

**Equivariant functions  
for the Möbius subgroups  
and applications**

by

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# Abstract

The aim of this thesis is to give a self-contained introduction to the hyperbolic geometry and the theory of discrete subgroups of  $\mathrm{PSL}_2(\mathbb{R})$ , and to generalize the work initiated in [16, 4] on equivariant functions to them. We show that there is a deep relation between the geometry of the group and some analytic and algebraic properties of these functions. In addition, we provide some applications of equivariant functions consisting of new results as well as providing new and simple proofs to classical results on automorphic forms.

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# Introduction

Let  $\mathbb{H} = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$  be the upper-half plane of the complex plane, and  $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{\pm I\}$ , where  $\text{SL}_2(\mathbb{R})$  is the group of real  $2 \times 2$  matrices with determinant one. The group  $\text{PSL}_2(\mathbb{R})$  acts on  $\mathbb{H}$  by Möbius transformations, *i.e.*

$$\gamma z = \frac{az + b}{cz + d}, \quad z \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).$$

Suppose that  $\Gamma$  is a subgroup of  $\text{PSL}_2(\mathbb{R})$  and  $\mathcal{F}(\mathbb{H})$  be the set of all functions from  $\mathbb{H}$  to  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Then we have an action of  $\Gamma$  on  $\mathcal{F}(\mathbb{H})$  given by

$$\gamma * h = \gamma^{-1}h\gamma.$$

The set of fixed points of this action is

$$\mathcal{E}(\Gamma) = \{h \in \mathcal{F}(\mathbb{H}), \gamma * h = h \quad \forall \gamma \in \Gamma\} = \{h \in \mathcal{F}(\mathbb{H}), \gamma h = h\gamma \quad \forall \gamma \in \Gamma\}.$$

An element  $h$  of  $\mathcal{E}(\Gamma)$  is called an *equivariant function* with respect to  $\Gamma$ , or a  $\Gamma$ -*equivariant function*.

In the papers [15] and [4], a serious work was initiated on equivariant functions for the modular group and its subgroups. Indeed, they establish the basic properties of these functions, and give many non trivial connections with other topics such as elliptic functions, modular forms, quasi-modular forms, differential forms and sections of line bundles. Our goal here is to go beyond the modular group and consider arbitrary discrete and non discrete subgroups of  $\text{PSL}_2(\mathbb{R})$  and generalize the theory of equivariant functions. As expected, the geometry of these groups will play a crucial role in our study.

In order to carry out this generalization, it was necessary to set the tools as well as a rich geometric terminology that will allow us to fully exploit the  $\Gamma$ -equivariance.

To build this terminology, the first chapter was entirely devoted to give a self-contained introduction to hyperbolic geometry and the theory of discrete subgroups of  $\mathrm{PSL}_2(\mathbb{R})$ .

The main idea of the hyperbolic geometry is to construct new types of geodesics, *i.e.* the shortest curves between two points, which will replace the straight lines in the Euclidean geometry. For example, the hyperbolic metric  $\rho$  is defined by

$$\rho(z, w) = \inf h(\gamma), \quad z, w \in \mathbb{H},$$

where

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt$$

is the length of the piecewise differentiable curve  $\gamma : I \rightarrow \mathbb{H}$  with  $I = [0, 1]$  with endpoints  $z, w$ , and defined by

$$\gamma(t) = \{v(t) = x(t) + iy(t), t \in I\},$$

The upper half-plane  $\mathbb{H}$  equipped with this metric is a model of the hyperbolic plane (we will see that the unit disc is also a model of the hyperbolic plane). The model  $(\mathbb{H}, \rho)$  has two fundamental properties, namely

1. Geodesics are the semicircles and the rays that are orthogonal to the real axis  $\mathbb{R}$ .
2. The group of real Möbius transformations, which is identified with  $\mathrm{PSL}_2(\mathbb{R})$ , together with the transformation  $(z \rightarrow -\bar{z})$  generate the group  $\mathrm{Isom}(\mathbb{H})$  of isometries of  $(\mathbb{H}, \rho)$ .

This second property will have a crucial role in the proof of one of the main results of Chapter 1, namely, the equivalence between the discreteness of a subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$ , endowed with the topology of  $\mathbb{R}^4$ , and the fact that  $\Gamma$  acts properly discontinuously on  $\mathbb{H}$ , *i.e.* for all  $x \in \mathbb{H}$ , the set  $\Gamma x$  has no accumulation point in  $\mathbb{H}$ , and the order of the stabilizer  $\Gamma_x$  of each point is finite. Discrete subgroups of  $\mathrm{PSL}_2(\mathbb{R})$  will be called *Fuchsian* groups. If  $\Gamma$  is a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ , its limit set  $\Lambda(\Gamma)$  is defined to be the set of all possible limit points of  $\Gamma$ -orbits  $\Gamma z$ ,  $z \in \mathbb{H}$ , for a Fuchsian group  $\Gamma$ . In fact,  $\Lambda(\Gamma)$  is a subset of  $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  and it plays a crucial role in the classification of Fuchsian groups. Indeed, we will see that there are three types of Fuchsian groups, namely

1.  $\Lambda(\Gamma) = \widehat{\mathbb{R}}$ . In this case  $\Gamma$  is called a Fuchsian group of the first kind.

2.  $\Lambda(\Gamma) \neq \widehat{\mathbb{R}}$  but contains at least three points. In this case  $\Gamma$  is called a Fuchsian group of the second kind.
3.  $\Lambda(\Gamma)$  contains at most two points. In this case  $\Gamma$  is called an elementary group (this definition is valid even if  $\Gamma$  is not Fuchsian).

It is along this classification that we explore the properties, like surjectivity, injectivity, and holomorphy of the equivariant functions when considered as functions from  $\mathbb{H}$  to the extended plane  $\widehat{\mathbb{C}}$ . For example, one of the main results of Chapter 2 asserts that under some local conditions, which are automatically satisfied by meromorphic functions, an equivariant function for a non elementary group  $\Gamma$  attains any point of the extended plane infinitely many times once its image contains a point in  $\Lambda(\Gamma)$ . In particular, a meromorphic equivariant function will have infinitely many non  $\Gamma$ -equivalent poles. We will also see that these points are in fact zeros of some special functions called quasi-automorphic forms.

A special attention is given to the classes of all meromorphic and holomorphic equivariant functions, which are respectively denoted by  $\mathcal{E}_m(\Gamma)$ ,  $\mathcal{E}^*(\Gamma)$ . Indeed, we provide a complete description of  $\mathcal{E}_m(\Gamma)$  when  $\Gamma$  is a non discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  by showing that  $\mathcal{E}_m(\Gamma)$  is either the commutator  $C_{\mathrm{PSL}_2(\mathbb{C})}(\Gamma)$  of  $\Gamma$ , or  $C_{\mathrm{PSL}_2(\mathbb{C})}(\Gamma)$  together with a special point in  $\widehat{\mathbb{C}}$ . When  $\Gamma$  is a Fuchsian group of the first kind, inspired by the work of M. Heins, [8], we show that  $\mathcal{E}(\Gamma) = \{id_{\mathbb{H}}\}$  using the celebrated theorem of Denjoy-Wolff.

The objective of Chapter 3 is to give explicit constructions of equivariant functions and to use the above results and classifications to give some interesting applications. Since we are working in the most general cases of subgroups of  $\mathrm{PSL}_2(\mathbb{R})$ , we are constrained to highlight some definitions. For  $z \in \mathbb{H}$ ,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ , we let

$$j_\gamma(z) = j(\gamma, z) = cz + d.$$

For any complex valued function  $f$  defined on  $\mathbb{H}$ , the slash operator of weight  $k$  on  $f$  is defined by

$$(f|_k\gamma)(z) = j_\gamma(z)^{-k} f(\gamma z).$$

A meromorphic function  $f$  defined on  $\mathbb{H}$  is called an unrestricted automorphic form of weight  $k \in \mathbb{R}$  and multiplier system (MS)  $\nu$  for a subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$  if it satisfies

$$f|_k\gamma = \nu(\gamma)f \text{ for all } \gamma \in \Gamma,$$

where  $\nu(\gamma)$  is a complex number of modulus 1. An unrestricted almost meromorphic automorphic form of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$  is a function  $f$  on  $\mathbb{H}$  which transforms like an automorphic form but, instead of being holomorphic, it is a polynomial in  $1/y$ ,  $y = \text{Im}(z)$ , with meromorphic coefficients. More precisely,  $f$  has the form

$$f(z) = \sum_{n=0}^p \frac{f_n(z)}{y^n}, \quad z \in \mathbb{H}, \quad (1)$$

where each  $f_n$  is a meromorphic function, and for all  $\gamma \in \Gamma$

$$f|_k \gamma = \nu(\gamma)f. \quad (2)$$

The constant term  $f_0$  of  $f$ , which is a meromorphic function, is called an unrestricted quasi-automorphic form of weight  $k$ , depth  $p$ , and MS  $\nu$  on  $\Gamma$ . More intrinsic definitions of quasi-automorphic forms will be provided in Chapter 3. The word *unrestricted* is removed from the above definitions if some growth conditions at some special points of the group, called *cusps*, are added.

If  $k$  is a nonzero real number and  $f$  is a nonzero unrestricted automorphic form on  $\Gamma$  of weight  $k$  and MS  $\nu$ , we attach to  $f$  the meromorphic function

$$h_f(z) = z + k \frac{f(z)}{f'(z)}. \quad (3)$$

Then  $h_f$  is a meromorphic equivariant function. Similarly, when  $f$  is a non zero unrestricted almost meromorphic automorphic form on  $\Gamma$  of weight  $k$  and MS  $\nu$ , the same formula gives rise to an equivariant function where the operator  $\frac{d}{dz}$  is replaced by  $\frac{\partial}{\partial x}$  (the two operators coincide on meromorphic functions). The class of equivariant functions constructed in this way is referred to as the class of rational equivariant functions.

More generally, any function  $g$  on  $\mathbb{H}$  verifying the following transformation property

$$(g|_2 \gamma)(z) = g + \frac{c}{cz + d}, \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad (4)$$

gives rise to an equivariant function given by

$$h_g = h_0 + \frac{1}{g}, \quad h_0 = id_{\mathbb{H}}.$$

The rational equivariant functions turn out to be extremely useful in studying automorphic forms for discrete and non discrete subgroups of  $\text{PSL}_2(\mathbb{R})$ . This is being illustrated

in Chapter 3 where we classify unrestricted automorphic forms for a non discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  in an elementary and elegant way. In fact, the question of classification of these forms has been raised by several authors [11, 2], and different methods have been used to provide the answer. Our method relying on equivariance is new and simple. A second application is to prove the new and unexpected result, asserting that an automorphic form of weight  $k \neq 0$ , and MS  $\nu$  on a Fuchsian group of the first kind  $\Gamma$  has infinitely many non  $\Gamma$ -equivalent critical points in  $\mathbb{H}$ . A third application is a corollary to the second one providing again a new and simple proof of the fact that the Fourier series expansion of an automorphic form cannot have only finitely many nonzero coefficients. This important result is classically proved in a non elementary way using the  $L$ -function of the automorphic form in question.

In conclusion, the theory of equivariant functions not only provides new objects that are interesting by themselves, but they allow us to prove new results in the theory of automorphic and modular forms and re-prove classical results in an elementary way. We believe that more applications of the theory of equivariant functions and connections with other fields in mathematics and theoretical physics can be established.

# Chapter 1

## Möbius transformations

The content of this chapter is entirely based on the references [1] and [9]. We will review the properties of Möbius transformations and the geometry of discrete groups.

### 1.1 Classification of Möbius transformations

Let  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  and  $a, b, c, d$  be complex numbers with  $ad - bc \neq 0$ , then the map  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  defined by

$$g(z) = \frac{az + b}{cz + d}$$

is called a *Möbius transformation*. If  $\mathcal{M}$  denotes the set of all Möbius transformations, then  $\mathcal{M}$ , equipped with the composition of maps, is a group. Moreover, the map  $A \rightarrow g_A$ , where

$$g_A(z) = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a group homomorphism  $\phi : \text{GL}_2(\mathbb{C}) \rightarrow \mathcal{M}$ , and

$$\text{Ker}(\phi) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \neq 0 \right\}.$$

In general, we shall be more concerned with the restriction of  $\phi$  to  $\text{SL}_2(\mathbb{C})$ . The kernel of this restriction is

$$\text{Ker}(\phi) \cap \text{SL}_2(\mathbb{C}) = \{-I, I\}$$

and therefore each  $g$  in  $\mathcal{M}$  is the projection of two matrices  $A$  and  $-A$ . Hence  $\mathcal{M}$  is isomorphic to  $\text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\{-I, I\}$ .

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C})$  and  $\text{tr}(A) = a + d$ , then the function

$$\frac{\text{tr}^2(A)}{|\det(A)|}$$

is invariant under the transformation  $A \rightarrow \lambda A$ ,  $\lambda \neq 0$ , and so it induces a function on  $\mathcal{M}$ , namely

$$\text{Tr}(g) = \frac{\text{tr}^2(A)}{|\det(A)|}$$

where  $A$  is any matrix which projects on  $g$ . Notice that  $\text{Tr}(g)$  is invariant under conjugation.

Since a non identity Möbius transformation  $g$  has either one fixed point or two distinct fixed points, then this gives us a simple way to classify them. However, it is more convenient to make a classification in term of conjugacy classes. To achieve this, we introduce some normalized Möbius transformations. For each  $k \neq 0$  in  $\mathbb{C}$ , we define  $m_k$  by

$$m_k(z) = kz \quad \text{if } k \neq 1$$

and

$$m_1(z) = z + 1.$$

Notice that for all  $k \neq 0$ , we have

$$\text{Tr}(m_k) = k + \frac{1}{k} + 2.$$

If  $g \neq I$  is any Möbius transformation, we denote by  $\alpha \in \widehat{\mathbb{C}}$  its fixed point if it has a unique one, and by  $\alpha$  and  $\beta$  in  $\widehat{\mathbb{C}}$ ,  $\alpha \neq \beta$ , if it has two distinct fixed points. Now let  $h$  be any Möbius transformation such that

$$h(\alpha) = \infty, \quad h(\beta) = 0, \quad h(g(\beta)) = 1 \text{ if } g(\beta) \neq \beta,$$

then

$$hgh^{-1}(\infty) = \infty, \quad hgh^{-1}(0) = \begin{cases} 0 & \text{if } g(\beta) = \beta \\ 1 & \text{if } g(\beta) \neq \beta. \end{cases}$$

If  $g$  fixes  $\alpha$  and  $\beta$ , then  $hgh^{-1}$  fixes 0 and  $\infty$  and  $hgh^{-1} = m_k$  for some  $k \neq 1$ . If  $g$  fixes  $\alpha$  only, then  $hgh^{-1}$  fixes  $\infty$  only and  $hgh^{-1}(0) = 1$  and thus  $hgh^{-1} = m_1$ . Therefore, any nonidentity Möbius transformation is conjugate to one of the standard form  $m_k$ .

**Definition 1.1.1.** Let  $g \neq I$  be any Möbius transformation. We say that

1.  $g$  is *parabolic* if and only if  $g$  is conjugate to  $m_1$  (equivalently  $g$  has a unique fixed point in  $\widehat{\mathbb{C}}$ );
2.  $g$  is *loxodromic* if and only if  $g$  is conjugate to  $m_k$  for some  $k$  satisfying  $|k| \neq 1$  ( $g$  has exactly two fixed points in  $\widehat{\mathbb{C}}$ );
3.  $g$  is *elliptic* if and only if  $g$  is conjugate to  $m_k$  for some  $k$  satisfying  $|k| = 1$  ( $g$  has exactly two fixed points in  $\widehat{\mathbb{C}}$ ).

It is convenient to subdivide the loxodromic class by reference to invariant discs rather than fixed points.

**Definition 1.1.2.** Let  $g$  be a loxodromic transformation. We say that  $g$  is *hyperbolic* if  $g(D) = D$  for some open disc or half-plane  $D$  in  $\widehat{\mathbb{C}}$ . Otherwise  $g$  is said to be *strictly loxodromic*.

Now, using the fact that  $Tr(g)$  is invariant under conjugation, we can give a complete classification of all Möbius transformations. Indeed, we have the following results.

