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PROBABILISTIC COMPLETION OF
NONDETERMINISTIC MODELS

By
Guy Beaulieu, B.Sc., M.S.

Thesis submitted to the
Faculty of Graduate and Postdoctoral Studies
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in partial fulfillment of the requirements for the
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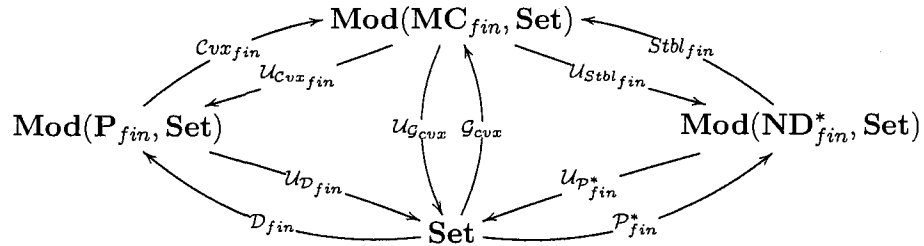
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Abstract

Motivated by Moggi's work [34] on how monads can be used to capture computational behavior, there has been a growing interest in finding monads which capture the precise computational effects generated when combining the theory of probabilistic choice and the theory of nondeterministic choice. The resulting theory, called here the theory of mixed choice [23, 31, 33], captures the interaction between the two choice operators by requiring appropriate distributivity axioms (probabilistic choice over nondeterministic choice), in addition to the operations and axioms from both individual choice theories.

Classically, the required monad has been defined by composing the adjunctions on the left hand side of the following diagram.



This requires the composition of the distributions functor \mathcal{D}_{fin} , which constructs free models of probabilistic choice over an arbitrary set X , and the finite convex set functor Cvx_{fin} , which constructs free models of mixed choice over an arbitrary convex set $(C, +_\lambda)$, as seen in [31, 33, 49]. This construction allows us to build a free mixed choice model from an arbitrary model for probabilistic (\mathbf{P}) choice, in such fashion as to preserve the \mathbf{P} choice model's probabilistic structure.

We consider the dual approach to the above algorithm. Through general categorical results from Barr and Wells [2], we observed that free models for mixed choice could also be constructed over arbitrary models of nondeterministic (**ND**) choice. This allows for an alternative algorithm for constructing models for mixed choice, which corresponds to going up the right hand side of the above commuting diagram. This requires the composition of the finite nonempty powerset functor \mathcal{P}_{fin}^* , which constructs free models of nondeterministic choice over an arbitrary set X , and the \vee -stable functor $Stbl_{fin}$, which constructs free models of mixed choice over an arbitrary semilattice (S, \vee) . This alternative approach allows for a wider range of mixed choice models to be studied: those that would arise from arbitrary semilattices, hence arbitrary nondeterministic models with a possible non-standard definition of nondeterminism.

By abstract categorical notions, it is known that the \vee -stable functor $Stbl_{fin}$ exists (the functor in the N-E corner of the above diagram). However, up until now there have been no concrete presentations of this functor, nor any notion of how to construct free mixed choice models over arbitrary semilattices. These concepts represent the main focus of this thesis. We motivate and develop a new property for convex sets generated over a set equipped with a semilattice structure. This notion is called \vee -*stability*. In essence, a \vee -stable convex set preserves the **ND** structure on its underlying set (i.e., if it contains an element s such that $x \vee y = s$, then it must also contain the convex subset generated by x and y).

Finally, we generalize many of our results in two directions. First, we construct the corresponding \vee -stable functors for *infinitary* mixed choice theories over **Set**. We then extend the entire development of the thesis to express posetal model categories (i.e over **Poset**) of our theories. This would include constructing models for mixed choice over models obtained from the convex (Plotkin), lower (Hoare), and upper (Smyth) variants of nondeterminism. Furthermore, we prove that the results in the literature obtained by following the left-hand side of the given diagram are equivalent to our results obtained by traversing its right-hand side.

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Dedication

I dedicate this work to my wonderful children Emma and Matteo. My life would not be complete without them. I've quickly realized that anything worth doing in life, as a parent, is done for your children. I hope that I can bring as much wonder and delight to their lives as they've brought to mine.

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Chapter 1

Introduction

Motivated by Moggi's work [34] on how monads can be used to capture computational behavior, there has been a growing interest in finding monads which capture specific computational effects. Among them, two of the most thoroughly studied types of computational effects are: probabilistic (**P**) choice [20, 22] and nondeterministic (**ND**) choice [37, 48].

As pointed out by Panangaden [36], discrete probabilistic combinations suffice to analyze much of the work in probabilistic process algebra. This motivates our focus on determining the combination of discrete probabilistic behavior with nondeterministic behavior. Furthermore, discrete probabilistic behavior is a computational effect which has a very simple description. For any weight $\lambda \in [0, 1]$, the **P** choice (\oplus_λ) between two terms represents a process which internally chooses its left hand side with probability λ and its right hand side with probability $1 - \lambda$. The theory for **P** choice has a very nice axiomatic theory, developed in detail in the Claire Jones' PhD thesis [20]. These axioms include associativity, commutativity and idempotence, up to possible reweightings. Therefore, from the basic assumptions, we can define an n -ary probabilistic choice operator of the form $\bigoplus_{(\lambda_i)_{i=1}^n}$, where $(\lambda_i)_{i=1}^n$ is a *finite probability density* (i.e a finite sequence of elements $\lambda_i \in (0, 1)$ such that $\sum_{i=1}^n \lambda_i = 1$). As we shall see in greater detail in the second chapter and as discussed in [16, 20, 21, 22, 38], the model categories for **P** choice are given by convex sets. Furthermore, the monads

associated to the different theories of **P** choice are given by variants of the distributions monad over **Set** and the probabilistic powerdomain monad over **Dom**.

We shall also consider an infinite variant of **P** choice. It is a well known result in analysis that for real-valued series, an uncountable sum converges only if at most countably many terms are nonzero. Since a **P** choice relies on combinations weighted by a *probability density* $(\lambda_i)_{i \in I}$, (i.e., a sequence of real numbers in $(0, 1]$ such that $\sum_{i \in I} \lambda_i = 1$), the above statement implies that we need at most consider *countable P choice*, $\bigoplus_{(\lambda_i)_I}$, where I is a countable indexing set. Models for this type of **P** choice are presented as superconvex sets as discussed in Section 2.1.3. This allows for a higher degree of generality in constructing our theories for mixed choice.

Nondeterministic behavior is an important factor in modelling distributed computation. There are many different flavors of **ND** choice, although every type of **ND** choice satisfies the same basic assumptions; namely that a nondeterministic choice operator must be associative, commutative and idempotent. Therefore, model categories for **ND** choice are usually presented as semilattice categories with possibly extra structure (such as ordered semilattices in the case of **ND** choice models over posets). Monads associated with **ND** choice theories have been found and are given by suitable powerset monads over **Set** and powerdomain monads over **Dom**, [37, 39, 48].

In this work, there are two types of **ND** choice theories we shall consider. The first is *external ND choice* $(+)$, which in addition to the usual axioms for nondeterminism includes a unit element. In this framework, the choice between terms is influenced by the current state of its environment. This is the standard type of nondeterminism for most concurrent computation such as the π -calculus [29]. However, external **ND** choice comes equipped with a unit element, denoted by 0 , for the choice operator. This is an undesirable property for the **ND** choice operator we wish to combine with **P** choice. We will discuss this in greater lengths in the second chapter, but in short the unit for **ND** choice does not fit in our framework for mixed choice and leads to many unwanted properties. Therefore, we consider a second type of **ND** choice: *internal ND choice* (\boxplus) , where \boxplus only satisfies the basic axioms for nondeterminism. The internal **ND** choice between terms is done through an internal scheduler and not influenced by its environment. This is the standard type of nondeterminism used

when combining **P** and **ND** choice.

We will also want to move beyond the usual finite operators used by classical **ND** choice theories and consider *unbounded internal ND choice* (\boxplus). This will allow for arbitrary nondeterministic combinations of terms, denoted as $\boxplus_{w \in W} A_w$, where W is an arbitrary indexing set. As pointed out by Mislove [30], one only has to look at how such an operator could be used in specification languages in order to determine its usefulness. For example, it can be used to specify a process whose behavior depends on the input of an arbitrary natural number: different inputs may lead to different behaviors. The use of uncountable **ND** choice in our framework is also motivated further by two important factors. First, the probabilistic completion of nondeterministic models with uncountably many nondeterministic choice operators has a simpler presentation and a greater degree of generality than its finite counterpart. In other words, it allows our definitions and results to be presented in a simple and concise notation. Second, let \boxplus denote **ND** choice and for $\lambda \in [0, 1]$, let $A \oplus_\lambda B$ denote the probabilistic choice operator. In our framework, the nondeterministic choice between terms A and B can be interpreted as the probabilistic choice of indeterminate weight of the constituents. This viewpoint is the same as the one proposed by Mislove [31] and also Tix [49] in their approach to creating a mixed choice theory. However, our use of uncountable nondeterministic choice operators is a new feature allowing us to express the above interpretation of nondeterminism formally as $A \boxplus B = \boxplus_{\lambda \in [0,1]} (A \oplus_\lambda B)$. The later is an equation derivable from our axioms defining a mixed choice theory, as discussed in Section 3.3.1. In the end, the use of unbounded **ND** choice enables the explicit statement of the relationship between **P** and **ND** choice in a mixed choice theory.

Recently, there has been an increasing interest in studying theories for *mixed choice*; the computational behavior combining both nondeterministic and probabilistic effects. The combination of these two theories seems to be essential in giving models for concurrent processes, see [14, 46, 52]. In the literature, the theory for mixed choice is most often presented as a finite Lawvere theory which admits two

types of binary operators: a single nondeterministic choice operator (**ND-Op**), denoted by \boxplus , and an uncountable family, indexed by $\lambda \in [0, 1]$, of probabilistic choice operators (**P-Ops**), denoted by \oplus_λ . It is also widely accepted that the axiomatic theory for mixed choice should admit an axiom which captures the interaction between the two types of choice operators. After careful consideration, it has become standard practice to require that the **P-Ops** distribute over the **ND-Op** [1, 35]. Given the required interaction between the choice operators, it becomes necessary to consider an **ND** choice operator without a unit: internal **ND** choice. Thus, in the theory of mixed choice, $A \boxplus B$ represents a term which makes an internal **ND** choice between A and B , as previously discussed. The standard axiomatic theory for mixed choice is obtained by combining the axioms for **P** choice \oplus_λ , the axioms for internal **ND** choice \boxplus , and the distributivity of **P-Ops** over **ND-Op** axiom.

There have been two typical approaches in constructing the monad associated to the computational behavior of mixed choice.

- (i) Combine the individual choice theories in order to construct the theory for mixed choice and then freely generate the corresponding monad.
- (ii) Modify the monads associated to **P** and **ND** choice in order to define a distributive law between them.

The first approach is seen in many recent works on combining nondeterministic and probabilistic choice, [15, 23, 31, 33, 49]. In recent works Power [41], and Hyland, Plotkin and Power [19] have developed machinery for combining Lawvere theories in order to capture the interaction of many types of computational effects; e.g. combining nondeterminism and probabilistic choice theories. There are three main ways of combining theories which are discussed in the above authors' works: the sum of theories, the tensor of theories and the distributive tensor of theories. In the first case, the sum of two theories is a theory which admits all the operators and axioms from the initial theories without adding anything new. In the second type of combination, the tensor of two theories merges the operators and equations from the initial theories and adds equations which force all the operators from one theory to commute with the operators of the second theory. The last kind of combination, the

distributive tensor, captures the theory we expect to obtain when combining **P** and **ND** choice. The distributive tensor between the Lawvere theories \mathbf{L} and \mathbf{L}' , denoted $\mathbf{L} \triangleright \mathbf{L}'$, is the theory which admits all the operations and axioms from \mathbf{L} and \mathbf{L}' but also imposes distributivity axioms which force every operation in \mathbf{L} to distribute over every operation in \mathbf{L}' . This describes exactly the standard structure for the theory of mixed choice. Once the correct theory for mixed choice has been established, we can freely generate its corresponding monad.

The second approach to combining **P** and **ND** choice is presented by Varacca [50], and further developed by Varacca and Winskel [51]. The goal of this approach is to modify one of the monads associated to **P** or **ND** choice in order to define a distributive law between the modified monad and the unmodified one. Due to the distributive law, the monads will compose into a monad whose theory will allow the operators to be appropriately distributive (**P**-Ops over **ND**-Op). To construct the monad which captures the behavior of mixed choice, one may consider taking the composition of the unmodified monads for each of the individual type of choice structures. This composition was shown to be inadequate by Varacca [50], due to the lack of a distributive law from the monad for **ND** choice over the monad for **P** choice or vice-versa. Thus, it is necessary to modify the definition of the initial monads in order to allow for the existence of a distributive law. The modification which Varacca proposes is the following: he remarks that weakening the axiomatic theory of **P** choice by omitting the idempotency axiom ($A \oplus_\lambda A = A$) leads to a distributive law which captures a variant of the theory of mixed choice. Therefore, he presents an alternate axiomatic theory for mixed choice which is obtained by omitting the idempotence axiom of **P** choice. Furthermore, he develops an indexed valuation powerdomain monad which can be composed with the Plotkin powerdomain monad in order to define a monad which captures the behavior of his theory of mixed choice.

Of the two approaches of combining **P** and **ND** choice theories, we shall be using the first one. We consider that any theory combining **P** and **ND** choice must admit the individual choice theories as subtheories. Therefore, it must contain each axiom from these individual choice theories. Modifying the monads in order to force a distributive law, will inadvertently alter the theory for one (or both) of the basic

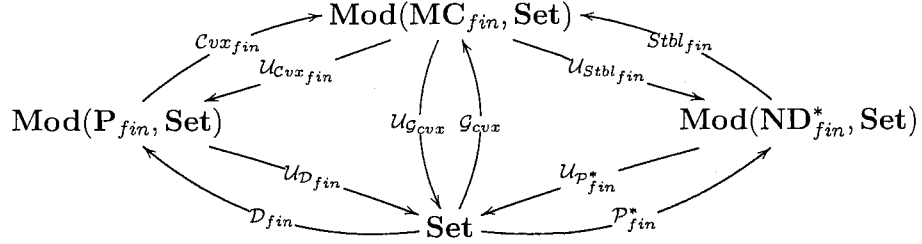


Figure 1: Mixed Choice Factored Through Probabilistic and Nondeterministic Choice

choice types. Nevertheless, Varacca et al. give many supportive arguments for their approach and it is not to be dismissed.

By appropriately combining our choice theories we have seen how we can obtain the theory for mixed choice. In order to construct the monad associated to our mixed choice theory over a suitable category \mathbf{C} , one may consider how to construct free mixed choice models from arbitrary models of one of the two choice structures over \mathbf{C} . This is the most widely used method in the literature. However, the focus has always been on constructing models for mixed choice from arbitrary models of probabilistic choice. This corresponds to following the left hand path of the commutative diagram in Figure 1 over \mathbf{Set} , where each pair of functors represents an adjunction following the schema; $* \dashv U_*$. Thus, we can consider the construction of free mixed choice models over \mathbf{C} to be given by the following two steps.

- (1) First, freely construct probabilistic models over the category \mathbf{C} . Thus for set-theoretical models one would use the *distributions functor*, \mathcal{D}_{fin} , and for domain-theoretical models one would use the *probabilistic powerdomain*, \mathcal{V} .
- (2) Then, freely construct a model for mixed choice over the probabilistic models obtained in (1). For set-theoretical models the *finite convex powerset functor*, \mathcal{Cvx}_{fin} is used and three different convex powerdomains are proposed for domain-theoretical models.

Since each step of the construction involves monads with composable adjunctions, the monad which captures the behavior of mixed choice arises from (the adjunction obtained by) composing the adjunctions from the above steps.

Using this algorithm, a monad for mixed choice can be defined over any category \mathbf{C} . However, we obtain a great deal more information than simply how to construct mixed choice models over an object $C \in \mathbf{C}$. The construction allows us to build a free mixed choice model from an arbitrary model for \mathbf{P} choice, in such a fashion as to preserve the \mathbf{P} choice model's probabilistic structure. In other words, given a model for \mathbf{P} choice, denoted $(C, +_\lambda)$, its image under the convex powerset functor, $\mathit{Cvx}_{fin}((C, +_\lambda))$, contains an isomorphic copy of $(C, +_\lambda)$. This arises due to the fact that the unit transformation, η , for the convex powerset monad is a \mathbf{P} choice isomorphism. This allows the construction of a much broader range of models for mixed choice.

In this thesis we wish to consider the dual approach to the above algorithm, which does not appear to have been considered. We first presented this result in our paper [4]. Through general categorical results from Barr and Wells [2], we observed that free models for mixed choice could also be constructed over arbitrary models of \mathbf{ND} choice. This allows for an alternative algorithm for constructing models for mixed choice, which corresponds to going up the right hand side of the commuting diagram from Figure 1.

- (1') First, freely construct \mathbf{ND} choice models over the category \mathbf{C} . Thus for set-theoretical models one would use the *non-empty powerset functor*, \mathcal{P}_{fin}^* , and one of the following types of powerdomains for domain theoretic models: the *convex (Plotkin) powerdomain* \mathcal{P} , the *lower (Hoare) powerdomain* \mathcal{H} , and the *upper (Smyth) powerdomain* \mathcal{S} .
- (2') Then, freely construct models for mixed choice over the \mathbf{ND} choice models obtained in (1'). For both set-theoretical and posetal models, we will present the monad over \mathbf{ND} choice models which accomplishes this in the fourth chapter.

This alternative approach allows for a wider range of mixed choice models to be studied: those that would arise from arbitrary semilattices, hence arbitrary non-deterministic models with a possible non-standard definition of nondeterminism. We will focus on giving a concrete definition of the functors on the right-hand

path. More specifically, we focus on how to explicitly compute the \vee -stable functor $Stbl_{fin} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$ (in the “northeast quadrant” of Figure 1) which assigns a model of mixed choice to an arbitrary nondeterministic model.

One of our main developments is the formulation of concrete definitions for a variety of \vee -stable functors, defined on set-theoretic (\mathbf{Set}) or posetal (\mathbf{Poset}) model categories, for various types and arities, of \mathbf{ND} and \mathbf{P} choice. These functors are left adjoint to their respective forgetful functors from the category of models of mixed choice to the category of models of nondeterminism. We consider mixed choice theories which admit \mathbf{P} -Ops of countable arity and \mathbf{ND} -Ops of possibly uncountable arity. Separately, each choice theory is associated to a monad. In our case, the \mathbf{P} choice theory is associated to the distributions monad and the \mathbf{ND} choice theory is associated to the non-empty powerset monad.

The second chapter offers a quick review of many of the important topics relevant to the theorems and constructions presented in latter chapters. We begin this chapter by highlighting some of the important notions of convexity and generalizing these ideas in order to define the notion of superconvexity (sets which are closed under countable convex combinations). These two notions are important since convexity (and superconvexity) capture the fundamental structure necessary in modelling the behavior of finite (countable) probabilistic processes. Secondly, we provide a quick overview on particular aspects concerning monads. This section includes important definitions and examples of particular monads and \mathbf{T} -algebras and includes results such as the correspondence between adjunctions and monads and an existence property for left adjoints important to our construction. Thirdly, we present the supporting material on Lawvere theories, their corresponding model categories and the correspondence between model categories over Lawvere theories and monads. Finally, we complete the chapter with a small resume on the different types of methods used to formally combine Lawvere theories.

The third chapter describes in greater depth many important details concerning our choice theories and their associated model categories in \mathbf{Set} and in \mathbf{Poset} . The first two sections recall the definitions of the theories for probabilistic choice and the

theories for nondeterministic choice. These sections include the definitions of many of their variants, their associated monads and their model categories over **Set** and **Poset**. The last section deals with the theory of mixed choice. We provide motivation behind our choice of the equational representation of the mixed choice theory, ranging from the distributivity of the **P**-Ops over the **ND**-Op, to the combination of countably infinite **P**-Ops with arbitrary **ND**-Ops. Finally, we discuss in detail the standard algorithm used to construct the monad associated to mixed choice models in **Set** and **Poset**.

The fourth chapter contains our main result on the free construction of models of mixed choice over arbitrary nondeterministic models. We present the concept of V -stable convex sets generated over an arbitrary V -semilattice. We define these new concepts and present theorems concerning their important properties. We provide motivating examples to support our reasoning on how they represent the correct extension over semilattices to models of mixed choice. Finally, using these new concepts, we present our concrete definition of the V -stable monad over models of nondeterministic choice, the monad which constructs free mixed choice models over arbitrary models of nondeterminism.

Finally, we include an appendix which presents an example of a process language with mixed choice capabilities, based on Milner's CCS [28], called the Mixed Choice CCS. We provide definitions for the different types of processes, the alternating operational semantics and a bisimulation relation for this calculus. In the end, we define a fully-abstract set theoretical model for the language.

Chapter 2

Preliminary Topics

This chapter provides a brief overview of the basic topics of convex sets and category theory used throughout the later chapters. The chapter is separated into many independent sections, each dealing with a particular fundamental concept. We omit giving the proofs of many of the theorems (unless the structure of the proof is the topic of discussion) and refer the reader to key references if more details are desired. The examples presented in this chapter are designed to introduce constructions (such as functors, monads and models) which will be relevant in later chapters.

We begin with a brief introduction to convex spaces. As noted by many authors working in developing models for mixed choice, such as Mislove, Ouaknine and Worrell [33], Varacca [50] and Keimel, Plotkin and Tix [23] to name a few, convexity is an important property for any model which captures probabilistic choice. Convexity will play a large role in our approach, since our aim is to construct appropriately behaved probabilistic models over arbitrary models for nondeterministic choice keeping the underlying **ND** structure intact. Indeed, we shall present such a construction in Chapter 4 by defining new properties on convex sets which will lead to our technique for constructing models for mixed choice.

2.1 Convex and Superconvex Spaces

Convex spaces are important when discussing models for probabilistic choice. Intuitively, convex combinations between objects x_1, x_2 in a particular convex space, denoted $x_1 \oplus_\lambda x_2$ (or $\lambda x_1 + (1 - \lambda)x_2$), where $\lambda \in [0, 1]$, are used to represent a probabilistic choice between the objects x_1 and x_2 , where x_1 will be chosen with probability λ and x_2 will be chosen with probability $(1 - \lambda)$. Many of the developments on finding models for probabilistic theories and how convexity relates to them can be found in [20, 21]. Moreover when dealing with models for mixed choice, it has been shown in [23, 33] that models of such theories are given by certain collections of convex subsets.

We shall also present a generalization on the definition of convex spaces. We will define a “convex” structure which allows *countable* convex combinations of its elements, which we shall call a *superconvex space*. In other words, we want to allow for possibly countable weighted sums of the form $\sum_{i \in I} \lambda_i x_i$, where $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geq 0$, formally denoted by $\sum_{(\lambda_i)_I} x_i$.

The presentation of each type of space shall be given in two parts. We shall begin by considering the class of convex (or superconvex) spaces which arise as subspaces of a real vector space (with the appropriate type of operations, i.e. countable sums for superconvex combinations) which satisfy a particular property, this special case of convex structure is due to Rodé [45]. This is the usual class of convex spaces analyzed in much of the literature [7, 12]. These subclasses of convex (or superconvex) spaces are interesting in their own right, since they are exactly the convex spaces obtained when dealing with free convex (or superconvex) spaces generated over a set. Secondly, we shall define convex (or superconvex) spaces as a set equipped with formal operations satisfying a specific set of equations. This formal definition is necessary in order to capture non-standard examples of convex (or superconvex) spaces.

2.1.1 Convex Structures in Real Vector Spaces

In this section we remind the reader of the basic definitions and useful results concerning convex structures in real vector spaces.

Affine Spaces

The theory of convex spaces is a subset of affine geometry. Many of the results and observations concerning convex spaces are inherited from affine spaces, as discussed in detail in Brøndsted [7], which together with Grünbaum [12] make up our basic references for this section.

Definition 2.1.1 (Affine Spaces). A subset A of a vector space V over \mathbb{R} is called an **affine space** if either $A = \emptyset$ or there exists a vector $\vec{v} \in V$ and a linear subspace L of V such that $A = \vec{v} + L$.

Example 2.1.2 (Affine spaces). Many examples of affine spaces can be found as subsets of \mathbb{R}^n . Standard examples include any straight line or any plane in \mathbb{R}^n . Notice that affine spaces unlike linear spaces do not require that the lines or planes pass through the origin. \triangle

Proposition 2.1.3. *A subset A of a real vector space V is an affine space if and only if*

$$(\forall a_1, a_2 \in A)(\forall \lambda \in \mathbb{R})\lambda a_1 + (1 - \lambda)a_2 \in A.$$

Definition 2.1.4 (Affine Combinations). Let V be a real vector space. An **affine combination** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in V$ is a linear combination of the form $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

Proposition 2.1.5. *Let A be an affine space. Any affine combination of points in A is again in A . In other words, given an n -family of points (x_1, x_2, \dots, x_n) in A and real scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$, then the affine combination $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$ is in A .*

Remark 2.1.6. The intersection of affine spaces is again an affine space. Thus any subset M of a real vector space V is contained in a minimal affine space.

Definition 2.1.7 (Affine Hull). Given a subset M of a real vector space V , define the **affine hull** of M in V , denoted by $\text{aff}(M)$, to be the smallest affine subspace of V containing the set M .¹

¹By affine subspace of V we mean a subset of V which is also an affine space.

Proposition 2.1.8. *Given any subset M of a real vector space V , $\text{aff}(M)$ is the set of all affine combinations of points in M .*

Definition 2.1.9 (Affinely Independent). [Brøndsted [7], p.7] An n -family of vectors $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ in a real vector space V is said to be **affinely independent** if a linear combination $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n$ with $\lambda_1 + \lambda_2 + \dots + \lambda_n = 0$ equals the zero vector implies that $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$.

Proposition 2.1.10. *Affine independence of the n -family of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is equivalent to the linear independence of at least one of or all of the following $(n-1)$ -families $(\vec{v}_1 - \vec{v}_i, \vec{v}_2 - \vec{v}_i, \dots, \vec{v}_{i-1} - \vec{v}_i, \vec{v}_{i+1} - \vec{v}_i, \dots, \vec{v}_n - \vec{v}_i)$, for all $1 \leq i \leq n$.*

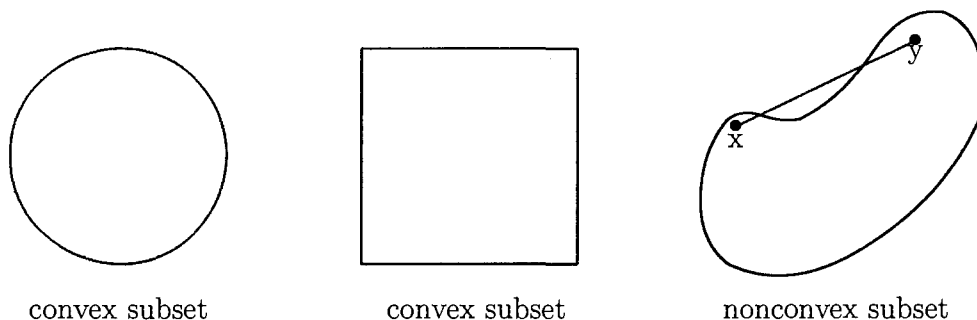
Definition 2.1.11 (Affine Basis/Dimension). An **affine basis** of an affine space A is an affinely independent n -family (x_1, x_2, \dots, x_n) of points in A such that $A = \text{aff}(\{x_1, x_2, \dots, x_n\})$. The **dimension** of a nonempty affine space A , denoted $\dim A$, is the dimension of the linear subspace L such that $A = \vec{v} + L$. By convention we let $\dim \emptyset = -1$.

Convex Spaces

A convex space is an affine space for which we require that the weights of the linear combinations be non-negative real scalars which sum up to 1. In what follows we present some of the definitions and results concerning convexity as presented by [7, 12].

Definition 2.1.12 (Convex Space). A subset C of a vector space V over \mathbb{R} is called a **convex space** if $\lambda_1 x_1 + \lambda_2 x_2$ belongs to C for all $x_1, x_2 \in C$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \geq 0$.

Example 2.1.13. Any subset S of the space \mathbb{R}^n which admits the following property: for every point x, y the line segment from x to y is contained in S , is a convex space. The figure below shows some basic examples and a non-example of convex spaces found in \mathbb{R}^2 . △

Figure 2: Examples and Non-example of convex subsets in \mathbb{R}^2 .

Definition 2.1.14 (Convex Combination). A **convex combination** of points x_1, x_2, \dots, x_n from a vector space V over \mathbb{R} is a linear combination $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$, where $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$ and $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$. We shall often write such a linear combination as:

$$(1) \quad x_1 + \lambda x_2 \quad \text{for binary combinations}$$

$$(2) \quad \sum_{(\lambda_i)_{i=1}^n}^C x_i \quad \text{for } n\text{-ary combinations}$$

to indicate that the linear combination is in fact a convex combination.

Notation 2.1.15. Here $\sum_{(\lambda_i)_{i=1}^n}^C$ is the usual notation for convex combinations found throughout [7]. In the rest of this thesis, we often omit the superscript C in $\sum_{(\lambda_i)_{i=1}^n}^C$ and denote convex combinations simply as $\sum_{(\lambda_i)_{i=1}^n}$.

Theorem 2.1.16. *A subset C of a vector space V over \mathbb{R} is convex if and only if C is closed under convex combinations. (I.e., given $x_1, x_2, \dots, x_n \in C$, $\sum_{(\lambda_i)_{i=1}^n} x_i \in C$)*

Given the notion of convex combinations, we can define the correct type of morphism between convex sets.

Definition 2.1.17 (Convex Space Morphism). Suppose we are given two convex spaces C in V and C' in V' . A **convex set morphism** is a map $f : C \rightarrow C'$ which preserves convex combinations. In other words, for an arbitrary convex combination $x_1 + \lambda x_2$ in C , $f(x_1 + \lambda x_2) = f(x_1) + \lambda f(x_2)$ is a convex combination in C' .

It is easy to see that the set of all convex spaces and the family of all convex space morphisms satisfy the required properties of a category.

Definition 2.1.18 (Category of Convex Spaces in Real Vector Spaces). The **category of convex spaces in real vector spaces**, denoted $\mathbf{Conv}_{\mathbb{R}}$, is the category of convex spaces in real vector spaces and convex space morphisms between them.

Next we show how to generate convex spaces over arbitrary subsets of any real vector space.

Remark 2.1.19. The intersection of any family of convex spaces in the vector space V over \mathbb{R} is a convex space. Thus, for any subset M of V we can define the smallest convex space which contains M .

Definition 2.1.20 (Convex Hull). Let M be a subset of a vector space V over \mathbb{R} . The **convex hull** of M (or the convex space **spanned** by M), denoted by $\text{conv}(M)$, is the smallest convex subspace of V containing M .²

Theorem 2.1.21. *For any subset M of the real vector space V , the convex hull $\text{conv}(M)$ is the set of all convex combinations of points in M .*

Theorem 2.1.22. [Brøndsted [7], p.13] *For any subset M of a real vector space V , the convex hull $\text{conv}(M)$ is the set of all convex combinations*

$$\sum_{(\lambda_i)_{i=1}^n} x_i$$

where (x_1, x_2, \dots, x_n) is an affinely independent n -family of points in M .

Remark 2.1.23. By Theorem 2.1.22 we do not have to take the convex combinations of all the points in M in order to construct $\text{conv}(M)$. It will be sufficient to only take the convex combinations on some affinely independent family in M . Unfortunately, no fixed family of points from M will suffice. This means that unlike affine spaces (and linear spaces) convex spaces can be generated by affinely independent families of varying size.

²By a convex subspace of V we mean a subset in V which is also a convex space.

Corollary 2.1.24 (Carathéodory's Theorem). [Brøndsted [7], p.14] *For any subset M of the vector space V over \mathbb{R} with $\dim(\text{aff}(M)) = n$, the convex hull $\text{conv}(M)$ is the set of all convex combinations of precisely $n + 1$ points from M .*

Convex Polytopes

We have seen that a general convex space in a real vector space does not admit a basis as do general affine and linear spaces. In this section, we discuss particular types of convex spaces that can be described by a fixed family of points, called *simplices*. For full details, see [7, 12].

Definition 2.1.25 (Polytope). A **convex polytope** P (or simply **polytope**), is defined to be the convex hull of a non-empty finite set of points in a vector space V over \mathbb{R} . (I.e., $P = \text{conv}(\{x_1, x_2, \dots, x_n\})$, where $x_1, x_2, \dots, x_n \in V$)

Definition 2.1.26 (Simplex). A **simplex** S is defined to be a polytope with the property that there exists an affinely independent family $\{x_1, x_2, \dots, x_n\}$ such that $S = \text{conv}(\{x_1, x_2, \dots, x_n\})$. The points x_1, x_2, \dots, x_n are called **vertices** of S .

Remark 2.1.27. Given that convex spaces do not have a basis as affine spaces do, we can still say that simplices have a kind of “convex basis”. In fact, if x_1, x_2, \dots, x_n are vertices of the simplex S , then by affine independence, each of the points in $\text{aff}(\{x_1, x_2, \dots, x_n\})$ has a unique representation as an affine combination of x_1, x_2, \dots, x_n ; in particular, each point in $\text{conv}(\{x_1, x_2, \dots, x_n\})$ has a unique representation as a convex combination of x_1, x_2, \dots, x_n . The following theorem states how simplices are the only type of convex spaces which have this kind of “convex basis”.

Theorem 2.1.28. *Let $M = \{x_1, x_2, \dots, x_n\}$ be a finite set of n points from a real vector space V such that the n -family (x_1, x_2, \dots, x_n) is affinely dependent. Then there are subsets M_1 and M_2 of M with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that*

$$\text{conv}(M_1) \cap \text{conv}(M_2) \neq \emptyset.$$

Thus, if $\text{conv}(M)$ is not a simplex, there exists a partition of M (i.e., subsets M_1 and M_2) such that there is an element in $\text{conv}(M)$ with a representation in both $\text{conv}(M_1)$ and $\text{conv}(M_2)$.

Corollary 2.1.29 (Radon's Theorem). [Brøndsted [7], p15] Let $M = \{x_1, x_2, \dots, x_n\}$ be a finite set of n points in V such that $n \geq \dim(V) + 2$. Then there are subsets M_1 and M_2 of M with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that

$$\text{conv}(M_1) \cap \text{conv}(M_2) \neq \emptyset.$$

2.1.2 Formal Convex Spaces

There exists a more general way of presenting convex spaces as a set equipped with formal set of operations satisfying certain equations. It was pointed out during the thesis defense that the theory of convex spaces in real vector spaces was insufficient to capture all possible models of the probabilistic choice theory. In truth, such convex sets only model the free probabilistic choice models. In order to capture the non-standard models of probabilistic choice we must consider a formal definition of convex spaces as a set equipped with formal operations satisfying the appropriate set of equations as presented by König [25].

Definition 2.1.30 (Formal Convex Space). A **formal convex space** is given by a tuple $(C, +_\lambda)$ where C is a set equipped with an $[0, 1]$ -indexed family of operators, denoted $+_\lambda$, satisfying the equations

$$\begin{aligned} \text{(P-Assoc)} \quad & (x +_{\lambda_1} y) +_{\lambda_2} z = x +_{\lambda_1 \lambda_2} \left(y +_{\frac{(1-\lambda_1)\lambda_2}{1-\lambda_1 \lambda_2}} z \right); & \lambda_1 \lambda_2 \neq 1 \\ \text{(P-Com)} \quad & x +_\lambda y = y +_{1-\lambda} x; \\ \text{(P-Idem)} \quad & x +_\lambda x = x; \\ \text{(P-One)} \quad & x +_1 y = x; \end{aligned}$$

Notation 2.1.31. Given a formal convex space $(C, +_\lambda)$, the formal combination $x +_\lambda y$ can be thought of as a process which chooses x with probability λ and y with probability $1 - \lambda$. It will also be useful to interpret n -ary convex combinations of elements in C using a single operation. Therefore, given the equations of a formal

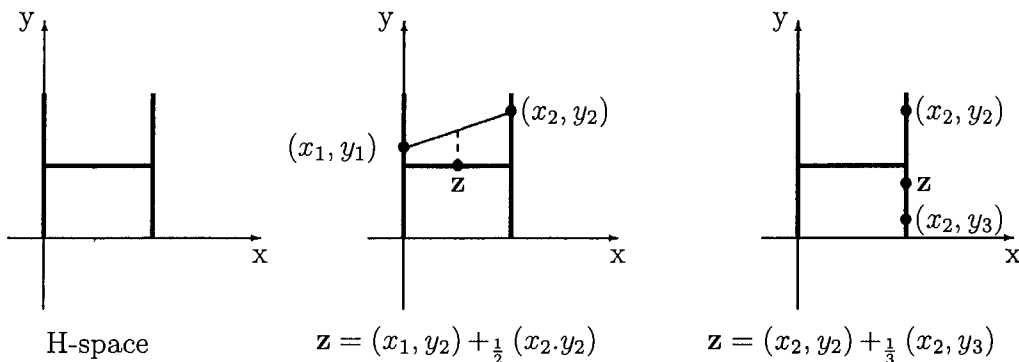


Figure 3: An example of an H-space and its convex combinations.

convex space, we can define operations of the form $\sum_{(\lambda_i)_{i=1}^n} x_i$, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \in (0, 1]$, and $x_i \in C$ for each $1 \leq i \leq n$. These operations, can be interpreted as a process which chooses the term x_i with probability λ_i , for any $1 \leq i \leq n$.

Example 2.1.32. An example of a non-standard convex space which cannot be expressed as the subspace of a particular real vector space is the *H-space*. The *H-space* is a subset of \mathbb{R}^n , consisting of two non-intersecting line segments joined together by a third segment between them (intuitively it has a shape isomorphic to the letter H). A particular *H-space* in \mathbb{R}^2 can be defined by $H = (\{0\} \times [0, 1]) \cup ((0, 1) \times \{\frac{1}{2}\}) \cup (\{1\} \times [0, 1])$, as shown in Figure 3. For any $\lambda \in (0, 1)$ and $(x_1, y_1), (x_2, y_2) \in H$ we define the convex combinations $+_\lambda$ as:

$$(x_1, y_1) +_\lambda (x_2, y_2) = \begin{cases} (0, (\lambda)y_1 + (1 - \lambda)y_2) & \text{if } x_1 = x_2 = 0 \\ (1, (\lambda)y_1 + (1 - \lambda)y_2) & \text{if } x_1 = x_2 = 1 \\ ((\lambda)x_1 + (1 - \lambda)x_2, \frac{1}{2}) & \text{otherwise} \end{cases}$$

In other words, when $x_1 \neq x_2$ we project the element $(x_1, y_1) +_\lambda (x_2, y_2) \in \mathbb{R}^2$ onto the middle segment of the *H*, as seen in Figure 3. Finally, we define the operations $+_1$ and $+_0$ by the appropriate projection operator. (I.e., $(x_1, y_1) +_1 (x_2, y_2) = (x_1, y_1)$ and $(x_1, y_1) +_0 (x_2, y_2) = (x_2, y_2)$. \triangle

We proceed by giving the formal analogs of many of the important definitions presented for convex spaces in real vector spaces.

Definition 2.1.33 (Formal Convex Combinations). Let $(C, +_\lambda)$ be a formal convex space, $(x_i)_{i=1}^n$ be an n -ary family of elements in C , and $(\lambda_i)_{i=1}^n$ be an n -ary family of elements in $(0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$. A **formal convex combination** of the elements $(x_i)_{i=1}^n$ weighted by $(\lambda_i)_{i=1}^n$ is given by any formal combination in $(C, +_\lambda)$ equivalent to $\sum_{(\lambda_i)_{i=1}^n} x_i$.

Definition 2.1.34 (Formal Convex Hulls). Let $(C, +_\lambda)$ be a formal convex space, and X be a subset of C . The **formal convex hull** in $(C, +_\lambda)$ generated by X , denoted by $\text{conv}(X)$, is given by the set of all formal convex combinations between the elements of X .

$$\text{conv}(X) = \left\{ \sum_{(\lambda_i)_{i=1}^n} x_i \mid x_i \in X, \lambda_i \in (0, 1], \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}$$

Definition 2.1.35 (Category of Formal Convex Spaces). The **category of formal convex spaces**, denoted **Conv**, is the category of formal convex spaces and convex space morphisms between them.

