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LA THÈSE A ÉTÉ
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TRANSIENT HEAT CONDUCTION AND THERMOELASTIC ANALYSIS
OF THREE-SPAN COMPOSITE HIGHWAY BRIDGE

by

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A thesis submitted to the School of Graduate Studies
at the University of Ottawa, in partial fulfillment of
the requirements for the degree of Doctor of Philosophy

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SYNOPSIS

An analytical investigation into the stresses and deformations induced by transient temperatures was carried out on a laboratory tested model of a three-span composite bridge. The 0.354-scale model of the prototype had a reinforced concrete deck acting compositely with its four supporting steel beams. The dimensions of the model are 53 ft. $1\frac{1}{2}$ in. long by 10 ft. $6\frac{1}{2}$ in. wide and a $2\frac{7}{16}$ in. thick slab.

A two-dimensional temperature distribution varying with time was produced in cross-sectional planes of the model bridge when its entire upper surface was rapidly cooled with melting ice. The decrease in surface temperature was maintained for an hour and seventeen minutes. The underside remained exposed to the air temperature that gradually declined from the initial temperature of 78.3°F to 71.8°F during the cooling test. Instrumentation of the bridge consisted of 41 thermocouples placed near the mid cross-sectional plane of the centre span, 40 electrical resistance strain gauges installed on the upper and lower surfaces of the steel beams at the five maximum moment sections, and 8 mechanical dial gauges used to measure uplift from the non-supporting piers at both ends. Temperature and strain data were automatically obtained using a data acquisition system. Measurements were taken at various time intervals over the cooling period. Within this temperature range, the ends of the bridge have deflected upwards very rapidly to a maximum value before the deflections gradually decreased towards steady temperature conditions. The method of analysis was based on the linear, small deformation theory of two-dimensional uncoupled quasi-static thermoelasticity.

In the transient heat conduction analysis, space discretization by finite elements leads to a system of first-order, linear differential equations. Central difference was used in the time-stepping solution to these differential equations. In the analysis for symmetrical cross ⁷ section, only an interior and a side portion need be used to obtain a solution for the temperature field. The temperature calculations were carried out with a regular mesh of linear temperature rectangular elements. For the temperature history of the cooling bridge, the solution was advanced 462 times with a time step of 10 seconds. The time-marching procedure involved repeated solution of 1652 and 882 simultaneous linear algebraic equations with a half bandwidth of 13. All pertinent material properties were considered independent of the temperatures throughout the cooling test. The changes of instantaneous temperature distributions in the bridge cross section are shown at various time intervals throughout the time history.

The thermoelastic analysis of the three-span continuous, composite bridge employed conforming displacement model elements leading to strain energy convergence of N^{-6} in terms of number of elements. The triangular plate element, being made of isotropic linearly elastic material, had constrained quintic displacements for bending and complete cubic polynomial expressions for in-plane displacements. The eccentric beam element included bending action, axial deformation, and St-Venant torsion. Only a quarter of the doubly symmetrical structure was analyzed with a uniform 7 x 30 grid of finite elements. There were a total of 2976 simultaneous linear algebraic equations with a half bandwidth of 108. The

finite element analysis of the bridge included uplift at the supports of the interior piers when its deck deformed into a doubly curved surface. Over the cooling period, calculation of the changes in the magnitude of the support reactions was required before uplift occurred.

The curves of the deflections and thermal stresses against time are presented with the thermal moments and forces acting on the bridge. The results indicate that the temperature changes at early times produced large thermal stresses, exceeding the concrete modulus of rupture. The strain diagrams with the free thermal contraction diagrams are included.

Disk files were used to reduce the storage demand of the FORTRAN programme on the computer. The resulting large systems of equations were stored in block form. Cholesky's decomposition method with backward and forward substitutions was used for repeated solution of these equations involving only different unknowns.

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The scaled model bridge was constructed and was tested at the Structures Laboratory of University of Ottawa. Appreciation and thanks are expressed to the laboratory staff and to Y.T. Chan who assisted during the casting of the concrete deck.

The IBM 360-65 system of the University of Ottawa, Computer Centre was used for the computations.

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NOTATION

Standard notation is used in this study. In certain cases, the same symbols denote a property in heat transfer and in structural analysis. The meaning that the symbol should have is evident from the context in which it is used

Latin Letters

[A]	Coefficient matrix for a system of differential equations
A_s	Area of eccentric stiffeners
a,b,c	Element dimensions
a_i, b_i	Coefficients of assumed polynomials
[C]	Capacity matrix for assembled structure
$[C_e]$	Element capacity matrix
C_t	Torsional constant for cross section
c	Specific heat
[D]	Matrix of elastic constants
D_f, D_m	Flexural and membrane rigidities
d_c	Slab thickness
d_s	Depth of wide-flange section
E	Modulus of elasticity
E_c	Elastic modulus of concrete
e	Eccentricity of the stiffeners
$F(m,n)$	Euler's beta function for exact integration
{F}	System load vector or known column matrix. Subscripts denote thermal loads for an element
$\{f_e\}$	Element matrix with specified nodal temperatures

x

NOTATION (Cont'd)

Latin Letters

G	Shear modulus of elasticity
$[H_e]$	Element convection matrix
h	Thickness of concrete slab; unit surface conductance, all used with subscripts indicating the involved lower surface
I	Functional; Second moment of stiffener's area
L_w	Warping torsional constant
$[K]$	Stiffness or conductance matrix for system. Subscripts indicate element matrix
k	Thermal conductivity of material
k_{ij}	Coefficient matrix of integral
l	Typical distance between nodes; length of beam element
$[M]$	Modal matrix
M^T	St. Venant torsion
M_t	Temperature moment
M^t	Twisting moment
M^w	Warping torsional moment
m_i, n_i	Exponents in i^{th} term of polynomial expression
p_i, q_i	
r_i, s_i	
N	Number of elements on a side
N_t	Temperature resultant for in-plane forces
$\{P\}$	Load vector for quadratic plane stress element
q	Intensity of distributed static load
$[R]$	Rotation matrix

NOTATION (Cont'd)

Latin Letters

R_w, R_t	Reaction due to bridge's weight and to temperatures.
r_x, r_y	Radius of curvature
S	First moment of the stiffener's area
s	Boundary surface
$[T]$	Nodal coordinate matrix
$[T_t]$	Nodal coordinate matrix for temperature load distributed in slab
t	Temperature; temperature change from datum temperature
t_g	Ambient air temperature
t_s	Surface temperature
U	Strain energy. Its subscripts denote the corresponding terms of the elements
$u, v, w,$	Displacements. Subscripts denote differentiation
V	Potential of external forces; Nodal displacement of quadratic plane stress element
x, y, z	Cartesian coordinates
X	Column vector of generalized displacements. Subscripts denote bridge displacements for different loading system
w	Thickness of web in structural steel section
w_0	Thermal curvature

NOTATION (Cont'd)

Greek Letters

α	Coefficient of thermal expansion
$\epsilon_x, \epsilon_y, \epsilon_z$	Strains in x, y and z directions
ϵ_0	Free thermal strain
γ_{xy}	Shearing strain
ξ, η	Local cartesian coordinates
θ	Time; angle between global and local coordinate axes; angle of twist
$\sigma_x, \sigma_y, \sigma_z$	Components of normal stress
τ_{xy}	Shear stress
Π	Total potential energy
κ	Thermal diffusivity
ρ	Density
λ, μ	Lamé's constants
λ	Eigenvalue
$[A]$	Diagonal matrix of exponential terms
ϕ	Arbitrary parameter in finite difference equation for time
ν	Poisson's ratio

Chapter 1

INTRODUCTION

1.1 Object

A study of the thermal behaviour of composite highway bridges, subjected to time-varying temperatures was undertaken using a 0.354-scale bridge model. The object was to establish an accurate method for predicting the redistributions of temperatures over a bridge cross section and the corresponding stresses and strains produced under transient temperature conditions in the environment. The three-span, continuous composite bridge used in this investigation is shown in cross section and elevation in Figure 1. The bridge, initially at laboratory temperature throughout, was cooled by suddenly exposing its entire upper surface to melting ice.

The thermoelastic analysis is uncoupled. The negligible effect of the coupling term on the temperature inside the bridge permits the heat conduction equation to be solved separately from the stress problem.

The variation in temperature over the complex cross section of the bridge required a two-dimensional heat conduction analysis. An exact solution for the temperatures is mathematically difficult to obtain. A finite element representation of the temperature distribution can easily be used for irregular cross sections composed of different materials. The temperature variation with time was approximated over even time intervals

by central finite difference expression. Instantaneous temperature distributions were calculated for successive time intervals. The time-stepping approach can involve boundary temperature changes. Within this temperature range, the heat transfer coefficients can be considered constant and the transient heat flow is described by a linear partial differential equation.

The structural response of the bridge to temperature changes at a given instant is determined from a knowledge of its internal temperature distribution. Its behaviour can be idealized as that of a rib stiffened plate. The three-span continuous bridge was represented as an assemblage of plane stress, plate bending, and eccentric beam elements. Their derivations were based on the assumption of small, quasi-static deformation theory for thin plates and shallow beams. These elements were made of isotropic and homogeneous materials stressed within their linear elastic range. Element rigidities were calculated for the uncracked section. It was assumed that the concrete could resist any tension stresses which would arise.

1.2 Nature of the Problem

North America has an extremely variable climate. The diurnal temperature fluctuation can generally be described as a wave function of time. The temperature usually rises during the daytime to a peak value and then drops to a nighttime low. It could rise again at dawn and the daily temperature fluctuations repeat themselves in a periodic manner. This cycle may be altered by the onset of warm or cold fronts. Highway bridges

are exposed to these temperature changes as well as to solar radiation, wind convection, rain and snow. Their exposure to weather conditions results in surface temperature changes and consequently nonuniform temperature distributions over bridge cross sections.

The maxima and minima of the annual temperature cycles vary for different climates. In summer, the bridge upper surface may be heated by the sun to a high of 140°F while the underside is exposed to ambient air at much lower temperatures. Or, there may just be side heating of the exterior girder. In cold weather, the temperature may reach lows far below 0°F.

The instant that the surrounding temperature is changed, the bridge will undergo a continuous redistribution of temperature varying with time. The temperature field in a bridge whose upper and lower surfaces are subjected to different temperature conditions is at least two-dimensional. The temperature varies primarily through the bridge depth. The variation of temperature across the slab width, due to the presence of the beams becomes negligible at sections away from the upper steel flange. At these points, the heat conduction in the slab is essentially unidirectional. There will be longitudinal temperature variations when the boundary conditions are nonuniform along the span or when the bridge has varying transverse cross section. As time passes with boundary temperatures remaining unchanged, temperatures within the bridge may eventually reach steady state condition.

Under atmospheric conditions, it is very unlikely that the surrounding temperature will be maintained sufficiently long to establish

steady conduction in the bridge. The low rate temperature response of concrete means a very slow penetration of temperature into the slab. Before the heat conduction can attain a steady state, a change in boundary temperatures will start another transient variation.

A bridge exposed to environmental conditions will continuously undergo various temperature distributions. The combination of a low and high thermal diffusivity of concrete and steel respectively has a significant effect on the magnitude of temperature gradients and the shape of the temperature distribution within the bridge. The steel beams respond quickly to sudden temperature changes. The concrete slab undergoes gradual adjustments in temperature. Various temperature gradients will exist within the exposed bridge.

The changes in temperature distribution from an unstressed reference state produce stresses and deformations. An accurate determination of the temperature history of the structure is required for the stress analysis.

The applicability of the proposed method for thermal analysis was established on a reduced-scale bridge model in the laboratory. Initially at uniform temperature its upper surface was suddenly and completely covered with melting ice. The temperature problem in which conditions are specified at the time of the temperature disturbance, is called initial-value problem. Its solution evolves from the initial values and reaches steady state after sufficient time has elapsed. Throughout the time history, the present study deals with time-dependent boundary conditions as would be the case in atmospheric conditions.

Thermal stresses are produced when the deformations are prevented from occurring freely, in a manner conforming with the temperature distribution. Constraints on these displacements are imposed by conditions prescribed on the boundary or by the continuity requirements of the material. There may be nonuniformity in the combined distribution of temperature change and coefficient of thermal expansion resulting from the temperature variation or from varying material properties. The displacements, proportional to their temperature change, will interfere with each other to form continuous single-valued functions. In stress analysis, the continuity restrictions are placed on the displacements by the compatibility equations. When there are no internal forces required to maintain continuity of displacements, a body free of external restraints will deform according to its temperature distribution. For such a temperature field, the thermal stresses are zero.

In conventional beam theory, the assumption that plane sections remain plane is made in place of the compatibility equations. Thermal stresses will result from temperatures that tend to warp the plane cross section. A necessary condition for zero stresses within a homogeneous and free body is a linear temperature distribution. Thermal stresses are caused by the nonlinearity in the temperature fields usually occurring in the transient heat flow. They reach a maximum during the early stages of the transient when temperature gradients are large.

In composite construction, there is interaction between the concrete slab and the steel beams. These materials have different coefficients of thermal expansion and undergo different temperature

variations. The temperature gradients in high conductivity steel are, as is known, smaller than those occurring in the concrete slab which has a lower coefficient of expansion. In this case, the combined effect in the slab and in the beams will reduce the composite structural action between them. The different thermal properties may have a reversed effect. The steel beams will quickly reach different temperatures while the temperature of the concrete slab remains unchanged. Combined with the high coefficient of thermal expansion, these temperature changes will produce large thermal stresses. Large stresses are also produced after a change in the upper surface temperature of the concrete slab. Before the temperature penetration reaches the steel beams, large temperature gradients will exist in the slab.

For statically indeterminate bridges, the external restraints provided at the supports will produce additional thermal stresses in the structure. The three-span continuous bridge cannot deflect freely with vertical displacement prevented by its supports.

In this study, both end supports were removed in order to measure and to visually observe the structural response of the bridge to temperature changes. The predicted and measured displacements were compared and the resulting stresses calculated.

Thermal stresses may overstress the exposed bridge when they are combined with stresses already existing as produced by dead load, live load, creep and shrinkage. The magnitude of the thermal stresses need not be very large to cause failure in the tensile zone of the concrete. These minute cracks may later develop into serious slab cracking. Under repeated

loading, additional cracks can be opened. The cracks, propagating throughout the bridge, would reduce the effective concrete section and would expose the reinforcing steel to corrosion. Due to excessive slab cracking, the bridge may become unserviceable. The importance of the thermal stresses warrants the extensive calculation necessary.

It is expected that the thermal loading applied in this investigation, will not stress the bridge beyond the yield point of the steel. The concrete stress was well below its compressive strength. The deflections when compared to the thickness of the slab were small. The proposed method of analysis was based on linear elastic behaviour of the materials. The stiffness of the concrete slab was computed based on the uncracked section. Under these conditions, the thermal effects can be additive to those resulting from dead and live loads.

Reduced section of concrete after cracking can be accounted in the calculation of thermal load and of elastic rigidity for the plate elements. Analysis can proceed in increments of the loading.

1.3 Review of Previous Research

This study is a continuation of the research in Reference (1) for the M.A.Sc. degree. Part of the thesis investigation appeared in Reference (2) by Berwanger and Symko. Thermal behaviour on bridge members, such as reinforced concrete T-beam and composite beam made of concrete slab and a steel beam, was investigated under various steady temperature conditions in the laboratory. The steady temperatures varying through the depth and

across the width were predicted using two different finite element models, linear temperature triangular and rectangular elements. The simply-supported beams were analyzed as plane strain problems. Their end surfaces were free of the initial restraints in the plane strain analysis. Linear strain rectangular elements were used to compute the strains and the thermoelastic stress distribution in the transverse cross section.

The basic theories of thermoelasticity leading to development of methods of analyses for composite bridges subjected to thermal loadings can be found in the textbooks of Timoshenko and Goodier (3), Gatewood (4), Boley and Weiner (5), Nowacki (6), Parkus (7) and Johns(8). The application of the thermoelastic theory to statically indeterminate beam problem involving simple and multiple spans is illustrated by Burgreen(9).

Zuk (10 and 11) has developed an analytical method, based on the theory of one-dimensional thermoelasticity, to determine the stresses in simple-span composite bridges made of homogeneous and isotropic materials. The temperature differentials between the upper and lower surfaces of the slab and beam were evaluated from observed data in Virginia, U.S.A.

Berwanger (12) has extended the analysis to reinforced concrete slab for composite beam members having simple and continuous spans. For continuous spans, the interior supports were removed and then the continuity was restored by applying a force. The deflections were calculated using the conjugate-beam method. Berwanger and Symko (13) have compared their results for a simple span composite beam.

The application of finite elements has been extended to thermoelastic analysis of plates. Gallagher, Padlog and Bijlaard (14) have

obtained a plane stress solution to an isotropic elastic plate having parabolic variations of both thickness and temperature across the width. In their analysis, the thickness of the linear strain rectangular element was constant but varied from element to element. The same plate problem was resolved by Weber (15) using linear strain rectangular elements with linearly varying thickness. Mufti and Harris (16), using a nonconforming cubic displacement triangular element for bending, have analyzed a simply supported square plate having 100°F temperature difference between its surfaces.

Plates with stiffeners have been mathematically modelled as plates having different flexural rigidities in two mutually perpendicular directions and were solved as orthotropic plates. The flexural rigidities provided by the stiffeners were uniformly distributed across the plate. The finite element method permitted a closer representation of the actual behavior of the structure. Mehra (17) has demonstrated the finite element displacement approach to simply-supported composite girder bridges subjected to static loads. The eccentrically stiffened plate problem has been analyzed using different types of elements. Gustafson (18) used rectangular elements having eight in-plane degrees of freedom and twelve out-of-plane. McBean (19) has suggested to use the 16 degrees of freedom conforming quadrilateral element for plate bending and the 16 degrees of freedom plane stress element. These elements, developed by Fraeijs de Veubeke, have nodes at each corner and at mid-sides. Wegmuller (20) has used the constant stress rectangular element (CSR) with the nonconforming rectangular plate-bending element (ACM) proposed by Adini, Clough and

Melosh. Mastrogeorgopoulos (21) has analyzed the steel girder in the composite bridge using quadratic stress triangular element (QST) in place of beam elements. His conforming but stiff triangular plate-bending element (HCT) was presented by Clough and Toucher. The proposed combination of elements adopted in this thermoelastic analysis of the continuous-span bridge has been presented by Desikan (22) and Bhalla (23) in their investigation of stiffened plates. There remains the problem of deriving the thermal load vectors for arbitrary temperature distribution.

The finite element method is applicable to problems that can be formulated in variational form. A variational principle exists for steady heat conduction. Steady temperature computations in linear problems using linear temperature elements of triangular and quadrilateral shape are outlined in Zienkiewicz and Cheung (24).

A variational principle is not available for the transient heat conduction equation in its original form. The governing differential equation can be transformed into a form which has a variational principle. Gurtin (25) has formulated a variational principle using convolution integrals. Wilson and Nickell (26) have applied the finite element method based on Gurtin's principle to linear problems in transient heat conduction. The special feature of this variational principle is that the initial conditions are included explicitly. The resulting system of integral equations at stationary values can be solved exactly by evaluating the eigenvalues or numerically by inserting time variation in a difference form. Wilson (27) has interpreted the system of finite element equations as the equilibrium of heat flow for each node of the finite element

idealization. Fujino and Ohsaka (28) formulated a finite element procedure using the variational principle for the Laplace transform of the equation governing heat flow. To obtain the inverse Laplace transform the resulting equations, after the variation, were uncoupled by using the nodal decomposition method. Visser (29) has derived a quasi-variational principle in which the time derivative was held constant during variation in temperature. The variational principle was formulated for only the spatial operations. The finite element formulation in the space variables yielded a system of first-order, linear, ordinary differential equations in the time variable. The exact solution of these semi-discrete equations was expressed in terms of an exponential matrix. Procedures for evaluating the exponential matrix by eigenvalues or in matrix series can be found in De Russo, Roy and Close (30). Zienkiewicz and Parekh (31) have used the method of weighted residuals-Galerkin, in place of variational principle, for the formulation of elements in space and time. The space elements employed in their analysis were the isoparametric type. Bruch and Zyvoloski (32) have also derived space-time elements based on the Galerkin method. Their differential equations resulting from only spatial discretization are identical to those obtained by variational means. Finlayson (33) has shown that, when the weighting and shape functions are identical, the variational method is equivalent to the Galerkin method. Lemmon and Heaton (34) have examined this relationship and the accuracy and stability of the methods including the finite difference method.

It has already been observed that all these methods lead to the same system of differential equations in time. No new computational

procedures were obtained. These differential equations can be integrated exactly by determining all the eigenvalues. Or, additional approximations are introduced with finite differences or Galerkin weighted residual procedure applied within each time interval. The use of the Galerkin process leading to a recurrence relation between values in a time increment has been demonstrated by Donea (35). In place of using successive time increments, the Galerkin procedure can be extended simultaneously to the whole time span divided into a finite number of elements or increments. This simultaneous application, by inter-relating the nodal temperature field at all time divisions, will lead to a larger system of equations.

Finite element solutions for a large number of transient flow problems have been obtained using various time integration methods. Mote (36) has used a time-marching method containing terms to second order of the matrix power series that define the exponential matrix. The integration time step can be selected for an allowable truncation error. This explicit method was applied to the heat conduction analysis of a rotating turbine disk. The exponential matrix for the exact solution of ordinary differential equations can be approximated using Padé rational matrix expansions as shown by Varga (37). Emery and Carson (38) have studied the accuracy of linear, quadratic, and cubic temperature elements for linear heat flow problems. They replaced the time derivative by finite difference approximations. Rybicki and Hoper (39) have taken into consideration thermal properties continuously varying with position. A higher order rectangular element, which was developed by Bogner, Fox and Schmit for plate bending, was used in space. The variation with time was

approximated by the backward-difference expression. Richardson and Shum (40) have applied Wilson and Nickell approach to problems involving nonlinear boundary conditions such as radiation. Nonlinearity also arises when the thermal conductivity is dependent on temperature. Aguirre-Ramirez and Oden (41) have demonstrated the application of the finite element method to nonlinear problems.

The development of the finite element method has been extended to transient flow problems related to water resources. Guymon (42) has transformed the spatial operators of the convective diffusion equation so that the variational procedure outlined for heat conduction can be followed. The resulting differential equations were solved numerically by Guymon, Scott and Herrmann (43) using the Adams-Moulton multistep method.

The computational effort in determining the eigenvalues for large systems of differential equations can be reduced by using the master-slave relation as called in the field of structural dynamics. This procedure is based on insight into the behaviour of the system. A number of temperature degrees of freedom can be eliminated by condensation in the matrices and fewer degrees of freedom are used to find the eigenvalues. Gallagher and Mallett (44) have presented this procedure in the heat conduction analysis by finite elements.

Knowledge regarding the properties of steel and concrete is required for accurately predicting the structural response of the bridge. The properties involved were the Young's modulus, Poisson's ratio, coefficient of thermal expansion, and the thermal diffusivity which combines the thermal conductivity and specific heat with mass density. The

steel properties are well known for the conditions in this analysis. The elastic and thermal properties coefficients for concrete are related to temperature, age, and its composition which includes the mix proportions, type of cement and aggregate, volume of air voids and moisture content. Berwanger (45) has investigated the behaviour of the modulus and coefficient of expansion of plain and reinforced concrete at different ages, temperature, and water contents. Further tests were carried out by Sarkar (46) and the results published by Berwanger and Sarkar (47).

The thermal conductivity of concrete varies mainly with its water content. Lentz and Monfore (48) have determined that its temperature dependence within the temperature range of 0°F to 750°F is not severe and the coefficient can be considered constant. Brewer (49) has correlated the conductivity with the unit weight for concrete from oven-dry to fully saturated. Other conductivity investigations undertaken were by Campbell-Allen and Thorne (50) and by Thompson (51).

Several studies of temperature effects in highway bridges have been made. A detailed literature survey regarding thermal calculations for bridge design is given by Reynolds and Emanuel (52). They concluded that there exist uncertainties as to the magnitudes and effects of thermally induced stresses and movements in bridges. Hunt and Cooke (53) have used finite difference methods based on the one-dimensional linear flow of heat to predict the unsteady temperature distributions over different thicknesses in the bridge cross sections. A concrete box girder bridge in the stress analysis was considered to be made of vertical and horizontal isotropic, homogeneous plates. The stresses were calculated by solving

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two-dimensional plane-strain problem in linear thermoelasticity. The resultants of the end restraints imposed by the plane strain assumption were superimposed. Hulsey (54) has modified a two-dimensional finite element programme presented by Wilson and Nickell to include nonlinear time-dependent boundary conditions in a stepwise manner. Temperatures inside a typical composite-girder bridge were predicted for the conditions recorded by a weather bureau. Part of an interior steel girder with slab was discretized into an irregular grid of linear temperature elements. The procedure in bringing together the slab and steel beam by satisfying compatibility of interface strains and curvatures was extended into a form of flexibility method. Following his analytical study, Emanuel and Wisch (55) have tested a model of a two-span composite bridge for thermal effects. The heating of the entire upper surface was made with a series of heat lamps. Thepchatri (56) has determined the thermal effects of daily atmospheric variations existing in Texas on three types of highway bridges such as concrete slab bridges and composite bridges having precast pretensioned concrete beams or steel beams. The bridges were analyzed as a uniform system of beams. The two-dimensional temperature distribution was obtained for a system of linear temperature elements by using a modified Wilson and Nickell method. The calculation of temperature stresses was based on elastic beam theory and the principle of superposition. As a joint project, Will (57) has used temperature data from field tests to investigate the thermal behaviour of a concrete slab bridge and a composite bridge with precast beams. The stresses corresponding to the computed internal temperatures were obtained from a finite element analysis employing plane stress and plate bending elements.

1.4 Scope of Present Investigation

This study included the measurement of temperatures and deformations during the cooling of the model bridge and, in addition, the derivation of mathematical solution for the time histories of temperatures, deformations and stresses. The model of a concrete-steel type bridge continuous over three spans was constructed in the laboratory at a 0.354 scale. The model bridge was tested under transient temperature condition with melting ice suddenly applied to its entire upper surface. The temperature of the air surrounding the lower surfaces has gradually decreased from its initial temperature. During the 77 minutes of cooling, the temperature and deformation readings were taken simultaneously at various time intervals, much more frequently at the early stages. Temperature measurements were confined to a transverse cross-sectional plane near the middle of the bridge. Forty-two thermocouples and forty electrical-resistance strain gauges were connected to a data acquisition system for rapid readings. The upper and lower surface strains in the four steel beams were measured at five different sections. Dial indicators were used to measure vertical displacement at each end of the four steel beams. The end piers were removed to allow displacement measurements for only temperature changes. Description of model, instrumentation and test procedure is contained in Chapter 2.

Chapter 3 is concerned with the determination of distribution of temperature in the cross-sectional plane of the bridge and its temperature history. The transient temperature distribution following the uniform temperature decrease on the entire upper surface was considered to be the

same for all transverse cross sections. The heat transfer coefficients

were taken as constants throughout the entire cooling period. A

variational principle was used in the space variables for the finite element formulation. Its application to the two-dimensional problem had reduced the partial-differential equation to a system of first-order, linear ordinary differential equations with time remaining as the independent variable. From symmetry, only two different parts of the transverse cross section needed to be modelled for the analysis of the entire temperature field. These subsections were represented as a system of equal-sized rectangular elements having linear temperature variation. The time derivatives, in turn, were replaced by finite differences using the Crank-Nicolson procedure. The temperature history was obtained in 462 time steps of 10 seconds. For the respective finite element models, each time step required solutions of 1652 and 882 simultaneous algebraic equations with the same half bandwidth of 13.

The procedure for solving large number of equations is explained near the end of the chapter. For symmetric systems of simultaneous equations, only the coefficients in the half bandwidth were stored in the form of blocks on disk files. The calculations were carried out with only two submatrices in the computer memory banks at a time. The solution routine was based on Cholesky's decomposition method with backward and forward substitutions repeated every time the system of equations was re-solved for the succeeding nodal temperatures.

