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# ON FINITE DIMENSIONAL ALGEBRAS ISOMORPHIC TO MONOMIAL ALGEBRAS

By

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July 1997

A Thesis

submitted to the School of Graduate Studies and Research  
in partial fulfillment of the requirements  
for the degree of  
Doctor of Philosophy in Mathematics<sup>1</sup>

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# Abstract

Let  $K$  be a field and  $\Gamma = (\Gamma_0, \Gamma_1)$  be a connected finite directed graph. Then  $K\Gamma$  denotes the path algebra of  $\Gamma$  over  $K$ . A two-sided ideal  $I$  of  $K\Gamma$  is *admissible* in case there exists a positive integer  $N \geq 2$  such that  $\langle \Gamma_1 \rangle^N \subseteq I \subseteq \langle \Gamma_1 \rangle^2$ . It is well-known that every connected split basic finite dimensional  $K$ -algebra  $A$  is isomorphic to the quotient  $K\Gamma/I$  of a path algebra  $K\Gamma$  by an admissible ideal  $I$  (cf. [16]). (The underlying connected finite directed graph  $\Gamma$  is usually called the *quiver* of  $A$ .) A quotient  $A = K\Gamma/I$  of path algebra by admissible ideal is called a (*finite dimensional*) *monomial algebra* if  $I$  is generated by a set of finite directed paths in  $\Gamma$ . In recent years the class of monomial algebras has been exposed to extensive study and significant advances have been made toward the understanding of the homological nature of this class of finite dimensional algebras (see [20], [21], [22] and [23]). We say that  $A = K\Gamma/I$  is of *monomial presentation type* if  $A$  is isomorphic to a monomial algebra over  $K$ . The primary purpose of this thesis is to identify the path algebra quotients  $K\Gamma/I$  of monomial presentation type under various conditions on the underlying finite directed graph  $\Gamma$  and/or the quotients  $K\Gamma/I$  themselves. Even though it is easy to determine whether or not a path algebra quotient is a monomial algebra, the identification of path algebra quotients of monomial presentation type is a fundamentally harder problem. Our study is of significance in that monomial algebras are known to be well behaved and many of their properties, such as homological ones, are algebra isomorphism invariants. The main tools to be utilized in the thesis are

the noncommutative Gröbner basis theory for path algebras and Saorín's change of variable method. In Chapter 2, we use these tools to find cases where path algebra quotients of monomial presentation type must already be monomial. We also study isomorphisms between monomial algebras and present a combinatorial criterion for two monomial algebras to be isomorphic. Chapter 3 is concerned with homological aspects of path algebra quotients of monomial presentation type. Specifically, we shall study minimality of the Anick-Green resolutions for the simple modules over path algebra quotients and discuss its relationship to the monomial presentation type. In the last chapter of the thesis, we devote ourselves to a particular class of quotients of path algebras – called *right monomial rings* – which were introduced by Burgess *et al.* in [6] as a generalization of monomial algebras. We shall answer an open question presented in [6] in some special cases, and provide a characterization of the monomial presentation type within the class of path algebra quotients which are right monomial rings by showing that they are precisely those gradable by the radical.

# Acknowledgements

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# **Dedication**

*To my father and mother*

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# Chapter 1

## Introduction

Let  $K$  be a field and  $\Gamma = (\Gamma_0, \Gamma_1)$  be a connected finite directed graph, where  $\Gamma_0$  and  $\Gamma_1$  are the sets of vertices and arrows in  $\Gamma$ , respectively. Let  $K\Gamma$  denote the path algebra of  $\Gamma$  over  $K$ . A two-sided ideal  $I$  of  $K\Gamma$  is called *admissible* if there exists a positive integer  $N \geq 2$  such that  $\langle \Gamma_1 \rangle^N \subseteq I \subseteq \langle \Gamma_1 \rangle^2$ . A basic fact in the theory of representations of finite dimensional algebras says that every connected split basic finite dimensional algebra  $A$  over  $K$  is isomorphic to the quotient  $K\Gamma/I$  of a path algebra  $K\Gamma$  by an admissible ideal  $I$  (cf. [16]). The underlying connected finite directed graph  $\Gamma$  is also called the quiver of  $A$ . Among the quotients of path algebras by admissible ideals is a class of finite dimensional algebras called monomial algebras. A quotient  $A = K\Gamma/I$  of path algebra by admissible ideal is called a (*finite dimensional*) *monomial algebra* if  $I$  is a monomial ideal in  $K\Gamma$ ; i.e.,  $I$  is generated by a set of finite directed paths in  $\Gamma$ . In recent years monomial algebras have been exposed to extensive study and significant advances have been made toward the understanding of the homological nature of this class of finite dimensional algebras (see [20], [21], [22] and [23]). We say that a path algebra quotient  $A = K\Gamma/I$  is of *monomial presentation type* if  $A$  is isomorphic to a monomial algebra  $\hat{A} = K\hat{\Gamma}/\hat{I}$ . Since two isomorphic finite dimensional algebras have the same quiver (up to isomorphism), it is without loss

of generality that we assume in the definition above that  $A$  and  $\hat{A}$  have the same underlying finite directed graph  $\Gamma$ . It is worth noting that in the literature “monomial algebra” sometimes means a path algebra quotient of monomial presentation type. We distinguish between these two classes of finite dimensional algebras because being a monomial algebra is not an algebra isomorphism invariant. Moreover, existing methods make the identification of monomial algebras algorithmic (Proposition 2.2.1) while no such algorithms for the monomial presentation type are known. The purpose of this thesis is to identify the path algebra quotients  $K\Gamma/I$  of monomial presentation type under various conditions on the underlying finite directed graph  $\Gamma$  and/or the quotients  $K\Gamma/I$  themselves. A simple example of a path algebra quotient which is not a monomial algebra but which is of monomial presentation type is given by the quotient of free algebra  $A = K\langle\alpha, \beta\rangle/I$  where  $I = \langle\alpha\beta, \beta^2, \alpha^2 - \beta\alpha, \alpha^3, \alpha^2\beta, \beta\alpha^2\rangle$ .

The thesis is composed of three parts. In Chapter 2, two basic tools are utilized to study characterizations of the monomial presentation type: one is the theory of noncommutative Gröbner bases for path algebras, introduced by Farkas, Feustel and Green in [10], which extends the known theory for the commutative polynomial rings (cf. [5]) and free associative algebras (cf. [24]), and the other is Saorín’s change of variable method (cf. [26]) for connecting two isomorphic presentations by a path algebra. After a brief introduction to Gröbner bases for path algebras, Section 2.2 focuses on several special types of finite directed graphs. Two main results are included in this section. The first result (Theorem 2.2.7) provides several conditions on the underlying finite directed graph  $\Gamma$  and the ideal  $I$  of relations such that  $A = K\Gamma/I$  is of monomial presentation type if and only if it is a monomial algebra. As one corollary to this result, we obtain that if  $\Gamma$  is square-free and  $A = K\Gamma/I$  is of Loewy length 3, then  $A$  is of monomial presentation type if and only if  $A$  is a monomial algebra. The second result of this section (Theorem 2.2.16) deals with a special type of underlying

finite directed graph  $\Gamma$  and gives a simple condition such that the path algebra quotient  $A = K\Gamma/I$  is not of monomial presentation type. In addition, various examples are given throughout this section. In Section 2.3 we prove that if  $A = K\Gamma/I$  is a one-sided serial algebra, then  $A$  is in fact a monomial algebra (Theorem 2.3.1). In the last section of Chapter 2, we consider the isomorphism problem between two monomial algebras and generalize an isomorphism theorem concerning local monomial algebras by Shirayanagi [28] to general monomial quotients of path algebras (Theorem 2.4.1). Moreover, we provide an example (Example 2.6) to show that this theorem does not generalize to pairs of isomorphic algebras where one is monomial and the other is binomial, thereby answering a question raised in [29].

Chapter 3 is concerned with the Anick-Green resolutions over quotients of path algebras and associated monomial algebras of these quotients. In [3], Anick and Green presented a combinatorial construction of projective resolutions for the simple modules over a quotient of path algebra. In general, as indicated in [3], these projective resolutions are not necessarily minimal. The authors showed, however, that if  $A = K\Gamma/I$  is a monomial algebra, then the Anick-Green resolutions over  $A$  with respect to any admissible ordering on the set  $B(\Gamma)$  of finite directed paths in  $\Gamma$  are indeed minimal. In Section 3.1 we explore relationships between a quotient of path algebra being of monomial presentation type and the minimality of the Anick-Green resolutions over this quotient. Some conditions are first provided such that the Anick-Green resolutions over a path algebra quotient are minimal or not minimal (Proposition 3.1.2 and Theorem 3.1.3). Then, we construct an example in which a quotient  $A = K\Gamma/I$  of path algebra is of monomial presentation type but the Anick-Green resolutions over  $A$  are not minimal with respect to any admissible ordering on  $B(\Gamma)$  (Example 3.2). The concept of associated monomial algebras of a quotient of path algebra with respect to an admissible ordering was also introduced in [3]. In Section 3.2 another example (Example 3.3) is constructed to give a negative answer

to the question: if  $A = K\Gamma/I$  is of monomial presentation type, is it isomorphic to its associated monomial algebra with respect to some admissible ordering over  $B(\Gamma)$ ?

The last Chapter of the thesis is devoted to a particular class of quotients of path algebras – called *right monomial rings*. One-sided monomial rings were introduced, as a generalization of monomial algebras, by Burgess *et al.* in [6]. As the authors showed, one-sided monomial rings constitute a large class of one-sided artinian rings which includes many important types of ring, such as one-sided (almost) serial rings and hereditary artinian rings. A basic question concerning this class of quotients of path algebras is how they are related to the monomial presentation type. If  $A = K\Gamma/I$  is a right monomial ring and of Loewy length 3, then, by [6, Proposition 4.3],  $A$  is of monomial presentation type. In Section 4.2 we generalize this result to Loewy length 4 and to the case where the underlying finite directed graph  $\Gamma$  is acyclic (Theorems 4.2.5 and 4.2.6). Section 4.3 involves some other results connecting right monomial rings and the monomial presentation type; in particular, it is proved that if  $\Gamma$  is square-free and strongly acyclic and if  $A = K\Gamma/I$  is a right monomial ring, then  $A$  is of monomial presentation type (Theorem 4.3.4 and Corollary 4.3.5). The last section of Chapter 4 provides a characterization of the monomial presentation type within the class of right monomial rings by means of gradability by the radical; specifically, we shall prove that a right monomial ring  $A$  is of monomial presentation type if and only if  $A$  is gradable by the radical (Theorem 4.4.2).

Throughout this thesis, unless otherwise stated,  $K$  will denote an arbitrary field and  $\Gamma = (\Gamma_0, \Gamma_1)$  an arbitrary connected finite directed graph. All algebras over  $K$  will be associative with identity. Given any nonempty subset  $X$  of an associative ring  $R$  with identity,  $X^*$  stands for the set  $\{x \in X \mid x \neq 0\}$  and  $\langle X \rangle$  for the two-sided ideal in  $R$  generated by  $X$ .

## Chapter 2

# Gröbner bases and change of variable method

The Gröbner basis theory for path algebras was introduced by Farkas, Feustel, and Green [10], as a generalization of the known theory for commutative polynomial rings (see [5]) and free associative algebras (see [24]). The change of variable method for connecting two isomorphic finite dimensional presentations by a path algebra was introduced by Saorín [26]. In this chapter, we shall make use of these two tools to study characterizations of quotients of path algebras of monomial presentation type.

### 2.1 Gröbner bases for path algebras

In this section we present a brief introduction to Gröbner bases for path algebras with emphasis on the description of the algorithm for computing Gröbner bases which we shall frequently use in the thesis. Our main references are [10] and [17].

Let  $K$  be a field. Let  $\Gamma = (\Gamma_0, \Gamma_1, o, t)$  be a connected finite directed graph, where  $\Gamma_0$  is the set of vertices in  $\Gamma$  (whose elements shall be denoted by letters  $u, v, w, \dots$ ),  $\Gamma_1$  is the set of arrows in  $\Gamma$  (whose elements shall be denoted by letters  $\alpha, \beta, \gamma, \dots$ ),

and  $o, t: \Gamma_1 \rightarrow \Gamma_0$  are the origin and terminus functions on  $\Gamma_1$ , respectively, defined via  $o(\alpha) =$  the original vertex of  $\alpha$ , and  $t(\alpha) =$  the terminal vertex of  $\alpha$ , for all  $\alpha \in \Gamma_1$ . We shall denote by  $B(\Gamma)$  the set of finite directed paths in  $\Gamma$ . The length of a finite directed path  $p \in B(\Gamma)$ , denoted by  $l(p)$ , is the number of arrows in  $p$ . A vertex  $v$  in  $\Gamma$  is regarded as a trivial finite directed path of length 0. Then  $\Gamma_0$  is a subset of  $B(\Gamma)$ . A finite directed path  $p \in B(\Gamma)$  of length  $s$  will be written in the following manner:

$$p = \alpha_1 \alpha_2 \dots \alpha_s$$

where  $\alpha_i \in \Gamma_1$  for  $1 \leq i \leq s$ , and  $t(\alpha_i) = o(\alpha_{i+1})$  for  $1 \leq i \leq s - 1$ . When  $s = 0$ ,  $p$  becomes a vertex in  $\Gamma$ . The functions  $o$  and  $t$  are extended to  $B(\Gamma)$  in a natural fashion; i.e., for any  $p = \alpha_1 \alpha_2 \dots \alpha_s \in B(\Gamma)$ ,  $o(p) = o(\alpha_1)$  and  $t(p) = t(\alpha_s)$ . In particular, we have that  $o(v) = t(v) = v$  for all  $v \in \Gamma_0$ . By adjoining a distinguished element 0 (called the zero element) to  $B(\Gamma)$ , the set  $B(\Gamma) \cup \{0\}$  is endowed with a semigroup with zero structure with multiplication defined in terms of the following concatenation rule: for any  $p = \alpha_1 \dots \alpha_r \in B(\Gamma)$  and  $q = \beta_1 \dots \beta_s \in B(\Gamma)$ .

$$pq = \begin{cases} \alpha_1 \dots \alpha_r \beta_1 \dots \beta_s & \text{if } t(\alpha_r) = o(\beta_1) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, by the definition above, for any  $v, w \in \Gamma_0$ ,  $vw = 0$  if  $v \neq w$  and  $vw = v$  if  $v = w$ . The semigroup algebra  $K[B(\Gamma)]$  of the semigroup  $B(\Gamma) \cup \{0\}$  over the field  $K$  is said to be the *path algebra* of  $\Gamma$  over  $K$ , denoted by  $K\Gamma$ . (The zero element of the semigroup algebra  $K[B(\Gamma)]$  is identified with that of the semigroup  $B(\Gamma) \cup \{0\}$ .) It is a simple fact that  $K\Gamma$  is an associative  $K$ -algebra with identity  $1 = \sum_{v \in \Gamma_0} v$  and  $\Gamma_0$  is a complete set of orthogonal primitive idempotents in  $K\Gamma$ .

In order to define Gröbner bases for path algebras, we first need the concept of admissible orderings on the set  $B(\Gamma)$  of finite directed paths in  $\Gamma$ .

Let  $<$  be a total ordering on  $B(\Gamma)$ . The ordering  $<$  is said to be an *admissible ordering* on  $B(\Gamma)$  if the following conditions are satisfied:

(1)  $<$  is a well-ordering on  $B(\Gamma)$ ;

(2) for any  $p, q, r, s \in B(\Gamma)$ ,  $p < q$  implies that  $rp < rq$  whenever  $rp, rq \neq 0$  and that  $ps < qs$  whenever  $ps, qs \neq 0$ ; (i.e.,  $<$  is compatible with the multiplication structure on  $B(\Gamma) \cup \{0\}$ );

(3) For any  $v, w \in \Gamma_0$  and any  $p \in B(\Gamma)$  with  $p = vpw$ ,  $v \leq p$  and  $w \leq p$ .

A typical example of an admissible ordering on  $B(\Gamma)$  is a *length-lexicographic ordering* on  $B(\Gamma)$  which is defined as follows: given any total ordering  $<$  on the pair  $(\Gamma_0, \Gamma_1)$  of sets, say  $v_1 < v_2 < \dots < v_{|\Gamma_0|}$  and  $\alpha_1 < \alpha_2 < \dots < \alpha_{|\Gamma_1|}$ , then  $<$  induces an admissible ordering, still denoted by  $<$ , on  $B(\Gamma)$  by defining, for any  $p, q \in B(\Gamma)$ ,  $p < q$  if either  $l(p) < l(q)$  or  $l(p) = l(q)$  but  $p$  precedes  $q$  in alphabetical order (reading left to right).

Fix an admissible ordering  $<$  on  $B(\Gamma)$ . Let  $f = \sum_{p \in B(\Gamma)} \lambda_p p$  be any element of  $K\Gamma$ . The *support* of  $f$  is defined to be

$$\text{Supp}(f) = \{p \in B(\Gamma) \mid \lambda_p \in K^*\}.$$

If  $f \in K\Gamma$  is nonzero, then the *tip* of  $f$ , denoted by  $\text{Tip}(f)$ , is the largest element of  $\text{Supp}(f)$  with respect to the ordering  $<$ . Given any subset  $S$  of  $K\Gamma$ , the *tip* of  $S$  is defined to be

$$\text{Tip}(S) = \{p \in B(\Gamma) \mid \exists f \in S^* \text{ such that } p = \text{Tip}(f)\}.$$

Given any finite directed paths  $p, q \in B(\Gamma)$ ,  $p$  is *divisible by*  $q$ , denoted by  $q \mid p$ , if there exist  $r, s \in B(\Gamma)$  such that  $p = rqs$ .

**Definition 2.1** Let  $<$  be an admissible ordering on the set  $B(\Gamma)$  of paths in  $\Gamma$  and  $I$  be a nonzero two-sided ideal in the path algebra  $K\Gamma$ . A nonempty subset  $\mathcal{G}$  of  $I^*$  is called a *Gröbner basis* (or Gröbner generating set) for  $I$  with respect to  $<$  if  $\langle \text{Tip}(I) \rangle = \langle \text{Tip}(\mathcal{G}) \rangle$ .

Equivalently,  $\mathcal{G}$  is a Gröbner basis for  $I$  if and only if for any nonzero element  $f$  in  $I$  there exists an element  $g$  in  $\mathcal{G}$  such that  $\text{Tip}(g) \mid \text{Tip}(f)$ . By convention, the empty set is the Gröbner basis for the zero ideal in  $K\Gamma$ .

Next, we discuss the reduction of elements in path algebras.

Let  $S$  be a nonempty subset of  $K\Gamma^*$  and  $f$  be an element of  $K\Gamma$ .  $f$  is said to be *reducible modulo  $S$*  if there exist  $h \in S$  and  $p, q \in B(\Gamma)$  such that  $p\text{Tip}(h)q \in \text{Supp}(f)$ ; otherwise,  $f$  is said to be *irreducible modulo  $S$* . In case  $f$  is reducible modulo  $S$ , the element

$$\bar{f} = f - (\text{Co}_{p\text{Tip}(h)q}(f)/\text{Lc}(h))phq$$

is a *simple reduction of  $f$  modulo  $S$* , denoted by  $f \rightarrow \bar{f} \bmod h$ , where  $\text{Co}_{p\text{Tip}(h)q}(f)$  denotes the coefficient of  $p\text{Tip}(h)q$  in  $f$  and  $\text{Lc}(h)$  the leading coefficient of  $h$ , i.e.,  $\text{Co}_{\text{Tip}(h)}(h)$ . The composition of a finite sequence of simple reductions,  $f \rightarrow \cdots \rightarrow g$ , is called a *reduction of  $f$  modulo  $S$* , denoted by  $f \rightarrow g \bmod S$ . We observe that  $<$  being a well-ordering ensures that every element  $f \in K\Gamma$  can be reduced to an irreducible element  $\bar{f}$  modulo  $S$ , denoted by  $\text{Red}_S(f)$ . In general, an irreducible reduction of  $f$  is not unique. However, if  $\mathcal{G}$  is a Gröbner basis for a two-sided ideal  $I$  with respect to  $<$ , then every element  $f$  in  $K\Gamma$  can be reduced to a unique irreducible element  $\bar{f} = \text{Red}_{\mathcal{G}}(f)$  modulo  $\mathcal{G}$  (see [10, Theorem 9]).

We now proceed to the criterion for Gröbner bases and the algorithm for computing Gröbner bases. Let  $f$  and  $g$  be any nonzero elements in  $K\Gamma$ . If there exist paths  $p$  and  $q$  of length  $\geq 1$  in  $B(\Gamma)$  such that  $\text{Tip}(f)p = q\text{Tip}(g)$ ,  $l(p) < l(\text{Tip}(g))$  and  $l(q) < l(\text{Tip}(f))$ , then the element

$$\text{Lc}(g)fp - \text{Lc}(f)qg$$

is called an *overlap relation* for  $f$  and  $g$ , denoted by  $o(f, g, p, q)$ . If  $f \neq g$  and if there exist  $p, q \in B(\Gamma)$  such that  $\text{Tip}(f) = p\text{Tip}(g)q$ , then the element

$$\text{Lc}(g)f - \text{Lc}(f)pgq$$

is called an *inclusion relation* for  $f$  and  $g$ , denoted by  $i(f, g, p, q)$ .

A nonzero element  $f$  of  $K\Gamma$  is called *uniform* if there exist vertices  $v$  and  $w$  in  $\Gamma$  such that  $f = vf = fw$ . We note that every two-sided ideal in  $K\Gamma$  can be generated by uniform elements in  $K\Gamma$ .

The following theorem provides a noncommutative version of Buchberger's Criterion in the commutative Gröbner basis theory and it is in fact a form of Bergman's Diamond Lemma (see [4]). Also see [24] for a proof in the free algebra case.

**Theorem 2.1.1 (Buchberger-Mora-Farkas-Green)** *Let  $<$  be an admissible ordering on  $B(\Gamma)$  and  $I$  be a nonzero two-sided ideal in  $K\Gamma$ . Suppose that  $\mathcal{G}$  is a generating set of  $I$  which consists of uniform elements. If the following two conditions are satisfied, then  $\mathcal{G}$  is a Gröbner basis for  $I$  with respect to  $<$ .*

(1) *every inclusion relation for any two elements  $f$  and  $g$  in  $\mathcal{G}$  reduces to 0 modulo  $\mathcal{G}$ ;*

(2) *every overlap relation for any two elements  $f$  and  $g$  (not necessarily distinct) in  $\mathcal{G}$  reduces to 0 modulo  $\mathcal{G}$ .*

### The Buchberger-Mora-Farkas-Green Algorithm

(\* An algorithm for computing a Gröbner basis for a finitely generated two-sided ideal in  $K\Gamma$ . The algorithm may be infinite. If there exists a finite Gröbner basis for the ideal, then the algorithm is finite. \*)

Input: a finite ordered set  $S = (f_1, \dots, f_m)$  of uniform elements in  $K\Gamma$  and an admissible ordering  $<$  on  $B(\Gamma)$ .

Output: a Gröbner basis  $\mathcal{G} = (g_1, g_2, \dots)$  for the two-sided ideal  $I = \langle S \rangle$  with respect to  $<$ .

**begin**

$\mathcal{G} := S;$

**repeat**

```

 $\mathcal{H} := \mathcal{G}$ ;
for all pairs  $(f, g) \in \mathcal{H} \times \mathcal{H}$  and all overlap and inclusion relations
     $r$  for  $f$  and  $g$  do
         $h := \text{Red}_{\kappa}(r)$ ;
        if  $h \neq 0$  then  $\mathcal{G} := \mathcal{G} \cup \{h\}$ 
until  $\mathcal{G} = \mathcal{H}$ 
end.
    
```

We note that in the above algorithm, we can calculate the reduction  $\text{Red}_{\kappa}(r)$  in a systematic way by using a noncommutative version of the Division Algorithm for commutative polynomial rings (see [17] and [24]).

Given an admissible ordering  $<$  on  $B(\Gamma)$  and a two-sided ideal  $I$  in  $KT$ , a tip  $p \in \text{Tip}(I)$  is called *minimal* if  $p$  is not divisible by any other element in  $\text{Tip}(I)$ . The *minimal tip* of  $I$ , denoted by  $\text{MinTip}(I)$ , is the set of all minimal tips in  $\text{Tip}(I)$ . The *nontip* of  $I$  is defined as

$$\text{NonTip}(I) = B(\Gamma) - \text{Tip}(I).$$

A nonzero element  $f$  of  $I$  is called *sharp* in  $I$  if

- (1)  $f$  is monic, i.e.,  $\text{Lc}(f) = 1$ , and
- (2)  $f - \text{Tip}(f) \in \text{Span}_K(\text{NonTip}(I))$ .

A sharp element  $f$  in  $I$  is called *minimal* if  $\text{Tip}(f)$  is a minimal tip in  $\text{Tip}(I)$ . The set of minimal sharp elements in  $I$  is written  $\text{MinSharp}(I)$ .

**Definition 2.2** Let  $<$  be an admissible ordering on  $B(\Gamma)$  and let  $\mathcal{G}$  be a Gröbner basis for a two-sided ideal  $I$  in  $KT$  with respect to  $<$ . Suppose that  $\mathcal{G}$  consists of uniform elements.  $\mathcal{G}$  is called *reduced* if the following conditions are satisfied:

- (1) every element  $f$  of  $\mathcal{G}$  is monic;
- (2) every element  $f$  of  $\mathcal{G}$  is irreducible modulo  $\mathcal{G} - \{f\}$ .

The following basic characterization of reduced Gröbner bases are due to Farkas, Feustel and Green [10].

**Theorem 2.1.2 (Farkas-Feustel-Green)** *Let  $<$  be an admissible ordering on  $B(\Gamma)$  and  $I$  be a nonzero two-sided ideal in  $KT$ . Then there exists a unique reduced Gröbner basis  $\mathcal{G}$  for  $I$  with respect to  $<$ ; and  $\mathcal{G} = \text{MinSharp}(I)$ .*

**Proof.** See [10, Theorem 13]. ■

### The reduced Gröbner basis algorithm

(\* An algorithm for computing a reduced Gröbner basis for a two-sided ideal  $I$  in  $KT$  with a finite Gröbner basis  $\mathcal{G}$ . \*)

Input: a finite Gröbner basis  $\mathcal{G}$  for a nonzero two-sided ideal  $I$  in  $KT$  with respect to an admissible ordering  $<$  on  $B(\Gamma)$ .

Output: the reduced Gröbner basis  $\mathcal{G}_{\text{red}}$  for  $I$  with respect to  $<$ .

**begin**

{ Minimalize  $\mathcal{G}$  }

$\bar{\mathcal{G}} := \mathcal{G}$ ;

**for all**  $g \in \mathcal{G}$  **do**

$f := (1/\text{Lc}(g))g$ ;

$\bar{\mathcal{G}} := (\bar{\mathcal{G}} - \{g\}) \cup \{f\}$ ;

**if**  $\text{Tip}(f) \in <\text{Tip}(\bar{\mathcal{G}} - \{f\})>$  **then**

$\bar{\mathcal{G}} := \bar{\mathcal{G}} - \{f\}$ ;

{ Reduce  $\bar{\mathcal{G}}$  }

$\mathcal{G}_{\text{red}} := \bar{\mathcal{G}}$ ;

**for all**  $\bar{g} \in \bar{\mathcal{G}}$  **do**

$\mathcal{G}_{\text{red}} := (\mathcal{G}_{\text{red}} - \{\bar{g}\}) \cup \{\text{Red}_{\bar{\mathcal{G}} - \{\bar{g}\}}(\bar{g})\}$

**end.**

Our aim is to find applications of the theory of Gröbner bases for path algebras in studying finite dimensional quotients of path algebras. At the conclusion of this section, we list in the following proposition several basic facts concerning finite dimensional quotients of path algebras as they relate to Gröbner bases. Refer to [10, Theorems 4 and 15] for a proof.

**Proposition 2.1.3** *Let  $I$  be a nonzero two-sided ideal in  $K\Gamma$  and let  $A = K\Gamma/I$  be finite dimensional over  $K$ . Then*

(1)  *$I$  has a finite Gröbner basis with respect to any admissible ordering on  $B(\Gamma)$ ; in particular,  $I$  has a finite set of generators.*

(2)  *$\text{NonTip}(I)$  modulo  $I$  forms a  $K$ -vector space basis for  $A$ .*

See Example 2.2 for an example of a Gröbner basis.

## 2.2 Changes of variable and characterizations of the monomial presentation type

In this section we shall find some characterizations of quotients of path algebras of monomial presentation type under certain conditions imposed on the underlying finite directed graph  $\Gamma$ .

Let  $K$  be a field and  $\Gamma$  be a (connected) finite directed graph. Let  $K\Gamma$  denote the path algebra of  $\Gamma$  over  $K$ . A two-sided ideal  $I$  of  $K\Gamma$  is said to be *admissible* if there exists some positive integer  $N \geq 2$  such that  $\langle \Gamma_1 \rangle^N \subseteq I \subseteq \langle \Gamma_1 \rangle^2$ . Let  $A = K\Gamma/I$  with  $I$  an admissible ideal. We know that  $A$  is a split basic finite dimensional  $K$ -algebra with basic set  $E = \{v+I \mid v \in \Gamma_0\}$  of primitive idempotents. The quotients of path algebras by admissible ideals comprise a very general class of finite dimensional algebras in that every connected split basic finite dimensional algebra is isomorphic

to such a quotient. A two-sided ideal  $I$  of  $K\Gamma$  is called *monomial* if it is generated by a set of finite directed paths in  $\Gamma$ .

**Definition 2.3** Let  $I$  be an admissible ideal of the path algebra  $K\Gamma$  and let  $A = K\Gamma/I$ .

(1)  $A$  is said to be a (*finite dimensional*) *monomial algebra* if  $I$  is a monomial ideal in  $K\Gamma$ .

(2)  $A$  is said to be of *monomial presentation type* if  $A$  is isomorphic to a monomial algebra  $\hat{A} = K\Gamma/\hat{I}$ , where  $\hat{I}$  is a monomial admissible ideal in  $K\Gamma$ .

The following simple fact about monomial ideals is important for our study of the monomial presentation type, so that we write down a proof for it here. This fact has been taken as a folklore in the literature, especially in the case of commutative polynomial rings.

