Pattern equivariant cohomology and deformations of tilings

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Contents

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1 Preliminaries

1.1 Notation

A tiling is a subdivision of $\mathbb{R}^n$ into tiles. These tiles are homeomorphic to the closed unit ball in $\mathbb{R}^n$, intersect only at their boundaries, and their union is $\mathbb{R}^n$. In addition to their geometric shapes, tiles may also have additional labeling, like colour or numbering. A **patch** is a finite subset of the tiles in a tiling. If A is a bounded subset of $\mathbb{R}^n$, then $[A]$ denotes the patch of all tiles that intersect A. The ***R*-patch** around $x \in \mathbb{R}^n$ is thus the intersection of the tiling and the ball of radius R. For $x \in \mathbb{R}^n$, and T a tiling, T - x is the same set of tiles translated by x.

This definition of tilings allow for a variety of arrangements and a large amount of shapes. Therefore, we restrict our attention to a specific type of tiling by making additional assumptions.

Definition 1.1. Suppose a tilling has a finite number of tiles, called prototiles, up to translation. Tilings whose prototiles are polytopes that meet full-edge to full-edge are called simple tilings.

Simple tilings also have a useful property in that there are, up to translation, a finite number of R-patches for any $R > 0$, a property called **finite local complexity (FLC)**. However, the topology of one tiling is fairly trivial. The space in which it lives, $\mathbb{R}^n$, is contractible, and its study is of little interest. Thus to compare two tilings, we create a metric on the set of all tiles.

1.2 Tiling Spaces

We say that two tilings are ϵ close if they agree on a ball of $1/\epsilon$ around the origin, up to a global translation of size ϵ or less. To satisfy the metric axioms, we say that the distance between two tilings is $\min \epsilon, 1$. Although we have defined it, the distance between two tilings isn't very useful (or sometimes even computable). Rather, it is used to induce the topological

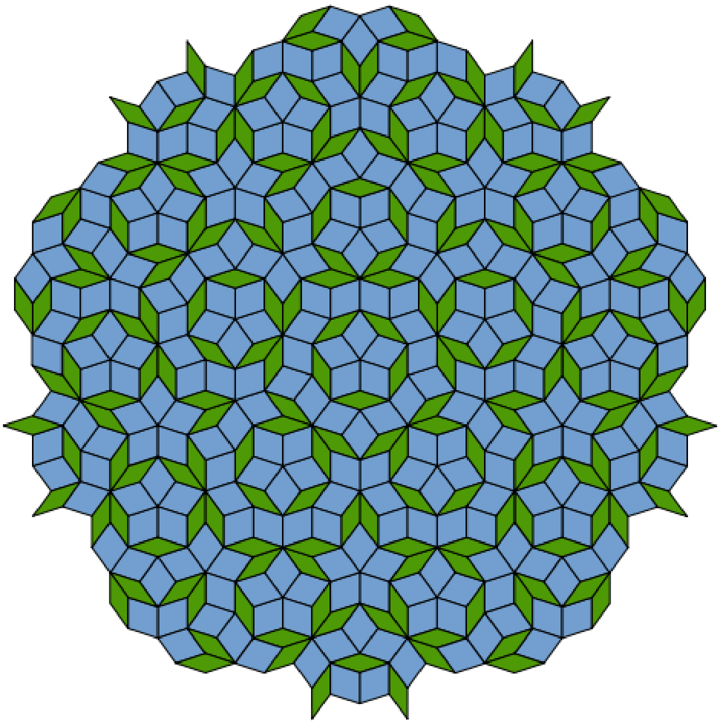

Figure 1: A patch of the Penrose tiling

space we want to study.

The second step in creating the topological space is to assign a group action to tilings. The group is simply $\mathbb{R}^n$ with addition as the operation, acting on a tiling by simple translation. The **orbit** of a tiling is the set $\mathcal{O}(T) = \{T - x | x \in \mathbb{R}^n\}$. Finally we consider the closure of the orbit induced by the metric. This space, Ω_T, called the **hull** of T or the tiling space associated with T, is going to be the main object of study when discussing tilings.

In topology, we wish to classify spaces up to equivalence. There are several different notions of equivalence, the weakest of which is a simple homeomorphism between hulls. We say that two tiling spaces Ω_T and $\Omega_{T'}$ are homeomorphic if there exists a continuous bijective function $f : \Omega_T \to \Omega_{T'}$ with continuous inverse $f^{-1} : \Omega_{T'} \to \Omega_T$. The map f is said to be a homeomorphism between Ω_T and $\Omega_{T'}$. Homeomorphisms preserve topology, but little else. The next notion of equivalence is that of topological conjugacy. Two tiling spaces are said to be **topological conjugates** if there exists a homeomorphism between them that also commutes with the group action.

Topological conjugacies play an important role in the dynamical properties of tiling spaces.

The last notion of equivalence is the strongest. Two tiling spaces are said to be **mutually locally derivable (MLD)** if there exists a topological conjugacy *f*, defined locally. More precisely, there exists a radius R such that the R-patch around the origin in T can be used to determine the patch of size 1 around the origin in $f(T)$ and vice versa. For a while, it was thought that topologically conjugate spaces were automatically MLD. This was later proven to be false through the use of shape deformations [1]. The purpose of this project was to examine why such shapes changes leads to tiling spaces that are topological conjugates, but not MLD.

1.3 Substitution Tilings

A tilling T of $\mathbb{R}^n$ is said to be **aperiodic** if for $x \in \mathbb{R}^n$, $T - x = T$ implies $x = 0$. Aperiodic tilings provide the most interesting source of tilings spaces. There are three main classes of tilings that come up when we wish to construct interesting tiling spaces: substitution tilings, cut-and-project method and local matching rules. We will only focus on substitution tilings, although it worth noting that there is some overlap between the three types.