The following definitions, due to König [25], will lead to an important theorem stated in [26] stating which type of formal convex spaces are equivalent to a convex structure in a real vector space. This proves why our example of the H -space is not equivalent to any convex spaces in a real vector space.

Definition 2.1.36 (Cancelable/Cancellation Law). Let $(C, +_\lambda)$ be a formal convex space.

- (a) Define $x \in C$ to be **cancelable** iff for all $\lambda \in (0, 1)$, one has

$$x +_\lambda y = x +_\lambda z \Rightarrow y = z, \forall y, z \in C.$$

- (b) The formal convex set $(C, +_\lambda)$ is said to fulfill the **cancellation law** iff all $x \in C$ are cancelable.

Theorem 2.1.37 (König, Wittstock [26]). *For a formal convex space $(C, +_\lambda)$ the following are equivalent*

- (a) There exists an injective convex morphism $\theta : (C, +_\lambda) \rightarrow V$ into a real vector space V .
- (b) $(C, +_\lambda)$ satisfies the cancellation law.

Finally, we present the definition of a polytope for formal convex spaces.

Definition 2.1.38 (Formal Polytopes). Given a formal convex space $(C, +_\lambda)$, a **formal convex polytope** P , is defined to be the formal convex hull of a non-empty finite set of points in C . (I.e., $P = \text{conv}(\{x_1, x_2, \dots, x_n\})$, where $x_1, x_2, \dots, x_n \in C$)

2.1.3 Superconvex Structures in Real Vector Spaces

In this final section on convexity, we consider a generalization on the definition of convex structures in real vector spaces. We will define a “convex” structure which allows countable convex combinations of its elements. In other words, we want to allow for possibly countable weighted sums of the form $\sum_{i \in I} \lambda_i x_i$, where $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geq 0$. In order to generalize, we assume that our real vector spaces V have a well-defined sum operation of countable arity and define the notion of *superconvex space* as follows.

Definition 2.1.39 (Superconvex Space). A subset C of a vector space V over \mathbb{R} is called a **superconvex space** if $\sum_{i \in I} \lambda_i x_i$ belongs to C where I is a countable indexing set and for all $i \in I$, $x_i \in C$ and $\lambda_i \in \mathbb{R}$, with $\lambda_i \geq 0$, such that $\sum_{i \in I} \lambda_i = 1$.

Example 2.1.40 (Superconvex Space). Consider the set of positive real numbers and infinity, $\mathbb{R}^+ \cup \{\infty\}$, where convex combinations of real numbers are defined as usual, countably infinite convex combinations (*superconvex combinations*), $\sum_{i \in I} \lambda_i x_i$, of real numbers $x_i \in \mathbb{R}$ equals r if the series converges to r and equals ∞ if the series diverges, and any combination which contains the element ∞ is equal to ∞ . Under these conditions, $\mathbb{R} \cup \{\infty\}$ is a superconvex structure. \triangle

Proposition 2.1.41. *If C is a superconvex subset of a real vector space V , C is also a convex subset of V .*

The proof of the above proposition is straightforward; in any superconvex space C , convex combinations of the form $\lambda x_1 + (1 - \lambda)x_2$, where $x_1, x_2 \in C$ and $0 \leq \lambda \leq 1$ must by definition belong to C .

Remark 2.1.42. Although any superconvex space C in a real vector space V can also be considered as a convex space of V , the converse does not hold. There exist convex subsets which are not superconvex. For example, consider the real number line, \mathbb{R} . The number line is convex under the usual definition of convex combinations (usual addition (+) and scalar multiplication (\cdot) rules). However, countably infinite convex combinations (*superconvex combinations*) of real numbers do not generally equal a real number. For instance the series of the form $\sum_{n \in \mathbb{N}} (\frac{1}{2^n})n!$ is divergent and is not equivalent to any real number.

It is well-known that if the infinite sum $\sum_{i \in I} \lambda_i$ converges, where $\lambda_i \in \mathbb{R}, \lambda_i > 0$ then the indexing set I must be countable. This implies that if we consider only combinations with non-zero weights, countable convex combinations are the highest arity for which we can extend our definition of convex combinations.

Definition 2.1.43 (Superconvex Combination). A **superconvex combination** of a countable-family of points $(x_i)_{i \in I}$ from a vector space V over \mathbb{R} is a countable weighted combination $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_i x_i + \dots$, where $\sum_{i \in I} \lambda_i = 1$ and for every $i \in I$, $\lambda_i \geq 0$. Following the notation proposed by Brøndsted [7], we let

$$\sum_{(\lambda_i)_I}^{SC} x_i$$

represent a linear combination which is in fact a superconvex combination.

Notation 2.1.44. Again by convention, as in Notation 2.1.15, we omit the superscript SC from $\sum_{(\lambda_i)_I}^{SC}$ and usually write superconvex combinations as $\sum_{(\lambda_i)_I}$.

Definition 2.1.45 (Superconvex Space Morphism). Suppose we are given two superconvex spaces C in V and C' in V' . A **superconvex space morphism** is a map $f : C \rightarrow C'$ which preserves superconvex combinations. In other words, for an arbitrary superconvex combination $\sum_{(\lambda_i)_I} x_i$ in C , $f(\sum_{(\lambda_i)_I} x_i) = \sum'_{(\lambda_i)_I} f(x_i)$ is a superconvex combination in C' .

It is easy to see that the set of all superconvex spaces and the family of all superconvex space morphisms satisfy the required properties of a category.

Definition 2.1.46 (Category of Superconvex Spaces). The **category of superconvex spaces in real vector spaces**, denoted $\mathbf{SConv}_{\mathbb{R}}$, is the category of superconvex spaces in real vector spaces and superconvex structure morphisms between them.

Next we show how to generate superconvex spaces over arbitrary subsets of any real vector space.

Remark 2.1.47. The intersection of any family of superconvex spaces in the vector space V over \mathbb{R} is a superconvex space. Thus, for any subset M of V we can define the smallest superconvex space which contains M . Indeed, suppose we are given an arbitrary indexing set W and an arbitrary family of superconvex spaces $(C_w)_{w \in W}$ of the real vector space V . Suppose we have a countable family of points $(x_i)_{i \in I}$ in $\bigcap_{w \in W} C_w$; then we know that $(x_i)_{i \in I}$ is a countable family of C_w for each $w \in W$. Since each C_w is superconvex, any superconvex combination of the countable family $(x_i)_{i \in I}$, i.e. $\sum_{(\lambda_i)_I} x_i$, is contained in C_w for each $w \in W$. Therefore $\sum_{(\lambda_i)_I} x_i \in \bigcap_{w \in W} C_w$, which implies that $\bigcap_{w \in W} C_w$ is a superconvex space.

Definition 2.1.48 (Superconvex Hull). Let M be a subset of a vector space V over \mathbb{R} . The **superconvex hull** of M , denoted by $\text{sconv}(M)$, is the smallest superconvex subset in V containing M .³

Theorem 2.1.49. *For any subset M of the real vector space V , the superconvex hull $\text{sconv}(M)$ is the set of all superconvex combinations of points in M .*

Proof: Consider the set of all superconvex combinations of points in M , denoted by S . Since $M \subseteq \text{sconv}(M)$ and $\text{sconv}(M)$ is a superconvex space, then S must be a subset of $\text{sconv}(M)$. However, S is also a superconvex space (a superconvex combination of superconvex combinations of points in M can be considered as a superconvex combination of points in M) and contains M , thus $\text{sconv}(M) \subseteq S$. Therefore, $S = \text{sconv}(M)$. \square

³By a superconvex subspace of V we mean a subset in V which is also a superconvex space.

Definition 2.1.50 (Superpolytope). A **superconvex polytope** P (or simply **superpolytope**), is defined to be the superconvex hull of a non-empty finite set of points in a vector space V over \mathbb{R} . (I.e., $P = \text{sconv}(\{x_1, x_2, \dots, x_n\})$, where $x_1, x_2, \dots, x_n \in V$)

2.1.4 Formal Superconvex Spaces

A second important remark that we pointed out during the thesis defense is that the theory of superconvex spaces which we have independently developed and presented above has been previously studied by a group of German researchers [6, 25, 26, 42]. However, their theory is based on a formal definition of superconvex spaces, given by a set equipped with formal operators satisfying some equations.

Definition 2.1.51 (Formal Superconvex Spaces). A **formal superconvex space** is given by a tuple $(C, \sum_{(\lambda_i)_I})$ where C is a set equipped with a countable family of operators, denoted $\sum_{(\lambda_i)_I}$, indexed by countable probability densities (i.e. countable sequences $(\lambda_i)_I$ of elements in $(0, 1]$ such that $\sum_{i \in I} \lambda_i = 1$) satisfying the equations

$$\begin{aligned} \text{(P-Ax1)} \quad & \sum_{(\lambda_i)_I} x_i = \sum_{((\lambda_{\phi(i)})_I)} x_{\phi(i)}, \quad \text{where } \phi : I \rightarrow I \text{ is a permutation.} \\ \text{(P-Idem)} \quad & \sum_{(\lambda_i)_I} x = x \end{aligned}$$

All corresponding definitions of superconvex combinations, superconvex hulls, and superconvex polytopes can be given for a formal superconvex space as done in Section 2.1.2. We let **SConv** denote the category of formal superconvex spaces and superconvex space morphisms.

As for formal convex spaces there is a very interesting theorem presented in [26] that characterizes which formal superconvex spaces are equivalent to a superconvex subspace of a real vector space.

Theorem 2.1.52. *For a formal superconvex space $(C, \sum_{(\lambda_i)_I})$ the following are equivalent.*

- (a) $(C, \sum_{(\lambda_i)_I})$ satisfies the cancellation law.

(b) $(C, \sum_{(\lambda_i)_I})$ is isomorphic to some superconvex subset $Y \subseteq V$ of a real vector space V .

2.2 Monads

Moggi [34] has proven that the categorical notion of a monad is an important tool for capturing computational behavior. The aim of our present work is to describe an alternate construction which captures the computational behavior of mixed choice. Therefore, the notion of monads and their properties are of central importance to the thesis.

Definition 2.2.1 (Monads). A **monad**, \mathbb{T} , on a category \mathbf{C} is a 3-tuple $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ where $\mathcal{T} : \mathbf{C} \rightarrow \mathbf{C}$ is a functor and $\eta : 1_{\mathbf{C}} \Rightarrow \mathcal{T}$ (**the unit**), $\mu : \mathcal{T}^2 \Rightarrow \mathcal{T}$ (**the multiplication**) are natural transformations satisfying the commutativity conditions $\mu \circ (\eta_{\mathcal{T}}) = 1_{\mathcal{T}} = \mu \circ (\mathcal{T}\eta)$, and $\mu \circ \mu_{\mathcal{T}} = \mu \circ \mathcal{T}\mu$, as shown in the following diagrams.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\mathcal{T}\eta} & \mathcal{T}^2 & \xleftarrow{\eta_{\mathcal{T}}} & \mathcal{T} \\ & \searrow & \downarrow \mu & \swarrow & \\ & & \mathcal{T} & & \end{array} \qquad \begin{array}{ccc} \mathcal{T}^3 & \xrightarrow{\mathcal{T}\mu} & \mathcal{T}^2 \\ \mu_{\mathcal{T}} \downarrow & & \downarrow \mu \\ \mathcal{T}^2 & \xrightarrow{\mu} & \mathcal{T} \end{array}$$

One important example of a monad is the *powerset monad*, \mathbb{P} . It (and its many variants) are the standard monads used to capture the computational behavior of many types of nondeterministic choice, as discussed in [40]. It will be a reoccurring construct in the following chapters.

Example 2.2.2 (Powerset Monad). The **powerset monad**, $\mathbb{P} = (\mathcal{P}, \eta, \mu)$, is the monad over the category **Set**, defined as follows:

(a) $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ is the **powerset functor**. For $X, Y \in \mathbf{Set}$, $f : X \rightarrow Y$ in **Set**, and $A \in \mathcal{P}(X)$,

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}, \quad (\text{the set of all subsets of } X),$$

$$\mathcal{P}f(A) = f[A] = \{f(a) \mid a \in A\}, \quad (\text{the direct image of } f \text{ on } A).$$

(b) $\eta : 1_{\mathbf{Set}} \Rightarrow \mathcal{P}$ is the **singleton transformation**. For $X \in \mathbf{Set}$ and $x \in X$,

$$\eta_X(x) = \{x\}.$$

(c) $\mu : \mathcal{P}^2 \Rightarrow \mathcal{P}$ is the **big union transformation**. For $X \in \mathbf{Set}$, $\mathcal{Y} \in \mathcal{P}^2(X)$,

$$\mu_X(\mathcal{Y}) = \bigcup_{Y \in \mathcal{Y}} Y.$$

Many variants of the powerset monad can be constructed by taking variations of the underlying functor \mathcal{P} , yet keeping the same definition (on an appropriately restricted domain) for the unit (η) and the multiplication (μ) transformations.

(a) The **non-empty powerset monad**, $\mathbb{P}^* = (\mathcal{P}^*, \eta, \mu)$. Let $X \in \mathbf{Set}$. The underlying (object part of the) functor is defined by:

$$\mathcal{P}^*(X) = \{Y \mid Y \in \mathcal{P}(X), Y \neq \emptyset\}.$$

(b) The **finite powerset monad**, $\mathbb{P}_{fin} = (\mathcal{P}_{fin}, \eta, \mu)$. Let $X \in \mathbf{Set}$. The underlying (object part of the) functor is defined by:

$$\mathcal{P}_{fin}(X) = \{Y \mid Y \in \mathcal{P}(X), |Y| < \infty\}.$$

(c) The **non-empty finite powerset monad**, $\mathbb{P}_{fin}^* = (\mathcal{P}_{fin}^*, \eta, \mu)$. Let $X \in \mathbf{Set}$. The underlying (object part of the) functor is defined by:

$$\mathcal{P}_{fin}^*(X) = \{Y \mid Y \in \mathcal{P}(X), 0 < |Y| < \infty\}.$$

△

A second monad we will repeatedly come across, is the monad which captures the computational behavior of probabilistic choice; the *distributions monad*, \mathbb{D}_{fin} . This construction arises from [20]. However, before presenting its formal definition, we define a few basic concepts about particular maps from a set X into $[0, 1]$, called *distributions*.

Definition 2.2.3 (Distributions/Support). Let X be a set.

- (a) A **distribution** (discrete probabilistic distribution) on X is a function which associates to each element of X a real value from the unit interval $[0, 1]$ such that the sum of the images is equal to 1. In other words, $d : X \rightarrow [0, 1]$ is such that $\sum_{x \in X} d(x) = 1$.
- (b) The **support** of a distribution d , denoted $\text{supp}(d)$, is the subset of X consisting of all the elements which are mapped to non-zero values. (I.e., $\text{supp}(d) = \{x \in X \mid d(x) \neq 0\}$).
- (c) For $x \in X$, the **Dirac distribution** (or **point-mass distribution**) over x , $\delta_x : X \rightarrow [0, 1]$, is the distribution whose support is $\{x\}$. In other words, δ_x is defined by:

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.2.4. Consider the set X and the distribution d on X .

- (a) The support of d must be either finite or countably infinite. Otherwise, the requirement that the sum of the images of the distribution must equal 1 cannot be satisfied, since the sum of an uncountable family of nonzero positive reals does not converge (and therefore cannot be 1). We say that a distribution d is **finite**, if it has a finite support, (i.e., $|\text{supp}(d)| < \infty$).
- (b) We can easily extend the definition of d to subsets Y of X by taking $d(Y)$ to represent the sum of the images of the elements of Y . Hence,

$$d(Y) = \sum_{y \in Y} d(y).$$

Example 2.2.5 (Distributions Monad). The **distributions monad**, $\mathbb{D}_{fin} = (\mathcal{D}_{fin}, \eta, \mu)$, is the monad over the category **Set**, defined as follows:

- (a) $\mathcal{D}_{fin} : \mathbf{Set} \rightarrow \mathbf{Set}$, is the **distributions functor**. For $X, Y \in \mathbf{Set}$, $f : X \rightarrow Y$ in **Set**, and $d \in \mathcal{D}_{fin}(X)$,

- (i) $\mathcal{D}_{fin}(X)$ is the set of all finite distributions over X . (i.e $\mathcal{D}_{fin}(X) = \{d : X \rightarrow [0, 1] \mid d(X) = 1 \text{ and } |supp(d)| < \infty\}$),
- (ii) $\mathcal{D}_{fin}f(d) = d_f : Y \rightarrow [0, 1]$, where

$$d_f(y) = \sum_{\{x \in X \mid f(x)=y\}} d(x).$$

- (b) The unit (η) is given by the **Dirac distribution transformation**. Let $X \in \mathbf{Set}$ and $x \in X$,

$$\eta_X(x) = \delta_x.$$

- (c) The multiplication (μ) is given by the **distribution flattening transformation**. Let $X \in \mathbf{Set}$ and $D \in (\mathcal{D}_{fin})^2(X)$,

$$\mu_X(D) = \sum_{(D(d))_{supp(D)}} \left(\sum_{(d(x))_{supp(d)}} \delta_x \right).$$

Remark 2.2.6. One can easily note that the multiplication of the distributions monad is simply given by integration.

△

Next we recall the notion of categories generated by a monad. The construction which interests us the most is the Eilenberg-Moore categories of our given monads. These will be shown to be isomorphic to the model categories capturing the choice theories we will be studying.

Definition 2.2.7 (\mathbb{T} -Algebras). Let $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ be a monad on a category \mathbf{C} . A **\mathbb{T} -algebra** is a pair (C, ξ) where for C in \mathbf{C} , $\xi : \mathcal{T}(C) \rightarrow C$ and the following equations hold:

$$\xi \circ \eta_C = id_C \quad \text{and} \quad \xi \circ \mathcal{T}\xi = \xi \circ \mu_C,$$

captured by the following diagrams.

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & \mathcal{T}(C) \\ & \searrow & \downarrow \xi \\ & & C \end{array} \qquad \begin{array}{ccc} \mathcal{T}^2(C) & \xrightarrow{\mu_C} & \mathcal{T}(C) \\ \mathcal{T}\xi \downarrow & & \downarrow \xi \\ \mathcal{T}(C) & \xrightarrow{\xi} & C \end{array}$$

If (D, ζ) is another \mathbb{T} -algebra, a morphism $f : (C, \xi) \rightarrow (D, \zeta)$ of \mathbb{T} -algebras is a morphism $f : C \rightarrow D$ in \mathbf{C} such that $f \circ \xi = \zeta \circ Tf$, as shown in the following diagram.

$$\begin{array}{ccc} T(C) & \xrightarrow{Tf} & T(D) \\ \xi \downarrow & & \downarrow \zeta \\ C & \xrightarrow{f} & D \end{array}$$

Proposition 2.2.8. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category \mathbf{C} . The \mathbb{T} -algebras and their morphisms form a category, written $\mathbf{C}^{\mathbb{T}}$, called the **Eilenberg-Moore category** of the monad.*

The Eilenberg-Moore categories of the family of powerset monads are associated to variations of semilattice categories. Below, we present a few variants of \vee -semilattice categories in order to illustrate some examples of the Eilenberg-Moore categories of particular powerset monads.

Since we shall be considering choice structures with infinite nondeterministic choice, we shall need to consider semilattices which have arbitrary joins, defined below as a *complete \vee -semilattice*

Definition 2.2.9 (\vee -Semilattice).

(a) A \vee -**semilattice** (S, \vee) , is an algebraic structure consisting of a set S with the binary operation \vee , called join, such that for all members x, y , and z of S , the following identities hold:

(i) Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$

(ii) Commutativity: $x \vee y = y \vee x$

(iii) Idempotency: $x \vee x = x$

(b) A map $f : (S, \vee) \rightarrow (S', \vee')$ is called a \vee -**semilattice morphism**, if it satisfies the following property:

$$f(x \vee y) = f(x) \vee' f(y)$$

(c) We denote by \vee -**SLat**, the **category of \vee -semilattices** and \vee -semilattice morphisms.

Definition 2.2.10 (Complete \vee -Semilattices).

- (a) A **complete \vee -semilattice** is a \vee -semilattice (S, \vee) such that the binary operation can be extended to arbitrary subsets of S . Thus, for any non-empty subset $T \subseteq S$, $\bigvee_{t \in T} t$ is well-defined. In other words, a complete \vee -semilattice is equivalent to a complete lattice which may or may not have a bottom element.
- (b) A **complete \vee -semilattice morphism** $f : (S, \vee) \rightarrow (S', \vee')$, is a \vee -semilattice morphism which preserves arbitrary non-empty joins (i.e., for any $\emptyset \neq T \subseteq S$, $f(\bigvee_{t \in T} t) = \bigvee_{t \in T} f(t)$).
- (c) We denote by $\vee\text{-CSLat}$, the **category of complete \vee -semilattices** and complete \vee -semilattice morphisms.

Definition 2.2.11 (Ordered \vee -Semilattices).

- (a) An **ordered \vee -semilattice** is a \vee -semilattice (S, \vee) equipped with a \vee -preserving order structure (\sqsubseteq) , denoted as $((S, \vee), \sqsubseteq)$ (\sqsubseteq need not be the order induced by the join structure of the semilattice). Thus, we require that the join structure $\vee : S \times S \rightarrow S$ be monotone:

$$\text{if } s \sqsubseteq t \text{ and } s' \sqsubseteq t', \text{ then } s \vee s' \sqsubseteq t \vee t'.$$

- (b) An **ordered \vee -semilattice morphism** $f : ((S, \vee), \sqsubseteq) \rightarrow ((S', \vee'), \sqsubseteq')$ is a \vee -semilattice morphism which also preserves \sqsubseteq (i.e., $s \sqsubseteq t \Rightarrow f(s) \sqsubseteq' f(t)$).
- (c) We denote by $\vee\text{-SLat}_{\leq}$, the **category of ordered \vee -semilattices** and ordered \vee -semilattice morphisms.

Example 2.2.12 (Ordered \vee -Semilattices).

- (a) Every \vee -semilattice can be viewed as an ordered \vee -semilattice. For any \vee -semilattice (S, \vee) , we can define an ordering induced by the \vee -structure (\leq_{\vee}) (i.e. $x \leq_{\vee} y$ iff $x \vee y = y$). Then, $((S, \vee), \leq_{\vee})$ is an ordered \vee -semilattice. Moreover, any \vee -semilattice morphism between (S, \vee) and (S', \vee') also preserves the induced orders (i.e., $x \leq_{\vee} y \Rightarrow f(x) \leq_{\vee'} f(y)$). Thus, $\vee\text{-SLat}$ is a subcategory of $\vee\text{-SLat}_{\leq}$.

- (b) There are many ordered \vee -semilattices whose order structure is different from the one induced from its inherent join structure. An easy example is the linear \vee -semilattice (S, \vee) where $S = \{x, y, z\}$, modelled by the left-hand semilattice structure in Figure 4 equipped with an ordering generated by $x \sqsubseteq z$ and $y \sqsubseteq z$. (I.e., the ordering generated by the right-hand semilattice structure in Figure 4). It is easy to verify that in this particular case $((S, \vee), \sqsubseteq)$ is indeed an ordered \vee -semilattice where $x \leq_{\vee} y$ but $x \not\sqsubseteq y$. This shows that $\vee\text{-SLat}$ is a proper

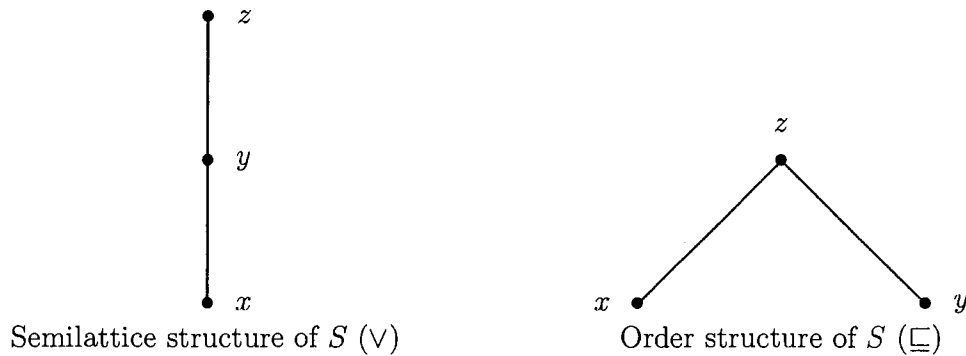


Figure 4: Example of an ordered \vee -semilattice with \sqsubseteq not generated by \vee subcategory of $\vee\text{-SLat}_{\leq}$.

△

Example 2.2.13 (Eilenberg-Moore Categories of Powerset Monads). The Eilenberg-Moore categories associated to the powerset monad and its many variations are equivalent to different types of semilattice categories. In the following chapters, we will come across the following:

- (a) The Eilenberg-Moore category of the non-empty finite powerset monad over \mathbf{Set} , $\mathbf{Set}^{\mathbf{P}^*_{fin}}$, is equivalent to the category of \vee -semilattices, $\vee\text{-SLat}$.
- (b) The Eilenberg-Moore category of the non-empty powerset monad over \mathbf{Set} , $\mathbf{Set}^{\mathbf{P}^*}$, is equivalent to the category of complete \vee -semilattices, $\vee\text{-CSLat}$.

- (c) The Eilenberg-Moore category of the non-empty finite powerset monad over \mathbf{Poset} , $\mathbf{Poset}^{\mathbb{P}_{fin}}$, is equivalent to the category of ordered *join*-semilattices, $\mathbf{V-SLat}_{\leq}$.

△

Example 2.2.14 (Eilenberg-Moore Categories of the Distribution Monad). The Eilenberg-Moore categories associated to the distribution monad and its countable variants are equivalent to the category \mathbf{Conv} of convex sets defined in Section 2.1.1 and \mathbf{SConv} of superconvex sets defined in Section 2.1.3, respectively. In other words, the category of \mathbb{D}_{fin} -algebras $\mathbf{Set}^{\mathbb{D}_{fin}^4}$ is equivalent to the category of convex sets \mathbf{Conv} and the category of \mathbb{D} -algebras $\mathbf{Set}^{\mathbb{D}}$ is equivalent to the category \mathbf{SConv} of superconvex sets.

Indeed, any \mathbb{D}_{fin} -algebra $(X, \alpha : \mathcal{U}_{\mathcal{D}_{fin}} \circ \mathcal{D}_{fin}(X) \rightarrow X)$ gives rise to a convex set $(X, +_{\lambda})$, where for $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ the convex combination $x_1 +_{\lambda} x_2$ is defined to be $\alpha(\lambda\delta_{x_1} + (1 - \lambda)\delta_{x_2})$. Recall that maps of the form δ_x are called *Dirac distribution*, they represent distributions whose support contains a single element (i.e., $\text{supp}(\delta_x) = \{x\}$). Moreover, any convex set C gives rise to a \mathbb{D}_{fin} -algebra (C, β) , where for a distribution $d \in \mathcal{U}_{\mathcal{D}_{fin}} \circ \mathcal{D}_{fin}(C)$, we define $\beta(d) = \sum_{x \in \text{supp}(d)} d(x)x$.

Similarly, by carefully extending our previous remarks to countably infinite convex combinations, we can find the correspondence between \mathbb{D} -algebras and superconvex sets. △

It will be important to recall the correspondence between monads and adjunctions. A large portion of our work is concerned with constructing the appropriate left adjoint to a particular forgetful functor in order to obtain a monad over the category of models of nondeterministic choice.

Theorem 2.2.15 (Huber; see Lambek and Scott [27]). *Let $\mathcal{U} : \mathbf{B} \rightarrow \mathbf{C}$ have a left adjoint $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{B}$ with adjunction morphisms $\eta : 1 \rightarrow \mathcal{U}\mathcal{F}$ and $\varepsilon : \mathcal{F}\mathcal{U} \rightarrow 1$. Then $\mathbb{T} = (\mathcal{U}\mathcal{F}, \eta, \mathcal{U}\varepsilon\mathcal{F})$ is a monad on \mathbf{C} .*

⁴Recall $\mathbf{C}^{\mathbb{T}}$ denotes the category of \mathbb{T} -algebras over \mathbf{C} .

The left adjoint which constructs free mixed choice models over arbitrary non-deterministic choice models exists due to an abstract categorical result. Before stating the result, we first introduce some necessary notions from Barr and Wells [2].

Definition 2.2.16 (Eilenberg-Moore Comparison Functor). Given a monad $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ on \mathbf{C} and an adjoint pair of functors, $\mathcal{F} \dashv \mathcal{U}$ where $\mathcal{U} : \mathbf{D} \rightarrow \mathbf{C}$ and $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$ such that $\mathcal{T} = \mathcal{U} \circ \mathcal{F}$. Then, there exists a functor $\Phi : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$, called the **Eilenberg-Moore comparison functor**, defined by: for $D \in \mathbf{D}$ and $f : D \rightarrow D'$ in \mathbf{D} ,

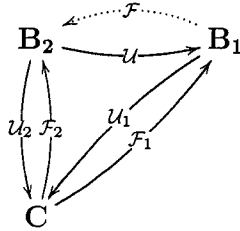
$$\begin{aligned}\Phi(D) &= (\mathcal{U}(D), \mathcal{U}\epsilon_D), \\ \Phi(f) &= \mathcal{U}(f).\end{aligned}$$

Definition 2.2.17 (Descent Type). A functor $\mathcal{U} : \mathbf{D} \rightarrow \mathbf{C}$ is of **descent type** if it satisfies the following properties:

- (a) \mathcal{U} has a left adjoint $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$,
- (b) Let \mathbb{T} be the monad induced by the adjunction $\mathcal{F} \dashv \mathcal{U}$. Then the Eilenberg-Moore comparison functor $\Phi : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ is full and faithful.

Then we obtain the following theorem on the existence of left adjoints.

Theorem 2.2.18 (Barr and Wells [2], Theorem 2 (b), p.132). *In the following diagram (not assumed to be commutative) of categories and functors, suppose that*



- (a) \mathcal{F}_2 is left adjoint to \mathcal{U}_2 ,
- (b) \mathcal{F}_1 is left adjoint to \mathcal{U}_1 ,
- (c) $\mathcal{U}_1 \circ \mathcal{U}$ is naturally isomorphic to \mathcal{U}_2 ,
- (d) \mathcal{U}_1 is of descent type, and
- (e) \mathbf{B}_2 has coequalizers.

Then \mathcal{U} has a left adjoint \mathcal{F} for which $\mathcal{F} \circ \mathcal{F}_1 \cong \mathcal{F}_2$.

2.3 Lawvere Theories

In this section, we recall the general approach of capturing specific structures characterized by the existence of one or several operations which are defined everywhere and satisfy axioms expressed by equalities. Thus, we shall recall some definitions and results concerning Lawvere theories and their possible combinations. In our case, we will focus our attention on well-known structures: the nondeterministic and probabilistic choice structures. Moreover, in the course of this thesis, it is our intention to work with a third choice structure, called mixed choice. This new choice structure is the result of a particular type of combination of the nondeterministic and probabilistic structures. We begin by recalling some basic definitions. For further references one may consult Barr and Wells [2] and Borceux [5].

We begin by recalling the definitions of finite and countable Lawvere theories, as presented in [19]. Let \aleph_0 denote a skeleton of the category of finite sets and all functions between them. So \aleph_0 has an object for each natural number n . Up to equivalence, \aleph_0 is the free category with finite coproducts on 1. Let \aleph_1 denote a skeleton of the category of countable sets and all functions between them. So \aleph_1 has an object for each natural number n and an object for \aleph_0 . Up to equivalence, \aleph_1 is the free category with countable coproducts on 1.

Definition 2.3.1 (Lawvere Theories).

- (a) A **Lawvere theory** consists of a small category \mathbf{L} with finite products together with a strict finite-product-preserving identity-on-objects functor $I : \aleph_0^{op} \rightarrow \mathbf{L}$ [2, 3].
- (b) A **Countable Lawvere theory** is given by the natural extension of a Lawvere theory in order to allow for countable operations, as stated in [18, 19]. A countable Lawvere theory consists of a small category \mathbf{L} with countable products together with a strict countable-product-preserving identity-on-objects functor $I : \aleph_1^{op} \rightarrow \mathbf{L}$.

The important Lawvere theories we shall be considering are presented in the following example.

Example 2.3.2 (Lawvere Theory of Nondeterministic Choice). The Lawvere theory \mathbf{ND}_{fin}^* for (binary) nondeterminism is the Lawvere theory freely generated by a binary operation $\boxplus : 2 \rightarrow 1$, subject to equations for associativity (**ND-Assoc**), commutativity (**ND-Com**) and idempotence (**ND-Idem**) (i.e., the Lawvere theory for a semilattice). \triangle

Example 2.3.3 (Lawvere Theory of Probabilistic Choice). The Lawvere theory \mathbf{P}_{fin} for probabilistic choice is freely generated by an uncountable number (indexed by $\lambda \in [0, 1]$) of binary operations $\oplus_\lambda : 2 \rightarrow 1$, subject to the equations for associativity, commutativity and idempotence in [16], that is

(i) (**P-Assoc**) For any $\lambda, \rho \in [0, 1]$ such that $\lambda\rho \neq 1$,

$$(A \oplus_\lambda B) \oplus_\rho C = A \oplus_{\lambda\rho} (B \oplus_{\frac{(1-\lambda)\rho}{1-\lambda\rho}} C)$$

(ii) (**P-Com**) For any $\lambda \in [0, 1]$,

$$A \oplus_\lambda B = B \oplus_{1-\lambda} A$$

(iii) (**P-Idem**) For any $\lambda \in [0, 1]$,

$$A \oplus_\lambda A = A$$

\triangle

Definition 2.3.4 (Algebraic Theory Model). Let \mathbf{L} be a (countable) Lawvere theory and \mathbf{C} be a category with finite (countable) products. A **model of \mathbf{L} in \mathbf{C}** is a functor $\mathcal{M} : \mathbf{L} \rightarrow \mathbf{C}$ which preserves finite (countable) products. We denote by $\mathbf{Mod}(\mathbf{L}, \mathbf{C})$, the model category of \mathbf{L} in \mathbf{C} . The model category $\mathbf{Mod}(\mathbf{L}, \mathbf{C})$ has as objects the set of models of \mathbf{L} in \mathbf{C} and as morphisms the set of all natural transformations between models.

Remark 2.3.5. An equivalent presentation for models of an arbitrary theory \mathbf{T} can be attained by using universal algebra. In lieu of describing a model as an appropriately (finite or countable) product-preserving functor from \mathbf{T} into \mathbf{C} , we shall present its underlying set (given by $F(1)$) and the set of generating operations (obtained by applying F to the operations generating \mathbf{T}).

Example 2.3.6 (Models of Nondeterministic Theories).

- (a) It is easy to verify that any model of \mathbf{ND}_{fin}^* in \mathbf{Set} is equivalent to a \vee -semilattice, (S, \vee) .
- (b) Consider the theory of arbitrary nondeterministic choice \mathbf{ND}^* (i.e., the theory which admits a nondeterministic choice operator of arbitrary arity). Models of \mathbf{ND}^* over \mathbf{Set} are given by complete \vee -semilattices, (S, \vee) .
- (c) Finally, one may consider the models of \mathbf{ND}_{fin}^* in \mathbf{Poset} , the category of partially ordered sets and order-preserving maps. These models are equivalent to ordered \vee -semilattices, $((S, \vee), \sqsubseteq)$.

△

Example 2.3.7 (Models of Probabilistic Choice).

- (a) It is easy to verify that any model of \mathbf{P}_{fin} in \mathbf{Set} is equivalent to a convex set, $(C, +_\lambda)$.
- (b) Consider the countable Lawvere theory of probabilistic choice \mathbf{P} (i.e., the theory which admits a probabilistic choice operator of countable arity). Models of \mathbf{P} over \mathbf{Set} are given by superconvex sets, $(C, \sum_{(\lambda_i)_I})$.

△

In order to state the next result we recall the definition of monads with rank, see Borceux [5].

Definition 2.3.8 (Regular Cardinal). An infinite cardinal α is regular when it satisfies

$$(|I| < \alpha \text{ and } (\forall i \in I) |X_i| < \alpha) \Rightarrow \left| \bigcup_{i \in I} X_i \right| < \alpha$$

where I and X_i are arbitrary sets.

Definition 2.3.9 (Filtered Category). Let α be a regular cardinal. A category \mathbf{C} is α -filtered when

- (i) there exists at least one object in \mathbf{C} ,
- (ii) given a set I with cardinality of I strictly less than α and a family $(C_i)_{i \in I}$ of objects in \mathbf{C} , there exists an object $C \in \mathbf{C}$ and morphisms $f_i : C_i \rightarrow C$ in \mathbf{C} ,
- (iii) given a set I with cardinality strictly less than α and a family $(f_i : C \rightarrow C')_{i \in I}$ in \mathbf{C} , there exists an object $C'' \in \mathbf{C}$ and a morphism $f : C' \rightarrow C''$ such that $f \circ f_i = f \circ f_j$, for all indices i, j .

Definition 2.3.10 (Filtered Colimits). Let α be a regular cardinal. By an α -filtered colimit in a category \mathbf{C} , we mean the colimit of a functor $F : \mathbf{D} \rightarrow \mathbf{C}$ where the category \mathbf{D} is α -filtered.

Definition 2.3.11 (Rank). A monad $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ on a category \mathbf{C} has **rank** α , for some regular cardinal α , when the functor $T : \mathbf{C} \rightarrow \mathbf{C}$ preserves α -filtered colimits. When $\alpha = \aleph_0$, thus when T preserves filtered colimits, one also says that \mathcal{T} is **finitary** or has **finite rank**.

The following result from Borceux [5], states the correspondence between \mathbb{T} -algebra categories of finitary monads over the category \mathbf{Set} and categories of algebraic theories \mathbf{T} , due to Lawvere.

Proposition 2.3.12. *There is an equivalence between;*

- (a) *the category $\mathbf{Mod}(\mathbf{L}, \mathbf{Set})$ of models of the Lawvere theory \mathbf{L} in \mathbf{Set} ;*
- (b) *the category of \mathbb{T} -algebras, the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{T}}$, where \mathbb{T} is a finitary monad on \mathbf{Set} .*

Example 2.3.13. As seen in the literature [19, 34, 50, 51], the model categories for our choice theories: nondeterministic \mathbf{ND}_{fin}^* and probabilistic \mathbf{P}_{fin} , are induced by the nonempty finite powerset monad \mathbb{P}_{fin}^* and the finite distributions monad \mathbb{D}_{fin} respectively. \triangle

2.4 Combining Lawvere Theories

There is an extensive literature dealing with how to combine Lawvere theories in order to create new theories. One may refer to [17, 18, 19, 41], for full details. Since we are interested in combining nondeterministic and probabilistic choice theories we shall give a brief description on the many ways this could be done.

2.4.1 The Sum of Lawvere Theories

The easiest method to combine two Lawvere theories is to consider their sum. As stated in many papers [18, 19, 41], this involves simply taking the coproduct between the two Lawvere theories which exists due to the following result. We present some of the definitions and results obtained in these papers.

Theorem 2.4.1. *The category of countable Lawvere theories is cocomplete.*

Remark 2.4.2 (Power [41]). An explicit construction of the sum is complicated as a general construction involves a transfinite induction, with inductive steps being given by a complicated coequalizer, cf [24].

An intuitive interpretation of the *sum of equational Lawvere theories* is given as follows. The sum of two equational theories \mathbf{L} and \mathbf{L}' , denoted by $\mathbf{L} + \mathbf{L}'$, is the Lawvere theory which admits all the operators found in both theories and which satisfies all their respective axioms.

Remark 2.4.3. This is still not as straightforward as it sounds, since we must still calculate the free Lawvere theory from the induced one. In other words, we wish to represent the model category for the sum of the two theories as a construction obtained using the individual theories. We now describe this process.