**Proposition 2.2.1** *Let  $I$  be a two-sided ideal in the path algebra  $K\Gamma$ . Then  $I$  is a monomial ideal if and only if the reduced Gröbner basis  $\mathcal{G}$  for  $I$  with respect to any admissible ordering on  $B(\Gamma)$  consists of paths in  $\Gamma$ .*

**Proof.** ( $\Rightarrow$ ) Fix any admissible ordering  $<$  on  $B(\Gamma)$ . Suppose that  $I = \langle S \rangle$ , where  $S$  is a set of paths in  $\Gamma$ . Clearly, every nonzero element of  $I$  has tip divisible by some path in  $S$ ; so  $S$  is a Gröbner basis for  $I$  with respect to  $<$ . Set  $\mathcal{G} = \{p \in S \mid p \text{ has no proper subpath in } S\}$ . Then  $\mathcal{G}$  is also a Gröbner basis and no element in  $\mathcal{G}$  can be reduced modulo the other elements in  $\mathcal{G}$ . Also, every element in  $\mathcal{G}$  is monic. Thus,  $\mathcal{G}$  is the reduced Gröbner basis with respect to  $<$  which consists of paths in  $\Gamma$ .

( $\Leftarrow$ ) This is obvious since every Gröbner basis for  $I$  is a set of generators in  $I$ . ■

From the fact above we see that in order to determine if  $A = K\Gamma/I$  is a monomial algebra it suffices to calculate the reduced Gröbner basis for  $I$  with respect to an

admissible ordering on  $B(\Gamma)$ . Thus the question of determining whether or not a given path algebra quotient is a monomial algebra has a satisfactory answer using Gröbner basis methods. “Monomial presentation type”, which is our topic of study, however, is a very different matter.

Another basic tool we shall use in this chapter is the change of variable method introduced by Saorín [26]. In the following,  $\Gamma_1^l$  will mean the set of paths of length  $l$  in  $\Gamma$  for  $l \geq 0$ .

**Definition 2.4** A  $K$ -algebra endomorphism  $T : K\Gamma \rightarrow K\Gamma$  is called a *change of variable* on  $K\Gamma$ , if the following conditions are satisfied:

- (1)  $T(v) = v$  for all  $v \in \Gamma_0$ ;
- (2) for any vertices  $v, w \in \Gamma_0$ , if  $v\Gamma_1 w = \{\alpha_1, \dots, \alpha_{m(v,w)}\}$ , then there exists some  $M \in \text{GL}_{m(v,w)}(K)$  such that (in the vector form)

$$T([\alpha_1, \dots, \alpha_{m(v,w)}]) \equiv [\alpha_1, \dots, \alpha_{m(v,w)}]M \pmod{\langle \Gamma_1^2 \rangle^{(m(v,w))}} .$$

**Proposition 2.2.2 (Saorín)** *Let  $\Gamma$  be a connected finite directed graph and let  $I$  and  $\hat{I}$  be admissible ideals of the path algebra  $K\Gamma$ . Suppose that  $\phi : K\Gamma/I \rightarrow K\Gamma/\hat{I}$  is a homomorphism of  $K$ -algebras and  $A = K\Gamma/I$  is of Loewy length  $L$ . Then the following statements are equivalent:*

- a)  $\phi$  is an isomorphism of  $K$ -algebras such that  $\phi(\Gamma_0 + I/I) = \Gamma_0 + \hat{I}/\hat{I}$ ;
- b) there exist an automorphism  $\sigma$  of  $\Gamma$  and a change of variable  $T$  on  $K\Gamma$  such that  $\hat{I} = \langle T(I^\sigma) \cup \Gamma_1^L \rangle$  and  $\phi$  is induced by  $T \circ \sigma$ , in the natural manner.
- b') there exist an automorphism  $\sigma$  of  $\Gamma$  and a change of variable  $T$  on  $K\Gamma$  such that  $\hat{I} = \langle T(I)^\sigma \cup \Gamma_1^L \rangle$  and  $\phi$  is induced by  $\sigma \circ T$ , in the natural manner.

**Proof.** See [26, Theorem 3]. ■

In fact, it is the following simple corollary to the proposition above that we shall frequently apply in the sequel.

**Corollary 2.2.3** *Let  $I$  be an admissible ideal in  $K\Gamma$  and let  $A = K\Gamma/I$  be of Loewy length  $L$ . If  $A$  is of monomial presentation type, then there exists a change of variable  $T$  on  $K\Gamma$  such that  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is a monomial ideal in  $K\Gamma$  and  $A$  is isomorphic to the monomial algebra  $\hat{A} = K\Gamma/\hat{I}$ . ■*

We start with several lemmas which lead to the description of a class of path algebra quotients in which “monomial” and “monomial presentation type” coincide.

Let  $A = K\Gamma/I$  be of Loewy length  $L$ . Let  $T$  be a change of variable on  $K\Gamma$ . We denote by  $T_*$  the  $K$ -algebra isomorphism from  $A = K\Gamma/I$  to  $\hat{A} = K\Gamma/\hat{I}$ , induced by  $T$ , where  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$ .

**Lemma 2.2.4** *Let  $A = K\Gamma/I$  be of Loewy length  $L$ . Let  $T_1$  and  $T_2$  be two changes of variable on the path algebra  $K\Gamma$ . Then*

- (1)  $T_2 \circ T_1$  is a change of variable on  $K\Gamma$ ;
- (2)  $(T_2 \circ T_1)_* = (T_2)_* \circ (T_1)_*$ .

**Proof.** (1) Clearly  $(T_2 \circ T_1)(v) = v$  for all  $v \in \Gamma_0$ . Since  $T_1$  and  $T_2$  are changes of variable on  $K\Gamma$ , it follows that for any vertices  $v$  and  $w$  in  $\Gamma$ , if  $v\Gamma_1 w = \{\alpha_1, \dots, \alpha_{m(v,w)}\}$ , then there exist invertible matrices  $M, N \in \text{GL}_{m(v,w)}(K)$  such that

$$T_1[\alpha_1, \dots, \alpha_m] \equiv [\alpha_1, \dots, \alpha_{m(v,w)}]M \pmod{\langle \Gamma_1^2 \rangle^{(m(v,w))}}$$

and

$$T_2[\alpha_1, \dots, \alpha_m] \equiv [\alpha_1, \dots, \alpha_{m(v,w)}]N \pmod{\langle \Gamma_1^2 \rangle^{(m(v,w))}} .$$

Thus one has that

$$(T_2 \circ T_1)[\alpha_1, \dots, \alpha_m] \equiv [\alpha_1, \dots, \alpha_{m(v,w)}]NM \pmod{\langle \Gamma_1^2 \rangle^{(m(v,w))}} .$$

This shows that  $T_2 \circ T_1$  is a change of variable on  $K\Gamma$ .

(2) By Proposition 2.2.2 there exist the following isomorphisms of  $K$ -algebras induced by  $T_1$  and  $T_2$ :

$$(T_1)_*: K\Gamma/I \rightarrow K\Gamma/\hat{I}$$

and

$$(T_2)_*: K\Gamma/\hat{I} \rightarrow K\Gamma/\langle T_2(\hat{I}) \cup \Gamma_1^L \rangle,$$

where  $\hat{I} = \langle T_1(I) \cup \Gamma_1^L \rangle$ . Now  $\langle T_2(\hat{I}) \cup \Gamma_1^L \rangle = \langle (T_2 \circ T_1)(I) \cup \Gamma_1^L \rangle$ . Hence  $(T_2 \circ T_1)_* = (T_2)_* \circ (T_1)_*$ . ■

A change of variable  $T$  on  $K\Gamma$  is called *scalar* if for any arrow  $\alpha \in \Gamma_1$ , there exists some nonzero scalar  $\lambda_\alpha$  in  $K$  such that  $T(\alpha) = \lambda_\alpha \alpha$ .

**Lemma 2.2.5** *Let  $A = K\Gamma/I$  be of Loewy length  $L$ . If  $T$  is a scalar change of variable on  $K\Gamma$ , then  $I$  is a monomial ideal if and only if  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is a monomial ideal.*

**Proof.** ( $\Rightarrow$ ) Suppose that  $I = \langle p_1, \dots, p_t \rangle$ , where each  $p_i$  is a path in  $\Gamma$ . Since  $T$  is scalar, we see that each  $T(p_i)$  is a scalar multiple of  $p_i$ . Hence  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is generated by  $\{p_1, \dots, p_t\} \cup \Gamma_1^L$ , which consists of paths in  $\Gamma$ .

( $\Leftarrow$ ) Suppose that  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is a monomial ideal. Let  $T^{-1}$  denote the inverse of  $T$ . Then  $T^{-1}$  is also a scalar change of variable on  $K\Gamma$ . Now

$$T^{-1}(\hat{I}) = \langle T^{-1}(T(I) \cup \Gamma_1^L) \rangle = \langle I \cup \Gamma_1^L \rangle = I.$$

By the proof of the necessary condition above,  $I$  is monomial ideal. ■

We say that a finite directed graph  $\Gamma$  is *square-free* if for any vertices  $v, w \in \Gamma_0$ , there exists at most one arrow from  $v$  to  $w$  in  $\Gamma$ .

**Lemma 2.2.6** *Let  $\Gamma$  be a square-free finite directed graph and let  $T$  be any change of variable on  $K\Gamma$ . Then there exist two changes of variable,  $T_1$  and  $T_2$ , on  $K\Gamma$  such that  $T = T_2 \circ T_1$ , where  $T_1$  is of the form*

$$T_1: \alpha \mapsto \alpha + H_\alpha, \quad \forall \alpha \in \Gamma_1$$

where  $H_\alpha \in \langle \Gamma_1^2 \rangle$ , and  $T_2$  is scalar.

**Proof.** Since  $\Gamma$  is square-free,  $T$  can be expressed as the following form

$$T: \alpha \mapsto \lambda_\alpha \alpha + G_\alpha, \quad \forall \alpha \in \Gamma_1$$

where  $\lambda_\alpha \in K^*$  and  $G_\alpha \in \langle \Gamma_1^2 \rangle$ . For any  $\alpha \in \Gamma_1$ , assume that  $G_\alpha = G_\alpha(\alpha_1, \dots, \alpha_s)$ , where  $\alpha_1, \dots, \alpha_s$  are all the arrows in  $\Gamma_1$  which appear in the components of  $G_\alpha$ ; that is,  $G_\alpha$  is a linear combination of paths made up of  $\alpha_1, \dots, \alpha_s$ . Let

$$H_\alpha = G_\alpha(\alpha_1/\lambda_{\alpha_1}, \dots, \alpha_s/\lambda_{\alpha_s}).$$

Define  $T_1, T_2: K\Gamma \rightarrow K\Gamma$  to be  $T_1(\alpha) = \alpha + H_\alpha$  and  $T_2(\alpha) = \lambda_\alpha \alpha$ , for all  $\alpha \in \Gamma_1$ . Then we see that  $T = T_2 \circ T_1$ . ■

Having provided the lemmas above, we now give the first theorem of this section. We recall that a two-sided ideal  $I$  in  $K\Gamma$  is *homogeneous* if  $I$  is generated by a set of length homogeneous elements in  $K\Gamma$ .

**Theorem 2.2.7** *Let  $\Gamma$  be a connected finite directed graph and  $I$  be an admissible ideal in  $K\Gamma$ . Let  $A = K\Gamma/I$ . Suppose that any one of the following conditions is satisfied.*

- (a) *For any arrow  $\alpha \in \Gamma_1$ , every path from  $o(\alpha)$  to  $t(\alpha)$  in  $\Gamma$ , other than  $\alpha$ , is of length  $\geq L - 1$ , where  $L \geq 3$  is the Loewy length of  $A$ .*
- (b)  *$\Gamma$  is square-free and  $I$  is a homogeneous ideal in  $K\Gamma$ .*
- (c)  *$\dim_K o(\alpha)At(\alpha) = 1$  for all arrows  $\alpha \in \Gamma_1$ .*

*Then  $A$  is of monomial presentation type if and only if  $A$  is a monomial algebra.*

**Proof.** ( $\Rightarrow$ ) Suppose that  $A$  is of monomial presentation type. Then by Corollary 2.2.3, there exists some change of variable on  $K\Gamma$

$$T: K\Gamma \longrightarrow K\Gamma$$

such that

$$\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$$

is a monomial ideal in  $K\Gamma$  and  $A$  is isomorphic to  $\hat{A} = K\Gamma/\hat{I}$ . We observe that under each of conditions (a), (b) and (c),  $\Gamma$  is square-free and hence, by Lemma 2.2.6,  $T = T_2 \circ T_1$  for some changes of variable on  $K\Gamma$ ,  $T_1$  and  $T_2$ , of the following forms

$$T_1: \alpha \longmapsto \alpha + H_\alpha, \quad \forall \alpha \in \Gamma_1$$

where  $H_\alpha \in \langle \Gamma_1^2 \rangle$ , and

$$T_2: \alpha \longmapsto \lambda_\alpha \alpha, \quad \forall \alpha \in \Gamma_1$$

where  $\lambda_\alpha \in K^*$ , respectively.

(a) By hypothesis, we have that  $H_\alpha \in \langle \Gamma_1^{L-1} \rangle$  for all  $\alpha \in \Gamma_1$ . Then from  $I \subseteq \langle \Gamma_1^2 \rangle$  it follows that for any  $f \in I$ ,

$$T_1(f) \equiv f \pmod{\langle \Gamma_1^L \rangle}.$$

Also, since  $\Gamma_1^L \subseteq I$ , it follows that

$$\langle T_1(I) \cup \Gamma_1^L \rangle = I,$$

and hence

$$\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle = \langle T_2(I) \cup \Gamma_1^L \rangle.$$

So by Lemma 2.2.5,  $I$  is a monomial ideal.

(b) Suppose that  $I = \langle g_1, \dots, g_t \rangle$ , with each  $g_i$  homogeneous. Then for each  $g_i$ ,

$$T_1(g_i) = g_i + G_i$$

where  $G_i \in \langle \Gamma_1^2 \rangle$  is the sum of the components of  $T_1(g_i)$  which are longer than  $g_i$ . Thus  $g_i$  is a homogeneous component of  $T_1(g_i)$ . Since  $T(g_i) \in \hat{I}$  and  $T_2$  is scalar, we see that  $T_1(g_i) \in \hat{I}$ . But  $\hat{I}$  is homogeneous, it follows that  $g_i \in \hat{I}$ . Hence  $I \subseteq \hat{I}$ . Noticing that  $\dim_K(A) = \dim_K(\hat{A})$ , we obtain that  $I = \hat{I}$  is monomial.

(c) Given any  $\alpha \in \Gamma_1$ , since  $\dim_K o(\alpha)At(\alpha) = 1$ , we see that every component in  $H_\alpha$  lies in  $I$ . Then for any  $f \in I$ ,

$$T_1(f) \equiv 0 \pmod{I}.$$

It follows that

$$\langle T_1(I) \cup \Gamma_1^L \rangle \subseteq I.$$

But  $\dim_K(A) = \dim_K(K\Gamma / \langle T_1(I) \cup \Gamma_1^L \rangle)$ , so  $I = \langle T_1(I) \cup \Gamma_1^L \rangle$  and hence

$$\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle = \langle T_2(I) \cup \Gamma_1^L \rangle.$$

Thus, by Lemma 2.2.5,  $I$  is a monomial ideal.

( $\Leftarrow$ ) This is obvious. ■

Let  $\Gamma$  be a finite directed graph. Using the terminology of [27],  $\Gamma$  is *strongly acyclic* in case for any vertices  $v, w \in \Gamma_0$ , if there is an arrow from  $v$  to  $w$  in  $\Gamma$ , then there is no path of length  $\geq 2$  from  $v$  to  $w$  in  $\Gamma$ .

The following two corollaries to Theorem 2.2.7(a) are immediate.

**Corollary 2.2.8** *Let  $\Gamma$  be a square-free, strongly acyclic finite directed graph and let  $A = K\Gamma/I$ . Then  $A$  is of monomial presentation type if and only if  $A$  is a monomial algebra. ■*

**Corollary 2.2.9** *Let  $\Gamma$  be a square-free finite directed graph. Suppose that  $A = K\Gamma/I$  is of Loewy length 3. Then  $A$  is of monomial presentation type if and only if  $A$  is a monomial algebra. ■*

Let us now look at several simple examples.

**Example 2.1** Let  $K$  be a field. Define  $\Gamma$  to be

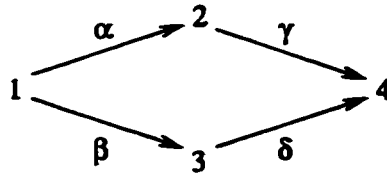


Figure 2.1

and let

$$\rho = \{\beta\delta - \alpha\gamma\}.$$

It is clear that  $\Gamma$  is square-free and strongly acyclic. Choose the following length-lexicographic ordering  $<$  on  $B(\Gamma)$ :

$$1 < 2 < 3 < 4 < \alpha < \beta < \gamma < \delta.$$

Then  $\rho$  is the reduced Gröbner basis for  $I = \langle \rho \rangle$  with respect to  $<$ . Thus  $I$  is not monomial and so, by Corollary 2.2.8,  $A = K\Gamma/I$  is not of monomial presentation type.

**Example 2.2** Let  $K$  be a field. Define  $\Gamma$  to be

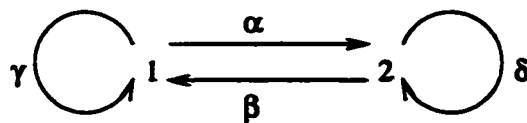


Figure 2.2

and let

$$\rho = \{\gamma^2 - \alpha\beta, \gamma\alpha - \alpha\delta, \delta\beta - \beta\gamma, \delta^2, \beta\alpha, \alpha\beta\gamma\}.$$

Let  $I = \langle \rho \rangle$ . It is easy to check that  $\Gamma$  is square-free and  $A = K\Gamma/I$  is of Loewy length 3. Choose the following length-lexicographic ordering  $<$  on  $B(\Gamma)$ :

$$1 < 2 < \alpha < \beta < \gamma < \delta.$$

By calculation, we see that  $\rho$  is the reduced Gröbner basis for  $I$  with respect to  $<$ . Thus  $I$  is not monomial and so, by Corollary 2.2.9,  $A = K\Gamma/I$  is not of monomial presentation type.

**Example 2.3** Let  $K$  be a field. Define  $\Gamma$  to be

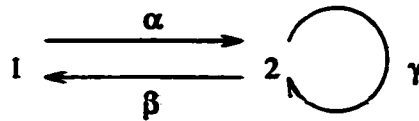


Figure 2.3

and let

$$\rho = \{\alpha\gamma^2 - \alpha\beta\alpha, \beta\alpha\beta, \beta\alpha\gamma, \gamma\beta\alpha, \gamma^2\beta, \gamma^3\}$$

and  $I = \langle \rho \rangle$ . Clearly,  $\Gamma$  is square-free. Choose the following length-lexicographic ordering  $<$  on  $B(\Gamma)$ :

$$1 < 2 < \alpha < \beta < \gamma.$$

We can easily check that  $\rho$  is the reduced Gröbner basis for  $I$  with respect to  $<$ . Thus  $I$  is not monomial. Since  $I$  is homogeneous, by Theorem 2.2.7(b)  $A = K\Gamma/I$  is not of monomial presentation type.

Let  $\Gamma$  be a finite directed graph. We recall (see, e.g., [27]) that a  $\mathbf{Z}$ -weight function  $d$  on  $\Gamma$  is a map from  $\Gamma_1$  to  $\mathbf{Z}$ . Such a function  $d$  induces a  $\mathbf{Z}$ -gradation on  $K\Gamma$  so that a path  $p = \alpha_1 \cdots \alpha_s \in B(\Gamma)$  is of degree  $d(\alpha_1) + \cdots + d(\alpha_s)$ . A two-sided ideal  $I$  in  $K\Gamma$  is  $d$ -homogeneous if it is generated by homogeneous elements with respect

to the induced  $\mathbf{Z}$ -gradation. A homogeneous ideal in  $K\Gamma$  is simply a  $d$ -homogeneous ideal in  $K\Gamma$  where  $d$  is constantly 1.

The following lemma shows that a reduced Gröbner basis for a  $d$ -homogeneous ideal consists of  $d$ -homogeneous elements.

**Lemma 2.2.10** *Let  $I$  be an admissible ideal in the path algebra  $K\Gamma$  and let  $d$  be a  $\mathbf{Z}$ -weight function on  $\Gamma$ . If  $I$  is  $d$ -homogeneous, then the reduced Gröbner basis  $\mathcal{G} = \text{MinSharp}(I)$  for  $I$  with respect to any admissible ordering on  $B(\Gamma)$  consists of  $d$ -homogeneous elements.*

**Proof.** Suppose that  $I = \langle f_1, \dots, f_m \rangle$ , where each  $f_i$  is a  $d$ -homogeneous uniform element in  $K\Gamma$ . We notice that for any two  $d$ -homogeneous uniform elements in  $I$ , every inclusion relation and every overlap relation for these two elements is still  $d$ -homogeneous. Then, in applying the Buchberger-Mora-Farkas-Green algorithm for computing a Gröbner basis for  $I$ , the  $d$ -homogeneity of elements in the intermediate generating sets is preserved through every step in the algorithm. Also, we see that when reducing the Gröbner basis obtained above to the reduced Gröbner basis  $\mathcal{G}$ , we still obtain  $d$ -homogeneous elements. Therefore, the reduced Gröbner basis  $\mathcal{G}$  for  $I$  with respect to any admissible ordering consists of  $d$ -homogeneous elements. ■

We recall that a finite dimensional  $K$ -algebra  $A$  is called *gradable by the radical* if there is a positive gradation  $A = \bigoplus_{n \geq 0} A_n$  on  $A$  such that  $(\text{Rad}A)^m = \bigoplus_{n \geq m} A_n$  for all  $m \geq 0$ .

**Proposition 2.2.11** *Let  $I$  be an admissible ideal in  $K\Gamma$  and let  $A = K\Gamma/I$  be of Loewy length  $L$ . Suppose that for any arrow  $\alpha$  in  $\Gamma$ , every path from  $o(\alpha)$  to  $t(\alpha)$  in  $\Gamma$  is of length 1 or  $\geq L - 1$ . If  $A$  is of monomial presentation type, then  $I$  is a homogeneous ideal in  $K\Gamma$ . Consequently, the reduced Gröbner basis for  $I$  with respect to any admissible ordering on  $B(\Gamma)$  consists of homogeneous elements.*

**Proof.** Since  $A = K\Gamma/I$  is isomorphic to a monomial algebra, it follows that  $A$  is gradable by the radical. By [27, Proposition 2.3], there exists a change of variable  $T$  on  $K\Gamma$  such that  $T$  induces the identity map on  $K\Gamma$  modulo  $\langle \Gamma_1^2 \rangle$  and that  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is a homogeneous ideal in  $K\Gamma$ . Then, by hypothesis, we see that

$$T(f) \equiv f \pmod{\langle \Gamma_1^L \rangle}$$

for all  $f \in I$ . Thus  $\hat{I} = I$  and so  $I$  is a homogeneous ideal in  $K\Gamma$ . The final assertion is from Lemma 2.2.10. ■

The following corollary to Proposition 2.2.11 is obvious.

**Corollary 2.2.12** *Let  $\Gamma$  be a strongly acyclic finite directed graph and  $I$  be an admissible ideal in  $K\Gamma$ . If  $A = K\Gamma/I$  is of monomial presentation type, then the reduced Gröbner basis for  $I$  with respect to any admissible ordering on  $B(\Gamma)$  consists of homogeneous elements.* ■

Here is a simple example.

**Example 2.4** Let  $K$  be a field. Define  $\Gamma$  to be

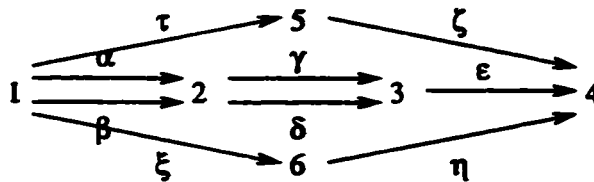


Figure 2.4

and let

$$\rho = \{\beta\gamma\epsilon - \alpha\delta\epsilon, \alpha\gamma\epsilon - \tau\zeta, \beta\delta\epsilon - \xi\eta\}.$$

Clearly,  $\Gamma$  is strongly acyclic. It is easy to check that  $\rho$  is the reduced Gröbner basis for  $I = \langle \rho \rangle$  with respect to the following length-lexicographic ordering on  $B(\Gamma)$ :

$$1 < 2 < 3 < 4 < 5 < 6 < \alpha < \beta < \gamma < \delta < \epsilon < \tau < \zeta < \xi < \eta.$$

Then, by Corollary 2.2.12,  $A = K\Gamma/I$  is not of monomial presentation type.

Let  $\Gamma$  be a connected finite directed graph. For any two vertices  $v$  and  $w$  in  $\Gamma$ , we denote by  $\Gamma_1(v, w)$  the set of arrows from  $v$  to  $w$  in  $\Gamma$ .

**Lemma 2.2.13** *Let  $I$  be an admissible ideal in  $K\Gamma$  and let  $A = K\Gamma/I$ . Suppose that  $T$  is a change of variable on  $K\Gamma$  satisfying the condition that for every pair of vertices  $v, w$  in  $\Gamma$ , there exist a permutation  $\sigma$  on  $\Gamma_1(v, w)$  and a system  $(\lambda_\alpha)_{\alpha \in \Gamma_1(v, w)}$  of nonzero scalars in  $K$  such that  $T(\alpha) = \lambda_\alpha \sigma(\alpha)$  for all  $\alpha \in \Gamma_1(v, w)$ . Then the following statements hold.*

- (1)  $A$  is isomorphic to  $\hat{A} = K\Gamma/T(I)$ ;
- (2)  $A$  is a monomial algebra if and only if  $\hat{A}$  is a monomial algebra.

**Proof.** (1) First, since  $I$  is admissible, we have that  $\Gamma_1^L \subseteq I$ , where  $L$  is the Loewy length of  $A$ . Then  $T(\Gamma_1^L) \subseteq T(I)$ . By hypothesis, we see that  $\langle T(\Gamma_1) \rangle = \langle \Gamma_1 \rangle$ . Then

$$\langle T(\Gamma_1^L) \rangle = \langle T(\Gamma_1)^L \rangle = \langle \Gamma_1^L \rangle,$$

and hence  $\langle T(I) \cup \Gamma_1^L \rangle = \langle T(I) \rangle = T(I)$ , since  $T$  is surjective. So, by Proposition 2.2.2,  $A$  is isomorphic to  $\hat{A} = K\Gamma/T(I)$ .

(2) By hypothesis, it is clear that if  $I$  is a monomial ideal, then so is  $T(I)$ . Let  $T^{-1}$  be the inverse of  $T$ , i.e.,  $T^{-1}(\alpha) = (1/\lambda_\alpha)\sigma^{-1}(\alpha)$ , for all  $\alpha \in \Gamma_1(v, w)$ . Then if  $T(I)$  is a monomial ideal,  $I = T^{-1}(T(I))$  is a monomial ideal. ■

In general, necessary conditions for the monomial presentation type are difficult to obtain. In the following proposition we find one in the case of a special underlying finite directed graph.

**Proposition 2.2.14** *Let  $\Gamma$  be the finite directed graph*

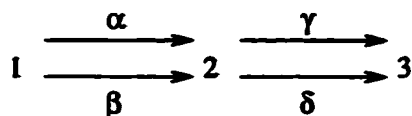


Figure 2.5

and  $I$  be an admissible ideal in  $K\Gamma$ . Let  $A = K\Gamma/I$ . Choose a length-lexicographic ordering  $<$  on  $B(\Gamma)$  such that  $\alpha > \beta > \gamma > \delta$ , and let  $\mathcal{G}$  be the reduced Gröbner basis for  $I$  with respect to  $<$ . If there exists some  $\lambda_0 \in K^*$  such that  $\alpha\gamma - \lambda_0\beta\delta$  is the only non-monomial in  $\mathcal{G}$ , then  $A$  is not of monomial presentation type.

**Proof.** Assume that  $A = K\Gamma/I$  is of monomial presentation type. We shall show that this gives rise to a contradiction. By Corollary 2.2.3, there exists a change of variable  $T$  on  $K\Gamma$  such that

$$\hat{I} = \langle T(\mathcal{G}) \rangle$$

is a monomial ideal and  $A \cong K\Gamma/\hat{I}$ . We here note that  $A$  is of Loewy length 3 and  $\Gamma_1^3 = 0$ . Since  $T$  is a change of variable on  $K\Gamma$ , there exist  $M, N \in GL_2(K)$  such that

$$T[\alpha, \beta] = [\alpha, \beta]M$$

and

$$T[\gamma, \delta] = [\gamma, \delta]N.$$

Write

$$M = \begin{bmatrix} \lambda_1 & \lambda_4 \\ \lambda_3 & \lambda_2 \end{bmatrix}, \quad N = \begin{bmatrix} \mu_1 & \mu_4 \\ \mu_3 & \mu_2 \end{bmatrix}.$$

We can assume that  $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ . In fact, if  $\lambda_1$  and  $\lambda_2$  are nonzero, then  $T$  can be decomposed into  $T = T_2 \circ T_1$ , where  $T_1$  and  $T_2$  are changes of variable on

$KT$  of the following forms, respectively:

$$\begin{cases} T_1(\alpha) = \alpha + (\lambda_3/\lambda_2)\beta \\ T_1(\beta) = (\lambda_4/\lambda_1)\alpha + \beta \end{cases}$$

and

$$\begin{cases} T_2(\alpha) = \lambda_1\alpha \\ T_2(\beta) = \lambda_2\beta. \end{cases}$$

Similarly for  $\mu_1$  and  $\mu_2$ . If, however,  $\lambda_1 = 0$  or  $\lambda_2 = 0$ , then we have that  $\lambda_3 \neq 0$ ,  $\lambda_4 \neq 0$  and

$$\begin{bmatrix} \lambda_1 & \lambda_4 \\ \lambda_3 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_4 & 0 \\ 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \lambda_2/\lambda_3 \\ \lambda_1/\lambda_4 & 1 \end{bmatrix}.$$

Thus  $T$  can be decomposed into  $T = T_3 \circ T_2 \circ T_1$ , where  $T_1$ ,  $T_2$  and  $T_3$  are changes of variable on  $KT$  of the following forms, respectively:

$$\begin{cases} T_1(\alpha) = \alpha + (\lambda_1/\lambda_4)\beta \\ T_1(\beta) = (\lambda_2/\lambda_3)\alpha + \beta, \end{cases}$$

$$\begin{cases} T_2(\alpha) = \beta \\ T_2(\beta) = \alpha \end{cases}$$

and

$$\begin{cases} T_3(\alpha) = \lambda_4\alpha \\ T_3(\beta) = \lambda_3\beta. \end{cases}$$

Similarly for  $\mu_1$  and  $\mu_2$ . So by Lemma 2.2.13,  $\hat{I} = \langle T(\mathcal{G}) \rangle$  is a monomial ideal if and only if  $\langle T_1(\mathcal{G}) \rangle$  is.

By hypothesis, we see that neither  $\alpha\delta$  nor  $\beta\gamma$  can be the tip of a non-monomial in  $\mathcal{G}$ . Then  $\mathcal{G}$  can be of only one of the following forms:

Case 1.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta\}$ ;

Case 2.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \alpha\delta\}$ ;

Case 3.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \beta\gamma\}$ ;

Case 4.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \alpha\delta, \beta\gamma\}$ .