Definition 1.2. Suppose we have a finite set of prototiles $\{T_1, ..., T_n\}$. Given a stretching factor $\lambda > 0$, a substitution σ is a function that stretches each prototile by a linear factor λ, and then replaces each stretched tile by a cluster of ordinary sized tiles that preserves the shape of the original tile. These clusters are called supertiles.

We can then apply the substitution on the supertile using the same rules. In general, the elements of the image of σ^i are called level i-supertiles. The condition that the substitution map preserves the shape of the tilings assures that applying the substitution again is well defined and has all the nice properties we want. The tiling space is then defined in the following manner. A tiling is in the space if and only if for any patch of any finite size around any point is found in some level i-supertile. For example, if a tiling has the sequence aaa or bbb, it isn't in the Fibonacci tiling space because no such sequence occurs in a supertile of any order.

Definition 1.3. The substitution matrix M keeps track of the population of different prototiles, with $M_{i,j}$ equaling the number of times that the $i-th$ prototile is found in the σ(j-th prototile).

A common example of a substitution tiling in 1-dimension is the Fibonacci tiling, which acts on the prototiles $\{a, b\}$ by $\sigma(a) = b, \sigma(b) = ab$. Because we are tiling the real line, the only acceptable prototiles are intervals, and so we can associate a and b with intervals of different lengths, with a being of length 1 and b being the length of the golden ration $(1+\sqrt{5})/2$. As we will see later on, substitution tiling are interesting because in many cases their topological properties are computable, which isn't always the case for general tiling spaces.

2 Tiling spaces as inverse limits

2.1 Inverse Limit Spaces

Definition 2.1. Suppose $\Gamma_0, \Gamma_1, ...$ are topological spaces, and for each natural number n, let $\pi_n : \Gamma_{n+1} \to \Gamma_n$ a continuous map. Consider the product space $\Pi\, \Gamma_i$ with the product topology. Thus, we have a space sequences $(x_0, x_1, ...)$ with each $x_n \in \Gamma_n$. We define the inverse limit space to be

$$\Gamma_\infty = \varprojlim(\Gamma, \pi) = \{(x_0, x_1, ...) \in \Pi\, \Gamma_i | \forall n, \pi_n(x_{n+1}) = x_n\} \tag{1}$$

The spaces Γ_n are called approximants of the inverse limits since, if you know x_n, then you immediately know $x_0, ..., x_{n-1}$. In the general case, we define the product topology to be the coarsest topology such that every canonical projection $p_i : \Gamma_\infty \to \Gamma_i$ is continuous. This is equivalent to saying that an open set in the product topology is of the form $\Pi\, U_i$, where U_i is open in Γ_i and $U_i \neq \Gamma_i$ for only a finite number of i.

The open sets in the inverse limit are defined in a similar manner, with the additional condition imposed by the maps π_n. As such, a general open set in the inverse limit topology is given by $\Pi\, U_i$, U_i is open in Γ_i, $U_{i+1} \subseteq \pi_i^{-1}(U_i)$ for all i and $U_{i+1} \neq \pi_i^{-1}(U_i)$ for a finite number of i . Informally the open sets in the inverse limit space are the ones such that, after some approximant Γ_k, one can no longer chose the open sets U_i as they are induced by the maps π_i.

The definition of an inverse limit space seems a bit unnecessary, and perhaps a bit clunky. Incredibly, its construction can also be done in the language of category theory through the universal property. The general construction of such an object is a bit technical, so we will stick to the inverse limit case. Let $\mathcal{C}$ be a category, I be a directed partially ordered set, and $(A_i)_{i \in I}$ a family of objects of $\mathcal{C}$ such that for every $i \leq j$, there exists a morphism

$$\pi_i^j : A_j \to A_i \tag{2}$$

that satisfy, for every $i \leq j \leq k$,

$$\pi_i^k = \pi_i^j \circ \pi_j^k \quad \text{and} \quad \pi_i^i = \mathrm{Id}_{A_i} \tag{3}$$

The **inverse limit** of this system, denoted $A_\infty \in \mathcal{C}$, exists if there are morphisms $\pi_i^\infty : A_\infty \to A_i$ such that for each $i \leq j$ the following diagram commutes.

$$\begin{tikzcd}
& A_\infty \arrow[dl, "\pi_j^\infty"'] \arrow[dr, "\pi_i^\infty"] & \\
A_j \arrow[rr, "\pi_i^j"'] & & A_i
\end{tikzcd}$$

The inverse limit is interesting because it is essentially unique in the following sense. Suppose there is another object $B_\infty \in \mathcal{C}$ and maps $\alpha_i^\infty : B_\infty \to A_i$ with the same commutative properties as the maps π_i^∞. Then there is a unique isomorphism $\phi : B_\infty \to A_\infty$ such that the following diagram commutes.

$$\begin{tikzcd}
& B_\infty \arrow[d, "\phi"] \arrow[ddl, bend right, "\alpha_j^\infty"'] \arrow[ddr, bend left, "\alpha_i^\infty"] & \\
& A_\infty \arrow[dl, "\pi_j^\infty"'] \arrow[dr, "\pi_i^\infty"] & \\
A_j \arrow[rr, "\pi_i^j"'] & & A_i
\end{tikzcd}$$

For our purposes, the objects A_i are the approximants Γ_i, the maps $\pi_i^j = \pi_i \circ \pi_{i+1} \circ ... \circ \pi_{j-1}$ and the maps π_i^∞ are simply the projections from the inverse limit to the i-th approximate. This construction allows us to forgo many technical details along the way by simply using the universal property.