Definition 2.4.4. Denote by $\mathbf{Mod}^*(\mathbf{L}, \mathbf{C})$ the identity-on-objects/fully-faithful factorization of the forgetful functor $\mathcal{U} : \mathbf{Mod}(\mathbf{L}, \mathbf{C}) \rightarrow \mathbf{Set}$. Thus we have the following

factorization induced by the product-preserving structure of $F \in \mathbf{Mod}(\mathbf{L}, \mathbf{C})$:

$$\begin{array}{ccc} \mathbf{Mod}(\mathbf{L}, \mathbf{C}) & \xrightarrow{U} & \mathbf{Set} \\ & \searrow & \nearrow \\ & \mathbf{Mod}^*(\mathbf{L}, \mathbf{C}) & \end{array}$$

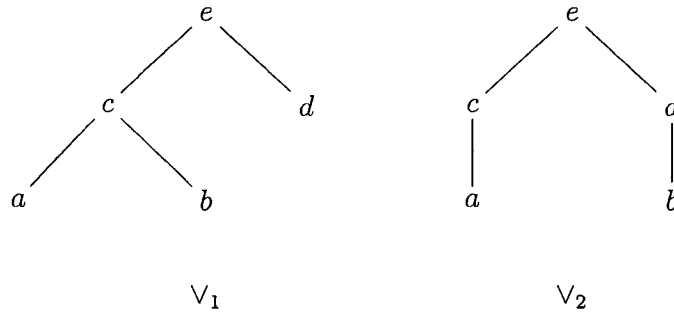
Remark 2.4.5. Thus, the objects of $\mathbf{Mod}^*(\mathbf{L}, \mathbf{C})$ are models of \mathbf{L} in \mathbf{C} , and arrows are simply given by arrows in \mathbf{C} .

Now, we can present the correspondence between the sum of the theories and the individual theories as given in Power [41].

Proposition 2.4.6 (Power [41]). *There is an equivalence of categories between $\mathbf{Mod}^*(\mathbf{L} + \mathbf{L}', \mathbf{C})$ and $\mathbf{Mod}^*(\mathbf{L}, \mathbf{Mod}^*(\mathbf{L}', \mathbf{C}))$.*

Example 2.4.7 (Models of $\mathbf{ND}_{fin}^* + \mathbf{P}_{fin}$).

- (a) Trivial model: Since any semilattice $(S, \vee) \in \mathbf{V-SLat}$ is also a model for probabilistic choice, where $(\forall \lambda \in (0, 1)) \oplus_\lambda = \vee$, then (S, \vee) is a model for the sum of \mathbf{P}_{fin} and \mathbf{ND}_{fin}^* , although this model satisfies more equations than required by the theory. For example, if $\oplus_\lambda = \vee$, we trivially have that the operators commute $(A \oplus_\lambda B) \vee C = (A \vee B) \oplus_\lambda C$, both distributivity axioms hold $(A \oplus_\lambda B) \vee C = (A \vee C) \oplus_\lambda (B \vee C)$ and $(A \vee B) \oplus_\lambda C = (A \oplus_\lambda C) \vee (B \oplus_\lambda C)$ and others.
- (b) Consider a set S with two semilattice structures, \vee_1 and \vee_2 . Then, by letting $\boxplus = \vee_1$ and defining $\oplus_\lambda = \vee_2$ for all $\lambda \in (0, 1)$, we have constructed another model for $\mathbf{P}_{fin} + \mathbf{ND}_{fin}^*$. For example, consider the set $S = \{a, b, c, d\}$ with the following semilattice operators:



Then $(S, \vee_1, \oplus_\lambda)$, where for each $\lambda \in (0, 1)$ we let $\oplus_\lambda = \vee_2$, is a model for $\mathbf{ND}_{fin}^* + \mathbf{P}_{fin}$.

- (c) Consider any set S equipped with a semilattice structure (S, \vee) and with a convex structure (S, \oplus_λ) , where the two choice operations are distinct (i.e., $\oplus_\lambda \neq \vee$). Then, it too is a model of $\mathbf{ND}_{fin}^* + \mathbf{P}_{fin}$. For example, consider $[0, 1]$, with $x \boxplus y = \max(x, y)$ and $x \oplus_\lambda y = \lambda x + (1 - \lambda)y$.

△

2.4.2 The Tensor of Lawvere Theories

The tensor of two Lawvere theories \mathbf{L} and \mathbf{L}' is the Lawvere theory which contains the operators and equations of both theories and adds equations so that all the operators from \mathbf{L} commute with the operators of \mathbf{L}' and vice-versa.

Definition 2.4.8. Given countable Lawvere theories \mathbf{L} and \mathbf{L}' the countable Lawvere theory $\mathbf{L} \otimes \mathbf{L}'$, called the **tensor product** of \mathbf{L} and \mathbf{L}' , is defined by the universal property of having maps of countable Lawvere theories from \mathbf{L} and \mathbf{L}' to $\mathbf{L} \otimes \mathbf{L}'$, with commutativity of all operations of \mathbf{L} with respect to all operations of \mathbf{L}' , i.e. given $f : m \rightarrow n$ in \mathbf{L} and $f' : m' \rightarrow n'$ in \mathbf{L}' , we demand commutativity of the diagram:

$$\begin{array}{ccc} m \times m' & \xrightarrow{m \times f'} & m \times n' \\ f \times m' \downarrow & & \downarrow f \times n' \\ n \times m' & \xrightarrow{n \times f'} & n \times n' \end{array}$$

As discussed in papers by Power [41] and Hyland et al [18, 19], we have the following results.

Proposition 2.4.9. *There is an extension of the tensor product \otimes defined above to a symmetric monoidal structure on the category of countable Lawvere theories.*

Proposition 2.4.10. *There is an equivalence of categories between $\mathbf{Mod}(\mathbf{L} \otimes \mathbf{L}', \mathbf{C})$ and $\mathbf{Mod}(\mathbf{L}, \mathbf{Mod}(\mathbf{L}', \mathbf{C}))$.*

Example 2.4.11 ($\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$). The Lawvere theory $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$ is generated by the operations and equations from \mathbf{ND}_{fin}^* and \mathbf{P}_{fin} , in addition to equations allowing the operations from \mathbf{ND}_{fin}^* and \mathbf{P}_{fin} to commute with each other.

Thus for each $\lambda \in [0, 1]$, we have the equation

$$(A \oplus_\lambda B) \boxplus (C \oplus_\lambda D) = (A \boxplus C) \oplus_\lambda (B \boxplus D). \quad (\otimes\text{-Com})$$

△

The following is our result showing that the only models of the theory arising from the tensor product between the theory of nondeterministic choice \mathbf{ND}_{fin}^* and the theory of probabilistic choice \mathbf{P}_{fin} must .

Proposition 2.4.12. $\text{Mod}(\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}, \mathbf{Set})$ is equivalent to the category $\vee\text{-SLat}$.

Proof: We begin with a series of four Lemmas.

Lemma 2.4.13. In the theory $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$ we can derive the following equation.

$$A \oplus_{\frac{1}{2}} B = A \boxplus B$$

Proof: We use the axioms of associativity, commutativity and idempotence defined for \mathbf{ND}_{fin}^* and \mathbf{P}_{fin} and the $(\otimes\text{-Com})$ from $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$.

$$\begin{aligned} A \boxplus B &= (A \boxplus B) \oplus_{\frac{1}{2}} (A \boxplus B) && (\mathbf{P}\text{-Idem}) \\ &= (A \boxplus B) \oplus_{\frac{1}{2}} (B \boxplus A) && (\mathbf{ND}\text{-Com}) \\ &= (A \oplus_{\frac{1}{2}} B) \boxplus (B \oplus_{\frac{1}{2}} A) && (\otimes\text{-Com}) \\ &= (A \oplus_{\frac{1}{2}} B) \boxplus (A \oplus_{\frac{1}{2}} B) && (\mathbf{P}\text{-Com}) \\ &= A \oplus_{\frac{1}{2}} B && (\mathbf{ND}\text{-Idem}) \end{aligned}$$

□

Lemma 2.4.14. In the theory $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$, the following inequalities hold, where the order is given by $A \leq_{\boxplus} B$ if and only if $A \boxplus B = B$. (I.e., \leq_{\boxplus} is induced by the semilattice structure given by \boxplus)

$$(a) \quad A \oplus_\lambda (A \boxplus B) \leq_{\boxplus} (A \boxplus B),$$

$$(b) (A \boxplus B) \oplus_\lambda B \leq_{\boxplus} (A \boxplus B).$$

Proof: It will be enough to show that $A \oplus_\lambda (A \boxplus B) \leq_{\boxplus} (A \boxplus B)$, since the second inequality follows by similar reasoning.

$$\begin{aligned} (A \oplus_\lambda (A \boxplus B)) \boxplus (A \boxplus B) &= (A \oplus_\lambda (A \boxplus B)) \boxplus ((A \boxplus B) \oplus_\lambda (A \boxplus B)) && \text{(P-Idem)} \\ &= (A \boxplus (A \boxplus B)) \oplus_\lambda ((A \boxplus B) \boxplus (A \boxplus B)) && \text{(\boxplus-Com)} \\ &= (A \boxplus B) \oplus_\lambda (A \boxplus B) && \text{(ND-Idem/Assoc)} \\ &= A \boxplus B && \text{(P-Idem)} \end{aligned}$$

$$\text{Hence, } A \oplus_\lambda (A \boxplus B) \leq_{\boxplus} (A \boxplus B) \quad \square$$

The next two Lemmas illustrate the key “left absorbing” property of \boxplus over \oplus_λ .

Lemma 2.4.15. *In $\text{ND}_{fin}^* \otimes \text{P}_{fin}$, we can derive, for $0 < \lambda \leq \frac{1}{2}$, the equations*

$$A \oplus_\lambda (A \boxplus B) = (A \boxplus B).$$

Proof:

$$\begin{aligned} A \oplus_\lambda (A \boxplus B) &= (A \oplus_{2\lambda} (A \boxplus B)) \oplus_{\frac{1}{2}} (A \boxplus B) && \text{(P-Assoc, P-Idem)} \\ &= (A \oplus_{2\lambda} (A \boxplus B)) \boxplus (A \boxplus B) && \text{(Lemma 2.4.13)} \\ &= (A \boxplus B) && \text{(Lemma 2.4.14)} \end{aligned}$$

□

Lemma 2.4.16. *In $\text{ND}_{fin}^* \otimes \text{P}_{fin}$, we can derive, for $\frac{1}{2} < \lambda < 1$, the equations*

$$A \oplus_\lambda (A \boxplus B) = (A \boxplus B).$$

Proof:

$$\begin{aligned} A \oplus_\lambda (A \boxplus B) &= (A \oplus_\lambda A) \oplus_\lambda (A \boxplus B) && \text{(P-Idem)} \\ &= A \oplus_{\lambda^2} (A \oplus_{\frac{\lambda}{1+\lambda}} (A \boxplus B)) && \text{(P-Assoc)} \\ &= A \oplus_{\lambda^2} (A \boxplus B) && (0 < \frac{\lambda}{1+\lambda} \leq \frac{1}{2}, \text{ Lemma 2.4.15}) \end{aligned}$$

Since $\lambda^{2^n} \rightarrow 0$ as $n \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that $\lambda^{2^N} \leq \frac{1}{2}$. Thus,

$$\begin{aligned} A \oplus_\lambda (A \boxplus B) &= A \oplus_{\lambda^{2^N}} (A \boxplus B) \\ &= A \boxplus (A \boxplus B) && \text{(Lemma 2.4.15)} \\ &= A \boxplus B && \text{(ND-Idem, ND-Assoc)} \end{aligned}$$

□

It is easy to see that $(S, \vee) \in \mathbf{V}\text{-SLat}$ is a model for $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$. Indeed, we define the probabilistic choice operators such that for all $\lambda \in (0, 1)$, $\oplus_\lambda = \boxplus = \vee$.

Conversely, suppose $(X, \boxplus, \oplus_\lambda)$ is a model for $\mathbf{ND}_{fin}^* \otimes \mathbf{P}_{fin}$. We show that it must be the case that for all $\lambda \in (0, 1)$, $\oplus_\lambda = \boxplus$. Indeed, we have that for $\lambda > \frac{1}{2}$,

$$A \oplus_\lambda B = A \oplus_{2\lambda-1} (A \oplus_{\frac{1}{2}} B) = A \oplus_{2\lambda-1} (A \boxplus B) = A \boxplus B$$

Similarly, for $\lambda \leq \frac{1}{2}$. □

2.4.3 The Distributive Tensor of Lawvere Theories

In this last section, we present the most important way of combining theories for the purpose of this thesis. It combines Lawvere theories in such a way as to introduce the axioms necessary when combining the theory of probabilistic and nondeterministic choice: distributivity of the probabilistic choice operators over the nondeterministic one. In his paper Power [41] introduces the idea that any discrete Lawvere theory, which includes finite and countable Lawvere theories can be combined in this fashion.

Definition 2.4.17. Given Lawvere theories \mathbf{L} and \mathbf{L}' , the Lawvere theory $\mathbf{L} \triangleright \mathbf{L}'$, called the **distributive tensor** of \mathbf{L} over \mathbf{L}' , is defined by the universal property of having maps of Lawvere theories from \mathbf{L} and \mathbf{L}' to $\mathbf{L} \triangleright \mathbf{L}'$ with all operations of \mathbf{L} distributing over all operations of \mathbf{L}' , but not conversely; note the asymmetry.

Remark 2.4.18. According to J. Power [41], it is possible to characterize the models of $\mathbf{L} \triangleright \mathbf{L}'$ in the category \mathbf{C} as the models of \mathbf{L} in the category $\mathbf{Mod}(\mathbf{L}', \mathbf{C})$. However, this is a complicated result which calls on operads and symmetric monoidal structure or more generally, multicategory structure of $\mathbf{Mod}(\mathbf{L}', \mathbf{C})$, which is beyond the scope of this thesis.

Definition 2.4.19 (Theory of $\mathbf{P}_{fin} \triangleright \mathbf{ND}_{fin}^*$). This Lawvere theory is generated by the operations from \mathbf{P}_{fin} and \mathbf{ND}_{fin}^* and their given equations in addition to the following equations stating that all the probabilistic choice operators distribute over the nondeterministic choice operator:

$$(x \boxplus y) \oplus_\lambda z = (x \oplus_\lambda z) \boxplus (y \oplus_\lambda z) \quad (\text{Dist}).$$

Example 2.4.20. Given $(C, +_\lambda)$ a convex set, let $\mathcal{Cvx}_{fin}(C) =$ the set of all finitely generated convex subsets of C (i.e., $\mathcal{Cvx}_{fin}(C) = \{\text{conv}(A) \subseteq (C, +_\lambda) \mid A \in \mathcal{P}_{fin}^*(C)\}$). Then, $(\mathcal{Cvx}_{fin}(C), \boxplus, \oplus_\lambda)$, is a model for $\mathbf{P}_{fin} \triangleright \mathbf{ND}_{fin}^*$, where $\text{conv}(A) \oplus_\lambda \text{conv}(B) = \text{conv}(\{a +_\lambda b \mid a \in A, b \in B\})$ and $\text{conv}(A) \boxplus \text{conv}(B) = \text{conv}(A \cup B)$. We show that this structure is indeed a free model in Section 3.3.2. \triangle

Remark 2.4.21. Plotkin has a result (perhaps unpublished) that the theory of combining \mathbf{ND}_{fin}^* and \mathbf{P}_{fin} in which *each* distributes over the other is the theory of bi-semilattices. (I.e., both operations are semilattice operations).

Chapter 3

Combining Theories

3.1 Theories of Probabilistic Choice

In this section we present two theories for probabilistic choice. The first theory we discuss is \mathbf{P}_{fin} , the finite Lawvere theory for probabilistic choice where only finite probabilistic combinations are considered. The second theory \mathbf{P} , is a countable Lawvere theory for probabilistic choice allowing for countably infinite combinations of terms. We will denote a probabilistic combination between terms A and B , as $A \oplus_\lambda B$, which will intuitively signify that the “process” it represents will choose term A with probability λ and term B with probability $(1 - \lambda)$.

Definition 3.1.1 (Finite Probabilistic Choice). The finite Lawvere theory of **probabilistic choice**, \mathbf{P}_{fin} , is the theory freely generated from an uncountable number of binary operations $\oplus_\lambda : 2 \rightarrow 1$, where $\lambda \in (0, 1)$, satisfying the following axioms:

$$\begin{aligned} \text{(P-Assoc)} \quad & (A \oplus_{\lambda_1} B) \oplus_{\lambda_2} C = A \oplus_{\lambda_1 \lambda_2} (B \oplus_{\frac{(1-\lambda_1)\lambda_2}{1-\lambda_1\lambda_2}} C); \\ \text{(P-Com)} \quad & A \oplus_\lambda B = B \oplus_{1-\lambda} A; \\ \text{(P-Idem)} \quad & A \oplus_\lambda A = A; \end{aligned}$$

Remark 3.1.2. In \mathbf{P}_{fin} , we will denote the left projection $\pi_L : 2 \rightarrow 1$ and the right projection $\pi_R : 2 \rightarrow 1$ as \oplus_1 and \oplus_0 , respectively. One can easily verify that by taking $\lambda_1, \lambda_2, \lambda \in [0, 1]$, all the equations in \mathbf{P}_{fin} still hold, except for (P-Assoc) when $\lambda_1 = \lambda_2 = 1$. In fact, \oplus_1 and \oplus_0 are the only probabilistic choice

operators which are associative on the nose (i.e., $(A \oplus_1 B) \oplus_1 C = A \oplus_1 (B \oplus_1 C)$ and $(A \oplus_0 B) \oplus_0 C = A \oplus_0 (B \oplus_0 C)$).

Suppose we wish to capture the following situation: For n terms A_1, A_2, \dots, A_n , we want that for $1 \leq i \leq n$, that the term A_i occurs with probability λ_i . We shall denote this situation by $\bigoplus_{(\lambda_i)_{i=1}^n} A_i$. However, there will be restrictions as to which sequences of the form $(\lambda_i)_{i=1}^n$ will be permitted. The sequences that will satisfy the required properties are called *probability densities*.

Definition 3.1.3 (Probabilistic Density).

- (a) A **probability density**, $(\lambda_i)_I$, is a countable sequence of elements in the interval $(0, 1]$ such that $\sum_{i \in I} \lambda_i = 1$. We will denote the set of all probability densities by Pwt .
- (b) A probability density, $(\lambda_i)_I \in Pwt$, is said to be **finite**, if I is a finite indexing set. We will denote the subset of all finite probability densities by $Pwt_{fin} \subseteq Pwt$.

Definition 3.1.4 (Countable Probabilistic Choice). The countable Lawvere theory of **probabilistic choice**, \mathbf{P} , is the theory freely generated from an uncountable number of operations of the form $\bigoplus_{(\lambda_i)_I} : |I| \rightarrow 1$, where I is a countable indexing set and $(\lambda_i)_I \in Pwt$, satisfying the following axioms:

$$\begin{aligned}
 (\mathbf{P}\text{-Ax1}) \quad \bigoplus_{(\lambda_i)_I} A_i &= \bigoplus_{(\rho_j)_J} \left(\bigoplus_{(\nu_k)_{K_j}} A_k \right), \text{ with } \{K_j \mid j \in J\} \text{ a partition of } I, \\
 &\quad \rho_j = \sum_{k \in K_j} \lambda_k \text{ and } \nu_k = \frac{\lambda_k}{\rho_j}. \\
 (\mathbf{P}\text{-Ax2}) \quad \bigoplus_{(\lambda_i)_I} A &= A.
 \end{aligned}$$

Remark 3.1.5. In the case of finite probabilistic combinations we can recover the finite axioms. We can derive $(\mathbf{P}\text{-Assoc})$ and $(\mathbf{P}\text{-Com})$ from ternary and binary instances of $(\mathbf{P}\text{-Ax1})$ respectively. Moreover, we can derive $(\mathbf{P}\text{-Idem})$ as a consequence of the binary instance of $(\mathbf{P}\text{-Ax2})$.

3.1.1 Probabilistic Model Categories in Set

For each set theoretical model category for the theories of probabilistic choice, we shall recall its associated monad. First, we begin with some minor technical definitions.

Definition 3.1.6 (Distributions/Support). Let X be an arbitrary set.

- (a) A function $d : X \rightarrow [0, 1]$ is called a **distribution over X** . A distribution $d : X \rightarrow [0, 1]$ is said to be **probabilistic** if the sum of its images is equal to 1 (i.e., $d(X) = \sum_{x \in X} d(x) = 1$).
- (b) The **support of d** , denoted $\text{supp}(d)$, is the set of elements in X which are mapped to nonzero numbers. Thus, for a distribution $d : X \rightarrow [0, 1]$, $\text{supp}(d) = \{x \in X \mid d(x) \neq 0\}$. A distribution, $d : X \rightarrow [0, 1]$, is said to be **finite**, if it has a finite support (i.e. $|\text{supp}(d)| < \infty$).
- (c) For an element $x \in X$, the **Dirac distribution over x** , denoted by $\delta_x : X \rightarrow [0, 1]$, is the finite probabilistic distribution whose support is the singleton $\{x\}$. Therefore, for any $x' \in X$,

$$\delta_x(x') = \begin{cases} 1 & \text{if } x = x', \\ 0 & \text{else.} \end{cases}$$

Proposition 3.1.7. *Any discrete distribution over X can be written as a convex combination of Dirac distributions over elements of its support. In other words, suppose we are given a distribution $d : X \rightarrow [0, 1]$ such that $\text{supp}(d) = \{x_i \in X \mid i \in I\}$ with I a countable indexing set, then $d = \sum_{i \in I} d(x_i) \delta_{x_i}$.*

Definition 3.1.8 (Probabilistic Distributions Monad). The **probabilistic distributions monad**, $\mathbb{D} = (\mathcal{D}, \eta, \mu) : \mathbf{Set}$, is given as follows:

- (a) The **probabilistic distributions functor** $\mathcal{D} : \mathbf{Set} \rightarrow \mathbf{Set}$,
 - (i) **On objects:** For $X \in \mathbf{Set}$,

$$\mathcal{D}(X) = \{d : X \rightarrow [0, 1] \mid d(X) = 1\}.$$

(ii) **On morphisms:** For $f : X \rightarrow X' \in \mathbf{Set}$ and $d \in \mathcal{D}(X)$ such that

$$d = \sum_{i \in I} \lambda_i \delta_{x_i},$$

$$\mathcal{D}f(d) = d_f = \sum_{i \in I} \lambda_i \delta_{f(x_i)}$$

(b) The unit transformation, $\eta : 1_{\mathbf{Set}} \Rightarrow \mathcal{D}$, is defined to be the **Dirac distribution transformation**. For each $X \in \mathbf{Set}$, we define η_X as follows:

$$\begin{aligned} \eta_X : X &\rightarrow \mathcal{D}(X) \\ x &\mapsto \delta_x \end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{D}^2 \Rightarrow \mathcal{D}$, is defined to be the **distributions flattening transformation**. For each $X \in \mathbf{Set}$ and $D : \mathcal{D}(X) \rightarrow [0, 1] \in \mathcal{D}^2(X)$, we define $\mu_X(D)$ as follows:

$$\mu_X(D) = \sum_{d \in \text{supp}(D)} D(d)d$$

Definition 3.1.9 (Finite Probabilistic Distributions Monad). The **finite probabilistic distributions monad** $\mathbb{D}_{fin} = (\mathcal{D}_{fin}, \eta, \mu) : \mathbf{Set}$, is defined as follows:

(a) The **finite probabilistic distributions functor** $\mathcal{D}_{fin} : \mathbf{Set} \rightarrow \mathbf{Set}$,

(i) **On objects:** For $X \in \mathbf{Set}$,

$$\mathcal{D}_{fin}(X) = \{d : X \rightarrow [0, 1] \mid d(X) = 1 \text{ and } d \text{ is finite}\}.$$

(ii) **On morphisms:** For $f : X \rightarrow X' \in \mathbf{Set}$ and $d \in \mathcal{D}(X)$ such that

$$d = \lambda_1 \delta_{x_1} + \lambda_2 \delta_{x_2} + \dots + \lambda_n \delta_{x_n},$$

$$\mathcal{D}_{fin}f(d) = d_f = \lambda_1 \delta_{f(x_1)} + \lambda_2 \delta_{f(x_2)} + \dots + \lambda_n \delta_{f(x_n)}$$

(b) The monad transformations, η and μ , are defined in the same way as shown for the probabilistic distribution monad.

Theorem 3.1.10.

(a) The algebra category $\mathbb{D}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{P}, \mathbf{Set})$.

(b) The algebra category $\mathbb{D}_{fin}\text{-Alg}$ is equivalent to the model category $\text{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$.

Proof: We shall prove the first assertion $\text{Mod}(\mathbf{P}, \mathbf{Set}) \cong \mathbb{D}\text{-Alg}$. We construct two inverse functors $\mathcal{F} : \text{Mod}(\mathbf{P}, \mathbf{Set}) \rightarrow \mathbb{D}\text{-Alg}$ and $\mathcal{G} : \mathbb{D}\text{-Alg} \rightarrow \text{Mod}(\mathbf{P}, \mathbf{Set})$:

(a) We begin by defining the functor $\mathcal{F} : \text{Mod}(\mathbf{P}, \mathbf{Set}) \rightarrow \mathbb{D}\text{-Alg}$:

$$\begin{array}{ccc} \mathcal{F} : \text{Mod}(\mathbf{P}, \mathbf{Set}) & \rightarrow & \mathbb{D}\text{-Alg} \\ (C, \sum_{(\lambda_i)_I}) & \mapsto & (C, \alpha^* : \mathcal{D}(C) \rightarrow C) \\ \downarrow f & \mapsto & \downarrow f \\ (C', \sum'_{(\lambda_i)_I}) & \mapsto & (C', (\alpha')^* : \mathcal{D}(C') \rightarrow C') \end{array}$$

For $d \in \mathcal{D}(C)$, we define $\alpha^*(d) = \sum_{(d(x))_{supp(d)}} x$ and show that it satisfies the required equations.

(i) For $D \in \mathcal{D}^2(C)$,

$$\begin{aligned} \alpha^* \circ \mathcal{D}\alpha^*(D) &= \alpha^* \left(\sum_{d \in supp(D)} D(d) \cdot \delta_{\alpha^*(d)} \right) \\ &= \alpha^* \left(\sum_{d \in supp(D)} D(d) \cdot \delta_{(\sum_{(d(x))_{supp(d)}} x)} \right) \\ &= \sum_{(D(d))_{supp(D)}} \left(\sum_{(d(x))_{supp(d)}} x \right) \\ &= \alpha^* \left(\sum_{(D(d))_{supp(D)}} \left(\sum_{(d(x))_{supp(d)}} \delta_x \right) \right) \\ &= \alpha^* \circ \mu_C(D) \end{aligned}$$

(ii) For $x \in C$,

$$\begin{aligned} \alpha^* \circ \eta_C(x) &= \alpha^*(\delta_x) \\ &= x \end{aligned}$$

Finally, we also verify that for any $f : (C, \sum_{(\lambda_i)_I}) \rightarrow (C', \sum'_{(\lambda_i)_I})$, then

$\mathcal{F}f = f$ is also an algebra map. Let $d = \sum_{i \in I} \lambda_i \delta_{x_i} \in \mathcal{D}(C)$.

$$\begin{aligned}
 (\alpha')^* \circ \mathcal{D}f(d) &= (\alpha')^*(d_f) \\
 &= \sum'_{(\lambda_i)_I} f(x_i) \\
 &= f\left(\sum_{(\lambda_i)_I} x_i\right) \\
 &= f(\alpha^*(d)) \\
 &= f \circ \alpha^*(d)
 \end{aligned}$$

(b) Next we define the functor $\mathcal{G} : \mathbb{D}\text{-Alg} \rightarrow \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$:

$$\begin{array}{ccc}
 \mathcal{G} : \mathbb{D}\text{-Alg} & \rightarrow & \mathbf{Mod}(\mathbf{P}, \mathbf{Set}) \\
 (X, \alpha : \mathcal{D}(X) \rightarrow X) & \mapsto & (X, \sum_{(\lambda_i)_I}^\alpha) \\
 \downarrow^g & & \downarrow^g \\
 (X', \alpha' : \mathcal{D}(X') \rightarrow X') & \mapsto & (X', \sum_{(\lambda_i)_I}^{\alpha'})
 \end{array}$$

For a countable family $(x_i)_I \in X$ and a probability density $(\lambda_i)_I$, we define the operator $\sum_{(\lambda_i)_I}^\alpha x_i$ as follows:

$$\sum_{(\lambda_i)_I}^\alpha x_i = \alpha\left(\bigoplus_{(\lambda_i)_I} \delta_{x_i}\right).$$

These operations are well-defined and satisfy the axioms for probabilistic choice due to the equations satisfied by α and the probabilistic choice operator $\bigoplus_{(\lambda_i)_I}$ defined on $\mathcal{D}(X)$.

(i) The operators satisfy the axiom (P-Ax1). Consider a probability density $(\lambda_i)_I \in Pwt$, for any partition $\{K_j \mid j \in J\}$ of I , a probability density $(\rho_j)_J$ and a family of probability densities $(\nu_k)_{K_j}$ such that $\rho_j = \sum_{k \in K_j} \lambda_k$

and $\nu_k = \frac{\lambda_k}{\rho_j}$. We have that

$$\begin{aligned}
\sum_{(\rho_j)_J}^{\alpha} \left(\sum_{(\nu_k)_{K_j}}^{\alpha} x_k \right) &= \alpha \left(\bigoplus_{(\rho_j)_J} \delta_{(\alpha(\bigoplus_{(\nu_k)_{K_j}} \delta_{x_k}))} \right) \\
&= \alpha \circ \mathcal{D}\alpha \left(\bigoplus_{(\rho_j)_J} \delta_{(\bigoplus_{(\nu_k)_{K_j}} \delta_{x_k})} \right) \\
&= \alpha \circ \mu_X \left(\bigoplus_{(\rho_j)_J} \delta_{(\bigoplus_{(\nu_k)_{K_j}} \delta_{x_k})} \right) \\
&= \alpha \left(\bigoplus_{(\rho_j)_J} \left(\bigoplus_{(\nu_k)_{K_j}} \delta_{x_k} \right) \right) \\
&= \alpha \left(\bigoplus_{(\lambda_i)_I} \delta_{x_i} \right) \\
&= \sum_{(\lambda_i)_I}^{\alpha} x_i
\end{aligned}$$

(ii) The operators satisfy the axiom (P-Ax2).

$$\begin{aligned}
\sum_{(\lambda_i)_I}^{\alpha} x &= \alpha \left(\bigoplus_{(\lambda_i)_I} \delta_x \right) \\
&= \alpha(\delta_x) \\
&= x
\end{aligned}$$

Remark 3.1.11. In what follows, we will need to prove similar results concerning operators generated for functors between different algebra categories and model categories. In future cases, we will omit the above justifications since they can be derived through easy modifications.

Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{P}, \mathbf{Set})$. Given any $g : (X, \alpha) \rightarrow (X', \alpha')$, a countable family $(x_i)_{i \in I}$ in X and a probability density $(\lambda_i)_I$, then

$$\begin{aligned}
g \left(\sum_{(\lambda_i)_I}^{\alpha} x_i \right) &= g \left(\alpha \left(\bigoplus_{(\lambda_i)_I} \delta_{x_i} \right) \right) \\
&= \alpha' \left(\mathcal{D}g \left(\bigoplus_{(\lambda_i)_I} \delta_{x_i} \right) \right) \\
&= \alpha' \left(\bigoplus_{(\lambda_i)_I} \delta_{g(x_i)} \right) \\
&= \sum_{(\lambda_i)_I}^{\alpha'} g(x_i)
\end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

$$(i) \mathcal{G} \circ \mathcal{F} = 1$$

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}(C, \sum_{(\lambda_i)_I}) &= \mathcal{G}((C, \alpha^*)) \\ &= (C, \sum_{(\lambda_i)_I}^{\alpha^*}) \end{aligned}$$

We can verify that $(C, \sum_{(\lambda_i)_I}) = (C, \sum_{(\lambda_i)_I}^{\alpha^*})$, since for a countable family $(x_i)_{i \in I} \in C$ and a probability density $(\lambda_i)_I$,

$$\begin{aligned} \sum_{(\lambda_i)_I}^{\alpha^*} x_i &= \alpha^*(\bigoplus_{(\lambda_i)_I} \delta_{x_i}) \\ &= \sum_{(\lambda_i)_I} x_i \end{aligned}$$

$$(ii) \mathcal{F} \circ \mathcal{G} = 1$$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}((X, \alpha)) &= \mathcal{F}((X, \sum_{(\lambda_i)_I}^{\alpha})) \\ &= (X, \alpha^*) \end{aligned}$$

However, $\alpha = \alpha^*$ since for $d \in \mathcal{D}(X)$

$$\begin{aligned} \alpha^*(d) &= \sum_{(d(x))_{\text{supp}(d)}}^{\alpha} x \\ &= \alpha(\bigoplus_{(d(x))_{\text{supp}(d)}} \delta_x) \\ &= \alpha(d) \end{aligned}$$

□

3.1.2 Probabilistic Model Categories in Poset

The probabilistic algebra functor over partially ordered sets is a particular case of the probabilistic powerdomain described in depth in Claire Jones' thesis [20].

Definition 3.1.12 (Probabilistic Algebra Monad). The **probabilistic algebra monad**, $\mathbb{V} = (\mathcal{V}, \eta, \mu) : \mathbf{Poset}$, is given as follows:

- (a) The **probabilistic algebra functor** $\mathcal{V} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

(i) **On objects:** For $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{V}((X, \sqsubseteq)) = (\{d : X \rightarrow [0, 1] \mid d(X) = 1, d \text{ is finite}\}, \preceq).$$

The order structure \preceq is called the **distributions order over** (X, \sqsubseteq) . For $d, d' \in \mathcal{V}(X)$,

$$d \preceq d' \iff (\forall Y \in \mathcal{P}_{fin}^*(X)) d(\uparrow_{\sqsubseteq} Y) \leq d'(\uparrow_{\sqsubseteq} Y),$$

where $\uparrow_{\sqsubseteq} Y$ denotes the set of all elements in X which are greater than some element in Y (i.e., $\uparrow_{\sqsubseteq} Y = \{x \in X \mid \exists y \in Y, y \sqsubseteq x\}$).

(ii) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $d \in \mathcal{V}(X)$,

$$\mathcal{V}f(d) = d_f \stackrel{def}{=} \sum_{x \in X} (d(x)) \delta_{f(x)}$$

- (b) The unit transformation, $\eta : \mathbf{1}_{\mathbf{Poset}} \Rightarrow \mathcal{V}$, is defined to be the Dirac distribution map for each $(X, \sqsubseteq) \in \mathbf{Poset}$.
- (c) The multiplication transformation, $\mu : \mathcal{V}^2 \Rightarrow \mathcal{V}$, is defined to be the probabilistic distributions flattening map for each $(X, \sqsubseteq) \in \mathbf{Poset}$. In truth the probabilistic distributions flattening map here can be interpreted as an integration.

Theorem 3.1.13. *The algebra category $\mathbb{V}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Poset})$.*

Proof: We show that the functors defined in the proof of theorem 3.1.10, can be extended to prove the above statement.

- (a) Given the functor \mathcal{F} as defined previously, we extend its definition on objects by:

$$\mathcal{F}((C, \sum_{(\lambda_i)_I}), \sqsubseteq) = ((C, \sqsubseteq), \alpha^* : \mathcal{V}((C, \sqsubseteq)) \rightarrow (C, \sqsubseteq)).$$

It remains to show that α^* preserves the distributions order over (C, \sqsubseteq) (i.e., if $d, d' \in \mathcal{V}((C, \sqsubseteq))$ such that $d \preceq d'$, then $\alpha^*(d) \sqsubseteq \alpha^*(d')$). This arises as a consequence of the Splitting Lemma presented in Jones' thesis [20], applied to convex combinations.

Theorem 3.1.14 (Claire Jones, [20]). *For two distributions $d = \sum_{x \in X} (\lambda_x) \delta_x$ and $d' = \sum_{y \in Y} (\rho_y) \delta_y$ such that $d, d' \in \mathcal{V}((C, \sqsubseteq))$, then the following are equivalent*

(i) $d \preceq d'$

(ii) *There exists non-negative real numbers $(t_{x,y})_{x \in X, y \in Y}$ such that*

* $(\forall x \in X) \sum_{y \in Y} t_{x,y} = \lambda_x$

* $(\forall y \in Y) \sum_{x \in X} t_{x,y} = \rho_y$

(for the general case of linear combinations (where we do not require that the sum of the coefficients is 1) we must relax this condition so that $\sum_{x \in X} t_{x,y} \leq \rho_y$)

* $(\forall x \in X, y \in Y)$ *If $t_{x,y} \neq 0$ then $x \sqsubseteq y$.*

(b) Given the functor \mathcal{G} as defined previously, we extend its definition on objects such that

$$\mathcal{G}(((C, \sqsubseteq), \alpha)) = ((C, +_{\lambda}^{\alpha}), \sqsubseteq)$$

It remains to show that the order preserves the convex combinations (i.e if $x_1 \sqsubseteq x'_1$ and $x_2 \sqsubseteq x'_2$, then $x_1 +_{\lambda}^{\alpha} x_2 \sqsubseteq x'_1 +_{\lambda}^{\alpha} x'_2$).

$$\begin{aligned} x_1 \sqsubseteq x'_1 \text{ and } x_2 \sqsubseteq x'_2 &\Rightarrow \delta_{x_1} \oplus_{\lambda} \delta_{x_2} \preceq \delta_{x'_1} \oplus_{\lambda} \delta_{x'_2} \\ &\Rightarrow \alpha(\delta_{x_1} \oplus_{\lambda} \delta_{x_2}) \sqsubseteq \alpha(\delta_{x'_1} \oplus_{\lambda} \delta_{x'_2}) \\ &\Rightarrow x_1 +_{\lambda}^{\alpha} x_2 \sqsubseteq x'_1 +_{\lambda}^{\alpha} x'_2 \end{aligned}$$

(c) The maps are inverses by similar reasoning to the proof of Theorem 3.1.10.

□

3.2 Theories for Nondeterministic Choice

There are many variants in considering the semantic behavior of a nondeterministic choice theory. The most widely used viewpoint is to consider it as an environmentally influenced schema of possible next states. In other words, the nondeterministic

choice between two terms, represents the processes' capacity of evolving as either one of its terms, which term is chosen depends on the environment's present needs. From this viewpoint, the choice structure admits a unit element, a process which has no possible next states called the *nil process*, denoted by 0 . Thus, having no possible next states, the nil process cannot be influenced by the demands made by the environment. Therefore, taking the nondeterministic sum between the nil process and any other process A is indistinguishable from the process A under any action taken by the environment (no matter what the environment tries to do it will never know the difference between the process $A + 0$ and A). We will call this viewpoint on nondeterminism as the *external ND choice* and define its Lawvere theory as follows.

Definition 3.2.1 (External ND Theory). The finite Lawvere theory of **external nondeterministic choice**, **ND**, is the theory freely generated by a nullary operation $0 : 0 \rightarrow 1$ and a binary operation $+$: $2 \rightarrow 1$ satisfying the axioms:

$$\begin{array}{ll} (+\text{-Assoc}) & (A + B) + C = A + (B + C); \\ (+\text{-Com}) & A + B = B + A; \end{array} \quad \begin{array}{ll} (+\text{-Idem}) & A + A = A; \\ (\text{Unit}) & A + 0 = A. \end{array}$$

However, since we are interested in combining probabilistic choice and nondeterministic choice, the theory of external **ND** choice is not an adequate interpretation for nondeterminism in our framework. From our viewpoint, we consider nondeterministic choice to be governed by an internal random scheduler (a separate entity which chooses the evolution between two processes). Hence, given such an *internal ND choice* between two states, the process will appear to randomly choose between the two independently of the environment's needs. If such a theory would have a unit, say the process U , then this would imply that the evolution of the process $A \boxplus U$ must be equivalent to the process A . However, in this framework $A \boxplus U$ may progress without any exterior influence as either A or U , implying that A must be equivalent to U . Therefore, the alleged unit would have to be equivalent to every process A in the model, which may arise in a system where each process already had the behavior of the unit as a required behavior (i.e., for any process A , A always has the option to terminate the program). This would mean that any process would already be predisposed to possibly behave as the unit. This is not a reasonable structure for our

framework, since any structure has at most one identity we would have to consider processes which share a common behavior to allow for a unit. Therefore, we define a second nondeterministic theory, the *internal ND theory*, which does not require the existence of a unit.