Moreover, by the choice of the ordering  $<$ , we have that  $\alpha\gamma > \alpha\delta > \beta\gamma > \beta\delta$ . We now discuss each case.

Case 1.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta\}$ . We have that

$$T(\alpha\gamma - \lambda_0\beta\delta) = [\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta]P_1,$$

where

$$P_1 = \begin{bmatrix} 1 - \lambda_0\lambda_4\mu_4 \\ \mu_3 - \lambda_0\lambda_4 \\ \lambda_3 - \lambda_0\mu_4 \\ \lambda_3\mu_3 - \lambda_0 \end{bmatrix}.$$

Noticing that the matrix  $N$  is invertible and  $\lambda_0 \neq 0$ , we have that

$$\begin{bmatrix} 1 - \lambda_0\lambda_4\mu_4 \\ \mu_3 - \lambda_0\lambda_4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \lambda_3 - \lambda_0\mu_4 \\ \lambda_3\mu_3 - \lambda_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

because in either case the contrary would show that  $\det N = 1 - \mu_3\mu_4 = 0$ . Then the column vector  $P_1$  has at least two nonzero components. This shows that the reduced Gröbner basis  $\text{MinSharp}(\hat{I})$  for  $\hat{I} = \langle T(\mathcal{G}) \rangle$  consists of a single non-monomial and hence  $\hat{I}$  is not a monomial ideal, a contradiction.

Case 2.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \alpha\delta\}$ . We have that

$$T[\alpha\gamma - \lambda_0\beta\delta, \alpha\delta] = [\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta]P_2,$$

where

$$P_2 = \begin{bmatrix} 1 - \lambda_0 \lambda_4 \mu_4 & \mu_4 \\ \mu_3 - \lambda_0 \lambda_4 & 1 \\ \lambda_3 - \lambda_0 \mu_4 & \lambda_3 \mu_4 \\ \lambda_3 \mu_3 - \lambda_0 & \lambda_3 \end{bmatrix}.$$

Successively performing the following column operations on  $P_2$ :

- (1) adding the  $\lambda_0 \lambda_4$  multiple of the second column,  $C_2$ , to the first column,  $C_1$ ,
- (2) adding the  $-\mu_4$  of  $C_1$  multiple to  $C_2$ ,

we obtain that

$$P_2 \longrightarrow R = [r_{ij}] = \begin{bmatrix} 1 & 0 \\ \mu_3 & 1 - \mu_3 \mu_4 \\ \lambda_3 - \lambda_0 \mu_4 (1 - \lambda_3 \lambda_4) & \lambda_0 \mu_4^2 (1 - \lambda_3 \lambda_4) \\ \lambda_3 \mu_3 - \lambda_0 (1 - \lambda_3 \lambda_4) & \lambda_3 (1 - \mu_3 \mu_4) + \lambda_0 \mu_4 (1 - \lambda_3 \lambda_4) \end{bmatrix},$$

where  $R$  is in lower-triangular form. Since  $M$  and  $N$  are invertible, it follows that  $1 - \lambda_3 \lambda_4 \neq 0$  and  $1 - \mu_3 \mu_4 \neq 0$ . If  $\mu_4 \neq 0$ , then we see that  $r_{32} \neq 0$  and so  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . Assume that  $\mu_4 = 0$ . If  $\lambda_3 \neq 0$ , then  $r_{42} \neq 0$  and  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ ; otherwise,  $r_{41} = -\lambda_0 (1 - \lambda_3 \lambda_4) \neq 0$  and  $C_1$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . This shows that  $\hat{I}$  is not a monomial ideal, a contradiction.

Case 3.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \beta\gamma\}$ . We have that

$$T[\alpha\gamma - \lambda_0\beta\delta, \beta\gamma] = [\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta]P_3,$$

where

$$P_3 = \begin{bmatrix} 1 - \lambda_0 \lambda_4 \mu_4 & \lambda_4 \\ \mu_3 - \lambda_0 \lambda_4 & \lambda_4 \mu_3 \\ \lambda_3 - \lambda_0 \mu_4 & 1 \\ \lambda_3 \mu_3 - \lambda_0 & \mu_3 \end{bmatrix}.$$

Successively performing the following column operations on  $P_4$ :

- (1) adding the  $\lambda_0\mu_4$  multiple of  $C_2$  to  $C_1$ ,
- (2) adding the multiple  $-\lambda_4$  of  $C_1$  to  $C_2$ ,

we obtain that

$$P_3 \longrightarrow R = [r_{ij}] = \begin{bmatrix} 1 & 0 \\ \mu_3 - \lambda_0\lambda_4(1 - \mu_3\mu_4) & \lambda_0\lambda_4^2(1 - \mu_3\mu_4) \\ \lambda_3 & 1 - \lambda_3\lambda_4 \\ \lambda_3\mu_3 - \lambda_0(1 - \mu_3\mu_4) & \mu_3(1 - \lambda_3\lambda_4) + \lambda_0\lambda_4(1 - \mu_3\mu_4) \end{bmatrix},$$

where  $R$  is in lower-triangular form. If  $\lambda_4 \neq 0$  then  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . Assume that  $\lambda_4 = 0$ . If  $\mu_3 \neq 0$ , then  $r_{42} \neq 0$  and  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ ; otherwise,  $r_{41} = -\lambda_0 \neq 0$  and  $C_1$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . This shows that  $\hat{I}$  is not a monomial ideal, a contradiction.

Case 4.  $\mathcal{G} = \{\alpha\gamma - \lambda_0\beta\delta, \alpha\delta, \beta\gamma\}$ . We have that

$$T[\alpha\gamma - \lambda_0\beta\delta, \alpha\delta, \beta\gamma] = [\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta]P_4,$$

where

$$P_4 = \begin{bmatrix} 1 - \lambda_0\lambda_4\mu_4 & \mu_4 & \lambda_4 \\ \mu_3 - \lambda_0\lambda_4 & 1 & \lambda_4\mu_3 \\ \lambda_3 - \lambda_0\mu_4 & \lambda_3\mu_4 & 1 \\ \lambda_3\mu_3 - \lambda_0 & \lambda_3 & \mu_3 \end{bmatrix}.$$

Successively performing the following column operations on  $P_4$ :

- (1) adding the  $\lambda_0\lambda_4$  multiple of  $C_2$  to  $C_1$ ,
- (2) adding the multiple  $-\mu_4$  of  $C_1$  to  $C_2$ ,
- (3) adding the multiple  $-\lambda_4$  of  $C_1$  to  $C_3$ ,
- (4) multiplying  $C_3$  by  $1/(1 - \lambda_3\lambda_4)$ ,
- (5) adding the  $-\lambda_3$  multiple of  $C_3$  to  $C_1$ ,

we obtain that

$$P_4 \longrightarrow R = [r_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ \mu_3 & 1 - \mu_3\mu_4 & 0 \\ -\lambda_0\mu_4 & \lambda_0\mu_4^2(1 - \lambda_3\lambda_4) & 1 + \lambda_0\lambda_4\mu_4 \\ -\lambda_0 & \lambda_3(1 - \mu_3\mu_4) + \lambda_0\mu_4(1 - \lambda_3\lambda_4) & \mu_3 + \lambda_0\lambda_4 \end{bmatrix}.$$

If  $r_{33} \neq 0$  and  $r_{43} \neq 0$ , then  $C_3$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . If  $r_{33} = 0$ , then  $\mu_4 \neq 0$ ; so  $r_{32} \neq 0$  and  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . Assume that  $r_{43} = 0$ . If  $r_{42} \neq 0$ , then  $C_2$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ ; otherwise, since  $\lambda_0 \neq 0$ ,  $C_1$  corresponds to a non-monomial in  $\text{MinSharp}(\hat{I})$ . This shows that  $\hat{I}$  is not a monomial ideal, a contradiction.

Therefore, we conclude that  $A$  is not of monomial presentation type. ■

Let  $\Gamma$  be a connected finite directed graph. We say that a subgraph  $\tilde{\Gamma}$  of  $\Gamma$  is *strongly full* if for any vertices  $v, w$  in  $\tilde{\Gamma}$ , every path from  $v$  to  $w$  in  $\Gamma$  is also a path in  $\tilde{\Gamma}$ .

**Proposition 2.2.15** *Let  $\Gamma$  be a connected finite directed graph and  $\tilde{\Gamma}$  be a strongly full subgraph of  $\Gamma$ . Let  $I$  be an admissible ideal in the path algebra  $K\Gamma$  with finite set  $\rho$  of uniform generators. Set  $\tilde{\rho} = \{f \in \rho \mid f = vfw \text{ for some vertices } v, w \in \tilde{\Gamma}\}$  and let  $\tilde{I}$  be the two-sided ideal generated by  $\tilde{\rho}$  in the path algebra  $K\tilde{\Gamma}$ . Then*

(1)  $\tilde{I}$  is admissible in  $K\tilde{\Gamma}$ .

(2) Given any admissible ordering  $<$  on  $B(\Gamma)$ , let  $\tilde{<}$  denote the restriction of  $<$  on  $B(\tilde{\Gamma})$ . Then the reduced Gröbner basis  $\tilde{\mathcal{G}}$  for  $\tilde{I}$  with respect to  $\tilde{<}$  is a subset of the reduced Gröbner basis  $\mathcal{G}$  for  $I$  with respect to  $<$ .

(3) if  $A = K\Gamma/I$  is a monomial algebra, then so is  $\tilde{A} = K\tilde{\Gamma}/\tilde{I}$ .

**Proof.** (1) Since  $I = \langle \rho \rangle$  is an admissible ideal in  $K\Gamma$ , it follows that there exists some integer  $N \geq 2$  such that  $\langle \Gamma_1 \rangle^N \subseteq I \subseteq \langle \Gamma_1 \rangle^2$ . Since  $\tilde{\rho}$  is a subset of  $\rho$ ,

each element of  $\bar{\rho}$  is a  $K$ -linear combination of paths of length at least 2, and hence  $\bar{I} = \langle \bar{\rho} \rangle_{K\bar{\Gamma}} \subseteq \langle \bar{\Gamma}_1 \rangle_{K\bar{\Gamma}}^2$ . Given any path  $p$  of length  $N$  in  $\bar{\Gamma}$ , suppose that  $p = vpw$  for some  $v, w \in \bar{\Gamma}_0$ . Then since  $p$  is also a path of length  $N$  in  $\Gamma$ , it follows that  $p \in I = (K\Gamma)\rho(K\Gamma)$ . Then  $p \in v(K\Gamma)\rho(K\Gamma)w \subseteq (K\bar{\Gamma})\bar{\rho}(K\bar{\Gamma}) = \bar{I}$ . So  $\langle \bar{\Gamma}_1 \rangle_{K\bar{\Gamma}}^N \subseteq \bar{I}$ . Thus  $\bar{I}$  is admissible in  $\bar{\Gamma}$ .

(2) Write  $\rho = \{f_1, \dots, f_s, h_1, \dots, h_t\}$  with  $\bar{\rho} = \{f_1, \dots, f_s\}$ . We shall now apply the Buchberger-Mora-Farkas-Green Algorithm. Without loss of generality, we can assume that  $\bar{\rho}$  is already the reduced Gröbner basis  $\bar{\mathcal{G}}$  for  $\bar{I}$  with respect to  $\bar{\prec}$ . We observe that for any  $f_i$  and  $h_j$ , any overlap relation or inclusion relation for  $f_i$  and  $h_j$ , if not zero, is not in  $K\bar{\Gamma}$ . Similarly, for any  $h_i$  and  $h_j$ , any overlap relation or any inclusion relation for  $h_i$  and  $h_j$ , if not zero, is not in  $K\bar{\Gamma}$ . Thus, after the first loop in the algorithm,  $\bar{\rho}$  remains as a subset of the new set  $\mathcal{G}_1$  of generators and no other element in  $\mathcal{G}_1$  lies in  $K\bar{\Gamma}$ . Repeating the same argument as above on  $\mathcal{G}_1$ , after a finite number of loops, we obtain a Gröbner basis  $\mathcal{G}_m$  for  $I$  with respect to  $\prec$  such that  $\bar{\rho}$  is a subset of  $\mathcal{G}_m$ . In the same fashion, when reducing  $\mathcal{G}_m$  to the reduced Gröbner basis  $\mathcal{G}$  for  $I$ ,  $\bar{\rho}$  remains as a subset of  $\mathcal{G}$ . The proof is complete.

(3) This is obvious from Proposition 2.2.1. ■

From Propositions 2.2.14 and 2.2.15 we obtain the following

**Theorem 2.2.16** *Let  $\Gamma$  be a connected finite directed graph and  $I$  be an admissible ideal in  $K\Gamma$ . Let  $\Gamma$  have a strongly full subgraph  $\bar{\Gamma}$  of the form*

$$\begin{array}{ccccc} & \xrightarrow{\alpha} & & \xrightarrow{\gamma} & \\ 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 \\ & \xrightarrow{\beta} & & \xrightarrow{\delta} & \end{array}$$

Figure 2.6

*Choose a length-lexicographic ordering  $\prec$  on  $B(\Gamma)$  such that  $\alpha \succ \beta \succ \gamma \succ \delta$ , and let*

$\mathcal{G} = \text{MinSharp}(I)$  be the reduced Gröbner basis for  $I$  with respect to  $<$ . Suppose that there exists some  $\lambda_0 \in K^*$  such that  $\alpha\gamma - \lambda_0\beta\delta$  is the only non-monomial in  $\mathcal{G}$  with each component path in  $\bar{\Gamma}$ . Then  $A = K\Gamma/I$  is not of monomial presentation type.

**Proof.** Assume that  $A = K\Gamma/I$  is of monomial presentation type. By Corollary 2.2.3, there exists a change of variable  $T$  on  $K\Gamma$  such that

$$\hat{I} = \langle T(\mathcal{G}) \cup \Gamma_1^L \rangle$$

is a monomial ideal and  $A \cong K\Gamma/\hat{I}$ , where  $L$  is the Loewy length of  $A$ . Let  $\bar{<}$  be the restriction of  $<$  on  $B(\bar{\Gamma})$  and  $\bar{T}$  be the restriction of  $T$  on  $K\bar{\Gamma}$ . Then we see that  $\bar{<}$  is an admissible ordering on  $B(\bar{\Gamma})$  and  $\bar{T}$  is a change of variable on  $K\bar{\Gamma}$ . By Proposition 2.2.15, the reduced Gröbner basis  $\text{MinSharp}(\bar{I})$  for  $\bar{I} = \langle T(\mathcal{G}) \cap K\bar{\Gamma} \rangle_{K\bar{\Gamma}} = \langle \bar{T}(\bar{\mathcal{G}}) \rangle_{K\bar{\Gamma}}$ , where  $\bar{\mathcal{G}} = \mathcal{G} \cap K\bar{\Gamma}$ , with respect to  $\bar{<}$  is a subset of the reduced Gröbner basis  $\text{MinSharp}(\hat{I})$  for  $\hat{I}$  with respect to  $<$ . Here we note that  $\bar{\Gamma}_1^L = 0$ . We observe that  $\bar{\mathcal{G}}$  is the reduced Gröbner basis for  $\langle \bar{\mathcal{G}} \rangle$  with respect to  $\bar{<}$  and  $\alpha\gamma - \lambda_0\beta\delta$  is the only non-monomial in  $\bar{\mathcal{G}}$ . By the proof of Proposition 2.2.14, we see that  $\text{MinSharp}(\bar{I})$  does not consist of monomials in  $K\bar{\Gamma}$ ; hence  $\text{MinSharp}(\hat{I})$  does not consist of monomials in  $K\Gamma$ ; i.e.,  $\hat{I}$  is not a monomial ideal in  $K\Gamma$ , a contradiction. This shows that  $A$  is not of monomial presentation type. ■

The following example shows that if the condition on the only non-monomial in Proposition 2.2.14 is not satisfied, then there exists some  $A = K\Gamma/I$  which is of monomial presentation type. One of the algebras in the example will appear again in Chapter 3 for other purposes.

**Example 2.5** Let  $K$  be a field. Define  $\Gamma$  to be the finite directed graph in Figure 2.7. Let

$$\rho_1 = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\delta, \beta\gamma - \beta\delta\},$$

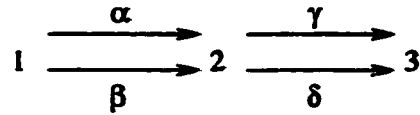


Figure 2.7

$$\rho_2 = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma\}.$$

Then

- (1)  $A_1 = K\Gamma/I_1$  is of monomial presentation type, where  $I_1 = \langle \rho_1 \rangle$ .
- (2) If  $\text{Char}(K) \neq 2$ , then  $A_2 = K\Gamma/I_2$  is of monomial presentation type, where  $I_2 = \langle \rho_2 \rangle$ .
- (3) If  $\text{Char}(K) = 2$ , then  $A_2 = K\Gamma/I_2$  is not of monomial presentation type, where  $I_2 = \langle \rho_2 \rangle$ .

First, we notice that both  $A_1$  and  $A_2$  are of Loewy length 3. Moreover, it is easy to see that the condition on the only non-monomial in Proposition 2.2.14 is not satisfied for  $I_1$  or  $I_2$ .

(1)  $\rho_1 = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\delta, \beta\gamma - \beta\delta\}$ . Define the following change of variable on  $K\Gamma$

$$T: [\alpha, \beta] \mapsto [\alpha, \beta] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad [\gamma, \delta] \mapsto [\gamma, \delta] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$T(\rho_1) = \{\alpha\gamma + \alpha\delta + \beta\gamma, \alpha\delta, \beta\gamma\},$$

and hence

$$\langle T(\rho_1) \rangle = \langle \alpha\gamma, \alpha\delta, \beta\gamma \rangle$$

is a monomial ideal. By Proposition 2.2.2,  $A_1 \cong K\Gamma/\langle T(\rho_1) \rangle$ , so  $A_1$  is of monomial presentation type.

(2)  $\text{Char}(K) \neq 2$  and  $\rho_2 = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma\}$ . Define the following change of variable on  $K\Gamma$

$$T: [\alpha, \beta] \mapsto [\alpha, \beta] \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad [\gamma, \delta] \mapsto [\gamma, \delta] \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$T(\rho_2) = \{2\alpha\gamma - 2\beta\delta, -2\alpha\gamma - 2\beta\delta\},$$

and hence

$$\langle T(\rho_2) \rangle = \langle \alpha\gamma, \beta\delta \rangle$$

is a monomial ideal. By Proposition 2.2.2,  $A_2 \cong K\Gamma / \langle T(\rho_2) \rangle$ , so  $A_2$  is of monomial presentation type.

(3)  $\text{Char}(K) = 2$  and  $\rho_2 = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma\}$ . Assume that  $A_2$  is of monomial presentation type. Then, by Corollary 2.2.3, there exists a change of variable  $T$  on  $K\Gamma$  such that  $\hat{I}_2 = \langle T(\rho_2) \rangle$  is a monomial ideal and  $A \cong K\Gamma / \hat{I}_2$ . Now there are some  $M, N \in \text{GL}_2(K)$  such that

$$T[\alpha, \beta] = [\alpha, \beta]M, \quad T[\gamma, \delta] = [\gamma, \delta]N.$$

As shown in the proof of Proposition 2.2.14, we can assume that  $M$  and  $N$  are of the following forms, respectively

$$M = \begin{bmatrix} 1 & \lambda_4 \\ \lambda_3 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & \mu_4 \\ \mu_3 & 1 \end{bmatrix}.$$

Then

$$T(\rho_2) = T[\alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma] = [\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta]P,$$

where

$$P = \begin{bmatrix} 1 - \lambda_4\mu_4 & \mu_4 - \lambda_4 \\ \mu_3 - \lambda_4 & 1 - \lambda_4\mu_3 \\ \lambda_3 - \mu_4 & \lambda_3\mu_4 - 1 \\ \lambda_3\mu_3 - 1 & \lambda_3 - \mu_3 \end{bmatrix}.$$

Choose a length-lexicographic ordering  $<$  on  $B(\Gamma)$  as follows:

$$\alpha > \beta > \gamma > \delta > 3 > 2 > 1.$$

Then we have that  $\alpha\gamma > \alpha\delta > \beta\gamma > \beta\delta$ . Let us now look at the reduced Gröbner basis  $\mathcal{G} = \text{MinSharp}(\hat{I}_2)$  for  $\hat{I}_2$  which, by assumption, consists of paths in  $\Gamma$ . The discussion is divided into the following two cases.

Case 1.  $1 - \lambda_4\mu_4 = 0$ . Since  $M$  and  $N$  are invertible,  $1 - \mu_3\mu_4 \neq 0$  and  $\lambda_3\lambda_4 - 1 \neq 0$ . Then  $\mu_3 - \lambda_4 \neq 0$  and  $\lambda_3 - \mu_4 \neq 0$ . We see that  $\lambda_4 = \mu_4$ , because otherwise the first column of  $P$  would correspond to a non-monomial in  $\mathcal{G} = \text{MinSharp}(\hat{I}_2)$  and hence  $\hat{I}_2$  would not be a monomial ideal, a contradiction. Thus  $\lambda_4 = \mu_4 = 1$ . Now we have that

$$P = \begin{bmatrix} 0 & 0 \\ \mu_3 - 1 & 1 - \mu_3 \\ \lambda_3 - 1 & \lambda_3 - 1 \\ \lambda_3\mu_3 - 1 & \lambda_3 - \mu_3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ \mu_3 - 1 & 0 \\ \lambda_3 - 1 & 0 \\ \lambda_3\mu_3 - 1 & \lambda_3 - \mu_3 + \lambda_3\mu_3 - 1 \end{bmatrix}.$$

Since  $\mu_3 - 1 \neq 0$  and  $\lambda_3 - 1 \neq 0$ , we see that the first column of  $P$  corresponds to a non-monomial in  $\mathcal{G} = \text{MinSharp}(\hat{I}_2)$ . Hence  $\hat{I}_2$  is not a monomial ideal, a contradiction.

Case 2.  $1 - \lambda_4\mu_4 \neq 0$ . Performing column operations on  $P$  by adding the  $-(\mu_4 - \lambda_4)$  multiple of the first column to the  $1 - \lambda_4\mu_4$  multiple of the second column, we obtain that

$$P \rightarrow \begin{bmatrix} 1 - \lambda_4\mu_4 & 0 \\ \mu_3 - \lambda_4 & (1 - \lambda_4^2)(1 - \mu_3\mu_4) \\ \lambda_3 - \mu_4 & (1 - \mu_4^2)(\lambda_3\lambda_4 - 1) \\ \lambda_3\mu_3 - 1 & k \end{bmatrix}$$

where

$$k = (\lambda_3 - \mu_3)(1 - \lambda_4\mu_4) - (\lambda_3\mu_3 - 1)(\mu_4 - \lambda_4).$$

Since  $1 - \lambda_4\mu_4 \neq 0$ , we see that  $(1 - \lambda_4^2)(1 - \mu_3\mu_4) = 0$  or  $(1 - \mu_4^2)(\lambda_3\lambda_4 - 1) = 0$ , i.e.,  $\lambda_4 = 1$  or  $\mu_4 = 1$ . This is because otherwise the second column of  $P$  would correspond to a non-monomial in  $\mathcal{G} = \text{MinSharp}(\hat{I}_2)$ , a contradiction. Let us say  $\lambda_4 = 1$ . Then  $\mu_3 - \lambda_4 = 0$  because otherwise the first column of  $P$  would correspond to a non-monomial in  $\mathcal{G} = \text{MinSharp}(\hat{I}_2)$ , a contradiction. Thus  $\mu_3 = 1$ . Now  $k = 2(\lambda_3 - 1)(1 - \mu_4) = 0$ . Hence  $\lambda_3\mu_3 - 1 = 0$  and so  $\lambda_3 = 1$ . This leads to a contradiction since  $1 - \lambda_3\lambda_4 \neq 0$ . The case  $\mu_4 = 1$  is similar. We complete our discussion.

## 2.3 One-sided serial algebras

Let  $K$  be a field. We recall that a basic finite dimensional  $K$ -algebra  $A$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents is *right (left) serial* if each indecomposable projective right (left)  $A$ -module  $e_iA$  ( $Ae_i$ ) is uniserial; i.e., the lattice of submodules of  $e_iA$  ( $Ae_i$ ) forms a chain. We shall prove in this section that a quotient  $A = K\Gamma/I$  of path algebra is a monomial algebra if it is one-sided serial.

**Theorem 2.3.1** *Let  $\Gamma$  be any connected finite directed graph and  $I$  be an admissible ideal in  $K\Gamma$ . If  $A = K\Gamma/I$  is right (left) serial, then  $A$  is a monomial algebra; i.e.,  $I$  is a monomial ideal in  $K\Gamma$ .*

We first prove a lemma.

**Lemma 2.3.2** *Let  $A$  be an arbitrary  $K$ -algebra and  $a, x$  be elements of  $A$ . Suppose that there are integers  $s_0 > s_1 > \dots > s_l \geq 0$  and nonzero scalars  $\lambda_1, \dots, \lambda_l$  in  $K$  such that  $ax^{s_0} + \lambda_1 ax^{s_1} + \dots + \lambda_l ax^{s_l} = 0$ . If there exists some integer  $N \geq 0$  such that  $ax^N = 0$ , then  $ax^{s_0} = 0$ . (Symmetrically for the expression  $x^{s_0}a + \lambda_1 x^{s_1}a + \dots + \lambda_l x^{s_l}a = 0$ .)*

**Proof.** Assume that  $ax^{s_0} \neq 0$ . We shall show that this leads to a contradiction. Without loss of generality, suppose that  $N$  is the minimal integer such that  $ax^N = 0$ .

Let  $t_0$  be the positive integer with  $s_0 + t_0 = N$ . Multiplying both sides of the equation

$$ax^{s_0} + \lambda_1 ax^{s_1} + \cdots + \lambda_l ax^{s_l} = 0,$$

by  $x^{t_0}$ , we get that

$$\lambda_1 ax^{s_1+t_0} + \cdots + \lambda_l ax^{s_l+t_0} = 0.$$

By repeatedly carrying on the same computation as above, we obtain that

$$\lambda_l ax^{s_l+t_0+\cdots+t_{l-1}} = 0,$$

where  $t_0, \dots, t_{l-1}$  are the integers satisfying the equations

$$s_{i-1} + t_0 + \cdots + t_{i-1} = N$$

for  $i = 1, \dots, l$ . From these equations and the relation  $s_0 > s_1 > \cdots > s_l \geq 0$  it follows that

$$s_l + t_0 + \cdots + t_{l-1} < N.$$

This contradicts the minimality of  $N$ , completing the proof. ■

**Proof of Theorem 2.3.1** Suppose that  $A$  is right serial. Then for any vertex  $v$  in  $\Gamma$ , the right  $A$ -module  $(v + I)(\text{Rad}A)/(v + I)(\text{Rad}A)^2$  is simple; so  $v$  can be the source of at most one arrow in  $\Gamma$  and hence  $\Gamma$  is a rooted tree or the union of an oriented cycle and finitely many disjoint rooted trees with each root on the cycle; i.e.,  $\Gamma$  is of one of the forms in Figure 2.8 (also see [7, p. 81]).

Choose an admissible ordering  $<$  on  $B(\Gamma)$  and let  $\mathcal{G}$  be the reduced Gröbner basis for  $I$  with respect to  $<$ . Given any vertices  $v$  and  $w$  in  $\Gamma$ , we observe that there exists at most one element in  $\mathcal{G}$  which has original vertex  $v$  and terminal vertex  $w$ . Indeed, if there are two different elements of this kind in  $\mathcal{G}$ , then the tip of one of these two elements will be a subpath of the tip of the other, which is impossible since  $\mathcal{G}$  is a reduced Gröbner basis. Now let  $g_{vw}$  be any element of  $\mathcal{G}$  with original vertex  $v$  and

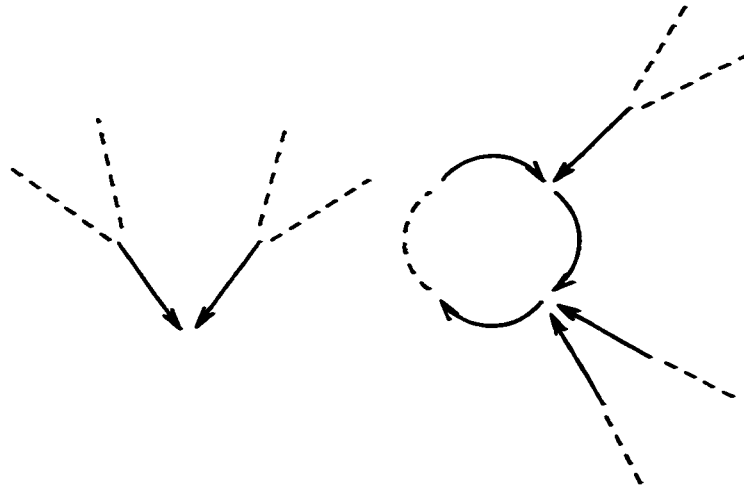


Figure 2.8

terminal vertex  $w$ . If  $w$  does not lie on the cycle, then there is at most one path from  $v$  to  $w$  and so  $g_{vw}$  is a monomial. Assume that  $w$  lies on the cycle. Then  $g_{vw}$  as a minimal sharp element in  $I$  can be expressed as

$$g_{vw} = p\sigma_w^{s_0} + \lambda_1 p\sigma_w^{s_1} + \cdots + \lambda_l p\sigma_w^{s_l},$$

where  $\lambda_1, \dots, \lambda_l \in K^*$ ,  $s_0 > s_1 > \cdots > s_l \geq 0$  ( $l \geq 0$ ),  $p$  is the unique shortest path from  $v$  to  $w$ , and  $\sigma_w$  is the directed cycle from  $w$  to itself. Since  $I$  is admissible, there exists some positive integer  $N$  such that  $p\sigma_w^N \in I$ . Thus, by Lemma 2.3.2, one has that  $p\sigma_w^{s_0} \in I$ . Hence,  $g_{vw}$  being a minimal sharp element in  $I$ ,  $l = 0$  and  $g_{vw} = p\sigma_w^{s_0}$  is a monomial. This shows that  $I$  is a monomial ideal and so  $A$  is a monomial algebra.

The proof for the left serial case is similar except that the arrows in  $\Gamma$  will point in the opposite direction. The symmetric version of Lemma 2.3.2 is applied. ■

## 2.4 Isomorphisms between monomial algebras

Let  $\Gamma$  be a finite directed graph and  $I$  be an admissible ideal in the path algebra  $K\Gamma$ . Suppose that  $A = K\Gamma/I$  is a monomial algebra, i.e.,  $I$  is a monomial ideal. Let  $P(A)$  denote the set of all nonzero paths in  $\Gamma$  modulo  $I$ . We know that  $P(A)$  is a  $K$ -vector space basis for  $A$  and  $P(A) \cup \{0\}$  forms a semigroup with zero under multiplication of paths. Given any  $p, q \in P(A)$ , we say that  $p \leq q$  in  $P(A)$  if there exists a path  $r$  in  $\Gamma$  such that  $q = pr$ . Then  $\leq$  forms a partial ordering on  $P(A)$ . The partially ordered set  $(P(A), \leq)$  is called the *path tree* of  $A$ .