2.2 Gähler's construction

Imagine that we have a finite set of prototiles and we wanted to tile $\mathbb{R}^n$. We would need an infinite set of instructions; the first instruction tells us how to place a tile at the origin, the second being how to place tiles around that first tile, ad infinitum. If we set the spaces Γ_n to be all possible 'instructions' on how to place a tile at the origin and the n-layers of tiles around it, we have something that resembles an inverse limit space! The map $\pi_n : \Gamma_{n+1} \to \Gamma_n$ is then the maps that forgets the instructions of the outermost ring. The condition $\pi_n(x_{n+1}) = x_n$ simply encodes the fact that the instructions agree on the first n rings. An point of the inverse limit $(x_0, x_1, ...)$ tells us how to tile all of $\mathbb{R}^n$ and thus represents a tiling [2].

The obvious problem is that instructions aren't topological spaces! If we wish to gleam an insight into the topology of the hull, we need to find a way to turn these instructions into topological spaces whose properties are computable. Let's start with the first set of instructions Γ_0. A point in Γ_0 instructs us how to put a tile at the origin. This involves picking a prototile and a point in said prototile to put at the origin. Thus, at first glance it would seem that Γ_0 is simply the set all prototiles.

But what if the origin lies on an edge of the tile? If two prototiles, let's say A and B, share an edge and that edge is on the origin, do we specify a point in the tile A or specify a point in the tile B? The answer is to identify the edge at which A meets B. So Γ_0 is the union of all prototiles with some identifications. If somewhere in the tiling two prototiles share an edge, then those two edges are identified. The same goes for vertices and higher dimensional counterparts.

Now that we have the first tile down, we need a way to we place the first layer around the center tile. To do this, we consider two prototiles in the tiling to be the same only if they have the same layer of tiles around them. In a similar manner, we could consider two prototiles to be the same only if they share the same 2 layers of tiles and so on. Tiles for which we consider the k-layers of tiles around them are called ***k*-collared** tiles. With these k-collared tiles, Γ_k's description becomes identical to Γ_0, except with k-collared tiles. The forgetful map is then just the map that ignores the outermost ring of tiles.

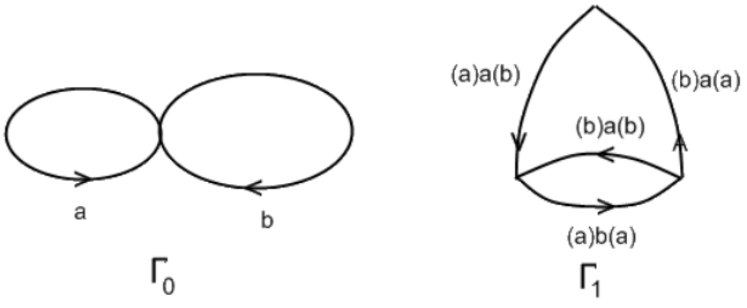

Figure 2: The first two Gähler approximants for the Fibonacci tiling space

2.3 Anderson-Putnam Construction

Although Gähler's construction gives us quite a bit of information about tiling spaces, it is fairly useless when it comes to actually computing topological invariants. In general, the spaces Γ_n and the maps π_n depend on

n and get successively more complicated as n grows. With substitutions tilings, this problem can be avoided. The Anderson-Putnam construction relies on the fact that the approximants Γ_n and the maps π_n are essentially the same.

Suppose we have a tiling space generated by substitution and let Γ_0 be as in the Gähler construction and in general Γ_n's is Γ_0 streched by a factor of λ^n. We consider Γ_n as containing one copy of each level n-supertile. As before, the forgetful map $f_n : \Gamma_{n+1} \to \Gamma_n$ restricts the attention to the level-n supertile. If we then rescale all the spaces to the same size as T_0, then all the maps f_n are very similar in the sense that they are induced by the substitution map σ. The result is that the tiling space is topologically the inverse limit of a single space with similar maps $\varprojlim(\Gamma_0, \tilde{\sigma}_n)$.

To properly prove first requires a concept introduced by Kellendonk which is called "forcing the border".

Definition 2.2. A substitution tiling T is said to force the border if there exists a positive integer n such that any two level-n supertiles of the same type have the same pattern of neighboring tiles.

Not all substitutions force the border. However, the trick introduced by Anderson and Putnam is to rewrite the substitution in terms of collared tiles. It can then be shown that substitution applied to these collared tiles always forces the border. The result is a very useful theorem about substitution tilings.

Theorem 2.3 ([3])**.** *Let Γ_0 and Γ_1 be as in the Gähler construction. In both cases, we denote the bonding map by σ. Ω_σ is always homeomorphic to $\varprojlim(\Gamma_1, \sigma)$ and if σ forces the border, then Ω_σ is homeomorphic to $\varprojlim(\Gamma_0, \sigma)$.*

3 Cohomology of tiling spaces

3.1 Direct Limit

Let G_i be a family of groups indexed by a partially ordered directed set I. Furthermore, suppose that for any pair $i \leq k \in I$, we have a group homomorphism $\rho^i_k : G_i \to G_k$ such that if $i \leq j \leq k$, $j \in I$ then

$$\rho^i_k = \rho^j_k \circ \rho^i_k \quad \text{and} \quad \rho^i_i = \mathrm{Id}_{G_i} \tag{4}$$

The **direct limit** of the groups, denoted G_∞ or $\varinjlim(\Gamma, \rho)$ is the disjoint union of all the G_i with the following equivalence relation. If $i \leq k$ and $x \in G_i$, then $x \sim \rho^i_k(x) \in G_k$. The direct limit also has a natural group structure. Namely, if $x \in G_i$, $y \in G_k$, $i \leq k$, and $j \leq k$ then x and y can be identified with $\rho^i_k(x)$ and $\rho^j_k(y)$ respectively. The product $x \cdot y$ is then defined to be $\rho^i_k(x) \cdot \rho^j_k(y) \in G_k$. It is easy to see that this multiplication is well defined and that G_∞ is in fact a group.