Chapter 4 describes the method for predicting the structural response of the bridge to transient temperatures by the use of the finite

element displacement approach. The analysis treated the model bridge as a flat thin plate with eccentric beam stiffeners. The structure was divided into a system of triangular plate elements for bending and stretching, and of eccentric beam elements with St-Venant torsional rigidity. The derivations of the elements were based on the assumptions of small deformation theory for thin plates and shallow beams. Isotropic, linearly elastic material for the beams and the slab was assumed. The elastic rigidities were computed for uncracked sections. The types of elements used were the 18-degree of freedom triangular bending element having constrained quintic displacement function, the quadratic plane stress triangular element having 6 degrees of freedom at each vertex and the eccentric beam element matching the in-plane and out-of-plane displacement functions of the plate elements. These elements are fully conforming and yield the same degree of accuracy in the strain energy. The thermal loading was represented by matrix vector derived for arbitrary temperature distributions. The double symmetry of the problem permitted modelling of only one-quarter of the three-span bridge. Finite element representation in a form of a 7×30 uniform grid was considered. The resulting stiffness matrix was in the order of 2976 and had a half bandwidth of 108. Uplift occurring at the interior supports when the bridge deflected in two orthogonal directions was considered in the finite element model. Prior to uplift, changes in reactions at the supports were computed.

The results and the effects of the transient temperatures on the structural behaviour of the bridge are discussed in Chapter 5. Conclusions based on this study are drawn in Chapter 6 which includes recommendations for further studies.

In Appendix C, the changing of the temperature distribution within the cross section of the cooled bridge is shown by plotting the temperature fields at various time intervals. The computed stresses and deformations are presented at same time intervals. The end deflections are compared throughout the cooling time.

CHAPTER 2

EXPERIMENTAL INVESTIGATION2.1 Outline of Test

A 0.354 scale model of a prototype bridge was built and tested for temperature varying with time. Measurements of temperatures, deflections and strains were made on the model. The steel-concrete composite bridge model of constant cross section along the three spans is shown in Figure 1. Its entire upper surface was suddenly covered with melting ice to produce two-dimensional temperature field varying with time over the transverse cross sections. The ice remained on the surface for a duration of an hour and seventeen minutes. The other surfaces were exposed to the ambient air in the laboratory whose temperature of 78.30F slightly declined to 71.80F. The continuous bridge was free of redundant constraints which would prevent thermal deflections. Uplift at the supports was not restrained. The bridge deck was permitted to bend into a dished surface. The readings from thermocouples, electrical resistance strain gauges and from displacement dial indicators were recorded simultaneously at various time intervals during the transient-state behaviour.

2.2 Scaled Model

The experimental investigation was undertaken on a reduced scale model of a highway bridge that spans the Red River near Ste-Agathe,

Manitoba. The model, shown in Figure 1, was made to a 0.354 scale which reduced the overall size to approximately 53 ft. long and 10½ ft. wide. Part of the bridge is shown from ground level in Figure 4.

The choice of the dimensional scale factor was based on the availability of standard wide-flange sections capable of representing the supporting steel beams in the bridge. A proper thickness for the concrete slab was required to assure temperature gradients of measurable and realistic magnitude. Small temperature differences usually exist between the outer faces in thin plates. The model slab was nearly 2½ in. thick. It is continuous in the transverse direction over the supporting longitudinal steel beams.

All structural details were reproduced, as shown in Figure 2.

Stringers

The 0.354 scale factor permitted the use of the standard W10x15 steel beams to model the W36x104 steel beams used in the prototype. The structural steel is of type CSA G40.12 with a specified minimum yield stress of 44 ksi and a specified minimum tensile strength of 65 ksi. The steel manufacturer was unable to ship the full length of the W10x15 beams. It was necessary to splice the 33 ft. 2-3/16 in. and 20 ft. 5½ in. beams. The bolted splices were placed at only one point of inflection in the middle span.

To provide for composite action, 3-1/8 in. long channel shear connectors, C 1.5 x 1.7 were welded to the upper surface of the flange. The shear connectors were not structurally redesigned for this bridge model

but were only scaled from the prototype. The same number of shear connectors was used in the model and the spacing is shown in Figure 2. The welding of the shear connectors had initially deflected the steel stringers from their original straight lines. The initial curvature is larger in the shorter beam. When the steel beams were erected, the average initial deflection at the end of the four longer beams was $1\frac{1}{2}$ inches as compared to 2 inches at the other end of the bridge.

Concrete Deck

The ready-mixed concrete for the slab was delivered by a local supplier. A normal weight concrete with a 28-day compressive strength of 4,000 psi, a 3 inch slump, 5% entrained air, a retarder for 2 hrs, and a $\frac{3}{8}$ " maximum size aggregate was specified. The concrete had a water-cement ratio of 0.54. The mix proportions of a cubic yard consisted of 610 lbs of normal type 10 Canada cement, 330 lbs of water, 1490 lbs of fine aggregates and 1670 lbs of coarse limestone aggregates. The coarse aggregates were extracted and crushed at the company's quarry in Aylmer, Quebec. The sand came from a deposit at Wilson's Corner, 15 miles north of Hull, Quebec. To provide workability in placing the concrete, 3 oz. of air-entraining admixture, Darex had been added to the mix. For retarding the setting time of concrete by 2 hrs., Deratard HC type ($3\frac{1}{2}$ oz per 100 lbs of cement) was used.

In the slab construction, its size was much too large for placing and finishing the concrete in one operation. Segmental construction was reproduced in the model. Such construction without shoring was divided

into five stages. The casting of the deck began on August 11, 1976 and continued until completed on September 9, 1976. The sequence with the casting dates is shown in Figure 2. The placing of the concrete terminated with construction joints at points of inflection. Lateral movement across these joints is prevented by maintaining the $\frac{1}{4}$ in. diameter reinforcing steel continuous.

The sequence of casting the deck was designed to minimize the negative bending moment in the concrete at the intermediate piers of the three-span continuous bridge. The steel stringers alone carry the slab's dead load before becoming composite. Negative bending moments would produce tension in the concrete slab where cracking can easily occur.

As soon as the fresh concrete had sufficiently hardened, the slab was covered with wet burlap. The burlap was kept continuously wet for a period of 7 days.

Concrete control cylinders of 3 x 6 inches were made for every half section cast in order to determine the compressive strength and modulus of elasticity at the day of the model test. Values of elastic modulus were determined from longitudinal strain measurements on the compression cylinders. The initial tangent modulus was computed using the method of least squares. The elastic modulus and compressive strength evaluations were based on one cylinder. The average modulus values for segments 1 to 5 were found to be 6.0, 7.8, 5.1, 6.7 and 6.0 (in terms of 10^6 psi) and the average ultimate cylinder strengths 6900, 7400, 6400, 7200 and 6800 psi, respectively.

Reinforcing steel

The structural action of the slab is essentially one-way. The loads are carried by the slab in the direction perpendicular to the traffic. Steel was provided in the longitudinal direction as temperature and shrinkage reinforcement. The reinforcing layout is shown in Figure 2. The tensile and compressive steel ratios shown in the figure are given for a doubly reinforced slab resisting positive and negative moments occurring between and at the steel beams respectively. The main reinforcement has tensile steel ratio varying from 0.0198 to 0.0314 and the compressive steel ratio from 0.0071 to 0.0136. For longitudinal reinforcement, the small steel ratio was 0.006.

The slab was reinforced with two different sizes of steel bars. The no. 3 deformed billet bars whose properties are defined in ASTM designation A615 have a 60 ksi specified minimum yield point and a minimum tensile strength of 90 ksi. The $\frac{1}{4}$ in. (no. 2) hot-drawn plain bars with specified structural grade had a yield point of 58,500 psi and ultimate tensile strength of 70,800 psi.

Bridge bearings

In bridge construction, one end is pinned. At that end, there is no longitudinal movement but only rotation. At the opposite end and at interior piers, the stringers are supported on a rocker and rollers to permit both rotation about their supports and longitudinal displacement. Details of the hinged and rocker-roller bearing are outlined in Figure 2. The bearings were not designed for the possibility of uplift.

The gap caused at both end supports by the initial curvature in the stringers was not completely closed with steel plates added onto the bearing plate. With the ends remaining unsupported, the upward deflections produced at the ends by temperature changes within the three-span bridge can be measured. Otherwise, measurements of the changes in the support reactions would have been required in order to determine the bridge behaviour before uplift occurred.

2.3 Instrumentation

All transient measurements were made with thermocouples, electrical resistance strain gauges, and displacement dial indicators. The measurement points are given in Figure 3. An electronic data acquisition system was used to automatically record readings of the large number of thermocouples and strain gauges at various time intervals. The B & F data acquisition system, model SY 256-7 can be seen in Figure 5. It was equipped with a paper-strip printer. The recording speed was reduced by using the paper-printer to twenty channels per second when in a one microvolt resolution mode. Simultaneously with the temperature and strain measurements, changes in the dial indicators were visually observed.

Temperature measurements

The recording capacity of the data acquisition system was limited to 42 thermocouples. Temperature measurements were concentrated only in the transverse cross section near the middle of the center span as shown in Figure 3a. The points, at which temperature measurements were made, are

given in Figure 3b. From this thermocouple arrangement, measurements at corresponding points can be compared. This would indicate whether a uniform thermal loading procedure was obtained resulting in a symmetric temperature distribution through the transverse cross section. With a limited number of thermocouples connected to the data acquisition system, uniformity of temperatures along the span was not verified.

An attempt was made to place thermocouples inside the concrete slab. It was difficult to maintain accurate location of thermocouples and there was the risk of inadequate compaction of fresh concrete. A form should have been designed to ensure fixed position of thermocouples that will give readings of temperatures at the desired points.

The butt-welded thermocouples were made of #25 gauge lead wires of copper and constantan. With the use of smaller diameter leads, a quicker response to fast-varying temperatures can be obtained.

The heat conduction in the leads was minimized by stripping the layer of insulation for more exposure of the bare thermocouple wires. In the thermocouple installation, the lead wires and the measuring junction were maintained on the same horizontal line where all points are at same temperature. There will be no heat conduction along the thermocouple wire at same temperature as its junction. A view of thermocouples installed between two supporting beams is given in Figure 3c.

The thermocouple potentials, based on a 1500F reference junction, were recorded in millivolts. The temperatures were measured to $\pm 0.05^\circ\text{F}$.

Strain measurements

Strains on the upper and lower surface of all four steel beams, were measured at five different sections where maximum and minimum bending moments are produced by a uniformly distributed gravity load. The locations of forty strain gauges are given in Figure 3a. The foil-type strain gauges were KYOWA model KFC 5-01-11. These 120Ω strain gauges had a gauge factor of 2.12 and were temperature compensated for steel having thermal expansion of 6.0 PPM/°F. The strain gauges were used in a single-active arm, three-wire lead system. All strain gauges were bounded with Mirco-Measurements M-Bond AE-10 adhesive cured six hours at 75°F. A protective coating, Micro-Measurements M-Coat A, was applied over each gauge installation followed by a waterproofing overcoat of M-Coat G for gauges only on the upper surface of the steel flange. The strain gauges were installed in May 1976. Readings were recorded in a least count of one microstrain.

Deflection measurements

Uplift at each end of the four stringers was observed visually by reading dial gauges with graduations of 0.01 mm. In Figure 5, a set of four dial indicators can be seen mounted at the end of the steel beams.

2.4 Test Procedure

Initially, the bridge in the laboratory was at uniform, room temperature of 78.3°F. Within two minutes, the upper surface was completely covered with melting ice as shown in Figure 5. It required 56

bags of ice cubes, each bag weighing 45 lbs. Simultaneous change in temperature on the entire upper surface will produce symmetric temperature fields over the geometrically symmetrical transverse cross section and same two-dimensional temperature distribution for every transverse section. After an elapsed time of 77 minutes, bare spots appeared on the surface as most of the ice had melted away and the cooling test was terminated. As the upper surface was no longer at uniform temperature there would be temperature variation along the span. It would take approximately 2 hrs. for the temperatures in the bridge to reach steady state, provided the cooling process continued.

During the cooling test, the large quantity of ice gradually reduced the ambient-air temperature to 71.8°F. The upper surface was not directly exposed to melting ice. It had a 4 mil polyethylene sheet to prevent water seeping into the concrete. Change in water content of concrete will affect the material properties. The swelling effect will evidently disrupt deflection and strain measurements. The deck surface temperature underneath the polyethylene sheet will exponentially approach a value of 36.7°F, in time. The deflection, strain, and temperature readings were recorded simultaneously at various time intervals during the 77-minute cooling period.

The cooling test on the bridge has been conducted in March 23, 1977. On the same day, 6 x 12 in. concrete cylinders were tested for compressive strength and modulus of elasticity.

CHAPTER 3 ,

TRANSIENT HEAT CONDUCTION ANALYSIS3.1 Temperature Problem

At zero time, the three-span composite bridge was cooled from the initial temperature state 78.3°F in the laboratory by suddenly covering its entire upper surface with melting ice for 77 minutes. During the cooling period, a steady temperature of the ambient air underneath the bridge was not maintained. It gradually decreased to a low of 71.8°F.

The most important temperature changes in the bridge occur through the depth and along the slab width. A transverse cross section in rectangular coordinate system x, y is shown in Figure 6. In the longitudinal direction z , the effect of end temperature and the temperature variations in the vicinity of the shear connectors may be neglected so that there is no heat flow along the span lengths. The temperature field to be determined is essentially two-dimensional, as $t(x, y, \theta)$ at an instant θ in time. Within this temperature range, the physical properties such as the thermal conductivity k , the specific heat c with specific weight ρ of the materials, concrete and steel, were taken to be constant. The thermal diffusivity of the pertinent materials is defined as $\kappa = \frac{k}{c\rho}$.

For temperature-independent properties, the transient heat conduction problem is linear. The governing partial-differential equation of the parabolic type with independent variables x, y and θ is given as

$$\frac{\partial}{\partial x} \left(k \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial t}{\partial y} \right) = c\rho \frac{\partial t}{\partial \theta} \quad (3.1)$$

The initial and boundary conditions are

a) initial temperature, $t = 78.3^{\circ}\text{F}$ at $\theta = 0$ throughout the section

b) specified temperature, $t = t_s(\theta)$ for $\theta > 0$ on the entire upper surface

At the time of sudden application of melting ice, the deck surface temperature was not instantaneously changed to and maintained at melting ice temperature of 32°F but was lowered to a time-varying temperature $t_s(\theta)$.

Some time was required for the temperatures to penetrate the 4 mil polyethylene sheet and reach the upper surface. The imposed surface temperature becomes a function of time $t_s(\theta)$. The polyethylene sheet was placed over the entire slab to prevent the concrete from being soaked and subsequent swelling.

c) convection to surrounding air, $q = h(t_g(\theta) - t)$

where q is the rate of convective heat transfer from a unit area surface through an air boundary layer whose outside slowly-varying temperature $t_g(\theta)$ is higher than the surface temperature t . The large quantity of ice cooling the upper surface had a considerable effect on the room temperature which continued to gradually decrease with time θ . The values of surface conductance h were considered to be uniformly distributed along vertical and horizontal surfaces and constant throughout the entire cooling period.

c) adiabatic surface or planes of symmetry, $\frac{dt}{dx} = 0$ for temperature gradient.

Equation (3.1) is the uncoupled heat conduction equation where the thermomechanical coupling term $(3\lambda + 2\mu)\alpha t_0 \frac{\partial \epsilon}{\partial \theta}$ has been excluded from the right side of the equation. The derivative $\frac{\partial \epsilon}{\partial \theta}$ denotes strain rate t_0 the reference temperature, α the linear coefficient of thermal expansion and λ, μ the Lamé's constants. When the coupling term is significant, the thermoelastic theory requires the simultaneous determination of temperature and deformation. The effect of the heat generated from deformation is extremely small on the temperature in the bridge that the coupling term can be disregarded. The temperature distributions were determined independently of the deformations.

The two-dimensional temperature distributions following the imposed initial condition were obtained by numerically solving the partial differential equation for successive time intervals required to cover the total elapsed time of the cooling process. The space-time dependent, partial-differential equation was first discretized by finite elements only in space while the time dependence remained continuous. This semi-discretization yielded a system of first-order, linear, ordinary differential equations in time. These equations can then be integrated directly or numerically. The numerical solution of the differential equations was based on the mid difference approximation in time. The transient temperatures at the boundaries were represented as a series of step changes corresponding to each successive time step.

3.2 Spatial Discretization by Finite Elements

The finite element method is based on variational principle in obtaining numerical solutions from stationary condition of functional. The

functional is defined in terms of the unknown function. The function that renders the functional stationary in value is the solution to the governing differential equation of the given problem.

At equilibrium state, the functional I is stationary. Its first variation vanishes, $\delta I = 0$. For the first variation of the functional to be equal to zero, the Euler-Lagrange equation in the functional variation δI must be satisfied. The functional at stationary conditions yields a system of equations from which approximate solutions are obtained. The solution to these equations will be the stationary function of the functional. These Euler-Lagrange equations are the equilibrium equations of a system. Its behaviour is described by differential equation of the problem.

The governing differential equations that are self-adjoint can be derived as Euler-Lagrange equation. When the operator of the differential equation is not self-adjoint, an extremum principle does not exist for the equation in its basic form. The functional is stationary without a maximum or minimum value, and in the general case no variational principle exist.

For the functional to be minimum, its second variation must be positive, $\delta^2 I > 0$ and the functional is said to be positive definite. The true function, that satisfy the Euler-Lagrange equation, gives the functional an absolute minimum value. The functional for an approximate function will be greater than the true value, $I_{\text{approx.}} > I_{\text{true}}$.

The true function is unknown. The solution can be expanded in terms of trial functions for which the functional has its stationary value near the true value. This function must satisfy the boundary conditions and should be continuous with its derivatives to one order less than the

highest derivative appearing in the functional. Material continuity is assured. The highest derivative need not be continuous but it must exist otherwise there is no meaning to the problem.

The heat conduction problem is expressed only in differential form. The differential equation (3.1) is not self-adjoint and there is no functional. A variational principle can be formulated for only the spatial operator which is self-adjoint. The time derivative was held fixed during variation. This approach is a quasi-variational principle. The first variation is

$$\delta I = \int_A \left[k \frac{\partial^2 t}{\partial x^2} + k \frac{\partial^2 t}{\partial y^2} - \rho c \frac{\partial t}{\partial \theta} \right] \delta t \, dx \, dy - \int_s h(t_g - t) \delta t \, ds = 0. \quad (3.2)$$

After space integration, the functional becomes

$$I = \frac{1}{2} \iint \left[k \left(\frac{\partial t}{\partial x} \right)^2 + k \left(\frac{\partial t}{\partial y} \right)^2 + \rho c \frac{\partial t^2}{\partial \theta} \right] dx \, dy + \int_s \frac{1}{2} h(t_g - t)^2 \, ds. \quad (3.3)$$

In the quasi-variational principle, the functional subject to variations in t is taken as stationary. Based on the true variational principle in the classical sense, the variational integral is not stationary. The variations in t were restricted as function only of x and y by maintaining the time derivative constant. The variation cannot be arbitrary and must remain the same for all time. Furthermore, the governing differential equation is not the Euler-Lagrange equation required for stationarity of the functional.

Application of the quasi-variational method is equivalent to the Galerkin method. The variational method can be regarded as a form of the weighted residual method. The Galerkin method requires the governing

differential equation. It is applicable when there may be no variational principle for the problem.

The finite element method provides a means for approximating the unknown temperature field throughout the cross section of the bridge by a piecewise construction. The cross-sectional area was subdivided as shown in Fig. 6 into a finite number of discrete elements. The elements are interconnected along their boundaries at assigned points that are called nodes. In each element a temperature function is assumed in terms of the temperature at the nodes. The solution at an instant in time is assumed in the form of piecewise function containing unknown nodal temperatures. A functional for each element can be written in terms of its temperature nodes so that the entire functional may be expressed as a summation of these functionals.

The variation of the functional is governed by nodal temperatures. The functional approximated by an assumed temperature function is considered, in quasi-variational principle, stationary with respect to t . The first variation vanishes for variation δt of all nodal temperature t_1

$$\delta I = \frac{\partial I}{\partial t_1} \delta t = 0. \quad (3.4)$$

The stationary condition yields, for each element, a system of linear ordinary equations of first order. These equations can now be expressed in matrix form

$$\frac{\partial I}{\partial t_1} = [C_e]\{\dot{t}\} + [K_e + H_e]\{t\} - \{f_e\} = 0. \quad (3.5)$$

The element capacity matrix $[C_e]$ was derived from the part of the functional associated with time while the space-dependent part gave the

conduction matrix $[K_e]$ for an element. The element convection matrix, denoted by $[H_e]$ is formed when the element surface is exposed to an ambient air temperature specified in $\{f_e\}$. The scalar, nodal temperature field is denoted by $\{t\}$ and $\{\dot{t}\}$ as its derivative.

Assembling the elements into a representation of the cross section and applying the boundary conditions

$$\sum ([C_e]\{\dot{t}\} + [K_e + H_e]\{t\} - \{f_e\}) = 0$$

the resulting system of differential equations in time can be written as

$$[C]\{\dot{t}\} + [K]\{t\} = \{F\}. \quad (3.6)$$

For the assembled structure, $[C]$ becomes the capacity matrix and $[K]$ the conduction matrix. Both matrices are symmetric and positive definite. Necessary and sufficient condition for positive definiteness is that all its eigenvalues be positive or that the determinant be positive. The specified nodal temperatures were transferred into $\{F\}$ which is also composed of the coefficients of $\{f_e\}$.

In this formulation, the time variable remained continuous. Only space domain was discretized. The problem was reduced to solution of the system of differential equations given in (3.6). These semi-discrete equations can be integrated exactly. Or, they are discretized by piecewise expansion of the temperature in time. The temperature history will consist of a sequence of approximations at the ends of successive time increments. The time-marching solution requires, at each time step, solution of a system of simultaneous algebraic equations.

3.3 Steady-Temperature State

Once the temperatures have reached steady state, the temperatures will remain constant everywhere with time. The rate of temperature change is zero. Equation (3.1) becomes an elliptical partial-differential equation

$$\frac{\partial}{\partial x} \left(k \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial t}{\partial y} \right) = 0. \quad (3.7)$$

A variational principle exists for steady state. The Euler-Lagrange equation is the governing equation (3.7) of steady heat conduction theory. Using the stationary valued principle, the functional was derived from the differential equation and the integration by parts yields

$$I = \frac{1}{2} \iint \left[k \left(\frac{\partial t}{\partial x} \right)^2 + k \left(\frac{\partial t}{\partial y} \right)^2 \right] dx dy + \int h \left(\frac{1}{2} t^2 - t_g t + \frac{1}{2} t_g^2 \right) ds. \quad (3.8)$$

As stated in the book by Finlayson (33) on page 224, this functional can be physically interpreted in terms of q the rate of heat flow. It is given as

$$I = \frac{1}{2} q (\Delta t_g) \quad (3.9)$$

where Δt_g is the temperature potential for heat flow.

This is a quadratic functional. The temperature and its derivatives appear in quadratic form. At equilibrium state which corresponds to steady-flow condition, the quadratic functional has an absolute minimum. Its second variation $\frac{1}{2} \delta^2 I$ is positive. For all admissible variations δt

$$\delta^2 I = \iint \left[k \left(\frac{\partial \delta t}{\partial x} \right)^2 + k \left(\frac{\partial \delta t}{\partial y} \right)^2 \right] dx dy + \int h \delta t^2 ds > 0. \quad (3.10)$$

Error bounds for the approximate solution can be deduced from variational principle. The functional for an approximate temperature field that satisfies the boundary condition will be greater than the true minimum. The rate q of heat flow in Equation (3.9) is then an upper bound to the exact value. Upper or lower bounds for temperature can be derived from the flow rate. In the finite element construction of the temperature field, the continuum theory requires that the temperatures be continuous across the element boundaries. The finite element solutions will then monotonically converge to the exact solution as the mesh size progressively decreases.

The Euler-Lagrange equations are the equilibrium equations of heat-flow. The amount of heat entering and leaving must be equal. Since the functional is quadratic, these equations in matrix representation as

$$[K]\{t\} = \{F\} \quad (3.11)$$

will be linear and symmetric. The matrix $[K]$ is positive definite for quadratic functional having an absolute minimum.

3.4 Exact Integration of First-Order Differential Equations

The nonhomogeneous system of differential equations (3.6) with constant coefficients can be re-written as

$$\{\dot{t}\} + [A]\{t\} = \{z\} \quad (3.12)$$

where the matrix $[A] = [C]^{-1}[K]$ and $\{z\} = [C]^{-1}\{F\}$.

By integration, the exact solution to these differential equation is

$$\{t\}_\theta = [e^{-A\theta}]\{t\}_{\theta=0} - [e^{-A\theta}][A]^{-1}\{z\} + [A]^{-1}\{z\}. \quad (3.13)$$

Should steady state prevail $\theta \rightarrow \infty$, the exponential matrix vanishes and the

temperature distribution becomes steady

$$\{t\}_{\infty} = [A]^{-1}\{z\} = [K]^{-1}\{F\}. \quad (3.14)$$

The exponential function of the matrix is defined as the infinite series

$$[\exp (-A\theta)] = [I] - [A]\theta + \frac{[A]^2\theta^2}{2} - \frac{[A]^3\theta^3}{6} + \frac{[A]^4\theta^4}{24} - \dots = \sum_{m=0}^{\infty} (-1)^m \frac{[A]^m\theta^m}{m!}. \quad (3.15)$$

Or, using the spectral decomposition, the exponential matrix is given as

$$[e^{-A\theta}] = [M][\Lambda][M]^{-1} \quad (3.16)$$

where each column in the modal matrix $[M]$ is the characteristic vectors $\{u_i\}$, commonly known as the eigenvectors of matrix $[A]$, corresponding to the characteristic values or eigenvalues λ_i . The diagonal matrix $[\Lambda]$ contains exponential terms $e^{\lambda_i\theta}$.

The transient solution takes the form

$$\sum_{i=1}^n c_i e^{\lambda_i\theta} \{u_i\}$$

where c_i are constants and n is the order of the matrix. The closed-form solution to the differential equations requires the solution of the generalized eigenproblem

$$[K + \lambda C]\{t\} = 0. \quad (3.17)$$

All computational effort is placed on finding the eigenvalues with their associated modes. The coefficient matrix $[A]$ by having all negative eigenvalues, is negative definite eigenvalues are real. The fact that these eigenvalues have negative real parts assures that the solution of the linear system $[C]\{\dot{t}\} + [K]\{t\} = \{F\}$ is asymptotically stable. The transient part of the solution tends to zero as $\theta \rightarrow \infty$. The temperatures are approaching equilibrium (steady condition) exponentially.

The applications of methods requiring eigenvalues are limited by computer storage and computation time. When high order matrices are involved, the exact integration procedures for these differential equations are abandoned in favour of numerical integration methods.

3.5 Numerical Integration of First-Order Differential Equations

The time span is divided into a number of finite increments $\Delta\theta$. The temperature $t(x,y)$ in the time history is obtained at the discrete intervals using recurrence relations for advancing the solution. A relation can be established between two successive values, t_n and t_{n+1} , or for several intervals. The approximate solution at the end of a time increment is formulated in terms of values at previous time. Starting from the initial condition, the computation can proceed for every successive time step in a time marching manner, until the complete temperature history is generated.

The temperature variation within a time increment $\Delta\theta$ may be assumed in finite difference form

$$t_{n+1} = t_n + [(1 - \phi)t_{n+1} + \phi t_n] \Delta\theta \quad (3.18)$$

where ϕ is a parameter ranging from 0 to 1 whose value indicates the type of difference equations. The finite difference substitution effects a transformation of the differential equations (3.6) into an algebraic form for the nodal temperatures. At each time step, the system of linear algebraic equations will require solution.

