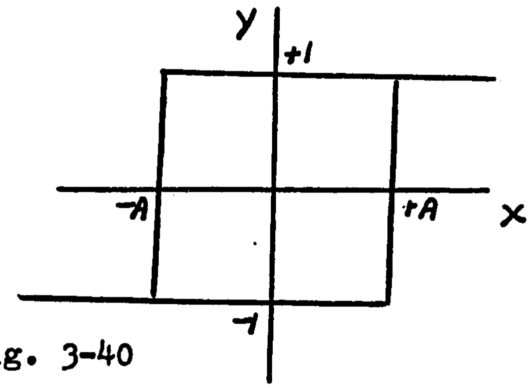

Fig. 3-40

$y(n) = -1$ if $x(n) < +A$ and $y(n-1) = -1$

$y(n) = +1$ if $x(n) \geqslant +A$ and $y(n-1) = -1$

$y(n) = -1$ if $x(n) \leqslant -A$ and $y(n-1) = +1$

$y(n) = +1$ if $x(n) > -A$ and $y(n-1) = +1$

Hence

$$y(n) = \delta(y(n-1)-1)\,\mathrm{sgn}(x(n)+A) + \delta(y(n-1)+1)\,\mathrm{sgn}(x(n)-A))$$

4. An orientated square loop:

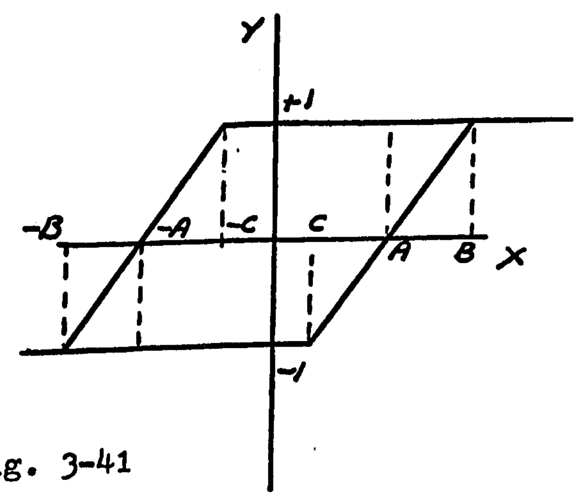

Fig. 3-41

with representation equation

$$y(n) = \delta(y(n-1)-1)\ ((x(n)-A)\ (u(x(n)+C)-u(x(n)-B)$$
$$+ \tfrac{1}{2}(sgn(x(n)+C)+sgn(x(n)-B))$$
$$+ \delta(y(n-1)+1)\ ((x(n)+A)\ (u(x(n)-C)-u(x(n)+B)$$
$$+ \tfrac{1}{2}(sgn(x(n)-C)+sgn(x(n)+B))$$

5. A double loop:

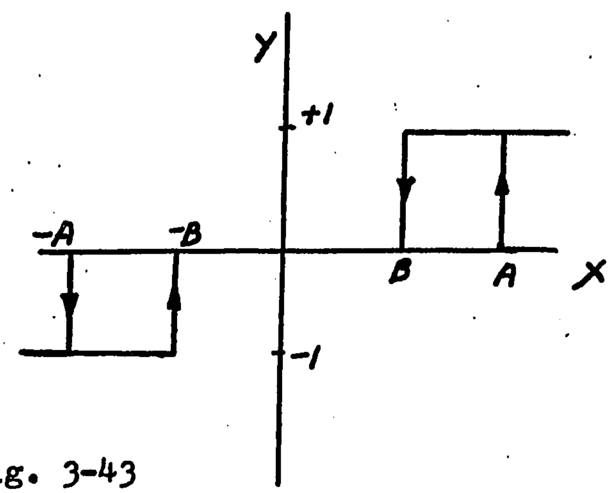

Fig. 3-43

with representation equation

$$y(n) = \delta\ (y(n-1)-1)(1/2)\ (sgn(x(n)-B)+sgn(x(n)+A))$$
$$+ \delta\ (y(n-1)+1)(1/2)\ (sgn(x(n)-A)+sgn(x(n)+B))$$
$$+ \delta\ (y(n-1))\ (1/2)\ (sgn(x-A)+sgn(x+A))$$

CONCLUSION

Different types of hysteresis models have been mentioned in this thesis: Middleton's model, Weygandt & Charp's model, Volterra's model, Kostitzin's model and Hayashi's model.

All the above models can be classified into two catergories: 1. Approximation by continuous operators, and, 2. Expression by discrete special functions.

We, also, proposed a model in each of these catergories. ——— All those models, while quite useful for simulation purposes, have many limitations. The complexity of the hysteresis phenomenon, observed in so many different branches of science, is the main reason. ——— We now believe that a more unified approach and a better model may be obtained from the study of the microscopic behaviour of materials. This is proposed for further study.

