

**YOUNG CHILDREN'S MATHEMATICAL SPATIAL REASONING IN A  
MONTESSORI CLASSROOM**

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## Abstract

The object of this research was to investigate young children's mathematical spatial reasoning in a Montessori classroom. Spatial reasoning is an important part of children's mathematical learning and development; however, opportunities for rich spatial reasoning are not readily available in the classroom. Rather, there is a focus on numeracy at the expense of geometry where activities for spatial development are usually found. Montessori designed a sensory curriculum around children's development, yet spatial reasoning in a Montessori classroom has not been fully investigated. This was a qualitative study using some tools of ethnography. The theoretical framework was Radford's *sensuous cognition* (2013, 2014) which allowed for an understanding of human development as cultural with the body essential to that development. The data, captured by video, were the children's *semiotic traces* (Bartolini Bussi and Baccaglini-Frank (2015, p. 393) which are the visible productions of the children's spatial reasoning such as their movements, text, drawings, and speech. The analysis found that the children had ample opportunities for engaging in challenging mathematical problems which required their spatial reasoning. These engaging activities resulted in the children using a wide range of spatial skills as they reasoned mathematically. The children's movement, the main semiotic trace generated by the children, was crucial to their spatial reasoning. This investigation concluded the pedagogical practices created a rich and dynamic environment for the children's spatial development. Practices included the use of well-designed mathematical manipulatives, engagement in the manner of guided play, co-operative learning with peers of mixed ages, extensive time for activities, and assessment based on observations of individual children.

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## **Chapter 1: Introduction**

### **The Question of the Children's Spatial Reasoning**

The goal of this research was to understand young children's spatial reasoning in a Montessori classroom. While recent research interest into spatial reasoning (Newcombe & Stieff, 2012; Sinclair & Bruce, 2015) emphasized its key role in the development of children and their success at school, as well as its contribution to the development of children's mathematical understanding (Cheng & Mix, 2014; Hawes, Tepylo & Moss, 2015; Mulligan & Mitchelmore, 2013; Wai, Lubinski, & Benbow, 2009), young children's spatial reasoning abilities have been, and still are, underestimated (Leavy, Pope, & Breatnach, 2018). Several reasons have contributed to this underestimation. To begin, there has been a narrow interpretation of the mathematics curriculum which tends to focus on numeracy more than on constructs such as spatial reasoning (Bruce & Hawes, 2015; Bruce et al., 2012; Ng & Sinclair, 2015a; Sinclair & Moss, 2012; Taylor & Hutton, 2013; Woolcott et al., 2020). Furthermore, the time allotted for engagement with ideas, concepts, and problems which are needed for the development of spatial reasoning has been limited (Clements et al., 1999; Hawes et al., 2015; Tepylo et al., 2015). This time constraint has also impacted the quality of engagement with mathematical thinking tools, both symbolic (mathematical language, conventions) and material (manipulatives, dynamic mathematical environments) which contribute importantly to children's spatial reasoning (Hallowell et al., 2015; Maschietto & Bartolini Bussi, 2009; Newcombe & Frick, 2010; Ng & Sinclair, 2015a; Sauter et al., 2012). In addition, researchers have demonstrated that research findings resulting from a focus on children's verbal productions may underestimate their actual knowledge (Kim et al., 2011; McGarvey et al., 2015; Sinclair & Moss, 2012). Finally, researchers have demonstrated

that the way the assessment of mathematical understanding has been carried out has provided us with a limited view of children's capabilities (Hallowell et al., 2015; Hawes et al., 2015), capabilities which include the ability to perform mental rotations (Bruce & Hawes, 2015; Hawes et al., 2015; Newcombe & Frick, 2010), and the ability to learn mathematics at a young age (Clements & Sarama, 2011; English & Mulligan, 2013; Lee & Ginsburg, 2009; Moss et al., 2016).

These findings have emerged as researchers have elaborated the designs of their studies and expanded their data sources in order to understand more fully how children reason spatially. For example, some researchers have investigated young children's use of drawings and gestures in expressing understanding (Kim et al., 2011; Thom & McGarvey, 2015). Others have investigated how the use of digital technology affects development (Bruce & Hawes, 2015; Kaur, 2015; Sinclair & Moss, 2012), and how manipulatives and other artifacts support children's spatial learning (Taylor & Hutton, 2013; Tepylo et al., 2015). All these approaches have lent research evidence to the conclusion that children are more capable than had been previously assumed.

Surprisingly, the educational environment which has not received much research attention in this regard is the Montessori environment. I say surprisingly since it has been over a hundred years that Maria Montessori, (1870-1952), a physician and social reformer, developed and implemented a sense-based curriculum designed around children's sensory expression. For Montessori, these sensory activities undertaken by the children were an exercise in reasoning (Montessori, 1913/2013, p. 212). As Montessori envisioned it, children's spontaneous activity in meaningful work is essential for learning, and consequently a key role for the teacher is to stimulate each child's interest through engaging activities with the use of materials. Activities

need to be open-ended, have multiple points of entry, and require children to use a variety of resources to solve the problems they encounter. In efforts to sustain the children's interest and concentration, Montessori (1914/1965) suggested that children are accorded as much time as they require to work, and that they themselves decide when their work is finished. Ideally, children should not be interrupted in their work and therefore Montessori eschewed scheduled periods or lessons. Montessori envisioned children having considerable autonomy in their movements as well. They may work on the floor, in the corridor, go room-to-room, or work outside. It must be noted, however, that there are considerable differences between schools as to how the various Montessori principles are instantiated (Lillard & Else-Quest, 2006).

### **Research Questions**

The goal of this research is to contribute to the mathematics education literature by investigating children's spatial reasoning in one Montessori classroom. The study is guided by the following research questions:

1. How is mathematical spatial reasoning experienced by young children as they participate in a Montessori mathematics curriculum?
2. In what ways do the Montessori materials allow the children to engage in spatial reasoning activities and actions?
3. How are the children's expressions of mathematical spatial reasoning revealed and communicated?

This research is important to me personally. Starting at age 3, each of my three children attended a Montessori school and over the years I observed in their various classrooms. Hearing my children's reasoning as they chatted to me about their activities with the materials was a stimulating experience. I later trained as a Montessori educator and my experiences of teaching

mathematics for ten years in a Montessori classroom, together with my understanding of the current challenges in mathematics education, have led me to think that the Montessori environment may have something to offer today in supporting young children's mathematical spatial reasoning. This study now gives me the opportunity to take on the role of researcher, rather than teacher, in investigating children's spatial reasoning.

### **Overview of The Document**

In Chapter 2, I define spatial reasoning as I have understood it, and review the literature on young children's spatial reasoning as presented through various studies in mathematics education and the psychological literature which have focused on young children's spatial reasoning and their mathematical development in the primary classroom. I also review the literature on children's play and their mathematical learning. In Chapter 3, I introduce the theoretical framework I have used and discuss Radford's sensuous cognition. In this discussion I have included an overview of Vygotsky's sociocultural theory and as well as an overview of theories of embodied cognition, particularly as it pertains to mathematics as described in the work of Lakoff and Nunez (2000) and Nunez, Edwards, and Matos (1999). Chapter 4 explains the methodology I have used for this qualitative study, for example, why I used video-recordings to collect the data, while Chapter 5, which is entitled The Learning Context, is a modification from the standard structure of a thesis. I considered this chapter on the learning context of the Montessori classroom necessary to support the reader's understanding of the milieu within which the children may reason spatially. The Montessori environment is different from a traditional classroom in numerous ways such as how the children are assessed and the lack of a scheduled lessons. This chapter is followed by the results in Chapter 6. The data in Chapter 6 have been presented in 11 tables and each table pertains to mathematical activities during which various

children reason spatially as they work with mathematical materials. Following each table is a narrative analysis of the data and then a shorter section focussing on the specific spatial reasoning revealed by the children in each table. In some cases, there is commentary on the analysis as well. The discussion, conclusion, and a section called post-analytic reflections are in the final chapter, Chapter 7, as are the limitations of the study. Included in this chapter are the possible contributions of this study, as well as suggestions for future areas of research.

## **Chapter 2: Review of the Literature**

In this review of the literature, I begin by defining spatial reasoning generally to provide an understanding for the term and how I use it. I review the literature which has been undertaken from various perspectives, including sociocultural and embodied theories. I do this to situate it as research that has attempted to investigate the challenges to the development of children's spatial reasoning. I briefly review the literature that explores some challenges to a spatial reasoning curriculum within mathematics education. I also offer a brief overview of some of the current research that has focused on Montessori learning environments which may be illustrative of the lack of investigations focusing explicitly on children's spatial reasoning in a Montessori setting. I consider the research on the role of play in children's mathematics learning as it is pertinent to this study as the children's activities in the Montessori classroom may be positioned as play (Samuelsson & Carlsson, 2008; Wager, 2013). I explore the notion of a gap in the literature where research on children's spatial reasoning in a Montessori classroom could be positioned.

### **Defining Spatial Reasoning**

Spatial reasoning is investigated in this study through the lens of the children's mathematical learning, but it is important to recognize the construct of spatial reasoning is found across disciplines (Newcombe & Frick, 2010; Woolcott et al., 2020). For example, the National Research Council of the National Academies (2006) addresses spatial reasoning from multiple perspectives such as in everyday life, at work, in the sciences, in geography, and in the geosciences (p. xi). However, while our experiences in the world are understood as "inherently spatial" (Lowrie et al., 2018, p. 1), a definition of spatial reasoning is contested (Uttal et al., 2013, p. 353).

In mathematics education the construct itself is referred to by various terminologies such as “spatial ability, spatial intelligence, or spatiality” (Mulligan, 2015, p. 513). Other terms used are “visuospatial reasoning”, “visualisation and visualising”, “spatial thinking” and “visual reasoning” (Sinclair et al., 2016, p. 696). There is debate as to the relationship of constructs such as visualization, visual-spatial reasoning, and spatial reasoning. The Spatial Reasoning Study Group (2015), a transdisciplinary research group, point out some authors use the terms interchangeably while others consider them distinct (p. 4). The Group (2015) use *spatial reasoning* as their preferred term and acknowledge the various meanings around the concept of spatial reasoning viewing it as a “multifaceted construct” (p. 1). Their focus is on a range of dynamic processes they see as inherent in spatial reasoning. For them, spatial reasoning “involves recognizing the dynamic or changing spatial relations among two or more objects or between one’s own moving body and objects or landmarks in the environment” (p. 5). Examples of these dynamic processes include locating, orienting, comparing, scaling, decomposing/recomposing, balancing, diagramming, transforming, and sensing. Mulligan et al. (2020) agree the term spatial reasoning is open to a wide interpretation of meanings but suggest skills such as visualization, mental rotation, symmetry, perspective-taking, locating, orienting, decomposing/recomposing and navigating are usually included in definitions (p. 286). Hawes et al. (2017) acknowledge that while some definitions of spatial reasoning imply an interaction with the physical environment where spatial skills such as navigation would be employed, theirs does not. In their study that looked at enhancing children’s spatial and numerical skills they use a definition of spatial reasoning that concerns “forming and manipulating visual-spatial mental images” (p. 236).

Some researchers draw attention to the role of the body in spatial reasoning: Authors Leavy et al. (2018) referring to young children's spatial awareness as they explore their immediate physical environment term this "spatial orientation" (p. 117). Owens (2020) offers an ecocultural perspective in her research on spatial reasoning where cultural experiences in the environment are examples of physical involvement that help develop spatial reasoning (p. 257). Kahn et al. (2015) suggest that spatial reasoning is the co-occurrence of sensation, perception, and the "situated movement of a body (or bodies) in the context of a goal-oriented situation" (p. 274). Despite the different terms and definitions used for spatial reasoning there is "converging agreement on the importance and malleability of visuospatial reasoning" (Sinclair et al. 2016, p. 696). What these terms do refer to is the ability to recognize and manipulate the spatial properties of objects and the spatial relations among objects (Mulligan, 2015, p. 513) and authors Moss et al. (2016) offer a definition of spatial reasoning in young children as "the ability to visualize, engage in perspective taking, situate oneself and other objects in space, and to mentally rotate figures" (p. 165).

As mentioned, spatial reasoning is considered an emergent process and while spatial reasoning may be discerned as a whole, it cannot be fully "comprehended by reducing it to its components" (Davis et al., 2015, p. 140). In the discussion above various spatial skills have been mentioned and despite the challenges this implies for investigation, specific spatial reasoning skills have been the subject of studies. For instance, decomposing and composing are spatial skills relating to the breaking down, or separating, of a composite shape into smaller units and building the shape back up. In two studies, Cross et al. (2012) and Jirout and Newcombe (2015), investigated these spatial skills with children's block building. Spatial visualization is a spatial skill involving a "multi-step processing of spatial information" (Casey, 2008, p. 272). It is a

process which involves reasoning associated with visual imagery but expressed and argued with spatial references” (Owens, 2015, p. 7). Spatial visualization involves imagining a shape then imagining it combined with other shapes to create a new shape (Casey et al., 2008). Spatial skills such as comparing are used in proportional reasoning to reason about the relationship of parts to the whole, i.e., combining parts to make a whole, (Casey et al., 2008) or in making sense of part-to-part relationships (Möhring, Newcombe, Levine, & Frick, 2016). An understanding of fractions, for instance, involves reasoning spatially about part-to-whole relationships. Another spatial skill is perspective-taking. This is the ability to imagine a viewpoint different from one’s own (Frick, Möhring, & Newcombe, 2014), and involves the use of spatial skills to make sense of object-to-object relations or self-to-object relations and the movements, or navigation, around these objects, or around self (Okamoto, Kotsopoulos, McGarvey, & Hallowell, 2015; van den Heuvel-Panhuizen, Iliade & Robitzsch, 2015). Symmetry is a spatial skill that is an important idea in mathematics and is relevant to reasoning about two- and three-dimensional figures (Ng & Sinclair, 2015a). Thinking symmetrically involves “recognizing equality, near equality, and balance” and is useful for problem solving mathematical and spatial problems (Owens, 2020, p. 279). However, researchers Ng and Sinclair (2015a) found that for young children, activities with symmetry often involve static instances such as finding the symmetry in a heart, for example, while the notion of symmetry as transformation involves three-dimensional figures and requires a dynamic approach such as with the use of digital technologies. Spatial scaling is a spatial skill used where children scale shapes across size differences. Hallowell et al. (2015) found children had difficulty with activities involving scaling.

Further spatial skills are combined with fine motor skills in activities that require manipulating puzzle pieces or objects into various positions within containers or openings (Bruce

& Hawes, 2015; Leavy, Pope, & Breatnach, 2018). Mental rotation is spatial skill that might be used in these instances (Bruce et al., 2015, p. 96). In mental rotation, two-dimensional and three-dimensional objects remain intact but are rotated through space around different axes (Bruce & Hawes, 2015; Lowrie et al., 2018). Mental rotation may be used in conjunction with composition and decomposition with two- and three- dimensional figures, in conjunction with symmetry and in conjunction with navigation tasks involving orientation, locating and map reading (Bruce & Hawes, 2015; Vander Heyden, Huizinga, & Jolles, 2017). All these emergent spatial skills are considered foundational to the development of children's spatial reasoning (Polly, Hill, & Julvanic, 2015).

For the purposes of this study, I understand mathematical spatial reasoning as those spatial reasoning skills used during the children's mathematical activities. Spatial reasoning is used when considering relationships between objects and relationships between objects and self. I use the term spatial reasoning in preference to other terms such as visuospatial reasoning as it allows for an inclusion of all the senses and does not highlight the visual sense. It is also the term used by the Spatial Reasoning Study Group and is increasingly used in mathematics education. I consider the body to play an essential role in the development of spatial reasoning. This allows spatial reasoning to be considered a specific area of non-verbal reasoning which "can be practiced with limited or no use of the eyes – with the hands, with the moving body and gestures" (Whiteley et al., 2015, p. 11). I consider spatial reasoning an emergent process not comprised of isolatable skills, but one that has many working in tandem (Whiteley et al., 2015, p. 5). Examples of spatial reasoning skills considered under my understanding of spatial reasoning are spatial visualization, mental rotation, symmetry, composition, decomposition and recomposition, orientation, spatial scaling, and comparison. Spatial reasoning is understood as a

dynamic process taking place over time and is used for problem solving in daily life as well as in mathematical problems. Finally, like any learning, I consider the development of spatial skills as dependent on the specific cultural context within which they develop. I am adopting this conception of spatial reasoning as it allows for an investigation that considers the children's movements and the role of their bodies in expressing spatial knowledge, as well as recognizing and including the contextual nature of their spatial experiences.

### **Current Interest in Spatial Reasoning in Mathematics Education**

In mathematics education research there has been renewed interest in young children's spatial reasoning. This renewed interest is in part due to research from the field of psychology which concludes the importance of spatial reasoning in children's mathematical development and other STEM subjects (Bruce et al., 2105; Hawes et al., 2022; Mulligan et al., 2020; Sinclair & Bruce, 2015; van den Heuvel-Panhuizen et al., 2015; Verdine et al., 2014; Wai et al., 2009). Spatial reasoning skills play a role in the development of patterning skills (Clements & Sarama, 2011) and spatial structuring which is an important part of the development of young children's number sense (Mulligan & Mitchelmore, 2013; van Nes & de Lange, 2007). However, despite the importance of spatial reasoning it is considered "an under-utilized bridging mechanism between real-world experiences and the mathematics curriculum" as suggested by the Spatial Reasoning Study Group (Woolcott et al., 2020, p. 2326).

Other studies in mathematics education have also drawn on the research in psychology where skills are considered malleable with variables such as gender and socioeconomic status possibly playing a role (Newcombe & Stieff, 2012; Uttal et al., 2013). Bruce and Hawes (2015) for example, conducted a classroom intervention with the goal of investigating its impact on children's mental rotation. The children were given two-dimensional mental rotation tasks

adapted from a widely used measure of children's spatial ability taken from the psychology literature (see Harris et al., 2013). The authors report on one team of 7 teachers and 42 students aged 4 to 8 years and found their data supported findings that mental rotation skills are malleable and can be improved, and at an earlier age than previously measured. Kostopoulos et al. (2015) use the same approach as Bruce and Hawes (2015) but include children's drawings and gestures which they suggest are helpful in eliciting children's underlying knowledge. Their research investigated how children 8 to 10 years of age represent motion in large-scale mapping tasks. Their results showed that children who struggled with mathematics drew fewer objects in their drawings, performed fewer gestures, and gave fewer verbal descriptions compared with the children who were more successful in mathematics. The authors suggest that children's drawings and gestures provide a useful lens for examining factors that may contribute to the difficulties some children have with tasks such as these (Bruce & Hawes, 2015, p. 461).

In mathematics education, research on the topic that is increasingly being termed spatial reasoning has frequently been undertaken within the parameters of the geometry curriculum (e.g., Sinclair et al., 2016). In response to the growing body of research on the importance of spatial reasoning for young children (e.g., Davis and the Spatial Reasoning Study Group, 2015; Newcombe & Stieff, 2012), researchers and educators are now suggesting expanding the geometry curriculum and considering the whole mathematics curriculum as offering opportunities for developing spatial abilities (Bruce et al., 2015; Clements & Sarama, 2011; Mulligan, 2015; Sinclair & Bruce, 2015). In keeping with this interest, an ICME-13 survey report in ZDM Mathematics Education examined the research on geometry education since 2008 (Sinclair et al., 2016). Some of the themes considered important include the development of theoretical perspectives, developments in understanding spatial reasoning, the role of diagrams

and gestures, the role of digital technologies, the teaching and learning of definitions, and three-dimensional and non-Euclidean geometries. Accompanying these important themes was a call for increased research. The themes of this report echo those of an earlier special issue of *ZDM Mathematics Education* (2015) which presents research findings from investigations on children's spatial reasoning, such as the use of drawings and diagrams (Thom & McGarvey, 2015), including digital technology (Clements & Sarama, 2011; Lai & White, 2014; Ng & Sinclair, 2015b), the importance of transformational skills (Bruce & Hawes, 2015), the use of imagination (van den Heuvel-Panhuizen et al., 2015), methods of assessment (Hallowell et al., 2015), and a broadening view of the geometry curriculum including concepts of space, composing and decomposing figures, symmetry, rotation skills, and classification (Moss et al., 2015; Sinclair & Bruce, 2015).

In a recent multidisciplinary research study, researchers from both the fields of psychology and education used a meta-analysis to investigate children's spatial reasoning activities in relation to their mathematics learning (Hawes et al., 2022). In their study, the authors are motivated by the accepted notion derived from decades of research that spatial reasoning is a contributing factor to the development of mathematical reasoning. The authors wished to investigate whether, what they termed spatial training, would be an effective way of improving mathematical abilities as historically this prediction was based on theoretical and correlational research (Hawes et al., 2022, p. 112). Spatial training may be understood as opportunities where children may engage in activities that require them to use various spatial reasoning skills. The authors point out that research which has investigated the relationship between spatial reasoning and mathematics has generally focused on the spatial skill of visualization but that there is a growing body of evidence that suggests other spatial skills such as scaling, proportional

reasoning, and “figure copying” are important in mathematical reasoning (Hawes et al., 2022, p. 113). This meta-analysis made important findings with regards to spatial reasoning. It established a causal relationship between spatial training and improved spatial reasoning, and between spatial training and mathematics performance. The analysis also found that the use of objects or materials used with the spatial training in some studies were more effective in training spatial thinking than those studies that did not use them as, for example, in studies using digital manipulatives for spatial training.

Lowrie et al. (2020), however, draw attention to the challenges that may arise when interpreting research undertaken in the fields of psychology and mathematics education which have investigated the relationship between spatial reasoning and mathematics learning. Citing Bruce et al. (2017), the authors point out that while mathematics educators and cognitive psychologists appear to be addressing similar issues, they operate independently from each other. There are a variety of reasons for this, such as the tools used in psychological spatial tests on one hand and the methods used in the “spatially rich mathematics tasks students encounter in school-based” assessments on the other hand (Lowrie et al., 2020, p. 184). Their commentary builds on observations by researchers Kahn, Francis, and Davis (2015) who suggest that much of the research emerging from the psychology literature focuses on mental processes, which are not visible, and on “the relationship between defined tasks and the outcomes of those invisible mental processes” and as such what is happening “elsewhere in the body” (Kahn et al., 2015, p. 270) is not part of any discussion.

Research using concepts of developmental levels builds on key theories such as those put forward by Piaget and Van Hiele who suggest that children must pass through one level to reach the next. Piaget’s stages are based on children’s ages while Van Hiele’s levels are instruction-

based, meaning he believed that supporting children with language appropriate to the task aided development from one level to the other (George, 2017). Some present-day research has also focused on levels and children's tasks as a means for understanding and improving children's spatial reasoning skills. One such study uses the construct of learning trajectories which is defined as a "conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (Sarama et al., 2011, p. 668). In this study, the authors develop and use a learning trajectory for young children's learning with regards to the measurement of length. The authors argue their findings support the construct of a learning trajectory in that the children "reliably moved through the levels of thinking in that progression" (p. 667). Currently, however, both Piagetian and Van Hiele theories have been considered by some researchers as essentialist and hierarchical (Kaur, 2015; Kim et al., 2011).

Alternative theoretical frameworks are now replacing the concept of appropriate development levels for spatial reasoning (see Clements & Sarama, 2009; Piaget & Inhelder, 1956/1985; Van Hiele, 1986). This renewed interest in young children's spatial reasoning is also related to the growing consideration of embodied cognition and the context of learning (Bruce et al., 2017, p. 145; Lakoff & Nunez, 2000; Radford, 2013, 2014). An increasing number of research studies draw on two theoretical strands: One strand is sociocultural theory emanating from the work of Vygotsky (1978) and others, such as found in the work of Roth and Radford (2011). This strand allows for an understanding of human development and spatial reasoning as occurring within a social context mediated by the use of objects and symbols. The other strand is embodied cognition which draws on the work of Lakoff and Nunez (2000) and Varela,

Thompson, and Rosch (1991). This perspective pays particular attention to the role of the body in human development. In the next two sections I examine each of these in turn.

From a sociocultural perspective spatial development emerges after meaningful social interaction with others in mathematical activity (van Oers, 2010). This lens draws our attention to the real person who has been “developed in and through the social relations” that have been entertained (Roth & Radford, 2011, p. 89). Recently, research from this perspective was undertaken to investigate children’s challenges in developing an inclusive mathematical definition for squares and rectangles. In this study the researchers focus on the children’s use of manipulatives, gestures, drawings, and written and spoken language (Bartolini Bussi & Baccaglini-Frank, 2015). They draw attention to the cultural role of the teacher and refer to the studies of Luria (1976) that showed geometric perception is dependent on the culture to which the children are exposed. The researchers introduce a robot programmed to trace paths with  $90^\circ$  turns and hypothesize that the robot, in its travels, might support children’s developing awareness of the four right-angled properties of rectangles. They found that in the initial stages of the study when children did not have exact terminology, the children employed gestures. Later, the children offered conjectures that were relatively advanced and invented a new term, the “squarized O.” The authors suggest their study demonstrated how the semiotic potential of the classroom might be exploited (Bartolini Bussi & Baccaglini-Frank, 2015, p. 404).

Two other studies, one by Ng & Sinclair (2015a) and one by Kaur (2015) use Sfard’s (2008) communicational approach but consider the children’s use of gestures as well. Ng and Sinclair (2015a) suggest children are capable of comprehending symmetry at a young age and they investigate this with children from a Grade 1/2 and Grade 2/3 classroom who explore reflectional symmetry using digital technology. The goal, in addition to noting the children’s

discourse, was to investigate their use of signs, the role of technology, and teacher mediation. The researchers found that the children were able to create meaning for themselves during their pencil and paper work and with the dynamic software. They also found that their research supported previous studies that showed the teacher's use of language is an important factor in shaping how children think about mathematics. In Kaur's study (2015) an intervention was undertaken in a Grade 2/3 classroom to find a solution to address the problems that arise in the teaching and learning of triangles. The goal was to investigate whether young children's use of digital technologies could help them develop an understanding of and reasoning about the properties of different triangles. The study showed that the children's routines moved from informal tool-based descriptions to more formal properties, and from a more particular discourse to a general discourse (p. 418). Kaur (2015) suggests that more generalized statements may be due to the teacher's use of exact nomenclature and the strong link, found in early childhood psychology literature, between object naming and object categorization (p. 419).

A perspective of embodied cognition suggests that cognitive development occurs in our bodies and with the use of our bodies (e.g., Lakoff & Johnson, 1980; Lakoff & Nunez, 2000; Varela et al., 1991). As far as mathematics is concerned, sensorimotor experiences greatly influence our development and our understanding of mathematics to the extent that "the only mathematics we can know is the mathematics that our bodies and brains allow us to know" (Lakoff & Nunez, 2000, p. 346). For instance, Kim et al. (2011) take gestures into account because "there is much of children's thinking that is not captured when we only consider their verbal expressions" (p. 209). In their study, which involved 23 second grade students, they investigate how gestures help children manage challenging ideas in geometry. Their paper presents three episodes of data they consider illustrate the relationship of children's spatial

thinking and gestures. The authors, Kim et al. (2011), found that the children's bodily movements and gestures supported their development of geometric concepts at a young age and that the children sometimes used gestures without speech as they explored new ideas (p. 232). This is consistent with the findings of other research (see Bartolini Bussi & Baccaglini-Frank, 2015; Kaur, 2015).

Researchers Thom and McGarvey (2015) ask the question whether drawings may serve as a means by which children become aware of geometric concepts and relationships rather than viewing the drawing as a product of that awareness (p. 465). The authors present three vignettes chosen as showing Grade 2 children drawing spontaneously during geometry lessons. The authors conclude that the children's drawings cannot be taken as evidence of their geometric understanding, but rather as opportunities for spontaneous, iterative drawing that "may serve as a space for geometric exploration and conceptual invention" (p. 479).

### **Children's Spatial Reasoning and the Context of the Classroom**

The review reveals there are challenges to implementing a rich spatial curriculum for children in school where spatial reasoning opportunities typically occur within the geometry curriculum (Sinclair et al., 2016). Despite the research on the crucial importance of spatial reasoning for children, the geometry curriculum does not necessarily support it (Mulligan, 2015; Okamoto et al., 2015). There is limited instructional time for geometry and the geometry curriculum is limited in scope (Moss, Hawes, Naqvi, & Caswell, 2015). Causes for this may be the artificial separation of different strands of mathematics where numeracy is privileged over geometry (Mulligan, 2015; Whiteley, Sinclair, & Davis, 2015). This creation of separate strands in the mathematics curriculum separates shape from number ignoring the "deep connections between the quantitative and the qualitative" (Whiteley et al., 2015, p. 7). In a recent study

Möhring, Newcombe, Levine, and Frick (2016) explicitly associate spatial proportional reasoning with a formal knowledge of fractions, that is, they connect the quantitative with the qualitative. Proportional reasoning concerns whole-part relationships and part-part relationships and is an important skill not only in mathematics but in other sciences and in everyday activities. The authors refer to the research of Mix, Levine, & Huttenlocher (1999) who suggested that the early focus on counting and whole number operations, that is, a focus on the “quantitative” (Whiteley et al., 2015, p. 7) may interfere with children’s ability to understand fractions and to perform calculations with fractions. According to Möhring et al. (2016) this gives rise to the question of whether children’s understanding of fractions presented numerically aligns with their sensitivity to proportional relationships presented in non-numerical ways, that is, spatially. They investigated the proportional reasoning and formal knowledge of fractions of 52 children aged 8 to 10 years. The children were required to reason about the amount of cherry juice in relation to the amount of water in a drink. The data pertaining to the cherry juice and water combination were presented in two separate graphs, one stacked and one side-by-side. Their research results showed that children could reason proportionally, younger than previously found, and that the children’s proportional reasoning skills were related to their knowledge of formal fractions.

Apparent within the studies are findings that other factors which support children’s spatial skills are not necessarily available to the children within the context of the classroom. The development of verbal language skills and the use of spatial language is considered important (Casey et al., 2008; Ng & Sinclair, 2015a; Oudgenoeg-Paz, Leseman, & Volman, 2015). Oral story-telling and other narratives may introduce spatial vocabulary but also provide a rich context for learning spatial language skills (Casey et al., 2008; Leavy, Pope, & Breatnach, 2018), and serve as an anchor for the children as they learn new concepts (Cross et al. 2012). Stories

encourage the use of imagination which is considered important in spatial reasoning, for spatial skills such as spatial visualization and mental rotation activities (Bartolini Bussi & Baccaglini-Frank, 2015). Paraphrasing Tahta, (1980) Sinclair & Bruce (2015) assert that without imagery there is no mathematics. Drawing, figure-copying skills, and tracing may support the children's spatial reasoning (Ginns et al., 2014; Hawes, 2022; Sinclair & Bruce, 2015; Thom & McGarvey, 2015). As far as the classroom is concerned, a carefully planned environment that allows for exploration (Polly et al., 2015; Wager, 2013) is considered valuable in supporting the children's spatial reasoning. Authors Oudgenoeg-Paz, Leseman, & Volman, (2015) suggest the children's explorations in the classroom are of considerable importance in the development of their motor milestones and spatial cognition and language. Explorations with spatial toys support the children's spatial reasoning (Jirout & Newcombe, 2015), however exploration with spatial materials should allow for a variety of shapes, and not be limited to traditional or prototypical shapes (Kaur, 2015; Verdine et al., 2014). Spatial reasoning activities that provide for sustained attention are important (Vander Heyden et al., 2016). Owens (2020) suggests that attention is important as when properties are noticed it can lead to the children reasoning about those properties and the relationships between them. Opportunities for movement with the body are considered important (Casey et al., 2008; Fisher et al., 2013; Leavy et al., 2018; Oudgenoeg-Paz et al., 2015). These studies suggest the importance of various considerations in the classroom for the development of spatial skills and draw our attention to the realization that the development of robust spatial skills may be constrained by the learning environment children find themselves in.

### **Research on Montessori and Spatial Reasoning**

In Montessori research the construct of spatial reasoning does not often appear in the literature, but there is research that could be considered to implicitly incorporate the notion. For

instance, there is research investigating children's understanding of place value (Chisnall & Maher, 2007; Laski et al., 2016; Reed, 2008), the development of fine motor skills (Bhatia et al., 2015; Chisnall & Maher, 2007; Ginns et al., 2015), and the acquisition of concepts of geometric shapes (Ongoren & Turcan, 2009). However, much of the research in Montessori education has focused largely on children's use of manipulatives in the classroom (Chisnall & Maher, 2007; Laski et al., 2015), and on the educational outcomes of a Montessori education (Dohrmann et al., 2007; Lillard & Else-Quest, 2006; Ruijs, 2017).

One study assessed 34 Montessori children and 28 non-Montessori children using a place-value task with the use of manipulatives. The authors concluded that conceptual development for numeracy may be enhanced with the use of manipulatives (Chisnall & Maher, 2007). In another study, researchers Laski et al. (2015) draw attention to a meta-analysis of 55 studies which investigated the use of manipulatives. The authors of the meta-analysis concluded the use of manipulatives is beneficial but only under certain conditions (see Carbonneau et al., 2013). The authors of the Montessori study, Laski et al. (2015), suggest that these certain conditions are found in the Montessori setting. A study of interest from the field of developmental psychology investigated both place value and the use of concrete manipulatives. In this study authors Mix et al. (2017) investigated whether the use of manipulatives, namely base 10 materials "grounded the symbols" for learning place value (p. 129). The investigation consisted of two experiments. Relevant to this study was the second experiment. Here the sample consisted of 68 children; half the children had been in a well-reputed Montessori school since age 3. The other half of the children had attended non-Montessori schools. All 68 children were divided into two age groups – kindergarten and Grade Two with equal numbers of Montessori and non-Montessori children in each group. The children were given three mathematical tasks to solve, one of which was a

place value task. The results showed that in the place value task the Montessori children had a clear advantage. The authors, Mix et al. (2017), suggest that the advantage may be due to the “distinguishing feature of Montessori mathematics instruction [which] is its early, consistent integration of concrete models” (p. 144). The advantage, however, emerged only with the older children and the authors suggest that this may be the “fruition of a long incubation period during which children gradually internalize their experiences with concrete models and link them to symbolic procedures” (p. 144).

With regards to research on Montessori outcomes, Ruijs (2017) investigated the outcomes of secondary school students who had won admission lotteries to Dutch Montessori schools. The author concluded that Montessori offered an alternative way for students to attain similar outcomes as non-Montessori students but was neither better nor worse. Another earlier study compared outcomes for Montessori and non-Montessori secondary students in the United States and found that the Montessori students had significantly higher scores on tests “associated with the math/science factor” (Dohrmann et al., 2007, p. 205). A further study investigated the learning outcomes of primary and elementary children who took part in tests designed to show academic and social skills. The authors concluded that Montessori education has a fundamentally different structure from traditional education and when strictly implemented it “fosters social and academic skills that are equal or superior to those fostered by a pool of other types of schools” (Lillard & Else-Quest, 2006, p. 1894).

### **Play and Mathematics Learning in the Classroom**

In this section of the literature review I explore the research regarding children’s play and mathematics learning and how, in the context of educational settings, play is defined. Much of the literature on play and learning emerges from studies undertaken in early childhood settings

where today play is considered a strength through which children develop a wide variety of important skills (Ilgaz, Hassinger-Das, Hirsh-Pasek, & Golinkoff, 2018; Lillard, 2013; Samuelsson & Carlsson, 2008; Samuelsson & Johansson, 2006). A distinction has been made between free, or spontaneous, play, and guided play. Some scholars suggest further concepts such as playful learning (Ilgaz et al., 2018; Lillard, 2013) or the concept of the “playing learning child” (Samuelsson & Carlsson, 2008, p. 1470). In free play children are not directed by the teacher but spontaneously select activities that appeal to them. Free play offers activities that are fun, joyful, voluntary, flexible, involve active engagement of the part of the child and have an element of make-believe (Lillard, 2013; Samuelsson & Johansson, 2006; Weisberg et al., 2013). The children’s play is considered their own with the teacher’s role to protect their playing and not to become involved in it. Guided play is a more directed approach with the teacher creating interesting and engaging activities that meet the teacher’s learning objectives for the children. The teacher initiates the learning process, and maintains focus on the learning goals, but the child directs his or her own learning within the context of the play. The children’s learning is supported by the teacher’s comments, observations, discussions, and reflections about the activities with the children (Fisher, Hirsh-Pasek, Golinkoff, & Newcombe, 2013). In both free play and guided play the child is in control of the activity, however in free play children are unlikely to achieve the learning goals the teacher has because of the freedom afforded them (van Oers, 2010). In guided play “the adult follows the child’s lead and allows the child to engage in discovery within the context of a prepared environment and with subtle scaffolding” (Weisberg et al. 2013, p., 105). In addition, the teacher introduces new content and new vocabulary

Play and learning have tended to be dichotomized in research and in practice (Björklund, Magnusson, & Palmér, 2018; Ilgaz et al. 2018; Samuelsson & Johansson, 2006; van Oers, 1994),

and this has been apparent in early mathematics education as well. Clements and Sarama (2014) call the distinction a false dichotomy and argue that children can play and learn at the same time. They maintain that in play children naturally engage in mathematical thinking (see van Hiele, 1999). Attempts have been made to erase the dichotomy of play and learning however a clear definition of play remains elusive (van Oers, 2013).

As mentioned earlier, a review of the literature revealed that much of the research on play comes from the field of early childhood education. At the grade level discussions of play appear to be replaced by notions of fun, games, or enjoyable activities. For instance, researchers suggest gamification to improve elementary children's engagement in geometry activities. Gamification uses elements of games such as pre-defined goals, players' actions defined by rules, and a quantifiable outcome a result of playing the game (e. g. Kamalodeen, Ramsawak-Jodha, Figaro-Henry, Jaggernauth, & Dedovets, 2021, p. 2). The authors suggest gamification could contribute to transforming "the dull learning environments in mathematics classrooms to smart ones" (p. 3). They suggest children may become more engaged in geometry when it is experienced through playing a game.

Whether play is free or guided it has been differentiated from direct instruction. In direct instruction the teacher plays an active, directive role and the children have a more passive role. Some researchers suggest a continuum where direct instruction is positioned at one end and free play at the other with guided play in the middle (Weisberg et al., 2013). In the psychological literature some studies suggest guided play is a suitable approach for engaging children in spatial reasoning activities (Fisher et al. 2013; Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2014). In one study, researchers Fisher et al. (2013) hypothesized that children in guided play would show an improved understanding of the features of shapes rather than relying on appearances.

Sixty children, aged between 4 and 5 years were divided equally into three groups based on pedagogical approach, namely guided play, didactic instruction, and free play. The children took part in a shape-sorting task consisting of 4 cards each of typical shapes, atypical shapes, and non-valid shapes. The children in the guided play group were taught the properties for each shape in a playful and exploratory way. The experimenter made a game of the activity suggesting the shapes had secrets and hid their shapes. The children had to find out what these secrets were and which were the real shapes. The experimenter supported them in this activity through questions and by encouraging the children to touch and trace the shapes. The children then used construction sticks to create new shapes for each of the shapes on the card, and to discuss how the constructed shapes and the card shapes were similar. In the free play cohort, the experimenter organized the cards according to their respective shapes so that triangles, for instance, whether typical or atypical, were in the same stack. After playing with the cards the children were given the construction sticks and told they could use them however they liked. In the experiment with didactic instruction the experimenter used similar wording and intonation as in the guided play but took on the role of discovering what the secrets of the shapes were and which shapes were real. The children observed and listened. The authors found that the children in the guided play showed improved ability to define the shapes and were willing to accept variations of valid shapes while rejecting the non-valid shapes. The children in the other two groups did not show this flexibility. The authors conclude that free play does not offer the children sufficient information to help them form specific concepts of shape, rather “appropriate scaffolding through dialogic enquiry and engagement facilitate geometric shape learning” as found in the guided play group (Fisher et al., 2013, p. 1877). This study considered the development of children’s spatial skills from the perspective of the pedagogy used, while a review by Verdine et

al., (2014) discussed the role of spatial toys and suggested that experiences with toys such as blocks, puzzle, and shapes have a “significant influence on the early development of spatial skills” (p. 7). Van Oers (2010), using sociocultural theory, views children’s mathematical thinking as emerging in the context of play. He suggests the learning needs to be meaningful, and to take place through participation in cultural practices, with the help of a more knowledgeable other such as the teacher.

Wager (2013) outlines what practices could support mathematics learning in a play-based classrooms such as found in Froebel, Montessori, and Reggio Emilio environments. A carefully planned environment with a wide range of materials that provoke engagement with mathematical ideas is necessary to support and encourage exploration. A variety of materials are suitable such as blocks, puzzles, and games. The teacher supports the children’s enquiry with observations of their mathematical activities and by supplying appropriate spatial terminology. A study by Casey et al. (2008) investigated the development of children’s spatial skills with interventions using block building activities. The researchers were also interested in examining the use of oral storytelling in the context of the children’s mathematics learning. They found empirical evidence for the benefits of using the narrative device of storytelling in teaching young children building skills with blocks.

While Montessori pedagogy may have the hallmarks of a play-based curriculum (e.g., Wager, 2013), Montessori spoke strongly against fantasy or pretend play and Lillard (2013) addresses this apparent contradiction when she suggests Montessori pedagogy may be considered *playful learning* which she positions between free play and guided play but spans both. Playful learning “is child centred, constructivist, affectively positive, and hands-on” and may involve fantasy and pretend play (p. 158). Playful learning in the Montessori classroom

involves the use of objects (known as materials) with which children play to learn (p. 163).

During lessons there is dialogue between the children, and the children and the teacher. Lillard

(2013) points out that in playful learning in the Montessori classroom the children's own

interests dictate their learning and children are free to choose not to take part in an activity.

Children may work alone or with others. There are no extrinsic rewards such as grades or

stickers of commendation as the playful learning itself is its own reward. "A well-functioning

Montessori classroom is full of deeply engaged children enjoying themselves" (p. 168).

Montessori termed what is understood as playful learning as *work* - the children's work and she

considered it a serious enterprise. Montessori made a distinction between fantasy and

imagination. Imagination was considered the basis of the elementary curriculum where children

could think about, or imagine events from different times and different places, for instance.

### **Summary**

The review of the literature highlights the vital importance of spatial reasoning, not only for its deeply intertwined relationship with mathematics, but for its value in human development in general (Cheng & Mix, 2014; Davis, Okamoto, & Whiteley, 2015). This review of the literature brings forward important points when considering the development of young children's spatial reasoning in the classroom. Some research addresses the classroom challenges that are identified, namely a focus on children's numeracy at the expense of other areas of the mathematics curriculum demonstrating a deep separation of the shape and number of mathematics, not enough time for prolonged engagement with work, a lack of engaging and challenging mathematical problems, a lack of other factors that support the children's spatial development such as an environment rich in spatial language, opportunities for imagination and exploration, and assessment that does not always acknowledge the various resources children

may use to demonstrate understanding, such as drawings and gestures. The review revealed the way children may engage in the classroom has a role to play in their spatial development. The review of play and playful learning showed there are different understandings of the term. Instead of seeking out a definition of play, authors Samuelsson and Johansson (2006) suggest it might be more worthwhile to consider the valuable characteristics of play that may be brought to children's learning such as its "unpredictability, its symbolic, communicative and social aspects, play as a process, play as children's experiencing joy and meaning, as well as its bodily form and its interconnection with children's life-worlds" (p. 52).

The review found that while developmental approaches to understanding children's spatial development, such as those put forward by Piaget, Pierre Van Hiele, Clements and Sarama, for example, have been highly influential, present research trends are moving away from these hierarchical levels. More recent research studies have used sociocultural and embodied understandings of human development which take children's gestures, drawings, movements into account, as well as considering the cultural and historical context of the children's lives. The review showed that what research has been undertaken in the Montessori environment focused mainly on educational outcomes and the use of manipulatives.

Considering the above review, there are a limited number of research studies undertaken within the Montessori educational community investigating either the Montessori pedagogy or children's learning experiences in the Montessori classroom. In particular, the realization that there is not a body of research on Montessori's contribution to young children's spatial reasoning is puzzling when considering Montessori's detailed sense-based curriculum. In a thorough and insightful review aimed at understanding how spatial reasoning is understood and developed across academic disciplines, some researchers present a detailed timeline showing the historical

emergence of the construct of spatial reasoning (Bruce et al., 2017). The education thread of the timeline is sparse and between Froebel (1782-1852) and Freudenthal (1905-1990), which is a period of roughly 150 years, there is no mention of Montessori. One of the stated goals of the paper is to “understand the gaps in research networks” that show the historical development of the construct of spatial reasoning, and to show that “holistic, problem-rooted inquiries that seek to integrate diverse expertise from across domains” are necessary for the “complex problems” found in the teaching and learning of mathematics (p. 145). Of further interest in their review is the consideration that change is the purpose of mathematics education research, as suggested by Freudenthal (1991), and as put forward by the authors themselves. Citing Choi and Pak (2006, p. 351) the authors reiterate the purpose of research to solve real world problems and to provide different perspectives on them (Bruce et al., 2017, p. 144). An investigation into children’s spatial reasoning in a Montessori environment might be considered especially timely today given the current research interest in spatial reasoning.

The goal of this research, therefore, is to investigate young children’s spatial reasoning in a Montessori classroom while considering the possibility that children in this setting may not experience the same challenges found in other classrooms as far as the development of their mathematical spatial reasoning is concerned. To undertake this research, the study will adopt a theoretical framework and methodology that considers human development as historically and culturally constituted and allows for a consideration of the role of the body in meaning-making.

### **Chapter 3: Theoretical Framework**

In order to elaborate and support my understanding of children's mathematic spatial reasoning as reasoning that is developed through children's activities with materials in a social milieu, I sought an integrated theory that would allow for these considerations. As such, I drew on the work of Radford (2013, 2014) as his theory of sensuous cognition offers an integration of sociocultural theory (Vygotsky, 1978, 1986) and its focus on the social setting, with aspects of theories of embodied cognition (Lakoff & Johnson, 1980; Lakoff & Nunez, 2000; Nunez, Edwards & Matos, 1999) and its focus on the role of the body. Before I discuss Radford's sensuous cognition in more detail, I review important concepts of Vygotsky's sociocultural theory. After this I consider important concepts found in a theory of embodied cognition. During my exploration of sensuous cognition, I discuss the ways Radford's understanding of human development may build on and differ from both Vygotsky's sociocultural theory, particularly with regards to the concept of mediation, and differ from concepts found in theories of embodied cognition.

#### **Vygotsky's Theory of Human Development**

There are four general concepts that form the basis of Vygotsky's theory of human development (Wertsch, 1985). These are the idea of a genetic development; the claim that mental processes are mediated by tools and signs, internalization and the development of perception and attention, and the zone of proximal development (Vygotsky, 1978; Wertsch, 1985).

#### ***Genetic Development***

When Vygotsky spoke of genetic development, he was not referring to stage theories of development where certain characteristics or functions of human activity can be attributed to certain periods of development, such as Piaget's (Kozulin, 1990; Lillard, 2005; Rogoff, 1990;

Vygotsky, 1978), but rather to his understanding that the “form of a phenomenon reflects the transformation it has undergone and the various factors that have entered into its development” (Wertsch, 1985, p. 18). These various factors are not limited to the development of the individual’s unique history, which Vygotsky called ontogenesis, but also derive from the genetic development of the species (phylogenesis); the genetic development of a culture (social history or anthrogenesis); and the genetic development of the moment-by-moment action of an individual undertaking a task (microgenesis) (Kozulin, 1990; Rogoff, 1990; Vygotsky, 1978; Wertsch, 1985). These domains of genetic development are each governed by “its own set of explanatory principles but ontogenesis involves the simultaneous, interrelated operation of more than one force of development” in that it involves both a natural or biological and a cultural or social line of development (Wertsch, 1985, p. 41). The natural or biological line of development is linked to elementary mental functions whereas the cultural line of development is linked to higher mental functions and its development requires mediation. Inextricably bound up with theory of genetic development is the idea that all higher development has its origins in the social milieu. Vygotsky’s (1978) general genetic law of cultural development asserts that any function in an individual’s development appears twice: First on the social plane (inter-psychological plane) and then on the psychological plane (intra-psychological plane) (Vygotsky, 1978; Wertsch, 1985). This was a key component of Vygotsky’s developmental theory and one set him apart from others who saw development as a process of the maturation of the individual (Kozulin, 1990).

### ***The Mediating Function of Tools and Signs***

Essential to Vygotsky’s theory of the social origins of intellectual development was his idea of the mediating function of psychological tools or signs, and “technical” tools (Wertsch, 1985,

p. 77). Examples of psychological signs are “language; various systems for counting; mnemonic techniques; algebraic symbols systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs; and so on” (Vygotsky, 1981a, p. 137, as cited in Wertsch, 1985, p. 79). Technical tools on the other hand, are instruments whose use is “directed toward the external world; it must stimulate some changes in the object; it’s a means of humans’ external activity, directed toward the subjugation of nature” (Vygotsky, 1960, p. 125, as cited in Wertsch, 1985, p. 78). Psychological tools are socially and historically rooted because they emerge as a result of interactions over time and not as a result of one individual’s invention or “independent interaction with nature”, nor can they be considered as instincts or unconditioned reflexes (Wertsch, 1985, p. 80). While signs bring about no changes in an object and are an internally directed means for regulating one’s behavior, a technical tool, on the other hand, is externally oriented and is a means by which humans can change and influence objects in their environment (Kozulin, 1990; Vygotsky, 1978, Wertsch, 1985). Vygotsky viewed the introduction of a psychological tool (language, for example) into a mental function (such as memory) as causing a fundamental transformation of that function (Wertsch, 1985, p. 79). The development of higher psychological functions is a result of the combination of tool and sign use in psychological activity (Vygotsky, 1978, p. 55).