In the time marching process, less integration steps are required to reach the desired time with larger increments. There are severe

restrictions imposed on the size of the time increment. The selection of the time step depends on the nature of the differential equation and the type of time integration method employed. Large time increments will increase the error in approximating the solution or they may lead to oscillation and instability in the solution.

Stability Analysis

From the mathematical point of view, this system of differential equations in (3.6) is said to be stiff. Stiffness is measured by its maximum-to-minimum eigenvalue ratio which is much greater than one. The transient solution $\sum c_i e^{\lambda_i \theta} u_i$ is described by widely varying exponential terms decaying with time. The reciprocals of the eigenvalues are known as time constants. The components $e^{\lambda_i \theta}$ with large negative eigenvalues will decay more rapidly as time progresses and will quickly vanish while the contributions from the others having smaller negative eigenvalues to the solution is still very significant. The effect of these rapidly decaying exponentials on the solution is apparent at early stages. The speed of response to temperature change is indicated by the decaying exponentials.

After a short time of the surface temperature disturbance, the cooling rates $\frac{dt}{d\theta}$ will diminish as the exponential terms with small time constants fade away. The temperatures near the suddenly exposed surface are very quickly changed. Changes in temperature deep inside the section begin after the rapidly decaying exponentials have vanished. With the remaining exponential terms, the rates of temperature change gradually decrease. The temperatures decrease exponentially with time and asymptotically approach a steady value.

The difficulty in solving these stiff differential equation numerically lies in the approximation of the series of exponential terms, each decaying at their own different time rate. The numerical methods must be able to approximate the rapid temperature changes at early times and then after the gradual changes. The truncation error introduced by omitting higher terms in the approximation can accumulate with each integration step. After some time, the error in the numerical solution can grow at a faster rate than the solution. The error will then become so large that the results are meaningless. The numerical solution will 'blow up' before the final integration step can be reached.

The cause of such unstable behaviour of the solution may be traced to numerical methods or to the inherently instability of the problem. The problem itself may be unstable. It is not the case here. The solution of the stiff differential equations is asymptotically stable. Instability may arise from the inability of the numerical method to predict the true behaviour of the system. The numerical solution, in addition to its approximation of the true solution, may contain extraneous or parasitic solution. This parasitic solution may decay as the computation proceeds in time and so does not cause numerical difficulties. Or, it can grow to catastrophic proportions, thus dominating the solution. Multi-step methods were not used to solve the first-order differential equations which are, in addition, stiff.

The stability of numerical solutions is determined from the eigenvalues of the coefficient matrix of the discrete equations. The criterion for stability is that the spectral radius of the coefficient matrix must be less than unity. The spectral radius is defined as the

the absolute value of the maximum eigenvalues, $\max_i |\lambda_i|$. All eigenvalues must lie within the unit circle. Only the lowest and highest eigenvalues are needed.

The explicit methods are conditionally stable. Their integration step size is restricted to ensure stability. The implicit methods have no restriction on the size of the integration step. Their numerical solutions remain bounded regardless of the step size used. These methods are unconditionally stable for any time increment. The accuracy of the approximation is deteriorated as time increment increases. For acceptable accuracy, the stiff differential equations require small time increments in their numerical integration.

The numerical methods for solving the first-order differential equations are considered as matrix approximations for the exponential matrix $e^{-[A]\Delta\theta}$. When a function cannot be conveniently evaluated or the power series converge too slowly, rational approximations may be used. The Padé matrix approximation of $e^{-[A]\Delta\theta}$ can be found in the book by Varga (37) as

$$e^{-[A]\Delta\theta} = \frac{[I - \frac{1}{2} \Delta\theta A + \frac{1}{12} \Delta\theta^2 A^2 - \dots]}{[I + \frac{1}{2} \Delta\theta A + \frac{1}{12} \Delta\theta^2 A^2 + \dots]} \quad (3.19)$$

Crank-Nicolson Method $\phi = \frac{1}{2}$

For $\phi = \frac{1}{2}$ in the difference equation (3.18), the solution $\{t\}_n$ is advanced to $\{t\}_{n+1}$ in a time interval $\Delta\theta$ by averaging their time derivative

$$\{t\}_{n+1} = \{t\}_n + \frac{1}{2} \Delta\theta (\{\dot{t}\}_n + \{\dot{t}\}_{n+1}) \quad (3.20)$$

The nodal temperatures $\{t\}_{n+1}$ are computed implicitly by using the derivative of the predicted values. This procedure is simply the

trapezoidal rule, or central difference method, or the modified implicit Euler method. The relation for $\{t\}$ in Equation (3.20) was used to reduce the ordinary differential equations (3.6) to a set of algebraic equations

$$([C] + \frac{1}{2}\Delta\theta[K])\{t\}_{n+1} = ([C] - \frac{1}{2}\Delta\theta[K])\{t\}_n + \Delta\theta\{F\} \quad (3.21)$$

which were solved for the nodal temperatures at each successive time interval. The exponential solution was approximated. This procedure can be recognized as the Padé (2,2) rational approximation which consists of only the first terms in Equation (3.19).

In implicit method, a matrix inversion is required to determine $\{t\}_{n+1}$. It is convenient to maintain a constant time-step and same coefficients of thermal properties throughout the time history.

There is no possibility of instability for any time increments. The eigenvalues λ of the problem $|[C - \frac{1}{2}\Delta\theta K] + \lambda[C + \frac{1}{2}\Delta\theta K]| = 0$ can be negative but never less than -1. The existence of negative eigenvalues will indicate stable oscillation in the solution. As the time step increases, more eigenvalues become negative and approach the negative limit. The solution loses accuracy. The eigenvalues will give a measure of the accuracy.

Other Methods

In the case $\phi = 1$, the temperatures at the end of a time step are based purely on their time derivative

$$\{t\}_{n+1} = \{t\}_n + \Delta\theta\{\dot{t}\}_{n+1} \quad (3.22)$$

The procedure for approximating the succeeding temperatures is known as the

pure implicit method, or the backward difference method and it corresponds to Padé (1,0) rational approximation. This method is absolutely stable. The eigenvalues of the matrix $[I] + \lambda[C + \Delta\theta K]$ cannot be negative. No oscillations appear in the solution for large time increments but the solution will continue to lose accuracy. The solution is not as accurate as the Crank-Nicolson solution.

When $\phi = 0$, the numerical procedure is the explicit Euler method, or the forward difference method. Its application corresponds to the Padé (0,1) approximation. The new temperatures are calculated from only the previous solution

$$\{t\}_{n+1} = \{t\}_n + \Delta\theta \{t\}_n \quad (3.23)$$

Explicit methods do not require the inversion of matrix. The time step can easily be changed in the step-by-step integration procedure. The time step is severely limited. For stability, the time step size in the coefficient matrix $[C - \Delta\theta K]$ must be reduced until all its eigenvalues lie inside the unit circle. With smaller time step taken, more integration steps are needed to reach the final point in time. Explicit methods should be avoided for solving stiff differential equations. They impose a severe restriction on the time step size to insure reasonable approximation to solution.

3.6 Properties of Linear Temperature Elements

A linear temperature variation within a rectangular and triangular element will ensure inter-element temperature compatibility required by the continuum theory. The temperatures varying linearly along

an edge are uniquely defined in terms of the nodal temperatures at each end. Between elements, the temperature gradients $\frac{dt}{dx}$ and $\frac{dt}{dy}$ fail to coincide at the interface. Physically, these derivatives when combined with the thermal conductivity of the material represent the rate of heat transfer. In the steady heat conduction, the amount of heat entering and leaving an element volume is equal at any time instant. The difference in these flow rates in transient conduction is the heat required to raise or lower the temperature of the volume according to the first law of thermodynamics. The continuity of temperature gradients across element boundaries will permit a closer analytical modelling of the thermal behaviour. This condition is not necessary. The continuity of gradient between elements does not guarantee heat balance when adjacent elements have different thermal conductivity.

Rectangular Element

The linear temperature rectangular element with nodal temperatures at each corner is shown in Figure 7. Its properties were derived in rectangular Cartesian coordinates x, y . The bilinear temperature function of x and y , assumed over each element, is of the form

$$t(x,y) = a_1 + a_2x + a_3y + a_4xy. \quad (3.24)$$

The temperature gradients, t_x and t_y vary linearly within the element but are incompatible at interface of adjacent elements.

The conduction matrix of the element is given as

$$[K_e] = \frac{k}{6ab} \begin{bmatrix} 2(a^2+b^2) & a^2-2b^2 & 2(a^2+b^2) & \text{Symmetric} \\ -a^2-b^2 & b^2-2a^2 & 2(a^2+b^2) & \\ b^2-2a^2 & -a^2-b^2 & a^2-2b^2 & 2(a^2+b^2) \end{bmatrix} \quad (3.25)$$

It should be noted that the sum of coefficients for any rows in matrix $[K_e]$ is equal to zero. In steady flow, the equilibrium equations for an element are satisfied when all its nodal temperatures are the same.

$$[C_e] = \frac{\rho c ab}{36} \begin{bmatrix} 4 & & & \text{Sym.} \\ 2 & 4 & & \\ 1 & 2 & 4 & \\ 2 & 1 & 2 & 4 \end{bmatrix} \quad (3.26)$$

When an element is exposed on the side l to an ambient air of temperature t_g , the convective effect $[H]\{t\} + \{f_e\}$ is as follows

$$hl \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} hl t_g \quad (3.27)$$

This convection matrix is consistent with the assumption made for the temperature distribution within the element. Alternately, the rate of heat convection, $q = hl(t - t_g)$, transmitted to the surface can be lumped. Regardless of the temperature distribution, the heat rate is distributed in equal portions to the two nodes on the convective side of the element. The lumped convective matrix is written as

$$hl \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} hl t_g \quad (3.28)$$

Triangular Element

The linear temperature triangular element shown in Figure 7 has a nodal temperature at each vertex and is conveniently derived in local coordinate system ξ, η . Its linear temperature function is assumed in the following form

$$t(\xi, \eta) = a_1 + a_2\xi + a_3\eta. \quad (3.29)$$

This element does not maintain temperature gradient continuity. Its first spatial derivatives are constant and vary from element to element in the analytical model. The function is invariant to rotation transformation of coordinates.

The conduction matrix of the element is given as

$$[K_e] = k \begin{bmatrix} \frac{c^2 + a^2}{2(a+b)c} & \text{Sym.} & \\ \frac{-c^2 + ab^2}{2(a+b)c} & \frac{c^2 + b^2}{2(a+b)c} & \\ \frac{-a}{2c} & \frac{-b}{2c} & \frac{a+b}{2c} \end{bmatrix} \quad (3.30)$$

The heat capacity matrix of the element will be

$$[C_e] = \frac{\rho c (a+b)c}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad (3.31)$$

3.7 Solution of the Temperature Problem

Idealization of Cross-Sectional Area

Full use is made of symmetry existing in this temperature problem. The transverse cross sections of the bridge made of four equally-spaced steel beams supporting a concrete deck are symmetric. Except for the shear connectors and the transverse reinforcing steel, the

cross sections are the same throughout the spans. Considering uniform boundary conditions, the temperature analysis was carried out for only the instrumented cross section which is near the middle of the interior span. Same two-dimensional temperature distribution was assumed for all cross sections. The finite element modelling of only two subsections of the full transverse cross section was sufficient. As shown in Figure 6, the part for side effect was the portion from the centerline of the exterior steel beam and the interior portion was represented by half of the slab-beam member.

The measured temperatures of the cooled surface and of surrounding air were used to determine the temperature distributions inside the bridge cross section. These temperatures at different points in the cross-sectional plane were plotted against time in Figure 8. The thermocouple readings indicated that the boundary temperatures are slightly uneven. From the thermocouple arrangement in Figure 3, it is shown that the temperatures are not exactly symmetric. The unsymmetric temperatures throughout the entire cross section can be computed using the same two finite element models. The problem is simply re-solved for the subsections subjected to different boundary temperatures. When temperatures are not uniform along the spans, several cross-sectional planes can be considered separately without extending the temperature analysis to three dimensions.

With slightly nonuniform temperature conditions as shown in Figure 8, only the temperatures corresponding to the exterior slab-beam member on the right side was used as input data for the calculation of temperature in the entire cross section. Under symmetric thermal loadings, the thermal stress analysis of the bridge can be reduced to one quadrant.

As shown in Figure 6, the interior and the exterior subsections of the bridge cross section were modelled with a regular network of linear temperature rectangular elements. Same element size was used in both models. There are a total of 1652 and 882 nodes. The nodal numbering system, leading to the narrowest half bandwidth of 13 in the structure matrices can be followed in Figure 6. It was necessary that the element be so small in order to preserve uniformity of the mesh. This is the largest element capable of representing the flange and web of the steel beam. Each round steel bar of longitudinal reinforcement was replaced with a rectangular bar of approximately the same area.

The rectangles in the finite element model can be subdivided into two right-angle triangles. The rectangular element, providing more accurate results, will be the only type of element used in this transient temperature analysis. The triangular element is better suited to handle curved boundaries and sections of irregular geometry. It is not necessary for this problem to use triangular elements. In the rectangular model, the finite element differential equations are not as stiff as those from the triangular model. The triangular finite element model will then require a smaller time step for equivalent accuracy. For the same time step in the Crank-Nicolson method, the spatial triangular discretization leads to more and larger negative eigenvalues, meaning larger numerical oscillations which are stable. The use of higher order elements such as the nine degrees of freedom triangular element or the sixteen degrees of freedom rectangular element will considerably increase the size of the structures matrices for this regular mesh. Furthermore, the differential equations will be stiffer. Higher order elements are efficient in coarse meshes where a smaller number of finite elements are used.

Time marching solution

The Crank-Nicolson procedure ($\phi = \frac{1}{2}$) outlined in Section 3.5 was used in the time integration. The 77-minute cooling period was covered in 462 intervals of 10 seconds. In the Crank-Nicolson formulation, all eigenvalues are believed to be negative. The resulting numerical oscillations were not sufficiently large to affect the results. It is felt that the maximum error is 0.5°F for temperatures in the most critical regions where rapid temperature changes occur. To avoid numerical oscillation, the time step must not exceed 0.01 second. Such a small step is impractical to cover a long time history.

In the solution at each time step, a constant time step was maintained and the thermal properties were taken to be constant. The computations were minimized by maintaining the same system of algebraic equations (3.21) for the nodal temperatures throughout their time history. Once the matrix $[C + \frac{1}{2}\Delta\theta K]$ was decomposed using Cholesky's method, the factored matrix was retained for solution at each time intervals. Only the backward and forward substitutions were required.

3.8 Solution to Large Systems of Algebraic Equations

The disadvantage of the finite element method is that it necessitates the use of high order matrices. Problems can be too large to be handled using the computer active core. The storage allocated for FORTRAN programme is less than 520K bytes on IBM 360/65 computer at the University of Ottawa. The storage demands of the finite element programme on the computer can be considerably reduced by the use of auxiliary storage units such as magnetic tape or disk file.

Only the upper band with the diagonal of the symmetric coefficient matrix was stored in blocks or submatrices on disk files. In the equation solving routine, not all the coefficients of the structure matrix are needed at a time during the arithmetical operations. A limited portion of the matrix is required at different stages of the operation. For particular stage of calculation, the required blocks or submatrices, in which the matrix was divided, are transferred into the core of the central processor unit of the computer. Once the calculations have been completed, the block in the computer core is replaced by a new block. The calculations are carried out through all these blocks. The computer programme dealing with large systems of simultaneous equations can be executed.

The size of the submatrices can be reduced to conserve core storage. The choice of the size of submatrices is based on the resulting number of read and write command statements and core storage allocated for the programme. Saving in core storage with reduction of the size of submatrices will result in longer execution time. Time is increased with more I/O statements. The storage should be utilized efficiently.

The coefficient matrices $[C + \frac{1}{2} \Delta \theta K]$ and $[C - \frac{1}{2} \Delta \theta K]$ each in the order of 1652 by a half bandwidth of 13 were subdivided into four rectangular submatrices of 412 by 13. Only two blocks of the matrices were required at any time during the execution of the FORTRAN routine. When the arithmetical operations in the first block have been completed, that submatrix was replaced by the other block in the computer memory. A new submatrix can now be transferred into core storage. The calculations, in Cholesky's decomposition method and in substitution procedures, proceed by

transferring required submatrices from disk files to the computer memory. In Cholesky's method, the matrix $[C + \frac{1}{2} \Delta \theta K]$ was destroyed and in its place the factored matrix was retained for repeated time integrations.

Advantage of I/O statements was taken to allow high speed transfer of data in and out of computer storage. Each data set was described with a DEFINE FILE statement. The submatrices were placed on disk as soon as they were formed in the finite element assemblage. The 412 by 13 submatrices were stored by columns. Each column constitutes a logical record and is described by an integer indicating its position.

3.9 Evaluation of Thermal Properties

For realistic representation of the structure, the finite element method requires the following thermal properties which are the thermal conductivity, diffusivity, and surface conductance. The thermal properties with the finite element mesh are shown in Figure 6. All these heat transfer properties can be taken as constant throughout the cooling period to facilitate the temperature analysis. These relatively small temperature changes have a negligible effect on the required properties for concrete and steel. The thermal capacity is essentially independent of temperature. There were no change in moisture content of concrete to affect the coefficients of conductivity and diffusivity. The values, which vary widely between concretes, were considered the same throughout the cross section. Properties measured on cast concrete specimen will only serve as a guide. Large variations of concrete properties may be expected. The constants were evaluated from readings taken on the bridge. The heat transfer coefficients for steel were assumed according to the specified standard of the beam.

The aim of this study is to determine the transient temperature distributions for the stress analyses. Only surface temperatures were measured. The values of the conductivity, the capacity and surface conductance can be changed in the finite element model until the corresponding surface temperatures are as close as possible to the measured temperatures. These calculations for such a model will require large computing time which accumulates with each trial.

The region of analysis was reduced to the middle portion between the steel beams. The heat conduction in that part of the slab can be said to be one-dimensional. The one-dimensional heat transfer in the slab was analyzed using nine linear temperature elements. The conductivity and capacity of concrete were determined together with the lower surface conductance by trial and error. All three coefficients are interrelated. This procedure was based on the time required for the temperatures to penetrate through the thick slab. A separate test can be established for determining these properties and could be carried out several times with various boundary conditions. In this test, only a portion of the surface needed to be covered with ice to assure a one-dimensional heat flow. If steady state prevailed, the relation between the conductivity and surface conductance becomes

$$\frac{k}{h} = \frac{d_c (t_s - t_g)}{\Delta t} \quad (3.32)$$

where Δt denotes the measured temperature drop between the outer surfaces of the slab with thickness d_c . It is seen that the solution to Equation (3.32) requires the flow rate.

Measurement of temperature provides sufficient data to determine the surface conductance for the vertical surface of the steel beams. The transient temperature only in the beam's web and lower flange was analyzed with a rectangular element model. The vertical surface conductance with surface conductance on lower flange can be estimated from the experimental data. From a heat transfer viewpoint, the steel web acts as a fin attached to a surface for the purpose of increasing the surface heat transmittance. In steady state, the vertical surface can be determined from a one-dimensional analytical treatment of the temperature in the steel web. The steady temperature distribution defined by the temperatures at the corner t_c of the web and the embedded flange in concrete, at mid-depth t_m , and at the bottom t_b is given in Jakob's (58) textbook on page 211. The vertical surface conductance is given as

$$h_v = k w \left[\frac{2}{d_s} \ln \left(\frac{t_c + t_b - 2t_g}{2t_m - 2t_g} \right) + \sqrt{\frac{t_c + t_b - 2t_g}{2t_m - 2t_g} - 1} \right]^2 \quad (3.33)$$

The steel beam of depth d_s and web thickness w has a thermal conductivity k of 31 BTU/hr-ft-°F.

The surface conductances on the surfaces were taken to be uniform and unchanged throughout the cooling duration. These values are uniform for an isothermal surface. The surface conductance is dependent on the temperature, velocity and physical properties of air surrounding the exposed surface and, as well as, on the surface temperature which varies with time. For the lower surface of the slab, the surface conductance value varies slightly and a constant average value may be used. There is a larger temperature variation along the depth of the steel beam. A closer

representation would be possible with varying vertical surface conductance only from element to element. The most crucial assumption made in the temperature analysis was that the surface conductance remains constant during the entire cooling period. Change in surface conductance involves the inversion of the coefficient matrix in the implicit formulations.

3.10 Solution Procedures

The steps involved in the computer programme for the temperature analysis are outlined as follows:

1. Calculation of element matrices $[K_e]$, $[C_e]$, $[H_e]$, and $\{f_e\}$
2. Assemblage of elements
 - structure matrices $[K]$, $[C]$ and $\{F\}$ are setup.
 - matrices $[C + \frac{1}{2} \Delta \theta K]$ and $[C - \frac{1}{2} \Delta \theta K]$ in algorithm are formed.
 - their submatrices of order 412 by 13 are stored on disk files.
3. Decomposition of matrix $[C + \frac{1}{2} \Delta \theta K]$
 - submatrices are transferred in memory core.
 - original matrix destroyed when replaced by factored matrix.
4. Multiplication of $[C - \frac{1}{2} \Delta \theta K] \{t\}$
 - submatrices are transferred in memory core.
5. Backward and forward substitutions
 - factored matrix is transferred from disk file.
6. Surface and air temperatures updated at each time step
 - step 4 and 5 are repeated 462 times.
7. Nodal temperatures on disk file for stress analysis.
8. Plot temperature field - running picture of isothermals.

Double precision was used throughout the computer programme. To execute the FORTRAN programme for the complete temperature history, in the 1652-node model, the CPU time required was 36 minutes. Storage allocated for the programme in the core of the central processor unit was 260K bytes and 12 cylinders on auxiliary unit. Once the matrix was decomposed, the solution for each increment was obtained within a computing time of 4 seconds.

CHAPTER 4

THERMOELASTIC ANALYSIS4.1 General

Throughout that cooling period, the bridge was subjected to time-varying thermal loads. The effects of the temperatures obtained from the heat conduction analysis may be translated into equivalent loads to determine the resulting stresses and deformations for the bridge. The analysis of the bridge to temperature loads was made with finite-element displacement models. The bridge was analyzed as an eccentrically-stiffened thin plate made of isotropic, homogeneous, and linearly elastic materials.

Basic assumptions

The reinforced concrete slab, in accordance with the general definition of thin plate, has a thickness-width ratio much less than one tenth. In the case where the thickness of the plate is relatively small to its other dimensions, the problem is reduced to one involving only two dimensions. A rectangular coordinate x and y were taken on the middle plane which bisects the thickness of the plate, and the z axis perpendicular to its faces. The stresses, σ_z normal to the plate are zero on both free upper and lower surfaces. Between the surfaces, the distribution of the stress σ_z in the thickness of the thin plate can be discarded.

The state of stresses in the bridge is quasi-static. The dynamic effects produced by sudden change in temperature are negligible. Based on

the stiffness of bridge, the thermal shock was not sufficiently large in intensity to induce vibrations. The inertia term was excluded from consideration in the stress-displacement calculations. The structural response involves time only when the temperatures change. The problem is one of statics.

The small-displacement theory applies to the thermal analysis of the bridge within this temperature range. Changes in temperature will cause the unrestrained plate to stretch or shrink. When the temperatures are distributed nonuniformly, the plate will bend or tend to deflect. According to the small-deflection theory for thin plates, there is no straining of the middle surface as a result of bending. The in-plane displacement and the normal deflection are independent of each other. The bending and membrane problems would be solved separately. In the analysis of beam-slab bridges having the girders as an integral part of the slab, their neutral axis is located below the middle surface. In bending, the plate mid-surface by not coinciding with the neutral axis will strain. The in-plane displacements and the lateral deflections are linked together through the displacements of the beam. The beams, in addition to resisting bending, provide torsional rigidity. This stress problem combines the theories of membrane, bending and torsion.

Kirchoff-Love's small-deflection theory is valid for lateral deflection limited to a fifth of the plate thickness. Stating this limitation in terms of span length, the deflection should be less than one fiftieth of the smaller span length. The slopes for small deflections are small compared to unity. For larger deflections, non-linear terms are included in the strain-deflection expressions to account for the straining.

of the middle surface. The in-plane and out-of-plane displacements of a thin plate are then described by two simultaneous, non-linear differential equations.

The lateral deflection is mainly due to bending. Deflection associated with shear is comparably small. The contribution of the shear strains γ_{xz} and γ_{yz} to deformation is omitted in the differential equation defining deflection. That assumption implies that, after bending, the angular distortion γ_{xz} and γ_{yz} from normal position of the initially straight lines normal to the surface may be regarded as negligible. Similar assumption is made in elementary beam theory known as Bernoulli-Euler law. The hypothesis states that sections originally plane remain so subsequent to bending. On the basis of plane sections remaining plane, the longitudinal strains vary linearly with their distance from the neutral axis.

Elastic properties

In stress-strain relationship, the assumption was made that the materials, steel and concrete, are homogeneous, isotropic, and linearly elastic (obey Hooke's law). The stresses and strains are proportional and are inter-related by two independent constants which are the modulus of elasticity, E and Poisson's ratio, ν . The steel beams exhibit linearly elastic behaviour provided that the stresses remain below the yield point. According to AISC Specifications, the modulus of elasticity is close to 30×10^6 psi at room temperature. The elastic modulus decreases linearly to 25×10^6 psi at 900°F. In addition, it gives the Poisson's ratio $\nu_s = 0.30$ and, over a temperature range from room temperature to 200°F, the coefficient of expansion $\alpha_s = 6.5 \times 10^{-6}$ in/in/°F. For the bridge

whose temperature varies from 78°F to 33°F, these physical properties for steel and for concrete remain constant.

The concrete due to its constituents is heterogeneous and, when reinforced with steel, anisotropic. It obeys Hooke's law only to some extent. Its stress-strain curve is approximately a straight line up to half the compressive cylinder strength. The compressive stresses produced by only the temperature changes in the bridge are well within the linear elastic range. Nearly the same slope describes the stress-strain curve for tensile stresses not exceeding half the flexural tension strength (modulus of rupture). According to ACI 435-201, the modulus of rupture ranges from 7.5 to 12 times the square root of the compressive strength in psi.

When the ultimate tensile value has been exceeded, cracks are formed in the concrete and the tension is resisted by the reinforcing steel. As these tension cracks progress towards the neutral axis under increasing load, there is a continual reduction in flexural rigidity. For a slab, this reduction of flexural rigidities may be uneven in the two mutually perpendicular directions. The concrete deck having its main reinforcement in one direction is an orthotropic plate. The flexural rigidities for the continuously supported deck varies along both longitudinal and transverse directions. Their variations will be further increased with tension cracks occurring in large bending moment regions. When cracks develop, the solution involves continual modification of the transformed section as the load is incrementally increased. It was expected that the bridge with the restraints removed at both ends, was partially relieved from its thermal stresses. Such extensive calculations may not be warranted for these temperatures. The steel reinforcement ratios along

the transverse direction are given in Figure 2. The analysis was simplified by considering an isotropic slab with same elastic rigidity for plain uncracked concrete section in all directions.

The elastic rigidity in the slab analysis was based on an averaged modulus of elasticity of 6.3×10^6 psi. The instantaneous modulus of elasticity for all five sections poured at different times was determined by measuring the strains in the 6" x 12" concrete cylinders on the same day of the test. A common value of 5.5×10^{-6} in/in/°F was used for the concrete coefficient of thermal expansion. These properties of concrete are dependent on age, temperature, mix proportion and amount of absorbed water. The modulus of elasticity and coefficient of thermal expansion could have been chosen from the tables prepared by Berwanger and Sarkar (47). The lower value of the Poisson's ratio ν_c , which varied from 0.15 to 0.20 for reinforced concrete, was used.