Let  $S_1$  and  $S_2$  be two multiplicative semigroups with zero. We recall that a map  $f$  from  $S_1$  to  $S_2$  is called a homomorphism of semigroups with zero if

- (1)  $f(xy) = f(x)f(y)$  for all  $x \in S_1$ , and
- (2)  $f(0) = 0$ .

In [28], Shirayanagi proved a “correspondence theorem” concerning local monomial algebras which states that two local monomial algebras  $A = K\Gamma/I$  and  $B = K\Gamma/J$  are isomorphic if and only if there exists an isomorphism  $\phi: P(A) \rightarrow P(B)$  of partially ordered sets such that (1)  $\phi$  restricts to a bijection from  $\Gamma_1 + I/I$  onto  $\Gamma_1 + J/J$ , where  $\Gamma_1$  is the set of arrows in  $\Gamma$ , and (2)  $\phi$  preserves nonzero products in  $P(A)$ . In the following proposition, we generalize Shirayanagi’s correspondence theorem to general monomial quotients of path algebras. The method used here is based on that in [28]. Here our statement of the theorem is equivalent to the correspondence theorem in the local case.

**Theorem 2.4.1** *Let  $A = K\Gamma/I$  and  $B = K\Gamma/J$  be any two monomial algebras with path semigroups  $P \cup \{0\}$  and  $Q \cup \{0\}$ , respectively, where  $P = P(A)$  and  $Q = P(B)$ . Then the following statements are equivalent:*

- (1)  $A$  and  $B$  are algebra isomorphic;
- (2)  $P \cup \{0\}$  and  $Q \cup \{0\}$  are isomorphic as semigroups with zero.

Furthermore, if the equivalent conditions above hold, then there exists an isomorphism of partially ordered sets from  $P$  onto  $Q$  which restricts to a bijection from  $\Gamma_1 + I/I$  onto  $\Gamma_1 + J/J$ .

**Proof.** (2)  $\Rightarrow$  (1). This is obvious since  $A$  and  $B$  are the semigroup algebras of the semigroups  $P \cup \{0\}$  and  $Q \cup \{0\}$  over  $K$ , respectively.

(1)  $\Rightarrow$  (2) Choose a length-lexicographic ordering  $<$  on the set  $B(\Gamma)$  of directed paths in  $\Gamma$ . Then  $P$  and  $Q$  become ordered sets by restricting  $<$  to  $P$  and  $Q$ . Let  $P_l$  and  $Q_l$  denote the subsets of paths of length  $l$  in  $P$  and  $Q$ , respectively, for  $l = 0, \dots, L-1$ , where  $L$  is the Loewy length of  $A$  and  $B$ . Then  $P = P_0 \cup \dots \cup P_{L-1}$  and  $Q = Q_0 \cup \dots \cup Q_{L-1}$  are disjoint unions. Moreover,  $P_0$  and  $Q_0$  are the basic sets of primitive idempotents in  $A$  and  $B$  corresponding to the vertices in  $\Gamma$ , respectively.

Suppose that  $\theta$  is a  $K$ -algebra isomorphism from  $A$  onto  $B$ . We can assume without loss of generality, by the Wedderburn-Malcev Theorem, that  $\theta(P_0) = Q_0$ . Define

$$\sigma_0: P_0 \rightarrow Q_0$$

by  $\sigma_0(e_i) = \theta(e_i)$  for all  $e_i \in P_0$ . Clearly,  $\theta$  is a  $K$ -vector space isomorphism. Let  $M = [m_{q,p}]_{|Q| \times |P|}$  be the representation matrix of  $\theta$  with respect to  $P$  and  $Q$ . We note that  $P_l \cup \dots \cup P_{L-1}$  and  $Q_l \cup \dots \cup Q_{L-1}$  are  $K$ -vector space bases for  $\text{Rad}^l A$  and  $\text{Rad}^l B$ , respectively, for all  $l = 0, \dots, L-1$ . Since  $\theta(\text{Rad}^l A) = \text{Rad}^l B$  for all  $l$ , it follows that  $M$  is a block lower-triangular matrix with diagonal blocks  $D_0, \dots, D_{L-1}$  of sizes  $|Q_0| \times |P_0|, \dots, |Q_{L-1}| \times |P_{L-1}|$ , respectively. Since  $M$  is invertible,  $\det M \neq 0$ ; in particular,  $\det D_1 \neq 0$ . Then there exists a bijection

$$\sigma_1: P_1 \rightarrow Q_1$$

such that  $m_{\sigma_1(\alpha), \alpha} \neq 0$  for all  $\alpha \in P_1$ . Such a bijection corresponds to a nonzero product in the defining sum of  $\det D_1$ . We note that  $\sigma_1(\alpha) = \sigma_0(e_i)\sigma_1(\alpha)\sigma_0(e_j)$ , if

$\alpha = e_i \alpha e_j$  for some  $e_i, e_j \in P_0$ . For  $2 \leq l \leq L - 1$ , we inductively define maps

$$\sigma_l: P_l \rightarrow Q_l$$

by  $\sigma_l(p) = \sigma_1(\alpha)\sigma_{l-1}(q)$ , for all  $p = \alpha q \in P_l$  with  $\alpha \in P_1$  and  $q \in P_{l-1}$ . In the following we shall prove that every  $\sigma_l$ ,  $1 \leq l \leq L - 1$ , is a well-defined bijection.

We proceed with induction on  $l$ . Clearly,  $\sigma_1$  is well-defined and bijective with  $m_{\sigma_1(\alpha), \alpha} \neq 0$  for all  $\alpha \in P_1$ . Assume that for some  $l \geq 1$ ,  $\sigma_l$  is well-defined and bijective with  $m_{\sigma_l(p), p} \neq 0$  for all  $p \in P_l$ . We now prove that  $\sigma_{l+1}$  is well-defined and bijective with  $m_{\sigma_{l+1}(p), p} \neq 0$  for all  $p \in P_{l+1}$ . To prove that  $\sigma_{l+1}$  is well-defined, we need to show that for any  $p = \alpha q$  in  $P_{l+1}$  with  $\alpha \in P_1$  and  $q \in P_l$ ,  $\sigma_1(\alpha)\sigma_l(q) \neq 0$ . This is because the expression  $p = \alpha q$  is unique. Since the following two sets

$$\{(\alpha, q) \mid \alpha q \neq 0, \alpha \in P_1, q \in P_l\}$$

and

$$\{(\sigma_l(\alpha), \sigma_l(q)) \mid \sigma_l(\alpha)\sigma_l(q) \neq 0, \alpha \in P_1, q \in P_l\}$$

are of the same cardinality which equals  $\dim_K(\text{Rad}^{l+1}A)$ , it suffices to prove that if  $p = \alpha q = 0$ , then  $\sigma_1(\alpha)\sigma_l(q) = 0$ . Assume that this is not true. Then from  $\theta(p) = \theta(\alpha)\theta(q) = 0$  it follows that

$$\sum_{x, y \in Q, xy = \sigma_1(\alpha)\sigma_l(q)} m_{x, \alpha} m_{y, q} = 0.$$

We note that  $m_{x, r} = 0$  whenever  $l(r) > l(x)$ . Thus the equation above has all terms zero except the term  $m_{\sigma_1(\alpha), \alpha} m_{\sigma_l(q), q}$ ; so it is zero also, a contradiction.

Next, we show that each  $\sigma_{l+1}$  is a bijection. Since  $|P_{l+1}| = |Q_{l+1}|$ , it suffices to show that  $\sigma_{l+1}$  is injective. Given any  $p_1 = \alpha_1 q_1 \neq p_2 = \alpha_2 q_2$  in  $P_{l+1}$ , where  $\alpha_i \in P_1$  and  $q_i \in P_l$ ,  $i = 1, 2$ , if  $\alpha_1 \neq \alpha_2$ , then  $\sigma_1(\alpha_1) \neq \sigma_1(\alpha_2)$  and hence  $\sigma_{l+1}(p_1) \neq \sigma_{l+1}(p_2)$ ; if  $q_1 \neq q_2$ , then by the induction assumption  $\sigma_l(q_1) \neq \sigma_l(q_2)$  and hence  $\sigma_{l+1}(p_1) \neq \sigma_{l+1}(p_2)$ .

Thirdly, from  $\theta(p) = \theta(\alpha q) = \theta(\alpha)\theta(q)$ , we obtain that

$$m_{\sigma_{l+1}(p), p} = \sum_{x, y \in Q, xy = \sigma_{l+1}(p)} m_{x, \alpha} m_{y, q} = m_{\sigma_l(\alpha), \alpha} m_{\sigma_l(q), q}.$$

Thus,  $m_{\sigma_{l+1}(p), p} \neq 0$ . This completes our induction.

Let  $\sigma: P \cup \{0\} \rightarrow Q \cup \{0\}$  be defined by  $\sigma(0) = 0$  and  $\sigma(p) = \sigma_l(p)$  for  $p \in P_l$ , where  $0 \leq l \leq L - 1$ . We show that  $\sigma$  is an isomorphism of semigroups with zero. Take any  $p_1, p_2$  in  $P \cup \{0\}$ , if  $p_1 = 0$  or  $p_2 = 0$ , then clearly  $\sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2) = 0$ . Now assume that  $p_1 \in P_l$  and  $p_2 \in P_k$ , where  $0 \leq l, k \leq L - 1$ . Suppose that  $p_1 p_2 \neq 0$ . If  $l = 0$  or  $1$ , then it is easy to see that  $\sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2)$ . Assume that the equation above holds for some  $l = s \geq 1$ . Then when  $l = s + 1$ , setting  $p_1 = \alpha_1 q_1$  with  $\alpha_1 \in P_1$  and  $q_1 \in P_s$ , we have that  $\sigma(p_1 p_2) = \sigma(\alpha_1)\sigma(q_1 p_2) = \sigma(\alpha_1)\sigma(q_1)\sigma(p_2) = \sigma(\alpha_1 q_1)\sigma(p_2) = \sigma(p_1)\sigma(p_2)$ . We observe that the following two sets

$$\{(p_1, p_2) \mid p_1 p_2 \neq 0, p_1 \in P_l, p_2 \in P_k\}$$

and

$$\{(\sigma(p_1), \sigma(p_2)) \mid \sigma(p_1)\sigma(p_2) \neq 0, p_1 \in P_l, p_2 \in P_k\}$$

are of the same cardinality which equals  $\dim_K(\text{Rad}^{l+k}A)$ . Hence if  $p_1 p_2 = 0$ , then  $\sigma(p_1)\sigma(p_2) = 0$ . This shows that  $\sigma$  is an isomorphism of semigroups with zero.

Now suppose that  $\phi: P \cup \{0\} \rightarrow Q \cup \{0\}$  is an isomorphism of semigroups with zero. We can easily check that  $\phi$  restricts to a bijection from  $\Gamma_1 + I/I$  onto  $\Gamma_1 + J/J$ . Then if  $p \in P$  is of length  $l$ , then so is  $\phi(p)$ . Given any  $p, q \in P$  with  $p \leq q$ , then there exists some path  $r$  in  $\Gamma$  such that  $q = pr$ . Thus  $\phi(q) = \phi(p)\phi(r)$  and this shows that  $\phi(p) \leq \phi(q)$ . On the other hand, assume that  $\phi(p) \leq \phi(q)$  in  $Q$ . Then there exists some path  $\phi(r)$  in  $\Gamma$  such that  $\phi(q) = \phi(p)\phi(r) = \phi(pr)$ . Hence  $q = pr$  in  $P$  since  $\phi$  is a bijection. This show that  $p \leq q$ . Consequently, we obtain that  $\phi$  is an isomorphism of partially ordered sets. ■

The corollary to Theorem 2.4.1 is straightforward.

**Corollary 2.4.2** *Let  $A = K\Gamma/I$  and  $B = K\Gamma/J$  be any two monomial algebras. Given any admissible ordering  $<$  on  $B(\Gamma)$  and suppose that  $\mathcal{G}_I$  and  $\mathcal{G}_J$  are the reduced Gröbner bases for  $I$  and  $J$  with respect to  $<$ , respectively. Then  $A$  is algebra isomorphic to  $B$  if and only if there exists a permutation  $\sigma$  on  $\Gamma_1$  such that  $\sigma(\mathcal{G}_I) = \mathcal{G}_J$ .*

■

In [29], Shirayanagi asked if his correspondence theorem could be generalized to local binomial algebras. (A path algebra quotient  $A = K\Gamma/I$  is called a *binomial algebra* if  $I$  is generated by a set of binomials and monomials in  $K\Gamma$ .) In the following, we shall construct an example which shows that the correspondence theorem need not hold in either direction if one of the two algebras in question is not monomial.

Suppose that  $A = K\Gamma/I$  is a path algebra quotient such that  $I$  is generated by a set of paths and/or differences of paths. Such a path algebra quotient is called a *diagram algebra* (see [13]). Let  $P(A)$  denote the set of equivalence classes of all nonzero paths in  $\Gamma$  modulo  $I$ . Then  $P(A) \cup \{0\}$  forms a multiplicative semigroup with zero by defining  $[p][q] = [pq]$  for  $[p], [q] \in P(A)$ . Given any  $[p], [q]$  in  $P(A)$ , we say that  $[p] \leq [q]$  in  $P(A)$  if there exists paths  $p_1, q_1, r$  in  $\Gamma$  such that  $[p_1] = [p]$ ,  $[q_1] = [q]$ , and  $q_1 = p_1r$ . Then  $\leq$  forms a partial ordering on  $P(A)$ . We refer to the partially ordered set  $(P(A), \leq)$  as the *path diagram* of  $A$ .

**Example 2.6** Let  $K$  be a field with  $\text{Char}(K) \neq 2$ . Define  $\Gamma$  to be

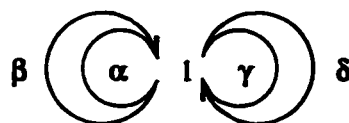


Figure 2.9

and let

$$\rho = \{\alpha^2, \alpha\beta, \beta\alpha, \beta^2, \alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma, \gamma\alpha, \gamma\beta, \gamma^2, \gamma\delta, \delta\alpha, \delta\beta, \delta\gamma, \delta^2\},$$

$$\bar{\rho} = \{\alpha^2, \alpha\beta, \beta\alpha, \beta^2, \alpha\gamma, \alpha\delta, \gamma\alpha, \gamma\beta, \gamma^2, \gamma\delta, \delta\alpha, \delta\beta, \delta\gamma, \delta^2\}.$$

Let  $A = K\Gamma/I$  and  $\bar{A} = K\Gamma/\bar{I}$ , where  $I = \langle \rho \rangle$  and  $\bar{I} = \langle \bar{\rho} \rangle$ .

First, we notice that the algebra  $A$  has the following path diagram  $P(A)$ :

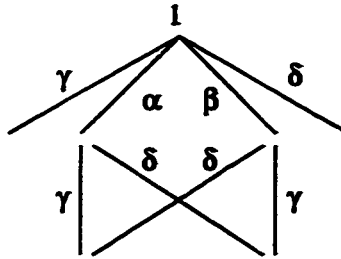


Figure 2.10

Clearly,  $A$  is of Loewy length 3. Define the following change of variable  $T$  on  $K\Gamma$

$$\begin{cases} T(\alpha) = \alpha - \beta \\ T(\beta) = \alpha + \beta \\ T(\gamma) = \gamma + \delta \\ T(\delta) = -\gamma + \delta. \end{cases}$$

Then

$$\langle T(\rho) \cup \Gamma_1^3 \rangle = \langle \alpha^2, \alpha\beta, \beta\alpha, \beta^2, \alpha\gamma, \beta\delta, \gamma\alpha, \gamma\beta, \gamma^2, \gamma\delta, \delta\alpha, \delta\beta, \delta\gamma, \delta^2 \rangle$$

is a monomial ideal. By Proposition 2.2.2,  $A \cong \hat{A} = K\Gamma/\hat{I}$ , where  $\hat{I} = \langle T(\rho) \cup \Gamma_1^3 \rangle$ , is of monomial presentation type. Now the monomial algebra  $\hat{A}$  has the path tree  $P(\hat{A})$  as in Figure 2.11. Observing the two path diagrams  $P(A)$  and  $P(\hat{A})$ , we see that there does not exist any order preserving bijection  $\phi$  from  $P(\hat{A})$  onto  $P(A)$  such that  $\phi$  restricts to bijections on  $\Gamma_0$  and  $\Gamma_1$  and preserves nonzero products in  $P(A)$ . This deals with one direction of the correspondence theorem.

On the other hand,  $\bar{A}$  is a monomial algebra with path tree  $P(\bar{A})$  as in Figure 2.12. We see that there does not exist any isomorphism  $\phi$  of semigroups with zero

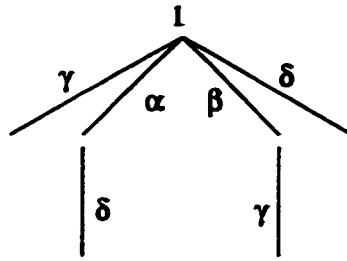


Figure 2.11

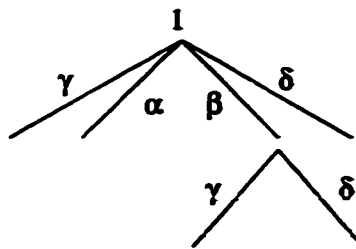


Figure 2.12

from  $P(\bar{A}) \cup \{0\}$  onto  $P(\hat{A}) \cup \{0\}$ . Then, by Theorem 2.4.1,  $\bar{A}$  is not isomorphic to  $\hat{A}$  and hence is not isomorphic to  $A$ . But we can construct an order preserving bijection  $\phi$  from  $P(\bar{A})$  to  $P(A)$  as follows:

$$\phi[1, \alpha, \beta, \gamma, \delta, \beta\gamma, \beta\delta] = [1, \alpha, \beta, \gamma, \delta, \beta\gamma, \beta\delta].$$

It is easy to see that  $\phi$  restricts to identity on  $\Gamma_0$  and  $\Gamma_1$  and preserves nonzero products in  $P(\bar{A})$ . This deals with the other direction of the correspondence theorem.

# Chapter 3

## The Anick-Green resolutions

This chapter is mainly concerned with a homological aspect of path algebra quotients of monomial presentation type. In the first section, the minimality of the Anick-Green resolutions for the simple modules over path algebra quotients is studied, and its relationship to the monomial presentation type is discussed. Section 3.2 deals with associated monomial algebras and provides a counter-example to a natural conjecture concerning its relationship to the monomial presentation type.

### 3.1 The Anick-Green resolutions

Anick and Green [3] presented a method of constructing projective resolutions (not necessarily minimal) for the simple modules over quotients of path algebras which reflects combinatorial properties of these quotients. In this section we shall study the minimality of the Anick-Green resolutions and its relationship to the monomial presentation type.

Let  $I$  be an admissible ideal in the path algebra  $K\Gamma$  and  $<$  be an admissible ordering on the set  $B(\Gamma)$  of finite directed paths in the connected finite directed graph  $\Gamma$ . Let  $A = K\Gamma/I$ . We shall denote by  $E = \{e_1, \dots, e_n\}$  the basic set of

primitive idempotents of  $A$  corresponding to the set  $\Gamma_0 = \{v_1, \dots, v_n\}$  of vertices in  $\Gamma$  and by  $S_i = e_i A / e_i(\text{Rad}A)$ ,  $i = 1, \dots, n$ , the corresponding simple right  $A$ -modules.

We start by describing the Anick-Green method of constructing projective resolutions for the simple modules over quotients of path algebras. To do this, we need first to define  $m$ -chains ( $m \geq -1$ ) on  $\text{MinTip}(I)$  (see [3]).

**Definition 3.1** Let  $I$  be an admissible ideal in  $K\Gamma$  and  $<$  be an admissible ordering on  $B(\Gamma)$ . Then

- (1) the sets of  $(-1)$ -chains and  $0$ -chains on  $\text{MinTip}(I)$  are  $\Gamma_0$  and  $\Gamma_1$ , respectively;
- (2) the set  $\Gamma_2$  of  $1$ -chains on  $\text{MinTip}(I)$  is  $\text{MinTip}(I)$ ;
- (3) for  $m \geq 2$ , the set  $\Gamma_{m+1}$  of  $m$ -chains on  $\text{MinTip}(I)$  is the set of paths  $p \in B(\Gamma)$  satisfying the following conditions:

- (i)  $p = p_1 p_2$  for some  $p_1 \in \Gamma_m$  and some  $p_2 \in \text{NonTip}(I) - \Gamma_0$ ;
- (ii) if  $p = p_1 p_2$  with  $p_1 \in \Gamma_m$  and if  $p_1 = q_1 q_2$  with  $q_1 \in \Gamma_{m-1}$ , then  $q_2 p_2 \in \text{Tip}(I)$ ;
- (iii)  $p$  has no proper left subpath which satisfies (i) and (ii) above.

See Example 3.1 for a calculation of some of these sets of chains.

As was shown in [3, Lemma 2.6], a consequence of the definition above is that every  $m$ -chain  $p \in \Gamma_{m+1}$  ( $m \geq 0$ ) may be factored uniquely as  $p = p_1 p_2$ , where  $p_1 \in \Gamma_m$  and  $p_2 \in \text{NonTip}(I) - \Gamma_0$ . The following result is due to Anick and Green [3].

**Proposition 3.1.1 (Anick and Green)** Let  $I$  be an admissible ideal in  $K\Gamma$  and let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents corresponding to  $\Gamma_0 = \{v_1, \dots, v_n\}$ . Suppose that  $<$  is an admissible ordering on  $B(\Gamma)$ . Let  $\Gamma_m$  ( $m \geq 0$ ) denote the set of  $(m-1)$ -chains on  $\text{MinTip}(I)$ . Set  $\Gamma_m^{(i)} = \Gamma_m \cap v_i B(\Gamma)$  for  $m \geq 0$  and  $1 \leq i \leq n$ . Let  $R = K\Gamma_0$  and  $S_i = e_i A / e_i(\text{Rad}A)$  for  $1 \leq i \leq n$ . Then for each  $i$  there exists an exact sequence of right  $A$ -modules

$$\dots \xrightarrow{\delta_3} K\Gamma_2^{(i)} \otimes_R A \xrightarrow{\delta_2} K\Gamma_1^{(i)} \otimes_R A \xrightarrow{\delta_1} K\Gamma_0^{(i)} \otimes_R A \xrightarrow{\delta_0} S_i \rightarrow 0 \quad (3.1.1)$$

satisfying that, for all  $m \geq 1$  and all  $p \in \Gamma_m^{(i)}$ ,

$$\delta_m(p \otimes e_{i(p)}) = p_1 \otimes q_1 + \lambda_2 p_2 \otimes q_2 + \cdots + \lambda_k p_k \otimes q_k \in \text{Span}_K(\Gamma_{m-1}^{(i)} \otimes_R \text{NonTip}(I))$$

where  $p = p_1 q_1$  and  $p_j q_j < p$  for all  $1 < j \leq k$ .

**Proof.** See [3, Theorem 2.7]. ■

We now describe the Anick-Green algorithm for computing projective resolutions for the simple modules over quotients of path algebras.

### The Anick-Green algorithm

(\* A recursive algorithm for computing the homomorphisms  $\delta_m$  ( $m \geq 0$ ) in the projective resolutions (3.1.1) for the simple modules  $S_i$ ,  $1 \leq i \leq n$ , over  $A = K\Gamma/I$ . \*)

Input: an admissible ordering  $<$  on  $B(\Gamma)$ , a simple right module  $S_i = e_i A / e_i(\text{Rad}A)$  over  $A$  where  $1 \leq i \leq n$ , and the sets  $\Gamma_m^{(i)}$  of  $(m-1)$ -chains with origin  $v_i$ , on  $\text{MinTip}(I)$  for  $m \geq 0$ .

Output: the sequence  $(\delta_m)_{m \geq 0}$  of  $A$ -homomorphisms in the resolution (3.1.1) for  $S_i$  and a sequence of  $K$ -vector space homomorphisms,  $\eta_m : \ker(\delta_{m-1}) \rightarrow K\Gamma_m^{(i)} \otimes_R A$ , for  $m \geq 0$ , where  $\delta_{-1} = 0$ . The  $\eta_m$  are  $K$ -vector space sections for the  $\delta_m$ .

**begin**

**if**  $m = 0$  **then**

$$\delta_m(v_i \otimes e_i) := e_i;$$

$$\eta_m(e_i) := v_i \otimes e_i;$$

**if**  $m = 1$  **then**

**for all**  $\alpha \in \Gamma_1^{(i)}$  **do**

$$\delta_m(\alpha \otimes e_{i(\alpha)}) := v_i \otimes \alpha;$$

**for all**  $p = \alpha q \in v_i \text{NonTip}(I) - \{v_i\}$  **with**  $\alpha \in \Gamma_1^{(i)}$  **do**

$$\eta_m(v_i \otimes p) := \alpha \otimes q ;$$

if  $m \geq 2$  then

for all  $p = p_1 p_2 \in \Gamma_m^{(i)}$  with  $p_1 \in \Gamma_{m-1}^{(i)}$  do

$$\delta_m(p \otimes e_{\iota(p)}) := p_1 \otimes p_2 - \eta_{m-1} \delta_{m-1}(p_1 \otimes p_2);$$

for all  $w = \sum_{j=1}^s \lambda_j p_j \otimes q_j \in \ker(\delta_{m-1})$  with each  $\lambda_j \neq 0$

and  $p_1 q_1 > p_j q_j$  for  $1 < j \leq s$  do

if  $w = 0$  then

$$\eta_m(w) := 0$$

else

find  $p' \in \Gamma_m^{(i)}$  and  $p'' \in \text{NonTip}(I)$  such that  $p' p'' = p_1 q_1$ ;

$$\eta_m(w) := \lambda_1 p' \otimes p'' + \eta_m(w - \lambda_1 \delta_m(p' \otimes p''))$$

end.

The existence of the factorization in the definition of  $\eta_m(w)$  is a crucial part of the proof in [3].

In the following proposition we provide a condition under which the Anick-Green resolutions over  $A$  are not minimal.

**Proposition 3.1.2** *Let  $I$  be an admissible ideal in  $K\Gamma$  and  $<$  be an admissible ordering on  $B(\Gamma)$ . Let  $A = K\Gamma/I$ . Suppose that  $\mathcal{G}$  is the reduced Gröbner basis for  $I$  with respect to  $<$  satisfying the following conditions:*

(1)  $\alpha p \in \mathcal{G}$  and  $\alpha p q \in \Gamma_3^{(i)}$ , for some  $\alpha \in \Gamma_1$  and  $p, q \in \text{NonTip}(I) - \Gamma_0$  and  $i \in \{1, \dots, n\}$ ;

(2) there exist  $k \in K^*$ ,  $p_0, q_0 \in B(\Gamma) - \Gamma_0$  such that  $p q - k p_0 q_0 \in \mathcal{G}$  and  $\alpha p_0 q_0 \in \mathcal{G}$ .

Then the Anick-Green resolution for  $S_i = e_i A / e_i(\text{Rad} A)$  is not minimal.

**Proof.** We calculate the image of  $\alpha p q \otimes e_{\iota(q)}$  under  $\delta_3$  in the Anick-Green resolution for  $S_i = e_i A / e_i(\text{Rad} A)$  as follows:

$$\delta_3(\alpha p q \otimes e_{\iota(q)})$$

$$\begin{aligned}
 &= \alpha p \otimes q - \eta_2 \delta_2(\alpha p \otimes q) \\
 &= \alpha p \otimes q - \eta_2(\alpha \otimes pq - (\eta_1 \delta_1(\alpha \otimes p))q) \\
 &= \alpha p \otimes q - \eta_2(\alpha \otimes pq - (\eta_1(v_i \otimes \alpha p))q) \\
 &= \alpha p \otimes q - \eta_2(\alpha \otimes pq) \quad (\text{since } \alpha p \in \mathcal{G}) \\
 &= \alpha p \otimes q - k\eta_2(\alpha \otimes p_0 q_0) \quad (\text{since } pq - kp_0 q_0 \in \mathcal{G}) \\
 &= \alpha p \otimes q - k\alpha p_0 q_0 \otimes e_{t(q_0)} - k\eta_2(\alpha \otimes p_0 q_0 - \delta_2(\alpha p_0 q_0 \otimes e_{t(q_0)})) \\
 &= \alpha p \otimes q - k\alpha p_0 q_0 \otimes e_{t(q_0)} - k\eta_2(\alpha \otimes p_0 q_0 - \alpha \otimes p_0 q_0 + \eta_1 \delta_1(\alpha \otimes p_0 q_0)) \\
 &= \alpha p \otimes q - k\alpha p_0 q_0 \otimes e_{t(q_0)} - k\eta_2 \eta_1(v_i \otimes \alpha p_0 q_0) \\
 &= \alpha p \otimes q - k\alpha p_0 q_0 \otimes e_{t(q_0)} \quad (\text{since } \alpha p_0 q_0 \in \mathcal{G}).
 \end{aligned}$$

This shows that  $\delta_3(\alpha p q \otimes e_{t(q)})$  is not in the radical of  $K\Gamma_2^{(i)} \otimes_R A$ , so the Anick-Green resolution for  $S_i$  is not minimal. ■

**Theorem 3.1.3** *Let  $I$  be an admissible ideal in  $K\Gamma$  and let  $A = K\Gamma/I$ . Given any admissible ordering  $<$  on  $B(\Gamma)$ , let  $\Gamma_m$  denote the set of  $(m-1)$ -chains on  $\text{MinTip}(I)$  with respect to  $<$ , where  $m \geq 0$ . If every path in  $\Gamma_m$  is of length  $m$  for all  $m \geq 0$ , then the Anick-Green resolutions for the simple modules  $S_i = e_i A / e_i(\text{Rad}A)$ ,  $i = 1, \dots, n$ , over  $A$  are minimal.*

**Proof.** Given any simple  $A$ -module  $S_i = e_i A / e_i(\text{Rad}A)$ , where  $1 \leq i \leq n$ , we need to prove that for all  $m \geq 1$  and all  $p \in \Gamma_m^{(i)}$ ,  $\delta_m(p \otimes e_{t(p)})$  lies in the radical of  $K\Gamma_{m-1}^{(i)} \otimes_R A$ . We proceed by induction on  $m$ .

