Smoothness in Codifferential Categories

Keith O'Neill

Thesis submitted to the Faculty of Graduate and Postdoctoral Studies in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

[1] The Ph.D. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

The Hochschild-Kostant-Rosenberg theorem, which relates the Hochschild homology of an algebra to its modules of differential n-forms, can be considered a benchmark for smoothness of an algebra. This notion is used here in the search for a conception of smoothness formulated in the context of codifferential categories, both commutative and noncommutative. Since it is desirable to adapt such a conception to the noncommutative context, the theory of this domain is developed considerably; a significant result in this direction establishes a connection between noncommutative codifferential categories and commutative ones.

This investigation necessitates, then, both the formulation of a notion of smoothness for T-algebras in codifferential categories and an adaptation of the Hochschild-Kostant-Rosenberg theorem to a wide variety of contexts which includes noncommutative ones. The former consideration fosters both the notion of a smooth monad, and a formulation of André-Quillen homology in codifferential categories; the latter engenders a highly adaptable version of the Hochschild-Kostant-Rosenberg theorem. Specifically, it is shown that for any algebra modality there exists a corresponding Hochschild-Kostant-Rosenberg theorem. This includes a version of the theorem for the free associative algebra monad, the conclusion of which is satisfied by noncommutative smooth algebras.

Dedications

This work is dedicated to Josie, Emmett, Nancy, and Barry.

Acknowledgement

I would like to thank NSERC for their financial support.

I would like to thank several people with whom I have shared many engaging and enlightening conversations. In particular, Robin Cockett, Geoff Crutwell, and Robert Seely. I must also thank Robin Cockett for providing several key ideas specific to this work.

I would like to thank Rory Lucyshyn-Wright for his many trenchant insights and for his inspiring work during our collaboration.

I must thank Richard Blute for his exemplary mentorship. He has attained - for me - a perfect balance between incisive criticism and steadfast encouragement. It is hard to imagine a better context for intellectual development.

Finally, I must acknowledge the debt I owe to Jean-Louis Loday; his work has enriched my life and inspired much of the work that follows.

Contents

Chapter 1

Introduction

Linear logic, as developed by Girard and articulated in [12], is a resource-conscious logic which serves as a refinement of both classical and intuitionistic logic in several ways; notably, greater control over weakening and contraction is obtained via the unitary operator !. It was given a categorical characterization by Seely in [21], wherein it was determined that certain types of symmetric monoidal categories provide categorical semantics for linear logic. Specifically, such categories must be equipped with a comonad in order to model !.

Later, Ehrhard constructed models for linear logic in which the hom-sets had a natural differentiation operator, hence initiating the study of differential linear logic as expounded in [8]. Differential categories, as described in [3], arise as models for differential linear logic. They are additive symmetric monoidal categories with coalgebra modalities - comonads which carry coalgebra structure - and, most notably, a differential combinator which computes derivatives of maps in adherence to the rules of differentiation familiar from elementary calculus. In this work, as in [4], the focus is instead on codifferential categories, the opposites of differential categories. The reason for this is that much of this work is inspired by results in the category of modules over a commutative ring, which has a novel and powerful interpretation as a codifferential category.

The impetus for this project originated from the desire to use the context of codifferential categories to develop a generalized version of the Hochschild-Kostant-Rosenberg theorem, which establishes an isomorphism between the Hochschild homology and the modules of differential n-forms of *smooth algebras* - algebras for which the corresponding affine scheme is locally well approximated by affine space. Consequently, it was germane to attempt to develop a notion of algebraic smoothness relative to a given codifferential category. Moreover, to realize the full potential of this project it is necessary to consider the possibility of carrying it out in the context of noncommutative versions of codifferential categories.

As one of a few steps taken in this work towards the accomplishment of this latter goal, a good deal of the contents of [4] have been reinterpreted in a framework for noncommutative codifferential categories developed by Robin Cockett in [5]. The prescience of this formulation is indicated by the effortlessness of the translation. In particular, it is shown that the free associative algebra monad, $\mathcal{F}$, plays, in the noncommutative framework, a role analogous to the symmetric algebra monad, $\mathcal{S}$, in the commutative one.

Several concerns arise immediately upon consideration of the HKR theorem in the context of codifferential categories. First, even in the commutative context, the terms involved in the mere statement of the HKR theorem must be defined in a general codifferential category. Putting the notion of smoothness aside momentarily, this necessitates both a delineation of Hochschild homology and of higher n-forms. Fortunately, a naive strategy suffices and the classical versions of these notions are readily transplanted into general commutative codifferential categories. Furthermore, the n-forms have a particularly convenient expression, which facilitates a homology theory closely analogous to algebraic de Rham cohomology as described in ([9], p.417); this is developed in Chapter 2.

Given these notions of n-forms and of Hochschild homology, one must then consider conditions on T-algebras under which these structures coincide. In addition to the inherent difficulty of developing a notion of smoothness which is appropriate for various contexts simultaneously, the problems are compounded by the unconventional nature of the modules of differential forms in a codifferential category. That is, it is not difficult to show that for a commutative algebra the modules of differential forms coincide with their first Hochschild homology groups, however in a general codifferential category it is not necessarily the case that the classical modules of differential forms coincide with the modules of T-differential forms, which are the appropriate differential forms for a codifferential category.

This latter concern may be addressed by positing additional assumptions about the monad for the codifferential category in which we are working. In fact, a good deal of work in codifferential categories has been devoted to the analysis of such an assumption known as property K [4],[2]. This property though, mandates that the symmetric algebra monad surjects onto the monad in the codifferential category and is hence is a more stringent condition than is desirable. Alternatively, we may ask that T emulate some of the most pertinent properties of the symmetric algebra monad; this approach gives rise to the notion of a smooth monad. For such a monad, the free algebras taken over flat objects satisfy the conclusion of the HKR theorem, but it is not necessarily the case that all of the T-modules of differential forms coincide with the classical modules of differential forms. The conditions for a smooth monad are given in Chapter 3.

Furthermore, in this chapter a connection is established between noncommutative codifferential categories and commutative ones. Specifically, it is shown that given a noncommutative codifferential category that, under certain conditions, one may construct a commutative codifferential category. A full analysis of the inheritance of properties by commutative codifferential categories from noncommutative codifferential categories would be desirable, but is not pursued here. However, it is shown that, in the case of the symmetric algebra monad, many of the smoothness properties can be derived by recourse to its relationship with the free associative algebra monad.

Given a smooth monad T, it is still necessary to develop a notion of smoothness for T-algebras with which it can be shown that the conclusion of the HKR theorem is satisfied. Since most any notion of smoothness will presume flatness, we know that given a smooth T-algebra, the n-forms for the free algebras over the underlying objects will satisfy the conclusion of the HKR theorem. Hence any potential notion of smoothness for a T-algebra is guided by the desire to relate the satisfaction of the conclusion of the HKR theorem for free T-algebras to the satisfaction of the HKR theorem for general T-algebras. The strategy used to approach this is an abstraction of André-Quillen homology.

One approach to the HKR theorem uses the fact that the non-trivial André-Quillen homology groups of a smooth algebra vanish. With this fact and the fact that the symmetric algebras over the underlying modules of the smooth algebras satisfy the conclusion of the HKR theorem, one can prove, using a spectral sequence argument, that the algebras themselves satisfy the conclusion of the HKR theorem. Therefore, if it is possible to formulate an appropriate notion of André-Quillen homology in a codifferential category, then any notion of smoothness for T-algebras should make the André-Quillen homology groups for such algebras vanish. The development of a putative theory of André-Quillen homology is relatively straightforward, however, in order to facilitate calculations, one must develop the appropriate notion of André-Quillen homology relative to any given T-algebra. These developments and the difficulties encountered are discussed in Chapter 4.

A crucial feature of the approach to the HKR theorem taken in this project is that the statement of the HKR theorem can be converted to a form that circumvents the construction of the n-forms for the algebra. Consequently, this new form of the HKR theorem is well adapted to contemplation in noncommutative codifferential categories, where the modules of n-forms are only tenuously connected to the Hochschild homology. This could potentially have interesting consequences in the realm of noncommutative geometry, as described by Connes in his seminal work [6]. In the theory of noncommutative geometry the HKR theorem is invoked to justify the utilization of Hochschild homology as a practical approximation to differential forms in the noncommutative realm ([15], p. 90). Because the classical HKR theorem requires the hypothesis of smoothness, it would seem that this new formulation of the HKR theorem, which is adaptable to the noncommutative context, would provide a benchmark for formulations of noncommutative smoothness.

As an example, it is shown that in the class of noncommutative codifferential categories in which the monad is the free associative algebra monad, the smooth algebras prescribed by [16] satisfy the conclusion of this new version of the HKR theorem. Furthermore, this approach to the theorem has interesting consequences in commutative codifferential categories. These developments are discussed in Chapter 5. The work concludes with an delineation of a strategy for proving an all-encompassing statement of the HKR theorem in any codifferential category.

The new material contained in this work will now be listed. Section 2.5 contains an entirely category-theoretic treatment of the wedge product and some of its important properties. Likewise, 2.6.3.12 takes a category-theoretic approach to prove a standard result which establishes an isomorphism between the first Hochschild homology group and the module of differential forms for a commutative algebra. In section 3.1 a proof that $\mathcal{F}$ carries a noncommutative codifferential structure is given; the contents of [4] are then translated to the setting of noncommutative codifferential categories in section 3.2. In section 3.3, a convenient construction of the T-n-forms for a T-algebra is given in 3.3.16, which furnishes the construction of the algebraic de Rham cohomology boundary map in the context of codifferential categories in 3.3.17. In section 4.1 the definition of a smooth monad is given, and subsequently an HKR-type theorem for the free T-algebras of a smooth monad is given in 4.1.3. The sequence of results 4.2.10 - 4.2.13 establishes a novel construction of the abelianization of an algebra, uses this construction to demonstrate a connection between noncommutative codifferential categories and commutative ones, and then finally demonstrates that the codifferential structure on $\mathcal{S}$ arises in this fashion. In 5.1.1-5.1.4 the T-modules of differential forms relative to T-algebra maps are defined and important properties are derived; this draws heavily from standard material, but takes a markedly different approach. The definition 5.1.5 and result 5.1.7 are natural translations of analogues from ordinary André-Quillen homology to the context of codifferential categories; the same could be said of 5.2.9 and 5.2.10. 6.1.1 is a novel formulation of the classical HKR theorem, and 6.1.2 is an immediate consequence of this formulation. 6.2.4 adapts this new formulation of the HKR theorem to the noncommutative setting; that is, it is an HKR theorem for noncommutative smooth algebras.

Chapter 2

Background

2.1 Standing Assumptions and Notational Conventions

Diagrammatic composition of morphisms is indicated by the use of semicolons. Internal reference numbers are unambiguous if one assumes that the final number indicates a definition, lemma, proposition, theorem, or an example.

Most of the categories considered in the present work are monoidal categories. In each case it is to be assumed that the tensor unit, k, is a commutative ring. Furthermore, it is assumed that the tensor product preserves reflexive coequalizers. That is, if

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

is a coequalizer and there exists a map $r : Y \longrightarrow X$ such that $r; f = r; g = 1_Y$, then for any other object B, both of the following diagrams are coequalizers

$$B \otimes X \underset{1_B \otimes g}{\overset{1_B \otimes f}{\rightrightarrows}} B \otimes Y \xrightarrow{1_B \otimes h} B \otimes Z$$

$$X \otimes B \underset{g \otimes 1_B}{\overset{f \otimes 1_B}{\rightrightarrows}} Y \otimes B \xrightarrow{h \otimes 1_B} Z \otimes B$$

Finally, in this work, a *flat module* is a k-module, which is the filtered colimit of free, finite dimensional modules - that is, the filtered colimit of modules of the form $\bigoplus_{i=1}^{n} k$. It follows that if the filtered colimits of the category are exact, then these flat modules have the usual flatness property: if a module V is flat, then for any monomorphism $f : A \longrightarrow B$ of modules,

$$1_V \otimes f : V \otimes A \longrightarrow V \otimes B$$

and

$$f \otimes 1_V : A \otimes V \longrightarrow B \otimes V$$

are monomorphisms as well. In an abelian category in which the filtered colimits are exact, the two notions coincide ([9], pp. 712-713).

2.2 Johnstone's Lemma

The following lemma, recorded by Peter Johnstone in his book *Topos Theory* (Lemma 0.17 p. 4 [13]), is used frequently - though sometimes only tacitly - in most work on codifferential categories, and especially in the present work. Under familiar circumstances it enables one to combine two reflexive coequalizers into one. This is often used to endow objects with associative algebra or module structure.

Lemma 1. *Consider the following diagram*

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\alpha_1"] & X_2 \arrow[r, "f_3"] \arrow[d, shift right, "\beta_2"'] \arrow[d, shift left, "\beta_1"] & X_3 \arrow[d, shift right, "\kappa_2"'] \arrow[d, shift left, "\kappa_1"] \\
Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "\alpha_3"'] & Y_2 \arrow[r, "g_3"] \arrow[d, "\beta_3"'] & Y_3 \arrow[d, "\kappa_3"] \\
Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2 \arrow[r, "h_3"] & Z_3
\end{tikzcd}$$

If $f_i;\beta_j = \alpha_j;g_i$ for $1 \leq i,j \leq 2$, and each of the columns and rows is a coequalizer, then the following diagram is also a coequalizer:

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "f_1;\beta_1"] \arrow[r, shift right, "f_2;\beta_2"'] & Y_2 \arrow[r, "g_3;\kappa_3"] & Z_3
\end{tikzcd}$$

In this work, the above lemma is most frequently applied by taking the tensor product of two reflexive coequalizers in parallel. This is the primary reason for the pervasive assumption that tensor products preserve reflexive coequalizers.

2.3 Modules of Differential Forms

A fundamental notion in this paper is that of a *derivation*, a k-linear map ∂ from a commutative k-algebra A to an A-module M, which satisfies the Leibniz rule:

$$\forall a, b \in A, \partial(ab) = a\partial(b) + b\partial(a)$$

For any such commutative k-algebra, one may construct a *universal derivation* $d_A : A \longrightarrow \Omega_{A/k}$ to a *universal module of differential forms*. This map is universal in the sense that for any other derivation $\partial : A \longrightarrow M$, there exists a unique morphism of modules $\partial^{\#} : \Omega_{A/k} \longrightarrow M$, which makes the following diagram commute

$$\begin{tikzcd} \Omega_{A/k} \arrow[r, "\partial^{\#}"] & M \\ A \arrow[u, "d_A"] \arrow[ur, "\partial"'] & \end{tikzcd}$$

One way to construct this object is via the quotient I/I^2, where I is the kernel of the multiplication map $A \otimes_k A \longrightarrow A$. Specifically, for $x \in A$, let

$$d_A(x) = x \otimes 1 - 1 \otimes x$$

Then

$$xd_A(y) + yd_A(x) = xy \otimes 1 - x \otimes y + yx \otimes 1 - y \otimes x$$

Since

$$(x \otimes 1 - 1 \otimes x)(y \otimes 1 - 1 \otimes y) = xy \otimes 1 - x \otimes y - y \otimes x + 1 \otimes xy \in I^2$$

the Leibniz rule follows, and further, any element $x \otimes y$ of I/I^2 has

$$\begin{aligned} x \otimes y &\equiv -y \otimes x + xy \otimes 1 + 1 \otimes xy \\ &\equiv -y \otimes x + xy \otimes 1 \\ &\equiv yd_A(x) \end{aligned}$$

Then $\partial^{\#}$ is the unique A-module map which sends $d_A(x)$ to $\partial(x)$.

If A is merely associative, there exists an analogous object $\Omega^{NC}_{A/k}$, the *noncommutative module of differential forms*, which is simply the kernel, I, of the multiplication map. It is an A-bimodule equipped with a universal *noncommutative derivation*; that is, a k-linear map ∂ from A to an A-bimodule satisfying the noncommutative Leibniz rule

$$\forall a, b \in A, \partial(ab) = a\partial(b) + \partial(a)b$$

The construction of this map is similar to that of the commutative case. More details can be found in, e.g., [11] Part II, 8.1.

2.4 Tensor Products

Throughout this work, it will be necessary to utilize tensor products of modules taken over various algebras, and furthermore, to invoke maps between such objects. The necessary facts and notation regarding these entities are collected here. In what follows, k is a commutative unital ring.

Definition 1. Let A be a k-algebra, M a right A-module with action ${}_M\bullet$, and N a left A-module with action $\bullet_N$. Then the *tensor product of M and N over A*, denoted $M \otimes_A N$ is defined by the following coequalizer diagram:

$$\begin{tikzcd} M \otimes A \otimes N \arrow[r, shift left, "{}_M\bullet\otimes 1"] \arrow[r, shift right, "1\otimes\bullet_N"'] & M \otimes N \arrow[r, "p_A"] & M \otimes_A N \end{tikzcd}$$

Since reflexive coequalizers are important in the work that follows, it is worthwhile to note that the existence of the map

$$M \otimes N \cong M \otimes k \otimes N \longrightarrow M \otimes A \otimes N$$

substantiates the fact that the above coequalizer is a reflexive coequalizer.

Some common facts about these tensor products are collected in the following lemma. Since this work is in essence category-theoretic, pains are taken to prove these facts category-theoretically.

Since the tensor product preserves reflexive coequalizers, it follows from the following diagram that if, for instance, B is a k-algebra and N is a right B-module with action ${}_N\bullet^B$, then $M \otimes_A N$ inherits a right B-module structure

$$\begin{tikzcd} M \otimes A \otimes N \otimes B \arrow[r, shift left, "{}_M\bullet\otimes 1_{N\otimes B}"] \arrow[r, shift right, "1_M\otimes\bullet^A_N\otimes 1_B"'] \arrow[d, "1_{M\otimes A}\otimes\bullet^B_N"'] & M \otimes N \otimes B \arrow[r, "p_A\otimes 1_B"] \arrow[d, "1\otimes\bullet^B_N"'] & (M \otimes_A N) \otimes B \arrow[d, dashed] \\ M \otimes A \otimes N \arrow[r, shift left, "{}_M\bullet\otimes 1_N"] \arrow[r, shift right, "1_M\otimes\bullet^A_N"'] & M \otimes N \arrow[r, "p_A"'] & M \otimes_A N \end{tikzcd}$$

Lemma 2. *Let A be an associative k-algebra. Then A inherits an A-bimodule structure and we have*

i) For any A-module N, $A \otimes_A N \cong N \cong N \otimes_A A$

ii) For M a right A-module, N an A-B-module, and P a left B-module, $M \otimes_A (N \otimes_B P) \cong (M \otimes_A N) \otimes_B P$

Proof:

i) Consider the following diagram:

$$\begin{tikzcd} A \otimes A \otimes N \arrow[r, shift left, "m_A\otimes 1"] \arrow[r, shift right, "1\otimes\bullet_N"'] & A \otimes N \arrow[r, "p_A"] \arrow[dr, "\bullet_N"'] & A \otimes_A N \arrow[d, dashed, "q"] \\ & & N \end{tikzcd}$$

The morphism q necessarily exists by the universal property of $A \otimes_A N$ and by virtue of the axioms of a module action. We also have a map $f = u_L; e_A \otimes 1_N; p_A : N \longrightarrow A \otimes_A N$. It will be shown that N has the same universal property as $A \otimes_A N$ by proving that $q; f$ is the identity, and that q is an epimorphism.

The latter fact is obvious, since $\bullet_N$ is an epimorphism. As for the former, observe the following calculation:

$$\begin{aligned}
& p_A; q; u_L; e_A \otimes 1_N; p_A \\
=& \bullet_N; u_L; e_A \otimes 1_N; p_A \\
=& u_L; e_A \otimes 1_N; 1_A \otimes \bullet_N; p_A \\
=& u_L; e_A \otimes 1_N; m_A \otimes 1_N; p_A \\
=& p_A
\end{aligned}$$

The last equality here is up to isomorphism. Since p_A is epimorphic, the result follows.

ii) Consider the following diagram:

$$\begin{tikzcd}
M \otimes A \otimes N \otimes B \otimes P \arrow[r, shift left, "1_{(M\otimes A)\otimes N}\bullet^B\otimes 1_P"] \arrow[r, shift right, "1_{(M\otimes A)}\otimes 1_N\otimes\bullet^B_P"'] \arrow[d, shift right, "1_M\otimes\bullet^A_N\otimes 1_{B\otimes P}"'] \arrow[d, shift left, "{}_M\bullet\otimes 1_{N\otimes B\otimes P}"] & M \otimes A \otimes N \otimes P \arrow[r, "1_{(M\otimes A)}\otimes p_B"] \arrow[d, shift right, "1_M\otimes\bullet^A_N\otimes 1_P"'] \arrow[d, shift left, "{}_M\bullet\otimes 1_{N\otimes P}"] & M \otimes A \otimes (N \otimes_B P) \arrow[d, shift right, "1_M\otimes\bullet^A_{N\otimes_B P}"'] \arrow[d, shift left, "{}_M\bullet\otimes 1_{N\otimes_B P}"] \\
M \otimes N \otimes B \otimes P \arrow[r, shift left, "1_{M\otimes N}\bullet^B\otimes 1_P"] \arrow[r, shift right, "1_M\otimes 1_N\otimes\bullet^B_P"'] \arrow[d, "p_A\otimes 1_{B\otimes P}"'] & M \otimes N \otimes P \arrow[r, "1_M\otimes p_B"] \arrow[d, "p_A\otimes 1_P"'] & M \otimes (N \otimes_B P) \arrow[d, "p_A"] \\
(M \otimes_A N) \otimes B \otimes P \arrow[r, shift left, "1_{M\otimes N}\bullet^B\otimes 1_P"] \arrow[r, shift right, "1_M\otimes 1_N\otimes\bullet^B_P"'] & (M \otimes_A N) \otimes P \arrow[r, "p_B"] & M \otimes_A N \otimes_B P
\end{tikzcd}$$

Since the tensor product preserves reflexive coequalizers, each of the rows and columns in this diagram is a coequalizer. It is clear that the object in the bottom right corner is endowed with the universal properties of both $M \otimes_A (N \otimes_B P)$ and $(M \otimes_A N) \otimes_B P$, and hence these two objects are isomorphic. Furthermore, Johnstone's Lemma shows that the bottom right square is a pushout, and that we have a coequalizer of the form:

$$\begin{tikzcd}
M \otimes A \otimes N \otimes B \otimes P \arrow[r, shift left] \arrow[r, shift right] & M \otimes N \otimes P \arrow[r] & M \otimes_A N \otimes_B P
\end{tikzcd}$$

with morphisms built from the obvious composites. ∎

The following proposition and its corollary will be used in an essential way in our work on n-forms.

Proposition 3. *Let M_1, M_2, and M_3 be right modules over, respectively, associative algebras A_1, A_2, and A_3 and with respective actions ${}_{M_1}\bullet$, ${}_{M_2}\bullet$, and ${}_{M_3}\bullet$. Also, let N_1, N_2, and N_3 be left modules over, respectively, A_1, A_2, and A_3 with respective actions $\bullet_{N_1}$, $\bullet_{N_2}$, and $\bullet_{N_3}$. Furthermore, let the following diagrams be reflexive coequalizers:*

$$\begin{tikzcd} M_3 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M_2 \arrow[r, "f"] & M_1 \end{tikzcd}$$

$$\begin{tikzcd} A_3 \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\gamma"'] & A_2 \arrow[r, "\alpha"] & A_1 \end{tikzcd}$$

$$\begin{tikzcd} N_3 \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & N_2 \arrow[r, "j"] & N_1 \end{tikzcd}$$

Then the following diagram is also a reflexive coequalizer

$$\begin{tikzcd} M_3 \otimes_{A_3} N_3 \arrow[r, shift left, "g\otimes k"] \arrow[r, shift right, "h\otimes l"'] & M_2 \otimes_{A_2} N_2 \arrow[r, "f\otimes j"] & M_1 \otimes_{A_1} N_1 \end{tikzcd}$$

Proof: For any algebra B, a right B-module M and a left B-module N with respective actions ${}_M\bullet$ and $\bullet_N$, the following diagram is a coequalizer diagram:

$$\begin{tikzcd} M \otimes B \otimes N \arrow[r, shift left, "{}_M\bullet\otimes 1_N"] \arrow[r, shift right, "1_M\otimes\bullet_N"'] & M \otimes N \arrow[r] & M \otimes_B N \end{tikzcd}$$

Hence we construct the following diagram:

$$\begin{tikzcd}
M_3 \otimes A_3 \otimes N_3 \arrow[r, shift left, "{}_{M_3}\bullet\otimes 1"] \arrow[r, shift right, "1\otimes\bullet_{N_3}"'] \arrow[d, shift right, "g\otimes\beta\otimes k"'] \arrow[d, shift left, "h\otimes\gamma\otimes l"] & M_3 \otimes N_3 \arrow[r, "p_3"] \arrow[d, shift right, "g\otimes k"'] \arrow[d, shift left, "h\otimes l"] & M_3 \otimes_{A_3} N_3 \arrow[d, shift right, "g\otimes k"'] \arrow[d, shift left, "h\otimes l"] \\
M_2 \otimes A_2 \otimes N_2 \arrow[r, shift left, "{}_{M_2}\bullet\otimes 1"] \arrow[r, shift right, "1\otimes\bullet_{N_2}"'] \arrow[d, "f\otimes\alpha\otimes j"'] & M_2 \otimes N_2 \arrow[r, "p_2"] \arrow[d, "f\otimes j"'] & M_2 \otimes_{A_2} N_2 \arrow[d, "f\otimes j"'] \\
M_1 \otimes A_1 \otimes N_1 \arrow[r, shift left, "{}_{M_1}\bullet\otimes 1"] \arrow[r, shift right, "1\otimes\bullet_{N_1}"'] & M_1 \otimes N_1 \arrow[r, "p_1"] & M_1 \otimes_{A_1} N_1
\end{tikzcd}$$

Each of the rows is a coequalizer diagram; since the tensor product preserves reflexive coequalizers, so are the two left-most columns. So if $q : M_2 \otimes_{A_2} N_2 \longrightarrow X$ is a morphism of A_2-bimodules, such that $g \otimes k; q = h \otimes l; q$, it follows that

$$\begin{aligned} g \otimes k; p_2; q &= p_3; g \otimes k; q \\ &= p_3; h \otimes l; q \\ &= h \otimes l; p_2; q \end{aligned}$$

Hence there is a unique $q' : M_1 \otimes N_1 \longrightarrow X$ such that $f \otimes j; q' = p_2; q$. Then,

$$\begin{aligned} f \otimes \alpha \otimes j;_{M_1} \bullet \otimes 1; q' &=_{M_2} \bullet \otimes 1; f \otimes fj; q' \\ &=_{M_2} \bullet \otimes 1; p_2; q \\ &= 1 \otimes \bullet_{N_2}; p_2; q \\ &= 1 \otimes \bullet_{N_2}; f \otimes j; q' \\ &= f \otimes \alpha j; 1 \otimes \bullet_{N_1}; q' \end{aligned}$$

Since $f \otimes \alpha \otimes j$ is an epimorphism, this calculation indicates that there must be a unique $q'' : M_1 \otimes_{A_1} N_1 \longrightarrow X$ such that $p_1; q'' = q'$. Finally,

$$\begin{aligned} p_2; f \otimes j; q'' &= f \otimes j; p_1; q'' \\ &= f \otimes j; q' \\ &= p_2; q \end{aligned}$$

Since p_2 is an epimorphism, it follows that q'' is the unique map such that $f \otimes j; q'' = q$ ∎

Corollary 4. *Let M_1, M_2, and M_3 be bimodules over the associative algebras A_1, A_2 and A_3, respectively. Furthermore let the following diagrams be reflexive coequalizer diagrams:*

$$\begin{tikzcd} M_3 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M_2 \arrow[r, "f"] & M_1 \end{tikzcd}$$

$$\begin{tikzcd} A_3 \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\gamma"'] & A_2 \arrow[r, "\alpha"] & A_1 \end{tikzcd}$$

Then for each n, the following diagram is also a reflexive coequalizer:

$$\begin{tikzcd} M_3^{\otimes_{A_3} n} \arrow[r, shift left, "g^{\otimes n}"] \arrow[r, shift right, "h^{\otimes n}"'] & M_2^{\otimes_{A_2} n} \arrow[r, "f^{\otimes n}"] & M_1^{\otimes_{A_1} n} \end{tikzcd}$$

Proof: If the statement is true for any n, then we may set N_1, N_2, and N_3 in the preceding proposition to $M_1^{\otimes_{A_1} n}$, $M_2^{\otimes_{A_2} n}$ and $M_3^{\otimes_{A_3} n}$, respectively, to establish the statement for $n + 1$. The result then follows by induction. ∎

2.5 The Wedge Product

The following material has a classical counterpart as described, for instance in [10], Appendix B.2. Although the added generality found here requires some work, the material in this section underlies much of the development that follows, and hence a loss of generality here would perpetuate itself into the remainder of the work.