**Theorem 1.1.1.** *Let  $f$  and  $g$  be two Möbius transformations, neither is the identity. Then  $Tr(f) = Tr(g)$  if and only if  $f$  and  $g$  are conjugate.*

**Theorem 1.1.2.** *Let  $g \neq I$  be any Möbius transformation. Then*

1.  $g$  is *parabolic* if and only if  $Tr(g) = 4$ ;
2.  $g$  is *elliptic* if and only if  $Tr(g) \in [0, 4)$ ;
3.  $g$  is *hyperbolic* if and only if  $Tr(g) \in (4, +\infty)$ ;
4.  $g$  is *strictly loxodromic* if and only if  $Tr(g) \notin [0, +\infty)$ .

We end this section by giving a result concerning the iterates of some Möbius transformations, namely the loxodromic and parabolic ones.

**Theorem 1.1.3.**

1. *Let  $g$  be a parabolic Möbius transformation with a fixed point  $\alpha$ . Then for all  $z \in \widehat{\mathbb{C}}$ ,  $g^n(z) \rightarrow \alpha$  as  $n \rightarrow \infty$ ; the convergence being uniform on compact subsets of  $\mathbb{C} - \{\alpha\}$ .*
2. *Let  $g$  be a loxodromic Möbius transformation. Then the fixed points  $\alpha$  and  $\beta$  of  $g$  can be labeled so that  $g^n(z) \rightarrow \alpha$  as  $n \rightarrow \infty$  for all  $z \in \widehat{\mathbb{C}} - \{\beta\}$ ; the convergence being uniform on compact subsets of  $\mathbb{C} - \{\beta\}$ . Here  $\alpha$  is called an attractive point and  $\beta$  a repulsive point.*

## 1.2 Hyperbolic geometry

### 1.2.1 The hyperbolic metric

Let  $\mathbb{H} = \{z = x + iy \in \mathbb{C}, \text{Im}(z) > 0\}$  be the Poincaré upper half-plane. When equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y},$$

$\mathbb{H}$  becomes a model of the *hyperbolic plane*. It can be used to calculate the length of curves in  $\mathbb{H}$  the same way the Euclidean metric  $\sqrt{dx^2 + dy^2}$  is used to calculate the length of curves on the Euclidean plane.

Let  $I = [0, 1]$  be the closed unit interval, and  $\gamma : I \rightarrow \mathbb{H}$  be a piecewise differentiable curve in  $\mathbb{H}$ ,

$$\gamma(t) = \{v(t) = x(t) + iy(t), t \in I\}.$$

The length of the curve  $\gamma$  is defined by

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt.$$

We define the *hyperbolic distance* between two points  $z, w \in \mathbb{H}$  by

$$\rho(z, w) = \inf h(\gamma),$$

where the infimum is taken over all piecewise differentiable curves  $\gamma$  connecting  $z$  and  $w$ .

**Proposition 1.2.1.** *The function  $\rho : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$  defined above is a distance.*

Let  $\text{SL}_2(\mathbb{R})$  be the group of real  $2 \times 2$  matrices with determinant one. If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$  and  $g_A(z) = \frac{az + b}{cz + d}$  is the associated Möbius transformation, then  $g_A$  maps  $\mathbb{H}$  into  $\mathbb{H}$ . Indeed, we can write

$$w = g_A(z) = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2}.$$

Therefore,

$$\text{Im}(w) = \frac{\text{Im}(z)}{|cz + d|^2}. \tag{1.1}$$

Thus  $\text{Im}(z) > 0$  implies  $\text{Im}(w) > 0$ .

Let  $\mathcal{M}^*$  denote the group of real Möbius transformations. As seen in the first section, the map  $\phi : \text{SL}_2(\mathbb{R}) \rightarrow \mathcal{M}^*$  given by  $A \rightarrow g_A$  is a surjective group homomorphism, and  $\text{Ker}(\phi) = \{-I, I\}$ . Thus  $\mathcal{M}^*$  can be identified with  $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{-I, I\}$ . Since the transformations in  $\mathcal{M}^*$  are continuous, we have the following result.

**Theorem 1.2.2.** *The group  $\text{PSL}_2(\mathbb{R})$  acts on  $\mathbb{H}$  by homeomorphisms.*

**Definition 1.2.1.** A transformation of  $\mathbb{H}$  onto itself is called an *isometry* if it preserves the hyperbolic distance in  $\mathbb{H}$ .

If  $\text{Isom}(\mathbb{H})$  denotes the group of all isometries of  $\mathbb{H}$ , then we have the following.

**Theorem 1.2.3.** *The group  $\text{PSL}_2(\mathbb{R})$  is contained in  $\text{Isom}(\mathbb{H})$ .*

*Proof.* Let  $T \in \text{PSL}_2(\mathbb{R})$  and  $\gamma : I \rightarrow \mathbb{H}$  be a piecewise differentiable curve given by  $z(t) = x(t) + iy(t)$ . Let

$$w = T(z) = \frac{az + b}{cz + d},$$

and set  $w(t) = T(z(t)) = u(t) + iv(t)$ . Differentiating, we obtain

$$\frac{dw}{dz} = \frac{1}{(cz + d)^2}. \tag{1.2}$$

By (1.1), we have

$$v = \frac{y}{|cz + d|^2}$$

and therefore

$$\left| \frac{dw}{dz} \right| = \frac{v}{y}.$$

Thus

$$h(T(\gamma)) = \int_0^1 \frac{\left| \frac{dw}{dt} \right|}{v(t)} dt = \int_0^1 \frac{\left| \frac{dw}{dz} \right| \left| \frac{dz}{dt} \right|}{v(t)} dt = \int_0^1 \frac{\left| \frac{dz}{dt} \right|}{y(t)} dt = h(\gamma).$$

The theorem follows. □

## 1.2.2 Geodesics

**Definition 1.2.2.** The shortest curves with respect to the hyperbolic metric are called *geodesics*.

**Theorem 1.2.4.** *The geodesics in  $\mathbb{H}$  are semicircles and the rays orthogonal to the real axis  $\mathbb{R}$ .*

*Proof.* Let  $z_1, z_2 \in \mathbb{H}$ . We begin by the case in which  $z_1 = ia, z_2 = ib$  with  $b > a$ . For any piecewise differentiable curve  $\gamma(t) = x(t) + iy(t)$  connecting  $z_1$  and  $z_2$ , we have

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt \geq \int_0^1 \frac{\left|\frac{dy}{dt}\right|}{y(t)} dt = \int_b^a \frac{dy}{y} = \ln\left(\frac{b}{a}\right),$$

which is exactly the hyperbolic length of the segment of the imaginary axis connecting  $ia$  and  $ib$ . Therefore the geodesic connecting  $z_1$  and  $z_2$  is the segment of the imaginary axis connecting them.

We now consider the case of arbitrary points  $z_1$  and  $z_2$ . Let  $L$  be the unique Euclidean semicircle or a straight line connecting them and suppose that  $L$  meets the real axis at a point  $\alpha$ . Then the transformation  $T(z) = \frac{-1}{z - \alpha} + \beta \in \text{PSL}_2(\mathbb{R})$  for an appropriate value of  $\beta$ , maps  $L$  into the positive imaginary axis. One can conclude using Theorem 1.2.3 and the above case.  $\square$

Let  $z$  and  $w$  be two points in  $\mathbb{H}$ , and let  $[z, w]$  denote the unique geodesic segment joining them. Then using the above theorem and the definition of the hyperbolic metric we get

**Corollary 1.2.5.** *If  $z$  and  $w$  are two distinct points in  $\mathbb{H}$ , then*

$$\rho(z, w) = \rho(z, u) + \rho(u, w)$$

*if and only if  $u \in [z, w]$ .*

As consequence we have

**Theorem 1.2.6.** *Any isometry of  $\mathbb{H}$ , and, in particular any transformation in  $\text{PSL}_2(\mathbb{R})$ , maps geodesics into geodesics.*

The *cross-ratio* of four distinct points  $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$  is defined by:

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

A key property of the cross-ratio is its invariance under Möbius transformations. This will be used to prove the following.

**Theorem 1.2.7.** *Let  $z, w \in \mathbb{H}$  be two distinct points with the geodesic joining  $z$  and  $w$  having endpoints  $z^*, w^* \in \mathbb{R} \cup \{\infty\}$ , and  $z \in [z^*, w]$ . Then*

$$\rho(z, w) = \ln[w, z^*, z, w^*].$$

*Proof.* Let  $T \in \text{PSL}_2(\mathbb{R})$  be the unique Möbius transformation which maps the geodesic joining  $z$  and  $w$  to the imaginary axis. Applying the transformations  $z \rightarrow kz$  ( $k > 0$ ) and  $z \rightarrow \frac{-1}{z}$  if necessary, we may assume that  $T(z^*) = 0$ ,  $T(w^*) = \infty$  and  $T(z) = i$ . Then  $T(w) = ri$  for some  $r > 1$ , and

$$\rho(T(z), T(w)) = \int_i^{ri} \frac{dy}{y} = \ln(r).$$

On the other hand,  $[ri, 0, i, \infty] = r$ , and so the theorem follows from the invariance of the cross-ratio under Möbius transformations.  $\square$

We shall give a compact formula for the hyperbolic distance involving the hyperbolic functions

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2}, \quad \cosh(x) = \frac{\exp(x) + \exp(-x)}{2}.$$

**Theorem 1.2.8.** *For  $z, w \in \mathbb{H}$ , we have*

$$\begin{aligned} \rho(z, w) &= \ln \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}, \\ \cosh(\rho(z, w)) &= 1 + \frac{|z - w|^2}{2\text{Im}(z)\text{Im}(w)}, \\ \sinh\left(\frac{1}{2}\rho(z, w)\right) &= \frac{|z - w|}{2(\text{Im}(z)\text{Im}(w))^{\frac{1}{2}}}. \end{aligned}$$

*Proof.* One can check that the three equations are equivalent. We will prove the third equation by first noticing that both sides are invariant under Möbius transformations. Therefore it is sufficient to check the formula when  $z = i$  and  $w = ir$  ( $r > 1$ ). In this case  $\rho(z, w) = \ln(r)$ . A simple calculation leads to the desired result.  $\square$

We now describe another model for the hyperbolic plane, namely the *unit disc*

$$\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}.$$

The Möbius transformation

$$f(z) = \frac{z - i}{z + i} \tag{1.3}$$

is a homeomorphism from  $\mathbb{H}$  onto  $\mathbb{U}$ , with the inverse map

$$f^{-1}(z) = -i \frac{z + 1}{z - 1}. \tag{1.4}$$

Thus  $\rho^*$  given for  $z, w \in \mathbb{U}$  by

$$\rho^*(z, w) = \rho(f^{-1}(z), f^{-1}(w))$$

is a metric on  $\mathbb{U}$ . However, as

$$\frac{2|f(z)|}{1 - |f(z)|^2} = \frac{1}{\operatorname{Im}(z)}, \tag{1.5}$$

we can identify  $\rho^*$  with the metric derived from the differential

$$ds = \frac{2|dz|}{1 - |z|^2}. \tag{1.6}$$

We will use  $\rho$  for  $\rho^*$ , and so  $f$  is an isometry of  $(\mathbb{H}, \rho)$  onto  $(\mathbb{U}, \rho)$ . We shall refer to these two models of the hyperbolic as the Poincaré models, and we shall frequently switch from one model to the other as each has its own particular advantage.

Note that the *principal circle*  $\Sigma = \{z \in \mathbb{C}, |z| = 1\}$  is the Euclidean boundary of  $\mathbb{U}$ ,  $\mathbb{R}$  the Euclidean boundary of  $\mathbb{H}$  and that  $f(\Sigma) = \mathbb{R}$ . Thus, in the model  $\mathbb{U}$ , the geodesics are segments of Euclidean circles orthogonal to the principal circle  $\Sigma$  and its diameters.

### 1.2.3 Isometries

The following Theorem identifies all isometries of the hyperbolic plane  $\mathbb{H}$ .

**Theorem 1.2.9.** *The group  $\operatorname{Isom}(\mathbb{H})$  is generated by the elements of  $PSL_2(\mathbb{R})$  together with the transformation  $z \rightarrow -\bar{z}$ . The group  $PSL_2(\mathbb{R})$  is a subgroup of  $\operatorname{Isom}(\mathbb{H})$  of index two.*

*Proof.* Let  $\phi$  be any isometry of  $\mathbb{H}$ . If  $I$  denotes the positive imaginary axis, then Theorem 1.2.6 shows that  $\phi(I)$  is a geodesic. We know that there exists  $g \in \text{PSL}_2(\mathbb{R})$  which maps  $\phi(I)$  onto  $I$ , and by applying the transformations  $z \rightarrow kz$  ( $k > 0$ ) and  $z \rightarrow -1/z$ , we may assume that  $g \circ \phi$  fixes  $i$  and maps the rays  $(i, \infty)$  and  $(0, i)$  onto themselves, and hence  $g \circ \phi$  fixes each point of  $I$ . Now let  $z = x + iy$ , and  $g \circ \phi = u + iv$ . For all positive  $t$ ,

$$\rho(z, it) = \rho(g \circ \phi(z), g \circ \phi(it)) = \rho(u + iv, it),$$

and by Theorem 1.2.8(3), we have

$$[x^2 + (y - tf)^2]v = [u^2 + (v - t)^2]y.$$

As this holds for all positive  $t$ , dividing both sides of the above equation by  $t^2$  and taking the limit as  $t \rightarrow \infty$ , we get  $v = y$  and  $x^2 = u^2$ . Thus

$$g \circ \phi = z \text{ or } -\bar{z}.$$

Since isometries are continuous, only one of the equations holds for all  $z$  in  $\mathbb{H}$ . The theorem follows from the fact that  $g \in \text{PSL}_2(\mathbb{R})$ .  $\square$

It is clear that  $\text{PSL}_2(\mathbb{R})$  consists of all holomorphic isometries of  $\mathbb{H}$ . If  $f$  is the map given by 1.3, then we have

**Corollary 1.2.10.** *The group of holomorphic isometries of  $\mathbb{U}$  is exactly  $f\text{PSL}_2(\mathbb{R})f^{-1}$ , and so it is a subgroup of  $\text{Isom}(\mathbb{U})$  of index two, and its elements have the following form:*

$$\frac{az + \bar{c}}{cz + \bar{a}}, \text{ where } a, c \in \mathbb{C}, a\bar{a} - c\bar{c} = 1.$$

## 1.3 Fuchsian groups

The group  $\text{GL}_2(\mathbb{C})$  endowed with the standard norm of  $\mathbb{C}^4$  is a topological space, and a subgroup  $\Gamma$  of  $\text{GL}_2(\mathbb{C})$  is called *discrete* if the subspace topology on  $\Gamma$  is the discrete

topology. Thus  $\Gamma$  is discrete if and only if for a sequence  $T_n$  in  $\Gamma$ ,  $T_n \rightarrow T \in \text{GL}_2(\mathbb{C})$  implies  $T_n = T$  for all sufficiently large  $n$ . It is not necessary to assume that  $T \in \Gamma$ . Indeed, in this case,

$$T_n(T_{n+1})^{-1} \rightarrow TT^{-1} = I.$$

Hence, for all sufficiently large  $n$  we have  $T_n = T_{n+1}$  and so  $T_n = T$ .

### 1.3.1 Discrete and properly discontinuous subgroups of $\text{PSL}_2(\mathbb{R})$

**Definition 1.3.1.** A discrete subgroup of  $\text{PSL}_2(\mathbb{R})$  is called a *Fuchsian* group.

Let  $(X, \rho)$  be a locally compact metric space, and let  $G$  be a group of isometries of  $X$ .

**Definition 1.3.2.** A family  $\{M_\alpha, \alpha \in A\}$  of subsets of  $X$  indexed by elements of a set  $A$  is called *locally finite* if for any compact subset  $K$  of  $X$ ,  $M_\alpha \cap K = \emptyset$  for all but a finite number of  $\alpha$  in  $A$ .

**Remark 1.3.1.** Some of the subsets  $M_\alpha$  may coincide but they are still considered different elements of the family.

**Definition 1.3.3.** For  $x \in X$  the family  $Gx = \{g(x), g \in G\}$  is called the  $G$ -orbit of  $x$ . Each point of  $Gx$  is contained with multiplicity equal to the cardinal of the stabilizer  $G_x = \{g \in G, g(x) = x\}$  of  $x$  in  $G$ .

**Definition 1.3.4.** We say that a group  $G$  acts properly discontinuously on  $X$  if the  $G$ -orbit of any point  $x$  in  $X$  is locally finite.