### ***Internalization and The Development of Perception and Attention***

In Vygotsky’s general genetic law of cultural development, the transfer of a function from the inter-psychological plane to the intra-psychological plane is a process Vygotsky (1978) called internalization and this occurs only as a result of prolonged development. Important to this process is “dynamic system of behaviour” involving perception, the sensory and motor systems and attention (p. 31). Vygotsky believed that attention was of particular importance as the ability

of a child to pay attention was a determining factor in the success or failure of any practical activity. Vygotsky referred to earlier studies by Binet (1890) and Stern (1924/1930) (as cited in Vygotsky, 1978, p. 32), which found that young two-year-old children, when shown a picture, limited their descriptions to naming the objects found in the picture, while the older children in the study described actions and relationships between the objects. Vygotsky (1978) replicated Stern's study but asked the children to communicate their understanding of the picture without talking. The two-year-old children "perceived the dynamic features of the picture and reproduced them with ease through pantomime" (p. 32). These findings demonstrated how the young children were able to communicate their spatial reasoning through movement, but not yet through language. When addressing the role of language in perception, Vygotsky (1978) noted that the "opposing tendencies implicit in the nature of visual perception and language" (p. 33) in that the "independent elements in a visual field are simultaneously perceived" while speech "requires sequential processing" as each "element is separately labeled and then connected in a sentence structure, *making speech essentially analytical*" (p. 33). These are important ideas when considering the spatial abilities of young children as well as the challenges of assessment for teachers. For Vygotsky, (1978) it is through the process of internalization of a function from the inter-psychological plane to the intra-psychological plane that the children are "capable of reconstructing their perception and thus freeing themselves from the given structure of the field. With the help of the indicative function of words, the child begins to master his attention" (p. 35). With this ability to pay attention and with the help of speech, the child is able to create a "time field that is just as perceptible and real to him as the visual one" and aided by memory the child can move back and forward in time (p. 36). These functions illustrate Vygotsky's "overarching principle of development" which Wertsch (1985) labelled "the principle of

decontextualization of mediational means” – the process whereby the meaning of signs and tools become less and less dependent on the unique context in which they are used (p. 33). These understandings of the dynamic interplay of the senses, attention, and movement, plus the analytic nature of language, allowed me to consider the development of the children’s spatial reasoning in a social setting.

### ***Zone of Proximal Development***

Another important idea of Vygotsky’s is the zone of proximal development (ZPD). Rogoff (1990) understands this zone as “a dynamic region of sensitivity to learning the skills of a culture, in which children develop through participation in problem solving with more experienced members of the culture” (p. 14). Vygotsky (1986) suggested two developmental levels in the zone of proximal development (ZDP): an actual developmental level which reflected mature functions, and the zone of proximal development which he considered was the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (p. 86)

The concept of the zone of proximal development afforded Vygotsky the opportunity to analyze higher mental functions as they occurred as the dynamic interplay of the senses, attention, and movement mentioned earlier, and not as an analysis of what Vygotsky (1978) called “fixed objects” (p. 61). Fixed objects were that data captured in a moment in time and subsequently they revealed only “fossilized behavior” and not the dynamic moment-by-moment processes he sought to capture (p. 61). The zone of proximal development (ZDP) also created a space for investigating the different kinds of maturing psychological functions (Kozulin, 1990;

Moll, 1990). Vygotsky (1978) used this space and what he termed double stimulation to assess higher psychological processes in action: He set up investigations where a child's success at a task could not be predicted. Typically, children had been expected to be successful at a task because "ready-made external artificial means" ensured the completion of a task (p. 74). Instead, Vygotsky gave the child a task beyond their existing skills, but the child was also given neutral objects to use or not. These objects were considered neutral because they were not necessary to the completion of the task but could be used by the child to help them in whatever manner they chose. This method Vygotsky termed double stimulation because the child had to deal with both the task and the neutral object. What was investigated was how, or if, the child incorporated the neutral object into his or her problem solving (p. 74). Observing the child's actions in this micro-genetic process allowed for a deeper understanding of the child's reasoning as he or she attempted to complete the task.

In mathematics education today the concept of the ZPD has evolved over time with varying considerations of the meaning of development, power, and the direction of learning (see Stott, 2016). Of recent interest is research that suggests tools in the zone of proximal development may be conceptualized differently, where the "consideration of tools as possible more knowledgeable others opens up discussion about the ways in which the alternation of the more knowledgeable others in an interaction could be conceptualized" (Abtahi, Graven, & Lerman, 2017, p. 276).

For Vygotsky the social origins, the mediation through signs and tools, and the genetic histories of individuals were the intricate roots of complex human functioning; not one single factor but many interacting over long periods of time. While Vygotsky's (1978) sociocultural perspective maintains that both biological and cultural strands are necessary for the development of what he termed higher mental processes (p. 46), it does not explicitly investigate the role of

the body and movement in development. The “core notion of Vygotsky’s work was shared labor, not biological structure” (Davis, 2009, p. 121). For their part, sociocultural theories focus on the cultural and historical foundation of “uniquely human” behaviour (Vygotsky, 1978, p. 19). This human development occurs “through enculturation into the practices of society; through the acquisition of society’s technology, its signs and tools; through education in all its forms” (Moll, 1990, p. 1). I turn to a discussion on embodied cognition and its consideration of the role of the body in human learning.

### **Theories of Embodied Cognition**

In *The World of Perception*, a collected volume of seven lectures, Merleau-Ponty (1948/2002) refutes the notion of a disembodied mind separate from the body. Rather the body is our only access to the world because the body is “embedded in those things” (p. 43). Varela, Thompson, and Rosch (1991) in their work on the embodied mind suggest Merleau-Ponty considered the environment and body as bound together “in reciprocal specification and selection” (p. 174). This binding together of the organism and the environment is a fundamental concept in the notion of embodiment which holds that our capacities for understanding are “rooted in the structures of our biological embodiment but are lived and experienced within a domain of consensual action and cultural history” (p. 149). Theories of embodied cognition emerged within the field of cognitive science in part in response to the dominant theory of cognitivism which saw the mind as separate from, and transcendent to, the body (Nunez et al., 1999). Cognitivism maintains a Cartesian dualism which does not view learning as taking place through biological experiences in a particular setting (Nunez et al., 1999). In part, theories of embodiment have emerged in response to theories of “situatedness” such as those presented in the work of Lave and Wenger (1991), and Rogoff (1990) who focus on the contextual and

cultural circumstances of human cognition. While these cultural aspects are a vital part of cognition, theorists such as Maturana and Varela (1987) and Varela et al. (1991) suggest the biological roots of cognition need to be investigated. Nunez et al. (1991) suggest a shift in focus is required to understand how human cognition is grounded in the body within a biological as well as a cultural setting. Theories of embodied cognition have found their way to the field of mathematics education and to discuss the theory I lean on the work of Nunez et al., (1999), and Nunez's work with Lakoff (Lakoff & Nunez, 2000) in mathematics education as well as the work of others such as Varela, Thompson, and Rosch (1991) who offer a broad theoretical approach to the subject. I now discuss how action and conceptual reasoning are understood in embodied cognition, the role of sensory and motor neural systems in that reasoning, and the idea of conceptual metaphors and their implications for mathematics learning. Finally, I discuss the myths regarding the nature of mathematics and how a theory of embodied cognition aims to dispel those notions.

### ***Embodied Cognition is Concerned with Embodied Action***

Rather than considering the mind as an abstract entity separate from the body, embodied cognition refers to the mind as *embodied* where the physical makeup of our bodies and brains and how we function daily in the world structure our thinking and reasoning (Lakoff & Nunez, 2000). This understanding requires a focus on the relationship between an organism and its experiences of the world. Davis (2009) explains this relationship as understanding life and learning “in terms of explorations of ever-evolving landscapes of possibility and of selecting (not necessarily consciously) actions that are adequate to situations (p. 154). Cognition, from this perspective, is about action, an *embodied action*, which is the interaction between a body which has sensory motor capabilities, and these sensory motor capabilities are themselves embedded in

a broader biological and cultural setting (Varela et al., 1991). *Action* refers to the understanding that the sensory motor systems, together with perception and action, are inseparable and evolve together (Varela et al., 1991). Put differently, cognition may be thought of as “enacting or bringing forth adaptive and effective behaviour” (Nunez et al., 1999). In their discussion on the embodied mind, Varela et al., (1991) also refer to the term *enaction* by which they mean two things; perception is perceptually guided action and concepts emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided (p. 173).

### ***Sensory Motor Neural Systems***

Our cognition considered as embodied action in the world is afforded through our sensorimotor capabilities. Using the sense of perception of colour as an illustration, a theory of embodied cognition rejects the view that the world is represented through visual images in the brain, or conversely that the world is a reflection of the mind’s own internal world. Instead, the perception of colour is a cultural and biological phenomenon, and the categories of colour are experiential (Varela et al., 1991). Lakoff et al., (2000) suggest that not only are sensory neural systems involved in mathematical thought, motor neural systems may be involved too. They refer to studies by Narayanan (1997) which showed that the “same neural structure used in the control of complex motor schemas can also be used in reason about events and actions” which would include reasoning about mathematics (Lakoff et al., 2000, p. 35). For example, a mathematical activity such as working with wooden triangles would require motor neural as well as sensory neural systems in the following manner. The body must ready itself first before performing a desired activity, for example, it must get up out of a chair in order to collect the materials. Then the materials must be reached for and grasped, put down and arranged. Now the purpose of the activity may begin. There are options of stopping and starting throughout the

process, or of being interrupted. When the activity is complete there is the option of repeating it. At completion there are consequences and results. According to Lakoff et al., (2000) these motor movements play a role in the development of mathematical concepts.

### ***Communication Using Conceptual Metaphors***

In a theory of embodied cognition our concepts emerge from our sensory-motor experiences in the world and these concepts are communicated using conceptual metaphors that are grounded in our physical experiences. Lakoff and Johnson (1980) have written extensively about how metaphors are a central process in everyday thought and are a basic means for abstract thought. Lakoff and Nunez (2000) discuss how the most fundamental mathematical ideas are inherently metaphorical in nature and are grounded in daily experiences. Of importance is the understanding “that much of the ‘abstraction’ of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor, often over the course of centuries” (p. 47). Rather than considering abstraction as thought disengaged from the body it has its roots in the physical world. For Lakoff and Nunez (2000) the symbols we use in mathematics are meaningful because they are attached to mathematical concepts that have arisen from our sensorimotor experiences in the world.

### ***The Myth of a Universal Transcendent Mathematics***

As mentioned earlier, a key objection to the cognitivism as a theory of a cognition is its understanding of the mind as an abstract entity separate from the body. Extending this premise in cognitivism allows reasoning, and therefore mathematical reasoning, to be considered as separate from the body. Lakoff and Nunez (2000) address these concepts in their work on embodiment and mathematics. They refer to the romance of mathematics where they discuss the various myths that surround its understanding. These myths exist despite the conceptual tension they

create, namely that while mathematics is considered real it is at the same time abstract and disembodied. Furthermore, although mathematics is considered abstract and disembodied, it is nonetheless found in the physical world where it creates an “underlying structure through which to access the truths the world is considered to hold” (p. xv). A theory of embodiment dispels the myth of mathematical reasoning as separate from the body, as well as the belief that mathematics represents a timeless truth. The authors argue that these myths are a legacy of the works of Plato and Descartes (Lakoff & Nunez, 2000). Human mathematics is the only kind of mathematics that human can know, and our mathematical reasoning arises from our sensory motor experiences in the world.

Theories of embodied cognition are found in the research in mathematics education. Research now considers children’s kinesthetic experience of the world and their interaction with materials as much more than “mere transitory” or secondary aspects of cognition (Radford, 2014, p. 350). Recent trends in theories of embodied cognition have allowed for a variety of ways to interpret human cognition such as seen in the work Bautista and Roth, (2012), the work of Lakoff and Nunez (2000) or the phenomenological work by Thom and Roth (2011) that “emphasize[s] the fleshy nature of thought” (Radford, 2014, p. 350).

### **Sensuous Cognition – An Integrated Approach**

I now turn to a discussion of the salient aspects of Radford’s sensuous cognition that guided my understanding of the children’s spatial reasoning. These are the nature of human reflection, the plasticity and the multimodal nature of the senses, the cultural development of the senses, and the use of artifacts. Within this section I refer to those aspects of Vygotsky’s sociocultural theory and aspects of embodied cognition that Radford suggests do not adequately address an understanding of human cognition as both embodied and situated. For instance, what Radford

(2014) believes theories of embodied cognition are challenged to do is to offer “a cogent account of the theoretical categories of the conceptual and the embodied, and their relationships” (p. 350). For Radford (2013) theories of embodied cognition may still maintain a dualistic approach to the mind, and it is “not enough to merely put the body and material culture back into thought” (p. 143). He suggests we need a different starting point for understanding cognition; one where the environment and cognition are not understood as being separate, but rather one where cognition is conceived of as a feature of physical bodies that have a capacity of “*responsive sensation*” (p. 141). For Radford (2013) sensuous cognition offers a non-dualist understanding of human development and it rests on the premise that cognition and the material culture in which individuals live, and grow are “intertwined entities” (p. 143). As a result, cognition can only be understood as a “culturally and historically constituted sentient form of creatively responding, acting, feeling, transforming, and making sense of the world” (p. 144).

### ***The Nature of Human Reflection***

In sensuous cognition Radford (2013) offers us the phrase “reflection” and asks us to consider it as a dynamic feature of humans as they respond to or reflect or act on their environment (p. 145). For him, reflection implies a relationship; something that is reflected upon, namely an object, and something that does the reflecting, a subject. By using the terms human reflection and responsive sensation in his explanation of human cognition, Radford (2013) brings our attention to the senses and the body’s relationship with the world and draws our attention away from a position that considers cognition as mental and internal. For him “reflection and action do not occur in two *separate* planes. They occur in the same plane – the plane of life” (p. 144). This relationship with the world is dialectical, meaning that “the objective world and its subject reflection co-evolve”, where, as Radford (2013) points out, the “mind can

only arise from the progressive complexity of processes of life” (p. 145). On the other hand, “more complex conditions of life require organisms to have the capacity to reflect reality through more complex forms of sensation” (p. 145). The co-evolving nature of the senses and the material world will be discussed further in the section on artifacts and the cultural development of the senses.

As far as the children’s spatial reasoning is concerned, the idea of a dynamic response is found in literature where it has been considered a “vital capacity for human action” (Davis & the Spatial Reasoning Study Group, 2015, p. 1). The Spatial Reasoning Study Group offers a preliminary list of these dynamic processes as they pertain to spatial reasoning, for example, locating, orienting, and diagramming, to name a few. Interestingly, the word sensing is included in the list but from the perspective of sensuous cognition all the dynamic processes listed are sensing. Contained within the idea of a dynamic process are other key concepts of sensuous cognition which are the plasticity and multimodality of the human senses.

### ***The Plasticity and Multimodality of Human Senses***

Given the co-evolution of the cultural world and the mind, human senses are of necessity plastic. Radford (2013) refers to the work of German social theorist Arnold Gehlen (1988) when he states that animals are born with specific instincts and highly developed senses which support their survival in their specific environments (p. 146). Humans, however, have highly unspecialized sensorial functions, yet due to the plasticity of these senses they are able to develop them whereby they can live in any number of environments. From a notion of the multimodal nature of sensuous cognition, different senses develop in an integrated manner not in isolation, and they collaborate in ways that are specific to human beings. This collaboration of the senses results in a “complex perception of reality” (p. 146). Some senses collaborate more

closely than others, such as touch and sight. The work of the Spatial Reasoning Study Group is based on an understanding of spatial reasoning as not an isolatable skill (Whiteley et al., 2015, p. 5). Thom et al. (2015) capture the sense of multimodality when they talk about the bodily experiences of a child (see Roth & Thom, 2011) as he explores the faces, edges, and vertices of two different rectangular prisms It is a

dynamic sensorial whole-body happenings-of sights, scents, emotions, feelings, sounds, vibrations, movement, velocity, angle, pressure, acceleration, momentum, friction, and so on – exist all-at-once and in ways that depend on their bodies to touch and move with the world as much as the world does with their bodies. (Thom et al., 2015, p. 75)

### ***The Cultural Development of the Senses***

The development of human senses that is made possible by their plasticity takes place within a social milieu and therefore sensory functions co-evolve with the culture of which they are a part. Referring to the cultural and biological strands in the development of cognition as suggested by Vygotsky, (1978), Radford (2013) states the “raw range of orienting-adjusting biological reactions are transformed into complex, historically-constituted forms of sensing” in a cultural setting (p. 147). Radford (2013) iterates that the formation of the senses is not a natural process but, quoting Marx, the “labor of the entire history of the world down to the present” (Marx, 1932/1988, pp. 108-109 as cited in Radford, p. 147)). In the field of spatial reasoning, research by Bartolini Bussi and Baccaglini-Frank (2015) draws on the studies of Luria (1976) which showed that geometric perception is dependent on the culture in which the children are living. Studies by researchers that have demonstrated children’s difficulties in identifying non-prototypical shapes (see Hannibal & Clements, 2000, Clements & Battista, 1992) may

demonstrate the children's limited exposure to those shapes as a contributing factor to their difficulties.

Radford, (2013) in his discussion on the cultural development of the senses, aims to eradicate the divide made in Vygotsky's distinction between natural and cultural lines of development as well as his distinction between elementary and higher mental functions. Vygotsky (1978) suggested that higher mental functions which were the outcome of cultural development could be attributed to the principle of decontextualization of mediational means which is the process where the meaning of psychological signs and tools become increasingly less dependent on the on specific context in which they have been used (Wertsch, 1985). This process today is understood as abstraction. While Vygotsky may have been unclear about the role of the body in development (Wertsch, 1985), Radford does not separate human development as deriving from cultural lines and biological lines.

### ***Artifacts and Materiality***

The development of new cultural forms of sensation in humans is inextricably intertwined with their use of artifacts. Not only are the senses culturally and historically developed, matter itself can be considered cultural and historical insofar as it "bears the traces of human labour and intellectual activity" (Radford, 2014, p. 353). For example, the children's use of mathematical material using their senses of touch and perception allows them to develop their mathematical form of perception to "distinguish between cultural categories of geometrical figures" (Radford, 2013, p. 148). In doing so the children have recourse to the material objects whose contours they explore. Radford (2013) considers that dispositions arise in humans through the use of artifacts whose use allows us to think, perceive, and to remember (p. 146).

From a Vygotskian perspective, artifacts mediate our thinking and experiences, but Radford (2013) goes beyond this understanding to view artifacts as a constitutive part of thinking and sensing - an understanding which emerges from a concept of mind as a property of matter. Our individual senses evolve intertwined not only with the other senses but also with the materiality of the objects in our surroundings. The materiality that shapes our senses is not reduced to inert matter, but matter that is already endowed with meaning such as “triangularity” or “quadrilaterality” (p. 149). It is this key role of artifacts in the constitution and evolution of forms of sensing and reflecting that Luria and Vygotsky underlined. Artifacts create dispositions through which to reflect, think, and remember and Radford (2013) gives us the example of electronic media where new opportunities for exploration create new ways for engaging in the world. As a result, the senses are transformed to respond in different ways to changes in the material world. In this research this understanding of matter has a bearing on the Montessori mathematics materials which the children use during their spatial reasoning activities. These culturally derived materials play a crucial role in the children’s culturally developed senses.

Radford (2013) claims that sensuous cognition questions the separation of mind and the world and uses new language to discuss this historical separation. He extends the Vygotskian notion of the mediation of artifacts as he considers they have a far greater role in the development of foundational part of cognition and behind this view lies the general concept of mind as a property of matter. For him, “sensation and its cultural transformation in sensing forms of action and reflection are understood to be interwoven with cultural artifacts and materiality at large” (p. 149).

In the research literature on mathematical spatial reasoning, we see cultural developments in the artifacts available to children, most recently with dynamic geometry environments (DGEs)

where children develop new sensibilities and new terminology for such actions such as dragging and tracing when creating a digital diagram using a sketch pad (Ng & Sinclair, 2015), or new terminology such as “squarized O” for a new shape described by a motorized and programmable bee-bot (Bartolini Bussi & Baccaglini-Frank, 2015, p. 398). In this study I observed an example of a culturally developed sensibility in the primary classroom in the prototypical orientation of triangles by both the teacher and children. The triangles were consistently oriented with an apex positioned above a base. This was the case whether the children were tracing the triangles or holding up wooden triangles in their hands.

### **Summary**

Sensuous cognition offers a perspective where sensation and its cultural transformation in sensing forms of action and reflection are understood to be interwoven with cultural artifacts and materiality at large. It presents human cognition from a monist perspective which eradicates the historical divide between the mind and the body (Radford, 2013, 2014). Important concepts of the dynamic processes of human sensation, the cultural development of the senses, their plasticity, and their multimodal way of operation, as well as the cultural co-evolution of these senses, offers a rich and dynamic theoretical framework for investigating young children’s spatial reasoning in a Montessori classroom. It also allows for Montessori’s understanding of the crucial importance of the body in development, particularly the refinement of the senses (Montessori, 1909/1964, 1946/2012), and for her understanding of development as culturally and historically situated (Montessori, 1909/1964, p. 217).

For my understanding of the children’s spatial reasoning, I use this integrated theoretical framework and focus on the social setting of the children’s learning and conceptualization of the artifacts from the perspective Radford puts forward as having a materiality and of playing a

constitutive role. I do not consider different planes of learning, nor ideas of external and internal processes.

## **Chapter 4: Methodology**

In this section I discuss why the choice of methodology was contingent on the theoretical framework used and how this effected the design of the research, including the choice and recruitment of the participants, the nature of the data sources and their collection, and the tools used for analysis.

### **Tools of Ethnography as Coherent with the Theoretical Framework**

Important to the selection of a methodology was its coherence with the theoretical lens through which to make sense of the children's spatial reasoning. For this study that lens encompassed Radford's sensuous cognition with its focus on the body (2013, 2014) and included the sociocultural theoretical perspective put forward by Vygotsky (1978). In mathematics education research, an ethnographic methodology is in keeping with the growing realization over the last 30 years that mathematics is a cultural endeavour (Bishop, 2002). Mathematics is increasingly viewed as knowledge that is socially constructed (Player-Koro, 2011, p. 326) and a means of communication with others and self (Sfard, 2008). This has led to the consideration of such ideas as human interactions, values and beliefs, the interaction between gestures, language, and mathematics, and the cultural roots of mathematics (Bishop, 2002; van Oers, 2010). From these perspectives spatial reasoning is positioned as learning that is contextual, mediated, and emergent with the body having a primary role (Thom et al., 2015). More specifically, spatial reasoning is considered a non-verbal reasoning and a non-isolatable skill (Whiteley et al., 2015).

The implications of these understandings of the children's spatial reasoning required a methodology that incorporated an extended period of time in the classroom in the research design. This extended period was important as it allowed the time necessary to make sense of the norms of the classroom when I observed the children as they selected materials and worked on

activities. The time allowed for observations of their interactions with others and with the teacher as well as their use of the mathematical materials, and it was a process to understand in what ways the children communicated their spatial reasoning where their spoken language was one aspect of their communication. The extended time also allowed for reflection on my part and this reflection is an integral part of an ethnographic endeavour (Eisenhart, 1988, 2001).

Observation as a methodological tool was familiar to me as it is the Montessori method for the teacher's assessment and planning. In assessment, for instance, observation is used for noticing which children are working, what they are working on, how they interact with others, how long they stay working with the material, whether they are concentrating deeply or not while doing an activity and which materials are being used, which forgotten. For planning, the teacher uses the observations to decide the lessons to give, which children should be grouped together, which lessons the children did not follow up, and what other mathematical materials may be used to engage them. Other ethnographic methods, such as in-depth discussions with the teacher about the children's work, employed over time (Eisenhart & Borko, 1993; Wolcott, 1999; Zaharlick, 1992), underpin the emergent design inherent in ethnographic methodologies. This feature allowed opportunities for contradictory behaviours and perspectives to emerge (Jeffrey & Troman, 2004, p. 545) and allowed for nuance and complexity (Campbell & Lassiter, 2015, p. 35). These features of ethnographic research required the researcher to inquire and to generate new questions in response to what was observed and understood (Zaharlick, 1992, p. 119). While aspects of my research design were true to an ethnographic methodology, there were areas where they differed. Ethnography typically involves an immersive and collaborative endeavour albeit often limited to the duration of the fieldwork (Campbell & Lassiter, 2015). I was not immersed in the practices of the classroom, nor did I collaborate with the teacher in any

deep sense of the word where outcomes would be affected. I was not a stranger to the environment rather I was deeply familiar and comfortable with the Montessori classroom setting, and knew the teacher, Linda, well. At the time of designing the research study I did not consider these divergences from a traditional ethnography as detracting from the methodology as an appropriate one for this study. The qualitative aspect of this methodology fit well both with an emergent understanding of children's learning and with the unscheduled nature of the children's school day, where children's activities and actions were not predicted and where the goal of the research was to develop an understanding the occurrences of the children's spatial reasoning in this setting and time. In keeping with the rationale of my methodology I spent approximately two months collecting data in the classroom. I started in early April 2019 and finished at the end of May 2019. I spent each day in the classroom, arriving at the same time as the children and leaving when they left. On the few days the children had field trips, I did not attend.

As there was reason to consider that certain Montessori principles, when instantiated in the classroom, may play a role in children's spatial reasoning, they were included in the methodological design. These key principles were an uninterrupted work cycle for the children, engaging and challenging activities, the children's autonomy and freedom of movement, a multi-age classroom, and the use of the Montessori didactic materials. The teacher's work of giving presentations, observing the children, teaching terminology, and the maintaining the classroom were also observed, again in terms of supporting children's spatial reasoning. I observed the teacher's use of mathematical materials to introduce, represent, explain, and model mathematical concepts. As the Montessori mathematics curriculum contains numerous materials for use in mathematical activities, I anticipated observing diverse forms of spatial reasoning as the children worked with these materials. These forms of spatial reasoning could include mental rotations and

transformations, physical transformations, two-dimensional and three-dimensional decomposing and recomposing, orienting, and locating.

### **The Setting**

In order to investigate young children's mathematical spatial reasoning in a Montessori environment, the setting was a Montessori primary classroom. It was important to this research that the Montessori schools considered for inclusion in the study instantiated the key pedagogical principles mentioned earlier. Since I wished to conduct the research in a primary classroom – the age group for which I am trained – I selected a Montessori school that had a primary program. Three schools in the area met with the criteria I had established.

I contacted the Montessori schools which I had selected based on their Montessori accreditation, their reputation within the Montessori community, and my own knowledge and experience. The Heads of School showed interest in their school's inclusion in this study. Once ethics approval was received, I approached one of these schools more formally and determined their interest in taking part in the study. I preferred this school based on my long-standing professional relationship with the primary teacher at the school. I met with the Head of the School to discuss the research project and explained my interest in spatial reasoning, and my history in the Montessori community. I outlined why I considered their school a good fit with the research goals. I answered questions about the length of time I anticipated the data collection to take, how I would be collecting data, and what the families could expect in terms of communication from me. Together we agreed that a presentation to the parents regarding the study would be helpful and informative and we set a date and time for when the parents would be free to attend. We also agreed that an introductory meeting with the children in the classroom was important and this was arranged for one morning after my presentation to the parents. After

this initial meeting with the Head, I met with the teacher, Linda. She was welcoming and invited me on a tour of the school and her classroom where I would be collecting the data. The school was in a house that had been remodelled and Linda's primary classroom was a new addition. In the classroom there was a kitchen area, a science area, and shelves containing materials and books for the various subject areas. There were child-sized tables and chairs arranged around the room. In the wide corridor off the classroom there were shelves with some of the mathematics material and a small table and two chairs. The rooms were bright with large windows looking out on gardens. There were two washrooms close to the classroom.

### **The Participants**

The participants of the study were the children in the primary class as well as Linda, the teacher, whose experiences were an important consideration in this research. The number of participants was dependent on number of children in the class who agreed to take part in the research study and who had received their parents' permission to participate. Typically, a Montessori classroom covers a three-year age range. This class covered a six-year age range with an unequal distribution of children from Grade 1 to Grade 6. The class comprised 17 children in total. Of the 17 possible participants, fifteen families agreed to their child's inclusion in the study. The cohort of participants comprised the following range: Grade 1: 8 children, Grade 2: 5 children, and Grade 4: 2 children.

### ***Meeting the Parents of the Children***

Using the research proposal as a guide, I created a PowerPoint presentation for the parents. The presentation was given before I requested consent from the families and children, and before any research was undertaken. At the meeting with the parents which was not attended by all the parents, I discussed spatial reasoning, how it is understood and its importance in the

learning of young children. I explained my interest in spatial reasoning and my reasons for undertaking the study. I discussed the research, informed them of what would take place, and I endeavoured to build their trust. I responded to any questions to allay any concerns they may have had. I informed the parents that I would be in the classroom for approximately two months. I assured the parents and the children of confidentiality and anonymity. I explained how each child would be identified by a pseudonym. The school's identity would be protected by a pseudonym as well. The teacher, Linda, chose not to use a pseudonym and she filled the appropriate box on her Teacher's Consent Form (see Appendix A for Teacher's Consent Form). Should any video-recordings or photographs be used in professional publications or professional presentations the identity of the child, or children, and the school, would continue to be protected by pseudonyms. For confidentiality purposes I would keep all data on a password protected computer and keep data-storage tools in a locked cabinet in my home. I explained that any data held by my supervisor, Dr. Barbara Graves, would also be stored on a password protected computer or data storage device. The data would be securely safeguarded for a period of up to ten years, after which it shall be destroyed.

I discussed the implications of their informed consent and their children's assent to take part in the study. I talked with the parents about the idea of informed consent and read through the Parental Consent Form explaining that there were two copies of the form. I mentioned that I would co-sign all completed forms and one copy of form was theirs to keep for their records. I showed the parents the children's Assent Form (See Appendix B for Children's Assent Form and Appendix C for Parents' Consent Form) and discussed how I would meet with the children to go through the form and to discuss the research. Some parents completed the Parental Consent Forms at the end of the presentation and gave them to me I assured the parents that they were

under no obligation to enrol their child in the study as participation was voluntary. I explained how children whose parents had not given permission for their child to take part in the study, would be treated respectfully and would be included in activities so that they did not feel left out, to the extent that it was possible. For instance, if I had a discussion with their child, I would not record it, nor refer to it in the research. I let the parents know that I would be meeting with their children in the classroom to introduce myself, discuss the study, explain the Assent Forms, and answer their questions.

### ***Meeting the Children***

One morning after meeting the parents, I met with the children. Linda greeted me at the door and invited me into the classroom. The children were seated on the floor awaiting my arrival. Linda introduced me and the children introduced themselves. Linda spoke with the children about research and why it would be undertaken. The children answered questions she asked about their experiences with any research, for instance, if their parents were researchers, and what they knew about it. I discussed the study and what they could anticipate daily when I was in the classroom. I read the Assent Form to them and discussed its purpose. We discussed the nature of the data I would be capturing and how I would do this. I showed them the tools I would use for this, namely a video-recorder and an audio recorder. I mentioned I would also be taking notes on my computer. Some children told me they did not want to be part of the study. Others were keen to be involved. Linda signed her consent form.

### **Data Sources**

I collected data, which when analyzed would help me understand how mathematical spatial reasoning was experienced by young children as they participated in a Montessori mathematics curriculum; how the children's expressions of mathematical spatial reasoning were revealed and

communicated; and how the children engaged with the Montessori mathematics materials. Drawing on the work of Radford (2009), Bartolini Bussi & Baccaglini-Frank (2015), and Montessori (1914/1965), I used the following sources of data; *semiotic traces* (Bartolini Bussi & Baccaglini-Frank, 2015) which are the “observable processes” produced by the children during the dynamic processes of spatial reasoning (p. 393). These traces included tracings, drawings, oral descriptions, written texts, and bodily gestures and movements. The semiotic traces are considered integral to the articulation process leading to the production of those thoughts (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2014, p. 898). These dynamic processes emerged within the *didactical cycle* (Bartolini Bussi & Baccaglini-Frank, 2015) which was the children’s cycle of work initiated by the teacher’s presentation with mathematical materials and continued with the children’s independent work. I drew on Radford’s (2009) term *space of learning* which he understands as the complex social space within which learning takes place. I expanded the notion to include observable instances of Montessori practices. These key practices were an uninterrupted work cycle for the children, engaging and challenging activities, the children’s autonomy and freedom of movement, a multi-age classroom, and the use of the Montessori didactic materials. I used the construct *space of learning* as a unit of measurement in my analysis. This follows the work of Ng & Sinclair (2015a) who themselves

[t]ake children’s talk and embodied actions in social settings to be the very unit of [their] analysis of children’s mathematical thinking. (p. 423)

### **The Tools Used to Collect Data**

In selecting the data collection tools for this study I considered both the sources of data and in what ways the unique features of a Montessori classroom might influence these choices. In brief, Montessori primary classrooms are arranged in a way not associated with traditional

classrooms. The tables are not uniformly sized, nor are they uniformly arranged. Instead, there are tables of varying sizes arranged informally in the room. The classroom is set-up around broad subject areas and there are areas for artwork, science work, and the preparation of food in a kitchen area. The various didactic materials and books for each subject area are visible on the open shelves. The Montessori materials on the shelves are laid out according to their order of presentation to the children. The shelves are low and so their contents are accessible to the children. House plants and artwork may decorate the room. Classes may have animal life such as fish, or a guinea pig. Children keep journals in which to write down the work they are doing, and these journals also serve as an account of the work they have done. At monthly conferences with the teacher, children individually account for their work over the previous month. Lessons are not scheduled with a timetable but are given based on the teacher's observations and the children's requests.

In considering the context of the classroom and the purpose of this study, which was to investigate the children's mathematical spatial reasoning within their Montessori classroom as I have defined it, I understood that I required tools for collecting data that fit with the context. Some of the requirements for the tools were that they needed to be relatively simple to use, available at any moment for use, easy to move around and not too intrusive in the classroom environment. The tools also needed to cohere with the sources of data and as such I chose the following for collecting the data I had anticipated would emerge from the children's activities, namely video-recordings, audio-recordings of the teacher and audio-recordings of the children, and an observation journal.

### ***Video-recordings***

Since the research concerned young children's mathematical spatial reasoning as they worked with materials in the classroom, and since the research literature suggests that spatial reasoning is a non-verbal and non-isolatable competence (Whiteley et al., 2015). I used video-recordings as the main tool of data collection. Discussions early on with the teacher gave me a better idea of how many presentations might be given but I intended to choose about four or five for detailed analysis (see Appendix D for criteria for selecting videos). Videos provided important visual data that is extremely useful not only for illustrative purposes but also as a reference when analyzing the children's independent follow-up work. Visual data were pertinent as this study was about investigating students' spatial reasoning which is concerned with relationships between objects in space.

### ***Audio-Recorded Conversations with the Teacher.***

The teacher's thoughts were an important consideration in understanding the children's spatial reasoning. Audio-recorded discussions with the teacher allowed for the capture of the teacher's thoughts and ideas about various aspects of the children's opportunities for mathematical spatial reasoning (see Appendix E for possible questions for discussions with the teacher). These aspects included learning what mathematical spatial concepts the teacher intended to present, what mathematical materials she would use, her reasons for choosing a particular presentation, and her reasons for choosing certain children to take part in the lesson. Audio-recordings allowed for the capture of the teacher's in-the-moment thoughts at the start of each didactical cycle, i.e., before each initial mathematics presentation, then after each presentation, and at different times during the didactical cycle. Recorded discussions of what follow-up activities the teacher would suggest to the children and what work she anticipated the

children would undertake supported my decision making on where to focus when videorecording. Given the dialectical nature of conversations audio-recordings created opportunities of emergent thinking to be captured and considered later.

### ***Audio-Recorded Conversations with Children.***

Audio-recorded conversations with the children were an important source of data where the children revealed their thinking during or after their spatial reasoning activities (see Appendix F for possible questions for discussions with the children). Conversations with the children were planned for two or three times during each didactical cycle, such as conversations with the children after the teacher's initial presentation with the mathematical materials, during and after their independent follow-on work, and after their reengagement with the teacher. It was anticipated that during these conversations the children would feel comfortable to freely talk about their thinking during the spatial reasoning activity given by the teacher, to talk about their experiences in their independent activities with the mathematical materials, and to talk about their interactions with their peers during their activities, their plans for follow-up work, and at the end of the didactic cycle, their experiences of the cycle. The use of audio-recorded conversations allowed the observation notes to be supplemented with the children's own thoughts.

### ***Observation Journal***

Detailed daily observation notes of the children and their activities in the classroom allowed me to keep track of the lessons presented by the teacher, Linda. Since there was no timetable of scheduled lessons in particular subject areas that I could refer to for this data. Observations of select periods of the didactical cycle (see Appendix G for criteria for observations) allowed me to record the children's actions when and if they chose the follow-up

work that emerged from any of the lessons given by Linda. The observation journal served as a place for documenting the semiotic traces indicative of the children's spatial reasoning as revealed in their bodily gestures, drawings, and written texts. In this way, the observations supplemented the video-recorded instances of the children's spatial reasoning. The observation journal served as a useful tool to generate reminders of questions I had for the teacher and children as well as providing a space to make notes of occurrences that impacted the routine of the classroom, such as unexpected visitors, or a fire drill. As a study using some tools of ethnography, the observation journal created the necessary space for the reflection and uncertainty that is part of the ethnographic process. The journal allowed for reflection on my observations, my assumptions, and my expectations. The journal presented me an opportunity to make visible through writing and drawing my own thought processes.

### **Analytic Framework**

The analytic framework was designed to allow for an analysis of the data pertaining to how mathematical spatial reasoning was experienced by young children as they participated in a Montessori mathematics curriculum; how the children's expressions of mathematical spatial reasoning were revealed and communicated; and how the children engaged with the Montessori mathematics materials. The framework was designed drawing on the works of Radford (2009), Bartolini Bussi & Baccaglini-Frank (2015), and Montessori (1914/1965). The following ideas from these works formed the main aspects of the data sources and analytic framework. These ideas were the semiotic traces, the didactical cycles, the space of learning, and key Montessori principles.

### ***Guide for Data Analysis***

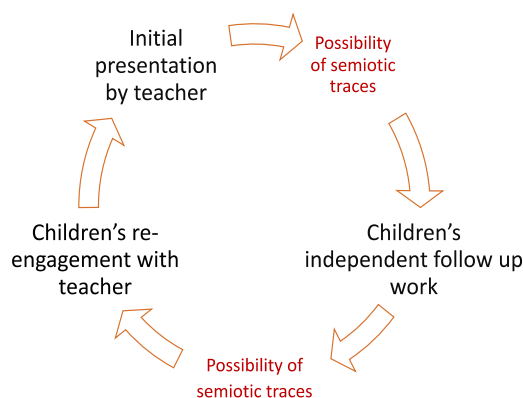
The key purpose of the analytic framework was to facilitate an analysis of the data to

understand children's spatial reasoning as revealed through the observable processes of their *semiotic traces*. As mentioned, the *semiotic traces* are nested within the *didactical cycle* which begins with, and includes, the presentations and continues with the children's follow-up work and ends with reengagement with the teacher. Within the didactical cycle is the *space of learning* which is unique to each recorded instance of the didactical cycle. In planning the methodology, the following was my guide for the analysis of the data.

*Didactical cycle.* This began with a teacher-initiated activity with mathematical materials, such as building block-type material known as the Binomial Cube, for example. Prior to the start of the cycle, however, I referred to the audio-recorded conversation I had with the teacher earlier in those instances when the presentation was discussed. This allowed me to reflect on the materials she intended to use, and her goals, and for me to consider what forms of spatial reasoning I could expect to be called into action by the activity. I made notes of the children she invited to the lesson and why. I gained an understanding of what follow-on work she considered possible for the children. When the teacher invited the children to the presentation, this constituted the start of the *didactical cycle* as well as a *space of learning* (see Figure 1).

## Figure 1

*The Didactical Cycle*



Although the semiotic traces produced by the children could not be predicted, I conducted an activity myself with material called the Binomial Cube to anticipate and experience what semiotic traces were possible in this instance. This allowed me to get a sense of what I might observe in the children's mathematical activities. I imagined the teacher introducing the material and inviting a discussion with the children. The children may have talked about the "squares" and "blue sides" and "long shapes." They may have talked about "breaking it down" and "building it up." They may have demonstrated a form of spatial thinking by using their fingers to show how they would hold the cube and reposition it so that one of its faces corresponds to a face the rectangular prism as they recompose the Cube. When recomposing the cube, they may have stood up from their chairs and walked around the table to view the Cube from a different perspective. With their bodily gestures, such as leaning into and focusing on the cube, they may have demonstrated their engagement. The students could have mimicked the teacher and reflected her hand and body movements as she decomposed the cube. The teacher could have demonstrate decomposing the material while giving terms such as "cube" "rectangular prism", etc. The teacher may have written the geometric names down on strips of paper for the children to read or write themselves. The children may have written down names as

well or created a drawing of the Cube. The aim of this imaginary exercise was to sensitize myself to the semiotic traces I anticipated would emerge from the children's spatial reasoning.

In analyzing the videorecording of the presentation, I observed for the teacher's bodily gestures, her movements with the material, her verbal communications with the children in order to enrich my understanding of the children's responses to the presentation. While I expected the follow-on work to offer a richer source of the children's semiotic traces, I observed for them during the presentation. The questions I asked myself are What forms of spatial reasoning (if any) are communicated during the presentation by the teacher? How did she communicate them? How did the children respond to the lesson? How did they communicate their thinking? Were there instances of spatial thinking by the children? How was this demonstrated? What forms were shown? Were the children enthusiastic about follow-on work? What ideas did they come up with? How did the teacher respond? During the cycle, I used my interpretation of the construct *space of learning* to analyze data regarding the instantiation, or not, of Montessori principles. For instance, how does the teacher engage the children's interest in the presentation? How do the children show their interest, or lack thereof? Who is invited to the presentation, and why?

Another part of the cycle is the children's individual or collective work emerging out of presentation. This was initiated by the children choosing to do the follow-on work. In the analysis I considered the *space of learning* by asking the following types of questions. Have the children chosen this work themselves? What, if anything, precipitated their choice? Are children working individually or have they invited peers? How did they extend the invitation? Are they moving freely around the classroom? How long have they worked on the activity for? Have they been interrupted? Considering their activities with the materials I observed to see what semiotic traces were being produced. Having identified earlier what forms of spatial reasoning I expect to

be generated by this mathematical activity, I observed for those. I asked myself what forms of spatial reasoning I was seeing and how I knew. The didactical cycle continued when the teacher reengaged with the children to discuss their work. Again, I observed, and video-recorded the communications and analyzed the data following the framework already discussed. The data was collected and analyzed over the entire didactical cycle.

### **Benefits and Risks**

I discussed with the parents and children that there were no foreseeable risks to their child's participation in the study, but should a child experience any discomfort every effort would be made on my part to minimize this discomfort and a child was free to stop taking part in activities at any time. The results of the study would not appear in any school records. The benefits were explained as an opportunity for children to be involved in the problem-solving activities and the follow-up conversations which would contribute to an enlarged understanding of young children's spatial reasoning in a Montessori classroom. The study would give the children an opportunity to be involved in a research project.

### **Trustworthiness and Validity**

The idea of rigor in qualitative research has long been understood in terms of credibility, dependability, and confirmability, but according to Toma is not a good fit for ethnographic research (2006, p. 2), and alternative standards have emerged. For instance, Lather (1993) views validity as an "incitement to discourse" and a construct that can neither be avoided nor resolved (p. 674). Toma (2006) echoes this by suggesting that the best ethnographic researchers can hope to do is attempt to approach rigor while being "more reflective and tolerant of ambiguity" (p. 3). As a researcher using some tools of ethnography, my task is to develop deep insights into children's spatial reasoning in a Montessori classroom which I clearly communicate or, in other

words, I need to deliver a “multi-faceted portrayal of a complex social situation” (Mok & Clarke, 2015), while at the same time being transparent and explicit as to how I came to those insights. My resources are the parameters of my theoretical framework as well as my methodology. In addition, the length of time in the classroom, the different sources of data, and the opportunities for reflexivity on my part guided me to reflect on my observations, acknowledge tensions, question my assumptions, and refine my questions. My ongoing discussions with the teacher allowed me to confirm what I had observed. Concentrated and detailed viewings of the video-recordings also created opportunities for questioning and reflection and for the emergence of new perspectives.

## **Chapter 5: The Learning Context**

### **Development of the Sensory Curriculum**

At the beginning of her career while working as a physician, Montessori was responsible for a large group of children who had largely been forgotten by society. These children were kept in a mental institute in Rome where they languished, their education wholly neglected. While researching how to help to these children, Montessori read the writings of Itard (1802), a physician, and Seguin (1866), a teacher who later became a physician, and was profoundly influenced by their work. Both had used a sensory curriculum for the institutionalized and marginalized children in their care by developing a series of exercises with materials for the children to use (Montessori, 1915/2008; Newman, 2010, p. 69; O'Donnell, 2007, p. 3). Itard and Seguin stressed the importance of movement in learning and believed these movements would be generated by the children when they chose to work with the sensory materials. Itard further believed that a nurturing environment would support the children's learning (Newman, 2010, p. 69). Montessori spent time working with these materials as she developed her own ideas.

Montessori's (1915/2008) sensory curriculum became the cornerstone of her pedagogical method as she came to believe that the children's movements, and in particular their movements with the materials, were foundational to their development (p. 270-271). Montessori (1946/2012) said the movement of the hands was important to this development, stating that "It is not that man must develop to work, but that man must work to develop" (p. 126). It was important for Montessori that the materials were interesting to the children. She believed that if the materials appealed to the children's kinesthetic sensibilities, they would respond by choosing to work with

them, thereby initiating that desired movement (1936/1966, p. 82). Montessori continued to develop her sensory curriculum, designing new materials and refining what was already in use.

### ***Montessori's Epistemology and Cosmic Education***

Within her sensory curriculum, Montessori developed an elementary curriculum for children ages 6 to 12 which she called *Cosmic Education*. Cosmic Education may be understood as the operational aspect of Montessori's worldview or "cosmic plan" (Grazzini, 2013, p. 107). Montessori's use of the term cosmic implied a comprehensive, holistic, ecological view of the universe which acknowledged the interconnectedness and interdependencies of life. It understood the universe as ordered (Grazzini, 2013).

A brief overview of Montessori's epistemology as well as her moral philosophy are warranted since her worldviews are instantiated in the children's learning environment through the classroom norms and the teacher's pedagogical practices. These norms and practices form part of the milieu within which the children's mathematical spatial reasoning takes place. Montessori considered the purpose of education foundational, that is, to prepare children for the complexity and seriousness of human life and she did not consider childhood as a phase to be passed through. Rather she saw it as an intense period in which a child is driven to "soak up their surroundings via a driven intensity from the faculty of awareness" (Coglan, 2016, p. 128). The education of young children was a crucial and serious endeavour and was linked to the idea of the development of the child's *character* or morals. Character was developed by creating optimal "contexts within which children freely choose and diligently pursue interesting work" (Frierson, 2021, p. 135). She proposed an educational approach that would allow children "to proceed in their own ways to begin to grasp the nature of the world around them through sensory education" (Colgan, 2016, p. 132).

Montessori viewed knowledge as hierarchical insofar that the senses form “an irreplaceable function in generating early conceptual knowledge” (Colgan, 2016, p. 129) which she considered foundational to the development of abstract, higher-order conceptual thinking. Hence the Montessori curriculum begins with a sensory curriculum following a hierarchical sequence which is organized and highly detailed with lessons and materials prepared across all subject areas (Lillard, 2007) resulting in a “depth of integration not found anywhere else” (Colgan, 2016, p. 128). Montessori’s beliefs required that interesting activities and unstructured time were part of the classroom routines. This may be a consideration as far as opportunities for the children’s spatial reasoning are concerned. Integral to the sensory curriculum are the Montessori materials and the nomenclature materials that accompany the various subject areas, and these are discussed after the next section. An important part of Montessori’s Cosmic Education for elementary children was a series of instructional narratives she termed *The Great Lessons*. These were considered foundational in creating an environment where the children could exercise their choices in interesting activities and hence develop their conceptual knowledge and moral character. These narrative lessons were considered such an essential component of the elementary curriculum that should a child join the class later in the year after the Great Lessons had already been told, the narratives would be repeated for that child. As such, the children’s spatial reasoning activities in the classroom need to be understood within the context of Montessori elementary curriculum, and therefore within the context of the Great Lessons.

### **The Great Lessons: An Invitation to Explore and Investigate**

The Great Lessons are one of the pedagogical tools for implementing Montessori’s vision of Cosmic Education in the elementary program. They are imaginative stories that present an

ordered view of the universe from its beginning to the present day. The purpose of the lessons is to provide the children an opportunity to view themselves in relation to the order understood to be found in the universe; to demonstrate the interconnectedness of the universe; to delineate the different subject areas the children will be exploring while at the same time showing how the subjects are connected to each other. They present the world as wondrous place worthy of exploration and investigation. They present the idea to the children that all life evolved to meet needs within the context of unique environments. In addition, the Great Lessons serve as a resource for the teacher as once the lessons have been given the teacher can refer to them with a child who may need some guidance in choosing work as the Great Lessons offer limitless topics for further investigation. The teacher could ask, for instance, what interested the child in the lessons then direct the child to further materials on the subject or decide what lesson to give to support the child's budding interests. In this way, the children's individual interests are nurtured. More directly, goal of these lessons is to support the children's engagement in experiential, sensorial work with materials in the classroom.