Stress, strain, and displacement relations

For thin plates, the Hookean stress-strain relations with t denoting small change in temperature from the stress-free state are expressed as follows

$$\sigma_x = \frac{E}{1 - \nu^2} [(\epsilon_x + \nu \epsilon_y) - (1 + \nu) \alpha t]$$

$$\sigma_y = \frac{E}{1 - \nu^2} [(\epsilon_y + \nu \epsilon_x) - (1 + \nu) \alpha t]$$

$$\tau_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy}$$

or, in matrix form

$$\{\sigma\} = [D]\{\epsilon - \epsilon_0\} \quad (4.1)$$

where $[D]$ is a matrix of elastic constants. In the small-displacement

theory of thermoelasticity, the products of small quantities such as displacement gradient and temperature changes are negligible. The thermal stress equations include time only through the transient temperatures.

The strain-displacement equations are linearized. The squares and products of gradients, considered very small when compared to the gradients themselves, were discarded. The relationship between strain and displacement is presented as

$$\epsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \quad (4.2a)$$

$$\epsilon_y = \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \quad (4.2b)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \quad (4.2c)$$

where u and v are displacement components in the x and y directions, respectively. The lateral deflections are denoted by w and the strains are given at distance z from the middle surface. The curvatures of the deflected midsurface in xz and yz planes respectively are given in the usual approximation as

$$\frac{1}{r_x} = \frac{\partial^2 w}{\partial x^2}, \quad \frac{1}{r_y} = \frac{\partial^2 w}{\partial y^2} \quad (4.3a)$$

and the twist of the surface with respect to the x and y axes is

$$\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \partial y} \quad (4.3b)$$

The curvature prior to the reduction is found in Timoshenko(59) as

$$\frac{1}{r_x} = \frac{\partial^2 w / \partial x^2}{[1 + (\partial w / \partial x)^2]^{3/2}} \quad (4.4)$$

For small deflections, the resulting slopes are small compared to unity and their square terms become negligible.

Within the assumptions stated, the in-plane displacements (u and v) together with lateral deflections were calculated using the finite element method. The strains were determined from these deformations and then, the stresses from the strains.

4.2 The Finite Element Method

In the finite element analysis by displacement method, the elements are based on the assumption of polynomial displacements. The entire displacement field of the bridge was constructed in a form of piecewise functions. The function taken in each element making up the bridge was defined in terms of unknown nodal displacements. The variational integral necessary for the finite element application is given by the principle of minimum potential energy (or of virtual work). The total potential energy Π is equal to the elastic strain energy U stored in the structure plus the potential energy V of the external forces.

$$\Pi = U - V \quad (4.5)$$

The potential energy Π is a minimum only for displacements satisfying boundary and equilibrium conditions. At equilibrium, its first variation vanishes for all admissible variations of deformation

$$\delta\Pi = \delta(U - V) = 0 \quad (4.6)$$

and it has a positive second variation, $\delta^2\Pi > 0$.

At the equilibrium point which is also stable equilibrium, the potential energy functional has an absolute minimum value. The strain energy, stored in a deformed linear elastic structure, is a quadratic positive definite function of the displacements with their derivatives. The strain energy is defined by

$$U = \frac{1}{2} \iiint \{\sigma\} \{\epsilon - \epsilon_0\} dx dy dz \quad (4.7)$$

The Euler equations of the variation $\delta\Pi$ are the differential equations of equilibrium considered at the nodal points. For all the elements in the assemblage, the equations of equilibrium corresponding to the nodal displacements form a symmetric set of linear algebraic equations that can be solved for the unknown nodal displacements. These equations, written in matrix form, are

$$\frac{\partial \Pi}{\partial \{w\}} = [K]\{w\} - \{F\} = 0 \quad (4.8)$$

where K denotes the structure stiffness matrix relating the displacement vector $\{w\}$ to the corresponding load vector $\{F\}$.

The governing differential equation of the problem is self-adjoint. It is recovered as the Euler equations. A functional can be derived from the differential formulation.

For the finite element formulation, there are a number of restrictions imposed on the assumed deformation field for elements to have good convergence properties. The elements, in their representation of the displacement field, must be compatible in terms of the displacements and their derivatives of one order less than the highest derivative appearing

in the strain energy expression. For requirement of completeness, the elements should be able to represent constant strain states and rigid body displacements. As the network of conforming elements is progressively re-divided into finer meshes, the total potential energy at equilibrium will monotonically converge from above to the true minimum. These elements then provide lower bounds on the exact strain energy and displacements.

The finite element model of the bridge was made up of triangular conforming elements of plane stress and plate bending with beam elements attached to the lower surface to represent the longitudinal steel beams. The strain energy of the bridge is expressed as the sum of the strain energies of these elements. From the strain energy expressions, the stiffness matrices and the thermal load vectors were derived for the elements. The plate bending element of triangular shape is based on a quintic polynomial with constraints imposed to enforce interelement continuity of normal slopes. With an accuracy comparable to the plate bending element, the conforming plane stress element has the two displacement components, u and v represented by a complete cubic polynomial. The beam elements, to be compatible with the plane stress and plate bending elements, have assumed same displacement patterns for in-plane and out-of-plane. Usually defined at the beam's neutral axis, the nodal displacements with their derivatives are transferred to the midsurface of the slab under the assumption of linear longitudinal strain through the depth. These beams resist rotation of the slab. The bending and torsion are uncoupled.

4.3 18-degree Plate Bending Element

This constraint-quintic displacement element for plate bending was developed by Cowper, Kosko, Lindberg and Olson (60), by Butlin and Ford (61), by Argyris et al. (62), and by Bell (63). The triangular element shown in Figure 7 has a total of 18 degrees of freedom with 6 at each vertex. The nodal degrees of freedom are w , w_x , w_y , w_{xx} , w_{xy} and w_{yy} . Its properties are derived in the rectangular local coordinates ξ, η and a transformation to the Cartesian coordinates x, y is made to assemble the elements. The element uses a quintic polynomial without the term $\xi^4\eta$ to represent the deflected shape.

$$w(\xi, \eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 + a_{11}\xi^4 + a_{12}\xi^3\eta + a_{13}\xi^2\eta^2 + a_{14}\xi\eta^3 + a_{15}\eta^4 + a_{16}\xi^5 + a_{17}\xi^3\eta^2 + a_{18}\xi^2\eta^3 + a_{19}\xi\eta^4 + a_{20}\eta^5 \quad (4.9)$$

The incomplete fifth-order polynomial may be written in summation form

$$w = \sum_{i=1}^{20} a_i \xi^{m_i} \eta^{n_i} \quad (4.10)$$

This element is fully conforming. The deflections with the slopes are continuous between elements. The quintic function of deflection along an edge is defined by six polynomial coefficients. These arbitrary constants are determined by three displacement parameters (deflection, slope and curvature) at each of the two nodal points. Considering two elements interconnected at the two nodal points, the quintic variations along the common edge are the same for both elements. The continuity of slopes along inter-element boundaries is restored by enforcing constraints on the quintic polynomial. The omission of the term $\xi^4\eta$ renders the normal

slope a cubic function in ξ along the base of the triangle where $\eta = 0$. This cubic variation of slope along an edge is uniquely defined by two generalized displacements (slope and twist) at each of the two vertices. To ensure that the normal slope is cubic along the remaining two edges of the triangle, constraints are imposed on the fifth-degree terms. The constraint equations are introduced in the last two rows of the coordinate function matrix $[T]$ listed in Appendix A.

The matrix $[T]$ defines the nodal positions in the expression for the nodal displacement parameters $\{w\} = [T] \{a\}$ in terms of the undetermined coefficients $\{a_1, a_2, \dots, a_{20}\}$ of the polynomial.

Rigid-body displacements of the element are represented by the first three terms, $a_1 + a_2\xi + a_3\eta$ in the quintic polynomial. According to Equation (4.3), these terms do not contribute to the strains (bending in the flexural problems). They represent three components of rigid-body displacements. The element is able, without any straining, to translate in the z -direction and rotate about two axes, the ξ -axis and η -axis. The rigid non-straining modes are confirmed by the presence of zero eigenvalue of the element stiffness matrix.

The zero strain state under rigid-body motion can be included among the states of uniform strain. In this case, the constant strain has a zero value. The uniform strain (curvature and twist) condition is accounted for in the next three terms of the deflection function.

For these convergence requirements, the displacement of an element is considered in two parts. Firstly, the element undergoes pure displacement as a rigid body without any bending. In the second part,

deflection takes place within the element simply supported at each vertex. With its nodal deflection equal to zero only at the vertices, no rigid-body motions are possible during bending.

The formulation of the element stiffness matrix with its corresponding thermal load vector is based on the strain energy expression. With the substitution of the stress components, the strain energy expression in Equation (4.7) can be written for plate bending elements in terms of its local deflection components

$$U = \frac{Eh^3}{12(1-\nu^2)} \iint \left\{ \begin{matrix} w_{\xi\xi} - w_o \\ w_{\eta\eta} - w_o \\ w_{\xi\eta} \end{matrix} \right\}^T \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix} \left\{ \begin{matrix} w_{\xi\xi} - w_o \\ w_{\eta\eta} - w_o \\ w_{\xi\eta} \end{matrix} \right\} d\xi d\eta \quad (4.11)$$

where the flexural rigidity is defined by $D_f = \frac{Eh^3}{12(1-\nu^2)}$ with h the plate thickness; and the symbol w_o for thermal curvature is equal to $\frac{12}{h^3} \alpha M_t$. The moment of temperature $t(z)$ varying in the thickness direction is given as $M_t = \int z t(z) dz$. The thermal load vector was derived from the part of the strain energy associated with temperature. The strain energy expression is conveniently separated in two parts, the thermal terms U_t , and the remaining bending strain terms U_b , such that

$$U = U_b + U_t \quad (4.12)$$

$$\text{where } U_b = \frac{1}{2} D_f \iint \left[w_{\xi\xi}^2 + w_{\eta\eta}^2 + 2\nu w_{\xi\xi} w_{\eta\eta} + (1-\nu) w_{\xi\eta}^2 \right] d\xi d\eta$$

$$\text{and } U_t = \frac{E\alpha}{1-\nu} \iint M_t (w_{\xi\xi} + w_{\eta\eta}) d\xi d\eta$$

The square term of M_t was excluded from the expression of U_t as it vanishes in the differentiations with respect to the nodal parameters.

The calculation of the element stiffness matrix $[K_b]$ in global coordinate system x, y can be carried out in the following manner

$$[K_b]_{18 \times 18} = D_f [R]^T [T^{-1}]^T [k] [T^{-1}] [R] \quad (4.13)$$

Following Reference (60), the double integrals may be evaluated in closed form

$$\iint \xi^m \eta^n d\xi d\eta = F(m, n) \quad (4.14)$$

$$\text{where } F(m, n) = c^{n+1} [a^{m+1} - (-b)^{m+1}] \frac{m!n!}{(m+n+2)!}$$

The integrals were carried out over the triangular area of the element whose dimensions a , b and c are shown in Figure 7. Their entries of the coefficient matrix $[k]$ are given in Appendix A, with the rotation matrix $[R]$ for rotational transformation of coordinates.

The thermal bending load vector for the element was found to be

$$\{F_t\} = \frac{E\alpha}{1-\nu} [R]_{18 \times 18}^T [T^{-1}]_{18 \times 20}^T [k]_{20 \times 3} [T_t^{-1}]_{3 \times 3} \{M_t\} \quad (4.15)$$

With temperatures varying across the width of the bridge, an assumed linear variation of temperature moment between the nodes is expressed as

$$M_t(\xi, \eta) = b_1 + b_2 \xi + b_3 \eta = \sum_{j=1}^3 b_j \xi^{r_j} \eta^{s_j} \quad (4.16)$$

from which the coordinate dimension matrix $[T_t]$ is established for the nodal values of M_t . The integrals over the triangular area are evaluated in the 20×3 matrix $[k]$ where its coefficients are

$$k_{ij} = m_i(m_i-1)F(m_i-2+r_j, n_i+s_j) + n_i(n_i-1)F(m_i+r_j, n_i-2+s_j) \text{ for } i=1, 20 \text{ \& } j=1, 3$$

The bridge is subjected to a distributed load q such as its weight. The element load vector is established in a consistent manner using the external potential energy expression $\iint q w d\xi d\eta$. The uniform-load vector is given by

$$\{P\} = [R]^T [T^{-1}]^T \{k\} \quad (4.17)$$

where $k_i = q \bar{F}(m_i, n_i)$ for $i = 1, 20$.

The error in the deflection function is of order ℓ^5 , where ℓ is a linear dimension of an element. This element will yield an error in strain energy of order ℓ^6 . The strain energy will have a convergence rate of N^{-6} where N is the number of elements.

4.4 Quadratic Plane Stress Element

The representation of the membrane state for this triangular element was originally suggested by Felippa (64). The in-plane displacements u and v are each expressed as complete cubic polynomials. For an arbitrary triangular element in local coordinates system shown in Figure 7, the ξ and η -direction displacements are designated by $\bar{u}$ and $\bar{v}$ respectively

$$\begin{aligned} \bar{u} &= b_1 + b_2 \xi + b_3 \eta + b_4 \xi^2 + b_5 \xi \eta + b_6 \eta^2 + b_7 \xi^3 + b_8 \xi^2 \eta + b_9 \xi \eta^2 + b_{10} \eta^3 \\ &= \sum_{i=1}^{10} b_i \xi^{m_i} \eta^{n_i} \end{aligned} \quad (4.18)$$

$$\begin{aligned} \bar{v} &= b_{11} + b_{12} \xi + b_{13} \eta + b_{14} \xi^2 + b_{15} \xi \eta + b_{16} \eta^2 + b_{17} \xi^3 + b_{18} \xi^2 \eta + b_{19} \xi \eta^2 + b_{20} \eta^3 \\ &= \sum_{i=11}^{20} b_i \xi^{p_i} \eta^{q_i} \end{aligned}$$

This is a conforming element with a total of 18 external and 2 condensed internal degrees of freedom. The 6 nodal degrees of freedom at each vertex are u , u_x , u_y , v , v_x and v_y and 2 displacements u and v at the centroid.

This cubic function ensures interelement continuity of displacements. The two nodal values, displacement and its derivative, available at each end of that side are sufficient for the unique cubic definition. Demanded by the principle of minimum potential energy, the continuity of displacement alone assures monotonic convergence. The membrane energy expression involves lower order derivatives than the energy expression for bending. In addition to conformity, the displacement fields satisfy rigid-body motions (two translations and a rotation) and constant strain conditions.

From the strain energy expression in Equation (4.7), the membrane part is

$$U = \frac{Eh}{2(1-\nu^2)} \iint \begin{bmatrix} u_\xi - \alpha \bar{t} \\ v_\eta - \alpha \bar{t} \\ u_\eta + v_\xi \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} u_\xi - \alpha \bar{t} \\ v_\eta - \alpha \bar{t} \\ u_\eta + v_\xi \end{bmatrix} dx dy \quad (4.19)$$

The mean value of arbitrary temperature distribution over the thickness is given as $\bar{t} = \frac{1}{h} \int t(z) dz$ which may vary in the planform direction. The integral quantity is denoted by N_t .

The membrane stiffness matrix with the thermal in-plane load vector can be generated in the same manner as previously described.

Separating the terms involving temperature, the strain energy U_m , only

in terms of the displacement gradients, and the thermal energy U_t are

$$U_m = D_m \iint (u_\xi^2 + v_\eta^2 + 2\nu u_\xi v_\eta + \frac{1}{2}(1-\nu)(u_\eta + v_\xi)^2) dx dy \quad (4.20a)$$

and, omitting the square of N_t which vanishes during the differentiation

$$U_t = - \frac{\alpha E}{1-\nu} \iint N_t (u_\xi + v_\eta) dx dy \quad (4.20b)$$

The stiffness matrix for the plane stress triangular element is obtained from the general expression

$$[K_m]_{20 \times 20} = D_m [R]^T [T^{-1}]^T [k] [T^{-1}] [R]. \quad (4.21)$$

The coefficient matrix $[k]$, the nodal coordinate matrix $[T]$ and the rotation matrix $[R]$ are listed in Appendix B.

The in-plane thermal force in the slab follows the distribution of the averaged temperature $\bar{t}$ varying in any manner. The thermal load vector was derived for a linear variation in the plane of an element.

$$N_t(\xi, \eta) = b_1 + b_2 \xi + b_3 \eta = \sum_{j=1}^3 b_j \xi^{r_j} \eta^{s_j} \quad (4.22)$$

For known values of N_t at the vertices, the in-plane thermal load vector acting on an element is given by

$$\{F_t\} = \frac{\alpha E}{1-\nu} [R]^T [T^{-1}]^T [k] [T^{-1}] \{N_t\} \quad (4.23)$$

where the coefficients in matrix $[k]$ are compiled from

$$k_{ij} = m_i F(m_i - 1 + r_j, n_i + s_j) + q_i F(p_i + r_j, q_i + s_j - 1) \quad \text{for } i=1, 20 \text{ \& } j=1, 3.$$

In the assembly of finite elements, the centroidal displacements u and v are not interrelated with the nodal displacement parameters of

adjacent elements. Prior to the assemblage, the internal node can be removed by the usual statical condensation process. The matrix equation of equilibrium for an element is written in partitioned form

$$\begin{bmatrix} K_o & K_{oc} \\ K_{oc}^T & K_c \end{bmatrix} \begin{Bmatrix} V \\ V_c \end{Bmatrix} = \begin{Bmatrix} P_o \\ P_c \end{Bmatrix} \quad (4.24)$$

The elimination of the centroidal displacement $\{V_c\}$ yields the reduced equilibrium equation $[K] \{V\} = \{P\}$ for an element having a stiffness matrix

$$\text{in condensed form} \quad [K]_{18 \times 18} = [K_o]_{18 \times 18} - [K_{oc}]_{18 \times 2} [K_c^{-1}]_{2 \times 2} [K_{oc}]_{2 \times 18}^T$$

$$\text{and, load vector} \quad \{P\} = \{P_o\} - [K_o]_{18 \times 2} [K_c^{-1}]_{2 \times 2} \{P_c\}$$

Only external degrees of freedom are then assembled into the structure stiffness matrix. A number of equations is reduced and the generality of the nodal numbering system is preserved.

The representation of the displacements u and v by complete cubic polynomials will be correct to order l^4 . These cubic functions lead to strain energy convergence proportional to N^{-6} which is the same rate in the triangular bending element.

4.5 Eccentric Beam Element

The steel beams are an integral part of the slab. They are parallel to the y axis. The continuity of v , w and w_y displacements along the adjoining edge between the beam and triangular elements of plane stress and plate bending are maintained. The same order polynomials were used for lateral deflection and in-plane displacements in the eccentric beam element. The beam element has at each end a node assigned not at its

neutral axis but in the middle surface of the concrete slab. The end node represents 5 degrees of freedom such as 2 in-plane (v and v_y) and 3 out-of-plane (w , w_y , and w_{yy}). The in-plane displacement in the spanwise direction is a cubic function

$$v = a_1 + a_2 y + a_3 y^2 + a_4 y^3 = \sum_{i=1}^4 a_i y^{m_i} \quad (4.25)$$

The quintic variation of the deflection is

$$w = a_5 + a_6 y + a_7 y^2 + a_8 y^3 + a_9 y^4 + a_{10} y^5 = \sum_{i=5}^{10} a_i y^{n_i} \quad (4.26)$$

In the shallow beam theory, the deformations due to transverse shear are neglected. From the strain-displacement equation, the longitudinal stresses for the beams at distance z from the midsurface of the slab are expressed in terms of displacement components as follows

$$\sigma_y = E(v_y - \alpha t(z) - z w_{yy}) \quad (4.27)$$

where $t(z)$ denotes an arbitrary temperature distribution in the beam.

Under plane section assumption, the beams and slab will deform with equal strain and curvature at the interface.

The strain energy in Equation (4.7) becomes

$$U = \frac{1}{2} E \iiint (v_y - \alpha t(z) - z w_{yy})^2 dz dx dy \quad (4.28)$$

For eccentric beam element, the strain energy without the temperature terms is

$$U_s = \frac{1}{2} E \int_0^L (A_s v_y^2 - 2 S v_y w_{yy} + I w_{yy}^2) dy \quad (4.29)$$

Here A_s is the cross-sectional area of the steel beam whose first and second moment of area with reference to midsurface of the deck are denoted

by S and I respectively. The axial and bending effects are coupled.

The element stiffness matrix for the stiffener is obtained from

$$[K_S]_{10 \times 10} = [T^{-1}]^T [k] [T^{-1}] \quad (4.30)$$

where in the matrix $[k]$ the coefficients from the integration are

$$k_{ij} = E A_S m_i m_j \frac{l^{m_i+m_j-1}}{m_i+m_j-1} - E S (m_i n_j (n_j-1) \frac{l^{m_i+n_j-2}}{m_i+n_j-2} + m_j n_i (n_i-1) \frac{l^{m_j+n_i-2}}{m_j+n_i-2}) \\ + E I n_i n_j (n_i-1) (n_j-1) \frac{l^{n_i+n_j-3}}{n_i+n_j-3}$$

The matrix $[T]$ contains the nodal coordinates.

For derivation of the thermal load vector, the strain energy associated with only the temperature is

$$U_t = \alpha E \int_0^l \int (-v_y t(z) + z w_{yy} t(z)) dy dA. \quad (4.31)$$

The integration over the stiffener area with respect to the middle surface yields

$$U_t = -\alpha E \int_0^l v_y N_t dy + \alpha E \int_0^l w_{yy} (M_t - e N_t) dy \quad (4.32)$$

The beam temperature distribution can be represented about its own neutral axis by the first temperature moment $M_t = \int z t(z) dA$ and the resultant temperature $N_t = \int t(z) dA$ at a distance e from the midsurface of the slab.

The thermal load vector for eccentric beam element becomes

$$\{F_t\} = \begin{Bmatrix} -\alpha E N_t \\ 0 \\ 0 \\ \alpha E (M_t - e N_t) \\ 0 \\ \alpha E N_t \\ 0 \\ 0 \\ -\alpha E (M_t - e N_t) \\ 0 \end{Bmatrix} \quad \text{corresponding to} \quad \begin{Bmatrix} v \\ v_y \\ w \\ w_y \\ w_{yy} \\ v \\ v_y \\ w \\ w_y \\ w_{yy} \end{Bmatrix} \quad (4.33)$$

4.6 Torsional Stiffness of Beam Element

The rigidly fixed beams in the slab-beam type bridge resist the slab twisting moments formed during bending. The integral beams stiffen the slab against rotation. Their transverse cross sections will rotate through an angle θ . The angle of twist for beams parallel to the x axis is equal to the slope w_y of the deflected slab. In the St-Venant theory of torsion, the twisting moment M_t is related to the angle of rotation by the expression

$$M_t = GC_t \frac{d\theta}{dy} \quad (4.34)$$

where $\frac{d\theta}{dy} = \frac{dw}{dy}$ which is the rate of change of rotation angle along the length of the beam. The symbol C_t denotes the torsional constant of the cross section. For wide flange shapes, a value of C_t is obtained by taking the sum of the torsional constants for the three thin plates making up the section. It is

$$C_t = \sum \frac{1}{3} \text{width} \times (\text{thickness})^3 \quad \text{for flanges and web.}$$

As taken from "Handbook of Steel Construction", the W 10x15 has a value equal to 0.105 in.⁴ for its torsional constant including rounded fillets.

The St-Venant torsion, often called pure or uniform torsion, is based on the premise that noncircular cross sections along a member are free to warp out of plane. Following the application of a torque, the transverse sections unless they are circular do not remain plane during deformation. This implies that there is axial deformation. For wide flange shapes the warping, or axial, displacement is distributed across the

flange width. The web deformation, being very small, is negligible. This warping deformation takes place when the flanges are free to bend in the y, z plane. It can be described by the bending of the flanges about the weak axis of the wide flange. With the flanges restrained against such bending, the following stresses will be induced. They are longitudinal stresses produced by the transverse resisting moment and shear stresses resulting from shear forces. The resisting warping moment in the flange is

$$M^w = -EI_w \frac{d^3\theta}{dy^3} \quad (4.35)$$

where I_w denotes the warping torsional constant for the cross section. Adding the warping torsion to the St-Venant torsion, the total torsional moment is

$$M^T = M^t + M^w. \quad (4.36)$$

Unless the slab curls to cylindrical surface, the twist will vary from section to section along the entire length of the beam. The warping displacement varies correspondingly. In nonuniform torsion, the tendency for sections to warp freely is prevented or resisted. The total torsional moment resisted by the section is M^T given in Equation (4.36).

Thin-walled open sections such as WF shapes are weak in resisting torsion on account of their low values of C_t . In their resistance to twist for the slab, the warping torsion M^w can be excluded. Only St-Venant torsion is considered. The torsion and bending effects are uncoupled. The strain energy of the twisted beam becomes

$$U_T = \frac{1}{2} G C_t \int_0^l \left(\frac{d\theta}{dy} \right)^2 dy. \quad (4.37)$$

For the angle of rotation equal to the slope w_y of the plate, same cubic function is used in the torsional element

$$\theta = \frac{\partial w}{\partial x} = a_1 + a_2 y + a_3 y^2 + a_4 y^3. \quad (4.38)$$

There are two degrees of freedom, w_x and w_{xy} at each end of the torsional beam element.

The resulting torsional stiffness matrix for the beam element is

$$[K_t] = GC_t \begin{bmatrix} \frac{6}{5l} & & & \\ & \frac{1}{10} & \frac{2l}{15} & \\ & -\frac{6}{5l} & -\frac{1}{10} & \frac{6}{5l} \\ & \frac{1}{10} & -\frac{2l}{30} & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix} \quad \text{Sym.} \quad (4.39)$$

4.7 Solution of the Thermoelastic Problem

Symmetry in the geometry of the bridge and in its temperature field which was taken as the same for all transverse cross sections facilitates the analysis. The finite element modelling of the three-span continuous bridge was reduced to only one quarter. Figure 9 depicts the bridge as a collection of equal-sized right-triangular elements of plane stress and bending plus eccentric beam elements attached to the lower surface. The assembly of all elements

$$\sum [K_m + K_b + K_s + K_t] \{X\} = \sum \{F_m + F_b + F_s\} = 0 \quad (4.40)$$

yields a set of linear equations of the form $[K]\{X\} = \{F\}$. The load vector $\{F\}$ represents the existing temperature distribution in the cross section. From the temperature distributions in the cooling history of the

bridge, a system of thermal load vectors was derived to determine the structural response.

In Figure 9, the quarter model of the bridge has 248 nodal points resulting from a uniform mesh division. The mesh line was divided into 31 equal parts along the span length and 7 laterally. The analysis involves a total of 2976 equations with a half bandwidth of 108. The lower band portion of the symmetric, positive definite matrix, K was partitioned into 24 submatrices of 108×125 . These submatrices were stored on disk, using 360 tracks. In the equation solver routine, only two submatrices were held in core at any particular stage of the operation.

For solution to the system of equations with a number of different load vectors, it is convenient to use Cholesky's decomposition method. The decomposed matrix is retained. When re-solving the system of equations for different load vectors only the backward and forward substitution procedures are simply repeated each time.

In the analysis of the three-span continuous bridge, the supports at both ends were removed to allow the bending of the slab to a dished surface. There are no reactions at the ends. The ends are free to deflect upwards when the upper surface was cooled. The bridge is only supported at the two intermediate piers and spanned transversely between the four steel beams. The steel beams, placed on rocker and rollers, are free to rotate, to translate longitudinally and to deflect only upwards. During the bending of the slab, the exterior beams will tend to lift from their supports. As the temperature gradient increases, these reactions at the supports diminish and may become zero when the curvatures are

sufficiently large. With the exterior beams completely lifted off their support, the bridge will be supported at only four points, of its interior steel beams.

