Homogeneous Projective Varieties of Rank 2 Groups

by

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Abstract

Root systems are a fundamental concept in the theory of Lie algebra. In this thesis, we will use two different kind of graphs to represent the group generated by reflections acting on the elements of the root system. The root systems we are interested in are those of type A_2, B_2 and G_2. After drawing the graphs, we will study the algebraic groups corresponding to those root systems. We will use three different techniques to give a geometric description of the homogeneous spaces G/P where G is the algebraic group corresponding to the root system and P is one of its parabolic subgroup. Finally, we will make a link between the graphs and the multiplication of basis elements in the Chow group $\mathrm{CH}(G/P)$.

Résumé

Les systèmes de racines sont un concept fondamental dans la théorie de l'algèbres de Lie. Dans cette thèse, nous utiliserons deux types de graphes permettant de représenter le groupe généré par les réflections qui agissent sur les éléments du système de racines. Les systèmes de racines qui nous intéressent sont ceux de type A_2, B_2 et G_2. Après avoir dessiné les graphes, nous étudierons les groupes algébriques correspondant à ces systèmes de racines. Nous utiliserons trois techniques différentes afin de donner une description géométrique des espaces homogènes G/P où G est un groupe algébrique correspondant au système de racines et P est l'un de ses groupes paraboliques. Finalement, nous ferons un lien entre les graphes et une multiplication des éléments de la base du groupe de Chow $\mathrm{CH}(G/P)$.

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Contents

Chapter 1

Introduction

The German mathematician Hermann Weyl was the first one to use the term Lie algebra (named after Sophus Lie) in lectures at the Institute for Advanced Study in 1933-1934. Since then, the theory of Lie algebra has greatly evolved and is now used in several different areas of mathematics, physics, computer science, and many more. In this thesis, we are mostly interested in root systems, which are a fundamental concept in the theory of Lie algebras. The root systems we are interested in are those of rank 2, e.g. A_2, B_2 and G_2. In our exposition we will follow [Bou], [EW] and [Hu].

A root system may be viewed as a collection of vectors in an Euclidian space satisfying some geometric properties. One of these properties is that the set of vectors is closed under the reflections through the hyperplanes perpendicular to these vectors. The group generated by these reflections is called the Weyl group and this will be the principal object of our studies. It is an interesting subject since the Weyl group has many properties that can be examined using different methods. A multitude of tools are available to study groups, but we will restrict ourselves to some particular ones from graph theory.

The Cayley graph, named after the British mathematician Arthur Cayley, is a graph that encodes the structure of a discrete group. By drawing the Cayley graph of the Weyl group, we will be able to exhibit some basic properties of the group. Unfortunately, we cannot interpret all the properties we want just by looking at the Cayley graph. Therefore, we will turn our interest toward another kind of graph, the Pieri graph. This type of graph uses a different formula to compute the number of arrows between vertices and contains information about the intersection product on cohomology of associated flag varieties.

In order to understand the usefulness of the Pieri graph, we must start to work with algebraic groups. More precisely, we are interested in the linear algebraic groups over an algebraically closed field of characteristic zero, which are in one-to-one correspondence with Lie algebras. Therefore, the study of the algebraic groups corresponding to certain root systems is a natural way to search for additional properties of the Pieri graph. The algebraic groups corresponding to the root systems of type A_2, B_2 and G_2 are respectively the groups $\mathrm{SL}_3(\mathbb{C})$, $\mathrm{SO}_5(\mathbb{C})$ and the group of automorphisms of the octonion algebra. As for Lie Algebra, the theory of algebraic groups is a vast area of study considered by mathematicians since the 19th century. Throughout this thesis, when it comes to algebraic groups, we will follow the books of [Bor], [Hu2] and [Spr].

Before making the link between algebraic groups and the Pieri graph, it is normal for us to seek a better understanding of the objects we are working with. In fact, for an algebraic group G, we are interested in the algebraic varieties of the form G/P for some closed subgroup P of G. The subgroup P satisfies some particular properties and is called a parabolic subgroup of G. In the Chapters 4, 5 and 6, we will study the varieties of the form G/P by giving a geometric description using three different methods.

We will begin by using a computational method. Since the groups $\mathrm{SL}_3(\mathbb{C})$ and $\mathrm{SO}_5(\mathbb{C})$ are matrix groups, linear algebra gives us plenty of tools to find a geometric description of G/P. In the case of $\mathrm{SO}_5(\mathbb{C})$, additional knowledge about bilinear forms will be necessary. Fortunately, [Lam] is a good source of information about the subject.

The next method involves viewing the algebraic groups as Chevalley groups. The lecture notes [St] explain how to construct an algebraic group G by starting with a root system using the Chevalley construction. Moreover, we can follow the work of [CG] to be able to find a geometric description of G/P. What is interesting about this method is that we can get a good description of G/P by looking at the Dynkin diagram of the corresponding root system.

The last method is motivated by representation theory [FH]. It is well-known that for any Lie algebra $\mathfrak{g}$, we can describe the algebra using the root space decomposition. Analogous to that, we get a decomposition for any representation V, called the weight space decomposition. Using these two decompositions, we will be able to elaborate a step-by-step method that will help us analyze some particular representations in order to give a geometric description. After all that work, we will have enough knowledge about the varieties of the form G/P.

We can now return on the track leading toward our main goal. The last tool needed will be some facts about Chow groups. Following [NSZ], we can find a basis for the group $\mathrm{CH}(G/P)$. Then, as we can see in [De], there is a connection between the multiplication of the basis elements of $\mathrm{CH}(G/P)$ and the Pieri graph of the root system corresponding to G. Just by looking at the Pieri graph, we will be able to compute right away the multiplication of certain basis elements of $\mathrm{CH}(G/P)$.

Chapter 2

Preliminaries

2.1 Root Systems

The primary goal of this chapter is to familiarize the reader with the basic notions that will be used throughout this thesis. All the tools necessary for the construction of our examples will be introduced here, but those examples will only be approached in the next chapter. Let us begin by establishing what is needed for our main definitions.

Let E be a fixed euclidian space, i.e., a finite-dimensional real vector space endowed with an inner product $(\cdot,\cdot)$. For $v \in E$, $v \neq 0$, we define s_v to be the orthogonal reflection in the hyperplane normal to v. We can write down the following explicit formula [EW, p. 109]:

$$s_v(x) = x - \frac{2(x,v)}{(v,v)}v, \forall x \in E. \tag{2.1}$$

Since we will often encounter the fraction $2(x,v)/(v,v)$, we will use this notation:

$$\langle x, v^\vee \rangle := \frac{2(x,v)}{(v,v)}$$

2.1.1 Remark. The symbol $\langle x, v^\vee \rangle$ is only linear with respect to the first variable, x.

Now that we know in which kind of space we work and that we have established the necessary notation, we can introduce our first definition.

2.1.2 Definition. [EW, p. 110][Hu, p. 42] A subset Φ of the euclidian space E is called a *root system* in E if the following axioms are satisfied:

(R1) Φ is finite, spans E, and does not contain 0.

(R2) If $\alpha \in \Phi$, the only scalar multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection s_α permutes the elements of Φ.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$.

The elements of Φ are called *roots*.

2.1.3 Definition. [Hu, p. 43] Let Φ_1, Φ_2 be two root systems with respective euclidian spaces E_1, E_2. We say that (E_1, Φ_1) and (E_2, Φ_2) are *isomorphic* if the exists a vector space isomorphism $\phi : E_1 \to E_2$ sending Φ_1 onto Φ_2.

It follows from the definition that $\phi(s_\alpha(\beta)) = s_{\phi(\alpha)}(\phi(\beta))$. Therefore, by (R3), we know that the group of automorphisms of Φ, $\mathrm{Aut}(\Phi)$, permutes the elements of Φ.

2.1.4 Lemma. [Bou, p. 142] $\mathrm{Aut}(\Phi)$ *is finite.*

Here is a direct consequence of (R4).

2.1.5 Lemma. [EW, p. 111] *Suppose that Φ is a root system in E. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then*

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}.$$

Using the previous lemma, we get the following possibilities when $\beta \neq \pm\alpha$ and $\|\beta\| \geq \|\alpha\|$:

$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	θ	$\frac{\|\beta\|^2}{\|\alpha\|^2}$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

By taking a good look at the table, we see that we have different values of $\langle \alpha, \beta^\vee \rangle$ depending on whether θ is acute or obtuse. This fact can help us in the proof of the following proposition.

2.1.6 Proposition. [EW, p. 112] *Let $\alpha, \beta \in \Phi$.*

(i) If the angle between α and β is strictly obtuse, then $\alpha + \beta \in \Phi$.

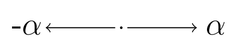
(ii) If the angle between α and β is strictly acute and $\|\beta\|^2 \geq \|\alpha\|^2$, then $\alpha - \beta \in \Phi$.

The next definition helps us in distinguishing root systems.

2.1.7 Definition. The *rank* of a root system Φ, denoted rk(Φ), is defined as the dimension of E, dim(E).

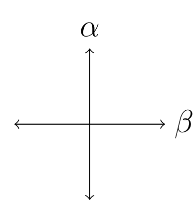
When $rk(\Phi) \leq 2$, we can describe the root system by drawing a picture. In the case where $rk(\Phi) = 1$, the condition (R2) gives us only one possibility. We denote that root system by A_1 and we get the following picture:

A_1:

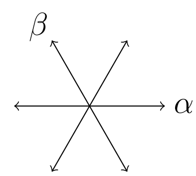


Rank 2 gives us more possibilities. In fact, one can check [EW, p. 112–113] that we have exactly four different root systems of rank 2. We can look in the table above for the angle θ between α and β as well as for the length ratio of the two roots. We can draw each picture as follows:

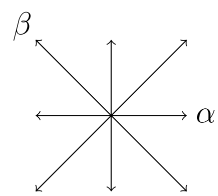
$A_1 \times A_1$:



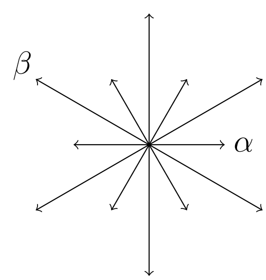
A_2:



B_2:



G_2:



Keep in mind that the pictures are important since those are the root systems that we will use as examples in the next chapters.

2.1.8 Remark. For $A_1 \times A_1$, the length of α and β does not need to be the same. Since the ratio is undetermined, both lengths are arbitrary.

The following definition gives another important characteristic of a root system.

2.1.9 Definition. [EW, p. 114] The root system Φ is *reducible* if Φ can be expressed as a disjoint union of two non-empty subsets $\Phi_1 \cup \Phi_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. If not, Φ is *irreducible*.

2.1.10 Remark. If such a decomposition exists, Φ_i is also a root system in $E_i = \text{span}(\Phi_i)$ for $i = 1, 2$.

2.1.11 Proposition. *$A_1 \times A_1$ is reducible.*

Proof: Let $\alpha \neq \pm\beta$ and $A_1 \times A_1 = \{\pm\alpha, \pm\beta\}$ like in the picture above. Let us write $A_1 \times A_1 = \Phi_1 \cup \Phi_2$ where $\Phi_1 = \{\pm\alpha\}$ and $\Phi_2 = \{\pm\beta\}$. We know that Φ_1 and Φ_2 are root systems of type A_1. Also, since the angle θ between α and β is $\pi/2$,

$$(\pm\alpha, \pm\beta) = \|\alpha\|\|\beta\| \cos\theta = 0.$$

Therefore, $A_1 \times A_1$ is reducible. $\square$

The next lemma tells us that we can write any root system as a disjoint union of irreducibles.

2.1.12 Lemma. [EW, p. 114][Hu, p. 57] *Let Φ be a root system in the real vector space E. We may write Φ as a disjoint union*

$$\Phi = \Phi_1 \cup \Phi_2 \cup ... \cup \Phi_k,$$

where each Φ_i is an irreducible root system in the space E_i spanned by Φ_i, and E is the direct sum of the orthogonal subspaces $E_1, E_2, ..., E_k$.

We will dedicate the next section to an important subset of Φ.

2.2 Base of a Root System

Since Φ spans E, any maximal linearly independent subset of Φ is a basis for Φ. In fact, we have a stronger definition for a base of a root system.

2.2.1 Definition. [EW, p. 115] A subset Δ of Φ is a *base* for the root system Φ if

(B1) Δ is a basis for E, and

(B2) every $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$, where all the non-zero coefficients k_α have the same sign.

We call the elements of Δ *simple roots*. Also, we say that a root $\beta \in \Phi$ is *positive* (respectively *negative*) with respect to Δ if the coefficients in (B2) are all positive (resp. negative). We denote by Φ^+ the set of all positive roots of Φ and by Φ^- the set of all negative roots. By property (B2), we can write Φ as a disjoint union of Φ^+ and Φ^-.

The following lemma is a direct consequence of the definition.

2.2.2 Lemma. [EW, Exercise 11.3] *If Δ is a base for the root system Φ, then the angle between any two distinct elements of Δ is obtuse.*

Proof: Let $\alpha, \beta \in \Delta$ and let $\alpha \neq \pm\beta$. Suppose the angle between α and β is strictly acute, then Proposition 2.1.4 tells us that $\alpha - \beta$ is a root but this contradicts (B2). □

2.2.3 Theorem. [EW, p. 116] *Every root system Φ has a base.*

2.2.4 Remark. A root system Φ can have different bases. For example, if Δ is a base for Φ, so is $\{-\alpha | \alpha \in \Delta\}$.

2.2.5 Proposition. [EW, Exercise 11.5] *Let Φ be a root system with base Δ. Take any $\phi \in \mathrm{Aut}(\Phi)$. Then, the set $\{\phi(\alpha) | \alpha \in \Phi\}$ is also a base of Φ.*

Proof: Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ be the base for our root system Φ. Let $\phi \in \mathrm{Aut}(\Phi)$. We need to show that $\{\phi(\alpha_1), \phi(\alpha_2), ..., \phi(\alpha_k)\}$ is also a base for Φ.

Since ϕ is an invertible linear map, $\{\phi(\alpha_1), \phi(\alpha_2), ..., \phi(\alpha_k)\}$ is also a basis for E so (B1) is satisfied. We only need to check if (B2) holds. Let $\beta \in \Phi$. Then, $\phi^{-1}(\beta) \in \Phi$. Since $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ is a base, we can write:

$$\phi^{-1}(\beta) = \sum_{i=1}^{k} c_i \alpha_i$$

where all the nonzero c_i have the same sign. Then,

$$\beta = \phi(\phi^{-1}(\beta)) = \phi\left(\sum_{i=1}^{k} c_i \alpha_i\right) = \sum_{i=1}^{k} c_i \phi(\alpha_i)$$

where all the nonzero c_i are integers of the same sign. Since the choice of β was arbitrary, (B2) holds. □

We defined in Section 2.1 the reflections in the hyperplane normal to the roots. In the next section we will introduce and study the group generated by those reflections.

2.3 Weyl Group of Φ

2.3.1 Definition. The *Weyl group* of a root system Φ, denoted $W(\Phi)$, is the group generated by the reflections s_γ where $\gamma \in \Phi$.

Note that we will call *simple reflections* the reflections in the hyperplanes normal to the simple roots.

The next proposition will be very useful in the next chapter.

2.3.2 Proposition. [EW, p. 119] *Suppose that $\beta \in \Phi$. There exists $g \in \langle s_\gamma | \gamma \in \Delta \rangle$ and $\alpha \in \Delta$ such that $\beta = g(\alpha)$.*

Therefore, we can recover all the roots of Φ by applying consecutive simple reflections to its simple roots.

Another fact we will need in order to find the Weyl group of a root system is that the Weyl group is a *Coxeter group* [Bou, p. 153]. It can be described by a presentation as follows:

$$W(A_2) = \langle s_{\alpha_i} | (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1, 1 \leq i, j \leq 2 \rangle$$

where m_{ij} is the order of the Weyl group element $s_{\alpha_i} s_{\alpha_j}$ and $m_{ij} = 1$ if $i = j$ or $m_{ij} \geq 2$ if $i \neq j$.

2.3.3 Definition. We say that an element of the Weyl group is of *length* n if it is the product of n simple reflections and not less.

We will denote by l the function that gives us the length of an element. By convention, $l(1) = 0$.

The following proposition gives us a way to recover all the reflections in the hyperplanes normal to the roots of Φ.

2.3.4 Proposition. [EW, Exercise 11.6] *Let $\alpha \in \Phi$ and $g \in W$, then $gs_\alpha g^{-1} = s_{g(\alpha)}$.*

Proof: For $\gamma \in \Phi$ we have

$$\begin{aligned}
gs_\alpha g^{-1}(\gamma) &= g(s_\alpha(g^{-1}(\gamma))) \\
&= g(g^{-1}(\gamma) - \langle g^{-1}(\gamma), \alpha^\vee \rangle \alpha) \\
&= g(g^{-1}(\gamma) - \langle \gamma, g(\alpha)^\vee \rangle \alpha) \text{ since } g^{-1} \text{ preserves the inner product} \\
&= \gamma - \langle \gamma, g(\alpha)^\vee \rangle g(\alpha) \\
&= s_{g(\alpha)}(\gamma) \quad \square
\end{aligned}$$

2.3.5 Corollary. [EW, p. 118–119] *Let Φ be a root system with base Δ. The Weyl group of Φ is generated by all reflections s_γ where $\gamma \in \Delta$.*

The previous proposition will be really helpful later because we will need to write down the reflections corresponding to the positive roots of Φ as products of simple reflections.

2.3.6 Lemma. [Hu, p. 43] *The Weyl group of a root system Φ is a normal subgroup of* $\mathrm{Aut}(\Phi)$.

2.3.7 Corollary. *$W(\Phi)$ is finite.*

Before going to the next section, we need another theorem that we will not prove.

2.3.8 Theorem. [EW, p. 223–226] *Let Φ be a root system and suppose that Δ_1 and Δ_2 are two bases of Φ. Then there exists an element g in the Weyl group $W(\Phi)$ such that $\Delta_2 = \{g(\alpha) | \alpha \in \Delta_1\}$.*

In other words, the theorem tells us that the Weyl group acts transitively on the set of bases of Φ. In the next section we will introduce the notion of the Cartan matrix.

2.4 Cartan Matrix, Dynkin Diagram and Fundamental Weights

2.4.1 Definition. [EW, p. 120] Let Φ be a root system with base Δ. We fix an ordering on the simple roots, say $(\alpha_1, \alpha_2, ..., \alpha_k)$. The *Cartan matrix* of Φ is the matrix $C = (c_{ij})$ where $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ for $\alpha_i, \alpha_j \in \Delta$.

We know from Theorem 2.3.8 that the Weyl group acts transitively on the collection of bases and that for any root $\gamma \in \Phi$ we have

$$\langle s_\gamma(\alpha), s_\gamma(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle.$$

Then, the Cartan matrix is independent of the chosen base. It only depends on the ordering of the simple roots.

2.4.2 Definition. [EW, p. 121] The *Dynkin diagram* of a root system Φ is a diagram with vertices labelled by the simple roots of Φ. Two vertices α_i and α_j ($i \neq j$) are joined by $d_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ many edges. If $d_{ij} > 1$, we draw an arrow pointing from the longer root to the shorter root.

We can find pictures of the Dynkin diagram of all root systems in [Bou, p. 197].

2.4.3 Definition. [Hu, p. 67] We will call *weights* all $\omega \in E$ such that $\langle \omega, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$.

2.4.4 Example. By (R4), all roots of Φ are weights.

2.4.5 Definition. [Hu, p. 67] The *fundamental weights* of a root system Φ with base Δ are weights $\omega_j \in E$ for $j = 1, .., |\Delta|$ such that for all $\alpha_i \in \Delta$,

$$\alpha_i = \sum_j c_{ij} \omega_j$$

where (c_{ij}) is the Cartan matrix of the root system. Equivalently, we have $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

2.5 Cayley Graph and Pieri Graph

In Section 2.1, we drew the pictures of the root system Φ. Now, let us find a way to depict the Weyl group of Φ.

2.5.1 Remark. Recall that the base of Φ is $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_k\}$. For $1 \leq i \leq k$, we will associate a different color to every i.

2.5.2 Definition. c.f. [GM, p. 45] The *Cayley graph* of $W(\Phi)$ is a colored directed graph where the vertices are the elements of $W(\Phi)$ and two vertices x and y are joined by a an arrow of color associated to i if $y = s_{\alpha_i} \cdot x$ for $i \in \{1, 2, ..., k\}$.