It is quite obvious that the direct limit and the inverse limit have many similar characteristics. In fact this is because they are dual in the categorical notion of the sense! The construction of the direct limit can also be done with the universal property much like the inverse limit, but with the direction of the morphisms inversed. There is also another family or morphisms $\rho^i_\infty : G_i \to G_\infty \; \forall i \in I$ such that if $i \leq j$ the following diagram commutes

$$\begin{tikzcd}
& G_\infty & \\
G_j \arrow[ur, "\rho^j_\infty"] & & G_i \arrow[ul, "\rho^i_\infty"'] \arrow[ll, "\rho^i_j"]
\end{tikzcd}$$

Like the inverse limit, the direct limit satisfies a universal property. If we have another object H_∞ with a family of morphisms $\chi^i_\infty : G_i \to H_\infty$ for all $i \in I$ and with the same commutative properties as the maps ρ^i_∞, then there is a unique isomorphism $\omega : G_\infty \to H_\infty$ such that the following diagram commutes

$$\begin{tikzcd}
& H_\infty & \\
& G_\infty \arrow[u, "\omega"] & \\
G_j \arrow[ur, "\rho^j_\infty"] \arrow[uur, bend left, "\chi_j^\infty"] & & G_i \arrow[ul, "\rho^i_\infty"'] \arrow[uul, bend right, "\chi_i^\infty"'] \arrow[ll, "\rho^i_j"]
\end{tikzcd}$$

Although in our construction we used groups, the same definition applies to rings, with ring homomorphisms instead of group homomorphisms are the morphisms used in the direct limit.

3.2 Simplicial Homology and Cohomology

While studying topological invariants, we would somehow like to characterize the number of "holes" in a space. But how do you study something that by definition that isn't there? The answer is with homology and its dual notion cohomology. We give a brief overview of simplicial homology, simplicial cohomology and then move on to something that is much more useful to studying tiling spaces, Čech cohomology.

Suppose S is a simplicial complex. We denote C_k the free abelian group generated by the set of k-simplicies in S. A general element of this group, called chains, can be written as a formal sum of k-simplicies

$$\sum_{i=0}^{N} c_i \zeta^i, \;\; c_i \in \mathbb{Z}, \;\; \zeta^i \in S \text{ the } i\text{-th } k\text{-simplex} \tag{5}$$

A basis element of C_k, a k-simplex, is a given as a $k+1$-tuple of vertices, or 0-simplices: $\zeta = \langle p_0, ..., p_k \rangle$. We also have boundary operators $\partial_k : C_k \to C_{k-1}$ which are homomorphisms defined by

$$\partial_k(\zeta) = \sum_{i=0}^{k} (-1)^i \langle p_0, ..., \hat{p_i}, ..., p_k \rangle \tag{6}$$

where $\langle p_0, ..., \hat{p_i}, ..., p_k \rangle$ is the oriented face of ζ obtained by removing the i-th vertex. The image of ∂_{k+1} forms a subgroup in $B_k \subset C_k$, whose elements are called boundaries. The kernel of ∂_k also forms a subgroup $Z_k \subset C_k$ which is said to consist of cycles. A simple computation shows that, for all k, $\partial_k \circ \partial_{k+1} = 0$ for any $k+1$ chain in C_{k+1}. So B_k is a subgroup of Z_k and it makes sense to speak of cycles that differ by boundaries. The k-th homology group H_k of S is then defined to be the quotient $H_k(S) = Z_k/B_k$. The idea is that rank of this group measures the k dimensional holes in the space.

There is also a dual notion of homology called cohomology. Consider the set of homomorphisms $\varphi : C_k \to G$ from the set of k-chains to an abelian group G. Then this set can be made into an abelian group where addition is defined in an obvious manner. The elements of this group C^k are called k-cochains. We then replace the boundary operator with the coboundary operator $\delta^{k+1} : C^k \to C^{k+1}$ defined as

$$\delta^{k+1}\varphi(\zeta) = \varphi(\partial_{k+1}\zeta) \tag{7}$$

for every $k+1$-chain ζ. Again it is obvious that the image of δ^k and the kernel of δ^{k+1} form subgroups of C^k, denoted B^k and Z^k respectively. Functions in B^k are called k-coboundaries and functions in Z^k are called k-cocycles. The k-th cohomology group $H^k(S;G)$ of S is then defined as $H^k(S;G) = Z^k/B^k$. It may seem that these theories answer the same problem. However cohomology carries an additional structure, the cup product, which turns it into a ring. This additional machinery isn't very useful for our purposes. Rather, we will use 1-cocycles to create new tilings from existing ones.

3.3 Čech cohomology

Simplicial cohomology is useful for simplicial complexes. This concept is then extended to spaces that can be approximated by simplicial complexes through the use of singular cohomology. As it turns out, Čech cohomology is a theory that allows us to compute topological invariants for a larger class of spaces.

Definition 3.1. Let X be a topological space and $\mathcal{U} = \{U_i\}$ an open cover. The nerve of $\mathcal{U}$, denoted $N(\mathcal{U})$, is a simplicial complex with an n-simplex for every non-empty intersection of $n+1$ open sets

.

For example, if $\mathcal{U} = X$, then the nerve is simply a point. The Čech cohomology of the cover $\check{H}^k(\mathcal{U})$ is the simplicial cohomology of $N(\mathcal{U})$. Of course, the cohomology of a single cover isn't useful because it depends on the choice of the cover.