REFERENCES

1. Ewing, J. A.
" Magnetic Induction in Iron and other related metals"
$3\frac{1}{2}$ edition, London, 1900

2. Warnier, Gregory H.
" Statistical Physics "
John Wiley and Sons Inc. , 1956

3. Bozorth, R. M.
"Ferromagnetism"
D. Van Nostrand Co. Inc.
Princeton, New Jersey, N.J. 1956

4. Giles, R.
" Mathematical Foundations of Thermodynamics "
Pergamom Press , 1964

5. Bridgman, P. W.
" Nature of Thermodynamics "
Harper Torchbook , 1925

6. Everett, D. H. and Whitton, W. I.
" A general approach to hysteresis "
Transactions of the Faraday Society, 48, 1952

7. Everett, D. H.
" A general approach to hysteresis " ,part II
Transactions of the Faraday Society, 50, 1954

8. Everett, D. H.
" A general approach to hysteresis " , part III
Transactions of the Faraday Society, 50, 1954

9. Everett, D. H.
" A general approach to hysteresis " , part IV
Transactions of the Faraday Society, 51, 1955

10. Enderby, J. A.
" The domain model of hysteresis "
Transactions of the Faraday Society, 51, 1955

11. Enderby, J. A.
" The domain model of hysteresis "
Transactions of the Faraday Society, 52, 1956

12. Gibson, J. E.
" Nonlinear automatic control "
McGraw Hill Book Co. , 1964

13. Graham, D. and McRuer, D.
" Analysis of nonlinear control systems"
John Wiley & sons Inc. , 1964

14. Sen, A. K.
" The complex equivalent gain of a memory type nonlinearity "
IEEE Transactions on Automatic Control, Apr., 1965

15. Weber, Ernst
" Nonlinear physical phenomena "
Symposium on Nonlinear Circuit Analysis
Polytechnic Institute of Brooklyn, Apr., 1953

16. Phillips, Aris
"Introduction to plasticity "
The Ronald Press, 1956

17. Cottrell, A. H.
" The mechanical properties of matter "
John Wiley and Sons Inc., 1964

18. Otheru, Sadamu

" On expressions of magnetic hysteresis characteristics"

AIEE Transactions, part III B, Power and Systems V-78-1959

19. Defigueiredo, Rui J. P.

" Forced oscillation in nonlinear systems with hysteresis "

IEEE Transactions on Circuit Theory, Mar., 1965 Vol. CT-12 No. 1

20. Leland, Harold R.

" Input and output Gaussian-correlation functions for some memory-type nonlinear systems with Gaussian input "

AIEE Transactions, July, 1960

21. Middleton, F. H.

" Analytic hysteresis functions "

Journal of Applied Physics, Supplement to Vol. 32, No. 3, Mar., 1961

22. Distler, R. J.
" Hysteresis loop analysis "
IEEE Transactions on Components Parts, Sept., 1963

23. Volterra, Vito
" Theory of functionals and of integral and integro-differential equations "
Dover Publications Inc. , New York, 1959

24. Kostitzin, V. A.
" Sur quelques applications des equations integrales au probleme de l'hysteresis magnetique "
International Mathematical Congress, 1928, Toronto

25. Weygandt, G. W. and Charp, S.
" Ageneral approach to the hysteresis loop "
Electrical Engineering, Vol. 61, pp. 387-8, 1942

26. Glinski, G. S. and Ahmed, N. U.
" Nonlinear Magnetics "
Technical Report 64-2, June 1964
Dept. of E.E., University of Ottawa

27. Glinski, G. S. and Tokumaru, H.
" Mathematical models for a tape wound core with square loop characteristics"
Technical Report 63-13, 1963
Dept. of E.E. , University of Ottawa

28. Artly, J. L.
" The evaluation of ferromagnetic materials in terms of the harmonic content of associated magnetic Fields "
Ph.D. Thesis, The Johns Hopkins University, Baltimore, 1956

29. Caughey, T. K.
" Equivalent linearization techniques "
The Journal of the Acoustical Society of America, Vol. 35, No. 11, 1963

30. Caughey, T. K.
" Sinusoidal excitation of a system with bilinear hysteresis "
Transactions of the ASME, Dec., 1960

31. Block, H. D.
" Periodic solutions of forced systems having hysteresis "
IRE Transactions on Circuit Theory, Dec. 1960

32. Brilliant, Martin B.
" Theory of the analysis of nonlinear systems "
Technical Report 345, MIT Research Lab. of Electronics 1958

33. George, D. A.
" Continuous nonlinear systems "
Technical Report 355, MIT research Lab. of Electronics 1959

34. Hayashi, Chihiro
" Nonlinear oscillation in physical systems "
McGraw-Hill Book Co., 1964

35. Gilli, L. and Meo, A. R.
" The behavior of rectangular hysteresis loop cores in every application "
Proceedings of the IEEE, Nov. 1963

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