Calculation of Changes in Reactions

The unknown reaction on the exterior beams at intermediate piers was determined by the use of the superposition method. With that support removed, the finite element equations are solved for the displacement vector $\{X_w\}$ of the bridge under its own weight. The deflection at the removed support is denoted, according to the finite element numbering system, as $X_w(1807)$. An imaginary unit force was applied in place of the support to produce a corresponding deflection $X_1(1807)$ in its deformation vector $\{X_1\}$. The reaction R_w is obtained as a ratio of $X_w(1807)$ to $X_1(1807)$. In linear elastic analysis, the initial displacement vector $\{X_1\}$ of the bridge under its own weight was calculated as

$$\{X_1\} = \{X_w\} + R_w\{X_1\} \quad (4.41)$$

For the bridge subjected to only temperature changes, the displacement vector $\{X_0\}$ was computed as

$$\{X_0\} = \{X_t\} + (R_t - R_w)\{X_1\} \quad (4.42)$$

where $\{X_t\}$ denotes the displacement vector produced by temperature changes in the bridge without its exterior support, which is R_t . The combined deflections, $X_w(1807)$ and $X_t(1807)$ in downward direction will indicate that the exterior steel beams remain supported. Knowing that

a unit force at the removed support produces a corresponding deflection $X_1(1807)$, the support reaction R_t varying for different temperature distributions can be calculated. This reaction becomes zero for large temperature gradients causing the bridge to deflect upwards at the ends of the intermediate piers.

From the known nodal displacement $\{X_0\}$, the strains can now be determined and, from the strains, the nodal stresses according to the stress-strain relations given in Equation (4.1). In using these types of elements, the nodal stress components are represented through the curvatures and in-plane gradients. The nodal stresses were obtained without averaging the stresses at a common node.

CHAPTER 5

DISCUSSION OF RESULTS5.1 Temperature History

The temperature history of the cooled bridge was computed at every 10 seconds for a duration of 77 minutes. The temperature fields at various instants in the bridge cross section are presented in Figures 10 to 18. Each figure shows temperature distributions evolved from a previous time. The dashed isothermal lines represent the temperatures at a time prior to that of the temperatures denoted by solid curve. The temperature variation through the slab thickness is shown for sections at mid-distance between steel beams, at quarter-points, at center line of steel beam, and at the slab edge.

The high concentration of isothermals and their shift with time indicate that large temperature changes occur at early times. Afterwards, the temperatures slowly approach steady state as shown in Figure 18. Steady conduction should be attained in approximately 2 hours. The time travel of, for instance, a 66°F isotherm can be followed in Figure 19.

Distribution of temperature on cross section

Approximately 7 minutes have elapsed before the temperatures have penetrated through the concrete slab and reached the embedded steel flange. Within that time, the temperature fields consisted of straight lines except for the slight temperature variation in the vicinity of the longitudinal

steel bars. The temperature fields are one-dimensional. Their temperatures vary in the thickness direction. When lower temperatures have reached the steel beam, noticeable variation of temperature appeared in the direction of the slab width. The temperature distribution becomes two-dimensional near the embedded steel flange. The temperature variations are shown for section with only isothermal upper surface. Its temperature gradients in the width direction would be larger when the upper surface is subject to convection. This variation along the width at further distance from the embedded flange fades away and the temperature distribution is effectively one-dimensional.

Noticeable temperature variation appeared in the vicinity of longitudinal reinforcing steel bars. Similar temperature variation can be expected near the transverse steel bars when considering spanwise temperature variation.

The temperature distribution through the slab thickness can be described by a parabola. The curve is very sharp at early times and large thermal moments are produced. The parabolic curve flattens out with time to a straight line at steady state. In the steel beam, the transient and steady temperatures are distributed nonlinearly across its depth.

The temperatures in the steel beam undergo rapid adjustment. Their temperature gradients are much smaller than those in the concrete slab. The effect of the difference in the rates of temperature change for concrete slab and steel beam will be discussed when the deflections of the bridge are examined.

The side portion of the bridge cross section in which the edge of the slab was exposed to ambient air was analyzed. The edge effect, as seen in the figures, was felt at distances approximately $1\frac{1}{2}$ times the slab thickness.

Significance test of finite element temperature

Statistical analysis was made to determine whether the predicted temperatures adequately represents the actual temperatures in the bridge. Only lower surface temperatures at the thermocouple points were compared. The ratio of the finite element temperatures to those registered by the thermocouples should be equal to or be very close to unit value. When the ratio is equal to unit value, the finite element and measured temperatures coincide. It was assumed that the ratios of corresponding temperatures are normally distributed. Their mean and the mean standard deviation was calculated as 1.0065 ± 0.00127 . Using the t test, the difference in the mean and unit value is not significant, not even at 0.05 percent level. The closeness of the predicted to measured temperatures was tested by calculating chi-square which had a value of 5.118. The probability, greater than 0.99, was obtained from the table with 67 degrees of freedom.

These test results indicate that temperatures were accurately obtained in the finite element analysis made with constant heat-transfer properties. The temperature distribution over the full transverse cross section may be determined by the analysis of subsections between planes of geometric symmetry as carried out in this study.

Accuracy and convergence of the temperature solution

It is felt that the method used in the temperature analysis yielded temperatures accurate to $\pm 0.5^\circ\text{F}$. No attempt was made to establish the error bounds of the transient solution at the 10 second intervals.

The finite element solution exhibited unusual oscillation at very early times, until the 30-second interval. The temperatures just slightly exceeded the initial temperature of 78.3°F . The first and second law of thermodynamics are violated. This early oscillatory behaviour is inherent in the solution for the finite element system of ordinary differential equations. This is not a stability problem from numerical time integration.

At steady state, the finite element solution will give an upper limit to the rate of heat transfer, according to Eq. 3.9. Most of the nodal temperatures will converge from below as the mesh is progressively refined. The elements in the 1652-node mesh can be further subdivided into four identical rectangular elements with two divisions in each direction. With mesh refinement to 6177 nodes, the accuracy of the nodal temperatures is improved by as much as 0.05°F on the lower bound and 0.02°F for upper bound temperature.

Higher accuracy is obtained with linear temperature element of rectangular shape than with the triangular element. It was not necessary to use triangular elements in modelling this cross section.

Embedded thermocouples

The temperature gradients in the transverse cross section varied from 20°F/in. at early times to 6°F/in. at steady condition. Measurement

locations for temperature inside the slab were not assured to within ± 0.025 in. for reliable readings. The temperature registered by the inside thermocouples stayed, as expected, between the temperatures of the outer surfaces. The time curves of the internal temperature can be defined by an exponentially decaying function.

With the measured data for the slab internal temperature, the type of temperature distribution through the slab thickness could be experimentally defined. Closer approximations of heat constants should be obtained by including the internal temperatures in the matching procedure.

Closure

Temperatures within the bridge can be predicted from observed temperatures on the surfaces. Several transverse cross sections along the bridge span can be instrumented to examine the spanwise temperature variation. Various sections can be analyzed for the different exposure.

5.2 Deflections

The effect of these transient temperature changes on the bridge can be seen in Figure 20 in which the bridge lateral deflections were plotted against time. The temperature loading histories, used as an input for stress analysis, are shown with the deflection curves. The thermal loading system was presented in the form of thermal moments for a portion of the slab between beams' centers and for a steel beam and in terms of average temperatures for the in-plane forces acting on the slab and steel beams. The temperature moments and mean temperatures were given in the instantaneous temperature field figures (10 to 18) at various sections across the width of the bridge.

In composite construction, the largest factor for the stress considerations seems to be the difference of heat transfer properties involved instead of the different coefficient of thermal expansion. The relatively slower response of slab to temperature changes allowed the thermal moments to increase very quickly to a peak value in approximately 6 minutes. At that time, lower temperatures have already reached the steel beam. It was the steel beams having higher coefficient of expansion that resisted the bending and contraction of the concrete slab. Within these 6 minutes, the deflection curves for the ends were nearly a straight line. With increasing thermal moment in the slab, the end deflections continued to increase at a diminishing rate reaching maximum value of 0.6 in. after 32 minutes of cooling. As steady-state conditions were approached, the deflections gradually decreased. The thermal moments in the steel beams increased continually as shown in the figure.

Under load, the slab of the bridge tends to deflect in two orthogonal directions. At the exterior support of interior piers, there was uplift between 3 minutes to 65 minutes times. The bridge weight has prevented further uplift at that support.

Comparison of finite element and measured deflections

As function of time, the observed end deflections of the beams were distributed in similar manner, with 0.23 in. maximum deflection occurring in approximately 47 minutes. Prior to the cooling test, measurements of uplift or of changes in reaction were not considered at the exterior supports provided by intermediate piers.

The ratios of the measured deflections to those obtained in the finite element analysis were calculated throughout the cooling period. The mean of the ratios was found to be 0.278 and the standard deviation of 0.068.

The difference in the actual and predicted may be attributable to the following factors which were overlooked. The effect of the ice load of 4.5 psf on the measured deflections was neglected. From finite element analysis, the weight of ice should produce a downward deflection at the end beams of approximately 0.10 inches. As the ice melts away, it was estimated that considerable decrease in ice load occurred after 20 minutes of cooling. The temperature fields may not be the same for every transverse cross section as assumed in the thermoelastic analysis. Surface temperatures observed at various points along the span would indicate whether uniform boundary conditions existed. The time of 2 to 3 minutes taken to completely cover the upper surface may be sufficient to produce spanwise temperature variations which were neglected. The finite element results show that the ends, in the first minute, deflect upwards to 0.0486 in. at center line and 0.0768 in. at the corners. During the cooling test water has leaked through the polyethylene sheet. The swelling of the concrete will have reduced the thermal deflections in the bridge.

5.3 Strains

After examining the results for deflection, similar comparison was expected between the measured and predicted strains in the steel beams. The strain measurements failed to indicate, at least, the pattern of

deformation in the bridge under transient temperature conditions. The automatic strain recorder was operating in the laboratory whose room temperature declined 6.5°F and humidity changed as well. These changes in the environment seem to have affected the system for recording strains.

The longitudinal strains calculated in the steel beams and in the concrete slab at the middle of the bridge and at the interior pier are shown in Figures 21 and 22 respectively. The strains were computed under the assumption that their distributions over the bridge depth are linear. The dashed straight line represents the strain distribution before the redundant reaction was considered. The shift from the dashed to solid line illustrates the redistribution of thermal stresses when external forces are applied.

It should be pointed out that the strains along the width and span are nearly constant as shown in the figures by the dashed lines for interior and exterior beam section at mid-span and at interior pier. There is strain variation near the ends of the bridge. In this case the bridge was free of external forces including its own weight. There were no reactions on the external beams at the interior pier. The long bridge having aspect ratio greater than 4 was bent to a cylindrical surface with single curvatures. The curvatures were nearly constant over the entire surface of the bridge except at its ends. When the weight of the bridge was applied, the curvatures and strains in both principal directions are redistributed as shown by the solid lines.

The shaded portion in the beams represents the longitudinal thermal stresses. The temperature compensating strain gauges will not register the thermal expansions. The strains to be measured are produced by the thermal stress.

5.4 Thermal Stresses

From the strain diagrams in Figures 21 and 22, the stress distribution in the steel beams can be seen for various temperature distributions as steady state is approached. The strains, bounded by the curve for the free thermal contraction and the dashed or solid straight line, were suppressed by the thermal stresses. Their distributions in the steel beams are described by the variation of these strains throughout the depth. The distributions of thermal stresses in the concrete slab are not bounded by that same straight line in the steel beam. They are given by the stress-strain expressions which includes strains in both principal directions.

When the bridge is cooled on the upper surface, tensile stresses are produced on the upper surface, compressive stresses at interface of concrete and of steel beam. The lower portion of the steel beams is usually in compression. The interior steel beams is stressed entirely in tension at the intermediate piers when the redundant reactions were included. A different redistribution of bending moments is expected in the bridge under its own weight. Its concrete slab spans in the transverse direction and is continuous over its four supports at the interior piers. The effect of temperature changes can be seen when restrictions are imposed on the thermal displacements of the bridge. Thermal stresses can partially relieve the bridge from its stresses or can increase the existing stresses.

The longitudinal stresses on the upper and lower surfaces of the slab and beams respectively are plotted against time in Figures 23 and 24. The transient-stress curves have similar shape.

The thermal stresses are expected to be lower in the bridge with non-supporting piers at both ends.

Stresses in steel beam

As shown in Figure 23, the peaks of the curves, i.e. maximum compression stresses on the beam upper surface, are reached in 8 minutes. Until that time, the beam temperature remained unchanged. The maximum compressive and tensile stresses of -1600 psi and 3300 psi respectively occurred in the exterior beam at the intermediate pier. The permissible stresses is 29 ksi for the G40.12 steel beams. The stresses then gradually decrease from their peak values to steady stresses indicated by the annotated mark at the 77-minute ordinate. The decrease in the stresses is related to the temperature penetration which, at this time, is in the steel beam. The variation of the upper and lower surface stresses along the beams is shown at 12 minutes in the figure.

Stresses in concrete slab

Figure 24 shows that the transient-stress curves for concrete above the beams have peaked very early, in 3 minutes. The maximum tensile stress of 1130 psi has just exceeded the thermal stress value for a completely restrained concrete slab. The segment of the slab over the interior piers was made of concrete with 7400 psi and 7200 psi compressive cylinder strength. The permissible tensile stress, which is 3 percent of compressive strength, has been exceeded by far and, as well as, the concrete-cracking stress of 10 times the square root of the compressive

strength. With cracks formed, the analysis must be modified according to the reduction in slab rigidity. This reduction affects the magnitude of the deflections which were overestimated.

The high tensile stresses were not maintained very long. After a rapid decrease during the next 30 minutes, the stresses continued to decrease at a slower rate towards the steady stress values indicated by the horizontal slash in the figure.

The concrete at the interface of the interior steel beams supported at the intermediate piers was stressed in compression to a maximum value of 345 psi. This value is substantially below the permissible stress of 40 percent its strength.

The transverse stresses were as high as 986 psi in tension and -330 psi in compression when longitudinal stresses have reached their peak values. The longitudinal stresses are greater than the stresses in the transverse direction. The steel beams provide flexural rigidity and resist the bending of the slab to temperature changes. In the transverse direction, resistance to the bending in the slab is provided by the redundant constraints at the interior supports.

5.5 Effects of Uniform Temperature Change

Uniform temperature change in the composite bridge has a smaller effect than the nonuniformly distributed temperatures over its cross section, especially the temperature changes at the early stages of the transient conditions. The bridge will bend in a single curvature when subjected to uniform temperature change. The longitudinal stresses and

strains are constant over the planform of the bridge, except near the ends. The transverse stresses are negligibly small.

For uniform temperature rise of 100°F , the ends of the bridge would deflect 0.53 in. in an upward direction. The concrete slab is stressed to -38 psi in compression at its upper surface and 154 psi in tension at the lower surface. Compressive stresses of 2289 psi are produced at the beam interface and tensile stresses of 1361 psi at the lower surface of the steel beams.

CHAPTER 6

CONCLUSIONS

A numerical procedure is presented for transient temperatures and thermoelastic analyses of slab-beam type highway bridges under time-varying temperature conditions. The application of the proposed method is described on a laboratory tested composite bridge model whose upper surface was rapidly cooled from its initial uniform temperature. A study was made on the behaviour of the bridge throughout its cooling period. The cooling test served to indicate certain conditions expected in the field. In general, the thermal behaviour of the prototype is similar to that of the reduced size model.

Solution for the two-dimensional temperature distributions in the cross-sectional profile of the bridge was obtained with accuracy. In view of the variation of instantaneous temperature along the slab width, a two-dimensional analysis is warranted especially in the vicinity of embedded steel flange and near the slab edges. Results indicate that the linear temperature rectangular element is of sufficient order to represent the temperature field in the transverse cross section. The numerical integration over succeeding time increments was carried out according to the Crank-Nicolson approximation. The proposed method of matching the calculated to observed temperatures on the lower surfaces can provide very close approximations of the coefficients for heat transfer properties involved in the temperature problem.

The solution for displacements in the cooled bridge was obtained from high order finite elements used in a sufficiently fine mesh. There is representation of the stresses in the finite element bridge model having curvatures as degrees of freedom. The thermoelastic analysis is based on small-deflection theory and on uncracked concrete section.

The structural response of the three-span bridge to time-varying temperatures was visualized by allowing uplift at the supports. The upward deflections of the bridge at the ends which were not supported initially were calculated to be as much as 0.6 inches in approximately 17 minutes from the time of the change in upper surface temperature. There was uplift for a certain time at the intermediate piers where the exterior steel beams were supported. The observed and finite element results described similar type of displacement pattern. The thermal stresses in the bridge with non-supporting piers at the ends were induced by temperature gradients and by the redundant constraints at the intermediate piers, produced by the weight of bridge. At early stages of the cooling period when large temperature gradients existed, the stresses in both principal directions have exceeded the tensile strength of concrete. The steel beams, being subjected to smaller and gradual temperature changes under these conditions, have experienced smaller stresses. The restraints, in the form of support reactions at the ends of the bridge prevent uplift and will further increase the existing stresses.

The proposed method for thermoelastic analysis is applicable to bridges subjected to a thermal environment that produces different temperature on the bridge surfaces. In variable climate, thermal stresses need to be considered.

Recommendations for further study

For closer representation of the actual bridge behaviour, the finite element model may include the construction joints existing in concrete slab, the diaphragms which provide certain amount of stiffness to the slab, cracked sections whenever tensile strength in concrete is exceeded, and the different flexural rigidities distributed in their respective orthogonal directions for the reinforced concrete slab. The finite element model for the structural analysis may be further refined by using for each segment of the slab its own modulus of elasticity and coefficient of thermal expansion.

Several transverse cross sections of the bridge can be instrumented with thermocouples for determining the spanwise temperature variations. The observed temperatures in these sections may then be input into the finite element model for temperature loads. Measurements from thermocouples accurately positioned inside the slab will confirm the distribution of temperature across slab thickness.

Difficulties were experienced in measuring the thermal strains in the bridge. Knowledge regarding the thermal strains is required when evaluating the strain measurements made in the field for nonthermal loads.

An alternate procedure for observing the structural response of the bridge to transient temperature changes may include load cells placed at all the supports. Changes in support reactions can be recorded.

The thermal analysis of the bridge may be extended to various temperature conditions. The proposed method can be applied to the analysis of the prototype for a complete understanding of the structural behaviour

under daily temperature fluctuations.

In the case of deep beams, plane stress elements may be used for their idealizations in place of the eccentric beam elements.

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APPENDIX A

Matrices for 18-Degree Triangular Plate-Bending Element

The matrices are defined in Ref.(60). The nodal coordinate matrix [T] is listed as

$$[T] = \begin{bmatrix} 1 & -b & 0 & b^2 & 0 & 0 & -b^3 & 0 & 0 & 0 & b^4 & 0 & 0 & 0 & 0 & -b^5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2b & 0 & 0 & 3b^2 & 0 & 0 & 0 & -4b^3 & 0 & 0 & 0 & 0 & 5b^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -b & 0 & 0 & b^2 & 0 & 0 & 0 & -b^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -6b & 0 & 0 & 0 & 12b^2 & 0 & 0 & 0 & 0 & -20b^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2b & 0 & 0 & 0 & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2b & 0 & 0 & 0 & 2b^2 & 0 & 0 & 0 & -2b^3 & 0 & 0 & 0 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & a^4 & 0 & 0 & 0 & 0 & a^5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 4a^3 & 0 & 0 & 0 & 0 & 5a^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 6a & 0 & 0 & 0 & 12a^2 & 0 & 0 & 0 & 0 & 20a^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2a & 0 & 0 & 0 & 2a^2 & 0 & 0 & 0 & 2a^3 & 0 & 0 & 0 \\ 1 & 0 & c & 0 & 0 & c^2 & 0 & c & 0 & c^3 & 0 & 0 & 0 & 0 & 0 & c^4 & 0 & 0 & 0 & c^5 \\ 0 & 1 & 0 & 0 & c & 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 & c^3 & 0 & 0 & 0 & c^4 \\ 0 & 0 & 1 & 0 & 0 & 2c & 0 & 0 & 0 & 3c^2 & 0 & 0 & 0 & 0 & 0 & 4c^3 & 0 & 0 & 0 & 5c^4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2c & 0 & 0 & 0 & 0 & 2c^2 & 0 & 0 & 0 & 0 & 2c^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2c & 0 & 0 & 0 & 0 & 3c^2 & 0 & 0 & 0 & 0 & 4c^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 6c & 0 & 0 & 0 & 0 & 12c^2 & 0 & 0 & 0 & 0 & 20c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5a^4c & 3a^2c^3-2a^4c & -2ac^4+3a^3c^2 & c^5-4a^2c^3 & 5ac^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5b^4c & 3b^2c^3-2b^4c & 2bc^4-3b^3c^2 & c^5-4b^2c^3 & -5bc^4 \end{bmatrix}$$

Its determinant, $\det T = -64(a+b)^{17}c^{20}(a^2+c^2)(b^2+c^2)$, will become singular only when the area of the triangular element vanishes.

In the 20x20 integral matrix [k], its coefficients were computed from the formula

$$k_{ij} = m_i n_j (m_i - 1)(n_j - 1) F(m_i + m_j - 4, n_i + n_j) + n_i n_j (n_i - 1)(m_j - 1) F(m_i + m_j, n_i + n_j - 4) + \{2(1-\nu)m_i n_j n_i n_j + \nu m_i n_j (m_i - 1)(n_j - 1) + \nu m_j n_i (m_j - 1)(n_i - 1)\} F(m_i + m_j - 2, n_i + n_j - 2) \quad \text{for } i = 1, 20 \text{ and } j = 1, 20$$

The rotation matrix [R], relating the displacements in the two coordinate systems,

is formed from

$$\begin{bmatrix} w_{\xi} \\ w_{\eta} \\ w_{\xi\xi} \\ w_{\xi\eta} \\ w_{\eta\eta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \\ 0 & 0 & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta \\ 0 & 0 & \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_{xx} \\ w_{xy} \\ w_{yy} \end{bmatrix}$$

where θ represents the angle between two coordinate axes.

APPENDIX B

Matrices for Quadratic Plane Stress Triangular Element

The matrices [T], [k], and [R] involved in the formulation of the QST element stiffness matrix are listed as follows.

The nodal coordinate matrix [T] is given as

$$[T] = \begin{bmatrix} T_1 & 0 \\ 0 & T_1 \\ T_2 & 0 \\ 0 & T_2 \\ T_3 & 0 \\ 0 & T_3 \\ T_4 & 0 \\ 0 & T_4 \end{bmatrix}$$

where

$$[T_1] = \begin{bmatrix} 1 & -b & 0 & b^2 & 0 & 0 & -b^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2b & 0 & 0 & 3b^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -b & 0 & 0 & b^2 & 0 & 0 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 \end{bmatrix}$$

$$[T_3] = \begin{bmatrix} 1 & 0 & c & 0 & 0 & 0 & c^2 & 0 & 0 & c^3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2c & 0 & 0 & 3c^2 \end{bmatrix}$$

$$[T_4] = \begin{bmatrix} 1 & \frac{a-b}{3} & \frac{c}{3} & \frac{(a-b)^2}{9} & \frac{(a-b)c}{9} & \frac{c^2}{9} & \frac{(a-b)^3}{27} & \frac{(a-b)^2c}{27} & \frac{(a-b)c^2}{27} & \frac{c^3}{27} \end{bmatrix}$$

The coefficients in the 20 x 20 integral matrix [k] were calculated

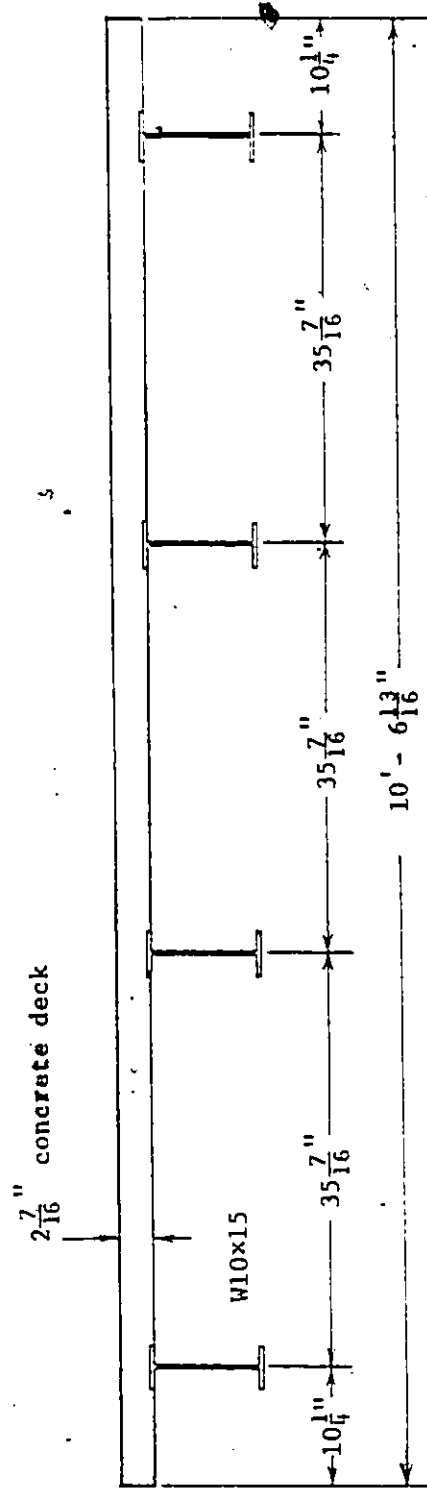
$$k_{ij} = m_i m_j F(m_i + m_j - 2, n_i + n_j) + q_i q_j F(p_i + p_j, q_i + q_j - 2) + \frac{1-\nu}{2} [n_i n_j F(m_i + m_j, n_i + n_j - 2) + p_i p_j F(p_i + p_j - 2, q_i + q_j)] \\ + [\frac{1-\nu}{2} n_j p_i + \nu m_j q_i] F(m_j + p_i - 1, n_j + q_i - 1) + [\frac{1-\nu}{2} n_i p_j + \nu m_i q_j] F(m_i + p_j - 1, n_i + q_j - 1)$$

$$p \text{ \& } q = 0 \text{ when } 1 \leq j \leq 10 \text{ and } m \text{ \& } n = 0 \text{ when } 1 \leq j > 10$$

The displacements in the two coordinate systems are related by the rotation matrix [R]. The coordinate transformation is made as follows

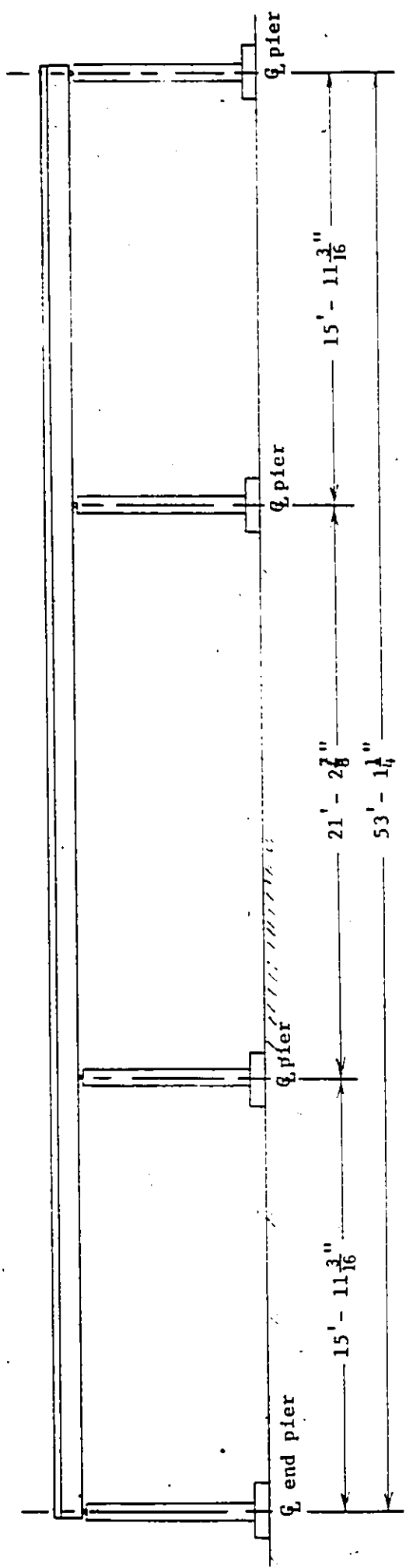
$$\begin{bmatrix} \bar{u} \\ \bar{u}_\xi \\ \bar{u}_\eta \\ \bar{v} \\ \bar{v}_\xi \\ \bar{v}_\eta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta & 0 & 0 \\ 0 & \cos^2\theta & \sin\theta\cos\theta & 0 & \sin\theta\cos\theta & \sin^2\theta \\ 0 & -\sin\theta\cos\theta & \cos^2\theta & 0 & -\sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & -\sin\theta\cos\theta & \sin^2\theta & 0 & \cos^2\theta & \sin\theta\cos\theta \\ 0 & \sin^2\theta & -\sin\theta\cos\theta & 0 & -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} u \\ u_x \\ u_y \\ v \\ v_x \\ v_y \end{bmatrix}$$

where θ represents the angle between two coordinate axes.