An open cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ if every open set in $\mathcal{V}$ is contained in an open set of $\mathcal{U}$. It is obvious that any two open covers share a common refinement, so the set of open covers forms a directed set. If $\mathcal{V}$ refines $\mathcal{U}$, there is a simplicial map between the nerves that induces a map $\mu_{\mathcal{U}\mathcal{V}} : \check{H}^k(\mathcal{U}) \to \check{H}^k(\mathcal{V})$ that is canonically defined.

Definition 3.2. Čech cohomology of X, denoted $\check{H}^k(X)$, is the direct limit of $(\check{H}^k(\mathcal{U}), \mu_{\mathcal{U}\mathcal{V}})$ where the limit is taken over all the open covers of X.

This is nice definition, but it's practically useless in terms of computability. For even the simplest spaces, there's no way to consider all open covers. Fortunately when it comes to inverse limits of certain topological spaces, the Čech cohomology becomes simpler, at least on a conceptual level.

Theorem 3.3 ([4])**.** *Let X be an inverse limit of a sequence of spaces Γ_i, where each Γ_i is a manifold or a finite CW complex, then the Čech cohomology of X is equal to the direct limit over the simplicial cohomologies of the approximants. That is to say*

$$\check{H}^k(X) = \varinjlim (H^k(\Gamma_n), \pi_n^*) \tag{8}$$

where the maps π_n^* are induced by the projection maps π_n.

3.4 Pattern equivariant cohomology

There is another cohomology theory that doesn't refer to the inverse limit structure of tiling spaces at all. Instead, it uses special differential forms and the exterior derivative to create cohomology groups. These differential forms aren't general, and are specific to each tiling.

Definition 3.4. Let P be a tiling of $\mathbb{R}^n$. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly P-equivariant with radius R if, for $x, y \in \mathbb{R}^n$, $[B_R(x) - x] = [B_R(y) - y]$ implies $f(x) = f(y)$.

In addition there is also the notion of weakly P-equivariant functions. A smooth function f is said to be **weakly P-equivariant** if it is the uniform limit of strongly P-equivariant functions and its derivative of all order are uniform limits of strongly P-equivariant functions. Note that the range of the strongly PE functions is unbounded as the approximation becomes better and better.

Given a tiling T of $\mathbb{R}^n$, we denote the strongly PE and weakly PE k-forms by $\Lambda^k_{s-P}(T)$ and $\Lambda^k_{w-P}(T)$ respectively. It is easy enough to check that the exterior derivative of a strongly or weakly pattern equivariant function is indeed a strongly or weakly pattern-equivariant function. We have that $d \circ d = 0$ for all forms. This results in the differential complex of strongly PE functions

$$\{0\} \xrightarrow{d} \Lambda^0_{s-P}(T) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k_{s-P}(T) \xrightarrow{d} \{0\}$$

The cohomology of the first complex is the strongly PE cohomology of the tiling, denoted $H^k_P(T)$. We will later on consider the mixed cohomology, that is the closed strongly PE forms modulo strongly PE forms that are the image under d of a weakly PE form.

Because PE functions are defined on all of $\mathbb{R}^n$, all closed forms are exact. What makes the cohomology non-trivial is that a closed PE form may not be the image of a PE form. Consider for example the 1-form dx. It is PE for all tilings, but it is the derivative of a 0-form x, which is never PE. Thus dx represents a non-zero class in $H^1_P(T)$.

Although weakly PE cohomology isn't well understood, strongly PE cohomology has very useful properties due to the following theorem by Kellendonk and Putnam.

Theorem 3.5 ([5])**.** *For a simple tiling T, we have that $\check{H}^k(\Omega_T, \mathbb{R}) \simeq H^k_P(T)$*

The proof presented in the paper relied on complicated machinery about foliated spaces, which we ignor. The intuitive idea is that every class in $\check{H}^k(\Omega_T, \mathbb{R})$ can be represented as a function that, in a sense, captures the structure of the tiling.

3.5 Cohomology of the Ammann-Beenker tiling

As mentioned earlier, the cohomology of substitution tilings is computable thanks to the Anderson-Putnam construction. We simply need to compute the direct limit of the spaces Γ_0 induced by the substitution map σ. That is to say, the map σ induces a homomorphism on the space of k-chains. We study the effect of this homomorphism on the (co)boundary and (co)kernel to compute the direct limit. In general, this can be quite tricky if the substitution map doesn't create a isomorphism on the chains. In this section we compute the cohomology of the Ammann-Beenker tiling, whose substitution is as follows:

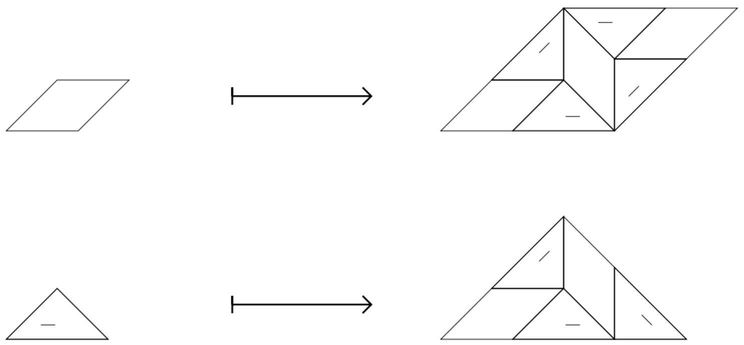

Figure 3: Substitution rule for the Ammann-Beenker tiling.

The other prototiles are substituted into the rotated versions of the supertiles. this tiling forces the border, so there is no need to collar the tiles to compute the cohomology.