Since the elementary curriculum consists of two three-year programs (ages 6-9 and ages 9-12) the children may hear the Great Lessons up to six times. Once they have heard the lessons the children free to choose whether to attend these repeated presentations and most do. The first of the Five Great Lessons is told on the first day of school to set the expectations for the elementary curriculum at the very beginning of the school year. The Great Lessons take approximately thirty minutes to be told. The children's attention is held by the dramatic delivery of the lessons as well as by the interesting objects that are shown and, in some Great Lessons, by the detailed timelines and interesting demonstrations. During the presentations, however, the

teacher responds to the verbal and body cues given by the children and should the teacher observe the children have lost interest, the story will be curtailed and continued another day. I describe the Great Lessons in more detail to give an overview of their content and their relationship to each other and to elaborate on how the teacher might use the stories in implementing the Montessori curriculum. I give examples of follow-up activities available to the children following the lessons. The photographs referenced in this section on the Great Lessons are my own or are from stock available on the internet.

There are five Great Lessons which are presented starting at the beginning of the school year and continuing over a course of approximately six weeks. Since they are considered foundational to the implementation of the curriculum the stories are told over a relatively short period of a month and a half. They comprise *The Creation Story*, *The Coming of Life*, *The Coming of Human Beings*, *The Story of Writing*, and *The Story of Numbers*. The stories are presented as serious but enthralling tales about the unimaginable forces which gave birth to the universe, the subsequent tableaux of the beginnings of the earth, and how human beings worked to survive and to make sense of their world. It is expected these stories will fill the children with a sense of wonder and awe as they hear about their amazing world. It is also hoped that the narratives serve as a stimulus and as an invitation to the children to delve enthusiastically into aspects of the stories that have caught their attention.

Figure 2<sup>1</sup> shows a hierarchical categorization of knowledge as presented through the elementary curriculum. Included are the Great Lessons in order of their telling. Branching off each story is a list of the main subject areas that are covered by further presentations in the classroom. Each of the stories serves as an introduction to the main subject areas of the

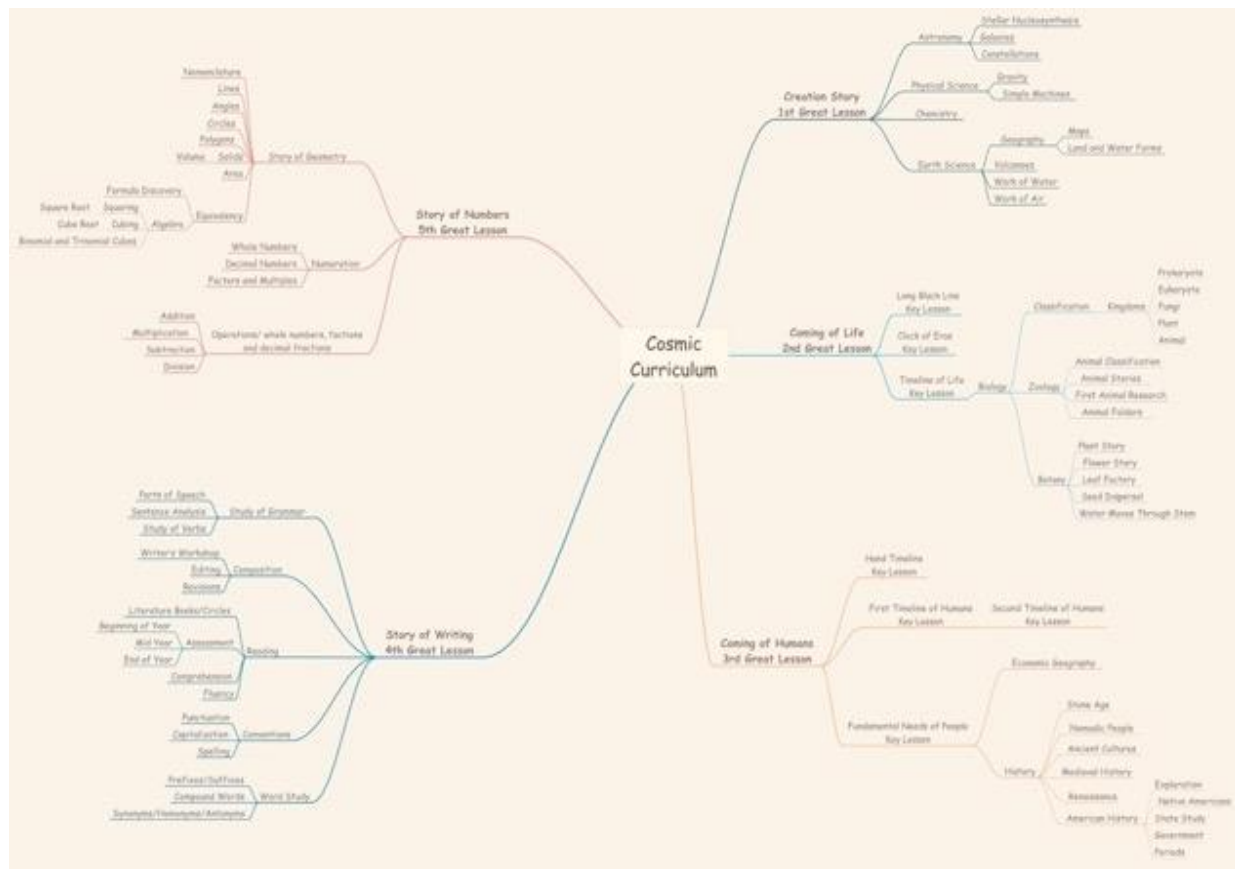
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<sup>1</sup> ©A. Winkler

elementary curriculum. For instance, *The Creation Story* serves to introduce the physical sciences, earth sciences, chemistry and astronomy, while *The Coming of Life* introduces botany, zoology, and taxonomy.

### Figure 2

## Diagram of the Cosmic Education Curriculum



## *The Creation Story*

*The Creation Story* is a narrative that forms an overarching structure from which the other narratives unfold. As in each of narratives, the teacher presents *The Creation Story* as an engrossing story. The children are told in the morning that *The Creation Story* will be presented that afternoon, and this generates an atmosphere of excitement and anticipation. The older

children already know the story and may talk about it with the younger children. In preparation, an area of the classroom is sectioned off where the story will be told. There are numerous science-like demonstrations that are part of the presentation and the necessary materials for them are prepared by the teacher.

As she begins the tale the teacher creates a dramatic atmosphere by asking the children to imagine a time when there was total darkness and unthinkable cold before the huge explosions of matter that signaled the beginnings of our universe (see Figure 3<sup>2</sup> for an impression of the beginnings of the universe). The teacher speaks with a serious demeanor using a compelling delivery which incorporates the use of literary devices such as dramatic visualization, imagery, and sensory detail, with variations in voice, including pauses to heighten the drama.

**Figure 3**

*The Beginnings of the Universe*



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<sup>2</sup> [www.thelearningark.com](http://www.thelearningark.com)

As the story unfolds a series of science-like experiments add to the drama. These demonstrations are intended to create sensory experiences for the children. The teacher, for example, discusses the effect of heat on different types of matter and gives a demonstration in which three items are subjected to heat such as a piece of wax, some lead-free solder, and a steel item such as a nut or bolt. When heated, the solder melts and becomes a liquid, the wax when heated first melts becoming a liquid and then evaporates as a gas, and the iron or steel bolt remains a solid. This is exciting for the children to watch.

Another demonstration creates a sensory impression of the different states of matter. Three containers, such as glasses, are used to contain examples of three different states of matter, one containing a solid (ice), one a liquid (water), and one a gas (air). The children are invited to pick up the glasses and examine the contents of each. There is a demonstration of how liquids settle according to their density (see Figure 4). In this demonstration molasses is poured into a test tube, water is then poured on top of the molasses, finally oil is poured over both the water and molasses. The molasses settles to the bottom, the water rests on top of the molasses and the oil rests on top of the water and molasses. The resulting image of the three liquids, clearly demarcated, is compelling for the children and forms a powerful impression the children may recall when they have further lessons on the density and states of matter (see Figure 4).

#### **Figure 4**

*Examples of Demonstrations*



Three states of matter, solid, liquid, gas



Liquids settle according to their weight

The story culminates dramatically with a volcano that suddenly erupts, surprising those children who have not heard the story before. The volcano represents the tumultuous period at the beginning of the creation of the earth when the earth was unstable. The eruption in the teacher's demonstration (see Figure 5)<sup>3</sup> results from a chemical reaction when baking soda is mixed with vinegar. These are a few examples of demonstrations given during this narrative.

### Figure 5

#### *Erupting Volcano*



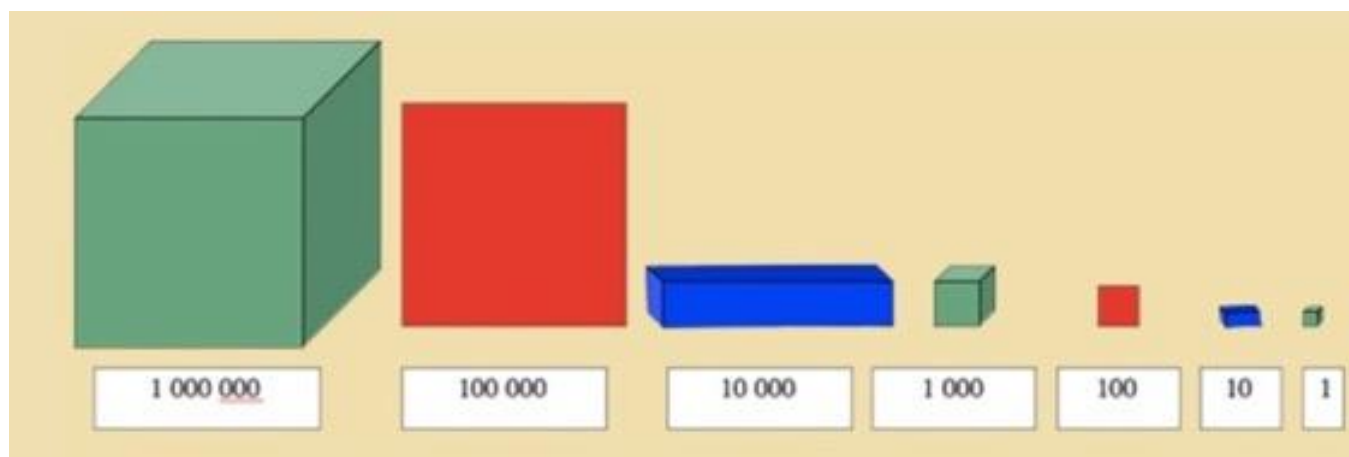
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<sup>3</sup> [www.thelearningark.com](http://www.thelearningark.com)

One of the teacher's roles is to support the children's learning by actively linking subject areas together. In a later discussion about the size of the sun in relation to the size of the earth, the teacher may link mathematics and astronomy using a mathematical material called the Wooden Hierarchical Material. In this diagram (see Figure 6<sup>4</sup> which shows a diagram of the Wooden Hierarchical Material) the large green cube on the left is used to represent the sun while the small green cube on the extreme right represents the earth.

**Figure 6**

*The Wooden Hierarchical Material*



The comparison between the size of the large cube and the small cube creates a powerful visual contrast for the children which is also accentuated by the difference in effort required by the children to move the two cubes. The unit cube is easily picked up between two fingers, while the cube representing a million requires more than one child to lift it.

After this Great Lesson the children are free to repeat any of the science-like demonstrations of which there are many. The story opens up subject areas that involve further

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<sup>4</sup> [www.gonzagarredi.com](http://www.gonzagarredi.com)

lessons by the teacher and opportunities for activities for investigating such concepts as the composition of the earth, three states of matter, attraction and gravity, saturation, supersaturation and crystallization, the work of water, and the work of air.

### ***The Coming of Life***

*The Coming of Life* is told approximately a week after *The Creation Story*. It is a tale about the emergence of life on earth starting with single-celled life forms in the sea evolving to a wide diversity of life on earth and is told dramatically to the children. It is accompanied by a three-metre-long timeline which is gradually unrolled as the teacher talks about the rise and demise of creatures during the different geological eras. The teacher slowly tells of life in the Age of Invertebrates, in the Age of Fish, then in the Age of Amphibians during the Paleozoic Era gradually leading to life in the Age of Reptiles during the Mesozoic period and finally on to the Age of Mammals in the Cenozoic Era, ending the story just before the appearance of humans. (Figure 7 shows the timeline, a fossilized fish and the end of the timeline showing a sketch of a human).

### **Figure 7**

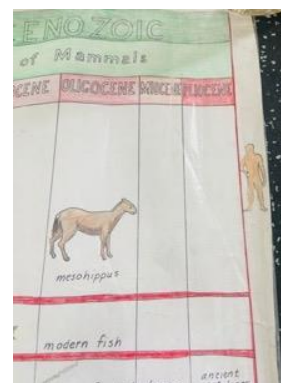
#### ***Timeline and Fossil***



Section of timeline



Photograph of fossil of a fish



End section with human

The human being is positioned outside the timeline and this placement creates a visual impression of the variety of life forms on earth and the length of time that has passed before humans appear.

Companion to the main timeline is the *Blank Timeline* which has the same format as the main timeline in terms of headings, lines, and divisions, but has no illustrations. It is designed to offer the children further opportunities to ponder about life on earth while they place cut-out illustrations copied from the original timeline in the appropriate ages and periods.

Figure 8<sup>5</sup> shows two children working with the blank timeline, choosing, and placing cut-out illustrations of dinosaurs. The main timeline is shown to the left of their workspace.

## Figure 8

*Working with the Blank Timeline*

<sup>5</sup> [www.berkshiremontessori.com](http://www.berkshiremontessori.com)



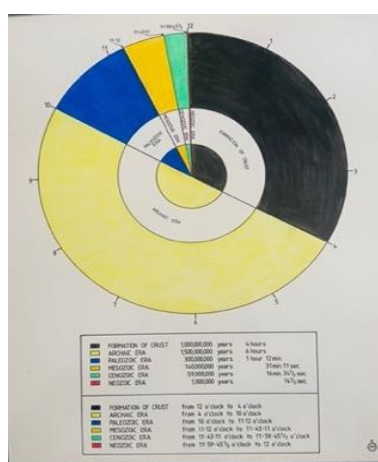
Timelines are an important aspect of the elementary classroom, and the children are first introduced to the concept of a timeline during *The Coming of Life*. The unrolling of the timeline presents time as dynamic and linear. There are several other timelines presented to the children during the course of these stories, and later the children are encouraged to use timelines as a format for presenting their own work on topics they have researched. Once the timeline is fully rolled out the teacher may start from the beginning again if there is interest on the part of the children, and this time she may focus on geological changes seen over time such as the rise of mountain ranges, ice ages, and climate changes. With the timelines, the children are exposed to the extensive vocabulary that a discussion about the timeline generates as the teacher names the animal and plant life and other features. The timeline may be left out for a couple of days so the children may come back to it should they wish.

As in *The Creation Story* the teacher supports the children's learning by linking subject areas together. For example, two presentations that emerge from this lesson connect mathematics to the geological time periods. Using an impressionistic chart to demonstrate a *Clock of Eras*, the teacher could invite the children to visualize the passage of geological time periods not in the

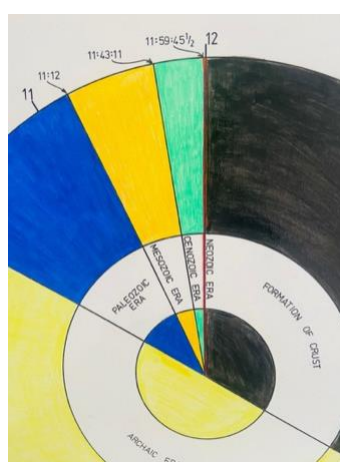
linear manner of the timeline, but in the circular manner as depicted on the chart (see Figure 9)<sup>6</sup>. The chart lists the times of the different geological periods then lists the equivalent amount of time when imagined as hours and minutes within a 12-hour period, for instance, the Neozoic Era was a million years, or 14 ½ seconds.

**Figure 9**

*Clock of Eras Chart with Enlarged Section in Right Image*



Clock of Eras chart



Enlarged section of chart

The image on the right of Figure 9 shows a closer examination of the *Clock of Eras* chart where it clearly shows geological time in time segments representing parts of the whole, which is 12 hours. The formation of the earth's crust takes 4/12 of the available time, for example. Using this chart, the children learn that proportionally humans have inhabited the earth for the last 9 seconds of the 12 hours. The teacher counts 9 seconds out loud for dramatic effect. The

<sup>6</sup> ©Association Montessori Internationale

photograph of a closer view of the *Clock of Eras* chart shows a thin red pencil line which represents the time humans have been on earth.

The second example of how the teacher could link subject areas together is with another visually dramatic presentation where time is presented linearly again with a material called the *Black Strip*. This is a 30-metre long by 30-centimetre-wide strip of black felt fabric rolled up around a wooden dowel. A small section of red cloth about 5cm wide is sewn on to the end width of the *Black Strip* (see Figure 10).

**Figure 10**

*The Black Strip*



This red strip corresponds with the red pencil line drawn on the *Clock of Eras* chart and represents the length of time humans have been on earth. The red fabric only becomes visible once the *Black Strip* has been unrolled in its entirety and this creates a stark visual contrast not only between the colours of black and red but also between the length of the black fabric and the length of the red fabric. The *Black Strip* serves the same purpose as the timelines in that it is

aimed at creating a strong sensorial impression around an idea that may be new and challenging for the children, which in this case, is the very short period humans have been on the earth in comparison with how long the earth has existed. As the 30-metres of black cloth are slowly unrolled, the teacher retells the story of *The Coming of Life* stopping at points to remind the children of when certain creatures appeared and when others died out. The photograph (see Figure 11)<sup>7</sup> shows two children working with the *Black Strip* on their own later after the presentation of *The Coming of Life*.

**Figure 11**

*Working with The Black Strip*



Activities for the children emerging from *The Coming of Life* are varied, such as investigations into geological time periods, into early life forms, perhaps with a focus on one life form such as trilobites, and more complex ideas such as evolution and climate change. The

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<sup>7</sup> [www.montessoriblog.org](http://www.montessoriblog.org)

children may use timelines to investigate topics of interest to them. They may make clay models of creatures and write a report and present it to the class to share what they had learned.

### ***The Coming of Human Beings***

Before starting the story on *The Coming of Human Beings*, the children are invited by the teacher to recall the sketch of the human at the very end of the *Timeline of Life* as a reminder of how much time had passed before humans appear. The children are then invited to imagine themselves as the first humans on earth and to consider how they may have lived and what they would have needed to survive; possibly by using a cave for shelter, gathering kindling and wood to make a fire, and by hunting for food. In the story there is strong emphasis on role of the hands in manipulating materials and making tools as humans fulfill their various needs (see Figure 12)<sup>8</sup>. The teacher may invite the children to create cave art with charcoal, or to carve some wood as a tool, or to collect bark for kindling.

**Figure 12**

*Cutting Bark with Tools*




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<sup>8</sup> ©Association Montessori Internationale

Photographs of relevance to the story that the children may examine are put out by the teacher such as the photograph of a fire made outdoors with twigs and brush. The teacher may talk with the children about what is necessary to make a fire successfully outdoors, and one that is good enough that food may be cooked on it. There may be a photograph of a selection of flint tools to show the children during the telling of this story (see Figure 13).

**Figure 13**

*Photographs of Interest*



Photograph of a fire



Photograph of flint tools

Objects of relevance to the story would be put out for the children to touch and examine. These could be reproductions of items such as a flint tool used for cleaning skin hides or a reproduction of a bone needle used for sewing hide. Lessons emerging from this story could be investigating the fundamental needs of humans, learning about the concept of geological time, the impact of the ice ages on life, movements of peoples, and a study of the child's own life, the country in which they live and how their human needs have been met.

*The Coming of Human Beings* is followed shortly afterwards with the *Hand Timeline*. This is similar to the *Black Strip* as it is a black felt timeline that is unrolled as the teacher talks

about how long humans have been on earth, working with their hands, making tools, building shelters, making clothing, all the time without leaving a written record such as we understand writing today. As the *Hand Timeline* unrolls, an illustration of a hand holding a flint tool is revealed to the children, emphasizing the importance of the hands for humans. The timeline continues to be unrolled by the teacher and at the end of the timeline is a red strip representing the short length of time we have had a record of writing. There is a further timeline, *The Timeline of Human Beings* which covers the earliest period of human life on earth, approximately 550,000 years, and which is presented to the children later.

The teacher may follow these presentations with more detailed lessons about, for example, the measurement and construct of time. The children will learn how to tell time, first the hours, then minutes, and seconds. They will learn about the language and notation for time, such as the concepts of *ante meridiem* and *post meridiem*. They will learn about the days of the week and the months of the year, including their etymology. With the aid of a timeline, they will learn about the concepts of B.C.E (before common era) and A.D. (*anno domini*). The children may make their own timelines of their family or of their life. They will also learn about different calendars and may create their own calendar or a class calendar.

### ***The Communication in Signs***

In *The Communication in Signs* the story begins with the teacher setting an imaginary scene with the first human beings. In this scene one person needs to communicate with another person who was not present. The teacher asks the children to imagine how they would solve this problem. Ideas are shared between the teacher and the children. The children are told about the thousands of years during which people communicated without writing but by drawing on rocks or on cave walls using local resources such as clay, ochre, and sticks to make the drawings (see

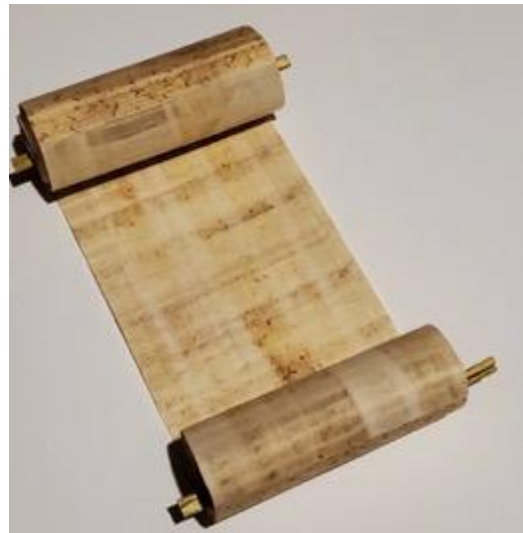
Figure 14) and how people continued to communicate in this way before they eventually started compiling the signs they had created over time that had meaning for them into a collection that became an alphabet.

**Figure 14**

*Ways of Communicating*



Drawing on cave wall



Egyptian roll of papyrus

At a point in the story the Egyptians are introduced as a people who lived in northern Africa on the banks of the Nile River. The children hear about the signs they used, hieroglyphs, and how they communicated on stone and on paper made from the plant papyrus found near the Nile River. The teacher may have a roll of papyrus to show the children so they can feel it and imagine how it was used. The children may be invited to try to make papyrus themselves and to write hieroglyphic signs.

The Phoenicians are introduced next, and the children hear about how these sea-faring traders sailed extensively around the lands of the Mediterranean with cargoes of ivory, spices, incense, silver, and glass. Their most valuable cargo, however, was a dye called Tyrian blue which was extracted from the shells of the murex sea snail (see Figure 15). As the Phoenicians had discovered this dye and learned how to extract it from the shells, they had exclusive access to it and controlled its sale.

**Figure 15**

*A Murex Shell*



The children hear that its harvesting was so labour intensive, and the quantity of shells needed so vast, that only royalty could afford it. The children learn from this story how the colour purple came to be associated with royalty. The implication of the story about the Phoenicians is they needed good bookkeeping skills and tools to successfully manage their trade.

While these details about the dye and royalty may seem unrelated to the development of writing, for the purposes of *The Communication in Signs* they provide a strong sensorial and intriguing context for the concept of writing which allows the children to develop their understanding within this rich framework. *The Communication in Signs* continues with discussions about how the Greeks and Romans communicated and unique aspects of their alphabets. The children are encouraged to investigate areas of interest to them after the story. The teacher may suggest ideas such as making pictographic stories, investigating any aspect of *The Communication in Signs* such as the Tyrian blue dye, or the making of papyrus. The teacher may have a selection of shells the children could crush to see if they could extract a different colour of dye (see Figure 16 which shows a basket of blue shells).

**Figure 16**

*Shells for Dye*



The children may explore writing on different surfaces, or different styles of writing such as copperplate and calligraphy, or different alphabets, or they may invent their own alphabets. They may borrow language material familiar to them from the 3-6-year-old class called the Sandpaper Letters. The children would have used this material tracing the sandpaper letters with their index finger as they learned the letters of the alphabet (see Figure 17 for the Sandpaper Letters, a Montessori material for learning the shapes of cursive letters of the alphabet)<sup>9</sup>.

**Figure 17**

*The Sandpaper Letters*



### ***The Story of Numbers***

This Great Lesson introduces the children to the ways people over time have used mathematics to solve real-life problems and to communicate mathematically about matters pertinent to their daily lives. The teacher may engage the children by asking them to imagine how early human hunters would communicate with each other about how many woolly

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<sup>9</sup> ©Association Montessori Internationale

mammoths they could see in the distance, for instance, or about how could tell each other about the number of red deer that had recently passed by. The story suggests they may have communicated quantity by keeping tally through notched marks on a stick, such as with this tally stick found in Scandinavia (see Figure 18)<sup>10</sup> or with stones, each representing an amount. The teacher may introduce reproductions of archaeological objects that have been found such as strings with knots, and shells strung together and used for counting.

**Figure 18**

*Scandinavian Tally Stick*



The ancient cultures of Mesopotamia are introduced, and the children learn how the Sumerians and Babylonians amongst others made use of local clay to make tablets on which they wrote their letters and numbers using a style of writing called cuneiform which related to the wedge-shaped stylus they used. The wedge could be positioned horizontally which represented 10, or vertically, which represented 1. The teacher may show the children a Montessori chart depicting the wedge-shaped numbers of the ancient Babylonian number system, (see Figure 19 for examples of the Montessori charts of Babylonian Numerals and Egyptian Numerals)<sup>11</sup>. The children hear that as important traders in the region, Mesopotamians needed to keep accurate

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<sup>10</sup> The British Museum

<sup>11</sup> ©Association Montessori Internationale

records of their transactions; what they sold and for how much, and this was accomplished using cuneiform numbers and letters.

## Figure 19

### *Charts Showing Numerals*

*Egyptian Numerals*

|           |                                                                                     |                       |
|-----------|-------------------------------------------------------------------------------------|-----------------------|
| 1         |                                                                                     | a staff               |
| 10        | ∩                                                                                   | a heel bone           |
| 100       | 9                                                                                   | a scroll              |
| 1,000     |    | a lotus flower        |
| 10,000    |   | a bent finger         |
| 100,000   |  | a tadpole             |
| 1,000,000 |  | a man in astonishment |
| 13,015    |  |                       |

Egyptian numerals

*Babylonian Numerals*

|         |      |
|---------|------|
| 1       | ▼    |
| 10      | ◀    |
| 60 • 10 | ◀◀   |
| 23      | ◀◀◀◀ |

Babylonian numerals

Another culture the children hear about in *The Story of Numbers* is the Egyptians. They learn how the Egyptians wrote their numbers differently from the Babylonians, and not on clay but on papyrus. Accompanying this part of the story about the Egyptians is a chart depicting Egyptian numbers written in hieroglyphs with an example of a quantity 13,015. The teacher

might point out the image of the astonished man which the Egyptians used to represent one million and discuss this unusual representation of an amount with the children.

The lesson continues by looking at the how the ancient Greeks represented their numbers using signs from their alphabet. The teacher might discuss the number five in Greek, *pent*, and relate it to the children's work on polygons. How the Romans represented number is discussed and the children are asked if they have seen Roman numerals anywhere around them. The teacher tells the children that the numbers they use are called Arabic numerals and explains how trade exposed people to other cultures' number systems. Finally, the concept of zero might be discussed and the children invited to think about writing numbers without using zero.

Activities for the children may include investigating numerals of various cultures, looking at the challenges faced by humans over time and how they used mathematics to solve problems, different counting systems, etc. Presentations given by the teacher following this story could include an introduction to the decimal system, the Commutative and Distributive Laws, squares and cubes of numbers, long multiplication, a geometrical form of multiplication, long division, multiples and factors, divisibility of numbers, fractions, decimal fractions, squaring, algebraic terms, cubing, square root, cube root, powers of numbers, non-decimal bases, measurement, ratio, proportion, negative numbers, etc. These lessons are given over the course of three to six years depending on the interest of the child and are given with mathematical materials. There are further timelines such as the Timeline of Mathematics which presents an overview of the history of mathematics. Other stories also emerge out of the Great Lesson on *The Story of Numbers*, and the story of how geometry got its name is an example of one. The teacher may compile stories about various cultures or individuals and within a narrative discuss how they have contributed to the field of mathematics. In this narrative on geometry the teacher

brings the children's attention to the ancient Egyptians and the challenge they had regarding the annual flooding of the Nile and how it caused confusion over each farmer's boundaries.

The Great Lessons initiate the sensory curriculum by providing a rich imaginative tapestry of ideas interconnected over time and across disciplines. According to Wenger (1998), "education must involve imagination in a central way" (p. 272). For him there are three aspects to educational imagination namely orientation, reflection, and exploration (p. 272-273). With orientation the children can locate themselves in time and space, with reflection they can consider themselves and their situations and can imagine other circumstances, and through exploration and experimentation the children can conceive of circumstances differently and experiment accordingly. The Great Lessons also serve as a framework for the elementary curriculum and as a pedagogical tool for the teacher when she introduces new mathematical concepts, for example. The narratives provide an opportunity for the introduction of new terminology and extensive vocabulary that is anchored in the context of the stories thereby possibly making the language more meaningful and relevant to the children as they are exposed to it. The stories leave the children with many unanswered questions but create boundless areas of interest for investigation. Importantly, they build a framework early in the school year which serves as a starting point for the various subject areas of the curriculum while at the same time they reinforce the interrelatedness and interconnectedness of the disciplines.

### **Supporting the Sensory Curriculum**

As far as the children's spatial reasoning is concerned important components of the sensory curriculum are the Montessori mathematical materials and the nomenclature booklets. The extensive array of mathematical material is made of wood or metal and is designed to be attractive to the senses. The material is colour-coded with colours consistently used across

materials. For instance, the materials for activities regarding the decimal system are colour-coded in green, blue, and red. Materials used for work with the concept of volume are blue. The materials are referred to by name. The children will be asked to collect the Box of Sticks, for instance, not knowing what lesson they would be having with it. The Box of Sticks is material used for various mathematical activities. It comprises 10 sets of sticks of different lengths and colours in their own compartments. The yellow sticks are the longest sticks (21cm) and the brown sticks the shortest. The sticks are graded in length by a difference of 2 cm. Another compartment contains one each of 10 different lengths of plain wood. All the sticks have a hole at each end as seen in the photograph (Figure 20).

**Figure 20**

*Box of Sticks Material*

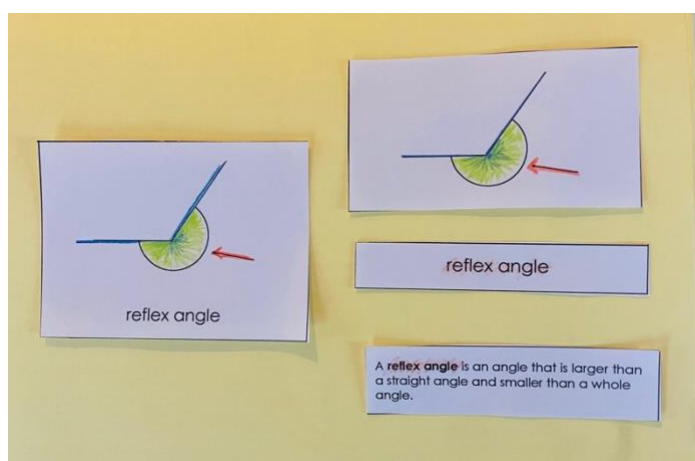


The nomenclature material which is either in booklet form or with cards and labels is an integral component in the elementary sensory curriculum. The nomenclature material serves as a reference source for the children to be used during their independent work and as they develop

their understanding of a new concept introduced by the teacher while using materials. Typically, the children's introduction to specific mathematical materials focuses on their sensorial experiences. The teacher may demonstrate how the material could be used, such as the Box of Sticks material when making different types of angles. An introductory lesson would include some new terminology, but the emphasis is on the children gaining experience using the materials. For instance, the nomenclature material for Types of Angles would be introduced after the children had made extensive use of the Box of Sticks material for creating different types of angles. The children may have had sensorial experiences using a selection of materials for making angles. The small teacher-made nomenclature booklets have a line-drawing illustration on the left side of the page, and a definition of the concept on the right side of the page. These booklets serve as a reference for the children to support their exploration of new mathematical concepts. The nomenclature materials are categorized according to subject. Figure 21 shows two formats of nomenclature material.

**Figure 21**

*Nomenclature Materials*



Nomenclature cards



Nomenclature booklet

## Chapter 6: Results

### *Managing the Data*

As mentioned earlier, I spent approximately two months in the classroom collecting data. This took place between April and May 2019. As instances of the children's spatial reasoning emerged at times I could not predict, I video-recorded many of the children's activities with mathematical materials in anticipation of capturing their semiotic traces. This meant that I was ready to video-record at any moment, and I kept a steady visual scan around the classroom and recorded any of the children's mathematical activities might give rise to semiotic traces. As a result, I recorded over 130 videos of varying lengths. Some were less than a minute long, some were as long as 25 minutes. At the end of each day, I identified those video-recordings I believed would offer a rich source of semiotic traces, and I deleted those that did not. I categorized the videos on the iPad by their date of recording. I copied all the videos for safe keeping on a password-protected external hard drive. I then categorized the videos according to the instance of the children's activity, that is, the space of learning. I further categorized the videos by materials used. Once identified, I compiled a chart that listed the date of the activity, the children involved, the materials used, and the perceived goal of the activity. I identified whether it was a presentation given by the teacher (a lesson), or a follow-up activity by the children (see Appendix H for a full chart of children's work with mathematical materials).

While I had anticipated using audio-recordings for data collection, I mainly limited their use to providing a supporting role to the video-recordings. As the children's interactions with the

teacher and with each other were recorded on the videos I used the audio recorder as a back-up. In instances where I video-recorded the teacher and the children from a distance, for example, I placed the audio-recorder close to the teacher so that I could capture what she was saying, and hopefully capture the children's responses. There were occasions during the analysis of the data that I used the audio-recordings to support my comprehension of the video-recordings. I did not have many conversations with the children, and when I did, I used the video-recorder rather than the audio-recorder. Audio-recordings were very useful in my discussions with Linda, the teacher. These took place outside regular school hours and there was no need for me to video-record the conversations.

From a body of approximately 130 videos, I made a preliminary selection of videos of children's work that interested me for the richness of the semiotic traces and whose quality was good. Videos that showed the children working with obvious interest and engagement were of interest to me as were videos showing activities that captured my curiosity where I found myself asking, "What is the child doing with that material?" That a child was engaged in an activity working with material was important to my selection criteria. In some videos, the children's work did not appear connected to any earlier lessons I had seen the teacher giving. This suggested the children had used their autonomy in choosing an activity and this was a further important criterion in the selection of videos. I watched each video several times to get a sense of what was being said and what was being done. I set the work in context, such as identifying the children, identifying the material being used, and identifying the purpose of the activity. I watched the selected videos on the iPad and using headphones, I transcribed the data, replaying each video many times, going over unclear parts to ensure I had transcribed it accurately. I worked from one selected video to another. I was particularly interested in the work done by one

child and began my analysis with the video of this activity. Even though this child was only active for about 15 minutes the semiotic traces were exceedingly rich, and as I analyzed and reviewed the videos repeatedly, deeper layers of spatial reasoning emerged. I gave myself time between the videos as I knew from experience that I needed time away from the work to reflect on what I was seeing and understanding.

I ultimately selected 11 videos for analysis and presentation in tables. While presenting the tables chronologically was an option, I chose different criteria. The tables were carefully ordered to facilitate the readers engagement with the data. Namely, I selected videos that showed various children working with a selection of materials. The purpose was to engage the reader and at the same time to generate a sense of the diversity of activities taking place in the classroom. Furthermore, I considered it interesting to offer those videos that showed children working with the same materials but with different purposes. For instance, in Tables 1, 2, 9, and 11, each child worked with the Box of Sticks material, but used the material in different ways. Conversely, some tables had analyzed videos of the children using different materials but for the same purposes. This is seen in Tables 1, 6, 8, and 9 where the children work in activities around the concept of triangles. It was important to present the tables in a sequence that while retaining the reader's interest also showed the variety and diversity of activities taking place in the classroom as far as the children's mathematical spatial reasoning was concerned. The order of the tables was made with these considerations in mind.

The tables included a combination of verbatim texts of the children, annotated descriptions of their actions, along with photographs captured from the videos. At the beginning of the tables I included selected photographs of the children whose activity is under discussion in the analysis. In addition to supporting the reader's categorization of the data, these photographs

serve as a means of adding to the rich visual tapestry already found in the analytic tables of the children as they go about their work in the classroom. There is a general discussion on the children's spatial reasoning in each table and this is followed by a more detailed examination of how the children are using spatial reasoning in their activities. In some instances, there is additional commentary such as with the work of one child which took extra efforts on my part to analyze.

### **Gail and Anisha**

The following tables and commentary concern work undertaken by Gail (Table 1) and Anisha (Table 2). Although the children worked separately, they both traced concentric circles on paper using the same materials (see Figure 22). What was of great interest to me was their spatial reasoning about the circles they had drawn in common. Gail reasoned about length while Anisha reasoned about angles. Gail did not speak of circles, and although Anisha used the word to introduce her work to me, she quickly said, "but I call it angles." What also drew me to the videos was Gail and Anisha's display of independence in choosing their activities, their organization of the necessary materials, their care as they traced each stick around creating a circle, and their perseverance in completing the challenging task they set themselves in drawing circles with sticks.

### **Figure 22**

*Drawing Circles with the Box of Sticks Material*



Gail tracing her circle



Anisha tracing her circle

Anisha started her activity in the morning. Her loud hammering caught my attention. To draw the circles Anisha had placed a large sheet of paper on a corkboard then hammered a nail to its centre. Anisha chose a coloured stick from the box. The sticks, which are of differing lengths, have a hole at each end. She put the hole of one end over the nail and inserted a pencil in the hole of the other end. She rotated the stick around the axis of the nail, tracing a circle. She made the largest circumference that could fit on the paper. After completing a series of circumferences, Anisha coloured in between the lines of the circumferences. The colours she chose corresponded to the colours of the sticks she had used making the circles. Later that day in the afternoon, Gail also drew concentric circles using the same materials. When Gail had finished making her circles, she, too, coloured in the spaces following the colours of the sticks she had used

**Table 1**

*Gail Makes Concentric Circles*

| Text | Images |
|------|--------|
|------|--------|

April 16

13:56:00 – 14:15:00

### Description of geometry material

The Box of Sticks material comprises 10 sets of sticks of different lengths and colours in their own compartments. The yellow sticks are the longest sticks (21cm) and the brown sticks the shortest. The sticks are graded in length by a difference of 2 cm. Another compartment contains one each of 10 different lengths plain wood. All the sticks have a hole at each end as seen in the photograph.

Gail sits on the floor with the Box of Sticks geometry material open next to her on a mat. She has a corkboard with a large sheet of paper on it. Gail hammers a nail into of the paper. Gail selects the stick she wishes to use.

Gail: First, we are going to use the longest one. Yup, this is the longest one (measuring the two sticks against each other and positioning the hole over the nail).

Gail: Now I need a lead pencil (getting up off the floor to get a pencil).

Gail: I'm going to need the second longest one. We are going to add. We need the second longest (choosing a stick).

Alison: What are you doing?

Gail: Well, we are measuring.

Alison: You are making circles?

### Box of Sticks Geometry Material



Gail positions the hole over the nail



Gail. Yeah (attaching the stick to the nail then moving it around in a circular movement).

Gail: It goes off the board (rotating the stick as it extends beyond the edge of the paper).

Gail determines the stick she wants by measuring each from the nail to the edge of the paper.

Gail: That goes off the board. Too short, too short! Not that one. Too long. I'll try this (picking up another stick and placing the hole over the nail).

She starts rotating the stick 180° by turning it to her left.

Gail. Nope! (rotating the stick which overlaps the edge of the paper).

Gail: See. We will go with this one.

Gail inserts a pencil in the hole. She moves the pencil to her left, drawing a line. At about 180° from where she started the pencil comes off the edge of the paper onto the corkboard. Gail continues. She changes the pencil to her right hand and traces back along the line. She continues drawing the pencil line until she reaches the start.

Gail. There! We will get the next one. Yellow! (attaching a shorter stick to the nail).

Gail traces another circle on the paper using the yellow stick. She retraces the lines a couple of times going back and forth. She completes a full circle. Gail removes the stick from the nail, and loosens the nail in the process.

Gail moves the stick around



Gail rotates the stick 180°



Gail changes the pencil to her right hand



She picks up the hammer and starts hammering the nail back in place.

Gail: Blue! (calling out the colour of stick she wants).

She chooses a bright pink stick and repeats the process drawing another concentric circle.

Gail picks up selection of coloured sticks placing a green, a light pink, and a black stick down on the paper. She adds a brown stick to the pile but keeps a purple stick in her hand. She has now put six sticks of different lengths and colours on the paper.

Gail: So. Pink! No, green!

Gail continues tracing circles with the different coloured sticks.

Gail: Pink! Next. And that's black (loosening the nail as she traces).

Gail: Loose (putting down the purple stick she had held for a while and reaching for the hammer).

She hammers the nail back in.

Gail: So, hold on. Is that black? Yeah, it was the black (placing the hole of the black stick over the nail and tracing it around for a couple of centimetres without the pencil).

Gail hammers the nail back in place



Gail repeats the process with a pink stick



Gail traces an arc with the black stick



Gail continues tracing using different sticks.

Gail: Oh! Um, red please! Orange please!

Gail places it on the nail and starts tracing with the orange stick.

Gail: Purple! (requesting the coloured stick).

Gail: The smallest piece (tracing a circle with the pencil and brown stick).

Other child: Oh, it's so cool (looking at Gail's work).

Gail: Now, take it out (touching the nail).

Other child: Take this out? (leaning over to touch the nail).

Gail. Yup, take the needle out.

Gail: You get it in the smallest hole (helping the other child position the claw of the hammer over the nail). Get it right here. Hold on. There (maneuvering the head of the hammer and removing the nail).

Gail: I'm just going to (getting up).

She comes back with another nail. She pushes both nails into the centre of the paper.

Gail traces using the orange stick and pencil



Gail traces with the brown stick and pencil



Gail uses the hammer to remove the nail



Gail: I need the hammer! So, we have two! (hammering the two nails into the centre of the paper).

Other child: Oh my gosh. Two needles! Why?

Gail: My dad told me. He is an electric. Now we are going to start colouring. So, this, - what colour do you want this? (drawing her finger all the way around the outermost circle.

Gail: What colour? What colour do you want? Yellow? (leaning over the coloured pencils).

Other child: Sure, yellow.

Gail: How about we copy this? (placing her hand on the Box of Sticks material).

Other child: Sure.

Gail: Right, I will be colouring in the lines. Okay? (colouring in the outermost circle).

Gail traces the outer circle with her finger



Gail colours in the outermost circle



### ***Commentary on Gail's work***

The data for this analysis were Gail's speech and her movements working with the materials. There was little discussion between us because when I asked Gail a question, I sensed an unease on her part and so I did not engage with her further. The other child appeared to be an observer, or helper, in the activity and as a result this the analysis is of data concerning Gail's reasoning. In analyzing Gail's reasoning about the circles, I found that it mainly revolved around

the concept of length. Gail demonstrated other instances of spatial reasoning such as when she located her materials on the shelves, oriented them on the floor, positioned a large sheet of paper on the corkboard aligning the sides and situated the nail at the centre of the paper. Gail selected her sticks to make the circle by length. It seemed she reasoned she needed the longest stick to draw the largest circle, and this was shown through the process of selecting her sticks. The criteria she appeared to use was to find the stick that measured from the nail in the centre to as close as possible to the edge of the paper. To circumscribe the largest circumference possible, which I took to be Gail's intention when she said she needed the longest stick, it was necessary for the nail to be positioned at the centre of the paper. Because the nail was not exactly at the centre of the paper, Gail made number of efforts in selecting the appropriate stick. One stick she had decided was the longest she realized was too short after trying it out. She chose another but decided it was too long as it came off the paper at some point of the rotation.

In choosing her sticks Gail also measured them against each other, aligning their lengths, and discriminating which was longer. She ordered the sticks when she said, "First we are going to use the longest one", and "I'm going to need the second longest one." She compared their lengths when she stated that one stick is "too long", and another is "too short." When she used the shortest stick to create the smallest circle she said, "the smallest piece" suggesting she had ordered the sticks from longest to shortest. She used mathematical language when she said, "We are going to add. We need the second longest." Gail's use of language suggested to me that her focus was on length. She also used language related to addition. I did not hear a reference to the circle. The only time I spoke with Gail was to ask her what she was doing, and she answered, "Well, we are measuring" which suggested that was how she understood her work.

I understood Gail's spatial reasoning mainly through her movements. She did not speak much. I viewed the video recording without audio so that I could focus on her movements and how they revealed her spatial reasoning. She appeared to know what steps she would follow to achieve her diagram of circles. In trying out the different sticks, she rotated each smoothly in an arc. For example, when she put on a stick that would not make the circle she wanted, she knew after rotating it about  $30^\circ$  that it was too short. She tested the range of the sticks in both directions, and she drew half a circle with one hand then transitioned the pencil to the other hand and continued drawing. This suggested to me that she was confident enough to stop the momentum of her work knowing how to pick it up again, at the same time transferring her pencil to the other hand. Gail went back and forth over the circle where necessary when the pencil lines were not dark enough. Once her first and largest circle was complete, she moved very quickly drawing the other circles using sticks of increasingly shorter lengths to create smaller circumferences. Her familiarity and understanding of the colour coding of the sticks might have been a help in this regard. After drawing four circumferences she speeded up her work by selecting the next six sticks she needed at one time.

### ***Spatial Reasoning Skills Used by Gail***

Gail used an integration of visualization and mental rotation skills in a multi-step process in her work of drawing her circles (Casey, 2008; Davis et al., 2015; Owens, 2015). She used mental transformation skills and transformational symmetry when she reasoned how she could use the linear sticks to create concentric circles. Her work may be considered a "notion of symmetry as transformation where a motion is used to transform one initial figure into another" (Ng & Sinclair, 2015). The composition of geometric shapes is considered foundational to spatial reasoning (Polly, Hill, & Vuljanic, 2015). Gail demonstrated the close mathematical and spatial

reasoning involved (Mulligan, 2015; Whiteley et al., 2015) when she spoke of the number concepts, “long”, “longer”, “short” (Hawes, et al. 2017), and stated that “We have to add.” The use of these words supported her spatial reasoning (Casey, 2008; Leavy et al. 2018). She used a pencil and paper to trace her circles, an activity shown to support children’s spatial reasoning (Ginns et al. 2014; Sarama & Clements, 2009).

Table 2 which follows, contains the data of my meeting with Anisha which took place about three weeks after she had drawn her circles. I had invited her to talk about her work with me and she had agreed.

**Table 2**

*Anisha Talks About Her Work with Concentric Circles*

| Text                                                                                                                                                                                                                                                                                                            | Images                                                                                                                                                                 |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>May 9<br/>09:15:00 - 09:18:00</p> <p>Anisha spread her sheet of paper out on the table and smoothed it flat.</p> <p>Anisha: Alright, so this is a poster of all kinds of circles (tracing around the outermost circumference with her index finger).</p> <p>She hesitates and is silent for six seconds.</p> | <p>Anisha traces the outer circle</p>  <p>Anisha squeezes her fingers together</p> |

Anisha: I call it angles (bringing her fingers together).

Anisha: So, what I do is I write where an acute angle is, and I have to draw a line to how it is an acute angle. Like a whole angle goes like that and then for a right angle, it goes like that (drawing her finger along the lines which define the angle).

Alison: So, which is the whole angle?

She repeats the action and draws her index finger down the pencil line bisecting the circles and ending the movement close her body).



Anisha draws her fingers along the lines



Anisha ends the movement close to her body



Anisha traces a straight line with her finger

Anisha: The whole angle is right down and then the right angle is like that (tracing the right angle) and the straight angle is like that (tracing the pencil line of the straight angle).



Alison: Okay, because I was sitting in the classroom and I heard this hammering and didn't know what was going on and then I turned and I saw you were hammering. So, what were you doing with the hammering?

Anisha indicates where the nail had been



Anisha: So, what I was doing, is I was taking, - I was taking a nail, and I was putting it right there on the board (putting her index finger at the centre of the paper where the nail had been inserted).

Anisha shows me the board she used

It's this board right here. Ahh! Not that board (putting the board back and picking up another one). This board.

I was nailing it on to this board, (showing me where she hammered the nail onto the board).

She moves back to the table and selects a yellow stick from the Box of Sticks on the shelf.

Anisha: So that way I can put one of these on top of the nail and then (aligning one end of the yellow stick with the centre and the other end at a point on the outer circumference) I take a pencil and I trace right here, and it goes (rotating the stick around the circumference).

Alison: All the way around?

Anisha: Yeah.

Alison: So how did you know what size to choose, how did you choose what were you doing?

Anisha: Oh, it goes from biggest to smallest (pointing to the circles she has drawn then turning to the shelf behind her)

Alison: Biggest to smallest?

Anisha: Yeah, see? (removing the lid from the Box of Sticks).



Anisha demonstrating how she rotated the stick



It goes bigger, a little bit smaller, a little bit smaller, a little bit smaller (touching the different lengths of sticks).