TRANSVERSE SECTION

$\frac{3}{4}$ in = 1 ft



ELEVATION

$\frac{3}{16}$ in = 1 ft

Fig. 1 The 0.354-scale model bridge

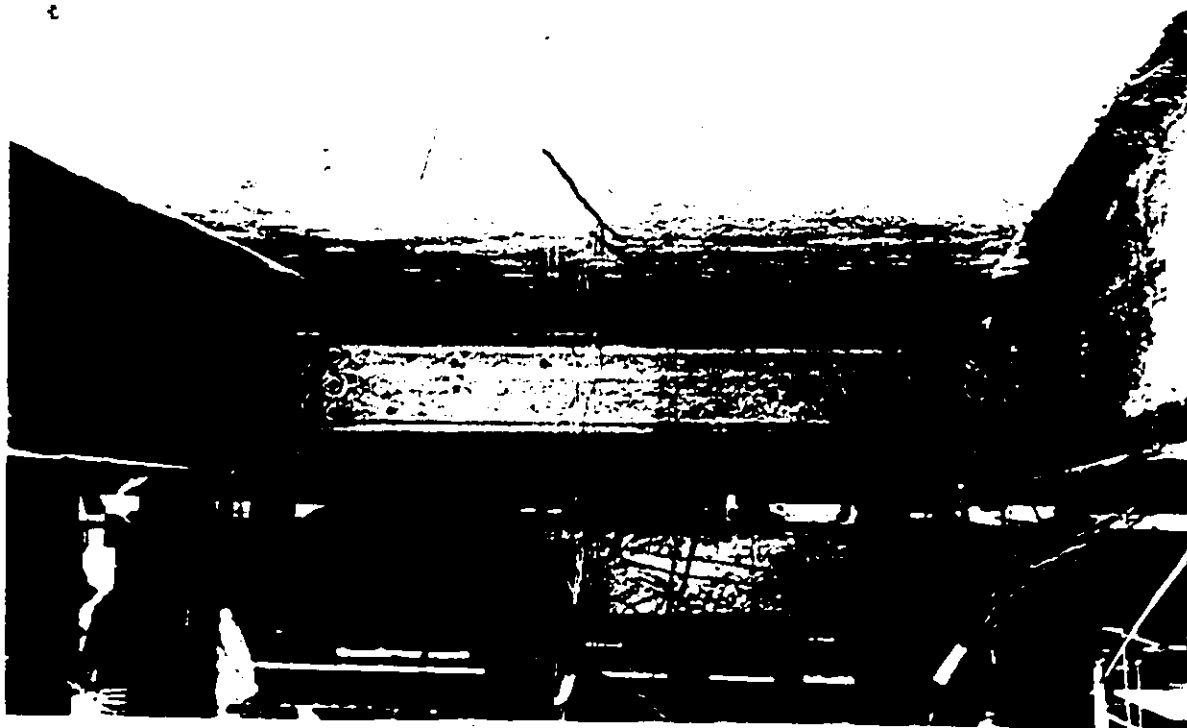
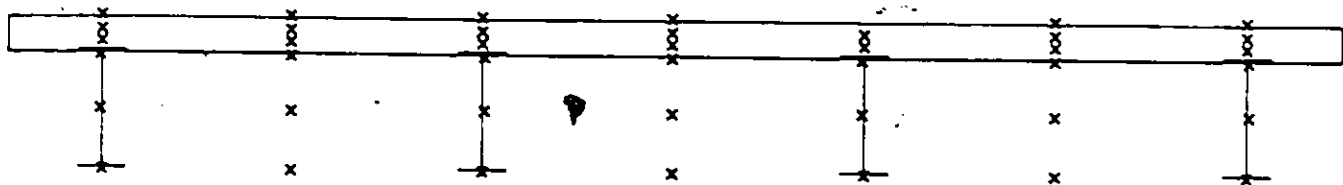
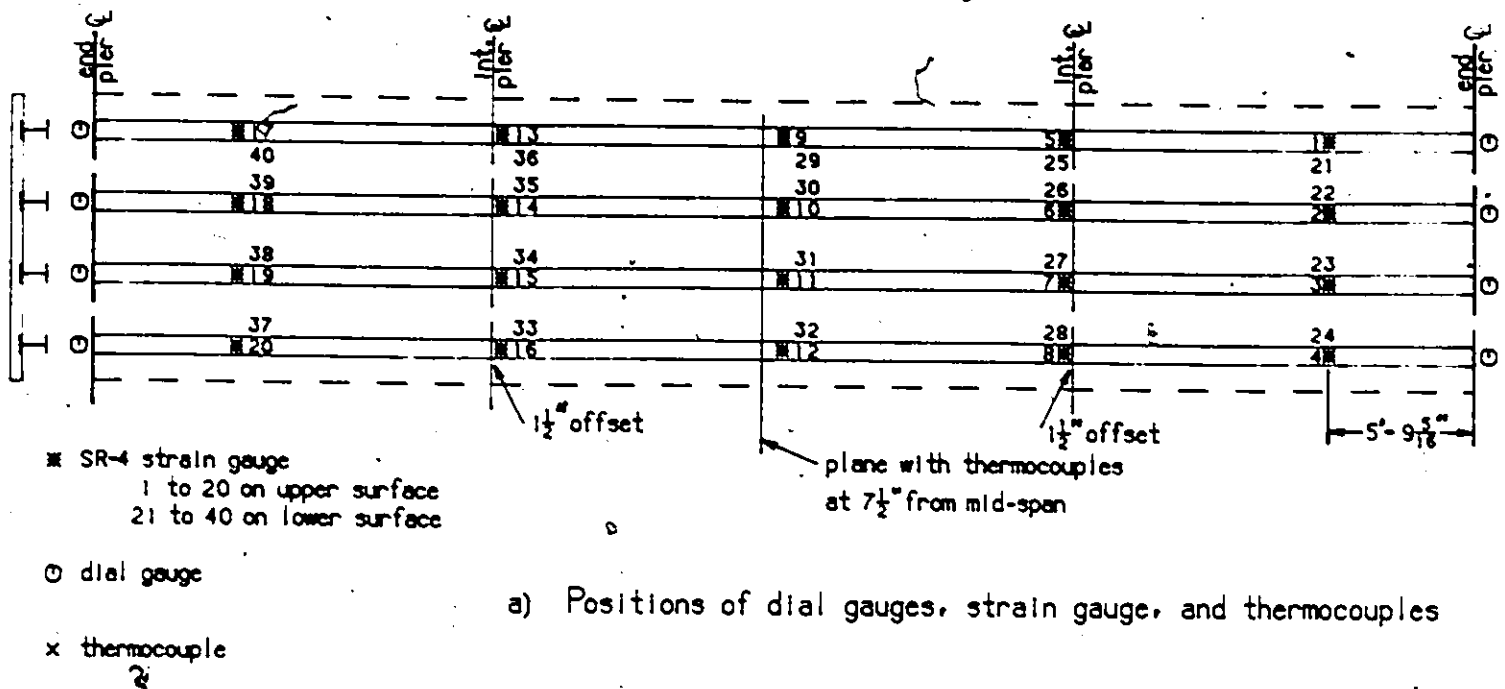


Fig. 3 Instrumentations



Fig. 4 Ground level view of the three-span composite bridge

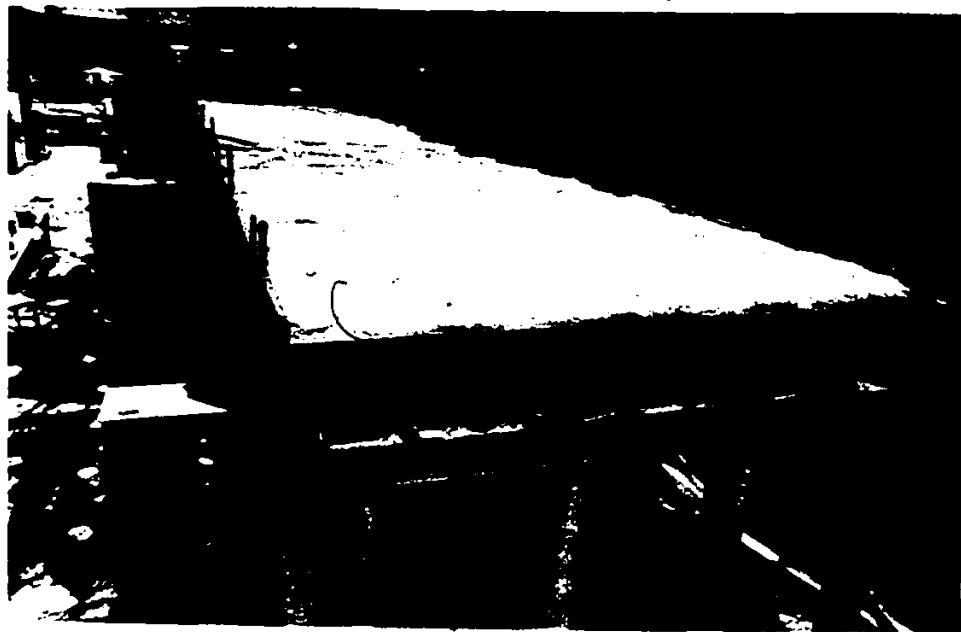
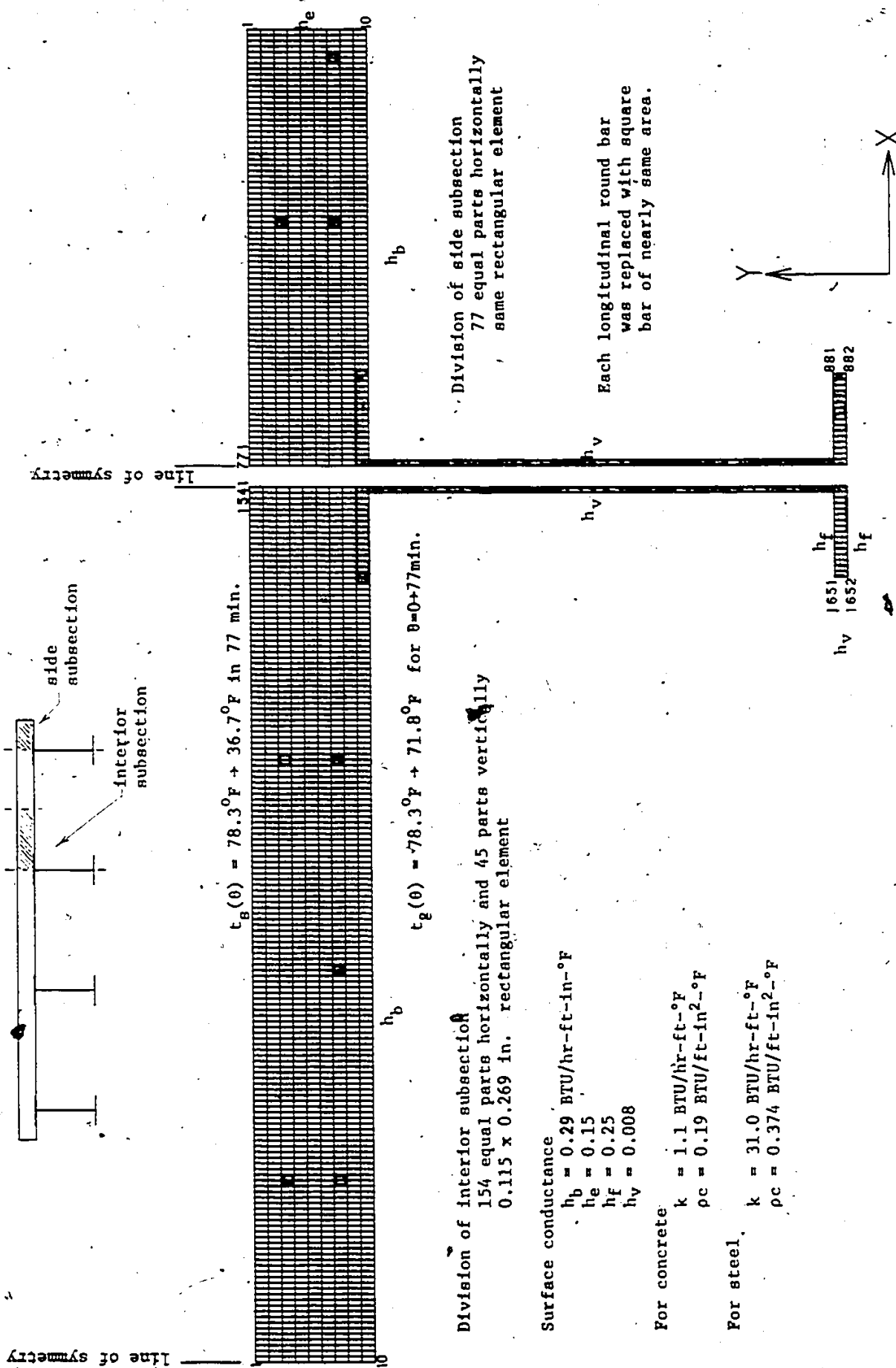
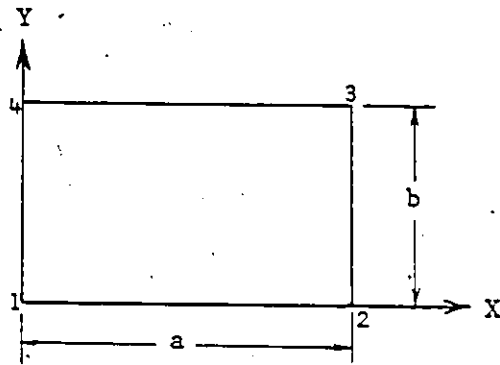


Fig. 5 Cooling test on bridge

Top: entire upper surface covered with melting ice cubes
 Front: 4 dial gauges mounted against the non-supporting end pier





a) Rectangular Element

Dimensions of triangle

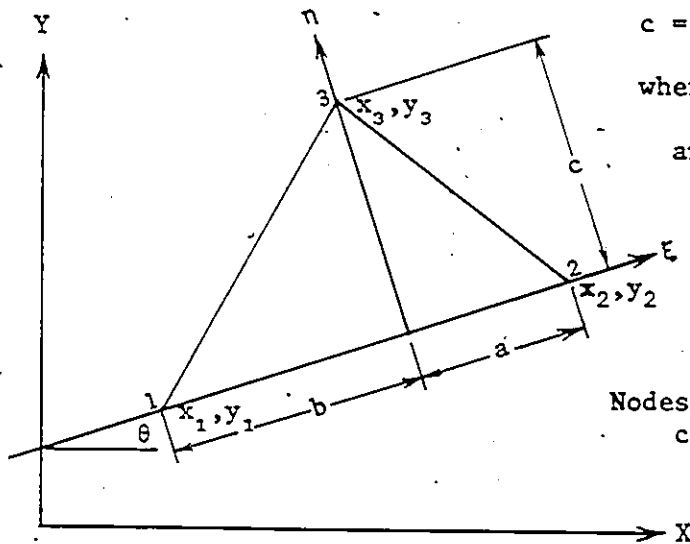
$$a = [(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1)]/r$$

$$b = [(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)]/r$$

$$c = [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]/r$$

$$\text{where } r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{and } \cos\theta = \frac{x_2 - x_1}{r} \quad \sin\theta = \frac{y_2 - y_1}{r}$$



Nodes are numbered in counter-clockwise order.

b) Triangular Element

Fig. 7 Rectangular and triangular elements.

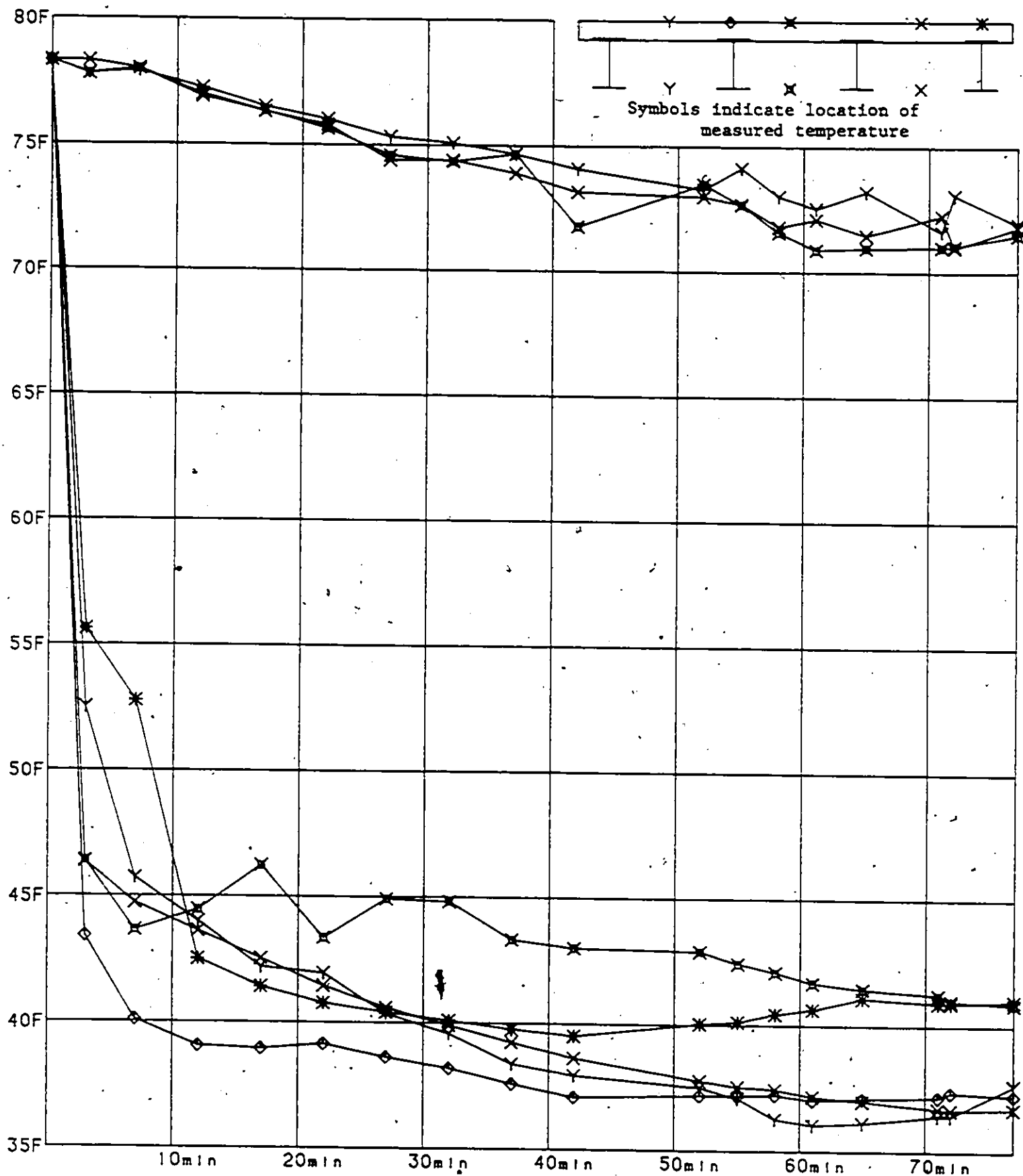


Fig. 8 Measured boundary temperatures during the 77-min. cooling period.

Division of 1/4 bridge

7 equal parts transversely
30 equal parts longitudinally

Total of 2976 d's of f. with a108 half bandwidth

At each node - 12 d's of f.

Material properties

$$E_c = 6.3 \times 10^6 \text{ psi}$$

$$P_g = 29 \times 10^6 \text{ psi}$$

$$\alpha_c = 5.5 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$

$$\alpha_g = 6.5 \times 10^{-6}$$

$$v_c = 0.15$$

e = 5.931 in

$$\left\{ \begin{array}{l} u \\ u_x \\ u_y \end{array} \right\} \quad \text{in-plane displacement} \quad \left\{ \begin{array}{l} v \\ v_x \\ v_y \end{array} \right\}$$

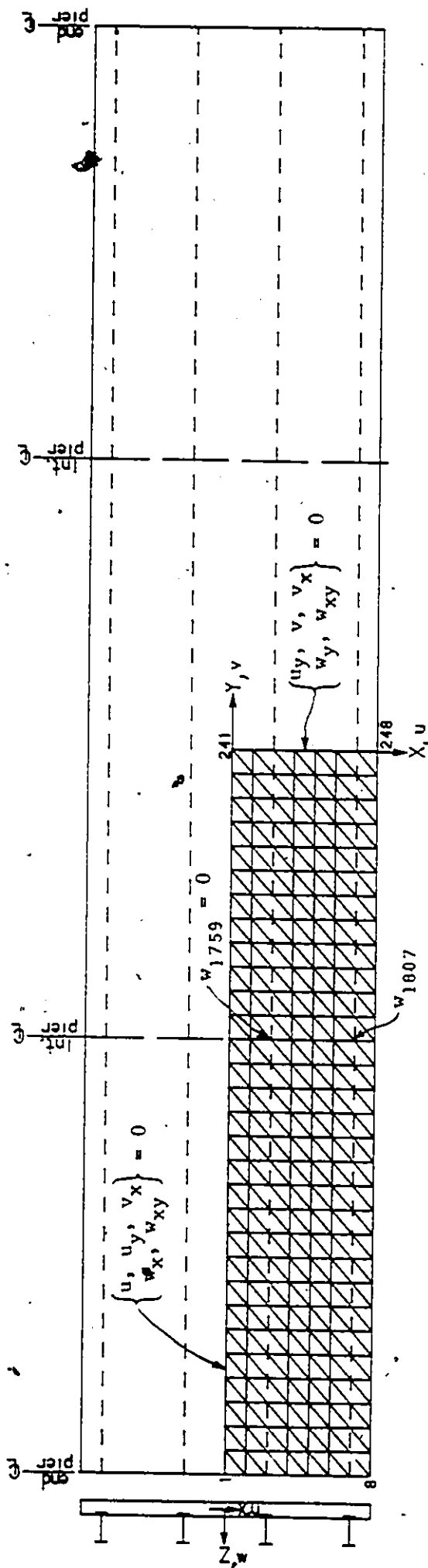


Fig. 9 Finite element analytical model of bridge.

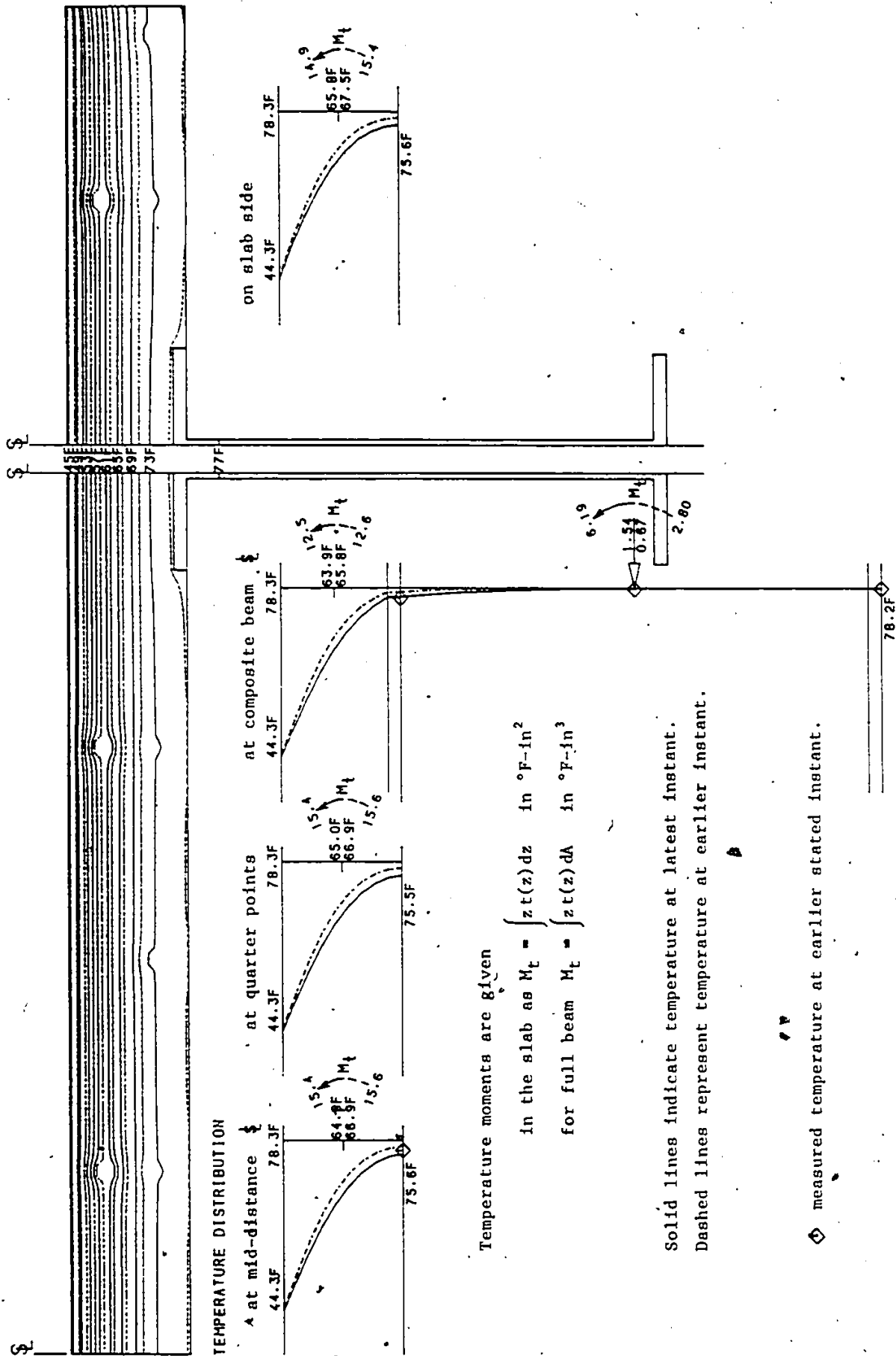


Fig. 10 Instantaneous temperature fields in cross section at 7 and 9 min.