2.5.3 Remark. We will always write the multiplication by elements of $W(\Phi)$ on the left.

2.5.4 Lemma. *Let $x, y \in W(\Phi)$ such that $l(y) = l(x) + 1$ and $\lambda \in \Phi^+$ such that $s_\lambda \cdot x = y$. This λ is unique.*

Proof: Let $\mu \in \Phi^+$ such that $s_\lambda \cdot x = y = s_\mu \cdot x$. Then, we have $s_\lambda = yx^{-1} = s_\mu$, so yx^{-1} is a reflection in a root. Since $s_\lambda = s_\mu$,

$$\begin{aligned} -\lambda &= s_\lambda(\lambda) = s_\mu(\lambda) = \lambda - \langle \lambda, \mu^\vee \rangle \mu, \\ &\Rightarrow \langle \lambda, \mu^\vee \rangle \mu = 2\lambda \neq 0, \\ &\Rightarrow \lambda \text{ and } \mu \text{ are linearly dependent}, \\ &\Rightarrow \lambda \in \mathbb{R}\mu, \\ &\Rightarrow \lambda = \pm\mu \text{ by (R3)}, \\ &\Rightarrow \lambda = \mu \text{ because } \lambda, \mu \in \Phi^+. \quad \square \end{aligned}$$

2.5.5 Remark. We will see in the next chapter that the λ need not be a simple root.

2.5.6 Lemma. *Let Φ be a root system with base $\Delta = \{\alpha_1, ..., \alpha_k\}$. Let $\lambda \in \Phi^+$, then $\langle \omega_i, \lambda^\vee \rangle \in \mathbb{N}$.*

Proof: Let $\lambda \in \Phi^+$. We know that

$$\begin{aligned} \langle \omega_i, \alpha_j^\vee \rangle &= \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \\ \Rightarrow (\omega_i, \alpha_j) &= \frac{\delta_{ij}(\alpha_j, \alpha_j)}{2}. \end{aligned}$$

We have to check that

$$\langle \omega_i, \lambda^\vee \rangle = \frac{2(\omega_i, \lambda)}{(\lambda, \lambda)}$$

is a positive integer. We know that $\langle \omega_i, \lambda^\vee \rangle \in \mathbb{Z}$ since ω_i is a weight. We also know that $(\lambda, \lambda) > 0$ so the only thing left to check is that $2(\omega_i, \lambda) \geq 0$. We have

$$\begin{aligned} 2(\omega_i, \lambda) &= 2(\omega_i, \sum_{j=1}^k c_j \alpha_j) = 2\left[\sum_{j=1}^k (\omega_i, c_j \alpha_j)\right] = 2\left[\sum_{j=1}^k c_j (\omega_i, \alpha_j)\right] \\ &= 2\left[\sum_{j=1}^k c_j \frac{\delta_{ij}(\alpha_j, \alpha_j)}{2}\right] = \sum_{j=1}^k c_j \delta_{ij} (\alpha_j, \alpha_j) = c_i(\alpha_i, \alpha_i) \geq 0. \quad \square \end{aligned}$$

2.5.7 Definition. The *Pieri graph* of $W(\Phi)$ is a colored directed graph where the vertices are the elements of $W(\Phi)$. Let $x, y \in W(\Phi)$ such that $l(y) = l(x) + 1$ and $\lambda \in \Phi^+$ such that $s_\lambda \cdot x = y$. This λ is unique by Lemma 2.5.4. For each $1 \leq i \leq k$ the number of arrows in the color associated to i going from x to y is given by the following formula :

$$\langle \omega_i, \lambda^\vee \rangle = \frac{2(\omega_i, \lambda)}{(\lambda, \lambda)} \tag{2.2}$$

where $i = 1, 2, ..., k$.

Before doing more complex examples in the next chapter, let us provide a simple example of a Cayley graph and a Pieri graph.

2.5.8 Example. Let us work with the root system A_1. We know that $A_1 = \{\pm\alpha\}$ so $W(A_1) = \{1, s_\alpha\}$. Therefore, we only have two vertices in both graphs. If we associate the color red to α, the Cayley graph of A_1 is the following:

$$1 \longleftrightarrow s_\alpha$$

One can check that the Pieri graph of A_1 is identical to its Cayley graph except for the double-sided arrow being a single arrow going from left to right. It looks like this:

$$1 \longrightarrow s_\alpha$$

In the next chapter, we will use most of the definitions and results we just introduced in order to draw the Cayley and Pieri graph of some root systems. By doing so, we will be able to see the differences between these two types of graphs.

Chapter 3

The Pieri Graph of the Root Systems $A_1 \times A_1, A_2, B_2$ and G_2

3.1 The Root System of $A_1 \times A_1$

The first example we will look at is the root system $A_1 \times A_1$. Note that we will go quickly through this example since it is pretty straightforward. Starting with the root system A_2, the examples will be more complex and we will need more results to find their Pieri graph. Before that, let us begin by defining a base of $A_1 \times A_1$.

We can write a base for the root system $A_1 \times A_1$ as the set $\Delta = \{\alpha, \beta\}$, where the angle between these two roots is $\pi/2$. One can realize $A_1 \times A_1$ in the 2-dimensional real vector space by choosing $\alpha = (1, 0)$ and $\beta = (0, 1)$ as its base.

3.1.1 Remark. We will use a different notation for the simple roots and will write $\{\alpha_1, \alpha_2\}$ as the base of the root system, where $\alpha_1 = \alpha$ and $\alpha_2 = \beta$.

By looking at the picture of $A_1 \times A_1$ in the first chapter, we know that the complete root system is $A_1 \times A_1 = \{\pm\alpha, \pm\beta\}$. We also have the following:

$$\langle \alpha, \beta^\vee \rangle = 0, \langle \beta, \alpha^\vee \rangle = 0,$$

and

$$\langle \alpha, \alpha^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2.$$

Using the equalities above, we can write the Cartan matrix of A_2 as:

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Now, let us find the group generated by the simple reflections.

We know that the Weyl group of a root system is generated by the simple reflections. Then, $W(A_1 \times A_1)$ is generated by s_α and s_β. One can check that $s_\alpha s_\beta = s_\beta s_\alpha$. Therefore, the Weyl group of $A_1 \times A_1$ is:

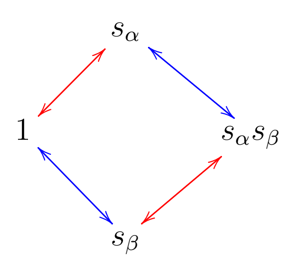
$$W(A_1 \times A_1) = \{1, s_\alpha, s_\beta, s_\alpha s_\beta\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

Now, let us find a way to picture those elements.

We will use Definition 2.5.1 to draw the Cayley graph of $W(A_1 \times A_1)$. We will begin by drawing a vertex for every element of the Weyl group and order them by length. Therefore, the first column will be the elements of length 0 and every column to the right will be of higher length. Then, we will draw arrows by following the rule in the definition.

3.1.2 Remark. Since we only have two elements in the base of $A_1 \times A_1$, we only need two different colors in its Cayley graph. We will join two vertices x and y by a red (respectively blue) arrow if $y = s_\alpha x$ (resp. $y = s_\beta x$).

We get the following Cayley graph for $W(A_1 \times A_1)$:



Now, we are interested in drawing the Pieri graph of $W(A_1 \times A_1)$. Remember that the fundamental weights appear in the formula needed to find the number of arrows between two vertices. Therefore, before being able to draw the Pieri graph, we have to solve the following system of equations to find the fundamental weights of $A_1 \times A_1$:

$$\begin{aligned} 2\omega_1 + 0\omega_2 &= \alpha = (1, 0), \\ 0\omega_1 + 2\omega_2 &= \beta = (0, 1), \\ \Rightarrow \omega_1 = (1/2, 0) &\text{ and } \omega_2 = (0, 1/2). \end{aligned}$$

Now that we found the fundamental weights, we need to check by which reflection in positive roots, s_λ, we have to multiply each vertex, x, such that $s_\lambda(x) = y$ and $l(y) = l(x) + 1$. Here is the complete list:

For 1, $s_\alpha \cdot 1 = s_\alpha$ and $s_\beta \cdot 1 = s_\beta$.

For s_α, $s_\beta \cdot s_\alpha = s_\beta s_\alpha$.

For s_β, $s_\alpha \cdot s_\beta = s_\alpha s_\beta$.

The final step is to compute

$$\langle \omega_i, \lambda^\vee \rangle = \frac{2(\omega_i, \lambda)}{(\lambda, \lambda)}$$

for $i = 1, 2$ and for every $\lambda \in \Phi^+$.

For α,

$$\langle \omega_1, \alpha^\vee \rangle = \frac{2(\omega_1, \alpha)}{(\alpha, \alpha)} = \frac{2(1/2, 0, 0) \cdot (1, 0, 0)}{(1, 0, 0) \cdot (1, 0, 0)} = \frac{2(1/2)}{1} = 1,$$

$$\langle \omega_2, \alpha^\vee \rangle = \frac{2(\omega_2, \alpha)}{(\alpha, \alpha)} = \frac{2(0, 1/2, 0) \cdot (1, 0, 0)}{(1, 0, 0) \cdot (1, 0, 0)} = \frac{2(0)}{1} = 0.$$

For β,

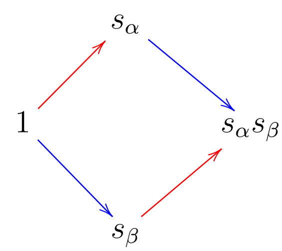
$$\langle \omega_1, \beta^\vee \rangle = \frac{2(\omega_1, \beta)}{(\beta, \beta)} = \frac{2(1/2, 0, 0) \cdot (0, 1, 0)}{(0, 1, 0) \cdot (0, 1, 0)} = \frac{2(0)}{1} = 0,$$

$$\langle \omega_2, \beta^\vee \rangle = \frac{2(\omega_2, \beta)}{(\beta, \beta)} = \frac{2(0, 1/2, 0) \cdot (0, 1, 0)}{(0, 1, 0) \cdot (0, 1, 0)} = \frac{2(1/2)}{1} = 1.$$

Now that all the necessary computations are done, we can draw the Pieri graph by starting by drawing the same vertices as for the Cayley graph and then draw the number of arrows corresponding to the formula.

3.1.3 Remark. Like for the Cayley graph, we will draw arrows of different colors depending on which fundamental weight is used in the formula, red arrows for ω_1 and blue arrows for ω_2.

Here is the Pieri graph for $A_1 \times A_1$:



Once again, we see that the Cayley graph and the Pieri graph are the same, up to the one-sided arrows versus two-sided arrows.

Now that we managed to draw the Pieri graph of $A_1 \times A_1$, let us do the same thing for the other root systems of rank 2.

3.2 The Root System A_2

We can write a base for the root system A_2 as the set $\Delta = \{\alpha, \beta\}$, where the angle between these two roots is $2\pi/3$. One can realize A_2 in the 3-dimensional real vector space by choosing $\alpha = (-1, 1, 0)$ and $\beta = (0, -1, 1)$ as its base. Another realization is given in the 2-dimensional Euclidean space, namely A_2 are the vertices of a regular hexagon, see figure on page 4.

We know from Proposition 2.3.2 that if we repeatedly apply simple reflections, i.e. reflections in simple roots, we can recover the full root system Φ. Therefore, by using Equation (2.1) we can find the complete root system of A_2. Since we know from the table in Section 2.1 that

$$\langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \alpha^\vee \rangle = -1,$$

and

$$\langle \alpha, \alpha^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2,$$

we can do the following calculations:

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta + \alpha,$$

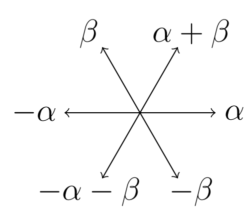
$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha + \beta,$$

$$\begin{aligned} s_\alpha(\alpha+\beta) = \alpha + \beta - \langle \alpha+\beta, \alpha^\vee \rangle \alpha &= \alpha + \beta - \langle \alpha, \alpha^\vee \rangle \alpha - \langle \beta, \alpha^\vee \rangle \alpha \\ &= \alpha + \beta - 2\alpha + \alpha \\ &= \beta, \end{aligned}$$

$$\begin{aligned} s_\beta(\alpha+\beta) = \alpha + \beta - \langle \alpha+\beta, \beta^\vee \rangle \beta &= \alpha + \beta - \langle \alpha, \beta^\vee \rangle \beta - \langle \beta, \beta^\vee \rangle \beta \\ &= \alpha + \beta + \beta - 2\alpha \\ &= \alpha. \end{aligned}$$

Since the action of s_α and s_β on the root $\alpha + \beta$ gives us respectively β and α, we have that the set of roots is invariant under the action of the Weyl group. Therefore, $A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$. Remember that we already drew the picture of the root system A_2 in Section 2.1. Let us add all the missing roots to the picture.

A_2:



Using the equalities above, we can write the Cartan matrix of A_2 as:

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let us find the group generated by the simple reflections.

3.3 Weyl Group of A_2

We know that the Weyl group of a root system is generated by the simple reflections. Let us find all the relations between the products of these reflections in order to find a presentation of the Weyl group of A_2. First, we have

$$s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)} = s_{\alpha+\beta} = s_\beta s_\alpha s_\beta.$$

Hence,

$$s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta = 1$$

so $s_\alpha s_\beta$ has order 3. Also, since s_{α_i} is a reflection, we have that $s_{\alpha_i} s_{\alpha_i}$ has order 1 for $i = 1, 2$. This means that $(s_{\alpha_i} s_{\alpha_j})^l \neq 1$ for $1 < l < 3$. Therefore, we have the following list of elements of the Weyl group according to their length:

Length 0: 1

Length 1: s_α, s_β

Length 2: $s_\alpha s_\beta, s_\beta s_\alpha$

Length 3: $s_\alpha s_\beta s_\alpha$

Now, we want to find all the reflections of positive roots. We already know that s_α and s_β are the reflections corresponding to α and β. There is only one more positive root left, $\alpha + \beta$. As we saw in the computations above, we have

$$s_{\alpha+\beta} = s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha^{-1} = s_\alpha s_\beta s_\alpha.$$

3.3.1 Remark. Since the length of an element of the Weyl group is defined as the minimal representation as a product of simple reflections, the length of $s_{\alpha+\beta} = s_\alpha s_\beta s_\alpha$ is 3.

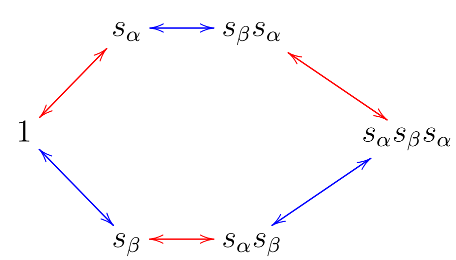
In the next section, we will draw the Cayley and Pieri graph of the Weyl group $W(A_2)$.

3.4 Cayley Graph and Pieri Graph of A_2

Now that we have all the elements of the Weyl group of A_2, we will use Definition 2.5.1 to draw the Cayley graph of $W(A_2)$.

3.4.1 Remark. Since we only have two elements in the base of A_2, we only need two different colors in its Cayley graph. We will join two vertices x and y by a red (respectively blue) arrow if $y = s_\alpha x$ (resp. $y = s_\beta x$).

We get the following Cayley graph for $W(A_2)$:



To draw the Pieri graph of $W(A_2)$, we have to solve the following system of equations to find the fundamental weights of A_2:

$$2\omega_1 - \omega_2 = \alpha = (-1, 1, 0),$$
$$-\omega_1 + 2\omega_2 = \beta = (0, -1, 1),$$
$$\Rightarrow \omega_1 = (-2/3, 1/3, 1/3) \text{ and } \omega_2 = (-1/3, -1/3, 2/3).$$

Now that we found the fundamental weights, we need to check by which reflection in positive roots, s_λ, we have to multiply each vertex, x, such that $s_\lambda(x) = y$ and $l(y) = l(x) + 1$. Here is the complete list:

For 1, $s_\alpha \cdot 1 = s_\alpha$ and $s_\beta \cdot 1 = s_\beta$.

For s_α, $s_\beta \cdot s_\alpha = s_\beta s_\alpha$ and $s_{\alpha+\beta} \cdot s_\alpha = s_\alpha s_\beta s_\alpha s_\alpha = s_\alpha s_\beta$.

For s_β, $s_\alpha \cdot s_\beta = s_\alpha s_\beta$ and $s_{\alpha+\beta} \cdot s_\beta = s_\beta s_\alpha s_\beta s_\beta = s_\beta s_\alpha$.

For $s_\alpha s_\beta$, $s_\beta \cdot (s_\alpha s_\beta) = s_\beta s_\alpha s_\beta$.

For $s_\beta s_\alpha$, $s_\alpha \cdot (s_\beta s_\alpha) = s_\alpha s_\beta s_\alpha$.

The final step is to compute

$$\langle \omega_i, \lambda^\vee \rangle = \frac{2(\omega_i, \lambda)}{(\lambda, \lambda)}$$

for $i = 1, 2$ and for every $\lambda \in \Phi^+$.

For α,

$$\langle \omega_1, \alpha^\vee \rangle = \frac{2(\omega_1, \alpha)}{(\alpha, \alpha)} = \frac{2(-2/3, 1/3, 1/3) \cdot (-1, 1, 0)}{(-1, 1, 0) \cdot (-1, 1, 0)} = \frac{2(1)}{2} = 1,$$

$$\langle \omega_2, \alpha^\vee \rangle = \frac{2(\omega_2, \alpha)}{(\alpha, \alpha)} = \frac{2(-1/3, -1/3, 2/3) \cdot (-1, 1, 0)}{(-1, 1, 0) \cdot (-1, 1, 0)} = \frac{2(0)}{2} = 0.$$

For β,

$$\langle \omega_1, \beta^\vee \rangle = \frac{2(\omega_1, \beta)}{(\beta, \beta)} = \frac{2(-2/3, 1/3, 1/3) \cdot (0, -1, 1)}{(0, -1, 1) \cdot (0, -1, 1)} = \frac{2(0)}{2} = 0,$$

$$\langle \omega_2, \beta^\vee \rangle = \frac{2(\omega_2, \beta)}{(\beta, \beta)} = \frac{2(-1/3, -1/3, 2/3) \cdot (0, -1, 1)}{(0, -1, 1) \cdot (0, -1, 1)} = \frac{2(1)}{2} = 1.$$

For $\alpha + \beta$,

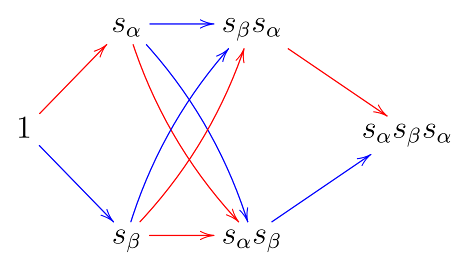
$$\langle \omega_1, (\alpha + \beta)^\vee \rangle = \frac{2(\omega_1, \alpha + \beta)}{(\alpha + \beta, \alpha + \beta)} = \frac{2(-2/3, 1/3, 1/3) \cdot (-1, 0, 1)}{(-1, 0, 1) \cdot (-1, 0, 1)} = \frac{2(1)}{2} = 1,$$

$$\langle \omega_2, (\alpha+\beta)^\vee \rangle = \frac{2(\omega_2, \alpha + \beta)}{(\alpha + \beta, \alpha + \beta)} = \frac{2(-1/3, -1/3, 2/3) \cdot (-1, 0, 1)}{(-1, 0, 1) \cdot (-1, 0, 1)} = \frac{2(1)}{2} = 1.$$

Now that all the necessary computations are done, we can draw the Pieri graph by starting by drawing the same vertices as for the Cayley graph and then draw the number of arrows corresponding to the formula.

3.4.2 Remark. Like for the Cayley graph, we will draw arrows of different colors depending on which fundamental weight is used in the formula, red arrows for ω_1 and blue arrows for ω_2.

Here is the Pieri graph for A_2:



For the first time, the Cayley and the Pieri graphs have a different number of arrows. Unlike the Cayley graph, the Pieri graph not only gives us a visual way to express the elements of $W(A_2)$ but also gives us additional information via the new colored arrows. In Chapter 7, we will find a way to use the new information.

Now that we managed to draw the Pieri graph of A_2, let us do the same thing for another root system.

3.5 The Root System B_2

For our third example, we will work with the root system B_2. We will see that everything will be a little more complex than in the case of A_2.

We can write a base for the root system B_2 as the set $\Delta = \{\alpha, \beta\}$, where the angle between these two roots is $3\pi/4$. One can realize B_2 in the 2-dimensional real vector space by choosing $\alpha = (1, 0)$ and $\beta = (-1, 1)$ as its base.

We can still recover all the roots by starting to apply simple reflections to the simple roots. Since we know from the table in Section 2.1 that

$$\langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \alpha^\vee \rangle = -2,$$

and

$$\langle \alpha, \alpha^\vee \rangle \ = \ \langle \beta, \beta^\vee \rangle = 2,$$

we can do the following calculations:

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta + 2\alpha,$$

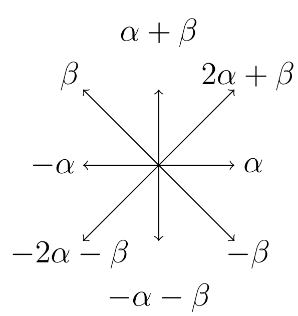
$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha + \beta,$$

$$\begin{aligned} s_\beta(2\alpha + \beta) = 2\alpha + \beta - \langle 2\alpha + \beta, \beta^\vee \rangle \beta &= 2\alpha + \beta - 2\langle \alpha, \beta^\vee \rangle \beta - \langle \beta, \beta^\vee \rangle \beta \\ &= 2\alpha + \beta + 2\beta - 2\beta \\ &= 2\alpha + \beta, \end{aligned}$$

$$\begin{aligned} s_\alpha(\alpha + \beta) = \alpha + \beta - \langle \alpha + \beta, \alpha^\vee \rangle \alpha &= \alpha + \beta - \langle \alpha, \alpha^\vee \rangle \alpha - \langle \beta, \alpha^\vee \rangle \alpha \\ &= \alpha + \beta - 2\alpha + 2\alpha \\ &= \alpha + \beta. \end{aligned}$$

Since we know that applying the same reflection twice is the same thing as applying the identity, we tried all the possibilities. Therefore, $B_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$. Remember that we already drew the picture of the root system B_2 in Section 2.1. We will update the picture with the addition of the missing roots.