Alison: Okay. So, then you drew them all around then you coloured them in, -

Anisha: Yeah (coming back to her work on the table)

Alison: And then when I saw your work before it didn't have the writing on it so you came back another day?

Anisha: Yeah, and I wrote it (making a writing motion with her hand on the paper).

Alison: Is this work finished now?

Anisha: No (looking at me).

Alison: Oh? What has to happen?

Anisha: No, well, the obtuse angle I have to fix (drawing her finger across the area of the circles in a straight line) and the reflex angle I think I have to do (putting her index finger on the paper outside the drawn circles and resting it there).

Alison: Do you know what these different types of angles are? What's a reflex angle? I don't think I remember.

Anisha: Not really, but I think I can tell you (getting up and going to the shelf with the nomenclature booklets).

Anisha: Types of Angles is in one of these books (going through the booklets on the shelf)

Type of Planes, Angles and its Parts (reading the titles of the booklets as she goes).

Alison: What are those books?

Anisha collecting a nomenclature book



Anisha: Angles and its Parts, Types of Angles there we go! (exclaiming as she retrieves the booklet she is looking for).

And so, in here it actually does not show reflex, I think (opening the booklet and going through the pages)

Alison: Oh, it doesn't?

Anisha: It doesn't show a reflex. But this is an acute angle (turning the booklet to me so that I can see the acute angle).

She uses her index finger to trace the lines of the acute angle then turns the page.

Anisha: A right angle (drawing her index finger down and along the lines drawn in the booklet).

Alison: And which is your right angle on your work?

Anisha: A right angle is right there (leaning over and tracing the lines on her work).

She returns to the book and turns the page.

Anisha: And an obtuse angle - I wrote it but I didn't put the lines (pointing to the obtuse angle in the booklet).

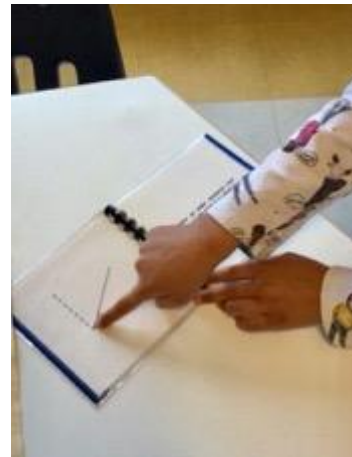
She comes to the definition of a straight angle in the booklet and goes back to her work

Anisha: Straight (drawing out the word and moving her finger along the line of the angle).

She turns the page to the definition of the whole angle.

And then a whole angle (going back to her work and drawing a line from left to right along the straight line).

Anisha showing me the acute angle



Anisha showing me the straight angle

Anisha: And that's it. There's no reflex (putting the booklets away).



### *Commentary on Anisha's work*

The data sources for my analysis of Anisha's work were her speech and her movements. Anisha appeared comfortable speaking with me which I understood by her willingness to share her work. Like Gail, Anisha did not talk about circles. In introducing her work to me Anisha said, "This is a poster of all kinds of circles" then said, "I call it angles." She did not use the word circle again but talked about angles for the remainder of the discussion, and therefore when considering her spatial reasoning I understood it in relation to angles. In preparation for her activity, Anisha had to locate the necessary materials from different parts of the classroom, locate a space on the classroom floor where she could position the corkboard, and orient the large sheet of paper on the centre of the corkboard. She needed to decide where the centre of the paper was so that she could hammer in the nail.

Anisha had added to her work since I had last seen it. She had drawn straight pencil lines from points on the circumference to the centre of the paper and she had written down the names of angles at various places on the circles. I commented that when I had originally seen her work there were no pencil lines. She explained that she had been to see Linda about finding where the angles were, and Linda had helped her. It seemed that Anisha reasoned that to find angles it was

necessary for her first to draw the series of circles and the angles were to be found therein. This was suggested when she said “what I have to do is write where an acute angle is, and I have to draw a line to how it is an acute angle” implying this needed to be done on the circles. It is also possible that since Linda presented a variety of angles, Anisha believed a variety of circles was needed. This could be the reason why Anisha, of her own volition, drew a series of circles. She may have understood a relationship between the variety of angles and the number of circles.

Anisha was confident in her terminology about the angles and there appeared to be a coherence in her movements and speech as she talked. This was seen as she named angles and used her finger to trace the sides of angles. Anisha comfortably answered me when I asked how she knew what size stick to choose. She pointed to the circles then showed me the Box of Sticks material. She said, “it goes from biggest to smallest”, then she said, “It goes bigger, a little bit smaller, a little bit smaller, a little bit smaller” which seemed to show that she had thought about length and could compare and order the sticks according to length.

### ***Spatial Reasoning Skills Used by Anisha***

Like Gail, Anisha used an integration of visualization and mental rotation skills in a multi-step process in her work of drawing her circles (Casey, 2008; Davis et al., 2015; Owens, 2015). She used mental transformation skills to reason how she could use the linear sticks to create concentric circles. She used spatial visualization skills to reason about angles (Gibson, Congdon, & Levine, 2015). Her reasoning about angles was a process that took time, in keeping with the research that shows the building an abstract of angle is difficult and takes time (Smith, King, & Hoyte, 2014). Anisha visualized to reason about different angles in keeping with research that found children can compare angles across multiple two-dimensional shapes by age 4 (Izard & Spelke, 2009). Of interest is research that shows that children, when reasoning about

angles, may focus on irrelevant properties such as the length of the sides of the angle (Gibson et al., 2015; Hallowell et al., 2014). Anisha's reasoning, however, was about dynamic angles defined as a turn or rotation, and this was challenge for her as it was hard to know where an angle began and ended (Smith, King, & Hoyte, 2014). That Anisha reasoned about angles as rotation was supported by her visual proof with the stick as she rotated it around the circles.

Anisha supported her spatial reasoning by using spatial terminology for the angles and used the nomenclature booklets (Bartolini Bussi & Baccaglini-Frank, 2015). She visually proved her claims about the angles with her body, using her fingers to establish the boundaries of each angle, then visually proved her angles again by first reading the definitions and then using her body to demonstrate where the angles were on the paper. She used tracing skills, and fine motor skills support her reasoning as she drew her circles (Ginns et al., 2014; Thom & McGarvey, 2015; Sarama & Clements 2009).

### ***Aspects of The Analysis of Both Sets of Work***

I developed my analysis of Anisha's work much earlier than I had Gail's. I had not considered analyzing Gail's work with the circles. I was fascinated by the fact that although Anisha had a sheet of concentric circles in front of her, she did not talk about circles. It became obvious to me that I could not accept the drawing as of circles if Anisha did not consider the drawing that way herself. I decided I needed to explore the genesis of her circles and remembering Gail's work on circles decided to look more closely at hers. Analyzing Gail's work, I came to realize that she did not mention circles either, and she referred to her work as measuring. As neither Anisha nor Gail referred to their work as concerning circles, which was how it patently looked to me, I decided it would be interesting to investigate how they came to

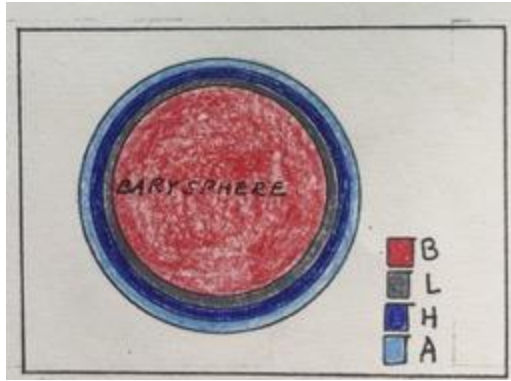
their understandings of what they were doing. My discussions with Linda helped track a possible genesis of these children's work.

### ***Linda Talks About Their Work***

Since both children were doing similar work, I had wondered if it was follow-up work to a lesson Linda had given. When I discussed this with Linda, she told me that there had not been a lesson. She had observed Gail and Anisha while they were making their circles and said she was curious at the time as to what each child was doing. Linda and I talked about what may have contributed to the initiation of the work. Linda suggested that experiences in the classroom over the course of the year may have served as an incubator for Anisha and Gail's ideas. She noted that Anisha had been drawing many circles, and over the past four months other children had been making circles. Linda commented that "Shan was doing some of the circles for other reasons." She suggested Anisha and Gail's interest was "probably playing into the Layers of the Earth [lesson], because ... some people started to do the layers of the earth." With this reference, Linda was explaining how in a Great Lesson called The Coming of the Universe the children are shown a large impressionistic poster called the Layers of the Earth. The poster shows 5 concentric circles with the spaces between the circumferences coloured in (Figure 23 shows a Montessori chart for the layers of the earth). Linda voiced her thoughts that this poster could have been a possible influence on Gail's and Anisha's interest in making concentric circles.

### **Figure 23**

*Chart of Layers of the Earth*



Linda thought that when material is brought out for a new lesson it reminds the children of previous lessons, they have had with it, and this might motivate them to engage with the material to practice their older lessons. Linda had a lesson in mind when she said this: In the Fall she had introduced the concept of angles using the Box of Sticks material. She had used two sticks attached at one end to paper with a nail. At the opposite she had inserted a pencil through the hole. Holding the pencil, Linda had gradually moved the stick to the right, creating and widening an angle measure. As she moved the stick, she had named types of angles as they occurred with the widening angle measure. Linda demonstrated an acute angle first (Figure 24 which is a photograph of a composition made by me).

**Figure 24**

*Angle Made with Sticks*



By the time Linda had reached a whole angle, a complete circle had been drawn. This had created a visual image of a circle made using the sticks. Linda offered the above examples as possible inspirations for Anisha and Gail's work.

### **Jennifer and Inez**

I selected the following video which showed two children, Jennifer and Inez, engaged in making a three-dimensional box out of paper (Table 3). I was curious as to what they were doing since the work was not familiar to me and I thought perhaps they were doing it as a craft activity, which turned out not to be the case. As Jennifer told me, their work was in response to a lesson given by Linda, but I did not manage to establish the topic.

**Table 3**

*Inez and Jennifer Make a 3D Paper Box*

| Text                         | Images               |
|------------------------------|----------------------|
| May 9<br>10:08:00 – 10:09:53 | The black wooden box |

Inez and Jennifer are making a three-dimensional paper box. They are replicating a little black wooden box that has two drawers. The black box is an item found in the language material for lessons on nouns. The box has two drawers. Inside the drawers is a series of paper cards on which the names of nouns may be written by students when they repeat the work of the lessons.

Inez and Jennifer have built three sides of the paper box as well as the bottom of the box and have inserted a divider for the drawers. Jennifer holds what appears to be the paper top of the box in her right hand.

I invite Inez and Jennifer to tell me about their work.

Jennifer: Well, this is like a box that you have to write stuff in, and you have to write nouns.

She puts down the paper top to pick up the black wooden box. She taps inside the opening from where the drawers have been removed.

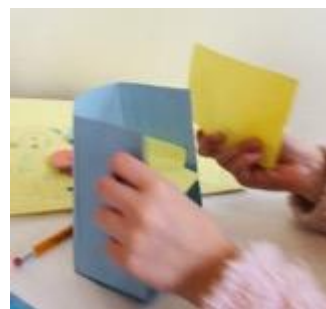
She places the black box down on the table and puts the drawers back in.

Inez: And we aren't quite done yet. There are two boxes, as you can see in these (pulling out a drawer from the black box).

Alison: Two drawers.



Three sides of the paper box are built



Jennifer taps the inside of the box



Inez: Yeah, two drawers, like a dressing drawer, and we are going to put a roof on top with a little door so you can peek in (placing her hand flat over the top of the paper box).

As Inez talks, Jennifer places the paper top on the box. When Jennifer removes her hand, Inez puts her hand back over the top, curling her fingers.

Alison: So, you are making your own one of these?

Inez and Jennifer: Yes.

Inez: And we are going to do this and put it on the sides (she places one side of the drawer flat on the surface of the table. Then she places each of the other three sides on the surface of the table. She puts the drawer back into the black box).

Alison: That's how you measured it?

Inez and Jennifer: Yeah.

Alison: Show me again what you did?

Inez: We were going to do this, (taking the drawer out again and placing the surface of each side of the drawer on the table.

The cards fall out of the drawer.

Alison: Why are you making your own box? Just for fun?

Inez: From the lesson.

Inez curls her fingers over the top



Inez places each side down



Inez places each side down again



Jennifer: We are trying to learn about it, and they want people to see how we can make things.

Inez: Yeah.

Jennifer: And Ms. Linda is also wanting us to learn how to use our fingertips, because normally you go like that. Ms. Linda is trying to make us learn the fingertip way.

Alison: What are you going to do with it?

Inez: We are going to decide who is going to take it home. I'm not. Actually, I may take it home.

Alison: Are you going to put cards in it?

Jennifer: Yes, we are making cards.

Jennifer cuts out a circle of paper from the middle of her piece of paper then places the paper on the top of the box.

Inez: It going to be awesome!

Jennifer demonstrates how to hold the pencil



Jennifer places the paper on top of the box



### ***Commentary on Inez and Jennifer's Work***

Inez and Jennifer had made most of their three-dimensional paper set of drawers when I started observing them. What was left to make was the top and the two drawers. Jennifer was in the process of gauging the size for the top by orienting a piece of paper over it. Watching this activity, I observed both children use a variety of spatial skills. In order to replicate the little wooden set with two drawers, Inez and Jennifer would initially have used spatial visualization. This type of reasoning allowed them to mentally picture the drawers being created in a different material, namely paper, and it allowed them to imagine what the object would look like upon completion. During their spatial visualization of the set of drawers, they would have used mental rotation skills to understand the object from different perspectives. This would be necessary for

them to comprehend the scope and dimensions of their project. In order to cut the pieces of paper in the correct dimensions to make the drawers, Jennifer and Inez needed to decompose the object into its constituent parts. For this they needed to transform a three-dimensional object into a two-dimensional shape, and once they had isolated the necessary parts of the two-dimensional shape, compose the parts into a three-dimensional object. Inez demonstrated how she did this with the drawer when she took it out of the wooden set, then placed each of its four sides on the table thereby isolating the surface area of each side. By visually establishing the surface area of the sides, she indicated that she had reasoned spatially and understood that she had decomposed the drawer and that four pieces were necessary for the sides. Inez did not demonstrate this with the bottom of the drawer.

Jennifer's movements suggested an understanding of the spatial attributes of depth, or length, when she put her fingers into the space for the drawers in the black wooden box. She compared the space there with the same space in the paper set of drawers and implied they needed to create paper drawers of the same dimensions. As she discussed the work with me, Jennifer likened their paper object to a box which suggested to me that while the wooden set could be called drawers, she was not prepared at this time to call the paper drawers the same name. Inez also referred to "two boxes" when discussing the paper representation that needed to be made, but while talking about the black wooden box, referred to "two drawers." The different language for the wooden drawers and paper box suggested that initially while the set of drawers was a known and familiar object, they did not see the paper object the same way. Later, Inez talked about adding a roof and a door, using terminology familiar to the home. When she talked about the being able to peek inside through a little cut-out door in the roof, it appeared that Inez

visualized the paper object as a house and had not yet settled on what the paper box would represent, namely a set of drawers or a house.

Both Jennifer and Inez used their hands in conjunction with their speech. When showing me the paper box, Jennifer put her fingers in the area where the paper drawers would be, and Inez curled her fingertips into a circle when talking about the circular hole that was to be cut into the top which would allow them to peek inside. Jennifer paid attention to how she held her pencil, visualizing how she holds it compared with how Linda would like her to hold it. When drawing comparisons between the paper box they are building and the black wooden box, Jennifer and Inez both touched the black box and the paper box, establishing a connection between the two.

### ***Spatial Reasoning Skills Used by Jennifer and Inez***

Jennifer and Inez used both visualization skills and mental transformation skills in building their paper set of drawers. They transformed shapes through folding and cutting (Leavy, Pope, Breatnach, 2018). Using mental rotation skills, they visualized a three-dimensional set of drawers using two-dimensional paper. The ability to visualize how to transform shapes in space is important in spatial reasoning (Bruce & Hawes, 2015). In explaining how she build the drawers, Inez mentally decomposed the three-dimensional figure back to a two-dimensional shape creating an imaginary net and gave visual proof for this when she placed each surface of the drawers on the paper, mentally unfolding a “cube-net” (Vander Heyden, Huizinga & Jolles, 2107). In support of their reasoning, Jennifer and Inez used spatial language and fine motor skills. They used their imaginations and explored.

### **Toby and Eric**

I chose the video of Toby and Eric's work with the blue volume material because I had not seen Linda presenting a lesson with this material so was curious about what they were doing (

Table 4

). The two boys, who were twins, were very animated in their discussions as they worked, and it was their noisy excitement that had drawn my attention to them in the first place. I established the genesis of this work by asking the children about it. Toby responded, "You know what, Emilio gave it to Mario, Mario gave it to Nicholas" Eric added, "and Nicholas gave it to me." Eric said, "I was the one who got it first, this year."

Their work went through different iterations of activity, and they worked collaboratively with Toby confirming a choice he made with Eric. While working, Toby and Eric interacted with other children. Nicholas called Toby over for help and Toby went to help him. Later Nicholas called across the room to say how far he had gone counting with the Large Bead Frame which was an activity he and Toby had been working on over many days. Anisha watched them working with the volume material and at one point warned them they would be in trouble with Linda if they lost any pieces.

**Table 4**

*Toby and Eric Work with the Blue Volume Material*

| Text                                                                                                                                                                                                                                                                        | Images |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| May 15<br>10:33:00 – 10:55:00<br><br>Toby has two pieces of the blue volume material out on his table, namely a blue square-based prism and a blue cube. The blue prism is twice the volume of the blue cube. He has a box of 2 cm <sup>3</sup> blocks. Using these blocks, |        |

he has built a cube of individual pieces near the small blue prism. He has also built a large prism using the 2 cm<sup>3</sup> blocks which he has placed next to the larger blue prism.

When I begin my observations, he is organizing the 2 cm<sup>3</sup> blocks in their box. The box has a lid and one side that folds down. Toby has used the lid as a tray for his work.

Turning to the cube he has built, Toby appears to count the individual 2 cm<sup>3</sup> blocks using his finger. He notices me observing him and stops working. After a few moments he begins to work again.

He turns to the box and folds up its side. He appears to count the 2 cm<sup>3</sup> blocks again. He stops, picks up the blue cube and brings it above the 2 cm<sup>3</sup> blocks. Toby hovers<sup>12</sup> the blue cube over the top of the blocks then touches the surfaces of the cube and blocks together.

Toby moves the blue cube to one side of the 2 cm<sup>3</sup> blocks and touches the surfaces of the sides. He puts the blue cube down next to the 2 cm<sup>3</sup> blocks which he straightens with his hand.

Nicholas, who is sitting close to him, calls him over with a question about Roman numerals and Toby gets up to speak with him. After some discussion, Toby returns to his work and Eric joins him.

Toby organizes the blocks in their box



Toby hovers the blue cube over the top of the wooden blocks



Toby touches the blue cube next to the wooden blocks



<sup>12</sup> The action of hovering material over other material is well-established in the primary Montessori classroom. Its purpose is to establish a relationship regarding attributes between two objects. In this case the attribute is surface area.

Toby starts dismantling the cube made of  $2\text{ cm}^3$  blocks. He removes three and places them back in the box. He continues dismantling the  $2\text{ cm}^3$  blocks and returning them to their box. Eric's and Toby's attention turns to arranging the  $2\text{ cm}^3$  blocks in the box.

Nicholas asks Toby two further questions about Roman numerals. They discuss the matter while Eric looks on. Toby returns to sorting the  $2\text{ cm}^3$  blocks. He picks up the last of the blocks.

Nicholas: I'm at 400, you know (calling across to Toby).

Eric starts dismantling the prism made of  $2\text{ cm}^3$  blocks.

Toby: Make it hollow (speaking to Eric).

When Toby said to make it hollow, I became confused as to what he was talking about and what he was referring to and so I asked them a question.

Alison: What are you boys doing?

Toby: You know like, you know what we are going to do, we are going to make these, (touching the blue prism), you know what we are actually going to do, we are going to make one of these (touching the blue prism again with his finger) but hollow.

Toby: Okay? (turning to Eric).

At this, Eric stops dismantling and starts rebuilding the tower with the  $2\text{ cm}^3$  blocks. They work to create a hollow section in the middle of the tower of blocks.

Toby arranges the wooden cubes in their box



Toby touches the blue prism



Eric: Put it right there, (pointing to a space and directing Toby where to place the  $2\text{ cm}^3$  block).

Toby positions the wooden block.

Toby: It was on here except it was sticking on my hands. It was sticking. That was really cool (referring to a block that stuck to his finger as he was positioning it).

Toby knocks down blocks from the prism and now starts building next to the blue cube taking the  $2\text{ cm}^3$  pieces from the dismantled tower.

Toby: I am going to count how many pieces (as he places the blocks down next to the blue cube).

Eric: Twenty-seven times two (speaking quietly to himself).

Toby counts aloud as he places the blocks.

Toby: Look nine, ten, eleven twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen. That's 18! Nineteen, 20, 21, 22, 23, 24, 25, 26. So this, I counted, is 27 pieces (moving the blue cube away, and straightening up the  $2\text{ cm}^3$  blocks).

Eric: Twenty-seven times two (Eric repeats his earlier comment).

Toby: Twenty-seven times two. Okay so - (picking up the blue cube).

Eric: Two twenties (interrupting Toby).

Toby: -forty, a seven and a seven (standing up to count while holding the blue cube).

Toby positions the block



Toby starts counting the blocks



Eric: Fourteen (responding to Toby's addition statement).

Toby: 54.

Eric: 58 (giving his answer).

Eric: Oh yeah, 54 (correcting himself).

Eric: 54.

Eric hovers the blue cube on top of the 2 cm<sup>3</sup> blocks, places it down briefly then moves it to the side of the blocks then places it down. He picks up the blue prism and places it on top of the blue cube. Toby takes the blue prism off the blue cube and moves it next to the cube made of 2 cm<sup>3</sup> blocks.

Toby: Do you think we can make ginormous stairs? (stretching his fingers between the 2 cm<sup>3</sup> blocks and the blue prism. Toby lifts up the blue prism.

Eric: Hey! Do you think we - I got a good idea (taking the blue prism from Toby).

He places the blue prism next to the blue cube and starts building in front of the prism using the blocks.

Toby: You want to know some cool thing we can do with this? (picking up the blue cube and walking behind Eric to the box of 2 cm<sup>3</sup> blocks).

Toby sits down and starts removing blocks from their container. Eric leaves his work and starts removing blocks from the box as well.

Toby: Stop! No! (talking to Eric as he takes out blocks).

Toby creates a space for the blue cube then fits it into the box.

Toby stretches his fingers between the blue prism and the wooden blocks



Toby squeezes the blue cube into the box



Toby: Let's put everything back.

Working together they cover up the blue cube with 2 cm<sup>3</sup> blocks.

Toby: Uh oh! It's covering up the village!

They continue working together putting the blocks back in around the top and sides of the blue cube.

Alison: Did Ms. Linda give you a lesson in this or you are just measuring?

Eric: It's volume.

Toby: It's volume. You know what, Emilio gave it [the lesson] to Mario, Mario gave it to Nicholas,

Eric: and Nicholas gave it to me.

Alison: So, Nicholas gave it to you?

Toby: Mario.

Alison: So, who got the first lesson?

Eric: I was the one who got it first, this year.

Alison: This year?

Toby: Last year they had it but they never practiced it, so they forgot.

Alison: Okay, so you are practicing it now?

Toby and Eric cover up the blue cube



Toby puts in the last 2 cm<sup>3</sup> block on top of the blue cube and lifts up the side flap of the box hiding the blue cube from view.

Toby: Uh oh! The blue square disappeared!

Eric opens the side and Toby closes it again.

Toby: It disappeared. You can't see it.

From my position I cannot tell if they can see it from their perspective when the wooden side flap is down.

Alison: Can you see it from your side? (leaning in for a better view).

Eric: Yes, you can see it.

Toby: Now it's two-dimensional, now it's two-dimensional, not three-dimensional (referring to square of the blue cube visible from the side view).

Eric: Now it's three-dimensional (unpacking some 2 cm<sup>3</sup> blocks from around the blue cube and removing it from the box).

Alison: What does that mean if it is three-dimensional?

Eric: It means you can touch it, not just like a ... (patting one of his hands on top the other one).

Toby: If it is three-dimensional you can stick a pencil through it (holding up the blue cube and pressing his finger along the side) and if it's two-dimensional you can't.

Eric: Two-dimensional that means each side is totally flat (patting two sides of the cube).

Toby and Eric talking about dimensions



Eric explaining two dimensional



Alison: Two dimensional?

Eric: Yeah. But three-dimensional is like a person.

Eric puts the blue cube back in the box.

Toby: Put it back in (helping Eric put the blue cube back in the box).

Eric: No, let's cover it up (taking the blue cube out again).

Toby: Huh? What are you talking about?

Eric: I've got a good idea. Put the cube there (placing the blue cube next to the blue prism).

He starts building 2  $\text{cm}^3$  blocks in front of the blue prism.

Toby: Hey! We are going to make an L! (commenting on Eric's work).

Eric: Yeah, we are going to make an L.

Toby: I'll make the square part (helping Eric by placing 2  $\text{cm}^3$  blocks in front of the blue cube).

Eric: We are making an L out of cubes.

Alison: An L out of cubes?

Eric: Yeah.

Toby: We are making an L standing. We are making a three-dimensional L.

They work together with 2  $\text{cm}^3$  blocks building up against the L made from the blue cube and blue prism.

Toby: One, two (talking quietly to himself).

Toby watching Eric work



Toby and Eric building next to the "L"



Toby works on the foot of the L while Eric builds the higher part of it, adding blocks one at a time and row by row.

Toby removes the blue cube and Eric removes the blue prism leaving the L made of blocks.

Toby: We made an L, a three-dimensional L (pointing to the shape).

Toby pushes the blue prism against the high part of the L and the blocks fall over.

Toby pointing to the “L”



### ***Commentary on Toby's and Eric's work***

Through careful analysis and reflection, I came to a different appreciation of the scope of Toby and Eric's work with the blue volume material and  $2\text{ cm}^3$  blocks than I first had. Initially I had understood their work as practicing with the material by building solids with the blocks to match the dimensions of the blue prism and blue cube, and then counting the blocks they had used. However, their activity involved more than this as they generated their own ideas for working with the material. It also became clear this was not the first time they had used the volume material as they referred to previous activities using it, such as the village they had created.

In my refreshed understanding of their activities, I decided to divide their work into three aspects; replicating the blue solids with the  $2\text{ cm}^3$  blocks, and addition and multiplication of  $2\text{ cm}^3$  blocks (these two aspects I termed practice). The third aspect was their imaginative work (which I termed play). I have presented the analysis under these three aspects.

**Practice: Replicating the Blue Solids - Building in Three-Dimensions.** When I first observed Toby, he had two blue volume pieces of material in front of him as well the solids he had made with blocks. What I understood to be practising was to compose solids made with the  $2\text{ cm}^3$  blocks in three dimensions, matching the length, height, and width of the blue solids then to decompose the solid built with blocks. To do this the two boys used spatial skills of composing and decomposing, as well as spatial visualization to compose the solids in three dimensions.

Separately they both made a movement, which I had termed hovering and which I interpreted as their establishing a relationship between the solids built of the blocks and the blue solids. This movement involved aligning the surface area and perimeter of the blue solids over, then alongside, the solids built of  $2\text{ cm}^3$  blocks. This required both Toby and Eric to spatially visualize their movement then to orient the blue solid and finally to visually compare the two objects. I later had a conversation with Linda who confirmed that building equivalent solids in three dimensions with the  $2\text{ cm}^3$  cubes was part of the practice.

**Counting the  $2\text{ cm}^3$  Blocks.** Toby twice appeared to count the blocks of the solid he had built when he touched them individually and twice, he stopped. I was curious as to why he stopped and wondered if he was not engaged with his work. When Eric joined him, he composed a cube with the  $2\text{ cm}^3$  blocks for the second time, and this time he announced he would count the pieces. He started counting aloud at the ninth block. Toby used his fingers to touch the cubes as he counted which may have given him visual and tactile support as he progressed. When Toby stated he had 27 pieces Eric said, “27 times 2.” I assumed Eric had visually reasoned the volume of the blue prism was double the volume of the blue cube and he represented this mathematically by his statement, implying there were  $27 \times 2$  of the small cubes in the equivalent prism. Eric said

“two twenties” suggesting it seemed, that he could add by increments of twenty or that he could multiply twenty by two. Toby appeared to add the two twenties together as shown when he said “Forty.” After this Toby added the units together. By adding this way both Eric and Toby decomposed the number 27 into categories of tens and units. They arrived at different answers and discussed the matter until they agreed on 54. By this exercise of counting, addition, and multiplication, Toby and Eric used mathematical statements to express their reasoning about their work.

**Play.** In attempting to understand Toby’s and Eric’s play with the material I removed the photographic images in my analysis and focused on their speech. I looked for instances where their language suggested new ideas for working with the material. I concentrated on phrases such as “I have a good idea,” “You know what?” and “Do you want to see something funny?” which I took as suggestions for new ways of working with the material. The following are phrases they used when trying out different concepts with the material.

**“Make it Hollow!”** When Toby and Eric began decomposing the tower consisting of the  $2\text{ cm}^3$  blocks, Toby suggested they make it hollow. When I heard this, I was unclear as to what they meant and asked what they were doing. Toby responded that they were going to make “one of these but hollow.” When he said “one of these” he touched the blue prism with his index finger. They rebuilt the prism leaving the centre empty of blocks to attain their goal of making it hollow (Figure 25 shows the “hollow” space) they needed to visualize in three dimensions what the hollow tower would look like.

**Figure 25**

*“Make it hollow”*



They needed to compare the relationship of what they had, namely the blue prism, with what they wanted to attain, the hollow tower, and then translate that conceptualization physically to meet that goal.

**"The Cube Has Disappeared!"** As they were working Eric said that he had a good idea. Toby did not respond to Eric's comment but instead came up with his own idea. He suggested they could do something cool, and he removed 2 cm<sup>3</sup> blocks from the materials box creating a space within the box. Toby put the blue cube inside the box and covered up the top of it with 2 cm<sup>3</sup> blocks. Toby lifted the side flap of the box and exclaimed, "The cube has disappeared!" In this part of their activity, Toby and Eric reasoned about the dimensions of space the cube would take up and showed this when they created a space in the box into which the cube would fit.

**"Now It's Two-Dimensional!"** In this section of their work, Toby and Eric played with changing dimensions. Toby looked at the surface area of the blue cube when it was inserted in the box of 2 cm<sup>3</sup> blocks and stated, "Now it's two-dimensional!" When Eric removed the 2 cm<sup>3</sup> blocks from the top of the box to reveal the blue cube he stated, "Now it's three-dimensional!" As I was curious, I asked what three-dimensional meant. Toby removed the blue cube and demonstrated three-dimensions by stating that you could stick a pencil through something three-dimensional. Using his finger as a pencil he pretended he was sticking it through the cube (Figure 26). Eric stated that a person is three-dimensional, while two-dimensional is "totally

flat.” Eric and Toby both appeared to reason that a three-dimensional solid has attributes of length, width, and height.

## Figure 26

### *Explaining Dimensions*



**“We Are Making a Three-Dimensional L!”** Eric told Toby that he had a good idea. He placed the blue cube next to the blue prism and started building with the  $2\text{ cm}^3$  blocks. Toby appeared to understand Eric’s intentions when he shouted out, “Hey! We are going to make an L” and added, “I’ll make the square part.” He said, “We are making an L standing. We are making a three-dimensional L.” Toby visualized the L before it was built, and he mentally decomposed the L such that there was a “square part.” When Toby said that the L was standing, he showed he had mentally rotated the L from a lying position to a standing position.

### *Aspects of The Analysis*

It was after numerous viewings of the data, that I came to understand the complexity of Toby and Eric’s work with the blue volume material. I had expected them to do the practice work with the material as this is what I was familiar with and was looking for. They also had implied that they were practising. I was not prepared for their imaginative work but through the process of analysis it was revealed. Their spatial reasoning with their imaginative work involved

working with concepts of empty space, three-dimensional space, as well as using mental rotation, comparison, composing and decomposing as they explored.

### ***Spatial Reasoning Skills Used by Toby and Eric***

The two boys used spatial visualization and mental rotation (Bruce & Hawes, 2015; Casey et al., 2008) as they built with their blocks. Block building supports children's developing spatial reasoning skills (Casey et al., 2008; Dionne et al., 2012; Jirout & Newcombe, 2015; Verdine et al., 2014). They used mental rotation skills in conjunction with spatial visualization to build different shapes, such as the cube, and the standing "L" and when they make their shape "hollow". As they composed and decomposed their blocks they used flips and turns to position individual blocks where they wanted them (Cross et al., 2012) and fitted the individual three-dimensional blocks into "matching aperture" (Bruce & Hawes, 2015, p. 333). When Toby hovered the blue cube over another cube built of individual blocks, he established a one-to-one correspondence (Jirout & Newcombe, 2015). They demonstrated proportional reasoning when Toby counted the number of individual cubes (part) that made up the larger (whole) built cube (Möhring, Newcombe, Levine, & Frick, 2016). By their recomposing of the cube Toby built up a part-whole relationship until he had counted all the cubes and established again, the whole-whole relationship of the built cube to the blue cube. He used number sense by counting the individual blocks combining this with his visual spatial reasoning and visually confirmed the one-to-one correspondence of the two cubes (Jirout & Newcombe, 2015). Toby and Eric demonstrated their mental rotation skills as they explained to me what the difference was between two-dimensional and three-dimensional figures.

**Luca**

The next video regards the work of Luca (Table 5). Luca joined the class in March when he was six. His first language was Spanish, and he knew a few words in English. My interest in Luca's work stemmed from the fact that he was so new to the class having been there only a couple of weeks. Linda explained to me that he was not used to choosing his own work and he asked her many times what he could do. He walked around the room observing the other children and familiarizing himself with the classroom and its routines. I observed him frequently as I was interested to observe if he would choose his own work and what he would select.

Linda began showing Luca the routines of the classroom, one of which was to keep a journal where he would record the start and end times of his activities. Luca was unfamiliar with telling the time. One of the ways Linda introduced the concept of time to Luca was with the wooden clock material. This standing clock comprised a clock face with a minute and an hour hand. With this clock, as the hands are moved around the clock face, the hour hand passes under the minute hand. There are numbered disks for the hours (Figure 27) which shows the clock with one disk inserted). The disks are stored in a lidded box-like container attached to the back of the clock. I video-recorded Luca working with the wooden clock material which showed him deeply engaged in his work.

**Figure 27**

*Luca with Wooden Clock*



**Table 5**  
*Time as Movement*

| Text                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Images                          |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| <p>April 16<br/>02:10:00 – 02:15:00</p> <p>Luca sits in front of a large wooden clock which has two moveable hands attached at the centre. There are 12 openings on the clockface where disks numbered for the hours can be inserted. Luca has placed 8 disks for the hours 1:00 to 8:00. When I start my observation, Luca has the clock hands at 2:25.</p> <p>Using his right hand, Luca moves the hour hand clockwise from the 2:00 but he stops when the disk in the 5:00 slot falls out. He replaces the disk with his right hand. He brings his left hand up from the table and places it on the minute hand. He puts his right hand back on the hour hand and brings it clockwise to the minute hand which is at the 5:00.</p> <p>He manoeuvres the hour hand under the minute hand which he is holding with his left. He transfers it to his left hand. He now uses his left hand to move the hour hand clockwise.</p> | <p>The clock hands are 2:25</p> |

While he is doing this with his left hand, he uses his right hand to move the minute hand from 5:00 to 6:00.

He continues to move the hour hand slowly clockwise with his left hand, pausing briefly at the 9:00 when he picks up the 9 disk in his right hand. He places the disk on the end of the minute hand. He continues moving the hour hand clockwise while with his right hand he moves the 9 disk along the minute hand then along the hour hand.

The disk arrives at the end of the hour hand as it reaches the 12:00 position.

With the disk now at the end of the hour hand he uses both hands to rotate the hour hand and the disk clockwise. When he reaches the minute hand at 6:00 he manoeuvres the hour hand under it with his left hand while he holds the disk in his right hand. He repositions the disk back on the hour hand once it has been manoeuvred under the minute hand.

He continues to move both the disk and hour hand to the 9:00 slot where his right hand inserts the disk. Taking the hour hand in his right hand, he moves it onward around the clock until he again

Luca moves the minute hand from 5 to 6



Luca stops the disk at 12:00 position



reaches the minute hand at the 6:00 position. He transfers the hour hand to his left hand as he manoeuvres it under the minute hand.

This time he uses his left hand to rotate the hour hand clockwise until he reaches 4:00 where he transfers it to his right hand. His right hand now moves the hour hand around and under the minute hand at 6:00. He passes it to his left hand and rotates the hour hand clockwise again, stopping when he reaches 5:00.

He manoeuvres the hour hand under the minute hand using both hands. With his left hand he starts moving the hour hand clockwise. As the hour hand passes 10:00 he picks up the 10 disk with his right hand.

He places it on the end of the minute hand and moves it up the minute hand and along the hour hand. He continues moving the hour hand clockwise. The disk reaches the end of the hour hand as the hour hand reaches 12:00.

He continues clockwise past 12:00 keeping the disk on the hour hand and using both hands. On reaching the minute hand at 6:00, he slides the hour hand under it while holding the 10 disk in his right hand.

Luca stops the hour hand when reaching 5



The disk and the hour hand are at the 12:00



Luca repositions the 10 disk on the hour hand once it has passed under the minute hand and using both hands brings the 10 disk to 10:00 where he inserts it.

After inserting the 10 disk and with his right hand he moves the hour hand clockwise while his left hand picks up the 11 disk. As he reaches the minute hand at 6:00 he uses only his right hand to manoeuvre the hour hand under it.

Then with his left hand he places the 11 disk on the minute hand and repeating earlier actions, moves the disk up the minute hand, and along the hour hand. As he reaches the 11 slot, he inserts the 11 disk using his left hand.

After inserting the 11 disk and with his right hand, Luca moves the hour hand clockwise. He reaches the minute hand at 6:00 and manoeuvres the hour hand under it while at the same time picking up the 12 disk with his left hand but immediately transfers it to his right hand.

As he rotates the hour hand, he places the 12 disk on it with his right hand. With two hands he moves both the disk and the hour hand from 6:00 to the 12:00. He inserts the 12 disk in the 12:00 slot with the number oriented down.

Luca inserts the 10 disk in slot



Luca inserts the 11 disk with his left hand



He rotates the 12 disk to orient the number. As he does this, he continues to move the hour hand with his left hand stopping after 1:00.

Once the 12 disk is oriented, Luca stops moving both his hands and takes them off the clock.

He picks up the lid for the container which holds the disks and is attached to the back of the clock. He places the lid on the container.

Luca rotates the 12 disk to orient it



Luca puts the lid on the container behind the clock



### ***Commentary on Luca working with the clock***

Luca chose to work independently with the clock on several occasions after Linda presented it to him first in March. The above data is from one of the times when Luca selected the clock for his work. Data for the analysis of Luca's spatial reasoning were drawn solely from my observations of the video recording of his movements as he did not speak. During this sequence of work Luca was deeply engaged in his activity, keeping his attention on the clock throughout the period. He worked systematically in an orderly progression, adding each disk from 9:00 to 12:00 in numeric sequence. Luca showed his focused connection to his work through his movements. Both of his hands were poised and ready to undertake the movements he required as he moved both clock hands around the face of the clock and selected and inserted the

disks where he deemed appropriate. It was with these constant movements of his hands that I searched for patterns and on which I based the analysis.

Since Luca had already set several disks in their slots by the time I began my observation, I assumed he was adding the numbered disks chronologically and this was the case. Luca had established a pattern of moving the hour hand around the clock face a few times. For instance, I had noticed his clock was set with the hands at 2:35 and that the next numbered disk needed was 9. When he resumed his work, Luca began by moving the hour hand from 2:00 all the way around the clock face, passing 2:00 again until he reached 9:00 where he inserted the 9 disk. When he started working on the next disk which would be 10:00, Luca moved the hour hand around the clock face four times before inserting it in 10:00 slot. For 11:00 and 12:00 disks he moved the hour hand around the clock face once each. These rotations around the clock face seemed to indicate that Luca reasoned about time as something that moved and moved in a circular fashion. It appeared that he reasoned the hour hand needed to travel a distance before he inserted a disk. That distance was variable as, for example, he circumscribed the clock face 4 times before he inserted the 10 disk, but only once each for the 11 and 12 disks. Luca consistently moved the hour hand in a clockwise direction and not in a counter clockwise direction suggesting his consistent intention to execute that movement. Luca did not move the minute hand often and it seemed that he required it at 6:00 which is, of course, the half-hour position. It appeared that spatially the minute hand served a function for Luca as on occasion he repositioned at the 6:00 which is where he kept it. From observations of his movements, it was not possible to ascertain whether Luca considered times as 9:30, 10:30, etc. The minute hand appeared to act as a divider between the left hand and right-hand aspects of the clock. It stopped Luca's action of turning the hour hand and required a different movement as he navigated the

hour hand under the minute hand to continue. In all instances of his moving his hands around the clock face except one, Luca transferred the hour hand from his right hand to his left hand after manoeuvring it under the minute hand. He continued the travel of the hour hand with his left hand. In one instance he used his right hand to manoeuvre the hour hand under the minute hand and continued using this hand until he inserted the 11 disk. In addition to Luca's circular motions around the clock face using his hands, his other consistent motion was his movements between 6:00 and 12:00. With each rotation of the hour hand, and as it reached the 6:00 position, Luca transferred it to his left hand, then picked up the next required disk in his right hand. He placed the disk on the minute hand still at 6:00 while his left hand took over moving the hour hand. He would then travel the disk along the minute hand and along the hour hand at the same time as the hour hand was being moved. The disk reached the end of the hour hand as the hour hand reached 12:00. Luca then carried on clockwise with the disk placed on the end of the hour hand before he put it in its slot. This movement of bringing the disk up first the minute hand from 6:00 then the hour hand as the hand reached 12:00 before inserting it in its slot, seemed an integral part of Luca's spatial reasoning with regards to telling the time as he performed the movement each time he prepared to insert a new disk.

### ***Spatial Reasoning Skills Used by Luca***

Luca used spatial visualization skills of orientation, and composition as he built up the hours of the clock. He used his body symmetrically as he managed his hands on the clock. He used fine spatial visualization skills as well as mental rotation skills to insert the disks in each slot (Bruce & Hawes, 2013; Leavy et al., 2018). He demonstrated his number sense when he inserted each disk in numerical order, thereby combining his reasoning with movement of his

body. Luca used proportional reasoning skills as he reasoned about each part belonged to the whole (Jirout & Newcombe, 2015).

### ***Aspects of the Analysis***

Luca's work with the clock revealed how extensively he used movement in his spatial reasoning. Luca's hands played a central role in his activity with the clock. Both hands were active in the work, each hand playing an equal role, as he transferred the hour hand from one hand to the other and rotated the hour hand while his other hand moved the disk in various locations. Luca's work with the clock powerfully suggested his understanding of time as a process. His hands worked smoothly in unison as he undertook the movements he considered necessary to account for the process of time. These were the rotation of the hour hand around the clock face, the movement of each disk along the two clock hands, the arrival of the disk at the end of the hour hand as it reached 12:00, and the maintenance the minute hand at 6:00. He used his orientation skills to move the hour hand clockwise around the clock, he oriented the hour hand at each hour he wished to change, he re-oriented the minute hand at 6:00 and re-oriented the 12 disk to its correct position in its slot.

### **Shan**

The next video I selected for presentation is of Shan's work with material called the Constructive Triangles (Table 6). Shan came and sat next to me on my second day in the classroom and started working with the triangle material. Shan was one of the older children in the classroom at age 10. He was a quiet, reserved boy and when he sat next to me, he did not say anything but it was apparent that he was showing his work to me. I video-recorded it and later through the analytic process became increasingly aware of what a piece of work rich in spatial reasoning he had undertaken, all the time remaining almost silent.

**Table 6**

*Shan Explores Relationships of Equivalence and Congruence with the Constructive Triangles Material.*

| Text                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Image                                                                                                               |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------|
| <p>April 12<br/>09:43:00 – 09:57:35</p> <p>Shan brings two wooden boxes to the table, sets them down and removes the lids. He takes out triangles of different shapes, colours and sizes. He arranges the triangles into four groups:</p> <ul style="list-style-type: none"> <li>isosceles triangles - grey, red, and yellow</li> <li>small equilateral triangles - red</li> <li>right-angled triangles – green</li> <li>a large equilateral triangle – grey</li> </ul> <p>Shan: Triangles (making a large circular motion above them with his hand).</p> <p>With his left hand he reaches across and picks up a green right-angled triangle that is positioned face down.</p> <p>Shan: So, right-angled triangle (turning it face up showing the green face and replacing it face down).</p> <p>He picks up a small red equilateral triangle.</p> <p>Shan: Equilateral triangle (putting it back)</p> <p>He picks up a grey isosceles triangle and puts it back without speaking. Shan continues to unpack the triangles placing a large yellow equilateral triangle on a large grey equilateral triangle. He removes more isosceles triangles from the box and puts them with the other isosceles triangles.</p> | <p>Shan sorts the triangles</p>  |

Alison: Why are you making them in piles?

Shan: I have to put the same ones together (moving his left hand in a circular motion over the triangles).

Alison: I see, you put the same ones together.

After unpacking he pushes the two empty boxes aside.

He selects a yellow isosceles triangle and puts it back, then picks up a large yellow equilateral triangle and places it in front of him.

He reaches for the small red equilateral triangles and places all four of them on the large yellow equilateral triangle, covering the area.

He slides the red triangles off and arranges them into a parallelogram.

Shan moves his hand over the triangles



Shan places four red equilateral triangles



Shan: So, *this (touching the red parallelogram)*  
is equivalent to that *(touching the large yellow*  
*equilateral triangle)*

because, see *(rearranging red triangles back on the*  
*yellow triangle)*

it takes up the same space *(moving his finger in a*  
*triangular movement above the triangle.*

Shan touches the parallelogram  
made with the triangles then the  
yellow triangles



Shan rearranges the red triangles  
on the yellow triangle



He slides the red triangles off the yellow triangle and makes a red parallelogram again.

Alison: What does it make? What's that shape?

Shan: Parallelogram (resting his hands on the parallelogram).

Shan breaks up the parallelogram and stacks the red triangles. He touches a few triangles then rests his hands on the table. He touches a red triangle then picks up a yellow isosceles triangle and places it in front of him. He stops. He picks up a green right-angled triangle.

He aligns one of the sides of the yellow isosceles triangle with a side of the green right-angled triangle.

He puts the yellow isosceles triangle away and keeps the green right-angled triangle.

He rests his hands on the table and looks at the triangles around him. He selects a second green right-angled triangle and aligns their longest sides creating a large equilateral triangle.

Then with a hand on each triangle he rotates them  $90^\circ$ , pauses a few seconds then continues to rotate them and forms a kite.

Alison: What's that shape now?

Shan aligns the sides of the yellow and green triangles



Shan: A kite (turning the apex of the kite towards him).

He brings the yellow equilateral triangle towards him and positions it next to the green kite.

Shan: These are equivalent (touching the two shapes).

He turns the green kite  $180^\circ$  then breaks it up and places each green right-angled triangle on the large yellow equilateral triangle.

Shan: It's the same (resting both hands on the green triangles).

Shan forms a kite



Shan touches the two shapes



Shan rests both hands on the green triangles



Shan slides the two green triangles off the yellow equilateral triangle and uses them to make an irregular polygon.

Shan: All of this (touching the green irregular polygon) and this (touching the yellow equilateral triangle) could also be equivalent (bringing both hands back to the green polygon).

Alison: What shape is that? (referring to the green irregular polygon. He does not answer).

As I speak, using both hands he rotates and aligns the shortest side of each green triangle and creates a large isosceles triangle.

Alison: What have we got here?

Shan does not answer for 30 seconds but rotates the newly-formed isosceles triangle  $90^\circ$  then flips it over, face down, while maintaining its shape.

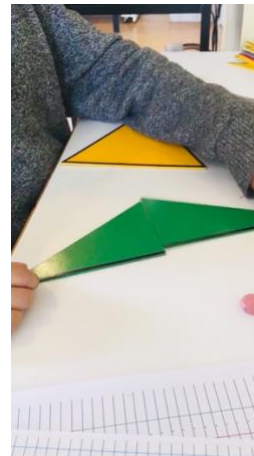
Shan: Isosceles (flipping the green triangles back over and breaking up the isosceles triangle).

He makes a further green polygon.

Shan makes an irregular polygon



Shan creates a large isosceles triangle



Shan: All of this (touching the green polygon)  
and this, (touching the large yellow equilateral  
triangle)  
could be equivalent.

He puts away the green right-angled triangles.

Shan selects the large grey equilateral triangle and puts it  
on his right side.

He takes three yellow isosceles triangles and makes an  
irregular polygon.

Shan: These are equivalent (waving his hand above the  
yellow isosceles and grey equilateral triangles).

He makes another irregular polygon with the three yellow  
isosceles triangles.

Shan: These (finishing his yellow irregular polygon).

Shan touches the green polygon



Shan makes a shape with the  
yellow isosceles triangles



He makes three more irregular polygons with the three yellow isosceles triangles pausing after making each shape.

He works for 36 seconds. He puts away the grey equilateral and yellow isosceles triangles.

With his left hand he picks up three small red equilateral triangles and makes a trapezoid, its longest side facing away from him. He turns the trapezoid  $90^\circ$ .

Using his left hand again, he picks up a green right-angled triangle. While holding the green triangle, he uses his right hand to break up the trapezoid, leaving a rhombus.