For any object M in an additive symmetric monoidal category, we may construct the n^{th} *exterior power* of M as the following coequalizer:

$$\begin{tikzcd} M^{\otimes n} \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "-1\otimes 1\otimes ...\otimes c"'] & M^{\otimes n} \arrow[r] & \bigwedge^n M \end{tikzcd}$$

More precisely, the right-most map projects onto $\bigwedge^n M$ and coequalizes the identity map with each map of the form $-1 \otimes ... \otimes c \otimes ... \otimes 1$, which transposes two adjacent factors in the tensor product and multiplies the resulting tensor product by -1. When, in the notation of the above definition, M is an A-bimodule for some associative algebra A and the tensor product $\otimes_A$, the resulting wedge product is denoted $\bigwedge_A^n M$

If the tensor product of our category preserves coequalizers, the wedge product is accompanied by some additional structure. First, observe the following diagram:

$$\begin{tikzcd} M^{\otimes p+q} \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "-1_{M^{\otimes p-2}}\otimes c\otimes 1_{M^{\otimes q}}"'] & M^{\otimes p+q} \arrow[r, "\pi_p^M\otimes 1_{M^{\otimes q}}"] \arrow[dr, "\pi_n^M"'] & \bigwedge^p M \otimes M^{\otimes q} \arrow[d, dashed, "m'"] \\ & & \bigwedge^{p+q} M \end{tikzcd}$$

Since the maps on the left side of this diagram are merely a restricted set of the maps which p_n^M coequalizes by definition, the indicated map m' indeed exists. Consequently, we may construct the following diagram

$$\begin{tikzcd} \bigwedge^p M \otimes M^{\otimes q} \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "-1_{M^{\otimes p+q-2}}\otimes c"'] & \bigwedge^p M \otimes M^{\otimes q} \arrow[r, "1^{\otimes p}\otimes\pi_q^M"] \arrow[dr, "m'"'] & \bigwedge^p M \otimes \bigwedge^q M \arrow[d, dashed, "m_\wedge"] \\ & & \bigwedge^{p+q} M \end{tikzcd}$$

It can be shown by precomposition with $\pi_p^M \otimes 1_{M^{\otimes q}}$ that m' coequalizes the maps on the left hand side of this diagram. Hence, such a map $m_\wedge$ exists. It is routine to verify that this map is defined independently of the order in which we take the colimits, and that, furthermore, $m_\wedge$ is associative.

The following lemma will be required in a subsequent chapter. Although it is a standard result - proven, for instance, in [10], Appendix B.2 - the proof given here is given in entirely category theoretic language.

Lemma 5. *Let V and W be modules in a category whose tensor product preserves coequalizers. Then*

$$\bigwedge^{n}(V \oplus W) \cong \bigoplus_{p+q=n} \bigwedge^{p} V \otimes \bigwedge^{q} W$$

Proof: Consider the following diagram

$$\begin{tikzcd}
(V \oplus W)^{\otimes n} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1\otimes 1\otimes \ldots\otimes c"'] & (V \oplus W)^{\otimes n} \arrow[r] \arrow[d, "\Sigma"'] & \bigwedge^{n}(V \oplus W) \arrow[d, dashed, "\phi"] \\
& \bigoplus_{p+q=n} V^{\otimes p} \otimes W^{\otimes q} \arrow[r, "\pi_p^V \otimes \pi_q^W"'] & \bigoplus_{p+q=n} \bigwedge^{p} V \otimes \bigwedge^{q} W
\end{tikzcd}$$

where Σ is the map which first distributes the direct sums across the tensor products as in

$$(V \oplus W)^{\otimes n} \cong (V^{\otimes n}) \oplus (V^{\otimes n-1} \otimes W) \oplus (V^{\otimes n-2} \otimes W \otimes V) \oplus \ldots \oplus (W^{\otimes n})$$

and then applies to each summand the appropriate shuffle map, $(-1)^{|\sigma|}\sigma$, which yields objects of the form $V^{\otimes p} \otimes W^{\otimes q}$. Note that $|\sigma|$ denotes the parity of the permutation σ here.

Consider the precomposition of $\Sigma; \pi_p^V \otimes \pi_q^W$ by $-1_{(V\oplus W)}^{\otimes i-1} \otimes c \otimes 1_{(V\oplus W)}^{\otimes n-i-1}$. For any of the summands with p Vs and q Ws, there are two possible scenarios to consider. First, if the summand contains the i^{th} instance of V and the $(i+1)^{st}$ instance of W or vice versa, the $-c$ factor will be coequalized with the identity by the accompanying factor of (-1) in the appropriate shuffle map. Second, if the summand contains the i^{th} and $(i+1)^{st}$ instances of either V or W, then the $-c$ factor will be coequalized with the identity by the projection to the respective wedge product.

It follows that such a ϕ exists. Now, consider the following diagram

$$\begin{tikzcd}
\bigoplus_{p+q=n} V^{\otimes p} \otimes W^{\otimes q} \arrow[r] \arrow[d, "\oplus i_0^{\otimes p} \otimes i_1^{\otimes q}"'] & \bigoplus_{p+q=n} \bigwedge^{p} V \otimes \bigwedge^{q} W \arrow[d, "\oplus \wedge^p i_o \otimes \wedge^q i_1"] \\
\bigoplus_{p+q=n} (V \oplus W)^{\otimes p} \otimes (V \oplus W)^{\otimes q} \arrow[r] \arrow[d, "a"'] & \bigoplus_{p+q=n} \bigwedge^{p}(V \oplus W) \otimes \bigwedge^{q}(V \oplus W) \arrow[d, "m_\wedge"] \\
(V \oplus W)^{\otimes n} \arrow[r] & \bigwedge^{n}(V \oplus W)
\end{tikzcd}$$

where a stands for "amalgamate" and is the obvious map. Let ψ be the composite comprised of the two right-hand maps. It will be shown that ϕ and ψ are inverse to one another.

First, consider the composite $\pi_n^{V\oplus W}; \phi; \psi = \Sigma; \oplus i_0^{\otimes p} \otimes i_1^{\otimes q}; a; \pi_n^{V\oplus W}$. Distributing the direct sums across the tensor product via the isomorphism which we will call t, we may evaluate this map on each summand individually. On a summand with p Vs and q Ws, this is

$$(-1)^{|\sigma|}\sigma; i_0^{\otimes p} \otimes i_1^{\otimes q}; t^{-1}; \pi_n^{V\oplus W} = (-1)^{|\sigma|}\sigma; t^{-1}; \pi_n^{V\oplus W} = \pi_n^{V\oplus W}$$

and so $\phi; \psi = 1$. Finally, consider

$$\oplus(\pi_p^V \otimes \pi_q^W); \psi; \phi = \oplus(i_0^{\otimes p} \otimes i_1^{\otimes q}; a; \Sigma; \pi_p^V \otimes \pi_q^W)$$

On a summand of the form $V^{\otimes p} \otimes W^{\otimes q}$ this is clearly just $\pi_p^V \otimes \pi_q^W$, and we are done. ∎

2.6 Homological Algebra

It is assumed that the reader has some level of familiarity with complexes and the homology and cohomology thereof. Notably, it is assumed that the reader is familiar with the derived functors, Tor and Ext, and their constructions via projective resolutions. Here, we will introduce all other technical tools and results from homological algebra which will be used in later chapters, and which, if introduced there, would divert the reader's attention too far from the main line of development. Unless otherwise noted, the material in the following sections draws from [22] and [17].

2.6.1 Presimplicial Structure

Many complexes that arise in practice have differential maps which may be conveniently described as alternating sums of simpler maps. Often, the structure underlying the sequence of objects which constitute such complexes is presimplicial:

Definition 6. A sequence of objects A_0, A_1, ... is said to have *presimplicial structure* when for each n, there exist maps $\partial_i : A_n \longrightarrow A_{n-1}$ for $i = 0, ..., n$ such that

$$\partial_j; \partial_i = \partial_i; \partial_{j-1} \text{ if } i < j$$

For the purposes of this paper, the following example of presimplicial structure is prototypical

Example 7. Let T be a monad and $< A, \nu >$ be a T-algebra. Then let $A_i = T^{i+1}A$, and

$$\partial_i = \begin{cases} T^i \mu_{T^{n-i-2}} & \text{if } i \leq n-1 \\ T^n \nu & \text{if } i = n \end{cases}$$

The fact that this sequence of objects has presimplicial structure then follows from naturality of μ and the T-algebra structure induced by ν and by μ. See 8.6.4 on pp. 280-281 in [22]

Given a sequence of objects A_i with presimplicial structure, we may define a sequence of maps $d^n : A_n \longrightarrow A_{n-1}$ by

$$d^n = \sum_{i=0}^{n} (-1)^i \partial_i$$

Consider the composite $d^n; d^{n-1}$. It is comprised of a sum of maps of the form $(-1)^{j+i}\partial_j; \partial_i$. When $j > i$, this is equal to $(-1)^{j+i}\partial_i; \partial_{j-1}$. Adding this term to the term $(-1)^{i+j-1}\partial_i; \partial_{j-1}$ which also occurs in this composite gives zero. Since this accounts for every term in the sum, it follows that $d^n; d^{n-1} = 0$

It follows that if a sequence of objects has presimplicial structure, the following diagram is a complex:

$$\ldots \longrightarrow A_2 \xrightarrow{d^2} A_1 \xrightarrow{d^1} A_0$$

A crucial feature of the above examples is that for each n, there is a map $\eta_{T^n} : T^n A \longrightarrow T^{n+1} A$, which interacts with the ∂_i maps in significant ways. For instance, suppose that the algebras for the monad T are constituted in such a way so that we may associate a complex to the canonical presimplicial structure and compute its homology. Then if $x \in ker(d^n)$, we have

$$\begin{aligned} 0 = &\eta_{T^n}(d^n(x))) \\ =&\eta_{T^n}(\mu_{T^{n-1}} - T\mu_{T^{n-2}} + \ldots + (-1)^n T^{n-1}\nu)(x) \\ =&(T\mu_{Tn-1} - T^2\mu_{T^{n-2}} + \ldots + (-1)^n T^n \nu)\eta_{T^{n+1}}(x) \\ \equiv&\mu_{T^n}(\eta_{T^{n+1}}(x)) = x \end{aligned}$$

where the penultimate equivalence is modulo the image of d^{n+1}. It follows that the homology of this complex vanishes.

Later in this work, we will apply additive functors to the category of T-algebras and T-algebra morphisms which underly these complexes. It's important to note that since the η maps are not typically algebra homomorphisms, the resultant complexes will not necessarily have trivial homology.

2.6.2 Double Complexes and Spectral Sequences

Consider the following diagram:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
A_{0,2} \arrow[d, "d^V"'] & A_{1,2} \arrow[l, "d^H"'] \arrow[d, "d^V"'] & A_{2,2} \arrow[l, "d^H"'] \arrow[d, "d^V"'] & \dots \arrow[l] \\
A_{0,1} \arrow[d, "d^V"'] & A_{1,1} \arrow[l, "d^H"'] \arrow[d, "d^V"'] & A_{2,1} \arrow[l, "d^H"'] \arrow[d, "d^V"'] & \dots \arrow[l] \\
A_{0,0} & A_{1,0} \arrow[l, "d^H"'] & A_{2,0} \arrow[l, "d^H"'] & \dots \arrow[l]
\end{tikzcd}$$

If $d^H; d^H = d^V; d^V = d^H; d^V + d^V; d^H = 0$, this diagram is referred to as a *double complex*. It may be noted that, as is the case with all of the double complexes we will consider here, this diagram is bounded below and on the left. It is possible to consider unbounded double complexes, but the extra work required is not needed here.

With the notation of the diagram above, when this diagram is a double complex, we may form a complex with maps

$$d^T : \bigoplus_{i+j=n} A_{i,j} \longrightarrow \bigoplus_{k+l=n-1} A_{k,l}$$

where $d^T = d^V + d^H$. It is easy to check that this is well-defined; the following computation proves that this yields a complex:

$$d^T; d^T = (d^V + d^H); (d^V + d^H) = d^V; d^V + (d^V; d^H + d^H; d^V) + d^H; d^H = 0$$

where the final equality follows from the three requirements fulfilled by a double complex. The homology of this complex is called the *total homology* of the double complex, and is denoted $Tot_*(A_{*,*})$

Example 8. Given two complexes, (C_*, d^C) and (D_*, d^D), one may form the *tensor product of two complexes* via a double complex. That is, one may construct the following diagram

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
C_0 \otimes D_2 \arrow[d, "1\otimes d^D"'] & C_1 \otimes D_2 \arrow[l, "d^C\otimes 1"'] \arrow[d, "-1\otimes d^D"'] & C_2 \otimes D_2 \arrow[l, "d^C\otimes 1"'] \arrow[d, "1\otimes d^D"'] & \dots \arrow[l] \\
C_0 \otimes D_1 \arrow[d, "1\otimes d^D"'] & C_1 \otimes D_1 \arrow[l, "d^C\otimes 1"'] \arrow[d, "-1\otimes d^D"'] & C_2 \otimes D_1 \arrow[l, "d^C\otimes 1"'] \arrow[d, "1\otimes d^D"'] & \dots \arrow[l] \\
C_0 \otimes D_0 & C_1 \otimes D_0 \arrow[l, "d^C\otimes 1"'] & C_2 \otimes D_0 \arrow[l, "d^C\otimes 1"'] & \dots \arrow[l]
\end{tikzcd}$$

Note that a -1 is applied to each vertical map in each odd-numbered column, where the left-most column is the 0^{th}. One may immediately confirm that this is a double complex. The complex corresponding to the total homology of this double complex is the tensor product of (C_*, d^C) and (D_*, d^D), and is denoted $C_* \otimes D_*$. That is,

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential map $d^C \otimes 1 + (-1)^q 1 \otimes d^D$

The following lemma, a particular instance of Proposition 1.0.12 on p. 6 of [17] allows us to compute the total homology of a complex under a very specialized - but important - circumstance:

Lemma 9. *Let $A_{*,*}$ be a double complex such that for all p, the homology of $A_{*,p}$ is 0, except possibly at $A_{0,p}$. Then $Tot_n(A_{*,*}) = H^V(H^H(A_{0,n}))$. Similarly, let $A_{*,*}$ be a first quadrant double complex such that for all q, the homology of $A_{q,*}$ is 0, except possibly at $A_{q,0}$. Then $Tot_n(A_{*,*}) = H^H(H^V(A_{n,0}))$*

Of course, it will not always be the case that such a vanishing condition will occur. More generally, we may take the homology of the rows (or columns) as the first step in the computation of the *spectral sequence associated to the double complex* $A_{*,*}$. The following is an adumbration of the related theory. See part 1 of [19] or Chapter 5 of [22] for further details.

Consider first, the modules of the form $E^1_{p,q}$ produced by taking the homology of the p^{th} column at the q^{th} row of $A_{*,*}$. The map $d^H : A_{p,q} \longrightarrow A_{p,q-1}$ induces a map $d^1 : E^1_{p,q} \longrightarrow E^1_{p,q-1}$, by the anti-commutativity property of double complexes. Since $d^H ; d^H = 0$, for each p, we arrive at a new complex:

$$\ldots \longleftarrow E^1_{p,q-1} \longleftarrow E^1_{p,q} \longleftarrow E^1_{p,q+1} \longleftarrow \ldots$$

Taking the homology of this complex at q gives us $E^2_{p,q}$. Note that if $[x] \in ker(d^1)$, $[x]$ is the homology class of a cycle in the vertical homology of $A_{*,*}$, which is sent, via d^H, to a boundary in the vertical homology of $A_{*,*}$. For $[x] \in E^2_{p,q}$ then, we may choose a representative $x \in A_{p,q}$ and send it to $y \in A_{p-1,q+1}$ such that $d^V(y) = d^H(x)$. Consider $z = d^H(y)$. Since

$$d^V(z) = d^V d^H(y) = d^H d^V(y) = d^H d^H(x) = 0$$

and

$$d^H(z) = d^H d^H(y) = 0$$

z is the representative of a class in $E^2_{p-2,q+1}$. One may check that the assignment $[x] \longrightarrow [z]$ is well-defined and furthermore that this assignment constitutes a differential map. We call this map $d^2 : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$.

A fundamental result in the study of such spectral sequences is that this process can be carried out ad infinitum, that is, that we may define $E^{r+1}_{p,q} := (kerd^r)/(Imd^r)$ and proceed to define maps of the form $d^{r+1} : E^{r+1}_{p,q} \longrightarrow E^{r+1}_{p-r-1,q+r}$. In the present case, note that eventually these maps will reach beyond the boundaries of the parameters p and q, and so for some r we will have $E^r_{p,q} \cong E^{r+1}_{p,q} \cong ... \cong E^{\infty}_{p,q}$.

Crucially, these modules, $E^{\infty}_{p,q}$, relate to the total homology of the complex $A_{*,*}$ as follows. First, there is an increasing filtration of the complex associated to the total homology of $A_{*,*}$:

$$(F_i A_{*,*})_n = \bigoplus_{\substack{p+q=n \\ p \leq i}} A_{p,q}$$

which induces an increasing filtration on the n^{th} homology group of the total homology of $A_{*,*}$, denoted $F_i(Tot_n(A_{*,*}))$ for $0 \leq i \leq n$. We have

Theorem 10. $E^{\infty}_{p,q} \cong F_p(Tot_{p+q}(A_{*,*}))/F_{p-1}(Tot_{p+q}(A_{*,*}))$

Observe that the decision to take the vertical homology of $A_{*,*}$ first in the development of the above spectral sequence was completely arbitrary. Indeed, it is possible to form an analogous spectral sequence, which bears the same relationship to the total homology of $A_{*,*}$ by first computing the horizontal homology. This final observation is critical to the present work.

2.6.3 Hochschild Homology

Hochschild homology is the homology theory with which we are predominantly concerned in the present work. On the surface, the Hochschild homology of an algebra is a series of invariants which elucidate properties of the algebra vaguely related to commutativity. Further investigation reveals that the Hochschild homology of an algebra is profoundly related to the differential structure associated with the algebra. The upshot of this is that Hochschild homology and the closely related cyclic homology become viable options for noncommutative geometers who require analogues to tangent spaces, cotangent spaces, and de Rham cohomology; in fact, this idea is mainstream in noncommutative geometry (see p. xiv in the introduction of [15]). We begin with the definition and a few rudimentary calculations and follow section 1.1 of [17].

Definition 11. Let A be an associative algebra and let $n > 0$. Then define $d_i : A^{\otimes n} \longrightarrow A^{\otimes n-1}$ as follows:

$$d_i = \begin{cases} 1^{\otimes i} \otimes m_A \otimes 1^{\otimes n-i-2} & 0 \leq i \leq n-2 \\ c; m_A \otimes 1^{\otimes n-2} & i = n-1 \end{cases}$$

where c in this definition transfers all of the constituents of the tensor product to the right, except for the final constituent, which is sent to the leftmost position. Finally, define $b = \sum_{i=0}^{n-1}(-1)^i d_i$. So, for instance, if we apply b to an element of the form $x \otimes y \otimes z$ we get an alternating sum of the form

$$xy \otimes z - x \otimes yz + zx \otimes y$$

Then the n^{th} *Hochschild homology* group of A, denoted $HH_n(A)$ is the n^{th} homology group of the following complex, known as the *Hochschild complex* for A:

$$\ldots \longrightarrow A^{\otimes 4} \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A$$

One may readily verify that the d_i maps endow the collection of objects $A^{\otimes n}$ with presimplicial structure, and therefore the above diagram is, indeed, a complex.

It is useful to first perform some low-dimensional computations. For instance, one may readily verify that $HH_0(A)$ is the same as the abelianization of A. The subsequent computation of $HH_1(A)$ is fundamental to much of the work that follows. Although this result is elementary in the theory of Hochschild homology, a proof is given here in order to emphasize the category theoretic perspective.

Lemma 12. *If A is a commutative k-algebra, $HH_1(A) \cong \Omega_{A/k}$*

Proof: Define the map d_{HH_1} to be the following composite:

$$A \xrightarrow{u} I \otimes A \xrightarrow{e_A \otimes 1_A} A \otimes A \xrightarrow{p} HH_1(A)$$

where p is the projection of $A \otimes A$ onto $coker(m_A \otimes 1_A - 1_A \otimes m_A + c_H; m_A \otimes 1_A)$. The following computation shows that d_{HH_1} obeys the Leibniz rule:

$$\begin{aligned} m_A; d_{HH_1} &= m_A; u_L; e_A \otimes 1_A; p \\ &= u_L; e_A \otimes 1_A \otimes 1_A; 1_A \otimes m_A; p \\ &= u_R \otimes 1_A; 1_A \otimes e_A \otimes 1_A; (m_A \otimes 1_A + c_H; m_A \otimes 1_A); p \\ &= u_R \otimes 1_A; 1_A \otimes e_A \otimes 1_A; (1_A \otimes 1_A \otimes 1_A + c); 1_A \otimes p; \bullet \\ &= (1_A \otimes d_{HH_1} + c; 1_A \otimes d_{HH_1}); \bullet \end{aligned}$$

It can be proven that $e_A; d_{HH_1} = 0$ and so d_{HH_1} is a derivation.

To show that d_{HH_1} is universal and hence that $HH_1(A)$ is the A-module of differential forms, consider a derivation $\partial : A \longrightarrow M$ to an A-module, M. Since

$$\begin{aligned} 1_A \otimes m_A; 1_A \otimes \partial; \bullet &= 1_A \otimes (1_A \otimes \partial + c; 1_A \otimes \partial); 1_A \otimes \bullet; \bullet \\ &= 1_A \otimes (1_A \otimes \partial + c; 1_A \otimes \partial); m_A \otimes 1_A; \bullet \\ &= (m_A \otimes 1_A + 1_A \otimes c; m_A \otimes 1_A); 1_A \otimes \partial; \bullet \end{aligned}$$

commutativity of A shows that $1_A \otimes \partial; \bullet$ coequalizes $1_A \otimes m_A$ and $m_A \otimes 1_A + c_H; m_A \otimes 1_A$. Hence there is a unique morphism of A-modules $f : HH_1(A) \longrightarrow M$ such that $p; f = 1_A \otimes \partial; \bullet$. Hence

$$\begin{aligned}
\partial &= u_L; e_A \otimes 1_A; m_A; \partial \\
&= u_L; e_A \otimes 1_A; (1_A \otimes \partial + c; 1_A \otimes \partial); \bullet \\
&= u_L; e_A \otimes 1_A; (1_A \otimes 1_A + c; 1_A \otimes 1_A); 1_A \otimes \partial; \bullet \\
&= u_L; e_A \otimes 1_A; (1_A \otimes 1_A + c; 1_A \otimes 1_A); p; f \\
&= u_L; e_A \otimes 1_A; p; f \\
&= d_{HH_1}; f
\end{aligned}$$

with the penultimate equality following from an application of the identity $u_L; e_A \otimes 1_A; m_A = 1_A$. ∎

There is a complex, which is closely related to the Hochschild complex, called *the bar complex*:

$$\ldots \longrightarrow A^{\otimes 4} \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2}$$

where $b' = \sum_{i=0}^{n-1} (-1)^i d_i$ with the notation used above in the description of the Hochschild complex. Note that b' is the same as b except that we have omitted d_n, the map which requires the symmetrical structure of the category.

Since the final map, d_n, is removed for each n, it is easy to see that this sequence maintains a presimplicial structure similar to that associated with the Hochschild complex. However, we may also consider maps $s : A^n \longrightarrow A^{n+1}$ where

$$s = u_L; e_A \otimes 1^{\otimes n}$$

hence inserting the unit as the first constituent of the tensor product. Since $s; d_0 = 1$ and $s; d_i = d_{i-1}; s$, we have $s; b' + b'; s = 1$ and so s is a *contracting homotopy* of the bar complex. It follows that the bar complex is acyclic. In more elementary terms, if $b'(x) = 0$,

$$x = b's(x) + sb'(x) = b's(x)$$

and so $x \in Im(b')$.

An important feature of the bar complex is that we may recover the Hochschild complex from it. Let A^e be the associative algebra formed by $A \otimes A^{op}$ - called the *enveloping algebra of* A - where $m_{A^{op}} = c; m_A$, and endow A with the obvious A^e-module structure. We have the following

Proposition 13. *Let A be an associative k-algebra. Then the Hochschild complex associated to A is formed by tensoring the bar complex associated to A with A over $\otimes_{A^e}$.*

The following proof is expressed in category theoretic language and is therefore somewhat unconventional.

Proof: We will first construct an explicit isomorphism $A \otimes_{A^e} A^{\otimes n} \cong A \otimes A^{\otimes n-2}$. For some semblance of clarity, let M be an A-bimodule and consider the following diagram:

$$\begin{tikzcd}
M \otimes A^e \otimes A^{\otimes n} \arrow[r, shift left, "\bullet_M \otimes 1"] \arrow[r, shift right, "1\otimes \bullet_{A^{\otimes n}}"'] & M \otimes A^{\otimes n} \arrow[r, "p_{A^e}"] \arrow[dr, "\bullet'_M"'] & M \otimes_{A^e} A^{\otimes n} \arrow[d, dashed, "q"] \\
 & & M \otimes A^{\otimes n-2}
\end{tikzcd}$$

Some explanation is in order. $\bullet_M$ is the map which acts on the right of M with the first component of A^e and on the left of M with the second component. $\bullet_A$ acts on the left of $A^{\otimes n}$ with the first component of A^e and on the right with the second. $\bullet'_M$ takes the first component of $A^{\otimes n}$ and acts on the right of M, and takes final component of $A^{\otimes n}$ and acts on the left of M.

One may readily verify that the two left-hand maps yield the same map when postcomposed with the diagonal map. Hence there is a map $q : M \otimes_{A^e} A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-2}$ which makes the triangle commute. Furthermore, there is a map

$$f = 1 \otimes u_L; 1 \otimes 1 \otimes u_R; 1 \otimes e_A \otimes 1 \otimes e_A; p_{A^e}$$

from $M \otimes A^{\otimes n-2}$ to $M \otimes_{A^e} A^{\otimes n}$. It is clear that $\bullet'_M$ is epi, and therefore so is q. Hence if it can be shown that $q; f = 1$, it will follow that q is the desired isomorphism.

Since p_{A^e} is epi, it is sufficient to show that $\bullet'_M; f = p_{A^e}; q; f = p_{A^e}$. This is best seen using the following diagram:

$$\begin{tikzcd}
M \otimes A^{\otimes n} \arrow[r, "\bullet'_M"] \arrow[d, "1\otimes 1\otimes u_R\otimes 1;1\otimes 1\otimes u_L\otimes 1"'] & M \otimes A^{\otimes n-2} \arrow[rr, "1\otimes u_L;1\otimes u_R"] & & M \otimes I \otimes A^{\otimes n-2} \otimes I \arrow[d, "1\otimes e_A\otimes 1\otimes e_A"] \\
M \otimes A \otimes I \otimes A^{n-2} \otimes I \otimes A \arrow[rr] \arrow[drr, "\cong"'] & & M \otimes A^{\otimes n+2} \arrow[r, "\bullet'_M"'] \arrow[d, "\bullet_A"'] & M \otimes A^{\otimes n} \arrow[d, "p_{A^e}"] \\
 & & M \otimes A^{\otimes n} \arrow[r, "p_{A^e}"'] & M \otimes_{A^e} A^{\otimes n}
\end{tikzcd}$$

The large square conveys the fact that result of invoking the tensor unit within the given tensor product is the same either before or after the application of $\bullet'_M$, and commutativity of the small square follows from the defining property of p_{A^e}

Now, if we consider the bar complex tensored over A^e with A, we have

$$\begin{tikzcd}
\ldots \arrow[r] & A \otimes_{A^e} A^{\otimes 4} \arrow[r, "1\otimes b'"] \arrow[d, "\cong"'] & A \otimes_{A^e} A^{\otimes 3} \arrow[r, "1\otimes b'"] \arrow[d, "\cong"'] & A \otimes_{A^e} A^{\otimes 2} \arrow[d, "\cong"'] \\
\ldots \arrow[r] & A^{\otimes 3} \arrow[r, "b"'] & A^{\otimes 2} \arrow[r, "b"'] & A
\end{tikzcd}$$

The fact that b is the appropriate map follows by first preceding each square with p_{A^e} and then performing a routine inspection. ∎

If A is flat, then the bar complex is a flat resolution over A^e of A. Hence by tensoring this complex with A over A^e and computing the homology of the resultant complex, we are computing $Tor_n^{A^e}(A, A)$. It follows that if A is flat, then $HH_n(A) \cong Tor_n^{A^e}(A, A)$.