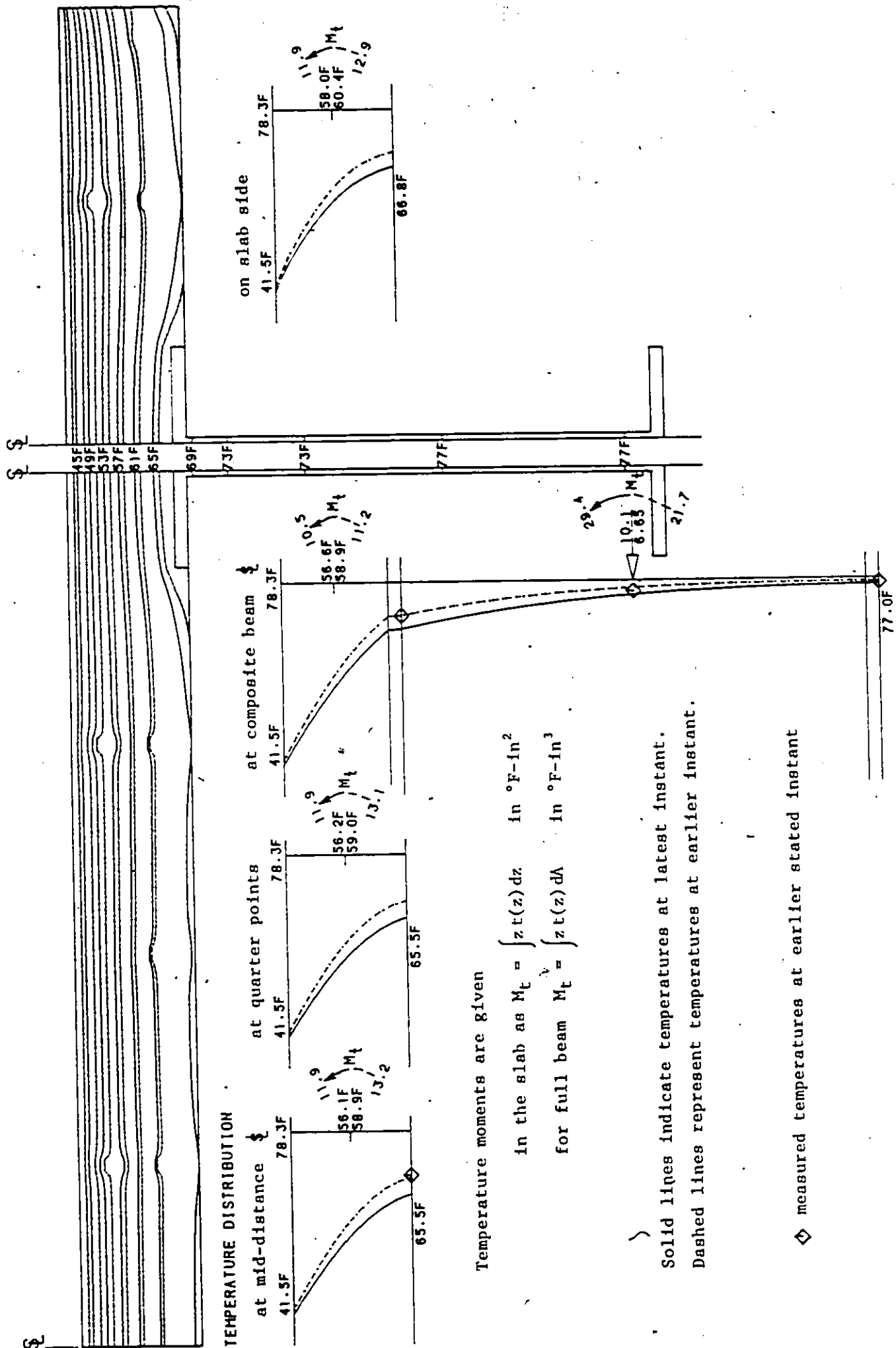


Fig. 12 Instantaneous temperature fields in cross section at 17 and 22 min.

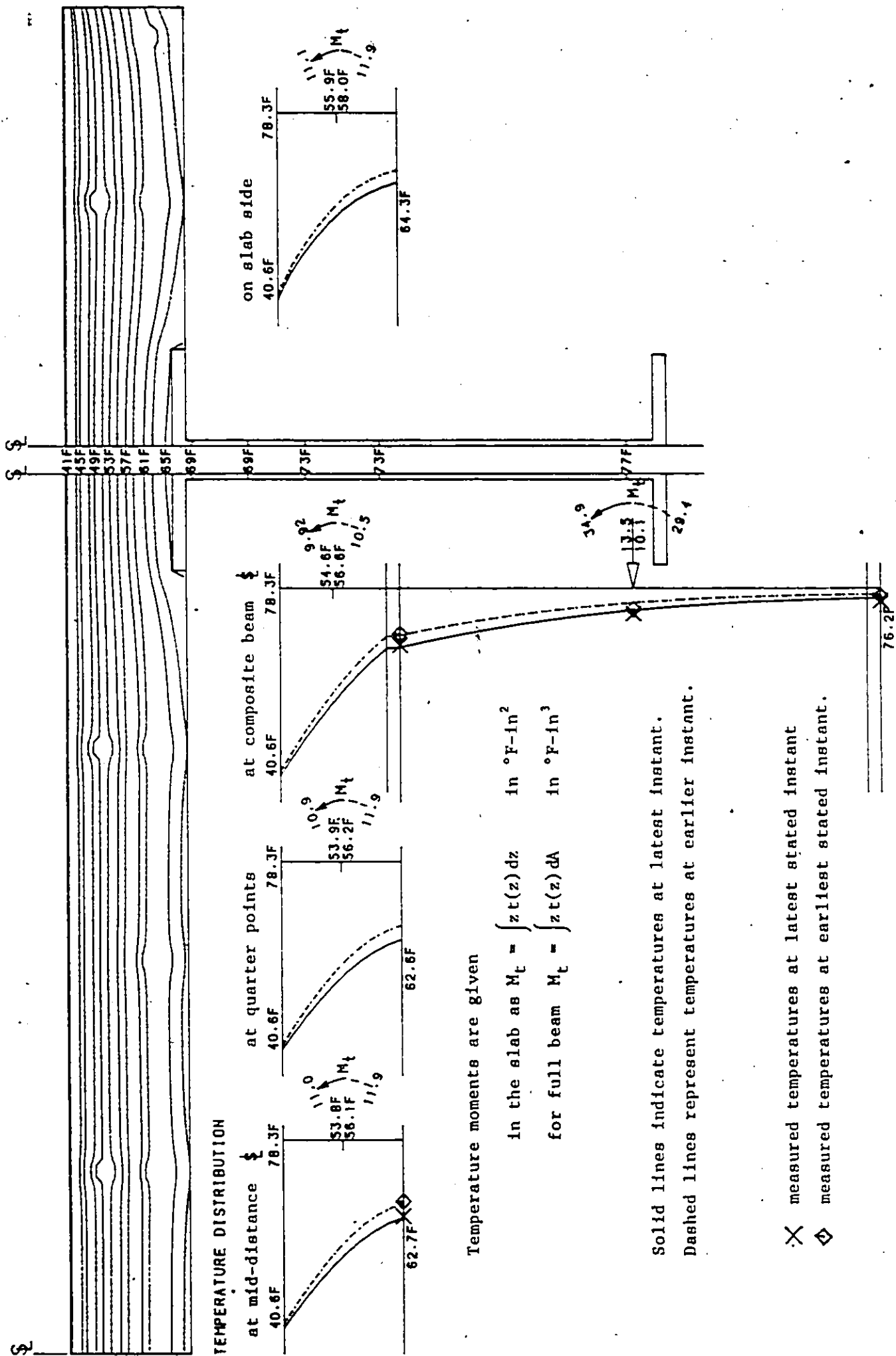


Fig. 13 Instantaneous temperature fields in cross section at 22 and 27 min.

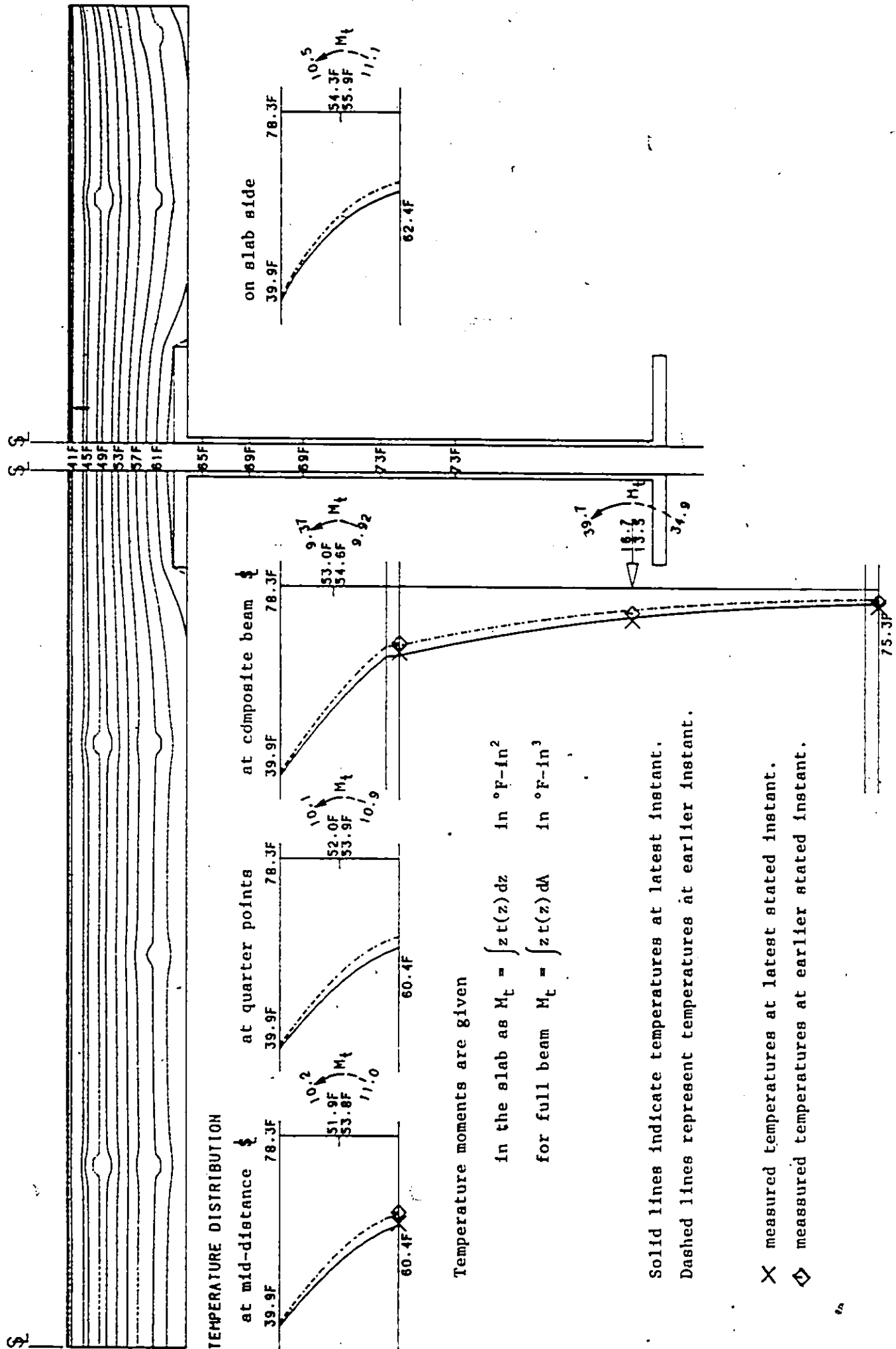


Fig. 14 Instantaneous temperature fields in cross section at 27 and 32 min.

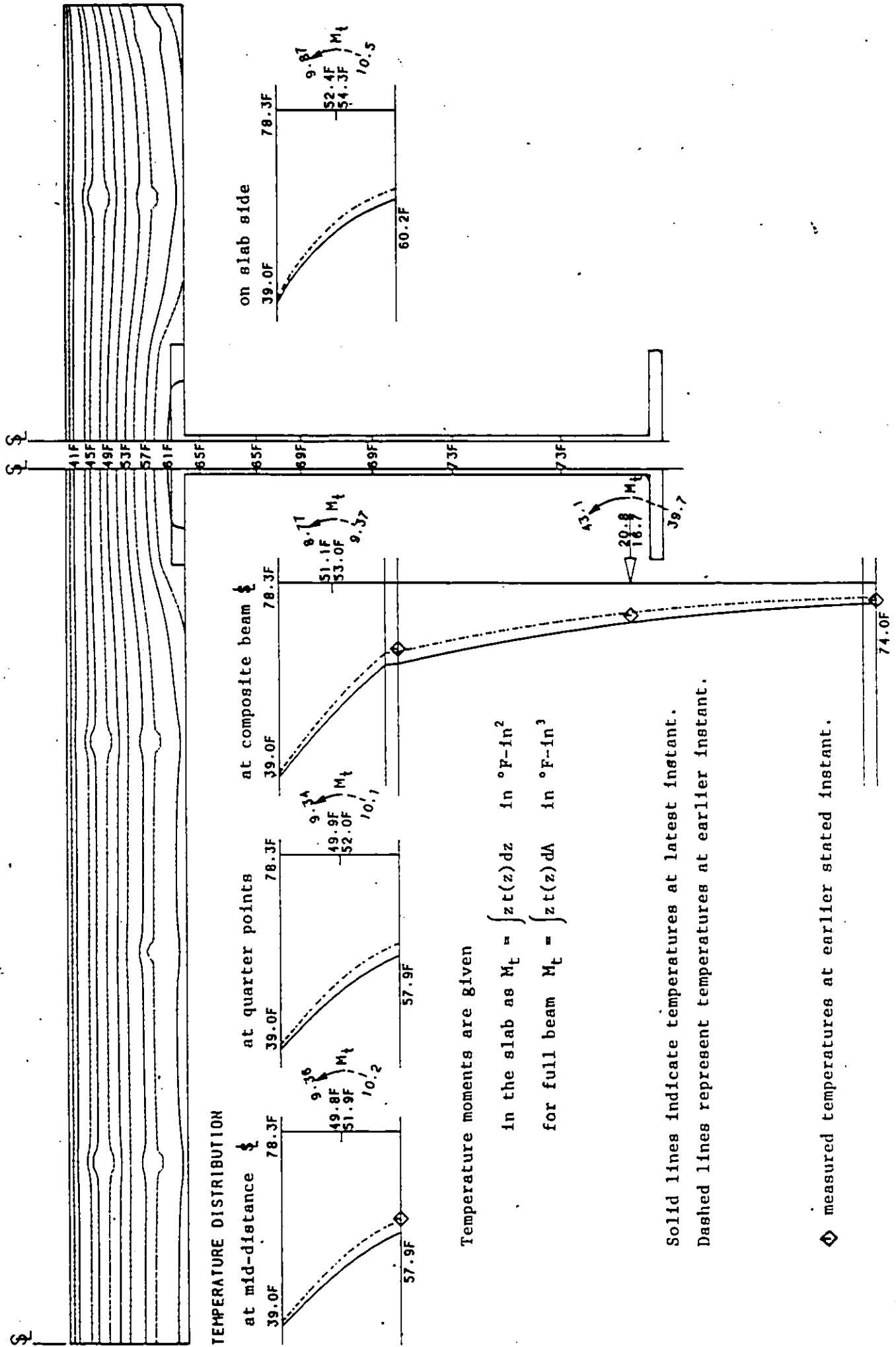


Fig. 15 Instantaneous temperature fields in cross section at 32 and 39 min.

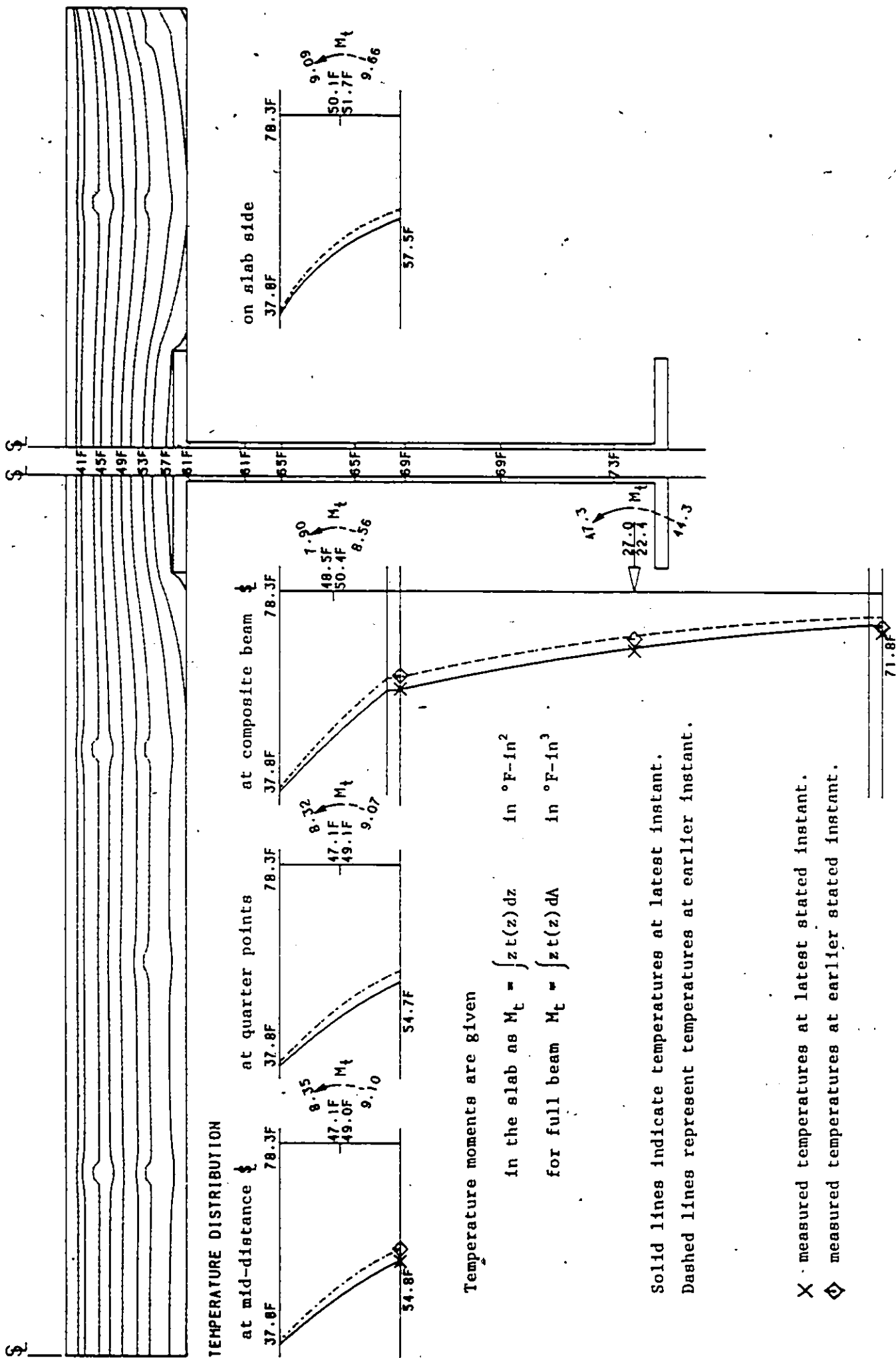
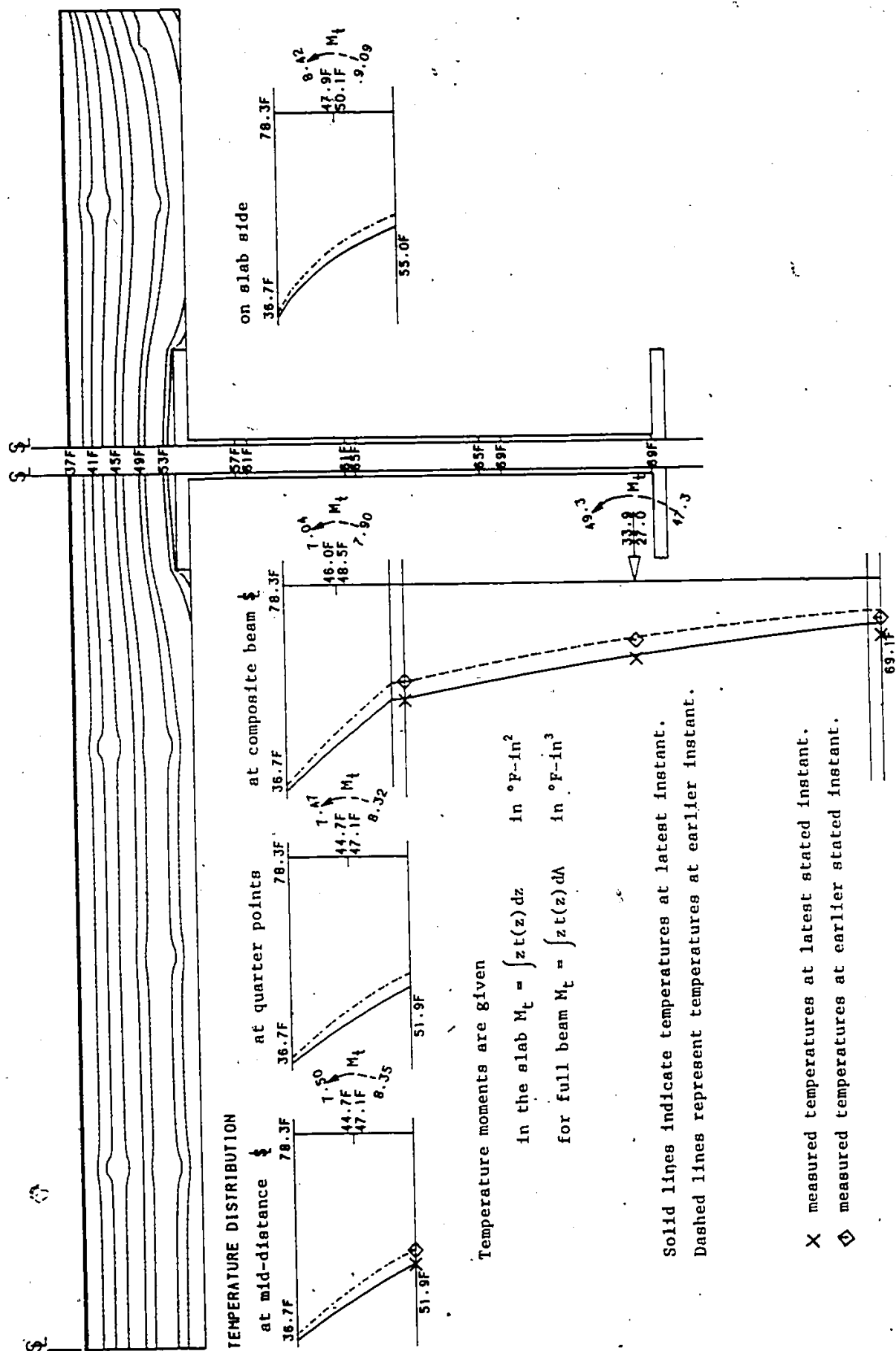


Fig. 16 Instantaneous temperature fields in cross section at 42 and 52 min.



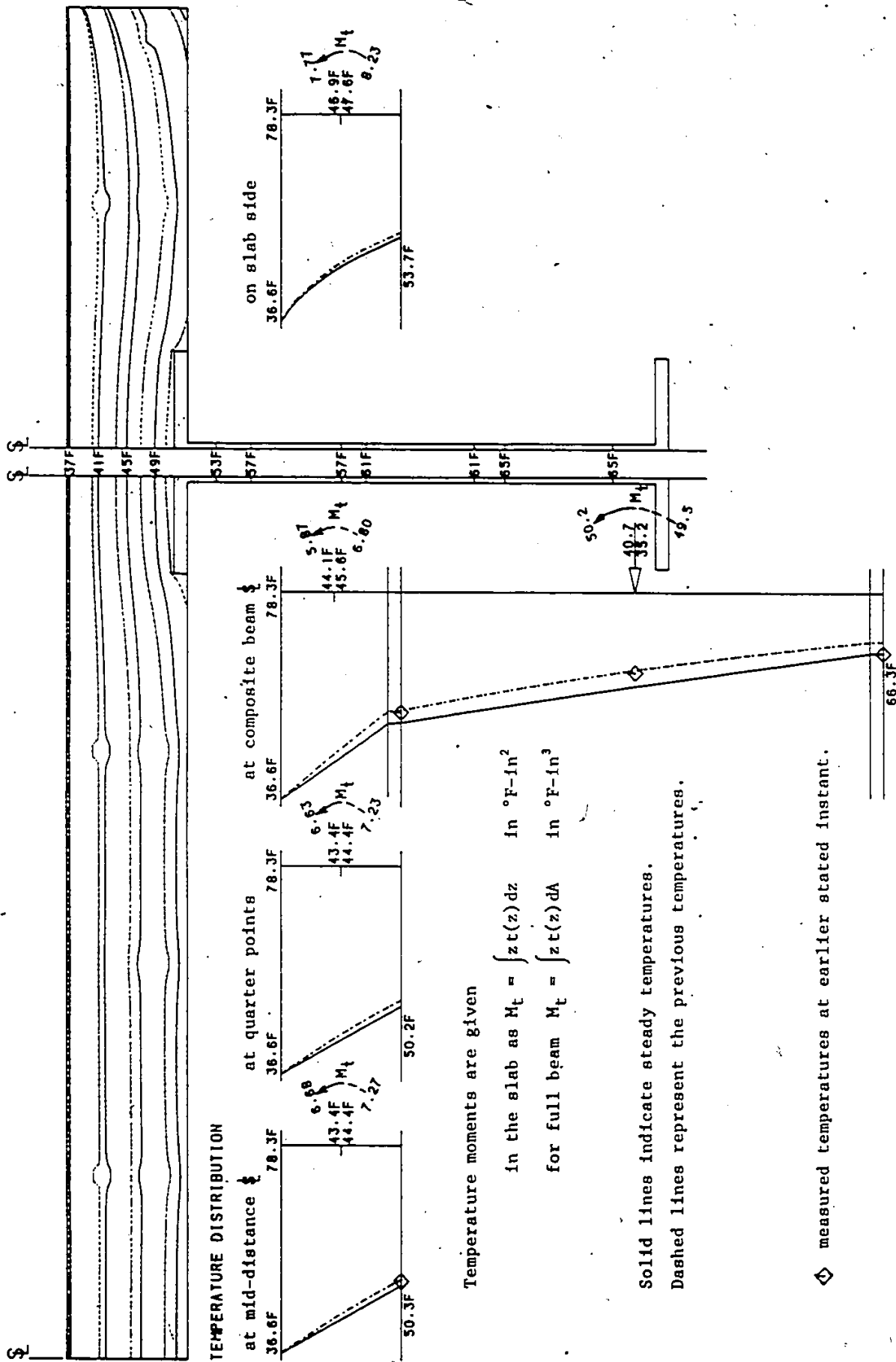


Fig. 18 Instantaneous temperature fields at 77-minute and steady state conditions.



cross-sectional profile of bridge

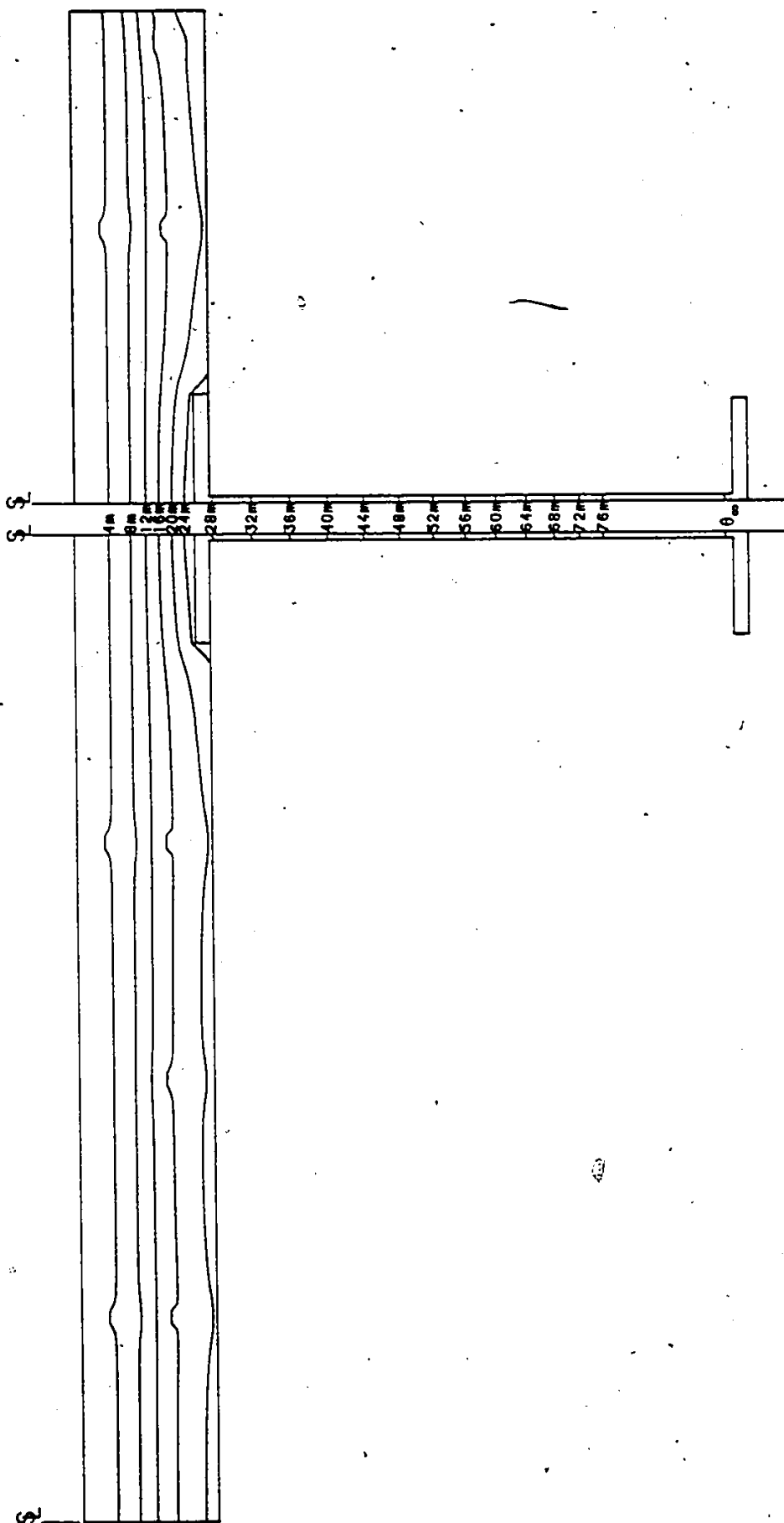


Fig. 19 Trace of 66°F isotherm at 4 min. intervals.