Shan: These are equivalent (placing the green triangle next to the red rhombus).

Alison: Oh yes? (questioning him)

He puts two red triangles on top of the green triangle.

Shan makes another irregular polygon



Shan places the green triangle next to the red rhombus



Shan puts red triangles on the green triangle



Shan: They don't fit on each other but they are still equivalent (removing the red triangles).

He forms the red triangles into a rhombus again.

Alison: Why do you think they are equivalent?

Shan: Because this is half of that (placing the green triangle on top of the grey triangle).

With his right hand he slides off the green triangle

and this is a quarter of that (at the same time placing a red triangle on the grey triangle with his left hand).

so, two quarters (placing two more red triangles on the grey triangle)

equal one half (lifting up the green triangle).

He puts the green and red triangles back in their groups.

Shan takes three yellow isosceles triangles and makes a large equilateral triangle. Moving his left hand over the

Shan places the green triangle on the grey



He places the red triangle on the grey triangle



pile of red equilateral triangles, he picks up two and brings them forward. He pauses and puts them back. He picks up two grey isosceles triangles and makes a rhombus.

He places a small red equilateral triangle on one half of the grey rhombus.

He places a second red triangle on the remaining grey portion of the rhombus. He rests his hands on the rhombus then slides the two red equilateral triangles off and puts the two grey isosceles triangles back in their group.

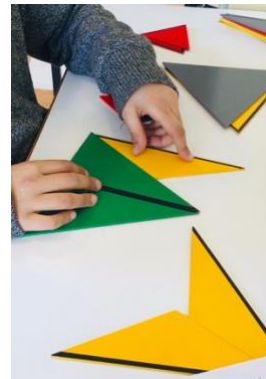
He rests his hands on the small red equilateral triangles and on the yellow isosceles triangles. He pauses. Then he picks up two green right-angled triangles and forms them into a large equilateral triangle. He pauses for 30 seconds.

He removes a yellow isosceles triangle from the large equilateral triangle and adds it to the two green triangles.

Shan places a small red triangle on the grey rhombus



Shan adds the yellow isosceles triangle to the green triangles



He picks up another yellow isosceles triangle and adds it to the shape.

He adds a third yellow isosceles triangle to the shape. He pauses for 20 seconds.

He picks up one red and two grey isosceles triangles and uses them.

Alison: Are you just experimenting or are you looking for something?

Shan: I'm looking for something (making an irregular polygon with one red and two grey isosceles triangles).

He leans over to touch the yellow and green irregular polygon.

Shan: These are equivalent (separating the yellow triangles from the green triangles then putting them back together again).

Shan adds another yellow isosceles triangle



Shan uses the red and grey triangles



Shan separates the yellow triangles from the green triangles



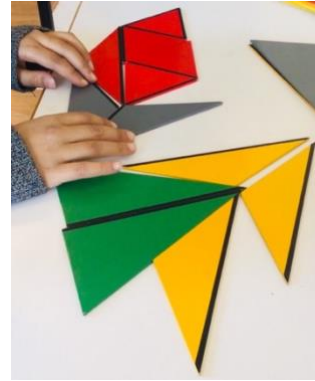
He adds a red parallelogram made with four small red equilateral triangles to his shape of one red and two grey isosceles triangles.

Shan: These are equivalent (placing his right hand on the green equilateral triangle while pointing to the red parallelogram with his left hand).

He slides the grey and red isosceles triangles away from the red parallelogram

because I just took these (holding a yellow isosceles over a grey isosceles triangle)  
 with this (putting both hands on the three yellow isosceles triangles)  
 and this (putting both hands on the grey and red isosceles triangles)  
 and they are the same (making a circular motion with his right hand over all the isosceles triangles)  
 and four of these (touching the four red equilateral triangles making up the parallelogram with his left hand)  
 equal two of these (placing both hands on the two green right-angled triangles).

Shan adds red triangles to his shape



Shan points to the red parallelogram



Shan holds the yellow triangle above the grey triangle



Alison: Interesting.

Shan pauses. He turns the four red triangles into a chevron-shape. He moves the two green triangles to the left of the red chevron-shape and puts them back in the shape of a large equilateral triangle. He pushes the isosceles triangles away.

Shan: These two are equivalent (touching the table below the large green triangle and the red chevron-shape)

Alison: Okay, why are they equivalent?

Shan: because four (placing two red triangles on the green triangles).

equal two of these (placing two more red triangles on the green triangles).

He slides the red equilateral triangles off while maintaining their shape as a large equilateral triangle.

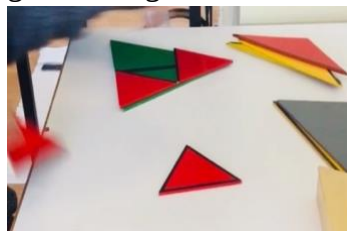
Using two hands he moves two red triangles from the base of the triangle to the apex changing the orientation of the triangle. He keeps his hands on the triangles.

Shan: They can be any shape (changing the red triangles around to form a parallelogram then facing his palm upwards).

Shan states the shapes are equivalent



Shan places red triangles on the green triangles



He moves the triangles again changing the red parallelogram to further irregular polygon.

Alison: Right, they are still equivalent.

Shan makes two more irregular polygons with the four red equilateral triangles then stacks them.

Alison: All sorts of different shapes.

He tidies up the other triangles. He pauses then makes an equilateral triangle with three yellow isosceles triangles. He makes a parallelogram with red equilateral triangles.

Shan: These are equivalent (placing his hand over the yellow equilateral triangle)

because the previous one I did (moving his hand in a backward motion)

four of these (placing his right hand on the four red equilateral triangles)

equal two of these (placing his left hand on the two green right-angled triangles)

and this (bringing the two green triangles over to his right and making an equilateral triangle)

and this (touching the yellow equilateral triangle) are the same (running his right hand above the yellow and green equilateral triangles)

so, this (his hand going from the yellow equilateral triangle)

and this (to the red parallelogram)

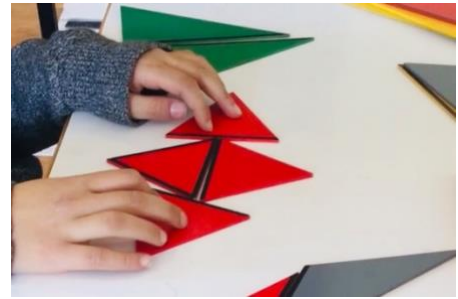
are the same (to the green equilateral triangle)

Alison: Nice. Very nice.

Shan: So, all three of these (running his hand a couple of times in a circular motion above the three shapes)

are equivalent (resting his hands on the table).

Shan changes the red triangles around



Shan moves his hand over the material



Alison: Are equivalent.

To the left of the three shapes, Shan picks up one red and two yellow isosceles triangles and makes an irregular polygon.

Shan: All this are equivalent because this (pointing to the yellow and red polygon).

is just three of those (pointing to the large yellow equilateral triangle made of three isosceles triangles).

Shan starts to make a fifth shape using one red and two grey isosceles triangles.

Shan: This is all equivalent (placing the red isosceles triangle next to the two grey isosceles triangles).

Alison: And you proved that. You proved it earlier, didn't you?

Shan: Yes. And all these three are equivalent (moving his hand above the red parallelogram, the yellow equilateral triangle and the green equilateral triangle)

Shan points to the yellow and red polygon



Shan moves his hand over the shapes



and everything is equivalent to this (separating the large yellow equilateral triangle into its component three isosceles triangles while touching the red and yellow isosceles triangles).

because it's three of the same pieces (running his hand above the other two irregular polygons made with isosceles triangles

Shan: So now we know that all of them are equivalent (running his hands over all five shapes).

Alison: That's great.

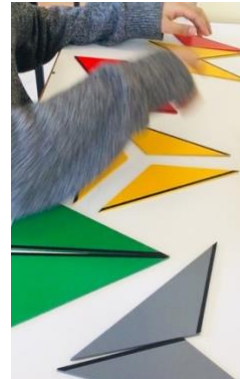
Shan starts gathering up the pieces.

Alison: So, do they go back in a certain way in the box?

Shan looks for a shape and turns it over.

Shan: The ones with the star on the back are for this box (putting his hand on the box and putting triangles back starting with the largest triangle).

Shan separates the large yellow equilateral triangle



Shan runs his hands over all five shapes



Shan finishes packing away the triangles.

Shan replaces the triangles in their box



### ***Commentary on Shan's Work***

In this fifteen-minute episode, Shan works with triangles and considers multiple components of equivalence. The concept of equivalence involves a relationship between two or more shapes that may differ with regards to the lengths or number of their sides, or to their angles, they however have the same area in common. The concept of congruence, which Shan refers to, involves two or more shapes that are the same size and the same shape. Shan's spatial exploration of equivalence appeared to focus on sensorial stimulation provided by visual and tactile data. He does not discuss lengths or numbers of sides, nor the measure of angles, but as his examples of equivalence became more complex, he introduced proportional reasoning to explain his mathematical spatial reasoning. Shan worked steadily for fifteen minutes, pausing at times when he appeared to be deciding what triangles to use. I have divided the episode into segments to underline the coherence of the data. There are seven segments each holding one or more instances of Shan establishing equivalence using the same triangles. For example, in the first segment he uses four small red equilateral triangles and a large yellow equilateral triangle to demonstrate equivalence.

As Shan begins, he unpacks the boxes of triangles and reveals spatial reasoning by sorting them according to size and shape. He sorts into four groups: two large equilateral, four small equilateral, ten isosceles, and two right-angled scalene triangles. After Shan has sorted the triangles, he begins his activity. Throughout the activity it was difficult to gauge whether Shan was following a sequence as by observing his hand movements his choice of triangles seemed exploratory. He would touch some without picking them up, chose some but put them back, and bring forward one or two but replace them after what looked like a brief consideration and exploration. I discuss the seven segments.

**Segment 1: Equivalence and congruence (red parallelogram and yellow equilateral triangle)** (See Figure 28 showing Shan comparing the red triangles with the yellow triangles in the first period of his work with the triangles).

**Figure 28**

*Work with Red and Yellow Triangles*



In the first segment, Shan appears to be investigating equivalence between a red parallelogram and a yellow equilateral triangle. When he takes four small red equilateral triangles and composes them on top of the large yellow equilateral triangle, it looks like he is

demonstrating an understanding of the concept of congruence. After establishing congruence, he transforms the red equilateral triangle into a parallelogram, then recomposes back on top of the yellow equilateral triangle. He does this again, decomposing and recomposing the four red triangles. That Shan alternates between making an equilateral triangle and a parallelogram with the small red triangles twice, suggests to me that he could be trying to establish a visual proof of equivalent area. That he understands the areas as equivalent is suggested by his speech when he states they are equivalent, verbally justifying his hitherto unspoken claim of equivalency. That he understands the areas of shapes as equivalent might also be shown by his movements when he touches each shape in turn implying a relationship between them. Together his movements and speech might offer another claim of equivalence when he composes the shapes and says that they take up the same space while his movements echo his words when he draws his finger in a triangular shape over them.

**Segment 2: Comparing lengths, and equivalence (green right-angled triangles, yellow isosceles, yellow equilateral triangle)** (See Figure 29 which is a snapshot of this segment of Shan's work with the yellow and green triangles).

### **Figure 29**

*Shan's Work with Green and Yellow Triangles*



At the beginning of this segment Shan appears to compare the sides of two triangles. He aligns a yellow isosceles triangle with a green right-angled triangle. (The sides are of differing lengths and he puts the yellow isosceles triangle away. Taking another green right-angled triangle, it looks like he is exploring making irregular polygons. Then he composes the two triangles as a large equilateral triangle which he rotates to form a kite. The apex faces away from him and he moves the kite to orient the apex towards him which I understand to imply that he is more comfortable with the orientation of the kite as typically presented. This apparent preference for a particular orientation is also seen in his positioning of the large equilateral triangles which he sets-up with the apex facing away from him. He uses his speech and movements to communicate that the two shapes are equivalent. As with the previous segment, he decomposes his shape to transform it to an equilateral triangle which he places on top of the yellow equilateral triangle. He says, "It's the same." With this movement Shan shows an understanding of conservation of area. Shan's speech is less here than in the previous segment. He does not say why it is the same, and it appears to me that he feels he has already established the reason. Neither does he decompose and recompose the shape twice, unlike in the first segment when he

made the parallelogram twice. I take this to signify that he did not need to display such an explicit understanding of equivalence as seen in the first segment. His movements are as they had been in the previous segment where he touched each shape in turn seeming to confirm a relationship, and he rested his hands on the two equilateral triangles when they were on top of each other. This movement suggested a finality or a completion to the assertion that the shapes were equivalent.

Previously, Shan proved equivalence using regular polygons, a parallelogram, and a kite. While he made an irregular polygon before he made the kite, he chose not to use it in his exploration of equivalence. However, after establishing congruence and equivalence between the kite and equilateral triangle Shan appears confident enough to take a chance to explore equivalence with irregular polygons. Despite this, when he compares an irregular polygon made with two green right-angled triangles his language seems more tentative. He says, “All of this, and this, could also be equivalent.” This is suggested by the word *could*. His movements do not look tentative, however, and he places his whole palm down on the irregular polygon thereby seeming to add conviction to his assertion. He continues investigating with the two right-angled triangles eventually making one large isosceles triangle. When I asked him what it was called, he pondered for 30 seconds before answering “isosceles.” As he speaks, he breaks the isosceles triangle down and makes two more irregular polygons. He uses the same language as he did earlier, saying, “All of this, and this, could be equivalent.”

**Segment 3: Equivalence (three yellow isosceles triangles and a large grey equilateral triangle)** (see Figure 30 which shows Shan’s work from Segment 3 where he uses the yellow and grey triangles).

**Figure 30**

*Shan's Work with Grey and Yellow Triangles*



In this brief segment Shan uses three yellow isosceles triangles and makes three irregular polygons with them. After making the first irregular polygon he says, “These are equivalent” and waves his hand over both the polygon and the grey equilateral triangle. When he makes the next irregular polygon with the three yellow isosceles triangles he merely says, “These” and rests his hands on the table. In this segment he does not attempt to show that the shapes are equivalent unlike his demonstrations in previous instances. In the first segment he demonstrated equivalence two times for the red parallelogram and equilateral triangle, in the second segment he demonstrated equivalent area once with the kite and the equilateral triangle. As previously suggested, this might illustrate that Shan does not think it necessary after having shown thoroughly how he arrives at equivalent areas in the first segment.

**Segment 4: Justifying equivalence using fractions and a mediator (red rhombus and green right-angled triangle)** (See Figure 31 which shows some of Shan's work with the red and green triangles).

**Figure 31**

*Work with Green and Red Triangles*



This segment marks a change where instead of using either the yellow or grey large equilateral triangles as the control for claiming equivalent areas with other shapes, Shan uses a green right-angled scalene triangle whose area is half that of the large equilateral triangles. He initially explores with three small red equilateral triangles, making a trapezoid whose longest side faces away from him. He turns the trapezoid  $90^\circ$ . As he is focusing on that, he appears to be deliberating his next move. He picks up a green right-angled triangle in his left hand, while still looking at the red trapezoid. He decides to move one of the small red equilateral triangles aside, which results in a rhombus. As he does this, he lays the green right-angled triangle down next to the rhombus. It seems that he had a shape in mind for comparison but was not sure how many red equilateral triangles he needed to establish congruence with the green right-angled triangle. As he places the right-angled triangle down, he says, “These are equivalent.” I answer, “Oh yeah?” in a questioning way because, unlike the previous examples of equivalence, this one does not lend itself to visually demonstrating congruency. Shan appears to acknowledge the challenge when he places the red equilateral triangles on the green right-angled triangle and seems to reason that while he cannot visually prove congruence, he understands spatially that they are equivalent. I encourage him to elaborate by asking him why he thinks they are equivalent.

At this point, Shan's reasoning seems to become more complex. It appears that Shan understands he cannot show equivalence by visual means and must find another way. For the first time, he calls on other mathematical spatial reasoning resources and introduces fractions and proportional reasoning to his justification. He brings back the large grey equilateral triangle to use as a mediator between the red rhombus and the green right-angled triangle. He establishes that the green right-angled triangle is half the area of the large grey equilateral triangle. Next, he demonstrates that each small red equilateral triangle is a quarter of the large grey equilateral triangle and proves the equivalence mathematically by stating that two quarters (two red equilateral triangles) equal one half (one green right-angled triangle).

**Segment 5: Establishing equivalence between parts of a whole first, and using fractions (red/grey polygon and yellow/green polygon).** (See Figure 32 which shows a photograph of the increasing complexity of Shan's work as he works with triangles of four different colours).

**Figure 32**

*Work With Triangles of Four Colours*



This period begins with Shan's activities giving the impression that he is exploring with the triangles as he composes a large equilateral triangle using three isosceles triangles and makes a rhombus with two more isosceles triangles. He puts two red small equilateral triangles on the rhombus but puts most of the material away without speaking. In this segment Shan increases the complexity of his work on equivalence using six isosceles, four equilateral, and two right-angled scalene triangles. He starts with two separate equilateral triangles, one made of two green right-angled triangles and one made of three yellow isosceles triangles. He experiments by taking first one, then two, then three of the yellow isosceles triangles from the large equilateral triangle and adding them to the green equilateral triangle. He ends up with an irregular polygon comprising five triangles. He begins working on a second irregular polygon, starting with three isosceles triangles (one red and two grey) then adding four small red equilateral triangles formed as a parallelogram. He now has two large irregular polygons; one formed with three isosceles and two right-angled triangles, the other formed with three isosceles and four small equilateral triangles. He places his right palm on the two green triangles formed as an equilateral triangle in one polygon and says "These are equivalent" at the same time touching the four small red equilateral triangles formed as a parallelogram in the other polygon. In other words, he is signifying equivalence between *parts* of each polygon. He looks like he signifies this again with the remaining parts of each irregular polygon when places both hands on the three yellow isosceles triangles, then on the three grey and red isosceles triangles. In this instance, the parts are triangles, and he states that they are the same. He makes a circular motion over the shapes which seems to support his claim. He refers to the parts of the irregular polygons again this time talking about the red parallelogram and the green equilateral triangle, and he chooses to discuss equivalence once more this time using the mathematical concept of proportional reasoning when

he states that four of these equal two of these. In this segment, Shan has approached equivalence using two different mathematical ideas, namely that of equivalence and that of fractions. He does not, however, speak about equivalence between the two irregular polygons as a whole.

**Segment 6: Equivalence using fractions and focusing on shape (red parallelogram and green equilateral triangle).** (In Figure 33 Shan focuses on red and green triangles).

**Figure 33**

*Focus on Green and Red Triangles*



In this segment, Shan pushes all the isosceles triangles to one side to focus on the red parallelogram which he has transformed into a chevron and the green equilateral triangle made with the two green right-angle triangles. He says, “These two are equivalent” and I ask him why. He uses mathematical terminology and says, “four of these equal two of these.” Shan brings his attention solely on the four red equilateral triangles formed as a large equilateral triangle and moves them around making a parallelogram, an equilateral triangle with its apex facing towards him, and another two irregular polygons. He states, “They can be any shape” with the underlying

assumption that they will still be equivalent to the green equilateral triangle made with the two right-angled triangles.

**Segment 7: Equivalence across five shapes (nine isosceles, two green right-angled, and four small red equilateral triangles).** (Figure 34 shows Shan's work with five different shapes using four different colours).

**Figure 34**

*Shan Works With Four Colours*



In this final segment we see the complexity of his work with equivalence. He establishes equivalence across five shapes; two equilateral triangles one made from three yellow isosceles and one made from two green right-angled triangles, one red parallelogram made from four small equilateral triangles, and two irregular polygons each made with three isosceles triangles. Until this point in his activity, Shan seems to propose using either the yellow and the grey large equilateral triangles as a control or mediator. In this last development he does not use either of them, but rather has two large equilateral triangles each composed of smaller triangles. I understand his omission of the large grey and yellow equilateral triangles as a sign that he is comfortable in what he is doing and does not need their presence to support what he will demonstrate. Unlike in his earlier demonstrations of equivalence, Shan does not decompose or

recompose any figures, nor does he attempt to establish congruence. The demonstration is restrained and seems to imply that Shan is only indicating what is necessary.

To begin, Shan claims equivalence between three shapes; the large yellow equilateral triangle composed of three isosceles triangles, the red parallelogram composed of four small red equilateral triangles, and the large green equilateral triangle composed of two right-angled triangles. He builds up his claim of equivalence between all three shapes by initially focusing on two shapes at a time, then claiming equivalence across all three. He first supports his claim by referring to earlier work “the previous one I did” where he established that “four of these” referring to the small red equilateral triangles in the parallelogram, “equal two of these” referring to the green right-angled triangles forming the large equilateral triangle. He moves on to the green equilateral triangle and the yellow equilateral triangle, saying “and this” as he takes the green right-angled triangles over next to the large yellow equilateral triangle “and this are the same” running his hand over the two. After maintaining that the green equilateral triangle is equivalent to the red parallelogram and equivalent to the yellow equilateral triangle, he claims that the red parallelogram and yellow equilateral triangle must therefore be equivalent to each other. He enforces this claim with his hand movements and his speech when he says, “So, this” and touches the yellow triangle, then goes directly to the red parallelogram and says, “and this.” He connects the red and yellow shapes with the green when he goes on to say, “are the same”, at the same time touching the green shape.

Using the words, “So, all three of these are equivalent”, Shan appears to suggest that having established equivalence between three shapes he is ready to move on to the next two shapes. He builds an irregular polygon with three isosceles triangles, and in saying “All this is equivalent because this is just three of those”, he implies this is a straightforward or obvious

matter of equivalence. This is shown in his use of the word “just.” He makes a fifth shape, again an irregular polygon made with three isosceles triangles. He states, “This is all equivalent.” He refers again to his earlier claim that the three original shapes were equivalent, then breaks up the large yellow equilateral triangle into its component three isosceles triangles and says that “everything is equivalent to this because its three of the same pieces.” He again reiterates the word “same” in this case referring to the isosceles triangles making up three of the five shapes. His final comments for the work are “So, now we know that all of them are equivalent” running his hand over all five shapes.

The types of spatial reasoning demonstrated by Shan in this episode are spatial discrimination according to various spatial properties such as size and shape. He demonstrates this when he categorizes the triangles at the start of his work and continues this spatial discrimination throughout the episode. Revealed through his speech and movements are spatial visualization and mental rotation as he imagines shapes such as the red equilateral triangles as a parallelogram and as a large equilateral triangle. He composes/decomposes/recomposes throughout the episode as he follows through on his spatial visualization and mental rotation. He reasons spatially when he demonstrates an understanding of the concepts of equivalence and congruence as revealed through his speech and movements.

Shan appeared to extend his spatial reasoning during the episode and to explore areas where he appears less confident. At the outset and in the first segment he is confident. But, when he chooses a yellow isosceles triangle and then chooses a green right-angled triangle, I understand that he has mentally visualized the sides as having the same length. However, when he compares using the actual manipulatives, he finds they do not have the same length and puts the yellow isosceles triangle away. As he does not speak during this, I infer his spatial reasoning

from his movements. As Shan explores equivalence with more challenging shapes -from regular to irregular - he speaks less and from this I infer he is in the process of developing his spatial reasoning and that language does not aid his development at this point. I increasingly observe his movements to reveal how he reasons spatially. When he established equivalence between regular shapes, such as the red parallelogram and the yellow equilateral triangle he did so quickly. As he moved from these to more elaborate irregular polygons, he took time in composing the polygons. In segment five he builds an elaborate demonstration of equivalence between two complicated irregular polygons. Not only does he attempt to establish equivalence between the two polygons as a whole but he attempts to establish equivalence between parts of each polygon.

### ***Spatial Reasoning Skills Used by Shan***

Shan built up a carefully orchestrated visual proof of the conservation of area over an increasing number of shapes in a multi-stepped demonstration. For this he used particular spatial reasoning skills, visualization, composition and decomposition, and mental rotation. He used proportional reasoning as he established the whole-to-whole relationships across different shapes (Möhring, et al. 2016). He spatially demonstrated a one-to-one correspondence in terms of area. Shan reasoned mathematically to confirm the spatial one-to-one correspondence of the shapes by discussing the relationship mathematically using the concept of fractions demonstrating the close link between spatial and mathematical thinking (Möhring et al., 2015).

### ***Aspects of The Analytic Process***

In order to understand the spatial reasoning of this particular child, Shan, during the episode where he explores instances of equivalence with triangles, I took various steps in the analytic process. As outlined in my proposal, I planned to collect the data through video- and audio-recordings. As Shan did not speak much and did not appear at ease communicating

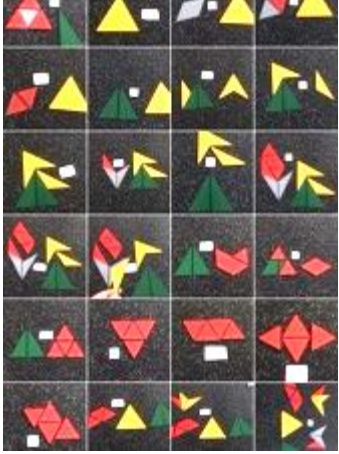
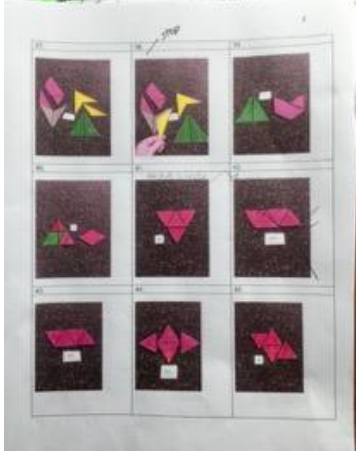
verbally, I considered a discussion with him might lead to discomfort on his part. I forewent a conversation with him for this reason.

After transcription and after selecting photographs to support the text, I drew up the analytic table. What was challenging when it came to the analysis of the data, was that Shan's work with the triangles moved steadily throughout the fifteen minutes that comprised the episode. He worked with several triangles he configured into numerous shapes. While I could analyze what he had said and done moment-by-moment, this aspect of the analysis did not offer an overview where I could step back and consider his work through a wider lens which I hoped would facilitate in discerning patterns and thereby offering an opportunity to draw further meaning. I took additional steps which allowed me to find a richer and deeper exposure of Shan's spatial reasoning. The additional steps I took are as follows: I purchased a miniature version of the Montessori material Shan used, namely the Constructive Triangles Boxes. Using the video data collected by me during Shan's work episode as a reference, I set up each shape Shan had composed with the various triangles. I labelled each shape with a number for ease of reference. I then photographed each shape. There were 48 different shapes composed by Shan during his fifteen-minute work episode. I made a compilation of the photographed triangle compositions from number 25 to number 48). After photographing each shape, I printed them out as a six-paged chart, each page showing nine examples of composed triangles. I printed the chart in black-and-white and using coloured pencils coloured in each triangle, matching its colour to the original triangle. I did this, rather than print them in colour, in order to allow me time to reflect on the triangles and their relationships to each other as I coloured. I made notes on the document writing down my observations and thoughts. Once I had the six-paged paper document I started, page by page, analyzing what Shan had done and what triangles he had used.

By looking at the different triangles he had used I was immediately able to divide the episode into segments and ended up with seven of them. As I have discussed in the analysis, I was able to understand that Shan had built up his demonstration of equivalence in a rigorous manner; from the beginning where he established his understanding of equivalence and congruence through repeated iterations using the same triangles, to the end where he demonstrated equivalence across five shapes. To further aid my analysis of the seven segments I generated another document into which I entered each segment. Under the heading of each segment, I divided the document into three columns. In the first column I listed the number of the sequence, the triangles used, and what shape Shan made. In the second column is listed only Shan's speech. In the third column I wrote down Shan's movements. The document filled five pages. The purpose of this document was to remove the visual images of the photographs and leave me with data in a textual form, in other words, Shan's speech and his movements captured as text. Having his speech written down in one column, with no other data, allowed me to focus on just his speech. In this way, I was able to notice his use of the word *could*, which as I suggested earlier, may possibly point to uncertainty on Shan's part as he was discussing a shape. By isolating his movement as written down in text I was able to take notice of the words that denoted spatial reasoning on his part, words such as rotates, aligns, flips, slides, and stacks. Figure 35 shows the three different documents I generated to aid my analysis.

**Figure 35**

*Supporting Documents*

|                                                                                   |                                                                                   |                                                                                     |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
|  |  |  |
| Compilation of Shan's triangles                                                   | Coloured in photocopied sheets of compilation                                     | Notes on Shan's movements and speech                                                |

Generating these two documents, in addition to the table, created additional work that took me about a week to do but they allowed me to consider the data from different perspectives and allowed me to triangulate my work since I was working from three data sources. When I went back to the table with the insights into Shan's spatial reasoning, I had made using the documents, I was able to make sense of Shan's spatial reasoning in a richer manner.

### Anisha

The following analytic table has data from Anisha's work finding the sum of angles in triangles (Table 7). This work emerged from a bigger lesson Linda had given the class on the Ancient Egyptians. Linda told them that one of the challenges Egyptian farmers had to contend with was the flooding of the Nile River. Because the floods destroyed the farm boundaries, land surveyors were required to measure and remap these boundaries. Amongst the tools the land surveyors used were long loops of rope that had knots marking regular intervals. The land surveyors stretched the knotted ropes over the farmland for measuring purposes. The ropes were stretched in a triangular shape. For the lesson, Linda had her own little piece of rope with knots, and with the children's help, demonstrated how to stretch it into a triangle. Figure 36

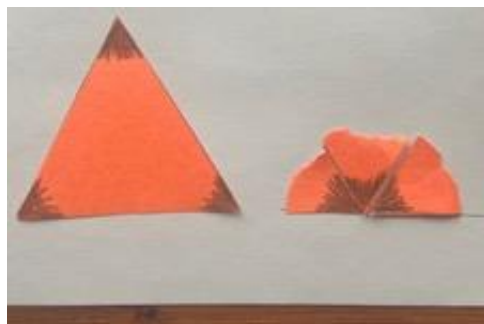
shows Maria holding the rope at the vertex and moving her index finger across the angle's measure.

**Figure 36**

*Maria Demonstrates an Angle*



After the story Linda invited Shan, Jennifer, Nina, and Anisha to a lesson where they would investigate angles further. It was interesting to observe Anisha interacting with the older students who were part of this follow-on lesson. She was very engaged in the social aspect of it, yet also contributed to the discussion by sharing her reasoning. Linda called the lesson *The Sum of Angles*. Linda explained to me that the goal of the lesson was to create an opportunity for the children to investigate the sum of angles in triangles and polygons. For the activity, the children traced various types and sizes of triangles and polygons on construction paper. They coloured in the angles before cutting out their shapes. Next, they tore off each angle then glued the angles, with the vertices converging on another piece of paper. Figure 37 shows a reconstruction of the activity using a traced equilateral triangle.

**Figure 37***Joining Vertices*

The first part of the image shows the equilateral triangle with its vertices coloured in and the second part shows the triangle decomposed and reconfigured into a straight angle with the vertices converging. The data that follow are from a discussion I had with Anisha three weeks after this lesson when I invited her to talk about the work she had done that day. In her explanation of her work, Anisha used the word *vertice* for the singular word, vertex.

**Table 7***Anisha Works with Paper Vertices*

| Text                                                                                                                                                                                                                                                                                                                                                                   | Images                                                                                                                   |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|
| <p>May 9, 2019<br/>09:19:08 – 09:21:40</p> <p>Anisha introduces her work to me. She has made and decorated a border around her work which is glued onto a sheet of green construction paper. She has written the date.</p> <p>Anisha: So, this is an angles poster. I call it an angles poster. So, what you do is you cut out triangles (using a cutting motion).</p> | <p>Anisha uses a cutting motion</p>  |

and then you've got to rip the corners (showing a ripping movement)

and then you've got to put the vertices all together (touching the different paper vertices) and then you see what kind of angle it makes, and you could write it down if you want, but most of the senior *elementary*s do the harder work (referring to the older students such as Nina and Shan who were both 10. Anisha had just turned 7).

Alison: You are not a senior elementary?

Anisha: No, I am a junior elementary.

Alison: I see you coloured in something. When did you decorate that? Was that before or after you ripped them?

Anisha: Oh this? (touching an angle). Oh, I colour the *vertice* first and then I rip it (making a colouring, then a tearing motion) because if I don't colour the vertices, how am I going to know where the *vertice* even is? (demonstrating emphasis with her left hand as she asks the question).

Anisha shows how she ripped the corners



Anisha touching the vertices



Anisha demonstrating emphasis



Anisha: Like this could be the *vertice*, this could be the *vertice*, this could be the *vertice* (touching the areas that could be considered a vertex).

I ask Anisha whether anything about this work surprised her.

Anisha: No. Miss Linda taught us this, and she also told us to cut out at least five triangles and put them into this triangles box which has quite a few triangles.

While talking, Anisha takes a box off a shelf and puts it on the table. She removes the lid and takes out some cut out paper triangles.

Alison: You used some and then you put these in for the next person?

Anisha: And then-, yeah (answering me)

Anisha: Look, see (showing me a cut-out paper triangle) you colour it in then you rip the vertices (showing me where the vertices had been coloured, then pretending to rip each of the three vertices).

Alison: What type of angles do you have in there? I mean triangles.

Anisha: Yeah, triangles (confirming my correction). This is a triangle that goes dow-, that has like a sort of sliding

Anisha demonstrating the vertices



shape (placing her index finger on one side of the triangle then running her finger up and down the same side).

Anisha goes to the Geometry Cabinet, pulls out a tray of triangles, then selects one.

Anisha: I think we have a type in here that looks exactly like that (superimposing the paper triangle on top of the blue triangle). Yeah.

Alison: That's the same? Oh, look at that!

Anisha: Yeah (aligning the right angle of the paper triangle with the right angle of the blue triangle).

She puts the blue triangle back into its frame and pushes the tray closed.

Alison: That's exactly the same kind. You had to put how many triangles in the box?

Anisha: I had to put five because Miss Linda asked for twenty (lifting up paper triangles with both hands) and there was four people. So I went, "Okay, it will be five."

Alison: And this was for the next people, was it?

Anisha: Yeah, next people who use it (putting the lid back on the box of paper triangles).

Anisha places her finger on the side of the triangle



Anisha aligning the right angles



Alison: Is this work finished for now?

Anisha: Yeah, it's my April cover then we make a scrap book out of it at end of the year (turning over her poster to show the other side of her work).

Anisha showing me her poster



### ***Commentary on Anisha's Work***

This analysis of Anisha's work has been divided into two sections: Anisha's reasoning about vertices, and Anisha's reasoning about triangles.

**Anisha Talks About Vertices.** At the start of our conversation Anisha clearly articulated the goal of her activity to me. She explained in steps what was needed to see “what kind of angle” could be made when the paper vertices of a triangle were reassembled as an angle. This task required Anisha to identify the vertices, to decompose a triangle by ripping off the vertices, then to recompose these vertices into a new angle. Since Anisha worked only with triangles, the result of each re-composition would be a straight angle of  $180^\circ$ . Anisha explained that the older children in the lesson, the senior elementary students, would do the hardest work which included measuring the degrees, and working with many-sided shapes. This was not required of Anisha. Her activity required her to reason about the changing positions of the vertices as she decomposed the triangles and then recomposed them. Anisha did not make any conjectures about the reconstructed angles as revealed when she said, “then you see what kind of angle it makes.” While Anisha called her work an angles poster and was clear that she would be making new

types of angles, Anisha's reasoning about the relationship between the vertices and the angles was not apparent. She did not use the words interchangeably but appeared to consider them the same entity. She is clear, however, that the vertices, as she referred to the angle, needed to be coloured in lest they were confused with other torn paper. This suggested that Anisha appreciated that each angle was integral to the goal of making a new angle.

**Anisha Talks About Triangles.** After Anisha had discussed how she made angles with the paper vertices, she turned her attention to the supply of paper triangles that were needed for other children to carry out the activity. She repeated the steps about ripping the vertices. In response to my question about what types of triangles she had, she referred to the paper triangle in her hand. It was a right-angled scalene triangle. She oriented the right angle of the triangle upwards, and the hypotenuse was parallel to the floor. She placed her finger on the longest side opposite the hypotenuse and said the triangle "went down", and thus used this term as a defining statement as to the type of triangle she had. She reasoned that there was a type of triangle "that looks exactly like that" in the Geometry Cabinet. She collected the blue triangle from the drawer then superimposed the paper triangle over it. She confirmed her reasoning when she said, "Yeah." While the paper triangle was the same type of triangle, it was not the same size having shorter sides, but Anisha appeared to reason they had the same angles.

Anisha used her hands as well as her speech to communicate action. Her fingers demonstrated cutting, colouring, and ripping motions as she spoke of her understanding of her work. When she reasoned about the superimposed triangles, she used words that denoted action, describing the triangle as going down and having a sliding shape. Later, Anisha reasoned mathematically when she decided she had to replace five triangles in the box, because Linda

needed 20 triangles replaced. There had been four students in the lesson so her responsibility was for five.

### ***Spatial Reasoning Skills Used by Anisha***

Anisha used visualization skills and mental transformation skills when she decomposed her two-dimensional triangle and reconfigured it so that it formed a different type of triangle (Bruce & Hawes, 2015). Using spatial scaling, Anisha established a one-to-one correspondence between the small paper scalene triangle and the much larger blue wooden one (Jirout & Newcombe, 2015). Unlike findings in the literature, (Hallowell et al., 2015; Hawes et al., 2017) she was not distracted by lengths of the side and was able to reason the properties of the triangles were the same (Kaur, 2015). However, she did not distinguish between the vertex and the area of the angle near the vertex. Anisha used mathematics when she multiplied numbers to work out how many paper triangles were necessary for the next children.

### **Nicholas and Toby**

One afternoon Linda gathered children, including Nicholas and Toby, for a lesson on triangles (Table 8). This lesson emerged from the story of How Geometry Got its Name. On the mat Linda set the Box of Sticks material and a tray of triangles from the Geometry Cabinet. She had made an isosceles, an equilateral, and a scalene triangle with the sticks. She spoke with the children about types and parts of triangles and encouraged a discussion on angles. The children were invited to look at the nomenclature booklet on types of triangles as they reasoned about the differences between them. Nicholas read aloud the definition of an equilateral triangle (Figure 38).

### **Figure 38**

*Nicholas Reads Aloud*



For follow-up work, Linda suggested to the children that they could make triangles with the Box of Sticks, or they could create pictures tracing triangles from the tray, or they could categorize a box of different plastic triangles she would give them. Nicholas decided to work with the tray of triangles and chose Toby to work with him. They moved to a nearby table to work. The following sequence was captured after Nicholas and Toby had started on their work. While all videos selected for analysis had to do with the children's spatial reasoning, I selected this video because of the pleasure it gave me watching Toby and Nicholas work together. The joy and enthusiasm they brought to their work was plain to see. Matthew had wanted to work with Nicholas on this activity, but Nicholas chose Toby.

**Table 8**

*Nicholas And Toby Work with The Blue Triangle Material*

| Text                                                                                                                                                                                                                                                                                                     | Image |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|
| <p>May 15<br/>15:06:00 - 15:42:00 (with 20 minutes spent away at a French lesson).</p> <p>The tray contains three isosceles triangles (acute, obtuse, and right-angled), two scalene triangles (obtuse, and right-angled) and one equilateral triangle. Each triangle has its own frame in the tray.</p> |       |

Nicholas sits at a table with the tray of six blue wooden triangles. He has traced an obtuse-angled isosceles triangle on a square of white paper. He colours in the traced triangle. Toby has gone to fetch more square paper. Toby's work is on the table as well. He has traced an equilateral triangle.

Nicholas replaces the obtuse-angled isosceles triangle and selects a right-angled isosceles triangle. He turns over the paper of his recent tracing then places the triangle down on the paper and traces it.

Toby returns holding a square of paper and his pencil. He takes a right-angled scalene triangle from its frame, places it on the square paper and traces all three sides. He returns the triangle to its frame and selects a right-angled isosceles triangle which he traces on another square of paper.

Nicholas gets up to collect the yellow and red measuring tool from the Box of Sticks. It is a tool for confirming right angles. It is a scalene triangle with angles of  $90^\circ$ ,  $60^\circ$ , and  $30^\circ$ .

The French teacher walks into the classroom.

Nicholas: Time for French! (as he continues tracing a triangle).

Toby continues to trace the sides of his triangle as well.

The French teacher leaves with a few children.

Nicholas colours in a traced triangle



Toby traces a triangle on a square of paper



A compilation showing the measuring tool on a right-angled triangle



Nicholas: This is a right (picking up the measuring tool and placing it inside his traced triangle).

He moves the measuring tool around as he tries to align its right angle with an angle in his drawing.

As Nicholas works with the measuring tool, Toby picks up his blue triangle and swivels it around over his traced triangle.

Toby: This isn't (responding to Nicholas).

He puts the triangle back in its frame and chooses another one.

Nicholas gets up and taking his sheet of paper walks to where Linda is standing with Inez.

Nicholas: Ms. Linda, is this a right angle?

Inez: Yes, that's definitely a right angle.

In the meantime at the table, Toby turns his sheet of paper over and places his next triangle on it. He traces the two longest sides of the triangle in one motion then lifts his pencil and traces the shortest side.

Nicholas measures a right angle



Toby swivels the blue triangle



Nicholas returns to the table and sits down. He leans over Toby's work and picks up the measuring tool.

He places the right angle of measuring tool over an angle in Toby's triangle.

Nicholas: It's not (shaking his head).

Toby: Let me see (snatching at the measuring tool).

They tussle over the measuring tool briefly. Toby takes the tool away from Nicholas who seems reluctant to part with it.

Toby places the measuring tool on his triangle. He aligns an angle of the tool with an angle of the tracing.

Nicholas: See! (touching the acute angle on the measuring tool with his finger tip and seeming to make a point to Toby).

Toby: What's this called? (referring to his traced triangle as he pushes the measuring tool away).

Nicholas: Ask Ms. Linda again (shrugging his shoulders).

Nicholas places the measuring tool over an angle



Toby aligns angles using the measuring tool



Nicholas talks with Toby



Toby: Just wait (leaning over and getting another triangle from the box).

They now have a nomenclature booklet on the table. They look at the page that has an illustration and definition of an equilateral triangle.

Nicholas turns over one of his sheets and looks at his tracing of an isosceles triangle then turns his attention to Toby.

Nicholas: You need help with this one (looking at the booklet and Toby's work).

Nicholas: What's this one called? (picking up a sheet of his own work and attempting to turn the page of the nomenclature booklet while Toby is still looking at the page).

Toby pushes the page back down to keep it open on the page for the equilateral triangle. Nicholas gets up and leaves the table.

Toby: Equilateral (sounding out the letters of the name of the triangle).

He writes the letters one by one looking first at the printed letters then copying each down.

The boys leave to attend their French lesson and are gone for approximately 20 minutes. After French they return to their table. Nicholas fetches a hole punch and some lengths of yarn. Toby continues to label each of his triangles with their names. Nicholas gathers his sheets of tracings together then

Nicholas shows Toby the booklet



Toby copies down the names of triangles



uses the hole punch to punch two holes on one side of the sheets.

Nicholas: Do you want to make a cover? (threading yarn through the two holes in his sheets of paper as Toby organizes his own sheets).

Toby gathers his sheets of paper together and turns the sheets over, seemingly to decide which hole to thread first. He brings the yarn through one hole.

Toby: Um, excuse me, how do you do a knot? (addressing me).

Alison: That's a good way to do it. You bring it through to the top, to the front.

Toby gathers his two ends of the yarn. He takes another sheet and threads the yarn through both of the holes.

Toby makes a booklet



Toby threads the yarn



### ***Commentary on Nicholas and Toby***

In this sequence of work, Toby and Nicholas use various spatial reasoning skills as they traced triangles from the geometry tray. For this analysis I have separated their work into two components: tracing the triangles and finding right-angles.

**Tracing Triangles.** Nicholas and Toby both selected different triangles to trace. They picked up each triangle from its frame using the little wooden knob found at the centre of the triangle. They oriented their triangles on the squares of paper they used. Both children oriented them in a way usually seen in school, namely with a base of the triangle running parallel to the bottom edge of the paper and with the apex of the triangle pointed to the top edge of the paper. This orientation of the triangles demonstrated how their spatial reasoning may be intricately connected to norms of the classroom. The boys traced the sides of each triangle thereby creating an outline which presented a new and different perspective of the triangle. Nevertheless, they appeared to consider the traced outlines as spatially the same as the wooden triangles. This was shown when they used the nomenclature booklet with its definitions and outlines of various triangles to label their own traced triangles. In other words, while the traced and wooden triangles looked different, they treated them as spatially the same.

**Finding Right Angles.** The second aspect of the activity was Nicholas' and Toby's search for right angles in the triangles they had traced. In her lesson, Linda had talked about right angles and had shown the children the measuring tool and how to use it with a triangle she made with sticks from the Box of Sticks material (see Figure 39) where Linda demonstrated how to use the measuring tool. She first measured the right angle, then placed the right angle of the measuring tool in the acute angle of the stick triangle. The photograph shows her establishing that the angle is not right angle.

### **Figure 39**

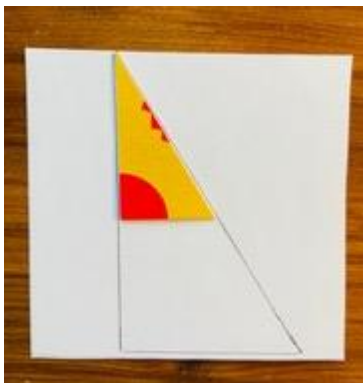
*Linda Demonstrates with Measuring Tool*



In his work, Nicholas oriented and then aligned the right angle of the measuring tool with the right angle of his scalene triangle whose angles were  $90^\circ$ ,  $60^\circ$  and  $30^\circ$ . He then used the  $30^\circ$  angle of the measuring tool and aligned it with the  $30^\circ$  angle of his tracing. (Figure 40 shows a reproduction of the alignment made by Nicholas). After this he used the tool to measure the  $60^\circ$  angle of the tracing. Although the two triangles had lengths that differed from each other, it appeared that Nicholas reasoned they were the same type of triangle when he aligned their identical angles. However, Nicholas needed further confirmation about his reasoning about his right angle and got it from Linda and Inez.

**Figure 40**

*Confirming an Acute Angle*



Instead of using the measuring tool, Toby used the blue right-angled isosceles triangle he had been tracing as a measuring tool. He slowly rotated it over the traced angles as he looked for a right angle. It was unclear whether Toby used only the right angle on the blue triangle, or whether he used the other two angles to measure as well. He reasoned that he did not have a right angle. Later, when he had the measuring tool, he aligned the  $60^\circ$  angle of the tool with the  $60^\circ$  angle of his traced equilateral triangle. To Nicholas, this proved it was not a right angle when he said to Toby, “See!” Toby for his part made no comment about the two angles both being  $60^\circ$ .

### ***Spatial Reasoning Skills Used by Toby and Nicholas***

Nicholas and Toby used fine motor skills during this activity when they oriented the triangles on the paper and when they placed the triangles back in their frames one-to-one correspondence sticking the wooden triangles in the containers (Bruce & Hawes, 2015; Leavy et al., 2018). They reasoned spatially as they established a one-to-one correspondence between their traced triangles and the wooden triangles, and then again when they compared their traced outlines those triangles in the nomenclature booklet. The tracing supported their spatial reasoning (Ginns et al., 2014). They used spatial visualization as they investigated right angles on their traced triangles and used mental rotation skills when they swivelled the wooden triangles around over their traced triangles and over the wooden opening. They compared the triangles again when they reasoned about whether their angles matched the angles on the angle measuring tool.

### **Jennifer and Matthew**

After Nicholas and Toby had chosen to work on tracing and naming triangles, Linda had invited other children from the lesson to choose follow-on work from that lesson. Some children continued making triangles with the Box of Stick material. Linda spoke to Matthew and

encouraged him to work longer with the Box of Sticks material. She asked Jennifer to work with him. The data from this video shows Jennifer and Matthew as they worked with the material (Table 9). They are sitting on the floor. On the mat are the triangles made by the children during Linda's lesson. Of interest to me was Linda's request to Jennifer to work with Matthew. Jennifer was 8 years old while Matthew was 6. I was curious to see how Jennifer would support Matthew. She helped him reason about the triangles as she pointed out their attributes and she drew his attention to aspects of the triangles she wished him to notice. She was patient and kind.

**Table 9**

*Jennifer Helps Matthew Work with Triangles*

| Text                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Images                                                                                                                                                                                                                                           |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>May 16</p> <p>15:08:00 – 15:18:00</p> <p>Jennifer picks up the booklet on triangles defined according to their sides. She opens the page on the definition for an isosceles triangle. Matthew moves closer to her.</p> <p>Jennifer: Right, so what about this. A triangle that has two sides equal is called isosceles (reading from the booklet and sounding out the word isosceles)</p> <p>Matthew reaches over to pick up two triangles from the pile. One is an isosceles triangle and one is a scalene triangle. He holds them up as Jennifer reads.</p> | <p>Jennifer reads to Matthew</p>  <p>Matthew holds up his triangles</p>  |

Jennifer: Not this one (shaking her head and taking the scalene triangle from Matthew).

She runs her finger down one side of the scalene then the other side. She puts it back on the pile.

Jennifer: That! (taking the isosceles triangle from Matthew and running her finger down one side of the triangle then another).

Jennifer: So, if it has two different colours, then yes, it is (referring to the green side and the yellow sides of the isosceles triangle in her hands).

Jennifer puts the isosceles triangle down and picks up another isosceles triangle. This one has two sides that are made with blue sticks and one side made with a yellow stick. She runs her finger down each of the longer sides in turn, then puts the triangle down.