2.6.4 Hochschild Cohomology

Although the present work is primarily concerned with Hochschild homology as it relates to smoothness, some crucial arguments about smoothness are phrased more naturally in the context of Hochschild cohomology. In particular, the second Hochschild cohomology groups characterize the square-zero extensions of an algebra, which are germane to accepted definitions of algebraic smoothness. The exposition here follows section 9.1 of [22].

Definition 14. Let A be a k-algebra and M be an A-bimodule. For any $n \in \mathbb{N}$, consider the maps $\partial^i : Hom_k(A^{\otimes n}, M) \longrightarrow Hom_k(A^{\otimes n+1}, M)$ defined as follows

$$\partial^i f(a_0, a_1, ...a_n) = \begin{cases} a_0 f(a_1, ..., a_n) & i = 0 \\ f(a_0, ..., a_{i-1}a_i, ...a_n) & 0 \leq i \leq n \\ f(a_0, ..., a_{n-1})a_n & i = n \end{cases}$$

and let $d^n = \Sigma_{i=0}^n (-1)^n \partial^i$. Then the *Hochschild cohomology complex* for A with coefficients in M is given as follows:

$$0 \longrightarrow M \xrightarrow{\partial^0 - \partial^1} Hom_k(A, M) \xrightarrow{d^1} Hom_k(A^{\otimes 2}, M) \longrightarrow ...$$

The homology of this complex is the *Hochschild cohomology of A with coefficients in M* and is denoted by $HC^n(A, M)$.

It must be confirmed, of course, that the ∂^i maps endow the above diagram with co-presimplicial structure, but this is routine.

As with Hochschild homology, Hochschild cohomology may be presented in terms of the Bar complex. That is, we may form the following complex:

$$0 \to Hom_{A^e}(A^{\otimes 2}, M) \xrightarrow{\mathcal{B}} Hom_{A^e}(A^{\otimes 3}, M) \xrightarrow{\mathcal{B}} Hom_{A^e}(A^{\otimes 4}, M) \to ...$$

where $\mathcal{B}$ sends f to $b'; f$. For any n, there is an isomorphism $Hom_{A^e}(A^{\otimes n+2}, M) \cong Hom_I(A^{\otimes n}, M)$ formed by sending $f \in Hom_{A^e}(A^{\otimes n+2}, M)$ to

$$\phi(f) = u_L; 1_I \otimes 1_{A^{\otimes n}} \otimes u_R; e_A \otimes 1_{A^{\otimes n}} \otimes e_A; f$$

and sending $g \in Hom_I(A^{\otimes n}, M)$ to $1_A \otimes g \otimes 1; \bullet_A$, using the notation of the previous section. One may easily confirm that these assignments are inverse to one another, once it is recognized that if g is an A^e-module map,

$$g = u_R \otimes 1_{A^{\otimes n}} \otimes u_L; 1 \otimes e_A \otimes 1_{A^{\otimes n}} \otimes e_A \otimes 1_A; 1 \otimes g \otimes 1; \bullet_M$$

With this it can be verified that for all n

$$\begin{tikzcd} Hom_{A^e}(A^{\otimes n+2}, M) \arrow[r, "\mathcal{B}"] \arrow[d, "\cong"'] & Hom_{A^e}(A^{\otimes n+3}, M) \arrow[d, "\cong"] \\ Hom_I(A^{\otimes n}, M) \arrow[r, "d^n"'] & Hom_I(A^{\otimes n+1}, M) \end{tikzcd}$$

Analogous to the Tor representation of Hochschild homology then, it follows that if A is projective, then $HC^n(A, M) \cong Ext_n^{A^e}(A, M)$.

For our purposes, the most important Hochschild cohomology group is the second. We require the following definition

Definition 15. An abelian extension of an algebra A by M is an extension of associative k-algebras

$$0 \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow 0$$

such that $E \cong M \oplus A$ and $M^2 = 0$. Two such extensions are said to be equivalent if there exists an algebra homomorphism $\phi : E \longrightarrow E'$, which commutes with 1_A and 1_M as in

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "1_M"'] & E \arrow[r] \arrow[d, "\phi"'] & A \arrow[r] \arrow[d, "1_A"'] & 0 \\ 0 \arrow[r] & M \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Theorem 16. *Let A be an associative k-algebra and M an A-bimodule. Then $HC^2(A, M) \cong Ext_{AB}(A, M)$*

2.6.5 Cyclic Homology

In noncommutative geometry, cyclic homology and cohomology tend to take precedence over Hochschild homology. However, in a few important ways, cyclic homology is manifested as an elaboration of Hochschild homology. Because of this, it is sometimes possible to derive properties of the Hochschild homology of an algebra through the examination of its cyclic homology. This provides the sole impetus for the inclusion of this section here, and hence, the exposition is somewhat brief; further details are found in chapter 2 of [17]. However, this lack of development should be understood as a temporary expedient only; it seems very likely that cyclic homology interacts with the structure of codifferential categories in significant ways. Evidence of this is discussed in the concluding chapter.

As with the Hochschild homology of an associative algebra A, we define the cyclic homology of such an algebra by defining maps between the various iterations of tensor products of A with itself, $A^{\otimes n}$. In addition to the Hochschild boundary map b, however, we will also need to consider the following family of endomorphisms.

Definition 17. Let A be an associative algebra. The n^{th} *cyclic group action* $t_n : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$ is defined on generators by

$$t_n(a_0, a_1, ..., a_n) = (-1)^n (a_n, a_0, ..., a_{n-1})$$

and extended by linearity.

It is common to drop the subscript n from t_n. In order to define cyclic homology, we also require the map $N = 1+t+...+t^n$. With this, we form the following diagram:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & \dots \arrow[l, "1-t"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & \dots \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & \dots \arrow[l, "1-t"']
\end{tikzcd}$$

Lemma 18. *The above diagram is a bicomplex.*

Indeed, it is known as the *cyclic bicomplex*, $CC(A)$. The proof of the above lemma is strictly technical and can be easily -but tediously - adapted to category theoretic language. With this, we define the n^{th} *cyclic homology group of* A, denoted $HC_n(A)$, by

$$HC_n(A) := Tot_n(CC(A))$$

This is not the only, nor the original, way to define cyclic homology. However, it is the most general, and furthermore, facilitates the proof of the following crucial result.

Theorem 19. *Let A be an associative k-algebra. Then there is a long exact sequence of the form*

$$... \longrightarrow HH_n(A) \longrightarrow HC_n(A) \longrightarrow HC_{n-2}(A) \rightarrow HH_{n-1}(A) \rightarrow ...$$

This long exact sequence follows in a straightforward way from the short exact sequence of bicomplexes

$$0 \longrightarrow CC(A)^{\{2\}} \longrightarrow CC(A) \longrightarrow CC(A)[2,0] \longrightarrow 0$$

where $CC(A)^{\{2\}}$ denotes the cyclic bicomplex with all but the first two columns zeroed, and $CC(A)[2,0]$ denotes the cyclic bicomplexes shifted two columns to the right. It is obvious that this sequence of bicomplexes is exact; it is only necessary to confirm that the total homology of the first two columns of the cyclic bicomplex for A is isomorphic to the Hochschild homology of A. This follows immediately from **2.6.2.9** and the observation that the bar complex is exact.

The fact that the cyclic bicomplex is a bicomplex requires that $(1-t);b = -b';(1-t)$. It follows that b induces a map $A^{\otimes n}/(1-t) \longrightarrow A^{\otimes n-1}/(1-t)$. Adopting the notation $C_n^\lambda(A) := A^{\otimes n+1}/(1-t)$, we get a complex:

$$\ldots \longrightarrow C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} C_{n-2}^\lambda(A) \longrightarrow \ldots$$

This is known as *Connes' complex* and its homology is denoted $H_n^\lambda(A)$. When k contains $\mathbb{Q}$ we have $H_n^\lambda(A) \cong HC_n(A)$, and, in fact, this is how cyclic homology was defined originally.

Chapter 3

Codifferential Categories

This chapter is largely expository in spirit, in the sense that the content distinctly resembles that of [4]. However, because the impetus for the present work is furnished significantly by formulations of noncommutative smoothness, this section will demonstrate the mostly nominal work of translating the contents of [4] into the context of *noncommutative codifferential categories*, as defined by Robin Cockett in [5].

However simple this translation is, the potential impact of a properly formulated theory of noncommutative codifferential categories should not be overlooked. On the contrary, the impact of this theory will likely be bolstered by the close connection with the theory of commutative codifferential categories. That is, similarities between the commutative theory and the noncommutative theory make the differences more salient; it is hoped that this will yield benefits as more advanced structures are developed in the general theory of codifferential categories.

Definition 1. An *algebra modality* on a symmetric monoidal category $\mathcal{C}$ consists of a monad (T, μ, η) on $\mathcal{C}$ and for each object C in $\mathcal{C}$, a pair of morphisms

$$m_{TC} : TC \otimes TC \longrightarrow TC$$

$$e_{TC} : I \longrightarrow TC$$

for each object TC, where I denotes the tensor unit. We ask that these maps endow TC with an associative algebra structure, and that e and m satisfy evident naturality conditions.

Note that we have already departed from the development of the commutative context, where it is mandated that the algebra structure on TC is commutative.

The significance of the following result to this work is disguised by its simplicity. The proof given is indicative of many of the computations necessitated by the structure of codifferential categories. Note that a more elegant approach is available, but that it obscures the explicit structure expatiated in the following. It is valuable to observe that these calculations utilize all of the axioms for a monad and for a T-algebra.

Lemma 2. *Let (T, μ, η) be an algebra modality on a monoidal category $\mathcal{C}$. Then any T-algebra $< C, \nu >$ inherits a canonical associative algebra structure, which makes ν an associative algebra morphism.*

Proof: Define a multiplication map and an identity map as follows:

$$m_C := \eta_C \otimes \eta_C; m_{TC}; \nu$$

$$e_C := e_{TC}; \nu$$

Then we have

$$\begin{aligned}
&\nu \otimes \nu; m_C \\
=&\nu \otimes \nu; \eta_C \otimes \eta_C; m_{TC}; \nu \\
=&\eta_{TC} \otimes \eta_{TC}; T\nu \otimes T\nu; m_{TC}; \nu \\
=&\eta_{TC} \otimes \eta_{TC}; m_{T^2C}; T\nu; \nu \\
=&\eta_{TC} \otimes \eta_{TC}; m_{T^2C}; \mu_C; \nu \\
=&\eta_{TC} \otimes \eta_{TC}; \mu_C \otimes \mu_C; m_{TC}; \nu \\
=&m_{TC}; \nu
\end{aligned}$$

so that if m_C is indeed an associative algebra multiplication map, it will follow from this, and from the definition of e_C, that ν is an associative algebra morphism. Using the fact that T-algebra structure maps are split epi, the following computation grants us associativity:

$$\begin{aligned}
&\nu \otimes \nu \otimes \nu; m_C \otimes 1_C; m_C \\
=&m_{TC} \otimes 1_{TC}; \nu \otimes \nu; m_C \\
=&m_{TC} \otimes 1_{TC}; m_{TC}; \nu \\
=&1_{TC} \otimes m_{TC}; m_{TC}; \nu \\
=&\nu \otimes \nu \otimes \nu; 1_C \otimes m_C; m_C
\end{aligned}$$

Finally, the fact that e_C is a right-hand unit is established by the following:

$$\begin{aligned}
&\nu \otimes e_C; m_C = 1_C \otimes e_{TC}; \nu \otimes \nu; m_C \\
=&1_C \otimes e_{TC}; m_{TC}; \nu \\
=&u_R; \nu \otimes 1_I
\end{aligned}$$

where $u_R : TC \longrightarrow TC \otimes I$ is the requisite isomorphism. The deduction that e_C is a left-hand unit is similar. ∎

The following definition was posited by Robin Cockett in [5].

Definition 3. A *noncommutative codifferential category* is an additive symmetric monoidal category with an algebra modality (T, μ, η) and a *noncommutative deriving transform*;i.e. a transformation natural in C

$$d_{TC} : TC \longrightarrow TC \otimes C \otimes TC$$

satisfying the following four equations:

(d1) $e; d = 0$ *(The derivative of a constant is zero)*

(d2) $m; d = (1 \otimes d); (m \otimes 1 \otimes 1) + (d \otimes 1); (1 \otimes 1 \otimes m)$ *(Leibniz rule)*

(d3) $\eta; d = e \otimes 1 \otimes e$ *(Derivative of a linear function is constant)*

(d4) $\mu; d = d; \mu \otimes d \otimes \mu; m \otimes 1 \otimes m$ *(Chain rule)*

Equations **d1** and **d2** dictate that the morphisms d_{TC} are derivations. As was the case for commutative codifferential categories, the question of universality of these derivations is paramount. Careful examination of the following example will elucidate the crux of this matter.

3.1 The Free Associative Algebra Monad

For any object C in a symmetric monoidal category $\mathcal{C}$ with countable direct sums, consider the object

$$\mathcal{F}C := I \oplus C \oplus C \otimes C \oplus ...$$

We may define an associative algebra structure on these objects in the usual way, by concatenation. That is,

$$m_{FC} : \mathcal{F}C \otimes \mathcal{F}C \longrightarrow \mathcal{F}C$$

is defined by taking $C^{\otimes m} \otimes C^{\otimes n}$ to $C^{\otimes m+n}$ and distributing over the sum. This map is manifestly associative. We may take the unit map $e_{FC} : I \longrightarrow \mathcal{F}C$ to be the insertion of I into the direct sum.

Now we may endow $\mathcal{F}C$ with a monad structure in the normal way, where $\mu_C : \mathcal{F}^2C \longrightarrow \mathcal{F}C$ is given on the direct sum

$$I \oplus \mathcal{F}C \oplus \mathcal{F}C \otimes \mathcal{F}C \oplus ...$$

by e_{FC} on I, and iterated multiplication on each $\mathcal{F}C^{\otimes n}$. η_C is the insertion of C into the direct sum. It is worth noting that since $\mathcal{F}$ is a monad which is also an algebra modality, any $\mathcal{F}$-algebra is equipped with an associative algebra structure.

Let us fix some notation for the following demonstration. Let $\langle C^{\otimes n} \rangle$ denote the n-fold tensor product $C^{\otimes n}$ which occurs as the $(n + 1)^{st}$ summand in the direct sum which defines the object $\mathcal{F}C$, and let $\langle j_n \rangle$ denote the insertion of $C^{\otimes n}$ into $\mathcal{F}C$, with $\langle j_0 \rangle = e_{FC}$. Furthermore, let $\langle\langle C^{\otimes n}\rangle\rangle$ denote the manifestation of $\langle C^{\otimes n}\rangle$ in any given occurrence of $\mathcal{F}C$ in $\mathcal{F}^2C$.

This notation is used, for example, to distinguish between three possible manifestations of $C^{\otimes 2}$ in the tensor product $\mathcal{F}C \otimes \mathcal{F}C$ inside $\mathcal{F}^2C$: one with $C^{\otimes 2}$ in the first factor and I in the second, one with I in the first factor and $C^{\otimes 2}$ in the second, and one with C in each factor. These would be denoted by $\langle\langle C^{\otimes 2}\rangle \otimes I\rangle$, $\langle I \otimes \langle C^{\otimes 2}\rangle\rangle$, and $\langle\langle C\rangle \otimes \langle C\rangle\rangle$.

Define $d^F_{FC} : \mathcal{F}C \longrightarrow \mathcal{F}C \otimes C \otimes \mathcal{F}C$ on the summands of $\mathcal{F}C$ as follows

i $I \longrightarrow \mathcal{F}C \otimes C \otimes \mathcal{F}C$ via 0

ii $\langle C\rangle \longrightarrow \mathcal{F}C \otimes C \otimes \mathcal{F}C$ via $u_L; 1_{FC} \otimes u_R; e_{FC} \otimes 1_C \otimes e_{FC}$

iii $\langle C^{\otimes n}\rangle \longrightarrow \mathcal{F}C \otimes C \otimes \mathcal{F}C$ via $\sum_{i=0}^{n-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{n-1-i}\rangle$

Observe that ii) is a special case of iii).

Proposition 4. *d^F is a deriving transform.*

Proof: It is routine to verify that d^F is natural. Furthermore, that d^F satisfies **d1** and **d3** is clear. Now, consider **d2**. Because both d^F and m_{FC} distribute across sums we may restrict our attention to products of the form

$$m_{FC} : \langle C^{\otimes m}\rangle \otimes \langle C^{\otimes n}\rangle \longrightarrow \langle C^{\otimes m+n}\rangle$$

Here,

$$m_{FC}; d^F_{FC} = \sum_{i=0}^{m+n-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{m+n-1-i}\rangle$$

On the other hand, on this same summand

$$\begin{aligned}
& d^F \otimes 1_{FC}; 1_{FC} \otimes 1_C \otimes m_{FC} + 1_{FC} \otimes d^F; m_{FC} \otimes 1_C \otimes 1_{FC} \\
=& [(\sum_{i=0}^{n-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{n-1-i}\rangle)] \otimes 1_{FC}; 1_{FC} \otimes 1_C \otimes m_{FC} \\
& + [1_{FC} \otimes (\sum_{i=0}^{m-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{m-1-i}\rangle)]; m_{FC} \otimes 1_C \otimes 1_{FC} \\
=& (\sum_{i=0}^{n-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{n+m-1-i}\rangle) + (\sum_{i=0}^{m-1} \langle j_{n+i}\rangle \otimes 1_C \otimes \langle j_{m-1-i}\rangle) \\
=& \sum_{i=0}^{n+m-1} \langle j_i\rangle \otimes 1_C \otimes \langle j_{n+m-1-i}\rangle = m_{FC}; d^F
\end{aligned}$$

Note that **d2** is trivial on summands of the form $I \otimes \langle C^{\otimes n} \rangle$ and $\langle C^{\otimes n} \rangle \otimes I$, so **d2** holds in general.

We must now prove that **d4**, the chain rule, holds for d^F. First observe that on I the chain rule holds trivially, by the definition of d^F. Next, on $\langle \mathcal{F}C \rangle$, $\mu; d^F_{FC} = d^F_{FC}$ and $d^F_{F^2C} = u_L; 1_{F^2C} \otimes u_R; e_{F^2C} \otimes 1_{FC} \otimes e_{F^2C}$; following this latter map by $\mu \otimes d^F \otimes \mu; m_{FC} \otimes 1_C \otimes m_{FC}$ then yields the desired result.

Finally, to show that the chain rule holds when restricted to summands of the form $\langle \mathcal{F}C^{\otimes n} \rangle$, first note that, by distributivity across sums, we may restrict our attention to summands of the form

$$\langle \langle C^{\otimes n_1} \rangle \otimes \ldots \otimes \langle C^{\otimes n_m} \rangle \rangle$$

Let $N = \sum_{i=1}^{m} n_i$. We must show that the following diagram is commutative:

$$\begin{tikzcd}
\langle \langle C^{\otimes n_1} \rangle \otimes \ldots \otimes \langle C^{\otimes n_m} \rangle \rangle \arrow[rr, "\mu"] \arrow[d, "d^F"'] & & \langle C^{\otimes N} \rangle \arrow[d, "d^F"] \\
\mathcal{F}^2 C \otimes \mathcal{F}C \otimes \mathcal{F}^2 C \arrow[r, "\mu \otimes d^F \otimes \mu"'] & \mathcal{F}C^{\otimes 2} \otimes C \otimes \mathcal{F}C^{\otimes 2} \arrow[r, "m_{FC} \otimes 1_C \otimes m_{FC}"'] & \mathcal{F}C \otimes C \otimes \mathcal{F}C
\end{tikzcd}$$

A typical summand on the left-hand side is

$$\langle \langle C^{\otimes n_1} \rangle \otimes \ldots \otimes \langle C^{\otimes n_k} \rangle \rangle \otimes \langle C^{\otimes n_{k+1}} \rangle \otimes \langle \langle C^{\otimes n_{k+2}} \rangle \otimes \ldots \otimes \langle C^{\otimes n_m} \rangle \rangle$$

This is sent, via $\mu \otimes d^F \otimes \mu$, to

$$\langle C^{\otimes n_1 + \ldots + n_k} \rangle \otimes [\sum_{i=0}^{n_{k+1}-1} \langle C^{\otimes i} \rangle \otimes C \otimes \langle C^{\otimes n_{k+1}-1-i} \rangle] \otimes \langle C^{\otimes n_{k+2} + \ldots + n_m} \rangle$$

and finally, this is sent, via $m_{FC} \otimes 1_C \otimes m_{FC}$, to

$$\sum_{i=0}^{n_{k+1}-1} \langle C^{\otimes n_1 + \ldots + n_k + i} \rangle \otimes C \otimes \langle C^{\otimes n_{k+2} + \ldots + n_m + n_{k+1} - 1 - i} \rangle$$

Summing over $0 \leq k \leq m-1$ then gives

$$d^F; \mu \otimes d^F \otimes \mu; m_{FC} \otimes 1_C \otimes m_{FC} = \sum_{i=0}^{N} \langle j_i \rangle \otimes 1_C \otimes \langle j_{N-1-i} \rangle$$

which is precisely $\mu; d^F$, and so **d4** holds. ∎

Hence there exists a derivation d_{FC} for each object of the form $\mathcal{F}C$. Furthermore, the target of this derivation is endowed with the structure of the free $\mathcal{F}C$-bimodule over C. As a result, for any derivation $\partial : \mathcal{F}C \longrightarrow M$, there exists a unique $\mathcal{F}C$-module morphism $\phi : \mathcal{F}C \otimes C \otimes \mathcal{F}C \longrightarrow M$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\eta^F_C"] & \mathcal{F}C \arrow[r, "d^F_{FC}"] \arrow[dr, "\partial"'] & \mathcal{F}C \otimes C \otimes \mathcal{F}C \arrow[d, "\phi"] \\ & & M \end{tikzcd}$$

Therefore if $\partial = d^F_{FC}; \phi$, then ϕ is unique in this capacity.

Now, for any associative algebra (A, m_A, e_A), and A-bimodule $(M, \bullet_L, \bullet_R)$, we may induce an associative algebra structure on the object $A \oplus M$ via direct sum of the maps

$$m_A : A \otimes A \longrightarrow A$$
$$\bullet_L : A \otimes M \longrightarrow M$$
$$\bullet_R : M \otimes A \longrightarrow M$$
$$0 : M \otimes M \longrightarrow M$$

With this algebra structure and a derivation $\partial : A \longrightarrow M$, the morphism $\begin{pmatrix} 1_A \\ \partial \end{pmatrix} : A \longrightarrow A \oplus M$ is an associative algebra morphism. It is easy to see that, conversely, if a map of the form $\begin{pmatrix} 1_A \\ \partial \end{pmatrix} : A \longrightarrow A \oplus M$ is an associative algebra morphism, then $\partial : A \longrightarrow M$ is a derivation.

Consider then, the diagram

$$\begin{tikzcd} C \arrow[r, "\eta^F_C"] & \mathcal{F}C \arrow[r, "\begin{pmatrix} 1_{FC} \\ d_{FC} \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1_{FC} \\ \partial \end{pmatrix}"'] & \mathcal{F}C \oplus \mathcal{F}C \otimes C \otimes \mathcal{F}C \arrow[d, "\begin{pmatrix} 1_{FC} \\ \phi \end{pmatrix}"] \\ & & \mathcal{F}C \oplus M \end{tikzcd}$$

It is clearly commutative, and since the triangle is composed of morphisms of associative algebras, the universal property of η mandates that the triangle is itself commutative. We have thus proved the following:

Proposition 5. $\Omega^{NC}_{FC/k} \cong \mathcal{F}C \otimes C \otimes \mathcal{F}C$

Since $\Omega^{NC}_{FC/k}$, as described in **2.3**, is isomorphic to the kernel of the map m_{FC}, there is an exact sequence of the form

$$0 \longrightarrow \mathcal{F}C \otimes C \otimes \mathcal{F}C \longrightarrow \mathcal{F}C \otimes \mathcal{F}C \xrightarrow{m} \mathcal{F}C \longrightarrow 0$$

It can furthermore be shown that the appropriate left-most nontrivial map is

$$1 \otimes \eta^F \otimes 1; (m \otimes 1 - 1 \otimes m)$$

where η^F is the monad unit.

3.2 Noncommutative T-derivations

An important distinction to be made between the preceding example and a general noncommutative codifferential category is the fact that the universal property of η^T does not apply to morphisms of associative algebras in general, but to those which are morphisms of T-algebras. That is, although we may form the commutative diagram

$$\begin{tikzcd}
C \arrow[r, "\eta^T_C"] & TC \arrow[r, "\begin{pmatrix} 1_{TC} \\ d_{TC} \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1_{TC} \\ \partial \end{pmatrix}"'] & TC \oplus TC \otimes C \otimes TC \arrow[d, "\begin{pmatrix} 1_{TC} \\ \phi \end{pmatrix}"] \\
 & & TC \oplus M
\end{tikzcd}$$

the universal property of η^T does not indicate that the triangle is itself commutative. It was precisely the analogous predicament in the commutative case which prompted the definition of a T-derivation in [4]. In order to adapt this definition to the non-commutative case, we require the following result:

Theorem 6. *Let $\mathcal{C}$ be a codifferential category with finite coproducts. Let $< A, \nu >$ be a T-algebra and M an A-bimodule. Then $< A \oplus M, \beta >$ is a T-algebra with structure map $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} : T(A \oplus M) \longrightarrow A \oplus M$ defined on the constituents of the target as follows:*

$$\begin{tikzcd}
T(A \oplus M) \arrow[r, "T\pi_1"] \arrow[dr, "\beta_1"'] & TA \arrow[d, "\nu"] \\
 & A
\end{tikzcd}$$

$$\begin{tikzcd}
T(A \oplus M) \arrow[rr, "d"] \arrow[d, "\beta_2"'] & & T(A \oplus M) \otimes (A \oplus M) \otimes T(A \oplus M) \arrow[d, "T\pi_1 \otimes \pi_2 \otimes T\pi_1"] \\
M & A \otimes M \otimes A \arrow[l, "\bullet_L \otimes 1; \bullet_R"] & TA \otimes M \otimes TA \arrow[l, "\nu \otimes 1 \otimes \nu"]
\end{tikzcd}$$

In order to prove this, even in the commutative case, one is confronted with the choice between unwieldy diagrams and abstruse calculations; this dilemma is compounded by the bimodule structure and concomitant additional summands required in the noncommutative setting. The latter option is chosen here, but one is free to simply refer to [4] - and translate appropriately - to convince oneself of the veracity of this theorem.