The first step is to create the first approximant Γ_0. We recall that this space is the disjoint union of all the prototiles modulo identifications if two translates of two prototiles share an edge or vertex somewhere in the tiling. This requires a bit of work and the resulting complex is impossible to draw, but in the end we have 20 faces, 16 edges and 1 vertex.

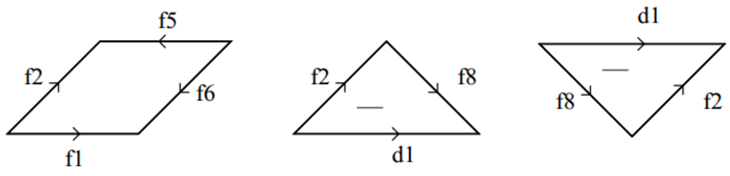

Figure 4: CW-complex for the first approximant Γ_0

Only the first three prototiles are drawn. The remaining 17 can be obtained by rotation of $\frac{n\pi}{4}$. For each rotation by $\frac{\pi}{4}$, the label on the edges increases the index by 1 modulo 8. As a results of this identification, we have that

$C^0 \simeq \mathbb{Z}, C^1 \simeq \mathbb{Z}^{16}$ and $C^2 \simeq \mathbb{Z}^{20}$. The fact that we only have 1 vertex implies that δ^1 is the zero map. We have also chosen an orientation for the edges. The boundary of a prototile is the signed sum of its edges, where we travel the boundary in a counter clockwise direction. Thus, we are left with the cochain complex

$$\{0\} \xrightarrow{\delta^0} \mathbb{Z} \xrightarrow{\delta^1} \mathbb{Z}^{16} \xrightarrow{\delta^2} \mathbb{Z}^{20} \xrightarrow{\delta^3} \{0\}$$

where δ^2 is the 20×16 matrix

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

When we mod out the images and kernels, we must make sure to take torsion into consideration. Counting ranks leads to the result if thre is no torsion. Fortunately, this is the case here. The rank of the matrix is 11, and so the dimension of the kernel is 5. So $H^0(\Gamma_0) = \ker \delta^1 \ / \ \text{im } \delta^0 = \mathbb{Z}/\{0\} = \mathbb{Z}$, $H^1(\Gamma_0) = \ker \delta^2 \ / \ \text{im } \delta^1 = \mathbb{Z}^5/\{0\} = \mathbb{Z}^5$, and $H^2(\Gamma_0) = \ker \delta^3 \ / \ \text{im } \delta^2 = \mathbb{Z}^{20}/\mathbb{Z}^{11} = \mathbb{Z}^9$.

Now that we have computed the cohomology of the zeroeth approximant, we need to inspect the action of the substitution on the first approximant. That is to say, what effect does the substitution map have on the kernels and images of the maps coboundary maps? The zeroth cohomology is seen to be unaffected because δ^1 is the zero map. The same goes for δ^3. Thus, the only cohomology groups that could be perturbed are the first and the second, due to the coboundary map δ^2. The substitution acts on the 1-cocycles through the following matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

This matrix is invertible under the integers and thus represents a isomorphisms between the 1-cocycles. The rank of the kernel and image of δ^2 is then conserved. Thus in all three cases, the groups in the direct limit are all the same and the homomorphisms between them are in fact isomorphisms. The direct limit is then isomorphic to the first group, and we have that $\check{H}^0(\Omega_{AB}, \mathbb{R}^2) \simeq \mathbb{Z}$, $\check{H}^1(\Omega_{AB}, \mathbb{R}^2) \simeq \mathbb{Z}^5$ and $\check{H}^2(\Omega_{AB}, \mathbb{R}^2) \simeq \mathbb{Z}^9$.

4 Deformations of tilings

4.1 Deformations as 1-cocycles on the approximants

Given a tiling, it is a natural question to ask how to create new ones. For a simple tiling, the place to start is its building blocks, the prototiles. To parameterize a prototile A, we need a collection of vectors $\{v_{A,n}\}$ in $\mathbb{R}^n$. That is to say, we have a shape function that inputes edges and outputs a vector in $\mathbb{R}^n$. Given that going around a prototile brings you back to where you started, these vectors can't be pricked arbitrarily, but rather are subject to the condition

$$\sum_n v_{A,j} = 0 \tag{9}$$

If we wish to deform a tiling to obtain a new one, we mustnt only consider changing prototiles independently. If two tiles share an edge, the corresponding vector must be the same. So if two prototiles share an edge anywhere in the tiling, they are identified so that the shape function can act of them in an appropriate manner. But this results in the first approximant Γ_0 in the Gähler space of the tiling. That is to say, shape deformations are simply vector valued 1-cochains on the approximant Γ_0. In general, we can also define deformations on k-collared prototile. Thus, shape deformations are then vector valued 1-cochains on Γ_k.

Shape deformations can't be general 1-cochains however. Namely, if we have a shape deformation φ,it must be a 1-cocycle. This is because for any 2 dimensional face A, we have

$$\delta^2\varphi(A) = \varphi(\partial_2 A) = 0 \tag{10}$$

since

$$\partial_2 A = \sum_n v_{A,j} = 0 \tag{11}$$

Note that all shape deformations do is take give us a new 1-skeleton, up to translation. They take as inputs the edges of prototiles (or k-collared tiles) and output a set of vectors. *A priori*, there is no reason that one should be able to turn this 1-skeleton into a tiling by adding in higher dimensional faces. But if such is the case, then we can build a new tiling from T, up to translation, a thus a new tiling space, which we denote $\Omega_{f(T)}$.