B_2:



Using the equalities above, we can write the Cartan matrix of B_2 as:

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

3.6 Weyl Group of B_2

We know that the Weyl group of B_2 is generated by the simple reflections and is also a Coxeter group, so it can be described by a presentation as follows:

$$W(B_2) = \langle s_{\alpha_i} | (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1, 1 \leq i, j \leq 2 \rangle$$

where m_{ij} is the order of $s_{\alpha_i}s_{\alpha_j}$. We know that $s_{\alpha_i}s_{\alpha_i}$ has order 1. Let us find the order of $s_{\alpha_i}s_{\alpha_j}$ for $i \neq j$. We have,

$$s_\beta s_\alpha s_\beta = s_{s_\beta(\alpha)} = s_{\alpha+\beta} = s_{s_\alpha(\alpha+\beta)} = s_\alpha s_{\alpha+\beta} s_\alpha = s_\alpha s_\beta s_\alpha s_\beta s_\alpha.$$

Hence,

$$s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha = 1$$

so $s_\beta s_\alpha$ has order 4. Then, we get that $(s_{\alpha_i}s_{\alpha_j})^l \neq 1$ for $1 < l < 4$. Therefore, we have the following list of elements of the Weyl group according to their length:

Length 0: 1

Length 1: s_α, s_β

Length 2: $s_\alpha s_\beta, s_\beta s_\alpha$

Length 3: $s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta$

Length 4: $s_\alpha s_\beta s_\alpha s_\beta$

We know that by using the simple reflections and the calculation we did to find all the positive roots of B_2, we can find the reflection of all positive roots. Here is the list:

$$s_{\alpha+\beta} = s_{s_\beta(\alpha)} = s_\beta s_\alpha s_\beta^{-1} = s_\beta s_\alpha s_\beta$$

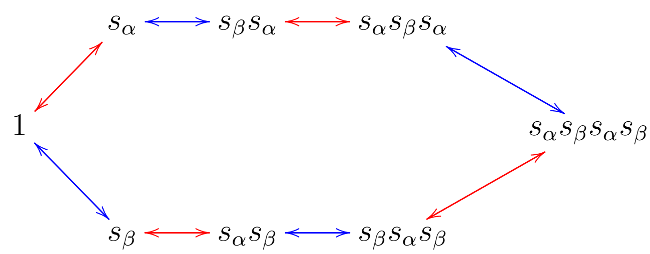
$$s_{2\alpha+\beta} = s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha^{-1} = s_\alpha s_\beta s_\alpha$$

3.7 Cayley Graph and Pieri Graph of B_2

Now that we have all the elements of the Weyl group of B_2, we can draw its Cayley graph.

3.7.1 Remark. We will use the same color of arrows in the Cayley and Pieri graph of B_2 as for those of A_2.

We get the following Cayley graph for $W(B_2)$:



Now, we want to draw the Pieri graph of $W(B_2)$. Before being able to do so, we have to solve the following system of equations to find the fundamental weights of B_2:

$$2\omega_1 - \omega_2 = (1, 0, 0),$$
$$-2\omega_1 + 2\omega_2 = (-1, 1, 0),$$
$$\Rightarrow \omega_1 = (1/2, 1/2, 0) \text{ and } \omega_2 = (0, 1, 0).$$

Now, let us find by which reflection in positive roots, s_λ, we have to multiply each vertex, x, such that $s_\lambda(x) = y$ and $l(y) = l(x) + 1$.

For 1, $s_\alpha \cdot 1 = s_\alpha$ and $s_\beta \cdot 1 = s_\beta$.

For s_α, $s_\beta \cdot s_\alpha = s_\beta s_\alpha$ and $s_{2\alpha+\beta} \cdot s_\alpha = s_\alpha s_\beta s_\alpha s_\alpha = s_\alpha s_\beta$.

For s_β, $s_\alpha \cdot s_\beta = s_\alpha s_\beta$ and $s_{\alpha+\beta} \cdot s_\beta = s_\beta s_\alpha s_\beta s_\beta = s_\beta s_\alpha$.

For $s_\alpha s_\beta$, $s_\beta \cdot (s_\alpha s_\beta) = s_\beta s_\alpha s_\beta$ and $s_{\alpha+\beta} \cdot (s_\alpha s_\beta) = s_\beta s_\alpha s_\beta s_\alpha s_\beta = (s_\beta s_\alpha s_\beta s_\alpha) s_\beta = (s_\alpha s_\beta s_\alpha s_\beta) s_\beta = s_\alpha s_\beta s_\alpha (s_\beta s_\beta) = s_\alpha s_\beta s_\alpha$.

For $s_\beta s_\alpha$, $s_\alpha \cdot (s_\beta s_\alpha) = s_\alpha s_\beta s_\alpha$ and $s_{2\alpha+\beta} \cdot (s_\alpha s_\beta) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha = (s_\alpha s_\beta s_\alpha s_\beta) s_\alpha = (s_\beta s_\alpha s_\beta s_\alpha) s_\alpha = s_\beta s_\alpha s_\beta (s_\alpha s_\alpha) = s_\beta s_\alpha s_\beta$.

For $s_\alpha s_\beta s_\alpha$, $s_\beta \cdot (s_\alpha s_\beta s_\alpha) = s_\beta s_\alpha s_\beta s_\alpha$.

For $s_\beta s_\alpha s_\beta$, $s_\alpha \cdot (s_\beta s_\alpha s_\beta) = s_\alpha s_\beta s_\alpha s_\beta$.

Finally, we will do all the computations needed to find the number of arrows between the vertices in the Pieri graph with the Formula (2.2).

For α,

$$\langle \omega_1, \alpha^\vee \rangle = \frac{2(\omega_1, \alpha)}{(\alpha, \alpha)} = \frac{2(1/2, 1/2, 0) \cdot (1, 0, 0)}{(1, 0, 0) \cdot (1, 0, 0)} = \frac{2(1/2)}{1} = 1,$$

$$\langle \omega_2, \alpha^\vee \rangle = \frac{2(\omega_2, \alpha)}{(\alpha, \alpha)} = \frac{2(0, 1, 0) \cdot (1, 0, 0)}{(1, 0, 0) \cdot (1, 0, 0)} = \frac{2(0)}{1} = 0.$$

For β,

$$\langle \omega_1, \beta^\vee \rangle = \frac{2(\omega_1, \beta)}{(\beta, \beta)} = \frac{2(1/2, 1/2, 0) \cdot (-1, 1, 0)}{(-1, 1, 0) \cdot (-1, 1, 0)} = \frac{2(0)}{2} = 0,$$

$$\langle \omega_2, \beta^\vee \rangle = \frac{2(\omega_2, \beta)}{(\beta, \beta)} = \frac{2(0, 1, 0) \cdot (-1, 1, 0)}{(-1, 1, 0) \cdot (-1, 1, 0)} = \frac{2(1)}{2} = 1.$$

For $\alpha + \beta$,

$$\langle \omega_1, (\alpha+\beta)^\vee \rangle = \frac{2(\omega_1, \alpha+\beta)}{(\alpha+\beta, \alpha+\beta)} = \frac{2(1/2, 1/2, 0) \cdot (0, 1, 0)}{(0, 1, 0) \cdot (0, 1, 0)} = \frac{2(1/2)}{1} = 1,$$

$$\langle \omega_2, (\alpha+\beta)^\vee \rangle = \frac{2(\omega_2, \alpha+\beta)}{(\alpha+\beta, \alpha+\beta)} = \frac{2(0, 1, 0) \cdot (0, 1, 0)}{(0, 1, 0) \cdot (0, 1, 0)} = \frac{2(1)}{1} = 2.$$

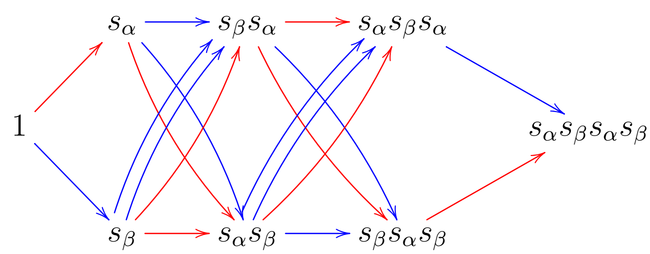
For $2\alpha + \beta$,

$$\langle \omega_1, (2\alpha+\beta)^\vee \rangle = \frac{2(\omega_1, 2\alpha+\beta)}{(2\alpha+\beta, 2\alpha+\beta)} = \frac{2(1/2, 1/2, 0) \cdot (1, 1, 0)}{(1, 1, 0) \cdot (1, 1, 0)} = \frac{2(1)}{2} = 1,$$

$$\langle \omega_2, (2\alpha+\beta)^\vee \rangle = \frac{2(\omega_2, 2\alpha+\beta)}{(2\alpha+\beta, 2\alpha+\beta)} = \frac{2(0, 1, 0) \cdot (1, 1, 0)}{(1, 1, 0) \cdot (1, 1, 0)} = \frac{2(1)}{2} = 1.$$

With all the computations above, we have all the data we need to be able to draw the Pieri graph of $W(B_2)$.

Here is the Pieri graph for B_2:



3.8 The Root System G_2

The root system G_2 is one of the five exceptional root systems. It will be the most complex of our three examples. We can write a base for the root system G_2 as the set $\Delta = \{\alpha, \beta\}$, where the angle between these two roots is $5\pi/6$. One can realize G_2 in the 3-dimensional real vector space by choosing $\alpha = (-1, 1, 0)$ and $\beta = (1, -2, 1)$ as its base.

We will recover the complete root system using the same method as before. Since we know from the table in Section 2.1 that

$$\langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \alpha^\vee \rangle = -3,$$

and

$$\langle \alpha, \alpha^\vee \rangle \quad = \quad \langle \beta, \beta^\vee \rangle = 2,$$

we can do the following calculations:

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta + 3\alpha,$$

$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha + \beta,$$

$$\begin{aligned} s_\alpha(\alpha + \beta) = \alpha + \beta - \langle \alpha + \beta, \alpha^\vee \rangle \alpha \quad &= \quad \alpha + \beta - \langle \alpha, \alpha^\vee \rangle \alpha - \langle \beta, \alpha^\vee \rangle \alpha \\ &= \quad \alpha + \beta - 2\alpha + 3\alpha \\ &= \quad 2\alpha + \beta, \end{aligned}$$

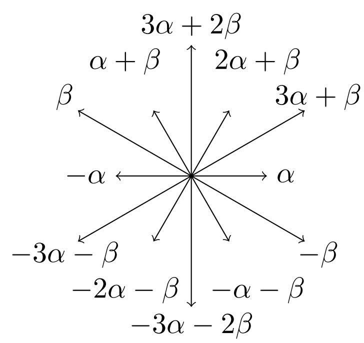
$$\begin{aligned} s_\beta(3\alpha+\beta) = 3\alpha+\beta-\langle 3\alpha+\beta,\beta^\vee\rangle\beta &= 3\alpha+\beta-3\langle\alpha,\beta^\vee\rangle\beta-\langle\beta,\beta^\vee\rangle\beta \\ &= 3\alpha+\beta+3\beta-2\beta \\ &= 3\alpha+2\beta, \end{aligned}$$

$$\begin{aligned} s_\alpha(3\alpha+2\beta) = 3\alpha+2\beta-\langle 3\alpha+2\beta,\alpha^\vee\rangle\alpha &= 3\alpha+2\beta-3\langle\alpha,\alpha^\vee\rangle\alpha-2\langle\beta,\alpha^\vee\rangle\alpha \\ &= 3\alpha+\beta-6\alpha+6\alpha \\ &= 3\alpha+2\beta, \end{aligned}$$

$$\begin{aligned} s_\beta(2\alpha+\beta) = 2\alpha+\beta-\langle 2\alpha+\beta,\beta^\vee\rangle\beta &= 2\alpha+\beta-2\langle\alpha,\beta^\vee\rangle\beta-\langle\beta,\beta^\vee\rangle\beta \\ &= 2\alpha+\beta+2\beta-2\beta \\ &= 2\alpha+\beta. \end{aligned}$$

Therefore, $G_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$. Let us complete the picture we drew in Section 2.1.

G_2:



Using the equalities above, we can write the Cartan matrix of G_2 as:

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

3.9 Weyl Group of G_2

We know that the Weyl group of G_2 is generated by the simple reflections. Remember that the Weyl group is a Coxeter group so it can be described by a presentation as follows:

$$W(G_2) = \langle s_{\alpha_i} | (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1, 1 \leq i, j \leq 2\rangle$$

where m_{ij} is the order of $s_{\alpha_i}s_{\alpha_j}$. It is clear that $m_{ii}=1$. We also have

$$\begin{aligned} s_\beta s_\alpha s_\beta = s_{s_\beta(\alpha)} = s_{\alpha+\beta} = s_{s_\alpha(2\alpha+\beta)} = s_\alpha s_{2\alpha+\beta} s_\alpha &= s_\alpha s_{s_\beta(2\alpha+\beta)} s_\alpha \\ &= s_\alpha s_\beta s_{2\alpha+\beta} s_\beta s_\alpha \\ &= s_\alpha s_\beta s_{s_\alpha(\alpha+\beta)} s_\beta s_\alpha \\ &= s_\alpha s_\beta s_\alpha s_{\alpha+\beta} s_\alpha s_\beta s_\alpha \\ &= s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha. \end{aligned}$$

Hence, the order of $s_\alpha s_\beta$ is equal to 6. This means that for $i \neq j$, we get that $(s_{\alpha_i}s_{\alpha_j})^l \neq 1$ for $1 < l < 6$. Therefore, we have the following list of elements of the Weyl group according to their length:

Length 0: 1

Length 1: s_α, s_β

Length 2: $s_\alpha s_\beta, s_\beta s_\alpha$

Length 3: $s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta$

Length 4: $s_\alpha s_\beta s_\alpha s_\beta, s_\beta s_\alpha s_\beta s_\alpha$

Length 5: $s_\alpha s_\beta s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta s_\alpha s_\beta$

Length 6: $s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$

We can also write the reflection of the positive roots as products of simple reflections. Here is the list:

$$s_{\alpha+\beta} = s_{s_\beta(\alpha)} = s_\beta s_\alpha s_\beta^{-1} = s_\beta s_\alpha s_\beta,$$

$$s_{3\alpha+\beta} = s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha^{-1} = s_\alpha s_\beta s_\alpha,$$

$$s_{3\alpha+2\beta} = s_{s_\beta s_\alpha(\beta)} = s_\beta s_\alpha s_\beta (s_\beta s_\alpha)^{-1} = s_\beta s_\alpha s_\beta s_\alpha^{-1} s_\beta^{-1} = s_\beta s_\alpha s_\beta s_\alpha s_\beta,$$

$$s_{2\alpha+\beta} = s_{s_\alpha s_\beta(\alpha)} = s_\alpha s_\beta s_\alpha (s_\alpha s_\beta)^{-1} = s_\alpha s_\beta s_\alpha s_\beta^{-1} s_\alpha^{-1} = s_\alpha s_\beta s_\alpha s_\beta s_\alpha.$$

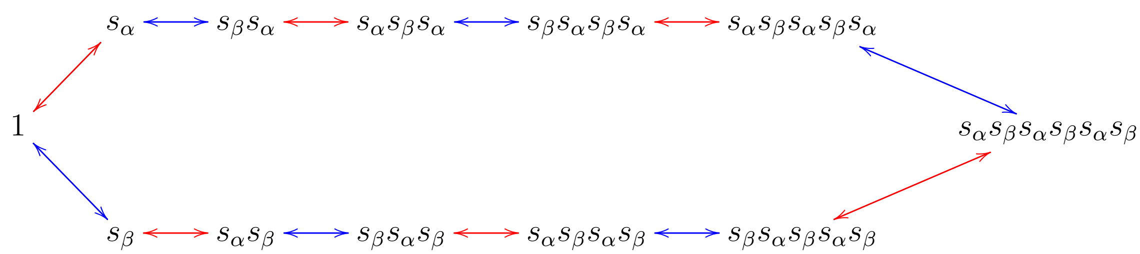
Furthermore, since the dihedral group of order 12, D_6, has the same presentation as the Weyl group of G_2 [DF, p. 26], they are isomorphic [DF, p. 39].

3.10 Cayley Graph and Pieri Graph of G_2

Now, the final part of this chapter is to draw the two graphs of $W(G_2)$. We will begin by drawing its Cayley graph.

3.10.1 Remark. Since we still only have two elements in the base of G_2, we can use the same colors as for A_2 and B_2 to color the arrows in both graphs.

We get the following Cayley graph for $W(G_2)$:



The last graph we need to draw is the Pieri graph of $W(G_2)$. We can solve the following system of equations to find the fundamental weights of G_2:

$$2\omega_1 - \omega_2 = (-1, 1, 0)$$
$$-3\omega_1 + 2\omega_2 = (1, -2, 1)$$
$$\Rightarrow \omega_1 = (-1, 0, 1) \text{ and } \omega_2 = (-1, -1, 2)$$

Now, we need to check by which reflection in positive roots, s_λ, we have to multiply each vertex, x, such that $s_\lambda(x) = y$ and $l(y) = l(x) + 1$.

For 1, $s_\alpha \cdot 1 = s_\alpha$ and $s_\beta \cdot 1 = s_\beta$.

For s_α, $s_\beta \cdot s_\alpha = s_\beta s_\alpha$ and $s_{3\alpha+\beta} \cdot s_\alpha = s_\alpha s_\beta s_\alpha s_\alpha = s_\alpha s_\beta$.

For s_β, $s_\alpha \cdot s_\beta = s_\alpha s_\beta$ and $s_{\alpha+\beta} \cdot s_\beta = s_\beta s_\alpha s_\beta s_\beta = s_\beta s_\alpha$.

For $s_\alpha s_\beta$, $s_\beta \cdot (s_\alpha s_\beta) = s_\beta s_\alpha s_\beta$ and $s_{2\alpha+\beta} \cdot (s_\alpha s_\beta) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\alpha s_\beta = s_\alpha s_\beta s_\alpha s_\beta s_\beta = s_\alpha s_\beta s_\alpha$.

For $s_\beta s_\alpha$, $s_\alpha \cdot (s_\beta s_\alpha) = s_\alpha s_\beta s_\alpha$ and $s_{3\alpha+2\beta} \cdot (s_\alpha s_\beta) = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\beta s_\alpha = s_\beta s_\alpha s_\beta s_\alpha s_\alpha = s_\beta s_\alpha s_\beta$.

For $s_\alpha s_\beta s_\alpha$, $s_\beta \cdot (s_\alpha s_\beta s_\alpha) = s_\beta s_\alpha s_\beta s_\alpha$ and $s_{3\alpha+2\beta} \cdot (s_\alpha s_\beta s_\alpha) = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha = (s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha) s_\beta s_\alpha = (s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta) s_\beta s_\alpha = s_\alpha s_\beta s_\alpha s_\beta s_\alpha (s_\beta s_\beta) s_\alpha = s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\alpha = s_\alpha s_\beta s_\alpha s_\beta$.

For $s_\beta s_\alpha s_\beta$, $s_\alpha \cdot (s_\beta s_\alpha s_\beta) = s_\alpha s_\beta s_\alpha s_\beta$ and $s_{2\alpha+\beta} \cdot (s_\beta s_\alpha s_\beta) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta = (s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta) s_\alpha s_\beta = (s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha) s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha s_\beta (s_\alpha s_\alpha) s_\beta = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\beta = s_\beta s_\alpha s_\beta s_\alpha$.

For $s_\alpha s_\beta s_\alpha s_\beta$, $s_\beta \cdot (s_\alpha s_\beta s_\alpha s_\beta) = s_\beta s_\alpha s_\beta s_\alpha s_\beta$ and $s_{\alpha+\beta} \cdot s_\alpha s_\beta s_\alpha s_\beta) = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta = (s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha) s_\beta = (s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta) s_\beta = s_\alpha s_\beta s_\alpha s_\beta s_\alpha (s_\beta s_\beta) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha$.

For $s_\beta s_\alpha s_\beta s_\alpha$, $s_\alpha \cdot (s_\beta s_\alpha s_\beta s_\alpha) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha$ and $s_{3\alpha+\beta} \cdot (s_\beta s_\alpha s_\beta s_\alpha) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha = (s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta) s_\alpha = (s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha) s_\alpha = s_\beta s_\alpha s_\beta s_\alpha s_\beta (s_\alpha s_\alpha) = s_\beta s_\alpha s_\beta s_\alpha s_\beta$.

The last step is to compute the formula (2.2) for $i = 1, 2$ and every positive root in G_2.