Definition 3.2.2 (Internal ND Theory). The finite Lawvere theory for **internal nondeterministic choice**, \mathbf{ND}_{fin}^* , is the theory freely generated by a binary operation $\boxplus : 2 \rightarrow 1$ satisfying the axioms:

$$\begin{aligned} (\boxplus\text{-Assoc}) \quad & (A \boxplus B) \boxplus C = A \boxplus (B \boxplus C); & (\boxplus\text{-Idem}) \quad & A \boxplus A = A; \\ (\boxplus\text{-Com}) \quad & A \boxplus B = B \boxplus A. \end{aligned}$$

It is also interesting to consider a theory of nondeterminism which allows for infinitary nondeterministic sums. As pointed out by Mislove [30], many program specification languages could do well to have this capability. It would allow the possibility to specify processes whose behavior differs depending on allowing an infinite number of inputs, such as natural numbers. That is, from the input the environment has an infinite number of possible inputs. Unbounded nondeterminism is also fundamental in obtaining proper refinement operators between models for Timed CSP [43, 44] and corresponding models for untimed CSP [8], for example. Indeed, to refine the process which, in an untimed setting is willing to do an action a and then normally terminate, one must distinguish between the process in the timed setting which can do an a at any given time (and hence, can postpone doing the a for an arbitrary amount of time) from the process which also could postpone doing the a for all time.

There are other reasons we will define an infinite theory for nondeterminism. For one, the free mixed choice models over nondeterministic models with uncountable nondeterministic choice operators have a simpler presentation and a greater degree of generality than their finite counterparts. Second, let $A \boxplus B$ denote nondeterministic choice between A and B and for $\lambda \in [0, 1]$, let $A \oplus_\lambda B$ denote the probabilistic choice operator which evaluates to A with probability λ and evaluates to B with probability $1 - \lambda$. In our framework, the nondeterministic choice between terms A and B can be interpreted as the probabilistic choice of indeterminate weight of the constituents. This viewpoint is the same as the one proposed by Mislove [31] and also Tix [49] in

their approach to creating a mixed choice theory. However, our use of uncountable nondeterministic choice operators is a new feature allowing us to express the above interpretation of nondeterminism formally as $A \boxplus B = \bigsqcup_{\lambda \in [0,1]} (A \oplus_{\lambda} B)$. This is an equation derivable from our axioms defining a mixed choice theory.

Definition 3.2.3 (Infinite ND Theory). The theory for **infinite nondeterministic choice**, ND^* , is the theory freely generated by **ND-operators** (**ND-Ops**) of the form $\bigsqcup_{w \in W} : |W| \rightarrow 1$, where W is an arbitrary non-empty indexing set, satisfying the axioms:

$$\begin{aligned} (\bigsqcup\text{-Ax1}) \quad \bigsqcup_{w \in W} A_w &= \bigsqcup_{u \in U} \left(\bigsqcup_{v \in V_u} A_v \right), \quad \text{where } \{V_u \mid u \in U\} \text{ is a partition of } W; \\ (\bigsqcup\text{-Ax2}) \quad \bigsqcup_{w \in W} A &= A. \end{aligned}$$

3.2.1 Nondeterministic Model Categories in Set

We recall the usual monads which correspond to the set theoretical models for the various nondeterministic theories.

Definition 3.2.4 (Powerset Monad). The **powerset monad**, $\mathbb{P} = (\mathcal{P}, \eta, \mu) : \mathbf{Set}$, is given as follows:

(a) The **powerset functor** $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$,

(i) **On objects**: For $X \in \mathbf{Set}$,

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}.$$

(ii) **On morphisms**: For $f : X \rightarrow X' \in \mathbf{Set}$ and $Y \in \mathcal{P}(X)$,

$$\mathcal{P}f(Y) = f[Y] := \{f(y) \mid y \in Y\}.$$

(b) The the unit transformation, $\eta : 1_{\mathbf{Set}} \Rightarrow \mathcal{P}$, is defined to be the **singleton map**.

For each $X \in \mathbf{Set}$, we define η_X as follows:

$$\begin{aligned} \eta_X : X &\rightarrow \mathcal{P}(X) \\ x &\mapsto \{x\} \end{aligned}$$

- (c) The multiplication transformation, $\mu : \mathcal{P}^2 \Rightarrow \mathcal{P}$, is defined to be the **big union map**. For each $X \in \mathbf{Set}$, we define μ_X as follows:

$$\begin{aligned} \mu_X : \mathcal{P}^2(X) &\rightarrow \mathcal{P}(X) \\ \mathcal{Y} &\mapsto \bigcup_{Y \in \mathcal{Y}} Y \end{aligned}$$

Below we define the functor part of the monads associated to the set theoretical model categories for the remaining variants of nondeterministic choice. In each case, the remaining monad structure is the singleton transformation (unit) and the big union transformation (multiplication). These are all variants of the usual covariant powerset functor (whose action on maps is direct image.)

Definition 3.2.5 (Variants of Powerset Monad).

- (a) **Finite Powerset Monad:** $\mathbb{P}_{fin} = (\mathcal{P}_{fin}, \eta, \mu) : \mathbf{Set}$.
For $X \in \mathbf{Set}$, $\mathcal{P}_{fin}(X) = \{Y \mid Y \subseteq_{fin} X\}$, where $Y \subseteq_{fin} X$ denotes that Y is a finite subset of X .
- (b) **Nonempty Powerset Monad:** $\mathbb{P}^* = (\mathcal{P}^*, \eta, \mu) : \mathbf{Set}$.
For $X \in \mathbf{Set}$, $\mathcal{P}^*(X) = \{Y \mid Y \subseteq X, Y \neq \emptyset\}$.
- (c) **Finite Nonempty Powerset Monad:** $\mathbb{P}_{fin}^* = (\mathcal{P}_{fin}^*, \eta, \mu) : \mathbf{Set}$.
For $X \in \mathbf{Set}$, $\mathcal{P}_{fin}^*(X) = \{Y \mid Y \subseteq_{fin} X, Y \neq \emptyset\}$.

Theorem 3.2.6. *We have the following equivalence of categories.*

- (a) *The algebra category $\mathbb{P}^*\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$.*
- (b) *The algebra category $\mathbb{P}_{fin}^*\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$.*
- (c) *The algebra category $\mathbb{P}_{fin}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{ND}, \mathbf{Set})$.*

Proof: We shall prove the first assertion, $\mathbb{P}^*\text{-Alg} \cong \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$, by constructing two inverse functors:

(a) We define a functor $\mathcal{F} : \text{Mod}(\text{ND}^*, \text{Set}) \rightarrow \mathbb{P}^*\text{-Alg}$.

$$\begin{array}{ccc} \mathcal{F} : \text{Mod}(\text{ND}^*, \text{Set}) & \rightarrow & \mathbb{P}^*\text{-Alg} \\ (S, \bigvee) & \mapsto & (S, \alpha^* : \mathcal{P}^*(S) \rightarrow S) \\ \downarrow f & \mapsto & \downarrow f \\ (S', \bigvee') & \mapsto & (S, (\alpha')^* : \mathcal{P}^*(S') \rightarrow S) \end{array}$$

For $T \in \mathcal{P}^*(S)$, we define $\alpha^*(T) = \bigvee_{t \in T} t$ and show that it satisfies the required equations.

(i) For $\mathcal{Y} \in (\mathcal{P}^*)^2(S)$,

$$\begin{aligned} \alpha^* \circ \mathcal{P}^* \alpha^*(\mathcal{Y}) &= \alpha^*(\{\alpha^*(Y) \mid Y \in \mathcal{Y}\}) \\ &= \bigvee_{Y \in \mathcal{Y}} \alpha^*(Y) \\ &= \bigvee_{y \in \bigcup_{Y \in \mathcal{Y}} Y} y \\ &= \alpha^*(\bigcup_{Y \in \mathcal{Y}} Y) \\ &= \alpha^* \circ \mu_S(\mathcal{Y}) \end{aligned}$$

(ii) For $s \in S$,

$$\begin{aligned} \alpha^* \circ \eta_S(s) &= \alpha^*({s}) \\ &= s \end{aligned}$$

Finally, we also verify that for any $f : (S, \bigvee) \rightarrow (S', \bigvee')$, then $\mathcal{F}f = f$ is also an algebra map. Suppose we are given $T \in \mathcal{P}^*(S)$.

$$\begin{aligned} (\alpha')^* \circ \mathcal{P}^* f(T) &= (\alpha')^*(f[T]) \\ &= \bigvee'_{t \in T} f(t) \\ &= f(\bigvee_{t \in T} t) \\ &= f(\alpha^*(T)) \\ &= f \circ \alpha^*(T) \end{aligned}$$

(b) We define a functor $\mathcal{G} : \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set}) \rightarrow \mathbb{P}^*\mathbf{-Alg}$.

$$\begin{array}{ccc} \mathcal{G} : \mathbb{P}^*\mathbf{-Alg} & \rightarrow & \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set}) \\ (X, \alpha) & \mapsto & (X, \bigvee^\alpha) \\ \downarrow^g & & \downarrow^g \\ (X', \alpha') & \mapsto & (X', \bigvee^{\alpha'}) \end{array}$$

For an arbitrary nonempty subset Y of X , we define the operator \bigvee^α as follows:

$$\bigvee_{y \in Y}^\alpha y = \alpha(Y).$$

These operations are well-defined and satisfy the axioms for nondeterministic choice due to the equations satisfied by α and the nondeterministic choice operator \bigvee defined on $\mathcal{P}^*(X)$. Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$. Given any $g : (X, \alpha) \rightarrow (X', \alpha')$ and a nonempty subset Y of S we have that

$$\begin{aligned} g\left(\bigvee_{y \in Y}^\alpha y\right) &= g(\alpha(Y)) \\ &= \alpha'(\mathcal{P}^*g(Y)) \\ &= \alpha'(g[Y]) \\ &= \bigvee_{y \in Y}^{\alpha'} g(y) \end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

(i) $\mathcal{G} \circ \mathcal{F} = 1$

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}(S, \bigvee) &= \mathcal{G}((S, \alpha^*)) \\ &= (S, \bigvee^{\alpha^*}) \end{aligned}$$

We can verify that $(S, \bigvee) = (S, \bigvee^{\alpha^*})$, since for a nonempty subset T of S ,

$$\begin{aligned} \bigvee_{t \in T}^{\alpha^*} t &= \alpha^*(T) \\ &= \bigvee_{t \in T} t \end{aligned}$$

$$(ii) \mathcal{F} \circ \mathcal{G} = 1$$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}((X, \alpha)) &= \mathcal{F}((X, \bigvee^\alpha)) \\ &= (X, \alpha^*) \end{aligned}$$

However, $\alpha = \alpha^*$ since for a nonempty subset T of S ,

$$\begin{aligned} \alpha^*(T) &= \bigvee_{t \in T}^\alpha t \\ &= \alpha(T) \end{aligned}$$

□

3.2.2 Nondeterministic Model Categories in Poset

By extending the underlying category from **Set** to **Poset**, we shall be able to interpret two additional nondeterministic theories. They arise by considering the possible orderings between a term A and a nondeterministic combination of terms containing A . The first theory is \mathbf{ND}_{fin}^* and is the theory which best describes “what actually happens”. The second and third theories differ only by the addition of an inequality axiom and are dual to one another. The second viewpoint (cf. Plotkin [37]) is to consider a term A to contain less information than any nondeterministic combination containing it; this viewpoint is useful in dealing with partial correctness properties and is given by the theory below.

Definition 3.2.7 (Lower (Hoare) ND Theory). The finite Lawvere theory for **lower nondeterministic choice**, \mathbf{H}_{fin} , is obtained by considering the theory \mathbf{ND}_{fin}^* , with the following extra axiom:

$$(H) \quad A \leq A \boxplus B$$

The third viewpoint presented by Plotkin [37] and streamlined by Smyth [48], is the dual to the second, wherein we deem the term A to be more precise than any nondeterministic combination containing it. This viewpoint is useful in dealing with total correctness properties. We shall define the theory below.

Definition 3.2.8 (Upper (Smyth) ND Theory). The finite Lawvere theory for **upper nondeterministic choice**, \mathbf{S}_{fin} , is obtained by considering the theory \mathbf{ND}_{fin}^* , with the following extra axiom:

$$(S) \quad A \geq A \boxplus B$$

Given the three viewpoints on nondeterministic theories we shall construct the three monads which are associated with their respective posetal model categories.

In order to define the convex (Plotkin) algebra functor, we require the following notions.

Definition 3.2.9 (Order Convex). Suppose we are given a poset (X, \sqsubseteq) , a subset Y of X is said to be **order-convex** if it satisfies the following property:

$$(\forall x \in X, y_1, y_2 \in Y) y_1 \sqsubseteq x \sqsubseteq y_2 \Rightarrow x \in Y.$$

Proposition 3.2.10. *Order-convex sets are closed under arbitrary intersections. In other words, given an arbitrary family of order-convex sets $(Y_w)_{w \in W}$ their intersection $\bigcap_{w \in W} Y_w$ is order-convex.*

Definition 3.2.11 (Order-Convex Closure). Suppose we are given the poset (X, \sqsubseteq) and M a subset X , the **order-convex closure** of M in X , denoted by \overline{M} , is the smallest order-convex subset of X which contains M .

Note that Proposition 3.2.10 implies the order-convex closure of a set is non-empty and well-defined.

Definition 3.2.12 (Convex Algebra Monad). The **convex algebra monad**, $\mathbb{P} = (\mathcal{P}, \eta, \mu) : \mathbf{Poset}$, is given as follows:

(a) The **convex algebra functor** $\mathcal{P} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

(i) **On objects:** For $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{P}((X, \sqsubseteq)) = ((\mathcal{P}(X), \sqcup), \sqsubseteq_{EM})$$

(1) $\mathcal{P}(X)$ is the set of all order-convex closures under \sqsubseteq of finitely generated subsets of (X, \sqsubseteq) (i.e $\mathcal{P}(X) = \{\overline{Y} \mid Y \in \mathcal{P}_{fin}^*(X)\}$, where \overline{Y} is the smallest order-convex set containing Y).

- (2) $\bar{\cup}$ is union closed under order-convexity (i.e., $\bar{Y}_1 \bar{\cup} \bar{Y}_2 = \overline{Y_1 \cup Y_2}$).
- (3) \sqsubseteq_{EM} is the **Egli-Milner order** over (X, \sqsubseteq) .

For $\bar{Y}_1, \bar{Y}_2 \in \mathcal{P}(X)$, $\bar{Y}_1 \sqsubseteq_{EM} \bar{Y}_2 \Leftrightarrow \downarrow Y_1 \subseteq \downarrow Y_2$ and $\uparrow Y_2 \subseteq \uparrow Y_1$.

Recall that $\downarrow A = \{x \in X \mid \exists a \in A, x \sqsubseteq a\}$, is the **lower set** of A in (X, \sqsubseteq) and $\uparrow A = \{x \in X \mid \exists a \in A, a \sqsubseteq x\}$, is the **upper set** of A in (X, \sqsubseteq) .

- (ii) **On morphisms:** Given the morphism $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\bar{Y} \in \mathcal{P}(X)$, then

$$\mathcal{P}f(\bar{Y}) = \overline{f[\bar{Y}]}.$$

- (b) The the unit transformation, $\eta : \mathbf{1}_{\mathbf{Poset}} \Rightarrow \mathcal{P}$, is defined to be the singleton map for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{P}((X, \sqsubseteq)) \\ x &\mapsto \{x\} \end{aligned}$$

- (c) The multiplication transformation, $\mu : \mathcal{P}^2 \Rightarrow \mathcal{P}$, is defined to be the order-convex closure of the big union map for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \mu_{(X, \sqsubseteq)} : \mathcal{P}^2((X, \sqsubseteq)) &\rightarrow \mathcal{P}((X, \sqsubseteq)) \\ \bar{Y} &\mapsto \overline{\bigcup_{Y \in \mathcal{Y}} Y} \end{aligned}$$

Definition 3.2.13 (Lower (Hoare) Nondeterministic Algebra Monad). The **lower ND algebra monad**, $\mathbb{H} = (\mathcal{H}, \eta, \mu) : \mathbf{Poset}$, is given as follows:

- (a) The **lower ND algebra functor** $\mathcal{H} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

- (i) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{H}((X, \sqsubseteq)) = ((\mathcal{H}(X), \cup), \sqsubseteq).$$

We define $\mathcal{H}(X)$ as the set of all lower sets of finitely generated subsets of (X, \sqsubseteq) (i.e $\mathcal{H}(X) = \{\downarrow Y \mid Y \in \mathcal{P}_{fin}^*(X)\}$).

(ii) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$, and $\downarrow_{\sqsubseteq} Y \in \mathcal{H}(X)$,

$$\mathcal{H}f(\downarrow_{\sqsubseteq} Y) = \downarrow_{\sqsubseteq'}(f[Y])$$

(b) The the unit transformation, $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{H}$, is defined to be the **downwards closed singleton map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{H}((X, \sqsubseteq)) \\ x &\mapsto \downarrow\{x\} \end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{H}^2 \Rightarrow \mathcal{H}$, is defined to be the **downwards closed big union map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \mu_{(X, \sqsubseteq)} : \mathcal{H}^2((X, \sqsubseteq)) &\rightarrow \mathcal{H}((X, \sqsubseteq)) \\ \downarrow_{\sqsubseteq} \mathcal{Y} &\mapsto \downarrow_{\sqsubseteq} \left(\bigcup_{\downarrow_{\sqsubseteq} Y \in \mathcal{Y}} Y \right) \end{aligned}$$

Definition 3.2.14 (Upper (Smyth) Nondeterministic Algebra Monad). The **upper ND algebra monad**, $\mathbb{S} = (\mathcal{S}, \eta, \mu) : \mathbf{Poset}$, is given as follows:

(a) The **upper ND algebra functor** $\mathcal{S} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

(i) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{S}((X, \sqsubseteq)) = ((\mathcal{S}(X), \cup), \supseteq).$$

We define $\mathcal{S}(X)$ as the set of all upper sets of finitely generated subsets of (X, \sqsubseteq) (i.e $\mathcal{S}(X) = \{\uparrow Y \mid Y \in \mathcal{P}_{fin}^*(X)\}$).

(ii) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$, and $\uparrow_{\sqsubseteq} Y \in \mathcal{S}(X)$,

$$\mathcal{S}f(\uparrow_{\sqsubseteq} Y) = \uparrow_{\sqsubseteq'}(f[Y])$$

(b) The the unit transformation, $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{S}$, is defined to be the **upwards closed singleton map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{S}((X, \sqsubseteq)) \\ x &\mapsto \uparrow\{x\} \end{aligned}$$

- (c) The multiplication transformation, $\mu : \mathcal{S}^2 \Rightarrow \mathcal{S}$, is defined to be the **upwards closed big union map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \mu_{(X, \sqsubseteq)} : \mathcal{S}^2((X, \sqsubseteq)) &\rightarrow \mathcal{S}((X, \sqsubseteq)) \\ \uparrow_{\supseteq} \mathcal{Y} &\mapsto \uparrow_{\sqsubseteq} \left(\bigcup_{\uparrow_{\sqsubseteq} Y \in \mathcal{Y}} Y \right) \end{aligned}$$

Theorem 3.2.15.

- (a) The algebra category $\mathbb{P}\text{-Alg}$ is equivalent to $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$.
 (b) The algebra category $\mathbb{H}\text{-Alg}$ is equivalent to $\mathbf{Mod}(\mathbf{H}_{fin}, \mathbf{Poset})$.
 (c) The algebra category $\mathbb{S}\text{-Alg}$ is equivalent to $\mathbf{Mod}(\mathbf{S}_{fin}, \mathbf{Poset})$.

Proof: We shall prove the first assertion $\mathbb{P}\text{-Alg} \cong \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$, since the second and third results follow using straightforward modifications.

- (a) We define a functor $\mathcal{F} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbb{P}\text{-Alg}$.

$$\begin{aligned} \mathcal{F} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) &\rightarrow \mathbb{P}\text{-Alg} \\ ((S, \vee), \sqsubseteq) &\mapsto ((S, \sqsubseteq), \alpha^* : \mathcal{P}((S, \sqsubseteq)) \rightarrow (S, \sqsubseteq)) \\ \downarrow^f &\mapsto \downarrow^f \\ ((S', \vee'), \sqsubseteq') &\mapsto ((S', \sqsubseteq'), (\alpha')^* : \mathcal{P}((S', \sqsubseteq')) \rightarrow (S', \sqsubseteq')) \end{aligned}$$

For $\bar{Y} \in \mathcal{P}((S, \sqsubseteq))$, we define the algebra structure map by $\alpha^*(\bar{Y}) = \bigvee_{y \in Y} y$ and verify that it is well-defined morphism in \mathbf{Poset} and that it satisfies the required equations.

- (i) **well-defined morphism in Poset.** Suppose that $\bar{Y} = \bar{W} \in \mathcal{P}(X)$, we have that

$$\begin{aligned} y \in Y &\Rightarrow y \in \bar{W} \\ &\Rightarrow (\exists w_1, w_2 \in W) w_1 \sqsubseteq y \sqsubseteq w_2 \\ &\Rightarrow \alpha^*(\bar{W}) \sqsubseteq y \vee \alpha^*(\bar{W}) \sqsubseteq \alpha^*(\bar{W}) \\ &\Rightarrow y \vee \alpha^*(\bar{W}) = \alpha^*(\bar{W}) \end{aligned}$$

Thus, we have that $\alpha^*(\bar{Y}) \vee \alpha^*(\bar{W}) = \alpha^*(\bar{W})$. Similarly, we obtain $\alpha^*(\bar{Y}) \vee \alpha^*(\bar{W}) = \alpha^*(\bar{Y})$, which implies that $\alpha^*(\bar{W}) = \alpha^*(\bar{Y})$.

(ii) α^* is a **Poset morphism**. Suppose $\bar{Y} \sqsubseteq_{EM} \bar{W}$, we have that $(\forall y \in Y)(\exists w_y \in W)y \sqsubseteq w_y$ (since $\downarrow Y \subseteq \downarrow W$) and $(\forall w \in W)(\exists y_w \in Y)y_w \sqsubseteq w$ (since $\uparrow W \subseteq \uparrow Y$). Thus, $\alpha^*(\bar{Y}) \sqsubseteq \bigvee_{y \in Y} w_y$ and $\bigvee_{w \in W} y_w \sqsubseteq \alpha^*(\bar{W})$, which implies that $\alpha^*(\bar{Y}) \sqsubseteq \alpha^*(\bar{W})$.

(iii) **The equation $\alpha^* \circ \mathcal{P}\alpha^* = \alpha^* \circ \mu_X$ holds.** For $\bar{\mathcal{Y}} \in \mathcal{P}^2(X)$,

$$\begin{aligned} \alpha^* \circ \mathcal{P}\alpha^*(\bar{\mathcal{Y}}) &= \alpha^*(\overline{\alpha^*[\bar{\mathcal{Y}}]}) \\ &= \bigvee_{y \in \bigcup_{Y \in \bar{\mathcal{Y}}} Y} y \\ &= \alpha^*(\overline{\bigcup_{Y \in \bar{\mathcal{Y}}} Y}) \\ &= \alpha^* \circ \mu_X(\bar{\mathcal{Y}}) \end{aligned}$$

(iv) **The equation $\alpha \circ \eta_X = 1$ holds.** For $x \in X$,

$$\begin{aligned} \alpha^* \circ \eta_X(x) &= \alpha^*({x}) \\ &= x \end{aligned}$$

Finally, we also verify that for any $f : ((S, \vee), \sqsubseteq) \rightarrow ((S', \vee'), \sqsubseteq')$, then $\mathcal{F}f = f$ is also an algebra map. Let $\bar{Y} \in \mathcal{P}(X)$.

$$\begin{aligned} (\alpha')^* \circ \mathcal{P}f(\bar{Y}) &= (\alpha')^*(\overline{f[\bar{Y}]}) \\ &= \bigvee'_{y \in Y} f(y) \\ &= f(\bigvee_{y \in Y} y) \\ &= f(\alpha^*(\bar{Y})) \\ &= f \circ \alpha^*(\bar{Y}) \end{aligned}$$

(b) We define a functor $\mathcal{G} : \mathbb{P}\text{-Alg} \rightarrow \text{Mod}(\text{ND}_{fin}^*, \text{Poset})$.

$$\begin{array}{ccc} \mathcal{G} : \mathbb{P}\text{-Alg} & \rightarrow & \text{Mod}(\text{ND}_{fin}^*, \text{Poset}) \\ ((X, \sqsubseteq), \alpha) & \mapsto & ((X, \vee^\alpha), \sqsubseteq) \\ \downarrow \mathcal{G} & & \downarrow \mathcal{G} \\ ((X', \sqsubseteq'), \alpha') & \mapsto & ((X', \vee^{\alpha'}), \sqsubseteq') \end{array}$$

For a elements $x_1, x_2 \in X$, we define the operator \vee^α as follows:

$$x_1 \vee^\alpha x_2 = \alpha(\overline{\{x_1, x_2\}}).$$

This operation is well-defined and satisfies the axioms for nondeterministic choice due to the equations satisfied by α and the properties of order-convex closures. Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{ND}^*, \mathbf{Poset})$. Given any $g : (X, \alpha) \rightarrow (X', \alpha')$ and elements $x_1, x_2 \in X$, then

$$\begin{aligned} g(x_1 \vee^\alpha x_2) &= g(\alpha(\overline{\{x_1, x_2\}})) \\ &= \alpha'(\mathcal{P}g(\overline{\{x_1, x_2\}})) \\ &= \alpha'(\overline{\{g(x_1), g(x_2)\}}) \\ &= g(x_1) \vee^{\alpha'} g(x_2) \end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

(i) $\mathcal{G} \circ \mathcal{F} = 1$

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}((S, \vee), \sqsubseteq) &= \mathcal{G}(((S, \sqsubseteq), \alpha^*)) \\ &= ((S, \vee^{\alpha^*}), \sqsubseteq) \end{aligned}$$

We can verify that $((S, \vee), \sqsubseteq) = ((S, \vee^{\alpha^*}), \sqsubseteq)$, since for any elements $x_1, x_2 \in S$

$$\begin{aligned} x_1 \vee^{\alpha^*} x_2 &= \alpha^*(\overline{\{x_1, x_2\}}) \\ &= x_1 \vee x_2 \end{aligned}$$

(ii) $\mathcal{F} \circ \mathcal{G} = 1$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(((X, \sqsubseteq), \alpha)) &= \mathcal{F}(((X, \vee^\alpha), \sqsubseteq)) \\ &= ((X, \sqsubseteq), \alpha^*) \end{aligned}$$

However, $\alpha = \alpha^*$ since for $\bar{Y} \in \mathcal{P}(X)$

$$\begin{aligned} \alpha^*(\bar{Y}) &= \bigvee_{y \in Y}^\alpha y \\ &= \alpha(\bar{Y}) \end{aligned}$$

□

3.3 Theories for Mixed Choice

3.3.1 Axiomatizing Mixed Choice

The goal of this section is to construct a suitable axiomatization for a theory which admits both a nondeterministic choice and a probabilistic choice operator. Clearly, the axioms from both ND_{fin}^* and P_{fin} must be found in any such theory for mixed choice. Next we discuss possible axioms capturing the interaction between these operations. Let us consider the following possible distributivity axioms:

$$\text{(Dist)} \quad (A \boxplus B) \oplus_\lambda C = (A \oplus_\lambda C) \boxplus (B \oplus_\lambda C);$$

$$\text{(Dist')} \quad (A \oplus_\lambda B) \boxplus C = (A \boxplus C) \oplus_\lambda (B \boxplus C).$$

If we examine (Dist') in the presence of (\boxplus -Idem),

$$\begin{aligned} (A \oplus_\lambda B) &= (A \oplus_\lambda B) \boxplus (A \oplus_\lambda B) \\ &= (A \boxplus (A \oplus_\lambda B)) \oplus_\lambda (B \boxplus (A \oplus_\lambda B)) \\ &= ((A \boxplus A) \oplus_\lambda (A \boxplus B)) \oplus_\lambda ((B \boxplus A) \oplus_\lambda (B \boxplus B)) \\ &= A \oplus_{\lambda^2} (B \oplus_{\frac{(1-\lambda)^2}{1-\lambda^2}} (A \boxplus B)) \end{aligned}$$

In particular we get the equation:

$$(A \oplus_{\frac{1}{2}} B) = A \oplus_{\frac{1}{4}} (B \oplus_{\frac{1}{3}} (A \boxplus B)) \quad (1)$$

The above derived equation (1), shows that the presence of (Dist') skews the probabilistic choices in a process. Since we interpret $(A \oplus_\lambda B)$ as a process which chooses A with probability λ and B with probability $1 - \lambda$, and interpret $A \boxplus B$ as a process which randomly chooses between A and B , we get that the left hand side of the above equation chooses the process A with probability $\frac{1}{2}$. However, the right hand side may choose the process A with probability between $\frac{1}{4}$ and $\frac{3}{4}$, depending on the choice of the random scheduler for $A \boxplus B$. Thus, (Dist') is not a desirable axiom for mixed choice, agreeing with [33].

On the other hand the presence of both (Dist) and (P-Idem) leads to the following

derived equation:

$$\begin{aligned}
A \boxplus B &= (A \boxplus B) \oplus_\lambda (A \boxplus B) \\
&= (A \oplus_\lambda (A \boxplus B)) \boxplus (B \oplus_\lambda (A \boxplus B)) \\
&= ((A \oplus_\lambda A) \boxplus (A \oplus_\lambda B)) \boxplus ((B \oplus_\lambda A) \boxplus (B \oplus_\lambda B)) \\
&= (A \boxplus B) \boxplus (A \oplus_\lambda B) \boxplus (B \oplus_\lambda A)
\end{aligned}$$

Which in particular, thanks to the idempotency of \boxplus , gives rise to the equation

$$(A \boxplus B) \boxplus (A \oplus_\lambda B) = A \boxplus B.$$

The derived equation $(A \boxplus B) \boxplus (A \oplus_\lambda B) = A \boxplus B$ captures our intuitive interpretation of the necessary behavior of the **ND** operator \boxplus in the presence of the **P** operators \oplus_λ ; in essence $A \boxplus B$ represents a process which can behave as $A \oplus_\lambda B$ for any weight $\lambda \in [0, 1]$. Thus, this derived equation does not lead to any behavioral conflicts between the operations. Intuitively, we shall consider the nondeterministic choice between A and B as the infinite nondeterministic sum over $\lambda \in [0, 1]$ of all possible probabilistic choices between A and B (i.e. $A \boxplus B = \bigsqcup_{\lambda \in [0,1]} (A \oplus_\lambda B)$) even in theories limited to finite nondeterministic combinations, which is consistent with general belief. Thus, we will choose **(Dist)** as our axiom of interaction for mixed choice, as do Mislove et al [33].

Definition 3.3.1 (Finite Mixed Choice). The finite Lawvere theory of **mixed choice**, \mathbf{MC}_{fin} , is the theory freely generated from the binary probabilistic choice operators, $\oplus_\lambda : 2 \rightarrow 1$, and the binary nondeterministic choice operator, $\boxplus : 2 \rightarrow 1$, satisfying **(P-Assoc)**, **(P-Com)**, **(P-Idem)**, **(\boxplus -Assoc)**, **(\boxplus -Com)**, **(\boxplus -Idem)** and

$$\text{(Dist)} \quad (A_1 \boxplus A_2) \oplus_\lambda A_3 = (A_1 \oplus_\lambda A_3) \boxplus (A_2 \oplus_\lambda A_3)$$

When considering a mixed choice theory which includes infinite nondeterministic choice operators (i.e., nondeterministic choice operators which allow infinitely many inputs) we can formally derive the equation which captures our intuition (i.e., $A \boxplus B = \bigsqcup_{\lambda \in I} (A \oplus_\lambda B)$). We define below a theory for mixed choice which allows for infinite nondeterministic choice operators and countably infinite probabilistic operations.

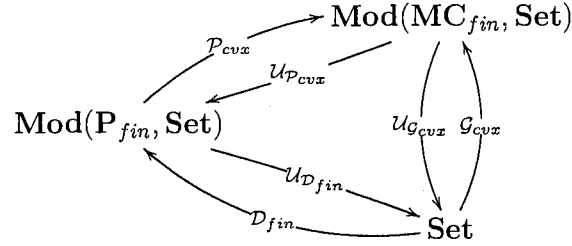


Figure 5: Mixed Choice Factored Through Probabilistic Choice

Definition 3.3.2 (Mixed Choice). The theory of **mixed choice**, MC , is the theory freely generated from the probabilistic choice operators, $\bigoplus_{(\lambda_i)_I} : |I| \rightarrow 1$ and the nondeterministic choice operators, $\boxplus_{w \in W} : |W| \rightarrow 1$, satisfying (P-Ax1) , (P-Ax2) , $(\boxplus\text{-Ax1})$, $(\boxplus\text{-Ax2})$, and

$$(\text{Dist}_\infty) \quad \bigoplus_{(\lambda_i)_I} \left(\boxplus_{w \in W_i} A_w \right) = \boxplus_{\bar{w} \in \prod_I W_i} \left(\bigoplus_{(\lambda_i)_I} A_{\pi_i(\bar{w})} \right)$$

3.3.2 Mixed Choice Model Categories in Set

In what follows, we describe the standard development in the literature of the mixed choice monad. This monad freely constructs set theoretical models for mixed choice. The method is based on following the left hand branch in Figure 5, the corresponding functors will be defined later in this section.

The main idea in constructing free mixed choice models over an arbitrary set X consists of finding its free probabilistic model and then freely adjoining an appropriately distributive nondeterministic choice operator to the free probabilistic model. The construction of the free probabilistic model over X is discussed in Section 3.1.1 and corresponded to the probabilistic distributions monad \mathbb{D}_{fin} . Therefore, our first objective is to define a construction which adjoins an appropriately distributive nondeterministic choice operator to any given model of probabilistic choice. Hence, we develop the *convex powerset functor* which generates the left adjunction illustrated in the top-left corner of Figure 5. Our motivations regarding the use of the convex powerset functor will be based on similar observations and arguments presented in

Plotkin's Pisa Notes, [40], when motivating the construction of the Plotkin Powerdomain.

It is known that for an arbitrary set $X \in \mathbf{Set}$, we capture the behavior of nondeterministic choice by considering particular collections of subsets Y of X . From our previous discussions, the choice of which subsets to consider depends on two factors: (i) the type of nondeterministic choice operator needed, and (ii) the structure (if any) on the set X we wish to preserve. For example for (i), we could consider such notions as internal, external, bounded or unbounded types of nondeterminism. For (ii), in this thesis we consider such structure as partial orders, convexity and semilattices.

Following the diagram above, we consider adding nondeterminism to a convex set. We wish to lift the convex set $(C, +_\lambda)$ in such a way that the convex structure is faithfully reflected in the lifting (i.e., that there is an isomorphic image of $(C, +_\lambda)$ embedded in the lifted object). In other words, we would like that the singleton map of the monad be an isomorphism.

This implies that

- (i) all singleton sets are part of the necessary class of subsets and
- (ii) that the convex combination of singletons is the singleton which contains the convex combination of their elements.

Suppose we have a finite subset $Y \subseteq X$, such that $|Y| > 2$. We want to explore the necessary requirements on a set W in order to allow it to model the nondeterministic choice between every element $y_i \in Y$ taking into consideration the axioms of mixed choice. The derived equation due to the interaction between **P** choice and **ND** choice

$$(\text{Abs}): \left(\bigsqcup_{y_i \in Y} y_i \right) \boxplus \left(\bigoplus_{(\lambda_i)_{i=1}^n} y_i \right) = \bigsqcup_{y_i \in Y} y_i,$$

implies that our proposed model set W must satisfy extra properties. This implies that to capture the nondeterministic behavior of a finite number of elements (i.e., to model $\bigsqcup_{y_i \in Y} y_i$) the set W must also contain all possible probabilistic combinations of those elements (i.e., for any $(\lambda_i)_{i=1}^n \in \mathit{Pwt}_{fin}$, $\bigoplus_{(\lambda_i)_{i=1}^n} y_i \in W$). Therefore, the subset W will model the nondeterministic behavior generated over Y if and only if

$\bigoplus_{(\lambda_i)_{i=1}^n} y_i \in W$ for every $(\lambda_i)_{i=1}^n \in Pwt_{fin}$. This requires that W be a *convex set*. Moreover to form a model for mixed choice we also require that the probabilistic operations on such convex subsets be done pointwise on the generators of the convex subsets. Finally, nondeterministic combinations of such convex subsets are given by taking the convex hull of the union of their generators.

Hence, we can construct a monad over models of probabilistic choice whose algebra category is equivalent to the model category of mixed choice as follows:

Definition 3.3.3 (Nonempty Convex Powerset Monad). The **nonempty convex powerset monad**, $Cvx_{fin} = (Cvx_{fin}, \eta, \mu) : \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$, is given as follows:

(a) The **nonempty convex powerset functor** $Cvx_{fin} : \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$,

(i) **On objects:** For $(C, +_\lambda) \in \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$,

$$Cvx_{fin}((C, +_\lambda)) = (Cvx_{fin}(C), \oplus_\lambda).$$

We denote by $Cvx_{fin}(C)$ the set of all convex hulls of nonempty finite subsets of C (i.e., $Cvx(C) = \{\text{conv}(A) \mid A \in \mathcal{P}_{fin}^*(C)\}$). Recall that $\text{conv}(A)$ denotes the convex hull of the subset A in $(C, +_\lambda)$. The probabilistic choice operator \oplus_λ is defined such that for $\text{conv}(A), \text{conv}(A') \in Cvx_{fin}(C)$ and $\lambda \in [0, 1]$,

$$\text{conv}(A) \oplus_\lambda \text{conv}(A') = \text{conv}(\{a +_\lambda a' \mid a \in A, a' \in A'\}).$$

(ii) **On morphisms:** For $h : (C, +_\lambda) \rightarrow (C', +'_\lambda) \in \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$ and $\text{conv}(A) \in Cvx_{fin}((C, +_\lambda))$,

$$Cvx_{fin}(h)(\overline{A}) = h[\text{conv}(A)]_{cvx} = \text{conv}(h[A]).$$

(b) The unit transformation, $\eta : 1_{\mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})} \Rightarrow Cvx_{fin}$, is the singleton map for each $(C, +_\lambda) \in \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$.

- (c) The multiplication transformation, $\mu : (\mathcal{Cvx}_{fin})^2 \Rightarrow \mathcal{Cvx}_{fin}$, is the **convex closed big union map** for each $(C, +_\lambda) \in \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Set})$.

$$\begin{aligned} \mu_{(C, +_\lambda)} : (\mathcal{Cvx}_{fin})^2((C, +_\lambda)) &\rightarrow \mathcal{Cvx}_{fin}((C, +_\lambda)) \\ \text{conv}(\mathcal{A}) &\mapsto \text{conv}(\bigcup_{\text{conv}(A) \in \mathcal{A}} A) \end{aligned}$$

We can also define a variant which considers superconvex combinations.

Definition 3.3.4 (Nonempty Superconvex Powerset Monad). The **nonempty superconvex powerset monad**, $\mathcal{SCvx} = (\mathcal{SCvx}, \eta, \mu) : \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$, is given as follows:

- (a) The **nonempty superconvex powerset functor** $\mathcal{SCvx} : \mathbf{Mod}(\mathbf{P}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$,

- (i) **On objects:** For $(C, \sum_{(\lambda_i)_I}) \in \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$,

$$\mathcal{SCvx}((C, \sum_{(\lambda_i)_I})) = (\mathcal{SCvx}(C), \bigoplus_{(\lambda_i)_I}).$$

We denote by $\mathcal{SCvx}(C)$ the set of all superconvex hulls of arbitrary nonempty subsets of C (i.e., $\mathcal{SCvx}(C) = \{\text{sconv}(A) \mid A \text{ a countable subset of } C\}$). Recall that $\text{sconv}(A)$ denotes the superconvex hull of the set A in $(C, \sum_{(\lambda_i)_I})$. The countable probabilistic choice operator $\bigoplus_{(\lambda_i)_I}$ is defined such that for a countable family of elements in $\mathcal{SCvx}(C)$, $(\text{sconv}(A_i))_{i \in I}$, and a probability density $(\lambda_i)_I \in \mathit{Pwt}$

$$\bigoplus_{(\lambda_i)_I} \text{sconv}(A_i) = \text{sconv}\left(\left\{ \sum_{(\lambda_i)_I} a_i \mid a_i \in A_i \right\}\right).$$

- (ii) **On morphisms:** For $h : (C, \sum_{(\lambda_i)_I}) \rightarrow (C', \sum'_{(\lambda_i)_I}) \in \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$ and $\text{sconv}(A) \in \mathcal{SCvx}(C)$,

$$\mathcal{SCvx}(h)(\text{sconv}(A)) = h[\text{sconv}(A)]_{\mathcal{SCvx}} = \text{sconv}(h[A]).$$

- (b) The unit transformation, $\eta : 1_{\mathbf{Mod}(\mathbf{P}, \mathbf{Set})} \Rightarrow \mathcal{SCvx}$, is the singleton map for each $(C, \sum_{(\lambda_i)_I}) \in \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$.
- (c) The multiplication transformation, $\mu : (\mathcal{SCvx})^2 \Rightarrow \mathcal{SCvx}$, is the **superconvex closed big union map** for each $(C, \sum_{(\lambda_i)_I}) \in \mathbf{Mod}(\mathbf{P}, \mathbf{Set})$.

$$\begin{aligned} \mu_{(C, \sum_{(\lambda_i)_I})} : (\mathcal{SCvx})^2((C, \sum_{(\lambda_i)_I})) &\rightarrow \mathcal{SCvx}((C, \sum_{(\lambda_i)_I})) \\ \text{sconv}(\mathcal{A}) &\mapsto \text{sconv}\left(\bigcup_{\text{sconv}(A) \in \mathcal{A}} A\right) \end{aligned}$$

Theorem 3.3.5.