There is also the question of invertibility. That is to say, is it possible to obtain T back from the new tiling through a similar procedure. There are situations in which this is obvious not possible. Intuitively, invertibility shouldn't be a problem if the new tiles differ a little from the new ones. We call shape functions admissible if they create a complete tiling space and is invertible.

4.2 Equivalence between deformed tilings

Now that we've laid down the foundations for deformations of tilings, we must ask ourselves what can we say about the hull of this new tiling. Recall that the projections π_i^j and π_n^∞ of the inverse limit structure induces group homomorphisms π_j^{*i} and π_∞^{*n} respectively in the direct limit of the Čech cohomology. Thus if f is a 1-cocycle on Γ_0, it defines a class $[f] \in H^1(\Gamma_0, \mathbb{R}^n)$ and $\pi_\infty^{*0}([f]) \in \check{H}^1(\Omega_T, \mathbb{R}^n)$.

Theorem 4.1 ([1]). *Suppose f and g are two admissible shape functions on Γ_0. Then if $\pi_\infty^{n*}([f]) = \pi_\infty^{n*}([g])$, $\Omega_{f(T)}$ and $\Omega_{g(T)}$ are mutually localy derivable.*

Proof. We must build *g(T)* from *f(T)* in a manner that is completely local. Recall that we have the projection map $\pi_0^{k-1} : \Gamma_k \to \Gamma_0$. If $\pi_\infty^{*0}([f]) = \pi_\infty^{*n}([g])$, then by the universal property we have, for some finite k, we have $\pi_{k-1}^{*0}[f] = \pi_{k-1}^{*0}[g]$, so $\pi_{k-1}^{*0}(f) - \pi_{k-1}^{*0}(g) = \delta^1\beta$ for some $\beta \in C^0(\Gamma_k, \mathbb{R}^n)$. Now every vertex $\mathbf{v}$ in every tiling in $\Omega_{f(T)}$ gets mapped to a unique vertex in Γ_k, the map being determined by a ball of size $(k+1)D$ around $\mathbf{v}$ where D is the diameter of the largest prototile. Moving each vertex by $-\beta(\mathbf{v})$ and interpolating the edges between the vertices converts a tiling in $\Omega_{f(T)}$ to a tiling in $\Omega_{g(T)}$. Because the conjugacy depends only on k-collared tiles, it is local. □

In the same paper, Sadun and Clarke also introduced asymptotically negligible cocycles, which lead to tiling spaces that are topological conjugates but not MLD. For a tiling $P \in \Omega_T$, a recurrence of size r is an ordered pair $(\mathbf{z_1}, \mathbf{z_2})$ of vertices in P such that $[B_r(\mathbf{z_1}) - \mathbf{z_1}] = [B_r(\mathbf{z_2}) - \mathbf{z_2}]$ but $[B_{r+\epsilon}(\mathbf{z_1}) - \mathbf{z_1}] \neq [B_{r+\epsilon(\mathbf{z_2})} - \mathbf{z_2}]$ for $\epsilon > 0$. If k is small enough so that the k-ring around any tile is contained in a ball of size r centered at that tile, then paths along edges from $\mathbf{z_1}$ to $\mathbf{z_2}$ lifts to cycles in $C_1(\Gamma_k)$, different paths leading to cycles that differ by boundaries. Therefore, if we have two paths p_1 and p_2 from $\mathbf{z_1}$ to $\mathbf{z_2}$, we have $p_2 - p_1 = \partial\beta$ where β is a 2-chain. So for a shape function φ, which is also a 1-cocyle,

$$\varphi(p_2 - p_1) = \varphi(\partial\beta) = 0 \tag{12}$$

$$\varphi(p_2) - \varphi(p_1) = 0 \tag{13}$$

$$\varphi(p_2) = \varphi(p_1) \tag{14}$$

Thus paths between two vertices are defined unambiguously. An element $\eta \in H^1(\Gamma_k, \mathbb{R}^n)$ is said to be asymptotically negligble if, for each ϵ >0, there exits a constant R_ϵ such that η applied to any recurrence of size greate than R_ϵ is smaller than ϵ in norm. The images of asymptotically neglible elements in $H^1(\Gamma_k, \mathbb{R}^n)$ under π_∞^{*k} are the asymptotic elements of $\check{H}^1(\Omega_T, \mathbb{R}^n)$.

Although the mathematical definition of asymptotically negligible cocycles is precise, it is at first glance not obvious what they do. Remember that our goal was to explicitely create topological conjugate tiling spaces that aren't MLD. The idea of mutual local derivability is that a finite patch of some size around a point in a tiling determines exactly the finite patch of size 1 around the same point in another tiling.

The role of asymptotically negligible functions is then to deform tilings ever so slightly so that if two points have the same patch around them, the two new points will have the same patch around them *up to a small translation.*

This translation becomes smaller and smaller as the size of recurrence becomes larger, but never the less is always there. In essence, this conjugacy becomes less and less local. The following theorem formalizes these ideas.

Theorem 4.2 ([1]). *Suppose f and g are two admissible shape functions on Γ_n associated wit a tiling space Ω_T where the orbit of T is dense. Then if $\pi_\infty^{n*}([f])-\pi_\infty^{n*}([g])$ is asymptotically negligble, $\Omega_{f(T)}$ and $\Omega_{g(T)}$ are topological conjugates, but generally not MLD.*

Proof. We give a proof identical to the one introduced by Sadun and Clarke in their original paper. The goal is build a conjugacy ϕ, which we do in stages. Note that there is a slight abuse in notion when using the shape function f. Although by definition it acts only on the edges of prototiles, for $x \in \Omega_T$ we also write $f(x)$ as the corresponding tiling in $\Omega_{f(T)}$.