First, we have that  $\delta_1(p \otimes e_{t(p)}) = v_i \otimes p \in (K\Gamma_0^{(i)} \otimes_R A)(\text{Rad}A)$ , for all  $p \in \Gamma_1^{(i)}$ . Choose any  $m \geq 2$  and any  $p \in \Gamma_m^{(i)}$ . By hypothesis, we can write  $p = \alpha_1 \cdots \alpha_m$ , where  $\alpha_j \in \Gamma_1$  for  $j = 1, \dots, m$ . Then by the recursive definition of  $\delta_m$ ,

$$\delta_m(p \otimes e_{t(p)}) = \alpha_1 \cdots \alpha_{m-1} \otimes \alpha_m - \eta_{m-1} \delta_{m-1}(\alpha_1 \cdots \alpha_{m-1} \otimes \alpha_m).$$

Set  $w_1 = \delta_{m-1}(\alpha_1 \cdots \alpha_{m-1} \otimes \alpha_m)$ . Then we have that

$$\delta_m(p \otimes e_{t(p)}) \equiv \eta_{m-1}(w_1) \pmod{(K\Gamma_{m-1}^{(i)} \otimes_R A)(\text{Rad}A)}.$$

By the induction assumption, we have that

$$\delta_{m-1}(\alpha_1 \cdots \alpha_{m-1} \otimes e_{t(\alpha_{m-1})}) \in (K\Gamma_{m-2}^{(i)} \otimes_R A)(\text{Rad}A),$$

so that

$$w_1 = \delta_{m-1}(\alpha_1 \cdots \alpha_{m-1} \otimes \alpha_m) \in (K\Gamma_{m-2}^{(i)} \otimes_R A)(\text{Rad}A)^2.$$

If  $w_1 = 0$ , there is nothing to do. Assume that  $w_1 \neq 0$ . Write

$$w_1 = \lambda_1 p_1 \otimes q_1 + \cdots + \lambda_s p_s \otimes q_s$$

where  $\lambda_j \in K^*$ ,  $p_j \in \Gamma_{m-2}^{(i)}$ ,  $q_j \in \text{NonTip}(I)$  with  $l(q_j) \geq 2$ , for all  $j = 1, \dots, s$ , and  $p_j q_j < p_1 q_1$ , for all  $j \neq 1$ . We denote the degree of  $w_1$  by  $\deg(w_1)$ , which is the length of the path  $p_1 q_1$  corresponding to the leading term  $\lambda_1 p_1 \otimes q_1$  of  $w_1$ . Then, by the definition of  $\eta_{m-1}$ ,

$$\eta_{m-1}(w_1) = \lambda_1 p'_1 \otimes q'_1 + \eta_{m-1}(w_1 - \lambda_1 \delta_{m-1}(p'_1 \otimes q'_1)),$$

for some  $p'_1 \in \Gamma_{m-1}^{(i)}$  and  $q'_1 \in \text{NonTip}(I)$  with  $p_1 q_1 = p'_1 q'_1$ . Set

$$w_2 = w_1 - \lambda_1 \delta_{m-1}(p'_1 \otimes q'_1).$$

Since  $l(p'_1) = m - 1$ , we see that  $l(q'_1) \geq 1$  and so that

$$\lambda_1 p'_1 \otimes q'_1 \in (K\Gamma_{m-1}^{(i)} \otimes_R A)(\text{Rad}A).$$

Thus

$$\eta_{m-1}(w_1) \equiv \eta_{m-1}(w_2) \pmod{(K\Gamma_{m-1}^{(i)} \otimes_R A)(\text{Rad}A)}.$$

By the induction assumption,  $\delta_{m-1}(p'_1 \otimes q'_1) \in (K\Gamma_{m-2}^{(i)} \otimes_R A)(\text{Rad}A)^2$ . Thus

$$w_2 = w_1 - \lambda_1 \delta_{m-1}(p'_1 \otimes q'_1) \in (K\Gamma_{m-2}^{(i)} \otimes_R A)(\text{Rad}A)^2.$$

If  $w_2 = 0$ , there is nothing to do. If not, then by the construction of  $\eta_{m-1}$ , we have that  $\deg(w_2) < \deg(w_1)$ . Repeatedly applying the argument above, we then obtain a finite sequence  $w_1, \dots, w_t, w_{t+1} = 0$  in  $(K\Gamma_{m-2}^{(i)} \otimes_R A)(\text{Rad}A)^2$  such that

$$\eta_{m-1}(w_1) \equiv \dots \equiv \eta_{m-1}(w_t) \equiv 0 \pmod{(K\Gamma_{m-1}^{(i)} \otimes_R A)(\text{Rad}A)},$$

where  $\deg(w_1) > \dots > \deg(w_t)$ . Consequently, we have that

$$\delta_m(p \otimes e_{t(p)}) \in (K\Gamma_{m-1}^{(i)} \otimes_R A)(\text{Rad}A).$$

The proof is complete. ■

**Proposition 3.1.4** *Let  $I$  be an admissible ideal in  $K\Gamma$  and  $<$  be an admissible ordering on  $B(\Gamma)$ . For  $m \geq 0$ , let  $\Gamma_m$  denote the set of  $(m-1)$ -chains on  $\text{MinTip}(I)$  with respect to  $<$ . If every path in  $\Gamma_2$  is of length 2, then every path in  $\Gamma_m$  is of length  $m$  for all  $m \geq 3$ .*

**Proof.** We do an induction on  $m$ . Choose any  $m \geq 2$  and assume that every path in  $\Gamma_l$  is of length  $l$ , for all  $l = 1, \dots, m$ . Take any  $p = q\alpha_1 \cdots \alpha_s \in \Gamma_{m+1}$ , where  $q \in \Gamma_m$  and  $\alpha_1 \cdots \alpha_s \in \text{NonTip}(I) - \Gamma_0$  with each  $\alpha_i \in \Gamma_1$ . Then, by the induction hypothesis,  $q$  is of length  $m$ . Now  $q = r\beta$ , for some  $r \in \Gamma_{m-1}$  and some  $\beta \in \text{NonTip}(I) - \Gamma_0$ . By the induction hypothesis,  $r$  is of length  $m-1$  and so  $\beta$  is of length 1, i.e.,  $\beta \in \Gamma_1$ . By the construction of chains, we know that  $\beta\alpha_1 \cdots \alpha_s$  has a right subpath lying in  $\Gamma_2$ . But  $\alpha_1 \cdots \alpha_s$  is a nontip element which has no subpath lying in  $\Gamma_2$ , so  $\beta\alpha_1 \cdots \alpha_s \in \Gamma_2$ . Thus  $s = 1$ . Since  $q$  is of length  $m$ , it follows that  $p$  is of length  $m+1$ . Our induction is complete. ■

From Proposition 3.1.4 and Theorem 3.1.3, the following corollary is immediate.

**Corollary 3.1.5** *Let  $I$  be an admissible ideal in  $K\Gamma$  and  $<$  be an admissible ordering on  $B(\Gamma)$ . Let  $A = K\Gamma/I$ . If every path in  $\text{MinTip}(I)$  is of length 2, then the Anick-Green resolutions for the simple modules  $S_i = e_i A_i / e_i(\text{Rad}A)$ ,  $i = 1, \dots, n$ , over  $A$  are minimal. ■*

It is worth mentioning that the result of the above corollary has been proved by Green and Huang in [18, Theorem 3] using different methods.

We look at an example.

**Example 3.1** Let  $K$  be a field. Define  $\Gamma$  to be

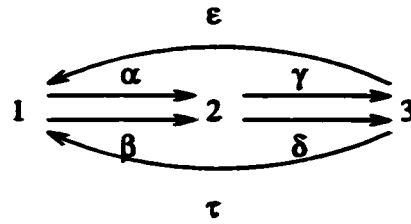


Figure 3.1

and let

$$\rho = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma - \beta\gamma, \gamma\epsilon - \delta\epsilon, \epsilon\alpha - \tau\alpha\}.$$

Let  $A = K\Gamma/I$ , where  $I = \langle \rho \rangle$ . We shall see that  $A$  is of monomial presentation type. There are in total 8 classes of admissible orderings on  $B(\Gamma)$ :

$$\langle_1: \beta\gamma > \alpha\gamma, \delta\epsilon > \gamma\epsilon, \tau\alpha > \epsilon\alpha;$$

$$\langle_2: \beta\gamma > \alpha\gamma, \delta\epsilon > \gamma\epsilon, \epsilon\alpha > \tau\alpha;$$

$$\langle_3: \beta\gamma > \alpha\gamma, \gamma\epsilon > \delta\epsilon, \tau\alpha > \epsilon\alpha;$$

$$\langle_4: \beta\gamma > \alpha\gamma, \gamma\epsilon > \delta\epsilon, \epsilon\alpha > \tau\alpha;$$

$$\langle_5: \alpha\gamma > \beta\gamma, \delta\epsilon > \gamma\epsilon, \tau\alpha > \epsilon\alpha;$$

$$\langle_6: \alpha\gamma > \beta\gamma, \delta\epsilon > \gamma\epsilon, \epsilon\alpha > \tau\alpha;$$

$$\langle_7: \alpha\gamma > \beta\gamma, \gamma\epsilon > \delta\epsilon, \tau\alpha > \epsilon\alpha;$$

$$\langle_8: \alpha\gamma > \beta\gamma, \gamma\epsilon > \delta\epsilon, \epsilon\alpha > \tau\alpha.$$

In the following we shall show that the Anick-Green resolutions for the simple right modules over  $A$  are not minimal with respect to any of the orderings  $\langle_1$  to  $\langle_7$  but minimal with respect to the ordering  $\langle_8$ . Let  $\mathcal{G}$  denote the reduced Gröbner basis

for  $I$  with respect to any admissible ordering on  $B(\Gamma)$ . We carry out the following computations.

(1) With respect to  $<_1$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma - \alpha\gamma, \delta\epsilon - \gamma\epsilon, \tau\alpha - \epsilon\alpha, \alpha\gamma\epsilon, \gamma\epsilon\alpha, \epsilon\alpha\gamma\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma, \delta\epsilon, \tau\alpha, \alpha\gamma\epsilon, \gamma\epsilon\alpha, \epsilon\alpha\gamma\}.$$

We see that  $\alpha\delta, \delta\epsilon - \gamma\epsilon, \alpha\gamma\epsilon \in \mathcal{G}$  and  $\alpha\delta\epsilon \in \Gamma_3^{(1)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_1 = e_1A/e_1(\text{Rad}A)$  is not minimal.

(2) With respect to  $<_2$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma - \alpha\gamma, \delta\epsilon - \gamma\epsilon, \epsilon\alpha - \tau\alpha, \alpha\gamma\epsilon, \tau\alpha\gamma\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma, \delta\epsilon, \epsilon\alpha, \alpha\gamma\epsilon, \tau\alpha\gamma\}.$$

As with (1),  $\alpha\delta, \delta\epsilon - \gamma\epsilon, \alpha\gamma\epsilon \in \mathcal{G}$  and  $\alpha\delta\epsilon \in \Gamma_3^{(1)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_1 = e_1A/e_1(\text{Rad}A)$  is not minimal.

(3) With respect to  $<_3$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma - \alpha\gamma, \gamma\epsilon - \delta\epsilon, \tau\alpha - \epsilon\alpha, \epsilon\alpha\gamma, \delta\epsilon\alpha\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma, \gamma\epsilon, \tau\alpha, \epsilon\alpha\gamma, \delta\epsilon\alpha\}.$$

We see that  $\delta\tau, \tau\alpha - \epsilon\alpha, \delta\epsilon\alpha \in \mathcal{G}$  and  $\delta\tau\alpha \in \Gamma_3^{(2)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_2 = e_2A/e_2(\text{Rad}A)$  is not minimal.

(4) With respect to  $<_4$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma - \alpha\gamma, \gamma\epsilon - \delta\epsilon, \epsilon\alpha - \tau\alpha, \tau\alpha\gamma\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \beta\gamma, \gamma\epsilon, \tau\alpha, \tau\alpha\gamma\}.$$

We see that  $\tau\beta, \beta\gamma - \alpha\gamma, \tau\alpha\gamma \in \mathcal{G}$  and  $\tau\beta\gamma \in \Gamma_3^{(3)}$ . Then by Proposition 3.1.2, the Anick-Green resolution for  $S_3 = e_3A/e_3(\text{Rad}A)$  is not minimal.

(5) With respect to  $<_5$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma - \beta\gamma, \delta\epsilon - \gamma\epsilon, \tau\alpha - \epsilon\alpha, \beta\gamma\epsilon, \gamma\epsilon\alpha\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma, \delta\epsilon, \tau\alpha, \beta\gamma\epsilon, \gamma\epsilon\alpha\}.$$

We see that  $\beta\delta, \delta\epsilon - \gamma\epsilon, \beta\gamma\epsilon \in \mathcal{G}$  and  $\beta\delta\epsilon \in \Gamma_3^{(1)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_3 = e_3A/e_3(\text{Rad}A)$  is not minimal.

(6) With respect to  $<_6$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma - \beta\gamma, \delta\epsilon - \gamma\epsilon, \epsilon\alpha - \tau\alpha, \beta\gamma\epsilon\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma, \delta\epsilon, \epsilon\alpha, \beta\gamma\epsilon\}.$$

As with (5),  $\beta\delta, \delta\epsilon - \gamma\epsilon, \beta\gamma\epsilon \in \mathcal{G}$  and  $\beta\delta\epsilon \in \Gamma_3^{(1)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_3 = e_3A/e_3(\text{Rad}A)$  is not minimal.

(7) With respect to  $<_7$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma - \beta\gamma, \gamma\epsilon - \delta\epsilon, \tau\alpha - \epsilon\alpha, \delta\epsilon\alpha\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma, \gamma\epsilon, \tau\alpha, \delta\epsilon\alpha\}.$$

As with (3),  $\delta\tau, \tau\alpha - \epsilon\alpha, \delta\epsilon\alpha \in \mathcal{G}$  and  $\delta\tau\alpha \in \Gamma_3^{(2)}$ . Then by Proposition 3.1.2 the Anick-Green resolution for  $S_2 = e_2A/e_2(\text{Rad}A)$  is not minimal.

(8) With respect to  $\langle \mathfrak{s} \rangle$ ,

$$\mathcal{G} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma - \beta\gamma, \gamma\epsilon - \delta\epsilon, \epsilon\alpha - \tau\alpha\}$$

and

$$\Gamma_2 = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma, \gamma\epsilon, \epsilon\alpha\}.$$

Since every path in  $\Gamma_2$  is of length 2, by Corollary 3.1.5 the Anick-Green resolutions for  $S_i = e_i A / e_i (\text{Rad} A)$ ,  $i = 1, 2, 3$ , are minimal.

We can easily check that  $A$  is of Loewy length 3. Define the following change of variable  $T$  on  $K\Gamma$

$$\begin{cases} T(\alpha) = \alpha + \beta \\ T(\beta) = \beta \\ T(\gamma) = \gamma + \delta \\ T(\delta) = \delta \\ T(\epsilon) = \epsilon + \tau \\ T(\tau) = \tau. \end{cases}$$

Then  $\langle T(\rho) \cup \Gamma_1^3 \rangle = \langle \hat{\rho} \rangle$ , where

$$\hat{\rho} = \{\alpha\delta, \beta\delta, \gamma\tau, \delta\tau, \epsilon\beta, \tau\beta, \alpha\gamma, \gamma\epsilon, \epsilon\alpha\}.$$

Thus, by Proposition 2.2.2  $A$  is isomorphic to the monomial algebra  $\hat{A} = K\Gamma / \langle \hat{\rho} \rangle$ . We note that  $\hat{\rho}$  is exactly  $\Gamma_2$  with respect to  $\langle \mathfrak{s} \rangle$ . So  $\hat{A}$  is also the associated monomial algebra of  $A$  with respect to  $\langle \mathfrak{s} \rangle$ . We shall discuss associated monomial algebras in the next section.

In [3, Corollary 2.9], Anick and Green proved that in every monomial algebra  $A = K\Gamma/I$ , the Anick-Green resolutions for its simple modules with respect to any admissible ordering on  $B(\Gamma)$  are minimal. We pose the following question: Given any  $A = K\Gamma/I$  of monomial presentation type, does there exist an admissible ordering  $\langle$

on  $B(\Gamma)$  such that the Anick-Green resolutions for the simple modules  $S_i$ ,  $1 \leq i \leq n$ , over  $A$  are minimal? In the following we give a counter-example in which a path algebra quotient  $A = K\Gamma/I$  is of monomial presentation type but the Anick-Green resolutions for its simple modules are not minimal with respect to *any* admissible ordering on  $B(\Gamma)$ .

**Example 3.2** Let  $K$  be a field. Define  $\Gamma$  to be

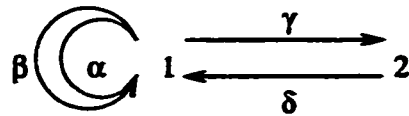


Figure 3.2

and let

$$\rho = \{\alpha^2, \beta\alpha, \beta^2, \gamma\delta, \delta\beta, \delta\gamma, \delta\alpha\beta, \alpha\beta\gamma - \alpha\gamma\}.$$

Let  $A = K\Gamma/I$ , where  $I = \langle \rho \rangle$ . We can easily check that  $A$  is of Loewy length 4.

Define the following change of variable  $T$  on  $K\Gamma$

$$\begin{cases} T(\alpha) = \alpha + \alpha\beta \\ T(\beta) = \beta \\ T(\gamma) = \gamma \\ T(\delta) = \delta. \end{cases}$$

Then we have that

$$\langle T(\rho) \cup \Gamma_1^4 \rangle = \langle \hat{\rho} \rangle,$$

where

$$\hat{\rho} = \{\alpha^2, \beta\alpha, \beta^2, \gamma\delta, \delta\beta, \delta\gamma, \delta\alpha\beta, \alpha\gamma\}.$$

Thus, by Proposition 2.2.2  $A$  is isomorphic to the monomial algebra  $\hat{A} = K\Gamma/\langle \hat{\rho} \rangle$ .

Given any admissible ordering  $<$  on  $B(\Gamma)$ , we have that  $\alpha\beta\gamma > \alpha\gamma$ . Calculating the

reduced Gröbner basis  $\mathcal{G}$  for  $I$  with respect to  $<$ , we obtain that  $\mathcal{G} = \rho \cup \{\delta\alpha\gamma\}$ . Then

$$\Gamma_2 = \{\alpha^2, \beta\alpha, \beta^2, \gamma\delta, \delta\beta, \delta\gamma, \delta\alpha\beta, \alpha\beta\gamma, \delta\alpha\gamma\}.$$

Now  $\delta\alpha\beta\gamma \in \Gamma_3^{(2)}$ . We calculate the image of  $\delta\alpha\beta\gamma \otimes e_2$  under  $\delta_3$  as follows:

$$\begin{aligned} \delta_3(\delta\alpha\beta\gamma \otimes e_2) &= \delta\alpha\beta \otimes \gamma - \eta_2\delta_2(\delta\alpha\beta \otimes \gamma) \\ &= \delta\alpha\beta \otimes \gamma - \eta_2(\delta \otimes \alpha\beta\gamma - (\eta_1\delta_1(\delta \otimes \alpha\beta))\gamma) \\ &= \delta\alpha\beta \otimes \gamma - \eta_2(\delta \otimes \alpha\gamma - (\eta_1(v_2 \otimes \delta\alpha\beta))\gamma) \\ &= \delta\alpha\beta \otimes \gamma - \eta_2(\delta \otimes \alpha\gamma) \\ &= \delta\alpha\beta \otimes \gamma - \delta\alpha\gamma \otimes e_2 - \eta_2(\delta \otimes \alpha\gamma - \delta_2(\delta\alpha\gamma \otimes e_2)) \\ &= \delta\alpha\beta \otimes \gamma - \delta\alpha\gamma \otimes e_2 - \eta_2\eta_1\delta_1(\delta \otimes \alpha\gamma) \\ &= \delta\alpha\beta \otimes \gamma - \delta\alpha\gamma \otimes e_2 \end{aligned}$$

Then we see that  $\delta_3(\delta\alpha\beta\gamma \otimes e_2)$  is not in the radical of  $K\Gamma_2^{(2)} \otimes_R A$ ; consequently, the Anick-Green resolution for  $S_2$  is not minimal.

## 3.2 Associated monomial algebras

Let  $I$  be an admissible ideal in  $K\Gamma$  and  $<$  be an admissible ordering on  $B(\Gamma)$ . We recall that  $\text{MinTip}(I)$  is the set of tips of minimal sharp elements in  $I$  with respect to  $<$ . The monomial algebra  $\bar{A} = K\Gamma / \langle \text{MinTip}(I) \rangle$  is called the *associated monomial algebra* of  $A = K\Gamma / I$  with respect to  $<$  (see [3]). Generally speaking, we wish to know the connections between  $A$  and its associated monomial algebra  $\bar{A}$ . Anick and Green [3, Propositions 3.1 and 3.2] proved that  $A$  and  $\bar{A}$  have the same Cartan matrix and that their global dimensions have the same bound in the sense that the global dimension of  $A$  is bounded by some integer  $d$  if and only if the global dimension of

$\bar{A}$  is bounded by  $d$ . In this section we answer the following question: Supposing that  $A_1 = K\Gamma/I_1$  and  $A_2 = K\Gamma/I_2$  are two isomorphic quotients of path algebra, does there exist an admissible ordering  $<$  on  $B(\Gamma)$  such that  $A_1$  and  $A_2$  have isomorphic associated monomial algebras  $\bar{A}_1$  and  $\bar{A}_2$  with respect to  $<$ , respectively? Indeed, we shall construct a counter-example in which a quotient  $A = K\Gamma/I$  of path algebra is of monomial presentation type but not isomorphic to any of its associated monomial algebras.

**Example 3.3** Let  $K$  be a field of characteristic  $\neq 2$ . Define  $\Gamma$  to be

$$\begin{array}{ccccc} & \xrightarrow{\alpha} & & \xrightarrow{\gamma} & \\ 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 \\ & \xrightarrow{\beta} & & \xrightarrow{\delta} & \end{array}$$

Figure 3.3

and let

$$\rho = \{\alpha\gamma - \beta\delta, \alpha\delta - \beta\gamma\}.$$

Then  $A = K\Gamma/\langle \rho \rangle$  is of monomial presentation type but not isomorphic to its associated monomial algebra  $\hat{A}$  with respect to any admissible ordering  $<$  on  $B(\Gamma)$ .

As we showed in Example 2.5,  $A$  is isomorphic to the monomial algebra  $\hat{A} = K\Gamma/\langle \hat{\rho} \rangle$ , where  $\hat{\rho} = \{\alpha\gamma, \beta\delta\}$ .

Given any admissible ordering  $<$  on  $B(\Gamma)$ , we have the following four cases:

Case 1.  $\alpha\gamma > \beta\delta, \alpha\delta > \beta\gamma$ . The associated monomial algebra of  $A$  with respect to  $<$  is  $\bar{A} = K\Gamma/\langle \bar{\rho} \rangle$ , where  $\bar{\rho} = \{\alpha\gamma, \alpha\delta\}$ ;

Case 2.  $\alpha\gamma > \beta\delta, \beta\gamma > \alpha\delta$ . The associated monomial algebra of  $A$  with respect to  $<$  is  $\bar{A} = K\Gamma/\langle \bar{\rho} \rangle$ , where  $\bar{\rho} = \{\alpha\gamma, \beta\gamma\}$ ;

Case 3.  $\beta\delta > \alpha\gamma, \alpha\delta > \beta\gamma$ . The associated monomial algebra of  $A$  with respect to  $<$  is  $\bar{A} = K\Gamma/\langle \bar{\rho} \rangle$ , where  $\bar{\rho} = \{\beta\delta, \alpha\delta\}$ ;

Case 4.  $\beta\delta > \alpha\gamma$ ,  $\beta\gamma > \alpha\delta$ . The associated monomial algebra of  $A$  with respect to  $\langle \cdot \rangle$  is  $\bar{A} = K\Gamma / \langle \bar{\rho} \rangle$ , where  $\bar{\rho} = \{\beta\delta, \beta\gamma\}$ .

We observe that  $\Gamma$  has the following four automorphisms as a finite directed graph:

$$\begin{aligned} \sigma_1: 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, \alpha \mapsto \alpha, \beta \mapsto \beta, \gamma \mapsto \gamma, \delta \mapsto \delta; \\ \sigma_2: 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, \alpha \mapsto \beta, \beta \mapsto \alpha, \gamma \mapsto \gamma, \delta \mapsto \delta; \\ \sigma_3: 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, \alpha \mapsto \alpha, \beta \mapsto \beta, \gamma \mapsto \delta, \delta \mapsto \gamma; \\ \sigma_4: 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, \alpha \mapsto \beta, \beta \mapsto \alpha, \gamma \mapsto \delta, \delta \mapsto \gamma. \end{aligned}$$

Assume that  $A$  is isomorphic to  $\bar{A} = K\Gamma / \langle \bar{\rho} \rangle$ . Since  $A$  is also isomorphic to  $\hat{A} = K\Gamma / \langle \hat{\rho} \rangle$ , by Proposition 2.2.2 there exist some automorphism  $\sigma$  of  $\Gamma$  and some change of variable  $T$  on  $K\Gamma$  such that

$$\langle T(\hat{\rho}^\sigma) \rangle = \langle \bar{\rho} \rangle. \quad (3.2.1)$$

Here we note that  $A$  is of Loewy length 3 and  $\Gamma_1^3 = \{0\}$ . Now

$$\hat{\rho}^{\sigma_1} = \hat{\rho}^{\sigma_4} = \{\alpha\gamma, \beta\delta\}$$

and

$$\hat{\rho}^{\sigma_2} = \hat{\rho}^{\sigma_3} = \{\alpha\delta, \beta\gamma\}.$$

Since  $T$  is a change of variable on  $K\Gamma$ , there exist  $M, N \in GL_2(K)$  such that

$$T[\alpha, \beta] = [\alpha, \beta]M$$

and

$$T[\gamma, \delta] = [\gamma, \delta]N.$$

Write

$$M = \begin{bmatrix} \lambda_1 & \lambda_4 \\ \lambda_3 & \lambda_2 \end{bmatrix}, \quad N = \begin{bmatrix} \mu_1 & \mu_4 \\ \mu_3 & \mu_2 \end{bmatrix}.$$

In the following, we shall consider only the first case and prove in this case that  $A$  is not isomorphic to its associated monomial algebra  $\bar{A}$ . The proof for the other cases is the same.

(i) Suppose that  $\tilde{\rho}^\sigma = \{\alpha\gamma, \beta\delta\}$ . Then

$$\begin{aligned} T(\alpha\gamma) &= (\lambda_1\alpha + \lambda_3\beta)(\mu_1\gamma + \mu_3\delta) \\ &= \lambda_1\mu_1\alpha\gamma + \lambda_1\mu_3\alpha\delta + \lambda_3\mu_1\beta\gamma + \lambda_3\mu_3\beta\delta, \\ T(\beta\delta) &= (\lambda_4\alpha + \lambda_2\beta)(\mu_4\gamma + \mu_2\delta) \\ &= \lambda_4\mu_4\alpha\gamma + \lambda_4\mu_2\alpha\delta + \lambda_2\mu_4\beta\gamma + \lambda_2\mu_2\beta\delta. \end{aligned}$$

Thus, from (3.2.1) one obtains the following relations

$$\begin{cases} \lambda_3\mu_1 = 0 \\ \lambda_3\mu_3 = 0 \\ \lambda_2\mu_4 = 0 \\ \lambda_2\mu_2 = 0. \end{cases}$$

Since  $N \in \text{GL}_2(K)$ , it follows from the system of equations above that  $\lambda_3 = 0$  and  $\lambda_2 = 0$ , which contradicts the fact that  $M \in \text{GL}_2(K)$ .

(ii) Suppose that  $\tilde{\rho}^\sigma = \{\alpha\delta, \beta\gamma\}$ . Then

$$\begin{aligned} T(\alpha\delta) &= (\lambda_1\alpha + \lambda_3\beta)(\mu_4\gamma + \mu_2\delta) \\ &= \lambda_1\mu_4\alpha\gamma + \lambda_1\mu_2\alpha\delta + \lambda_3\mu_4\beta\gamma + \lambda_3\mu_2\beta\delta, \\ T(\beta\gamma) &= (\lambda_4\alpha + \lambda_2\beta)(\mu_1\gamma + \mu_3\delta) \\ &= \lambda_4\mu_1\alpha\gamma + \lambda_4\mu_3\alpha\delta + \lambda_2\mu_1\beta\gamma + \lambda_2\mu_3\beta\delta. \end{aligned}$$

Thus, from (3.2.1) one obtains the following relations

$$\begin{cases} \lambda_3\mu_4 = 0 \\ \lambda_3\mu_2 = 0 \\ \lambda_2\mu_1 = 0 \\ \lambda_2\mu_3 = 0. \end{cases}$$

Again since  $N \in \text{GL}_2(K)$ , it follows from the system of equations above that  $\lambda_3 = 0$  and  $\lambda_2 = 0$ , which contradicts the fact that  $M \in \text{GL}_2(K)$ . Therefore,  $A$  is not isomorphic to its associated monomial algebra  $\bar{A}$ .

# Chapter 4

## Right monomial rings

As a generalization of monomial algebras, one-sided monomial rings were introduced by Burgess *et al.* [6]. As the authors showed in the article, one-sided monomial rings include many important classes of rings, such as monomial algebras, hereditary artinian rings and one-sided (almost) serial rings. Associated with every one-sided monomial ring there is a monomial algebra which turns out to possess many common homological properties with the original one-sided monomial ring; for example, they have the same projective dimensions of the corresponding simple modules. Furthermore, the authors conjectured in the article that if a split basic finite dimensional algebra  $A$  is a one-sided monomial ring, then it is of monomial presentation type. One main task of this chapter is to demonstrate the truth of this conjecture in some special cases.

### 4.1 Tree subsets and right monomial rings

In this section we shall recall some basic notions and results concerning one-sided monomial rings (see [6]). Although one-sided monomial rings, as indicated in [6], are artinian rings which need not be algebras over a field, we shall throughout the present

chapter confine ourselves on the class of quotients of path algebras by admissible ideals.

Let  $K$  be a field and  $\Gamma$  be a connected finite directed graph. Let  $I$  be an admissible ideal of the path algebra  $K\Gamma$  and let  $A = K\Gamma/I$ . Throughout this chapter we shall denote by  $E = \{e_1, \dots, e_n\}$  any fixed basic set of orthogonal primitive idempotents of  $A$ . A typical choice of such a basic set is the set of idempotents of  $A$  corresponding to the vertices  $v_1, \dots, v_n$  in  $\Gamma$ .  $\text{Rad}A$  will denote the Jacobson radical of  $A$  and  $S_i = e_iA/e_i(\text{Rad}A)$ ,  $i = 1, \dots, n$ , the simple right  $A$ -modules corresponding to  $E$ .