- (a) The algebra category $\mathbf{SCvx-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$.
- (b) The algebra category $\mathbf{Cvx}_{fin}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$.

Proof: We shall prove the first assertion $\mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) \cong \mathbf{SCvx-Alg}$: We construct two inverse functors $\mathcal{F} : \mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) \rightarrow \mathbf{SCvx-Alg}$ and $\mathcal{G} : \mathbf{SCvx-Alg} \rightarrow \mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$:

- (a) We begin by defining the functor $\mathcal{F} : \mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) \rightarrow \mathbf{SCvx-Alg}$:

$$\begin{aligned} \mathcal{F} : \mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) &\rightarrow \mathbf{SCvx-Alg} \\ (M, \bigvee, \sum_{(\lambda_i)_I}) &\mapsto ((M, \sum_{(\lambda_i)_I}), \alpha^* : \mathcal{SCvx}((M, \sum_{(\lambda_i)_I})) \rightarrow (M, \sum_{(\lambda_i)_I})) \\ \downarrow^{\mathcal{F}} &\mapsto \downarrow^{\mathcal{F}} \\ (M', \bigvee', \sum'_{(\lambda_i)_I}) &\mapsto ((M', \sum'_{(\lambda_i)_I}), (\alpha')^* : \mathcal{SCvx}((M', \sum'_{(\lambda_i)_I})) \rightarrow (M', \sum'_{(\lambda_i)_I})) \end{aligned}$$

For $\text{sconv}(A) \in \mathcal{SCvx}((M, \sum_{(\lambda_i)_I}))$, we define $\alpha^*(\text{sconv}(A)) = \bigvee_{a \in A} a$ and show that it is a well-defined morphism and satisfies the required equations.

- (i) α^* is well-defined. Suppose that $\text{sconv}(A) = \text{sconv}(B) \in$

$\mathcal{SCvx}((M, \sum_{(\lambda_i)_I}))$, we have that

$$\begin{aligned} a \in A &\Rightarrow a \in \text{sconv}(B) \\ &\Rightarrow \exists b_i^a \in B, (\lambda_i^a)_{i \in I} \in \text{Pwt}. \quad a = \sum_{(\lambda_i^a)_I} b_i^a \\ &\Rightarrow \bigvee_{a \in A} a = \bigvee_{a \in A} \left(\sum_{(\lambda_i^a)_I} b_i^a \right) \\ &\Rightarrow \left(\bigvee_{a \in A} a \right) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(B)) \end{aligned}$$

Thus, we have that $\alpha^*(\text{sconv}(A)) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(B))$. Similarly, we obtain $\alpha^*(\text{sconv}(A)) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(A))$, which implies that $\alpha^*(\text{sconv}(A)) = \alpha^*(\text{sconv}(B))$.

- (ii) α^* is a **Mod(P, Set) morphism**. Suppose we are given a countable family of elements of $\mathcal{SCvx}((M, \sum_{(\lambda_i)_I}))$, $(\text{sconv}(A_i))_{i \in I}$, and a probability density $(\lambda_i)_I \in \text{Pwt}$:

$$\begin{aligned} \alpha^*\left(\bigoplus_{(\lambda_i)_I} \text{sconv}(A_i)\right) &= \alpha^*\left(\text{sconv}\left(\left\{ \sum_{(\lambda_i)_I} a_i \mid a_i \in A_i \right\}\right)\right) \\ &= \bigvee_{\vec{a} \in \prod_{i \in I} A_i} \left(\sum_{(\lambda_i)_I} \pi_i(\vec{a}) \right) \\ &= \sum_{(\lambda_i)_I} \left(\bigvee_{a \in A_i} a \right) \\ &= \sum_{(\lambda_i)_I} \alpha^*(\text{sconv}(A_i)) \end{aligned}$$

- (iii) **The equation $\alpha^* \circ \mathcal{SCvx}(\alpha^*) = \alpha^* \circ \mu_{(M, \sum_{(\lambda_i)_I})}$ holds.** For $\text{sconv}(\mathcal{A}) \in \mathcal{SCvx}^2((M, \sum_{(\lambda_i)_I}))$,

$$\begin{aligned} \alpha^* \circ \mathcal{SCvx}(\alpha^*)(\text{sconv}(\mathcal{A})) &= \alpha^*(\text{sconv}(\alpha^*[\mathcal{A}])) \\ &= \bigvee_{a \in \bigcup_{\text{sconv}(A) \in \mathcal{A}} A} a \\ &= \alpha^*(\text{sconv}(\bigcup_{\text{sconv}(A) \in \mathcal{A}} A)) \\ &= \alpha^* \circ \mu_{(M, \sum_{(\lambda_i)_I})}(\text{sconv}(\mathcal{A})) \end{aligned}$$

- (iv) **The equation $\alpha \circ \eta_{(M, \sum_{(\lambda_i)_I})} = 1$ holds.** For $m \in M$,

$$\begin{aligned} \alpha^* \circ \eta_{(M, \sum_{(\lambda_i)_I})}(m) &= \alpha^*({m}) \\ &= m \end{aligned}$$

Finally, we also verify that for any $f : (M, \bigvee, \sum_{(\lambda_i)_I}) \rightarrow (M', \bigvee', \sum'_{(\lambda_i)_I})$, then $\mathcal{F}f = f$ is also an algebra map. Let $\text{sconv}(A) \in \mathcal{SCvx}((M, \sum_{(\lambda_i)_I}))$.

$$\begin{aligned}
(\alpha')^* \circ \mathcal{SCvx}(f)(\text{sconv}(A)) &= (\alpha')^*(\text{sconv}(f[A])) \\
&= \bigvee'_{a \in A} f(a) \\
&= f\left(\bigvee_{a \in A} a\right) \\
&= f(\alpha^*(\text{sconv}(A))) \\
&= f \circ \alpha^*(\text{sconv}(A))
\end{aligned}$$

(b) Next we define the second functor $\mathcal{G} : \mathbf{SCvx}\text{-Alg} \rightarrow \mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$:

$$\begin{array}{ccc}
\mathcal{G} : \mathbf{SCvx}\text{-Alg} & \rightarrow & \mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) \\
((C, \sum_{(\lambda_i)_I}), \alpha) & \mapsto & (C, \bigvee^\alpha, \sum_{(\lambda_i)_I}^\alpha) \\
\downarrow^g & & \downarrow^g \\
((C', \sum'_{(\lambda_i)_I}), \alpha') & \mapsto & (C', \bigvee^{\alpha'}, \sum_{(\lambda_i)_I}^{\alpha'})
\end{array}$$

For an arbitrary non-empty subset Y of C , a countable family $(x_i)_I \in X$ and a probability density $(\lambda_i)_I$, we define the operators \bigvee^α and $\sum_{(\lambda_i)_I}^\alpha$ as follows:

$$\begin{aligned}
\bigvee_{y \in Y}^\alpha y &= \alpha(\text{sconv}(Y)) \\
\sum_{(\lambda_i)_I}^\alpha x_i &= \alpha(\text{sconv}(\{\sum_{(\lambda_i)_I} x_i\})).
\end{aligned}$$

These operations are well-defined and satisfy the axioms for mixed choice due to the equations satisfied by α , the probabilistic choice operator $\sum_{(\lambda_i)_I}$ and the properties of superconvex hulls. Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$. Given any $g : ((C, \sum_{(\lambda_i)_I}), \alpha) \rightarrow ((C', \sum'_{(\lambda_i)_I}), \alpha')$, an arbitrary non-empty subset Y of C , a countable family $(x_i)_{i \in I}$ in C and a

probability density $(\lambda_i)_{i \in I}$, then

$$\begin{aligned} g\left(\bigvee_{y \in Y}^{\alpha} y\right) &= g(\alpha(\text{sconv}(Y))) \\ &= \alpha'(\mathcal{SCvx}(g)(\text{sconv}(Y))) \\ &= \alpha'(\text{sconv}(g[Y])) \\ &= \bigvee_{y \in Y}^{\alpha'} g(x_i) \end{aligned}$$

$$\begin{aligned} g\left(\sum_{(\lambda_i)_I}^{\alpha} x_i\right) &= g(\alpha(\text{sconv}(\{\sum_{(\lambda_i)_I} x_i\}))) \\ &= \alpha'(\mathcal{SCvx}(g)(\text{sconv}(\{\sum_{(\lambda_i)_I} x_i\}))) \\ &= \alpha'(\text{sconv}(\{\sum_{(\lambda_i)_I}' g(x_i)\})) \\ &= \sum_{(\lambda_i)_I}^{\alpha'} g(x_i) \end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

(i) $\mathcal{G} \circ \mathcal{F} = 1$

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}(M, \bigvee, \sum_{(\lambda_i)_I}) &= \mathcal{G}(((M, \sum_{(\lambda_i)_I}), \alpha^*)) \\ &= (M, \bigvee^{\alpha^*}, \sum_{(\lambda_i)_I}^{\alpha^*}) \end{aligned}$$

We can verify that $(M, \bigvee, \sum_{(\lambda_i)_I}) = (M, \bigvee^{\alpha^*}, \sum_{(\lambda_i)_I}^{\alpha^*})$, since for an arbitrary non-empty subset Y of M , a countable family $(x_i)_I \in M$ and a probability density $(\lambda_i)_I$,

$$\begin{aligned} \bigvee_{y \in Y}^{\alpha^*} &= \alpha^*(\text{sconv}(Y)) \\ &= \bigvee_{y \in Y} y \\ \sum_{(\lambda_i)_I}^{\alpha^*} x_i &= \alpha^*(\text{sconv}(\{\sum_{(\lambda_i)_I} x_i\})) \\ &= \sum_{(\lambda_i)_I} x_i \end{aligned}$$

(ii) $\mathcal{F} \circ \mathcal{G} = 1$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}((C, \sum_{(\lambda_i)_I}, \alpha)) &= \mathcal{F}((C, \bigvee^\alpha, \sum_{(\lambda_i)_I}^\alpha)) \\ &= ((C, \sum_{(\lambda_i)_I}^\alpha), \alpha^*) \end{aligned}$$

However, $(C, \sum_{(\lambda_i)_I}) = (C, \sum_{(\lambda_i)_I}^\alpha)$ and $\alpha = \alpha^*$ since for a countable family $(x_i)_I \in C$, a probability density $(\lambda_i)_I$, and $\text{sconv}(A) \in \mathcal{SCvx}((C, \sum_{(\lambda_i)_I}^\alpha))$.

$$\begin{aligned} \sum_{(\lambda_i)_I}^\alpha x_i &= \alpha(\text{sconv}(\{\sum_{(\lambda_i)_I} x_i\})) \\ &= \alpha \circ \eta_{(C, \sum_{(\lambda_i)_I})}(\sum_{(\lambda_i)_I} x_i) \\ &= \sum_{(\lambda_i)_I} x_i \end{aligned}$$

$$\begin{aligned} \alpha^*(\text{sconv}(A)) &= \bigvee_{a \in A}^\alpha a \\ &= \alpha(\text{sconv}(A)) \end{aligned}$$

□

From the finite distributions monad and the non-empty convex powerset monad we can finally construct the monad over **Set** which freely constructs models for mixed choice: the *geometrically convex monad*. This corresponds to the functor on the right-hand side in Figure 5.

Definition 3.3.6 (Finite Geometrically Convex Monad). The **finite geometrically convex monad**, $\mathbb{G}_{cvx} = (\mathcal{G}_{cvx}, \eta, \mu) : \mathbf{Set}$, is given as follows:

(a) The **finite geometrically convex functor** $\mathcal{G}_{cvx} : \mathbf{Set} \rightarrow \mathbf{Set}$,

(i) **On objects:** For $X \in \mathbf{Set}$,

$$\mathcal{G}_{cvx}(X) = (\mathcal{G}_{cvx}(X), \boxplus, \oplus_\lambda)$$

We denote by $\mathcal{G}_{cvx}(X)$ the set of all convex hulls of nonempty finite subsets of $\mathcal{D}_{fin}(X)$ (i.e., $\mathcal{G}_{cvx}(X) = \{\text{conv}(A) \mid A \in \mathcal{P}_{fin}^*(\mathcal{D}_{fin}(X))\}$). The operators \boxplus and \oplus_λ are defined such that for $\text{conv}(A), \text{conv}(B) \in \mathcal{G}_{cvx}(X)$ and a $\lambda \in I$,

$$\text{conv}(A) \boxplus \text{conv}(B) = \text{conv}(A \cup B)$$

$$\text{conv}(A) \oplus_\lambda \text{conv}(B) = \text{conv}(\{\lambda a + (1 - \lambda)b \mid a \in A, b \in B\}).$$

(ii) **On morphisms:** For $f : X \rightarrow X' \in \mathbf{Set}$ and $\text{conv}(A) \in \mathcal{G}_{cvx}(X)$,

$$\mathcal{G}_{cvx}f(\text{conv}(A)) = \text{conv}(f[A]_{cvx}).$$

(b) The unit transformation, $\eta : \mathbf{1}_{\mathbf{Set}} \Rightarrow \mathcal{G}_{cvx}$, is the **singleton Dirac distribution map** for each $X \in \mathbf{Set}$.

$$\begin{aligned} \eta_X : X &\rightarrow \mathcal{G}_{cvx}(X) \\ x &\mapsto \{\delta_x\}. \end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{G}_{cvx}^2 \Rightarrow \mathcal{G}_{cvx}$, is the **convex set flattening map** for each $X \in \mathbf{Set}$.

$$\begin{aligned} \mu_X : \mathcal{G}_{cvx}^2(X) &\rightarrow \mathcal{G}_{cvx}(X) \\ \text{conv}(\mathcal{A}) &\mapsto \boxplus_{D \in \mathcal{A}} \left(\bigoplus_{(D(\text{conv}(A)))_{\text{supp}(D)}} \text{conv}(A) \right) \end{aligned}$$

In essence, the convex set flattening map takes a convex set generated by a finite set of distributions over $\mathcal{G}_{cvx}(X)$ (i.e., a convex set of the form $\text{conv}(\{D_1, D_2, \dots, D_n\})$ where $D_i : \mathcal{G}_{cvx}(X) \rightarrow [0, 1]$), flattens each distribution $D^b = \bigoplus_{(D(\text{conv}(A)))_{\text{supp}(D)}} \text{conv}(A)$, and computes the convex hull generated by the flattened distributions.

Once more, we can define a similar monad for models of arbitrary mixed choice as follows.

Definition 3.3.7 (Geometrically Convex Monad). The **geometrically convex monad**, $\mathbb{G}_{scvx} = (\mathcal{G}_{scvx}, \eta, \mu) : \mathbf{Set}$, is given as follows:

(a) The geometrically convex functor $\mathcal{G}_{scvx} : \mathbf{Set} \rightarrow \mathbf{Set}$,

(i) On objects: For $X \in \mathbf{Set}$,

$$\mathcal{G}_{scvx}(X) = (\mathcal{G}_{scvx}(X), \boxplus, \bigoplus_{(\lambda_i)_I})$$

We denote by $\mathcal{G}_{scvx}(X)$ the set of all superconvex hulls of arbitrary nonempty subsets of $\mathcal{D}(X)$ (i.e., $\mathcal{G}_{scvx}(X) = \{\text{sconv}(A) \mid A \in \mathcal{P}^*(\mathcal{D}(X))\}$). The operators \boxplus and $\bigoplus_{(\lambda_i)_I}$ are defined such that for an arbitrary nonempty family, $(\text{sconv}(A_w))_{w \in W}$, a countable family, $(\text{sconv}(A_i))_{i \in I}$, and a probability density $(\lambda_i)_{i \in I}$,

$$\begin{aligned} \boxplus_{w \in W} \text{sconv}(A_w) &= \text{sconv}\left(\bigcup_{w \in W} A_w\right) \\ \bigoplus_{(\lambda_i)_I} \text{sconv}(A_i) &= \text{sconv}\left(\left\{\sum_{i \in I} \lambda_i a_i \mid a_i \in A_i\right\}\right). \end{aligned}$$

(ii) On morphisms: For $f : X \rightarrow X' \in \mathbf{Set}$ and $\text{sconv}(A) \in \mathcal{G}_{scvx}(X)$,

$$\mathcal{G}_{scvx}f(\text{sconv}(A)) = \text{sconv}(f[C]_{scvx}).$$

(b) The unit transformation, $\eta : 1_{\mathbf{Set}} \Rightarrow \mathcal{G}_{scvx}$, is the singleton Dirac distribution map for each $X \in \mathbf{Set}$. For $X \in \mathbf{Set}$,

$$\begin{aligned} \eta_X : X &\rightarrow \mathcal{G}_{scvx}(X) \\ x &\mapsto \{\delta_x\} \end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{G}_{scvx}^2 \Rightarrow \mathcal{G}_{scvx}$, is the **superconvex flattening map** for each $X \in \mathbf{Set}$.

$$\begin{aligned} \mu_X : \mathcal{G}_{scvx}^2(X) &\rightarrow \mathcal{G}_{scvx}(X) \\ \text{sconv}(\mathcal{A}) &\mapsto \boxplus_{D \in \mathcal{A}} \left(\bigoplus_{(D(\text{sconv}(A)))_{\text{supp}(D)}} \text{sconv}(A) \right) \end{aligned}$$

Theorem 3.3.8.

(a) The algebra category $\mathbb{G}_{scvx}\text{-Alg}$ is equivalent to the model category $\text{Mod}(\text{MC}, \text{Set})$.

(b) The algebra category $\mathbb{G}_{cvx}\text{-Alg}$ is equivalent to the model category $\text{Mod}(\text{MC}_{fin}, \text{Set})$.

Proof: We shall prove the first assertion $\text{Mod}(\text{MC}, \text{Set}) \cong \mathbb{G}_{scvx}\text{-Alg}$: We construct two inverse functors $\mathcal{F} : \text{Mod}(\text{MC}, \text{Set}) \rightarrow \mathbb{G}_{scvx}\text{-Alg}$ and $\mathcal{G} : \mathbb{G}_{scvx}\text{-Alg} \rightarrow \text{Mod}(\text{MC}, \text{Set})$:

(a) We begin by defining the functor $\mathcal{F} : \text{Mod}(\text{MC}, \text{Set}) \rightarrow \mathbb{G}_{scvx}\text{-Alg}$:

$$\begin{aligned} \mathcal{F} : \text{Mod}(\text{MC}, \text{Set}) &\rightarrow \mathbb{G}_{scvx}\text{-Alg} \\ (M, \bigvee, \sum_{(\lambda_i)_I}) &\mapsto (M, \alpha^* : \mathcal{G}_{scvx}(M) \rightarrow M) \\ \downarrow f &\mapsto \downarrow f \\ (M', \bigvee', \sum'_{(\lambda_i)_I}) &\mapsto (M', (\alpha')^* : \mathcal{G}_{scvx}(M') \rightarrow M') \end{aligned}$$

For $\text{sconv}(A) \in \mathcal{G}_{scvx}(M)$, we define $\alpha^*(\text{sconv}(A)) = \bigvee_{a \in A} (\sum_{(a(m))_{\text{supp}(a)}} m)$ and show that it is a well-defined map and satisfies the required equations.

(i) α^* is well-defined. Suppose that $\text{sconv}(A) = \text{sconv}(B) \in \mathcal{G}_{scvx}(M)$, we have that

$$\begin{aligned} a \in A &\Rightarrow a \in \text{sconv}(B) \\ &\Rightarrow \exists b_i^a \in B, (\lambda_i^a)_{i \in I} \in \text{Pwt.} \quad a = \sum_{(\lambda_i^a)_I} b_i^a \\ &\Rightarrow \sum_{(a(m))_{\text{supp}(a)}} m = \sum_{(\lambda_i^a)_I} (\sum_{(b_i^a(m))_{\text{supp}(b_i^a)}} m) \\ &\Rightarrow \bigvee_{a \in A} (\sum_{(a(m))_{\text{supp}(a)}} m) = \bigvee_{a \in A} (\sum_{(\lambda_i^a)_I} (\sum_{(b_i^a(m))_{\text{supp}(b_i^a)}} m)) \\ &\Rightarrow (\bigvee_{a \in A} (\sum_{(a(m))_{\text{supp}(a)}} m)) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(B)) \end{aligned}$$

Thus, we have that $\alpha^*(\text{sconv}(A)) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(B))$. Similarly, we obtain $\alpha^*(\text{sconv}(A)) \vee \alpha^*(\text{sconv}(B)) = \alpha^*(\text{sconv}(A))$, which implies that $\alpha^*(\text{sconv}(A)) = \alpha^*(\text{sconv}(B))$.

(ii) **The equation $\alpha^* \circ \mathcal{G}_{scvx} \alpha^* = \alpha^* \circ \mu_X$ holds.** For $\text{sconv}(\mathcal{A}) \in \mathcal{G}_{scvx}^2(X)$,

$$\begin{aligned}
\alpha^* \circ \mathcal{G}_{scvx}(\alpha^*)(\text{sconv}(\mathcal{A})) &= \alpha^*(\text{sconv}(\alpha^*[\mathcal{A}]_{scvx})) \\
&= \bigvee_{D \in \mathcal{A}} \left(\sum_{(D(\text{sconv}(A)))_{\text{supp}(D)}} \alpha^*(\text{sconv}(A)) \right) \\
&= \alpha^* \left(\bigoplus_{D \in \mathcal{A}} \left(\bigoplus_{(D(\text{sconv}(A)))_{\text{supp}(D)}} \text{sconv}(A) \right) \right) \\
&= \alpha^* \circ \mu_X(\text{sconv}(\mathcal{A}))
\end{aligned}$$

(iii) **The equation $\alpha \circ \eta_X = 1$ holds.** For $x \in X$,

$$\begin{aligned}
\alpha^* \circ \eta_X(x) &= \alpha^*({\delta_x}) \\
&= x
\end{aligned}$$

Finally, we also verify that for any $f : (M, \bigvee, \sum_{(\lambda_i)_I}) \rightarrow (M', \bigvee', \sum'_{(\lambda_i)_I})$, then $\mathcal{F}f = f$ is also an algebra map. Let $\text{sconv}(A) \in \mathcal{G}_{scvx}(M)$.

$$\begin{aligned}
(\alpha')^* \circ \mathcal{G}_{scvx} f(\text{sconv}(A)) &= (\alpha')^*(\text{sconv}(f[A]_{scvx})) \\
&= \bigvee'_{d \in A} \left(\sum'_{(d(m))_{\text{supp}(d)}} f(m) \right) \\
&= f \left(\bigvee_{d \in A} \left(\sum_{(d(m))_{\text{supp}(d)}} m \right) \right) \\
&= f(\alpha^*(\text{sconv}(A))) \\
&= f \circ \alpha^*(\text{sconv}(A))
\end{aligned}$$

(b) Next we define the second functor $\mathcal{G} : \mathbb{G}_{scvx}\text{-Alg} \rightarrow \text{Mod}(\text{MC}, \text{Set})$:

$$\begin{array}{ccc}
\mathcal{G} : \mathbb{G}_{scvx}\text{-Alg} & \rightarrow & \text{Mod}(\text{MC}, \text{Set}) \\
(X, \alpha) & \mapsto & (X, \bigvee^\alpha, \sum_{(\lambda_i)_I}^\alpha) \\
\downarrow \mathcal{G} & & \downarrow \mathcal{G} \\
(X', \alpha') & \mapsto & (X', \bigvee^{\alpha'}, \sum_{(\lambda_i)_I}^{\alpha'})
\end{array}$$

For an arbitrary non-empty subset Y of X , a countable family $(x_i)_I \in X$ and a

probability density $(\lambda_i)_I$, we define the operators \bigvee^α and $\sum_{(\lambda_i)_I}^\alpha$ as follows:

$$\begin{aligned}\bigvee_{y \in Y}^\alpha y &= \alpha(\text{sconv}(\{\delta_y \mid y \in Y\})) \\ \sum_{(\lambda_i)_I}^\alpha x_i &= \alpha(\text{sconv}(\{\sum_{i \in I} \lambda_i \delta_{x_i}\})).\end{aligned}$$

These operations are well-defined and satisfy the axioms for mixed choice due to the equations satisfied by α , the probabilistic choice operator $\sum_{(\lambda_i)_I}$ and the properties of superconvex hulls. Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$. Given any $g : (X, \alpha) \rightarrow (X', \alpha')$, an arbitrary non-empty subset Y of X , a countable family $(x_i)_{i \in I}$ in X and a probability density $(\lambda_i)_{i \in I}$, then

$$\begin{aligned}g(\bigvee_{y \in Y}^\alpha y) &= g(\alpha(\text{sconv}(\{\delta_y \mid y \in Y\}))) \\ &= \alpha'(\mathcal{G}_{\text{scvx}}g(\text{sconv}(\{\delta_y \mid y \in Y\}))) \\ &= \alpha'(\text{sconv}(g[Y]_{\text{scvx}})) \\ &= \bigvee_{y \in Y}^{\alpha'} g(y) \\ g(\sum_{(\lambda_i)_I}^\alpha x_i) &= g(\alpha(\text{sconv}(\{\sum_{i \in I} \lambda_i \delta_{x_i}\}))) \\ &= \alpha'(\mathcal{G}_{\text{scvx}}g(\text{sconv}(\{\sum_{i \in I} \lambda_i \delta_{x_i}\}))) \\ &= \alpha'(\text{sconv}(\{\sum_{i \in I} \lambda_i \delta_{g(x_i)}\})) \\ &= \sum_{(\lambda_i)_I}^{\alpha'} g(x_i)\end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

(i) $\mathcal{G} \circ \mathcal{F} = 1$

$$\begin{aligned}\mathcal{G} \circ \mathcal{F}(M, \bigvee, \sum_{(\lambda_i)_I}) &= \mathcal{G}((M, \alpha^*)) \\ &= (M, \bigvee^{\alpha^*}, \sum_{(\lambda_i)_I}^{\alpha^*})\end{aligned}$$

We can verify that $(M, \bigvee, \sum_{(\lambda_i)_I}) = (M, \bigvee^{\alpha^*}, \sum_{(\lambda_i)_I}^{\alpha^*})$, since for an arbitrary non-empty subset Y of M , a countable family $(x_i)_I \in M$ and a probability density $(\lambda_i)_I$,

$$\begin{aligned} \bigvee_{y \in Y}^{\alpha^*} y &= \alpha^*(\text{sconv}(\{\delta_y \mid y \in Y\})) \\ &= \bigvee_{y \in Y} y \\ \sum_{(\lambda_i)_I}^{\alpha^*} x_i &= \alpha^*(\text{sconv}(\{\sum_{i \in I} \delta_{x_i}\})) \\ &= \sum_{(\lambda_i)_I} x_i \end{aligned}$$

(ii) $\mathcal{F} \circ \mathcal{G} = 1$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}((X, \alpha)) &= \mathcal{F}((X, \bigvee^{\alpha}, \sum_{(\lambda_i)_I}^{\alpha})) \\ &= (X, \alpha^*) \end{aligned}$$

However, $\alpha = \alpha^*$ since for $\text{sconv}(A) \in \mathcal{SCvx}((C, \sum_{(\lambda_i)_I}^{\alpha}))$.

$$\begin{aligned} \alpha^*(\text{sconv}(A)) &= \bigvee_{d \in A}^{\alpha} (\sum_{(d(x))_{\text{supp}(d)}}^{\alpha} x) \\ &= \alpha(\text{sconv}(A)) \end{aligned}$$

□

3.3.3 Mixed Choice Model Categories in Poset

As we have seen in Section 3.2.2, the presence of an order structure enables the definition of three possible types of nondeterministic choice: convex (Plotkin), lower (Hoare) and upper (Smyth). In this section we will present analogous versions of mixed choice each derived by combining the probabilistic choice theory and either the convex, lower or upper nondeterministic choice theory, resulting in the usual, the lower and the upper mixed choice theories, respectively. Many of these definitions and results arise from the work of Keimel, Plotkin and Tix [23], however their focus

was to obtain domain theoretical models. In what follows we adapt their definitions to discuss posetal models for all three variants of the theory of mixed choice.

The convex (Plotkin) nondeterministic choice theory captures the classical notion of nondeterminism independent of the presence of an order structure. This means that the **ND** operator and the equations it satisfies are no different than those used when dealing with set theoretical models of nondeterministic choice. In truth, when considered on sets endowed with the discrete order, all three types of posetal **ND** theories (Plotkin, Hoare and Smyth) collapse to the same theory considered over **Set**. Similarly, combining the probabilistic choice theory with the convex nondeterministic choice theory gives rise to the previously considered mixed choice theory. Thus it admits the same mixed choice operators and equations as we have been considering for models of mixed choice over **Set**. Below we present our posetal adaptation of the biconvex powerdomain monad, called the *biconvex algebra monad* which is associated to the model category for the regular mixed choice theory over **Poset**.

Definition 3.3.9 (Biconvex Algebra Monad). The **biconvex algebra monad**, $\mathbb{P}\mathcal{V} = (\mathcal{P}\mathcal{V}, \eta, \mu) : \mathbf{Poset}$ is given as follows:

(a) The **biconvex algebra functor** $\mathcal{P}\mathcal{V} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

(i) **On objects:** For $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{P}\mathcal{V}((X, \sqsubseteq)) = ((\mathcal{P}\mathcal{V}(X), \overline{\boxplus}, \overline{\boxplus}_\lambda), \preceq_{EM}).$$

We denote by $\mathcal{P}\mathcal{V}(X)$ the set of order-convex closures under \preceq of finitely generated convex subsets of $\mathcal{V}((X, \sqsubseteq))$ (i.e., $\mathcal{P}\mathcal{V}(X) = \{\overline{C} \mid C \in \mathcal{G}_{cvx}(X)\}$, where \overline{C} is the smallest order-convex, convex subset of $\mathcal{V}(X)$ containing C). The operators $\overline{\boxplus}$ and $\overline{\boxplus}_\lambda$ are the order-convex closures of \boxplus and \boxplus_λ defined for \mathcal{G}_{cvx} . For $\overline{C}_1, \overline{C}_2 \in \mathcal{P}\mathcal{V}(X)$, we have that

$$\begin{aligned} \overline{C}_1 \overline{\boxplus} \overline{C}_2 &= \overline{(C_1 \boxplus C_2)}, \\ \overline{C}_1 \overline{\boxplus}_\lambda \overline{C}_2 &= \overline{(C_1 \boxplus_\lambda C_2)}. \end{aligned}$$

Finally, the order structure \preceq_{EM} is the *Egli-Milner order over* $\mathcal{V}((X, \sqsubseteq))$.

(ii) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\bar{C} \in \mathcal{PV}(X)$,

$$\mathcal{PV}f(\bar{C}) = \overline{f[C]_{cvx}}.$$

(b) The unit transformation, $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{PV}$, is define to be the singleton Dirac distribution map for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{PV}((X, \sqsubseteq)) \\ x &\mapsto \{\delta_x\} \end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{PV}^2 \Rightarrow \mathcal{PV}$, is the **order-convex flattening map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$.

$$\begin{aligned} \mu_{(X, \sqsubseteq)} : \mathcal{PV}^2((X, \sqsubseteq)) &\rightarrow \mathcal{PV}((X, \sqsubseteq)) \\ \bar{C} &\mapsto \overline{\left(\bigsqcup_{D \in \text{gen}(C)} \left(\bigoplus_{(D(\bar{C}))_{\text{supp}(D)}} C \right) \right)} \end{aligned}$$

In essence, the order-convex flattening map is defined similarly to the convex set flattening map presented in Definition 3.3.6 of the finite geometrically convex monad. In other words, given $\overline{\text{conv}(\{D_1, D_2, \dots, D_n\})}$ where $D_i : \mathcal{PV}(X) \rightarrow [0, 1]$, we flatten each generating distribution (i.e., $D^b = \bigoplus_{(D(\text{conv}(A)))_{\text{supp}(D)}} \text{conv}(A)$), compute the convex hull over the set of flattened distributions ($\text{conv}(\{D_1^b, \dots, D_n^b\})$) and take its order-closure ($\overline{\text{conv}(\{D_1^b, \dots, D_n^b\})}$).

The lower (Hoare) nondeterministic choice theory is described as capturing the angelic view of nondeterminism, see Keimel, Plotkin and Tix [23]. This means that the **ND** operator and the equations it satisfies contain those presented for the usual nondeterministic choice theory, but in addition it admits the inequality: $A \leq A \boxplus B$. Therefore, when defining the theory which combines probabilistic choice and lower nondeterministic choice, our axiomatic theory must also admit the required inequality.

Definition 3.3.10 (Lower Mixed Choice Theory). The finite Lawvere theory for **lower mixed choice**, $\mathbf{H-MC}_{fin}$, is obtained by considering the theory \mathbf{MC}_{fin} , under the extra condition:

$$(H) \quad A \leq A \boxplus B$$

Below we present our posetal adaptation of the convex lower powerdomain monad, called the *convex lower ND algebra monad*, which is associated to the model category for the lower mixed choice theory over **Poset**.

Definition 3.3.11 (Convex Lower Nondeterministic Algebra Monad). The **convex lower ND algebra monad**, $\mathbb{H}\mathcal{V} = (\mathcal{H}\mathcal{V}, \eta, \mu) : \mathbf{Poset}$ is given as follows:

(a) The **convex lower ND algebra functor** $\mathcal{H}\mathcal{V} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,

(i) **On objects:** For $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{H}\mathcal{V}((X, \sqsubseteq)) = ((\mathcal{H}\mathcal{V}(X), \boxplus^\downarrow, \oplus_\lambda^\downarrow), \sqsubseteq),$$

We denote by $\mathcal{H}\mathcal{V}(X)$ the set of lower sets under \preceq (the distributions order) of finitely generated convex subsets of $\mathcal{V}((X, \sqsubseteq))$ (i.e., $\mathcal{H}\mathcal{V}(X) = \{\downarrow_{\preceq} C \mid C \in \mathcal{G}_{cvx}(X)\}$, where $\downarrow_{\preceq} C$ is the set of all distributions in $\mathcal{V}(X)$ which are smaller than some distribution in C). The operators \boxplus^\downarrow and $\oplus_\lambda^\downarrow$ are the appropriate closures of \boxplus and \oplus_λ . For $\downarrow_{\preceq} C_1, \downarrow_{\preceq} C_2 \in \mathcal{H}\mathcal{V}(X)$, we have that

$$\begin{aligned} \downarrow_{\preceq} C_1 \boxplus^\downarrow \downarrow_{\preceq} C_2 &= \downarrow_{\preceq} (C_1 \boxplus C_2), \\ \downarrow_{\preceq} C_1 \oplus_\lambda^\downarrow \downarrow_{\preceq} C_2 &= \downarrow_{\preceq} (C_1 \oplus_\lambda C_2). \end{aligned}$$

(ii) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\downarrow_{\preceq} C \in \mathcal{H}\mathcal{V}(X)$,

$$\mathcal{H}\mathcal{V}f(\downarrow_{\preceq} C) = \downarrow_{\preceq'} (f[C]_{cvx}).$$

(b) The unit transformation, $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{H}\mathcal{V}$, is defined to be the **downclosed singleton Dirac distribution map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$. It is obtained by taking the down-closure (under the distributions order \preceq) on the image of the singleton Dirac distributions map.

$$\begin{aligned} \eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{H}\mathcal{V}((X, \sqsubseteq)) \\ x &\mapsto \downarrow_{\preceq} \{\delta_x\}. \end{aligned}$$

- (c) The multiplication transformation, $\mu : \mathcal{H}\mathcal{V}^2 \Rightarrow \mathcal{H}\mathcal{V}$, is defined to be the **down-closed flattening map** for each $(X, \sqsubseteq) \in \mathbf{Poset}$. This map is an easy modification of the order-convex flattening map given above. In particular, we take the down-closure (under the distributions order \preceq) on the image of the convex set flattening map.

$$\begin{aligned} \mu_{(X, \sqsubseteq)} : \mathcal{H}\mathcal{V}^2((X, \sqsubseteq)) &\rightarrow \mathcal{H}\mathcal{V}((X, \sqsubseteq)) \\ \downarrow_{\preceq} \mathcal{C} &\mapsto \downarrow_{\preceq} \left(\bigsqcup_{D \in \text{gen}(\mathcal{C})} \left(\bigoplus_{(D \downarrow_{\preceq} \mathcal{C})_{\text{supp}(D)}} C \right) \right) \end{aligned}$$

The upper (Smyth) nondeterministic choice theory is described as capturing the demonic view of nondeterminism, see Keimel, Plotkin and Tix [23]. This means that the ND operator and the equations it satisfies contain those presented for the usual nondeterministic choice theory, but in addition it admits the inequality: $A \geq A \boxplus B$. Therefore, when defining the theory which combines probabilistic choice and upper nondeterministic choice, our axiomatic theory must also admit the required inequality.

Definition 3.3.12 (Upper Mixed Choice Theory). The finite Lawvere theory for **upper mixed choice**, $\mathbf{S-MC}_{fin}$, is obtained by considering the theory \mathbf{MC}_{fin} , under the extra condition:

$$(S) \quad A \geq A \boxplus B$$

Below we present our posetal adaptation of the convex upper powerdomain monad, called the *convex upper ND algebra monad*, which is associated to the model category for the convex upper mixed choice theory over \mathbf{Poset} .

Definition 3.3.13 (Convex Upper Nondeterministic Algebra Monad). The **convex upper ND algebra monad**, $\mathbf{SV} = (\mathcal{S}\mathcal{V}, \eta, \mu) : \mathbf{Poset}$ is given as follows:

- (a) The **convex upper ND algebra functor** $\mathcal{S}\mathcal{V} : \mathbf{Poset} \rightarrow \mathbf{Poset}$,
- (i) **On objects:** For $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{S}\mathcal{V}((X, \sqsubseteq)) = ((\mathcal{S}\mathcal{V}(X), \boxplus^\dagger, \oplus_\lambda^\dagger), \supseteq),$$

We denote by $\mathcal{S}\mathcal{V}(X)$ the set of upper sets under \preceq (the distributions order) of finitely generated convex subsets of $\mathcal{V}((X, \sqsubseteq))$ (i.e., $\mathcal{S}\mathcal{V}(X) =$

$\{\uparrow_{\leq} C \mid C \in \mathcal{G}_{cvx}(X)\}$, where $\uparrow_{\leq} C$ is the set of all distribution in $\mathcal{V}(X)$ which are larger than some distribution in C). The operators \boxplus^{\uparrow} and $\oplus_{\lambda}^{\uparrow}$ are the appropriate closures of \boxplus and \oplus_{λ} . For $\uparrow_{\leq} C_1, \uparrow_{\leq} C_2 \in \mathcal{SV}(X)$, we have that

$$\begin{aligned}\uparrow_{\leq} C_1 \boxplus^{\uparrow} \uparrow_{\leq} C_2 &= \uparrow_{\leq} (C_1 \boxplus C_2), \\ \uparrow_{\leq} C_1 \oplus_{\lambda}^{\uparrow} \uparrow_{\leq} C_2 &= \uparrow_{\leq} (C_1 \oplus_{\lambda} C_2).\end{aligned}$$

(ii) For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\uparrow_{\leq} C \in \mathcal{SV}(X)$,

$$\mathcal{SV}f(\uparrow_{\leq} C) = \uparrow_{\leq'} (f[C]_{cvx}).$$

(b) The unit transformation, $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{SV}$, is defined to be the **upclosed singleton Dirac distribution map**. Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\begin{aligned}\eta_{(X, \sqsubseteq)} : (X, \sqsubseteq) &\rightarrow \mathcal{SV}((X, \sqsubseteq)) \\ x &\mapsto \uparrow_{\leq} \{\delta_x\}.\end{aligned}$$

(c) The multiplication transformation, $\mu : \mathcal{SV}^2 \Rightarrow \mathcal{SV}$, is defined to be the **upclosed flattening map**. Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\begin{aligned}\mu_{(X, \sqsubseteq)} : \mathcal{SV}^2((X, \sqsubseteq)) &\rightarrow \mathcal{SV}((X, \sqsubseteq)) \\ \uparrow_{\leq} \mathcal{C} &\mapsto \uparrow_{\leq} \left(\boxplus_{D \in \text{gen}(\mathcal{C})} \left(\bigoplus_{(D(\uparrow_{\leq} C))_{\text{supp}(D)}} C \right) \right)\end{aligned}$$

Theorem 3.3.14.