For α,

$$\langle \omega_1, \alpha^\vee \rangle = \frac{2(\omega_1, \alpha)}{(\alpha, \alpha)} = \frac{2(-1,0,1) \cdot (-1,1,0)}{(-1,1,0) \cdot (-1,1,0)} = \frac{2(1)}{2} = 1,$$

$$\langle \omega_2, \alpha^\vee \rangle = \frac{2(\omega_2, \alpha)}{(\alpha, \alpha)} = \frac{2(-1,-1,2) \cdot (-1,1,0)}{(-1,1,0) \cdot (-1,1,0)} = \frac{2(0)}{2} = 0.$$

For β,

$$\langle \omega_1, \beta^\vee \rangle = \frac{2(\omega_1, \beta)}{(\beta, \beta)} = \frac{2(-1,0,1) \cdot (1,-2,1)}{(1,-2,1) \cdot (1,-2,1)} = \frac{2(0)}{6} = 0,$$

$$\langle \omega_2, \beta^\vee \rangle = \frac{2(\omega_2, \beta)}{(\beta, \beta)} = \frac{2(-1,-1,2) \cdot (1,-2,1)}{(1,-2,1) \cdot (1,-2,1)} = \frac{2(3)}{6} = 1.$$

For $\alpha + \beta$,

$$\langle \omega_1, (\alpha+\beta)^\vee \rangle = \frac{2(\omega_1, \alpha+\beta)}{(\alpha+\beta, \alpha+\beta)} = \frac{2(-1,0,1) \cdot (0,-1,1)}{(0,-1,1) \cdot (0,-1,1)} = \frac{2(1)}{2} = 1,$$

$$\langle \omega_2, (\alpha+\beta)^\vee \rangle = \frac{2(\omega_2, \alpha+\beta)}{(\alpha+\beta, \alpha+\beta)} = \frac{2(-1,-1,2) \cdot (0,-1,1)}{(0,-1,1) \cdot (0,-1,1)} = \frac{2(3)}{2} = 3.$$

For $2\alpha+\beta$,

$$\langle\omega_1,(2\alpha+\beta)^\vee\rangle = \frac{2(\omega_1,2\alpha+\beta)}{(2\alpha+\beta,2\alpha+\beta)} = \frac{2(-1,0,1)\cdot(-1,0,1)}{(-1,0,1)\cdot(-1,0,1)} = \frac{2(2)}{2} = 2,$$

$$\langle\omega_2,(2\alpha+\beta)^\vee\rangle = \frac{2(\omega_2,2\alpha+\beta)}{(2\alpha+\beta,2\alpha+\beta)} = \frac{2(-1,-1,2)\cdot(-1,0,1)}{(-1,0,1)\cdot(-1,0,1)} = \frac{2(3)}{2} = 3.$$

For $3\alpha+\beta$,

$$\langle\omega_1,(3\alpha+\beta)^\vee\rangle = \frac{2(\omega_1,3\alpha+\beta)}{(3\alpha+\beta,3\alpha+\beta)} = \frac{2(-1,0,1)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)} = \frac{2(3)}{6} = 1,$$

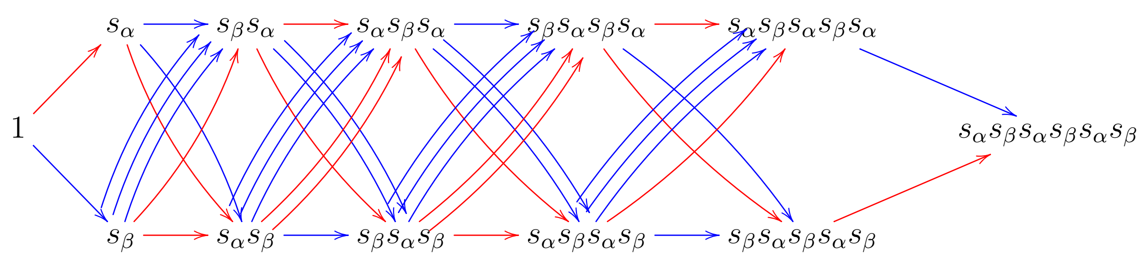
$$\langle\omega_2,(3\alpha+\beta)^\vee\rangle = \frac{2(\omega_2,3\alpha+\beta)}{(3\alpha+\beta,3\alpha+\beta)} = \frac{2(-1,-1,2)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)} = \frac{2(3)}{6} = 1.$$

For $3\alpha+2\beta$,

$$\langle\omega_1,(3\alpha+2\beta)^\vee\rangle = \frac{2(\omega_1,3\alpha+2\beta)}{(3\alpha+2\beta,3\alpha+2\beta)} = \frac{2(-1,0,1)\cdot(-1,-1,2)}{(-1,-1,2)\cdot(-1,-1,2)} = \frac{2(3)}{6} = 1,$$

$$\langle\omega_2,(3\alpha+2\beta)^\vee\rangle = \frac{2(\omega_2,3\alpha+2\beta)}{(3\alpha+2\beta,3\alpha+2\beta)} = \frac{2(-1,-1,2)\cdot(-1,-1,2)}{(-1,-1,2)\cdot(-1,-1,2)} = \frac{2(6)}{6} = 2.$$

We have all the information we need to draw the Pieri graph for G_2. Here it is:



Chapter 4

Geometric Description using a Computational Method

4.1 Preliminaries

In this section, we will work over an algebraically closed field $\mathbb{F}$ of characteristic different from 2.

4.1.1 Definition. [DF, p. 195] Let G be a group. For any subset $X \subset G$ we will denote by $\langle X \rangle$ the subgroup of G generated by X. Also, we denote by $[g, h] = g^{-1}h^{-1}gh$ the *commutator* of elements g and h in G. The series

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots$$

where $G^{(i)} = [G^{(i-1)}, G^{(i-1)}] = \langle [g, h] | g, h \in G^{(i-1)} \rangle$ is called the *derived series* of G.

4.1.2 Definition. [DF, p. 195] A group G is *solvable* if and only if $G^{(n)} = 1$ for some $n \geq 0$.

We will now follow the work of [Lam, Ch.1] on bilinear forms while using a slightly different notation.

4.1.3 Definition. Let V be a finite-dimensional vector space over $\mathbb{F}$ and $Q : V \times V \to \mathbb{F}$ be a bilinear form. The bilinear form Q is said to be *symmetric* if $Q(v, w) = Q(w, v)$ for all $v, w \in V$. Also, Q is *non degenerate* if for all $v \in V, v \neq 0$, there exists $w \in W$ such that $Q(v, w) \neq 0$.

We denote by V^* the dual space of a vector space V. A bilinear form $Q : V \times V \to \mathbb{F}$ defines a map $Q^* : V \to V^*$ by $Q^*(w)(v) = Q(v, w)$ for any $w \in V$.

4.1.4 Remark. The bilinear form Q is non degenerate if and only if Q^* is injective. Furthermore, since V and V^* are finite-dimensional vector spaces of the same dimension, this map is injective if and only if it is invertible.

After choosing a basis of V, we can and will identify V with $\mathbb{F}^n$ and the chosen basis with the standard basis $\{e_1, e_2, ..., e_n\}$. Any bilinear form is of the form $Q(v, w) = v^T M w$ for a unique matrix M, namely the matrix with (ij)-entry $Q(e_i, e_j)$

4.1.5 Remark. If V is a vector space with basis $\{v_1, v_2, ..., v_n\}$, then its dual space V^* has a dual basis $\{\phi_1, \phi_2, ..., \phi_n\}$. The element ϕ_j of the dual basis is defined as the unique linear map from V to $\mathbb{F}$ such that

$$\phi_j(v_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

4.1.6 Proposition. *Let $\{v_1, ..., v_n\}$ be a basis of V and $\{\phi_1, ..., \phi_n\}$ be a basis for its dual space. The matrix for Q with respect to $\{v_1, ..., v_n\}$ is the same as the matrix for Q^*, the transformation matrix from $\{v_1, ..., v_n\}$ to $\{\phi_1, ..., \phi_n\}$.*

Proof: Let $M = [Q^*]_{v_1,...,v_n}^{\phi_1,...,\phi_n}$ be the transformation matrix from $\{v_1, ..., v_n\}$ to $\{\phi_1, ..., \phi_n\}$. Then,

$$Q^*(v_j) = \sum_{k=1}^{n} M_{kj}\phi_k$$

and we have $Q(v_i, v_j) = Q^*(v_j)(v_i) = M_{ij}$. □

By Remark 4.1.4, we get the following result.

4.1.7 Corollary. *A bilinear form Q is non degenerate if and only if M is invertible.*

4.1.8 Definition. c.f. [Hu, p. 15] A *flag* F in an n-dimensional vector space V is a sequence of subspaces of V

$$F = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k \subset V),$$

where $\dim(V_i) = d_i$ for $0 < d_1 < \cdots < d_k < n$. If $d_i = i$ and $k = n - 1$, then F is called a *complete flag*. Otherwise, F is called a *partial flag*.

4.1.9 Definition. Let $\{e_1, e_2, ..., e_n\}$ be the standard basis of $\mathbb{F}^n$. The flag

$$F_0 = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V),$$

where $V_i = \text{span}\{e_1, ..., e_i\}$ for $1 \leq i \leq n$, is called the *standard flag.*

4.1.10 Definition. [H, p. 63–64] The *Grassmanian* $Gr(d, n)$ is the set of all d-dimensional linear subspaces of $\mathbb{F}^n$. For an arbitrary vector space V, we may also view the Grassmanian $Gr(d, V)$ as a subvariety of $\mathbb{P}(\bigwedge^d V)$, the projective space of one-dimensional subspaces of $\bigwedge^d V$.

The last definitions needed will be about algebraic groups. Let G be an algebraic group over $\mathbb{F}$ in the sense of [Spr, p. 21].

4.1.11 Definition. [Spr, p. 102] A *Borel subgroup* of G is a closed, connected, solvable subgroup of G, which is maximal for these properties.

4.1.12 Definition. [Spr, p. 102] A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.

From now on, we will work over $\mathbb{C}$ since it is an algebraically closed field of characteristic different from 2 and is needed for the next results. We will now introduce equivalent definitions for the two kind of subgroups defined above. It is a known fact in the theory of Lie algebras that a choice of Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$ determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\}$ and Φ is the corresponding root system [Bou2, p. 76]. Let us begin by introducing the notion of a Borel subalgebra.

4.1.13 Definition. [Bou2, p. 88] Let $\mathfrak{g}$ be a semisimple Lie algebra and let Φ^+ be the corresponding set of positive roots. The Lie subalgebra given by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

is called a *Borel subalgebra.*

One can check that this is a maximal solvable subalgebra of $\mathfrak{g}$ [Bou2, p. 88].

4.1.14 Lemma. [FH, p. 383] *Let G be a Lie group with semisimple Lie algebra $\mathfrak{g}$. The connected subgroup B of G with Lie algebra $\mathfrak{b}$ is a Borel subgroup.*

4.1.15 Lemma. [FH, p. 384] *A subgroup P of G such that the quotient G/P can be realized as the orbit of the action of G on $\mathbb{P}V$ for some representation V of G is a parabolic subgroup.*

The Lie algebra $\mathfrak{p}$ corresponding to the parabolic subgroup contains the Borel subalgebra $\mathfrak{b}$ and is invariant under the action of B on $\mathfrak{g}$. It can be described as

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha$$

for some subset $T(\Sigma)$ of Φ [Bou2, p. 91]. In fact, let Δ be the set of simple roots of Φ and let $\Sigma \subset \Delta$, then $T(\Sigma) = \Phi^+ \cup \Sigma^-$, where Σ^- is a subset of Φ of the roots that can be written as sums of negatives of the roots in Σ. This means that the parabolic subgroups of the simple group G are in one-to-one correspondence with the subsets of Δ, i.e., with the subsets of the nodes in the Dynkin diagram [Spr, p. 147].

We can now begin to look at the geometric description of the homogeneous spaces G/P for the algebraic groups corresponding to the root system of type A_2, B_2, and G_2. We will approach this problem by 3 different methods. We will begin by a purely computational method, that will give us some answers in the case of A_2 and B_2. Then, we will take a look at the Dynkin diagrams of our root systems with the help of Chevalley groups. Finally, for the last method we will be using representation theory. The combination of these 3 methods will give us a description of the homogeneous spaces G/P for all parabolic subgroups of the given algebraic group.

4.2 Root System A_2

The algebraic group corresponding to the root system A_2 is $\mathrm{SL}_3(\mathbb{C}) := \{A \in M_3(\mathbb{C}) | \det(A) = 1\}$ [Bor, p. 254]. We want to find a geometric description of the homogeneous spaces $\mathrm{SL}_3(\mathbb{C})/P$, where P is a parabolic subgroup.

First of all, let us find a Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$ for $n \geq 1$.

4.2.1 Lemma. [Spr, Exercise 2.1.5(4)] *Let* $B = \{A \in \mathrm{SL}_n(\mathbb{C}) | B_{ij} = 0$ *for* $i > j\}$*, the subgroup of upper triangular matrices of* $\mathrm{SL}_n(\mathbb{C})$*. Then* B *is a solvable subgroup of* $\mathrm{SL}_n(\mathbb{C})$*.*

Proof: We need to show that B is a solvable subgroup of $\mathrm{SL}_n(\mathbb{C})$.

Claim: $B^{(k)} = I_n$, the $n \times n$ unit matrix, for some $k \geq 0$.

If we can prove this, then B is solvable by Definition 4.1.2. Let $X, Y \in B$. Then, $X_{ij} = Y_{ij} = 0$ for $i > j$. Let $X^{-1} = \{A_{ij}\}$ and $Y^{-1} = \{C_{ij}\}$ be the inverses of X and Y respectively. Then, since $XX^{-1} = I_n$, we have $X_{ii}A_{ii} = 1$ for any $i = 1, ..., n$. Therefore, we have $A_{ii} = X_{ii}^{-1}$ and

$$(X^{-1}Y^{-1}XY)_{ii} = 1.$$

Thus, the diagonal entries of any element of $B^{(1)}$ are all equal to 1. Now, let us take a look at the entries on the super-diagonal, i.e. entries with $j = i+1$, of the elements of $B^{(1)}$. Let $X, Y \in B^{(1)}$. Then,

$$\begin{aligned} 0 = (XX^{-1})_{i,i+1} &= \sum_{r=1}^{n} X_{ir}A_{r,i+1} \\ &= X_{ii}A_{i,i+1} + X_{i,i+1}A_{i+1,i+1} \\ &= 1 \cdot A_{i,i+1} + X_{i,i+1} \cdot 1 \end{aligned}$$

and we have $A_{i,i+1} = -X_{i,i+1}$ for $i = 1, ..., n-1$. This means that

$$\begin{aligned} (X^{-1}Y^{-1}XY)_{i,i+1} &= (X^{-1}Y^{-1})_{ii}(XY)_{i,i+1} + (X^{-1}Y^{-1})_{i,i+1}(XY)_{i+1,i+1} \\ &= (X_{i,i+1} + Y_{i,i+1}) + (A_{i,i+1} + C_{i,i+1}) \\ &= X_{i,i+1} + Y_{i,i+1} - X_{i,i+1} - Y_{i,i+1} = 0 \end{aligned}$$

Therefore, for any $X \in B^{(2)}$, the diagonal entries are all equal to 1 and all the entries on the super-diagonal are equal to 0.

By induction, suppose the elements of $B^{(k+1)}$ have the entries from $(i, i+1)$ to $(i, i+k)$ all equal to zero. Let us show that for the elements of $B^{(k+2)}$ we also have that all the entries $(i, i+k+1)$ are equal to zero.

Let $X \in B^{(k+1)}$. Then,

$$\begin{aligned} 0 = (XX^{-1})_{i,i+k+1} &= \sum_{r=1}^{n} X_{ir} A_{r,i+k+1} \\ &= X_{ii} A_{i,i+k+1} + X_{i,i+k+1} A_{i+k+1,i+k+1} \\ &= 1 \cdot A_{i,i+k+1} + X_{i,i+k+1} \cdot 1 \end{aligned}$$

and we have $A_{i,i+k+1} = -X_{i,i+k+1}$. Therefore, $(X^{-1}Y^{-1}XY)_{i,i+k+1} = 0$ for all $X, Y \in B^{(k+1)}$.

We can conclude that $B^{(n)} = [B^{(n-1)}, B^{(n-1)}] = I_n$, hence B is solvable. $\square$

Let us find the geometric description of $\mathrm{SL}_n(\mathbb{C})/B$.

Let $\mathcal{F}$ be the set of all complete flags in $V = \mathbb{C}^n$. We have a natural action of $\mathrm{SL}_n(\mathbb{C})$ on $\mathcal{F}$ defined as:

$$A \cdot F = (\{0\} = A(V_0) \subset A(V_1) \subset \cdots \subset A(V_n) = \mathbb{C}^n),$$

where $A \in \mathrm{SL}_n(\mathbb{C})$ and A acts on V_i by matrix multiplication.

4.2.2 Lemma. *The action of* $\mathrm{SL}_n(\mathbb{C})$ *on the set of complete flags is transitive.*

Proof: Let

$$F_1 = (\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n)$$

and

$$F_2 = (\{0\} \subset U_1 \subset U_2 \subset \cdots \subset U_n)$$

be two complete flags in $\mathbb{C}^n$. We need to find $A \in \mathrm{SL}_n(\mathbb{C})$ such that $A \cdot F_1 = F_2$. Let $\{v_1, ..., v_n\}$ be an ordered basis of V_n such that for every $i = 1, ..., n$, $\{v_1, ..., v_i\}$ is a basis of V_i. Let $\{u_1, ..., u_n\}$ be a basis of U_n such that for every $i = 1, ..., n$, $\{u_1, ..., u_i\}$ is a basis of U_i. Then, it is enough to find $A \in \mathrm{SL}_n(\mathbb{C})$ such that $Av_i = ku_i$ for every $i = 1, ..., n$ and $k \in \mathbb{C}$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{C}$, if we find $C \in \mathrm{SL}_n(\mathbb{C})$ such that $Ce_i = cu_i$ for $c \in \mathbb{C}\backslash\{0\}$, similarly we can also find $D \in \mathrm{SL}_n(\mathbb{C})$ such that $De_i = dv_i$ for $d \in \mathbb{C}$ and we will have

$$u_i = \frac{1}{c}(Ce_i) = \frac{d}{c}(CD^{-1}v_i).$$

Therefore $A = CD^{-1}$ fulfills our requirement. Let E be the $n \times n$ matrix given by $E = (u_1, u_2, ..., u_n)$. Then, $Ee_i = u_i$ and we know that $\det(E) \neq 0$ since $\{u_1, u_2, ..., u_n\}$ is a basis of $\mathbb{C}^n$, hence the u_i's are linearly independent. The determinant of E may not be equal to 1. Suppose $\det(E) = a \neq 1$. Then, if we take $C = (u_1, u_2, ..., u_n/a)$ we have $\det(C) = 1$. Similarly we can define $D = (v_1, v_2, ..., v_n/b)$ where $b = \det((v_1, v_2, ..., v_n))$ and we get $A = CD^{-1} \in \mathrm{SL}_n(\mathbb{C})$. □

It is clear that for any $V_i = \mathrm{span}\{e_1, ..., e_i\}$, $B \cdot V_i = V_i$. Hence, B is contained in the isotropy group of F_0. Let G_{F_0} be the isotropy group of F_0. Suppose $X \in G_{F_0}$ such that $X \notin B$. Then, $X_{ij} \neq 0$ for $i > j$. Then, $X \cdot V_j \subseteq V_i$ which contradicts the fact that G_{F_0} is the isotropy group of F_0. Therefore, $B = G_{F_0}$ and since $\mathrm{SL}_n(\mathbb{C})$ acts transitively on the complete flags, we have an isomorphism between $\mathcal{F}$ and $\mathrm{SL}_n(\mathbb{C})/B$. Furthermore, since $\mathcal{F}$ is complete [Bor, p. 136], then $\mathrm{SL}_n(\mathbb{C})/B$ is also complete.

4.2.3 Proposition. [Spr, p. 101] *A closed subgroup P of G is a parabolic subgroup if and only if G/P is complete.*

Therefore, B is a solvable parabolic subgroup of $\mathrm{SL}_n(\mathbb{C})$. If B is a parabolic subgroup, then by definition, it contains a Borel subgroup B'. Also, B' is a maximal solvable subgroup and B is solvable. Hence, $B = B'$. Therefore, we found the geometric description of the homogeneous space $\mathrm{SL}_n(\mathbb{C})/B$ where B is a Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$.

Now that we know how to describe $\mathrm{SL}_3(\mathbb{C})/B$, let us find the other parabolic subgroups for A_2. Since there are only two simple roots in A_2, we know from the description of parabolic subgroups at the end of Section 4.1 that there are only two proper parabolic subgroups. For example, we have a parabolic subalgebra for the set of roots $T(\Sigma) = \{\alpha, -\alpha, \beta, \alpha + \beta\}$ so it has dimension 6 (number or roots in $T(\Sigma)$ plus two for the dimension of the Cartan subalgebra). We also know that the parabolic subgroups must contain the Borel subgroup B. Then, if we look at the matrices that are upper triangular and have at least one non-zero entry under the diagonal, we get the two following subgroups.

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}) \right\},$$

and

$$P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}) \right\}.$$

Now that we know that P_1 and P_2 are maximal parabolic subgroups of $\mathrm{SL}_3(\mathbb{C})$, let us find a way to describe $\mathrm{SL}_3(\mathbb{C})/P_1$ and $\mathrm{SL}_3(\mathbb{C})/P_2$. We can define an action of $\mathrm{SL}_3(\mathbb{C})$ on $Gr(1,3)$ as follows:

$$\begin{array}{rcl} \mathrm{SL}_3(\mathbb{C}) \times Gr(1,3) & \to & Gr(1,3) \\ (A, \mathrm{span}\{x\}) & \mapsto & \mathrm{span}\{A \cdot x\} \end{array}$$

where $\cdot$ is the usual matrix multiplication.