A finite directed graph  $\mathcal{M}$  with a distinguished vertex  $0$  is said to be a *module diagram* if the following conditions are satisfied:

- (1)  $\mathcal{M}$  is square-free; (i.e., for any vertices  $x, y$  in  $\mathcal{M}$ , there is at most one arrow  $x \rightarrow y$  in  $\mathcal{M}$ ;) )
- (2)  $\mathcal{M}$  is strongly acyclic; (i.e., for any vertices  $x, y$  in  $\mathcal{M}$ , if there is an arrow  $x \rightarrow y$  in  $\mathcal{M}$ , then there is no path of length  $\geq 2$  from  $x$  to  $y$  in  $\mathcal{M}$ ;) )
- (3) for any nonzero vertex  $x$  in  $\mathcal{M}$ , there is an arrow  $x \rightarrow 0$  in  $\mathcal{M}$  if and only if there is no arrow  $x \rightarrow y \neq 0$  in  $\mathcal{M}$ .

With an abuse of notation, we shall also denote by  $\mathcal{M}$  the set of vertices in the module diagram  $\mathcal{M}$ . The symbol  $\mathcal{M}^*$  will stand for  $\mathcal{M} - \{0\}$ .

Let  $\mathcal{M}$  be a module diagram. A subgraph  $\mathcal{U}$  of  $\mathcal{M}$  is a *subdiagram* of  $\mathcal{M}$ , denoted by  $\mathcal{U} \leq \mathcal{M}$ , if for any vertex  $x$  in  $\mathcal{U}$ ,  $x \rightarrow y$  in  $\mathcal{M}$  implies  $x \rightarrow y$  in  $\mathcal{U}$ . The *radical* of  $\mathcal{M}$  is defined as

$$\text{Rad}\mathcal{M} = \{y \in \mathcal{M} \mid x \rightarrow y \text{ in } \mathcal{M} \text{ for some } x \in \mathcal{M}\},$$

which is a subdiagram of  $\mathcal{M}$ . The set  $\mathcal{M} - \text{Rad}\mathcal{M}$  is called the *top* of  $\mathcal{M}$  and denoted by  $\mathcal{M}^\tau$ . For  $m \geq 1$ , the *m-th radical* of  $\mathcal{M}$  is defined recursively as

$$\text{Rad}^m\mathcal{M} = \text{Rad}(\text{Rad}^{(m-1)}\mathcal{M}),$$

where  $\text{Rad}^0 \mathcal{M} = \mathcal{M}$ . The set of all subdiagrams of  $\mathcal{M}$  forms a complete lattice under intersection and union of subdiagrams, which will be denoted by  $\mathcal{L}(\mathcal{M})$ . Given any subset  $S \subseteq \mathcal{M}$ , then  $S$  generates a smallest subdiagram  $\mathcal{U}(S)$  of  $\mathcal{M}$  containing  $S$ . The module diagram  $\mathcal{M}$  is called *local* if it is generated by one vertex.

Let  $M$  be a finitely generated right  $A$ -module. A module diagram  $\mathcal{M}$  together with a pair of maps  $\delta: \mathcal{L}(\mathcal{M}) \rightarrow \mathcal{L}(M)$ , where  $\mathcal{L}(M)$  is the lattice of submodules of  $M$ , and  $\lambda: \mathcal{M}^* \rightarrow \{1, \dots, n\}$ , which is called a *labelling* on  $\mathcal{M}^*$ , is said to be a *diagram for  $M$*  if the following conditions are satisfied:

- (M0)  $\delta$  is an injective homomorphism of lattices;
- (M1)  $\text{Card}(\mathcal{M}^*) = c(M)$ , where  $c(M)$  is the composition length of  $M$ ;
- (M2)  $\delta(\text{Rad } \mathcal{U}) = \text{Rad}(\delta(\mathcal{U}))$  for all  $\mathcal{U} \in \mathcal{L}(\mathcal{M})$ ;
- (M3)  $\delta(\mathcal{V})/\delta(\mathcal{U}) \cong S_{\lambda(x)}$  whenever  $\mathcal{U} < \mathcal{V}$  in  $\mathcal{L}(\mathcal{M})$  and  $\mathcal{V} = \mathcal{U} \cup \{x\}$ .

Let  $M \neq 0$  be a finitely generated right  $A$ -module. A nonempty finite subset  $\mathcal{X}$  of  $M^*$  is said to be a *normed subset* of  $M$  if the following conditions are satisfied:

- (1) for any  $x \in \mathcal{X}$ , there exists  $j \in \{1, \dots, n\}$  such that  $xe_j = x$  (such an element is called *normed*);
- (2) for any  $x, y \in \mathcal{X}$ ,  $x \neq y$  implies  $xA \neq yA$ .

Let  $\mathcal{X}$  be a normed subset of  $M$ . Then  $\mathcal{X}$  can be associated to a module diagram  $\mathcal{M}(\mathcal{X})$  in such a way that  $\mathcal{M}(\mathcal{X})$  has  $\mathcal{X}$  as the set of nonzero vertices and for any  $x, y \in \mathcal{X}$ ,  $x \rightarrow y$  in  $\mathcal{M}(\mathcal{X})$  if and only if

- (i)  $yA \subset xA$ , and
- (ii) for any  $z \in \mathcal{X}$ ,  $yA \subseteq zA \subseteq xA$  implies  $z = x$  or  $z = y$ .

In the sequel, we shall identify a normed subset  $\mathcal{X}$  with its associated module diagram  $\mathcal{M}(\mathcal{X})$ .

**Definition 4.1** Let  $\mathcal{X}$  be a normed subset of a nonzero finitely generated right  $A$ -module  $M$ . The subset  $\mathcal{X}$  is called a *tree subset* of  $M$  if the following conditions are satisfied.

(1)  $(\mathcal{X}, \delta, \lambda)$  is a diagram for  $M$ , where  $\delta: \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(M)$  is defined by  $\delta(\mathcal{U}) = \mathcal{U}A$  for all  $\mathcal{U} \in \mathcal{X}$  and  $\lambda: \mathcal{X} \rightarrow \{1, \dots, n\}$  by  $\lambda(x) = j$  if  $x \in \mathcal{X}$  and  $x = xe_j$ ;

(2)  $\mathcal{X} = \mathcal{U}(x_1) \dot{\cup} \dots \dot{\cup} \mathcal{U}(x_m)$  is a disjoint union of local subdiagrams such that each  $\mathcal{U}(x_i)$  is a rooted tree with root  $x_i$  in the sense that for any  $x \in \mathcal{U}(x_i)$  there exists a unique path from  $x_i$  to  $x$  in  $\mathcal{U}(x_i)$ .

The following characterization of tree subsets is found in [6, Proposition 1.2].

**Proposition 4.1.1** *Let  $\mathcal{X}$  be a normed subset of a finitely generated right  $A$ -module  $M$  of Loewy length  $L$ . Then  $\mathcal{X}$  is a tree subset of  $M$  if and if  $\mathcal{X}$  can be expressed as a disjoint union  $\mathcal{X} = \mathcal{Y}_0 \dot{\cup} \dots \dot{\cup} \mathcal{Y}_{L-1}$  such that*

$$(1) M = \bigoplus_{y \in \mathcal{Y}_0} yA;$$

(2) for each  $l$  with  $0 \leq l \leq L-1$  and each  $x \in \mathcal{Y}_l$ , there exists a subset  $\mathcal{Y}_{lx} \subseteq \mathcal{Y}_{l+1}$  such that  $\mathcal{Y}_{l+1} = \bigcup_{x \in \mathcal{Y}_l} \mathcal{Y}_{lx}$  and  $x(\text{Rad}A) = \bigoplus_{y \in \mathcal{Y}_{lx}} yA$ .

Furthermore, under these conditions,  $M(\text{Rad}^l A) = \bigoplus_{y \in \mathcal{Y}_l} yA$  for all  $l = 1, \dots, L-1$ .

We now state the definition of a right monomial ring (see [6, Definition 2.2]).

**Definition 4.2** Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and let  $\mathcal{X}$  be a right tree subset of  $A$  (i.e., a tree subset of  $A$  as right  $A$ -module) with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$ . The pair  $(A, \mathcal{X})$  is said to be a *right monomial ring* with respect to  $E$  if every element of  $\mathcal{X}$  satisfies the annihilator condition with respect to  $\mathcal{X}$  in the sense that for any  $x \in e_i \mathcal{X} e_j$ ,  $1 \leq i, j \leq n$ , there exists some subdiagram  $\mathcal{A}(x) \leq e_j \mathcal{X}$  such that  $\text{rann}_{e_j \mathcal{A}}(x) = \mathcal{A}(x)A$ , where  $\text{rann}_{e_j \mathcal{A}}(x)$  denotes the right annihilator of  $x$  in  $e_j A$ .

The following result, which also comes from [6], provides a sufficient condition such that there exists a right monomial ring  $(A, \mathcal{Y})$  where  $\mathcal{Y} \cup \{0\}$  is a multiplicative semigroup in  $A$ .

**Proposition 4.1.2** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Let  $\mathcal{Y}_1 \subseteq \bigcup_{i,j=1}^n e_i(\text{Rad}A)e_j - \{0\}$  be a finite set. Define  $\mathcal{Y}_0 = E$  and  $\mathcal{Y}_l = \mathcal{Y}_1^l - \{0\}$  for  $l = 1, \dots, L-1$ , where  $L$  is the Loewy length of  $A$ . Assume the following two conditions are satisfied:*

(1) *if  $x_1, \dots, x_r$  and  $y_1, \dots, y_s$  belong to  $\mathcal{Y}_1$  and  $x_1 \cdots x_r = y_1 \cdots y_s \neq 0$ , then  $r = s$  and  $x_i = y_i$  for all  $i = 1, \dots, r$ ;*

(2)  *$(\text{Rad}A)^l = \bigoplus_{y \in \mathcal{Y}_l} yA$  for all  $l = 1, \dots, L-1$ .*

*Set  $\mathcal{Y} = \bigcup_{l=0}^{L-1} \mathcal{Y}_l$ . Then  $\mathcal{Y} \cup \{0\}$  is a multiplicative semigroup in  $A$  such that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ .*

**Proof.** See [6, Proposition 4.1]. ■

A finite (multiplicative) semigroup  $\mathcal{S}$  with zero is said to be an *algebra semigroup* (see [13]) if there is a set  $\{e_1, \dots, e_n\}$  of nonzero orthogonal idempotents in  $\mathcal{S}$  such that

(1)  $\mathcal{S} = \bigcup_{i,j=1}^n e_i \mathcal{S} e_j$ , and

(2)  $\mathcal{N} = \mathcal{S} - \{e_1, \dots, e_n\}$  is a nilpotent ideal of  $\mathcal{S}$ .

The nilpotent ideal  $\mathcal{N}$  is called the *radical* of the algebra semigroup  $\mathcal{S}$ . The semigroup algebra  $K[\mathcal{S}^*]$  of  $\mathcal{S}$  over  $K$  is called the *diagram algebra* of  $\mathcal{S}$  over  $K$  (see [13]).

Given any algebra semigroup  $\mathcal{S}$  with radical  $\mathcal{N}$ , then  $\mathcal{S}$  induces a module diagram  $\mathcal{M}(\mathcal{S})$  in such a way that  $\mathcal{M}(\mathcal{S})$  has  $\mathcal{S}$  as the set of vertices,  $x \rightarrow y \neq 0$  in  $\mathcal{M}(\mathcal{S})$  if and only if  $xa = y \neq 0$  for some  $a \in \mathcal{N} - \mathcal{N}^2$ , and  $0 \neq x \rightarrow 0$  in  $\mathcal{M}(\mathcal{S})$  if and only if  $x \neq 0$  and  $xa = 0$  for all  $a \in \mathcal{N} - \mathcal{N}^2$ . Furthermore,  $(\mathcal{M}(\mathcal{S}), \delta, \lambda)$  forms a diagram for the right  $K[\mathcal{S}^*]$ -module  $K[\mathcal{S}^*]_{K[\mathcal{S}^*]}$ , where  $\delta: \mathcal{L}(\mathcal{M}(\mathcal{S})) \rightarrow \mathcal{L}(K[\mathcal{S}^*]_{K[\mathcal{S}^*]})$  is defined via  $\delta(\mathcal{U}) = K\mathcal{U}$  and  $\lambda: \mathcal{M}(\mathcal{S}) \rightarrow \{1, \dots, n\}$  via  $\lambda(x) = j$  if  $x = xe_j$ , where  $\{e_1, \dots, e_n\}$  is the set of orthogonal idempotents in  $\mathcal{S}$ .

The following result is due to Fuller (see [13, Theorem 4.2]).

**Proposition 4.1.3** *Let  $A = K\Gamma/I$ . Then  $A$  is of monomial presentation type if and only if  $A$  is isomorphic to the diagram algebra  $K[\mathcal{Y}]$  for some algebra semigroup  $\mathcal{Y} \cup \{0\}$  satisfying the condition that if  $x_1, \dots, x_r, y_1, \dots, y_s \in \mathcal{N} - \mathcal{N}^2$  with  $x_1 \cdots x_r = y_1 \cdots y_s \neq 0$ , where  $\mathcal{N}$  is the radical of  $\mathcal{Y} \cup \{0\}$ , then  $r = s$  and  $x_i = y_i$  for all  $i = 1, \dots, r$ .*

## 4.2 Loewy length 4 and acyclic finite directed graphs

Let  $A = K\Gamma/I$  be a quotient of path algebra by admissible ideal. In [6, Proposition 4.3], it is shown that if there is a right tree subset  $\mathcal{X}$  of  $A$  such that  $(A, \mathcal{X})$  is a right monomial ring and if  $A$  is of Loewy length 3, then  $A$  is of monomial presentation type. Furthermore, the authors conjectured that the result holds for any Loewy length. In this section we shall extend the result of these authors to the cases of Loewy length 4 and of acyclic finite directed graph.

We start with several lemmas.

**Lemma 4.2.1** *Let  $A = K\Gamma/I$  and  $\mathcal{X}$  be a tree subset of a finitely generated right  $A$ -module  $M$ . Then  $\mathcal{X}$  is a  $K$ -vector space basis for  $M$ .*

**Proof.** Since  $\mathcal{X}$  is a tree subset of  $M$ , it follows that  $c(M) = \text{Card}(\mathcal{X})$ . On the other hand, since  $A$  is a split basic  $K$ -algebra, every simple right  $A$ -module is one dimensional, and so we have that  $\dim_K M = c(M)$ . Then  $\dim_K M = \text{Card}(\mathcal{X})$ . We need only to show that  $\mathcal{X}$  is  $K$ -linearly independent. Since  $M = x_1A \oplus \cdots \oplus x_mA$ , where  $\mathcal{X}^\tau = \{x_1, \dots, x_m\}$ , with each  $\mathcal{U}(x_i)$  a tree subset of  $x_iA$ , it suffices to show that each  $\mathcal{U}(x_i)$  is  $K$ -linearly independent. Now

$$\mathcal{U}(x_i) = \{x_i\} \cup (\text{Rad} \mathcal{U}(x_i))^\tau \cup \cdots \cup (\text{Rad}^{(L_i-1)} \mathcal{U}(x_i))^\tau,$$

where  $L_i$  is the Loewy length of  $x_i A$ . Suppose that we are given

$$\lambda^{(0)} x_i + \sum_{w \in (\text{Rad } \mathcal{U}(x_i))^\tau} \lambda_w^{(1)} w + \cdots + \sum_{w \in (\text{Rad}^{(L_i-1)} \mathcal{U}(x_i))^\tau} \lambda_w^{(L_i-1)} w = 0 \quad (4.2.1)$$

where  $\lambda^{(0)}, \lambda_w^{(1)}, \dots, \lambda_w^{(L_i-1)}$  are scalars in  $K$ . Then  $\lambda^{(0)} x_i \in (\text{Rad } \mathcal{U}(x_i)) A = \text{Rad}(x_i A)$ , and so by Nakayama's lemma,  $\lambda^{(0)} = 0$ . Set  $(\text{Rad } \mathcal{U}(x_i))^\tau = \{w_1, \dots, w_t\}$ . Then

$$\text{Rad}(x_i A) = w_1 A \oplus \cdots \oplus w_t A = \mathcal{U}(w_1) A \oplus \cdots \oplus \mathcal{U}(w_t) A. \quad (4.2.2)$$

From (4.2.1) and (4.2.2) we see that for all  $j = 1, \dots, t$ ,

$$\lambda_{w_j}^{(1)} w_j + \sum_{z \in \mathcal{U}(w_j) \cap (\text{Rad}^2 \mathcal{U}(x_i))^\tau} \lambda_z^{(2)} z + \cdots + \sum_{z \in \mathcal{U}(w_j) \cap (\text{Rad}^{(L_i-1)} \mathcal{U}(x_i))^\tau} \lambda_z^{(L_i-1)} z = 0$$

and hence  $\lambda_{w_j}^{(1)} w_j \in (\text{Rad } \mathcal{U}(w_j)) A = \text{Rad}(w_j A)$ ; hence  $\lambda_{w_j}^{(1)} = 0$  for all  $j$ . Applying induction on  $l$ ,  $1 \leq l \leq L_i - 1$ , and the same argument as above, we obtain that  $\lambda_w^l = 0$ , for all  $l$  and all  $w \in (\text{Rad}^l \mathcal{U}(x_i))^\tau$ . Thus  $\mathcal{U}(x_i)$  is  $K$ -linearly independent. The proof is complete. ■

**Lemma 4.2.2** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and suppose that  $\mathcal{X}$  is a right tree subset of  $A$  with*

$$E \subseteq \mathcal{X} \quad \text{and} \quad \mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}.$$

*Given any nonzero element  $x \in e_i A e_j$  (not necessarily an element of  $\mathcal{X}$ ) where  $1 \leq i, j \leq n$ , if  $x$  satisfies the annihilator condition with respect to  $\mathcal{X}$ , i.e.,  $\text{rann}_{e_j A}(x) = A(x)A$  for some  $A(x) \leq e_j \mathcal{X}$ , then*

$$x(\text{Rad } A) = \bigoplus_{\substack{y \in e_j(\text{Rad } \mathcal{X})^\tau \\ xy \neq 0}} xy A.$$

**Proof.** Consider the  $A$ -homomorphism  $\theta : e_j A \rightarrow xA$  by  $\theta(e_j) = x$ . Then  $\theta$  restricts to an  $A$ -epimorphism

$$\theta_1 = \theta|_{e_j(\text{Rad } A)} : e_j(\text{Rad } A) \rightarrow x(\text{Rad } A).$$

Since

$$\ker \theta = \text{rann}_{e_j A}(x) = \mathcal{A}(x)A \subseteq \text{Rad}(e_j \mathcal{X})A = e_j(\text{Rad}A),$$

it follows that  $\ker \theta_1 = \mathcal{A}(x)A$ . Set  $\mathcal{Y} = \text{Rad}(e_j \mathcal{X}) - \mathcal{A}(x)$  and let  $\mathcal{Z}$  be the subdiagram of  $\text{Rad}(e_j \mathcal{X})$  generated by  $\mathcal{Y}$ . Then  $\theta_1$  restricts to an  $A$ -epimorphism

$$\theta_2 = \theta_1 |_{\mathcal{Z}A}: \mathcal{Z}A \rightarrow x(\text{Rad}A)$$

with kernel  $(\mathcal{Z} \cap \mathcal{A}(x))A$ . Thus  $\theta_2$  induces an  $A$ -isomorphism

$$\tilde{\theta}_2: \bar{\mathcal{Z}}A \rightarrow x(\text{Rad}A),$$

where

$$\bar{\mathcal{Z}} = (\mathcal{Z} - \mathcal{A}(x)) + (\mathcal{Z} \cap \mathcal{A}(x))A \subseteq A/(\mathcal{Z} \cap \mathcal{A}(x))A.$$

From the definition of  $\mathcal{Y}$  and  $\mathcal{Z}$ , we have that

$$\mathcal{Z}A = \bigoplus_{y \in \mathcal{Y}^r} yA.$$

It follows that

$$\bar{\mathcal{Z}}A = \sum_{y \in \mathcal{Y}^r} (y + (\mathcal{Z} \cap \mathcal{A}(x))A)A. \quad (4.2.3)$$

We claim that the sum in the equation (4.2.3) is direct. In fact, suppose that we are given

$$(y_1 + (\mathcal{Z} \cap \mathcal{A}(x))A)a_1 + \cdots + (y_t + (\mathcal{Z} \cap \mathcal{A}(x))A)a_t = 0,$$

where  $\mathcal{Y}^r = \{y_1, \dots, y_t\}$  and  $a_i \in A$  for all  $i = 1, \dots, t$ . Then

$$y_1 a_1 + \cdots + y_t a_t \in (\mathcal{Z} \cap \mathcal{A}(x))A.$$

Now we have that

$$\mathcal{Z} \cap \mathcal{A}(x) = (\mathcal{U}(y_1) \cap \mathcal{A}(x)) \dot{\cup} \cdots \dot{\cup} (\mathcal{U}(y_t) \cap \mathcal{A}(x))$$

is a disjoint union. It follows that

$$(\mathcal{Z} \cap \mathcal{A}(x))A = (\mathcal{U}(y_1) \cap \mathcal{A}(x))A \oplus \cdots \oplus (\mathcal{U}(y_t) \cap \mathcal{A}(x))A.$$

Thus for each  $y_i \in \mathcal{Y}^\tau$ ,  $y_i a_i \in (\mathcal{U}(y_i) \cap \mathcal{A}(x))A \subseteq (\mathcal{Z} \cap \mathcal{A}(x))A$ , which shows that the sum in (4.2.3) is direct. So

$$x(\text{Rad}A) = \widetilde{\theta}_2(\widetilde{\mathcal{Z}}A) = \bigoplus_{y \in \mathcal{Y}^\tau} xyA = \bigoplus_{\substack{y \in e_j(\text{Rad}\mathcal{X})^\tau \\ xy \neq 0}} xyA. \blacksquare$$

**Lemma 4.2.3** *Let  $A = K\Gamma/I$  be of Loewy length  $L$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Suppose that  $\mathcal{X}$  is a right tree subset of  $A$  with*

$$E \subseteq \mathcal{X} \quad \text{and} \quad \mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}.$$

Set

$$\mathcal{Y} = E \cup \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_{L-1},$$

where  $\mathcal{Y}_l = ((\text{Rad}\mathcal{X})^\tau)^l - \{0\}$  for  $l = 1, \dots, L-1$ . If every element of  $\mathcal{Y}$  satisfies the annihilator condition with respect to  $\mathcal{X}$ , then

(a)

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A$$

for all  $l = 1, \dots, L-1$ ;

(b) for any  $l$  with  $1 \leq l \leq L-1$ , if  $x_1 \cdots x_l = y_1 \cdots y_l \neq 0$  with each  $x_i$  and  $y_i$  in  $(\text{Rad}\mathcal{X})^\tau$ , then  $x_i = y_i$  for all  $i = 1, \dots, l$ .

(c)  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ .

**Proof.** (a) We do an induction on  $l$ . Since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows that

$$\text{Rad}A = \bigoplus_{x_1 \in (\text{Rad}\mathcal{X})^\tau} x_1 A.$$

Assume that

$$(\text{Rad}A)^{l-1} = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1} A,$$

where  $l \geq 2$ . Then

$$\begin{aligned} (\text{Rad}A)^l &= \left( \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1} A \right) (\text{Rad}A) \\ &= \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1} (\text{Rad}A) \\ &= \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} \left( \bigoplus_{\substack{x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A \right) \quad (\text{by Lemma 4.2.2}) \\ &= \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A. \end{aligned}$$

(b) From (a) we see that for any  $l$ ,  $1 \leq l \leq L - 1$ , if  $x_1, \dots, x_l, y_1, \dots, y_l$  lie in  $(\text{Rad}\mathcal{X})^\tau$  and  $x_1 \cdots x_l = y_1 \cdots y_l \neq 0$ , then  $x_i = y_i$  for  $i = 1, \dots, l$ . Thus, (b) follows.

(c) This is from (a), (b) and Proposition 4.1.2. ■

The following proposition provides a characterization of the monomial presentation type in terms of multiplicative semigroups and annihilator conditions with respect to tree subsets.

**Proposition 4.2.4** *Let  $A = K\Gamma/I$ . The following statements are equivalent:*

- (1) *A is of monomial presentation type;*
- (2) *There exist a basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents of A and a right tree subset  $\mathcal{X}$  of A with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$ , such that  $\mathcal{X} \cup \{0\}$  is a multiplicative semigroup in A and that  $(A, \mathcal{X})$  is a right monomial ring with respect to E;*
- (3) *There exist a basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents of A and a right tree subset  $\mathcal{X}$  of A with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$ , such that  $\mathcal{X} \cup \{0\}$*

is a multiplicative semigroup in  $A$  and that every element of  $(\text{Rad}\mathcal{X})^\tau$  satisfies the annihilator condition with respect to  $\mathcal{X}$ .

**Proof.** (1)  $\Rightarrow$  (2). Suppose that  $A$  is isomorphic to  $\hat{A} = K\Gamma/\hat{I}$ , where  $\hat{I}$  is a monomial admissible ideal in  $K\Gamma$ . Let  $P$  denote the set of nonzero paths in  $\Gamma$  modulo  $\hat{I}$ . Then it is easy to check that  $\bar{P} \cup \{0\}$  is a multiplicative semigroup in  $\hat{A}$  such that  $(\hat{A}, \bar{P} \cup \{0\})$  is a right monomial ring with respect to  $\bar{\Gamma}_0$ , where  $\bar{P}$  and  $\bar{\Gamma}_0$  are the images of  $P$  and  $\Gamma_0$  under the natural map  $K\Gamma \rightarrow \hat{A}$ , respectively. Let  $\bar{P}$  and  $\bar{\Gamma}_0$  correspond to  $\mathcal{X}$  and  $E$ , respectively, under the given  $K$ -algebra isomorphism. Then  $\mathcal{X} \cup \{0\}$  is a multiplicative semigroup in  $A$  such that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ .

(2)  $\Rightarrow$  (3). This is obvious from the definition of a right monomial ring.

(3)  $\Rightarrow$  (1). Since  $\mathcal{X}$  is a right tree subset of  $A$ , by Lemma 4.2.1  $\mathcal{X}$  is a  $K$ -vector space basis for  $A$ ; hence from the condition that  $\mathcal{X} \cup \{0\}$  is a multiplicative semigroup in  $A$  it follows that  $A = K[\mathcal{X}]$  is the semigroup algebra of  $\mathcal{X} \cup \{0\}$  over  $K$ . Again since  $\mathcal{X}$  is a right tree subset of  $A$ ,  $\text{Rad}\mathcal{X}$  is a nilpotent ideal of the semigroup  $\mathcal{X} \cup \{0\}$ . Thus  $A = K[\mathcal{X}]$  is the diagram algebra of  $\mathcal{X} \cup \{0\}$  over  $K$ .

Suppose that

$$x_1 \cdots x_r = y_1 \cdots y_s \neq 0 \tag{4.2.4}$$

where  $x_i, y_j \in (\text{Rad}\mathcal{X})^\tau$ ,  $1 \leq i \leq r$ , and  $1 \leq j \leq s$ . We show by induction on  $r$  that  $r = s$  and  $x_i = y_i$  for all  $i = 1, \dots, r$ . If  $r = 1$ , then  $s = 1$  and  $x_1 = y_1$  since  $x_1 \notin (\text{Rad}\mathcal{X})^2$ . Suppose that  $r \geq 2$ . Then  $s \geq 2$ . Since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows that

$$\text{Rad}A = \bigoplus_{x \in (\text{Rad}\mathcal{X})^\tau} xA.$$

Then from (4.2.4) we see that  $x_1 = y_1$ . Since  $x_1$  satisfies the annihilator condition with respect to  $\mathcal{X}$ , there exists some module subdiagram  $\mathcal{A}(x_1)$  of  $e_j\mathcal{X}$  such that

$$x_2 \cdots x_r - y_2 \cdots y_s \in \mathcal{A}(x_1)A,$$

where  $e_j$  is such that  $x_1 = x_1 e_j$ . Assume that  $x_2 \cdots x_r \neq y_2 \cdots y_s$ . Since  $\mathcal{X} \cup \{0\}$  is closed under multiplication,  $x_2 \cdots x_r$  and  $y_2 \cdots y_s$  lie in  $e_j \mathcal{X}$ . We note that  $e_j \mathcal{X}$  is  $K$ -linearly independent and  $\mathcal{A}(x_1)$  is a  $K$ -vector space basis for  $\mathcal{A}(x_1)A$ . It follows that

$$x_2 \cdots x_r \in \mathcal{A}(x_1) \quad \text{and} \quad y_2 \cdots y_s \in \mathcal{A}(x_1).$$

Thus  $x_1 \cdots x_r = y_1 \cdots y_s = 0$ , contradicting (4.2.4). So we conclude that  $x_2 \cdots x_r = y_2 \cdots y_s$ . By the induction hypothesis, we obtain that  $r = s$  and  $x_i = y_i$  for all  $i = 2, \dots, r$ . Finally, by Proposition 4.1.3,  $A$  is of monomial presentation type. ■

We are now ready to prove the main results of this section.

**Theorem 4.2.5** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Suppose that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ . If  $A$  is of Loewy length 4, then there exists a right tree subset  $\mathcal{Y}$  of  $A$  such that  $\mathcal{Y} \cup \{0\}$  is a multiplicative semigroup in  $A$  and that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ .*

**Proof.** Define

$$\mathcal{Y} = E \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3,$$

where  $\mathcal{Y}_l = ((\text{Rad } \mathcal{X})^\tau)^l - \{0\}$  for  $l = 1, 2, 3$ . Then  $\mathcal{Y} \cup \{0\}$  forms a multiplicative semigroup in  $A$ . We shall now prove that every element of  $\mathcal{Y}$  satisfies the annihilator condition with respect to  $\mathcal{X}$  and then, by Lemma 4.2.3,  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ . Since  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$  and  $A$  is of Loewy length 4, it suffices to prove that the annihilator condition holds for every element of  $\mathcal{Y}_2$ .