- (a) The algebra category $\mathbf{PV}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$.
- (b) The algebra category $\mathbf{HV}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{H}\text{-MC}_{fin}, \mathbf{Poset})$.
- (c) The algebra category $\mathbf{SV}\text{-Alg}$ is equivalent to the model category $\mathbf{Mod}(\mathbf{S}\text{-MC}_{fin}, \mathbf{Poset})$.

Proof: We shall prove the first assertion $\mathbb{P}\mathcal{V}\text{-Alg} \cong \text{Mod}(\text{MC}_{fin}, \text{Poset})$, since the second and third results follow using straightforward modifications.

(a) We define a functor $\mathcal{F} : \text{Mod}(\text{MC}_{fin}, \text{Poset}) \rightarrow \mathbb{P}\mathcal{V}\text{-Alg}$.

$$\begin{array}{ccc} \mathcal{F} : \text{Mod}(\text{MC}_{fin}, \text{Poset}) & \rightarrow & \mathbb{P}\mathcal{V}\text{-Alg} \\ ((M, \vee, +_\lambda), \sqsubseteq) & \mapsto & ((M, \sqsubseteq), \alpha^* : \mathcal{P}\mathcal{V}((M, \sqsubseteq)) \rightarrow (M, \sqsubseteq)) \\ \downarrow^f & \mapsto & \downarrow^f \\ ((M', \vee', +'_\lambda), \sqsubseteq') & \mapsto & ((M', \sqsubseteq'), (\alpha')^* : \mathcal{P}((M', \sqsubseteq')) \rightarrow (M', \sqsubseteq')) \end{array}$$

For $\overline{C} \in \mathcal{P}\mathcal{V}((M, \sqsubseteq))$, we define the algebra structure map by $\alpha^*(\overline{C}) = \bigvee_{d \in \text{gen}(C)} (\sum_{(d(m))_{\text{supp}(d)}} m)$, where $\text{gen}(C)$ is the set of finite generators of the convex set C . Next we verify that α^* is a well-defined morphism in Poset and that it satisfies the required equations.

(i) α^* is well-defined. Suppose that $\overline{C}_1 = \overline{C}_2 \in \mathcal{P}\mathcal{V}((M, \sqsubseteq))$, we have that

$$\begin{aligned} d \in \text{gen}(C_1) &\Rightarrow d \in \overline{C}_2 \\ &\Rightarrow (\exists d_1, d_2 \in C) d_1 \preceq d \preceq d_2 \\ &\Rightarrow \sum_{(d_1(m))_{\text{supp}(d_1)}} m \sqsubseteq \sum_{(d(m))_{\text{supp}(d)}} m \sqsubseteq \sum_{(d_2(m))_{\text{supp}(d_2)}} m, \\ &\Rightarrow \alpha^*(\overline{C}_2) \sqsubseteq (\sum_{(d(m))_{\text{supp}(d)}} m) \vee \alpha^*(\overline{C}_2) \sqsubseteq \alpha^*(\overline{C}_2) \\ &\Rightarrow (\sum_{(d(m))_{\text{supp}(d)}} m) \vee \alpha^*(\overline{C}_2) = \alpha^*(\overline{C}_2) \end{aligned}$$

Thus, we have that $\alpha^*(\overline{C}_1) \vee \alpha^*(\overline{C}_2) = \alpha^*(\overline{C}_2)$. Similarly, we obtain $\alpha^*(\overline{C}_1) \vee \alpha^*(\overline{C}_2) = \alpha^*(\overline{C}_1)$, which implies that $\alpha^*(\overline{C}_1) = \alpha^*(\overline{C}_2)$.

(ii) α^* a Poset morphism. Suppose $\overline{C}_1 \sqsubseteq_{EM} \overline{C}_2$, we have that $(\forall d \in \text{gen}(C_1)) (\exists D \in C_2) d \preceq D$ (since $\downarrow_{\preceq} C_1 \subseteq \downarrow_{\preceq} C_2$) and $(\forall D' \in \text{gen}(C_2)) (\exists d' \in C_1) d' \preceq D'$ (since $\uparrow_{\preceq} C_2 \subseteq \uparrow_{\preceq} C_1$). Thus, $\alpha^*(\overline{C}_1) \sqsubseteq \bigvee_{d \in \text{gen}(C_1)} (\sum_{(D(m))_{\text{supp}(D)}} m)$ and $\bigvee_{D' \in \text{gen}(C_2)} (\sum_{(d'(m))_{\text{supp}(d')}} m) \sqsubseteq \alpha^*(\overline{C}_2)$, which implies that $\alpha^*(\overline{C}_1) \sqsubseteq \alpha^*(\overline{C}_2)$.

(iii) **The equation $\alpha^* \circ \mathcal{PV}\alpha^* = \alpha^* \circ \mu_{(X, \sqsubseteq)}$ holds.** For $\bar{C} \in \mathcal{PV}^2(X)$,

$$\begin{aligned} \alpha^* \circ \mathcal{PV}\alpha^*(\bar{C}) &= \alpha^*(\overline{\alpha^*[C]_{cvx}}) \\ &= \bigvee_{D \in \mathcal{C}} \left(\sum_{(D(\bar{C}))_{\text{supp}(D)}} \alpha^*(\bar{C}) \right) \\ &= \alpha^*(\overline{\bigoplus_{D \in \mathcal{C}} \left(\bigoplus_{(D(\bar{C}))_{\text{supp}(D)}} C \right)}) \\ &= \alpha^* \circ \mu_{(X, \sqsubseteq)}(\bar{C}) \end{aligned}$$

(iv) **The equation $\alpha \circ \eta_{(X, \sqsubseteq)} = 1$ holds.** For $x \in X$,

$$\begin{aligned} \alpha^* \circ \eta_{(X, \sqsubseteq)}(x) &= \alpha^*(\{\delta_x\}) \\ &= x \end{aligned}$$

Finally, we also verify that for any $f : ((M, \vee, +_\lambda), \sqsubseteq) \rightarrow ((M', \vee', +'_\lambda), \sqsubseteq')$, then $\mathcal{F}f = f$ is also an algebra map. Let $\bar{C} \in \mathcal{PV}(X)$.

$$\begin{aligned} (\alpha')^* \circ \mathcal{PV}f(\bar{C}) &= (\alpha')^*(\overline{f[C]_{cvx}}) \\ &= \bigvee'_{d \in \text{gen}(C)} \left(\sum'_{(d(x))_{\text{supp}(d)}} f(x) \right) \\ &= f \left(\bigvee_{d \in \text{gen}(C)} \left(\sum_{(d(x))_{\text{supp}(d)}} x \right) \right) \\ &= f(\alpha^*(\bar{C})) \\ &= f \circ \alpha^*(\bar{C}) \end{aligned}$$

(b) We define a functor $\mathcal{G} : \mathbb{PV}\text{-Alg} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$.

$$\begin{array}{ccc} \mathcal{G} : \mathbb{PV}\text{-Alg} & \rightarrow & \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset}) \\ ((X, \sqsubseteq), \alpha) & \mapsto & ((X, \vee^\alpha, +_\lambda^\alpha), \sqsubseteq) \\ \downarrow \mathcal{G} & & \downarrow \mathcal{G} \\ ((X', \sqsubseteq'), \alpha') & \mapsto & ((X', \vee^{\alpha'}, +_\lambda^{\alpha'}), \sqsubseteq') \end{array}$$

For a elements $x_1, x_2 \in X$, we define the operators \vee^α and $+_\lambda^\alpha$ as follows:

$$x_1 \vee^\alpha x_2 = \alpha(\overline{\{\delta_{x_1}, \delta_{x_2}\}})$$

$$x_1 +_\lambda^\alpha x_2 = \alpha(\overline{\{\lambda\delta_{x_1} + (1 - \lambda)\delta_{x_2}\}})$$

These operations are well-defined and satisfy the axioms for mixed choice due to the equations satisfied by α and the properties of order-convex closures. Moreover, we show that $\mathcal{G}g = g$ is a morphism in $\mathbf{Mod}(\mathbf{MC}, \mathbf{Poset})$. Given any $g : ((X, \sqsubseteq), \alpha) \rightarrow ((X', \sqsubseteq'), \alpha')$ and elements $x_1, x_2 \in X$, then

$$\begin{aligned} g(x_1 \vee^\alpha x_2) &= g(\alpha(\overline{\{\delta_{x_1}, \delta_{x_2}\}})) \\ &= \alpha'(\mathcal{PV}g(\overline{\{\delta_{x_1}, \delta_{x_2}\}})) \\ &= \alpha'(\overline{\{\delta_{g(x_1)}, \delta_{g(x_2)}\}}) \\ &= g(x_1) \vee^{\alpha'} g(x_2) \end{aligned}$$

$$\begin{aligned} g(x_1 +_\lambda^\alpha x_2) &= g(\alpha(\overline{\{\lambda\delta_{x_1} + (1-\lambda)\delta_{x_2}\}})) \\ &= \alpha'(\mathcal{PV}g(\overline{\{\lambda\delta_{x_1} + (1-\lambda)\delta_{x_2}\}})) \\ &= \alpha'(\overline{\{\lambda\delta_{g(x_1)} + (1-\lambda)\delta_{g(x_2)}\}}) \\ &= g(x_1) +_\lambda^{\alpha'} g(x_2) \end{aligned}$$

(c) Finally, we verify that the two functors are inverses.

(i) $\mathcal{G} \circ \mathcal{F} = 1$

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}((M, \vee, +_\lambda), \sqsubseteq) &= \mathcal{G}(((M, \sqsubseteq), \alpha^*)) \\ &= ((M, \vee^{\alpha^*}, +_\lambda^{\alpha^*}), \sqsubseteq) \end{aligned}$$

We can verify that $((M, \vee, +_\lambda), \sqsubseteq) = ((M, \vee^{\alpha^*}, +_\lambda^{\alpha^*}), \sqsubseteq)$, since for any elements $m_1, m_2 \in M$

$$\begin{aligned} m_1 \vee^{\alpha^*} m_2 &= \alpha^*(\overline{\{\delta_{m_1}, \delta_{m_2}\}}) \\ &= m_1 \vee m_2 \end{aligned}$$

$$\begin{aligned} m_1 +_\lambda^{\alpha^*} m_2 &= \alpha^*(\overline{\{\lambda\delta_{m_1} + (1-\lambda)\delta_{m_2}\}}) \\ &= m_1 +_\lambda m_2 \end{aligned}$$

(ii) $\mathcal{F} \circ \mathcal{G} = 1$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}((X, \sqsubseteq), \alpha) &= \mathcal{F}((X, \vee^\alpha, +^\alpha), \sqsubseteq) \\ &= ((X, \sqsubseteq), \alpha^*) \end{aligned}$$

However, $\alpha = \alpha^*$ since for $\bar{C} \in \mathcal{PV}(X)$

$$\begin{aligned} \alpha^*(\bar{C}) &= \bigvee_{d \in \text{gen}(C)}^\alpha \left(\sum_{(d(x))_{\text{supp}(d)}}^\alpha x \right) \\ &= \alpha(\bar{C}) \end{aligned}$$

□

Chapter 4

Mixed Choice via Nondeterminism

4.1 The Existence of an Alternate Path to Models of Mixed Choice

In this section we show that there exists an alternate path in the construction of models for mixed choice. In the previous chapter we have obtained models for mixed choice by essentially analyzing how one would add the appropriate kind of nondeterminism to models of probabilistic choice. This has the advantage of not only permitting us to construct models for mixed choice over any adequately structured category \mathbf{C} , but also allows us to construct mixed choice models over arbitrary \mathbf{C} -models of probabilistic choice.

We now consider a dual approach: we interchange the order of freely adjoining the choice operations from the standard development presented in Chapter 3. Consider the diagram in Figure 6, of which the left hand triangle was used in Figure 5. The alternative path we shall study in this chapter is represented by following the right-hand path in the diagram.

As in our previous discussion concerning probabilistic extensions to mixed choice, we wish to obtain the same result by freely adjoining appropriately distributive probabilistic choice operators to \mathbf{C} -models of nondeterministic choice. In order to do so, we need to construct a left adjoint to the forgetful functor $\mathcal{U}_{Stbl_{fin}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{C}) \rightarrow$

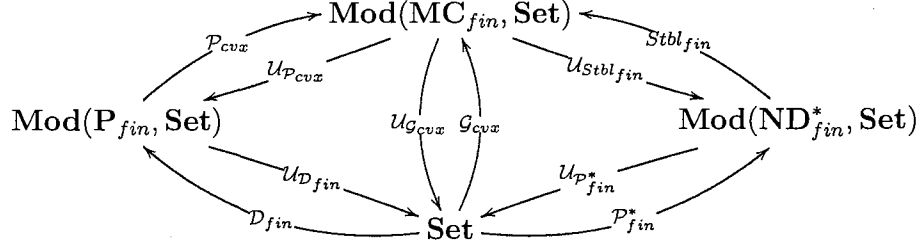
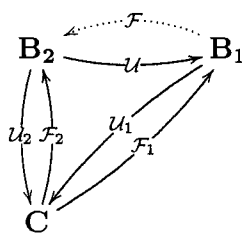


Figure 6: Mixed Choice Factored Through Probabilistic and Nondeterministic Choice

$\text{Mod}(\text{ND}_{fin}^*, \mathbf{C})$. The existence of such a left adjoint arises due to the following theorem from Barr and Wells [2, p.133].

Theorem 4.1.1. *Consider the diagram below satisfying the given conditions,*



- (a) \mathcal{F}_2 is left adjoint to \mathcal{U}_2 ,
- (b) \mathcal{F}_1 is left adjoint to \mathcal{U}_1 ,
- (c) $\mathcal{U}_1 \circ \mathcal{U}$ is naturally isomorphic to \mathcal{U}_2 ,
- (d) \mathcal{U}_1 is of descent type, and
- (e) \mathbf{B}_2 has coequalizers.

Then \mathcal{U} has a left adjoint \mathcal{F} for which $\mathcal{F} \circ \mathcal{F}_1 \cong \mathcal{F}_2$.

Proof: (Barr and Wells [2]). If $\mathcal{F}_1(C)$ is an object in the image of \mathcal{F}_1 , then we have

$$\begin{aligned} \mathbf{B}_1(\mathcal{F}_1(C), \mathcal{U}(B_2)) &\cong \mathbf{C}(C, \mathcal{U}_1 \mathcal{U}(B_2)) \\ &\cong \mathbf{C}(C, \mathcal{U}_2(B_2)) \\ &\cong \mathbf{B}_2(\mathcal{F}_2(C), B_2) \end{aligned}$$

which shows that $\mathcal{F}_2(C)$ represents the functor $\mathbf{B}_1(\mathcal{F}_1(C), \mathcal{U}(-))$. Moreover, the Yoneda Lemma can easily be used to show that maps in \mathbf{B}_1 between objects in the image of \mathcal{F}_1 give rise to morphisms in \mathbf{B}_2 with the required naturality properties. Thus we get a functor \mathcal{F} defined at least on the full subcategory whose objects are the images of \mathcal{F}_1 . It is easily extended to all of \mathbf{B}_1 by letting

$$\mathcal{F}_1(C) \rightrightarrows \mathcal{F}_1(C') \longrightarrow B_1$$

be a coequalizer and defining $\mathcal{F}(B_1)$ so that

$$\mathcal{F}_2(C) \rightrightarrows \mathcal{F}_2(C') \longrightarrow \mathcal{F}(B_1)$$

is as well. The universal mapping property of coequalizers gives, for any object $B_2 \in \mathbf{B}_2$ the diagram below in which both rows are equalizers,

$$\begin{array}{ccccc} \mathbf{B}_2(\mathcal{F}(B_1), B_2) & \longrightarrow & \mathbf{B}_2(\mathcal{F}_2(C'), B_2) & \rightrightarrows & \mathbf{B}_2(\mathcal{F}_2(C), B_2) \\ & & \downarrow \cong & & \downarrow \cong \\ \mathbf{B}_1(B_1, \mathcal{U}(B_2)) & \longrightarrow & \mathbf{B}_1(\mathcal{F}_1(C'), \mathcal{U}(B_2)) & \rightrightarrows & \mathbf{B}_1(\mathcal{F}_1(C), \mathcal{U}(B_2)) \end{array}$$

from which the adjointness follows. \square

Corollary 4.1.2. *The forgetful functor $\mathcal{U}_{Stbl_{fin}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ has a left adjoint.*

Proof: We have the following diagram

$$\begin{array}{ccc} & \xleftarrow{\dots Stbl_{fin} \dots} & \\ \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set}) & \xrightarrow{\mathcal{U}_{Stbl_{fin}}} & \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set}) \\ \uparrow \mathcal{U}_{\mathcal{G}_{cvx}} & \nearrow \mathcal{U}_{\mathcal{P}_{fin}^*} & \\ \mathbf{Set} & \xleftarrow{\mathcal{P}_{fin}^*} & \end{array}$$

satisfying all the required properties stated in Theorem 4.1.1. \square

Although Corollary 4.1.2 states the existence of the left adjoint $Stbl_{fin}$, the proof of Theorem 4.1.1 is only partially constructive. The proof clearly states that the image of a free object $(\mathcal{P}_{fin}^*(X), \cup) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ under the functor $Stbl_{fin}$ is isomorphic to $\mathcal{G}_{cvx}(X) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$. However, the image of a non-free object is less trivial. It requires that we understand how to compute coequalizers in $\mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$. Hence, to define $Stbl_{fin}$ one must determine how to calculate coequalizers in $\mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$.

In the following section we shall present the necessary material in order to give a concrete presentation of the functor $Stbl_{fin}$.

4.2 \vee -Stable Convex Sets

Our goal is to construct the free mixed choice model from any given model of nondeterminism. We shall motivate our construction of the \vee -stable convex functor based on similar observations and arguments presented in Plotkin's Pisa Notes, [40], when motivating the construction of the Plotkin Powerdomain.

We've already presented the structure of a free mixed choice model constructed from a set X without any particular structure, by presenting the finite geometrically convex functor, \mathcal{G}_{cvx} , in Section 3.3.2. It consists of taking the collection of all finitely generated convex subsets of $(\mathcal{D}_{fin}(X), \oplus_\lambda)$. Moreover, in that same section, we have also presented the construction of free mixed choice models from convex sets $(C, +_\lambda)$, using the convex powerset functor, Cvx_{fin} . It consists of taking the finitely generated convex subsets in $(C, +_\lambda)$. In the second construction, *it is important to note that the unit transformation is a convex set isomorphism*. In other words, there exists an isomorphic copy of $(C, +_\lambda)$ within the free mixed choice model. Moreover, the free model is also the smallest model for mixed choice which contains the image of the unit transformation. We shall be keeping these properties in mind when determining the appropriate definition for our \vee -stable monad.

Now consider a semilattice (S, \vee) . By studying key properties present due to the presence of the \vee structure, we shall motivate the need to consider a particular type of convex subset as our model. Intuitively, a convex subset represents a mixed choice process whose behavior is determined by the nondeterministic combination of its generating terms, since the unit of the monad must be a semilattice homomorphism. In particular suppose a convex subset $C \subseteq \mathcal{D}_{fin}(S)$ contains a set of distributions of the form $\{\delta_t \mid t \in T\}$; then C captures the nondeterministic behavior induced from $\bigvee_{t \in T} t \in S$. However, except in trivial cases, there are many (possibly uncountable) ways to describe the same nondeterministic behavior in a given semilattice. For example, in any powerset semilattice, different unions of subsets can lead to exactly the same set. (e.g. $\{a, b\} \cup \{c\} = \{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$).

Definition 4.2.1. Suppose we are given a semilattice (S, \vee) and an particular element

$s \in S$. Then, $\mathcal{F}_s = \{T \mid T \subseteq_{fin} S \text{ and } \bigvee T = s\}$ denotes the set of all **finite** \vee -**decompositions** of s .

Since every $T' \in \mathcal{F}_{(\bigvee T)}$ describes equivalent *nondeterministic behaviors* (in other words they evaluate under \vee to the same element in S), then we must have that $\{\delta_{t'} \mid t' \in T'\} \subseteq C$ for each $T' \in \mathcal{F}_{\bigvee T}$ because each such set captures the same ND behavior as $\{\delta_t \mid t \in T\}$.

More generally, consider a convex set C containing $\mathcal{D}_{fin}(T_1) \oplus_\lambda \mathcal{D}_{fin}(T_2)$, where $T_i \subseteq_{fin} S$ for $i = 1, 2$. Thus C captures the nondeterministic behavior induced from T_1 with probability λ and captures the nondeterministic behavior induced from T_2 with probability $1 - \lambda$. From our previous discussion, we know that capturing the ND behavior induced by T_i implies that it must capture the ND behavior induced by T'_i where $T'_i \in \mathcal{F}_{(\bigvee T_i)}$, for $i = 1, 2$. Therefore for all $T'_1 \in \mathcal{F}_{(\bigvee T_1)}$ and $T'_2 \in \mathcal{F}_{(\bigvee T_2)}$, C must contain $\mathcal{D}_{fin}(T'_1) \oplus_\lambda \mathcal{D}_{fin}(T'_2)$. Given our observations, we shall require that convex subsets in our model be \vee -stable.

Definition 4.2.2 (\vee -Stable). Suppose we are given a semilattice (S, \vee) . A convex set $C \in \mathcal{G}_{conv}(S)$ is said to be \vee -stable, if whenever there exists a family, $(T_i)_I$, of finite subsets of S such that $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \subseteq C$ then $(\forall T'_i \in \mathcal{F}_{(\bigvee T_i)}) \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq C$.

However, to completely capture mixed choice, the set of all \vee -stable convex subsets is still too large. We shall consider a minimal set of \vee -convex subsets which allows us to satisfy the following condition: *there exists an isomorphic copy of the semilattice (S, \vee) in the induced mixed choice model*. Hence, in the sections below, we will consider the set of \vee -stable convex subsets of $\mathcal{G}_{conv}(S)$ generated by the isomorphic image of (S, \vee) under the operations $\tilde{\boxplus}$ and $\widetilde{\oplus}_\lambda$, the usual operations closed under \vee -stability.

4.2.1 Completion of Convex Sets over Semilattices

In this section we define, on an arbitrary semilattice (S, \vee) , a closure operation called the \vee -stable completion on arbitrary convex subsets of the free convex set over S . This closure operation is very important in defining the left adjoint of

$\mathcal{U}_{Stbl_{fin}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$. It generates the right collection of convex subsets needed in order to build a mixed choice model which faithfully reflects the nondeterministic behavior of its underlying semilattice. The set of arbitrary convex subsets of the free convex subset over S is given by the underlying set of the image of the functor $Cvx : \mathbf{Set} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$. We define this functor below.

Definition 4.2.3 (Arbitrary Convex Set Functor). The **arbitrary convex set functor** $Cvx : \mathbf{Set} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$ is defined as follows:

(a) On objects: Let $X \in \mathbf{Set}$,

$$Cvx(X) = (Cvx(X), \boxplus, \oplus_\lambda)$$

- (i) The underlying set $Cvx(X)$ is the set of all arbitrary convex subsets of $\mathcal{D}_{fin}(X)$. Equivalently, we have that $Cvx(X) = \{C \mid C \subseteq_{cvx} \mathcal{D}_{fin}(X)\}$, where $C \subseteq_{cvx} \mathcal{D}_{fin}(X)$ signifies that C is a convex subset of $\mathcal{D}_{fin}(X)$.
- (ii) The nondeterministic choice operation \boxplus is the convex hull operation.

$$\begin{aligned} C \boxplus C' &= \text{the smallest convex subset of } \mathcal{D}_{fin}(X) \text{ containing } C \text{ and } C' \\ &= \text{conv}(C \cup C') \end{aligned}$$

- (iii) The probabilistic choice operations are defined such that

$$C \oplus_\lambda C' = \{d \oplus_\lambda d' \mid d \in C, d' \in C'\}$$

(b) On morphisms: Let $f : X \rightarrow Y \in \mathbf{Set}$ and $C \in Cvx(X)$,

$$Cvx f(C) = f[C]_{cvx},$$

recall that $f[C]_{cvx} = \{d_f \mid d \in C\}$.

Remark 4.2.4. It is important to note that for any $X \in \mathbf{Set}$, that the free mixed choice model $\mathcal{G}_{cvx}(X)$, defined in Section 3.3.2, is a mixed choice submodel of $Cvx(X)$. In other words, $\mathcal{G}_{cvx}(X) \subseteq Cvx(X)$, and the nondeterministic and probabilistic operations in $\mathcal{G}_{cvx}(X)$ are the appropriate restrictions on the corresponding operations in $Cvx(X)$.

In our previous discussion, our observations on which convex subsets will be necessary in order to build this left adjoint focused heavily on specific properties involving convex subsets generated by finite subsets of S (*faces of $\mathcal{D}_{fin}(S)$*), and their probabilistic combinations (*facial elements of $\mathcal{D}_{fin}(S)$*).

Remark 4.2.5. Consider a finite semilattice (S, \vee) , where n is the cardinality of S . The convex set $\mathcal{D}_{fin}(S)$ is isomorphic to the n -simplex whose vertices are given by the elements of S . Thus, a convex set of the form $\mathcal{D}_{fin}(T)$ for $T \in \mathcal{P}_{fin}^*(S)$ is geometrically represented by an actual face of the n -simplex.

Definition 4.2.6 (Faces and Facial Elements). Given a semilattice (S, \vee) ,

- (a) A convex subset, $\mathcal{D}_{fin}(T) = \{d : T \rightarrow [0, 1] \mid \text{supp}(d) < \infty, d(T) = 1\} \in \text{Cvx}(S)$, where T is a finite subset of S , is called a **face of $\mathcal{D}_{fin}(S)$** . Note that if (S, \vee) is a finite semilattice, then for any subset $T \subseteq S$, $\mathcal{D}_{fin}(T)$ corresponds to the face generated by the elements of T in the n -simplex generated by the elements of S , where n is the cardinality of the set S .
- (b) A **facial element of $\mathcal{D}_{fin}(S)$** , F , is a convex subset obtained by taking probabilistic combinations of faces of $\mathcal{D}_{fin}(S)$ in the mixed choice model $(\text{Cvx}(S), \boxplus, \oplus_\lambda)$. The set of all facial elements, $\mathcal{F}^{elt}(S)$, is a convex subset of $\text{Cvx}(S)$, generated by the faces of $\mathcal{D}_{fin}(S)$.

$$\mathcal{F}^{elt}(S) = \left\{ \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \mid (\lambda_i)_I \in \text{Pwt}_{fin}, T_i \in \mathcal{P}_{fin}^*(S) \right\} \subseteq \text{Cvx}(S).$$

Example 4.2.7. Consider the two semilattices S_1 and S_2 with the same underlying set $\{a, b, c, d\}$ but with different semilattice structures, \vee_1 and \vee_2 respectively, as shown in Figure 7. Since the semilattices have the same underlying set, their set of faces will be identical. Some of the faces of $\mathcal{D}_{fin}(\{a, b, c, d\})$, are shown in Figure 8. Since we are working with a finite underlying set, $\{a, b, c, d\}$, we shall often denote a convex subset of $\mathcal{D}_{fin}(\{a, b, c, d\})$, by the geometrical object it represents. For example, any convex set generated by a single point mass distribution shall be denoted by its corresponding point in the tetrahedron. Since the semilattices have the same set of faces they generate the same set of facial elements (i.e $\mathcal{F}^{elt}(S_1) = \mathcal{F}^{elt}(S_2)$).

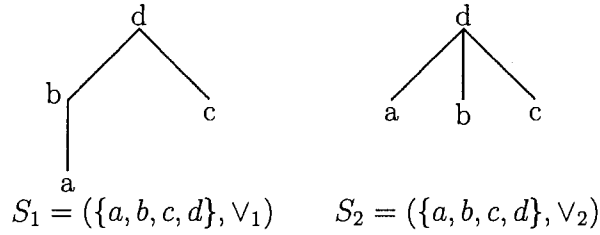


Figure 7: Semilattices S_1 and S_2

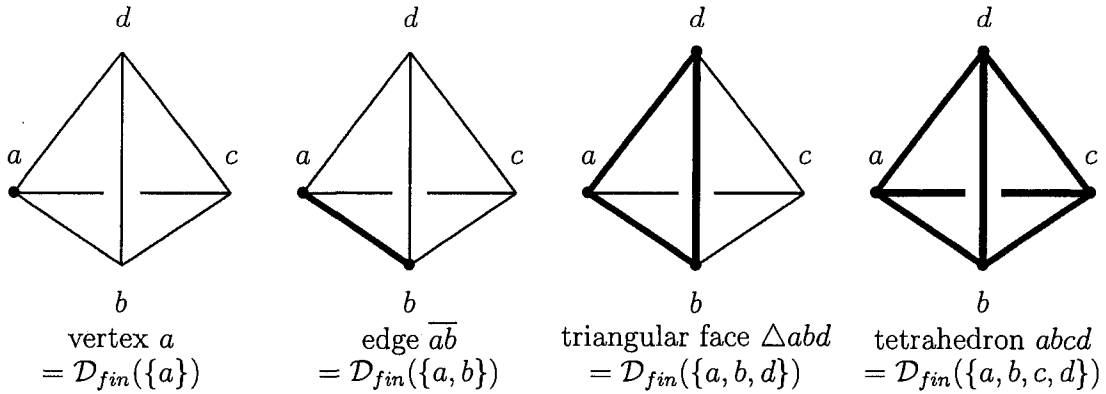
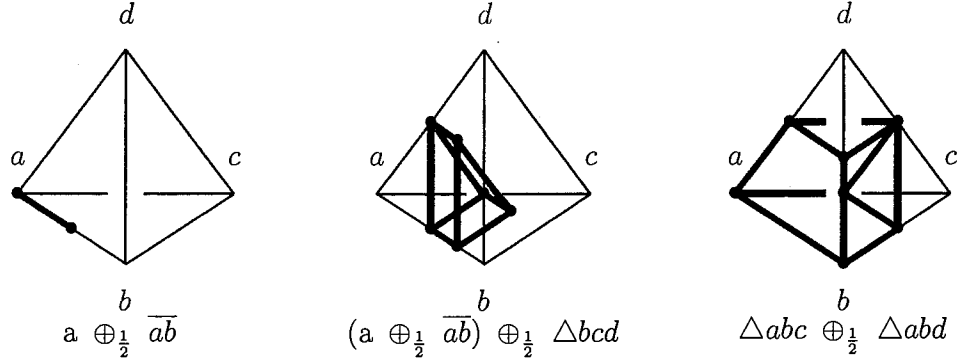


Figure 8: Examples of the four types of faces of $\mathcal{D}_{fin}(\{a, b, c, d\})$

In Figure 9, we give the geometrical representation of three different combination of faces. The first is obtained by taking the fair (probabilistic weight of $\frac{1}{2}$) combination of the vertex a , ($= \mathcal{D}_{fin}(\{a\})$) and the edge \overline{ab} , ($= \mathcal{D}_{fin}(\{a, b\})$). Thus we obtain

$$\begin{aligned}
 \mathcal{D}_{fin}(\{a\}) \oplus_{\frac{1}{2}} \mathcal{D}_{fin}(\{a, b\}) &= \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{2}} a, a \oplus_{\frac{1}{2}} b\}) \\
 &= \mathcal{D}_{fin}(\{a, a \oplus_{\frac{1}{2}} b\}).
 \end{aligned}$$

The second is obtained by taking the fair combination of the previous facial element


 Figure 9: Three examples of facial elements of $\mathcal{D}_{fin}(\{a, b, c, d\})$

$\mathcal{D}_{fin}(\{a, a \oplus_{\frac{1}{2}} b\})$ and the triangle $\triangle bcd$, ($= \mathcal{D}_{fin}(\{b, c, d\})$). In this case we obtain¹

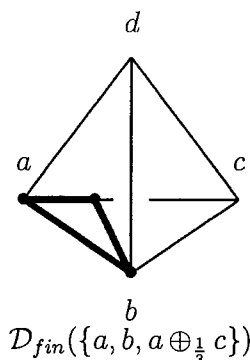
$$\begin{aligned}
 \mathcal{D}_{fin}(\{a, a \oplus_{\frac{1}{2}} b\}) \oplus_{\frac{1}{2}} \mathcal{D}_{fin}(\{b, c, d\}) &= \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{2}} b, a \oplus_{\frac{1}{2}} c, a \oplus_{\frac{1}{2}} d, (a \oplus_{\frac{1}{2}} b) \oplus_{\frac{1}{2}} b, \\
 &\quad (a \oplus_{\frac{1}{2}} b) \oplus_{\frac{1}{2}} c, (a \oplus_{\frac{1}{2}} b) \oplus_{\frac{1}{2}} d\}) \\
 &= \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{2}} b, a \oplus_{\frac{1}{2}} c, a \oplus_{\frac{1}{2}} d, a \oplus_{\frac{1}{4}} b, \\
 &\quad \bigoplus_{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})} (a, b, c), \bigoplus_{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})} (a, b, d)\})
 \end{aligned}$$

The final example is the fair combination of the triangle $\triangle abc$, ($= \mathcal{D}_{fin}(\{a, b, c\})$), and the triangle $\triangle abd$, ($= \mathcal{D}_{fin}(\{a, b, d\})$). We obtain

$$\begin{aligned}
 \mathcal{D}_{fin}(\{a, b, c\}) \oplus_{\frac{1}{2}} \mathcal{D}_{fin}(\{a, b, d\}) &= \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{2}} a, a \oplus_{\frac{1}{2}} b, a \oplus_{\frac{1}{2}} d, b \oplus_{\frac{1}{2}} a, b \oplus_{\frac{1}{2}} b, b \oplus_{\frac{1}{2}} d, \\
 &\quad c \oplus_{\frac{1}{2}} a, c \oplus_{\frac{1}{2}} b, c \oplus_{\frac{1}{2}} d\}) \\
 &= \mathcal{D}_{fin}(\{a, a \oplus_{\frac{1}{2}} d, b, b \oplus_{\frac{1}{2}} d, c \oplus_{\frac{1}{2}} a, c \oplus_{\frac{1}{2}} b, c \oplus_{\frac{1}{2}} d\})
 \end{aligned}$$

In Figure 10 we give an example of an element in $\mathcal{G}_{cvx}(\{a, b, c, d\})$ which is not a facial element. This shows that in general, $\mathcal{G}_{cvx}(S) \neq \mathcal{F}^{elt}(S)$. Consider the convex set $\mathcal{D}_{fin}(\{a, b, a \oplus_{\frac{1}{3}} c\})$. If this convex set were to be written as a probabilistic combination of faces of $\mathcal{D}_{fin}(\{a, b, c, d\})$, then to obtain the vertex a we must only consider combinations containing the faces: $a, \overline{ab}, \overline{ac}, \overline{ad}, \triangle abc, \triangle abd, \triangle acd$, and $abcd$. Moreover, if we must also obtain the vertex b , we must narrow our combination to one consisting of the faces: $\overline{ab}, \triangle abc$, and $abcd$. Since our convex subset does not contain

¹recall from (def 1.2.3) that $\bigoplus_{(\lambda_i)_I} (a_i)$ represents the convex combination of the a_i each with respective weight λ_i .

Figure 10: An element of $\mathcal{G}_{cvx}(\{a, b, c, d\})$ which is not a facial element

any probabilistic combinations containing the vertex d , it follows that it must be some combination of the faces \overline{ab} and $\triangle abc$. However,

$$\begin{aligned} \mathcal{D}_{fin}(\{a, b\}) \oplus_{\lambda} \mathcal{D}_{fin}(\{a, b, c\}) &= \mathcal{D}_{fin}(\{a \oplus_{\lambda} a, a \oplus_{\lambda} b, a \oplus_{\lambda} c, b \oplus_{\lambda} a, b \oplus_{\lambda} b, b \oplus_{\lambda} c\}) \\ &= \mathcal{D}_{fin}(\{a, b, a \oplus_{\lambda} c, b \oplus_{\lambda} c\}) \\ &\neq \mathcal{D}_{fin}(\{a, b, a \oplus_{\frac{1}{3}} c\}) \text{ for any choice of } \lambda \end{aligned}$$

△

As we have discussed, a model which contains a particular representation of a ND behavior must contain all other equivalent representations of that ND behavior. Moreover, if our model contains $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \in \mathcal{F}^{elt}(S)$, then it must also contain $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\{\bigvee T_i\})$, since they have equivalent ND behaviors. Thus, our model must contain the largest convex set containing all ND behaviors equivalent to $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\{\bigvee T_i\})$. We call such a convex set the \vee -expansion of the facial element.

Definition 4.2.8 (\vee -Expansion). Suppose we are given a semilattice (S, \vee) and a facial element $F \in \mathcal{F}^{elt}(S)$ of the form $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i)$, where $T_i \in \mathcal{P}_{fin}^*(S)$ and $(\lambda_i)_I \in Pwt_{fin}$. Then the \vee -expansion of F , denoted F^{\downarrow} , is defined as the following convex set:

$$F^{\downarrow} = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\downarrow_{\vee}(\bigvee T_i)) \in \mathcal{Cvx}(S).$$

From our previous observations we have seen one way of determining when a convex subset is \vee -stable. Recall, that a convex set $C \in \mathcal{Cvx}(S)$ is \vee -stable if whenever

$\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \subseteq C$ then $(\forall T'_i \in \mathcal{F}_{(\vee T_i)}) \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq C$. In what follows, we shall give a more concise definition for the \vee -stability property.

Proposition 4.2.9. *Let $C \in \mathcal{Cvx}(S)$. Then the following are equivalent:*

(a) *If $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \subseteq C$ then $(\forall T'_i \in \mathcal{F}_{(\vee T_i)}) \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq C$.*

(b) *For every facial element F of $\mathcal{D}_{fin}(S)$,*

$$F \subseteq C \Rightarrow F^\perp \subseteq C.$$

Proof:

(i) Suppose that C is \vee -stable. Suppose we are given a facial element $F \in \mathcal{F}^{elt}(S)$ of the form $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i)$ such that $F \subseteq C$. Then, by property (a), for every choice of $T'_i \in \mathcal{F}_{(\vee T_i)}$, $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq C$. We claim that this implies that $F^\perp \subseteq C$. Let $d \in F^\perp$, then $d = \sum_{i \in I} \lambda_i d_i$ such that $d_i \in \mathcal{D}_{fin}(\downarrow_{\vee}(\vee T_i))$. Thus, we have that $d \in \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(supp(d_i))$ where $supp(d_i) \subseteq (supp(d_i) \cup \{\vee T_i\}) \in \mathcal{F}_{(\vee T_i)}$. Therefore, $d \in \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(supp(d_i)) \subseteq \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}((supp(d_i) \cup \{\vee T_i\})) \subseteq C$.

(ii) Suppose that C satisfies (b) and that a facial element $F \in \mathcal{F}^{elt}(S)$ of the form $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i)$ exists such that $F \subseteq C$. We must show that for every $T'_i \in \mathcal{F}_{(\vee T_i)}$ that $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq C$. However, if $T'_i \in \mathcal{F}_{(\vee T_i)}$ then $T'_i \subseteq \downarrow_{\vee}(\vee T_i)$. Hence, $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T'_i) \subseteq F^\perp \subseteq C$. □

Therefore, using the above proposition, we can provide a more concise definition for a \vee -stable convex set. This new definition is easier to work with than our initial formulation.