Since  $X$  is locally compact, a group  $G$  acts properly discontinuously on  $X$  if and only if each orbit has no accumulation point in  $X$ , and the order of the stabilizer of each point is finite. The first condition, however, is equivalent to the fact that each orbit of  $G$  is discrete. Indeed, if otherwise  $g_n(x) \rightarrow y \in X$ , then  $\rho(g_n(x), g_{n+1}(x)) \rightarrow 0$ . But since  $g_n$  is an isometry, we have  $\rho(g_n^{-1}g_{n+1}(x), x) \rightarrow 0$ , which implies that  $x$  is an accumulation point for its orbit  $Gx$  and hence  $Gx$  is not discrete.

The main result of this subsection is to show that a subgroup  $\Gamma$  of  $\text{PSL}_2(\mathbb{R})$  is Fuchsian if and only if it acts properly discontinuously on  $\mathbb{H}$ .

**Proposition 1.3.1.** *Let  $z_0 \in \mathbb{H}$  and  $K$  a compact subset of  $\mathbb{H}$ . Then the set*

$$E = \{T \in PSL_2(\mathbb{R}), T(z_0) \in K\}$$

*is compact.*

*Proof.* The map  $\phi_0 : PSL_2(\mathbb{R}) \rightarrow \mathbb{H}$  given by

$$T \rightarrow T(z_0) = \frac{az_0 + b}{cz_0 + d}, T = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a continuous map and  $E = \phi_0^{-1}(K)$  which is closed as it is the inverse image of the closed set  $K$ . We now show that  $E$  is bounded. As  $K$  is bounded there exists  $M_1 > 0$  such that

$$|T(z_0)| = \left| \frac{az_0 + b}{cz_0 + d} \right| < M_1, \text{ for all } T \in E.$$

Since  $K$  is compact in  $\mathbb{H}$ , there exists  $M_2 > 0$  such that

$$\frac{\text{Im}(z_0)}{|cz_0 + d|^2} = \text{Im} \left( \frac{az_0 + b}{cz_0 + d} \right) \geq M_2.$$

Hence

$$|cz_0 + d| \leq \sqrt{\frac{\text{Im}(z_0)}{M_2}}.$$

Thus

$$|az_0 + b| \leq M_1 \sqrt{\frac{\text{Im}(z_0)}{M_2}}.$$

It follows that  $a, b, c, d$  are bounded. □

**Theorem 1.3.2.** *Let  $\Gamma$  be a subgroup of  $PSL_2(\mathbb{R})$ . Then  $\Gamma$  is a Fuchsian group if and only if  $\Gamma$  acts properly discontinuously on  $\mathbb{H}$ .*

*Proof.* We first show that a Fuchsian group acts properly discontinuously on  $\mathbb{H}$ . Let  $z \in \mathbb{H}$  and  $K$  be a compact subset of  $\mathbb{H}$ . The above proposition shows that the set  $\{T \in \Gamma, T(z) \in K\} = \{T \in PSL_2(\mathbb{R}), T(z) \in K\} \cap \Gamma$  is finite since it is the intersection of a compact and a set without accumulation point. Hence  $\Gamma$  acts properly discontinuously. Conversely, suppose  $\Gamma$  acts properly discontinuously, but it is not a discrete subgroup of  $PSL_2(\mathbb{R})$ . Then there exists a sequence  $\{T_k\}$  of distinct elements of  $\Gamma$  such that  $T_k \rightarrow I$ . Since the set of fixed points of the sequence  $\{T_k\}$  is countable, we can find  $x \in \mathbb{H}$  which

is not fixed by any element  $T_k$ . Then  $\{T_k(x)\}$  is a sequence of points distinct from  $x$  which converges to  $x$ . Hence  $x$  is accumulation point of  $\Gamma x$ . Therefore  $\Gamma$  does not act properly discontinuously; a contradiction.  $\square$

Now, using the above theorem and the fact that the elements of  $\mathrm{PSL}_2(\mathbb{R})$  are isometries of the hyperbolic plane  $\mathbb{H}$ , we get the following result.

**Theorem 1.3.3.** *Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ . Then the following statements are equivalent:*

1.  $\Gamma$  acts properly discontinuously on  $\mathbb{H}$ ;
2. For any compact set  $K$  in  $\mathbb{H}$ ,  $T(K) \cap K \neq \emptyset$  for only finitely many  $T \in \Gamma$ ;
3. Any point  $x \in \mathbb{H}$  has a neighborhood  $V$  such that  $T(V) \cap V \neq \emptyset$  implies  $T(x)=x$ .

In the next section we will need the description of discrete subgroups of the additive group  $\mathbb{R}$ , and  $\mathbb{S}^1$  the multiplicative group of complex numbers of modulus 1, and it is more convenient to give this description here accompanied with some examples of Fuchsian group.

**Lemma 1.3.4.** *We have*

1. Any non-trivial discrete subgroup of  $\mathbb{R}$  is infinite cyclic.
2. Any discrete subgroup of  $\mathbb{S}^1$  is finite cyclic.

*Proof.* For the first assertion, let  $\Gamma$  be a discrete subgroup of  $\mathbb{R}$ , then there exists a smallest positive element  $x \in \Gamma$ , otherwise  $\Gamma$  would not be discrete. The set  $\{nx, n \in \mathbb{Z}\}$  is a subgroup of  $\Gamma$ . Suppose there exists  $y \in \Gamma$ ,  $y \neq nx$ . We may assume  $y > 0$ , otherwise we take  $-y$  which is also in  $\Gamma$ . There exists an integer  $k \geq 0$  such that  $kx < y < (k+1)x$ , so  $y - kx < x$  and  $(y - kx) \in \Gamma$  which contradicts the choice of  $x$ .

As for the second assertion, let  $\Gamma$  be a discrete subgroup of  $\mathbb{S} = \{z \in \mathbb{C}, z = \exp(i\theta)\}$ , then by discreteness there exists  $z = \exp(i\theta_0)$  with the smallest argument  $\theta_0$  and for some  $m \in \mathbb{Z}$ ,  $m\theta_0 = 2\pi$  otherwise we get a contradiction with the choice of  $\theta_0$ .  $\square$

Our first examples of Fuchsian group are given by the following theorem.

**Theorem 1.3.5.** *We have*

1. *All hyperbolic and parabolic cyclic subgroups of  $PSL_2(\mathbb{R})$  are Fuchsian groups.*
2. *An elliptic cyclic subgroup of  $PSL_2(\mathbb{R})$  is a Fuchsian group if and only if it is finite.*

**Example 1.3.1.** The modular group  $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{-I, I\}$  and its subgroups are Fuchsian groups.

### 1.3.2 Properties of Fuchsian groups

We first investigate the commuting of elements of a Fuchsian group. If  $G$  is any group and  $g \in G$ , then the centralizer of  $g$  in  $G$  is

$$C_G(g) = \{h \in G, hg = gh\}.$$

**Lemma 1.3.6.** *If  $ST = TS$  then  $S$  maps the fixed-point set of  $T$  to itself.*

Now, suppose that  $T = z + 1$  and  $S \in C_{PSL_2(\mathbb{R})}(T)$ , then  $S(\infty) = \infty$ . Thus  $S(z) = az + b$ , and  $ST = TS$  yields  $a = 1$ . Hence

$$C_{PSL_2(\mathbb{R})}(T) = \{z \rightarrow z + k, k \in \mathbb{R}\}.$$

If  $T$  is elliptic or hyperbolic then, after conjugation, we can suppose that  $T$  has a standard form  $m_k$ ,  $k \neq 1$ , and a similar argument as above gives the following.

**Theorem 1.3.7.** *Two non-identity elements of  $PSL_2(\mathbb{R})$  commute if and only if they have the same fixed-point set.*

**Theorem 1.3.8.** *The centralizer in  $PSL_2(\mathbb{R})$  of a hyperbolic (resp. parabolic, elliptic) element of  $PSL_2(\mathbb{R})$  consists of all hyperbolic (resp. parabolic, elliptic) elements with the same fixed-point set, together with the identity element.*

The next result enables us to characterize all abelian Fuchsian groups.

**Theorem 1.3.9.** *Let  $\Gamma$  be a Fuchsian group all of whose non-identity elements have the same fixed-point set. Then  $\Gamma$  is cyclic.*

*Proof.* Since the nature of an element is given by the type of its fixed-point set, then all elements of  $\Gamma$  must be of the same type. Suppose all elements of  $\Gamma$  are hyperbolic. Then by choosing a conjugate group we may assume that all elements of  $\Gamma$  fix 0 and  $\infty$ . Thus  $\Gamma$  is a discrete subgroup of  $\Omega = \{z \rightarrow kz, k > 0\}$  which is isomorphic as a topological group to  $\mathbb{R}^*$ , the multiplicative group of positive real numbers. But  $\mathbb{R}^*$  is isomorphic as a topological group to  $\mathbb{R}$  via the isomorphism  $x \rightarrow \ln(x)$ . Hence by Lemma 1.3.4,  $\Gamma$  is infinite cyclic. Similarly, if  $\Gamma$  contains a parabolic element, then  $\Gamma$  is an infinite cyclic group containing only parabolic elements. Suppose  $\Gamma$  contains an elliptic element. In  $\mathbb{U}$ , the unit disc model,  $\Gamma$  is a discrete subgroup of orientation-preserving isometries of  $\mathbb{U}$ . By choosing a conjugate group we may assume that all elements of  $\Gamma$  have 0 as a fixed point, and therefore all elements of  $\Gamma$  are of the form  $z \rightarrow e^{i\theta}z$ . Thus  $\Gamma$  is isomorphic to a subgroup of  $\mathbb{S}^1$  and so it is discrete if and only if the corresponding subgroup of  $\mathbb{S}^1$  is discrete. Now the assertion follows from Lemma 1.3.4.  $\square$

**Corollary 1.3.10.** *Every abelian Fuchsian group is cyclic.*

*Proof.* By Theorem 1.3.7, all non-identity elements in an abelian Fuchsian group have the same fixed-point set. The result follows immediately from the above theorem.  $\square$

### 1.3.3 Elementary groups

It is clear that  $\mathrm{PSL}_2(\mathbb{R})$  acts on  $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ , and so we can extend its action on  $\mathbb{H}$  to  $\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ .

**Definition 1.3.5.** A subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$  is called elementary if there exists a finite  $\Gamma$ -orbit in  $\widehat{\mathbb{H}}$ .

Let  $g, h \in \mathrm{PSL}_2(\mathbb{R})$ , the element  $[g, h] = ghg^{-1}h^{-1} \in \Gamma$  is called the *commutator* of  $g$  and  $h$ . Let

$$\mathrm{tr}[g, h] = \mathrm{tr}[A, B], \quad g, h \in \mathrm{PSL}_2(\mathbb{R}) \text{ represented by } A \text{ and } B.$$

This is a well-defined function of  $g$  and  $h$ , since two representatives of a Möbius transformation in  $\mathrm{PSL}_2(\mathbb{R})$  only differ by a sign.

**Theorem 1.3.11.** *Let  $\Gamma$  be a subgroup of  $PSL_2(\mathbb{R})$  containing only elliptic elements besides the identity. Then all elements of  $\Gamma$  have the same fixed point in  $\mathbb{H}$  and hence  $\Gamma$  is an abelian elementary group.*

*Proof.* We shall prove that all elliptic elements in  $\Gamma$  must have the same fixed point. We will work in the unit disc model. After a suitable conjugation, we may assume that an element  $g \neq I$  of  $\Gamma$  fixes 0, hence

$$g = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}.$$

Let  $h \in \Gamma$ ,  $g \neq h$ , then

$$h = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$$

and we have  $|tr[g, h]| = 2 + 4|c|^2(\text{Im}(u))^2$ . Since  $\Gamma$  does not contain hyperbolic elements,  $|tr[g, h]| \leq 2$ . Hence, either  $\text{Im}(u) = 0$  or  $c = 0$ . If  $\text{Im}(u) = 0$  then  $u = \bar{u} \in \mathbb{R}$  and  $g = I$ ; a contradiction. It follows that  $c = 0$ , and so  $h$  fixes 0. Therefore all elements of  $\Gamma$  have the same fixed point. Since 0 is a  $\Gamma$ -orbit,  $\Gamma$  is elementary, and by Theorem 1.3.7 it is abelian.  $\square$

**Corollary 1.3.12.** *Any Fuchsian group containing besides the identity only elliptic elements is a finite cyclic group.*

Let  $\Gamma$  be an elementary Fuchsian subgroup of  $PSL_2(\mathbb{R})$ , and suppose that  $\Gamma\alpha$  is finite, with  $\alpha \in \widehat{\mathbb{H}}$ .

*Case 1.*  $\Gamma\alpha$  contains a single point  $\alpha$ .

If  $\alpha \in \mathbb{H}$ , then all elements of  $\Gamma$  are elliptic and by the above corollary  $\Gamma$  is a finite cyclic group. If  $\alpha \in \mathbb{R} \cup \{\infty\}$ , then  $\Gamma$  cannot have elliptic elements. We shall show that all elements of  $\Gamma$  are either hyperbolic or parabolic. Assume the opposite, and suppose that  $\alpha = \infty$ ,  $g(z) = lz$  ( $l > 1$ ) and  $h(z) = z + k$  (since  $g$  and  $h$  have only one common fixed point,  $k \neq 0$ ). Then  $g^{-n}hg^n(z) = z + l^{-n}k$ . Since  $l > 1$ , the sequence  $g^{-n}hg^n \rightarrow I$  and this contradicts the discreteness of  $\Gamma$ . Now, if  $\Gamma$  contains only parabolic elements, then by Theorem 1.3.9 it is an infinite cyclic group.

Suppose  $\Gamma$  contains only hyperbolic elements. We will prove that in this case their second fixed points must also coincide, and so  $\Gamma$  will fix two points in  $\widehat{\mathbb{R}}$ . Suppose  $f(z) = lz$  ( $l > 1$ ) (it fixes 0 and  $\infty$ ) and

$$g(z) = \frac{az + b}{cz + d}$$

which fixes 0 but not  $\infty$ . Then  $b = 0$ ,  $c \neq 0$ , and  $d = 1/a$ . Then  $[f, g]$  is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t = \frac{c}{a} \left( \frac{1}{l} - 1 \right).$$

Since  $c \neq 0$  we obtain a parabolic element in  $\Gamma$ ; a contradiction.

*Case 2.* Now suppose  $\Gamma\alpha$  consists of two points 0 and  $\infty$  (as we may assume). An element of  $\Gamma$  either fixes each of them or interchanges them. A parabolic element can neither fix nor interchange two points since it is conjugated to  $m_1$ . Hence  $\Gamma$  does not contain any parabolic elements, and by the same argument all hyperbolic elements must have the same fixed point set. If  $\Gamma$  contains only hyperbolic elements, then it is cyclic by Theorem 1.3.9. If it contains only elliptic elements, it is finite cyclic by the above corollary. If  $\Gamma$  contains both hyperbolic and elliptic elements, it must contain an elliptic element of order 2 interchanging the common fixed points of the hyperbolic elements and then  $\Gamma$  is generated by  $g(z) = lz$  ( $l > 1$ ) and  $h(z) = -1/z$ .

*Case 3.* Suppose now  $\Gamma\alpha$  contains at least three points. Since the parabolic elements can have only one fixed point or infinite orbits, and hyperbolic elements can have only two fixed points or infinite orbits,  $\Gamma$  must contain only elliptic elements and therefore is a finite cyclic group.

**Remark 1.3.2.** Suppose  $\Gamma$  is an elementary subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ , not necessarily discrete, with a finite orbit  $\Gamma\alpha$ ,  $\alpha \in \widehat{\mathbb{H}}$ . If  $\Gamma\alpha$  contains at least three points, then  $\Gamma$  contains only elliptic elements and therefore they have the same fixed point, see Theorem 1.3.11.

We summarize the above discussion in the following theorem.

**Theorem 1.3.13.** *Any elementary Fuchsian group is either cyclic or is conjugate in  $\mathrm{PSL}_2(\mathbb{R})$  to a group generated by  $g(z) = lz$  ( $l > 1$ ) and  $h(z) = -1/z$ .*

The next two results give some necessary conditions for a group to be or not an elementary group.