Matthew: Here's one more (picking up another isosceles triangle and holding the shortest side up).

Jennifer takes it away from him and places it on the pile of triangles she has sorted).

Jennifer: And this one (picking up a smaller isosceles triangle).

She turns the page of the booklet and moves closer to Matthew.

Jennifer discusses triangles



Jennifer sorts Matthew's triangles



Jennifer: You are going to read this one (holding the page in front of Matthew and using her finger to mark the words).

Jennifer reads the words aloud. Matthew repeats the words after her.

Jennifer: Skyleen (sounding out the word scalene).

Matthew: All these ones (he picks up four triangles then puts them down). Look! You forgot this one (picking up a very small isosceles triangle and showing Jennifer).

Jennifer picks up a very small equilateral triangle and shows him. She puts it down.

Jennifer: Do you want me to read and you make the triangles? Do you want to make a book? No?

Matthew: No, I want to make these (picking up the very small equilateral triangle).

Jennifer: Okay, but then we have to clean it all up. Okay? (dismantling an equilateral triangle). Matthew, when the big hand is on the 4:00 then we have to clean up, okay? (pointing to the clock on the wall). That's going to be fifteen minutes. Oh wait. Yes, that will be fifteen minutes.

Matthew: My mom's coming today.

Jennifer: Ooh Matthew! Do you want to show her this? (picking up three triangles).

Jennifer starts disassembling more triangles.

Jennifer supports Matthew in his reading



Matthew shows Jennifer a triangle



Jennifer points to the clock



|                                                                           |  |
|---------------------------------------------------------------------------|--|
| Nicholas: How do you write acute? (coming over and speaking to Jennifer). |  |
|---------------------------------------------------------------------------|--|

### ***Commentary on Jennifer's work with Matthew***

This analysis of the data is of Jennifer's spatial reasoning as she encouraged Matthew in his reasoning. The piles of triangles she had to work with were from the earlier lesson just given by Linda, so Jennifer did not make the triangles, but rather classified them. She picked up the booklet that classified triangles according to sides which indicated her intention to identify triangles in the booklet then to find corresponding triangles in the pile. From the booklet she selected the isosceles triangle and read its definition as a triangle having two sides of equal length. As she read, Matthew picked up an isosceles and a scalene triangle. Looking at them in Matthew's hands, Jennifer identified each of them and so was able to correct Matthew's choices. She read aloud that an isosceles triangle had sides of equal length, but when she explained the idea of an isosceles triangle to Matthew, she said there were two colours. In this way, she equated the colour of the sticks with their length demonstrating a confidence that if a stick were a certain colour, it would be a certain length. Jennifer possibly reasoned that it would be easier for Matthew to identify the colour of the sticks rather than the length of the sticks. Jennifer accompanied her speech with movement as she ran her fingers down each side of the same-coloured lengths. She chose another smaller isosceles triangle, and again pointed out the same-coloured lengths while moving her finger up and down the similarly coloured sides. Matthew chose a small isosceles triangle and presented it to Jennifer with its shortest side uppermost, an orientation not usually seen in school. This suggested that Matthew was not yet enculturated in the norms of his classroom as to the typical orientation of triangles.

When Jennifer moved on to discuss scalene triangles, Matthew identified an isosceles triangle in the pile that they had not discussed, and he suggested that Jennifer had missed it. Thus, he demonstrated a differentiation between the individual isosceles triangles rather than generalizing the concept of them. Jennifer offered Matthew a choice of activities. She suggested she could continue reading to him and he could make triangles, or he could make a booklet. Matthew was drawn to a tiny equilateral triangle and said he wanted to make that. Jennifer demonstrated she had reasoned about time when she looked at the clock and told Matthew they needed to clean up at 4:00 and that it would be in fifteen minutes time.

In this vignette, Jennifer had her own understanding of the different types of triangles classified according to sides. In her communication with Matthew, however, it seemed that she considered her own understanding and then adapted it to what she perceived was Matthew's understanding. This is shown when after she reads a definition of an isosceles triangle, Matthew picks up a scalene triangle. Jennifer finds another way to define an isosceles triangle and so refers to the colours of the sides. As mentioned earlier, this showed her experience working with the different lengths of coloured sticks and therefore her confidence in her assertion.

### ***Spatial Reasoning Skills Used by Jennifer When Helping Matthew***

Jennifer used spatial visualization when working with the triangles (Hallowell et al., 2015; Kaur, 2015). She used spatial scaling when she reasoned about different-sized, but proportionally equivalent, triangles (Hallowell et al., 2015; Jirout & Newcombe, 2015). Research has shown that the sorting of shapes can reveal children's understanding of the properties of shape (Leavy et al., 2018). Jennifer was not distracted by the lengths of the sides and appeared not to take those into consideration (Hallowell et al., 2015). She did, however, consider whether there were two sides of the same colour when considering the

triangles and may have used this to help her reason. She supported Matthew's reasoning when she suggested he use the colours to help him distinguish the different triangles.

### **Declan and Maria**

The following data are from a lesson Linda gave the children where she introduced the concept of degrees (Table 10). The lesson began with a story of the Ancient Babylonians and how they used their understanding of the rotation of the stars in developing ideas in the measurements of time and space. Linda used material called the Montessori protractor and two trays of fraction pieces. In addition, she had books, an atlas, and a globe. Figure shows Linda and the children at the beginning of the lesson as she introduces the Montessori protractor, which can be seen in her hand. The circular Montessori protractor is divided by increments of ten marked in the longer lines. The shorter lines represent increments of five. The longer lines are marked with numbers beginning at zero and progressing to 350. The two trays of fraction pieces have a whole, or unit piece, then pieces divided into halves, thirds, etc., up to tenths (see Figure 41 where Linda introduces the Montessori protractor and included are a close-up photograph of a black protractor and an old green protractor with a red fraction piece inside it).

### **Figure 41**

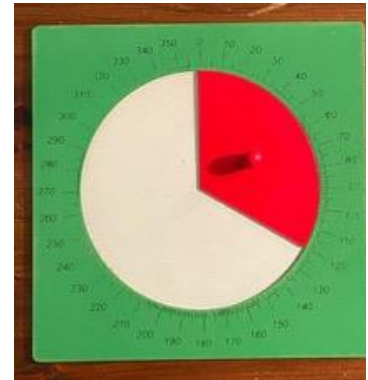
*Linda Introduces the Protractor*



Linda introduces the protractor



The Montessori protractor.



Protractor with red fraction piece.

After the story, Linda showed the children how to measure an angle by inserting a fraction piece into the Montessori protractor, first aligning one side with zero, then reading the number aligned to the second side. This gave them the number of degrees. The fraction piece measures  $120^\circ$ . At the close of the lesson Linda invited the children to take turns using the protractor to measure different fractions from the trays. She also suggested follow-up activities. The data are from the ending of the lesson and the work chosen by Maria and Declan. Declan and Maria had not planned to work together but Linda suggested that they do.

**Table 10***Degrees Of “One-Eighty Hundreds”*

| Text                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | Image                                                                                                                                                                                                                                               |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>23 April<br/>10:08:00 – 10:25:00</p> <p>Eric measures a fraction piece using the Montessori protractor. Viewing the fraction as an angle, he has one side of the angle lined up at the zero mark and the other side lined up at the 180° mark.</p> <p>Linda: Okay, while you are waiting for the protractor, will you please put up your hand and tell me what you are going work on. You can take the atlas out; there are the books on the Babylonians; you can take the fractions out and trace them; there are the constellations that you can do.</p> <p>Declan: I am going to trace these (picking up a tray of fraction pieces).</p> <p>Linda: Would you like to do it with Maria?</p> <p>Nicholas: I am going to take the book on constellations (picking up the book and moving to another table with Toby).</p> <p>Maria and Declan agree to work together. They set up the two trays of fractions on a table, collect paper and red coloured pencils.</p> <p>Maria gets the Montessori protractor from Eric. She chooses the same shape Eric had measured, the angle of 180°. She</p> | <p>Eric uses the Montessori protractor</p>  <p>Maria sets up her work</p>  |

puts the angle into the Montessori protractor, aligning the sides with zero and 180°. Maria removes the angle from the Montessori protractor and positions it horizontally on a sheet of paper. She traces the outline, then puts the angle back into the Montessori protractor.

Maria: One eighty hundreds (reading a number off the protractor).

She colours in her tracing with a red pencil. While Maria is busy, Declan chooses a fraction piece that is a third of the whole. He puts it in Montessori protractor to measure it, removes it and traces it on paper. He gets up to collect more coloured pencils. He returns and starts colouring.

Maria: You are not allowed to change the colour (commenting on Declan's coloured pencil).

Maria and Declan continue to work, choosing different fraction pieces, then measuring, tracing, and colouring them.

Maria: Eighty (reading a number from the protractor).

Declan: This one is seventy (holding up his fraction piece).

He places another fraction piece in the protractor.

Declan: Oh my goodness! This is so odd. This one is 35! (exclaiming as he reads the measurement).

Maria: I am going to sharpen the pencils (getting up).

Anisha comes over to see what Declan is referring to. She picks up his fraction piece, looks at it, then puts it back in the protractor and measures the piece herself. She leaves without speaking. Declan returns the piece to its frame and chooses another one which he places in the protractor.

Declan: This is not 80, it is 82, 83, 84, 85, oh my goodness, it is 88. This one is 89!

Maria and Declan work together



As Declan manoeuvres the piece, his focus is on reading the number of degrees of the angle and not on the side he believes is aligned to zero. As he refines his alignment of the piece, he comes closer to accurately establishing the number of degrees which is 90.

Maria returns bringing the pencils she has sharpened. She selects a fraction piece and puts it in the protractor. She aligns one side of the angle with zero, then reads the number of degrees where the other side stops.

Maria: It says 40 (reading the number while holding the fraction piece in place).

In response to Maria, Declan gets up out of his seat and moves closer to Maria.

He leans over Maria's work and inspects the fraction in the protractor. He stands upright suddenly.

Declan: Wait! (lifting the 40° fraction piece out of the protractor).

Declan places the fraction in its frame which is in the fraction tray,

then he takes it from the frame and places it down on his tracing of the same piece.

Maria measures a fraction piece



Declan inspects the fraction piece



Declan: Yes, it's 40 (handing it back to Maria who takes the fraction, places it on paper and traces it).

Declan and Maria continue measuring, tracing, and colouring various fractions.

Maria writes the number of degrees and the degree symbol inside her angles.

Declan: (looking at his watch) Every clock is different. This is the real time at home. Every clock is different.

Maria traces and colours seven fraction pieces in total then creates a decorative border around her work. She adds the date, day, and year. She and Declan put the trays of fraction material away.

Declan measures the fraction piece



Maria's traced work



### ***Commentary on Maria and Declan***

In this lesson, Linda presented ideas that link fractions, angles, and degrees. This created an opportunity for the children to consider the relationship between angles, degrees, and fractions as they worked with the material. Linda also presented the lesson in the context of the ideas' historical genesis. This further gave the children the chance to reason about the relationships between degrees, the shape of the world, the Babylonians and their solutions for a measurement of time and space. While Toby and Nicholas chose to draw the constellations used by the Babylonians in their measurements of degrees, Declan and Maria chose to investigate the number of degrees in selected fraction pieces.

For this task, Maria and Declan needed to orient each fraction piece they chose in the Montessori protractor. They had to accurately align one side of the angle against the  $0^\circ$  of the protractor and then to ascertain where the other side of the angle reached. This activity required them to reason about the fraction piece as an angle and to perceive its edges as equivalent to the sides of an angle. The activity created an opportunity for the children to think about an angle in terms of measurement in degrees. On paper they traced the outline of their fractions which presented a different perspective for them.

When Declan asked Maria to wait so that he could confirm that her angle was  $40^\circ$ , he confirmed the angle in three different ways. He confirmed the degrees when he measured the angle in the protractor, he then measured in its fraction frame, and he confirmed it against his own tracing of the  $40^\circ$ . In this manner, Declan drew a relationship, and established congruency between the fraction, the degrees, and the angle of Maria's piece.

### ***Spatial Reasoning Skills Used by Declan and Maria***

Research has shown children have difficulty with fractions and that children must be able to reason proportionally to reason about fractions (Möhring et al., 2016) In their work, Declan and Maria reasoned proportionally as they compared fractions parts to the whole of the Montessori protractor. They reasoned mathematically when they measured the fractions pieces and assigned a numerical value to them in degrees. They used mental rotation skills as they inserted the pieces into the protractor (Bruce & Hawes, 2015). Their tracings and spatial language support their spatial reasoning (Ginns et al., 2014; Leavy et al., 2018).

### **Mario and Others**

Early one morning Linda invited a visiting student to a lesson on polygons. Linda showed the visitor how to write the name of the lesson and the date in the journal she had been given.

Using the Box of Sticks material Linda demonstrated how to connect two sticks together using a brass fastener (Figure 42 shows a back and a front view of two sets of sticks connected with brass fasteners).

**Figure 42**

*Example of Joined Sticks*



For the purposes of the lesson, Linda appeared to consider each stick as a side of the polygon. Linda joined four sticks together in a chain, then joined the two ends of the chain together to form her polygon. She introduced the polygon as a closed shape. Linda handed the work over to the visitor and asked Inez to work with her to make as many polygons as they could using all the sticks in the box. As Inez worked with the visiting child, she suggested making a rainbow polygon and the child suggested making one as long as the room. Linda returned later and, looking at the closed shapes the girls had assembled using a selection of sticks, suggested they find out the names of the polygons they had made. Inez fetched a nomenclature booklet, *Types of Polygons*, to show the visiting child. Over the next couple of days, the Box of Sticks material lay out on the mat on the floor and at various times children worked with the material, adding sides until there was a long chain of joined sticks. At one point, Anisha joined Maria and showed her the fastest way to join the sticks with the brass fasteners. On another day, Linda borrowed the Box of Sticks from Maria to give a lesson on the positions of two lines. Maria

attended this lesson and worked on both the polygons and the positions of two lines lesson when she got the material back. Nina joined Maria later where they worked together adding sticks to create a long chain (Figure 43 shows Maria and Nina working together). Later Mario took over this work and the following data were collected when Mario and Evan added more sides to the already long chain as they built their polygon (Table 11).

**Figure 43**  
*Nina and Maria Work Together*



**Table 11**  
*Making a Polygon and Adding Sides*

| Text                                                                                                                                                                            | Images |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| April 11<br>10:15:00 -10:17:39<br><br>Evan stands, holding one end of the long chain of sticks.<br>Mario sits on the floor and passes sticks of the chain<br>through his hands. |        |

Mario: I just want to find the end (moving his hand from one stick to the next).

He sees the end stick and reaches over to pick it up. He stands up. Without saying anything to each other, Mario and Evan each walk backwards in opposite directions. Together they pull the chain of sticks taut and it lifts off the floor. Eric comes to watch. Mario walks backwards into the kitchen area of the classroom.

Mario: You come closer. We need it from the counter (directing Evan to walk towards him).

Eric walks forward. The chain becomes slack and rests on the floor.

Mario: Now go back (now directing Evan to walk backwards which tautens the chain of sticks and lifts it high). Stop! Whoa! Don't detach it (as Eric stops walking). Now we need to join it.

Mario and Evan walk towards each other. Each boy holds an end of the chain of sticks and a brass fastener. The bulk of the chain lies on the floor. Eric stands close by and Toby comes to watch as well.

Mario: Here, Evan (handing Evan a brass fastener).

Toby: Wait, are you going to use that two pin? (looking at the brass fastener in Evan's hand).

Mario: No, this one because it is smoother (holding his brass fastener).

Looking for the end of the chain of sticks



Allowing the chain to rest on the floor



Toby: Evan, it's more smoother (watching as Evan and Mario join the ends of the chain).

Mario: Let's see (picking up part of the polygon from the floor). Whoa! Let's see how long, let's see, I'll count, all right Evan?

Mario starts to count.

One, two, three, four (while moving his hands one over the other with each stick he counts).

Toby joins in counting aloud with him.

Mario: Forty-one, forty-two. There's forty-one sticks! (finishing his counting after which he drops the polygon to the floor).

Toby: There's forty-one!

Toby reaches forward and picks up the polygon from the floor.

Toby: A forty-one-a-gon! It's a forty-one-a-gon (speaking excitedly).

Evan: Mario, I'll clean up all the sticks. I'll clean up all the sticks (getting down on the floor).

Mario and Evan join the chain



Counting each stick of the chain



|                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                              |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------|
| <p>Toby: It's a forty-one-a-gon (repeating his word).</p> <p>Eric: It's actually a hexohaxagon (correcting Toby).</p> <p>Toby: (repeating his original claim) It's a forty-one-a-gon.</p> <p>Eric: It's a hexo – (challenging Toby)</p> <p>Toby: It's a forty-one-a-gon (speaking over Eric).</p> <p>Mario unfastens the sticks one by one and tosses them on to the mat. Evan sits on the floor and sorts them into the Box of Sticks container.</p> | <p>Toby naming the “Forty-one-a-gon”</p>  |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------|

### ***Commentary on Mario's work***

For the purpose of his activity, Mario appeared to understand each stick as a side of a polygon. Linda had implied each stick was a side when she gave the lesson on the concept of a polygon. Mario's goal, when he continued using the materials left on the mat by other children, seemed that he wished to use all the sticks in the box to create the largest polygon he could. While neither Mario nor Evan used language such as sides when they stretched out the chain as long as possible, it appeared they were assessing the lengths of sides they had available for the perimeter of the polygon. Later Mario said that he needed to find the end of the chain so that the ends could be joined. This echoed Linda's assertion that a polygon is a closed space.

An important aspect of creating the polygon was when the ends were joined, and all four boys crowded around to watch Mario and Evan. This was the moment of transformation when the polygon came into being. Mario appeared to find it important to count how many sticks they had used to make the polygon and appeared to take delight in counting up to forty-two. When Toby made up the name for the type of polygon, i.e., forty-one-a-gon, he revealed that he understood the shape as a polygon. Eric created a different name for the shape, but he showed he

too had accepted the shape as a polygon. Mario appeared to visualize a polygon made with all the sticks and spatially he knew that to grow his polygon he needed to add to the number of sides. Toby understood this categorization of polygons by their number of sides when he called it a forty-one-a-gon. All the boys accepted that the chain of sticks had to be joined to create the closed space required of a polygon.

### ***Spatial Reasoning Skills Used by Mario and Others***

Mario and the other children used spatial visualization skills to create their polygon. They also used mental rotation skills to transform the linear sticks into a closed curved shape resembling a circle. This dynamic process of transformation where shapes “can be continuously transformed” (Sinclair & Moss, 2012, p. 28) may support the children’s skills in identifying shapes according to their properties and not according to their prototypicality such as has been found in the research (Hallowell et al., 2015; Kaur, 2015). This activity echoes the activities of Gail and Anisha where they used sticks to draw their concentric circles.

### **Summary**

In this chapter I analyzed eleven video-recordings which were selected based on the rich examples of the children’s semiotic traces they offered. These semiotic traces such as the children’s speech, writings, drawings, and movements had emerged during their spatial reasoning as they worked with the mathematical materials. A commentary at the end of each analyzed video served to communicate my findings. The integrated nature of Radford’s (2013, 2014) theory of sensuous cognition formed a solid theoretical framework for my understanding of the data as I had defined spatial reasoning. To recall, I understand mathematical spatial reasoning as those reasoning skills used during the children’s mathematical activities. It is used when considering relationships between objects and relationships between objects and self. I

consider the body to play an essential role in the development of spatial reasoning. This allows spatial reasoning to be considered a specific area of non-verbal reasoning which “can be practiced with limited or no use of the eyes – with the hands, with the moving body and gestures” (Whiteley et al., 2015, p. 11). I consider spatial reasoning an emergent process not comprised of isolatable skills, but one that has many working in tandem (Whiteley et al., 2015, p. 5).

## **Chapter 7: Discussion and Conclusion**

In this chapter I discuss the children's spatial reasoning in the Montessori classroom in light of the analytical work I have undertaken with the data and within the context of the literature on spatial reasoning and my theoretical framework. In answering the research questions, I consider the children's spatial reasoning while at the same time I revisit the implications of the sociocultural foundations of sensuous cognition which are based on Vygotsky's work (1978, 1986) and were integral to my interpretation of the data. I include the implications of the sensuous aspects of Radford's theory which allowed me to pay attention to how the children used their bodies and their senses in the generation of the semiotic traces that were indicative of their spatial reasoning. To recall, the research questions are as follows

1. How is mathematical spatial reasoning experienced by young children as they participate in a Montessori mathematics curriculum?
2. In what ways do the Montessori materials allow the children to engage in spatial reasoning activities and actions?
3. How are the children's expressions of mathematical spatial reasoning revealed and communicated?

In this section I present the findings of my research. I address the first research question that inquired into the children's spatial reasoning experiences in the following sections, namely, how the classroom was arranged as a learning community, how the children's active participation was expected, how time was allowed for participation in activities, how the use of stories and spatial terminology were used to support spatial reasoning and, finally, how the teacher's assessment was observation and discussion. The second and third research questions

are addressed together since they are intimately linked. These two questions asked in what ways the Montessori materials allowed the children to engage in spatial reasoning activities and actions, and how the children's spatial reasoning was revealed and communicated. The findings are discussed in the following sections: how the children engaged in guided play with the materials, how the children's spatial reasoning skills were culturally developed, how the children generated their semiotic traces, and the role of the mathematics materials in their spatial reasoning.

### **Addressing the First Research Question**

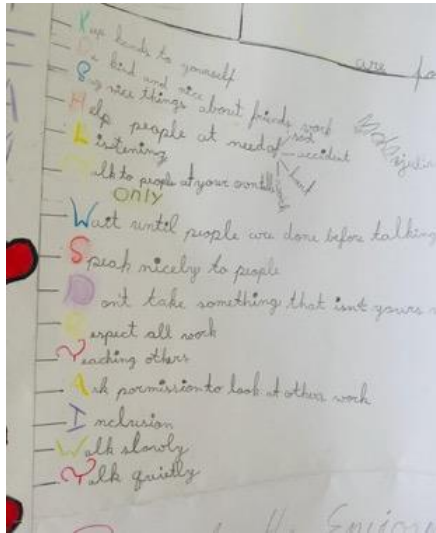
Research on spatial reasoning found the environment within which children learn has an important role in the development of their spatial skills (Bruce et al., 2015; Mulligan et al., 2020; van Oers, 2010). Wager (2013) suggests that an environment conducive to mathematics learning needs to be prepared and planned. The sociocultural foundations of sensuous cognition allowed me to consider this context of the children's mathematics activities where, through enculturation into the practices of the classroom (Moll, 1990), they were able to develop their spatial reasoning. Therefore, in addressing the first research question, I explore how the children's experiences were supported by the cultural practices of the classroom such as in the carefully-organized environment, in the children's active participation in their work with the mathematical materials, in the time allowed for the development of their spatial reasoning, in the contextual support offered by stories and narratives, and in the role the teacher had in giving lessons, teaching spatial terminology and assessing the children.

The extensive opportunities for spatial reasoning arose in part from the teacher's pedagogical choices and from the norms instantiated in the classroom which were communicated

within this planned environment. The Montessori environment was planned in the following ways:

***The Classroom was Arranged as a Learning Community***

Following Lave and Wenger (1991) the classroom may be considered a learning community whose organization was seen in the physical arrangement of the room, in the arrangements and care of the materials on the shelves, and in the teacher's guidance of the "children's learning towards established goals." (Lillard, 2013, p. 160).) The classroom was arranged in a home-like way to facilitate social interaction between the children with furniture such as comfortable chairs for relaxing and an area for preparing food. Artwork and interesting objects, plants, and pets contributed to the sense of a family setting. The Montessori materials were arranged on open shelves according to subject. The mixed ages of the children supported a collaborative, non-competitive way of learning. Values considered important were communicated at the beginning of school year when the teacher met with the children to establish the community guidelines. At this meeting the teacher explained her role and responsibilities and invited the children to consider their roles and responsibilities. Her role, she told the children, was to observe, to give lessons with material based on those observations and to model the appropriate behaviour she wished to see in the classroom. The children's responsibilities were to practice with the materials, to be courteous and helpful to each other, and together with the teacher, to take care of the contents of the classroom, such as the various materials, the furnishings, the pets, and plants. The guidelines covered three areas: care of others, care of the environment, and care of oneself. Once agreed upon, the guidelines were written on a chart by the children and signed by each of them. The chart was hung on a classroom wall (Figure 44).

**Figure 44***Classroom Guidelines*

The various components of the prepared environment were foundational to the learning community and allowed the children to engage in “sympractical activity” where individual “forms of consciousness cannot be understood separately from collective consciousness so that practices inherently are shared” (Roth & Radford, 2011, p. 141). The establishment of the learning community highlights a view of learning as “situated” where “newcomers ... move toward full participation in the sociocultural practices of a community” (Lave & Wenger, 1991, p. 29). We see Eric demonstrating his enculturation into the classroom community when he said to me, referring to the blue volume material, “Last year they had it [the lesson] but they never practiced it, so they forgot.” These ideas are brought forward here to illustrate how the teacher’s and children’s shared responsibilities were made explicit, and how from the outset of the school year the new children were brought into the practices of the classroom community.

The social foundations of the children’s learning are an integral part of a Vygotskian understanding of human development and therefore apply to the development of their spatial reasoning within the community of the classroom. In his genetic development Vygotsky

suggested that all higher development takes place “on four historical levels: the development of the species (*phylogeny*), the *history* of human beings since their emergence as a distinct species, *ontogeny* (the history of individual children), and *microgenesis*, the development of particular psychological processes.” (Cole, 1990, p. 92). Social processes are part of each of these levels. In sensuous cognition, Radford (2014) focuses in part on a “cultural and historical” understanding of the conceptual world but considers the body in human development in detail (p. 349). This is discussed in the section on the cultural development of the children’s senses and spatial reasoning.

### ***The Children’s Active Participation Was Expected***

The children were encouraged to work freely and spontaneously but within the guidelines of the classroom community. Their self-initiated and active participation in activities with mathematical materials supported their spatial reasoning (Fisher et al., 2013; Leavy et al., 2018; Smith et al., 2014; van Oers, 2010). The social values instantiated in the guidelines freed the teacher from managing the children as a group and allowed her to focus on individual or small group lessons and on her observations of children while at the same time modelling engagement in work. The guidelines served the children in a few ways as they offered the children a degree of independence from the teacher which allowed them to pursue avenues of work of interest to them. The freedom to choose their activities allowed the children time for reflection and deep exploration with materials. The shared responsibilities created the space for children to teach and support their peers in their spatial reasoning. For instance, Jennifer helped the younger child, Matthew, when he was reasoning about triangles.

The children’s actively selected meaningful work with mathematical materials according to their interests and did not look to the teacher for either approval or confirmation of

their choices. Their interests were stimulated by the Great Lessons, by challenging work with the mathematical materials, and by opportunities to work with others and by observing others at work. The children's active participation in the mathematical cultural practices of the classroom (van Oers, 2010) was also supported by the largely unscheduled day, as mentioned, and importantly by the routines the children used for managing their time, such as their daily journals. The children learned how to write in their journals putting the date, the start and finish time of their work and their activity. Figure 45 shows Maria working with the Box of Sticks material and in front of her is her journal. Her first entry was for 9:00 a.m. and she had written "French", at 10:15 a.m. she had written "help toddlers pet hamster", then she had "snack" at 10:45 a.m. and "polygons" at 11:57 a.m.

**Figure 45**

*Maria Writes in Her Journal*



The first lessons Linda gave Luca, who joined the class in April when I was collecting data, were how to tell the time, the days of the week, and how to write in his journal. Figure 46 shows Luca working with the clock material as he practised telling the time. The development of these skills was essential in helping Luca, and the other children, actively participate and become accountable for their use of time.

**Figure 46**

*Luca Works With the Clock*



Lave and Wenger (1991) make a distinction between a teaching curriculum and a learning curriculum. A teaching curriculum supplies and controls the resources for learning and is not viewed from the position of the learner. A learning curriculum, in contrast, consists of “situated opportunities for the improvisational development of new practice” and may be considered a “characteristic of a community” (p. 97). Implicit in Lave and Wenger’s understanding of a learning curriculum is the idea of the agency of the learners in selecting their work. The Montessori curriculum may be considered a learning curriculum which is positioned from the viewpoint of the learner. The children’s abilities to select their own work played a

considerable role in supporting their spatial reasoning as it heightened their engagement with the various mathematical materials. The children's engagement and exploration with materials are discussed later.

The classroom guidelines allowed for another Vygotskian construct, the zone of proximal development, which we can consider in the instance of Jennifer's work with Matthew. To recall, the zone of proximal development is

[t]he distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (Vygotsky, 1978, p. 86)

In the example with Jennifer and Matthew, Jennifer serves as the more capable peer as she guides Matthew in his reasoning about triangles. The concept of the zone of proximal development raises a question about the role of time in the children's spatial learning.

While the children choose activities with the mathematical materials for themselves, observation of others at work played an important part in their participation in the classroom practices. The younger children often observed the older children working with material they had not yet been introduced to, and peers observed each other working. Children would leave their own activities to go and watch others, or they would call friends over to show them their work. Observing other children at work was an acceptable use of time and these observations were further source of the interest necessary for engagement with the mathematical materials. This "legitimate peripheral participation" (Lave & Wenger, 1991, p. 29) not only served as a way for children to become integrated into the cultural practices of the classroom, but it also offered the children

opportunities for “educational imagination” (Wenger, 1998, p. 272) where they could imagine and anticipate future work with themselves using the material.

Sometimes children would ask, or be invited, to join an activity and may later possibly take over that work themselves. For example, Mario’s polygon work with the Box of Sticks material began with a lesson the teacher gave on polygons to Inez and a visiting child. Inez and the visiting child worked with the material making different kinds of polygons. The work was left out on a mat and Maria took it over the next day spending the morning making polygons. Nina joined her for a while, and they used the material to repeat the lesson the teacher had given with the material on The Position of Two Lines. Various children worked with her for short periods, coming and going. The following day Mario and Eric picked up the sticks Maria had left out on the mat and continued working with the material, creating the largest polygon they could using all the sticks. This way of working created a social “community of practice” (Wenger, 1998, p. 6) which was fluid in terms of who the children worked with and what activity they worked on.

Allowing the children many ways of participating increased their opportunities for spatial reasoning as it exposed the children to the wide array of activities undertaken by their peers. These activities could serve as a source of interest for the children who were observing. The different ways of participation allowed the children to engage in activities in a more relaxed manner where they were not required to explain their reasoning to the teacher nor to complete their work to a deadline.

### ***Time Was Allowed for Participation and Activities***

Spatial reasoning is considered a dynamic process that requires time (Casey et al., 2008; Polly et al., 2015; Smith et al., 2014). In his work on human development Vygotsky (1978)

differentiated between fossilized behaviours which are “processes that have gone through a very long stage of historical development” and have become “automated” (p. 63), and the dynamic processes found within the zone of proximal development. As Vygotsky (1978) stated, “the zone of proximal development enables us to propound a new formula, namely that the only ‘good learning’ is that which is in advance of development” (p. 89) and based on this conception of learning the Montessori classroom might offer the children many opportunities for learning in the zone of proximal development. That the children do not speak much during their spatial reasoning activities might point to a consideration that they are working in the zone of proximal development where they are not ready to articulate what they are learning. The other consideration is Vygotsky’s (1978) finding in his study replicating the work of Stern (1924/193) that very young children could communicate in pantomime but had difficulty doing so with speech, which Vygotsky considered “essentially analytical” (p. 33).

Implicit in the concept of dynamic processes is the idea that learning is active and takes place over time. In the Montessori classroom this time was secured by the lack of a timetable as the days were unscheduled except for French classes taken outside the classroom twice a week. Research on children’s spatial reasoning has found that children have a limited time for engagement in mathematical ideas and problems which are a necessary part of their spatial development (Clements, Swaminathan, Hannibal, & Sarama, 1999; Hawes, LeFevre, Chang, & Bruce, 2015; Tepylo et al., 2015). In the Montessori classroom the children had the time they needed and could continue with their activity over a couple of days or weeks if they wished. They decided for themselves when they were finished. For example, Nicholas worked with the Large Bead Frame on most of the days I observed in the classroom. He built up sheets and sheets of work of which he was proud and which he showed to other children. He sometimes worked on

his own and other times worked with different children, such as Toby. Nicholas's interest in his work with the Large Bead Frame was apparent from the length of time he spent with the material, his concentration when he was working, and his animation and excitement when he spoke with others about his work.

### ***Stories and Spatial Terminology Supported Spatial Reasoning***

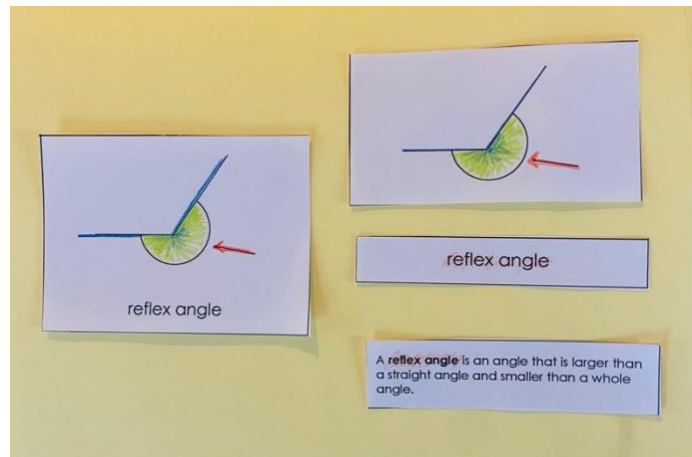
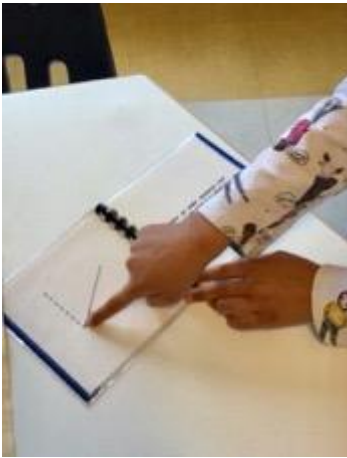
Research into spatial reasoning has found that the use of oral stories and narratives benefit young children's spatial work (Casey et al., 2008; Cross et al., 2012). Other research found that stories generate a rich source of imaginative ideas for the children allowing them to connect their experiences with the mathematical ideas expressed in the stories (Leavy et al., 2018). They suggest that stories are a good source of spatial terminology which may be introduced to the children. The Great Lessons offered an extremely rich source of spatial ideas which were expressed visually through the linear timelines, and the Black Strip, through the circular Clock of Eras charts, for example, and through ideas and concepts stemming from the Great Lessons. Many more stories emerged from each of the Great Lessons, and some brought forth specific mathematical and spatial concepts couched within the story. This was seen in the story of the ancient Egyptian farmers who had to reinstate their farm boundaries on the occasions that the Nile River flooded. Emerging from this story, the children were introduced to the idea of mathematics as a problem-solving approach, to different types of angles, to Pythagoras the Pythagorean Theorem, and to other mathematical concepts. These stories offered a contextual framework for the children as they learned new mathematical ideas (Cross et al., 2012).

In planning and preparing for spatial learning the teacher played a key role in giving presentations with mathematical materials, in observing the children and in supplying spatial vocabulary (Kaur, 2005; Moss et al., 2015; Wager, 2013). Supplying pertinent spatial vocabulary

is considered an important role for the teacher as the use of rich spatial language is necessary for the children's developing spatial reasoning (Bartolini Bussi et al., 2015; Leavy et al. 2018). As pointed out by Lave and Wenger (1991), "The importance of language should not ... be overlooked. Language is a part of practice, and it is in practice that people learn" (p. 85). While work with the material was considered of the utmost importance, nomenclature was introduced at the beginning of each lesson, for instance when Linda introduced the Montessori protractor. The children were not expected to immediately remember the nomenclature nor to give a definition of the spatial concepts to which they had been introduced, however, we see Anisha's confident use of spatial terminology when she names different types of angles, and we see Shan stating that "this is equivalent to that because it takes up the same space." Terminology for spatial concepts was brought to the children's attention more intentionally after the children had extensive use with a material, such that they could clearly articulate what they were doing. This use of accurate terminology amid discussion allowed for a bridging of the children's mathematical experiences with "formal mathematical language" (Leavy et al., 2018, p. 129). The definitions could be discussed at this point and the children could then consider these definitions in light of their reasoning with the material. Figure 47 shows examples of nomenclature material.

### **Figure 47**

*Examples of Nomenclature*



Nomenclature

Nomenclature cards

The nomenclature booklets with their naming and definitions of spatial concepts were an obvious way the cultural practices were introduced to the children. Bartolini Bussi and Baccaglini-Frank (2015) focus on the cultural role of the teacher and refer to the studies of Luria (1976) which showed that geometric perception is dependent on the culture to which the children are exposed, and in this instance, as introduced through the nomenclature booklets.

### ***The Teacher's Assessment was Observation and Discussion***

Another role for the teacher was assessment of the children's spatial reasoning. In the classroom the children worked for themselves, not for the teacher. In other words, they were motivated by their own interests in the materials and not by consequences given by the teacher, Linda. The teacher observed, not always certain what a child was doing with the material, but she did not disturb them unless they were careless with the objects, or they asked for help. The

children were not corrected, rather the teacher took observations notes to reflect on later.

Assessment of the children's mathematical spatial reasoning skills was subtle and part of a broader holistic assessment of each child's participation in the classroom. This form of dynamic assessment is in keeping with the notion of the zone of proximal as it acknowledges learning as a dynamic process over time.

Research has demonstrated that the way the assessment of mathematical understanding has been carried out has provided a limited view of children's spatial capabilities (Hallowell et al., 2015; Hawes et al., 2015). Researchers have also demonstrated that research findings resulting from a focus on children's verbal productions may underestimate their actual spatial knowledge (Kim et al., 2011; McGarvey et al., 2015; Sinclair & Moss, 2012), as well underestimate children's abilities to learn mathematics at a young age (Clements & Sarama, 2011; English & Mulligan, 2013; Lee & Ginsburg, 2009, Moss et al., 2016). Assessment of the children's mathematical understanding in the classroom fell within this broader assessment of the children and their activities. The children were not called upon to tell the teacher what they knew or what they had learned. At the beginning of new lessons, the teacher would talk about the material she was about to present and notice how the children responded. From these informal discussions the teacher would assess the children's thinking about a topic. However, the children's verbal productions were only a small part of assessment. As mentioned, the teacher's observations were her main assessment tool. It appeared to me that the questions that concerned Linda in her assessment of each child were "Does that child have enough work to do?" "Has that child done work in all the subject areas?" "What are the next lessons I should be giving this child?" "How is this child working with other children?" "Is the child engaged and concentrating with their work?", "Is the child struggling with the use of any mathematical material?"

The teacher also met with the children monthly for individual conferences where they discussed what had been done, what work was finished, what needed finishing, where the child needed help, and what lessons the child would like. They would use the child's journal as a source of information for the conference. I heard a child requesting a lesson she had seen Linda give to someone else. Some children met more frequently with the teacher, depending on their needs.

As regards the children's spatial experiences, the Montessori environment played a role in fostering them. The classroom was a carefully planned learning community that encouraged cooperation, respect for each other, respect for work and respect for the materials. The children were encouraged to engage actively in their work and to take the time they needed in their mathematical activities.

### **Addressing the Second and Third Research Questions**

The second and third research questions address the children's engagement with the mathematical materials and how they revealed and communicated their spatial reasoning. These two questions are discussed together as they are intimately related. I first consider the way the children engaged with the materials, namely in a manner which may be considered guided play. I discuss the cultural development of the children's senses in their spatial reasoning endeavours and include a discussion of Radford's conception of the multimodal aspect of sense development as well as the plasticity of the children's senses. I incorporate how the children revealed and communicated their spatial reasoning, their use of the mathematical materials, and include Radford's concept of human reflection when I consider the children's movement.

### ***The Children Engaged in Guided Play with the Materials***

Research on the development of children's mathematical reasoning has found that the way the children engage in mathematical activities plays a role (Leavy et al., 2018; Wager,

2013). Spatial play has been associated with the development of spatial skills (Jirout & Newcombe, 2015) and Hawes et al. (2017) point to the “playful and imaginative nature of spatial thinking.” (p. 236). Earlier in the discussion I focused on the active engagement of the children in the classroom, while here I consider how we may understand that active engagement as guided play. Guided play is child-directed but may take several different directions as the children engage with the materials and one another. The teacher supports and enhances the children’s explorations by having conversations with them about their work, suggesting further work or alternative choices, and opening up new ideas (Fisher et al., 2013; Weisberg et al., 2013). The children’s guided play allows for exploration which Oudgenoeg-Paz et al. (2015) consider a core mechanism in support of the development of spatial reasoning and spatial language. Anisha and Gail used exploration in their spontaneous, self-initiated work when tracing circles. The teacher supported them gradually over the course of their work, for instance, when Anisha asked Linda to show her where the angles were in the circles. The children’s explorations with the material facilitated work that was not limited to traditional shapes (Kaur, 2015) and opened avenues for a variety of experiences with the materials. We see this when Toby calls a shape a “forty-one-agon.”

Van Oers (2010) suggests that from a sociocultural perspective children’s mathematical thinking emerges in the context of play and as such children should be allowed “a high degree of freedom with regard to how they want to carry out or elaborate the rule-governed activity.” (p. 29). Toby and Eric explored and used their imaginations while they built with the blue volume material. They created a village which at one point Eric accused Toby of knocking down implying an infringement of the tacit rules of their activity. They built what they named a “standing L” using the spatial language resources available to them to identify their creation.

Mario and his friends used their imaginations to explore with the sticks adding them on and coming increasingly closer to a circle-like shape as they continued to build. This process allowed Mario to develop his spatial thinking when the repeated action of adding linear sticks moved the shape closer to a circle as he curved his whole shape. This was shown when he said he was going to make a circle. This use of the children's "mathematical imaginations" in their work with the mathematical materials was important to their reasoning especially to their mental rotation skills (Bruce et al., 2015, p. 96).

### ***The Children's Spatial Reasoning Skills were Culturally Developed***

The cultural development of the children's spatial reasoning skills took place in the classroom as they used their senses and engaged with the materials. Authors Möhring et al. (2016) in their research on spatial proportional reasoning and a formal knowledge of fractions suggest that the way fractions are introduced in school affects children's fraction understanding. Kaur (2015) reported that children's exposure to prototypical shapes, offered to them within the culture of the classroom, created a difficulty in their "understanding of a square as a rectangle" (p. 408). In the classroom Anisha had been introduced to an angle as a turn or as a rotation (Smith et al. 2014) and this is apparent when Anisha referred to her concentric circles as angles. In making a 3-dimensional paper box Jennifer and Inez modelled their work on a little wooden box and as Jennifer talked with me about her work, her attention was drawn to the pencil she held, and she changed the topic to say that the teacher wanted them to learn "how to use our fingertips" when holding the pencil (Figure 48). Here the teacher's instruction influenced how she perceived and touched a pencil. Eric's movement of hovering<sup>13</sup> the blue cube over the cube of 2 cm<sup>3</sup> blocks is an example of his membership into a well-established Montessori practice for

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<sup>13</sup> Hovering is a specific Montessori term for an action where children hold an object over another searching for similar attributes

establishing a relationship regarding the attributes of two objects. This action is an example of the development of his spatial reasoning as Eric has learned how to perceive and to hold the blue cube in a certain manner over the smaller cubes (Figure 48).

**Figure 48**

*Cultural Practices of the Classroom*



Jennifer demonstrates her grip.



Toby “hovers” the cube.

A foundational premise of Radford’s (2014) sensuous cognition is that the conceptual cannot be understood as separate from sensation. He understands sensation a “phylogenetically evolved feature of living organisms through which they reflect and respond to or act on their environment” (p. 352). Organisms themselves are material and consequently their thinking and actions do not occur in two different planes, “[T]hey occur in the same plane – the plane of life” (p. 352). Radford (2014) states that the development of the senses is not a natural process but the ‘labor of the entire history of the world down to the present’ (Marx 1932/1988, pp. 108-109; as cited in Radford, p. 352). Radford (2014), however, makes explicit that a focus on sensation does not limit the mind to “the realm of pure senses or the materiality of the world”, but rather to a

recognition that our “relations to the world ... are an entanglement of both the material world and ideational culture” (p. 353). Implicit, therefore, in the cultural shaping of the senses – and spatial reasoning, is the historical nature of this shaping. Radford (2013), leaning heavily on the work of Vygotsky, states that the “raw range of orienting-adjusting biological reactions” are transformed into complex, historically constituted forms of sensing in a cultural setting (p. 147). The process is dependent on the work of others over time. This understanding allowed me to investigate the children’s spatial reasoning as historically constituted. We see this on an individual, microgenetic, level when Toby and Eric talked with me about the genesis of their work with the blue cubes. Referring to the lesson, Toby told me that “Emilio gave it to Mario, Mario gave it to Nicholas” and Eric added, “And Nicholas gave it to me. I was the one who got it first this year.” Each child interpreted the lesson within the context of the classroom and taught the next child what they had understood through their interpretation. They used their senses, culturally developed within the classroom, to show the work to the next child and to interpret it for themselves.

Another example of the historical aspect of the cultural development of the children’s spatial reasoning skills is the instance where Jennifer helped Matthew identify types of triangles made with the Box of Sticks material. She held an isosceles triangle made with two yellow sticks and one green stick. Referring to the colours of the sticks she said, “So, if it has two different colours, then yes, it is [an isosceles triangle]” and by this conditional statement demonstrated her spatial reasoning had developed within the history of this Montessori classrooms where the colours of mathematical materials are consistent over time. Jennifer used her knowledge of that consistency as a resource to confidently state that if the colours of two sticks were the same then

the triangle must be an isosceles triangle (see Figure 49 where Jennifer explains her reasoning to Matthew).

**Figure 49**

*Jennifer Explains*



Jennifer's use of her various senses and her knowledge of the classroom practices allowed her to “distinguish between cultural categories of geometrical figures” (Radford, 2013, p. 148).

### ***The Semiotic Traces of the Children's Spatial Reasoning***

Another feature of Radford's theory which has facilitated my interpretation of the children's spatial reasoning is the notion of the plasticity and multimodal nature of the senses in sensuous cognition. The notion is in keeping with the Spatial Reasoning's Study Group stance that spatial reasoning “is not an isolatable competence” (Whiteley et al., 2015, p. 5). This understanding allowed me to analyze the semiotic traces pertaining to the children's spatial reasoning as produced through more than one sense, as is seen in the examples given above, and as evolving over time. Johansson et al. (2014) found that young children who were developing their verbal fluency used other means to communicate their understanding, such as gesture. Shan

did not speak much but used gestures in his communication to me when he placed his hands over the Constructive Triangle materials he had selected as equal in area. Drawings communicated the children's spatial reasoning and were a way for children to explore and become aware of spatial concepts (McGarvey et al., 2015). Gail and Anisha's drawings of the circles communicated their spatial ideas; Gail considered her work as measuring, while Anisha considered her work as angles. This is seen also in the work of Declan and Maria as they reasoned about the measurement of degrees when they used the Montessori protractor and traced the fraction pieces. They used their integrated senses of touch, hearing, and perception to generate their drawings and speech when they reasoned and communicated with each other during their activity (Figure 50 shows Declan and Maria discussing their work together).

**Figure 50**

*Declan and Maria at Work*



Ginns et al. (2016) investigated tracing as a means for developing spatial reasoning. Tracing offers multiple sensory modalities such as “visual, auditory, tactile, and kinesthetic” (p. 161) which support the children's spatial reasoning. Sarama and Clements (2009) found that activities with pens and pencils supported children's spatial reasoning. Nicholas and Toby traced the variously shaped blue triangles as they were learning to discriminate between the triangles.

Gail and Anisha traced their circles and while Luca did not use a pencil, he traced with his hands as he moved them around the clock face. Anisha traced the shape of the acute angle with her finger as she read the definition to me from the nomenclature booklet.

### ***The Montessori Mathematics Materials in the Classroom***

Not only do we see a cultural shaping of the senses in the examples given but when we think about the materials, we must consider their cultural development and their role in the semiotic traces of the children's spatial reasoning. The environment was rich with materials available to the children to use and these materials played a powerful role in the children's spatial reasoning (Leavy et al., 2018). From a sociocultural perspective the availability of these mathematical materials supported a special feature of human perception which "is the perception of real objects" which meant the objects perceived were endowed with "sense and importance" (Vygotsky, 1978, p. 33). While Vygotsky referred to the sense and meaning endowed in objects, Radford (2014) refers to the materiality of the objects that shape our senses. "Matter [is] already endowed with meaning" such as "triangularity" and "quadrilaterality" (p. 149). In the study conducted by Bartolini Bussi and Baccaglini Frank (2015) the children imaginatively generate new spatial terminology when they refer to a "squarized O", a term they have created to give mathematical meaning to their robot. This imbuing of meaning and the generation of spatial terminology is seen in Toby's creation of the word "forty-one-agon", in Inez and Jennifer's reference to the "roof" of their 3-dimensional box, and in Toby and Eric's reference to the "standing L" they built. Radford (2014) suggests that not only do artifacts mediate human thinking and experiences in the world "they are a constitutive part of thinking and sensing" an understanding that emerges from a concept of the mind as a property of matter (p. 149).) This

understanding is in keeping with views that consider “context [such as the materials] as inseparable from human actions in cognitive events or activities” (Rogoff, 1990, p. 27).