Proof: We confirm that β_1 and β_2 behave appropriately with regard to the T-algebra axioms individually. First, observe that

$$\begin{aligned}
&\mu;\beta_1 = \mu;T\pi_1;\nu\\
=&T^2\pi_1;\mu;\nu\\
=&T^2\pi_1;T\nu;\nu\\
=&T(T\pi_1;\nu);\nu\\
=&T(\beta_1);\nu\\
=&T\beta;T\pi_1;\nu = T\beta;\beta_1
\end{aligned}$$

and

$$\eta;\beta_1 = \eta;T\pi_1;\nu = \pi_1;\eta;\nu = \pi_1$$

This confirms that β_1 behaves properly with respect to μ and η. Now, for β_2:

$$\begin{aligned}
&\mu;\beta_2 = \mu;d;T\pi_1\otimes\pi_2\otimes T\pi_1;\nu\otimes 1\otimes\nu;\bullet_M\\
=&d;\mu\otimes d\otimes\mu;m\otimes 1\otimes m;T\pi_1\otimes\pi_2\otimes T\pi_1;\nu\otimes 1\otimes\nu;\bullet_M\\
=&d;\mu\otimes d\otimes\mu;(T\pi_1;\nu)^{\otimes 2}\otimes\pi_2\otimes(T\pi_1;\nu)^{\otimes 2};m_A\otimes 1\otimes m_A;\bullet_M\\
=&d;\mu\otimes d\otimes\mu;(T\pi_1;\nu)^{\otimes 2}\otimes\pi_2\otimes(T\pi_1;\nu)^{\otimes 2};1\otimes\bullet_M\otimes 1;\bullet_M
\end{aligned}$$

We shall break this calculation into parts:

$$\begin{aligned}
&\mu\otimes d\otimes\mu;(T\pi_1;\nu)^{\otimes 2}\otimes\pi_2\otimes(T\pi_1;\nu)^{\otimes 2};1\otimes\bullet_M\otimes 1\\
=&T^2\pi_1\otimes d\otimes T^2\pi_1;\mu\otimes T\pi_1\otimes\pi_2\otimes T\pi_1\otimes\mu;\nu^{\otimes 2}\otimes 1\otimes\nu^{\otimes 2};1\otimes\bullet_M\otimes 1\\
=&T^2\pi_1\otimes d\otimes T^2\pi_1;T\nu\otimes T\pi_1\otimes\pi_2\otimes T\pi_1\otimes T\nu;\nu^{\otimes 2}\otimes 1\otimes\nu^{\otimes 2};1\otimes\bullet_M\otimes 1\\
=&T\beta_1\otimes\beta_2\otimes T\beta_1;\nu\otimes 1\otimes\nu\\
=&T\beta\otimes\beta\otimes T\beta;T\pi_1\otimes\pi_2\otimes T\pi_1;\nu\otimes 1\otimes\nu
\end{aligned}$$

Precomposing with d and postcomposing with $\bullet_M$ thus gives:

$$\begin{aligned}
&d;T\beta\otimes\beta\otimes T\beta;T\pi_1\otimes\pi_2\otimes T\pi_1;\nu\otimes 1\otimes\nu;\bullet_M\\
=&T\beta;d;T\pi_1\otimes\pi_2\otimes T\pi_1;\bullet_M\\
=&T\beta;\beta_2
\end{aligned}$$

as desired. Finally, we check that $\eta; \beta_2 = \pi_2$:

$$\begin{aligned}\eta; \beta_2 =& \eta; d; T\pi_1 \otimes \pi_2 \otimes T\pi_1; \nu \otimes 1 \otimes \nu; \bullet_M \\ =& u; e \otimes 1 \otimes e; T\pi_1 \otimes \pi_2 \otimes T\pi_1; \nu \otimes 1 \otimes \nu; \bullet_M\end{aligned}$$

$$=u; 1 \otimes \pi_2 \otimes 1; e \otimes 1 \otimes e; \bullet_M$$
$$=\pi_2$$

∎

Definition 7. We denote this T-algebra by $W(A, M) =< A \oplus M, \beta^{AM} >$

It is evident that with such a T-algebra structure on $A \oplus M$, the map $\pi_1 : A \oplus M \longrightarrow A$ is a morphism of T-algebras. Hence $W(A, M)$ is manifestly an object in $\mathcal{C}^T/A$. Accordingly, we make the following pair of definitions, inspired by [1].

Definition 8.

- Let $< A, \nu >$ be a T-algebra. Let M be an A-module. A *Beck T-derivation* for A valued in M is a T-algebra map

$$A \longrightarrow W(A, M)$$

in the slice category $\mathcal{C}^T/A$.

- A *T-derivation* is a morphism $\partial : A \longrightarrow M$ such that

$$< 1, \partial >: A \longrightarrow A \oplus M$$

is a Beck T-derivation.

One may confirm that these two notions are in bijective correspondence with one another. The notions of *universal T-derivation* and *universal Beck T-derivations* are the obvious ones; the corresponding module is called the *module of T-differential forms* and for a T-algebra $< A, \nu >$, it is denoted $\Omega^T_{A,\nu}$, or Ω^T_A when no ambiguity will occur.

At this point it should be obvious that the strategy pursued is to utilize the freeness of $TC \otimes C \otimes TC$ in order to factor TC derivations after precomposition with η_C and then to use the universal property of η_C along with the correspondence between T-derivations and T-algebra morphisms. Before we can do this though, we need to show that the canonical derivations d_T are themselves T-derivations. This is accomplished by the following series of propositions.

Proposition 9. *Let $< A, \nu >$ be a T-algebra, and let M be an A-bimodule. A morphism $\partial : A \longrightarrow M$ is a T-derivation if and only if the following diagram commutes.*

$$\begin{tikzcd} TA \arrow[r, "T\begin{pmatrix}1_A \\ \partial\end{pmatrix}"] \arrow[d, "\nu"'] & T(A \oplus M) \arrow[d, "\beta_2"] \\ A \arrow[r, "\partial"'] & M \end{tikzcd}$$

Proof: The requirement that $< 1_A; \partial >: A \longrightarrow A \oplus M$ is a T-algebra morphism is fulfilled when the given diagram is commutative, and additionally, when the following diagram is commutative.

$$\begin{tikzcd} TA \arrow[r, "T\begin{pmatrix}1_A\\ \partial\end{pmatrix}"] \arrow[d, "\nu"'] & T(A \oplus M) \arrow[d, "\beta_1"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

Since

$$T\begin{pmatrix}1_A\\ \partial\end{pmatrix}; \beta_1 = T\begin{pmatrix}1_A\\ \partial\end{pmatrix}; T\pi_1; \nu$$
$$= T(1_A); \nu = \nu$$

the result follows. ∎

Proposition 10. *Let $< A, \nu >$ be a T-algebra, and let M be an A-bimodule. A morphism $\partial : A \longrightarrow M$ is a T-derivation if and only if the following diagram commutes.*

$$\begin{tikzcd} TA \arrow[rr, "\nu"] \arrow[d, "d"'] & & A \arrow[d, "\partial"] \\ TA \otimes A \otimes TA \arrow[r, "\nu \otimes \partial \otimes \nu"'] & A \otimes M \otimes A \arrow[r, "\bullet_M"'] & M \end{tikzcd}$$

Proof: Given this premise, consider $T\begin{pmatrix}1_A\\ \partial\end{pmatrix}; \beta_2$

$$= T\begin{pmatrix}1_A\\ \partial\end{pmatrix}; d; T\pi_1 \otimes \pi_2 \otimes T\pi_1; \nu \otimes 1 \otimes \nu; \bullet_M$$
$$= d; T\begin{pmatrix}1_A\\ \partial\end{pmatrix} \otimes \begin{pmatrix}1_A\\ \partial\end{pmatrix} \otimes T\begin{pmatrix}1_A\\ \partial\end{pmatrix}; T\pi_1 \otimes \pi_2 \otimes T\pi_1; \nu \otimes 1 \otimes \nu; \bullet_M$$
$$= d; \nu \otimes \partial \otimes \nu; \bullet_M$$

and the result follows from the previous proposition. ∎

It immediately follows from the chain rule that the morphisms, d^T, in a codifferential category are T-derivations. We now have the following:

Theorem 11. *For any object C in a codifferential category $\mathcal{C}$, d^T_{TC} is a universal T-derivation; consequently, $TC \otimes C \cong \Omega^T_{TC}$*

Given the discussion at the beginning of the chapter, an informal explanation should suffice. The point is that T-derivations correspond to certain T-algebra morphisms, the Beck T-derivations. Hence if two T-derivations are equal when preceded by η^T, the corresponding Beck T-derivations are equal when preceded by η^T, and the universal property of η^T mandates that those Beck T-derivations must be equal. This means that the original T-derivations were equal.

It should be emphasized that the preceding theorem is indicative of the privileged position of the chain rule axiom. Without it, the axiomatic derivations in a codifferential category would not necessarily have any universal property.

With these universal T-derivations on the free T-algebras of a codifferential category at hand, one may construct universal T-derivations for any T-algebra, as in the following.

Theorem 12. *Let $< A, \nu >$ be a T-algebra in a codifferential category. Then the module of T-differential forms for A, denoted Ω^T_A is constructed as the following coequalizer:*

$$\begin{tikzcd} \Omega^T_{T^2A} \arrow[r, shift left, "\Omega^T_{\mu^T}"] \arrow[r, shift right, "\Omega^T_{T\nu}"'] & \Omega^T_{TA} \arrow[r, "\Omega^T_\nu"] & \Omega^T_{<A,\nu>} \end{tikzcd}$$

Here we have used the fact that precomposing a T-derivation with a morphism of T-algebras, f, yields another T-derivation, along with the universal property of d^T to construct the morphisms of the form Ω^T_f.

The next theorem provides a very convenient way to view codifferential categories.

Theorem 13. *Let $\mathcal{C}$ be an additive symmetric monoidal category for which the free associative algebra monad $\mathcal{F}$ exists. Assume that $\mathcal{C}$ has reflexive coequalizers and that these are preserved by the tensor product. Then $\mathcal{C}$ is a codifferential category if*

i $\mathcal{C}$ *is equipped with a monad* T

ii there exists a monad morphism $\lambda : \mathcal{F} \longrightarrow T$

iii for each $C \in Ob(\mathcal{C})$ *there is a transformation* $d^T_{TC} : TC \longrightarrow TC \otimes C \otimes TC$ *such that*

a. the diagram

$$\begin{tikzcd} \mathcal{F}C \arrow[r, "\lambda_C"] \arrow[d, "d^F"'] & TC \arrow[d, "d^T"] \\ \mathcal{F}C \otimes C \otimes \mathcal{F}C \arrow[r, "\lambda_C \otimes 1_C \otimes \lambda_C"'] & TC \otimes C \otimes TC \end{tikzcd}$$

commutes

b. *The chain rule axiom holds for* d^T

The crux of the matter is the observation that given these conditions, any T-derivation is an $\mathcal{F}$-derivation. This ensures that axioms **d1** and **d2** hold. To this end, note that any T-algebra $< A, \nu >$ inherits an $\mathcal{F}$-algebra structure $< A, \lambda_A; \nu >$. If $d^T_A : A \longrightarrow M$ is a T-derivation, we have

$$\begin{aligned}
&\lambda_A; \nu; d^T \\
=&\lambda_A; d^T; \nu \otimes d^T \otimes \nu; \bullet_M \\
=&d^F; \lambda_A \otimes 1 \otimes \lambda_A; \nu \otimes d^T \otimes \nu; \bullet_M \\
=&d^F; (\lambda_A; \nu) \otimes d^T \otimes (\lambda_A; \nu); \bullet_M
\end{aligned}$$

as required. Axiom **d3** is established similarly. Axiom **d4** is provided in the premise of the above theorem precisely because this type of strategy will not work. d^T does not inherit adherence to the given chain rule from $\mathcal{F}$; this is what distinguishes it as a T-derivation.

Again, the formal proof has been foregone. For the full details, one may see [4] for the proof of the following analogous result, which is also important for the present work.

Theorem 14. *Let $\mathcal{C}$ be an additive symmetric monoidal category for which the symmetric algebra monad $\mathcal{S}$ exists. Assume that $\mathcal{C}$ has reflexive coequalizers and that these are preserved by the tensor product. Then $\mathcal{C}$ is a commutative codifferential category if*

i *$\mathcal{C}$ is equipped with a monad T*

ii *there exists a monad morphism $\lambda : \mathcal{S} \longrightarrow T$*

iii *for each $C \in Ob(\mathcal{C})$ there is a transformation $d^T_{TC} : TC \longrightarrow TC \otimes C$ such that*

a. *the diagram*

$$\begin{tikzcd}
\mathcal{S}C \arrow[r, "\lambda_C"] \arrow[d, "d^S"'] & TC \arrow[d, "d^T"] \\
\mathcal{S}C \otimes C \arrow[r, "\lambda_C \otimes 1_C"'] & TC \otimes C
\end{tikzcd}$$

commutes

b. *The chain rule axiom holds for* d^T

3.3 n-forms and De Rham Cohomology

In differential geometry, de Rham cohomology utilizes easily computable aspects of the differential structure of a smooth manifold in order to derive global properties. Namely, one builds a complex out of the higher differential forms on the manifold and computes the homology of the complex. Because such higher differential forms - or *n-forms* - are built from copies of the modules of differential forms and little else, the machinery required to compute an algebraic version of de Rham Cohomology is readily available to algebraic geometers. Here it will be shown that the n-forms and de Rham cohomology for a T-algebra in a codifferential category have a simple description, and furthermore that the construction of the module of n-forms for a T-algebra closely resembles the construction of its module of differential forms.

It can be observed that both the construction of de Rham cohomology and the subsequent introduction to the theory of connections described here are not pertinent to the main line of development. There are two reasons for their inclusion: first, it was necessary to develop a convenient description of the n-forms and it is interesting to observe how easily the other constructions follow from this description; second, the theory of de Rham cohomology is closely connected to cyclic homology ([17] Theorem 3.4.12, p. 106) and future work may require this additional development.

For each T-algebra $< A, \nu >$ in a commutative codifferential category, we may define the *module of T-differential n-forms* by

$$\Omega_A^{(n)T} := \bigwedge_A^n \Omega_A^T$$

If no ambiguity will result, we will simply denote these by Ω_A^n. As a first step towards a convenient construction for the n-forms of a T-algebra, we prove the following

Proposition 15. *Let $< A, \nu >$ be a T-algebra in a codifferential category. Then for any n the following diagram is a reflexive coequalizer diagram:*

$$\begin{tikzcd} \Omega_{T^2A}^{\otimes^n_{T^2A}} \arrow[r, shift left, "\Omega_\mu^{\otimes^n_{T^2A}}"] \arrow[r, shift right, "\Omega_{T\nu}^{\otimes^n_{T^2A}}"'] & \Omega_{TA}^{\otimes^n_{TA}} \arrow[r, "\Omega_\nu^{\otimes^n_{TA}}"] & \Omega_{A,\nu}^{\otimes^n_A} \end{tikzcd}$$

Proof: This is an immediate consequence of **3.2.12** and **2.1.1** ∎

Now, consider the following diagram:

$$
\begin{tikzcd}
\Omega^{\otimes^n_{T^2A}}_{T^2A} \arrow[r, shift left, "\Omega_\mu^{\otimes^n_{T^2A}}"] \arrow[r, shift right, "\Omega_{T\nu}^{\otimes^n_{T^2A}}"'] \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "\cdots\quad -1\otimes 1\otimes\ldots\otimes c"] & \Omega^{\otimes^n_{TA}}_{TA} \arrow[r, "\Omega_\nu^{\otimes^n_{TA}}"] \arrow[d, shift right=2] \arrow[d, shift left=2, "\cdots\quad -1\otimes 1\otimes\ldots\otimes c"] & \Omega^{\otimes^n_A}_{A,\nu} \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "\cdots\quad -1\otimes 1\otimes\ldots\otimes c"] \\
\Omega^{\otimes^n_{T^2A}}_{T^2A} \arrow[r, shift left, "\Omega_\mu^{\otimes^n_{T^2A}}"] \arrow[r, shift right, "\Omega_{T\nu}^{\otimes^n_{T^2A}}"'] \arrow[d] & \Omega^{\otimes^n_{TA}}_{TA} \arrow[r, "\Omega_\nu^{\otimes^n_{TA}}"] \arrow[d] & \Omega^{\otimes^n_A}_{A,\nu} \arrow[d] \\
\bigwedge^n_{T^2A}\Omega_{T^2A} \arrow[r, shift left, "\bigwedge^n_{T^2A}\Omega_\mu"] \arrow[r, shift right, "\bigwedge^n_{T^2A}\Omega_{T\nu}"'] & \bigwedge^n_{TA}\Omega_{TA} \arrow[r, "\bigwedge^n_{TA}\Omega_\nu"] & \bigwedge^n_A\Omega_{A,\nu}
\end{tikzcd}
$$

All of the obvious squares commute, each column is a coequalizer diagram, and the top two rows are coequalizers. It follows that the bottom row is a coequalizer. We have just proven

Theorem 16. *Let $< A,\nu >$ be a T-algebra in a codifferential category. Then the following diagram is a coequalizer:*

$$
\begin{tikzcd}
\Omega^n_{T^2A} \arrow[r, shift left, "\Omega^n_\mu"] \arrow[r, shift right, "\Omega^n_{T\nu}"'] & \Omega^n_{TA} \arrow[r, "\Omega^n_\nu"] & \Omega^n_{A,\nu}
\end{tikzcd}
$$

When we consider a free algebra $< TA,\mu_A >$ in a codifferential category whose tensor product preserves arbitrary coequalizers, we may arrive at a more explicit representation of Ω^n_{TA} as shown in the following diagram

$$
\begin{tikzcd}
(TA\otimes A)^{\otimes^n_{TA}} \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "-1\otimes 1\otimes\ldots\otimes c"'] \arrow[d, "\cong"'] & (TA\otimes A)^{\otimes^n_{TA}} \arrow[r] \arrow[d, "\cong"'] & \bigwedge^n_{TA}(TA\otimes A) \arrow[d, "\cong"'] \\
TA\otimes A^{\otimes^n} \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "-1_{TA}\otimes 1\otimes 1\otimes\ldots\otimes c"'] & TA\otimes A^{\otimes^n} \arrow[r] & TA\otimes\bigwedge^n A
\end{tikzcd}
$$

Having expressed the higher n-forms in this way, we are properly equipped to define de Rham cohomology in a codifferential category. This, of course, requires the definition of a series of maps $\delta^n_A : \Omega^n_{A,\nu} \longrightarrow \Omega^{n+1}_{A,\nu}$ for each T-algebra $< A,\nu >$. This will be borne out somewhat indirectly and will make essential use of our explicit representation of Ω^n_{TA}; it is unclear how one would proceed otherwise.

Consider the following diagram

$$\begin{tikzcd}
TA \otimes A^{\otimes n} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1_{TA}\otimes 1\otimes 1\otimes \ldots \otimes c"', "\vdots"] & TA \otimes A^{\otimes n} \arrow[r, "p_n"] \arrow[d, "d_{TA}\otimes 1^{\otimes n}"'] & TA \otimes \bigwedge^n A \\
TA \otimes A^{\otimes n+1} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1_{TA}\otimes 1\otimes 1\otimes \ldots \otimes c"', "\vdots"] & TA \otimes A^{\otimes n+1} \arrow[r, "p_{n+1}"] & TA \otimes \bigwedge^{n+1} A
\end{tikzcd}$$

That all of the maps emanating from the upper left hand corner and terminating through p_{n+1} are equal is true upon inspection. Hence there is a unique map $\delta^n_{TA} : TA \otimes \bigwedge^n A \longrightarrow TA \otimes \bigwedge^{n+1} A$ such that $p_n; \delta^n_{TA} = d_{TA} \otimes 1^{\otimes^n}; p_{n+1}$.

Observe that this definition of a boundary map for de Rham cohomology is closely modeled after the classical algebraic de Rham cohomology boundary map, which takes an n-form $adx_1 \bigwedge dx_2 \bigwedge ... \bigwedge dx_n$ to $da \bigwedge dx_1 \bigwedge ... \bigwedge dx_n$([9] p. 417). This formal similarity notwithstanding, we require the following lemma:

Lemma 17. $\delta_{TA}; \delta_{TA} = 0$

Proof: We may examine the following commutative diagram:

$$\begin{tikzcd}
TA \otimes A^{\otimes n} \arrow[r, "p_n"] \arrow[d, "d_{TA}\otimes 1^{\otimes^n}"'] & TA \otimes \bigwedge^n A \arrow[d, "\delta^n_{TA}"] \\
TA \otimes A^{\otimes n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{TA}\otimes 1^{\otimes^{n+1}}"'] & TA \otimes \bigwedge^{n+1} A \arrow[d, "\delta^{n+1}_{TA}"] \\
TA \otimes A^{\otimes n+2} \arrow[r, "p_{n+2}"] & TA \otimes \bigwedge^{n+2} A
\end{tikzcd}$$

Since p_n is an epimorphism, if it can be shown that the counterclockwise composite of the large rectangle is 0, the lemma will follow. To this end, consider the precedence of this composite by $m_{TA} \otimes 1_A^{\otimes^n}$ - an epimorphism.

It is sufficient to consider the case where $n = 0$. That is, consider the map $m_{TA}; d_{TA}; d_{TA} \otimes 1_A : TA \otimes TA \longrightarrow TA \otimes A \otimes A$. It is convenient to label each instance of A that arises. Two applications of the Leibniz rule produce the following sequence of maps:

$$\begin{array}{c} TA_1 \otimes TA_2 \\ \downarrow{\scriptstyle 1\otimes d_{TA}+c;1\otimes d_{TA}} \\ TA_1 \otimes TA_2 \otimes A_2 + TA_2 \otimes TA_1 \otimes A_1 \\ \downarrow{\scriptstyle (1\otimes d_{TA}\otimes 1+c\otimes 1;1\otimes d_{TA}\otimes 1);m_{TA}\otimes 1_{A\otimes A}} \\ TA_3 \otimes A_2^{\otimes^2} + TA_3 \otimes A_1 \otimes A_2 + TA_3 \otimes A_1^{\otimes_2} + TA_3 \otimes A_2 \otimes A_1 \end{array}$$

Here TA_3 arises from the multiplication of TA_1 and TA_2; commutativity of the algebra makes their order irrelevant. It is now evident that postcomposition of $m_{TA}; d_{TA}; d_{TA} \otimes 1_A$ with $1 \otimes c$ changes nothing. Therefore the composite $m_{TA} \otimes 1_A^{\otimes^n}; d_{TA} \otimes 1_A^{\otimes^n}; d_{TA} \otimes 1_A^{\otimes^{n+1}}; p_{n+2}$ is 0. Since $m_{TA} \otimes 1_A^{\otimes^n}$ is epi, the result follows. ∎

As an immediate corollary, we may construct the de Rham boundary map δ^n_A for a T-algebra $< A, \nu >$ via the following diagram:

$$\begin{tikzcd}
\Omega^n_{T^2A} \arrow[r, shift left, "\Omega^n_\mu"] \arrow[r, shift right, "\Omega^n_{T\nu}"'] \arrow[d, "\delta^n_{T^2A}"'] & \Omega^n_{TA} \arrow[r, "\Omega^n_\nu"] \arrow[d, "\delta^n_{TA}"'] & \Omega^n_{A,\nu} \arrow[d, dashed, "\delta^n_A"'] \\
\Omega^{n+1}_{T^2A} \arrow[r, shift left, "\Omega^{n+1}_\mu"] \arrow[r, shift right, "\Omega^{n+1}_{T\nu}"'] & \Omega^{n+1}_{TA} \arrow[r, "\Omega^{n+1}_\nu"] & \Omega^{n+1}_{A,\nu}
\end{tikzcd}$$

Since

$$\begin{aligned} \Omega^n_\nu; \delta^n_A; \delta^{n+1}_A &= \delta_{TA}; \Omega^{n+1}_\nu; \delta^{n+1}_A \\ &= \delta^n_{TA}; \delta^{n+1}_{TA}; \Omega^{n+2}_\nu \\ &= 0 \end{aligned}$$

it follows that $\delta_A; \delta_A = 0$

3.3.1 Connections

One benefit of the introduction of de Rham cohomology in codifferential categories is the possibility of also defining *connections* in this context. In this brief section, both connections and *curvature* are defined in a familiar way ([17], 8.1), which only requires the definitions of n-forms and the de Rham boundary map given in the previous section.

Definition 18. *Let $< A, \nu >$ be a T-algebra and M an A-module. A* connection *on M is a linear map:*

$$\nabla^n : M \otimes_A \Omega^n_A \longrightarrow M \otimes_A \Omega^{n+1}_A$$

defined for all $n \geq 0$ and such that the following diagram commutes:

$$\begin{tikzcd}
M \otimes_A \Omega^p_A \bigwedge_A \Omega^q_A \arrow[r, "\nabla^p \bigwedge_A id + (-1)^p id \bigwedge_A \delta^q_A"] \arrow[dr, "\nabla^{p+q}"'] & M \otimes_A \Omega^{p+q+1}_A \arrow[d, "id"] \\
& M \otimes_A \Omega^{p+q+1}_A
\end{tikzcd}$$

Proposition 19. *For any connection $\nabla : M \otimes_A \Omega^*_A \longrightarrow M \otimes_A \Omega^{*+1}_A$ and any p, q, we have*

$$\nabla^{p+q}; \nabla^{p+q+1} = \nabla^p \bigwedge_A id; \nabla^{p+1} \bigwedge_A id$$

When restricted to M, this composite is denoted $R : M \longrightarrow M \otimes_A \Omega^2_A$ and is called the *curvature* of ∇.

Proof:

$$\begin{aligned}
\nabla^{p+q}; \nabla^{p+q+1} &= (\nabla^p \bigwedge_A id + (-1)^p id \bigwedge_A \delta^q_A); \nabla^{p+q+1} \\
&= (\nabla^p; \nabla^{p+1}) \bigwedge_A id + (-1)^{p+1} \nabla^p \bigwedge_A \delta^q + (-1)^p \nabla^p \bigwedge_A \delta^q + id \bigwedge_A (\delta^q; \delta^{q+1}) \\
&= (\nabla^p; \nabla^{p+1}) \bigwedge_A id
\end{aligned}$$

∎

Chapter 4

Properties of Algebra Modalities

Because of the chain rule, the T-derivations in a codifferential category are inextricably linked to the monad T. To clarify, although most of the axioms for T-derivations are satisfied as a consequence of the relationship between T and $\mathcal{S}$ (or T and $\mathcal{F}$ in noncommutative codifferential categories), the chain rule axiom must be established independently. This is evinced by comparing the two equivalent definitions of codifferential categories given in the previous chapter.

It must be observed, however, that in the case where T satisfies Property K - that is, when there is a surjection of monads $\mathcal{F} \twoheadrightarrow T$ - it can be demonstrated that any $\mathcal{S}$-derivation from a T-algebra satisfies the chain rule associated to T. Hence, the essential structure of a codifferential category may be determined, at least partially, by the relationship between its algebra modality T and the algebra modalities $\mathcal{F}$, $\mathcal{S}$, or any other monad which may exist in the codifferential category or closely related codifferential categories.

Apropos to this, in the following sections various properties of monads and their relevance to differential structure are investigated. In the first section, essential properties of the symmetric algebra monad are axiomatized, which enable one to deduce a specialized HKR-type theorem for the free algebras of a monad which adheres to the axioms. Such monads are called *smooth monads*, and occupy a middle-ground between general algebra modalities, and monads with property K.

In the second section of this chapter, a connection between noncommutative codifferential categories and commutative codifferential categories is elucidated. An example of this phenomenon occurs between categories with the free associative algebra monad, and corresponding categories with the symmetric algebra monad. It is shown that $\mathcal{S}$ inherits some of its smoothness properties from its relationship with $\mathcal{F}$.

4.1 Smooth Monads

The inspiration for the following set of axioms comes primarily from the proof that for any flat k-module, V, $HH_n(\mathcal{S}V) \cong \Omega^n_{\mathcal{S}V/k}$ for all n, found in [17]. The utility of these axioms lies in the resultant ability to generalize from the finite to the infinite, as will be demonstrated in the proof of the following theorem. Such a strategy is ubiquitous in proofs of HKR-type theorems; one may observe the proof of the Connes-Hochschild-Kostant-Rosenberg theorem as in Theorem 8.17, p. 363, [11] where similar properties are utilized.

Definition 1. An algebra modality, T, in a codifferential category $\mathcal{C}$ is said to be *smooth* if the following conditions hold:

i $\forall A, B \in Ob(\mathcal{C})$, $TA \otimes TB \cong T(A \oplus B)$ via the canonical map

$$\phi := Ti_0 \otimes Ti_1; m_{T(A\oplus B)}$$

ii T preserves filtered colimits

iii T preserves flatness

iv The following sequence is exact

$$0 \longrightarrow Tk \otimes k \otimes Tk \xrightarrow{1\otimes\eta\otimes 1;(m\otimes 1-1\otimes m)} Tk \otimes Tk \xrightarrow{m} Tk \longrightarrow 0$$

Remark 2. Observe the formal similarity between condition iv) and the exact sequence which defined $\Omega^{NC}_{FA/k}$ in section **3.1**. When $T = S$, this property is satisfied because $\mathcal{S}k \cong \mathcal{F}k$.