**Theorem 1.3.14.** *A non-elementary subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$  contains infinitely many hyperbolic elements, no two of which have a common fixed point.*

*Proof.* We begin first by showing that  $\Gamma$  has a hyperbolic element. Suppose  $\Gamma$  does not contain hyperbolic elements. If  $\Gamma$  contains only elliptic elements (and  $I$ ), then by

Theorem 1.3.11 it is elementary. Hence  $\Gamma$  contains a parabolic element  $f$  fixing, say,  $\infty$ . So we can take  $f(z) = z + 1$ . Let

$$g(z) = \frac{az + b}{cz + d}$$

be any element in  $\Gamma$ . Then

$$f^n g(z) = \frac{(a + nc)z + (b + nd)}{cz + d}.$$

Thus,

$$\text{Tr}(f^n g) = (a + d + nc)^2.$$

Since all elements in the group are either elliptic or parabolic, we have

$$0 \leq (a + d + nc)^2 \leq 4$$

for all  $n$  and this implies that  $c = 0$ . But then  $g$  also fixes  $\infty$ , so that  $\infty$  is fixed by all elements in  $\Gamma$  and hence  $\Gamma$  is elementary, a contradiction. Now, let  $T$  be a hyperbolic element of  $\Gamma$  with fixed points  $\alpha = \infty$  and  $\beta = 0$ , as we may assume. Since  $\Gamma$  is not elementary, there is some  $S \in \Gamma$  which does not leave  $\{\alpha, \beta\}$  invariant.

Suppose first that the sets  $\{\alpha, \beta\}$  and  $\{S(\alpha), S(\beta)\}$  do not intersect. In this case, the elements  $T$  and  $T_1 = STS^{-1}$  both are hyperbolic and have no common fixed point. The sequence  $\{T^n T_1 T^{-n} \mid n \in \mathbb{Z}\}$  consists of hyperbolic elements with fixed points  $T^n S(\alpha)$  and  $T^n S(\beta)$  which are pairwise different. If the sets  $\{\alpha, \beta\}$  and  $\{S(\alpha), S(\beta)\}$  have one point of intersection, say  $\alpha$ , then  $P = [T, T_1]$  fixes  $\alpha$  (because  $T$  and  $T_1$  do). Since  $\alpha = \infty$ , we have

$$T = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \text{ and } T_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}.$$

An easy computation shows that  $\text{tr}(P) = \text{tr}([T, T_1]) = 2$ , hence  $P$  is parabolic. Now, since  $\alpha$  cannot be  $\Gamma$ -invariant, there exists  $U \in \Gamma$  not fixing  $\alpha$  and therefore  $Q = UPU^{-1}$  is parabolic and does not fix  $\alpha$ . It follows that  $Q$  and  $T$  (or  $Q$  and  $T_1$ ) have no common fixed points. Let  $\gamma$  be the fixed point of  $Q$  and  $\beta_1$  the second fixed point of  $T_1$  ( $\beta_1 \neq \beta$ ). Then by Theorem ??,  $Q^n(\alpha)$  and  $Q^n(\beta)$  (or  $Q^n(\alpha)$  and  $Q^n(\beta_1)$ ) converge to  $\gamma$  when  $n \rightarrow \infty$ . Since  $\gamma$  is different from  $\alpha$  and  $\beta$  (or  $\beta_1$ ), the elements  $Q^n T Q^{-n}$  (or  $Q^n T_1 Q^{-n}$ ) which are hyperbolic will have no common fixed point for a large  $n$ , and so the problem is reduced to the first case.  $\square$

**Theorem 1.3.15.** *If  $\Gamma$  is a subgroup of  $PSL_2(\mathbb{R})$  containing no elliptic elements, then it is either elementary or discrete.*

*Proof.* Assume  $\Gamma$  is non-elementary. Then by the above theorem it contains a hyperbolic element  $h$ . We may assume that  $h$  is given by the matrix

$$\begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}, \quad u > 0.$$

In order to prove that  $\Gamma$  is discrete we have to show that for any sequence  $g_n \rightarrow I$  ( $g_n \in \Gamma$ ),  $g_n = I$  for sufficiently large  $n$ . Let

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

be such a sequence. An easy calculation shows that, since  $g_n \rightarrow I$ ,

$$\text{tr}[h, g_n] = 2 - b_n c_n \left(u - \frac{1}{u}\right) \rightarrow 2 \text{ as } n \rightarrow \infty.$$

The fact that  $\Gamma$  contains no elliptic elements implies that  $|\text{tr}[h, g_n]| \geq 2$  and therefore

$$b_n c_n \leq 0$$

for a sufficiently large  $n$ . Suppose  $f_n = [h, g_n]$  is given by a matrix

$$f_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}.$$

As  $n \rightarrow \infty$ , we have  $f_n \rightarrow I$ , since  $g_n \rightarrow I$ . Therefore by the same argument as above, we obtain for sufficiently large  $n$

$$B_n C_n \leq 0.$$

But

$$\text{tr}[h, f_n] = 2 - B_n C_n \left(u - \frac{1}{u}\right)^2 = 2 + b_n c_n (1 + b_n c_n) \left(u - \frac{1}{u}\right)^4 \rightarrow 2 \text{ as } n \rightarrow \infty,$$

hence

$$b_n c_n \geq 0$$

for sufficiently large  $n$ , and so there exists  $N > 0$  such that for  $n > N$  we have  $b_n c_n = 0$ . Hence for  $n > N$ ,  $h$  and  $g_n$  have a common fixed point which is 0 if  $b_n = 0$  and  $\infty$  if

$c_n = 0$ . Now, by the above theorem, three hyperbolic elements  $h_1, h_2$ , and  $h_3$  are in  $\Gamma$  no two of which have a common fixed point. Hence, for  $n > N$ ,  $g_n$  has a common fixed point with each of  $h_1, h_2$ , and  $h_3$  and no two of which coincide. Therefore  $g_n$  has three fixed points, and thus  $g_n = Id$ .  $\square$

Before going further, we recall the so-called Selberg's Lemma [17].

**Lemma 1.3.16. (*Selberg's lemma*)** *Let  $G$  be a finitely generated subgroup of  $GL_n(\mathbb{C})$ . Then  $G$  contains a normal subgroup of finite index which contains no non-trivial element of finite order.*

The following theorem gives a criterion for discreteness but we skip the proof since it is long and needs more material.

**Theorem 1.3.17.** *A non-elementary subgroup  $\Gamma$  of  $PSL_2(\mathbb{R})$  is discrete if and only if for each  $T$  and  $S$  in  $\Gamma$ , the group  $\langle T, S \rangle$  is discrete.*

We end this subsection with the following theorem which is a summary of several earlier results.

**Theorem 1.3.18.** *Let  $\Gamma$  be a non-elementary subgroup of  $PSL_2(\mathbb{R})$ . Then the following are equivalent:*

1.  $\Gamma$  is discrete;
2.  $\Gamma$  acts properly discontinuously;
3. The fixed points of elliptic elements do not accumulate in  $\mathbb{H}$ ;
4. The elliptic elements do not accumulate to  $I$ ;
5. Each elliptic element has a finite order.

*Proof.* If  $\Gamma$  contains no elliptic elements, then the theorem follows from Theorem 1.3.15, so we suppose that  $\Gamma$  has elliptic elements. By Theorem 1.3.2 we get the equivalence between (1) and (2). (2)  $\Rightarrow$  (3) and (1)  $\Rightarrow$  (4) follow from the definition.

(4)  $\Rightarrow$  (5): If (5) fails, as usual we can suppose that  $\Gamma$  contains an element of the form  $T(z) = e^{2\pi i\theta}z$ , where  $\theta$  is irrational. Since the cyclic group generated by  $T$  has no element of a smallest argument (otherwise  $T$  will have a finite order) one can construct a sequence  $\{T^{a_n}, a_n \in \mathbb{Z}\}$  such that  $T^{a_n} \rightarrow I$  as  $n \rightarrow \infty$ , which is impossible.

(3)  $\Rightarrow$  (5): Suppose (5) does not hold. Then  $\Gamma$  contains an element  $T$  of infinite order, and as above, we construct a sequence  $T^{a_n}, a_n \in \mathbb{Z}$  such that  $T^{a_n} \rightarrow I$  when  $n \rightarrow \infty$ . Let  $x$  be the fixed point of  $T$ . Since  $\Gamma$  is not elementary, there exists  $S \in \Gamma$  such that  $S(x) \neq x$ . Then the sequence  $T^{a_n}S(x)$  which consists of elliptic fixed points of the following sequence  $(T^{a_n}S)T(T^{a_n}S)^{-1}$  converges to  $x$  which is a contradiction since  $T^{a_n} \rightarrow I$  as  $n \rightarrow \infty$ .

(5)  $\Rightarrow$  (1): We view  $\Gamma$  as a group of matrices and let  $S, T \in \Gamma$  and  $\Gamma_0 = \langle T, S \rangle$ . By Selberg's Lemma  $\Gamma_0$  contains a subgroup  $\Gamma_1$  of finite index which contains no elliptic elements of finite order. Hence by our hypothesis, it contains no elliptic elements. Thus by Theorem 1.3.15,  $\Gamma_1$  is discrete. Since  $\Gamma_1$  is of finite index in  $\Gamma_0$ ,  $\Gamma_0$  is also discrete. The discreteness of  $\Gamma$  follows from Theorem 1.3.17.  $\square$

### 1.3.4 Fundamental region

Let  $X$  be a locally compact metric space, and  $G$  be a group of homeomorphisms acting properly discontinuously on  $X$ .

**Definition 1.3.6.** A closed region  $F \subset X$  is called a fundamental region for  $G$  if

1.  $\bigcup_{T \in G} T(F) = X$ ;
2.  $Int(F) \cap T(Int(F)) = \emptyset$  for all  $T \in G - \{Id\}$ , where  $Int(F)$  is the interior of  $F$ .

The family  $\{T(F), T \in G\}$  is called a tessellation of  $X$ .

The existence of a fundamental region for a Fuchsian group will be guaranteed by the following. Let  $\Gamma$  be an arbitrary Fuchsian group and let  $p \in \mathbb{H}$  not fixed by any element of  $\Gamma - \{Id\}$ . Such point exists since by Theorem 1.3.18, the fixed points of elliptic elements of  $\Gamma$  do not accumulate in  $\mathbb{H}$ . We define the *Dirichlet region* for  $\Gamma$  centered at  $p$  to be

the set

$$D_p(\Gamma) = \{z \in \mathbb{H}, \rho(z, p) \leq \rho(z, T(p)) \text{ for all } T \in \Gamma\}.$$

Since the hyperbolic metric is invariant under  $\text{PSL}_2(\mathbb{R})$ , this region can be defined by

$$D_p(\Gamma) = \{z \in \mathbb{H}, \rho(z, p) \leq \rho(T(z), p) \text{ for all } T \in \Gamma\}.$$

**Theorem 1.3.19.** *Let  $\Gamma$  be a Fuchsian group and let  $p \in \mathbb{H}$  not fixed by any element of  $\Gamma - \{Id\}$ . Then  $D_p(\Gamma)$  is a connected fundamental region for  $\Gamma$ .*

**Example 1.3.2.** It can be shown that the set

$$F = \{z \in \mathbb{H}, |z| \geq 1, |Re(z)| \leq \frac{1}{2}\}$$

is a fundamental region for the modular group  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . Moreover  $F = D_p(\Gamma)$  for any  $p = ki$ , where  $k > 1$ .

### 1.3.5 The limit set of a Fuchsian group

**Definition 1.3.7.** Let  $\Gamma$  be a Fuchsian group. Then the set of all possible limit points of  $\Gamma$ -orbits  $\Gamma z, z \in \mathbb{H}$  is called the *limit set* of  $\Gamma$  and denoted by  $\Lambda(\Gamma)$ .

Since  $\Gamma$  is discrete, the  $\Gamma$ -orbits do not accumulate in  $\mathbb{H}$ . Hence  $\Lambda(\Gamma) \subset \widehat{\mathbb{R}}$ . Also note that for an elementary group the limit set consists of at most two points.

In what follows we give the main properties of  $\Lambda(\Gamma)$ , where  $\Gamma$  is a Fuchsian group.

**Proposition 1.3.20.** *Let  $x, y, z \in \mathbb{R}$  be three distinct points with  $x \in \Lambda(\Gamma)$ . Then  $x$  is a limit point either for  $\Gamma y$  or for  $\Gamma z$ .*

**Theorem 1.3.21.** *If  $\Lambda(\Gamma)$  contains more than one point, it is the closure of the set of fixed points of the hyperbolic transformations of  $\Gamma$ .*

**Theorem 1.3.22.** *If  $\Lambda(\Gamma)$  contains more than two point, then either*

1.  $\Lambda(\Gamma) = \widehat{\mathbb{R}}$ , or
2.  $\Lambda(\Gamma)$  is a perfect subset of  $\widehat{\mathbb{R}}$ .

By the above theorem, one can classify Fuchsian groups as follows.

1. If  $\Lambda(\Gamma) = \widehat{\mathbb{R}}$ , then  $\Gamma$  is called a Fuchsian group of the first kind.
2.  $\Lambda(\Gamma) \neq \widehat{\mathbb{R}}$ , then  $\Gamma$  is called a Fuchsian group of the second kind.

# Chapter 2

## Equivariant functions

The notion of equivariant functions and forms was introduced in [14] and [4, 15] for the modular group and its subgroups. In this chapter, we generalize this notion to an arbitrary subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  (discrete or not). As we will not impose any conditions on the behavior of these functions on the cusps, we will consider only the equivariant functions instead of equivariant forms. We will mainly study the mappings properties of equivariant functions with the purpose of classifying them in some cases.

### 2.1 Equivariant functions

Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  and  $\mathcal{F}(\mathbb{H})$  be the set of all functions from  $\mathbb{H}$  to  $\widehat{\mathbb{C}}$ . Then we have an action of  $\Gamma$  on  $\mathcal{F}(\mathbb{H})$  given by

$$\gamma * h = \gamma^{-1}h\gamma.$$

The set of fixed points of this action is

$$\mathcal{E}(\Gamma) = \{h \in \mathcal{F}(\mathbb{H}), \gamma * h = h \ \forall \gamma \in \Gamma\} = \{h \in \mathcal{F}(\mathbb{H}), \gamma h = h\gamma \ \forall \gamma \in \Gamma\}.$$

**Definition 2.1.1.** An element  $h$  of  $\mathcal{E}(\Gamma)$  is called an *equivariant function* with respect to  $\Gamma$ , or a  $\Gamma$ -*equivariant function*.

$\mathcal{E}_m(\Gamma)$  and  $\mathcal{E}^*(\Gamma)$  will respectively denote the set of all meromorphic, holomorphic equivariant function.

**Examples 2.1.1.**

1. There are two trivial examples of equivariant functions, the identity map and the complex conjugation which will be respectively denoted by  $h_0$  and  $h_1$ .
2. If  $\Gamma$  is an abelian group, then  $\Gamma$  is contained in  $\mathcal{E}(\Gamma)$ .

Infinitely many non trivial examples will be constructed in the next chapter.

It turns out that these equivariant functions have very interesting mapping properties. One major question that arises is about the size of the image set of an equivariant function. Our purpose is to give some necessary conditions for an equivariant function to be surjective, and since commuting with the action of the group is not a trivial property, it is natural that some of these conditions will involve some characteristics of the group. We will see the nature of the limit set  $\Lambda(\Gamma)$  of  $\Gamma$  will play an important role in this context.

**Definition 2.1.2.**

1. Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ , we define  $Fix(\Gamma)$  to be the set of parabolic or hyperbolic fixed points of  $\Gamma$ . Notice that  $Fix(\Gamma)$  is contained in  $\widehat{\mathbb{R}}$ .
2. Two points  $z, w \in \widehat{\mathbb{C}}$  are said to be  $\Gamma$ -*equivalent* if there exists  $\gamma \in \Gamma$  such that  $w = \gamma(z)$ .
3. Let  $h \in \mathcal{F}(\mathbb{H})$  and  $z \in \mathbb{H}$  such that  $h(z) = \infty$ . We say that  $h$  is continuous at  $z$  if  $1/h$  is continuous at  $z$ .
4. Let  $h \in \mathcal{F}(\mathbb{H})$  and  $z \in \mathbb{H}$ . We say  $h$  is *locally open* at  $z$  if there is an open neighborhood  $V$  of  $z$  such that the restriction of  $h$  to  $V$  is a continuous open map.

**Theorem 2.1.1.** *Let  $\Gamma$  be a discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ ,  $h \in \mathcal{E}(\Gamma)$ ,  $x \in Fix(\Gamma)$ ,  $\gamma \in \Gamma$  with  $\gamma x = x$ . Suppose that  $x = h(w)$  for some  $w \in \mathbb{H}$ , and that  $h$  is locally open at  $w$ . If  $z \in \widehat{\mathbb{C}}$  is not in the set of fixed points of  $\gamma$ , then  $h^{-1}(\{z\})$  contains infinitely many non  $\Gamma$ -equivalent points.*

*Proof.* Since  $x \in \text{Fix}(\Gamma) \subset \widehat{\mathbb{R}}$ ,  $w$  is not an elliptic point. Indeed, if  $\gamma w = w$  for some elliptic element  $\gamma$  in  $\Gamma$ , then

$$x = h(w) = h(\gamma w) = \gamma h(w) = \gamma x,$$

and the elliptic element  $\gamma$  will have a fixed point in  $\widehat{\mathbb{R}}$ , which is impossible. Using Theorem 1.3.3 and the fact that  $w$  is not elliptic, we get an open neighborhood  $V$  of  $x$  containing no  $\Gamma$ -equivalent points. After intersecting  $V$  with the open neighborhood obtained from the local openness of  $h$  at  $w$ , we may assume that  $h(V)$  is an open neighborhood of  $x$ .