Let us pick a nonzero element $y = (y_1, y_2, y_2)^T \in \mathbb{C}^3$. If $y_1 \neq 0$, there exist $A = (y, e_2, e_3/y_1)$ such that $A \in \mathrm{SL}_3(\mathbb{C})$ and $A \cdot e_1 \in \mathrm{span}\{y\}$, where $\{e_1, ..., e_3\}$ is the standard basis of $\mathbb{C}^3$. If $y_1 = 0$, either $y_2 \neq 0$ or $y_3 \neq 0$. If $y_2 \neq 0$, $A = (y, e_3, e_1/y_2)$ and $A \cdot e_1 \in \mathrm{span}\{y\}$. If $y_3 \neq 0$, $A = (y, e_1, e_2/y_3)$ and $A \cdot e_1 \in \mathrm{span}\{y\}$. Therefore, $\mathrm{SL}_3(\mathbb{C}) \cdot e_1 = Gr(1,3)$ is the unique $\mathrm{SL}_3(\mathbb{C})$-orbit of the action of $\mathrm{SL}_3(\mathbb{C})$ on $Gr(1,3)$. For $p \in P_1$, we have $\mathrm{span}\{p \cdot e_1\} = \mathrm{span}\{e_1\}$. Therefore, by the same reasoning as for B, P_1 is the isotropy group of the action of $\mathrm{SL}_3(\mathbb{C})$ on $Gr(1,3)$, hence $\mathrm{SL}_3(\mathbb{C})/P_1 \simeq Gr(1,3)$.

Similarly, we can show that $G/P_2 \simeq Gr(2,3)$.

By definition of P_1 and P_2, we see that

$$\dim(\mathrm{SL}_3(\mathbb{C})/P_1) = \dim(\mathrm{SL}_3(\mathbb{C})/P_2) = 2$$

4.2.4 Lemma. *Up to isomorphism, there is only one maximal parabolic subgroup of* $\mathrm{SL}_3(\mathbb{C})$.

Proof: Let

$$\begin{array}{rcl} \phi : P_1 & \to & U \subseteq \mathrm{SL}_3(\mathbb{C}) \\ A & \mapsto & (A^{-1})^T \end{array}$$

This is clearly an isomorphism from P_1 to the subgroup of matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}.$$

By matrix multiplication, we see that $\phi(A)(\mathbb{C}e_2+\mathbb{C}e_3) \subseteq (\mathbb{C}e_2+\mathbb{C}e_3)$. Then, the elements of U leave $\mathbb{C}e_2+\mathbb{C}e_3$ invariant. We also know that the elements of P_2 leave $\mathbb{C}e_1+\mathbb{C}e_2$ invariant. Then, if we can find a matrix $V \in \mathrm{SL}_3(\mathbb{C})$ sending $\mathbb{C}e_2+\mathbb{C}e_3$ to $\mathbb{C}e_1+\mathbb{C}e_2$, we are done. Let

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then, for $x, y \in \mathbb{C}$, we have

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}.$$

Therefore, V sends $\mathbb{C}e_2+\mathbb{C}e_3$ to $\mathbb{C}e_1+\mathbb{C}e_2$ and we have an isomorphism

$$\begin{aligned} \psi : P_1 &\to P_2, \\ A &\mapsto V(A^{-1})^T V^{-1}. \quad \square \end{aligned}$$

Therefore, since $\psi \in \mathrm{Aut}(\mathrm{SL}_3(\mathbb{C}))$ and $\psi(P_1) = P_2$, $\mathrm{SL}_3(\mathbb{C})/P_1 \simeq \mathrm{SL}_3(\mathbb{C})/P_2$. Also, since $Gr(1,3)$ is the projective variety of all lines in $\mathbb{C}^3$, it is isomorphic to the projective space $\mathbb{P}^2$. Then, we have the following picture to describe the geometry of $\mathrm{SL}_3(\mathbb{C})/P_i$ for $i = 1, 2$:

$$\begin{array}{ccccc} & & \mathrm{SL}_3(\mathbb{C})/P_1 & & \mathrm{SL}_3(\mathbb{C})/P_2 \\ & & \wr| & & \wr| \\ \mathbb{P}^2 & \simeq & Gr(1,3) & \simeq & Gr(2,3) \end{array}$$

4.3 Root System B_2

The algebraic group corresponding to the root system B_2 [Bor, p. 257] is the special orthogonal group

$$\mathrm{SO}_n(\mathbb{C}) = \{A \in \mathrm{SL}_n(\mathbb{C}) | A^T A = I_n\}$$

for $n = 5$.

We can also define this group as the subgroup of $\mathrm{SL}_n(\mathbb{C})$ preserving a bilinear form

$$Q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

defined as

$$Q(v, w) = v^T M w$$

for $v, w \in \mathbb{C}^5$ and $M = I_5$. The two definitions are equivalent since we have

$$\begin{aligned} Q(Av, Aw) &= Q(v, w), \\ \Leftrightarrow (Av)^T M (Aw) &= v^T M w, \\ \Leftrightarrow v^T A^T M A w &= v^T M w, \\ \Leftrightarrow A^T M A &= M. \end{aligned}$$

Furthermore, Q is a non degenerate symmetric bilinear form. It is symmetric since we have

$$Q(v, w) = v^T M w = v^T w = w^T v = Q(w, v)$$

and non degenerate since M is invertible.

Before trying to find the parabolic subgroups of $\mathrm{SO}_5(\mathbb{C})$, let us introduce some results that will allow us to change to another non degenerate symmetric bilinear form.

4.3.1 Definition. [Lam, p. 2] We say that two bilinear forms Q and Q' are in the same *equivalence class* if for M and M' being the representation in the standard basis of $\mathbb{C}^n$ of Q and Q' respectively, we have $M' = C^T M C$ for $C \in \mathrm{GL}_n(\mathbb{C})$.

4.3.2 Theorem. [Lg, p. 575] *Let E be a nonzero finite dimensional vector space over $\mathbb{C}$. Let Q be a non degenerate symmetric bilinear form on E. Then there exists an orthogonal basis of E.*

This means that any non degenerate symmetric bilinear form is diagonalizable. Then, since in an algebraically closed field of characteristic different from 2 we can multiply the diagonal entries by any square and stay in the same equivalence class, we get the following result.

4.3.3 Corollary. *If Q and Q' are quadratic forms on $\mathbb{C}^n$, represented with respect to the standard basis of $\mathbb{C}^n$ by matrices M and M', then there exists an invertible matrix C such that $C^T M C = M'$.*

Let us look at what happens to the special orthogonal group when we change the bilinear form. Let

$$\mathrm{SO}_n(M, \mathbb{C}) = \{A \in \mathrm{SL}_n(\mathbb{C}) | A^T M A = M\}$$

where M is an invertible symmetric matrix. Then, we have the following result.

4.3.4 Lemma. *The algebraic group* $\mathrm{SO}_n(I, \mathbb{C})$ *is isomorphic to* $\mathrm{SO}_n(M, \mathbb{C})$.

Proof: From 4.3.3, there exist an invertible matrix C such that $I = C^T M C$. Hence, we also have $(C^{-1})^T C = M$. Let

$$\begin{array}{rcl} \psi : \mathrm{SO}_n(I, \mathbb{C}) & \to & \mathrm{SO}_n(M, \mathbb{C}), \\ A & \mapsto & CAC^{-1}. \end{array}$$

This map is an homomorphism of algebraic groups from $\mathrm{SO}_n(I, \mathbb{C})$ to $\mathrm{SO}_n(M, \mathbb{C})$ since

$$\psi(A)\psi(B) = (CAC^{-1})(CBC^{-1}) = CABC^{-1} = \psi(AB)$$

and

$$\begin{array}{rcl} \psi(A)^T M \psi(A) = (CAC^{-1})^T M (CAC^{-1}) & = & (C^{-1})^T A^T C^T M C A C^{-1} \\ & = & (C^{-1})^T A^T A C^{-1} \\ & = & (C^{-1})^T C^{-1} = M. \end{array}$$

In fact, it is an isomorphism since we have the following inverse map:

$$\begin{array}{rcl} \psi^{-1} : \mathrm{SO}_n(M, \mathbb{C}) & \to & \mathrm{SO}_n(I, \mathbb{C}), \\ A & \mapsto & C^{-1}AC. \quad \square \end{array}$$

These results allow us to change the matrix M in the definition of Q to any matrix such that Q is a non degenerate symmetric bilinear form. From now on, we will use the following matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can now replace the old definition of $\mathrm{SO}_5(\mathbb{C})$ by the new one and view $\mathrm{SO}_5(\mathbb{C})$ as the subgroup of $\mathrm{SL}_5(\mathbb{C})$ such that any element $A \in \mathrm{SO}_5(\mathbb{C})$ satisfies the property that $A^T M A = M$. We do this since in the new realization the description of Borel and parabolic subgroups becomes easier. We are now interested in the homogeneous spaces $\mathrm{SO}_5(\mathbb{C})/P$.

In the case of $\mathrm{SO}_5(\mathbb{C})$, we will not use straightforward computations as we did for $\mathrm{SL}_3(\mathbb{C})$. We will follow the ideas of [Mak] by using some properties of reductive groups.

4.3.5 Definition. [Spr, p. 43] A linear algebraic group T is a *torus* if it is isomorphic to a group of diagonal matrices. A *maximal torus* is a torus which is maximal in the set-theoretical sense.

4.3.6 Definition. [Mak, p. 4] Let T be a maximal torus of an algebraic group G. We call *characters* the elements of $\mathrm{Hom}(T, \mathbb{G}_m)$, where $\mathbb{G}_m$ is the multiplicative group.

4.3.7 Proposition. [Spr, p. 132] *Let G be a connected, reductive, linear algebraic group and T be a maximal torus of G. For a character α there exists an isomorphism u_α of $\mathbb{G}_a$, the additive group, onto a unique closed subgroup U_α of G such that $t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T$ and all $x \in \mathbb{C}$.*

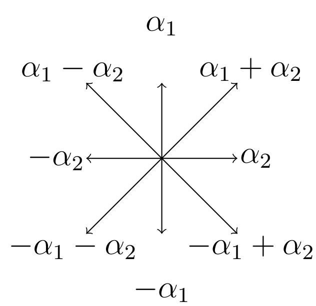
Let us begin by finding the maximal torus T of $\mathrm{SO}_5(\mathbb{C})$. Let t be a diagonal matrix with diagonal entries $t_{ii} = t_i$ for $i = 1, ..., 5$. This matrix must satisfy the condition $tMt = M$ and must have determinant 1 to be in $\mathrm{SO}_5(\mathbb{C})$. We have that

$$tMt = \begin{pmatrix} 0 & 0 & 0 & 0 & t_1t_5 \\ 0 & 0 & 0 & -t_2t_4 & 0 \\ 0 & 0 & t_3^2 & 0 & 0 \\ 0 & -t_4t_2 & 0 & 0 & 0 \\ t_5t_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $t_1t_5 = t_2t_4 = 1$ and $t_3 = \pm 1$. Since the determinant of t must be equal to 1, we have $\det(t) = (t_1t_5)(t_2t_4)t_3 = t_3$ and $t_3 = 1$. Then, we have that

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} | t_1, t_2 \in \mathbb{C} \right\}$$

is a torus of G and it is clear that it is maximal with these properties. Let us take the characters α_i such that for any $t \in T$, we have $\alpha_i(t) = t_i$ and $-\alpha_i(t) = t_i^{-1}$ for $i = 1, 2$. One can show that the set of characters $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\}$ form a root system of type B_2 that has the following root diagram



with the set of positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 - \alpha_2, \alpha_1 + \alpha_2\}$.

4.3.8 Proposition. [Spr, p. 138] *Let Φ^+ be the set of positive roots. Then, T and U_α with $\alpha \in \Phi^+$ generate a Borel subgroup of G.*

This means that if we can find the U_α for $\alpha \in \Phi^+$, we will find a Borel subgroup of G. We want to define the maps $u_\alpha : \mathbb{G}_a \to G$. For example, if for any $x \in \mathbb{C}$ we define

$$u_{\alpha_2}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & \frac{1}{2}x^2 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then we will get that

$$tu_{\alpha_2}(x)t^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_2x & \frac{t_2^2}{2}x^2 & 0 \\ 0 & 0 & 1 & t_2x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = u_{\alpha_2}(t_2x) = u_{\alpha_2}(\alpha_2(t)x),$$

for any $t \in T$. Similarly, we can take

$$u_{\alpha_1}(x) = \begin{pmatrix} 1 & 0 & x & 0 & -\frac{1}{2}x^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, u_{\alpha_1-\alpha_2}(x) = \begin{pmatrix} 1 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$u_{\alpha_1+\alpha_2}(x) = \begin{pmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, since we know that the subgroups U_α are the image of the maps u_α for $\alpha \in \Phi^+$, it is clear that $B = \langle T, U_\alpha | \alpha \in \Phi^+ \rangle$ is the subgroup of upper triangular matrices of $\mathrm{SO}_5(\mathbb{C})$.

We can also find all the other parabolic subgroups just by looking at the subgroups of $\mathrm{SO}_5(\mathbb{C})$ containing B. A maximal parabolic subgroup will be generated by the maximal torus T, the subgroups U_α for $\alpha \in \Phi^+$ and also by U_β where β is the negative of the simple root corresponding to the parabolic subgroup. For example, by looking at the root diagram of the characters, we see that P_1 corresponds to the character α_2. Also, one can check that for

$$u_{-\alpha_2}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & \frac{1}{2}x^2 & x & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

we have $tu_{-\alpha_2}(x)t^{-1} = u_{-\alpha_2}(t_2^{-1}x) = u_{-\alpha_2}(-\alpha_2(t)x)$ for any $t \in T$. Hence,

$$P_1 = \left\{ \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{SO}_5(\mathbb{C}) \right\}.$$

Similarly, we get that P_2 which correspond to the character $\alpha_1 + \alpha_2$ is

$$P_2 = \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \in \mathrm{SO}_5(\mathbb{C}) \right\}.$$

Now we know what are the parabolic subgroups P, we want to find the geometric description of the homogeneous spaces $\mathrm{SO}_5(\mathbb{C})/P$.

4.3.9 Definition. A subset W of V is called *isotropic* if there exist $x \in W$ such that $Q(x, x) = 0$. A subset W of V is called *totally isotropic* if for any $w \in W$, $Q(w, w) = 0$.

From Theorem 4.1 in [Lam, p. 12] we get that any vector space V endowed with a symmetric bilinear form Q splits into an orthogonal sum of subspaces called a *Witt decomposition*. From this decomposition, we get an integer m called the *Witt index* with the property $m = 1/2 \cdot \dim(W)$, where W is a summand of the Witt decomposition of V. Therefore, $m \leq 1/2 \cdot \dim(V)$.

4.3.10 Lemma. [Lam, p. 13] *Let V be a vector space over a field of characteristic different from 2. If Q is a non degenerate symmetric bilinear form, the Witt index m of V equals the dimension of any maximal totally isotropic subspace of V.*

From this lemma, we may deduce that a maximal flag of totally isotropic subspace is as follows

$$\{(0 \subset V_1 \subset V_2 \subset \cdots \subset V_{[n/2]} \subset \mathbb{C}^n | Q(V_i, V_i) = 0\}.$$

Since for any $A \in \mathrm{SO}_5(\mathbb{C})$ we have $Q(A \cdot V_i, A \cdot V_i) = Q(V_i, V_i) = 0$, then $\mathrm{SO}_5(\mathbb{C})$ preserves the totally isotropic flags. We can also show by doing work similar as in Section 4.2 that the Borel subgroup B of $\mathrm{SO}_5(\mathbb{C})$ is the isotropy group of the standard totally isotropic flag. Therefore,

$$\mathrm{SO}_5(\mathbb{C})/B \simeq \{(0 \subset V_1 \subset V_2 \subset \mathbb{C}^5 | Q(V_i, V_i) = 0\}.$$

Similarly, we get that P_1 is the isotropy group of the action of $\mathrm{SO}_5(\mathbb{C})$ on the variety of totally isotropic lines in $\mathbb{C}^5$ and that P_2 is the isotropy group of the action of $\mathrm{SO}_5(\mathbb{C})$ on the variety of totally isotropic planes in $\mathbb{C}^5$. Therefore, we have isomorphisms of varieties

$$\mathrm{SO}_5(\mathbb{C})/P_1 \simeq \text{variety of totally isotropic lines in } \mathbb{C}^5$$

and

$$\mathrm{SO}_5(\mathbb{C})/P_2 \simeq \text{variety of totally isotropic planes in } \mathbb{C}^5.$$

In the next chapter, we will use a new method that will help us understand why we get these results.

Chapter 5

Geometric Description using Chevalley Groups

In this chapter, we will follow the work of [CG] to give a geometric description of the homogeneous spaces G/P.

Let G be a semisimple algebraic group over $\mathbb{C}$. Fix a representation of G on a finite-dimensional vector space V. Let V be a fundamental irreducible representation of G, i.e. the highest weight of V is a fundamental weight. For each vertex α_i of the Dynkin diagram, we choose a nonzero proper subspace V_i of V that is invariant under the maximal parabolic P_i. By [CG, Prop 3.4], a subspace satisfying this property exists.

We will define our algebraic group as a *Chevalley group* constructed from a root system Φ using the definition in [St, p. 21]. Let $x_\alpha : \mathbb{G}_a \to G$ as α ranges over the roots in Φ be an homomorphism subject to some relations described in [St, p. 66]. Let $\mathfrak{X}_\alpha = \{x_\alpha(t) | t \in \mathbb{C}\}$. Then, G is generated by the $\mathfrak{X}_\alpha$ as α ranges over the roots in Φ.

Fix a root $\gamma \in \Delta$, where Δ is a base of the root system Φ, and let ω be the corresponding fundamental weight. Let V be the irreducible representation with highest weight ω and highest weight vector $v^+ \in V$.

5.0.1 Definition. For a simple root $\alpha_i \in \Delta \backslash \{\gamma\}$, we define the α_i-*component* of Δ to be the connected component of γ in $\Delta \setminus \{\alpha_i\}$. In other words, the α_i-component is the subset of Δ containing all the simple roots such that there is a path connecting them to γ in the Dynkin diagram without the vertex corresponding to α_i.

By convention, the γ-component of Δ is the empty set. Now fix $\alpha_i \in \Delta$. For each root $\delta \in \Phi$, write $\delta = \sum_{\alpha_k \in \Delta} c_k \alpha_k$ for integers c_k.

5.0.2 Definition. We define the α_i-*height* of δ to the $\sum c_k$ as α_k ranges over the simple roots not in the α_i-component of Δ.

Write L_{α_i} for the subgroup of G generated by the $\mathfrak{X}_\delta$ as δ ranges over the roots such that the α_i-height of δ equal 0. Then, L_{α_i} is a simple algebraic group whose Dynkin diagram is the α_i-component [CG, p. 6].

We set V_{α_i} to be the subspace $L_{\alpha_i} \cdot v^+$ spanned by the L_{α_i}-orbit of v^+.

5.0.3 Remark. We have that L_γ is trivial and V_γ is the line tv^+.

5.0.4 Proposition. [CG, Prop. 3.4] *For every $\alpha_i \in \Delta$, the subspace V_{α_i} is a nonzero proper subspace of V stabilized by P_{α_i}.*

5.0.5 Proposition. [CG, Prop. 3.5] *For $\alpha_i \in \Delta \setminus \{\gamma\}$, the subspace V_{α_i} is a fundamental irreducible representation of L_{α_i}.*

Since we know the Dynkin diagram of L_{α_i}, we can find the dimension of V_{α_i} by looking at Table 2 in [Bou2, p. 214].

5.0.6 Definition. We say that a vertex in a graph is *terminal* if it is joined to at most one other vertex.

5.0.7 Definition. We call a subspace $X \subseteq V$ in the $G(\mathbb{C})$-orbit of V_{α_i} a *α_i-space.*

5.0.8 Proposition. [CG, Prop. 3.7] *Let X be a α_i-space in V (with $\alpha_i \neq \gamma$). Suppose that the α_i-component of Δ is of type A and γ is a terminal vertex of the α_i-component. Then, every 1-dimensional subspace of X is a γ-space.*

We now have all the tools we need to take a look at the first example.

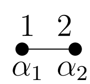
5.1 Type A_2

For our first example we will follow [CG, p. 9] and we will work on the case of A_2. Then, we have $G \simeq \mathrm{SL}_3(\mathbb{C})$ and let us take the standard representation $V = \mathbb{C}^3$ corresponding to the simple root α_1.

Let V_{α_1} be the 1-dimensional subspace of V spanned by the highest weight v^+. We know from Proposition 5.0.5 that V_{α_2} is a fundamental irreducible representation of L_{α_2}. We also know that the Dynkin diagram of L_{α_2} is the α_2-component so it is of type A_1. Hence, from the table 2 in [Bou2, p. 214],

Fundamental weight	A_1	A_2	B_2	G_2
ω_1	2	3	5	7
ω_2	-	3	4	14

we have that $\dim(V_{\alpha_2}) = 2$. Let us draw the Dynkin diagram of A_2 and write the dimension of V_{α_i} above the vertex α_i.



We proved in Chapter 4 that $\mathrm{SL}_3(\mathbb{C})$ acts transitively on the i-dimensional subspaces of $V = \mathbb{C}^3$ for all $i = 1, 2$. Then, the α_i-spaces are precisely the i-dimensional subspaces of $\mathbb{C}^3$. Since P_i is the stabilizer of the V_{α_i}, we get the same geometric description for G/P_i as in Chapter 4.