Theorem 3. *Let $\mathcal{C}$ be a codifferential category with small filtered colimits in which filtered colimits preserve exactness - that is, the filtered colimit of an exact sequence is an exact sequence - and the tensor product preserves colimits in each variable. Let A be a flat k-module, and T a smooth algebra modality. Then for all n there is a natural isomorphism $HH_n(TA) \cong \Omega^{n_T}_{TA}$.*

The proof of this theorem will follow the proof of the analogous proposition, which is specific to $\mathcal{S}$, as in Theorem 3.2.2, p. 94, [17]. Here, the properties which enable one to demonstrate the above isomorphism have been isolated and encapsulated in the definition of smooth monads. First, this exemplifies one of the guiding principles of the present work: that the HKR theorem is an appropriate yardstick for smoothness. Second, it highlights the role that each of the putative smoothness properties play in the analysis of smoothness.

To foster a more detailed analysis while at the same time maintaining coherent articulation, the proof will be broken into a series of lemmas. First, we require a construction.

For any k-module V in a commutative codifferential category with algebra modality T, and any n, consider the series of maps $TV \otimes V^{\otimes n} \otimes TV \longrightarrow TV \otimes V^{\otimes n-1} \otimes TV$ given by

$$\delta_i = 1_{TV} \otimes 1_V^{\otimes i} \otimes \eta_V^T \otimes 1_V^{\otimes n-i-1} \otimes 1_{TV}; (c; m_{TV} \otimes 1_V^{\otimes n-1} \otimes 1_{TV} - c; 1_{TV} \otimes 1_V^{\otimes n-1} \otimes m_{TV})$$

where $0 \leq i \leq n-1$ and the c maps are the appropriate symmetry maps. This is most easily deciphered when $n = 1$ and

$$\delta_0 = 1_{TV} \otimes \eta_V^T \otimes 1_{TV}; (m_{TV} \otimes 1_{TV} - 1_{TV} \otimes m_{TV})$$

That is, first we insert V into TV via η^T, and then we take the difference between multiplying it with the TV on the left and with that on the right. For larger values of n, each δ_i performs an analogous operation using the $i+1^{st}$ instance of V.

Lemma 4. *The maps δ_i endow the sequence of objects $TV \otimes V^{\otimes n} \otimes TV$ with pres-implicial structure.*

This follows by noticing that if $i < j$, the j^{th} entry in $V^{\otimes n}$ shifts to the $(j-1)^{st}$ position after the action of δ_i. One must also invoke commutativity of the algebra TV. Hence, if we define

$$\delta = \sum_{i=0}^{n-1} (-1)^i \delta_i$$

then the following diagram is a complex.

$$\ldots \longrightarrow TV \otimes V^{\otimes n} \otimes TV \xrightarrow{\delta} TV \otimes V^{\otimes n-1} \otimes TV \to \ldots$$
$$\ldots \longrightarrow TV \otimes V \otimes TV \xrightarrow{\delta} TV \otimes TV$$

Furthermore, observe that if we postcompose δ with the projection onto $TV \otimes \wedge^n V \otimes TV$, p_n, then for $1 \leq i \leq n-1$

$$\delta; p_n = -\tau_i; \delta; p_n$$

where τ_i is the transposition of the i^{th} instance of V with the $(i+1)^{st}$. This should be confirmed scrupulously for the case $n = 2$, but the remainder of cases follow easily thereafter. It follows that there is an induced map

$$\delta : TV \otimes \wedge^n V \otimes TV \longrightarrow TV \otimes \wedge^{n-1} V \otimes TV$$

which makes

$$\ldots \longrightarrow TV \otimes \wedge^n V \otimes TV \xrightarrow{\delta} TV \otimes \wedge^{n-1} V \otimes TV \to \ldots$$
$$\ldots \longrightarrow TV \otimes V \otimes TV \xrightarrow{\delta} TV \otimes TV$$

a complex. We denote this complex by $C^{sm}_*(TV)$

Lemma 5. *If T is a smooth monad, $C^{sm}_*(T(V \oplus W)) \cong C^{sm}_*(TV) \otimes C^{sm}_*(TW)$*

Proof: First note that for any n

$$\begin{aligned} & T(V \oplus W) \otimes \wedge^n (V \oplus W) \otimes T(V \oplus W) \\ & \cong TV \otimes TW \otimes \bigoplus_{p+q=n} (\wedge^p V \otimes \wedge^q W) \otimes TV \otimes TW \\ & \cong \bigoplus_{p+q=n} C^{sm}_p(TV) \otimes C^{sm}_q(TW) \end{aligned}$$

It remains to be shown that such isomorphisms induce chain maps. That is, that for all n the following diagram commutes

$$\begin{tikzcd} C^{sm}_n(T(V \oplus W)) \arrow[r, "\delta^{V \oplus W}"] \arrow[d, "\cong"'] & C^{sm}_{n-1}(T(V \oplus W)) \arrow[d, "\cong"] \\ \bigoplus_{p+q=n} C^{sm}_p(TV) \otimes C^{sm}_q(TW) \arrow[r, "\delta^{V \otimes W}"'] & \bigoplus_{r+s=n-1} C^{sm}_r(TV) \otimes C^{sm}_s(TW) \end{tikzcd}$$

where $\delta^{V \otimes W}$ is the usual differential map bestowed on the tensor product of two complexes.

First, consider the case where $n = 1$. On the summand

$$TV \otimes V \otimes TV \otimes TW \otimes TW$$

we have

$$\begin{aligned} & \delta^{V \otimes W}; 1 \otimes c \otimes 1; \phi \otimes \phi \\ = & \delta^V_0 \otimes 1_{TW} \otimes 1_{TW}; 1 \otimes c \otimes 1; \phi \otimes \phi \\ = & \delta^V_0 \otimes 1_{TW} \otimes 1_{TW}; 1 \otimes c \otimes 1; (Ti_0 \otimes Ti_1; m_{T(V \oplus W)})^{\otimes 2} \\ = & \delta^V_0 \otimes 1_{TW} \otimes 1_{TW}; Ti_0^{\otimes 2} \otimes Ti_1^{\otimes 2}; 1 \otimes c \otimes 1; m^{\otimes 2}_{T(V \oplus W)} \end{aligned}$$

To examine this further, we have

$$\delta^V_0; Ti_0 \otimes Ti_0$$

$$
\begin{aligned}
&= 1_{TV} \otimes \eta_V \otimes 1_{TV}; (m_{TV} \otimes 1_{TV} - 1_{TV} \otimes m_{TV}); Ti_0 \otimes Ti_0 \\
&= 1_{TV} \otimes \eta_V \otimes 1_{TV}; Ti_0 \otimes Ti_0 \otimes Ti_0; (m_{T(V\oplus W)} \otimes 1 - 1 \otimes m_{T(V\oplus W)}) \\
&= Ti_0 \otimes i_0 \otimes Ti_0; 1 \otimes \eta_{V\oplus W} \otimes 1; (m_{T(V\oplus W)} \otimes 1 - 1 \otimes m_{T(V\oplus W)}) \\
&= Ti_0 \otimes i_0 \otimes Ti_0; \delta^{V\oplus W}
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta^{V\oplus W} \otimes 1 \otimes 1; 1 \otimes c \otimes 1; m \otimes m \\
=&(1 \otimes \eta \otimes 1; (m \otimes 1 - 1 \otimes m)) \otimes 1 \otimes 1; 1 \otimes c \otimes 1; m \otimes m
\end{aligned}
$$

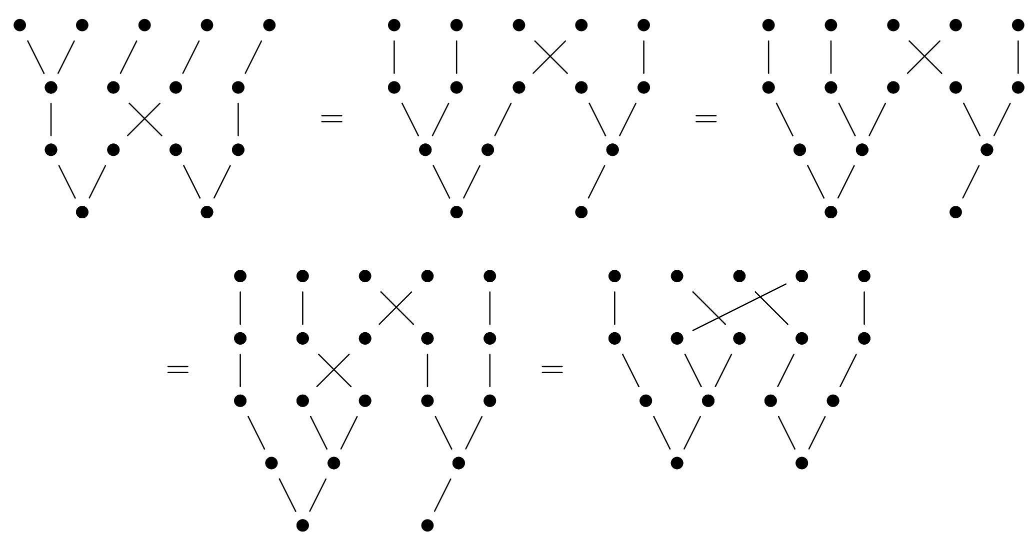
We may examine

$$m \otimes 1 \otimes 1 \otimes 1; 1 \otimes c \otimes 1; m \otimes m$$

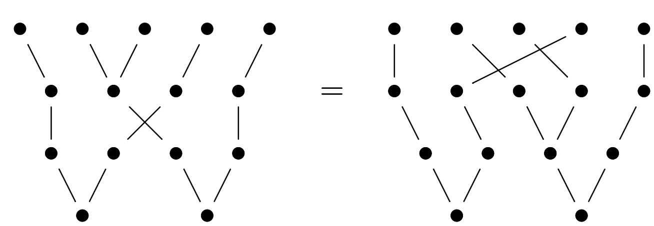
and

$$1 \otimes m \otimes 1 \otimes 1; 1 \otimes c \otimes 1; m \otimes m$$

using the following respective sequences of string diagrams. Here $\bullet$ represents an instance of $T(V \oplus W)$, each row represents a tensor product, the joining of wires represents a multiplication, and the crossing of wires represents an interchange of positions.



and for $1 \otimes m \otimes 1 \otimes 1; 1 \otimes c \otimes 1; m \otimes m$,



so that

$$\begin{aligned}
&(1\otimes\eta\otimes 1;(m\otimes 1-1\otimes m))\otimes 1\otimes 1;1\otimes c\otimes 1;m\otimes m\\
=&1\otimes 1\otimes c\otimes 1;1\otimes c\otimes 1\otimes 1;1\otimes(1\otimes\eta\otimes 1;(m\otimes 1-1\otimes m))\otimes 1;m\otimes m\\
=&1\otimes 1\otimes c\otimes 1;1\otimes c\otimes 1\otimes 1;1\otimes\delta^{V\oplus W}\otimes 1;m\otimes m\\
=&1\otimes 1\otimes c\otimes 1;1\otimes c\otimes 1\otimes 1;m\otimes 1\otimes m;\delta^{V\oplus W}
\end{aligned}$$

Hence

$$\begin{aligned}
&\delta^{V\otimes W};1\otimes c\otimes 1;\phi\otimes\phi\\
=&1\otimes 1\otimes c\otimes 1;1\otimes c\otimes 1\otimes 1;Ti_0\otimes Ti_1\otimes i_0\otimes Ti_0\otimes Ti_1;m\otimes 1\otimes m;\delta^{V\oplus W}\\
=&1\otimes 1\otimes c\otimes 1;1\otimes c\otimes 1\otimes 1;\phi\otimes i_0\otimes\phi;\delta^{V\oplus W}
\end{aligned}$$

as required. Similarly, on $TV\otimes TV\otimes TW\otimes W\otimes TW$, we find that

$$\begin{aligned}
&\delta^{V\otimes W};1\otimes c\otimes 1;\phi\otimes\phi\\
=&1\otimes c\otimes 1\otimes 1;1\otimes 1\otimes c\otimes 1;\phi\otimes i_1\otimes\phi;\delta^{V\oplus W}
\end{aligned}$$

In order to complete the proof, we may precompose the square in question with the epimorphic map

$$1\otimes p_n^{V\oplus W}\otimes 1 : T(V\oplus W)\otimes(V\oplus W)^{\otimes n}\otimes T(V\oplus W)\longrightarrow C_n^{sm}(T(V\oplus W))$$

The commutativity of the resultant square then follows exactly as if we were equipped with bases for our modules, but the proof is much more tedious here. Essentially, given the above calculation, one needs only to show that the signs of the relevant shuffle maps match up when performed either before or after the relevant δ maps. ∎

Note that the proof of the above lemma utilizes only property i) of smoothness for a monad.

Lemma 6. *If V is a free finite-dimensional k-module - i.e. a module of the form $\bigoplus_{i=1}^n k$, denoted here by k^n - and T is a smooth monad, then $C_*^{sm}(TV)$ is a finite, flat resolution of TV*

Proof: Since V is free and finite dimensional, we may assume that $V\cong I^n$ for some n. The lemma is true for $n=1$ by the definition of a nonsingular monad. Suppose then, that for any free I-module V' of dimension $<n$, the conclusion of the lemma holds. Then $C_*^{sm}(TV)\cong C_*^{sm}(T(k^n))\cong C_*^{sm}(T(k^{n-1}))\otimes C_*^{sm}(Tk)$ so that $C_*^{sm}(TV)$ is isomorphic to the tensor product of two flat resolutions. Explicitly, the

homology of this complex is the total homology of the following double complex:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \\
C_2^{sm}(T(k^{n-1}))\otimes Tk\otimes Tk \arrow[d] & C_2^{sm}(T(k^{n-1}))\otimes Tk\otimes k\otimes Tk \arrow[l] \arrow[d] & \dots \arrow[l] \\
C_1^{sm}(T(k^{n-1}))\otimes Tk\otimes Tk & C_1^{sm}(T(k^{n-1}))\otimes Tk\otimes k\otimes Tk \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Since k and Tk are both flat, taking the homology of each column results in a single non-zero row of the form

$$T(k^{n-1})\otimes Tk\otimes Tk \longleftarrow T(k^{n-1})\otimes Tk\otimes k\otimes Tk \longleftarrow 0$$

by hypothesis. Since $T(k^{n-1})$ is flat, this sequence is exact, and has 0^{th} homology group $T(k^{n-1})\otimes Tk \cong T(k^n)$. Since we have computed the total homology of the above bicomplex, the proof is complete. ∎

Of course, once we have a systematic way to produce flat resolutions, we have a means by which to compute Tor functors, and hence a convenient means by which to compute Hochschild homology. This is the method by which the crux of the above theorem is proven in the following:

Lemma 7. *If V is a free finite-dimensional k-module and T is a smooth monad in a codifferential category $\mathcal{C}$ in which the tensor product preserves coequalizers, then for all n $HH_n(TV)\cong \Omega^{nT}_{TV}$*

Proof: Since $HH_n(TV)\cong Tor_n^{TV\otimes TV}(TV,TV)$, the Hochschild homology of TV may be computed using the complex $C_*^{sm}(TV)\otimes_{TV\otimes TV} TV$, by the above lemma. Observe the following commutative diagram:

$$\begin{tikzcd}
TV\otimes\wedge^n V\otimes TV\otimes_{TV\otimes TV} TV \arrow[r, "\delta"] \arrow[d, "\cong"'] & TV\otimes\wedge^{n-1} V\otimes TV\otimes_{TV\otimes TV} TV \arrow[d, "\cong"] \\
TV\otimes\wedge^n V \arrow[r, "0"'] & TV\otimes\wedge^{n-1}V
\end{tikzcd}$$

Since this persists at every stage of the complex $C_*^{sm}(TV)\otimes_{TV\otimes TV} TV$, the result follows. ∎

Before proceeding with the proof of the theorem, we require one last, technical lemma

Lemma 8. *If T preserves filtered colimits, then so does Ω^{nT}*

Proof: Let $A = colim_i A_i$ be a filtered colimit of T-algebras over the filtered category I with maps $f_{i,j} : A_i \longrightarrow A_j$. Since T preserves filtered colimits, it follows that A inherits a T-algebra structure $< A, \alpha >$. Now, for each pair $\{i, j\}$, there exists a morphism of A_i-modules $\Omega^T_{A_i} \longrightarrow \Omega^T_{A_j}$. Let M be the filtered colimit $colim_i \Omega^T_{A_i}$. It will be proven that $M \cong \Omega^T_A$.

First, consider the category consisting of objects $TA_i \otimes A_j$ and maps of the form $Tf_{i,k} \otimes f_{j,l} : TA_i \otimes A_j \longrightarrow TA_k \otimes A_l$. It can be shown using the filtration properties of I that the full subcategory consisting of objects $TA_i \otimes A_i$ is final in the larger category, and hence $colim_i TA_i \otimes colim_j A_j = TA \otimes A$ is the filtered colimit $colim_i TA_i \otimes A_i$. Similarly, $T^2 A \otimes TA$ is the filtered colimit $colim_i T^2 A_i \otimes TA_i$. Since Ω^T_A may be constructed via the following coequalizer

$$T^2A \otimes TA \underset{\Omega^T_{T\alpha}}{\overset{\Omega^T_\mu}{\rightrightarrows}} TA \otimes A \xrightarrow{\Omega^T_\alpha} \Omega^T_A$$

the result follows immediately from the commutativity of colimits. ∎

Now, for the first time explicitly using property ii) of smoothness, we have the following:

Proof: (**Proof of Theorem 3**)
Since flat k-modules are filtered colimits of free finite-dimensional k-modules, and T, HH_n, and Ω^{nT} each preserve filtered colimits, the result is immediate. ∎

For the purposes of the above material, the condition that T preserves flatness could be relaxed to something like "Tk is flat." The reason to dismiss this added generality is that it is desirable that the above isomorphism is preserved when we apply T again. That is, an immediate corollary of the above theorem is

Corollary 9. *Given the premises of the above theorem, for any i and any n, $HH_n(T^i V) \cong \Omega^{nT}_{T^i V}$*

4.2 Symmetrization of an Algebra Modality

Here we commence with a general lemma regarding the construction of a commutative algebra from an associative algebra. It is important to understand that this process is distinct from the usual abelianization of a multiplicative structure. Here, we take a coequalizer in a category which includes objects other than algebras.

Lemma 10. *Let $\mathcal{C}$ be a symmetric monoidal category in which $\otimes$ preserves reflexive coequalizers in both variables. Let A be an associative algebra, and let the following diagram be a coequalizer:*

$$A^{\otimes 3} \underset{1\otimes m_A;m_A}{\overset{1\otimes c;1\otimes m_A;m_A}{\rightrightarrows}} A \xrightarrow{q} C$$

Then this diagram is moreover a reflexive coequalizer, C is a commutative algebra, and for any algebra morphism $f : A \longrightarrow X$ where X is a commutative algebra there is a unique algebra morphism $g : C \longrightarrow X$ which factors f through q.

Before the commencement of the proof, a note on the statement of this lemma is probably in order. One may wonder whether it is possible to formulate C as the following coequalizer

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, shift left, "c;m_A"] \arrow[r, shift right, "m_A"'] & A \arrow[r, "q"] & C \end{tikzcd}$$

This is tempting, however the means by which one would produce even an A-module structure on C are dubious. For instance, the obvious and, perhaps, only strategy to pursue is to consider the following diagram:

$$\begin{tikzcd} A \otimes A^{\otimes 2} \arrow[r, shift left, "1\otimes c;1\otimes m_A"] \arrow[r, shift right, "1\otimes m_A"'] & A \otimes A \arrow[r, "1\otimes q"] \arrow[dr, "m_A;q"'] & A \otimes C \arrow[d, dashed] \\ & & C \end{tikzcd}$$

The top row is a coequalizer because we assume that the tensor product preserves reflexive coequalizers. If it can be shown that the precomposition of the diagonal map by either of the two left maps is the same, then there is a unique map $A \otimes C \longrightarrow C$ which makes the triangle commute. Although this may appear to be the case, consider the following analysis, which uses elemental notation for the sake of brevity.

If $a\otimes b\otimes c \in A\otimes A^{\otimes 2}$, then the two left-hand maps send this element, respectively, to $a \otimes cb$ and $a \otimes bc$. The diagonal map then inserts, respectively, acb and abc into C via q. Hence, we must show that $q(abc) = q(acb)$ always.

We only know that q does not distinguish between precomposition by $c; m_A$ and precomposition by m_A. We may quickly tabulate the strings of equalities that this prescribes:

$$\begin{tikzcd}[column sep=small] & & & q(abc) \arrow[dll, dash, "="'] \arrow[drr, dash, "="] & & & \\ & q(cab) \arrow[dl, dash, "="'] \arrow[dr, dash, "="] & & & & q(bca) \arrow[dl, dash, "="'] \arrow[dr, dash, "="] & \\ q(bca) & & q(abc) & & q(abc) & & q(cab) \end{tikzcd}$$

It appears we have encountered an insurmountable impasse. This suggests that q must inherit knowledge of associativity; the formulation in the lemma accomplishes this, as will be demonstrated in the following.

Proof: First, it is clear that this coequalizer is reflexive, with the two left maps having a mutual left inverse in $u_R; u_R; 1 \otimes e_A \otimes e_A$. Since $\otimes$ preserves reflexive coequalizers, the top row of the following diagram is a coequalizer

$$\begin{tikzcd}
A \otimes A^{\otimes 3} \arrow[r, shift left, "1\otimes 1\otimes c;1\otimes 1\otimes m_A;1\otimes m_A"] \arrow[r, shift right, "1\otimes 1\otimes m_A;1\otimes m_A"'] \arrow[d, "m_A\otimes 1\otimes 1"'] & A \otimes A \arrow[r, "1\otimes q"] \arrow[d, "m_A"'] & A \otimes C \arrow[d, dashed, "\bullet"] \\
A^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_A;m_A"] \arrow[r, shift right, "1\otimes m_A;m_A"'] & A \arrow[r, "q"] & C
\end{tikzcd}$$

Here, the rightmost vertical map is the unique map such that $1 \otimes q; \bullet = m_A; q$. Its uniqueness follows from the universal property of the top coequalizer and the commutativity of the two leftmost squares, established as in the following calculation:

$$\begin{aligned}
&1 \otimes 1 \otimes c; 1 \otimes 1 \otimes m_A; 1 \otimes m_A; m_A \\
&= 1 \otimes 1 \otimes c; 1 \otimes m_A \otimes 1; 1 \otimes m_A; m_A \\
&= 1 \otimes 1 \otimes c; 1 \otimes m_A \otimes 1; m_A \otimes 1; m_A \\
&= 1 \otimes 1 \otimes c; m_A \otimes 1 \otimes 1; m_A \otimes 1; m_A \\
&= m_A \otimes 1 \otimes 1; 1 \otimes c; 1 \otimes m_A; m_A
\end{aligned}$$

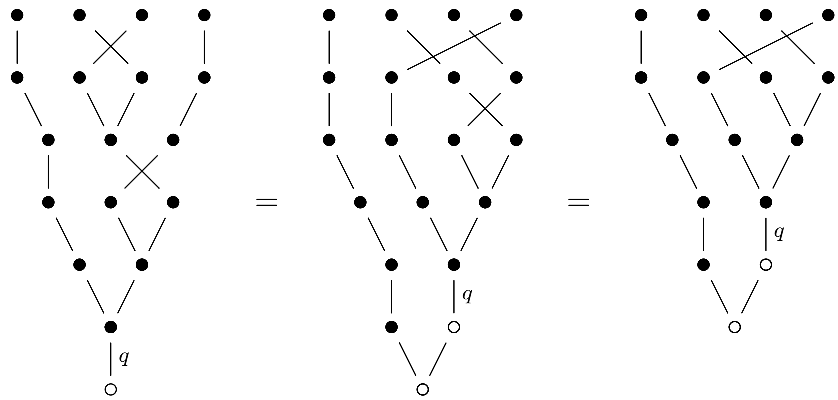
The second square is similar, but simpler. Next then, observe the following diagram:

$$\begin{tikzcd}
A^{\otimes 3} \otimes C \arrow[r, shift left, "1\otimes c\otimes 1;1\otimes m_A\otimes 1;m_A\otimes 1"] \arrow[r, shift right, "1\otimes m_A\otimes 1;m_A\otimes 1"'] & A \otimes C \arrow[r, "1\otimes q"] \arrow[dr, "\bullet"'] & C \otimes C \arrow[d, dashed, "m_C"] \\
& & C
\end{tikzcd}$$

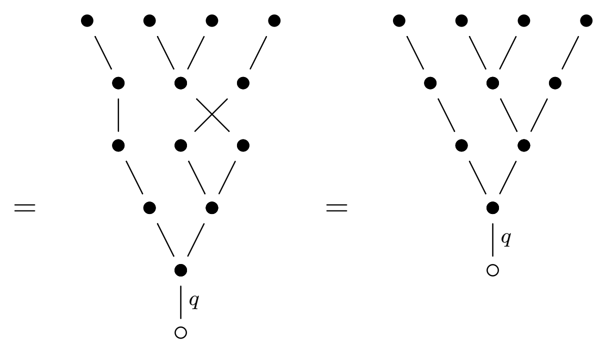
The top row is a coequalizer because $\otimes$ preserves coequalizers. If it can be shown that the result of postcomposing either of the two left maps with $\bullet$ is the same, we will have a unique map m_C such that $1 \otimes q; m_C = \bullet$. We have the following computation:

$$\begin{aligned}
&1 \otimes 1 \otimes 1 \otimes q; 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; m_A \otimes 1; \bullet \\
&= 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; m_A \otimes 1; 1 \otimes q; \bullet \\
&= 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; m_A \otimes 1; m_A; q \\
&= 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; 1 \otimes m_A; m_A; q \\
&= 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; 1 \otimes c; 1 \otimes m_A; m_A; q \\
&= 1 \otimes 1 \otimes c; 1 \otimes m_A \otimes 1; m_A \otimes 1; m_A; q
\end{aligned}$$

Due to frequency with which we must reason about symmetry maps, the subsequent calculations are best derived using a type of string diagram. Here $\circ$ will represent the object C and $\bullet$ will represent the object A. The rest of the diagrammatic notation is identical to that of **4.1.5**.



Observe the use of the defining property of q in this last equality. Continuing:



again using the defining property of q. Hence we have that

$$\begin{aligned}
&1 \otimes 1 \otimes 1 \otimes q; 1 \otimes c \otimes 1; 1 \otimes m_A \otimes 1; m_A \otimes 1; \bullet \\
&= 1 \otimes m_A \otimes 1; 1 \otimes m_A; m_A; q \\
&= 1 \otimes m_A \otimes 1; m_A \otimes 1; m_A; q \\
&= 1 \otimes 1 \otimes 1 \otimes q; 1 \otimes m_A \otimes 1; m_A \otimes 1; \bullet
\end{aligned}$$

$1 \otimes 1 \otimes 1 \otimes q$ can be constructed as a coequalizer map, and so is epi. This means that there is a unique map $m_C : C \otimes C \longrightarrow C$ such that $q \otimes 1; m_C = \bullet$. We then have $q \otimes q; m_C = 1 \otimes q; \bullet = m_C; q$. From this we have that if C has an algebra structure, then q is an algebra morphism.

Now, it follows from **2.1.1** that $q \otimes q \otimes q$ is a coequalizer map, and hence epi. Therefore the following computation shows that m_C has the associativity property:

$$\begin{aligned}
&q \otimes q \otimes q; 1 \otimes m_C; m_C \\
&= 1 \otimes m_A; m_A; q \\
&= m_A \otimes 1; m_A; q
\end{aligned}$$

$$= q \otimes q \otimes q; m_C \otimes 1; m_C$$

We may let e_C be defined as the composite $e_A; q$. Then clearly $e_C \otimes 1_C; m_C = 1_C$ and $1_C \otimes e_C; m_C = 1_C$. So (C, m_C, e_C) is an associative algebra.