Suppose that  $\gamma$  is parabolic and  $z \in \widehat{\mathbb{C}} - \{x\}$ . Using Theorem ??, there exists  $N > 0$  such that  $\gamma^n z \in h(V)$  for all  $n > N$ . Write  $\gamma^n z = h(w_n)$  for some  $w_n \in V$ , then

$$z = h(\gamma^{-n} w_n) \text{ for all } n > N.$$

Suppose that for some  $m, n > N$  we have

$$\gamma^{-n} w_n = \alpha \gamma^{-m} w_m \text{ with } \alpha \in \Gamma,$$

then  $\gamma^{m-n} = \alpha$  since  $V$  contains no  $\Gamma$ -equivalent points, and so  $\alpha x = x$ . Also, we have  $h(\gamma^{-n} w_n) = h(\alpha \gamma^{-m} w_m)$ , which implies that  $\gamma^{-n} h(w_n) = \alpha \gamma^{-m} h(w_m)$ . This gives  $z = \alpha z$  and hence  $z = x$ , which is impossible since  $\{x\}$  is the set of fixed point of  $\gamma$ .

Suppose that  $\gamma$  is hyperbolic. If  $x$  is a repulsive point of  $\gamma$ , then it is an attractive one for  $\gamma^{-1}$ , which is also a hyperbolic element of  $\Gamma$  verifying all the hypotheses of the theorem, and so we can always assume that  $x$  is an attractive point for  $\gamma$ . Now, use Theorem ?? and mimic the proof of the first part to get the desired result.  $\square$

If in the above theorem  $\Gamma$  was not an elementary group, then by Theorem 1.3.14, it will contain infinitely many hyperbolic elements; no two of which have a common fixed point. Now, take any hyperbolic element  $\gamma_1 \in \Gamma - \{\gamma\}$ , then  $\gamma_1$  will verify the conditions of the theorem. Since  $\gamma$  and  $\gamma_1$  have common fixed point, the pre-image of any fixed point of  $\gamma$  will contain infinitely many non  $\Gamma$ -equivalent points. Therefore, we have

**Corollary 2.1.2.** *Let  $\Gamma$  be a non elementary subgroup of  $PSL_2(\mathbb{R})$ ,  $h \in \mathcal{E}(\Gamma)$ ,  $x \in \text{Fix}(\Gamma)$  and  $\gamma \in \Gamma$  with  $\gamma x = x$ . Suppose that  $x = h(w)$  for some  $w \in \mathbb{H}$ , and that  $h$  is locally open at  $w$ . Then  $h : \mathbb{H} \rightarrow \widehat{\mathbb{C}}$  is surjective. More precisely, for any  $z \in \widehat{\mathbb{C}} - h^{-1}(\{z\})$  contains infinitely many non  $\Gamma$ -equivalent points.*

We saw in the proof of the theorem that  $w$  is not an elliptic element, hence if  $\Gamma$  is a Fuchsian group, we can speak about the Dirichlet region of  $\Gamma$  centered at  $w$ , and the following result is directly deduced from the proof of the above theorem.

**Corollary 2.1.3.** *Let  $\Gamma$  be a Fuchsian subgroup of  $PSL_2(\mathbb{R})$ ,  $h \in \mathcal{E}(\Gamma)$ ,  $x \in \text{Fix}(\Gamma)$  and  $\gamma \in \Gamma$  with  $\gamma x = x$ . Suppose that  $x = h(w)$  for some  $w \in \mathbb{H}$ , and that  $h$  is locally open at  $w$ . If  $\langle \gamma \rangle$  is the cyclic group generated by  $\gamma$  and  $D_w(\Gamma)$  is the Dirichlet region of  $\Gamma$  centered at  $w$ , then:*

1. *If  $\gamma$  is parabolic, then*

$$\widehat{\mathbb{C}} = \bigcup_{\alpha \in \langle \gamma \rangle} \alpha D_w(\Gamma).$$

2. *If  $\gamma$  is hyperbolic with fixed points  $x, y$ , then*

$$\widehat{\mathbb{C}} - \{y\} = \bigcup_{\alpha \in \langle \gamma \rangle} \alpha D_w(\Gamma).$$

If  $\Gamma$  is a non elementary Fuchsian group, then by Theorem 1.3.21,  $\Lambda(\Gamma)$  is the closure of the set of fixed points of the hyperbolic transformations of  $\Gamma$ . Hence if the image of  $h$  contains one point in  $\Lambda(\Gamma)$ , then it will contain infinitely many hyperbolic fixed points of  $\Gamma$ . As a consequence, we have

**Corollary 2.1.4.** *Let  $\Gamma$  be a non elementary Fuchsian subgroup of  $PSL_2(\mathbb{R})$ ,  $h \in \mathcal{E}(\Gamma)$ ,  $x \in \Lambda(\Gamma)$  and  $x = h(w)$  for some  $w \in \mathbb{H}$ . Suppose that  $h$  is locally open at  $w$ . Then  $h : \mathbb{H} \rightarrow \widehat{\mathbb{C}}$  is surjective, and for any  $z \in \widehat{\mathbb{C}}$  the set  $h^{-1}(\{z\})$  contains infinitely many non  $\Gamma$ -equivalent points.*

## 2.2 Holomorphic equivariant functions

Holomorphic functions form a rich class in  $\mathcal{F}(\mathbb{H})$  and so it is natural to investigate holomorphic equivariant functions and to ask whether it is possible to classify them. It turns out that in most of the cases the answer is yes. For example, the discreteness of the set of zeros of a holomorphic, or more generally meromorphic, equivariant function  $h$ , will guarantee its triviality, i.e.  $h = h_0$ . The use of other properties of holomorphic functions will allow us to give a simple classification (when it is possible) of elements of  $\mathcal{E}^*(\Gamma)$ .

## 2.2.1 Meromorphic equivariant functions of non discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$

Our aim here is to give a complete description of  $\mathcal{E}_m(\Gamma)$  and  $\mathcal{E}^*(\Gamma)$ .

We begin by the following useful lemma, which can be directly deduced from the definition of an equivariant function.

**Lemma 2.2.1.** *Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ ,  $h \in \mathcal{E}(\Gamma)$ , and  $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ . Then*

$$\alpha^{-1}h\alpha \in \mathcal{E}(\alpha^{-1}\Gamma\alpha).$$

In our investigation of  $\mathcal{E}_m(\Gamma)$ , we start first with the case when  $\Gamma$  is a non discrete non elementary subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ .

**Theorem 2.2.2.** *Let  $\Gamma$  be a non discrete non elementary subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ . Then  $\mathcal{E}_m(\Gamma) = \{h_0\}$ .*

*Proof.* Let  $h \in \mathcal{E}_m(\Gamma)$ . By Theorem 1.3.17 and Theorem 1.3.18,  $\Gamma$  contains an elliptic element  $\gamma$  of infinite order. Since the action of  $\mathrm{PSL}_2(\mathbb{R})$  on  $\mathbb{H}$  is transitive and using the above lemma, we can suppose that  $\gamma$  fixes  $i$ . Since the Möbius transformation

$$f(z) = \frac{z-i}{z+i}$$

is a homeomorphism from  $\mathbb{H}$  onto  $\mathbb{U}$ , the map

$$g = fhf^{-1} : \mathbb{U} \rightarrow \widehat{\mathbb{C}}$$

commutes with the action of the group  $\Gamma_1 = f\Gamma f^{-1}$  on the unit disc  $\mathbb{U}$ . Moreover, the element  $\alpha = fhf^{-1} \in \Gamma_1$  is elliptic of infinite order and fixing 0. Hence, by Corollary 1.2.10, we have

$$\alpha(z) = kz, \quad k = e^{2\pi i\theta} \quad (\theta \text{ is irrational}).$$

As in the proof of Theorem 1.3.18, we can construct a sequence  $\{\alpha^{a_n}, a_n \in \mathbb{N}\}$  such that  $\alpha^{a_n} \rightarrow I$  as  $n \rightarrow \infty$ .

Since  $\alpha \in \Gamma_1$ , we have  $g\alpha = \alpha g$ , *i.e.*

$$g(kz) = kg(z) \text{ for all } z \in \mathbb{U}.$$

Differentiating both sides yields

$$g'(kz) = g'(z) \text{ for all } z \in \mathbb{U}.$$

Now, take any point  $x \in \mathbb{U} - \{0\}$  which is not a pole of  $g$ , and consider the meromorphic function  $G(z) = g'(z) - g'(x)$  and the sequence  $x_n = \alpha^{an}x = k^{an}x$ . We have

$$G(x) = 0, G(x_n) = 0, \text{ and } x_n \rightarrow x \text{ as } n \rightarrow \infty.$$

Therefore,  $G$  is meromorphic on  $\mathbb{U}$  with its set of zeros having an accumulation point in  $\mathbb{U}$ . It follows that  $G$  must be a constant function, that is, for some  $a, b \in \mathbb{C}$

$$g(z) = az + b, z \in \mathbb{U},$$

while the commuting with  $\alpha$  implies that  $b = 0$ .

Since  $\Gamma$  is not elementary,  $\Gamma_1$  contains an element  $\beta$  which fixes neither 0 nor  $\infty$ . If  $y$  is a fixed point of  $\beta$ , then  $ay = g(y)$  is also a fixed point of  $\beta$  since  $g$  commutes with  $\beta$ . Iterating this, one shows that  $a^2y$  is also a fixed point of  $\beta$ . Therefore, two of the complex numbers  $y, ay, a^2y$  must be equal. Since  $y$  is neither 0 nor  $\infty$ , one easily sees that  $a = \pm 1$ . The case  $a = -1$  is to be excluded because otherwise we would have for all  $\gamma_1 \in \Gamma_1$

$$\gamma_1(-z) = -\gamma_1(z), z \in \mathbb{U},$$

and for  $z = 0$ , we would get  $\gamma_1(0) = 0$  for all  $\gamma_1 \in \Gamma_1$ , which is impossible since  $\Gamma_1$  is not elementary. Hence  $g$  is equal to the identity map, that is,  $h = h_0$ .  $\square$

We now focus on the case when  $\Gamma$  is a non discrete elementary subgroup of  $\text{PSL}_2(\mathbb{R})$ .

**Theorem 2.2.3.** *Let  $\Gamma$  be a non discrete elementary subgroup of  $\text{PSL}_2(\mathbb{R})$ . If  $\alpha \in \widehat{\mathbb{H}}$  is such that, among all the finite  $\Gamma$ -orbits,  $\Gamma\alpha$  has the least number of points, then  $\Gamma\alpha$  contains at most two points. Moreover,*

1. *If  $\Gamma\alpha$  contains one point, then  $\mathcal{E}_m(\Gamma) = \{\alpha\} \cup C_{\text{PSL}_2(\mathbb{C})}(\Gamma)$ .*
2. *If  $\Gamma\alpha$  contains two points, then  $\mathcal{E}_m(\Gamma) = C_{\text{PSL}_2(\mathbb{C})}(\Gamma)$ .*

*Proof.* Let  $\Gamma$  be a non discrete elementary subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  with a finite orbit  $\Gamma\alpha$ ,  $\alpha \in \widehat{\mathbb{H}}$ . If  $\Gamma\alpha$  contains at least three points, then by Remark 1.3.2,  $\Gamma$  contains only elliptic elements, therefore  $\Gamma$  will have an orbit consisting of one point, see Theorem 1.3.11. Thus, it suffices to treat the cases when the orbit  $\Gamma\alpha$  contains one or two points.

Suppose that  $\Gamma\alpha$  contains one point. If  $\alpha \in \mathbb{H}$ , then all elements of  $\Gamma$  are elliptic fixing  $\alpha$ . Thus the constant function  $h(z) = \alpha$  is in  $\mathcal{E}(\Gamma)_m$ . We can suppose that we are working in the unit disc model and that  $\alpha = 0$ . Hence any  $\gamma \in \Gamma$  has the form  $\gamma(z) = kz$  for some  $k \in \mathbb{C}$ . Let  $h \in \mathcal{E}_m(\Gamma)$ . Since  $\Gamma$  is not discrete, we can construct a sequence  $\{\gamma_n\}$  of elements of  $\Gamma$  such that  $\gamma_n \rightarrow I$  as  $n \rightarrow \infty$ , and using the same method as in the proof of the above theorem, we find that for some  $a \in \mathbb{C}$ ,

$$h(z) = az, \quad z \in \mathbb{U}.$$

Hence, in the upper half plane model, we have

$$\mathcal{E}_m(\Gamma) = \{\alpha\} \cup C_{\mathrm{PSL}_2(\mathbb{C})}(\Gamma).$$

If we now suppose that  $\alpha \in \widehat{\mathbb{R}}$ , then we can take  $\alpha = \infty$  and so any  $\gamma \in \Gamma$  will have the following form

$$\gamma(z) = az + b, \quad z \in \mathbb{H}$$

for some  $a, b \in \mathbb{R}$ . As in the proof of the above theorem, we get

$$h(z) = cz + d, \quad z \in \mathbb{U}$$

for some  $c, d \in \mathbb{C}$ . If  $c = 0$ , then  $h$  is the constant function  $\alpha$ , otherwise, if  $c \neq 0$ , then  $h$  a Möbius transformation commuting with all elements of  $\Gamma$ . Hence

$$\mathcal{E}_m(\Gamma) = \{\alpha\} \cup C_{\mathrm{PSL}_2(\mathbb{C})}(\Gamma).$$

Now, Suppose that  $\Gamma\alpha$  contains two points, say  $\alpha, \beta$ . Since a parabolic element  $\gamma$  is conjugated to  $m_1$ , then each orbit of the group  $\langle \gamma \rangle$  generated by  $\gamma$  consists either of one element (the orbit of the fixed point of  $\gamma$ ) or infinitely many elements. Hence  $\Gamma$  contains only elliptic or hyperbolic elements. If all elements of  $\Gamma$  are elliptic, then by Theorem 1.3.11,  $\Gamma$  will have an orbit consisting of one point, which contradicts the fact that all the finite  $\Gamma$ -orbits must have at least two points. Therefore  $\Gamma$  contains a hyperbolic element.

Since a hyperbolic element  $\gamma$  is conjugated to  $m_k$ ,  $|k| \neq 1$ , then each orbit of the group  $\langle \gamma \rangle$  generated by  $\gamma$  consists either of one element (the orbit of a fixed point of  $\gamma$ ) or infinitely many elements. The orbits  $\langle \gamma \rangle \alpha$  and  $\langle \gamma \rangle \beta$  are both finite since  $\langle \gamma \rangle \alpha \subset \Gamma\alpha$ , and  $\langle \gamma \rangle \beta \subset \Gamma\beta$  and  $\Gamma\alpha = \Gamma\beta$  is finite. Therefore  $\alpha, \beta$  are the fixed points of any hyperbolic element  $\gamma$ .

As usual, we may assume that  $\alpha = \infty$ , and  $\beta = 0$ . Since  $\Gamma\infty$  consists of  $\infty$  and  $0$ , then an element  $\gamma \in \Gamma$  will either fixes  $\infty$  and  $0$ , and so it is a hyperbolic element of the form

$$\gamma(z) = az \quad \text{for all } z \in \mathbb{H}, \text{ for some } a \in \mathbb{R},$$

or it interchanges them, and so it will have the following form

$$\gamma(z) = b/z \quad \text{for all } z \in \mathbb{H}, \text{ for some } b \in \mathbb{R} - \{0\},$$

hence it is an elliptic element of order two.