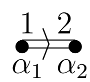
5.2 Type B_2

For the case of B_2, some additional properties will appear. If V_γ satisfies a certain property $\mathcal{P}$ and this property is G-invariant, then by definition, all γ-spaces will satisfy this property. Furthermore, if we can prove that G acts transitively on the d-dimensional spaces of V satisfying $\mathcal{P}$, the γ-spaces are precisely those d-dimensional spaces.

Let us start by taking V to be the standard representation of $SO_5(\mathbb{C})$ and fix the root $\alpha_1 \in \Delta$. We know from Chapter 4 that there exist a G-invariant non degenerate bilinear form Q on V. Let v^+ be the highest weight vector of V with highest weight ω. Then, for any t in the maximal torus T, we have

$$Q(v^+, v^+) = Q(t \cdot v^+, t \cdot v^+) = Q(\omega(t)v^+, \omega(t)v^+) = \omega(t)^2 Q(v^+, v^+).$$

Since ω is not the trivial character, $\omega(t) \neq 0$ so $Q(v^+, v^+) = 0$. Hence, V_{α_1} is isotropic. Moreover, by looking at the Dynkin diagram of B_2, we see that L_{α_2} is of type A_1 so $\dim(V_{\alpha_2}) = 2$ and by Proposition 5.0.5, the α_2-spaces are also isotropic. We have the following Dynkin diagram with the dimensions of V_{α_i}.



Now, since we have from [CG, p. 12] that $\mathrm{SO}_n(\mathbb{C})$ acts transitively on the m-dimensional isotropic spaces of V for $m < n$, we have that

$$G/P_i \simeq \text{ } i\text{-dimensional isotropic spaces of } V.$$

5.3 Type G_2

The case of G_2 will be similar to B_2. We will follow the work of [CG, p. 25] with the help of results from [SV].

5.3.1 Definition. [SV, p. 4] A *composition algebra* C over a field k is a not necessarily associative algebra over k with identity element e such that there exists a non degenerate quadratic form N on C which permits composition, i.e., such that

$$N(xy) = N(x)N(y), \forall x, y \in C.$$

The quadratic form N is called the *norm* of C.

5.3.2 Definition. [SV, p. 14] A composition algebra C of dimension 8 that is not commutative nor associative is called an *octonion algebra*.

5.3.3 Definition. [SV, p. 19] An octonion algebra with isotropic norm is called a *split octonion algebra*.

5.3.4 Example. The two-dimensional split composition algebra over $\mathbb{C}$ is $\mathbb{C} \oplus \mathbb{C}$ with norm defined as $N((x, y)) = xy$ for $x, y \in \mathbb{C}$.

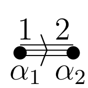
Let C be a split octonion algebra. Let G be the group of automorphisms of C. Then, we have that G is of type G_2 [SV, p. 32]. Let V be the standard representation of G. We can define a G-invariant bilinear product on V by

$$Q(x, y) := N(x + y) - N(x) - N(y).$$

We can take the following basis for the split octonions $\{e, i, j, k, l, li, lj, lk\}$. Then, the multiplication of octonions is completely determined by the following multiplication table [CS, p. 66]:

e	i	j	k	l	li	lj	lk
i	-e	k	-j	-li	l	-lk	lj
j	-k	-e	i	-lj	lk	l	-li
k	j	-i	-e	-lk	-lj	li	l
l	li	lj	lk	e	i	j	k
li	-l	-lk	lj	-i	e	k	-j
lj	lk	-l	-li	-j	-k	e	i
lk	-lj	li	-l	-k	j	-i	e

The vector space V may be viewed as the trace zero elements in the split octonion algebra, i.e. the elements $v \in C$ such that $Q(v, e) = 0$. We can still find the dimension of the spaces with Proposition 5.0.5. We get the following diagram



From the same argument as for B_2, we get that the bilinear product is identically zero on the α_1-spaces, and from Proposition 5.0.8, also on the α_2-spaces. Afterward, we can prove that G acts transitively on the 1-dimensional spaces of V on which the bilinear product is identically zero using the following result.

5.3.5 Proposition. [SV, p. 17] *Let C be a composition algebra and let D and D' be subalgebras of the same dimension. Then, every linear isomorphism from D onto D' can be extended to an automorphism of C.*

Let C be the octonion algebra. Let $D = \langle a \rangle$ and $D' = \langle b \rangle$ where $a, b \in V$ such that the bilinear product is identically zero on D and D'. We want to find an element $g \in G = \text{Aut}(C)$ such that $g(a) = b$. Let us take a linear isomorphism $\mu : D \to D'$ such that $\mu(a) = b$. Then, from the proposition above, μ can be extended to an automorphism μ' of C. Therefore, $g = \mu'$.

Similarly, we can show that G acts transitively on the 2-dimensional spaces of V on which the bilinear product is identically zero using Proposition 5.3.3.

Therefore, we have that G/P_i is isomorphic to the i-dimensional spaces of V on which the bilinear product is identically zero.

Chapter 6

Geometric Description by Representation Theory

In this section, we will approach the problem of giving a geometric description of the homogeneous spaces using representation theory as in [FH].

Let $V = \Gamma_\lambda$ be an irreducible representation of a linear algebraic group G with highest weight λ, and consider the action of G on the projective space $\mathbb{P}V$. Let $p \in \mathbb{P}V$ be the point corresponding to the weightspace with weight λ.

6.0.1 Lemma. [FH, p. 388] *The orbit $G \cdot p$ is the unique closed orbit of the action of G on $\mathbb{P}V$.*

We know from Section 4.1 that the stabilizer P_λ of p is a parabolic subgroup. In fact, P_λ is the parabolic subgroup corresponding to the subset of simple roots that are perpendicular to the weight λ, i.e. $\langle \lambda, \alpha^\vee \rangle = 0$. This means that we are looking for irreducible representations with highest weight perpendicular to some simple roots. Recall that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, so the fundamental weight ω_1 is perpendicular to the simple root α_2 and vice-versa. In fact, we are looking for irreducible representations with highest weight being scalar multiples of the fundamental weights.

We can establish a step-by-step method that will help us find these representations for a given Lie algebra $\mathfrak{g}$.

Step 1: Find a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acting diagonally on $\mathfrak{g}$ by the adjoint representation. A subalgebra $\mathfrak{h}$ with these properties is called a *Cartan subalgebra* of $\mathfrak{g}$.

Step 2: Let $\mathfrak{h}$ act on $\mathfrak{g}$ by the adjoint representation and find a decomposition of $\mathfrak{g}$. Since $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$, we get the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

where $\alpha \in \mathfrak{h}^*$ such that for any $H \in \mathfrak{h}$ and any $X \in \mathfrak{g}_\alpha$ if and only if

$$\mathrm{ad}(H)(X) = [H, X] = \alpha(H)X$$

holds for all $H \in \mathfrak{h}$.

We will call *roots* the non-zero $\alpha \in \mathfrak{h}^*$ with $\mathfrak{g}_\alpha \neq 0$ and *root spaces* the $\mathfrak{g}_\alpha \neq 0$. The set of all roots is denoted $\Phi \subset \mathfrak{h}^*$. One can check that the roots defined this way coincide with the roots found in Chapter 2 and have all the same properties [EW, Example 11.2]. The lattice $\Lambda_\Phi \subset \mathfrak{h}^*$ generated by the roots α is called the *root lattice* and will help us to draw root the diagram. Since we will work with the Lie algebras having root system A_2, B_2 and G_2, we already know what the root diagram should look like. We also get the following result:

6.0.2 Lemma. [Hu, p. 39] *If $\alpha, \beta, \alpha + \beta \in \Phi$, then the adjoint action of $\mathfrak{g}_\alpha$ maps $\mathfrak{g}_\beta$ onto $\mathfrak{g}_{\alpha+\beta}$.*

Step 3: We must fix an ordering on the roots by finding a linear functional

$$\iota : \Lambda_\Phi \to \mathbb{R}$$

such that $\iota(\alpha) \neq 0$ for all $\alpha \in \Phi$. Then, we can define $\Phi^+ = \{\alpha \in \Phi | \iota(\alpha) > 0\}$ and $\Phi^- = \{\alpha \in \Phi | \iota(\alpha) < 0\}$. This gives us the decomposition

$$\Phi = \Phi^+ \cup \Phi^-.$$

We can now define the simple roots as the set of roots which can't be written as a sum of two positive roots and then get the same root diagram as in Section 2.1.

Step 4: Let V be an irreducible finite-dimensional representation of $\mathfrak{g}$. There is a similar direct sum decomposition for V given by

$$V = \bigoplus_{\mu} V_\mu,$$

where $\mu \in \mathfrak{h}^*$ and $V_\mu = \{v \in V | H(v) = \mu(H)v, \forall H \in \mathfrak{h}\}$.

We call *weights* the non-zero $\mu \in \mathfrak{h}^*$ with $V_\mu \neq 0$ and *weight spaces* the non-zero V_μ. The dimension of a weight space V_μ is called the *multiplicity* of the weight μ in V. We get the following result.

6.0.3 Lemma. [FH, p. 165] *The action of $\mathfrak{g}_\alpha$ on V_μ sends it to $V_{\alpha+\mu}$.*

Now, we want to find a lattice for the weights like the one we found for the roots. To do so, we must recall some facts about the representations of $\mathfrak{sl}_2(\mathbb{C})$.

6.0.4 Lemma. [EW, p. 97][Hu, p. 37] *For any root α of a semisimple Lie algebra $\mathfrak{g}$, the subalgebra*

$$s_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

We know that $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_{-\alpha}$ to $\mathfrak{h}$ via the adjoint action so one can show that there exists a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$. This element H_α will be called a *distinguished element* of $\mathfrak{h}$. Since $s_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$, we know that all eigenvalues of the action of H_α on finite-dimensional representations must be integers. Then, for every eigenvalue $\mu \in \mathfrak{h}^*$ of a representation of $\mathfrak{g}$, $\mu(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. We will call *weight lattice* of $\mathfrak{g}$ and denote by Λ_W the set of all linear functional $\mu \in \mathfrak{h}^*$ such that $\mu(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. Then, all weights of all representations of $\mathfrak{g}$ will lie in Λ_W. In particular, $\Lambda_\Phi \subset \Lambda_W$ since the $\lambda \in \Lambda_\Phi$ are weights in the adjoint representation.

Step 5: Let us start with a definition.

6.0.5 Definition. [FH, p. 202] Let V be any representation of $\mathfrak{g}$. A nonzero vector $v \in V$ that is both an eigenvector for the action of $\mathfrak{h}$ and in the kernel of the action of $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi^+$ is called a *highest weight vector* of V. A weight μ such that $H(v) = \mu(H) \cdot v$ for all $H \in \mathfrak{h}$ is called a *highest weight* of that representation.

6.0.6 Lemma. [FH, p. 204] *Any irreducible representation V of $\mathfrak{g}$ is generated by the images of its highest weight vector v under successive applications of root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$.*

6.0.7 Lemma. [FH, p. 204] *Every vertex of the convex hull of the weights of V is conjugate to the highest weight μ under the Weyl group.*

These two results are enough to find all the weights of a representation given that we know the highest weight. We get the convex hull by applications of the Weyl group on the highest weight and then find all the roots lying inside the convex hull by applications of the action of the root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$.

Using the step-by-step method stated above, we will be able to find the weight space decomposition of any irreducible representation. Then, we will find the unique orbit for this representation and this will be isomorphic to G/P_λ. We will denote by $\Gamma_{a,b}$ an irreducible representation with highest weight $\mu = a\omega_1 + b\omega_2$. Thus, we are looking for the representations $\Gamma_{a,0}$ and $\Gamma_{0,b}$.

6.1 Representations of A_2

We have to find a Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. We are looking for a subspace $\mathfrak{h}$ that acts diagonally on $\mathfrak{sl}_3(\mathbb{C})$. In fact, we can take $\mathfrak{h}$ as the 2-dimensional subspace of all diagonal matrices in $\mathfrak{sl}_3(\mathbb{C})$. Now, we want to find the decomposition

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha.$$

Let $M = (m_{ij})$ be a 3×3 matrix and $D = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$. Basic matrix computation give us that the (ij)-th entry of the commutator

$$[D, M]_{ij} = (a_i - a_j)m_{ij}.$$

Then, the eigenvectors of the action of $\mathfrak{h}$ on $\mathfrak{sl}_3(\mathbb{C})$ must be scalar multiples of the 3×3 matrices E_{ij} with (ij)-th entry equal to 1 and all other entries equal to 0. The matrices E_{ij} generate all the eigenspaces of the action of $\mathfrak{h}$

on $\mathfrak{g}$.

We have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} | a_1 + a_2 + a_3 = 0 \right\}$$

and the linear functionals $\alpha \in \mathfrak{h}^*$ appearing in the direct sum decomposition will be the six functionals $L_i - L_j$ for $1 \leq i, j \leq 3$ such that

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i \text{ and } L_1 + L_2 + L_3 = 0.$$

Then, the root space $\mathfrak{g}_{L_i - L_j}$ for $i \neq j$ will be spanned by E_{ij} and we have

$$[H, X] = (a_i - a_j)X = (L_i - L_j)(H)X$$

for all $H \in \mathfrak{h}$ and all $X \in \mathfrak{g}_{L_i - L_j}$. Then, we have the following decomposition of our Lie algebra:

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{L_i - L_j \in \mathfrak{h}^*} \mathfrak{g}_{L_i - L_j}$$

where the sum is over all $i \neq j$.

Now that we know the root space decomposition of $\mathfrak{sl}_3(\mathbb{C})$, we want to give the weight space decomposition of some of its irreducible representations. To do so, we need to define a linear functional on the root lattice

$$\iota : \Lambda_\Phi \to \mathbb{R}$$

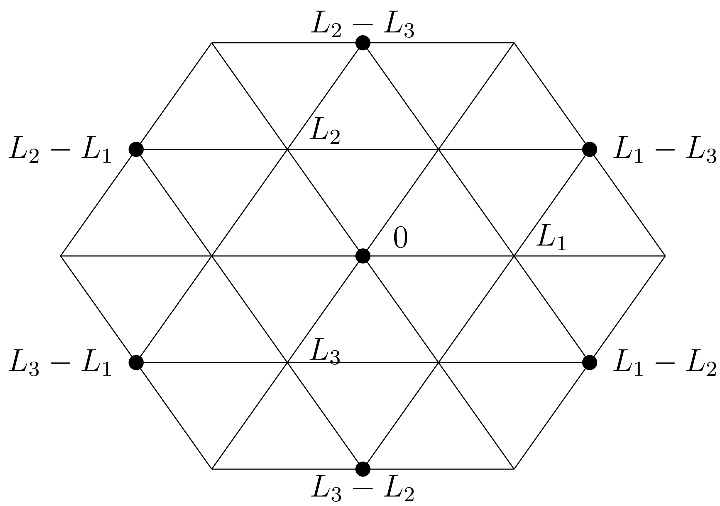
that will help us find the highest weight vector of a given representation. Remember that we need $\iota(\alpha) \neq 0$ for all root α in the corresponding root system $\Phi = A_2$. We can define the linear functional as

$$\iota(a_1 L_1 + a_2 L_2 + a_3 L_3) = a a_1 + b a_2 + c a_3$$

where $a + b + c = 0$ and $a > b > c$. Then, the root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ such that $\iota(\alpha) > 0$ are exactly $\mathfrak{g}_{L_1 - L_3}, \mathfrak{g}_{L_2 - L_3}$, and $\mathfrak{g}_{L_1 - L_2}$. The set of positive roots is

$$\Phi^+ = \{L_1 - L_3, L_2 - L_3, L_1 - L_2\}$$

and the simple roots are $\alpha_1 = L_1 - L_2$ and $\alpha_2 = L_2 - L_3$. We get the same root diagram as in Section 2.1



Now, we want to define the weight lattice Λ_W. Let us look at the subalgebra

$$s_{L_i-L_j} = \mathfrak{g}_{L_i-L_j} \oplus \mathfrak{g}_{L_j-L_i} \oplus [\mathfrak{g}_{L_i-L_j}, \mathfrak{g}_{L_j-L_i}].$$

We have the following distinguished elements

$$H_{ij} = [E_{ij}, E_{ji}] = E_{ij}E_{ji} - E_{ji}E_{ij} = E_{ii} - E_{jj}$$

since by the following computation

$$[E_{ii} - E_{jj}, E_{ij}] = [E_{ii}, E_{ij}] - [E_{jj}, E_{ij}] = E_{ij} + E_{ij} = 2E_{ij} = (L_i - L_j)(H_{ij})E_{ij}$$

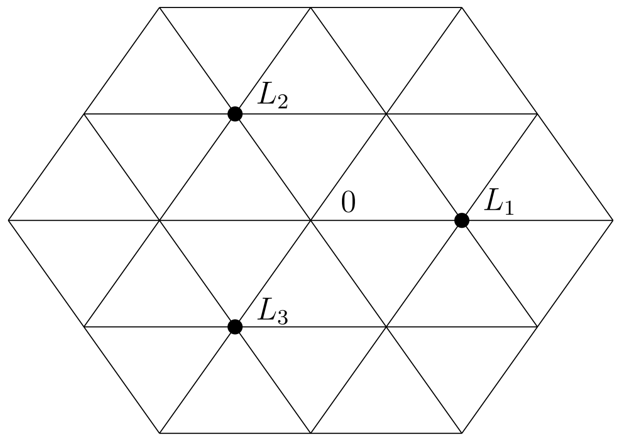
we get that $(L_i - L_j)(H_{ij}) = 2$ and $H_{ij} \in s_{L_i-L_j}$.

Then, we know that the eigenvalues for the action of H_{ij} are integers and are integral linear combinations of the L_i's. Hence, $\Lambda_W = \langle L_1, L_2, L_3 \rangle$.

We can now start looking at the weight space decomposition of the representation with given highest weights. Recall that the fundamental weights are $\omega_1 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 = L_1$ and $\omega_2 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 = L_1 + L_2$. We will denote by $\Gamma_{a,b}$ the representation with highest weight $\mu = a\omega_1 + b\omega_2$. Let us start by looking at the standard representation of $\mathfrak{sl}_3(\mathbb{C})$ on $V \simeq \mathbb{C}^3$. For $H \in \mathfrak{h}$,

$$He_i = a_i e_i = L_i(H)e_i.$$

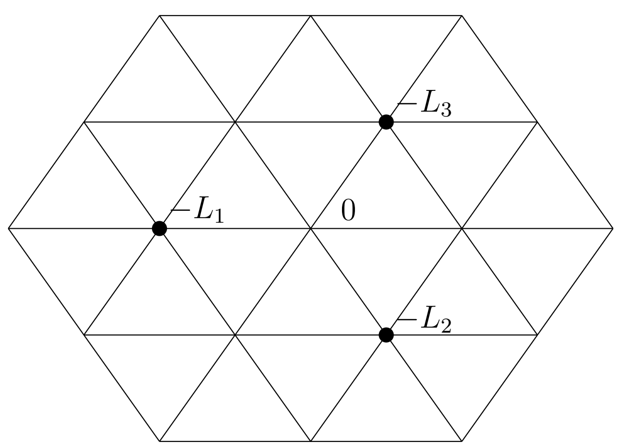
Hence, the eigenvectors for the action of $\mathfrak{h}$ are the elements e_1, e_2, and e_3 of the standard basis of $\mathbb{C}^3$ with eigenvalues L_1, L_2, and L_3. Therefore, we have the following weight diagram:



By looking at the diagram, we see that the weight L_1 is killed by the action of $\mathfrak{g}_{L_1-L_2}, \mathfrak{g}_{L_2-L_3}$, and $\mathfrak{g}_{L_1-L_3}$. We can also see that if we start with the highest weight $L_1 = \omega_1$, we can get the two other weights by the following actions of the root spaces:

$$\begin{aligned}\mathfrak{g}_{-\alpha_1} \cdot V_{L_1} &= \mathfrak{g}_{-(L_1-L_2)} \cdot V_{L_1} = V_{L_2}, \\ \mathfrak{g}_{-\alpha_2} \cdot V_{L_2} &= \mathfrak{g}_{-(L_2-L_3)} \cdot V_{L_2} = V_{L_3}.\end{aligned}$$

Therefore, this is the representation $\Gamma_{1,0}$ with highest weight $L_1 = \omega_1$. Next, we may look at the dual representation V^*. We have $\mu \in V^*$ such that $X(\mu(v)) = -\mu(X(v))$ for any $X \in \mathfrak{g}$ and any $v \in V$. Then, V^* has eigenvalues being the negative of those of V. Then, the weight diagram of V^* is:

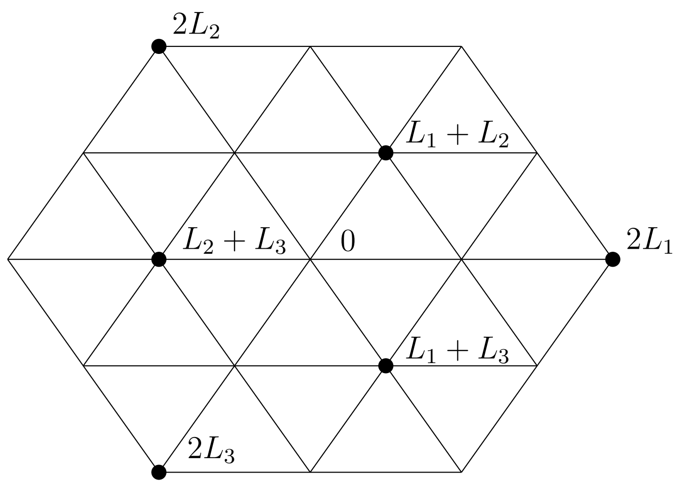


Once again, we see that this is the representation $\Gamma_{0,1}$ with highest weight $\omega_2 = -L_3$.