Definition 4.2.10 (\vee -Stable). Given a semilattice (S, \vee) , a convex set $C \in \mathcal{Cvx}(S)$ is said to be \vee -stable if it satisfies the following property: for any facial element $F \in \mathcal{F}^{elt}(S)$,

$$F \subseteq C \Rightarrow F^\perp \subseteq C$$

We will denote by $Stbl_{(S, \vee)}$ the set of \vee -stable convex subsets in $\mathcal{Cvx}(S)$.

Example 4.2.11. Consider our semilattices S_1 and S_2 from Example 4.2.7. We list below some examples of convex sets in $Cvx(\{a, b, c, d\})$ which are stable with respect to one of the join structures but not the other.

- (a) The face $\mathcal{D}_{fin}(\{b\})$ is \vee_2 -stable. However, it is not \vee_1 -stable, since $\mathcal{D}_{fin}(\{b\}) \subseteq \mathcal{D}_{fin}(\{b\})^\downarrow = \mathcal{D}_{fin}(\downarrow_{\vee_1} \{b\}) = \mathcal{D}_{fin}(\{a, b\}) \not\subseteq \mathcal{D}_{fin}(\{b\})$
- (b) The face $\mathcal{D}_{fin}(\{a, b\})$ is \vee_1 -stable. However, it is not \vee_2 -stable, since $\mathcal{D}_{fin}(\{a, b\}) \subseteq \mathcal{D}_{fin}(\{a, b\})^\downarrow = \mathcal{D}_{fin}(\downarrow_{\vee_2} \{a \vee_2 b\}) = \mathcal{D}_{fin}(\{a, b, c, d\}) \not\subseteq \mathcal{D}_{fin}(\{a, b\})$.
- (c) Given an arbitrary $\lambda \in (0, 1)$, the convex subspaces $C_\lambda = \mathcal{D}_{fin}(\{a, b\}) \oplus_\lambda \mathcal{D}_{fin}(\{c, d\})$ of $\mathcal{D}_{fin}(\{a, b, c, d\})$ (Figure 11 depicts a particular case) are \vee_1 -stable but are not \vee_2 -stable. They are not \vee_2 -stable because the face $\mathcal{D}_{fin}(\{a, b\}) \subseteq C_\lambda$, but $\mathcal{D}_{fin}(\{a, b\})^\downarrow = \mathcal{D}_{fin}(\{a, b, c, d\}) \not\subseteq C_\lambda$.

△

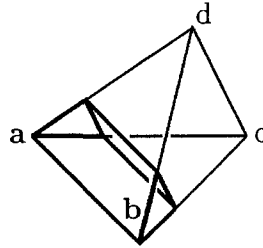


Figure 11: A particular case of a convex set of the form $\mathcal{D}_{fin}(\{a, b\}) \oplus_\lambda \mathcal{D}_{fin}(\{c, d\})$

Proposition 4.2.12. *Suppose we are given a semilattice (S, \vee) . Then $Stbl_{(S, \vee)}$ is closed under arbitrary non-empty intersections.*

Proof: Let W be an arbitrary indexing set and for each $w \in W$, $C_w \in Stbl_{(S, \vee)}$, such that $\bigcap_{w \in W} C_w \neq \emptyset$.

- (i) $\bigcap_{w \in W} C_w \in Cvx(S)$.

Let $d, d' \in \bigcap_{w \in W} C_w$. Thus, for every $w \in W$, $d, d' \in C_w$ and thus for all $\lambda \in [0, 1]$, $d \oplus_\lambda d' \in C_w$. This implies that for each $\lambda \in [0, 1]$, $d \oplus_\lambda d' \in \bigcap_{w \in W} C_w$.

(ii) $\bigcap_{w \in W} C_w$ is \vee -stable.

Suppose for some facial element $F \in \mathcal{F}^{elt}(S)$ that $F \subseteq \bigcap_{w \in W} C_w$. Thus, for every $w \in W$, we have $F \subseteq C_w$. Since the C_w are \vee -stable, we obtain $F^\downarrow \subseteq C_w$ for all $w \in W$. Therefore, $F^\downarrow \subseteq \bigcap_{w \in W} C_w$.

□

The above proposition allows us to define a closure operation based on \vee -stability. That is, we will define the \vee -stable closure for any convex subset $C \in \mathcal{Cvx}(S)$ as the smallest \vee -stable convex set, $C^\dagger \in \mathcal{Cvx}(S)$, which contains C . It can be obtained by taking the intersection of all \vee -stable convex sets $D \in \mathcal{Stbl}_{(S, \vee)}$ such that $C \subseteq D$,

$$C^\dagger = \bigcap_{\{D \in \mathcal{Stbl}_{(S, \vee)} \mid C \subseteq D\}} D.$$

Definition 4.2.13 (\vee -Stable Closure). Given a semilattice (S, \vee) and a convex subset $C \in \mathcal{Cvx}(S)$, the \vee -stable closure of C , denoted by C^\dagger , is the smallest \vee -stable convex set in $\mathcal{Stbl}_{(S, \vee)}$ containing C .

Example 4.2.14. For any semilattice (S, \vee) , we have the following standard \vee -closures.

(a) The \vee -closure of a face $\mathcal{D}_{fin}(T)$, $\mathcal{D}_{fin}(T)^\dagger$, is equal to its \vee -expansion $\mathcal{D}_{fin}(T)^\downarrow = \mathcal{D}_{fin}(\downarrow_\vee(\bigvee T))$. It is easy to see that $\mathcal{D}_{fin}(T)^\downarrow = \mathcal{D}_{fin}(\downarrow_\vee(\bigvee T)) \subseteq \mathcal{D}_{fin}(T)^\dagger$. For the other inclusion, we notice that the only facial elements which are contained in a face are faces generated by subsets of T .

(b) More generally, we have that for a facial element that

$$\left(\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \right)^\dagger = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\downarrow_\vee(\bigvee T_i)) = \bigoplus_{(\lambda_i)_I} (\mathcal{D}_{fin}(T_i))^\downarrow.$$

△

Example 4.2.15. (a) Consider the semilattice S_2 from Figure 7 and the convex sets C_λ from Example 4.2.11. Any \vee_2 -stable convex set containing a C_λ inherently contains the face $\mathcal{D}_{fin}(\{a, b\})$. Thus, since it is \vee_2 -stable, it must contain the \vee_2 -expansion of $\mathcal{D}_{fin}(\{a, b\})$, given by $\mathcal{D}_{fin}(\{a, b\})^\downarrow =$

$\mathcal{D}_{fin}(\{a, b, c, d\})$. Therefore, the only \vee_2 -stable convex set containing any of the C_λ is $\mathcal{D}_{fin}(\{a, b, c, d\})$. This implies that for any $\lambda \in (0, 1)$, $C_\lambda^\dagger = \mathcal{D}_{fin}(\{a, b, c, d\})$.

- (b) *The stable closure of a finitely generated convex subset is not necessarily finitely generated.* Consider the simple semilattice $S = (\{a, b, c, d\}, \vee)$ and a finitely generated convex set $C_0 = \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{3}} b, a \oplus_{\frac{1}{2}} b, c\}) \in \mathcal{G}_{cvx}(\{a, b, c, d\})$ shown in Figure 12. It is easy to see, as shown in Figure 12, that the facial element $F_1 = \bigoplus_{(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})} (a, b, \overline{ac}) \subseteq C_0$. Thus, C_0^\dagger must contain $F_1^\downarrow = \bigoplus_{(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})} (a, b, abcd) = \mathcal{D}_{fin}(\{a \oplus_{\frac{1}{2}} b, a \oplus_{\frac{1}{4}} b, \bigoplus_{(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})} (a, b, c), \bigoplus_{(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})} (a, b, d)\})$, because C_0^\dagger is \vee -stable. Moreover, since C_0^\dagger is a convex set which contains both C_0 and F_1^\downarrow , it must contain their convex hull $C_0 \boxplus F_1^\downarrow = C_1$.

Now, it is easy to see that the facial element $F_2 = \bigoplus_{(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})} (a, b, \overline{ac}) \subseteq C_1$, thus for similar reasons as given previously $C_2 = F_2^\downarrow \boxplus C_1$ must be a convex subset of C_0^\dagger .

Continuing this particular type of construction, we can recursively construct the convex sets $C_n = F_n^\downarrow \boxplus C_{n-1}$, which must all be contained in C_0^\dagger . In the end, we obtain the convex set $C_\infty \subseteq C_0^\dagger$, where C_∞ is shown in Figure 12.

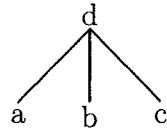
Similarly, we apply this construction starting with the facial element $F_{-1} = \bigoplus_{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} (a, b, \overline{bc})$ to obtain a $C_{-\infty} \subseteq C_0^\dagger$.

Finally, we can deduce that since $C_\infty \subseteq C_0^\dagger$ and $C_{-\infty} \subseteq C_0^\dagger$ and that $C_\infty \boxplus C_{-\infty} \subseteq C_0^\dagger$. Therefore, $(C_\infty \boxplus C_{-\infty})^\dagger = C_0^\dagger$ and is depicted in Figure 13.

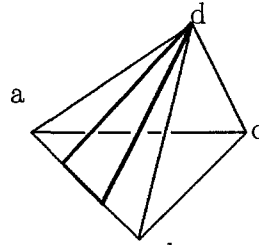
△

Proposition 4.2.16. *Given a semilattice (S, \vee) , the operation of forming \vee -stable closures is a closure operation on convex subsets in $Cvx(S)$.*

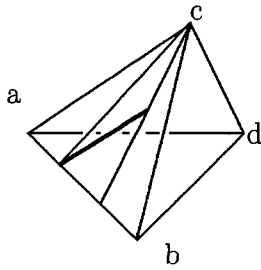
- (a) $C \subseteq C^\dagger$ (extensivity),
- (b) $(C^\dagger)^\dagger = C^\dagger$ (idempotence),
- (c) if $C \subseteq D$, then $C^\dagger \subseteq D^\dagger$ (monotonicity).



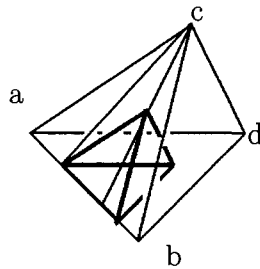
The semilattice $S = (\{a, b, c, d\}, \vee)$



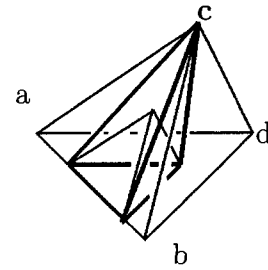
The convex subset C_0 .



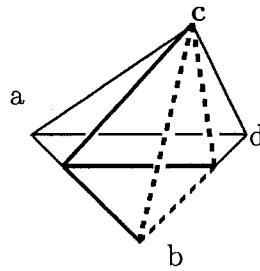
The convex subset F_1 .



The facial expansion of F_1 ,
(i.e. F_1^\downarrow).



The convex subset C_1 .



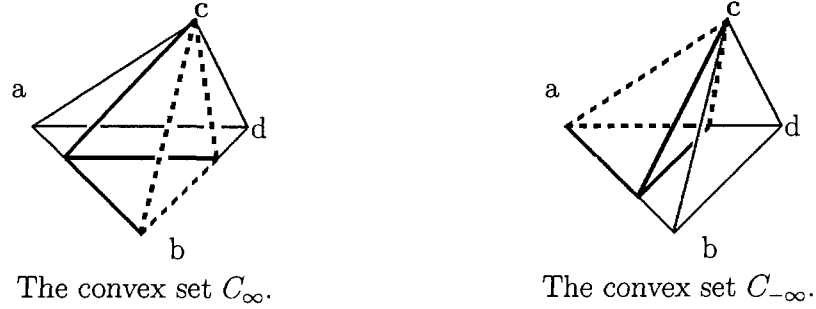
The convex set C_∞ .

Figure 12: The recursive construction of C_∞

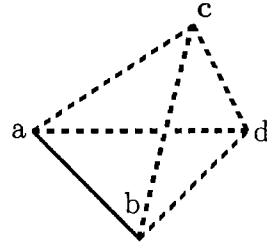
Lemma 4.2.17. *Given a semilattice (S, \vee) , a convex set $C \in \text{Cvx}(S)$ and a facial element $F \in \mathcal{F}^{\text{elt}}(S)$ such that $F \subseteq C$, then $(F^\downarrow \boxplus C)^\dagger = C^\dagger$.*

Proof: Since $F \subseteq C \subseteq C^\dagger$, we have that $F^\downarrow \subseteq C^\dagger$ by the definition of \vee -stable convex sets. Hence, $F^\downarrow \boxplus C \subseteq C^\dagger$ and thus, $(F^\downarrow \boxplus C)^\dagger \subseteq C^\dagger$. Finally, we have that $C \subseteq F^\downarrow \boxplus C$, therefore $C^\dagger \subseteq (F^\downarrow \boxplus C)^\dagger$. \square

Suppose we are given a convex set $C \in \text{Cvx}(S)$ such that there exists a facial



The closure of the convex hull of the above subsets is equal to



The \vee -stable closure of C_0 , (i.e. C_0^\dagger).

Figure 13: A finitely generated convex set whose stable closure is not finitely generated

element $F \in \mathcal{F}^{elt}(S)$ such that $F \subseteq C$. The convex set $F^\downarrow \boxplus C$ is constructed such that $C \subseteq F^\downarrow \boxplus C$ and by Lemma 4.2.17, $(F^\downarrow \boxplus C)^\dagger = C^\dagger$. We could continue and make a similar observation on $F^\downarrow \boxplus C$, by finding a facial element $F' \in \mathcal{F}^{elt}(S)$ such that $F' \subseteq F^\downarrow \boxplus C$. Thus we get that $((F')^\downarrow \boxplus (F^\downarrow \boxplus C))^\dagger = (F^\downarrow \boxplus C)^\dagger = C^\dagger$. We will say a convex set obtained using the above algorithm is a *partial \vee -stable closure* of C .

Definition 4.2.18 (Partial \vee -Stable Closures). Given a semilattice (S, \vee) and a convex set $C \in \text{Cvx}(S)$, the set of all partial \vee -stable closures of C , $C^{\dagger_{ptl}}$, is given by the following recursive construction.

- (a) $C \in C^{\dagger_{ptl}}$,
- (b) If $Y \in C^{\dagger_{ptl}}$ and there exists $F \in \mathcal{F}^{elt}(S)$ such that $F \subseteq Y$, then $F^\downarrow \boxplus Y \in C^{\dagger_{ptl}}$.

Proposition 4.2.19 (Properties of $C^{\dagger ptl}$).

- (a) For each $Y \in C^{\dagger ptl}$, $Y \in Cvx(S)$,
- (b) For each $Y \in C^{\dagger ptl}$, $C \subseteq Y$,
- (c) $C^{\dagger ptl}$ is closed under \boxplus in $Cvx(S)$ (i.e., If $Y, Y' \in C^{\dagger ptl}$, then $Y \boxplus Y' \in C^{\dagger ptl}$).

Proof: We only need to prove (c) since the first two assertions follow from the definition of partial \vee -stable closures.

Given $Y, Y' \in C^{\dagger ptl}$, we will show that $Y \boxplus Y' \in C^{\dagger ptl}$ by structural induction on Y .

- (i) **Base Case:** Given that $Y = C$, by part (b) we know that $C \subseteq Y'$. Hence $C \boxplus Y' = Y' \in C^{\dagger ptl}$.
- (ii) **Induction Step:** Suppose that $Y = Y'' \boxplus F^\downarrow$, where $Y'' \in C^{\dagger ptl}$, $F \in \mathcal{F}^{elt}(S)$, $F \subseteq Y''$ and $Y'' \boxplus Y' \in C^{\dagger ptl}$ by induction hypothesis.

$$\begin{aligned} Y \boxplus Y' &= (Y'' \boxplus F^\downarrow) \boxplus Y' \\ &= (Y'' \boxplus Y') \boxplus F^\downarrow \end{aligned}$$

where $F \subseteq Y'' \boxplus Y'$ and $Y'' \boxplus Y' \in C^{\dagger ptl}$. Hence, $Y \boxplus Y' = (Y'' \boxplus Y') \boxplus F^\downarrow \in C^{\dagger ptl}$ by definition. □

Theorem 4.2.20. If C is a convex set in $Cvx(S)$ then $C^\dagger = \bigcup_{Y \in C^{\dagger ptl}} Y$.

Proof: We will show that $\bigcup_{Y \in C^{\dagger ptl}} Y$ is the smallest \vee -stable convex set in $Cvx(S)$ which contains C .

- (a) **It is a convex.** Given $d, d' \in \bigcup_{Y \in C^{\dagger ptl}} Y$ and a $\lambda \in [0, 1]$, we must show that $d \oplus_\lambda d' \in \bigcup_{Y \in C^{\dagger ptl}} Y$. Since $d, d' \in \bigcup_{Y \in C^{\dagger ptl}} Y$, then there exists $Y_d, Y_{d'} \in C^{\dagger ptl}$, such that $d \in Y_d$ and $d' \in Y_{d'}$. By Prop 4.2.19, part(c), $Y_d \boxplus Y_{d'} \in C^{\dagger ptl}$ such that $d, d' \in Y_d \boxplus Y_{d'}$. By Prop 4.2.19, part (a), $Y_d \boxplus Y_{d'}$ is a convex set in $Cvx(S)$, thus $d \oplus_\lambda d' \in Y_d \boxplus Y_{d'} \subseteq \bigcup_{Y \in C^{\dagger ptl}} Y$.

- (b) **It contains C** , by Prop 4.2.19, part(b).
- (c) **It is \vee -stable**. Given a facial element $F \in \mathcal{F}^{elt}(S)$ such that $F \subseteq \bigcup_{Y \in C^{\dagger_{ptl}}} Y$ we must show that $F^\downarrow \subseteq \bigcup_{Y \in C^{\dagger_{ptl}}} Y$. Note that F is a finitely generated convex set by construction. Let $gen(F)$ be the set of generators for the convex set F . Thus, $\forall d \in gen(F)$ there exists $Y_d \in C^{\dagger_{ptl}}$ such that $d \in Y_d$. By Prop 4.2.19, part(c), we know that $\bigoplus_{d \in gen(F)} Y_d \in C^{\dagger_{ptl}}$. Since $F \subseteq \bigoplus_{d \in gen(F)} Y_d$, we have by the recursive definition of partial \vee -stable closures that $F^\downarrow \boxplus \left(\bigoplus_{d \in gen(F)} Y_d \right) \in C^{\dagger_{ptl}}$. Hence, $F^\downarrow \subseteq \bigcup_{Y \in C^{\dagger_{ptl}}} Y$.
- (d) **It is minimal**. Suppose there exists another \vee -stable convex set D in $Cvx(S)$ containing C . We need to show that $\bigcup_{Y \in C^{\dagger_{ptl}}} Y \subseteq D$. We shall prove by structural induction that if $Y \in C^{\dagger_{ptl}}$ then $Y \subseteq D$.
- (i) **Base Case:** Let $Y = C$, then by definition $Y = C \subseteq D$.
- (ii) **Induction Step:** Suppose $Y = F^\downarrow \boxplus Y'$ where $Y' \in C^{\dagger_{ptl}}$ and $F \in \mathcal{F}^{elt}(S)$ such that $F \subseteq Y'$. By induction hypothesis we assume that $Y' \subseteq D$. Moreover, we have that $F \subseteq Y' \subseteq D$; thus $F^\downarrow \subseteq D$ because D is \vee -stable. Therefore, $Y \subseteq D$ since D is convex.

□

We want to consider a minimal set of \vee -stable convex subsets. We have already discussed that this set should consist of the smallest mixed choice model generated by $\left\{ \mathcal{D}_{fin}(T)^\dagger \mid T \in \mathcal{P}_{fin}^*(S) \right\}$, together with the \vee -closure of the usual convex set operations. In the following section we show that we have indeed constructed the free mixed choice model from a semilattice.

4.2.2 Probabilistic Completion of Nondeterministic Models

In this section we prove that the set obtained by taking the \vee -stable closure on all the finitely generated convex sets in $\mathcal{G}_{cvx}(S)$ under the appropriate \vee -stable closed nondeterministic and probabilistic operations forms a mixed choice model.

Proposition 4.2.21 (The Mixed Choice Model $Stbl_{fin}(S)$). *Suppose we are given a semilattice (S, \vee) . The set of all \vee -stable convex subsets obtained by taking the \vee -stable closure on all the finitely generated convex subsets in $\mathcal{G}_{cvx}(S)$, $Stbl_{fin}(S) \subseteq Stbl_{(S, \vee)}$, together with the operations $\tilde{\boxplus}$ and $\widetilde{\oplus}_\lambda$, the \vee -stable closed convex set operations on $\mathcal{G}_{cvx}(S)$, forms a mixed choice model $(Stbl_{fin}(S), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)$.*

Before we can prove that $(Stbl_{fin}(S), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)$ is in fact a mixed choice model we prove some technical lemmas which we shall require.

Lemma 4.2.22. *Suppose we are given a semilattice (S, \vee) . Let $C \in Stbl_{(S, \vee)}$ be a \vee -stable convex set. Then for any $C' \in Cvx(S)$ and $F \in \mathcal{F}^{elt}(S)$ such that $C' \oplus_\lambda F \subseteq C$ we have that $C' \oplus_\lambda F^\downarrow \subseteq C$.*

Proof: Consider an arbitrary $d \in C' \oplus_\lambda F^\downarrow$, then $d = \lambda d' + (1 - \lambda)f$, where $d' \in C'$ and $f \in F^\downarrow$. Note that $\{d'\} = \bigoplus_{(d'(s))_{supp(d')}} \{\delta_s\}$; therefore $\{d'\} \oplus_\lambda F \in \mathcal{F}^{elt}(S)$. Since $C' \oplus_\lambda F \subseteq C$, we have that $\{d'\} \oplus_\lambda F \subseteq C$. Since C is \vee -stable, we have that $(\{d'\} \oplus_\lambda F)^\downarrow = \{d'\}^\downarrow \oplus_\lambda F^\downarrow \subseteq C$. Finally, we can deduce that $d \in C$, since $d' \in (\{d'\}^\downarrow)^\downarrow$ so $d = \lambda d' + (1 - \lambda)f \in (\{d'\}^\downarrow)^\downarrow \oplus_\lambda F^\downarrow \subseteq C$. \square

Lemma 4.2.23. *Let $C, C' \in Cvx(S)$,*

$$(a) \ C^\dagger \widetilde{\oplus}_\lambda (C')^\dagger = (C \oplus_\lambda C')^\dagger,$$

$$(b) \ C^\dagger \tilde{\boxplus} (C')^\dagger = (C \boxplus C')^\dagger.$$

Proof:

(a) We show that $(C^\dagger \oplus_\lambda (C')^\dagger)^\dagger \subseteq (C \oplus_\lambda C')^\dagger$ and $(C^\dagger \oplus_\lambda (C')^\dagger)^\dagger \supseteq (C \oplus_\lambda C')^\dagger$.

(\subseteq) It will be enough to show that $C^\dagger \oplus_\lambda (C')^\dagger \subseteq (C \oplus_\lambda C')^\dagger$; hence we need to show that for every $Y \in C^{\dagger pti}$ and $Y' \in (C')^{\dagger pti}$, $Y \oplus_\lambda Y' \subseteq (C \oplus_\lambda C')^\dagger$.

We proceed by simultaneous structural induction on Y and Y' .

(i) **Base Case:** Let $Y = C$ and $Y' = C'$, then $Y \oplus_\lambda Y' = C \oplus_\lambda C' \subseteq (C \oplus_\lambda C')^\dagger$.

(ii) **Induction Step:** Let $Y = F^\downarrow \boxplus Y_1$, and $Y' = (F')^\downarrow \boxplus Y_2$, where $Y_1 \in C^{\dagger pti}$, $Y_2 \in (C')^{\dagger pti}$, $F, F' \in \mathcal{F}^{elt}(S)$ such that $F \subseteq Y_1$ and

$F' \subseteq Y_2$. By I.H we assume that $Y_1 \oplus_\lambda Y_2 \subseteq (C \oplus_\lambda C')^\dagger$.

$$\begin{aligned} Y \oplus_\lambda Y' &= (F^\downarrow \boxplus Y_1) \oplus_\lambda ((F')^\downarrow \boxplus Y_2) \\ &= (F^\downarrow \oplus_\lambda (F')^\downarrow) \boxplus (F^\downarrow \oplus_\lambda Y_2) \boxplus (Y_1 \oplus_\lambda (F')^\downarrow) \boxplus (Y_1 \oplus_\lambda Y_2) \end{aligned}$$

Since $(F \oplus_\lambda F'), (F \oplus_\lambda Y_2)$ and $(Y_1 \oplus_\lambda F')$ are subsets of $Y_1 \oplus_\lambda Y_2 \subseteq (C \oplus_\lambda C')^\dagger$ and since $(C \oplus_\lambda C')^\dagger$ is \vee -stable, we have that $(F^\downarrow \oplus_\lambda (F')^\downarrow), (F^\downarrow \oplus_\lambda Y_2)$ and $(Y_1 \oplus_\lambda (F')^\downarrow)$ are contained in $(C \oplus_\lambda C')^\dagger$. Thus, $Y \oplus_\lambda Y' \subseteq (C \oplus_\lambda C')^\dagger$.

(\supseteq) Since $C \subseteq C^\dagger$ and $C' \subseteq (C')^\dagger$, we have that $C \oplus_\lambda C' \subseteq C^\dagger \oplus_\lambda (C')^\dagger$. Therefore, $(C \oplus_\lambda C')^\dagger \subseteq (C^\dagger \oplus_\lambda (C')^\dagger)^\dagger$.

(b) We show that $(C^\dagger \boxplus (C')^\dagger)^\dagger \subseteq (C \boxplus C')^\dagger$ and $(C^\dagger \boxplus (C')^\dagger)^\dagger \supseteq (C \boxplus C')^\dagger$.

(\subseteq) It will be enough to show that $C^\dagger \boxplus (C')^\dagger \subseteq (C \boxplus C')^\dagger$. Hence, that for every $Y \in C^{\dagger pt}$ and $Y' \in (C')^{\dagger pt}$, $Y \boxplus Y' \subseteq (C \boxplus C')^\dagger$. We proceed by simultaneous structural induction on Y and Y' .

(i) **Base Case:** Let $Y = C$ and $Y' = C'$, then $Y \boxplus Y' = C \boxplus C' \subseteq (C \boxplus C')^\dagger$.

(ii) **Induction Step:** Let $Y = F^\downarrow \boxplus Y_1$, and $Y' = (F')^\downarrow \boxplus Y_2$, where $Y_1 \in C^{\dagger pt}$, $Y_2 \in (C')^{\dagger pt}$, $F, F' \in \mathcal{F}^{elt}(S)$ such that $F \subseteq Y_1$ and $F' \subseteq Y_2$. By I.H we assume that $Y_1 \boxplus Y_2 \subseteq (C \boxplus C')^\dagger$.

$$\begin{aligned} Y \boxplus Y' &= (F^\downarrow \boxplus Y_1) \boxplus ((F')^\downarrow \boxplus Y_2) \\ &= (F^\downarrow \boxplus (F')^\downarrow) \boxplus (Y_1 \boxplus Y_2) \end{aligned}$$

Since F and F' are subsets of $Y_1 \boxplus Y_2 \subseteq (C \boxplus C')^\dagger$ and since $(C \boxplus C')^\dagger$ is \vee -stable, we have that F^\downarrow and $(F')^\downarrow$ are contained in $(C \boxplus C')^\dagger$. Thus, $Y \boxplus Y' \subseteq (C \boxplus C')^\dagger$.

(\supseteq) Since $C \subseteq C^\dagger$ and $C' \subseteq (C')^\dagger$, we have that $C \boxplus C' \subseteq C^\dagger \boxplus (C')^\dagger$. Therefore, $(C \boxplus C')^\dagger \subseteq (C^\dagger \boxplus (C')^\dagger)^\dagger$.

□

Proof: [Prop 4.2.21] We first show that the operations $\widetilde{\boxplus}$ and $\widetilde{\oplus}_\lambda$ are well-defined. Suppose $\lambda \in [0, 1]$, $A^\dagger = B^\dagger$ and $(A')^\dagger = (B')^\dagger$, where $A, A', B, B' \in \mathcal{G}_{cvx}(S)$.

(a) We show that $A^\dagger \widetilde{\oplus}_\lambda (A')^\dagger \in Stbl_{fin}(S)$.

$$\begin{aligned} A^\dagger \widetilde{\oplus}_\lambda (A')^\dagger &= (A^\dagger \oplus_\lambda (A')^\dagger)^\dagger \\ &= (A \oplus_\lambda A')^\dagger, \quad \text{by Lemma 4.2.23} \end{aligned}$$

Thus, since $A \oplus_\lambda A' \in \mathcal{G}_{cvx}(S)$, we have that $A^\dagger \widetilde{\oplus}_\lambda (A')^\dagger \in Stbl_{fin}(S)$.

(b) We show that $A^\dagger \widetilde{\boxplus} (A')^\dagger \in Stbl_{fin}(S)$.

$$\begin{aligned} A^\dagger \widetilde{\boxplus} (A')^\dagger &= (A^\dagger \boxplus (A')^\dagger)^\dagger \\ &= (A \boxplus A')^\dagger, \quad \text{by Lemma 4.2.23} \end{aligned}$$

Thus, since $A \boxplus A' \in \mathcal{G}_{cvx}(S)$, we have that $A^\dagger \widetilde{\boxplus} (A')^\dagger \in Stbl_{fin}(S)$.

(c) We show that $A^\dagger \widetilde{\oplus}_\lambda (A')^\dagger = B^\dagger \widetilde{\oplus}_\lambda (B')^\dagger$.

$$\begin{aligned} A^\dagger = B^\dagger, (A')^\dagger = (B')^\dagger &\Rightarrow A^\dagger \oplus_\lambda (A')^\dagger = B^\dagger \oplus_\lambda (B')^\dagger \\ &\Rightarrow (A^\dagger \oplus_\lambda (A')^\dagger)^\dagger = (B^\dagger \oplus_\lambda (B')^\dagger)^\dagger \\ &\Rightarrow A^\dagger \widetilde{\oplus}_\lambda (A')^\dagger = B^\dagger \widetilde{\oplus}_\lambda (B')^\dagger \end{aligned}$$

(d) We show that $A^\dagger \widetilde{\boxplus} (A')^\dagger = B^\dagger \widetilde{\boxplus} (B')^\dagger$.

$$\begin{aligned} A^\dagger = B^\dagger, (A')^\dagger = (B')^\dagger &\Rightarrow A^\dagger \boxplus (A')^\dagger = B^\dagger \boxplus (B')^\dagger \\ &\Rightarrow (A^\dagger \boxplus (A')^\dagger)^\dagger = (B^\dagger \boxplus (B')^\dagger)^\dagger \\ &\Rightarrow A^\dagger \widetilde{\boxplus} (A')^\dagger = B^\dagger \widetilde{\boxplus} (B')^\dagger \end{aligned}$$

We show that all the mixed choice axioms are satisfied. Given Lemma 4.2.23, the axioms are satisfied since \boxplus and \oplus_λ satisfy them. Recall that, in what follows, the elements C_i are in $Stbl_{fin}(S)$. Thus they are \vee -stable, implying that $C_i = C_i^\dagger$.

$$\text{(ND-ASSOC)} \quad C_1 \widetilde{\boxplus} (C_2 \widetilde{\boxplus} C_3) = (C_1 \widetilde{\boxplus} C_2) \widetilde{\boxplus} C_3.$$

$$\begin{aligned} C_1 \widetilde{\boxplus} (C_2 \widetilde{\boxplus} C_3) &= (C_1 \boxplus (C_2 \boxplus C_3))^\dagger \\ &= ((C_1 \boxplus C_2) \boxplus C_3)^\dagger \\ &= (C_1 \widetilde{\boxplus} C_2) \widetilde{\boxplus} C_3 \end{aligned}$$

$$(\text{ND-COM}) \quad C_1 \tilde{\boxplus} C_2 = C_2 \tilde{\boxplus} C_1.$$

$$\begin{aligned} C_1 \tilde{\boxplus} C_2 &= (C_1 \boxplus C_2)^\dagger \\ &= (C_2 \boxplus C_1)^\dagger \\ &= C_2 \tilde{\boxplus} C_1 \end{aligned}$$

$$(\text{ND-IDEM}) \quad C \tilde{\boxplus} C = C.$$

$$\begin{aligned} C \tilde{\boxplus} C &= (C \boxplus C)^\dagger \\ &= C^\dagger \\ &= C \end{aligned}$$

$$(\text{P-ASSOC}) \quad C_1 \widetilde{\oplus}_{\lambda_1} (C_2 \widetilde{\oplus}_{\lambda_2} C_3) = (C_1 \widetilde{\oplus}_{\lambda_1 \lambda_2} C_2) \widetilde{\oplus}_{\frac{(1-\lambda_1)\lambda_2}{1-\lambda_1\lambda_2}} C_3.$$

$$\begin{aligned} C_1 \widetilde{\oplus}_{\lambda_1} (C_2 \widetilde{\oplus}_{\lambda_2} C_3) &= (C_1 \oplus_{\lambda_1} (C_2 \oplus_{\lambda_2} C_3))^\dagger \\ &= ((C_1 \oplus_{\lambda_1 \lambda_2} C_2) \oplus_{\frac{(1-\lambda_1)\lambda_2}{1-\lambda_1\lambda_2}} C_3)^\dagger \\ &= (C_1 \widetilde{\oplus}_{\lambda_1 \lambda_2} C_2) \widetilde{\oplus}_{\frac{(1-\lambda_1)\lambda_2}{1-\lambda_1\lambda_2}} C_3 \end{aligned}$$

$$(\text{P-Com}) \quad C_1 \widetilde{\oplus}_\lambda C_2 = C_2 \widetilde{\oplus}_{(1-\lambda)} C_1.$$

$$\begin{aligned} C_1 \widetilde{\oplus}_\lambda C_2 &= (C_1 \oplus_\lambda C_2)^\dagger \\ &= (C_2 \oplus_{(1-\lambda)} C_1)^\dagger \\ &= C_2 \widetilde{\oplus}_{(1-\lambda)} C_1 \end{aligned}$$

$$(\text{P-IDEM}) \quad C \widetilde{\oplus}_\lambda C = C.$$

$$\begin{aligned} C \widetilde{\oplus}_\lambda C &= (C \oplus_\lambda C)^\dagger \\ &= C^\dagger \\ &= C \end{aligned}$$

$$(\text{DIST}) \quad (C_1 \tilde{\boxplus} C_2) \widetilde{\oplus}_\lambda C_3 = (C_1 \widetilde{\oplus}_\lambda C_3) \tilde{\boxplus} (C_2 \widetilde{\oplus}_\lambda C_3).$$

$$\begin{aligned} (C_1 \tilde{\boxplus} C_2) \widetilde{\oplus}_\lambda C_3 &= (C_1 \boxplus C_2) \oplus_\lambda C_3)^\dagger \\ &= ((C_1 \oplus_\lambda C_3) \boxplus (C_2 \oplus_\lambda C_3))^\dagger \\ &= (C_1 \widetilde{\oplus}_\lambda C_3) \tilde{\boxplus} (C_2 \widetilde{\oplus}_\lambda C_3) \end{aligned}$$

□

We have associated to any semilattice (S, \vee) a mixed choice model $(Stbl_{fin}(S), \widetilde{\boxplus}, \widetilde{\boxplus}_\lambda)$. We use this correspondence in the definition of our functor.

Definition 4.2.24 (\vee -Stable Convex Functor). The \vee -stable convex functor, $Stbl_{fin} : \text{Mod}(\text{ND}_{fin}^*, \text{Set}) \rightarrow \text{Mod}(\text{MC}_{fin}, \text{Set})$ is defined as follows:

(a) **On objects:** For $(S, \vee) \in \text{Mod}(\text{ND}_{fin}^*, \text{Set})$,

$$Stbl_{fin}((S, \vee)) = (Stbl_{fin}(S), \widetilde{\boxplus}, \widetilde{\boxplus}_\lambda),$$

(b) **On morphisms:** For $f : (S, \vee) \rightarrow (S', \vee') \in \text{Mod}(\text{ND}_{fin}^*, \text{Set})$ and $A^\dagger \in Stbl_{fin}(S)$,

$$Stbl_{fin}f(A^\dagger) = (f[A]_{cvx})^\dagger.$$

Lemma 4.2.25 (Functoriality of $Stbl_{fin}$). Given $f : (S, \vee) \rightarrow (S', \vee') \in \text{Mod}(\text{ND}_{fin}^*, \text{Set})$ and $A^\dagger \in Stbl_{fin}(S)$, then

(a) $(f[A^\dagger]_{cvx})^\dagger = (f[A]_{cvx})^\dagger.$

(b) $Stbl_{fin}f : Stbl_{fin}((S, \vee)) \rightarrow Stbl_{fin}((S', \vee'))$ is well-defined.

(c) $Stbl_{fin}$ is a functor.

Proof:

(a) We prove that $(f[A^\dagger]_{cvx})^\dagger \subseteq (f[A]_{cvx})^\dagger$ and $(f[A^\dagger]_{cvx})^\dagger \supseteq (f[A]_{cvx})^\dagger.$

(\subseteq) We shall prove by structural induction that for any $Y \in A^{\dagger pt}$, $f[Y]_{cvx} \subseteq f([A]_{cvx})^\dagger.$

(i) **Base Case:** For $Y = A$, we have that

$$f[A]_{cvx} \subseteq (f[A]_{cvx})^\dagger.$$

(ii) **Induction Step:** Suppose that $Y = F^\downarrow \boxplus Y'$, where $Y' \in A^{\dagger pt}$, $F \in \mathcal{F}^{elt}(S)$ and $F \subseteq Y'$. By induction hypothesis, we have that $f[Y']_{cvx} \subseteq (f[A]_{cvx})^\dagger.$ Note that $f[F]_{cvx} \in \mathcal{F}^{elt}(S')$, thus by \vee' -stability we know that $(f[F]_{cvx})^\downarrow \subseteq (f[A]_{cvx})^\dagger.$ Given that $f[F^\downarrow]_{cvx} \subseteq (f[F]_{cvx})^\downarrow,$ we have that $f[Y]_{cvx} \subseteq (f[A]_{cvx})^\dagger.$

(⊇)

$$\begin{aligned} A \subseteq A^\dagger &\Rightarrow f[A]_{cvx} \subseteq f[A^\dagger]_{cvx} \\ &\Rightarrow (f[A]_{cvx})^\dagger \subseteq (f[A^\dagger]_{cvx})^\dagger \end{aligned}$$

(b) Suppose $A, B \in \mathcal{G}_{cvx}(S)$ such that $A^\dagger = B^\dagger$. We need to show that $Stbl_{fin}f(A^\dagger) = Stbl_{fin}f(B^\dagger)$.

$$\begin{aligned} A^\dagger = B^\dagger &\Rightarrow f[A^\dagger]_{cvx} = f[B^\dagger]_{cvx} \\ &\Rightarrow (f[A^\dagger]_{cvx})^\dagger = (f[B^\dagger]_{cvx})^\dagger \\ &\Rightarrow (f[A]_{cvx})^\dagger = (f[B]_{cvx})^\dagger \\ &\Rightarrow Stbl_{fin}f(A^\dagger) = Stbl_{fin}f(B^\dagger) \end{aligned}$$

(c) $Stbl_{fin}$ is functorial. We need to show that $Stbl_{fin}$ satisfies the following properties.