First, since  $(A, \mathcal{X})$  is a right monomial ring, it follows that every element of  $(\text{Rad } \mathcal{X})^\tau$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . Then by Lemma 4.2.2

we have that

$$\begin{aligned} (\text{Rad}A)^2 &= \left( \bigoplus_{x \in (\text{Rad}\mathcal{X})^\tau} xA \right) \text{Rad}A \\ &= \bigoplus_{\substack{x, y \in (\text{Rad}\mathcal{X})^\tau \\ xy \neq 0}} xyA. \end{aligned}$$

On the other hand, since  $\mathcal{X}$  is a right tree subset of  $A$ , we have that

$$(\text{Rad}A)^2 = \bigoplus_{w \in (\text{Rad}^2\mathcal{X})^\tau} wA.$$

Thus

$$\bigoplus_{\substack{x, y \in (\text{Rad}\mathcal{X})^\tau \\ xy \neq 0}} xyA = \bigoplus_{w \in (\text{Rad}^2\mathcal{X})^\tau} wA. \quad (4.2.5)$$

Given any  $xy \in \mathcal{Y}_2$  with  $x, y \in (\text{Rad}\mathcal{X})^\tau$ , then by (4.2.5) there exist  $a_1, \dots, a_s \in A$  such that

$$xy = w_1a_1 + \dots + w_sa_s \quad (4.2.6)$$

where  $w_1, \dots, w_s \in (\text{Rad}^2\mathcal{X})^\tau$ . Also since  $(A, \mathcal{X})$  is a right monomial ring, each  $w_i$  in (4.2.6) satisfies the annihilator condition with respect to  $\mathcal{X}$ ; i.e., there exists some subdiagram  $\mathcal{A}(w_i)$  of  $e_{j(w_i)}\mathcal{X}$ , where  $w_i = w_i e_{j(w_i)}$ , such that

$$\text{rann}_{e_{j(w_i)}A}(w_i) = \mathcal{A}(w_i)A.$$

Since  $A$  is of Loewy length 4 and each  $w_i$  belongs to  $(\text{Rad}A)^2$ , by (4.2.6) and Lemma 4.2.1 we can choose each  $a_i$  such that  $a_i = e_{j(w_i)}a_i e_{j(y)}$  and

$$a_i = \lambda_i e_{j(w_i)} + \sum_{z \in e_{j(w_i)}(\text{Rad}\mathcal{X})^\tau e_{j(y)}} \mu_z z,$$

where  $y = ye_{j(y)}$  and  $\lambda_i, \mu_z \in K$ . Given any  $a \in \text{rann}_{e_{j(y)}A}(xy)$ , then

$$xya = w_1a_1a + \dots + w_sa_sa = 0$$

and hence  $w_i a_i a = 0$ , for all  $i = 1, \dots, s$ , since the sum  $\sum_{w \in (\text{Rad}^2 \mathcal{X})^r} wA$  is direct. On the other hand, noticing that  $\alpha \in \text{Rad}A$ , we have that

$$\begin{aligned} w_i a_i a &= \lambda_i w_i a + \sum_{z \in e_{j(w_i)} (\text{Rad} \mathcal{X})^r e_{j(y)}} \mu_z w_i z a \\ &= \lambda_i w_i a + \sum_{z \in e_{j(w_i)} (\text{Rad} \mathcal{X})^r e_{j(y)}} \mu_z 0 \quad (\text{since } (\text{Rad}A)^4 = 0) \\ &= \lambda_i w_i a. \end{aligned}$$

Thus we obtain that

$$\text{rann}_{e_{j(y)}A}(xy) = \bigcap_{\substack{1 \leq i \leq s \\ \lambda_i \neq 0}} \text{rann}_{e_{j(w_i)}A}(w_i) = \bigcap_{\substack{1 \leq i \leq s \\ \lambda_i \neq 0}} A(w_i)A = \left( \bigcap_{\substack{1 \leq i \leq s \\ \lambda_i \neq 0}} A(w_i) \right) A.$$

Here we note that for each  $i$  with  $\lambda_i \neq 0$ ,  $e_{j(w_i)} = e_{j(y)}$ . Since an intersection of subdiagrams of  $e_{j(y)}\mathcal{X}$  is a subdiagram, we see from the equation above that  $xy$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . The proof is complete. ■

**Theorem 4.2.6** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Suppose that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ . If  $\Gamma$  is acyclic, then there exists a right tree subset  $\mathcal{Y}$  of  $A$  such that  $\mathcal{Y} \cup \{0\}$  is a multiplicative semigroup in  $A$  and that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ .*

**Proof.** Define

$$\mathcal{Y} = E \cup \mathcal{Y}_1 \cup \dots \cup \mathcal{Y}_{L-1},$$

where  $\mathcal{Y}_l = ((\text{Rad} \mathcal{X})^r)^l - \{0\}$  for  $l = 1, \dots, L-1$  and  $L$  is the Loewy length of  $A$ . Then  $\mathcal{Y} \cup \{0\}$  forms a multiplicative semigroup in  $A$ . In the following we shall show that

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad} \mathcal{X})^r \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A$$

for all  $l = 1, \dots, L-1$  and hence, by Proposition 4.1.2,  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ .

We proceed by induction on  $l$ . First, since  $\mathcal{X}$  is a right tree subset of  $A$ , we have that

$$\text{Rad}A = \bigoplus_{x_1 \in (\text{Rad}\mathcal{X})^\tau} x_1 A.$$

Assume that

$$(\text{Rad}A)^{l-1} = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1} A,$$

where  $l \geq 2$ . Again since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows that

$$(\text{Rad}A)^{l-1} = \bigoplus_{w \in (\text{Rad}^{l-1}\mathcal{X})^\tau} wA.$$

We note that every direct summand in the two direct sum decompositions above for  $(\text{Rad}A)^{l-1}$  is local and so indecomposable. Then for any  $x_1 \cdots x_{l-1}$  in  $\mathcal{Y}_{l-1}$  with  $x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau$ , by the Krull-Schmidt theorem, there exists some  $w \in (\text{Rad}^{l-1}\mathcal{X})^\tau$  such that the right  $A$ -modules  $wA$  and  $x_1 \cdots x_{l-1}A$  are isomorphic. Suppose that  $\theta: wA \rightarrow x_1 \cdots x_{l-1}A$  is such an  $A$ -isomorphism. Then  $wA$  and  $x_1 \cdots x_{l-1}A$  have the same projective cover and so  $w$  and  $x_1 \cdots x_{l-1}$  have the same right type; i.e., there exists some  $j$ ,  $1 \leq j \leq n$ , such that  $w = we_j$  and  $x_1 \cdots x_{l-1} = x_1 \cdots x_{l-1}e_j$ . Thus  $\theta(w) = x_1 \cdots x_{l-1}a$  for some  $0 \neq a \in e_j A e_j$ . But  $\Gamma$  is acyclic, so  $a = \lambda e_j$  for some  $\lambda \in K^*$ . Then from  $\theta(w) = \lambda x_1 \cdots x_{l-1}$  it follows that

$$\text{rann}_{e_j A}(w) = \text{rann}_{e_j A}(x_1 \cdots x_{l-1}).$$

This shows that  $x_1 \cdots x_{l-1}$  satisfies the annihilator condition with respect to  $\mathcal{X}$  since  $w$  does so. Then by Lemma 4.2.2

$$x_1 \cdots x_{l-1}(\text{Rad}A) = \bigoplus_{\substack{x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A$$

for all  $x_1 \cdots x_{l-1}$  in  $\mathcal{Y}_{l-1}$  and hence

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1}(\text{Rad}A)$$

$$= \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad } \mathcal{X})^r \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A.$$

Our induction is complete. ■

From Proposition 4.2.4, Theorem 4.2.5 and Theorem 4.2.6, we at once obtain the following

**Corollary 4.2.7** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Suppose that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ .*

- (1) *If  $A$  is of Loewy length 4, then  $A$  is of monomial presentation type.*
- (2) *If  $\Gamma$  is acyclic, then  $A$  is of monomial presentation type. ■*

In [6, Example 4.5], it is shown that there exists a quotient of path algebra which has a right tree subset but is not a right monomial ring. As an application of the corollary above, we can now provide an another example of this kind. We shall use the following result due to Saorín (see [27, Proposition 2.3]) in the example.

**Proposition 4.2.8** *Let  $I$  be an admissible ideal in the path algebra  $K\Gamma$  and let  $A = K\Gamma/I$  be of Loewy length  $L$ . Then  $A$  is gradable by the radical if and only if there exists a change of variable  $T$  on  $K\Gamma$  such that  $T(\alpha) \equiv \alpha \pmod{\langle \Gamma_1^2 \rangle}$  for all arrows  $\alpha$  in  $\Gamma$  and that  $\hat{I} = \langle T(I) \cup \Gamma_1^L \rangle$  is a homogeneous ideal in  $K\Gamma$ .*

**Example 4.1** Let  $K$  be a field. Define  $\Gamma$  to be

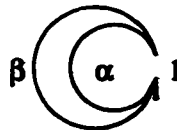


Figure 4.1

and let

$$\rho = \{\alpha^3, \alpha\beta\alpha, \alpha\beta^2 - \alpha^2, \beta\alpha\beta, \beta^3\}.$$

Let  $A = K\Gamma/I$ , where  $I = \langle \rho \rangle$ . It is easy to check that  $A$  is of Loewy length 4. In the following we show that  $A$  has a right tree subset  $\mathcal{X}$  but is not gradable by the radical. Then since a path algebra quotient of monomial presentation type is gradable by the radical, from (1) in Corollary 4.2.7 it follows that  $A$  is not a right monomial ring.

Define

$$\mathcal{X} = \{1\} \cup \{\alpha, \beta\} \cup \{\alpha\beta, \beta\alpha, \beta^2\} \cup \{\alpha\beta^2, \beta^2\alpha\}.$$

It is straightforward to check that  $\mathcal{X}$  is a right tree subset of  $A$ . Moreover,  $\mathcal{X} \cup \{0\}$  is a multiplicative semigroup in  $A$ . Assume that  $A$  is gradable by the radical. Then, by Proposition 4.2.8, there is a change of variable  $T$  on  $K\Gamma$  of the following form

$$T(\alpha) = \alpha + \phi$$

$$T(\beta) = \beta + \psi$$

where  $\phi, \psi \in \langle \Gamma_1^2 \rangle$ , such that  $\hat{I} = \langle T(\rho) \cup \Gamma_1^4 \rangle$  is a homogeneous ideal in  $K\Gamma$ , and by Proposition 2.2.2,  $A$  is algebra isomorphic to  $\hat{A} = K\Gamma/\hat{I}$ . Now we have that

$$\hat{I} = \langle \alpha^3, \alpha\beta\alpha, \lambda\beta^2\alpha + (\lambda - 1)\alpha\beta^2 + \alpha^2, \beta\alpha\beta, \beta^3 \rangle + \langle \Gamma_1^4 \rangle$$

where  $\lambda$  is the coefficient of  $\beta^2$  in  $\phi$ . Since  $\hat{I}$  is homogeneous, we see that  $\alpha^2$  and  $\lambda\beta^2\alpha + (\lambda - 1)\alpha\beta^2$  are in  $\hat{I}$ . Thus

$$\hat{I} = \langle \alpha^2, \alpha\beta\alpha, \lambda\beta^2\alpha + (\lambda - 1)\alpha\beta^2, \beta\alpha\beta, \beta^3 \rangle + \langle \Gamma_1^4 \rangle.$$

We calculate the  $K$ -vector space dimensions of  $A$  and  $\hat{A}$  to obtain that  $\dim_K A = 8$  and  $\dim_K \hat{A} = 7$ . This contradicts the fact that  $A$  is isomorphic to  $\hat{A}$ . Therefore,  $A$  is not gradable by the radical.

We observe that in the proof of Theorem 4.2.5 and Theorem 4.2.6, the multiplicative semigroup  $\mathcal{Y} \cup \{0\}$  generated by the top of the radical of the original tree subset  $\mathcal{X}$  is exactly such that  $(A, \mathcal{Y})$  is a right monomial ring. The following example shows that this is in general not the case. Our algebra will be local and of Loewy length 5.

**Example 4.2** Let  $K$  be a field with  $\text{Char}(K) \neq 2$ . Define  $\Gamma$  to be

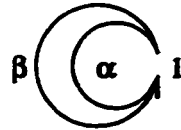


Figure 4.2

and let

$$\rho = \{\alpha^3, \alpha^2\beta, \beta\alpha^2, \beta\alpha\beta, \beta^3, \alpha\beta^2\alpha - \alpha\beta\alpha\}.$$

Let  $A = K\Gamma/I$ , where  $I = \langle \rho \rangle$ . Define

$$\begin{aligned} \mathcal{X} = & \{1\} \cup \{\alpha - \alpha\beta - \beta\alpha, \beta\} \cup \{\alpha^2, \alpha\beta, \beta\alpha, \beta^2\} \\ & \cup \{\alpha\beta^2, \beta^2\alpha\} \cup \{\alpha\beta\alpha\}. \end{aligned}$$

In the following we shall show that  $\mathcal{X}$  is a right tree subset of  $A$  which is such that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E = \{1\}$ .

(1)  $\mathcal{X}$  is a right tree subset of  $A$ .

By Proposition 4.1.1 and the definition of the set  $\rho$  of relations, we need only to check that the following two equations hold:

$$\text{Rad}A = (\alpha - \alpha\beta - \beta\alpha)A \oplus \beta A \tag{4.2.7}$$

$$(\alpha - \alpha\beta - \beta\alpha)(\text{Rad}A) = \alpha^2 A \oplus \alpha\beta A \tag{4.2.8}$$

To do this, we first notice that  $\text{Rad}A = \alpha A \oplus \beta A$ . Then  $\alpha - \alpha\beta - \beta\alpha \in \text{Rad}A$  and so  $(\alpha - \alpha\beta - \beta\alpha)A + \beta A \subseteq \text{Rad}A$ . On the other hand, we have that  $\alpha =$

$(\alpha - \alpha\beta - \beta\alpha)(1 + \beta + \beta^2) + \beta\alpha$ , and so  $\text{Rad}A \subseteq (\alpha - \alpha\beta - \beta\alpha)A + \beta A$ . Suppose that  $(\alpha - \alpha\beta - \beta\alpha)a + \beta b = 0$ , where  $a, b \in A$ . Then it follows from  $\alpha A \cap \beta A = 0$  that  $(\alpha - \alpha\beta)a = 0$  and  $\beta\alpha a + \beta b = 0$ , with the former equation implying that  $a \in \text{Rad}A$ . Thus  $\beta\alpha a = 0$  and  $\beta b = 0$ . This shows that the equation (4.2.7) holds. From (4.2.7) we have that  $(\alpha - \alpha\beta - \beta\alpha)(\text{Rad}A) = (\alpha^2 - \alpha\beta\alpha)A + (\alpha\beta - \alpha\beta^2)A = \alpha^2 A + \alpha\beta A$  since  $1 - \beta$  is invertible. From the definition of  $\rho$  we see that  $\alpha^2 A \cap \alpha\beta A = 0$ ; hence the equation (4.2.8) holds.

(2) Each element of  $\mathcal{X}$  satisfies the annihilator condition with respect to  $\mathcal{X}$ .

Since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows from Lemma 4.2.1 that  $\mathcal{X}$  is a  $K$ -vector space basis for  $A$ . Take any element in  $A$

$$\begin{aligned} f &= \lambda_0 + \lambda_1(\alpha - \alpha\beta - \beta\alpha) + \lambda_2\beta + \lambda_3\alpha^2 + \lambda_4\alpha\beta + \lambda_5\beta\alpha + \lambda_6\beta^2 \\ &\quad + \lambda_7\alpha\beta^2 + \lambda_8\beta^2\alpha + \lambda_9\alpha\beta\alpha. \end{aligned}$$

We carry out the following calculations:

$$\begin{aligned} (\alpha - \alpha\beta - \beta\alpha)f &= \lambda_0(\alpha - \alpha\beta - \beta\alpha) + \lambda_1(\alpha^2 - \alpha\beta\alpha) + \lambda_2(\alpha\beta - \alpha\beta^2) \\ &\quad + \lambda_6\alpha\beta^2 + \lambda_8\alpha\beta\alpha \\ \beta f &= \lambda_0\beta + \lambda_1(\beta\alpha - \beta^2\alpha) + \lambda_2\beta^2 + \lambda_5\beta^2\alpha \\ \alpha^2 f &= \lambda_0\alpha^2 \\ \alpha\beta f &= \lambda_0\alpha\beta + \lambda_2\alpha\beta^2 + \lambda_5\alpha\beta\alpha \\ \beta\alpha f &= \lambda_0\beta\alpha \\ \beta^2 f &= \lambda_0\beta^2 + \lambda_1\beta^2\alpha \\ \alpha\beta^2 f &= \lambda_0\alpha\beta^2 + \lambda_1\alpha\beta\alpha \\ \beta^2\alpha f &= \lambda_0\beta^2\alpha \\ \alpha\beta\alpha f &= \lambda_0\alpha\beta\alpha. \end{aligned}$$

From the relations above and the definition of  $\rho$  we obtain that

$$\begin{aligned} \text{rann}_A(\alpha - \alpha\beta - \beta\alpha) &= \{\alpha^2, \alpha\beta, \beta\alpha\}A \\ \text{rann}_A(\beta) &= \{\alpha^2, \alpha\beta, \beta^2\}A \\ \text{rann}_A(\alpha^2) &= \{\alpha, \beta\}A \\ \text{rann}_A(\alpha\beta) &= \{\alpha - \alpha\beta - \beta\alpha, \beta^2\}A \\ \text{rann}_A(\beta\alpha) &= \{\alpha, \beta\}A \\ \text{rann}_A(\beta^2) &= \{\beta, \alpha^2, \alpha\beta\}A \\ \text{rann}_A(\alpha\beta^2) &= \{\beta, \alpha^2, \alpha\beta\}A \\ \text{rann}_A(\beta^2\alpha) &= \{\alpha, \beta\}A \\ \text{rann}_A(\alpha\beta\alpha) &= \{\alpha, \beta\}A. \end{aligned}$$

This shows that each element of  $\mathcal{X}$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . Consequently,  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E = \{1\}$ .

We now have the following two different elements in the semigroup  $\mathcal{Y} \cup \{0\}$  generated by  $(\text{Rad}\mathcal{X})^\tau$ :

$$y_1 = (\alpha - \alpha\beta - \beta\alpha)\beta(\alpha - \alpha\beta - \beta\alpha) = -\alpha\beta^2\alpha$$

and

$$y_2 = (\alpha - \alpha\beta - \beta\alpha)\beta^2(\alpha - \alpha\beta - \beta\alpha) = \alpha\beta^2\alpha,$$

which are  $K$ -linearly dependent. This shows that the semigroup  $\mathcal{Y} \cup \{0\}$  is not a right tree subset of  $A$  and hence is not such that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E = \{1\}$ .

Finally, we observe that  $A$  is of monomial presentation type by applying the following change of variable  $T$  on  $K\Gamma$ :

$$\begin{aligned} T(\alpha) &= \alpha \\ T(\beta) &= \beta + \beta^2. \end{aligned}$$

The corresponding monomial algebra is  $\hat{A} = K\Gamma/\hat{I}$ , where  $\hat{I} = \langle \hat{\rho} \rangle$ , and

$$\hat{\rho} = \{\alpha^3, \alpha^2\beta, \beta\alpha^2, \beta\alpha\beta, \beta^3, \alpha\beta\alpha\}.$$

### 4.3 Further sufficient conditions

Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and suppose that  $A$  has a right tree subset  $\mathcal{X}$  with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i\mathcal{X}e_j - \{0\}$ . This section will provide some other conditions such that  $A = K\Gamma/I$  is of monomial presentation type. In particular, as a corollary to one of these results we shall obtain that if  $\Gamma$  is square-free and strongly acyclic, then  $A$  is of monomial presentation type.

**Proposition 4.3.1** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and suppose that  $A$  has a right tree subset  $\mathcal{X}$  with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i\mathcal{X}e_j - \{0\}$ . If for any  $x \in (\text{Rad}\mathcal{X})^r$  there exists at most one  $y \in (\text{Rad}\mathcal{X})^r$  satisfying  $xy \neq 0$ , then  $A$  is of monomial presentation type.*

**Proof.** Notice that it is not assumed that  $\mathcal{X}$  satisfies the annihilator condition. Since  $\mathcal{X}$  is a right tree subset of  $A$ , we have that

$$\text{Rad}A = \bigoplus_{x_1 \in (\text{Rad}\mathcal{X})^r} x_1A. \quad (4.3.1)$$

Assume that

$$(\text{Rad}A)^{l-1} = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^r \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1}A,$$

where  $l \geq 2$ . Then

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^r \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1}(\text{Rad}A).$$

By hypothesis, there exists at most one  $x_l \in (\text{Rad}\mathcal{X})^\tau$  such that  $x_{l-1}x_l \neq 0$  for any  $x_{l-1}$  in the equation above. Then from (4.3.1) it follows that

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A.$$

Thus, by Proposition 4.1.2 the multiplicative semigroup  $\mathcal{Y} \cup \{0\}$  generated by the top  $(\text{Rad}\mathcal{X})^\tau$  of  $\mathcal{X}$  is such that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ , and hence by Proposition 4.2.4,  $A$  is of monomial presentation type. ■

We recall that a two-sided artinian ring is *right locally distributive* if each of its indecomposable projective right modules has a distributive lattice of submodules.

**Proposition 4.3.2** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. If  $A$  is right locally distributive and  $(\text{Rad}A)_A$  is a direct sum of local right ideals of  $A$ , then  $A$  is of monomial presentation type.*

**Proof.** First we have that

$$(\text{Rad}A)_A = e_1(\text{Rad}A) \oplus \cdots \oplus e_n(\text{Rad}A).$$

Since  $(\text{Rad}A)_A$  is a direct sum of local right ideals, by the Krull-Schmidt Theorem each direct summand  $e_i(\text{Rad}A)$  is also a direct sum of local right ideals. Then there exists a finite subset  $\mathcal{T}_i$  of each  $e_i A$  with

$$\mathcal{T}_i \subseteq e_i(\text{Rad}A) - e_i(\text{Rad}A)^2 \quad \text{and} \quad \mathcal{T}_i = \bigcup_{j=1}^n \mathcal{T}_i e_j - \{0\},$$

such that

$$e_i(\text{Rad}A) = \bigoplus_{x \in \mathcal{T}_i} xA.$$

Set  $\mathcal{T} = \bigcup_{1 \leq i \leq n} \mathcal{T}_i$ . Define

$$\mathcal{Y} = E \cup \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_{L-1},$$

where  $\mathcal{Y}_l = \mathcal{T}^l - \{0\}$  for  $l = 1, \dots, L - 1$ , and  $L$  is the Loewy length of  $A$ . Then  $\mathcal{Y} \cup \{0\}$  forms a multiplicative semigroup in  $A$ .

Take any  $x_1 = x_1 e_j \in \mathcal{T}$ , where  $1 \leq j \leq n$ . Then

$$x_1(\text{Rad}A) = \sum_{\substack{x_2 \in \mathcal{T}_j \\ x_1 x_2 \neq 0}} x_1 x_2 A. \quad (4.3.2)$$

Suppose that we are given

$$\sum_{\substack{x_2 \in \mathcal{T}_j \\ x_1 x_2 \neq 0}} x_1 x_2 a_{x_2} = 0,$$

where  $a_{x_2} \in A$ . Let  $G_{x_1}$  be the kernel of the  $A$ -homomorphism  $\theta: e_j A \rightarrow x_1 A$  defined by  $\theta(a) = x_1 a$  for  $a \in e_j A$ . Then

$$\sum_{\substack{x_2 \in \mathcal{T}_j \\ x_1 x_2 \neq 0}} x_2 a_{x_2} \in G_{x_1}.$$

Since  $A$  is right locally distributive, it follows that

$$\left( \bigoplus_{\substack{x_2 \in \mathcal{T}_j \\ x_1 x_2 \neq 0}} x_2 A \right) \cap G_{x_1} = \bigoplus_{\substack{x_2 \in \mathcal{T}_j \\ x_1 x_2 \neq 0}} (x_2 A \cap G_{x_1}).$$

Thus we see that  $x_2 a_{x_2} \in G_{x_1}$ , i.e.,  $x_1 x_2 a_{x_2} = 0$ , for all  $x_2 \in \mathcal{T}_j$ . This shows that the sum in (4.3.2) is direct for all  $x_1 \in \mathcal{T}$ ; hence

$$(\text{Rad}A)^2 = \bigoplus_{\substack{x_1, x_2 \in \mathcal{T} \\ x_1 x_2 \neq 0}} x_1 x_2 A.$$

Applying the same argument as above, we inductively obtain that

$$(\text{Rad}A)^l = \bigoplus_{\substack{x_1, \dots, x_l \in \mathcal{T} \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A$$

for all  $l = 1, \dots, L - 1$ . Thus, by Proposition 4.1.2  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ , and hence by Proposition 4.2.4  $A$  is of monomial presentation type. ■

**Proposition 4.3.3** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents corresponding to the set  $\Gamma_0$  of vertices in  $\Gamma$ . Suppose that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$  such that the right tree subset  $\mathcal{X}$  consists of paths in  $\Gamma$  modulo  $I$ . Then  $A$  is of monomial presentation type.*

**Proof.** Since  $\mathcal{X}$  consists of paths in  $\Gamma$  modulo  $I$ , it follows that  $(\text{Rad}\mathcal{X})^\tau = \Gamma_1 + I/I$ , where  $\Gamma_1$  is the set of arrows in  $\Gamma$ . Define

$$\mathcal{Y}_l = ((\text{Rad}\mathcal{X})^\tau)^l - \{0\}, \quad l = 1, \dots, L-1,$$

where  $L$  is the Loewy length of  $A$ . Clearly we have that  $(\text{Rad}\mathcal{X})^\tau = \mathcal{Y}_1$ . Assume that

$$(\text{Rad}^k \mathcal{X})^\tau = \mathcal{Y}_k = \{x_1 \cdots x_k \mid x_1, \dots, x_k \in (\text{Rad}\mathcal{X})^\tau, x_1 \cdots x_k \neq 0\}$$

for all  $1 \leq k \leq l$ , where  $l \geq 1$ . Then each element of  $\mathcal{Y}_l$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . Hence by Lemma 4.2.2

$$\begin{aligned} (\text{Rad}A)^{l+1} &= \left( \bigoplus_{w \in (\text{Rad}^l \mathcal{X})^\tau} wA \right) \text{Rad}A \\ &= \bigoplus_{w \in (\text{Rad}^l \mathcal{X})^\tau} w(\text{Rad}A) \\ &= \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l (\text{Rad}A) \\ &= \bigoplus_{\substack{x_1, \dots, x_{l+1} \in (\text{Rad}\mathcal{X})^\tau \\ x_1 \cdots x_{l+1} \neq 0}} x_1 \cdots x_{l+1} A. \end{aligned}$$

On the other hand, we have that

$$(\text{Rad}A)^{l+1} = \bigoplus_{w \in (\text{Rad}^{l+1} \mathcal{X})^\tau} wA.$$

By hypothesis, each  $w$  in  $(\text{Rad}^{l+1} \mathcal{X})^\tau$  is a path in  $\Gamma$  modulo  $I$  which, from the induction assumption, must be of length  $l+1$ . Thus we see from the equations above that

$$(\text{Rad}^{l+1} \mathcal{X})^\tau = \mathcal{Y}_{l+1} = \{x_1 \cdots x_{l+1} \mid x_1, \dots, x_{l+1} \in (\text{Rad}\mathcal{X})^\tau, x_1 \cdots x_{l+1} \neq 0\}.$$

Then by induction we obtain that

$$\mathcal{X} \cup \{0\} = E \cup \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_{L-1} \cup \{0\}$$

is a multiplicative semigroup in  $A$ ; so, by Proposition 4.2.4  $A$  is of monomial presentation type. ■

**Theorem 4.3.4** *Let  $A = K\Gamma/I$  be of Loewy length  $L \geq 3$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and let  $A$  have a right tree subset  $\mathcal{X}$  with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$ . Suppose that  $\Gamma$  satisfies the condition that for any arrow  $\alpha$  in  $\Gamma$ , provided that  $o(\alpha)$  is a non-source in  $\Gamma$ , then every path in  $\Gamma$  from  $o(\alpha)$  to  $t(\alpha)$ , other than  $\alpha$ , if any, is of length  $\geq L - 1$ . Then  $A$  is of monomial presentation type.*

**Proof.** Let  $x$  be any element of  $(\text{Rad } \mathcal{X})^\tau$ . Since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows that

$$x(\text{Rad } A) = x\left(\bigoplus_{y \in (\text{Rad } \mathcal{X})^\tau} yA\right) = \sum_{\substack{y \in (\text{Rad } \mathcal{X})^\tau \\ xy \neq 0}} xyA = \bigoplus_{w \in (\text{Rad } \mathcal{U}(x))^\tau} wA,$$

where  $\mathcal{U}(x)$  is the subdiagram of  $\mathcal{X}$  generated by  $x$ . Let us say

$$x(\text{Rad } A) = xy_1A + \cdots + xy_rA = w_1A \oplus \cdots \oplus w_sA \quad (4.3.3)$$

where  $y_1, \dots, y_r \in (\text{Rad } \mathcal{X})^\tau$  are different with  $xy_i \neq 0$  for each  $i$ , and  $w_1, \dots, w_s \in (\text{Rad } \mathcal{U}(x))^\tau$ . We claim that the set  $\{xy_1, \dots, xy_r\}$  is  $K$ -linearly independent modulo  $x(\text{Rad } A)^2$ . To show this, suppose that

$$\lambda_1 xy_1 + \cdots + \lambda_r xy_r \in x(\text{Rad } A)^2 = xy_1(\text{Rad } A) + \cdots + xy_r(\text{Rad } A),$$

where  $\lambda_1, \dots, \lambda_r$  are scalars in  $K$ . Since  $y_1, \dots, y_r$  have the same origin, which is the terminus of  $x$ , by hypothesis they are of different right types, i.e.,  $e_{j(y_k)} \neq e_{j(y_l)}$  whenever  $k \neq l$ , where  $e_{j(y_k)}, e_{j(y_l)}$  are such that  $y_k = y_k e_{j(y_k)}$  and  $y_l = y_l e_{j(y_l)}$ . Hence

$$\lambda_k xy_k \in xy_1(\text{Rad } A) + \cdots + xy_r(\text{Rad } A),$$

for all  $k = 1, \dots, r$ . Now assume that  $\lambda_k \neq 0$  for some  $k$ . Then

$$xy_k = xy_1a_1 + \dots + xy_ra_r \quad (4.3.4)$$

for some  $a_l \in e_{j(y_l)}(\text{Rad}A)e_{j(y_k)}$ ,  $l = 1, \dots, r$ . Let  $x = xe_{j(x)}$ . Then from (4.3.4) we see that  $j(x)$  is a non-source in  $\Gamma$  and there is an arrow from  $j(x)$  to  $j(y_k)$ ; hence, by hypothesis  $y_la_l \in (\text{Rad}A)^{L-1}$  and so  $xy_la_l = 0$ , for all  $l = 1, \dots, r$ . Thus  $xy_k = 0$ , a contradiction, which shows that  $\{xy_1, \dots, xy_r\}$  is  $K$ -linearly independent modulo  $x(\text{Rad}A)^2$ . Then from this and (4.3.3) it follows that  $r = s$ . Without loss of generality, we can assume that  $y_k$  and  $w_k$  are of the same right type for all  $k = 1, \dots, r$ . Then, for any fixed  $k$ , from (4.3.3) we have that

$$w_k = xy_1b_1 + \dots + xy_k(\mu_k + b_k) + \dots + xy_rb_r,$$

for some  $b_1, \dots, b_r \in \text{Rad}A$  with  $b_l = e_{j(y_l)}b_le_{j(w_k)}$  for each  $l$ , and some  $\mu_k \in K^*$ . Again since  $j(x)$  is a non-source vertex in  $\Gamma$  and there is an arrow from  $j(x)$  to  $j(y_k)$ , by hypothesis  $y_lb_l \in (\text{Rad}A)^{L-1}$  and hence  $xy_lb_l = 0$  for all  $l = 1, \dots, r$ . It follows that  $w_k = \mu_k xy_k$  and  $xy_kA = w_kA$ . Then

$$x(\text{Rad}A) = \bigoplus_{\substack{y \in (\text{Rad}\mathcal{X})^r \\ xy \neq 0}} xyA$$

for all  $x \in (\text{Rad}\mathcal{X})^r$  and hence

$$(\text{Rad}A)^2 = \bigoplus_{\substack{x, y \in (\text{Rad}\mathcal{X})^r \\ xy \neq 0}} xyA.$$

Assume that

$$(\text{Rad}A)^{l-1} = \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}\mathcal{X})^r \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1}A$$

where  $l \geq 2$ . For each component  $x_1 \cdots x_{l-1}A$  in the direct sum above, applying the exactly same argument as above with  $x(\text{Rad}A)$ , we obtain that

$$x_1 \cdots x_{l-1}(\text{Rad}A) = \bigoplus_{\substack{x_l \in (\text{Rad}\mathcal{X})^r \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_lA.$$

Thus

$$\begin{aligned}
 (\text{Rad}A)^l &= \bigoplus_{\substack{x_1, \dots, x_{l-1} \in (\text{Rad}A)^r \\ x_1 \cdots x_{l-1} \neq 0}} x_1 \cdots x_{l-1} (\text{Rad}A) \\
 &= \bigoplus_{\substack{x_1, \dots, x_l \in (\text{Rad}A)^r \\ x_1 \cdots x_l \neq 0}} x_1 \cdots x_l A.
 \end{aligned}$$

Consequently, by Proposition 4.1.2, the multiplicative semigroup  $\mathcal{Y} \cup \{0\}$  generated by  $(\text{Rad}A)^r$  is such that  $(A, \mathcal{Y})$  is a right monomial ring with respect to  $E$ . Therefore, by Proposition 4.2.4  $A$  is of monomial presentation type. ■

We give a simple example.