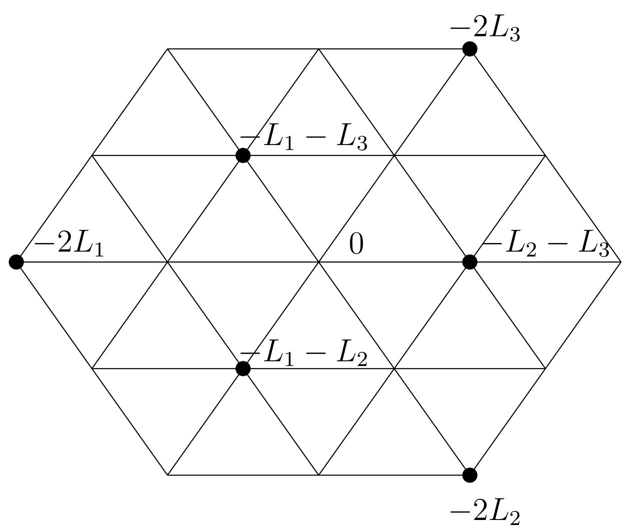
Let us look at the weight diagram of the symmetric power $\operatorname{Sym}^2 V$. The eigenvectors will be of the form $e_i e_j$ with eigenvalues $L_i + L_j$ since

$$H(e_i e_j) = H(e_i)e_j + e_i H(e_j) = L_i(H)e_i e_j + e_i L_j(H) e_j = (L_i + L_j)(H) e_i e_j.$$

Therefore, $\operatorname{Sym}^2 V$ has weights equal to pairwise sums of the weights of V. Then, its weight diagram is



Similarly, we have the following weight diagram for $\operatorname{Sym}^2 V^*$



By looking at the diagrams for $\operatorname{Sym}^2 V$ and $\operatorname{Sym}^2 V^*$, we see that they have highest weight $2L_1$ and $-2L_3$ respectively and every weight is of multiplicity 1. Hence, $\operatorname{Sym}^2 V = \Gamma_{2,0}$ and $\operatorname{Sym}^2 V^* = \Gamma_{0,2}$. This observation leads to the following general result.

6.1.1 Lemma. [FH, p. 182] *If the representation V has highest weight vector v with highest weight μ, then the representation $\operatorname{Sym}^n V$ has highest weight vector v^n with highest weight $n\mu$.*

Now that we know that the two representations $\operatorname{Sym}^n V = \Gamma_{n,0}$ and $\operatorname{Sym}^n V^* = \Gamma_{0,n}$ have highest weights being scalar multiples of the fundamental weights, we can start looking at the homogeneous spaces.

Let $G = \mathrm{SL}_3(\mathbb{C})$ and $W = \operatorname{Sym}^n V$. Let $p \in \mathbb{P}W$ is the point corresponding to the eigenspace with eigenvalue $\mu = n\omega_1 = nL_1$ being the highest weight of W. Then,

$$p = \{w \in W | Hw = nL_1(H)w\} = \langle e_1^n \rangle.$$

Therefore, since the action of G on $\mathbb{C}^3$ is transitive and $g \cdot v^n = (g \cdot v)^n$, we have by Lemma 6.0.1 that

$$G \cdot p = \{v^n\}_{v \in V}$$

is the unique closed orbit of the action of G on $\mathbb{P}W$. We also have

$$\{v^n\}_{v \in V} \simeq \mathbb{P}V \simeq \mathbb{P}^2$$

since $\{v^n\}_{v \in V}$ is the image of the injective map from $\mathbb{P}V$ to $\mathbb{P}W$ called the Veronese embedding [FH, p. 154]. Therefore, since $\langle \omega_1, \alpha_2^\vee \rangle = 0$, the maximal parabolic subgroup corresponding to ω_1 is P_2 and we have

$$G/P_2 \simeq \mathbb{P}^2.$$

Similarly, we get that $G/P_1 \simeq \mathbb{P}^2$. We managed to get the same result as on Page 39 using techniques of representation theory. We can now use the same method for the two other root systems.

6.2 Representations of B_2

We already know that $\mathrm{SO}_5(\mathbb{C})$ can be viewed as the group of matrices preserving a given non degenerate symmetric bilinear form Q. In this section, we will define the non degenerate bilinear form as:

$$Q(x,y) = x^T M y$$

with

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know from Section 4.3 that the group of matrices in $\mathrm{SL}_5(\mathbb{C})$ preserving Q is isomorphic to the group we used before. The Lie algebra $\mathfrak{so}_5(\mathbb{C})$ is defined as the space of matrices X satisfying the relation $X^T M + MX = 0$. Then,

$$so_5(\mathbb{C}) = \left\{ \begin{pmatrix} A & B & C \\ D & -A^T & E \\ -C^T & -E^T & 0 \end{pmatrix} \middle| \begin{array}{c} A, B, D \in M_2(\mathbb{C}), C, E \in \mathbb{C}^2 \\ \text{and } B, D \text{ skew-symmetric} \end{array} \right\}.$$

We take the Cartan subalgebra $\mathfrak{h}$ to be the diagonal matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have $\mathfrak{h} = \langle E_{ii} - E_{i+2,i+2} \rangle$ for $i = 1, 2$. We denote $H_i = E_{ii} - E_{i+2,i+2}$ and we take as basis for $\mathfrak{h}^*$ the dual basis L_j defined by $L_j(H_i) = \delta_{ij}$.

Let us see how H_i and H_j acts on E_{ij}. We have

$$[H_i, E_{ij}] = [E_{ii}, E_{ij}] - [E_{i+2,i+2}, E_{ij}] = E_{ij},$$

$$[H_j, E_{ij}] = [E_{jj}, E_{ij}] - [E_{j+2,j+2}, E_{ij}] = -E_{ij},$$

and $[H_k, E_{ij}] = 0$ for $k \neq i$ and $k \neq j$. The same is true for $E_{j+2,i+2}$ so

$$X_{ij} = E_{ij} - E_{j+2,i+2}$$

is an eigenvector for the action of $\mathfrak{g}$ with eigenvalue $L_i - L_j$.

We can also see that

$$Y_{ij} = E_{i,j+2} - E_{j,i+2}$$

and

$$Z_{ij} = E_{i+2,j} - E_{j+2,i}$$

are eigenvectors for the action of $\mathfrak{h}$ with eigenvalues $L_i + L_j$ and $-L_i - L_j$ respectively. Finally,

$$U_i = E_{i,5} - E_{5,i+2}$$

and

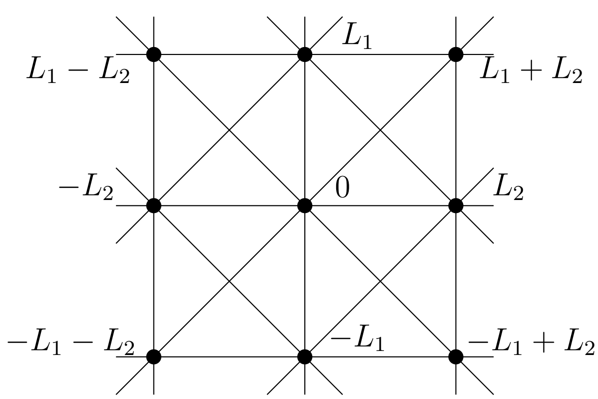
$$V_i = E_{i+2,5} - E_{5,i}$$

are eigenvectors for the action of $\mathfrak{h}$ with eigenvalues $+L_i$ and $-L_i$ respectively. Therefore the set of roots $\Phi = \{\pm L_1, \pm L_2, \pm L_1 \pm L_2, \pm(L_1 - L_2)\}$ is of type B_2.

Now, to fix an ordering on the roots, we define a linear functional

$$\begin{aligned} \iota : \Lambda_\Phi &\rightarrow \mathbb{R} \\ a_1 L_1 + A_2 L_2 &\mapsto c_1 a_1 + c_2 a_2 \end{aligned}$$

for $c_1 > c_2 > 0$. Then, the set of positive roots is $\Phi^+ = \{L_1, L_2, L_1 - L_2, L_1 + L_2\}$ and the simple roots are $L_2 = \alpha_1$ and $L_1 - L_2 = \alpha_2$. As in Section 2.1, we get the following root diagram



Now, we want to find the weight lattice. To do so, we have to find the distinguished elements of the subalgebras

$$s_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

Let us start with the root $L_1 - L_2$ with corresponding root space generated by X_{12}. We have

$$\begin{aligned} [X_{12}, X_{21}] &= [E_{12} - E_{43}, E_{21} - E_{34}] \\ &= [E_{12}, E_{21}] - [E_{12}, E_{34}] - [E_{43}, E_{21}] + [E_{43}, E_{34}] \\ &= E_{11} - E_{22} - 0 - 0 + E_{44} - E_{33} = H_1 - H_2. \end{aligned}$$

Then, the distinguished element $H_{L_1-L_2}$ is a scalar multiple of $H_1 - H_2$. We need $(L_1 - L_2)(H_{L_1-L_2}) = 2$ and we have

$$\begin{aligned} [H_1 - H_2, X_{12}] &= [H_1 - H_2, E_{12} - E_{43}] \\ &= [H_1, E_{12}] - [H_1, E_{43}] - [H_2, E_{12}] + [H_2, E_{43}] \\ &= E_{12} - E_{43} + E_{12} - E_{43} \\ &= 2X_{12} = (L_1 - L_2)(H_1 - H_2)X_{12}. \end{aligned}$$

Therefore, $H_{L_1-L_2} = H_1 - H_2$. For the root $L_1 + L_2$, we have

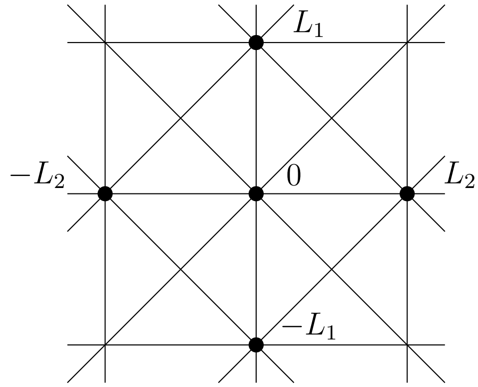
$$[Y_{12}, Z_{12}] = -H_1 - H_2$$

and

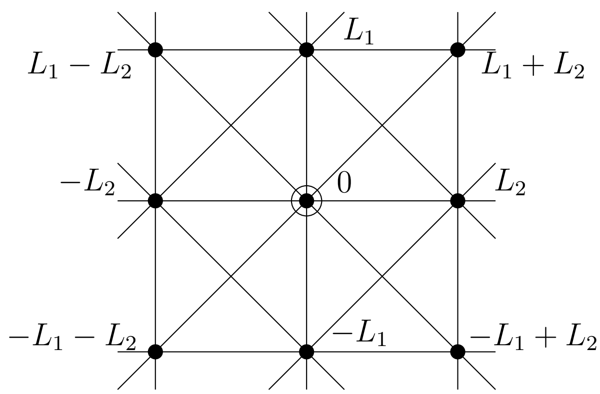
$$[-H_1 - H_2, Y_{12}] = -2Y_{12} = (L_1 + L_2)(-H_1 - H_2)Y_{12}.$$

Then, $H_{L_1+L_2} = H_1 + H_2$. Similarly, we have $H_{L_i} = 2H_i$. Therefore, the distinguished elements are $\{\pm H_1 \pm H_2, \pm H_1, \pm H_2\}$. It is clear that $L_i(H_\alpha) \in \mathbb{Z}$ for any $\alpha \in \Phi$ and we also have that $\frac{1}{2}(L_1 + L_2)(H_\alpha) \in \mathbb{Z}$. Then, Λ_W is generated by L_1, L_2, and $\frac{1}{2}(L_1 + L_2)$.

We are now ready to look at the representations of $so_5(\mathbb{C})$. Recall that from Section 3.7, the fundamental weights for the root system B_2 are $\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2 = \frac{1}{2}(L_1 + L_2)$ and $\omega_2 = \alpha + \beta = L_1$. By definition of the Cartan subalgebra $\mathfrak{h}$, it is clear that the weight diagram for the standard representation $V = \mathbb{C}^5$ is

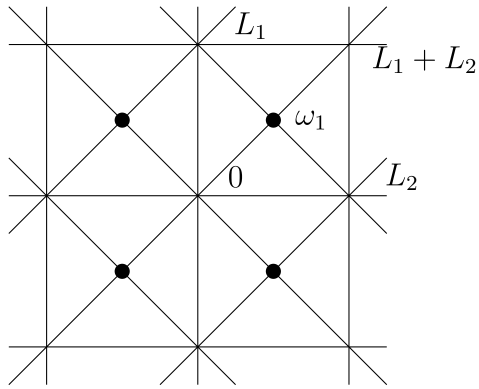


We see that V has highest weight $L_1 = \omega_2$. If we start with the weight L_1 and apply reflections by $s_{\alpha_1} = s_{L_1}$ and $s_{\alpha_2} = s_{L_1-L_2}$, we get the convex hull with corners being the weights $L_1, L_2, -L_1$ and $-L_2$. Then, by consecutive actions of $\mathfrak{g}_{-L_2}$ and $\mathfrak{g}_{-(L_1-L_2)}$ on the weight spaces, we find all the weights. Therefore, we have that $V = \Gamma_{0,1}$. Let us now take a look at the representation $\bigwedge^2 V$. It has weights equal to the pairwise sums of distinct weights of V. Its weight diagram is



Since $V = \text{span}\{L_1, L_2, L_0, L_{-1}, L_{-2}\}$, then $V \bigwedge V = \text{span}\{L_i \bigwedge L_j | i < j\}$. Then, since $L_{-2} \bigwedge L_2$ and $L_{-1} \bigwedge L_1$ are basis elements of $\bigwedge^2 V$ they must be linearly independent. Therefore, the weight 0 has multiplicity 2 in $\bigwedge^2 V$.

By looking carefully at the weight diagram, we see that $\bigwedge^2 V$ has highest weight $L_1 + L_2 = 2\omega_1$ and that it has the same diagram as $\text{Sym}^2 W$ for a representation W with highest weight ω_1 and weight diagram



This is in fact the representation $\Gamma_{1,0}$ but we can't use our usual method of starting with the representation V to construct this representation.

We can now start looking at the orbits of the action of $G = \mathrm{SO}_5(\mathbb{C})$ on $\mathbb{P}V$ for the irreducible representations. Let us begin by looking at the standard representation $V = \Gamma_{0,1}$ with highest weight ω_2. The corresponding maximal parabolic subgroup is P_1 which corresponds to the simple root α_1. We know from Section 4.1 that in the decomposition of the parabolic subalgebra corresponding to α_1 we have $T(\Sigma) = \{\alpha_1, -\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. Then, $\dim(P_1) = 7$. We have that

$$\dim(G/P_1) = 10 - 7 = 3$$

and by looking at the weight diagram of V we see that $\dim(V) = 5$. Hence, we have

$$G/P_1 \subset \mathbb{P}V \simeq \mathbb{P}^4$$

which means that G/P_1 is an hypersurface in $\mathbb{P}^4$. Since we know from Chapter 4 that G preserves a symmetric bilinear form, G/P_1 is a quadric hypersurface in $\mathbb{P}^4$ [H, p. 33].

For P_2, since we can only find the weight diagram of $\Gamma_{1,0}$ by doing backward reasoning starting with the adjoint representation, we need another method to construct $\Gamma_{1,0}$ and find G/P_2. We could look at the representations of C_2 to find that the representation $\Gamma_{1,0}$ for B_2 coincides with the standard representation of C_2 and conclude, as in [FH, p. 389], that

$$G/P_2 \simeq \mathbb{P}^3.$$

6.3 Representations of G_2

In the case of G_2, we will do the analysis without a precise description of the Lie algebra $\mathfrak{g}_2$. We will use the approach of [FH, Section 22.1]. We will start with the root system from Chapter 2 and construct its corresponding Lie algebra.

Let α_1 and α_2 be the usual simple roots. We choose X_1 (respectively X_2) to be an eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_1 (respectively X_2). We choose Y_1 and Y_2 similarly such that they have eigenvalues $-\alpha_1$ and $-\alpha_2$ respectively. We take

$$H_1 = [X_1, Y_1] \text{ and } H_2 = [X_2, Y_2].$$

We can then rescale Y_1 and Y_2 such that for $H_1 \in [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \subset \mathfrak{h}$ and $H_2 \in [\mathfrak{g}_{\alpha_2}, \mathfrak{g}_{-\alpha_2}] \subset \mathfrak{h}$ we have

$$\alpha_1(H_1) = \alpha_2(H_2) = 2.$$

Then, we have

$$[H_i, X_i] = 2X_i \text{ and } [H_i, Y_i] = -2Y_i.$$

Now, we need to find generators for the other root spaces. We do that by looking at the root diagram of G_2. If we fix

$$\alpha_3 = \alpha_1 + \alpha_2, \qquad \alpha_4 = 2\alpha_1 + \alpha_2$$
$$\alpha_5 = 3\alpha_1 + \alpha_2, \qquad \alpha_6 = 3\alpha_1 + 2\alpha_2.$$

Then, we have

$$\begin{aligned}
\alpha_3 &= \alpha_1 + \alpha_2 \\
\alpha_4 &= 2\alpha_1 + \alpha_2 = \alpha_1 + \alpha_3 \\
\alpha_5 &= 3\alpha_1 + \alpha_2 = \alpha_1 + \alpha_4 \\
\alpha_6 &= 3\alpha_1 + 2\alpha_2 = \alpha_2 + \alpha_5.
\end{aligned}$$

We can then choose

$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3]$$
$$X_5 = [X_1, X_4], \qquad X_6 = [X_2, X_5].$$

and define Y_3, Y_4, Y_5, and Y_6 similarly. The elements H_i, X_j, and Y_j for $i = 1, 2$ and $j = 1, ..., 6$ form a basis for the 14-dimensional Lie algebra $\mathfrak{g}_2$ [FH, p. 335]. Here, $\{H_1, H_2\}$ is a basis for $\mathfrak{h}$ and X_j (respectively Y_j) is a generator of the root space $\mathfrak{g}_{\alpha_j}$ (respectively $\mathfrak{g}_{-\alpha_j}$).

The next step is to find the multiplication table for those basis elements. After a fairly good amount of computations and some rescaling we get the following multiplication table [FH, p. 346].

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							$H_1 + 3H_2$	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1 + 3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											$H_1 + H_2$	0	$-Y_2$
Y_5												X_2	0
X_6													$H_1 + 2H_2$

Now, we would like to find the distinguished elements of the subalgebras s_{α_i}. If we take $H_i = [X_i, Y_i]$, we know from the multiplication table that

$$H_3 = H_1 + 3H_2, \qquad H_4 = 2H_1 + 3H_2,$$
$$H_5 = H_1 + H_2, \qquad H_6 = H_1 + 2H_2.$$

We can also see that

$$[H_i, X_i] = 2X_i \text{ and } [H_i, Y_i] = -2Y_i.$$

Therefore, the H_i's are the distinguished elements and the set of distinguished elements is generated by H_1 and H_2. Then, since we have

$$\alpha_1(H_1) = 2, \qquad \alpha_1(H_2) = -1,$$
$$\alpha_2(H_1) = -3, \qquad \alpha_2(H_2) = 2,$$

and H_1,H_2 generate all the distinguished elements, it means that the weight lattice is generated by α_1 and α_2. In fact, since Λ_Φ is also generated by α_1 and α_2, $\Lambda_\Phi = \Lambda_W$.