If all elements of  $\Gamma$  have the first form, then the orbit of  $0$  consists of one point which is absurd. Hence  $\Gamma$  contains both hyperbolic and elliptic elements. Since  $\Gamma$  is not discrete, we have a sequence  $\{\gamma_n\}$  of elements of  $\Gamma$  such that  $\gamma_n \rightarrow I$  as  $n \rightarrow \infty$ . If  $\{\gamma_n\}$  contains infinitely many elliptic elements, then it has a subsequence  $\{\gamma_{n_m}\}$  with

$$\gamma_{n_m}(z) = b_{n_m}/z \quad \text{for all } z \in \mathbb{H}, \text{ for some } b_{n_m} \in \mathbb{R} - \{0\},$$

such that  $\gamma_{n_m} \rightarrow I$  as  $m \rightarrow \infty$ . Then, for all  $z \in \mathbb{H}$  we have  $\gamma_{n_m}z \rightarrow z$  as  $m \rightarrow \infty$ , *i.e.*  $b_{n_m} \rightarrow z^2$  as  $m \rightarrow \infty$  for all  $z \in \mathbb{H}$ , which is absurd. Hence  $\{\gamma_n\}$  contains only finitely many elliptic elements, thus we may assume that all elements in  $\{\gamma_n\}$  are hyperbolic. Finally, as in the proof of the preceding theorem, we find that for some  $c \in \mathbb{C}$

$$h(z) = cz \quad \text{for all } z \in \mathbb{U}.$$

and the commutativity with the elliptic element shows that  $c = \pm 1$ . Hence

$$\mathcal{E}_m(\Gamma) = C_{\text{PSL}_2(\mathbb{C})}(\Gamma).$$

□

## 2.2.2 Holomorphic equivariant functions of Fuchsian group of the first kind

Unlike the classification of holomorphic equivariant functions in the non discrete case, the case of Fuchsian groups presents more challenges and the proof is more elaborate. In addition to the equivariance and the holomorphy of the functions, we have to use some properties of the equivariance group, namely the limit set.

The fact that the limit set of a Fuchsian group of the first kind  $\Gamma$  is  $\widehat{\mathbb{R}}$  enables us to show that  $\mathcal{E}^*(\Gamma) = \{h_0\}$ . But in the elementary or in the second kind case, the situation is more difficult since the limit set of Fuchsian groups of the second kind is complicated, and it is almost trivial for an elementary Fuchsian groups (consisting at most of two points).

**Theorem 2.2.4.** *If  $\Gamma$  is a Fuchsian group of the first kind, then  $\mathcal{E}^*(\Gamma) = \{h_0\}$ .*

*Proof.* Suppose that  $\Gamma$  is a Fuchsian group of the first kind, and let  $h$  be in  $\mathcal{E}^*(\Gamma)$ . Since  $h$  is an open map, and  $\Lambda(\Gamma) = \widehat{\mathbb{R}}$ ,  $h$  cannot take any real value, otherwise it will have infinitely many poles, see Corollary 2.1.4. Hence either  $h$  maps the upper half plane into itself, or it maps it into the lower half plane  $\mathbb{H}^-$ .

Suppose first that  $h$  maps the upper half-plane  $\mathbb{H}$  into itself and that  $h \neq h_0$ . According to the theorem of Denjoy-Wolff, [3],  $h$  is either an elliptic element of  $\mathrm{PSL}_2(\mathbb{R})$  or the iterates  $h_n = h \circ \dots \circ h$ ,  $n$  times, which are also in  $\mathcal{E}^*(\Gamma)$ , converge uniformly on compact subsets of  $\mathbb{H}$  to a point  $p \in \widehat{\mathbb{H}}$ . Since  $h$  commutes with  $\Gamma$ ,  $h$  cannot be an elliptic element of  $\mathrm{PSL}_2(\mathbb{R})$ , otherwise all elements of  $\Gamma$  will have the same fixed point set according to Theorem 1.3.7 and therefore  $\Gamma$  will be an elementary group which is not the case. If the iterates  $h_n$  of  $h$  converge to a point  $p \in \widehat{\mathbb{H}}$ , then since  $\gamma h_n = h_n \gamma$ , we have  $\gamma p = p$  for all  $\gamma \in \Gamma$ . This impossible since  $\Gamma$  is not an elementary group. Consequently, the only equivariant function that maps  $\mathbb{H}$  into itself is  $h_0$ .

We now suppose that  $h$  maps the upper half-plane  $\mathbb{H}$  into the lower half-plane  $\mathbb{H}^-$ . But then we can extend  $h$  to a holomorphic function  $\tilde{h}$  defined on  $\mathbb{H} \cup \mathbb{H}^-$  by

$$\tilde{h}(z) = \overline{h(\bar{z})}, \quad z \in \mathbb{H}^-,$$

and it is easy to see that the restriction of  $\tilde{h} \circ \tilde{h}$  to  $\mathbb{H}$  is an equivariant function that

maps  $\mathbb{H}$  into itself. By the above discussion, we have

$$\tilde{h} \circ \tilde{h}(z) = z \text{ for } z \in \mathbb{H}.$$

This implies that  $h$  is bijective. Therefore, the function  $-h$  is an automorphism of  $\mathbb{H}$  and so is an element of  $\mathrm{PSL}_2(\mathbb{R})$ . Therefore  $h$  is a Möbius transformation commuting with  $\Gamma$ . Since  $h \circ h = h_0 = id$ ,  $h$  is elliptic of order 2. After a suitable conjugation, we can assume that  $h = m_k$  with  $k^2 = 1$ . If  $k = -1$  then for all  $\gamma \in \Gamma$  we have

$$\gamma(-z) = -\gamma(z) \text{ for all } z \in \widehat{\mathbb{C}}.$$

In particular,

$$\gamma(0) = 0 \text{ for all } \gamma \in \Gamma,$$

and hence  $\Gamma$  is elementary which is not the case. Thus  $k = 1$ , and so  $h = h_0$ . We conclude that  $\mathcal{E}^*(\Gamma) = \{h_0\}$ .  $\square$

Note that the unique property of  $\Gamma$  that was used above is the fact that  $\Gamma$  is non elementary. We deduce

**Corollary 2.2.5.** *If  $\Gamma$  is a non elementary Fuchsian group and  $h \in \mathcal{E}^*(\Gamma) - \{h_0\}$ , then the image of  $h$  meets the real line.*

# Chapter 3

## Construction and applications

In this chapter, we make a connections between automorphic forms and equivariant functions. This allows us to establish three major applications of equivariant functions to the theory of automorphic forms, namely we provide new and very simple proofs to two classical theorems on automorphic forms as well as a major new result on their critical points.

### 3.1 Automorphy

#### 3.1.1 Unrestricted automorphic forms

In this section we follow the treatment of [13], and [10].

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ ,  $z \in \mathbb{H}$  and  $k$  be a real number. If  $j_\gamma(z) = j(\gamma, z) = cz + d$ , then for any complex valued function  $f$  defined on  $\mathbb{H}$ , the slash operator of weight  $k$  on  $f$  is defined by:

$$(f|_k\gamma)(z) = j_\gamma(z)^{-k} f(\gamma z).$$

**Definition 3.1.1.** An *automorphic factor* (AF)  $\mu$  of weight  $k \in \mathbb{R}$  for a subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$  is a map  $\mu : \Gamma \times \mathbb{H} \rightarrow \mathbb{C}^\times$  satisfying

1. For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and  $z \in \mathbb{H}$ ,  $|\mu_\gamma(z)| = |j_\gamma(z)|^k$ .
2. For all  $\alpha, \gamma \in \Gamma$  and  $z \in \mathbb{H}$ ,  $\mu_{\alpha\gamma}(z) = \mu_\alpha(\gamma z)\mu_\gamma(z)$ .
3. For all  $\gamma \in \Gamma$  and  $z \in \mathbb{H}$ ,  $\mu_{-\gamma}(z) = \mu_\gamma(z)$

**Remark 3.1.1.** In what follows we will identify an element  $\gamma \in \text{PSL}_2(\mathbb{R})$  with its representative  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ , since the third condition says that the automorphic factor  $\mu$  is well defined on  $\Gamma \times \mathbb{H}$ , where  $\Gamma$  is a subgroup of  $\text{PSL}_2(\mathbb{R})$ .

**Example 3.1.1.** For any  $k \in 2\mathbb{Z}$ , the function  $j_\gamma^k : \text{PSL}_2(\mathbb{R}) \times \mathbb{H} \rightarrow \mathbb{C}^\times$  is an automorphic factor of weight  $k$ .

Since a holomorphic function on  $\mathbb{H}$  of constant modulus must be constant, then from the above definition we have

$$\mu_\gamma(z) = \mu(\gamma, z) = \nu(\gamma)j_\gamma$$

for all  $\gamma \in \Gamma$  and  $z \in \mathbb{H}$ , where  $\nu(\gamma)$  depends only on  $\gamma$  and

$$|\nu(\gamma)| = 1.$$

The factor  $\nu(\gamma)$  is called a *multiplier*, and the function  $\nu$  defined on  $\Gamma$  is called a *multiplier system (MS)* of weight  $k$  for  $\Gamma$ . Note that  $\nu(I) = 1$  and  $\nu(-I) = e^{-i\pi k}$ , and for any  $\alpha, \gamma \in \Gamma$  we have

$$\nu(\alpha\gamma) = \sigma(\alpha, \gamma)\nu(\alpha)\nu(\gamma), \text{ where } \sigma(\alpha, \gamma) = \frac{j(\alpha, \gamma z)j(\gamma, z)}{j(\alpha\gamma, z)}, \quad (|\sigma(\alpha, \gamma)| = 1).$$

**Lemma 3.1.1.** Let  $L \in \text{SL}_2(\mathbb{R})$ ,  $\mu$  an AF of weight  $k \in \mathbb{R}$  for a subgroup  $\Gamma$  of  $\text{PSL}_2(\mathbb{R})$  and  $\nu$  its associated MS. If  $\Gamma^L = L^{-1}\Gamma L$  then we have an AF  $\mu^L$  for  $\Gamma^L$  defined by

$$\mu^L(L\gamma L^{-1}, z) = \frac{\mu(\gamma, Lz)j(L, z)}{j(L, L\gamma L^{-1}z)}, \quad \gamma \in \Gamma, \quad z \in \mathbb{H},$$

and its associated MS is given by

$$\nu^L(L\gamma L^{-1}, z) = \nu(\gamma) \frac{\sigma(\gamma, L)}{\sigma(L, L\gamma L^{-1})}.$$

Moreover,  $(\mu^{L_1})^{L_2} = \mu^{L_1 L_2}$  for any  $L_1, L_2 \in \text{SL}_2(\mathbb{R})$ .

**Definition 3.1.2.** Let  $\mu$  be an automorphic factor of weight  $k \in \mathbb{R}$  for a subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{R})$  and let  $\nu$  be the associated multiplier system. A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is called an *unrestricted automorphic form* for  $\Gamma$  of weight  $k$ , with automorphic factor  $\mu$  (or, equivalently, with multiplier system  $\nu$ ) if it satisfies

1.  $f$  is meromorphic on  $\mathbb{H}$ .
2.  $f|_k \gamma = \nu(\gamma)f$  for all  $\gamma \in \Gamma$ .

Denote by  $M'(\Gamma, k, \nu)$  the  $\mathbb{C}$ -vector space of all unrestricted automorphic forms for  $\Gamma$  of weight  $k$  and MS  $\nu$ .

**Theorem 3.1.2.** *Let  $f \in M'(\Gamma, k, \nu)$  and  $L, L_1, L_2 \in \mathrm{SL}_2(\mathbb{R})$ . We have*

1.  $f|_k L \in M'(\Gamma^L, k, \nu^L)$ .
2.  $f|_k L_1 L_2 = \sigma(L_1, L_2)(f|_k L_1)L_2$ .

Suppose that  $\gamma$  is a parabolic element of  $\Gamma$  with a fixed point  $x$ , and let

$$L = L(x) = \begin{pmatrix} 0 & -1 \\ 1 & -x \end{pmatrix}.$$

Then  $L \in \mathrm{SL}_2(\mathbb{R})$ ,  $Lx = \infty$ , and  $U_L = L^{-1}\gamma L$  is an element of  $\Gamma^L$  fixing  $\infty$ , and so

$$L^{-1}\gamma L(z) = z + n_L, \text{ for some } n_L \in \mathbb{R}.$$

Since  $U_L$  is a translation we have

$$\nu^L(U_L) = \nu(\gamma) = e^{2\pi i k_L} \text{ with } 0 \leq k_L < 1.$$

From the automorphy of  $f_L$  on  $\Gamma_L$  we get

$$(f|_k L)(z + n_L) = e^{2\pi i k_L} (f|_k L)(z), \quad z \in \mathbb{H}.$$

It follows that the function

$$f_L^*(z) = e^{-2\pi i k_L z / n_L} (f|_k L)(z), \quad z \in \mathbb{H},$$

is periodic of period  $n_L$ . Hence, there is a unique meromorphic function  $F_L$  defined on  $\mathbb{U} - \{0\}$ , such that

$$f_L^*(z) = F_L(t),$$

where  $t = e^{-2\pi iz/n_L}$  is called a *local uniformizing variable*. If  $f_L$  is holomorphic on  $\{z \in \mathbb{H}, \text{Im}(z) > \eta\}$ , for some  $\eta \geq 0$ , then  $F_L$  is holomorphic on  $\{t \in \mathbb{U}, 0 < |t| < e^{-2\pi\eta/n_L}\}$  and has a convergent Laurent series in this neighborhood of 0

$$(f|_k L)(z) = e^{2\pi i k_L z/n_L} \sum_{m=N_L}^{\infty} a_m(L) e^{2\pi i m z/n_L}, \quad N_L \in \mathbb{Z} \cup \{\infty\}.$$

We say that  $f$  is meromorphic, holomorphic, or a cusp function at the cusp  $x$ , if respectively  $N_L \neq \infty$ ,  $N_L \geq 0$ ,  $N_L > 0$

**Definition 3.1.3.** Suppose that  $f \in M'(\Gamma, k, \nu)$ . Then  $f$  is called an *automorphic form* of weight  $k$  and MS  $\nu$  for  $\Gamma$  if it is meromorphic at every cusp of  $\Gamma$ . The  $\mathbb{C}$  vector space of these functions is denoted by  $M(\Gamma, k, \nu)$ .

Suppose that  $f$  is holomorphic on  $\mathbb{H}$  and that  $f \in M(\Gamma, k, \nu)$ , then  $f$  is called an *entire automorphic form*, respectively a *cusp form* of weight  $k$  and MS  $\nu$  for  $\Gamma$ , if  $f$  is a holomorphic, respectively a cusp function at all cusps of  $\Gamma$ . The  $\mathbb{C}$ -vector space of entire automorphic forms, cusp forms are respectively denoted by  $EM(\Gamma, k, \nu)$ ,  $S(\Gamma, k, \nu)$ . In all cases, when  $k = 0$  and  $\nu = 1$ , the word form is replaced by the word function.

If the context is clear, the reference to the group and the multiplier system will be omitted.

Examples of automorphic forms of a Fuchsian group of any type can be found in [6]. Here, we content ourselves to give examples in the case of the modular group and some of its special subgroups.

### 3.1.2 Congruence subgroups of $\text{PSL}_2(\mathbb{Z})$

Since the limit set of a Fuchsian group is closed and contains the parabolic points, the limit set of  $\text{PSL}_2(\mathbb{Z})$  is equal to  $\mathbb{R} \cup \{\infty\}$  as the set of cusps of  $\text{PSL}_2(\mathbb{Z})$  is  $\mathbb{Q} \cup \{\infty\}$ . Hence  $\text{PSL}_2(\mathbb{Z})$  and all its subgroups of finite index are Fuchsian groups of the first kind.

Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, P = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then  $S$  is elliptic of order 2 with fixed point  $i$ , and  $P$  is elliptic of order 3 and fixes  $\rho = e^{2\pi i/3}$ . Moreover  $\mathrm{PSL}_2(\mathbb{Z})$  is generated by  $S$  and  $T$ , or equivalently, by  $S$  and  $P$ .

An important class of subgroups of  $\mathrm{PSL}_2(\mathbb{Z})$  is the so-called *congruence subgroups*. A subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{Z})$  is called a congruence subgroup if it contains some principal congruence subgroup  $\Gamma(n)$ , for some positive integer  $n$ , given by

$$\Gamma(n) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}), \gamma \equiv \pm I \pmod{n}\} / \{\pm I\},$$

and the smallest such  $n$  is called the *level* of  $\Gamma$ .

One easily shows that  $\Gamma(n)$  is a normal subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$  of finite index and that the conjugates in  $\mathrm{PSL}_2(\mathbb{Z})$  of a congruence subgroup are also congruence subgroups.