First, pick a tiling x in Ω_T that has a vertex at the origin. For each vertex $\mathbf{v}$ in x, there is a path $p_{\mathbf{v}}$ from the origin to $\mathbf{v}$ along edges. Each vertex $\mathbf{v}$ in x also has a corresponding vertex in $f(x)$, the location of which is precisely $f(p_{\mathbf{v}})$. At the corresponding vertex in $\phi(f(x))$, we place $g(p_{\mathbf{v}})$. Note that the paths from the origin to $\mathbf{v}$ isn't unique, but different paths differ by boundaries. Now that the vertices of $\phi(f(x))$ specified, constructing the edges, faces and prototiles is simple.

This defines $\phi(f(x))$. For $\mathbf{z} \in \mathbb{R}^n$, we define $\phi(f(x) - \mathbf{z}) = \phi(f(x)) - \mathbf{z}$. Because we assumed the orbit of T was dense, to extend ϕ to the hull we need to prove it is uniformly continuous on the orbit of x. Suppose $\mathbf{z_1}$ and $\mathbf{z_2}$ are vertices in x such that $x - \mathbf{z_1}$ and $x - \mathbf{z_2}$ agree on a large ball around the origin. This means that $\phi(x) - g(p_{\mathbf{z_1}})$ and $\phi(x) - g(p_{\mathbf{z_1}})$ agree on a large ball around the origin. Because $f - g$ is asymptotically negligible, $f(p_{\mathbf{z_1}}) - f(p_{\mathbf{z_2}})$ is close to $g(p_{\mathbf{z_1}}) - g(p_{\mathbf{z_2}})$, so $\phi(x - \mathbf{z_1}) = \phi(x) - f(p_{\mathbf{z_1}})$ is close to $\phi(x - \mathbf{z_2}) = \phi(x) - f(p_{\mathbf{z_2}})$ agrees on a large ball around the radius up to a translation of size $(f(p_{\mathbf{z_2}}) - f(p_{\mathbf{z_1}})) - (g(p_{\mathbf{z_2}}) - g(p_{\mathbf{z_1}}))$. Because this translation can be made arbitrarily small by increasing the size of the recurrence (independent of $\mathbf{z_1}$ and $\mathbf{z_2}$), ϕ is uniformally continuous.

To see that ϕ is invertible under a map $\phi' : \Omega_{g(T)} \to \Omega_{f(T)}$, simply repeat the same procedure with the role of f and g and with $\phi(x)$ as the reference tile.

□

4.3 Deformations as closed 1-forms

Since shape functions are 1-cocycles on the approximants, we can also use the pattern-equivariant cohomology and represent them as strongly pattern-equivariant forms. The question is, which 1-forms represent the asymptotically negligble elements? Theorem 4.2 relied on the fact that the size of the translation between to balls can be made arbitrarily small as we increase the size of the recurrence. This reminds us of the definition of weakly PE functions which can be approximated as strongly PE-functions. That is to say, the error of the approximation becomes smaller and smaller as the range of the strongly PE function increases, leading in the end to a non-local conjugacy.

Theorem 4.3 ([6])**.** *An element* $\eta \in \check{H}^1(\Omega_T, \mathbb{R}^n)$ *is asymptotically negligible if and only if the associated strongly pattern-equivariant 1-form is in* $B^1_{w-p}(\mathbb{R}^n, \mathbb{R}^n)$
$\cap\, Z^1_{s-p}(\mathbb{R}^n, \mathbb{R}^n)/B^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$

Proof. We give an idea as to why the theorem is true instead of the whole proof. It is also worth nothing that, *a priori*, PE functions are defined on $\mathbb{R}^n$. For our purposes, this simply means that PE functions acts on the vertices of the tiles and gives us a new 1-skeleton.

Remember that pattern equivariant 1-forms are defined on all of $\mathbb{R}^n$. Thus by Poincaré's lemma, any such PE form can be written as dF, where F is a smooth function from $\mathbb{R}^n \to \mathbb{R}^n$, not necessarily pattern equivariant. Suppose dF is strongly PE with range R and that $x, y \in \mathbb{R}^n$ are two vertices in a tiling T such that $[B_R(x) - x] = [B_R(y) - y]$. We have $dF(x) = dF(y)$ and if our deformation in question is simply $Id_{\mathbb{R}^n} + dF$ then the displacement of the two vertices are the same. In general, we can't say anything about the patch around $x + dF(x)$ and $y + dF(y)$ in the new tiling.

But suppose F is strongly pattern equivariant, with radius r large compared to R. Then if we integrate from y to x, we get

$$\int_y^x dF = F(x) - F(y) \tag{15}$$

which vanishes if the patches around y and x agree on a ball of size r. This is analogous to Theorem 4.1, where shape functions that lead to MLD topological tiling spaces are precisely those that agree on all the k-collared tiles, modulo a global translation. In this instance, dF is the translation and it leads to MLD tilings spaces if the integration along recurrences of a large size disappear, that is to say, F is strongly PE.

Suppose now that $dF \in B^1_{w-p}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-p}(\mathbb{R}^n, \mathbb{R}^n)$, which means F is weakly PE. Thus, $\forall \epsilon > 0$, there exists R_ϵ such that $F = \varphi_\epsilon + \psi_\epsilon$ where φ_ϵ is PE with range R_ϵ and $||\psi_\epsilon||_\infty < \epsilon$. Computing the previous integral

$$\int_y^x dF = (\varphi_\epsilon(x) + \psi_\epsilon(x)) - (\varphi_\epsilon(y) + \psi_\epsilon(y)) \tag{16}$$

Thus, regardless of the size of the recurrence, there will always be a small error in the translation. We can clearly see that this is similar to asymptotically negligible cocycles defined on the approximants. □

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