**Example 4.3** Let  $K$  be a field. Define  $\Gamma$  to be

$$\begin{array}{ccccc}
 & \xrightarrow{\alpha(1)} & & & \\
 & \xrightarrow{\alpha(2)} & & & \\
 1 & \xrightarrow{\quad \cdot \quad} & 2 & \xrightarrow{\beta} & 3 \\
 & \xrightarrow{\quad \cdot \quad} & & & \\
 & \xrightarrow{\alpha(\nu)} & & & 
 \end{array}$$

Figure 4.3

and let

$$\rho = \{\alpha(1)\beta - \alpha(2)\beta, \dots, \alpha(1)\beta - \alpha(\nu)\beta\}$$

where  $\nu \geq 2$ . Let  $A = K\Gamma/I$ , where  $I = \langle \rho \rangle$ .

We see that  $A$  is of Loewy length 3 and  $\Gamma$  satisfies the condition in Theorem 4.3.4.

Define

$$\mathcal{X} = \{e_1, e_2, e_3, \alpha(1), \alpha(1) - \alpha(2), \dots, \alpha(1) - \alpha(\nu), \beta, \alpha(1)\beta\}.$$

Then  $\mathcal{X}$  is a right tree subset of  $A$ . So by Theorem 4.3.4,  $A$  is of monomial presentation.

**Corollary 4.3.5** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents and let  $A$  have a right tree subset  $\mathcal{X}$  with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$ . If  $\Gamma$  is square-free and strongly acyclic, then  $A$  is of monomial presentation type.*

**Proof.** This is obvious from the definition of a square-free and strongly acyclic finite directed graph and Theorem 4.3.4 ■

## 4.4 Gradation by the radical

The task of this last section of the present chapter is to present a characterization of path algebra quotients of monomial presentation type within the class of right monomial rings. We shall show that they are exactly those gradable by the radical.

Let us start with the following lemma.

**Lemma 4.4.1** *Let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Then  $A$  is of monomial presentation type if and only if there exists a right tree subset  $\mathcal{X}$  of  $A$  with  $E \subseteq \mathcal{X}$  and  $\mathcal{X} = \bigcup_{i,j=1}^n e_i \mathcal{X} e_j - \{0\}$  such that the following conditions are satisfied:*

- (a)  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ ;
- (b) the  $K$ -vector space direct sum decomposition

$$A = \text{Span}_K E \oplus \text{Span}_K (\text{Rad} \mathcal{X})^\tau \oplus \dots \oplus \text{Span}_K (\text{Rad}^{L-1} \mathcal{X})^\tau$$

is a positive gradation for  $A$ , where  $L$  is the Loewy length of  $A$ .

**Proof.** ( $\Rightarrow$ ) Suppose that  $A$  is of monomial presentation type. Then there exists a monomial admissible ideal  $\hat{I}$  in  $K\Gamma$  such that  $A$  is algebra isomorphic to  $\hat{A} = K\Gamma/\hat{I}$ . Let  $P$  denote the set of nonzero paths in  $\Gamma$  modulo  $\hat{I}$ . We can easily check that  $\bar{P}$  is a right tree subset of  $\hat{A}$  such that  $(\hat{A}, \bar{P})$  is a right monomial ring with respect to  $\bar{\Gamma}_0$ ,

where  $\bar{P}$  and  $\bar{\Gamma}_0$  are the images of  $P$  and  $\Gamma_0$  under the natural map  $K\Gamma \rightarrow K\Gamma/\hat{I}$ , respectively. Moreover,  $\hat{A}$  has the following positive gradation

$$\hat{A} = \text{Span}_K \bar{\Gamma}_0 \oplus \text{Span}_K \bar{P}_1 \oplus \cdots \oplus \text{Span}_K \bar{P}_{L-1}$$

where  $P_l$  is the set of nonzero paths of length  $l$  in  $\Gamma$  modulo  $\hat{I}$  for  $l = 1, \dots, L - 1$ . Let  $F$  and  $\mathcal{X}$  correspond to  $\bar{\Gamma}_0$  and  $\bar{P}$ , respectively, under the given  $K$ -algebra isomorphism. Without loss of generality, we can assume that  $E = F$ . In fact, by the Wedderburn-Malcev Theorem, there exists an inner automorphism  $\sigma$  of  $A$  such that  $E = \sigma(F)$ . Then  $\mathcal{X}$  is a right tree subset of  $A$  such that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$  and that the  $K$ -vector space direct sum decomposition

$$A = \text{Span}_K E \oplus \text{Span}_K (\text{Rad} \mathcal{X})^\tau \oplus \cdots \oplus \text{Span}_K (\text{Rad}^{L-1} \mathcal{X})^\tau$$

is a positive gradation for  $A$ .

( $\Leftarrow$ ) Define

$$\mathcal{Y} = E \cup \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_{L-1},$$

where  $\mathcal{Y}_l = ((\text{Rad} \mathcal{X})^\tau)^l - \{0\}$  for  $l = 1, \dots, L - 1$ . Then  $\mathcal{Y} \cup \{0\}$  is a multiplicative semigroup in  $A$ . In the following we shall show that every element of  $\mathcal{Y}$  satisfies the annihilator condition with respect to  $\mathcal{X}$  and hence, by Lemma 4.2.3 and Proposition 4.2.4,  $A$  is of monomial presentation type.

First, since  $(A, \mathcal{X})$  is a right monomial ring, every element of  $\mathcal{Y}_1 = (\text{Rad} \mathcal{X})^\tau$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . Given any  $l \geq 2$  and any  $x_1 \cdots x_l \in \mathcal{Y}_l$  with each  $x_i$  in  $(\text{Rad} \mathcal{X})^\tau$ , then by the condition (b) we have that  $x_1 \cdots x_l \in \text{Span}_K (\text{Rad}^l \mathcal{X})^\tau$ , so that

$$x_1 \cdots x_l = \sum_{w \in e_{i(x_1)} (\text{Rad}^l \mathcal{X})^\tau e_{j(x_l)}} \lambda_w w$$

for some  $\lambda_w \in K$ , where  $x_1 = e_{i(x_1)} x_1$  and  $x_l = x_l e_{j(x_l)}$ . Since the right  $A$ -modules

$wA$  are independent when  $w$  runs over  $(\text{Rad}^l \mathcal{X})^\tau$ , it follows that

$$\text{rann}_{e_j(x_l)\mathcal{X}}(x_1 \cdots x_l) = \bigcap_{\substack{\lambda_w \neq 0 \\ w \in e_i(x_1)(\text{Rad}^l \mathcal{X})^\tau e_j(x_l)}} \text{rann}_{e_j(x_l)A}(w).$$

Since  $(A, \mathcal{X})$  is a right monomial ring, each  $w$  in the equation above satisfies the annihilator condition with respect to  $\mathcal{X}$ . Hence  $x_1 \cdots x_l$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . Thus, we obtain that every element of  $\mathcal{Y}$  satisfies the annihilator condition with respect to  $\mathcal{X}$ . The proof is complete. ■

Now we prove the main result of this section.

**Theorem 4.4.2** *Let  $\Gamma$  be any connected finite directed graph and let  $A = K\Gamma/I$  with basic set  $E = \{e_1, \dots, e_n\}$  of primitive idempotents. Suppose that  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ . Then  $A$  is of monomial presentation type if and only if  $A$  is gradable by the radical.*

**Proof.** ( $\Leftarrow$ ) Suppose that  $A$  is gradable by the radical. By definition there exists a positive gradation for  $A$ :

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{L-1} \tag{4.4.1}$$

such that

$$(\text{Rad}A)^l = \bigoplus_{m \geq l} A_m$$

for all  $l = 0, 1, \dots, L-1$ , where  $L$  is the Loewy length of  $A$ . By hypothesis,  $\mathcal{X}$  is a right tree subset of  $A$ ; then by Lemma 4.2.1  $\mathcal{X}$  is a  $K$ -vector space basis for  $A$  and so

$$A = KE \oplus K(\text{Rad}\mathcal{X})^\tau \oplus \cdots \oplus K(\text{Rad}^{L-1}\mathcal{X})^\tau.$$

Moreover, we have that

$$(\text{Rad}A)^l = (\text{Rad}^l \mathcal{X})A = \bigoplus_{m \geq l} K(\text{Rad}^m \mathcal{X})^\tau$$

for all  $l$ . Here  $E = (\text{Rad}^0 \mathcal{X})^\tau$ . Now define a map  $\theta: \mathcal{X} \rightarrow A$  by

$$\theta(x) = \text{the } l\text{-th component of } x \text{ relative to the } \mathbf{Z}\text{-gradation (4.4.1)}$$

for all  $x \in (\text{Rad}^l \mathcal{X})^\tau$ ,  $l = 0, 1, \dots, L - 1$ . Then  $\theta$  extends to a  $K$ -vector space endomorphism on  $A$  such that

$$\theta(K(\text{Rad}^l \mathcal{X})^\tau) \subseteq A_l \tag{4.4.2}$$

for all  $l$ . Let  $\mathcal{Z} = \theta(\mathcal{X})$  and  $\mathcal{Z}_l = \theta((\text{Rad}^l \mathcal{X})^\tau)$  for  $l = 0, 1, \dots, L - 1$ . It is clear from the definition of  $\theta$  that for any  $x \in (\text{Rad}^l \mathcal{X})^\tau$  with  $0 \leq l \leq L - 1$ ,

$$x \equiv \theta(x) \pmod{(\text{Rad}A)^{l+1}}.$$

We proceed with the following steps.

Step 1.  $\theta$  is a  $K$ -vector space automorphism of  $A$  and  $\theta(K(\text{Rad}^l \mathcal{X})^\tau) = A_l$  for all  $l$ .

Given any  $x_1, x_2$  in  $\mathcal{X}$  with  $x_1 \in (\text{Rad}^l \mathcal{X})^\tau$  and  $x_2 \in (\text{Rad}^k \mathcal{X})^\tau$  where  $0 \leq l, k \leq L - 1$ , if  $l \neq k$ , then clearly  $\theta(x_1) \neq \theta(x_2)$ . Suppose that  $l = k$ . We have that

$$\begin{aligned} x_1 &\equiv \theta(x_1) \pmod{(\text{Rad}A)^{l+1}} \\ x_2 &\equiv \theta(x_2) \pmod{(\text{Rad}A)^{l+1}}. \end{aligned}$$

If  $\theta(x_1) = \theta(x_2)$ , then  $x_1 - x_2 \in (\text{Rad}A)^{l+1}$ . But

$$(\text{Rad}A)^{l+1} \cap K(\text{Rad}^l \mathcal{X})^\tau = 0, \tag{4.4.3}$$

so  $x_1 = x_2$ . This shows that  $\theta|_{\mathcal{X}}$  is an injection. To prove that  $\theta$  is a  $K$ -vector space automorphism on  $A$ , it suffices to show that  $\mathcal{Z}$  is  $K$ -linearly independent; in fact, by (4.4.1) and (4.4.2) we need only to show that each  $\mathcal{Z}_l$  is  $K$ -linearly independent. Now suppose that

$$\lambda_1 z_1^{(l)} + \dots + \lambda_t z_t^{(l)} = 0,$$

where  $\lambda_i \in K$  and  $z_i^{(l)} \in \mathcal{Z}_l$  for  $i = 1, \dots, t$ . Then

$$\lambda_1 \theta^{-1}(z_1^{(l)}) + \dots + \lambda_t \theta^{-1}(z_t^{(l)}) \equiv 0 \pmod{(\text{Rad}A)^{l+1}}.$$

Since each  $\theta^{-1}(z_i^{(l)})$  lies in  $(\text{Rad}^l \mathcal{X})^\tau$  which is  $K$ -linearly independent, by (4.4.3) we obtain that  $\lambda_i = 0$  for all  $i$ .

Noticing that  $\dim_K(A_l) = \dim_K K(\text{Rad}^l \mathcal{X})^\tau$  for all  $l$ , from (4.4.2) we then have that  $\theta(K(\text{Rad}^l \mathcal{X})^\tau) = A_l$  for all  $l$ .

Step 2. Let  $F = \theta(E) = \mathcal{Z}_0 = \{f_1, \dots, f_n\}$ . Then  $F$  is a basic set of primitive idempotents of  $A$ .

We have that

$$A = KE \oplus \text{Rad}A = KF \oplus \text{Rad}A.$$

By the Wedderburn-Malcev Theorem, there exists an inner automorphism  $\sigma$  of  $A$ , induced by some element  $1 - y$  where  $y \in \text{Rad}A$ , such that  $KF = \sigma(KE)$ . Then

$$\sigma(e_i) = (1 - y)^{-1} e_i (1 - y) \equiv e_i \pmod{\text{Rad}A}$$

for all  $i = 1, \dots, n$ . On the other hand,

$$e_i \equiv \theta(e_i) = f_i \pmod{\text{Rad}A},$$

for all  $i$ . It follows that

$$\sigma(e_i) \equiv f_i \pmod{\text{Rad}A}.$$

But both  $\sigma(e_i)$  and  $f_i$  are in  $KF$ , so  $f_i = \sigma(e_i)$  for all  $i$ ; hence  $F = \sigma(E)$  is a basic set of primitive idempotents of  $A$ .

Step 3.  $\mathcal{Z} = \bigcup_{i,j=1}^n f_i \mathcal{Z} f_j - \{0\}$  and  $\mathcal{Z}$  is a normed subset of the right  $A$ -module  $A$  relative to the basic set  $F$ .

Let  $z$  be any element of  $\mathcal{Z}$ . Let  $x = \theta^{-1}(z)$ . Suppose that  $x = e_i x e_j \in \mathcal{X}_l$  for

some  $l$  and some  $i, j$ . Then  $z \in \mathcal{Z}_l$ . Now

$$\begin{aligned} x &\equiv z \pmod{(\text{Rad}A)^{l+1}} \\ e_i &\equiv f_i \pmod{\text{Rad}A} \\ e_j &\equiv f_j \pmod{\text{Rad}A}. \end{aligned}$$

Then

$$x = e_i x e_j \equiv f_i z f_j \pmod{(\text{Rad}A)^{l+1}}$$

and so

$$z \equiv f_i z f_j \pmod{(\text{Rad}A)^{l+1}}.$$

Since  $z$  and  $f_i z f_j$  are both in  $A_l$ , it follows that  $z = f_i z f_j$ . This shows that  $\mathcal{Z} = \bigcup_{i,j=1}^n f_i \mathcal{Z} f_j - \{0\}$ .

Given any different  $z_1, z_2$  in  $\mathcal{Z}$  with  $z_1 \in \mathcal{Z}_l$  and  $z_2 \in \mathcal{Z}_k$  where  $0 \leq l, k \leq L-1$ , if  $l \neq k$ , then it is clear that  $z_1 A \neq z_2 A$ . Let  $l = k$ . Assume that  $z_1 A = z_2 A$ . Then  $z_2 = z_1 a$  for some  $a = a_0 + a_1 + \dots + a_{L-1} \in A$  with each  $a_i$  in  $A_i$ . Since  $z_1, z_2 \in A_l$ , it follows that  $z_2 = z_1 a_0$ . Assume that  $a_0 = \lambda_1 f_1 + \dots + \lambda_n f_n$  for some scalars  $\lambda_i$  in  $K$ . Then  $z_2 = \lambda_j z_1 f_j = \lambda_j z_1$ , where  $f_j$  is such that  $z_1 = z_1 f_j$ . But by Step 1,  $\mathcal{Z}$  is  $K$ -linearly independent, this leads to a contradiction, which shows that  $z_1 A \neq z_2 A$ . Thus,  $\mathcal{Z}$  is a normed subset of  $A$ , as right  $A$ -module.

Step 4.  $\mathcal{Z}$  is a right tree subset of  $A$ .

Clearly we have that

$$\mathcal{Z} = F \dot{\cup} \mathcal{Z}_1 \dot{\cup} \dots \dot{\cup} \mathcal{Z}_{L-1}$$

is a disjoint union and  $A = \bigoplus_{i=1}^n f_i A$ . We shall apply Proposition 4.1.1. Given any  $l$  with  $0 \leq l \leq L-1$ , assume that  $\mathcal{Z}_l = \{z_1, \dots, z_s\}$ . Then  $\theta^{-1}(z_i) \in (\text{Rad}^l \mathcal{X})^r$  for all  $i = 1, \dots, s$ . Since  $\mathcal{X}$  is a right tree subset of  $A$ , it follows that

$$\theta^{-1}(z_i)(\text{Rad}A) = u_{i1}A \oplus \dots \oplus u_{it_i}A \tag{4.4.4}$$

where  $u_{ij} \in (\text{Rad}^{l+1} \mathcal{X})^\tau$  for all  $i, j$ , and

$$(\text{Rad}^{l+1} \mathcal{X})^\tau = \{u_{1j} \mid 1 \leq j \leq t_1\} \dot{\cup} \cdots \dot{\cup} \{u_{sj} \mid 1 \leq j \leq t_s\}$$

is a disjoint union. Let  $w_{ij} = \theta(u_{ij})$  for all  $i, j$ . Then

$$\mathcal{Z}_{l+1} = \theta((\text{Rad}^{l+1} \mathcal{X})^\tau) = \{w_{1j} \mid 1 \leq j \leq t_1\} \dot{\cup} \cdots \dot{\cup} \{w_{sj} \mid 1 \leq j \leq t_s\}$$

is a disjoint union. For any  $z_i \in \mathcal{Z}_l$  and any  $a \in A_1$ , by (4.4.4)

$$\theta^{-1}(z_i)a \equiv \lambda_1 u_{i1} + \cdots + \lambda_{t_i} u_{it_i} \pmod{(\text{Rad}A)^{l+2}} \quad (4.4.5)$$

for some scalars  $\lambda_j$  in  $K$ . Now

$$\begin{aligned} \theta^{-1}(z_i) &\equiv z_i \pmod{(\text{Rad}A)^{l+1}} \\ u_{ij} &\equiv w_{ij} \pmod{(\text{Rad}A)^{l+2}} \end{aligned}$$

for all  $j = 1, \dots, t_s$ . Thus it follows from (4.4.5) that

$$z_i a \equiv \lambda_1 w_{i1} + \cdots + \lambda_{t_i} w_{it_i} \pmod{(\text{Rad}A)^{l+2}}.$$

But  $z_i a, w_{i1}, \dots, w_{it_i} \in A_{l+1}$ , so

$$z_i a = \lambda_1 w_{i1} + \cdots + \lambda_{t_i} w_{it_i}.$$

This shows that

$$z_i A_1 \subseteq w_{i1} A + \cdots + w_{it_i} A.$$

Since  $A$  is gradable by the radical, we have that  $A_k = A_1^k$  for all  $k = 1, \dots, L-1$ .

Thus

$$z_i A_k \subseteq w_{i1} A + \cdots + w_{it_i} A$$

for all  $k$ , and so

$$z_i (\text{Rad}A) \subseteq w_{i1} A + \cdots + w_{it_i} A.$$

On the other hand, for each  $u_{ij}$  in (4.4.4) we have that

$$u_{ij} = \theta^{-1}(z_i)b$$

for some  $b = b_1^{(i,j)} + \dots + b_{L-1}^{(i,j)} \in \text{Rad}A$  with each  $b_k^{(i,j)}$  in  $A_k$ . Then

$$w_{ij} \equiv z_i b_1^{(i,j)} \pmod{(\text{Rad}A)^{l+2}}.$$

But  $w_{ij}, z_i b_1^{(i,j)} \in A_{l+1}$ , so  $w_{ij} = z_i b_1^{(i,j)}$  and hence

$$w_{i1}A + \dots + w_{it_i}A \subseteq z_i(\text{Rad}A).$$

Thus we have that

$$z_i(\text{Rad}A) = w_{i1}A + \dots + w_{it_i}A,$$

for all  $z_i \in \mathcal{Z}_l$ .

Next, we need to show that the sum above is direct for each  $z_i$ . Fixing  $i$ , it suffices to prove that for any  $k$  with  $0 \leq k \leq L-1$  and any  $a_1, \dots, a_{t_i} \in A_k$ ,

$$w_{i1}a_1 + \dots + w_{it_i}a_{t_i} = 0 \tag{4.4.6}$$

implies that  $w_{ij}a_j = 0$  for all  $j$ . Assume that we are given (4.4.6). Then

$$u_{i1}a_1 + \dots + u_{it_i}a_{t_i} \equiv 0 \pmod{(\text{Rad}A)^{l+k+2}}.$$

But

$$\begin{aligned} (\text{Rad}A)^{l+k+2} &= \left( \bigoplus_{u \in (\text{Rad}^{l+1}\mathcal{X})^r} uA \right) (\text{Rad}A)^{k+1} \\ &= \bigoplus_{u \in (\text{Rad}^{l+1}\mathcal{X})^r} u(\text{Rad}A)^{k+1}. \end{aligned}$$

Thus each  $u_{ij}a_j$  is in  $u_{ij}(\text{Rad}A)^{k+1}$  and so each  $w_{ij}a_j = \theta(u_{ij})a_j$  is in  $(\text{Rad}A)^{k+l+2}$ . On the other hand,  $w_{ij}a_j \in A_{l+k+1}$ . Hence  $w_{ij}a_j = 0$  for all  $j = 1, \dots, t_i$ . Consequently, by Proposition 4.1.1  $\mathcal{Z}$  is a right tree subset of  $A$ .

Step 5.  $(A, \mathcal{Z})$  is a right monomial ring with respect to  $F$ .

Take any  $l$  with  $0 \leq l \leq L-1$  and any  $z \in \mathcal{Z}_l$ . We need to prove that  $z$  satisfies the annihilator condition with respect to  $\mathcal{Z}$ . Suppose that  $z = f_i z f_j$  for some  $i, j$ . Then  $\theta^{-1}(z) = e_i \theta^{-1}(z) e_j$ . Since  $(A, \mathcal{X})$  is a right monomial ring with respect to  $E$ ,  $\theta^{-1}(z)$  satisfies the annihilator condition with respect to  $\mathcal{X}$ ; i.e., there exists a subdiagram  $\mathcal{A}(\theta^{-1}(z))$  of  $e_j \mathcal{X}$  such that

$$\text{rann}_{e_j A}(\theta^{-1}(z)) = \mathcal{A}(\theta^{-1}(z))A.$$

Let  $\mathcal{B}(z) = \theta(\mathcal{A}(\theta^{-1}(z)))$ . From the proof in Step 4 we see that  $\mathcal{B}(z)$  is a subdiagram of  $f_j \mathcal{Z}$ . We claim that  $\text{rann}_{f_j A}(z) = \mathcal{B}(z)A$ . In fact, given any  $w \in \mathcal{B}(z)$ , say  $w \in \mathcal{Z}_k$  for some  $k$ , then  $\theta^{-1}(w) \in \mathcal{A}(\theta^{-1}(z))$  and  $\theta^{-1}(z)\theta^{-1}(w) = 0$ . Thus

$$zw \equiv 0 \pmod{(\text{Rad}A)^{k+l+1}}.$$

But  $zw \in A_{l+k}$ , so  $zw = 0$ . This shows that  $\mathcal{B}(z)A \subseteq \text{rann}_{f_j A}(z)$ . On the other hand, given any  $a \in \text{rann}_{f_j A}(z)$ , then  $za = 0$ . Let  $a = \lambda_1 z_1 + \dots + \lambda_t z_t$  for some  $\lambda_s \in K^*$  and some different  $z_s \in f_j \mathcal{Z}$ ,  $s = 1, \dots, t$ . Then

$$\lambda_1 z z_1 + \dots + \lambda_t z z_t = 0.$$

Without loss of generality, we can assume that all  $z_s$  are in  $\mathcal{Z}_k$  for some  $k$ . Then we have that

$$\lambda_1 \theta^{-1}(z)\theta^{-1}(z_1) + \dots + \lambda_t \theta^{-1}(z)\theta^{-1}(z_t) \equiv 0 \pmod{(\text{Rad}A)^{k+l+1}}.$$

But

$$\begin{aligned} (\text{Rad}A)^{l+k+1} &= \left( \bigoplus_{u \in (\text{Rad}^l \mathcal{X})^r} uA \right) (\text{Rad}A)^{k+1} \\ &= \bigoplus_{u \in (\text{Rad}^l \mathcal{X})^r} u(\text{Rad}A)^{k+1}. \end{aligned}$$

So

$$\lambda_1 \theta^{-1}(z) \theta^{-1}(z_1) + \cdots + \lambda_t \theta^{-1}(z) \theta^{-1}(z_t) \equiv 0 \pmod{\theta^{-1}(z)(\text{Rad}A)^{k+1}}$$

and then

$$\sum_{i=1}^t \lambda_i \theta^{-1}(z) \theta^{-1}(z_i) = \theta^{-1}(z) \sum_{w \in \text{Rad}^{k+1} \mathcal{X}} \mu_w w$$

for some scalars  $\mu_w$  in  $K$ . Thus

$$\sum_{i=1}^t \lambda_i \theta^{-1}(z_i) - \sum_{w \in \text{Rad}^{k+1} \mathcal{X}} \mu_w w \in \text{rann}_{e_j A}(\theta^{-1}(z)) = \mathcal{A}(\theta^{-1}(z))A.$$

Since  $\mathcal{X}$  is  $K$ -linearly independent and  $\mathcal{A}(\theta^{-1}(z)) \subseteq \mathcal{X}$  is a  $K$ -vector space basis for  $\mathcal{A}(\theta^{-1}(z))A$ , it follows that each  $\theta^{-1}(z_i)$  is in  $\mathcal{A}(\theta^{-1}(z))$ . Thus each  $z_i$  is in  $\mathcal{B}(z)$  and so

$$a = \lambda_1 z_1 + \cdots + \lambda_t z_t \in \mathcal{B}(z)A.$$

This shows that  $\text{rann}_{f_j A}(z) \subseteq \mathcal{B}(z)A$ . Consequently,  $\text{rann}_{f_j A}(z) = \mathcal{B}(z)A$  and so  $z$  satisfies the annihilator condition with respect to  $\mathcal{Z}$ . The proof of Step 5 is complete.

Finally, by Lemma 4.4.1  $A$  is of monomial presentation type.

( $\Rightarrow$ ) This is obvious since every monomial algebra is gradable by the radical. ■

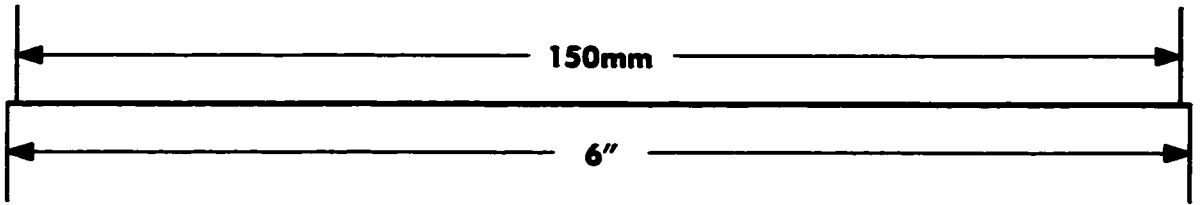
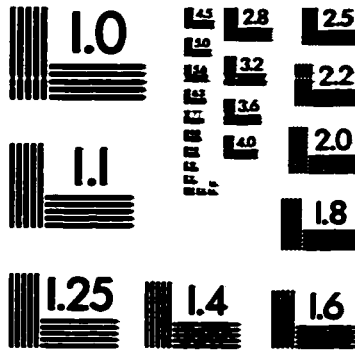
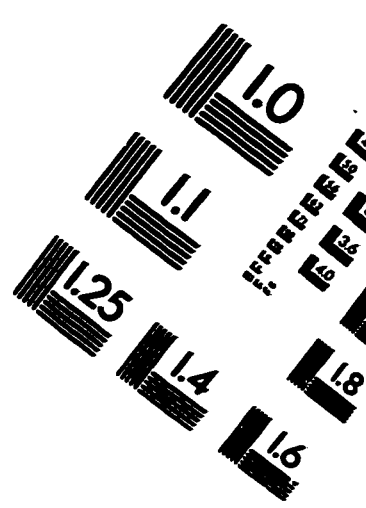
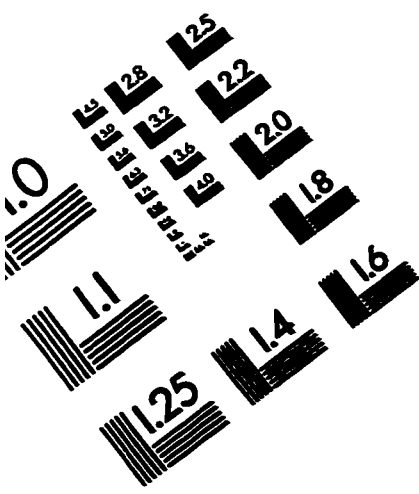
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