**Example 3.1.2.** Here we give some important and well known examples of congruence subgroups of  $\mathrm{PSL}_2(\mathbb{Z})$ .

$$\Gamma_1(n) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}), \gamma \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{n}\} / \{\pm I\},$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \pmod{n} \right\} / \{\pm I\},$$

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), b \equiv 0 \pmod{n} \right\} / \{\pm I\}.$$

It is clear that  $\Gamma(n) \subset \Gamma_1(n) \subset \Gamma_0(n)$  and that  $\Gamma_0(n), \Gamma_1(n), \Gamma^0(n)$  are of level  $n$ . Note that  $\Gamma_0(n)$  is conjugate to  $\Gamma^0(n)$  by  $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$

### 3.1.3 Examples of modular forms

When speaking about the modular group  $\mathrm{PSL}_2(\mathbb{Z})$  and its subgroups, the word automorphic in the definition of any type of automorphy is replaced by the word modular.

We start our examples by the well known Eisenstein series. They are defined for every even integer  $k \geq 2$  and  $z \in \mathbb{H}$  by

$$G_k(z) = \sum'_{m,n} \frac{1}{(mz+n)^k},$$

where the symbol  $\sum'$  means that the summation is over the pairs  $(m, n) \neq (0, 0)$ . Note that this series is not absolutely convergent for  $k = 2$ . We normalize Eisenstein series by letting

$$E_k = 2\zeta(k)G_k$$

to get the following representation

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \quad q = e^{2\pi iz}.$$

Here  $B_k$  is the  $k$ -th *Bernoulli* number and  $\sigma_{k-1}(n) = \sum_{d|n} d^k$ . The most familiar Eisenstein series are:

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

For  $k \geq 4$ , the series  $E_k$  are entire modular forms of weight  $k$ . The Eisenstein series  $E_2$  is holomorphic on  $\mathbb{H}$  and at the cusps, but it is not a modular form as it does not satisfy the modularity condition. The Eisenstein series  $E_2$  is an example of a *quasimodular form* and plays an important role in the construction of equivariant functions as will be seen later on. Moreover,  $E_2$  satisfies

$$E_2(z) = \frac{1}{2\pi i} \frac{\Delta'(z)}{\Delta(z)}, \tag{3.1}$$

where  $\Delta$  is the weight 12 cusp form for  $\mathrm{PSL}_2(\mathbb{Z})$  given by

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

The Eisenstein series satisfy the Ramanujan relations

$$\frac{6}{\pi i} E_2' = E_2^2 - E_4, \tag{3.2}$$

$$\frac{3}{2\pi i} E_4' = E_4 E_2 - E_6, \quad (3.3)$$

$$\frac{1}{\pi i} E_6' = E_6 E_2 - E_4^2. \quad (3.4)$$

The Dedekind  $j$ -function given by

$$j(z) = \frac{E_4^3 - E_6^2}{\Delta}$$

is a modular function and it generates the function field of modular functions for  $\mathrm{PSL}_2(\mathbb{Z})$ .

An example of an entire modular form with a non-trivial multiplier system is the Dedekind eta-function  $\eta(z) = \Delta(z)^{1/24}$ . It is a weight  $1/2$  entire modular form for  $\mathrm{PSL}_2(\mathbb{Z})$  with the multiplier systems given by

$$\nu(T) = e^{i\pi/12}, \quad \nu(S) = e^{-i\pi/4}, \quad \nu(P) = e^{-i\pi/8}.$$

Now, we give an example of entire modular forms on congruence subgroups with non-trivial multiplier systems, namely the Jacobi theta functions. They are defined by

$$\theta_2(z) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

They are entire holomorphic modular forms of weight  $\frac{1}{2}$  for the conjugate congruence subgroups  $\Gamma_0(2)$ ,  $\Gamma_0^P(2)$ ,  $\Gamma_0^{P^2}(2)$  respectively. Their associated multiplier systems are  $u, v$  and  $w$  respectively and are defined by

$$v(-I) = u(-I) = w(-I) = -i,$$

$$v(T^2) = u(PT^2P^{-1}) = w(P^2T^2P^{-2}) = 1,$$

$$v(S) = u(PSP^{-1}) = w(P^2SP^{-2}) = e^{-i\pi/4}.$$

Moreover, these modular forms do not vanish on  $\mathbb{H}$  and satisfy the Jacobi identity

$$\theta_2^4 + \theta_4^4 = \theta_3^4.$$

Furthermore, the following relations hold between the theta functions,  $E_4$ ,  $\Delta$  and  $\eta$ :

$$\begin{aligned}\Delta(z) &= (2^{-1}\theta_2(z)\theta_3(z)\theta_4(z))^8 \\ E_4(z) &= \frac{1}{2}(\theta_2^8(z) + \theta_3^8(z) + \theta_4^8(z)) \\ \theta_2(z) &= \frac{2\eta(4z)^2}{\eta(2z)} \\ \theta_3(z) &= \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} \\ \theta_4(z) &= \frac{\eta(z)^2}{\eta(2z)}.\end{aligned}\tag{3.5}$$

### 3.1.4 Almost meromorphic and quasi automorphic forms

All the results of this subsection are a straightforward generalization of those obtained in [7], where the authors consider the case of holomorphic functions, the trivial MS, and the full modular group  $\mathrm{PSL}_2(\mathbb{Z})$ .

Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ ,  $k \in \mathbb{R}$ , and  $\nu$  be a MS for  $\Gamma$ . An *unrestricted almost meromorphic automorphic form* of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$  is a function  $f$  on  $\mathbb{H}$  which transforms like an automorphic form but, instead of being holomorphic, is a polynomial in  $1/y$ ,  $y = \mathrm{Im}(z)$ , with meromorphic coefficients. More precisely,  $f$  has the form

$$f(z) = \sum_{n=0}^p \frac{f_n(z)}{y^n}, \quad z \in \mathbb{H},\tag{3.6}$$

where each  $f_n$  is a meromorphic function, and for all  $\gamma \in \Gamma$ ,

$$f|_k\gamma = \nu(\gamma)f.\tag{3.7}$$

We denote by  $\widehat{M}'(\Gamma, k, p, \nu)$  the space of such forms, and by  $\widehat{M}'(\Gamma, k, \nu) = \bigcup_p \widehat{M}'(\Gamma, k, p, \nu)$  the space of all unrestricted almost meromorphic automorphic forms of weight  $k$  and MS

$\nu$ . When  $k$  runs over  $\mathbb{Z}$ , the set  $\widehat{M}'_*(\Gamma, \nu) = \bigoplus_k \widehat{M}'(\Gamma, k, \nu)$  will be called the *graded algebra of all unrestricted almost meromorphic automorphic forms for the MS  $\nu$* .

The space  $\widetilde{M}'(\Gamma, k, p, \nu)$  of constant terms  $f_0$  of  $f$  as  $f$  runs over  $\widehat{M}'(\Gamma, k, p, \nu)$  is called the space of *unrestricted quasi-automorphic forms* of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$ . The following result shows that unrestricted quasi-automorphic forms can be defined intrinsically.

**Theorem 3.1.3.** [7] *A function  $f$  is in  $\widetilde{M}'(\Gamma, k, p, \nu)$  if and only if  $f$  is meromorphic on  $\mathbb{H}$ , and for all  $\gamma \in \Gamma$ ,  $z \in \mathbb{H}$*

$$(f|_k \gamma)(z) = \nu(\gamma) \sum_{i=0}^p f_n(z) \left( \frac{c}{cz+d} \right)^n, \quad (3.8)$$

where each  $f_n$  is a meromorphic function of  $\mathbb{H}$ .

The quasi-automorphic polynomial attached to  $f$  is defined by

$$P_{f,z}(X) := \sum_{i=0}^p f_n(z) X^n. \quad (3.9)$$

If we set  $\gamma = I$  in (3.8), we find that  $f_0 = f$ . Let  $\psi : \widehat{M}'(\Gamma, k, p, \nu) \rightarrow \widetilde{M}'(\Gamma, k, p, \nu)$  be the map defined by  $f \rightarrow f_0$ . Then by the above theorem  $\psi$  is a surjective homomorphism. In [7], it is shown that if  $f \in \widetilde{M}'(\Gamma, k, p, \nu)$ , then the function  $P_{f,z}(1/2iy)$  is an element of  $\widehat{M}'(\Gamma, k, p, \nu)$ , and it is clear that its constant term is  $f$ . If in addition  $\Gamma$  is a non elementary Fuchsian group, then  $\psi$  is also injective. Hence  $\psi$  is an isomorphism, and so the unrestricted quasi-automorphic forms inherit the structure of the unrestricted almost meromorphic automorphic forms. The set  $\widetilde{M}'(\Gamma, k, \nu) = \bigcup_p \widetilde{M}'(\Gamma, k, p, \nu)$  will denote the space of all unrestricted quasi-automorphic forms of weight  $k$  and MS  $\nu$ . When  $k$  runs over  $\mathbb{Z}$ , the set  $\widetilde{M}'_*(\Gamma, \nu) = \bigoplus_k \widetilde{M}'(\Gamma, k, \nu)$  is a graded algebra that will be called the *graded algebra of all unrestricted quasi-automorphic forms for the MS  $\nu$* .

Let  $f$  be a nonzero entire automorphic form in  $f \in EM(\Gamma, k, \nu)$ ,  $k \neq 0$ . The derivative  $df = f'$  satisfies

$$(f'|_k \gamma)(z) = \nu(\gamma) (kc(cz+d)^{k+1} f(z) + (cz+d)^{k+2} f'(z)), \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad (3.10)$$

that is,  $df = f' \in \widehat{M}'(\Gamma, k+2, 1, \nu)$  and

$$P_{f',z}(X) = f'(z) + kf(z)X.$$

If  $L_f$  denote the logarithmic derivative of  $f$ , *i.e.*

$$L_f = f'/f, \tag{3.11}$$

then

$$(L_f|_2\gamma)(z) = L_f(z) + \frac{kc}{cz+d}, \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{3.12}$$

Thus  $L_f \in \widehat{M}'(\Gamma, 2, 1, \nu = 1)$ , and

$$P_{L_f,z}(X) = L_f(z) + kX.$$

The non holomorphic function  $\phi(z) = \frac{i}{2y}$  is called the *absolute* quasi-automorphic form of weight 2, depth 1 with trivial MS, since it satisfies

$$(\phi|_2\gamma)(z) = \phi(z) + \frac{c}{cz+d}, \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}). \tag{3.13}$$

Let  $\phi_f = L_f - k\phi$ , then  $\phi_f$  is an unrestricted almost meromorphic automorphic form of weight 2 depth 1 and the trivial MS, and  $\psi(\phi_f) = L_f$ . Furthermore, we have the following structure theorem.

**Theorem 3.1.4.** *With the same notation as above, we have*

1.  $\widehat{M}'(\Gamma, k, p, \nu) = \bigoplus_{n=0}^{n=p} M'(\Gamma, k-2n, \nu) \cdot \phi_f^n$ .
2.  $\delta \left( \widehat{M}'(\Gamma, k, p, \nu) \right) \subset \widehat{M}'(\Gamma, k+2, p, \nu)$ , where  $\delta F = dF - kF\phi$ . The space

$$\widehat{M}(\Gamma, k, p, \nu) = \bigoplus_{n=0}^{n=p} M(\Gamma, k-2n, \nu) \phi_f^n,$$

*is called the space of almost meromorphic automorphic forms of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$ .*

3.  $d \left( \widetilde{M}'(\Gamma, k, p, \nu) \right) \subset \widetilde{M}'(\Gamma, k+2, p, \nu)$ .

4. If  $\Gamma$  is a non elementary Fuchsian subgroup, then

$$\widetilde{M}'(\Gamma, k, p, \nu) = \bigoplus_{n=0}^{n=p} M'(\Gamma, k - 2n, \nu) L_f^n.$$

The space

$$\widetilde{M}(\Gamma, k, p, \nu) = \bigoplus_{n=0}^{n=p} M(\Gamma, k - 2n, \nu) L_f^n,$$

is called the space of meromorphic quasi-automorphic forms of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$ .

If  $f$  has no zeros in  $\mathbb{H}$ , then  $L_f$  is holomorphic on  $\mathbb{H}$  and we have the following definitions:

The spaces

$$\begin{aligned} \widehat{EM}(\Gamma, k, p, \nu) &= \bigoplus_{n=0}^{n=p} EM(\Gamma, k - 2n, \nu) \phi_f^n, \\ \widehat{S}(\Gamma, k, p, \nu) &= \bigoplus_{n=0}^{n=p} S(\Gamma, k - 2n, \nu) \phi_f^n, \end{aligned}$$

are respectively called the space of almost holomorphic automorphic forms, almost cusp forms of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$ . If, in addition,  $\Gamma$  is a non elementary Fuchsian subgroup, then the spaces

$$\begin{aligned} \widetilde{EM}(\Gamma, k, p, \nu) &= \bigoplus_{n=0}^{n=p} EM(\Gamma, k - 2n, \nu) L_f^n, \\ \widetilde{S}(\Gamma, k, p, \nu) &= \bigoplus_{n=0}^{n=p} S(\Gamma, k - 2n, \nu) L_f^n, \end{aligned}$$

are respectively called the space of quasi-automorphic forms, quasi-cusp forms of weight  $k$ , depth  $p$  and MS  $\nu$  on  $\Gamma$ .

**Example 3.1.3.** The group  $\Gamma = \text{PSL}_2(\mathbb{Z})$  is a non elementary Fuchsian group of the first kind. If  $f = \Delta$  is the weight 12 cusp form for  $\text{PSL}_2(\mathbb{Z})$ , then  $f$  does not vanish on  $\mathbb{H}$ , and by (3.1) we have

$$L_f = 2\pi i E_2.$$

Using the fact that the graded algebra of entire modular forms with the trivial MS on  $\text{SL}_2(\mathbb{Z})$  is

$$\mathbb{C}[E_4, E_6],$$

see [13], we get

$$\begin{aligned}\widetilde{EM}_*(\mathrm{PSL}_2(\mathbb{Z}), \nu = 1) &= \mathbb{C}[E_4, E_6, E_2], \\ \widehat{EM}_*(\mathrm{PSL}_2(\mathbb{Z}), \nu = 1) &= \mathbb{C}[E_4, E_6, E_2^*],\end{aligned}$$

where

$$E^* = E_2 - \frac{6i}{\pi} \phi \in \widehat{EM}(\mathrm{PSL}_2(\mathbb{Z}), 2, 1, \nu = 1),$$

and  $\widetilde{EM}_*(\mathrm{PSL}_2(\mathbb{Z}), \nu = 1)$ ,  $\widehat{EM}_*(\mathrm{PSL}_2(\mathbb{Z}), \nu = 1)$  are respectively the graded algebra of entire quasi-modular forms, and almost holomorphic modular forms with the trivial MS on  $\mathrm{PSL}_2(\mathbb{Z})$ .

**Remark 3.1.2.** From now on, the trivial MS will be omitted from notations.

## 3.2 Construction of equivariant functions

For a detailed discussion on equivariant functions for the modular group (construction and structure) see [4].

Let  $\Gamma$  be a subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ ,  $k \in \mathbb{R} - \{0\}$ , and suppose that  $f$  is a nonzero unrestricted automorphic form on  $\Gamma$  of weight  $k$  and MS  $\nu$ . We attach to  $f$  the meromorphic function

$$h(z) = h_f(z) = z + k \frac{f(z)}{f'(z)} = z + \frac{k}{L_f(z)}. \quad (3.14)$$

**Proposition 3.2.1.** [15, 18] *The function  $h$  is an equivariant function, i.e., it satisfies*

$$h(\gamma \cdot z) = \gamma \cdot h(z) \text{ for all } \gamma \in \Gamma, z \in \mathbb{H}. \quad (3.15)$$

*Proof.* Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ . Then, we have

$$\begin{aligned}h(\alpha \cdot z) &= \frac{az + b}{cz + d} + \frac{\nu(\gamma) k (cz + d)^k f(z)}{\nu(\gamma) [ck(cz + d)^{k+1} f(z) + (cz + d)^{k+2} f'(z)]} \\ &= \frac{(az + b)(ckf(z) + (cz + d)f'(z)) + kf(z)}{(cz + d)(ckf(z) + (cz + d)f'(z))} \\ &= \frac{akf(z) + (az + b)f'(z)}{ckf(z) + (cz + d)f'(z)}\end{aligned}$$

using the identity  $ad - bc = 1$ . In the meantime, we have

$$\begin{aligned}\alpha \cdot h(z) &= \frac{ah(z) + b}{ch(z) + d} \\ &= \frac{(az + b)f'(z) + akf(z)}{(cz + d)f'(z) + ckf}.\end{aligned}$$

□

In fact, the key property in the above proof was the form of the quasi-automorphic polynomial attached to  $\frac{1}{k}L_f$ , *i.e.*,

$$P_{\frac{1}{k}L_f, z}(X) = \frac{1}{k}L_f(z) + X,$$

in the sense that any function  $g$  on  $\mathbb{H}$  verifying the following property

$$(g|_2\gamma)(z) = g + \frac{c}{cz + d}, \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad (3.16)$$

gives arise to an equivariant function given by

$$h_g = h_0 + \frac{1}{g}, \quad h_0(z) = z.$$

Hence, if  $\mathcal{Q}(\Gamma)$  denotes the set of all the functions satisfying (3.16) in addition to the constant function  $g_0 = \infty$ , and if we extend the definition of the quasi-automorphic polynomial to any element  $g \in \mathcal{Q}(\Gamma)$  by

$$P_{g, z}(X) = g(z) + X,$$

then  $\mathcal{M}'(\Gamma, 2, 1) \subset \mathcal{Q}(\Gamma)$ , and  $h_{g_0} = h_0$ . Moreover, if  $g \in \mathcal{Q}(\Gamma)$ , and  $f$  is any invariant function under the weight 2 slash operator on  $\Gamma$ , that is

$$f|_2\gamma = f, \quad \text{for all } \gamma \in \Gamma,$$

then one can easily check that

$$f + g \in \mathcal{Q}(\Gamma).$$

Conversely, if we associate to each element  $h \in \mathcal{E}(\Gamma)$  the function defined by

$$\widehat{h} = \frac{1}{h - h_0},$$

then  $\widehat{h} \in \mathcal{Q}(\Gamma)$ , and  $\widehat{h}_0 = g_0$ , and clearly we have the following result.