The mathematical materials in the Montessori classroom are culturally derived and as such are imbued with meanings by others, historically and currently, for example, we see materiality endowed with Luca’s meaning in his work with the wooden clock. Luca responded to the various features of the mathematical material with his own culturally derived understanding of its purpose. We see Erica and Luca when they used the mathematical material for an activity finding the greatest common multiple (GCM). When Eric said to Luca, “Get a ten, get a blue” it suggested that to Eric the blue material was imbued with the meaning of the quantity of ten. In the Montessori classroom the materials are endowed with “sense and importance” (Vygotsky, 1978, p. 33) and were known by their names and referred to as such, for example, The Box of Sticks or The Constructive Triangle Boxes. This consistent naming reinforced their importance and meaning for the children as well as supported the children’s ability to categorize the materials. For example, when I spoke with the two children, Toby and Eric, who were working with the Blue Volume Material and asked them what they were doing, they both answered, “It’s volume.” It appeared to me they were naming the material rather than naming the activity they were engaged in which evolved into imaginative play. This showed a relationship the children develop with the materials.

Research into spatial reasoning acknowledges the importance of objects in children’s spatial development (Bruce & Hawes, 2015; Casey et al., 2008; Fisher et al., 2013; Hawes et al., 2022; Jirout & Newcombe, 2015; Verdine et al., 2014; Wager, 2013). The materials in the classroom created an environment rich with objects for the children to explore and were designed to develop the children’s reasoning. In this regard Montessori built on the work of Itard and

Seguin who had stressed the importance of movement in learning and believed these movements would be generated by the children when they chose to work with the sensory materials.

Montessori spent considerable time designing additional materials and adding these to the sensory curriculum for the elementary children. For instance, with the Wooden Hierarchical Material the children could explore ideas of a hierarchy, the notion of a number system based on ten, compare the dimensions of the different cubes and the colour-coding of this material could stimulate the children's perception. The size of cube representing a million required heavy lifting and navigating on the part of the children to move it around the classroom. They laughed and struggled under its weight recalling the playful nature of spatial reasoning (Hawes et al., 2017). The kinesthetic aspect of the children's sensorial experiences with this material allowed them to physically experience the weight and dimension of the cube in a visceral manner, and to have these experiences in various parts of the body, such as the pull on their arms and the pressure on their backs, and the weight on their feet when carrying this heavy and awkward weight. They experience what parts of the body are engaged in safely navigating the large cube around the classroom such as using their eyes to look to find a route, where to plant their feet as they move, and how to twist and turn around any obstacles, all the while being cognisant of the other children holding the cube and how their movements affect their own movements. They must carefully place down the cube without dropping it and work together to do this. Figure 51 shows the children carrying the cube into the classroom.

**Figure 51**

*Children Laughing While Carrying the Heavy Material*



For Montessori, movement formed “a very fundamental basis in the education of the little child” (1936/1966, p. 198). Activities the Montessori classroom were predicated on the use of materials which were designed in part to generate interest and engagement and to encourage movement. From a Vygotskian perspective, the Montessori focus on the children’s movement and engagement with materials may be considered as mediated action. Here I consider the notion of action as a semiotic trace in the children’s spatial reasoning, and I do this aware of Wertsch’s (1998) caution that

any attempt to reduce the account of mediated action to one or the other of these elements runs the risk of destroying the phenomenon under observation. But from time to time, it may be productive to abstract these moments, or aspects, as part of an analytic strategy. (p. 25).

Implicit in the term action, is movement and I turn to Radford (2014) for a fuller description of this in his idea of “responsive sensation” and “human reflection” (p. 352). These two terms may be considered to encompass the children’s semiotic traces as they worked with the mathematics materials. Human reflection, for Radford, is not a passive activity but a dynamic feature of humans as they respond to, reflect or act on their environment creating a dialectical relationship with the material world. This idea of a dynamic response is found in the literature where spatial reasoning has been considered a “vital capacity for human action” (Spatial Reasoning Study Group, 2015, p. 3). In my analysis of the data, I was struck by the movement of the children as they worked, often quietly, deeply engaged in their activities. In particular, in analyzing their spatial reasoning through the semiotic traces of their movements, I was struck by how their movements were not limited to gestures but involved their whole bodies, whether they were leaning into their work, standing up and sitting down, moving around their materials, or moving their arms, torsos, or heads.

The realization that the children’s movements had been the major source of semiotic traces in my analysis of the data led me to research movement further, which I did with the work of Maxine Sheets-Johnstone (2018) whose consideration of kinesthesia as a sensory modality was important. Sheets-Johnstone (2009) is a philosopher whose early research was on the phenomenology of dance. In her recent research she has focused extensively on the role of movement and on the body as a “lived” body (p. 48). To her the lived and animated body is in essence the same as the physical body (2020, p. 30). The term *animation* is one Sheets-Johnstone (2009) considers the fundamental, essential “and properly descriptive concept” for understanding animate life (p. 375). This animation is present from the moment we come into the world and this movement, according to Sheets-Johnstone (2009), means we are “cognitively

attuned in a sense making manner discovering ourselves and our surrounding world in and through ... our bodies from the very beginning” (p. 382), a description which closely aligns with understandings of spatial reasoning. Sheets-Johnstone (2020) argues that “postural, embodied, and purely spatial descriptions of the lived body ... fail to accord with the spatial-temporal and kinesthetic nature of the lived body” (p. 28). Where Radford (2014) looks to outline an “embodied ... conception of cognition” (p. 349), Sheets-Johnstone (2009) instead suggests that,

When we strip the lexical band-aid ‘*embodiment*’ off the more than 350 year-old wound inflicted by the Cartesian split of mind and body, we find *animation*, the foundational dimension of the living. (p. 375)

The bodies of the children are animated and moving (Sheets-Johnstone, 2020, p. 28). Sheets-Johnstone (2008) calls movement our mother tongue from both an evolutionary and developmental perspective (p. 8) and that the dynamics of our touching, feeling, and moving bodies are basic to life. For her, kinesthesia is a forgotten sensory modality. Our understanding of our senses of sound, touch, taste, sight, and smell without considering kinesthesia as part of the multimodal nature of the senses, limits our understanding of sensuous cognition. The notion of kinesthesia allows us to understand the actions that are required around the use of the senses. Sheets-Johnstone (2018) points out that if we discuss the senses alone, we do not have an awareness of our initiating movement, nor an awareness of the spatio-temporal dynamics of our movement, and “no awareness of the qualitative nature of those dynamics” (p. 11). In reality, the children’s bodily dynamics are at the forefront of their spatial reasoning.

It was through my analysis of Luca and his work with the wooden clock that I realized Sheets-Johnstone’s understanding of the lived body would help to better understand Luca’s spatial reasoning. Initially I had not considered analyzing the videorecording of his work with

the clock as there were none of the semiotic traces, I had come to expect such as speech, tracings, drawings, written language, or discernible gestures. However, as I reviewed the videorecording many times, I came to see his touches were the semiotic traces generated by his spatial reasoning, and the deeper into the analysis of his work I went the more I was astounded by the complexity of his reasoning. Luca's hands worked together and played a central role in his activity with the clock. Each hand played its role as he passed the hour hand from one hand to the other, then rotated the hour hand while his other hand moved the disk in various locations. It was from Luca's movement alone that I was able to make sense of his spatial reasoning about time. Had I discounted his kinesthesia I would have missed rich instances of spatial reasoning. Shan's work with the Constructive Triangle Boxes is another example of how the concept of kinesthesia as a semiotic trace allowed me to understand his work. Shan worked extensively for fifteen minutes with the material, not speaking, and generating no written texts or drawings. Given that he did not draw nor speak, his movements were an important way to understand his reasoning. It was only in the in-depth analysis of his kinesthetic movements with the materials that the sophistication of his spatial reasoning was revealed.

The children's consistent use of the materials over time built up a repertoire of spatial kinesthetic experiences with their bodies that can be considered a resource for both the children and the teacher. These resources can be called upon at a later time for further lessons when Linda introduces more advanced concepts such as gravity, mass, or balance for instance. The children's bodily sensations of holding the Wooden Hierarchical Material, or the bodily sensations of balancing the blocks such as Toby and Eric's work with the Blue Volume Material, or the bodily sensations of tracing different triangles with a pencil as was done by Toby and Nicholas in their

triangle work with the Blue Triangles, were foundational to their spatial experiences in the classroom.

In the next section of the discussion, I reflect on aspects of the study I gradually became aware of that caused me to reflect and to change my perspective. I discuss the adaptations I made in response to these emergent perspectives.

### ***Pedagogical Practices and Classroom Norms***

I found my training as a Montessori educator inhibited my ability to interrupt the children while they were engaged in activities. On the other hand, even though a Montessori educator myself, prior to the start of the study I did not process the implications of how Montessori educators teach and as a result my expectations were unrealistic. To elaborate, while preparing for this study, I had anticipated that during her mathematical lessons Linda would clearly identify spatial concepts to the children. Prior to each lesson, I had expected to talk with Linda about the spatial concepts she would present, and after the lesson I envisaged the children undertaking follow-on work in a clearly discernable manner. In reality, pedagogical practices of the classroom where Linda's lessons were based on her observations and the children had autonomy in their activities, created an environment where lessons and children's follow-up work could not be predicted and as such it was mainly through my observations that I would become aware of children reasoning spatially as they worked with materials. The children were given opportunities for extensive sensorial experiences with materials before spatial definitions and nomenclature were introduced by Linda. In other words, the introduction of spatial concepts was a process given over a long period of time beginning with the introduction of different sensorial materials.

Linda's role as the teacher was different from what is normally understood as a teacher's role. Her role was not to teach to the whole class, but to observe the classroom, give lessons based on her observations, and to manage the individualized programs of each child. As the lessons emerged from observations Linda did not have a tightly planned schedule of lessons but rather was flexible when and to whom she gave a lesson. Not all children had daily lessons as they needed time to work undisturbed. Linda would speak with children independently should they have questions or difficulties. In Linda's discussions with me about the lessons she did not identify the mathematical concept that she was aiming to introduce but rather talked about the material she was using and what she expected the children to do with the material. She talked about what lessons had preceded a lesson and what lessons would follow but named mathematical materials, not mathematical concepts. Since the children's richest learning was considered to take place when the children worked with material in the social environment of the classroom, Linda presented the mathematical materials she believed would capture the children's interest so that they would want to work with the materials themselves. For instance, the concept of degrees was introduced with a story of the Babylonians. The children then explored the materials that Linda had introduced during the lessons in a variety of ways.

Another pedagogical practice that I considered had affected my collection of data was how the assessment of the children was managed. The teacher's assessment tool was observations and individual conferences with each child, generally once a month. The conferences served as an accounting by the child for their management of time, and an accounting on Linda's part for the lessons she had planned and had given. Children were not expected to account to Linda what they knew about a subject, although through conversations during a lesson Linda could assess what a child had understood. As a result, the children

sometimes appeared surprised and confused when I asked any questions as they were not used to be questioned. Many of the children were in the first year and possibly did not have the vocabulary necessary to talk about their spatial reasoning work.

### ***Adaptations for Data Collection and Data Analysis***

The nature of the classroom practices and my experiences as a Montessori teacher both required me to adapt my approach to data collection and data analysis in the following ways: My reluctance to disturb the children in addition to the unscheduled nature of lessons precluded me from setting up a video-recorder on a stand. Instead, I used my iPad, which was much less intrusive and always ready for use. The children became used to it on my lap or held near my face when I video-recorded and it seemed to me that from their perspective it was not always obvious if, or whom, I was filming. It was a more subtle tool than the video-recording camera and stand, and the quality of videos was excellent. Since there were a limited number of lessons given and since children could select their own work, I could not plan the recording of videos, unless the teacher specifically told me about a lesson she was planning to give, and I could not anticipate observing instances of the children's spatial reasoning. This meant it was necessary for me to scan the classroom frequently to catch such emerging instances. For example, I began recording two children as they created a three-dimensional paper figure, then turned and saw another two children working with the blue Volume Material. I decided to switch my recording to the two children working with the blue Volume Material and was rewarded with rich data, now part of my analyzed videos. I discuss the data collection tools in more detail.

The videos offered important visual data that was extremely useful not only for illustrative purposes but also as a reference when I analyzed the children's work. They allowed me to investigate and analyze the children's movements, a consideration of importance to this

study. In addition, the video-recordings also afforded me the opportunity of surveying the classroom context, or the children's space of learning (Radford, 2009) as individual or small groups of children worked with the materials. Importantly, the video-recordings not only captured their voices but their movements too. For instance, when I asked Anisha a question about her work on angles, she answered with words, but she also moved her fingers over her drawing of angles to communicate with me and this enriched my understanding of her spatial reasoning. When I spoke with Toby, another child, about his work and asked him what the word two-dimensional meant to him, his hand and finger movements with the blue blocks he was working with added considerably to my understanding of his spatial reasoning, (Figure 52) which would not have been available as a resource had I used only audio-recordings.

**Figure 52**

*Toby Speaks About Dimensions*



While I had initially considered using an audio-recorder to record conversations with the children this proved unnecessary as the video-recorder captured what conversations they had with each other and their responses to me. I used the audio-recorder for two purposes, namely as back-up support for the video-recordings when, for instance, Linda was giving a presentation and I was too far away to clearly hear all that the children said. In these cases, I placed the audio-

recorder close to Linda. I also used the audio-recorder to record the conversations I had with Linda in the classroom after the children had left. There was no need to video-record these conversations, and audio-recording our conversations worked well. Linda and I met to talk after the end of the school day as Linda did not like to disturb the rhythm of the day with the children by talking during class time. The conversations I had with Linda provided much insight into understanding what I had observed and Linda's comments about the children and my observations were valued.

The concepts I used in my analytic framework which included the didactical cycle, the semiotic traces, and the space of learning, supported my analysis of the data well. As an example, I use a lesson where Linda represented the concept of triangles using mathematical materials. This lesson followed on from a previous lesson on triangles given a month earlier. This earlier lesson was couched in the story of ancient Egyptian farmers and the annual flooding of the Nile River which erased their farm boundaries. The problem Linda presented to the children was how the ancient Egyptian farmers coped with the loss of their farm boundaries each year and how the use of geometrical tools worked as a solution to their problem. During the telling of the story, Linda briefly referred to other mathematical concepts such as the Pythagorean Theorem, ratios, the Babylonian's numbers system, but did not elaborate. In this next lesson on triangles Linda used mathematical materials called the Box of Sticks, and a tray of blue wooden triangles from the Geometry Cabinet to talk about triangles in more detail. Prior to the start of the didactical cycle, I had met with Linda and we discussed this lesson. This allowed me to reflect on the materials she intended to use, and her goals, and allowed me to consider what forms of spatial reasoning I could expect to be called into action by the activities. I noted

which children she planned to invite and why. I gained an understanding of what follow-on work she considered possible for the children.

When she invited the children to the presentation, this constituted the start of the didactical cycle as well as a space of learning. The semiotic traces produced by the children cannot be predicted, but using the example of this lesson, the following semiotic traces were observed. When Linda held up an equilateral triangle made with the Box of Sticks material and asked the children about it, they answered with “three sides” and “three angles.” Their bodily gestures, such as leaning over the material, raising their hands into the air in response to a question, and smiling, the children demonstrated their engagement in the lesson. Linda presented the triangle with the base parallel to the floor to discuss parts of the triangle. When a child, Gwen, shouted out “I know! I know!” she demonstrated her interest and her spatial thinking since she considered she knew the answer to Linda’s question about vertices. When showing the children an isosceles triangle, Linda ran two fingers up and down the sides of the triangle that had the same length then Linda ran her finger between two of the sides of the triangle demonstrating an angle and asked the children what this distance was called. The children called out together that it was an angle. When Linda demonstrated a right angle, one child called out that it was a left angle, demonstrating her spatial reasoning. Linda continued discussing triangles and their parts asking in what way triangles were similar and in what way they were different. In response, a child, Gwen moved over to touch the triangles in Linda’s hand and gave a verbal definition of three types of triangles.

Linda introduced a nomenclature book on types of triangles, which Nicholas read aloud to the others. The children were asked to consider the root word of “equilateral.”

Linda invited the children to each make scalene triangles with the Box of Sticks material. Here, the semiotic traces demonstrated by the children were their movements as they reached over to select different lengths of sticks, their talking amongst themselves about different triangles, and in the case of Toby the use of his whole body when he stood up and said he was an isosceles triangle. He said, “Look! Is this one? One is straight and one is out!” as he referred to the position of his legs (Figure 53 shows Toby making a triangle with his body).

**Figure 53**

*Toby Makes an Isosceles Triangle*



Using this lesson also as an example of the analytic process, when reviewing the videorecording of the presentation I observed for the teacher’s bodily gestures, her movements with the material, her verbal communications with the children in order to enrich my understanding of the children’s responses to the presentation. The children’s engagement in independent follow-on work offered a richer source of semiotic traces. The questions I asked myself were: What forms of spatial reasoning (if any) were communicated during the presentation by the teacher? How did she communicate them? How did the children respond to the lesson? How did they communicate their thinking? Were there instances of spatial thinking by the children? How

was this demonstrated? What forms were shown? Were the children enthusiastic about their follow-on work? What ideas did they come up with? How did the teacher respond? During the cycle, I used my interpretation of the construct space of learning to analyze data regarding the instantiation, or not, of Montessori principles. For instance, how did the teacher engage the children's interest in the presentation? How did the children show their interest, or lack thereof? Who was invited to the presentation, and why? Were the children uninterrupted in their follow-on work? Was this work chosen by them?

Not all questions I posed were relevant to each instance of the didactical cycle. For instance, in the lesson on triangles, the children did not freely choose the work but continued working with the Box of Sticks material and the blue wooden triangles at Linda's urging. She gave them a choice of work, but it was limited to follow-up work from the lesson. So, Toby and Nicholas took the blue triangle material to their seating area and traced shapes on paper, labelled them and made a booklet. These constituted some of the semiotic traces indicative of their spatial reasoning. Linda asked Jennifer to support Matthew in making triangles with the Box of Sticks material and in this instance, Jennifer created semiotic traces through her speech and the movements of her hands as she made and explained the different triangles to Matthew.

In the analysis I considered the space of learning by asking the following types of questions not all of which were relevant, however the questions formed a guideline for inquiry. The following are examples of questions I posed: Have the children chosen this work themselves? What, if anything, precipitated their choice? Are children working individually or have they invited peers? How did they extend the invitation? Are they moving freely around the classroom? How long have they worked on the activity for? Have they been interrupted? Considering their activities with the materials I will observe to see what semiotic traces are being

produced. Having identified earlier what forms of spatial reasoning I could expect to be generated by this mathematical activity, I observed for those. I asked myself what forms of spatial reasoning I was seeing and how I knew. The didactical cycle continued when the teacher reengaged with the children to discuss their work. Again, I observed and video recorded as necessary then analyzed the data following the framework already discussed.

### **Summary of Discussion and Conclusion**

The research questions posed for this investigation into children's mathematical spatial reasoning in a Montessori classroom served me well as they gave me space to consider a wide variety of factors both expected and unexpected. The study revealed that the children's experiences of spatial reasoning took place in a structured learning community which was organized in terms of its layout, its content, and in terms of the expectations for community behaviours. The children and the teacher were expected to work, to respect one another, and to take care of themselves and others, and to treat the contents of the environment with respect. Within this environment the children were encouraged to participate actively in their selection of activities meaningful to them and to use what time they needed. The teacher supported their spatial learning through observations, giving lessons, telling stories, and offering spatial terminology. The study found that the children engaged with the materials in spontaneous way, generally following their interests or the interests of their peers and occasionally guided by the teacher, Linda. Their use of the material was circumscribed by the values of the community guidelines which required the children to treat the materials with respect and to respect others who were working with the materials. This deep respect for the materials and one another's work created an atmosphere in the classroom that recognized the children's work as a serious endeavour where each child was accountable for their own behaviour. The children engaged

seriously with the materials but that did not preclude their use of their imaginations to create various scenarios, such as when Toby and Eric talked about the village when working with the blue volume material. The children communicated and revealed their spatial reasoning through the semiotic traces they generated as they reasoned. Their speech revealed their reasoning but most of their reasoning was revealed through the movement of their bodies as they engaged with the materials. The materials played a crucial role in the children's mathematical spatial reasoning.

The integrated approach of Radford's sensuous cognition with Vygotsky's sociocultural theory provided me with an excellent theoretical lens through which to understand the children's spatial reasoning in the Montessori classroom. Sociocultural theory allowed me to investigate the cultural development of their spatial reasoning. The theory facilitated my understanding of the classroom as a community and the shared experiences of the members of that community. Radford's sensuous cognition offered a compelling and exciting perspective for a monistic view of cognition that did not find itself straddling concepts of mind and body. The depth and quantity of movements indicative of the children's spatial reasoning in this study were not anticipated. Sheets-Johnstone's work on movement and kinesthesia offered a powerful way for re-examining movement and for positioning it to the fore of the children's spatial activities. The ethnographic aspects of this qualitative study allowed me the time that proved to be crucial for my emerging insights and reflections that were necessary to reveal the depth of the children's mathematical spatial reasoning as they worked in the classroom.

This study found that the children in this Montessori classroom had extensive opportunities for engaging in rich and varied activities that supported their mathematical spatial development. The key finding of the study, however, is the recognition of the overwhelming

importance of movement in their spatial reasoning. The extensive range and the beautiful quality of the mathematical materials played a crucial role in the children's movements. A further important finding is the recognition that the way the children engaged with the materials is through *guided play*. Opportunities for guided play are typically limited to preschool classrooms and are not available to primary children, except perhaps in subjects such as art. That the children were eager and able to learn complex mathematics through guided play up to age 12, as in the case of Shan, is a finding worthy of further consideration. Another important finding is a recognition of the diversity of spatial activities taking place in the classroom at any moment and that this diversity was made possible, in part, by both the multi-age nature of the Montessori classroom and by the extensive time allowed for children to work with the materials in the company of older and younger peers.

### **Post-Analytic Reflections and Possible Contributions of Study**

As discussed in the literature review, defining spatial reasoning is challenging as its definition is unsettled. However, as a researcher going through the analytic process, I have learned of the overwhelming importance of movement in the children's spatial reasoning. With this new insight, I reflected further on the Spatial Reasoning Study Group's understanding of spatial reasoning as dynamic processes. Their use of action words closely aligns with the actions of the children I observed. For instance, their use of the action word, *comparing*, can be understood as a spatial reasoning skill that is used for one-to-one correspondence and proportional reasoning. Other action words such as *composing*, and *decomposing* are terms to refer to spatial reasoning skills employed when building up and breaking down objects into composite parts. *Scaling* is another action term which refers to the spatial skill employed when reasoning about similar geometric shapes across size differences, for example. The importance of

the terminology used by the Spatial Reasoning Study Group is that it closely ties spatial reasoning to movement which is a key finding in this study. Using these action words in further analyses regarding children's spatial reasoning would offer consistency and clarity to a challenging construct.

This research contributes to the literature on spatial reasoning in mathematics education in several ways. It responds to the call from the Spatial Reasoning Study Group (2015) for further research that investigates the dissonance between how spatial reasoning is neglected in young children's mathematics education and the growing awareness of its importance in society (Whiteley et al., 2015). In this study, the participants' opportunities for spatial reasoning were not neglected. By considering the children's engagement with the mathematical materials a priority, Linda generated endless opportunities for the children's emerging spatial reasoning. This research may offer ideas in answering some of the other concerns brought forward regarding the opportunities for spatial reasoning in the classroom. These are the narrow focus on numeracy (Bruce & Hawes, 2015; Bruce et al., 2012; Ng & Sinclair, 2015a; Sinclair & Moss, 2012; Taylor & Hutton, 2013), limited time for engagement with challenging open-ended problems (Hallowell et al., 2015; Maschietto & Bartolini Bussi, 2009; Newcombe & Frick, 2010; Ng & Sinclair, 2015a; Sauter et al., 2012), assessments that do not always account for the variety of ways spatial reasoning is communicated (Hallowell et al., 2015; Hawes et al., 2015), and an underestimation of the abilities of young children to learn mathematics (Clements & Sarama, 2011; English & Mulligan, 2013; Lee & Ginsburg, 2009; Moss et al., 2016). The participants in this study had few time constraints on their mathematical activities and could remain with their work as long as they were engaged and challenged. Assessment was through observations by the teacher, and discussions with the children. The quality of the mathematical materials, refined

over decades and developed to be used with increasingly complex mathematical concepts, meant they were used by the children over many years. This study also responds to the call for further research into other educational systems, of which Montessori method is an example.

This study contributes to the research that recognizes that spatial reasoning is found across all academic subjects and adds to the call for cross-disciplinary research (Bruce, et al., 2017) as well as for research in other settings “beyond mathematics and other STEM fields.” (Davis et al., 2015, p. 146). The Montessori classroom offers opportunities for spatial reasoning in all subject areas as materials, opportunities for movement, and other pedagogical practices that support spatial reasoning are available in these different subject areas. Another possible contribution is a deeper understanding of the role and use of materials designed for young children’s spatial reasoning as well as an understanding of how children engage with those materials in the process of developing their spatial reasoning. The spatial activities available with the Montessori mathematical materials allow for an integration of “number and shape, arithmetic and geometry, or computational and spatial” (Whiteley et al., 2015, p. 8). The children explore sensorially with the materials coming to understand the materials’ spatiality, but later and over time, the mathematical relationships and patterns in the materials’ design become a focus of interest and mathematical reasoning. This research contributes to an understanding of how the children reason about mathematical concepts in a manner where number and shape have not been artificially separated. This research may be considered a contribution to the Montessori community which is an under-researched and insular pedagogy existing outside mainstream education and university curricula. In turn, this research brings findings of children’s spatial reasoning in a Montessori classroom which is rich with didactic materials developed over decades, and in some instances centuries, to current mathematics education research.

Finally, Radford's (2013; 2014) integrated theory of sensuous cognition, in addition to incorporating the work of Sheets-Johnstone (2009, 2018, 2020), constitutes a broad framework that allows for a consideration of both the cultural nature of mathematical reasoning, as well as the immense importance of the body and movement in that reasoning. For instance, sensuous cognition (and sociocultural theory) holds that our senses are culturally developed, and this brings to our attention the challenges for the children in creating two-dimensional representations of three-dimensional objects. These are learned skills (Francis & Whiteley, 2015, p. 134) that need to be taught in the classroom. This wider integrated theoretical approach as well as a strong focus on movement allows for the revelation of children's spatial reasoning and mathematical capabilities at a younger age than does a reliance on children's verbal productions. The focus on movement in this framework facilitates a deeper and more nuanced understanding of the mathematical capabilities and spatial reasoning of young children, abilities that have been underestimated.

### **Limitations of the Study**

Before the study began, I was cognisant of certain factors that might play a role in my research such as my new role as a researcher rather than one of a Montessori educator. I was also uncertain how my relationship with the teacher would unfold with my holding the position of researcher as opposed to that of a colleague which was how the teacher knew me. Over the course of the study, I found some of these concerns were warranted. That I had taught in a primary Montessori classroom for ten years impacted my role as a researcher. The challenges arose because of my Montessori training and knowledge of key pedagogical practices usually found in a Montessori classroom. One of these practices was to support the children's concentration in their work by protecting individual or groups of children from interruptions and

distractions. I understood my reluctance to disturb them as arising from my adherence to this principle, but according to Wolcott (1999) “for anyone naturally reticent or shy, traits not uncommon among ethnographers, ... it is too easy to withdraw and become an onlooker especially when the role you are trying to take [of observer] seems to support exactly that behaviour” and I considered this could apply to me (p. 48). Wolcott (1999) goes on to say that there is “nothing wrong with a researcher taking an essentially passive role and remaining uninvolved” (p. 49), but this behaviour on my part impacted my collection of data as after a few days in the classroom, I realized I had not had a conversation with any of the children and to do so I would need to approach them and interrupt them. Although I obtained the teacher’s permission to speak with the children it was still hard for me to do this, and I did not completely overcome my deep reluctance to disturb them. As a result, I possibly had less discussions with the children than I had planned and hoped for.

Part of the practice of allowing the children to work undisturbed required keeping loud noises, busy movements, and other distractions out of the classroom. For visitors to the classroom there is an observation chair where they may sit and watch the children without disturbing them. Visitors are expected to stay in their observation area. When I arrived, Linda showed me to the chair she had set aside for me. I sat in it and did not move much. The lack of movement and my child-sized chair caused me physical discomfort but nevertheless I was not at ease when I occasionally moved around the classroom. Fortunately, the classroom was small, and I had a good view of activities going on unless they were blocked by a low shelf, for instance. Another challenge was my viewing the children through the lens of an experienced teacher rather than that of a researcher. This perspective often led me to assume I knew what the children were doing. It was only through the process of detailed analysis of the children’s

semiotic traces that I became aware that in more than one instance the children's reasoning was not what I initially thought.

My relationship with Linda spanned fifteen years and I had been unsure in what ways this relationship would influence my role as a researcher. This proved not to be a challenge and I perceived no issues that caused either Linda or me to be uncomfortable. We shared a mutual respect and were used to sharing ideas about the children, about lessons, and about pedagogical issues. It was helpful to have discussions with Linda as she questioned my understandings and interpretations and so served as a catalyst in my reflections and changing perceptions. Linda had personal health concerns which ultimately resulted in her not completing the academic year. This meant that I stopped collecting data prematurely by two weeks. However, I had enough data, and I was able to meet with Linda later in June at the end of the academic year and then again in October. She invited me to contact her as I needed and so she continued being a resource for me. An extension on my ethics certificate allowed me to see Linda these other times.

### **Areas for Future Research**

This research revealed the critical importance of movement in the children's spatial reasoning and this movement was not limited to gestures but was a full range of body movements as the children moved, carried, and worked with manipulatives around the classroom. Research on fostering movement in the classroom would be helpful to building up opportunities for movement. There is considerable research on mathematics manipulatives and their use, see for example, Carbonneau et al. (2013) who offer a meta-analysis of the efficacy of teaching mathematics with "concrete manipulatives" (p. 269). Further research could focus on how manipulatives stimulate and support children's movements, and therefore their spatial reasoning, as opposed to focusing on their efficacy for teaching mathematics. Researchers have

called for a “[m]ore nuanced understanding of the interplay of numeracy and spatiality” (Davis et al., 2015, p. 148) and the Montessori mathematical materials may offer an avenue for further investigation into this area. Research that continues to examine the relationship between mathematics and spatial reasoning could build on the findings of this study which demonstrated a close link between two forms of reasoning, qualitative (shape) and quantitative (number) (Whiteley et al., 2015, p. 8).

Studies on the role of narrative stories in supporting children’s mathematical activities is another area for further research (see Casey et al., 2008). The Montessori lessons emerged from historical stories such as the story of the Egyptians and their farms along the Nile River, or the Babylonians and their work with calendars. Figures 54 and 55 show examples of the children’s work emerging from these narratives.

**Figure 54**

*Children’s Work Emerging from Narratives*



Making angles and drawing constellations

**Figure 55***Working with Pythagorean Theorem Material*

These stories, such as the story of the ancient Egyptians or the story of the Babylonians, emerge from the Great Lesson on the Story of Numbers. This story itself is nested within the other Great Lessons that preceded it the beginning of the year. Since the stories are couched in historical contexts of specific times and places, they offer a multidisciplinary approach to executing the curriculum as other aspects of the context may be considered such as the economic geography of the ancient Egyptians at the time, or the climate that caused the flooding, or what the weather conditions were, or how the Egyptians communicated about the flooding, or who Pythagoras was and what connection he had to the Egyptians. Since the children had Great Lessons about communication in the story called The Communication in Signs, or the Great Lesson about how humans meet their needs in the Story of The Coming of Human Beings, their attention has been drawn to these overarching concepts such as human communication and how humans fulfill their various needs for shelter, food, etc. and their awareness to these ideas may serve as a useful tool for investigating other times and places.

Given the complexity of spatial reasoning which includes the verbal, the gestural, the kinesthetic, research on how best to support teachers' assessments of the children's spatial reasoning could be an important area of study. Linda's daily observations and record-keeping of

individual children through these observations taken over a period of three years meant she had a detailed knowledge and catalogue of each child's work. This form of assessment meant that the teacher did not rely solely on the children's verbal accounting of their mathematical reasoning. However, exposing the children to spatial language, as was done using the nomenclature materials and the teacher's use of spatial terminology, may support the children's efforts to talk spatially about their work. Further research on how to incorporate spatial terminology into the children's mathematical activities in the classroom could be another area for investigation.

This work used theories encompassing sociocultural theory, theories of embodied cognition and a theory of sensuous cognition. There is much research in mathematics education that draws on these theories from different perspectives and in different areas of interest. The realization emerging from the analysis of the data of the essential "oneness" of spatial reasoning and movement points to theories such as Sheets-Johnstone's (2009, 2018, 2020), as equipped to contribute to a fuller understanding of movement as foundational to conceptual development. Sheets-Johnstone's (2018) focus on the primacy of movement shifts the focus from the brain as the "source of judgments, insights, and feats of all kinds" (p. 5) to the body which is not "just alive, but moving" (2020, p., 28). Sheets-Johnstone's theories offer a possible framework for a fuller and deeper understanding of what constitutes children's mathematical spatial reasoning.

### **Concluding Thoughts**

The development of young children's spatial reasoning abilities is of great importance not only for the sake of developing better spatial reasoning skills and better mathematical reasoning skills, but for the development of the whole child. Spatial reasoning pertains to the child's real and actual experiences in the world and are demonstrated to be key to the development of concepts, in other words, key to the development of the children themselves. This brings us back

to Montessori's understanding of education as an education for life. Philosopher Nel Noddings (2013), in her book *Education and democracy in the 21<sup>st</sup> century*, offers a view of education quite like the Montessori experience. She discusses educating the whole person within a setting that engenders respect and caring for each other and where responsibility towards one another is pledged (p. 119). Dialogue, basic to critical thinking, as well as opportunities for practice, working with peer groups, exercising choice in activities, and a "competent caring" teacher who both observes and advances the children's intellectual interest and social competence (p. 121), all offer "a better way" for education (p. 119). While the Montessori environment is a rich source for the children's spatial reasoning, it is also an environment where the children learn to exercise their agency. This agency in the Montessori pedagogy is, according to philosopher Patrick Frierson, (2021), "the basis of individual character, the proper object of respect, and the foundation of social solidarity" (p. 153) and a foundation for life in the world.

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## **Appendix A**

### **Teacher's Consent Form**

**Project title:** Young children's mathematical spatial reasoning in a Montessori classroom

**Contact information:**

Ms. Alison Goss, PhD candidate, Faculty of Education, Dr. Barbara Graves, Professor,  
Faculty of Education, University of Ottawa, 613-5800 extension 2225, bgraves@uottawa.ca.

**Invitation to Participate:** I have been invited to participate in a research project conducted by Alison Goss under the supervision of her supervisor, Dr. Barbara Graves. This research is part of Alison Goss's thesis project for a Doctor of Philosophy in Education which she is undertaking at the University of Ottawa.

**Purpose of the Study:** The purpose of the study is to investigate young children's mathematical spatial reasoning in a Montessori classroom.

**Participation:** I understand that Alison Goss will be in the classroom for approximately two months on a daily basis. My position and responsibilities as the Montessori classroom teacher will remain the same. Alison Goss will collect data using video-recordings, photographs, audio-recordings and observations. She will audio-record discussions with me about the presentations I anticipate giving to the children with the use of the Montessori materials, my pedagogical goals

for these presentations, my observations about children's work and their follow-on work. The purpose of this is for Alison Goss to gain a deeper understanding of the children's mathematical spatial reasoning. Alison Goss will engage the children in conversations as well. These, too, will be audio-recorded. The conversations will revolve around the children's choices regarding their independent follow-on work, the mathematical materials they are working with, their reasoning about the concepts that have been introduced, their plans for further work. Alison Goss anticipates having three to four conversations with me with each cycle of work around a particular mathematical concept and mathematical material, as well as three to four conversations with children engaged in this work.

Alison Goss will make video-recordings of the initial mathematical presentations given by me aimed at teaching a mathematical concept and also aimed at encouraging the children to use the Montessori mathematics and geometry materials. She will also video-record and photograph the children's work and productions as they engage in the follow-on work emerging from the initial mathematical presentation. These productions could include the children's bodily gestures, their drawings, their writings, and their verbal communications.

Alison Goss will keep detailed observations notes of the children working with the mathematical materials following the initial presentation with the materials. The purpose of these observations notes is to record how the children are working, their engagement with the work and with each other, their discussions, and their spatial reasoning.

**Assessment of risks:** My participation in this study entails no foreseeable risks. However, if I experience any discomfort, Alison Goss has assured me that she will make every effort to minimize this discomfort. I may decide to stop participating at any time.

**Benefits:** Being involved in this research project, I will contribute to an enlarged understanding of young children's spatial reasoning in a Montessori classroom.

**Privacy of participants:** I have received assurance from Alison Goss that the information I share will remain strictly confidential. My anonymity cannot be guaranteed, however, as others within the school community, including the Head of School, will know which teacher has participated. I may choose to use my name in reports of the research or I may choose a pseudonym. The school and the children will be identified by pseudonyms.

I understand, and consent, that video-recordings or digital photographs may be used in presentations or publications to illustrate instances of spatial reasoning as demonstrated by the children and as understood through their verbal communications, bodily movements, and writings and drawings.

I understand that in the case of children and their parents not consenting to their taking part in the study, they will not be video-recorded or digitally photographed, but will nonetheless take part in lessons and activities as usual.

**Confidentiality and conservation of data:** The data will be used for the completion of the thesis project as well as for dissemination in research journals and presentations. I have been assured that the video and audio-recordings, and photographs will be kept in a secure manner at

the researchers' home during the research, and upon completion of the project will be stored on Alison Goss' and Dr. Barbara Graves' password protected computers. The data will be kept securely for a period of up to ten years.

**Voluntary Participation:** I am under no obligation to participate and if I choose to participate, I can withdraw from the study at any time. All data gathered until the time of withdrawal will be destroyed.

**Acceptance:** I, \_\_\_\_\_, agree to participate in the above research study conducted by Alison Goss as part of her thesis project, at the Faculty of Education, University of Ottawa under the supervision of Dr. Barbara Graves.

If I have any questions about the study, I may contact Alison Goss or Dr. Barbara Graves.

**BOX** I consent to the use of video-recordings and digital photographs in presentations and publications.

**BOX** I choose to use a pseudonym in this study.

If I have any questions regarding the ethical conduct of this study, I may contact the Office for Ethics in Research, University of Ottawa, Tabaret Hall, 550 Cumberland Street, Room 154, Ottawa, ON K1N 6N5

Tel.: (613) 562-5387

Email: [ethics@uottawa.ca](mailto:ethics@uottawa.ca)

There are two copies of the consent form, one of which is mine to keep.

---

Participant's name

Signature:

Date:

Alison Goss

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Researcher's name

Signature:

Date:

## **Appendix B**

### **Children's Assent Form**

Hello

My name is Alison Goss and I am a student at the University of Ottawa. I'm studying children's mathematical spatial reasoning in a Montessori classroom. I will be in your classroom for about two months. I will be making video-recordings of your teacher giving you math lessons with the materials. I will be audio-recording conversations I have with your teacher. I will observe you and the other students work. I will have conversations with you and some of these conversations will be audio-recorded. I will photograph and video-record some of your activities.

You are invited to be part of this study. If you agree to take part your identity will be protected through the use of a pseudonym. If you do not want to be a part of the study, that is fine. If you join and then change your mind and decide not to be a part of the study any more that is also fine. If you leave the study, I will not use any video-recordings of you and I will not write down or record what you say and do. You will, however, just carry on in the classroom as usual. You can do your usual lessons, join with the others who are in the study, and talk with me at any time.

The video and audio-recordings will be kept in a safe place at my home during the research, and when I am finished the project, they will be stored on Dr. Barbara Graves' and my password protected computers. The results of this study will not impact your schoolwork or appear in any school records.

## **Appendix C**

### **Parents' Consent Form**

**Project title:** Young children's mathematical spatial reasoning in a Montessori classroom.

**Names of researchers and contact information:**

Ms. Alison Goss, PhD candidate, Faculty of Education, University of Ottawa. Dr. Barbara Graves, PhD, Professor, Faculty of Education, University of Ottawa, 562-5800, extension 2225, bgraves@uottawa.ca.

**Invitation to Participate:** My child has been invited to participate in a research project conduction by Alison Goss under the supervision of Dr. Barbara Graves as part of Alison Goss' thesis project in a Doctor of Philosophy in Education at the University of Ottawa.

**Purpose of the Study:** The goal of this research is to understand young children's spatial reasoning in a Montessori classroom.

**Participation:** Alison Goss will be present in my child's classroom for approximately two months in order to make observations of the children working with the mathematical materials. She will make a video-recording of each mathematical lesson given by the teacher to the students which is aimed at supporting the children's developing spatial reasoning. The purpose of this video-recording is to create a visual reference to support Alison Goss' understanding and analysis when the children themselves work with the mathematical material. Alison Goss will audio-record conversations with the teacher in order to gain insights into her pedagogical choices, her

understanding of the children's spatial reasoning, and her observations about their follow-up work. Alison Goss will audio-record conversations with the children as well. The conversations will revolve around the children's choices regarding their independent follow-on work, the mathematical materials they are working with, their reasoning about the concepts that have been introduced, their plans for further work. Alison Goss anticipates having three to four conversations with children in each cycle of work emerging from the initial presentation.

She will also video-record and photograph the children's work and productions as they engage in the follow-on work emerging from the initial mathematical presentation. These productions could include the children's bodily gestures, their drawings, their writings, and their verbal communications. The video-recordings and photographs may be used when disseminating the results at academic conferences and teacher workshops. Alison Goss will keep detailed observations notes of the children working with the mathematical materials following the initial presentation with the materials. The purpose of these observations notes is to record how the children are working, their engagement with the work and with each other, their discussions, and their spatial reasoning.

The activities will be held at my child's school, in their classroom, and with their teacher. Also, the results of the study will not appear in any school records.

**Assessment of risks:** My child's participation in this study entails no foreseeable risks. However, if my child experiences any discomfort, Alison Goss has assured me that she will make

every effort to minimize this discomfort. My child may decide to stop the activities and discussions at any time.

**Benefits:** My child will be involved in the problem-solving activities and the follow-up conversations which will contribute to an enlarged understanding of young children's spatial reasoning in a Montessori classroom.

**Privacy of participants:** My child has received assurance from Alison Goss that the information they share will remain strictly confidential. My child will be identified by a pseudonym in this project. The school will be identified by a pseudonym as well. Video-recordings and digital photographs may be used in professional publications or at professional presentations. In these settings my child's identity will be protected by the use of a pseudonym.

**Confidentiality and conservation of data:** The data will be used for the completion of the thesis project as well as for dissemination in research journals and presentations. My child has been assured that the video and audio-recordings, and photographs will be kept in a secure manner at the researchers' home during the research, and upon completion of the project will be stored on Alison Goss' and Dr. Barbara Graves' password protected computers. The data will be securely safeguarded for a period up to ten years along with the other data collected for the thesis.

**Voluntary Participation:** My child is under no obligation to participate and if they choose not to participate, they can withdraw from the study at any time and refuse to answer any questions, without suffering any negative consequences. If my child chooses to withdraw, all

interviews gathered until the time of withdrawal will be destroyed. Also, should I or my child decline consent to participate in the study, the child will stay in the classroom and work on the mathematics activities. Alison Goss anticipates between 1 and 4 students taking part in each activity. These small groupings give her flexibility in that not every student in the classroom needs to be part of the various activities. Furthermore, groupings are organic and fluid insofar that they do not require a certain number of specific children, and may take place at a time conducive to the children. In addition, if there are children in an activity that have not consented, Alison Goss can limit the video-recording to those children in the group who have given consent.

Should a student wish to withdraw from the study and has been part of an activity that has been video-recorded, Alison Goss will assess each instance of recording to ascertain whether it is possible to use sections of the video-recording with this particular student's involvement erased, or whether the entire video-recording must be ignored for research purposes.

**Acceptance:** I, \_\_\_\_\_, the parent of \_\_\_\_\_ agree that my child participates in the above research study conducted by Alison Goss as part of her thesis project, at the Faculty of Education, University of Ottawa under the supervision of Dr. Barbara Graves

If my child and I have any questions about the study, we may contact Alison Goss and Dr. Barbara Graves. If we have any questions regarding the ethical conduct of this study, we may contact the Office for Ethics in Research, University of Ottawa, Tabaret Hall, 550 Cumberland Street, Room 154, Ottawa, ON K1N 6N5, telephone 613-562-5387, email: [ethics@uottawa.ca](mailto:ethics@uottawa.ca)

There are two copies of the consent form, one of which is mine to keep.

---

Participant's name:

Signature:

Date:

Alison Goss

---

Researcher's name:

Signature:

Date:

## **Appendix D**

### **Criteria for Selection of Video-recorded Presentations**

1. That the goal of the presentation is to create an activity with mathematical materials that will result in the children reasoning spatially, however articulated by the teacher.
2. That the video shows a clear example of the teacher using mathematical materials to support spatial reasoning, including her communications with the children.
3. That the video creates a valuable illustrative reference clearly showing the start and finish of the presentation.
4. That the video-recorded presentation fits into the sequence of the didactical cycle, that is, the children go on to engage in follow-up work.
5. That the quality of the video is good insofar that the images are clear and the conversations can be heard.

## **Appendix E**

### **Possible Questions for Generating Discussions with the Teacher**

#### **Prior to presentation**

1. From your observations, what lessons in geometry are you planning for the children?
2. What criteria will you use to select children for the lesson?
3. What resources will you be using for your next geometry lesson? What mathematical materials? And what concept are you hoping the children will be able to explore in this lesson?
4. What follow-up work may arise from this lesson?
- 5.

#### **After presentation**

1. How did you feel the lesson went? How were your goals met?
2. What did you think about the children's communications regarding the concept you introduced?
3. Were the children engaged? How do you know?
4. Do you think they will engage in follow-up work? What makes you think so?

#### **After children have engaged in follow-up work (or not)**

1. What strikes you about the children's follow-up work?
2. The children did not engage in follow-up work. Why do you think this is the case?
3. In what ways was this work challenging for the children?
4. How will you assess what they have communicated?
5. What resources did the children use that you identified?

## **Appendix F**

### **Possible Questions to Stimulate Discussions with the Children**

#### **After presentation**

1. What did you enjoy about the lesson you have just had?
2. What did you know about the topic before you had the presentation?
3. When have you used those materials before?
4. What was important about this lesson?
5. How do you plan to do follow-up work? What ideas do you have?

#### **After or during follow-up work**

1. What made you choose this material, these resources, for your follow-up work?
2. How did you choose who to work with?
3. What have you been discussing?
4. What have you understood working with this material?
5. Can you tell me about your drawings/writing, etc.?

#### **Reengagement with teacher**

1. How did you enjoy the lesson and work with this material?
2. Have you worked on it for as long as you wanted, or do you have other ideas of things you would like to do?
3. What is your thinking about this work now compared with the beginning of the presentation?

4. Have you worked with all the people you wanted to work with?

## Appendix G

### Observation Criteria for Didactical Cycle and Space of Learning

*Initial presentation by teacher* (This is video-recorded for illustrative purposes)

What is the name of the initial geometry presentation? What spatial concept does the teacher hope to introduce with the help of the materials? How does the teacher invite the children to the lesson? What is their response to the invitation? How does the teacher use the mathematical material? What forms of spatial reasoning (if any) are communicated during the presentation by the teacher? How did she communicate them? How did the children respond to the lesson? What traces are generated that could be considered illustrative of spatial reasoning? How did they communicate their thinking? How was this demonstrated? What forms were shown? Were the children enthusiastic about follow-on work? What ideas did they come up with? How did the teacher respond?

*Children's individual/collective activity with materials following initial presentation*

How is spatial follow-up work initiated? By teacher, by child individually, by group of children? How does the child engage other children? How long after the initial presentation does follow-up spatial work begin? What resources do the children use for this follow-up work? Are children afforded as much time as they like to work? Are they protected from interruptions by others? What are the spatial semiotic traces produced by the children during this work? What spatial reasoning do these semiotic traces convey and how? What ends the children's work?

*Children's individual/collective reengagement with teacher following independent work*

Much the same questions from above will be asked again, but what is of interest is the teacher discussion has, or I subsequently have, with the children around their communications of spatial reasoning, in other words, around what they have produced.