To see that C is a commutative algebra, consider the following:

$$\begin{aligned}
&q \otimes q; c; m_C \\
&= c; q \otimes q; m_C \\
&= c; m_A; q \\
&= c; u_L; e_A \otimes 1 \otimes 1; m_A \otimes 1; m_A; q \\
&= u_L; e_A \otimes 1 \otimes 1; 1 \otimes c; 1 \otimes m_A; m_A; q \\
&= u_L; e_A \otimes 1 \otimes 1; 1 \otimes m_A; m_A; q \\
&= u_L; e_A \otimes 1 \otimes 1; m_A \otimes 1; m_A; q \\
&= m_A; q = q \otimes q; m_C
\end{aligned}$$

It follows that $c; m_C = m_C$ and so C is a commutative algebra.

Now, if $f : A \longrightarrow X$ is a morphism of algebras, where X is commutative, then

$$\begin{aligned}
&1 \otimes c; 1 \otimes m_A; m_A; f \\
=&1 \otimes c; f \otimes f \otimes f; 1 \otimes m_X; m_X \\
=&f \otimes f \otimes f; 1 \otimes c; 1 \otimes m_X; m_X \\
=&f \otimes f \otimes f; 1 \otimes m_X; m_X \\
=&1 \otimes m_A; m_A; f
\end{aligned}$$

and so f factors uniquely through q by the universal property of the coequalizer. ∎

Lemma 11. *Let $\mathcal{C}$ be a category such that $\otimes$ preserves reflexive coequalizers, and let T be an endofunctor which has as its image a subcategory of the category of associative algebras in $\mathcal{C}$. Then if for each $C \in Ob(\mathcal{C})$, $T_S C$ is defined to be the following coequalizer*

$$TC^{\otimes 3} \underset{1 \otimes m_{TC}; m_{TC}}{\overset{1 \otimes c; 1 \otimes m_{TC}; m_{TC}}{\rightrightarrows}} TC \xrightarrow{\lambda_C} T_S C$$

then $T_S C$ is an endofunctor which has as its image a subcategory of the category of commutative algebras in $\mathcal{C}$, and λ is a natural transformation.

Proof: This follows immediately from the previous lemma and the fact that for any morphism $f : C \longrightarrow D$ we may construct the following diagram:

$$\begin{tikzcd}
TC^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{TC};m_{TC}"] \arrow[r, shift right, "1\otimes m_{TC};m_{TC}"'] \arrow[d, "(Tf)^{\otimes 3}"'] & TC \arrow[r, "\lambda_C"] \arrow[d, "Tf"] & T_SC \arrow[d, dashed, "T_Sf"] \\
TD^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{TD};m_{TD}"] \arrow[r, shift right, "1\otimes m_{TD};m_{TD}"'] & TD \arrow[r, "\lambda_D"] & T_SD
\end{tikzcd}$$

The two left squares commute because Tf is an algebra morphism. It follows that there is a unique morphism which makes the right square commute, this we call T_Sf. That this map assignment is functorial follows from this uniqueness.

Finally, by the previous lemma, the coequalizer which defines T_SC is a reflexive coequalizer for each C. Hence $\lambda\otimes\lambda$ is, in particular, epimorphic, and we may compute

$$\begin{aligned}
&\lambda \otimes \lambda; T_Sf \otimes T_Sf; m_{T_S} \\
=&Tf \otimes Tf; m_T; \lambda \\
=&m_T; Tf; \lambda \\
=&\lambda \otimes \lambda; m_{T_S}; T_Sf
\end{aligned}$$

and deduce that the following diagram commutes

$$\begin{tikzcd}
T_SC \otimes T_SC \arrow[r, "T_Sf\otimes T_Sf"] \arrow[d, "m_{T_S}"'] & T_SD \otimes T_SD \arrow[d, "m_{T_S}"] \\
T_SC \arrow[r, "T_Sf"'] & T_SD
\end{tikzcd}$$

∎

Obviously, in the present work we are concerned with the case in which T is an algebra modality in a codifferential category. However, it does not appear to be the case that if T has a monad structure, then T_S in the above lemma will inherit one. Namely, it is not clear how one would endow T_S with a monad multiplication map. However, there is at least one prominent example where this does occur and where, furthermore, λ is a monad morphism. The following theorem examines the consequences of this phenomenon.

Theorem 12. *Let T be the algebra modality associated to a noncommutative codifferential category $\mathcal{C}$. For any $B \in Ob(\mathcal{C})$, consider the following diagram, defined to be a coequalizer:*

$$\begin{tikzcd}
TB^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{TB};m_{TB}"] \arrow[r, shift right, "1\otimes m_{TB};m_{TB}"'] & TB \arrow[r, "\lambda_B"] & T_SB
\end{tikzcd}$$

If λ is a morphism of monads such that $T\lambda$ is epimorphic, then T_S is the algebra modality associated to a commutative codifferential category.

Proof: First, by the above lemma, proving that T_S is an algebra modality only requires the proof that μ^{T_S} is an algebra morphism. First note that $\lambda; \mu^{T_S}$ is a T-algebra structure map, since

$$\begin{aligned}
&T(\lambda; \mu^{T_S}); (\lambda; \mu^{T_S})\\
=&T\lambda; \lambda; T_S\mu^{T_S}; \mu^{T_S}\\
=&T\lambda; \lambda; \mu^{T_S}; \mu^{T_S}\\
=&\mu^T; (\lambda; \mu^{T_S})
\end{aligned}$$

and

$$\begin{aligned}
&\eta^T; (\lambda; \mu^{T_S})\\
=&\eta^{T_S}; \mu^{T_S} = 1
\end{aligned}$$

By **3.2** this means that $\lambda; \mu^{T_S}$ is also an associative algebra morphism. Hence

$$\begin{aligned}
&\lambda \otimes \lambda; \mu^{T_S} \otimes \mu^{T_S}; m_{T_S}\\
=&m_T; \lambda; \mu^{T_S}\\
=&\lambda \otimes \lambda; m_{T_S}; \mu^{T_S}
\end{aligned}$$

Since $\lambda \otimes \lambda$ is epimorphic, it follows that μ^{T_S} is an algebra morphism. Next, we must construct a suitable map $d^{T_S} : T_SB \longrightarrow T_SB \otimes B$. We do this by first considering the following diagram:

$$\begin{tikzcd}
 & TB \otimes B \otimes TB \arrow[r, "1\otimes c; m_{TB}\otimes 1; \lambda\otimes 1"] & T_SB \otimes B \\
TB^{\otimes 3} \arrow[r, shift left, "1\otimes c; 1\otimes m_{TB}; m_{TB}"] \arrow[r, shift right, "1\otimes m_{TB}; m_{TB}"'] & TB \arrow[u, "d^T"] \arrow[r, "\lambda_B"'] & T_SB \arrow[u, dashed, "d^{T_S}"']
\end{tikzcd}$$

It will be shown that the two paths from $(TB)^{\otimes 3}$ to $T_SB \otimes B$ are equal, and hence the universal property of λ_B induces a map d^{T_S}. Consider the following computation:

$$\begin{aligned}
&1 \otimes c; 1 \otimes m; m; d^T; 1 \otimes c; m \otimes 1; \lambda \otimes 1\\
=&1 \otimes c; 1 \otimes m; (1 \otimes d^T; m \otimes 1 \otimes 1 + d^T \otimes 1; 1 \otimes 1 \otimes m); 1 \otimes c; m \otimes 1; \lambda \otimes 1
\end{aligned}$$

This produces two summands with which we can deal individually. Note that $1 \otimes c; m_{TB} \otimes 1; \lambda \otimes 1 = \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1$ and so the computation

$$\begin{aligned}
&1 \otimes c; 1 \otimes m; d^T \otimes 1; 1 \otimes 1 \otimes m; 1 \otimes 1 \otimes \lambda\\
=&1 \otimes c; 1 \otimes m; d^T \otimes 1; 1 \otimes 1 \otimes \lambda \otimes \lambda; 1 \otimes 1 \otimes m_{T_S}
\end{aligned}$$

$$
\begin{aligned}
&=1 \otimes c; 1 \otimes \lambda \otimes \lambda; 1 \otimes m_{T_S}; d^T \otimes 1; 1 \otimes 1 \otimes \lambda \otimes 1; 1 \otimes 1 \otimes m_{T_S} \\
&=1 \otimes \lambda \otimes \lambda; 1 \otimes m_{T_S}; d^T \otimes 1; 1 \otimes 1 \otimes \lambda \otimes 1; 1 \otimes 1 \otimes m_{T_S} \\
&=1 \otimes m; d^T \otimes 1; 1 \otimes 1 \otimes m; 1 \otimes 1 \otimes \lambda
\end{aligned}
$$

is sufficient to dispense with one of the summands. The point here is that c and d^T don't interact in any significant way.

To construct d^{T_S}, it is now sufficient to show that

$$
\begin{aligned}
&1 \otimes c; 1 \otimes m; 1 \otimes d^T; m \otimes 1 \otimes 1; 1 \otimes c; m \otimes 1; \lambda \otimes 1 \\
=&1 \otimes m; 1 \otimes d^T; m \otimes 1 \otimes 1; 1 \otimes c; m \otimes 1; \lambda \otimes 1
\end{aligned}
$$

or, equivalently, that

$$
\begin{aligned}
&\lambda \otimes 1; 1 \otimes (c; m; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1); m_{T_S} \otimes 1 \\
=&\lambda \otimes 1; 1 \otimes (m; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1); m_{T_S} \otimes 1
\end{aligned}
$$

Since the tensor product preserves reflexive coequalizers, we may eliminate $\lambda \otimes 1$ from the prefix. It will be shown that

$$
\begin{aligned}
&c; m; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1 \\
=&m; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

By the noncommutative Leibniz rule, this is equivalent to

$$
\begin{aligned}
&c; (1 \otimes d^T; m \otimes 1 \otimes 1 + d^T \otimes 1; 1 \otimes 1 \otimes m); \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1 \\
=&(1 \otimes d^T; m \otimes 1 \otimes 1 + d^T \otimes 1; 1 \otimes 1 \otimes m); \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

It will be shown that

$$
\begin{aligned}
&c; 1 \otimes d^T; m \otimes 1 \otimes 1; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1 \\
=&d^T \otimes 1; 1 \otimes 1 \otimes m; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

and that

$$
\begin{aligned}
&1 \otimes d^T; m \otimes 1 \otimes 1; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1 \\
=&c; d^T \otimes 1; 1 \otimes 1 \otimes m; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

and the construction of d^{T_S} will be complete. Since $c; c = 1$, we need only prove one of these equalities. Compute as follows:

$$
\begin{aligned}
&c; 1 \otimes d^T; m \otimes 1 \otimes 1; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1 \\
=&d^T \otimes 1; c_{4,1}; m \otimes 1 \otimes 1; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

$$
\begin{aligned}
&= d^T \otimes 1; c_{4,1}; \lambda \otimes \lambda \otimes 1 \otimes \lambda; 1 \otimes 1 \otimes c; m_{T_S} \otimes 1 \otimes 1; m_{T_S} \otimes 1 \\
&= d^T \otimes 1; c_{4,1}; 1 \otimes 1 \otimes c; \lambda \otimes \lambda \otimes \lambda \otimes 1; 1 \otimes m_{T_S} \otimes 1; m_{T_S} \otimes 1 \\
&= d^T \otimes 1; c_{2,4}; \lambda \otimes \lambda \otimes \lambda \otimes 1; c_{3,1}; 1 \otimes m_{T_S} \otimes 1; m_{T_S} \otimes 1 \\
&= d^T \otimes 1; c_{2,4}; \lambda \otimes \lambda \otimes \lambda \otimes 1; 1 \otimes m_{T_S} \otimes 1; c \otimes 1; m_{T_S} \otimes 1 \\
&= d^T \otimes 1; c_{2,4}; \lambda \otimes \lambda \otimes \lambda \otimes 1; 1 \otimes m_{T_S} \otimes 1; m_{T_S} \otimes 1 \\
&= d^T \otimes 1; 1 \otimes 1 \otimes m; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1
\end{aligned}
$$

This final equality follows by cycling backward through the equalities with the crucial symmetrization map removed. The penultimate equality is the crux of the matter, and uses the fact that $T_S B$ is a commutative algebra in an essential way. Now, by definition, since T is a noncommutative algebra modality, we have a morphism of monads $\rho : \mathcal{F} \longrightarrow T$. Observe the following diagram:

$$
\begin{tikzcd}
TB^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{TB};m_{TB}"] \arrow[r, shift right, "1\otimes m_{TB};m_{TB}"'] & TB \arrow[r, "\lambda_B"] & T_S B \\
\mathcal{F}B^{\otimes 3} \arrow[u, "\rho^{\otimes 3}"] \arrow[r, shift left, "1\otimes c;1\otimes m_{\mathcal{F}B};m_{\mathcal{F}B}"] \arrow[r, shift right, "1\otimes m_{\mathcal{F}B};m_{\mathcal{F}B}"'] & \mathcal{F}B \arrow[u, "\rho"] \arrow[r, "\omega_B"] & \mathcal{S}B \arrow[u, dashed, "\sigma"']
\end{tikzcd}
$$

Since ρ, λ and ω are morphisms of monads and, in particular, ω is an epimorphism, it is routine to verify that σ is a morphism of monads. It follows then, that in order to show that $T_S B$ is the monad associated to a commutative codifferential category with canonical derivations d^{T_S}, we must show that σ and d^{T_S} behave in the correct way according to the characterization of a codifferential category in **3.2.14**.

First, consider a map $f : B \longrightarrow C$. We must prove that $T_S f; d^{T_S}_{T_S C} = d^{T_S}_{T_S B}; T_S f \otimes f$:

$$
\begin{aligned}
&\lambda_B; T_S f; d^{T_S} \\
&= Tf; \lambda_C; d^{T_S} \\
&= Tf; d^T; 1 \otimes c; m_{TC} \otimes 1; \lambda_C \otimes 1 \\
&= d^T; Tf \otimes f \otimes Tf; 1 \otimes c; m_{TC} \otimes 1; \lambda_C \otimes 1 \\
&= d^T; 1 \otimes c; m_{TB} \otimes 1; Tf \otimes f; \lambda_C \otimes 1 \\
&= d^T; 1 \otimes c; m_{TB} \otimes 1; \lambda_B \otimes 1; T_S f \otimes f \\
&= \lambda_B; d^{T_S}; T_S f \otimes f
\end{aligned}
$$

The fact that λ is epimorphic affirms the naturality of d^{T_S}.

Next we show that d^{T_S} obeys the chain rule:

$$\mu^{T_S}; d^{T_S} = d^{T_S}; \mu \otimes d^{T_S}; m_{T_S} \otimes 1$$

We utilize the fact that $T\lambda$ is epimorphic:

$$T\lambda; \lambda; \mu^{T_S}; d^{T_S}$$

$$
\begin{aligned}
&=\mu^T; \lambda; d^{T_S}\\
&=\mu^T; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1\\
&=d^T; \mu \otimes d^T \otimes \mu; m \otimes 1 \otimes m; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S} \otimes 1\\
&=d^T; \mu \otimes d^T \otimes \mu; \lambda^{\otimes 2} \otimes 1 \otimes \lambda^{\otimes 2}; 1 \otimes 1 \otimes c \otimes 1; 1 \otimes m_{T_S} \otimes c; ...\\
&\quad ...m_{T_S} \otimes 1 \otimes 1; m_{T_S} \otimes 1
\end{aligned}
$$

$$
\begin{aligned}
&=d^T; (\mu;\lambda) \otimes 1 \otimes (\mu;\lambda); 1 \otimes (\lambda; d^{T_S}) \otimes 1; 1 \otimes 1 \otimes c; m_{T_S} \otimes 1 \otimes 1; m_{T_S} \otimes 1\\
&=d^T; T\lambda \otimes \lambda \otimes T\lambda; (\lambda;\mu) \otimes d^{T_S} \otimes (\lambda;\mu); m_{T_S} \otimes c; m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes d^{T_S} \otimes 1; \mu \otimes 1 \otimes 1 \otimes \mu; 1 \otimes 1 \otimes c; m_{T_S} \otimes 1 \otimes 1; ...\\
&\quad ...m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes d^{T_S} \otimes 1; 1 \otimes 1 \otimes c; \mu \otimes 1 \otimes \mu \otimes 1; 1 \otimes m_{T_S} \otimes 1; ...\\
&\quad ...m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes d^{T_S} \otimes 1; 1 \otimes 1 \otimes c; \mu \otimes 1 \otimes \mu \otimes 1; 1 \otimes c \otimes 1; ...\\
&\quad ...1 \otimes m_{T_S} \otimes 1; m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes d^{T_S} \otimes 1; 1 \otimes 1 \otimes c; 1 \otimes c \otimes 1; \mu \otimes \mu \otimes 1 \otimes 1...\\
&\quad ...m_{T_S} \otimes 1 \otimes 1; m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes d^{T_S} \otimes 1; 1 \otimes 1 \otimes c; 1 \otimes c \otimes 1; m_{T_S^2} \otimes 1 \otimes 1; ...\\
&\quad ...\mu \otimes 1 \otimes 1; m_{T_S} \otimes 1\\
&=T\lambda; d^T; \lambda \otimes 1 \otimes \lambda; 1 \otimes c; m_{T_S^2} \otimes 1; \mu \otimes d^{T_S}; m_{T_S} \otimes 1\\
&=T\lambda; \lambda; d^{T_S}; \mu \otimes d^{T_S}; m_{T_S} \otimes 1
\end{aligned}
$$

This, combined with the fact that $T\lambda$ and λ are epimorphic, demonstrates that the chain rule holds.

Finally, we must show that for each B, the following diagram commutes

$$
\begin{tikzcd}
\mathcal{S}B \otimes B \arrow[r, "\sigma \otimes 1"] & T_S B \otimes B \\
\mathcal{S}B \arrow[u, "d^S"] \arrow[r, "\sigma"'] & T_S B \arrow[u, "d^{T_S}"']
\end{tikzcd}
$$

Since $\omega : \mathcal{F}B \longrightarrow \mathcal{S}B$ is epimorphic, demonstrating the commutativity of the above diagram reduces to demonstrating the commutativity of the outer rectangle of

$$\begin{tikzcd}
\mathcal{F}B \otimes B \otimes \mathcal{F}B \arrow[r, "1\otimes c; m_{FB}\otimes 1; \omega\otimes 1"] & \mathcal{S}B \otimes B \arrow[r, "\sigma\otimes 1"] & T_S B \otimes B \\
\mathcal{F}B \arrow[u, "d^F"] \arrow[r, "\omega"'] & \mathcal{S}B \arrow[u, "d^S"] \arrow[r, "\sigma"'] & T_S B \arrow[u, "d^{T_S}"']
\end{tikzcd}$$

But the commutativity of the outer rectangle here is equivalent to the commutativity of the outer rectangle of

$$\begin{tikzcd}
\mathcal{F}B \otimes B \otimes \mathcal{F}B \arrow[r, "\rho\otimes 1\otimes\rho"] & TB \otimes B \otimes TB \arrow[r, "1\otimes c; m_{TB}\otimes 1; \lambda\otimes 1"] & T_S B \otimes B \\
\mathcal{F}B \arrow[u, "d^F"] \arrow[r, "\rho"'] & TB \arrow[u, "d^T"] \arrow[r, "\lambda"'] & T_S B \arrow[u, "d^{T_S}"']
\end{tikzcd}$$

which follows from the established commutativity of the two smaller rectangles. ∎

Although the condition that $T\lambda$ be epimorphic is only used here to establish the chain rule, the significance of this condition is not diminished. On the contrary, since the chain rule is the crucial condition which distinguishes T-derivations from $\mathcal{S}$ (or $\mathcal{F}$) derivations, the condition that $T\lambda$ is epimorphic appears to be of paramount importance in this connection between noncommutative codifferential categories and commutative ones.

Proposition 13. *The symmetric algebra monad $\mathcal{S}$ is isomorphic to the symmetrization of the free associative algebra monad $\mathcal{F}$. That is, for any $C \in Ob(\mathcal{C})$, there is a coequalizer of the form:*

$$\begin{tikzcd}
\mathcal{F}C^{\otimes 3} \arrow[r, shift left, "1\otimes c; 1\otimes m_{FC}; m_{FC}"] \arrow[r, shift right, "1\otimes m_{FC}; m_{FC}"'] & \mathcal{F}C \arrow[r] & \mathcal{S}C
\end{tikzcd}$$

Proof: By 4.2.10, it follows that the coequalizer objects depicted in these diagrams, namely objects of the form $\mathcal{F}_S C$, are commutative algebras in the image of an endofunctor $\mathcal{F}_S$. If it can be established that these objects $\mathcal{F}_S C$ have the universal property, which defines $\mathcal{S}$, then the proposition will follow. That is, it needs to be shown that given any object $C \in \mathcal{C}$ and any map $f : C \longrightarrow X$ such that X is a commutative algebra, there is a unique morphism of algebras $f' : \mathcal{F}_S C \longrightarrow X$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_S C \arrow[r, "f'"] & X \\
C \arrow[u, "\eta^F;\lambda"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Since X is, in particular, an associative algebra, there is a unique morphism of algebras, g, such that

$$\begin{tikzcd}
\mathcal{F}C \arrow[r, "g"] & X \\
C \arrow[u, "\eta^F"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. It follows from this and **4.2.10** that there is indeed a morphism of algebras $f' : \mathcal{F}_S C \longrightarrow X$ such that $\lambda; f' = g$ and hence $\eta^F; \lambda; f' = \eta^F; g = f$. By the universal properties of η^F and λ, this map is unique. ∎

Call the map constructed above $\sigma : \mathcal{F} \longrightarrow \mathcal{S}$. Since it is a morphism of associative algebras, there is a unique map ψ such that

$$\begin{tikzcd}
\mathcal{F}^2 C \arrow[r, "\psi"] & \mathcal{S}C \\
\mathcal{F}C \arrow[u, "\eta^F"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Clearly, $\psi = \mu^F; \sigma$. Observe also though that

$$\begin{aligned}
&\eta^F; \mathcal{F}\sigma; \sigma; \mu^S \\
=&\sigma; \eta^F; \sigma; \mu^S \\
=&\sigma; \eta^S; \mu^S \\
=&\sigma
\end{aligned}$$

where the equality $\eta^S = \eta^F; \sigma$ follows from the last part of the above proof. Hence $\mu^F; \sigma = \mathcal{F}\sigma; \sigma; \mu^S$. Therefore σ is a morphism of monads.

Since $\mathcal{F}$ is constructed as a direct sum of iterated tensor products and iterated tensor products of reflexive coequalizers are reflexive coequalizers, $\mathcal{F}\sigma$ is itself a coequalizer map, and therefore an epimorphism. Hence $\mathcal{S}$ inherits codifferential structure from $\mathcal{F}$ as in the above theorem.

Normally $\mathcal{S}$ is characterized as the free tensor algebra over the symmetrized tensor product. Using such a description, it is a simple enough matter to demonstrate that $\mathcal{S}$ is a nonsingular monad - although property i) is not immediate. However, it is worthwhile to examine the extent to which the properties of nonsingular monads are demonstrably possessed by $\mathcal{S}$ as a result of the coequalizer construction given above. One could use this, for instance, to posit a definition for *smooth noncommutative monads*.

Proofs of the following in the traditional setting can be found in e.g. [9], Proposition A2.2, pp. 576-578 and [10] B.2.

Proposition 14. $\mathcal{S}(A \oplus B) \cong \mathcal{S}A \otimes \mathcal{S}B$

Proof: Consider the map $f : A \oplus B \longrightarrow \mathcal{S}A \otimes \mathcal{S}B$ defined on A as the following composite:

$$A \xrightarrow{\eta^S_A} \mathcal{S}A \xrightarrow{u_R} \mathcal{S}A \otimes I \xrightarrow{1 \otimes e_{SA}} \mathcal{S}A \otimes \mathcal{S}B$$

and similarly, on B as

$$B \xrightarrow{\eta^S_B} \mathcal{S}B \xrightarrow{u_L} I \otimes \mathcal{S}B \xrightarrow{e_{SB} \otimes 1} \mathcal{S}B \otimes \mathcal{S}B$$

Since $\mathcal{S}A \otimes \mathcal{S}B$ is equipped with a commutative algebra structure, there is a unique map $f' : \mathcal{S}(A \oplus B) \longrightarrow \mathcal{S}A \otimes \mathcal{S}B$ such that $\eta^S_{A \oplus B}; f' = f$.

Next, we may construct a map $g : \mathcal{S}A \otimes \mathcal{S}B \longrightarrow \mathcal{S}(A \oplus B)$ with the following composite:

$$\begin{tikzcd}
\mathcal{S}A \otimes \mathcal{S}B \arrow[r, "\mathcal{S}(i_B) \otimes \mathcal{S}(i_A)"] \arrow[dr, "g"'] & \mathcal{S}(A \oplus B) \otimes \mathcal{S}(A \oplus B) \arrow[d, "m_{S(A \oplus B)}"] \\
 & \mathcal{S}(A \oplus B)
\end{tikzcd}$$

It will be shown that $f'; g = 1_{\mathcal{S}(A \oplus B)}$ and $g; f' = 1_{\mathcal{S}A \otimes \mathcal{S}B}$. To show that $f'; g = 1$, it will be shown that $f; g = \eta^S_{A \oplus B}$; we will then invoke the universal property of η^S.

First, note that $\eta^F_{A \oplus B}$ is defined on A as $\eta^F_A; \mathcal{F}(i_B)$ and defined on B as $\eta^F_B; \mathcal{F}(i_A)$. Hence $\eta^S_{A \oplus B} = \begin{pmatrix} \eta^F_A; \mathcal{F}(i_A) \\ \eta^F_B; \mathcal{F}(i_B) \end{pmatrix}; \lambda_{A \oplus B} = \begin{pmatrix} \eta^S_A; \mathcal{S}(i_A) \\ \eta^S_B; \mathcal{S}(i_B) \end{pmatrix}$. Calculate as follows:

$$\begin{aligned}
f; g &= \begin{pmatrix} \eta^S_A; u_R; 1 \otimes e_{SA} \\ \eta^S_B; u_L; e_{SB} \otimes 1 \end{pmatrix}; \mathcal{S}(i_A) \otimes \mathcal{S}(i_B); m_{S(A \oplus B)} \\
&= \begin{pmatrix} \eta^S_A; \mathcal{S}(i_A); u_R; 1 \otimes e_{S(A \oplus B)} \\ \eta^S_B; \mathcal{S}(i_B); u_L; e_{S(A \oplus B)} \otimes 1 \end{pmatrix}; m_{S(A \oplus B)} \\
&= \begin{pmatrix} \eta^S_A; \mathcal{S}(i_A) \\ \eta^S_B; \mathcal{S}(i_B) \end{pmatrix} = \eta^S_{A \oplus B}
\end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{F}(A\oplus B)^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{F(A\oplus B)};m_{F(A\oplus B)}"] \arrow[r, shift right, "1\otimes m_{F(A\oplus B)};m_{F(A\oplus B)}"'] & \mathcal{F}(A\oplus B) \arrow[r, "\lambda_(A\oplus B)"] \arrow[d, "h"'] & \mathcal{S}(A\oplus B) \arrow[d, "f'"] \\
 & \mathcal{F}A\otimes\mathcal{F}B \arrow[r, "\lambda_A\otimes\lambda_B"'] & \mathcal{S}A\otimes\mathcal{S}B
\end{tikzcd}$$

Here h is the unique morphism of algebras such that

$$\eta^F_{A\oplus B}; h = \begin{pmatrix}\eta^F_A; u_R; 1\otimes e_{FB}\\ \eta^F_B; u_L; e_{FA}\otimes 1\end{pmatrix}$$

Since the four maps which comprise the square are morphisms of associative algebras, it follows that the square is commutative if it is commutative when precomposed with $\eta^F_{A\oplus B}$. To this end, observe

$$\begin{aligned}
&\eta^F_{A\oplus B}; h; \lambda_A\otimes\lambda_B\\
=&\begin{pmatrix}\eta^F_A; u_R; \lambda_A\otimes e_{SB}\\ \eta^F_B; u_L; e_{SA}\otimes\lambda_B\end{pmatrix}\\
=&\eta^S_{A\oplus B}; f'\\
=&\eta^F_{A\oplus B}; \lambda_{A\oplus B}; f'
\end{aligned}$$

Suppose we have $q : \mathcal{F}(A\oplus B) \longrightarrow X$ such that $1\otimes c; 1\otimes m_{F(A\oplus B)}; m_{F(A\oplus B)}; q = 1\otimes m_{F(A\oplus B)}; m_{F(A\oplus B)}; q$. Then there exists a unique map $t : \mathcal{S}(A\oplus B) \longrightarrow X$ such that $q = \lambda_{A\oplus B}; t$. Hence $\lambda_{A\oplus B}; f'; g; t = q$. If it can be shown that $g; t$ is the unique map for which this latter equation holds, it will follow that the following diagram is a coequalizer diagram:

$$\begin{tikzcd}
\mathcal{F}(A\oplus B)^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{F(A\oplus B)};m_{F(A\oplus B)}"] \arrow[r, shift right, "1\otimes m_{F(A\oplus B)};m_{F(A\oplus B)}"'] & \mathcal{F}(A\oplus B) \arrow[r, "\lambda_(A\oplus B);f'"] & \mathcal{S}A\otimes\mathcal{S}B
\end{tikzcd}$$

and hence $\mathcal{S}A\otimes\mathcal{S}B \cong \mathcal{S}(A\oplus B)$. This will hold if it can be shown that $\lambda_{A\oplus B}; f'$ is an epimorphism - or equivalently, that $h; \lambda_A\otimes\lambda_B$ is an epimorphism. Since by **2.1.1**, $\lambda_A\otimes\lambda_B$ is a coequalizer map, we need only to show that h is an epimorphism.