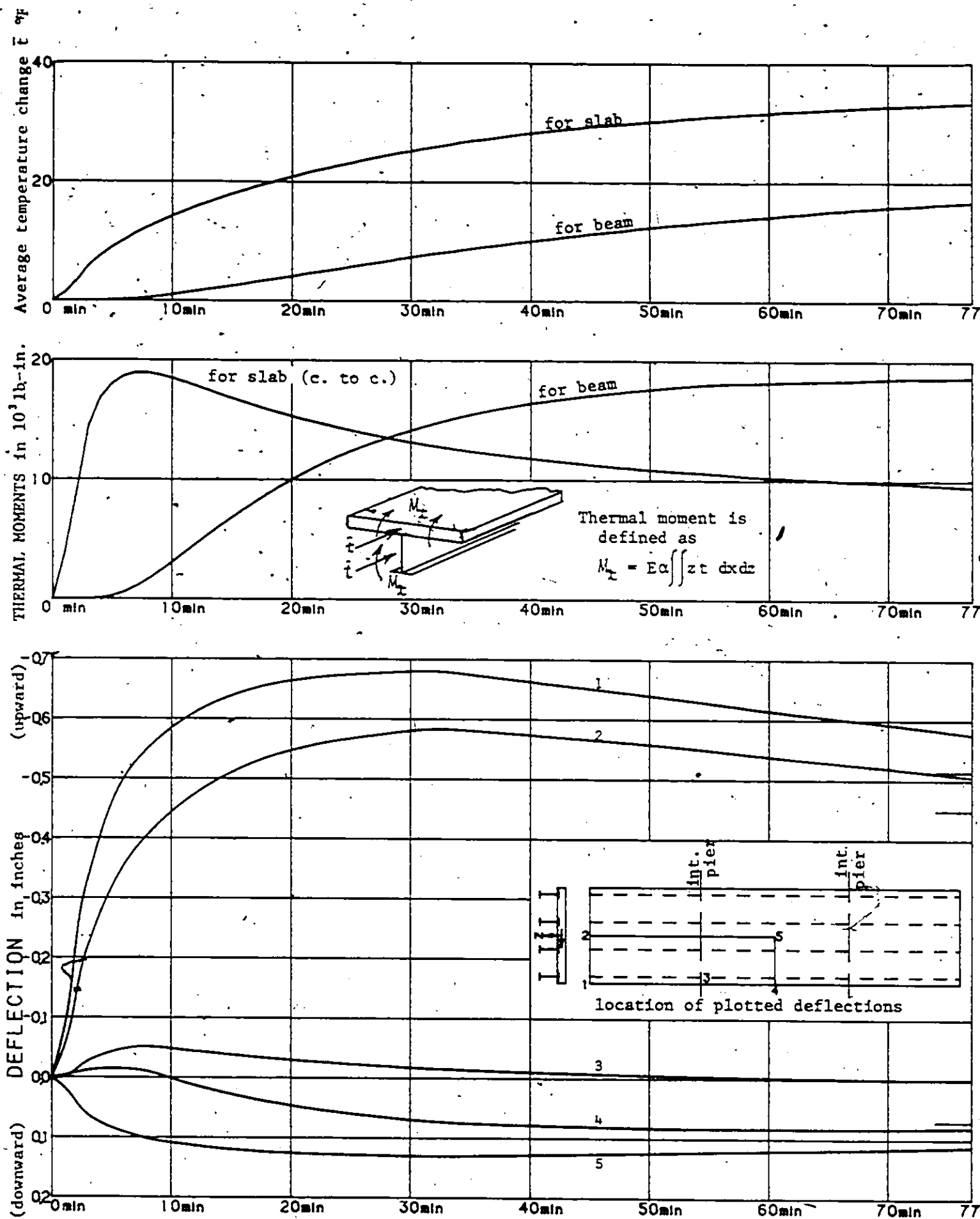


Fig. 20 Calculated bridge deflections with temperature loading history.

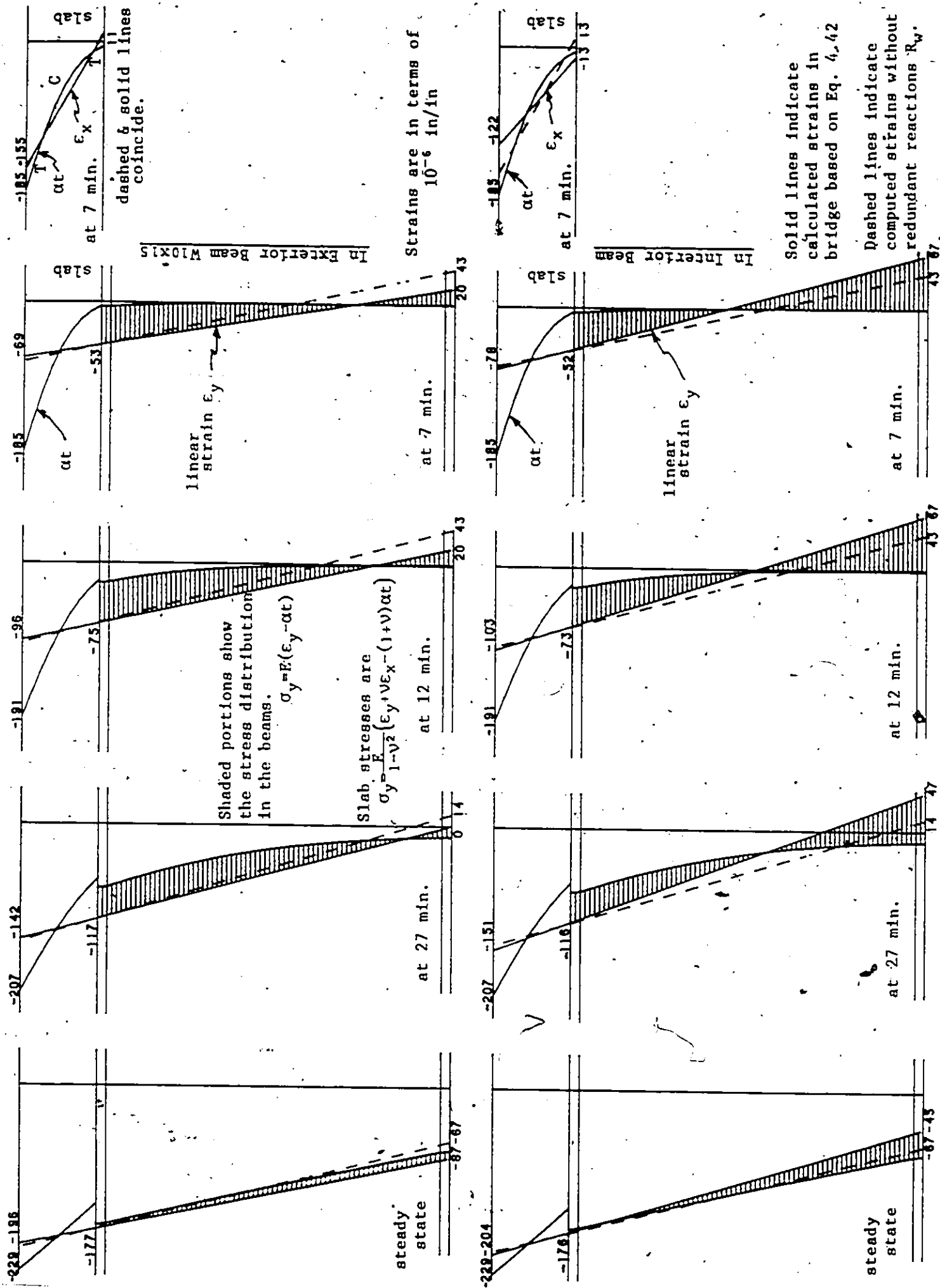


Fig. 21 Instantaneous strain distributions in section at mid-central span.

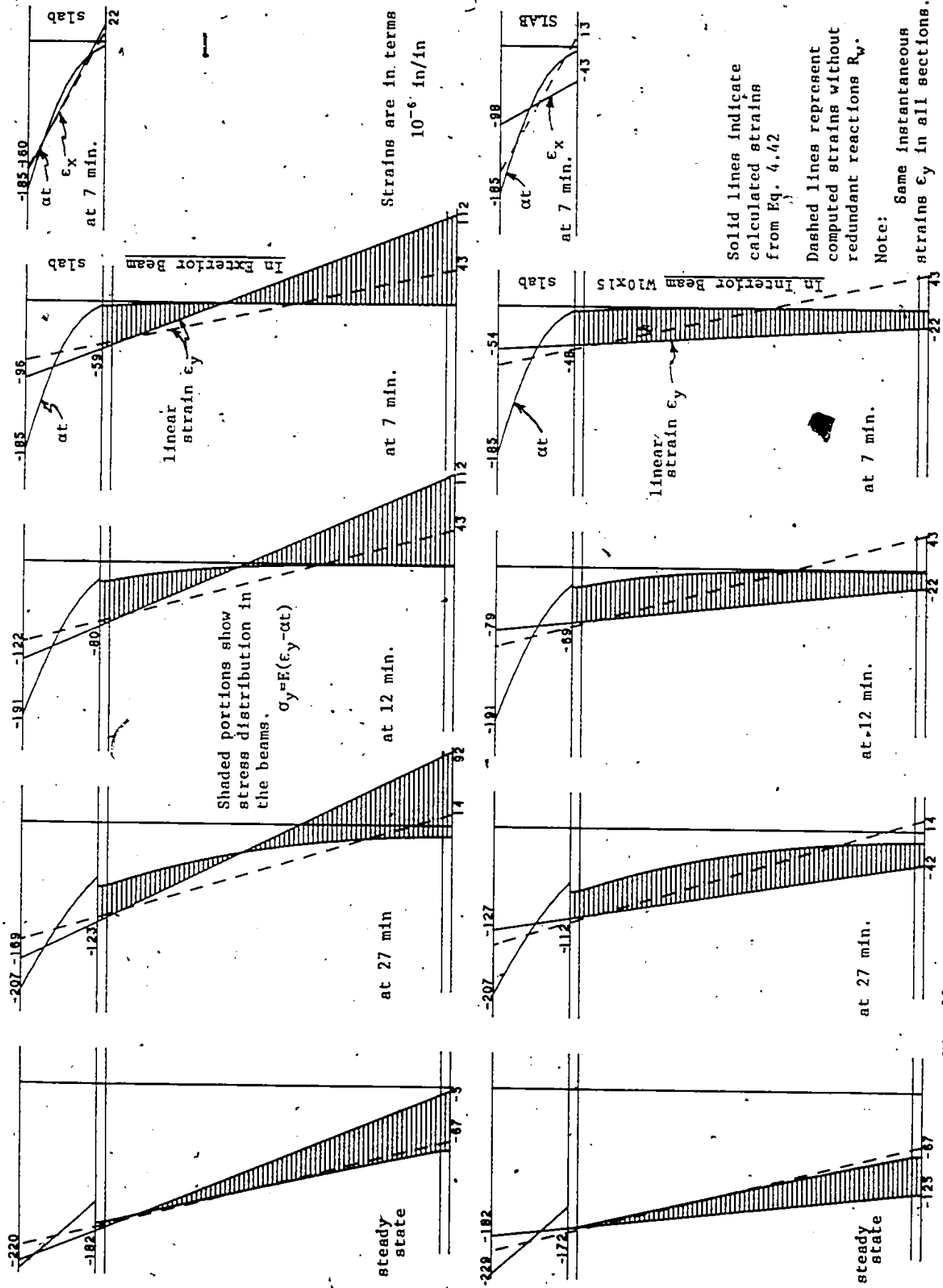


Fig. 22 Instantaneous strain distributions in section at interior pier.

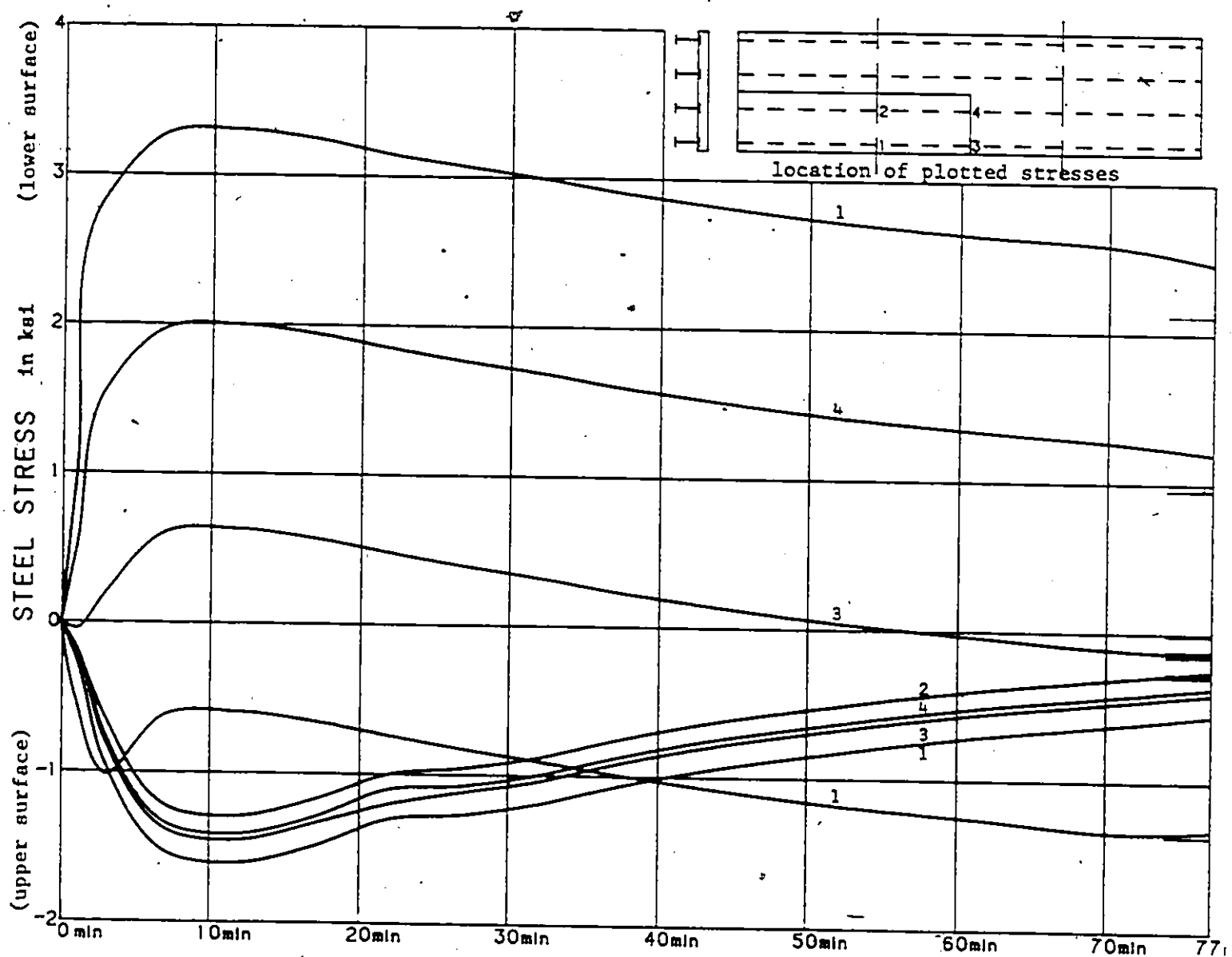
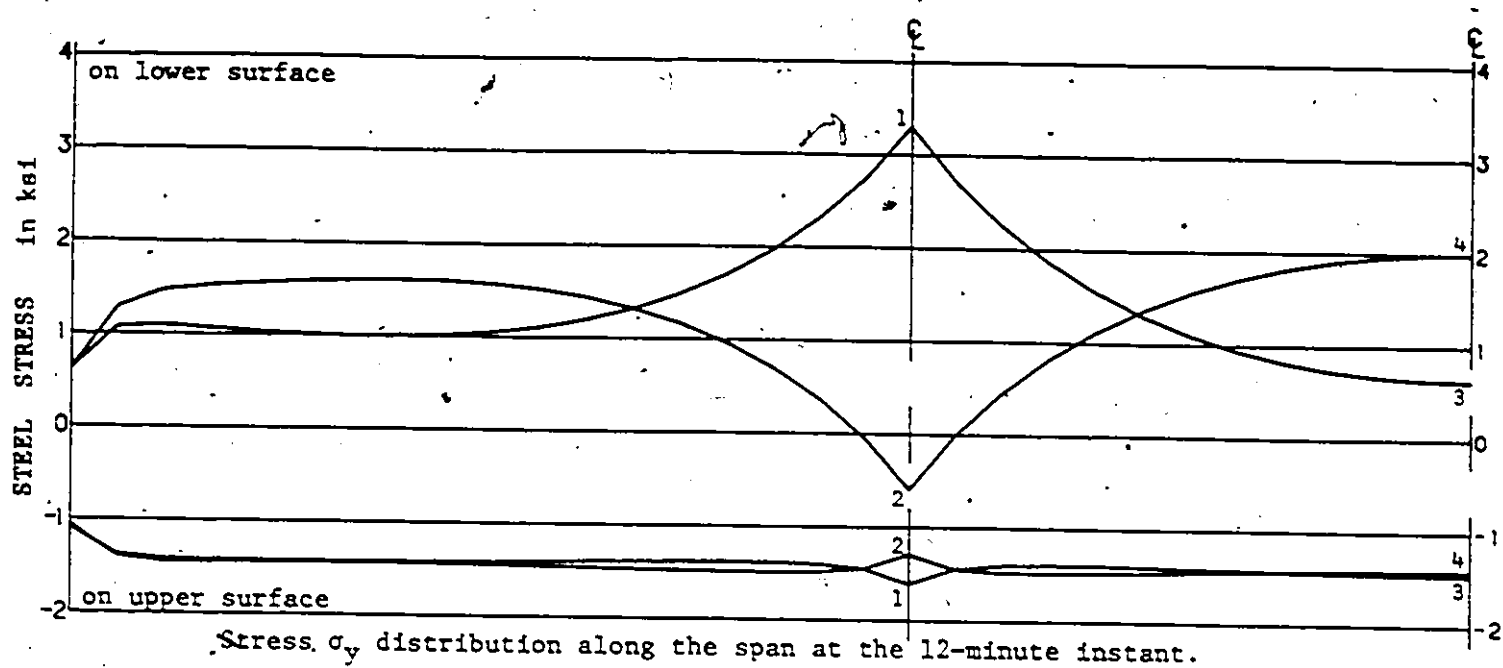


Fig. 23 Longitudinal thermal stresses in steel beams.

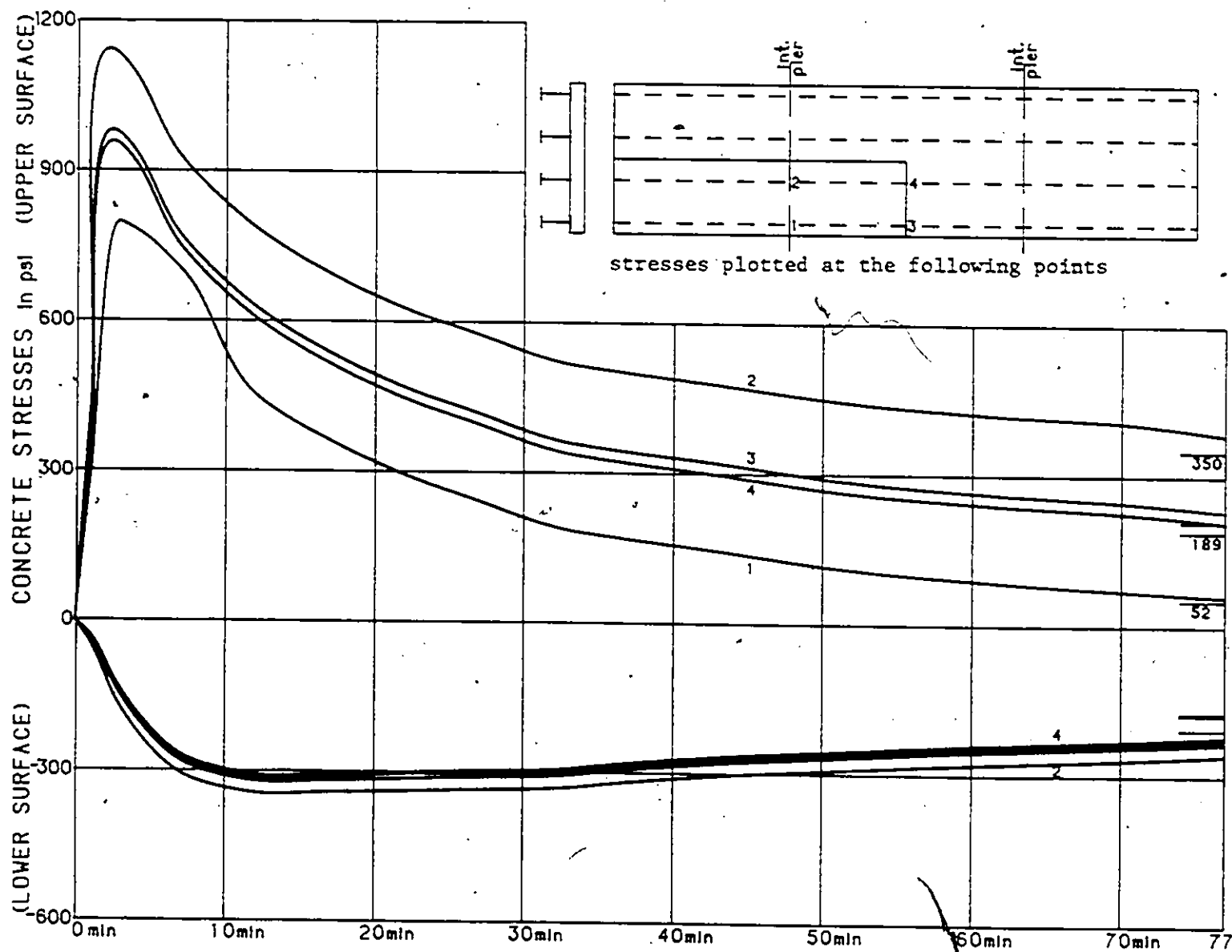
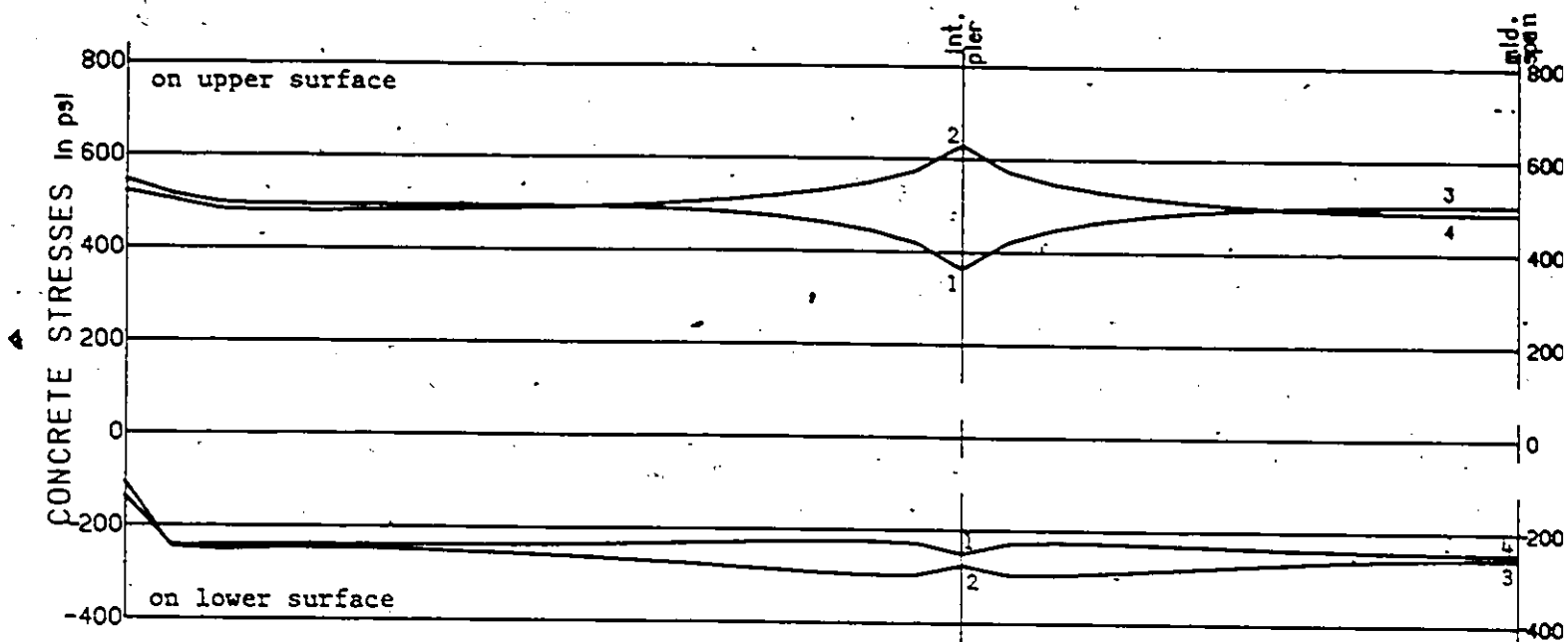


Fig. 24 Longitudinal thermal stresses in concrete slab.