**Proposition 3.2.2.** [4] *We have a one-to-one correspondence between  $\mathcal{Q}(\Gamma)$  and  $\mathcal{E}(\Gamma)$  given by the two mutually inverse maps*

$$g \rightarrow h_g, \quad h \rightarrow \widehat{h}.$$

**Examples 3.2.1.**

1. If  $g = \phi$  the absolute quasi-automorphic form of weight 2, depth 1, then  $h_g = h_1$ . (recall that  $h_1$  is the complex conjugation).
2. Via the above bijections, the set  $\mathcal{E}_m(\Gamma)$  of meromorphic equivariant functions corresponds to  $\widetilde{M}(\Gamma, 2, 1)$  the space of unrestricted quasi-automorphic forms of weight 2 depth 1 on  $\Gamma$ .
3. For concrete examples, one can take

$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}), \text{ and } g = \frac{i\pi}{6}E_2.$$

Then any meromorphic equivariant function  $h$  of  $\mathcal{E}_m(\Gamma)$  has the form

$$h = h_0 + \frac{1}{f + g},$$

where  $f$  is a unrestricted modular forms of weight 2. If  $f$  is an unrestricted almost modular forms of weight 2 and depth 1, then we still have the equivariance but the meromorphy is replaced  $C^\infty$  differentiability (over a suitable connected open subset of  $\mathbb{H}$ ), for example

$$h = h_0 + \frac{1}{E_2^* + \frac{i\pi}{6}E_2}.$$

Note that the shift from the two classes is provided by replacing  $f$  by  $f + F\phi$ , where  $F$  is an unrestricted modular function for  $\mathrm{PSL}_2(\mathbb{Z})$ .

4. We have seen in (3.12) that the logarithmic derivative of an unrestricted automorphic form  $f$  is an unrestricted quasi-automorphic form of weight 2 and depth 1. If  $f$  is an unrestricted almost automorphic form, then after extending  $d/dz$  (and the logarithmic derivative) to the class of unrestricted almost automorphic forms by  $\frac{\partial}{\partial x}$  (the two operators coincide on meromorphic functions), one can mimic the steps

of (3.10) and (3.12) to get the same result. More explicitly, we have the following: If  $f \in M'(\Gamma, k, p, \nu)$ ,  $k \neq 0$ , then

$$h = h_0 + \frac{k}{L_f} \in \mathcal{E}_m(\Gamma).$$

If  $f \in \widetilde{M}'(\Gamma, k, p, \nu)$ ,  $p \neq 0$ , and  $P_{f,z}(X) = \sum_{i=0}^p f_n(z)X^n$ , then the function defined by

$$g(z) = P_{f,z}\left(\frac{1}{2iy}\right), \quad z \in \mathbb{H}$$

is an element of  $\widehat{M}'(\Gamma, k, p, \nu)$  and  $h_0 + \frac{k}{L_g} \in \mathcal{E}(\Gamma)$ . Notice that

$$h(z) = z + k \frac{\sum_{i=0}^p f_n(z) \left(\frac{1}{2iy}\right)^n}{\sum_{i=0}^p f'_n(z) \left(\frac{1}{2iy}\right)^n}, \quad z \in \mathbb{H}.$$

**Definition 3.2.1.** After [4], the class of equivariant functions constructed above is referred to as the class of *rational* equivariant functions.

## 3.3 Applications

### 3.3.1 Classification of unrestricted automorphic forms for non discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$

The classification of automorphic forms for a non discrete (non elementary) subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  was raised by many authors, see [11, 2] and the references therein. Here we provide a new and simple proof of the complete classification in the non discrete case, in the non elementary case and give the general form of these automorphic form in the elementary case.

Let  $\Gamma$  be a non discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  and suppose that  $f \in M'(\Gamma, k, \nu)$ ,  $k \neq 0$ , be an unrestricted automorphic form of weight  $k$  and MS  $\nu$ . From Theorem 2.2.2 and Theorem 2.2.3, we know that an element  $h$  of  $\mathcal{E}_m(\Gamma)$  is either a constant or a Möbius transformation . Hence, if  $f$  is not the zero function, then

$$h_f = h_0 + \frac{k}{L_f} \in \mathcal{E}_m(\Gamma),$$

and so  $L_f = \frac{f'}{f}$  will be the quotient of a polynomial of degree one over a polynomial of degree two. If for example  $h_f$  is a Möbius transformation represented by  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$ , then

$$\frac{az + b}{cz + d} = z + k \frac{f}{f'}, \quad z \in \mathbb{H}.$$

Hence

$$\frac{f'(z)}{f(z)} = k \frac{cz + d}{-cz^2 + (a - d)z + b}, \quad z \in \mathbb{H}.$$

Thus an elementary computation will give the form of  $f$ . But in the most interesting case, namely when  $\Gamma$  is not an elementary group, we have a more precise result.

**Theorem 3.3.1.** *If  $\Gamma$  is a non elementary non discrete subgroup  $\Gamma$  of  $\text{PSL}_2(\mathbb{R})$ , then we have*

1. *If  $k \neq 0$ , then  $M'(\Gamma, k, \nu) = \{0\}$ .*
2. *If  $k = 0$ , then  $M'(\Gamma, k, \nu) = \mathbb{C}$ .*

*Proof.* Suppose that  $k \neq 0$ , and that there is an element  $f \in M'(\Gamma, k, \nu) - \{0\}$ . Then

$$h_f = h_0 + \frac{k}{L_f} \in \mathcal{E}_m(\Gamma).$$

Meanwhile, using Theorem 2.2.2, we have  $\mathcal{E}_m(\Gamma) = \{h_0\}$ . Hence

$$\frac{k}{L_f} = 0,$$

which is impossible. Thus, for a non elementary non discrete subgroup  $\Gamma$  of  $\text{PSL}_2(\mathbb{R})$ ,  $M'(\Gamma, k, \nu) = \{0\}$ .

If  $k = 0$ , and  $f \in M'(\Gamma, 0, \nu)$ , then the derivative  $f'$  of  $f$  is an element of  $M'(\Gamma, 2, \nu)$ . Hence  $f' = 0$  and  $f$  is a constant function.  $\square$

**Remark 3.3.1.** If  $\Gamma$  is elementary, then after a suitable conjugation in  $\text{SL}_2(\mathbb{C})$ , one can suppose that an element of  $\Gamma$  either fixes  $\infty$  or interchanges it with zero, and so any element  $\gamma$  of the commutator of  $\Gamma$  will be of the form

$$\gamma z = az + b, \text{ or } \gamma z = a/z, \quad a, b \in \mathbb{C}.$$

Using the above method will give the form of  $f$ , see [2, 11].

### 3.3.2 Zeros of weight 2 depth 1 Unrestricted quasi-automorphic forms and critical points of Unrestricted automorphic forms

We begin with the following result which is a direct consequence of Corollary 2.1.4.

**Theorem 3.3.2.** *Let  $\Gamma$  be a Fuchsian group of the first kind, and suppose that  $h \in \mathcal{E}_m(\Gamma) - \{h_0\}$ . Then for any  $z \in \widehat{\mathbb{C}}$ ,  $h_g^{-1}(\{z\})$  contains infinitely many non  $\Gamma$ -equivalent points. In particular, for  $z = \infty$ , we find that  $h$  has infinitely many non  $\Gamma$ -equivalent poles in  $\mathbb{H}$ .*

*Proof.* Let  $h \in \mathcal{E}(\Gamma)_m - \{h_0\}$ . If  $h$  has no poles in  $\mathbb{H}$ , then it will be a holomorphic equivariant function for  $\Gamma$ . But Theorem 2.2.4 says that the unique holomorphic  $\Gamma$ -equivariant function is  $h_0$  which contradicts our assumption. Thus  $h$  has at least one pole  $x \in \mathbb{H}$ .

Since  $\Gamma$  is not elementary, it contains an element  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $c \neq 0$ , otherwise all elements of  $\Gamma$  will fix  $\infty$ , and so  $\Gamma$  will be elementary. By the equivariance property we have

$$h(\gamma x) = \gamma h(x) = \gamma \infty = \frac{a}{c} \in \widehat{\mathbb{R}} = \Lambda(\Gamma).$$

Since  $h$  is an open function, the theorem follows from Corollary 2.1.4 □

**Theorem 3.3.3.** *Let  $\Gamma$  be a Fuchsian group of the first kind, and  $g \in \widetilde{M}'(\Gamma, 2, 1)$  be a non zero unrestricted quasi-automorphic forms of weight 2 and depth 1 on  $\Gamma$ . Then  $g$  has infinitely many non  $\Gamma$ -equivalent zeros in  $\mathbb{H}$ .*

*Proof.* If  $g \in \widetilde{M}'(\Gamma, 2, 1)$  is a non zero unrestricted quasi-automorphic forms of weight 2 depth 1 on  $\Gamma$ , then

$$h_g = h_0 + \frac{1}{g}$$

is a meromorphic equivariant function. Since  $g$  is meromorphic, we have  $h_g \neq h_0$ , and it is clear that the poles of  $h_g$  are the zeros of  $g$ . The result follows from the above theorem. □

The importance of the above results lies in the following.

**Theorem 3.3.4.** *Let  $\Gamma$  be a Fuchsian group of the first kind, and  $f \in M'(\Gamma, k, p, \nu)$ ,  $k \neq 0$ , be an unrestricted automorphic form of weight  $k$  and MS  $\nu$ . Then  $f$  has infinitely many non  $\Gamma$ -equivalent critical points in  $\mathbb{H}$ .*

*Proof.* This follows from the fact that

$$h_f = h_0 + k \frac{f}{f'} \in \mathcal{E}_m(\Gamma),$$

and that the zeros of  $f$  are not poles of  $h_f$  (they are fixed points of  $h_f$ ). Thus the poles of  $h_f$  are exactly the zeros of the derivative  $f'$  of  $f$ .  $\square$

Suppose that  $f$  is holomorphic in  $\mathbb{H}$ , and that  $\gamma$  is a parabolic element of  $\Gamma$  with a fixed point  $x$ , and Let

$$L = L(x) = \begin{pmatrix} 0 & -1 \\ 1 & -x \end{pmatrix}.$$

Then

$$(f|_k L)(z) = e^{2\pi i k_L z / n_L} \sum_{m=N_L}^{\infty} a_m(L) e^{2\pi i m z / n_L}, \quad N_L \in \mathbb{Z} \cup \{\infty\}.$$

In [12], they give a negative answer to the following question: when  $\Gamma$  is a finite index subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$ ,  $k=2$ ,  $\nu = 1$ ,  $x = \infty$ , and  $N_L > 0$ , is it possible that  $a_m(L) = 0$  for  $m \gg 0$ ? Their method uses some analytic properties of the *Rankin-Selberg zeta function* of  $f|_k L$ , defined for  $\Re(s) > 2$  by

$$R_{(f|_k L)}(s) = \sum_{m=1}^{\infty} \frac{a_m(L)}{m^{-s}}.$$

They also give another proof using the theory of *vector-valued modular form*. In our case, the answer is a simple consequence of the above theorem, and this without the restrictions on the group, the weight, and the MS required in their work.

**Theorem 3.3.5.** *Let  $\Gamma$  be a Fuchsian group of the first kind, and  $f \in M'(\Gamma, k, p, \nu)$ ,  $k \neq 0$ , be an unrestricted automorphic form of weight  $k$  and MS  $\nu$ . With the above notations, if  $a_m(L) = 0$  for  $m \gg 0$ , then  $f = 0$ .*

*Proof.* Suppose the converse is true, then  $f|_k L$  will be a nonzero rational function of the local uniformizing parameter  $e^{2\pi i z / n_L}$ , hence it cannot have infinitely many non  $L^{-1}\Gamma L$ -equivalent zeros, which is absurd, since  $L^{-1}\Gamma L$  is a Fuchsian group of the first kind, and  $f|_k L \in M'(L^{-1}\Gamma L, k, p, \nu)$ .  $\square$

We end this chapter by giving some examples involving some standard modular forms and their critical points, see [14] for more details.

**Examples 3.3.1.** 1. As a consequence of (3.1), we recover some results of [5]: The Eisenstein series  $E_2$  (and hence  $\Delta'$ ) has infinitely many non equivalent zeros in the strip  $-1/2 < \text{Re}(z) \leq 1/2$ . Moreover, all these zeros are simple since  $E_2$  and  $E_2'$  cannot vanish at the same time because of (3.2) and the fact that  $E_4$  vanishes only at the orbit of the cubic root of unity  $\rho$ . Similarly, the zeros of  $E_4'$  are also simple because of (3.3) and the fact that  $E_6$  vanishes only at the orbit of  $i$ . Using the same argument and (3.4), the zeros of  $E_6'$  are all simple. We also note that  $E_2$  is real on the axis  $\text{Re}(z) = 0$  and  $\text{Re}(z) = 1/2$ , and one can show that it has a unique zero on each axis given approximately by  $i 0.5235217000$  and  $1/2 + i 0.1309190304$ , see [5]. Also,  $E_4'$  and  $E_6'$  are purely imaginary on both axis and have zeros on  $\text{Re}(z) = 1/2$  given respectively by  $1/2 + i 0.4086818600$  and  $1/2 + i 0.6341269863$ .

2. Using (3.1) and (3.5), we get the following formulas for the derivative of the theta functions:

$$\frac{1}{4\pi i} \frac{\theta_2'(z)}{\theta_2(z)} = 4E_2(4z) - E_2(2z), \quad (3.17)$$

$$\frac{24}{\pi i} \frac{\theta_3'(z)}{\theta_3(z)} = 5E_2(2z) - E_2(z) - 4E_2(4z), \quad (3.18)$$

$$\frac{1}{2\pi i} \frac{\theta_4'(z)}{\theta_4(z)} = E_2(z) - E_2(2z). \quad (3.19)$$

It is interesting to notice that each of the above combinations with  $E_2$ ,  $E_4$  and  $E_6$  vanishes infinitely many times at inequivalent points in a vertical strip in  $\mathbb{H}$ .

3. Let  $F$  be any meromorphic function on  $\mathbb{C}$ . then the function

$$f(z) = j'(z)F'(j(z)) + \frac{i\pi}{6}E_2(z), \quad z \in \mathbb{H},$$

is an unrestricted quasi-modular form of weight 2 depth 1 for  $\text{PSL}_2(\mathbb{Z})$ . Hence it has infinitely many non equivalent zeros in the strip  $-1/2 < \text{Re}(z) \leq 1/2$ . Note that the term

$$j'(z)F'(j(z)), \quad z \in \mathbb{H},$$

corresponds to the change of variable  $w = j(z)$ ,  $w \in \mathbb{C}$ , and that even if the function  $F(w)$  does not vanish on  $\mathbb{C}$  (e.g.  $F(w) = e^w$ ), making this change of variable and performing a perturbation of  $F$  by a fixed weight 2 depth 1 quasi-modular form

leads to the intriguing facts: there are infinitely many zeros, and most importantly, they are non  $\Gamma$ -equivalent whatever change of variable we perform. A phenomenon that deserves to be investigated further.

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