Upon direct inspection, we see that $\mathcal{F}A\otimes\mathcal{F}B$ is the direct sum of every possible object of the form $A^{\otimes n}\otimes B^{\otimes m}$, where m and n are natural numbers; when $m = n = 0$, this object is I. If it can be proven that h surjects onto each summand, then the proposition will be established. First, by the definition of an algebra homomorphism, h surjects onto I; that is, the I summand in $\mathcal{F}(A\oplus B)$ is mapped identically to I in $\mathcal{F}A\otimes\mathcal{F}B$. As for any other object $A^{\otimes n}\otimes B^{\otimes m}$, we may consider these in the context of the image of the maps:

$$(m_{FA\otimes FB})^{n+m}; proj : (\mathcal{F}A\otimes\mathcal{F}B)^{n+m} \longrightarrow A^{\otimes n}\otimes B^{\otimes m}$$

That is, since multiplication is just the concatenation of vector products, and the maps η^F_A and η^F_B simply map, respectively, A and B identically into their representative summands in, respectively again, $\mathcal{F}A$ and $\mathcal{F}B$, the composite

$$\begin{tikzcd}
(A\otimes I)^{\otimes n}\otimes(I\otimes B)^{\otimes m} \arrow[r, "(\eta^F_A\otimes e_{FA})^n\otimes(e_{FB}\otimes\eta^F_B)"] \arrow[dr] & (\mathcal{F}A\otimes\mathcal{F}B)^{n+m} \arrow[d, "(m_{FA\otimes FB})^{n+m};proj"] \\
 & A^{\otimes n}\otimes B^{\otimes m}
\end{tikzcd}$$

is an epimorphism.

The maps $\eta^F_A\otimes e_{FB}\ :\ A\otimes I \longrightarrow \mathcal{F}A\otimes\mathcal{F}B$ may be realized as the following composite:

$$A \xrightarrow{i_A} A\oplus B \xrightarrow{\begin{pmatrix} id_A \\ 0 \end{pmatrix}} A\oplus B \xrightarrow{\eta_{A\oplus B}} \mathcal{F}(A\oplus B) \xrightarrow{h} \mathcal{F}A\otimes\mathcal{F}B$$

and the maps $e_{FA}\otimes\eta^F_B : I\otimes B \longrightarrow \mathcal{F}A\otimes\mathcal{F}B$ may be realized thusly:

$$B \xrightarrow{i_B} A\oplus B \xrightarrow{\begin{pmatrix} 0 \\ id_B \end{pmatrix}} A\oplus B \xrightarrow{\eta_{A\oplus B}} \mathcal{F}(A\oplus B) \xrightarrow{h} \mathcal{F}A\otimes\mathcal{F}B$$

It follows that $h^{\otimes n+m};(m_{FA\otimes FB})^{m+n} = (m_{F(A\oplus B)})^{m+n};h$ is an epimorphism, so that h is an epimorphism. ∎

Proposition 15. *The following sequence is exact*

$$0 \longrightarrow \mathcal{S}k\otimes k\otimes\mathcal{S}k \xrightarrow{\eta^S\otimes 1\otimes\eta^S;(m\otimes 1-1\otimes m)} \mathcal{S}k\otimes\mathcal{S}k \xrightarrow{m} \mathcal{S}k \longrightarrow 0$$

Proof: This follows from **3.1.5** if we prove that $\mathcal{F}k\cong\mathcal{S}k$. To this end, consider the following diagram

$$\begin{tikzcd}
\mathcal{F}k^{\otimes 3} \arrow[r, shift left, "1\otimes c;1\otimes m_{Fk};m_{Fk}"] \arrow[r, shift right, "1\otimes m_{Fk};m_{Fk}"'] & \mathcal{F}k \arrow[r, "\sigma"] \arrow[dr, "id"'] & \mathcal{S}k \arrow[d, dashed] \\
 & & \mathcal{F}k
\end{tikzcd}$$

wherein the right-hand map exists and factors $id_{\mathcal{F}k}$ if $\mathcal{F}k$ is a commutative algebra; in this case σ_k will be an epimorphism with a (diagrammatic) right inverse, and hence an isomorphism. Since the multiplication on $\mathcal{F}k$ is the concatenation of direct sums of the tensor unit, this is immediate.

∎

Proposition 16. *If $colim_j A_j$ is a filtered colimit of k-modules, $\mathcal{S}(colim_j A_j) \cong colim_j \mathcal{S}(A_j)$*

Proof: In light of the coequalizer construction of $\mathcal{S}$, this follows immediately from the fact that $\mathcal{F}$ preserves filtered colimits. ∎

Although $\mathcal{S}$, as normally constructed, does preserve flatness, it is not immediate that - even with our relaxed conception of flatness - one can show that $\mathcal{S}$ preserves flatness by virtue of the relationship with $\mathcal{F}$ exhibited in **4.2.13**. That is, if in a general symmetric monoidal category, one were to construct $\mathcal{S}$ using **4.2.13** as a definition, it is not clear that such an $\mathcal{S}$ would preserve flatness.

Chapter 5

André-Quillen Homology and Smoothness

André-Quillen homology is a homology theory designed for commutative rings[20]. Since the complex on which André-Quillen homology is computed is comprised of modules of differential forms, codifferential categories are an ideal context in which to extend this theory.

The upshot of this extension is that it imparts to the theory of codifferential categories a practical means by which to measure smoothness. To clarify, in the traditional context of André-Quillen (co)-homology, the theory may be used to conveniently characterize smoothness via vanishing conditions. In this section, a definition of smooth T-algebras is posited analogously; it is then shown that with such a definition at hand, a version of the HKR theorem may be proven.

5.1 André-Quillen Homology

Before engaging in André-Quillen homology proper, it will be necessary to extend the theory of codifferential categories a little further. The astute reader will notice that in the notation that is used for modules of differential forms, the underlying ring is omitted. This is simply a matter of convenience and will be utilized when context deems this practice appropriate. However, powerful tools of André-Quillen homology require an increased level of flexibility, and necessitate the construction of modules of differential forms relative to variable algebras. Furthermore, the power that the Jacobi-Zariski long exact sequence affords in the utilization of this theory designates such considerations as the crux of the matter. Throughout the entirety of this section, unless otherwise noted, the superscript T is dropped from Ω^T. Also, throughout this section, we require that the tensor product preserves arbitrary coequalizers, and not just reflexive ones.

Definition 1. *Let $f : A \longrightarrow B$ be a morphism of T-algebras in a codifferential category. Define the module of differential forms of B over f, denoted $\Omega_{B/A}$ to be the cokernel of the map $1_B \otimes_A \Omega_f ; \bullet_B : B \otimes_A \Omega_A \longrightarrow \Omega_B$. That is, the sequence*

$$B \otimes_A \Omega_A \xrightarrow{1_B \otimes_A \Omega_f ; \bullet_B} \Omega_B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

is exact.

Lemma 2. *$\Omega_{B/A}$ defined above inherits a B-module structure*

Proof: Observe the following diagram

$$\begin{tikzcd}
B \otimes B \otimes_A \Omega_A \arrow[r, "1_B \otimes 1_B \otimes_A \Omega_f ; 1_B \otimes \bullet_B"] \arrow[d, "m_B \otimes 1"'] & B \otimes \Omega_B \arrow[r] \arrow[d, "\bullet_B"'] & B \otimes \Omega_{B/A} \arrow[d, dashed] \\
B \otimes_A \Omega_A \arrow[r, "1_B \otimes_A \Omega_f ; \bullet_B"] & \Omega_B \arrow[r] & \Omega_{B/A}
\end{tikzcd}$$

The left-hand square commutes since Ω_f is a morphism of modules and $\bullet_B$ is a B-module action. Since the top row is a coequalizer, there is necessarily a map on the right which makes the right-hand square commute. It is routine to verify that this map is a B-module action. ∎

It is important to take into account that the construction of the B-module structure on $\Omega_{B/A}$ invokes the condition that the tensor product preserves coequalizers.

Note also that the notation $\Omega_{B/A}$ obscures the significance of the map f. Some authors choose to use notation such as Ω_f for these objects. However, the evident conflict that this causes with the notation for the maps named Ω_f coupled with the contextual clarification of any possible ambiguity settle this dilemma.

Observe that this definition accords with the short Jacobi-Zariski exact sequence associated to the sequence of maps $I \longrightarrow A \longrightarrow B$. This is, in fact, the objective. It remains to be shown that the short Jacobi-Zariski exact sequence can be constructed for any sequence of T-algebra maps $A \longrightarrow B \longrightarrow C$. First though, we must exhibit a universal property of the B-modules $\Omega_{B/A}$ which facilitates its use.

Proposition 3. *Let $f : A \longrightarrow B$ be a morphism of T-algebras in a codifferential category, let $\phi_{A/B} : \Omega_B \longrightarrow \Omega_{B/A}$ be the cokernal of the map $1_B \otimes_A \Omega_f ; \bullet_B$, and consider the B-derivation $\delta_f := d_B ; \phi_{A/B} : B \longrightarrow \Omega_B \longrightarrow \Omega_{B/A}$. Then $f ; \delta_f = 0$ and for any B-derivation $D : B \longrightarrow M$ such that $f ; D = 0$ there exists a unique B-module map $h : \Omega_{B/A} \longrightarrow M$ such that $\delta_f ; h = D$.*

Proof: First observe the commutativity of the following diagram:

$$\begin{tikzcd}
B \otimes_A \Omega_A \arrow[r, "1_B \otimes_A \Omega_f;\bullet_B"] & \Omega_B \\
I \otimes A \cong A \arrow[u, "e_B \otimes d_A;p"] \arrow[r, "f"'] & B \arrow[u, "d_B"']
\end{tikzcd}$$

p here denotes the projection onto the tensor product over A. It follows that $f;\delta_f$ is the same as following the clockwise route of this diagram and following by $\phi_{A/B}$, which is of course 0.

Now, let $D : B \longrightarrow M$ be a B-derivation such that $f;D = 0$. In particular, since D is a derivation from B to a B-module there is a unique morphism of B-modules $g : \Omega_B \longrightarrow M$ such that $d_B; g = D$. Consider then the following elaboration of the previous diagram

$$\begin{tikzcd}
B \otimes_A \Omega_A \arrow[r, "1_B \otimes_A \Omega_f;\bullet_B"] & \Omega_B \arrow[r, "\phi_{B/A}"] \arrow[rr, bend left, "g"] & \Omega_{B/A} \arrow[r, dashed] & M \\
I \otimes A \cong A \arrow[u, "e_B \otimes d_A;p"] \arrow[r, "f"'] & B \arrow[u, "d_B"'] \arrow[ur, "\delta_f"'] \arrow[urr, bend right, "D"'] & &
\end{tikzcd}$$

If it can be shown that $1_B \otimes_A \Omega_f; \bullet_B; g = 0$, then there must exist a unique B-module map $h : \Omega_{B/A} \longrightarrow M$ such that $\phi_{B/A}; h = g$. The desired result will follow from the definition of δ_f and the uniqueness of maps g and h. To this end:

$$\begin{aligned}
e_B \otimes d_A; p; 1_B \otimes_A \Omega_f; \bullet_B; g &= e_B \otimes d_A; p; 1_B \otimes_A \Omega_f; 1_B \otimes_A g; \bullet \\
&= e_B \otimes d_A; 1_B \otimes_A (\Omega_f; g); \bullet \\
&= 1_I \otimes (d_A; \Omega_f; g); e_B \otimes 1_M; \bullet \\
&= u_L^{-1}; d_A; \Omega_f; g
\end{aligned}$$

Since the first expression above is 0 by definition, it follows that $d_A; \Omega_f; g = 0$. The universal property of d_A determines that $\Omega_f; g = 0$ and so it follows that $1_B \otimes_A \Omega_f; \bullet_B; g = 1_B \otimes_A \Omega_f; 1_B \otimes_A g; \bullet = 0$.

∎

The universal property identified in the proposition above defines objects such as $\Omega_{B/A}$ in an intuitive way; the coequalizer diagram merely indicates their construction in a codifferential category. With such objects constructed, we are now able to ask whether the Jacobi-Zariski exact sequence holds:

Proposition 4. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms of T-algebras in a codifferential category. Then the following sequence is exact:*

$$C \otimes_B \Omega_{B/A} \xrightarrow{1_C \otimes_B \Omega_{g/A};\bullet_C} \Omega_{C/A} \xrightarrow{\Omega_{C/f}} \Omega_{C/B} \longrightarrow 0$$

Proof: First, let us consider the nature of these maps. Observe the following diagram:

$$\begin{tikzcd}
 & \Omega_{B/A} \arrow[r, "\Omega_{g/A}"] & \Omega_{C/A} \arrow[r, "\Omega_{C/f}"] & \Omega_{C/B} \\
A \arrow[r, "f"'] & B \arrow[u, "\delta_f"] \arrow[r, "g"'] & C \arrow[u, "\delta_{f;g}"'] \arrow[ur, "\delta_g"'] &
\end{tikzcd}$$

Since g is a T-algebra morphism, $g; \delta_{f;g}$ is a B-derivation, and by definition $f; g; \delta_{f;g} = 0$ so that by the universal property of δ_f, $\Omega_{g/A}$ is the unique B-module map making the square in the above diagram commute. Similarly, since $f; g; \delta_g = f; 0 = 0$, $\Omega_{C/f}$ is the unique C-module map making the triangle commute. The implicit naming conventions for these maps should be self-evident.

Forming a commutative diagram analogous to that of the first diagram of the previous proof, it follows that if $q : \Omega_{C/A} \longrightarrow M$ is a C-module map to some C-module M such that $1_C \otimes_B \Omega_{g/A}; \bullet_B; q = 0$, then $g; \delta_{f;g}; q = 0$. Hence $\delta_{f;g}; q$ factors through δ_g via some C-module map h:

$$\begin{tikzcd}
 & & M \\
 & \Omega_{C/A} \arrow[ur, "q"] \arrow[r, "\Omega_{C/f}"] & \Omega_{C/B} \arrow[u, dashed, "h"'] \\
B \arrow[r, "g"'] & C \arrow[u, "\delta_{f;g}"] \arrow[ur, "\delta_g"'] &
\end{tikzcd}$$

The universal property of $\delta_{f;g}$ necessitates that $q = \Omega_{C/f}; h$

To complete the proof it is required to show that $\Omega_{C/f}$ is epi. Note that this can be done very easily if we assume that the Hom sets are equipped with additive inverses. In general though, we may observe the following:

$$\begin{tikzcd}
\Omega_{C/A} \arrow[rr, "\Omega_{C/f}"] & & \Omega_{C/B} \\
& C \arrow[ul, "\delta_{f;g}"'] \arrow[ur, "\delta_g"] \arrow[dl, "d_C"'] \arrow[dr, "d_C"] & \\
\Omega_C \arrow[uu, "\phi_{C/A}"] \arrow[rr, "1_{\Omega_C}"'] & & \Omega_C \arrow[uu, "\phi_{C/B}"']
\end{tikzcd}$$

Each triangle commutes by definition, and so the square commutes because of the universal property of d_C. The fact that $\phi_{C/B}$ is epi implies here that $\Omega_{C/f}$ is epi also. ∎

The above exact sequence is referred to as the short Jacobi-Zariski exact sequence to distinguish it from the long Jacobi-Zariski exact sequence, which is comprised of André-Quillen homology groups and extends the above short exact sequence. Consider then, the following definition:

Definition 5. Let $< B, \omega >$ be a T-algebra in a codifferential category. The n^{th} *André-Quillen homology group of* B, denoted $D_n^T(B)$ is the n^{th} homology group of the following complex:

$$\ldots \longrightarrow B \otimes_{T^3B} \Omega_{T^3B} \longrightarrow B \otimes_{T^2B} \Omega_{T^2B} \longrightarrow B \otimes_{TB} \Omega_{TB}$$

The maps here are the alternating sums $(\sum_{i=0}^{n}(-1)^i \Omega_{T^{n-i}\mu T^i}) + \Omega_{T^n \nu}$ tensored on the left with the identity map.

Similarly, the *higher André-Quillen homology groups*, denoted $D_n^{(p)T}(B)$ for $p > 1$ are the homology groups of the following complex

$$\ldots \longrightarrow B \otimes_{T^3B} \Omega^p_{T^3B} \longrightarrow B \otimes_{T^2B} \Omega^p_{T^2B} \longrightarrow B \otimes_{TB} \Omega^p_{TB}$$

A glaring omission here is that of the definition of the relative André-Quillen homology groups $D_n^T(B|A)$ for a morphism of T-algebras $f :< A, \alpha > \longrightarrow < B, \beta >$. This would require, analogous to the case $T = \mathcal{S}$, an appropriate monad T_A defined relative to A, which is equipped with codifferential structure. Contingent on the eventual discovery of such monads and subsequent definition of relative André-Quillen homology, we have the following

Conjecture 6. *Let* $f :< A, \alpha > \longrightarrow < B, \beta >$ *and* $g :< B, \beta > \longrightarrow < C, \kappa >$ *be morphisms of* T*-algebras in a codifferential category. Then there is a long exact sequence of the following form:*

$$\to D_{n+1}^T(C|B) \to C \otimes_B D_n^T(B|A) \to D_n^T(C|A) \to D_n^T(C|B) \to D_{n-1}^T(C|A) \to$$

The obstacles to realizing this formulation are manifold. Unfortunately, the utilization of this conjecture is sorely missed in the computation of André-Quillen homology groups. A more thorough discussion of these problems is deferred to the concluding chapter of this work.

Fortunately, there are still instances where the computation of André-Quillen groups is tractable.

Proposition 7. *For all A in a codifferential category, $D_n^T(TA) = 0$ for $1 \leq n$*

This will be proven for the case $n = 1$; the remainder of cases follow analogously. First, it must be recognized that since we are calculating André-Quillen homology in a codifferential category, the modules of differential forms in which we are interested are T-modules of differential forms. Hence at each stage of the André-Quillen homology complex for an object A, we have an object of the form $A \otimes_{T^nA} \Omega_{T^nA} = A \otimes_{T^nA} T^nA \otimes T^{n-1}A \cong A \otimes T^{n-1}A$. This explicit construction is helpful in computations that follow, and is refined by the following lemma:

Lemma 8. *Let A be an object in a codifferential category, and let $(\mu_i^j)_A$ be the map $T^jA \longrightarrow T^iA$ which is the composite $\mu_{T^{j-1}A}; ...; \mu_{T^iA}$. Then the following diagram is a coequalizer:*

$$TA \otimes T^nA \otimes \Omega_{T^nA} \rightrightarrows TA \otimes \Omega_{T^nA} \xrightarrow{1 \otimes (\mu_1^n)_A \otimes 1; m_{TA} \otimes 1} TA \otimes T^{n-1}A$$

The upper and lower left maps in the above lemma are the actions of T^nA on TA and Ω_{T^nA}, respectively. A slightly more general lemma which replaces TA by a general T-algebra is possible, but the statement is cumbersome, the proof is analogous to the proof of this lemma, and the more general lemma is superfluous for our purposes here.

Proof: The equivalence of the two paths is computed as follows:

$$\begin{aligned}
&(\mu_1^n)_A \otimes 1 \otimes 1; 1 \otimes (\mu_1^n)_A \otimes 1_{\Omega_{T^nA}}; m_{TA} \otimes 1; 1 \otimes (\mu_1^n)_A \otimes 1_{T^{n-1}A}; m_{TA} \otimes 1_{T^{n-1}A} \\
=&(\mu_1^n)_A \otimes (\mu_1^n)_A \otimes 1_{\Omega_{T^nA}}; m_{TA} \otimes 1; 1 \otimes (\mu_1^n)_A \otimes 1_{T^{n-1}A}; m_{TA} \otimes 1 \\
=&m_{T^nA} \otimes 1_{\Omega_{T^nA}}; (\mu_1^n)_A \otimes (\mu_1^n)_A \otimes 1_{T^{n-1}A}; m_{TA} \otimes 1 \\
=&m_{T^nA} \otimes 1_{\Omega_{T^nA}}; m_{T^nA} \otimes 1_{T^{n-1}A}; (\mu_1^n)_A \otimes 1_{T^{n-1}A} \\
=&1 \otimes m_{T^nA} \otimes 1_{T^{n-1}A}; m_{T^nA} \otimes 1_{T^{n-1}A}; (\mu_1^n)_A \otimes 1_{T^{n-1}A} \\
=&1 \otimes m_{T^nA} \otimes 1_{T^{n-1}A}; (\mu_1^n)_A \otimes (\mu_1^n)_A \otimes 1_{T^{n-1}A}; m_{TA} \otimes 1 \\
=&(\mu_1^n)_A \otimes 1 \otimes 1; 1 \otimes m_{T^nA} \otimes 1_{T^{n-1}A}; 1 \otimes (\mu_1^n)_A \otimes 1_{T^{n-1}A}; m_{TA} \otimes 1_{T^{n-1}A}
\end{aligned}$$

Since $(\mu_1^n)_A \otimes 1 \otimes 1$ is epi, the equivalence has been confirmed.

Now, since $1 \otimes (\mu_1^n)_A \otimes 1; m_{TA} \otimes 1$ is an epimorphism, if it can be shown that the projection $p_n : TA \otimes \Omega_{T^nA} \longrightarrow TA \otimes_{T^nA} \Omega_{T^nA}$ factors through this map, the proof will be complete. To this end:

$$1_{TA} \otimes (\mu_1^n)_A \otimes 1; m_{TA} \otimes 1; 1 \otimes u_L; 1 \otimes e_{T^nA} \otimes 1; p_n$$

$$
\begin{aligned}
&=1 \otimes 1 \otimes u_L; 1 \otimes 1 \otimes e_{T^nA} \otimes 1; 1 \otimes (\mu_1^n)_A \otimes 1 \otimes 1; m_{TA} \otimes 1 \otimes 1; p_n \\
&=1 \otimes 1 \otimes u_L; 1 \otimes 1 \otimes e_{T^nA} \otimes 1; 1 \otimes \bullet_n; p_n \\
&=p_n
\end{aligned}
$$

∎

Proof: (**Proof of Proposition 5**)
Consider the following complex, closely related to the André-Quillen complex for TA over k:

$$
\begin{tikzcd}
\ldots \arrow[r] & TA \otimes \Omega_{T^4A} \arrow[r] & TA \otimes \Omega_{T^3A} \arrow[r] & TA \otimes \Omega_{T^2A}
\end{tikzcd}
$$

Clearly this complex surjects onto the André-Quillen complex for TA. Consider the rightmost map of this complex, $1 \otimes \Omega_{\mu_T} - 1 \otimes \Omega_{T\mu}$. Following it by $1 \otimes \Omega_{T^2\eta}$ gives

$$
\begin{aligned}
&1 \otimes \Omega_{\mu_T}; 1 \otimes \Omega_{T^2\eta} - 1 \otimes \Omega_{T\mu}; 1 \otimes \Omega_{T^2\eta} \\
=&1 \otimes \Omega_{T^3\eta}; 1 \otimes \Omega_{\mu_{T^2}} - 1 \otimes \Omega_{T^3\eta}; 1 \otimes \Omega_{T\mu_T} \\
=&1 \otimes \Omega_{T^3\eta}; (1 \otimes \Omega_{\mu_{T^2}} - 1 \otimes \Omega_{T\mu_T})
\end{aligned}
$$

Modulo the image of the map $1 \otimes \Omega_{\mu_{T^2}} - 1 \otimes \Omega_{T\mu_T} + 1 \otimes \Omega_{T^2\mu}$, this last map is equivalent to $1 \otimes \Omega_{T^3\eta}; 1 \otimes \Omega_{T^2\mu} = 1 \otimes 1$. Hence the first homology group of the above complex is trivial.

Consider the following diagram:

$$
\begin{tikzcd}
\ldots \arrow[r] & TA \otimes_{T^4A} \Omega_{T^4A} \arrow[r] & TA \otimes_{T^3A} \Omega_{T^3A} \arrow[r] & TA \otimes_{T^2A} \Omega_{T^2A} \\
\ldots \arrow[r] & TA \otimes T^3A \arrow[r] \arrow[u, "\cong"] & TA \otimes T^2A \arrow[r] \arrow[u, "\cong"] & TA \otimes TA \arrow[u, "\cong"] \\
\ldots \arrow[r] & TA \otimes \Omega_{T^4A} \arrow[r] \arrow[u, "1\otimes(\mu_1^4)\otimes 1; m_{TA}\otimes 1"] & TA \otimes \Omega_{T^3A} \arrow[r] \arrow[u, "1\otimes(mu_1^3)\otimes 1; m_{TA}\otimes 1"] & TA \otimes \Omega_{T^2A} \arrow[u, "1\otimes(mu_1^3)\otimes 1; m_{TA}\otimes 1"]
\end{tikzcd}
$$

The middle row is comprised of the maps of the top row composed with the appropriate isomorphisms, and therefore the middle row is a complex which is equivalent to the that of the top row.

Let us utilize elemental notation briefly for the sake of clarity; the translation to category-theoretic language is routine. If $a \otimes b$ is an element of the kernel of the map $TA \otimes T^2A \longrightarrow TA \otimes TA$, then there is an $x \otimes y \in TA \otimes \Omega_{T^3A}$ which maps to it, and we have $(1 \otimes (\Omega_{\mu_T} - \Omega_{T\mu}); 1 \otimes \mu \otimes 1; m_{TA} \otimes 1)(x \otimes y) = 0$. According to the discussion above, $x \otimes y$ is homologous to $(1 \otimes (\Omega_{\mu_T} - \Omega_{T\mu}); 1 \otimes \Omega_{T^2\eta})(x \otimes y)$. Applying $1 \otimes (\mu_1^3) \otimes 1; m_{TA} \otimes 1$ then gives

$$
1 \otimes (\Omega_{\mu_T} - \Omega_{T\mu}); 1 \otimes \Omega_{T^2\eta}; 1 \otimes (\mu_1^3) \otimes 1; m_{TA} \otimes 1
$$

$$
\begin{aligned}
=&1\otimes(\Omega_{\mu_T}-\Omega_{T\mu});1\otimes T^2\eta\otimes T\eta;1\otimes\mu_T\otimes 1;1\otimes\mu\otimes 1;m_{TA}\otimes 1\\
=&1\otimes(\Omega_{\mu_T}-\Omega_{T\mu});1\otimes(\mu;T\eta;\mu)\otimes 1;m_{TA}\otimes 1;1\otimes T\eta\\
=&1\otimes(\Omega_{\mu_T}-\Omega_{T\mu});1\otimes\mu\otimes 1;m_{TA}\otimes 1;1\otimes T\eta
\end{aligned}
$$

Applying this map to $x\otimes y$ produces 0, which is therefore homologous to $a\otimes b$ ∎

In a similar manner, one may prove that $D_n^{(m)T}(TA)=0$ for any $m\geq 1$. The critical observation is that

$$\sum_{i=0}^{n}(-1)^i T^i\mu_{T^{n-i}};T^{n+1}\eta=T^{n+2}\eta;\sum_{i=0}^{n}(-1)^i T^i\mu_{T^{n-i}}$$

and that, crucially, $T^n\eta$ is a morphism of T-algebras.