We can now start looking at the representations of $\mathfrak{g}_2$ with highest weight being the fundamental weights. Recall that from Section 3.10, the fundamental weights are

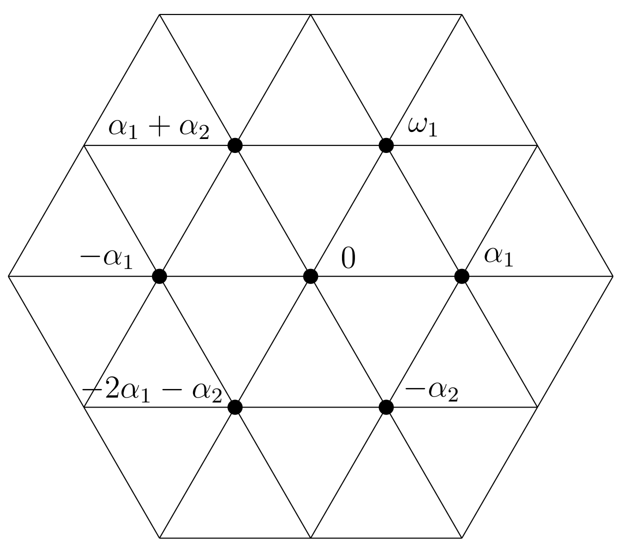
$$\omega_1 = 2\alpha_1 + \alpha_2 \text{ and } \omega_2 = 3\alpha_1 + 2\alpha_2.$$

Let us start by looking at the representation $\Gamma_{1,0}$ with highest weight ω_1. We find the weights on the boundary of the convex hull by applications of the following reflections on ω_1:

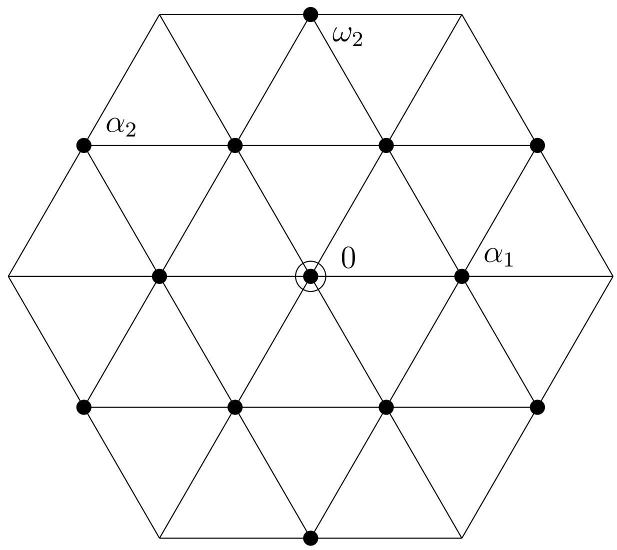
$$s_{\alpha_1}(\omega_1) = \alpha_1 + \alpha_2, s_{\alpha_2}(\omega_1) = \alpha_1, s_{\alpha_1}(\alpha_1) = -\alpha_1,$$

$$s_{\alpha_2}(\alpha_1 + \alpha_2) = -\alpha_2, s_{\alpha_1}(-\alpha_2) = -2\alpha_1 - \alpha_2.$$

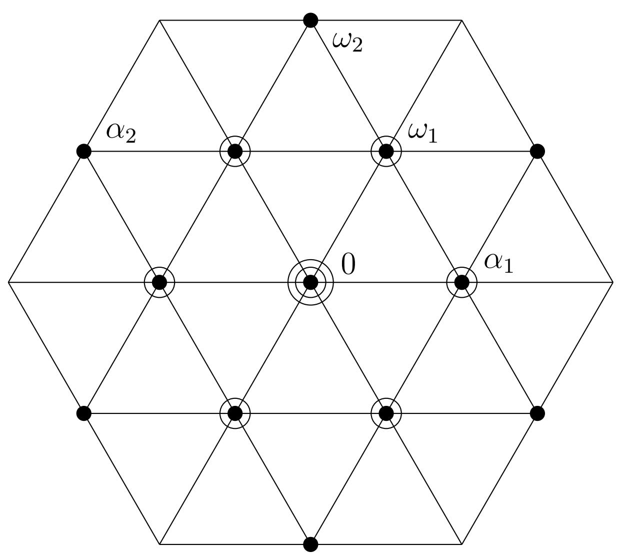
Then, we can find the remaining weights by action of the root spaces $\mathfrak{g}_{-\alpha_1}$ and $\mathfrak{g}_{-\alpha_2}$ on the weight spaces. Therefore, the weight diagram of $\Gamma_{1,0}$ is



We see that $V = \Gamma_{1,0}$ is 7-dimensional since each weight space is one-dimensional. We will call it the standard representation of $\mathfrak{g}_2$. Now, we want to look at the representation $\Gamma_{0,1}$. By finding the weights in the convex hull and then by actions of the root spaces of the negative simple roots, we get the following weight diagram for this representation

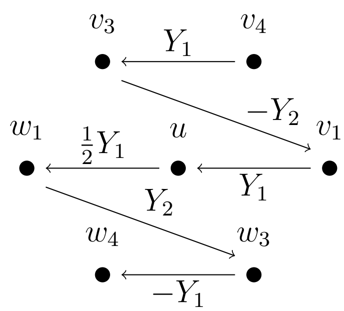


Here the multiplicity of 0 is 2 and the dimension of $\Gamma_{0,1}$ is 14. By looking at the multiplication table, we see that $[X_i, X_6] = 0$ for all $i = 1, ..., 6$, hence the highest weight of the adjoint representation is $\alpha_6 = 3\alpha_1 + 2\alpha_2 = \omega_2$. This means that $\Gamma_{0,1}$ is the adjoint representation. Next, we can consider the representation $\bigwedge^2 V$ with weight diagram



We know from Weyl's Theorem [Hu, p. 28] that the representation $\bigwedge^2 V$ is completely reducible. By looking at the weight diagram, we see that by taking out the weights of $\Gamma_{0,1}$ from the weight diagram of $\bigwedge^2 V$, we are left with the weight diagram of V. Hence, $\bigwedge^2 V \simeq \Gamma_{0,1} \oplus V$. This means that $\Gamma_{0,1}$ is contained in $\bigwedge^2 V$. Since $\Gamma_{0,1} \subset \bigwedge^2 V$ and any irreducible representation $\Gamma_{a,b} \subset \mathrm{Sym}^a V \oplus \mathrm{Sym}^b \Gamma_{0,1}$, it means that any irreducible representation is contained in some tensor power of V [FH, 353].

Next, we want to check how $\mathfrak{g}_2$ acts on the standard representation. Let us pick a highest weight vector $v_4 \in V_{\alpha_4}$. This will be our first basis element of V. To find the other basis elements, we will apply consecutive actions of Y_1 and Y_2 on v_4. We may look at the following diagram to understand the action of Y_1 and Y_2.



We choose the basis vectors of V as follows:

$$v_4, v_3 = Y_1(v_4), v_1 = -Y_2(v_3), u = Y_1(v_1),$$

$$w_1 = \frac{1}{2}Y_1(u), w_3 = Y_2(w_1), \text{ and } w_4 = -Y_1(w_3).$$

By looking at the multiplication table above, we get the following relations between the generators of the root spaces:

$$\begin{aligned} X_3 &= [X_1, X_2], & Y_3 &= -[Y_1, Y_2], \\ X_4 &= \frac{1}{2}[X_1, X_3], & Y_4 &= -\frac{1}{2}[Y_1, Y_3], \\ X_5 &= -\frac{1}{3}[X_1, X_4], & Y_5 &= \frac{1}{3}[Y_1, Y_4], \\ X_6 &= -[X_2, X_5], & Y_6 &= [Y_2, Y_5]. \end{aligned}$$

Using these relations and the definitions of our basis elements, we can find how each generators act on the basis elements. The following table gives us all the results.

	v_4	v_3	v_1	u	w_1	w_3	w_4
Y_1	v_3	0	u	$2w_1$	0	$-w_4$	0
Y_2	0	$-v_1$	0	0	w_3	0	0
Y_3	$-v_1$	u	0	$2w_3$	w_4	0	0
Y_4	u	$-w_1$	w_3	$2w_4$	0	0	0
Y_5	w_1	0	$-w_4$	0	0	0	0
Y_6	w_3	$-w_4$	0	0	0	0	0
X_1	0	v_4	0	$2v_1$	u	0	w_3
X_2	0	0	v_3	0	0	w_1	0
X_3	0	0	v_4	$2v_3$	0	u	$-w_1$
X_4	0	0	0	$2v_4$	$-v_3$	v_1	$-u$
X_5	0	0	0	0	v_4	0	v_1
X_6	0	0	0	0	0	v_4	$-v_3$

We can see from the table that basis elements different from u are always sent to another basis element up to sign and that $X_i(u) = 2v_i$ and $Y_i(u) = 2w_i$ for $i = 1, 3, 4$.

Now we need a new definition.

6.3.1 Definition. Let V be a representation of a Lie algebra $\mathfrak{g}$ over a field k. Then a bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to k$$

is *invariant* if

$$\langle X(v), w \rangle + \langle v, X(w) \rangle = 0$$

for all $X \in \mathfrak{g}, v, w \in V$.

What we want to prove at this point is that the action of $\mathfrak{g}_2$ on the standard representation V preserves a bilinear form, i.e. there exist an invariant bilinear form. We get the following result from the definition.

6.3.2 Lemma. *Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form. Then, for any $v_\lambda \in V_\lambda$ and any $v_\mu \in V_\mu$, we have*

$$\langle v_\lambda, v_\mu \rangle = 0$$

if $\lambda + \mu$ is non-zero.

Proof: Let $v_\lambda \in V_\lambda$ and $v_\mu \in V_\mu$. Then, for any $H \in \mathfrak{h}$, we have

$$\begin{aligned} \langle H(v_\lambda), v_\mu\rangle + \langle v_\lambda, H(v_\mu)\rangle &= 0 \\ \Rightarrow \langle \lambda(H)(v_\lambda), v_\mu\rangle + \langle v_\lambda, \mu(H)(v_\mu)\rangle &= 0 \\ \Rightarrow (\lambda+\mu)(H)\langle v_\lambda, v_\mu\rangle &= 0 \end{aligned}$$

Then, $\langle v_\lambda, v_\mu\rangle = 0$ if $(\lambda+\mu)(H) \neq 0$, hence if $\lambda+\mu$ is non-zero. □

Suppose that $\langle\cdot,\cdot\rangle$ is an invariant bilinear form for the standard representation of $\mathfrak{g}_2$. Suppose that

$$\langle v_4, w_4\rangle = 1$$

and let us find $\langle v_i, w_i\rangle$ for $i = 0, 1, 3$ using the invariance. We have,

$$\langle v_3, w_3\rangle = \langle Y_1(v_4), w_3\rangle = -\langle v_4, Y_1(w_3)\rangle = -\langle v_4, -w_4\rangle = 1,$$

$$\langle v_1, w_1\rangle = \langle -Y_2(v_3), w_1\rangle = \langle v_3, Y_2(w_1)\rangle = \langle v_3, w_3\rangle = 1,$$
$$\langle u, u\rangle = \langle Y_1(v_1), u\rangle = -\langle v_1, 2w_1)\rangle = -2.$$

Now, suppose that we have a bilinear form $\langle\cdot,\cdot\rangle$ such that for $i = 1, 3, 4$

$$\langle v_i, w_i\rangle = 1, \langle u, u\rangle = -2,$$

and $\langle v_\lambda, v_\mu\rangle = 0$ if $\lambda+\mu$ is non-zero. Let us prove that this bilinear form is invariant.

Let us start by writing down everything we get from the properties above in a table. Note that since the weight diagram for $\bigwedge^2 V$ is equal to the root diagram of $\mathfrak{g}_2$ up to multiplicity, $\langle v_i, v_j\rangle = 0$ for $i \neq -j$. Then, we get the following table.

	v_4	v_3	v_1	u	w_1	w_3	w_4
v_4	0	0	0	0	0	0	1
v_3	0	0	0	0	0	1	0
v_1	0	0	0	0	1	0	0
u	0	0	0	-2	0	0	0
w_1	0	0	1	0	0	0	0
w_3	0	1	0	0	0	0	0
w_4	1	0	0	0	0	0	0

Then, we only need to do the computations for the action of all the basis elements of $\mathfrak{g}_2$ on all pairs of basis elements of V. We will not do all the computations here but we can look at some examples.

$$\langle Y_1(v_4), w_3\rangle + \langle v_4, Y_1(w_3)\rangle = \langle v_3, w_3\rangle + \langle v_4, -w_4\rangle = 1 + (-1) = 0$$

$$\langle Y_1(v_4), w_1\rangle + \langle v_4, Y_1(w_1)\rangle = \langle v_3, w_1\rangle + \langle v_4, 0\rangle = 0$$

$$\langle X_4(v_3), u\rangle + \langle v_4, X_4(u)\rangle = \langle 0, u\rangle + \langle v_3, 2v_4\rangle = 0$$

Therefore, the action of $\mathfrak{g}_2$ on V preserves a symmetric bilinear form. With this last property about the representations of $\mathfrak{g}_2$, we can start looking at the geometric description of the homogeneous spaces for G_2.

Recall that the dimension of G_2 is 14. By looking at the definition of the parabolic subalgebra and the fact that $\mathfrak{g}_2$ has 6 positive roots, we get

$$\mathfrak{p}_i = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha,$$

where $T(\Sigma) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -\alpha_i\}$ for $i = 1, 2$. Then, $\dim(P_i) = 9$ and

$$\dim(G_2/P_i) = 14 - 9 = 5.$$

Let us begin by looking at the homogeneous space G_2/P_2. We know that the representation with highest weight ω_1 is the standard representation V and it has dimension 7. The corresponding maximal parabolic subgroup is P_2 since $\langle \omega_1, \alpha_2^\vee\rangle = 0$. Hence, we have

$$G_2/P_2 \subseteq \mathbb{P}V \simeq \mathbb{P}^6.$$

Therefore, G_2/P_2 is an hypersurface in $\mathbb{P}^6$. Furthermore, since the action of $\mathfrak{g}_2$ on V preserves a symmetric bilinear form, G_2/P_2 is a quadric hypersurface in $\mathbb{P}^6$.

The case of G_2/P_1 is more complicated. We know that the the representation with highest weight ω_2, $W = \Gamma_{0,1}$, is contained in $\bigwedge^2 V$. From the work in [LM, p. 149], we get that

$$G_2/P_1 = Gr(2, 7) \cap \mathbb{P}W.$$

Chapter 7

Chow Groups

In this section, we will make a link between the Pieri graph of the root systems Φ and the Chow groups $\mathrm{CH}(G/B)$ for G an algebraic group in the sense of [Spr, p. 21] with root system Φ. We will only give a brief description of a Chow group. The main resource for information about Chow groups will be [Fu, Chapter 1].

7.0.1 Definition. c.f. [J2, p. 9–10] A *category* $\mathfrak{C}$ is composed of a class of objects, denoted $\mathrm{ob}(\mathfrak{C})$, such that:

(a) $\forall A, B \in \mathrm{ob}(\mathfrak{C})$, there exist a set $\mathrm{Hom}_{\mathfrak{C}}(A, B)$, called the *set of morphism* from A to B, such that $\mathrm{Hom}_{\mathfrak{C}}(A, B) \cap \mathrm{Hom}_{\mathfrak{C}}(C, D) = \emptyset$ if $(A, B) \neq (C, D)$

(b) $\forall A, B, C \in \mathrm{ob}(\mathfrak{C})$, there exist a composition law:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathfrak{C}}(A, B) \times \mathrm{Hom}_{\mathfrak{C}}(B, C) & \to & \mathrm{Hom}_{\mathfrak{C}}(A, C) \\ (f, g) & \mapsto & g \cdot f \end{array}$$

This composition law must satisfy the following conditions:
(i) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$
(ii) $\forall A \in \mathrm{ob}(\mathfrak{C})$, there exist a morphism $\mathrm{id}_A : A \to A$, called the *identity morphism*, such that if $B \xrightarrow{u} A \xrightarrow{v} C$, then $\mathrm{id}_A \cdot u = u$ and $v \cdot \mathrm{id}_A = v$.

7.0.2 Definition. [J2, p. 19] Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two categories. A *functor* μ from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ maps any object $A \in \mathfrak{C}_1$ to a unique object $\mu(A) \in \mathfrak{C}_2$ and for any $A, B \in \text{ob}(\mathfrak{C}_1)$, there exist a map

$$\begin{aligned} \text{Hom}_{\mathfrak{C}_1}(A,B) &\to \text{Hom}_{\mathfrak{C}_2}(\mu(A),\mu(B)) \\ f &\mapsto \mu(f) \end{aligned}$$

such that:
(a) $\mu(\text{id}_A) = \text{id}_{\mu(A)}$
(b) For all $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathfrak{C}_1$, $\mu(g \cdot f) = \mu(g) \cdot \mu(f)$

Consider a contravariant functor

$$\begin{aligned} \text{CH}^i : \text{ Smooth Varieties over } \mathbb{C} &\to \text{ Graded Commutative Rings} \\ X &\mapsto \text{CH}^i(X) \end{aligned}$$

where for $X \to Y$ we have $\text{CH}^i(Y) \to \text{CH}^i(X)$.

Also, if $Z \to X$ is a proper map, $\text{CH}_i(Z) \to \text{CH}_i(X)$ where $\text{CH}_i(Y) \simeq \text{CH}^{\dim(Y)-i}(Y)$.

The CH^i functor satisfies the two following axioms:

(1) Homotopy Invariance:

$$\begin{aligned} X \times_{\mathbb{C}} \mathbb{A}^1 &\to X \\ \text{CH}^i(X) &\overset{\simeq}{\to} \text{CH}^i(X \times_{\mathbb{C}} \mathbb{A}^1) \end{aligned}$$

(2) Localization:

For $U \subset X$ open and $Z = X \setminus U$ closed, we have the following exact sequence

$$\text{CH}^{\dim(Z)-(\dim(X)-i)}_{\dim(X)-i}(Z) \to \text{CH}^i_{\dim(X)-i}(X) \to \text{CH}^i(U) \to 0.$$

7.0.3 Example. [Fu, p. 23] $\text{CH}^{n-k}(\mathbb{P}^n) = \mathbb{Z}$ for $k \leq n-1$. In particular, $\text{CH}(\mathbb{P}^1) = \mathbb{Z}$.

Now, we can follow the work of [NSZ] to get to the point where we make a link between the Pieri graph and $\text{CH}(G/B)$. Let G be a split simple linear algebraic group over $\mathbb{C}$. Let X be a projective G-homogeneous variety, that is, $X \simeq G/P$, where P is a parabolic subgroup of G. The abelian group structure of CH(X) is well-known. By the Bruhat decomposition [FH, p. 396–397], X has cellular decomposition given by

$$X = \coprod_{w \in W} B \cdot w' \cdot x$$

where w' is a representative of w and x is a $\mathbb{C}$-rational point on X. If X is of dimension d, we can define a sequence

$$X = X_0 \supset X_1 \supset \cdots \supset X_d$$

by

$$X_i = \coprod_{w \in W^{\Theta}, l(w) \leq d-i} B \cdot w' \cdot x$$

and we have

$$X_i \setminus X_{i-1} \simeq \coprod_{w \in W^{\Theta}, l(w) = d-i} \mathbb{A}_{\mathbb{C}}^{l(w)} \text{ and } X_d = x.$$

We fix a maximal split torus T in G and a Borel subgroup B of G containing T defined over $\mathbb{C}$. We denote by Φ the root system of G, by Δ the set of simple roots of Φ corresponding to B, by W the Weyl group, and by S the corresponding set of simple reflections.

Let $P = P_{\Theta}$ be a parabolic subgroup corresponding to a subset $\Theta \subset \Delta$, i.e., $P = BW_{\Theta}B$, where $W_{\Theta} = \langle s_{\theta}, \theta \in \Theta \rangle$. Denote

$$W^{\Theta} = \{w \in W | \forall s \in S, l(ws) = l(w) + 1\},$$

where l is the length function. The pairing

$$\begin{array}{rcl} W^{\Theta} \times W_{\Theta} & \to & W \\ (w, v) & \mapsto & wv \end{array}$$

is a bijection and $l(wv) = l(w) + l(v)$. Then, we have that W^{Θ} consists of all representatives in the cosets W/W_{Θ} which have minimal length. We denote ${}^{\Theta}W$ the set of all representatives of maximal length. There is a bijection $W^{\Theta} \to {}^{\Theta}W$ given by $v \mapsto vw_{\theta}$, where w_{θ} is the longest element of W_{Θ}. The longest element of W^{Θ} corresponds to the longest element w_0 of the Weyl group.

To a subset Θ of the finite set Δ we associate an oriented labeled graph, which we call a *Hasse diagram* and denote by $\mathcal{H}_W(\Theta)$. This graph is constructed as follows. The vertices of this graph are the elements of W^Θ. There is an arrow from a vertex w to a vertex w' labeled with i if and only if $l(w) < l(w')$ and $w' = s_i w$ where s_i is a simple reflection. We see that the diagram $\mathcal{H}_W(\emptyset)$ coincides with the Cayley graph associated to the pair (W, S).

Now consider the Chow ring of the projective variety G/P_Θ. It is well known that $\mathrm{CH}(G/P_\Theta)$ is a free abelian group [De, p. 69] with basis given by varieties $[X_w]$, called *Schubert classes*, that correspond to the vertices w of the Hasse diagram $\mathcal{H}_W(\Theta)$. The degree of the basis element $[X_w]$ is equal to the minimal number of arrows needed to connect the respective vertex w and longest element w_θ. The multiplicative structure of $\mathrm{CH}(G/P_\Theta)$ depends only on the root system of G and the diagram $\mathcal{H}_W(\Theta)$.

7.0.4 Lemma. [NSZ, Cor. 3.6] *Let B be a Borel subgroup of G and P one of its parabolic subgroup. Then $\mathrm{CH}(G/P)$ is a subring of $\mathrm{CH}(G/B)$. The generators of $\mathrm{CH}(G/P)$ are $[X_w]$, where $w \in {}^\Theta W \subset W$. The cycle $[X_w]$ in $\mathrm{CH}(G/P)$ has codimension $l(w_0) - l(w)$.*

Hence, in order to compute $\mathrm{CH}(G/P)$ it is enough to compute $\mathrm{CH}(X)$, where $X = G/B$ is the variety of complete flags. We now have all the tools to make a link between the Pieri graph and $\mathrm{CH}(X)$.

In order to multiply two basis elements of $\mathrm{CH}(X)$ one of which is of codimension 1, we use the following formula called the Pieri formula [De, Cor. 2 of 4.4]

$$[X_{s_\alpha w_0}][X_w] = \sum_{\beta \in \Phi^+, l(s_\beta w) = l(w)+1} \langle \omega_\alpha, \beta^\vee \rangle [X_{s_\beta w}],$$

where the sum runs through the set of positive roots $\beta \in \Phi^+$, s_α denotes the simple reflection corresponding to α and ω_α is the fundamental weight corresponding to α. Here $[X_{s_\alpha w_0}]$ is the element of codimension 1. Hence, the coefficients can be found just by looking at the number of arrows of the color corresponding to α between w and $s_\beta w$ in the Pieri graph.

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