## Appendix H

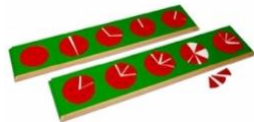
Chart Of Children's Work with Mathematical Materials

| <i>April 9</i> | <i>Children</i>         | <i>Activity</i>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | <i>Material</i>                                                                                         | <i>Notes and observations</i>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|----------------|-------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Lesson         | Inez and visiting child | <p><i>Introduction to polygons</i> by “building figures with straight lines.” Introduction to, and etymology of terminology, e.g. <i>figure, interior surface, exterior surface, sides, quadrilateral, polygon</i>. Beginning classification of polygons. Introduction of nomenclature cards on polygons.</p> <p><i>Follow-up</i><br/>Repeat lesson by building polygons with the material and label them. Find polygons in the Montessori materials, in the classroom, in books, at home, etc. Collect and classify polygons, draw polygons and make a booklet, copy nomenclature booklet.</p> | <p>Box of Sticks</p>  | <p>After the lesson Linda says “I’ll leave you to make lots and lots of pieces together. See what is the biggest polygon you can make” Linda points to a closed figure and says, “This is a polygon”, then she opens the figure and says, “This isn’t a polygon.” Linda directs Inez to work with the visiting child for the morning. Later the visiting child shows the polygons she has made to Linda who redirects her to Inez. Child suggests making a polygon “as long as the room.” Inez says, “Let’s make a rainbow one.”</p> |
| Lesson         | Anisha, Maria,          | <p><i>Position of two lines in the same plane</i>. Purpose is to provide “sensorial impressions” and initial definitions for parallel, divergent, and convergent lines. Introduction of etymology and terminology, e.g. <i>co-planar lines, parallel, divergent, convergent</i>. Lesson is set in a narrative of two children and their emotions as they come</p>                                                                                                                                                                                                                               | Box of Sticks, paper figures                                                                            | <p>Maria was working on her polygons, assembling sticks into a long line when Linda wanted the Box of Sticks material for this lesson. Maria ends up having this work and her own work in front of her and she works on both. Anisha comes and joins her. Various other students come and work with the material over a couple of days.</p>                                                                                                                                                                                          |

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|         |               | <p>together, walk side by side, or move away from each other.</p> <p><i>Follow-up</i><br/>Repeat the lesson, and tell story to themselves or to others, label different pairs of lines, make a booklet.</p>                                                                                                                                                                                                                                                                                                   |                                |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| Lesson  | Shan and Nina | <p><i>Review of operations with fractions.</i> This lesson is in preparation for a lesson on decimal fractions with the fraction skittles. The lesson does not get beyond the review of operations of fractions and a review of terminology for fractions, e.g. <i>numerator</i>, <i>improper fractions</i>. Linda uses the box of fraction pieces.</p> <p><i>Follow-up</i><br/>Linda leaves Shan and Nina on their own to continue working with operations. She comes and checks on them every so often.</p> | <p>Box of fraction pieces</p>  | <p>Linda tells the students that this is a whole review to make sure they know their operations. She invites the two students to write down their own questions. Nina struggles and calls herself “bad.” She goes to use the fraction material but Linda says she does not need it. Nina says, “I never thought I would choose such a hard question!”</p> <p>Another day, Nina comes back to her fraction work, but although she has the material out, she does not use it.</p> |
| April 9 |               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |

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| Activity | Eric and Evan                                   | <p><i>Long number chains.</i><br/>The children take the chains off their hooks and lay them on the ground on a mat. They count off the multiples and label each multiple with a card printed with the number of the multiple. In this activity they are counting out and labelling the long chain of nine.</p> <p><i>Follow-up work</i><br/>Linda is not sure what these activities are following-up. The chains are used for a number of activities, including making squares of numbers and cubes of numbers. There are short chains which when folded make the square of a number and there are long chains which when folded and stacked make the cube of a number</p> |                                        | <p>They use the Large Bead Frame to count as well. Linda finds this unusual. Evan counts off the multiples on the chain but at each multiple he confirms his number by counting the same amount on the Large Bead Frame. They work from 8.45 a.m. until 10.20 a.m. with a 20-minute break for French. Linda is concerned this is busy work. The Bead Cabinet is found in the classroom for the 3 to 6-year-old children and there would have been plenty of opportunity for activities with both the short and long chains. Evan is new to the classroom and to Montessori so would not have had that opportunity.</p> |
| Lesson   | Luca, Jennifer, Evan, Maria, and visiting child | <p><i>Introduction to the Geometric Hierarchy of Numbers.</i> This material is presented to the children as a geometric form of the hierarchy of number for 1, 10, 100, 1,000, 10,000, 100,000 and 1,000 000. The precise design of the material makes it possible to compare by size the decimal categories, and to visualize through the colour-coded system the concept of decimal value and the families of number<sup>14</sup></p>                                                                                                                                                                                                                                    | <p>Wooden Hierarchical Material</p>  | <p>Children laugh and struggle with the heavy material. It requires three children to carry the unit for the million family – the largest piece of material. One child says, “So heavy! I don’t think I can manage!” Linda tells them the names of each item, such as unit of the simple family (1), or the ten of the thousand family (10,000). Using the material, she draws comparisons.</p>                                                                                                                                                                                                                        |

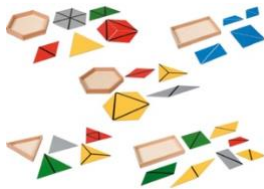
<sup>14</sup> [www.gonzagarredi.com](http://www.gonzagarredi.com)

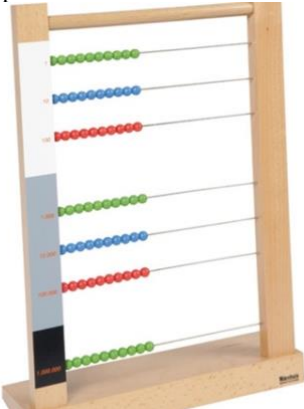
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|          |                     | <p><i>Follow-up work</i></p> <p>Repeat the presentation, compare sizes and families of numbers. Repeat lesson independently of teacher. Use metre stick to measure and imagine what the tens and the hundreds of the million family would look like. Linda presents the material without the notational cards which are the numbers cards for labelling each unit, ten, or hundred, in the decimal system up to 1,000,000.</p> |                                                                                                                                                   |                                                                                                                                                                                                                                                                                         |
| Activity | Maria, Nina, Anisha | <p><i>Follow-up of lesson on polygons and position of two lines.</i> Children are working on both at the same time. Follow-up from previous day's lesson. Building polygons with Box of Sticks material, repeating lesson of the position of two lines, reading nomenclature book on polygons, writing down name of booklet in journal.</p>                                                                                    | <p>Box of Sticks, Figures for story, nomenclature booklets</p>  | <p>Maria works with the characters and tells the story of them coming together and moving apart. She picks up a label that says divergent and reads it. Later Anisha comments the line segment Maria is making is longer than her leg. She measures it against her leg. Nina says, "We are going to name this masterpiece!" Linda says, "How many sides does it have?"</p> |
| Activity | Nina                | <p><i>Review of equivalence</i></p> <p>Nina is drawing circles and other shapes on construction paper.</p>                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                   | <p>Nina says she is doing a review of equivalence. Linda comes to sit with Nina to see what she is doing.</p>                                                                                                                                                                                                                                                              |
| Activity | Evan                | <p><i>Follow-up working with fraction trays to find equivalence fractions.</i></p> <p>Evan removes fraction pieces from the tray in order to find equivalent pieces to fill the spot.</p>                                                                                                                                                                                                                                      | <p>Fraction insets</p>                                        | <p>Linda directs Evan to find some work and has 10 seconds to do so before she does it for him. Evan takes out the two fraction trays.</p>                                                                                                                                                                                                                                 |
| Activity | Eric and Toby,      | <p><i>Follow-up activities</i></p>                                                                                                                                                                                                                                                                                                                                                                                             | <p>Long number chains</p>                                                                                                                         | <p>The children have positioned themselves too near to the doorway. There is a</p>                                                                                                                                                                                                                                                                                         |

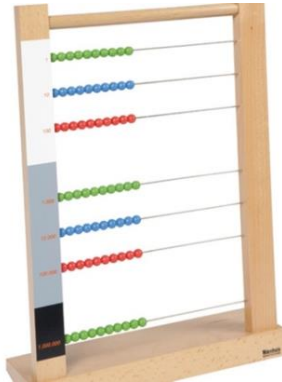
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|                 | then Eric with Nicholas and Evan | Working with the number chains (7 chain, then 9 chain, later the 10 chain)                                                                                                                                                                                                                                  |                                                                                                                                                       | discussion with others as to where a good place for their mat would be.                                                                                                        |
|                 |                                  |                                                                                                                                                                                                                                                                                                             |                                                                                                                                                       |                                                                                             |
| <b>April 11</b> |                                  |                                                                                                                                                                                                                                                                                                             |                                                                                                                                                       |                                                                                                                                                                                |
| Activity        | Nicholas with Eric               | <p><i>Follow-up activity with the Large Bead Frame</i></p> <p>Large Bead Frame is a material for operations in the decimal system. The colours of the beads correspond to the colours of other Montessori material for the decimal system, namely, green for units, blue for tens and red for hundreds.</p> | <p>Large Bead Frame used in conjunction with Notational Paper</p>  |                                                                                            |
| Lesson          | Shan and Nina                    | <b>Expanded notation: Powers and decomposing large numbers</b>                                                                                                                                                                                                                                              | White board and math sheet                                                                                                                            | Linda writes numbers on the whiteboard and Shan and Nina write the amounts in expanded notation. They do addition and subtraction. Linda leaves them on their own to practice. |

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| Activity | Mario,<br>Evan, Nina, Eric<br>Toby | Follow-up of the lesson on the introduction to polygons                                                                                                                                                                                                                                                                                                                                                                                                                      | Box of Sticks<br>             | Building the biggest polygon possible using all the sticks in the box. They stretch it out and Eric says, “it’s a forty-one-a-gon.” “agon” is Greek suffix for “angle.” Mario says he is trying to make a circle.                                                   |
| Activity | Evan and<br>Eric                   | Follow up-counting long chains (7 chain)                                                                                                                                                                                                                                                                                                                                                                                                                                     | Chain material                                                                                                  |                                                                                                                                                                                                                                                                     |
| Activity | Shan                               | Follow-up: Arithmetic trinomial material<br>This cube represents the equation $(a + b + c)^3$ in concrete form ... the component pieces are color-coded to represent hierarchical values from 1 unit to 1 million. This makes it possible for the child to equate an arithmetical value to each of the cubes and prisms <sup>15</sup><br><br>In this activity, Shan decomposes the box of arithmetic trinomial material, sorts, then labels each piece. He then puts it back | Arithmetic trinomial cube<br> | I am not sure what this is follow-up to, but Shan appears to want to show me that he can do it. This is an intricate piece of material and Shan works with it proficiently.<br> |
| Lesson   | Luca and<br>Matthew                | Addition with the Stamp Game<br><br>This material is found in the younger classes as well. It is an early material for operations.                                                                                                                                                                                                                                                                                                                                           | Stamp Game<br>              | Luca is new to the school and English is not his first language. He appears excited but a little overwhelmed by the classroom.                                                                                                                                      |

<sup>15</sup> [www.gonzgarredi.com](http://www.gonzgarredi.com)

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| <b>April 12</b> |                   |                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                             |                                                                                                                                                                                            |
| Activity        | Nicholas and Eric | <p><b>Follow-up activity: Large Bead Frame</b></p> <p>Nicholas and Eric continue to work on the Large Bead Frame, adding to their growing list of numbers on the Notational Paper.</p>                                                                                                                                                                                                            | Large Bead Frame and Notational Paper                                                                                       |                                                                                                         |
| Activity        | Evan              | <p><b>Counting long chains (7 chain)</b></p> <p>Evan is working on the long chains again.</p>                                                                                                                                                                                                                                                                                                     | Chains from Bead Cabinet                                                                                                    |                                                                                                                                                                                            |
| Activity        | Shan              | <p><b>Follow-up activity: Exercises in congruence, equivalence and similarity.</b></p> <p>The purpose of the Constructive Triangle Boxes is to give the child practical experience in plane geometry. The set consists of 2 rectangular boxes, 1 triangular box, 1 small hexagonal box, and 1 large hexagonal box, each containing triangles of differing size, shape, and colour<sup>2</sup></p> | <p>Constructive Triangle Boxes (3)</p>  | <p>Shan comes and sits beside me again to show me his work with the Constructive Triangle boxes.</p>  |

|          |                   |                                                                                                                                                                                                                                  |                                                                                                                                     |                                                                                                                                                                                                   |
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| Activity | Gail and Jennifer | Follow-up activity: Subtraction using the Large Bead Frame and a subtraction booklet.                                                                                                                                            | Blank subtraction booklet and Large Bead Frame                                                                                      |                                                                                                                                                                                                   |
| Lesson   | Nicholas and Evan | Lesson: "Making Fifteen-Tens." Exploring place value.                                                                                                                                                                            | Large Bead Frame and Notational Paper<br>         |                                                                                                                                                                                                   |
| Lesson   | Gail and Matthew  | Lesson: Introduction to the Commutative Law. This law as well as the Distributive Law is presented to children in a sensory way at a young age. Ideally, they are comfortable with numbers 0 – 9, and know their addition facts. | Box of Bead Bars, and teacher made material.<br> |                                                                                                               |
| Lesson   | Anisha and Mario  | Lesson: "Making Fifteen-Tens." Exploring place value<br>This is the same lesson Linda gave Nicholas and Evan                                                                                                                     | Large Bead Frame and Notational Paper                                                                                               |                                                                                                                                                                                                   |
| April 15 |                   |                                                                                                                                                                                                                                  |                                                                                                                                     |                                                                                                                                                                                                   |
| Activity | Nicholas          | Follow-up: Counting on the Large Bead Frame with Notational Paper                                                                                                                                                                | Large Bead Frame, Notational Paper                                                                                                  | Nicholas started counting from 1 and he is now at 433. He is working on his own. He chose this work about an hour after he came to class. Later he tells Maria and Gail that he is almost at 500. |

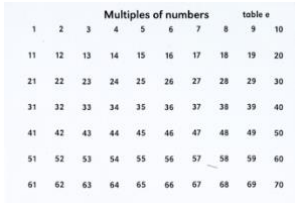
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| Activity   | Nicholas and Evan | Follow-up of telling the time. The two boys are reading the time on the clock                                                                                                  | Large analogue clock on the wall                                                                                          | Nicholas tells Evan he used to say, “10, 20, 30, 35” when he was telling the time but now he has “got it.”                                                                                                                                                                                                               |
| Lesson     | Gail and Jennifer | Lesson: Reading large numbers on the Large Bead Frame, writing notation                                                                                                        | Large Bead Frame, Notational Paper<br> | Linda gives the lesson to the two girls then leaves them to work on their own. They stop their activity for a bit then come back to it. Gail becomes a little confused as to which side to put the beads that have been counted.<br> |
| Key Lesson | Whole class       | Lesson: “How Geometry Got its Name”<br>This is a key lesson for the whole class. It is considered key because it is set in a narrative which introduces new ideas and offers a | Rope, atlas, books.                                                                                                       | Linda tells the story with the goal of getting the children to think about the challenges for farmers along the Nile River which flooded annually. She asks them to imagine the disagreements that could arise with farmers disputing whose land was                                                                     |

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|                 |                        | <p>variety of follow-up work that is not contained within one subject area. The story is about the flooding of the Nile River during the period of the ancient Egyptians and how land surveyors called “harpedonaptai, which translates into “rope stretchers” used ropes to create right-angled triangles to measure the land and assign it back to the owners. There is debate about whether these <i>harpedonaptai</i> were familiar with the Pythagorean theorem.</p> <p><b>Follow-up activities:</b> Make triangles, find out more about the Egyptians, repeat the lesson, examine life around the Nile River.</p> |  | <p>whose, and what they did to solve the problems. The children note that the right-angled scalene triangles create a rectangle when one is flipped and their hypotenuse are aligned.</p>                                                       |
| Lesson          | Nina, Shan, and Anisha | <p><b>Lesson: Investigating the sum of angles in triangles.</b> This lesson arose immediately after Linda’s lesson on How Geometry Got its Name. The focus of the lesson was to give the children opportunities to explore the notion that the sum of angles in triangles is <math>180^\circ</math>.</p>                                                                                                                                                                                                                                                                                                                | Metal fraction pieces, construction paper, pencils, glue                           | <p>This is the activity that Anisha discussed with me on May 9 when I asked her about it. Shan pays a great deal of attention to lining his angles up.</p>  |
| <b>April 16</b> |                        |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                                    |                                                                                                                                                                                                                                                 |
| Activity        | Evan and Eric          | <b>Follow-up: Telling time</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Analogue clock on wall                                                             | <p>Evan has a large-faced wrist watch on his arm. Linda told me that children were allowed wrist watches as long as they were analogue watches. Evan compares the time on his wrist watch with the time of the clock on the wall.</p>           |

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| Activity                    | Eric             | Evan and<br><b>Counting long chains (e, 8, and 4 chains)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | Long chains, and Large Bead frame                                                                       | When there is an exchange in place value the children have been taught to hold the sound of the name of the number until the change is made, so I am hearing Evan say “s-i-i-i-x” as he changes beads on the Large Bead Frame. According to Linda the children do not need the Large Bead Frame for this work but are obviously enjoying it as they use the Frame in conjunction with the chains every time.                                                                                                                                                                                                                                                                                        |
| Activity                    | Inez             | <b>Follow-up – Unknown math work at this time</b><br>Inez invites me to come and see the mathematics work she is doing. I assume it is a powers of numbers chart. I find out later that it is not. It is number charts from other cultures. She trims her chart.                                                                                                                                                                                                                                                                              | Chart paper                                                                                             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| Activity and later a Lesson | Gail and Matthew | <b>Follow-up activity: The commutative law.</b> Linda speaks to Gail and Matthew and reminds them of this lesson they recently had. She says that she is going to check on it. They rush to get the Box of Bead Bars. This work is challenging for them and they move on to using the bead bars for addition and start writing in an addition booklet.<br><br>Linda joins them and gives them a <b>lesson on the Distributive Law.</b> Gail and Matthew do not appear very interested in the lesson and shortly after the start of the lesson | Box of Bead Bars<br> | Gail and Matthew set up the material for doing activities using the Commutative Law. They are not comfortable with it and are experiencing some challenges. Gail spends considerable time directing Matthew. Matthew is finding it hot sitting in the sun and complains. Gail says she will swop with him but then he has to do the work.<br>Linda comes to watch them. Matthew is challenged to conserve numbers and must count every bead on every bead bar.<br>He plays with the paper brackets, making them “walk.” Gail says Matthew is being silly and Matthew says Gail is being bossy. He also says the work is boring. Gail suggests other work (the Tone Bars). They pack this work away. |

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|          |                  | Gail is interrupted to leave for French. She comes back and she and Matthew are not engaged in the work.                                                                                                                                               |                                                                                                                                                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| Activity | Eric and Evan    | The boys bring out another long chain (5, 3), later the 4 chain.                                                                                                                                                                                       | Long chains                                                                                                                                     |                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| Activity | Anisha           | <p>Follow-up from lesson on introduction to the concept of an angle.</p> <p>Anisha draws concentric circles using sticks from the Box of Sticks.</p>                                                                                                   | <p>Box of Sticks, chart paper, coloured pencil crayons.</p>  | <p>Anisha draws concentric circles, similar to the work done by Gail and Matthew. I was drawn to the noise of her hammering, but at the time did not pay attention to what she was doing until Linda later drew her work to my attention.</p>                                                                                                                                                                                                           |
| Activity | Gail and Matthew | <p>Follow-up from lesson on angles: Drawing concentric circles.</p> <p>This work may have arisen out of the lesson Linda gave in the fall on the concept of an angle, and it may also have arisen out of Anisha's work drawing concentric circles.</p> | Box of Sticks, chart paper                                                                                                                      | <p>Gail and Matthew appear to have been inspired by Anisha's work on concentric circles, but when I ask Linda about it she felt that Gail and Matthew had been drawing circles long before Anisha. In my observation notes I make the comment that they "take up the same work that [Anisha] has been doing with the circle on paper with the hammer and nails."</p> <p>Gail spends time carefully selecting the stick that will create the largest</p> |

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|          |       |                                                            |                                   | <p>circumference while still fitting on the paper. She tries out a few sticks before she is happy with her choice. In the Box of Sticks there is a compartment containing the different lengths of sticks but are not coloured. Gail chooses to use these.</p> <p>When she is finished Gail cuts out her circle and puts it in her work folder.</p> <p>Gail chooses the colours and does the colouring-in. She directs Matthew to get the colours that she chooses.</p> |
| Activity | Luca  | Follow-up on telling time: Finding the hours of the clock. | <p>Wooden manipulative clock</p>  | <p>Luca uses a stamp pad and a stamp depicting a clock face without hands. He turns the hands on the clock, then stamps the clock face onto paper, then fills in the big and second clock hands. This is the first work that I have seen that Luca has chosen on his own. He sings as he works.</p> <p>He counts on his fingers.</p> <p>When he is finished, he tidies up his work and puts it away.</p>                                                                |
| Activity | Maria | Follow-up: Addition on the Large Bead Frame                | Large Bead Frame, addition booklets                                                                                 | <p>Although Maria has the Large Bead Frame out, she is not using it to count. Rather she is using her fingers. She collects her sheets together and is carefully trimming them. She rolls up her work into a tight roll. She ties an elastic band around it and takes it outside the classroom.</p>                                                                                                                                                                     |

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| Activity | Anisha               | Follow-up from Types of Angles lesson – filling in angles on her drawing of concentric circles                                                     |                     | Linda said Anisha came to ask her where the angles were in her concentric circles. Linda did not know what she was referring to but eventually understood Anisha to be thinking about the angles lesson they had had in the fall. |
| Lesson   | Nicholas, Evan, Toby | Review: Investigating multiples of numbers. The review is in preparation of the lesson Finding the Lowest Common Multiple.                         | Multiples paper<br> | They circle multiples of seven then circle multiples of one, circling every number.                                                                                                                                               |
| Lesson   | Evan, Toby, Nicholas | Finding the Lowest Common Multiple.<br><br>Linda gives the lesson on mat on the floor, showing the children how to find the Lowest Common Multiple | Pegs and Peg Board                                                                                    |                                                                                                                                                                                                                                   |

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| <b>April 18</b> |                  |                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                       |                                                                                                                                                                                |
| Activity        | Luca             | Follow-up activity: Telling the time                                                                                                                                                                                                                                                                                                                                                                                                           | Wooden clock material                                                                                                                                                                                                                                                                                                 | Luca is writing the date and time in his journal. He goes up to the wall to read the clock. Other children walk to and from the clock checking the time for their journals.    |
| Activity        | Eric and Evan    | <p>Follow-up activity: Assembling the Paper Decanomial.</p> <p>This work is a follow-up from work with the sensorial decanomial material. The Montessori material consists of squares and rectangles colour-coded to match the bead bars and to match the long and short number chains. The decanomial is based on the multiplication table known as the Table of Pythagoras. The material allows children to explore patterns in numbers.</p> | <p>Paper Decanomial</p> <p>The paper decanomial consists of paper teacher-made material. It is not colour-coded in the same way, and rather has yellow paper squares for the squares of numbers, and white paper rectangles.</p>  | <p>The children are sorting the pieces for the paper decanomial which they will then set up horizontally and vertically. The pieces are thin and small and easy to mislay.</p> |
| Activity        | Luca and Matthew | Box of Sticks. Linda suggests to Matthew that he get out the Box of                                                                                                                                                                                                                                                                                                                                                                            | Box of Sticks.                                                                                                                                                                                                                                                                                                        | Matthew gets out the Box of Sticks, sets out a mat near the fish tank, folding over one                                                                                        |

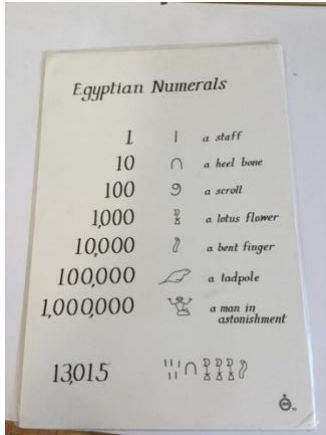
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|          |       | Sticks and give Luca a lesson because “Luca doesn’t know how to do it.”                                                                                                                                                                                                                                                                                                                                                                                                             |                              | side. He brings out the pinboard. Then he packs everything away without giving Luca a lesson of any kind. |
| Activity | Shan  | <b>Follow-up work:</b> I take a photo of Shan’s work on earthquakes. His drawing features convergent, and divergent lines.                                                                                                                                                                                                                                                                                                                                                          |                              |                                                                                                           |
| Lesson   | Mario | <b>Using the 24-hour clock.</b> Linda gives this lesson to Mario on his own. Linda acknowledges to Mario that he knows how to tell the time. She tells him that “midi” from “midday” is French for “middle.” She says they use the 24-hour clock in France, and you would not get confused about whether it was a.m. or p.m. if you use the 24-hour clock. Linda says he can use the 24-hour clock in his journal if he likes. Mario said he was very keen to learn 24 -hour clock. | <p>Wooden clock material</p> | There is a piece missing from the clock which bothers Linda and she talks to me about it.                 |

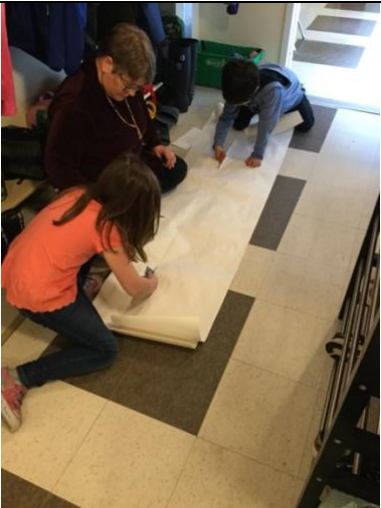
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| Activity   | Evan                        | Counting with the long chains.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | Long chains                                                                                                                                 | Linda asks Evan if knows his multiples. She asks him what a common multiple of 4 and 6 is. He says he does not know what a common multiple is. Linda reminds him of the lesson he had yesterday. Linda asks him about 3 and 6.                                                                                                                                                                                                                                                                                                                                                                    |
| April 23   |                             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| Key Lesson | Whole class                 | <p><b>Story of the Babylonians</b> – linking angles to degrees, and measuring angles. Also links clock, and the year. Linda asks “Why were angles important to them? Because they had to build buildings, roads, steps, because they could measure and be accurate.”</p> <p><b>Notation of a degree.</b> Linda suggests the children can trace the angles and she shows them how when it has been traced it can be coloured in and the amount can be written in. Linda says the “bubble” sign tells you it is an angle.</p> <p><b>Follow-up work:</b> Using the atlas, reading the book on the Babylonians, trace and colour fractions, constellations.</p> <p>Declan and Maria trace fraction pieces, Nicholas and Eric draw constellations</p> | <p>Red metal fraction pieces, Montessori protractor</p>  | <p>Linda puts half a fraction circle into the protractor and asks what kind of angle it is. Declan says, “When there was a whole, I was going to say it had to be a whole, but I thought we had to say 360.”</p> <p>The children count the degrees from zero to 180 quietly to themselves. Linda checks and everyone gets 180. She then inserts a fraction piece that creates a right angle.</p> <p>Linda says the right angle is a very special angle. Linda asks about an acute angle. Toby says an acute angle is half a right angle. Linda is not sure. Mario measures and says it is so.</p> |
| Activity   | Anisha, Maria, Declan, Nina | Follow-up of Story of the Babylonians lesson: Tracing,                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Montessori protractor, metal fraction pieces                                                                                                | Maria and Declan work together. Maria arranges the trays to suit her. She chooses a                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |

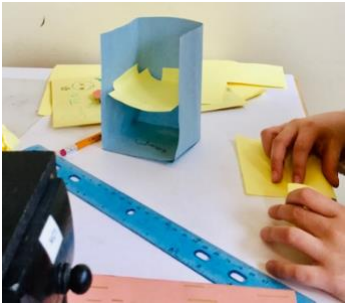
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|          |                   | colouring, measuring fractions in degrees                                                       |                                                          | <p>half circle, measures it in the Montessori protractor then takes it over to her paper and positions it horizontally. She traces the semi-circle. Declan chooses <math>\frac{1}{3}</math> piece and puts it into the protractor to measure it. Maria measures again and says, "One eighty hundreds." She starts colouring in the angle measure. She says to Declan that he is not allowed to change colours. Maria chooses another piece and says it is "eighty." Declan says "This is so odd – this one is 35!" Anisha comes over to confirm his measuring. Declan taking another piece, says, "This is not 80, it is 82, 83, 84, 85, oh my god, it is 88, this one is 89!" Maria and Declan continue, comparing and confirming sizes.</p> <p>Declan says, "Every clock is different. This is the real time at home. Every clock is different."</p> |
| Activity | Nicholas and Eric | Follow-up of Story of the Babylonians lesson: Drawing constellations.                           | <p>Book of constellations, Montessori square paper</p>  | <p>The two boys are copying constellations out of a book of constellations. Nicholas says he has drawn the Big Dipper. They discuss what other constellations they should draw.</p> <p>Nicholas shows his pages to Linda. He is going to make a booklet. Eric says "What about your cover?" and Nicholas answers he does not need one. He makes a cover after all.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| Activity | Nina              | <p>Follow-up: Operations with fractions.</p> <p>Nina is multiplying and dividing fractions.</p> | Cut-out fraction pieces                                                                                                                    | <p>Linda invites me to listen to Nina do her fraction work. Nina says the work "was heard, very hard." She says she will try to do some more on her own before she goes to</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |

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|          |                                   |                                                                                                                                                                                                                                                                                                                                       |           | “Miss Linda for some help.” She does not use the fraction pieces.                                                                                                                                     |
| Activity | Luca                              | Follow-up: Working with the stamp game                                                                                                                                                                                                                                                                                                | Stamp game.                                                                                 | Luca uses his fingers to count.                                                                                                                                                                       |
| Lesson   | Nicholas, Declan, Evan, and Maria | Introduction to the Pentatonic Scale and Diatonic Scale. Linda asks the children to remember their work with pentagons and asks what they think “pentatonic” means. In the lesson, the children have to choose between higher and lower notes. Linda asks Nicholas to play a scale. Linda talks a little about the history of scales. | Bells<br> | Linda asks the children some questions about polygons, such as, what a four-sided polygon is called. Mario answers that it is a quadrilateral. Linda talks about “tetra” as also meaning 4.           |
| Activity | Eric                              | Assembling the paper decanomial                                                                                                                                                                                                                                                                                                       | Paper decanomial                                                                            |                                                                                                                                                                                                       |
| April 25 |                                   |                                                                                                                                                                                                                                                                                                                                       |                                                                                             |                                                                                                                                                                                                       |
| Activity | Matthew                           | Follow-up to lesson on time: Telling the time                                                                                                                                                                                                                                                                                         | Large analogue clock                                                                        |                                                                                                                                                                                                       |
| Activity | Nicholas                          | Follow-up to review on multiples: Finding multiples                                                                                                                                                                                                                                                                                   | Multiple paper                                                                              |                                                                                                                                                                                                       |
| Activity | Nicholas, Evan                    | Follow-up to the Story of Numbers (Communication in signs. Investigating numbers in different cultures                                                                                                                                                                                                                                | Number charts for Egyptians, Romans.                                                        | Evan says to Nicholas, “You can do Egyptian numbers and I will do Greek numbers.” Evan directs Nicholas, “You keep going with ones until you get to tens then you do a triangle.” Nicholas says, “one |

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|                 |                        |                                                                                                                                                                                                                                              |                                                   | triangle, two triangle, three triangle.” Evan says, “Soon you will know how to do it, [Nicholas].” Later Evan says, “I’m sorry [Nicholas], but you are not doing it right. It’s just a part of life, I mean school. I’m at 235, no 234.” Evan says, “I might have to get something.” He gets out the Greek number chart and attaches it to the wall in front of them. |
| Activity        | Nicolas                | Follow-up work of the decanomial<br><br>I meet with Nicholas to discuss with him his creative work on the decanomial. He has drawn and labelled various bead bars.                                                                           | Nicholas’s decanomial work                        |                                                                                                                                                                                                                                                                                                                                                                       |
| <b>April 26</b> |                        |                                                                                                                                                                                                                                              |                                                   |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Evan, Toby, and Declan | Investigating numbers in different cultures: Writing Greek numbers                                                                                                                                                                           | Greek number chart                                |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Luca                   | Counting the long number chains (5 chain)                                                                                                                                                                                                    | Long number chains                                |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Luca and Matthew       | Counting the long number chains (4 chain)                                                                                                                                                                                                    | Long number chains                                |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Evan and Eric          | Counting the long number chains (9 chain)                                                                                                                                                                                                    | Long number chains                                |                                                                                                                                                                                                                                                                                                                                                                       |
| Lesson          | Luca, Matthew, Gail    | Presenting notation with the Wooden Hierarchical Material.<br>This is a lesson, but also a follow-up lesson to the earlier one where the material was presented. Here the children are presented the notation to connect symbol to quantity. | Notational cards and Wooden Hierarchical Material |                                                                                                                                                                                                                                                                                                                                                                       |
| <b>April 29</b> |                        |                                                                                                                                                                                                                                              |                                                   |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Evan, Nicholas         | Writing Greek numbers                                                                                                                                                                                                                        | Greek number chart                                |                                                                                                                                                                                                                                                                                                                                                                       |
| Activity        | Luca                   | Counting the long number chains (5 chain)                                                                                                                                                                                                    | Long number chains                                |                                                                                                                                                                                                                                                                                                                                                                       |

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| Activity        | Eric           | Counting the long number chains                                                | Long number chains                                                                 |                                      |
| <b>April 30</b> |                |                                                                                |                                                                                    |                                      |
| Activity        | Eric and Evan  | Writing down Egyptian numbers                                                  | Egyptian numbers chart                                                             |                                      |
|                 |                |                                                                                |  |                                      |
| Activity        | Luca           | Counting long number chains (5 and 3)                                          | Long number chains                                                                 |                                      |
| Activity        | Shan           | Writing down Roman numbers                                                     | Roman numbers chart                                                                |                                      |
| <b>May 1</b>    |                |                                                                                |                                                                                    |                                      |
| Activity        | Nicholas       | Multiples of numbers (9 and 10)                                                | Multiples of numbers paper                                                         |                                      |
| <b>May 2</b>    |                |                                                                                |                                                                                    |                                      |
| Activity        | Declan, Anisha | Measuring angles                                                               | Metal fraction insets                                                              | Follow up from lesson on Babylonians |
| Activity        | Evan and Mario | Writing Egyptian numbers                                                       | Egyptian numbers chart                                                             |                                      |
| Activity        | Mario          | Measuring with tape measure. He compares feet to centimetres and measures Luca | Tape measure                                                                       |                                      |
| Activity        | Gail           | Follow-up from lesson on personal time lines. Draws a personal time line       | Paper                                                                              |                                      |

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| Activity     | Gail and Matthew | Follow-up: Counting with the Large Bead Frame                        | Large Bead Frame and notational Paper |                                                                                     |
| Activity     | Maria            | Adding                                                               | Blank addition booklet                |                                                                                     |
| <b>May 6</b> |                  |                                                                      |                                       |                                                                                     |
| Activity     | Evan and Matthew | Counting the long number chains (9 chain)                            | Long number chains                    |                                                                                     |
| Activity     | Luca             | Counting the long number chain (5 chain)                             | Long number chains                    |                                                                                     |
| Activity     | Gail             | Writing numbers on Notational Paper                                  |                                       |                                                                                     |
| <b>May 9</b> |                  |                                                                      |                                       |                                                                                     |
| Activity     | Gail             | Telling the time                                                     | Analogue clock on wall                |                                                                                     |
| Activity     | Anisha           | Follow-up on lesson on angles: Discussing different types of angles. | Her drawing of concentric circles     | I have a discussion with Anisha about her work with concentric circles and angles.  |

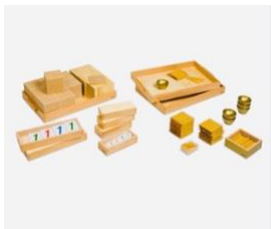
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| Activity | Luca                               | Counting the long chains (4 chain)                                                                                             | Long number chains                                                                                 | Luca sets up the chain in a circle                                                          |
| Activity | Inez and Jennifer                  | Creating a three-dimensional paper figure                                                                                      | Paper<br>        | The two girls are creating a box with drawers copying another box.                          |
| Lesson   | Shan, Inez, Nina and another child | Introduction to decimal chequerboard with felt squares.<br><br>This is a sensory exercise where the children identify and name | Decimal chequerboard, cut felt squares representing decimal numbers from 1 million to 1 millionth. | A very tricky exercise where children have to read where Linda has placed the felt squares. |

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|        |                        | decimal places in whole numbers and in decimal fractions.                                                                                                          | <p>Linda builds up the area, beginning with the unit, and getting the children to name that, then she will move on to 10 units, a tenth of a unit, 100 units, a 100<sup>th</sup> of a unit, etc.</p>  |                                                                                                             |
| Lesson | Shan and another child | <b>Review of Pythagorean Theorem.</b> In preparation for another lesson, Linda reviews the Theorem of Pythagoras using the Pythagoras Plate. After the review, she | Pythagorean Theorem plate, Constructive Triangles Boxes                                                                                                                                                                                                                                  | Linda says that this is a challenging presentation. Linda said she had only presented it twice in 19 years. |

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|               |      | <p>presents a lesson on building different shapes on the sides of the right-angled triangle.</p> <p>Lesson: Building shapes, other than squares, on the sides of the right-angled triangle.</p> |   |                                                                            |
| <b>May 10</b> |      |                                                                                                                                                                                                 |                                                                                                                                                                          |                                                                            |
| Activity      | Shan | <p>Following yesterday's lesson, using different shapes with the Pythagorean Theorem</p> <p>Shan presents his work to me. He attempts to build various shapes</p>                               | Pythagorean Theorem plate,<br>Constructive Triangle Boxes                                                                                                                | I invited Shan to present to me the lesson Linda had given the day before. |

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|          |                               | on the legs of the right-angled scalene triangle. He works with rhombi, trapezium, hexagons, pentagons. |                       |                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Activity | Declan and Eric               | Follow-up: Distributive Law                                                                             | Box of Bead Bars<br> | This is a follow-up activity from a lesson Linda has just given. Declan says they are going to make the biggest question ever. Eric says to Declan, "Are you sure you want to do this, I don't." Inez says "Do you know this is a Grade 5 lesson?" Eric says, "I'm going to quit soon." Declan says, "Linda is counting on us to do this." Eric replies, "It's too hard." They stop. Linda later says you just need some enthusiastic children for a lesson to take off. |
| Activity | Gail, later joined by Matthew | Follow-up: Box of Sticks, putting sticks of different lengths together.                                 | Box of Sticks                                                                                           | Not sure what she is doing, but she later tells Declan she is making a long chain. Gail bosses Matthew around telling him to "Get to work!" They pack their work away without completing anything. They did not appear interested.                                                                                                                                                                                                                                       |

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| Activity | Maria, Luca and another visiting child | Multiples of numbers                                                    | Multiple paper                                                                     | Maria and Luca are Spanish speaking, as is the visiting child. All three are working in Spanish.                                                                                                                                                                                                                                                                                                                                                                            |
| Activity | Declan and Eric                        | Writing Greek numbers                                                   | Greek number chart                                                                 | Declan invites Eric to work with him. He says, "I'm at 300 and something" Eric says, "I'm at the 200s."                                                                                                                                                                                                                                                                                                                                                                     |
| Activity | Toby                                   | Counting a long number chain                                            |                                                                                    |                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| May 13   |                                        |                                                                         |                                                                                    |                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| Activity | Evan and Toby                          | Counting long number chain (7 chain). They have the 7 cube out as well. |                                                                                    | Drawing lines on graph paper. Not sure what they are doing. They have drawn squares and rectangles to do the decanomial on paper, writing 7, 14, 21, 28, 35 on the horizontal plane and on the vertical plane 42, 49, 56, 63, 70. They have the chain and labels spread out, also squares of numbers, and cubes of numbers, as well as LBF, but are doing the multiplication in their heads. They do not strike me as very engaged. Linda encourages them to get their work |

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|               |          |                                                                                                                                                                                                                                                                                         |                                                                                                                          | done. Linda speaks to me about their work as asks if I am getting anything out of it. She sees it as busy work.                                                                                                                                        |
| Activity      | Matthew  | Box of Sticks                                                                                                                                                                                                                                                                           | Box of Sticks                                                                                                            | Matthew took out the work himself, independently of Gail. Unusual for him. He has made a square with the sticks. He says to Eric that it is like an axe and chops with it. Packs his work away after 10 minutes                                        |
| Activity      | Matthew  | Box of Beads                                                                                                                                                                                                                                                                            | Box of bead bars                                                                                                         | Material taken out at Linda's direction that he find some work. Does not do anything with the beads and puts them back.                                                                                                                                |
| Activity      | Jennifer | Telling the time                                                                                                                                                                                                                                                                        | Analogue clock on wall                                                                                                   | Jennifer looks to the clock and tells some children that their snack time is up after she has worked out how much time they have used.                                                                                                                 |
| Activity      | Nicholas | Numbers of different cultures                                                                                                                                                                                                                                                           | Egyptian Numbers chart                                                                                                   | Nicholas is working on his Egyptian numbers. He asked Mario a question with regard to Egyptian numbers and got reprimanded for disturbing him. Nicholas notes he has "cutted" the bottom of his numbers roll. He rolls up his list of numbers tightly. |
| Activity      | Nicholas | Large Bead Frame and Notational Paper                                                                                                                                                                                                                                                   | Notational Paper                                                                                                         | Nicholas is making a booklet of all the notational papers he has completed. He punched a hole in one corner and tied it up with string. Linda suggested to Nicholas that he show this work to me. He comes over but is not keen so I leave it.         |
| <b>May 14</b> |          |                                                                                                                                                                                                                                                                                         |                                                                                                                          |                                                                                                                                                                                                                                                        |
| Lesson        | Mario    | <p>Divisibility by 2 with the golden bead material.</p> <p>The Golden Bead material introduces the child to the decimal system. The set includes golden bead units, golden bead bars of 10 beads each, golden bead squares of a hundred units, golden bead cubes, and number cards.</p> | <p>Golden Bead decimal material</p>  |                                                                                                                                                                                                                                                        |

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| Activity | Matthew and Maria      | Telling time                               |  | Matthew has jumped up three or four times to come to the clock to check what time it is so he can write it in his journal. He discusses the time with Maria. Maria points to the “12” on the clock using a stick. |
| Activity | Matthew                | Box of Sticks                              | Box of Sticks                                                                      | Does not keep this work out long at all and I cannot discern that he used any of the material. He goes to join Eric and Evan but Linda sends him away saying that it is not his work.                             |
| Activity | Maria                  | Telling time                               | Analogue clock on the wall                                                         | Jumping up to tell time so that she can write it accurately in her journal.                                                                                                                                       |
| Activity | Eric and Evan          | Counting on the long bead chain (10 chain) | Long bead chains                                                                   | They have the chain out on a mat but are not getting much work done as they are chatting.                                                                                                                         |
| Lesson   | Anisha, Luca, and Toby | Greatest Common Multiples                  | Peg board and pegs                                                                 | Linda sets out pegs for 9 and 4, then for 7 and 8.                                                                                                                                                                |
| Activity | Mario                  | Numbers from other cultures                | Number chart                                                                       | Mario has a very long list of numbers written on his number chart. All rolled up.                                                                                                                                 |

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| Activity | Nina and Mario    | Telling time                                                                                                                                                                                                                                               | Analogue clock on the wall                                                                              | Nina and Mario tell the children at the snack table that they have to clean up as they only have one minute left.                                                                       |
| Activity | Maria             | Playing music on the tone bars                                                                                                                                                                                                                             | Tone bars and number charts                                                                             | Maria reads the numbers from her sheet of paper and plays them on the tone bars. She asks Gail how to play “Mary has a little lamb” and Gail reads out the numbers to her. Maria plays. |
| Lesson   | Gail and Nicholas | <b>Personal timelines</b><br>The students have a lesson on how to create their own personal timelines. They have had other lessons using timelines, such as the first and second timelines of human beings, a timeline of the coming of life in the world. | Long chart paper<br> |                                                                                                                                                                                         |

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| Activity | Nina                          | Telling time                                                     | Analogue clock on the wall                                                                                                           | Nina goes to Nicholas and says that he has three more minutes for snack.                                                                                                                                                                                                                                                                            |
| Activity | Eric                          | Telling time                                                     | Analogue clock on the wall                                                                                                           | Eric tells Evan that his time is up.                                                                                                                                                                                                                                                                                                                |
| Activity | Mario, Nicholas, Inez, Declan | Numbers in different cultures                                    | Roman number chart                                                                                                                   | Mario tells Inez that Declan is never going to catch up with him. He says he is at 265 and he knows Declan stopped at 260. They discuss where Nicholas is. The discussion continues. Mario tells Evan that he is now at 311.                                                                                                                        |
| Activity | Maria                         | Telling time                                                     | Analogue clock on the wall                                                                                                           | She looks at the clock on the wall appearing to tell the time.                                                                                                                                                                                                                                                                                      |
| Activity | Eric                          | Tape measure?                                                    | Tape measure                                                                                                                         | Eric is measuring various objects and people. He measures Evan's height then measures his own. He says they are both 119 cm. Matthew wants to get measured. He takes over the tape and measures various items.                                                                                                                                      |
| Lesson   | Shan and another child        | Rate, principal, interest over time                              | Charts                                                                                                                               |                                                                                                                                                                                                                                                                                                                                                     |
| Activity | Maria                         | Telling time                                                     | Analogue clock on the wall                                                                                                           | Maria is using her pencil to point to the clock on the wall to aid her tell time.                                                                                                                                                                                                                                                                   |
| May 15   |                               |                                                                  |                                                                                                                                      |                                                                                                                                                                                                                                                                                                                                                     |
| Activity | Anisha, Declan                | Finding the Greatest Common Multiple                             | Pegboard and pegs                                                                                                                    | Anisha chooses this work and picks the numbers 3 and 7. Declan comes over and asks what she is doing. She softly writes the name in his journal and he goes over it in pencil. He asks himself what the time is. Linda says to me that she has to keep an eye on Anisha teaching Declan.                                                            |
| Activity | Eric and Toby                 | Building with the white volume material and blue volume material | White volume cubes 2 cm and blue volume cube<br> | I am not sure what they are doing. He says they are building but are making it higher. Check video. Eric talks about 2D and 3D. He says 2D is flat and 3D you can push a pencil through it. I ask what the lesson was but Eric answers that he was the first one to get the lesson and he gave it to someone else who also gave it to someone else. |

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| Activity | Matthew and Jennifer                             | Large Bead Frame                                                                                                                                              | Large Bead Frame                                                                                            |                                                                                                                                                                                                                                                                                                                                     |
| Activity | Luca                                             | Telling time                                                                                                                                                  | Wooden clock material<br> | Working on the hours of the clock                                                                                                                                                                                                                                                                                                   |
| Activity | Toby                                             | Counting the long chains (10 chain)                                                                                                                           | Long chain material                                                                                         |                                                                                                                                                                                                                                                                                                                                     |
| Activity | Eric and Toby                                    | Paper decanomial                                                                                                                                              | Paper decanomial                                                                                            | They are not engaged in this work. There are papers from the work on the floor, Toby is making noises and writhing in his seat.                                                                                                                                                                                                     |
| Lesson   | Eric, Nicholas, Anisha, Matthew, Jennifer, Mario | Triangles classified according to sides.<br>Tracing angles from geometry cabinet, making booklets, classifying angles according to sides, according to angles | Box of Sticks, tray of triangles from the Geometry Cabinet, right angle measurer, booklets                  | Children have made different types of triangles using the sticks. Eric checks his spelling for <i>isosceles</i> in the nomenclature booklet. He starts making his own booklet.<br>Jennifer and Matthew classify angles, Nicholas and Eric trace triangles from the geometry cabinet, Mario classifies triangles according to sides. |

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| Lesson        | Declan     | Addition with the stamp game                                                                                       | Stamp game                                                                         | Linda is directing him and Matthew who is on another piece of work. Linda talks about <i>addends</i> to Declan.                                                                                                        |
| Activity      | Toby       | Counting the long chains (10 chain)                                                                                | Long chains                                                                        |                                                                                                                                                                                                                        |
| Lesson        | Matthew    | Commutative law of multiplication. This is a follow-up lesson from the one Linda originally gave Gail and Matthew. | Box of Bead Bars                                                                   | Linda re-presents this lesson to Matthew and directs him step by step.                                                                                                                                                 |
| <b>May 16</b> |            |                                                                                                                    |                                                                                    |                                                                                                                                                                                                                        |
| Activity      | Luca       | Telling time /Counting the long chains (6 chain, then 4 chain then 3 chain)                                        | Analogue clock on the wall.<br>Long chains                                         | Luca is looking at the clock to tell the time for his journal entry. He also has the long chain of 6 out on a mat                                                                                                      |
| Activity      | Nicholas   | Numbers in other cultures                                                                                          | Number chart                                                                       | Nicholas is rolling his number chart. Linda discusses it with him. Nicholas says it needs a title and also needs decorating. He says that it will take so long.                                                        |
| Activity      | Evan       | Counting the long chains (7 chain)                                                                                 |                                                                                    |                                                                                                                                                                                                                        |
| Activity      | Luca       | Telling time                                                                                                       | Large analogue clock on the wall                                                   | Luca gets his journal out of his box. Opens his journal and looks at the clock. He gets up and walks to the clock. He points with his fingers in a circular motion, talks to himself when walking back to his journal. |
| Activity      | Toby, Evan | Counting the long chains (6 chain)                                                                                 | Long chains                                                                        | Evan wants Toby to come and work with him on the 6 chain. Toby does not want to.                                                                                                                                       |
| Activity      | Luca       | Playing notes on the tone bars:<br>Reading numbers                                                                 | Tone bars                                                                          | Luca is reading numbers from a laminated sheet to play notes on the tone bars.                                                                                                                                         |

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| Activity | Evan and Declan         | Telling time                                                                                                                               | Analogue clock on the wall                                                         | Evan tells Declan he cannot stay long for his snack because his time is almost up. Gail tells Evan what time she started her snack. |
| May 22   |                         |                                                                                                                                            |                                                                                    |                                                                                                                                     |
| Lesson   | Nina and Inez           | Decimal Fractions (follow-up lesson?) subtraction                                                                                          | Decimal Fraction material?                                                         | Linda showing the two girls how to read decimal numbers                                                                             |
| Activity | Mario                   | Follow-up. Cutting circles.<br>The circles will be part of Mario's food chain he is making for his work on a food chain for sea creatures. | Construction paper                                                                 |                                                                                                                                     |
| Activity | Gail                    | Follow-up? Gail is using a tape measure for some reason. I cannot see what she is doing with it.                                           |                                                                                    |                                                                                                                                     |
| Activity | Anisha, Mario, and Toby | Follow-up: Divisibility by two.<br>The students are working with this material. They are choosing                                          | Golden Bead material                                                               |                                                                                                                                     |

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|  |  | numbers and answering either “yes” or “no” as to whether the number is divisible by two or not. |  |  |
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