In the case where $< A,\alpha >$ is a general T-algebra, it is not clear that any such map exists. The computation of the André-Quillen homology of general T-algebras therefore requires more subtle considerations.

5.2 HKR Theorem

The following definition is certainly provisional. Indeed, a primary goal for future work in this area should be to make the following definition a theorem, and to develop a much more constructive definition. For the time being though, this definition is sufficiently powerful, general, and parsimonious.

Definition 9. Let $< A,\alpha >$ be a T-algebra in a commutative codifferential category. Then A is said to be *T-smooth* if it is flat as a k-module and if $D_n^{(m)T}(A)=0$ for all $m,n\in\mathbb{N}$ such that $1\leq m,n$.

Some motivation is in order. Consider the following bicomplex:

$$
\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
(TA)^{\otimes 3} \arrow[d, "b"] & (T^2A)^{\otimes 3} \arrow[l] \arrow[d, "-b"'] & (T^3A)^{\otimes 3} \arrow[l] \arrow[d, "b"'] & \ldots \arrow[l] \\
(TA)^{\otimes 2} \arrow[d, "b"] & (T^2A)^{\otimes 2} \arrow[l] \arrow[d, "-b"'] & (T^3A)^{\otimes 2} \arrow[l] \arrow[d, "b"'] & \ldots \arrow[l] \\
TA & T^2A \arrow[l] & T^3A \arrow[l] & \ldots \arrow[l]
\end{tikzcd}
$$

Here the n^{th} row is the canonical simplicial resolution of $A^{\otimes n}$, and hence its homology is isomorphic to $A^{\otimes n}$ at the first column, and 0 elsewhere. Therefore the n^{th} homology group of this bicomplex is $HH_n(A)$.

On the other hand, if we compute the vertical homology of a given column, we are clearly computing the Hochschild homology of a given T^nA. If we are to pursue the computation of the spectral sequence associated to this bicomplex then, $E^2_{p,q}$ will be the q^{th} homology group of the following complex:

$$\dots \longrightarrow HH_p(T^3A) \longrightarrow HH_p(T^2A) \longrightarrow HH_p(TA)$$

where the morphisms arise as the application of the functor HH_p to the canonical simplicial resolution of A. Under the conditions of **4.1.3**, this is equivalent to the complex

$$\dots \longrightarrow \Omega^{(p)T}_{T^3A} \longrightarrow \Omega^{(p)T}_{T^2A} \longrightarrow \Omega^{(p)T}_{TA}$$

Thus, we have the following

Theorem 10. *Let $\mathcal{C}$ be a codifferential category in which filtered colimits preserve exactness and the tensor product preserves colimits and let T be a smooth monad. If $< A, \nu >$ is a T-smooth T-algebra, then $\Omega^{nT}_{A,\nu} \cong HH_n(A)$ for all n.*

Proof: By the above discussion, there is a convergent spectral sequence with

$$E^2_{pq} = D^{(q)T}_p(A) = H_p(\Omega^{qT}_{T^pA})$$

and converging to $HH_{p+q}(A)$. Since $D^{(q)T}_p(A) = 0$ when $p > 0$ by hypothesis and $D^{(q)T}_0(A) \cong \Omega^{qT}_{A,\nu}$ by **3.3.16**, the result follows. ∎

Chapter 6

A More General HKR Theorem

In the study of noncommutative geometry, the HKR theorem is invoked as a justification for the use of cyclic (co-)homology as the correct analogue to commutative de Rham cohomology. This tacitly suggests that smooth algebras are the fundamental objects of study for algebraic differential structure and that the failure of an algebra to satisfy the conclusion of the HKR theorem is an exhibition of an aberration; otherwise, there is no reason to believe that Hochschild homology should be used to interpret the differential structure of algebras in general. It is thus the task of a noncommutative algebraic geometer to decide which noncommutative algebras are sufficiently analogous to commutative smooth algebras to be considered the fundamental objects of study for noncommutative differential structure.

In this chapter, a reformulation of the HKR theorem is described, which facilitates the appraisal of smoothness in a more general context. Specifically, for any given algebra modality T - commutative or not - it phrases the HKR theorem in terms of a comparison between the Hochschild homology of a given T-algebra and the Hochschild homology of related free T-algebras. It is suggested here that this new type of HKR theorem could serve as a litmus test for any putative notion of algebraic smoothness.

6.1 The Classical HKR Theorem Reconsidered

Here, the new approach to the HKR theorem is motivated by the consideration of the original HKR theorem explicitly elaborated in the context of the codifferential category of k-modules with algebra modality $\mathcal{S}$. First, observe the following diagram:

$$\begin{tikzcd} \Omega^n_{S^2A} \arrow[r, shift left, "\Omega^n_\mu"] \arrow[r, shift right, "\Omega^n_{S\nu}"'] \arrow[d] & \Omega^n_{SA} \arrow[r, "\Omega^n_\nu"] \arrow[d] & \Omega^n_{A,\nu} \arrow[d] \\ HH_n(S^2A) \arrow[r, shift left, "HH_n(\mu)"] \arrow[r, shift right, "HH_n(S\nu)"'] & HH_n(SA) \arrow[r, "HH_n(\nu)"] & HH_n(A) \end{tikzcd}$$

Here S is the symmetric algebra monad and $< A, \nu >$ is an S-algebra. **3.3.16** tells us that the top row is a coequalizer diagram. If A is flat as a k-module, we know that the two left-most vertical maps are isomorphisms. It follows then that for A flat, the bottom row is a coequalizer if and only if $\Omega^n_{A,\nu} \cong HH_n(A)$. We may thus rephrase the HKR theorem:

Theorem 1. *If A is a flat k-module and a smooth commutative algebra with S-algebra structure map ν, then for any n the following diagram is a coequalizer diagram:*

$$\begin{tikzcd} HH_n(S^2A) \arrow[r, shift left, "HH_n(\mu)"] \arrow[r, shift right, "HH_n(S\nu)"'] & HH_n(SA) \arrow[r, "HH_n(\nu)"] & HH_n(A) \end{tikzcd}$$

Pertinent here is the fact that this statement of the HKR theorem makes no reference to the differential forms for the algebra A. As a result, we may replace S by any algebra modality T - commutative or not - and ask under what conditions the corresponding collection of diagrams are coequalizer diagrams. In the next section it will be shown that if we replace S with F, the corresponding diagrams will be coequalizers when A is noncommutative smooth.

First though, a curious fact about the above restatement of the HKR theorem is that its conclusion is satisfied in cases of algebras for which the original HKR theorem's conclusion may not have been satisfied. Namely, the conclusion of the HKR theorem can be shown to hold for a free commutative algebra SA only when the underlying k-module A is flat. As we will see momentarily, flatness is not required for the satisfaction of the restatement's conclusion:

Proposition 2. *Let A be a k-module. Then the following diagram is a coequalizer diagram:*

$$\begin{tikzcd} HH_n(S^3A) \arrow[r, shift left, "HH_n(\mu_{SA})"] \arrow[r, shift right, "HH_n(S\mu)"'] & HH_n(S^2A) \arrow[r, "HH_n(\mu)"] & HH_n(SA) \end{tikzcd}$$

Proof: Consider the following diagram for any natural number, n:

$$\begin{tikzcd}
S^3A^{\otimes n-1} \arrow[r, "\mu_{SA}^{\otimes n-1}-(S\mu)^{\otimes n-1}"] & S^2A^{\otimes n-1} \arrow[r, "\mu^{\otimes n-1}"] & SA^{\otimes n-1} \arrow[r] & 0 \\
S^3A^{\otimes n} \arrow[u, "b"] \arrow[r, "\mu_{SA}^{\otimes n}-(S\mu)^{\otimes n}"] & S^2A^{\otimes n} \arrow[u, "b"] \arrow[r, "\mu^{\otimes n}"] & SA^{\otimes n} \arrow[u, "b"] \arrow[r] & 0 \\
S^3A^{\otimes n+1} \arrow[u, "b"] \arrow[r, "\mu_{SA}^{\otimes n+1}-(S\mu)^{\otimes n+1}"] & S^2A^{\otimes n+1} \arrow[u, "b"] \arrow[r, "\mu^{\otimes n+1}"] & SA^{\otimes n+1} \arrow[u, "b"] \arrow[r] & 0
\end{tikzcd}$$

Each row is exact. Taking homology vertically calculates the n^{th} Hochschild homology groups of - from left to right - S^3A, S^2A, and SA. It will be shown that the bottom row of the following commutative diagram is exact:

$$\begin{tikzcd}
S^3A^{\otimes n} \arrow[r, "\mu_{SA}^{\otimes n}-(S\mu)^{\otimes n}"] & S^2A^{\otimes n} \arrow[r, "\mu^{\otimes n}"] & SA^{\otimes n} \arrow[r] & 0 \\
ker(b) \arrow[u] \arrow[r] & ker(b) \arrow[u] \arrow[r] & ker(b) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

The proposition will then follow by commutativity of colimits.

The fact that the above diagram is exact at $ker(b : S^2A^{\otimes n} \longrightarrow S^2A^{\otimes^{n-1}})$ is elementary. What remains to be proven then is that the induced map

$$(ker(b : S^2A^{\otimes n} \longrightarrow S^2A^{\otimes^{n-1}})) \longrightarrow (ker(b : SA^{\otimes n} \longrightarrow SA^{\otimes^{n-1}}))$$

is surjective. This is immediate though, since $S\eta^{\otimes^n}; \mu^{\otimes^n} = id$ and the following diagram is commutative:

$$\begin{tikzcd}
S^2A^{\otimes n-1} & SA^{\otimes n-1} \arrow[l, "S\eta^{\otimes n-1}"'] \\
S^2A^{\otimes n} \arrow[u, "b"] & SA^{\otimes n} \arrow[l, "S\eta^{\otimes n}"'] \arrow[u, "b"]
\end{tikzcd}$$

∎

The apparent discrepancy here - that a commutative algebra, A, may satisfy the conclusion of this new HKR theorem but not the original - can be resolved when one recognizes that the equivalence between the two theorems is contingent on precisely the fact that the associated free $\mathcal{S}$-algebras, $\mathcal{S}A$ and $\mathcal{S}^2A$, satisfy the conclusion of the original HKR theorem. This new approach to the HKR theorem should therefore not be thought of as any less stringent, but rather more amenable to a flexible notion of smoothness. The fact that any free $\mathcal{S}$-algebra satisfies the revised HKR theorem accords well with the intuitive notion that smooth algebras should correspond to affine schemes which look locally like affine space.

6.2 Noncommutative Smoothness

There are a few approaches to noncommutative smoothness, but here we will focus on that of Kontsevich and Rosenberg, as expounded in [16]. Although their paper lists a number of equivalent characterizations of smoothness, one that stands out is the following:

Definition 3. A k-algebra A is said to be *noncommutative smooth* if for any algebra B, a two-sided nilpotent ideal I of B and any algebra homomorphism $f : A \longrightarrow B/I$ there exists an algebra homomorphism $g : A \longrightarrow B$ such that $f = g; pr$, where $pr : B \longrightarrow B/I$ is the obvious map.

Contrast this with the following common characterization of smoothness for commutative algebras given in, e.g., **9.3.2** of [**22**], which is arrived at by modifying the above definition so that A and B are stipulated to be commutative algebras. A somewhat subtle point here is that this definition of noncommutative smoothness does not encompass the commutative smooth algebras. For instance, every polynomial ring is smooth in the commutative sense, but polynomial rings in two or more varialbes will fail to be noncommutative smooth. The relationship between the two notions, however, is encapsulated in the comparison between the following theorem and **6.1.1**.

Theorem 4. *If $< A, \nu >$ is a noncommutative smooth k-algebra where k is a field, then for each n the following diagram is a coequalizer diagram:*

$$HH_n(F^2A) \underset{HH_n(F\nu)}{\overset{HH_n(\mu)}{\rightrightarrows}} HH_n(FA) \xrightarrow{HH_n(\nu)} HH_n(A)$$

Before commencing with the portion of this proof that is entirely new, it should be explained that a good deal of this result follows trivially after an appropriation of some relevant literature.

For instance, it can be shown that for $n \geq 2$, $HH_n(\mathcal{F}V) = 0$ for any k-module V (Proposition 3.1.4, [17]). Hence if it can be shown that when A is smooth, $HH_n(A) = 0$ for $n \geq 2$, then the above theorem will only require consideration of the cases $n = 0$ and $n = 1$.

To prove that for a smooth algebra, A, $HH_n(A) = 0$ for $n \geq 2$, first note that the following two conditions are equivalent to smoothness [7]:

i A has cohomological dimension ≤ 1 with respect to Hochschild cohomology

ii Ω_A^{NC} is a projective bimodule over A

These equivalences essentially follow from the fact that for any A-module M, the square-zero extensions of A by M are characterized by $HC^2(A, M)$, that $HC^n(A, M) \cong Ext_n^{A^e}(A, M)$, and that the following sequence is exact:

$$0 \longrightarrow \Omega^{NC}_{A/k} \longrightarrow A \otimes A \xrightarrow{m_A} A \longrightarrow 0$$

That is, these conditions are all equivalent to the declaration that

$$\Omega^{NC}_{A/k} \longrightarrow A \otimes A$$

is a resolution for A which is projective over A^e. But then this also provides us a projective resolution with which to calculate $Tor_n^{A^e}(A, A) = HH_n(A)$. The necessary conclusion immediately follows.

Proof: The fact that the above diagram is a coequalizer for $n = 0$ is a consequence of the commutativity of colimits.

Now, let $n = 1$. Consider Connes' long exact sequence applied to the algebra A:

$$... \longrightarrow HH_n(A) \longrightarrow HC_n(A) \longrightarrow HC_{n-2}(A) \longrightarrow HH_{n-1}(A) \longrightarrow ...$$

Since $HH_n(A) = 0$ for $n > 1$, we obtain the exact sequence:

$$0 \to HC_3(A) \longrightarrow HC_1(A) \to 0 \to HC_0(A) \longrightarrow HH_1(A) \longrightarrow HC_1(A) \to 0$$

The first consequence of this to note is that $HC_1(A) \cong HC_3(A)$. The following diagram will be shown to be exact:

$$HC_3(F^2A) \xrightarrow{HC_3(\mu) - HC_3(F\nu)} HC_3(FA) \xrightarrow{HC_3(\nu)} HC_3(A) \longrightarrow 0$$

The only contentious point here is exactness at $HC_3(A)$. For this, consider any element $a_0 \otimes a_1 \otimes a_2 \otimes a_3 \in A^{\otimes^4}$ and its counterpart denoted in the exact same way in $FA^{\otimes^4}$. Obviously, $\nu(a_0) \otimes ... \otimes \nu(a_3) = a_0 \otimes ... \otimes a_3$ - here we have neglected to emphasize that these are actually classes of elements, though without any loss of rigor. It must be shown that $a_0 \otimes ... \otimes a_3$ is a cycle. Since k is a field, we may consider Connes' complex for cyclic homology; in particular, we examine the behavior of the element $a_0 \otimes ... \otimes a_3$ under the map:

$$b : FA^{\otimes^4}/(1-t) \longrightarrow FA^{\otimes^3}/(1-t)$$

First note that b sends $a_0 \otimes a_1 \otimes a_2 \otimes a_3$ to

$$(a_0 \otimes a_1) \otimes a_2 \otimes a_3 - a_0 \otimes (a_1 \otimes a_2) \otimes a_3 + a_0 \otimes a_1 \otimes (a_2 \otimes a_3) - (a_3 \otimes a_0) \otimes a_1 \otimes a_2$$

This is, of course, equivalent to the following element in $FA^{\otimes 3}$:

$$a_0 \otimes (a_1 \otimes a_2) \otimes a_3 - a_3 \otimes (a_0 \otimes a_1) \otimes a_2$$

Modulo $1-t$, this is equivalent to

$$a_0 \otimes a_1 \otimes a_2 \otimes a_3 - a_0 \otimes a_1 \otimes a_2 \otimes a_3 = 0$$

So $a_0 \otimes ... \otimes a_3$ is in the kernel of b, and it follows that the above diagram is exact. Applying Connes' long exact sequence to FA and F^2A gives us that $HC_3(FA) \cong HC_1(FA)$ and $HC_3(F^2A) \cong HC_1(F^2A)$, respectively. It follows that the following sequence is also exact:

$$HC_1(F^2A) \xrightarrow{HC_1(\mu)-HC_1(F\nu)} HC_1(FA) \xrightarrow{HC_1(\nu)} HC_1(A) \longrightarrow 0$$

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
HC_0(F^2A) \arrow[rr, "HC_0(\mu)-HC_0(F\nu)"] \arrow[d] & & HC_0(FA) \arrow[r, "HC_0(\nu)"] \arrow[d] & HC_0(A) \arrow[r] \arrow[d] & 0 \\
HH_1(F^2A) \arrow[rr, "HH_1(\mu)-HH_1(F\nu)"] \arrow[d] & & HH_1(FA) \arrow[r, "HH_1(\nu)"] \arrow[d] & HH_1(A) \arrow[r] \arrow[d] & 0 \\
HC_1(F^2A) \arrow[rr, "HC_1(\mu)-HC_1(F\nu)"] \arrow[d] & & HC_1(FA) \arrow[r, "HC_1(\nu)"] \arrow[d] & HC_1(A) \arrow[r] \arrow[d] & 0 \\
0 & & 0 & 0 &
\end{tikzcd}$$

The top and bottom (non-trivial) rows are exact, as are each of the columns. What needs to be shown is that the middle row is exact. Once again, exactness of the middle row at $HH_1(A)$ is all that is left. It is a routine diagram chase to prove that $HH_1(\nu)$ is surjective; this will be carried out now for the sake of completeness.

Let a be an element of $HH_1(A)$, which maps to $b \in HC_1(A)$. Since $HC_1(\nu)$ is surjective, there is an $x \in HC_1(FA)$ such that $HC_1(\nu)(x) = b$. Exactness of the middle column gives us a $y \in HH_1(FA)$, which maps to x. Since $HH_1(\nu)(y)$ and a are both mapped to b, there is a $z \in HC_0(A)$, which maps to $a - HH_1(\nu)(y)$ and a $q \in HC_0(FA)$ such that $HC_0(\nu)(q) = z$. Also denote by q this element's image in $HH_1(FA)$. $HH_1(\nu)(q+y)$ is then equal to $a - HH_1(\nu)(y) + HH_1(\nu)(y) = a$, and surjectivity is established. ∎

A few comments on the requirement that k be a field are necessary. First, this allows us to use the equivalence $Tor_n^{A^e}(A, A) \cong HH_n(A)$ unreservedly. If we were to generalize this theorem for a general commutative ring k, we would have to account for this. This is not a great impediment though; for we could either substitute the assumption that k is a field for the assumption that A is flat, or we could use the common convention of defining Hochschild homology by this Tor functor.

The greater difficulty lies in the utilization of cyclic homology above. When $k \supseteq \mathbb{Q}$, the definition of cyclic homology substantially simplifies and consequently facilitates the computation of the Hochschild homology group in question. It is hoped that only a subtle adaptation of the above proof would suffice in this more general case. In this spirit of hope, the following is put forth

Conjecture 5. *If $< A, \nu >$ is a noncommutative smooth k-algebra where k is a commutative ring, then for each n the following diagram is a coequalizer diagram:*

$$\begin{tikzcd} HH_n(F^2A) \arrow[r, shift left, "HH_n(\mu)"] \arrow[r, shift right, "HH_n(F\nu)"'] & HH_n(FA) \arrow[r, "HH_n(\nu)"] & HH_n(A) \end{tikzcd}$$

Chapter 7

Conclusion

Throughout this work, the impetus has been the desire to develop a characterization of smoothness that is customizable for a given category of T-algebras. That is, it was hoped that given a monad and its T-algebras, we could give a criterion for a given T-algebra to be T-smooth. This was, perhaps, too ambitious, but the difficulties incurred produced other interesting questions, some of which had intriguing answers. For instance, although it would have been preferable to have a fully realized version of André-Quillen homology, even broaching this investigation necessitated developing the modules of T-differential forms relative to a T-algebra morphism; this latter object has particularly elegant descriptions. Also, the definition of a smooth monad, which was based on a prototypical commutative algebra modality, resulted from the fact that the investigation of smoothness for T-algebras should, presumably, be facilitated by an investigation of the properties of monads, and led to the comparison between noncommutative algebra modalities and commutative ones, which was quite successful. Finally, the requirement imposed in this work that any definition of smoothness should be accompanied by an appropriate HKR theorem fostered the statement of the HKR theorem which is adaptable to the noncommutative realm.

That being said, a number of sections in this work are laden with threads which ought to be followed. The most obvious one leads to a fully realized André-Quillen homology theory for codifferential categories which simulates the classical André-Quillen homology relative to a general algebra map. Without this, one is precluded from defining and, subsequently, evaluating the exactness of the long Jacobi-Zariski sequence. Consequently, it would have been utterly premature to posit a definition of T-smoothness which did not invoke a vanishing condition on André-Quillen homology.

To explain this last point, in the case of smooth algebras in the category of modules over a commutative ring, one can prove (Theorem 25.2, p. 194, [18]) that the short Jacobi-Zariski sequence is left exact. It then follows from the long Jacobi-Zariski exact sequence that $D_1(A) = 0$. Furthermore, using **5.1.7** in the cases $T = \mathcal{S}$ and $T = \mathcal{S}_{\mathcal{A}}$, it follows - again by the long Jacobi-Zariski exact sequence - that $D_n(A) = 0$

for $n \geq 1$. A similar deduction holds for the higher André-Quillen homology groups, and so the HKR theorem follows as described in section **5.2**.

As mentioned in section **5.1**, the challenge of this resides in finding analogues to $\mathcal{S}_A$ in a general codifferential category. More specifically, in order to construct a long Jacobi-Zariski exact sequence in a codifferential category, one needs first to construct presimplicial resolutions for objects of the form $\Omega^T_{B/A}$. One approach to this is, given a T-algebra $< A, \nu >$ and a morphism of T-algebras $f :< A, \nu > \longrightarrow < B, \omega >$, construct the pushout

$$\begin{tikzcd} TB \arrow[r] & T_A B \\ \mathcal{S}B \arrow[u] \arrow[r] & \mathcal{S}_A B \arrow[u] \end{tikzcd}$$

This appears, at first, to be a tenable solution. For instance, one may construct an ostensibly suitable map $T_A B \longrightarrow T_A B \otimes_A B$ which is prospectively a T_A-derivation. Unfortunately, it is not at all clear that one can in this way construct a monad, nor is it immediate that $T_A B$ will inherit a commutative algebra structure. Without a monad structure, it is impossible to specify a codifferential categorical structure. Furthermore, it is difficult to see how one would circumvent monad structure to define a presimplicial resolution of $\Omega^T_{B/A}$ directly.

Another example of a deficiency which involves the monad structure is found in the statement of **4.2.12**. Here the monad structure is built into the premise of the theorem, when naturally it would be desirable to have it arise as a consequence of the coequalizer construction. It is instructive to examine the construction of $\mathcal{S}$ from $\mathcal{F}$ via this coequalizer to see where the difficulty lies. The point is that the universal property which defines $\mathcal{S}$ is precisely encapsulated by this coequalizer construction; the $\mathcal{S}$-algebra structures can be expressed directly using properties of the monoidal structure, which is precisely what is invoked in the construction of this coequalizer.

To develop this work, one should search for examples of noncommutative codifferential categories and subsequently calculate the prescribed coequalizer. It would be interesting to encounter examples both where commutative codifferential categorical structure results, and where it does not. Furthermore, it would be interesting to investigate the conditions on a given commutative codifferential category which allow it to be manifested by the application of this construction to some noncommutative codifferential category, and moreover, whether such a noncommutative codifferential category is necessarily unique.

Finally, the conjecture with which this work closes alludes to further work to be done in the examination of noncommutative smoothness. One approach to this could potentially determine a proof of the HKR theorem, which applies to any codifferential category, regardless of whether or not it is commutative. Observe that in the discussion preceding **5.2.10**, the conditions of **4.1.3** were invoked in order to transform the E^1 groups of the spectral sequence in such a way that enabled the utilization of André-Quillen homology. The reasons for this were entirely pragmatic; classical André-Quillen homology is well-developed and powerful. But we could have instead attempted to directly calculate the homology of complexes of the form

$$\ldots \longrightarrow HH_n(T^3A) \longrightarrow HH_n(T^2A) \longrightarrow HH_n(TA)$$

to arrive at the second stage of the spectral sequence. If it could be shown that all of the homology groups vanished, except at $HH_n(TA)$, then it would follow that for all n

$$HH_n(T^2A) \underset{HH_n(T\nu)}{\overset{HH_n(\mu)}{\rightrightarrows}} HH_n(TA) \xrightarrow{HH_n(\nu)} HH_n(A)$$

is a coequalizer. Although this project would be difficult, we are not entirely bereft of tools. Consider for instance, the Jacobi-Zariski long exact sequence for Hochschild homology, found in [14].

Bibliography

[1] J. Beck. *Triples, Algebras and Cohomology.* Thesis, (1967). *can be found in:* Reprints in Theory and Applications of Categories

[2] R. Blute, R. Cockett, T. Porter, R. Seely. Kähler categories, Cahiers de Topologie et Geometrie Differentielle 52, pp.253-268, (2011)

[3] R. Blute, R. Cockett, R. Seely. Differential categories, Mathematical Structures in Computer Science 16, pp. 1049-1083,(2006).

[4] R. Blute, R.B.B Lucyshyn-Wright, K. O'Neill. Derivations in codifferential categories, preprint, (2015).

[5] R. Cockett. Lectures on noncommutative Kähler categories. Preprint (2014).

[6] A. Connes. *Noncommutative Geometry.* Academic Press, 1994. http://www.alainconnes.org/docs/book94bigpdf.pdf

[7] J. Cuntz, D. Quillen. Algebra extensions and nonsingularity. Journal of the American Mathematical Society, 8 Number 2, pp. 251-289, (1995).

[8] T. Ehrhard, L. Regnier. The differential λ-calculus. Theoretical Computer Science 309, pp. 1-41, (2003).

[9] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer-Verlag, (2004).

[10] W. Fulton, J. Harris. *Representation Theory: A First Course.* Springer-Verlag, (1991).

[11] H. Figueroa, J. M. Gracia-Bondía, J. C. Várilly. *Elements of Noncommutative Geometry.* Birkhäuser, (2001).

[12] J.-Y. Girard. Linear logic. Theoretical Computer Science 50, pp. 1-102, (1987).

[13] P.T. Johnstone *Topos Theory.* Academic Press, (1977).

[14] A. Kaygun. Jacobi-Zariski exact sequence for Hochschild homology and cyclic (co)homology, Homology, Homotopy and Applications, Vol. 14, pp. 65-78, (2012).

[15] M. Khalkhali. *Basic Noncommutative Geometry.* European Mathematical Society, (2015).

[16] M. Kontsevich, A. Rosenberg. Noncommutative smooth spaces, The Gelfand Mathematical Seminars, pp. 85-108, (1998).

[17] K.-L. Loday. *Cyclic Homology.* Springer-Verlag, (1998).

[18] H. Matsumura. *Commutative Ring Theory.* Cambridge University Press, (1986).

[19] J. McCleary. *A User's Guide to Spectral Sequences.* Cambridge University Press, (2001).

[20] D. Quillen. On the (co-)homology of commutative rings. Proc. Symp. on Categorical Algebra, pp. 65-87, (1970).

[21] R.A.G.Seely. Linear logic, *-autonomous categories and cofree coalgebras, Contemporary Mathematics, Volume 92, (1989).

[22] C.A. Weibel. *An Introduction to Homological Algebra.* Cambridge University Press, (1997).