(i) Given the identity function $id_{(S, \vee)} : (S, \vee) \rightarrow (S, \vee) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ we have that

$$\begin{aligned} Stbl_{fin}id_{(S, \vee)}(A^\dagger) &= (id_{(S, \vee)}[A]_{cvx})^\dagger \\ &= A^\dagger \\ &= id_{Stbl_{fin}((S, \vee))}(A^\dagger) \end{aligned}$$

(ii) Given two functions $f : (S, \vee) \rightarrow (S', \vee')$ and $g : (S', \vee') \rightarrow (S'', \vee'') \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ we have that

$$\begin{aligned} Stbl_{fin}(f \circ g)(A^\dagger) &= ((f \circ g)[A]_{cvx})^\dagger \\ &= (f[g[A]_{cvx}]_{cvx})^\dagger \\ &= Stbl_{fin}f((g[A]_{cvx})^\dagger) \\ &= (Stbl_{fin}f \circ Stbl_{fin}g)(A^\dagger) \end{aligned}$$

□

Theorem 4.2.26. *The \vee -stable convex functor $Stbl_{fin} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$ is left adjoint to the forgetful functor $\mathcal{U}_{Stbl_{fin}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$. Thus it forms a monad $Stbl_{fin} = (\mathcal{U}_{Stbl_{fin}} \circ Stbl_{fin}, \eta, \mu)$ over $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$. We include the adjunction and monad structures:*

(a) **The unit**(η): for $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ and $s \in S$,

$$\eta_{(S, \vee)}(s) = \{\delta_s\}^\dagger.$$

(b) **The counit**(ε): for $(M, \vee, +_\lambda) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$ and $N^\dagger \in Stbl_{fin}(M)$,

$$\varepsilon_{(M, \vee, +_\lambda)}(N^\dagger) = \bigvee_{d \in \text{gen}(N)} \left(\sum_{(d(m))_{\text{supp}(d)}} m \right).$$

(c) **The multiplication**(μ): for $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Set})$ and $\mathcal{N}^\dagger \in Stbl_{fin}(Stbl_{fin}(S))$,

$$\mu_{(S, \vee)}(\mathcal{N}^\dagger) = \widetilde{\bigoplus_{D \in \text{gen}(\mathcal{N})}} \left(\widetilde{\bigoplus_{(D(A^\dagger))_{\text{supp}(D)}} A^\dagger} \right).$$

(Note that the every $C \in \text{supp}(D)$ is equal to A^\dagger for some $A \in \mathcal{G}_{\text{conv}}(S)$.)

Lemma 4.2.27. *Given a model of mixed choice $(M, \vee, +_\lambda)$, the probabilistic choice operator $+_\lambda$ preserves the ordering \leq_\vee generated by the semilattice structure. In other words, for $m, m', n, n' \in M$ and any $\lambda \in (0, 1)$*

$$m \leq_\vee m' \text{ and } n \leq_\vee n' \text{ implies } m +_\lambda n \leq_\vee m' +_\lambda n'$$

Proof: By initial hypothesis we have that $m \vee m' = m'$ and $n \vee n' = n'$. We need to show that $(m +_\lambda n) \vee (m' +_\lambda n') = (m' +_\lambda n')$.

First we show that $(m +_\lambda n') \leq_\vee (m' +_\lambda n')$, by using the distributivity of $+_\lambda$ over \vee .

$$\begin{aligned} m' +_\lambda n' &= (m' \vee m) +_\lambda n' \\ &= (m' +_\lambda n') \vee (m +_\lambda n') \end{aligned}$$

Similarly, we can prove that $(m' +_\lambda n) \leq_\vee (m' +_\lambda n')$. Thus, we get that

$$\begin{aligned} m' +_\lambda n' &= (m' \vee m) +_\lambda (n' \vee n) \\ &= (m' +_\lambda n') \vee (m' +_\lambda n) \vee (m +_\lambda n') \vee (m +_\lambda n) \\ &= (m' +_\lambda n') \vee (m +_\lambda n) \end{aligned}$$

□

Proof: [Theorem 4.2.26] We must show that the unit and counit maps are well defined and satisfy the necessary conditions.

(a) The unit is a well-defined semilattice morphism.

(i) **Well-defined:** Consider $s, s' \in S$ such that $s = s'$, clearly $\eta_{(S, \vee)}(s) = \eta_{(S, \vee)}(s')$.

(ii) **Semilattice morphism:** Given a semilattice (S, \vee) and $s, s' \in S$,

$$\begin{aligned} \eta_{(S, \vee)}(s \vee s') &= \{\delta_{s \vee s'}\}^\dagger \\ &= \{\delta_s, \delta_{s'}\}^\dagger \\ &= \{\delta_s\}^\dagger \tilde{\boxplus} \{\delta_{s'}\}^\dagger \\ &= \eta_{(S, \vee)}(s) \tilde{\boxplus} \eta_{(S, \vee)}(s') \end{aligned}$$

(b) The counit is a well-defined mixed choice morphism.

(i) **Well-defined:** Let $N^\dagger, (N')^\dagger \in \text{Stbl}_{fin}(M)$, such that $N^\dagger = (N')^\dagger$. We must show that $\varepsilon_{(M, \vee, +_\lambda)}(N^\dagger) = \varepsilon_{(M, \vee, +_\lambda)}((N')^\dagger)$.

In order to simplify notation we introduce the following “flattening” p-choice morphism $(\cdot)^b : (\mathcal{D}_{fin}(M), \oplus_\lambda) \rightarrow (M, +_\lambda)$. let $d \in \mathcal{D}_{fin}(M)$, then $(d)^b = \sum_{(d(m))_{\text{supp}(d)}} m$. Indeed, $(\cdot)^b$ is a p-choice morphism since, for any $d, d' \in \mathcal{D}_{fin}(M)$,

$$\begin{aligned} (d \oplus_\lambda d')^b &= \sum_{(d \oplus_\lambda d'(m))_{\text{supp}(m)}} m \\ &= \left(\sum_{(d(m))_{\text{supp}(d)}} m \right) +_\lambda \left(\sum_{(d'(m'))_{\text{supp}(d')}} m' \right) \\ &= (d)^b +_\lambda (d')^b \end{aligned}$$

For any $d \in \text{gen}(N')$ we show that $(d)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$ where \leq_v is the order generated by the semilattice structure in $(M, \vee, +_\lambda)$.

Given a $d \in \text{gen}(N')$, since $d \in N' \subseteq (N')^\dagger = N^\dagger$, there exists a partial completion $Y \in N^{\dagger \text{pt}}$ such that $d \in Y$. We show by structural induction on $Y \in N^{\dagger \text{pt}}$ that $(\forall d \in Y)(d)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}((N)^\dagger)$.

- (1) **Base Case:** $Y = N$. If $d \in Y = N$, then $d = \bigoplus_{(\lambda_i)_I} d_i$ where $|I| < \infty$, and for each $i \in I, d_i \in \text{gen}(N)$. Since $(d)^b = \sum_{(\lambda_i)_I} (d_i)^b$ and for each $i \in I, (d_i)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$, we have that $(d)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$.
- (2) **Structural Case:** Let $Y = Y' \boxplus F^\downarrow$, where $F \in \mathcal{F}^{\text{elt}}(M), Y' \in N^{\dagger \text{pt}}, F \subseteq Y'$ and by induction hypothesis the property holds for any $d' \in Y'$ (i.e., for any $d' \in Y', (d')^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$). Since $d \in Y$, there exists $\rho \in [0, 1], d' \in Y'$ and $f \in F^\downarrow$ such that $d = d' \oplus_\rho f$.

Claim 1. Say $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{\text{fin}}(N_i)$, where $|I| < \infty$, and for each $i \in I, N_i \subseteq_{\text{fin}} M$, then the distribution defined as $f^* = \bigoplus_{(\lambda_i)_I} \delta_{(\vee N_i)} \in F^\downarrow$ has the following property:

$$(f^*)^b = \sum_{(\lambda_i)_I} (\vee N_i) \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger).$$

Indeed, let $\vec{n} \in \prod_I N_i$ and consider the distributions of the form $f_{\vec{n}} = \bigoplus_{(\lambda_i)_I} \delta_{(\pi_i(\vec{n}))} \in F$. Since $F \subseteq Y'$, then by the induction hypothesis we have that for each $\vec{n} \in \prod_I N_i, (f_{\vec{n}})^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$. This implies that $\vee_{\vec{n} \in \prod_I N_i} (f_{\vec{n}})^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$. By the distributivity property of $+_\lambda$ over \vee from $(M, \vee, +_\lambda)$, we have that

$$(f^*)^b = \bigvee_{\vec{n} \in \prod_I N_i} (f_{\vec{n}})^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger).$$

Claim 2. $(\forall f \in F^\downarrow) (f)^b \leq_v (f^*)^b$.

Indeed, consider $f \in F^\downarrow$, then $f = \bigoplus_{(\lambda_i)_I} \delta_{m_i}$ where for each $i \in I, m_i \leq_v \vee N_i$. Therefore, we have that $(f)^b \leq_v (f^*)^b$.

Therefore,

$$(d)^b = (d')^b +_\rho (f)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$$

Thus, $(d)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$, $\forall d \in \text{gen}(N')$, which implies that

$$\varepsilon_{(M, \vee, +_\lambda)}((N')^\dagger) = \bigvee_{d \in \text{gen}((N'))} (d)^b \leq_v \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger)$$

Similarly, we can prove that $\varepsilon_{(M, \vee, +_\lambda)}(N^\dagger) \leq_v \varepsilon_{(M, \vee, +_\lambda)}((N')^\dagger)$. Hence, we have that $\varepsilon_{(M, \vee, +_\lambda)}$ is well-defined.

(ii) **Mixed choice morphism:** we must show that the counit is a mixed choice morphism. Given a mixed choice model $(M, \vee, +_\lambda)$ and $N^\dagger, (N')^\dagger \in \text{Stbl}_{\text{fin}}(M)$,

(1) A nd-choice morphism.

$$\begin{aligned} & \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger \boxplus (N')^\dagger) \\ &= \varepsilon_{(M, \vee, +_\lambda)}((N \boxplus N')^\dagger), \text{ by Lemma 4.2.23} \\ &= \bigvee_{(d(m))_{d \in \text{gen}(N \boxplus N')}} (d)^b \\ &= \left(\bigvee_{d' \in \text{gen}(N)} (d')^b \right) \vee \left(\bigvee_{d'' \in \text{gen}(N')} (d'')^b \right), \text{ since } \text{gen}(N \boxplus N') = \text{gen}(N) \cup \text{gen}(N') \\ &= \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger) \vee \varepsilon_{(M, \vee, +_\lambda)}((N')^\dagger) \end{aligned}$$

(2) A p-choice morphism.

$$\begin{aligned} & \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger \oplus_\lambda (N')^\dagger) \\ &= \varepsilon_{(M, \vee, +_\lambda)}((N \oplus_\lambda N')^\dagger), \text{ by Lemma 4.2.23} \\ &= \bigvee_{d \in \text{gen}(N \oplus_\lambda N')} (d)^b \\ &= \bigvee_{d' \in \text{gen}(N)} \bigvee_{d'' \in \text{gen}(N')} ((d')^b +_\lambda (d'')^b), \\ & \text{ since } d \in \text{gen}(N \oplus_\lambda N') \Rightarrow d = d' +_\lambda d'' \text{ for some } d' \in \text{gen}(N) \text{ and } d'' \in \text{gen}(N'). \\ &= \left(\bigvee_{d' \in \text{gen}(N)} (d')^b \right) +_\lambda \left(\bigvee_{d'' \in \text{gen}(N')} (d'')^b \right), \text{ by distributivity} \\ &= \varepsilon_{(M, \vee, +_\lambda)}(N^\dagger) +_\lambda \varepsilon_{(M, \vee, +_\lambda)}((N')^\dagger) \end{aligned}$$

(c) The required equations are satisfied. Given a semilattice (S, \vee) and $A^\dagger \in \mathit{Stbl}_{fin}(S)$,

$$\begin{aligned} & \varepsilon_{(\mathit{Stbl}_{fin}(S), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)} \circ \mathit{Stbl}_{fin} \eta_{(S, \vee)}(A^\dagger) \\ &= \varepsilon_{(\mathit{Stbl}_{fin}(S), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)}((\eta_{(S, \vee)}[A]_{cvx})^\dagger) \\ &= \widetilde{\boxplus}_{d \in \mathit{gen}(\eta_{S, \vee}[A]_{cvx})} \left(\widetilde{\oplus}_{(d(C))_{supp(d)}} C \right) \end{aligned}$$

However, we have that $\mathit{gen}(\eta_{S, \vee}[A]_{cvx}) = \{d_{\eta_{(S, \vee)}} \mid d \in \mathit{gen}(A)\}$, where

$$d_{\eta_{(S, \vee)}} = \widetilde{\oplus}_{(d(s))_{supp(d)}} \delta_{(\eta_{(S, \vee)}(s))} = \widetilde{\oplus}_{(d(s))_{supp(d)}} \delta_{(\{\delta_s\}^\dagger)}.$$

Hence,

$$\begin{aligned} \widetilde{\boxplus}_{d \in \mathit{gen}(\eta_{S, \vee}[A]_{cvx})} \left(\widetilde{\oplus}_{(d(C))_{supp(d)}} C \right) &= \widetilde{\boxplus}_{d \in \mathit{gen}(A)} \left(\widetilde{\oplus}_{((d(s))_{supp(d)})} \{\delta_s\}^\dagger \right) \\ &= A^\dagger \end{aligned}$$

Given a mixed choice model $(M, \vee, +_\lambda)$ and $m \in M$,

$$\begin{aligned} & \mathcal{U}_{\mathit{Stbl}_{fin}} \varepsilon_{(\mathit{Stbl}_{fin}(M), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)} \circ \eta_{(M, \vee)}(m) \\ &= \mathcal{U}_{\mathit{Stbl}_{fin}} \varepsilon_{(\mathit{Stbl}_{fin}(M), \tilde{\boxplus}, \widetilde{\oplus}_\lambda)}(\{\delta_m\}^\dagger) \\ &= m \end{aligned}$$

□

Proposition 4.2.28. *The category $\mathit{Stbl}_{fin}\text{-Alg}$ is equivalent to the category $\mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set})$. Thus, Stbl_{fin} constructs the free mixed choice models over semilattices.*

Proof: $\mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set}) \cong \mathit{Stbl}_{fin}\text{-Alg}$: We construct two inverse functors $\mathcal{F} : \mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set}) \rightarrow \mathit{Stbl}_{fin}\text{-Alg}$ and $\mathcal{G} : \mathit{Stbl}_{fin}\text{-Alg} \rightarrow \mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set})$:

(a) We begin by defining the functor $\mathcal{F} : \mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set}) \rightarrow \mathit{Stbl}_{fin}\text{-Alg}$:

$$\begin{array}{ccc} \mathcal{F} : \mathit{Mod}(\mathit{MC}_{fin}, \mathit{Set}) & \rightarrow & \mathit{Stbl}_{fin}\text{-Alg} \\ (M, \vee, +_\lambda) & \mapsto & ((M, \vee), \alpha_{(M, \vee)} : \mathit{Stbl}_{fin}((M, \vee)) \rightarrow (M, \vee)) \\ \downarrow^f & & \downarrow^f \\ (M', \vee', +_{\lambda'}) & \mapsto & ((M', \vee'), \alpha_{(M', \vee')} : \mathit{Stbl}_{fin}((M', \vee')) \rightarrow (M', \vee')) \end{array}$$

where

$$\alpha_{(M, \vee)} = \varepsilon_{(M, \vee, +_\lambda)}.$$

Note $\mathcal{F}(f) = f$ (the latter considered as an algebra map).

Indeed, $\alpha_{(M, \vee)}$ is well-defined and satisfies the necessary equations:

- (i) **It is well-defined.** We have already shown that the counit map is well-defined.
- (ii) **It satisfies the required equations.** Let $(M, \vee, +_\lambda) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$ and $\mathcal{N}^\dagger \in \mathbf{Stbl}_{fin}(\mathbf{Stbl}_{fin}(M))$.

$$\begin{aligned} \alpha_{(M, \vee)} \circ \mathbf{Stbl}_{fin} \alpha_{(M, \vee)}(\mathcal{N}^\dagger) &= \alpha_{(M, \vee)}((\alpha_{(M, \vee)}[\mathcal{N}]_{cvx})^\dagger) \\ &= \widetilde{\bigsqcup}_{D \in \mathit{gen}(\alpha_{(M, \vee)}[\mathcal{N}]_{cvx})} \left(\bigoplus_{(D(C))_{\mathit{supp}(D)}} C \right) \\ &= \widetilde{\bigsqcup}_{D \in \mathit{gen}(\mathcal{N})} \left(\bigoplus_{(D(C))_{\mathit{supp}(D)}} \alpha_{(M, \vee)}(C) \right) \\ &= \alpha_{(M, \vee)} \left(\widetilde{\bigsqcup}_{D \in \mathit{gen}(\mathcal{N})} \left(\bigoplus_{(D(C))_{\mathit{supp}(D)}} C \right) \right) \\ &= \alpha_{(M, \vee)} \circ \mu_{(M, \vee)}(\mathcal{N}^\dagger) \end{aligned}$$

Let $m \in M$,

$$\begin{aligned} \alpha_{(M, \vee)} \circ \eta_{(M, \vee)}(m) &= \alpha_{(M, \vee)}(\{\delta_m\}^\dagger) \\ &= m \end{aligned}$$

Moreover, we also verify that for any $f : (M, \vee, +_\lambda) \rightarrow (M', \vee', +'_\lambda)$, then f is also an algebra map. Let $\mathcal{N}^\dagger \in \mathbf{Stbl}_{fin}(M)$.

$$\begin{aligned} \alpha_{(M', \vee')} \circ \mathbf{Stbl}_{fin} f(\mathcal{N}^\dagger) &= \alpha_{(M', \vee')}((f[\mathcal{N}]_{cvx})^\dagger) \\ &= \bigvee'_{d \in \mathit{gen}(f[\mathcal{N}]_{cvx})} \left(\sum'_{(d(m))_{\mathit{supp}(d)}} m \right) \\ &= \bigvee'_{d \in \mathit{gen}(\mathcal{N})} \left(\sum'_{(d(m))_{\mathit{supp}(d)}} f(m) \right) \\ &= f \left(\bigvee_{d \in \mathit{gen}(\mathcal{N})} \left(\sum_{(d(m))_{\mathit{supp}(d)}} m \right) \right) \\ &= f \circ \alpha_{(M, \vee)}(\mathcal{N}^\dagger) \end{aligned}$$

(b) Next we define the second functor $\mathcal{G} : \mathbf{Stbl}_{fin}\text{-Alg} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set})$:

$$\begin{array}{ccc} \mathcal{G} : \mathbf{Stbl}_{fin}\text{-Alg} & \rightarrow & \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Set}) \\ ((S, \vee), \alpha : \mathbf{Stbl}_{fin}((S, \vee)) \rightarrow (S, \vee)) & \mapsto & (S, \vee, \oplus_\lambda^\alpha) \\ \downarrow^g & & \downarrow^g \\ ((S', \vee'), \alpha : \mathbf{Stbl}_{fin}((S', \vee')) \rightarrow (S', \vee')) & \mapsto & (S', \vee', \oplus_\lambda^{\alpha'}) \end{array}$$

where for $s, s' \in S$,

$$s \oplus_\lambda^\alpha s' = \alpha(\{\delta_s \oplus_\lambda \delta_{s'}\}^\dagger).$$

These operations are well-defined and satisfy the axioms for mixed choice due to the equations satisfied by α , \vee and \oplus_λ and the fact that $s_1 \vee s_2 = \alpha(\eta_{(S, \vee)}(s_1 \vee s_2)) = \alpha(\{\delta_{s_1 \vee s_2}\}^\dagger)$.

Finally, we verify that the two functors are inverses.

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}((M, \vee, +_\lambda)) &= \mathcal{G}(((M, \vee), \alpha_{(M, \vee)})) \\ &= (M, \vee, \oplus_\lambda^{\alpha_{(M, \vee)}}) \end{aligned}$$

We can verify that $(M, \vee, +_\lambda) = (M, \vee, \oplus_\lambda^{\alpha_{(M, \vee)}})$, since

$$\begin{aligned} m \oplus_\lambda^{\alpha_{(M, \vee)}} m' &= \alpha_{(M, \vee)}(\{\delta_m \oplus_\lambda \delta_{m'}\}^\dagger) \\ &= m +_\lambda m' \end{aligned}$$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(((S, \vee), \alpha)) &= (\mathcal{F}((S, \vee, \oplus_\lambda^\alpha)) \\ &= ((S, \vee), \alpha_{(S, \vee)}) \end{aligned}$$

However, $\alpha = \alpha_{(S, \vee)}$ since

$$\begin{aligned} \alpha_{(S, \vee)}(A^\dagger) &= \bigvee_{d \in \text{gen}(A)} ((\bigoplus_{(d(s)) \text{supp}(d)}^\alpha) s) \\ &= \bigvee_{d \in \text{gen}(A)} (\alpha(\{\bigoplus_{(d(s)) \text{supp}(d)} \delta_s\}^\dagger)) \\ &= \alpha(\widetilde{\bigoplus_{d \in \text{gen}(A)} \{\bigoplus_{(d(s)) \text{supp}(d)} \delta_s\}^\dagger}) \\ &= \alpha(A^\dagger) \end{aligned}$$

□

Theorem 4.2.29. *We can factorize the geometrically convex powerset functor, \mathcal{G}_{cvx} , through the finite non-empty powerset functor \mathcal{P}_{fin}^* by using the \vee -stable convex functor $Stbl_{fin}$. In other words, $\mathcal{G}_{cvx} \cong Stbl_{fin} \circ \mathcal{P}_{fin}^*$.*

Proof: We define two inverse natural transformations:

- (a) $\phi : (Stbl_{fin} \circ \mathcal{P}_{fin}^*) \Rightarrow \mathcal{G}_{cvx}$, where for an $X \in \mathbf{Set}$, the mixed choice morphism $\phi_X : (Stbl_{fin} \circ \mathcal{P}_{fin}^*)(X) \rightarrow \mathcal{G}_{cvx}(X)$ is defined as follows. If $C^\dagger \in Stbl_{fin}(\mathcal{P}_{fin}^*(X))$, we define

$$\phi_X(C^\dagger) = \bigsqcup_{d \in gen(C)} \left(\bigoplus_{(d(A))_{supp(d)}} (\mathcal{D}_{fin}(A)) \right).$$

- (i) ϕ_X is well-defined. Suppose $C_1^\dagger, C_2^\dagger \in Stbl_{fin}(\mathcal{P}_{fin}^*(X))$ such that $C_1^\dagger = C_2^\dagger$. We must show that $\phi_X(C_1^\dagger) = \phi_X(C_2^\dagger)$. Let $d \in gen(C_2)$. This implies that $d \in C_1^\dagger$ and thus, that there exists a $Y \in C_1^{\dagger pt}$ such that $d \in Y$. We shall show by structural induction on Y that for all $d \in Y$, $\bigoplus_{((d(A))_{supp(d)})} \mathcal{D}_{fin}(A) \subseteq C_1^\dagger$.

Base Case: Let $Y = C_1$, then $d = \bigoplus_{(\rho_j)_J} d_j$ where $|J| < \infty$, and for each $j \in J$, $d_j \in gen(C_1)$. Thus

$$\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) = \bigoplus_{(\rho_j)_J} \left(\bigoplus_{(d_j(A_j))_{supp(d_j)}} \mathcal{D}_{fin}(A_j) \right) \subseteq \phi_X(C_1^\dagger)$$

Induction Step: Let $Y = Y' \boxplus F^\downarrow$, where $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(T_i) \in \mathcal{F}^{elt}(\mathcal{P}_{fin}^*(X))$, $Y' \in C_1^{\dagger pt}$, $F \subseteq Y$ and that by induction hypothesis we have that for every $d' \in Y'$, $\bigoplus_{(d'(A))_{supp(d')}} \mathcal{D}_{fin}(A) \subseteq \phi_X(C_1^\dagger)$. Since, $d \in Y = Y' \boxplus F^\downarrow$, then there exists $\rho \in [0, 1]$, $d' \in Y'$ and $f \in F^\downarrow$ such that $d = d' \oplus_\rho f$. Next we show that $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\bigcup T_i) \subseteq \phi_X(C_1^\dagger)$. For every $\vec{t} \in \prod_I T_i$, consider the distribution $f_{\vec{t}} = \bigoplus_{(\lambda_i)_I} \delta_{\pi_i(\vec{t})} \in F$. Since $F \subseteq Y'$, we have that for every $\vec{t} \in \prod_I T_i$ that $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\pi_i(\vec{t})) \subseteq \phi_X(C_1^\dagger)$. Therefore, by the convexity of $\phi_X(C_1^\dagger)$, we have that $\bigsqcup_{\vec{t} \in \prod_I T_i} \left(\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\pi_i(\vec{t})) \right) =$

$\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\bigcup T_i) \subseteq \phi_X(C_1^\dagger)$. Finally, we see that for any $f = \bigoplus_{(\lambda_i)_I} \delta_{S_i} \in F^\dagger$, where $S_i \subseteq \bigcup T_i$, that $\bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(S_i) \subseteq \bigoplus_{(\lambda_i)_I} \mathcal{D}_{fin}(\bigcup T_i) \subseteq \phi_X(C_1^\dagger)$. Therefore, since $d = d' \oplus_\rho f$ our result follows.

(ii) ϕ_X is a natural transformation. We prove that the necessary equations hold. Let $f : X \rightarrow Y \in \mathbf{Set}$ and $C^\dagger \in Stbl_{fin}(X)$.

$$\begin{aligned}
\phi_Y \circ Stbl_{fin} \mathcal{P}_{fin}^* f(C^\dagger) &= \phi_Y((\mathcal{P}_{fin}^* f[C]_{cvx})^\dagger) \\
&= \bigsqcup_{d' \in gen(\mathcal{P}_{fin}^* f[C]_{cvx})} \left(\bigoplus_{(d'(B))_{supp(d')}} \mathcal{D}_{fin}(B) \right) \\
&= \bigsqcup_{d \in gen(C)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(f[A]) \right) \\
&= \mathcal{G}_{cvx} f \left(\bigsqcup_{d \in gen(C)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) \right) \right) \\
&= \mathcal{G}_{cvx} f \circ \phi_X(C^\dagger)
\end{aligned}$$

(iii) ϕ_X is a mixed choice morphism. Let $C_1^\dagger, C_2^\dagger \in Stbl_{fin}(\mathcal{P}_{fin}^*(X))$. It preserves the ND operations.

$$\begin{aligned}
&\phi_X(C_1^\dagger \boxplus C_2^\dagger) \\
&= \phi_X((C_1 \boxplus C_2)^\dagger) \\
&= \bigsqcup_{d \in gen(C_1 \boxplus C_2)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) \right) \\
&= \left(\bigsqcup_{d \in gen(C_1)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) \right) \right) \boxplus \left(\bigsqcup_{d' \in gen(C_2)} \left(\bigoplus_{(d'(B))_{supp(d')}} \mathcal{D}_{fin}(B) \right) \right) \\
&= \phi_X(C_1^\dagger) \boxplus \phi_X(C_2^\dagger)
\end{aligned}$$

It preserves the P operations.

$$\begin{aligned}
&\phi_X(C_1^\dagger \oplus_\lambda C_2^\dagger) \\
&= \phi_X((C_1 \oplus_\lambda C_2)^\dagger) \\
&= \bigsqcup_{d \in gen(C_1 \oplus_\lambda C_2)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) \right) \\
&= \left(\bigsqcup_{d \in gen(C_1)} \left(\bigoplus_{(d(A))_{supp(d)}} \mathcal{D}_{fin}(A) \right) \right) \oplus_\lambda \left(\bigsqcup_{d' \in gen(C_2)} \left(\bigoplus_{(d'(B))_{supp(d')}} \mathcal{D}_{fin}(B) \right) \right) \\
&= \phi_X(C_1^\dagger) \oplus_\lambda \phi_X(C_2^\dagger)
\end{aligned}$$

- (b) $\psi : \mathcal{G}_{cvx} \Rightarrow (Stbl_{fin} \circ \mathcal{P}_{fin}^*)$, where for an $X \in \mathbf{Set}$, the mixed choice morphism $\psi_X : \mathcal{G}_{cvx}(X) \rightarrow (Stbl_{fin} \circ \mathcal{P}_{fin}^*)(X)$ on $C \in \mathcal{G}_{cvx}(X)$ is given by

$$\psi_X(C) = (\eta_X[C]_{cvx})^\dagger.$$

where $\eta : 1_{\mathbf{Set}} \Rightarrow \mathcal{U}_{\mathcal{P}_{fin}^*} \circ \mathcal{P}_{fin}^*$ is the unit for the finite powerset monad.

- (i) ψ_X is well-defined. This is immediate, from the definition.
- (ii) ψ is a natural transformation. This follows from the naturality of η .
Let $f : X \rightarrow Y \in \mathbf{Set}$ and $C \in \mathcal{G}_{cvx}(X)$, then

$$\begin{aligned} Stbl_{fin} \mathcal{P}_{fin}^* f \circ \psi_X(C) &= Stbl_{fin} \mathcal{P}_{fin}^* f ((\eta_X[C]_{cvx})^\dagger) \\ &= (\mathcal{P}_{fin}^* f [\eta_X[C]_{cvx}]_{cvx})^\dagger \\ &= ((\mathcal{P}_{fin}^* f \circ \eta_X)[C]_{cvx})^\dagger \\ &= ((\eta_Y \circ f)[C]_{cvx})^\dagger \\ &= \psi_Y((f[C]_{cvx})^\dagger) \\ &= \psi_Y \circ \mathcal{G}_{cvx} f(C). \end{aligned}$$

- (iii) ψ_X is a mixed choice morphism. Let $C_1, C_2 \in \mathcal{G}_{cvx}(X)$.

It preserves ND operations

$$\begin{aligned} \psi_X(C_1 \boxplus C_2) &= (\eta_X[C_1 \boxplus C_2]_{cvx})^\dagger \\ &= (\eta_X[C_1]_{cvx} \boxplus \eta_X[C_2]_{cvx})^\dagger \\ &= (\eta_X[C_1]_{cvx})^\dagger \tilde{\boxplus} (\eta_X[C_2]_{cvx})^\dagger \\ &= \psi_X(C_1) \tilde{\boxplus} \psi_X(C_2) \end{aligned}$$

It preserves P operations.

$$\begin{aligned} \psi_X(C_1 \oplus_\lambda C_2) &= (\eta_X[C_1 \oplus_\lambda C_2]_{cvx})^\dagger \\ &= (\eta_X[C_1]_{cvx} \oplus_\lambda \eta_X[C_2]_{cvx})^\dagger \\ &= (\eta_X[C_1]_{cvx})^\dagger \widetilde{\oplus}_\lambda (\eta_X[C_2]_{cvx})^\dagger \\ &= \psi_X(C_1) \widetilde{\oplus}_\lambda \psi_X(C_2) \end{aligned}$$

(c) These natural transformations are inverses.

(i) Let $C \in \mathcal{G}_{cvx}(X)$.

$$\begin{aligned}
\phi_X \circ \psi_X(C) &= \phi_X((\eta_X[C]_{cvx})^\dagger) \\
&= \bigsqcup_{d \in \text{gen}(\eta_X[C]_{cvx})} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} \mathcal{D}_{fin}(A) \right) \\
&= \bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(x))_{\text{supp}(d)}} \mathcal{D}_{fin}(\{x\}) \right) \\
&= C
\end{aligned}$$

(ii) Let $C^\dagger \in \text{Stbl}_{fin}(\mathcal{P}_{fin}^*(X))$.

$$\begin{aligned}
\psi_X \circ \phi_X(C^\dagger) &= \psi_X\left(\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} \mathcal{D}_{fin}(A) \right) \right) \\
&= \widetilde{\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} \psi_X(\mathcal{D}_{fin}(A)) \right)} \\
&= \widetilde{\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} (\eta_X[\mathcal{D}_{fin}(A)]_{cvx})^\dagger \right)} \\
&= \widetilde{\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} (\mathcal{D}_{fin}(\{\{a\} \mid a \in A\}))^\dagger \right)} \\
&= \widetilde{\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(A))_{\text{supp}(d)}} \{\delta_A\}^\dagger \right)} \\
&= C^\dagger
\end{aligned}$$

□

4.3 Extending to Infinitary Operations

Recall the extension to theories which included infinitary nondeterministic choice operators and countable probabilistic choice operators discussed in Chapter 3. By combining these two infinitary theories together we defined the infinite mixed choice theory, to which we also associated the monad \mathbb{G}_{scvx} . This monad was again defined by following the left-hand path of our usual diagram slightly modified to reflect the new infinite operations.

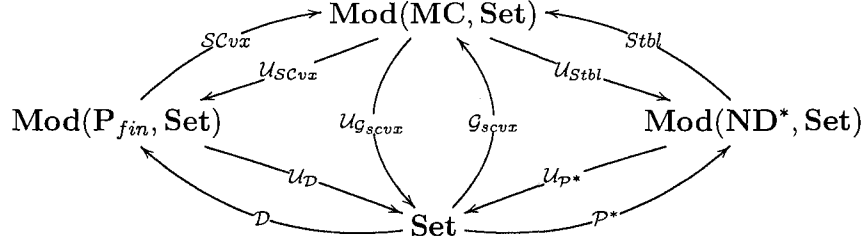


Figure 14: Combining infinite nondeterministic and probabilistic theories: two possible approaches.

In this section we show how we can extend the results obtained in the previous section to define a \vee -stable superconvex functor, $\mathcal{Stbl} : \text{Mod}(\text{ND}^*, \text{Set}) \rightarrow \text{Mod}(\text{MC}, \text{Set})$: the functor which constructs free infinite mixed choice models from arbitrary semilattices.

We begin by extending/modifying the basic definitions seen in the previous section to compensate for the presence of the infinite operations. In the instance of faces, since our semilattice now admits infinite ND combinations, any ND behavior may arise as an arbitrary ND combination of elements. Thus, we now allow faces to be generated by arbitrary subsets of the semilattice. Moreover, the definition of facial elements must be modified to compensate for the possible superconvex combinations of faces.

Definition 4.3.1 (Faces/Facial Elements). Given a semilattice $(S, \vee) \in \text{Mod}(\text{ND}^*, \text{Set})$, the model category of arbitrary nondeterministic algebras.

- (a) An element of $\mathcal{G}_{scvx}(S)$ of the form $\mathcal{D}(T)$, where $T \in \mathcal{P}^*(S)$, is called a **face** of $\mathcal{D}(S)$.
- (b) A **facial element** of $\mathcal{D}(S)$, F , is a superconvex combination of faces of $\mathcal{D}(S)$ in $\mathcal{G}_{scvx}(S)$. We shall denote the set of all facial elements of $\mathcal{G}_{scvx}(S)$ by $\mathcal{F}^{elt}(S)$.

We have previously discussed the importance that, if a convex set captures a particular probabilistic combination of ND behaviors each associated to a particular ND combination of elements, then it must capture the probabilistic combination of ND behaviors arising from any equivalent ND combinations (\vee -expansion). For

the infinitary case, we need to make modifications in order to allow for superconvex combinations and ND behaviors arising from arbitrary ND combinations.

Definition 4.3.2 (\vee -Expansion). Given a facial element $F \in \mathcal{F}^{elt}(S)$ of the form $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}(T_i)$, the \vee -**expansion** of F , denoted by F^\downarrow ², is defined as the following superconvex set :

$$F^\downarrow = \bigoplus_{(\lambda_i)_I} \mathcal{D}(\downarrow_{\vee} \bigvee T_i).$$

Given the infinite versions of our definitions, we can now define the notion of \vee -stability for the infinite case: \vee -*stability*.

Definition 4.3.3 (\vee -Stable). Given a semilattice $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$, a superconvex set $C \in \mathcal{G}_{scvx}(S)$ is said to be \vee -**stable** if it satisfies the following property: for any facial element $F \in \mathcal{F}^{elt}(S)$,

$$F \subseteq C \implies F^\downarrow \subseteq C$$

We will denote the set of all \vee -stable superconvex sets in $\mathcal{D}(S)$ as $Stbl(S) \subseteq \mathcal{G}_{scvx}(S)$.

Proposition 4.3.4. *Suppose we are given a semilattice (S, \vee) , $Stbl(S)$ is closed under arbitrary non-empty intersections.*

Proof: Let W be an arbitrary indexing set and for each $w \in W$, $C_w \in Stbl(S)$, such that $\bigcap_{w \in W} C_w \neq \emptyset$.

- (i) ($\bigcap_{w \in W} C_w \in \mathcal{G}_{scvx}(S)$) Let $(\lambda_i)_I \in Pwt$ and for each $i \in I$, $d_i \in \bigcap_{w \in W} C_w$. Thus, for every $w \in W$, and for each $i \in I$, $d_i \in C_w$ and thus $\bigoplus_{(\lambda_i)_I} d_i \in C_w$ (since C_w is a superconvex set). This implies that for any $(\lambda_i)_I \in Pwt$, $\bigoplus_{(\lambda_i)_I} d_i \in \bigcap_{w \in W} C_w$.
- (ii) ($\bigcap_{w \in W} C_w$ is \vee -stable) Suppose for some facial element $F \in \mathcal{F}^{elt}(S)$ that $F \subseteq \bigcap_{w \in W} C_w$. Thus, for every $w \in W$, we have $F \subseteq C_w$. Since the C_w are \vee -stable, we obtain $F^\downarrow \subseteq C_w$ for all $w \in W$. Therefore, $F^\downarrow \subseteq \bigcap_{w \in W} C_w$.

□

²We use the same notation F^\downarrow as defined previously for finitary theories. It will always be clear from the context which version is meant.

The above proposition allows us to define a closure operation based on \vee -stability (as we used its finite counterpart to define the \vee -stable closure). That is, we will define the \vee -stable closure of any superconvex set $C \in \mathcal{G}_{scvx}(S)$, to be the smallest \vee -stable superconvex set, C^\dagger in $\mathcal{G}_{scvx}(S)$, which contains C . It can be obtained by taking the arbitrary intersection of all \vee -stable superconvex sets $D \in Stbl(S)$ such that $C \subseteq D$ (i.e., $C^\dagger = \bigcap_{\{D \in Stbl(S) \mid C \subseteq D\}} D$).

Definition 4.3.5 (\vee -Stable Closure). Given a semilattice $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$ and a superconvex set $C \in \mathcal{G}_{scvx}(S)$, the \vee -stable closure of C , denoted by C^\dagger ³, is the smallest \vee -stable superconvex set in $\mathcal{G}_{scvx}(S)$ containing C .

Finally, we have all the necessary definitions to define the \vee -stable superconvex functor $Stbl$.

Definition 4.3.6 (\vee -Stable Superconvex Functor). The \vee -stable convex functor, $Stbl : \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$ is defined as follows:

(a) **On objects:** For $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$,

$$Stbl((S, \vee)) = (Stbl(S), \widetilde{\boxplus}, \widetilde{\bigoplus_{(\lambda_i)_I}}),$$

where $\widetilde{\boxplus}$ and $\widetilde{\bigoplus_{(\lambda_i)_I}}$ are the \vee -stable closed counterparts of \boxplus and $\bigoplus_{(\lambda_i)_I}$ from $\mathcal{G}_{scvx}(S)$.

(i) For any arbitrary family of \vee -stable superconvex sets $(C_w)_W$ in $Stbl(S)$,

$$\widetilde{\boxplus}_{w \in W} C_w = (\boxplus_{w \in W} C_w)^\dagger.$$

(ii) For any countable family of \vee -stable superconvex sets $(C_i)_I$ in $Stbl(S)$ and any probability density $(\lambda_i)_I \in Pwt$,

$$\widetilde{\bigoplus_{(\lambda_i)_I}} C_i = (\bigoplus_{(\lambda_i)_I} C_i)^\dagger.$$

³We use the same notation C^\dagger as defined previously for finitary theories. It will always be clear from the context which version is meant.

- (b) **On morphisms:** For $f : (S, \vee) \rightarrow (S', \vee') \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$ and $C \in \mathbf{Stbl}(S)$,

$$\mathbf{Stbl}f(C) = (f[C]_{scvx})^\dagger.$$

where $f[C]_{scvx} = \{d_f \mid d \in C\}$.

We have the following results hold in the infinitary case, for similar reasons as the finite case. In fact, the proofs for these results are easier than their finite counterparts, because we do not have to restrict ourselves to finitary combinations and finitely generated convex sets. This greatly simplifies the proofs that our maps are well-defined.

Theorem 4.3.7. *The \vee -stable convex functor $\mathbf{Stbl} : \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$ is left adjoint to the forgetful functor $\mathcal{U}_{\mathbf{Stbl}} : \mathbf{Mod}(\mathbf{MC}, \mathbf{Set}) \rightarrow \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$. Thus it forms a monad $\mathbf{Stbl} = (\mathcal{U}_{\mathbf{Stbl}} \circ \mathbf{Stbl}, \eta, \mu)$ over $\mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$. We include the adjunction and monad structures:*

- (a) **The unit**(η): for $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$ and $s \in S$,

$$\eta_{(S, \vee)}(s) = \{\delta_s\}^\dagger.$$

- (b) **The counit**(ε): for $(M, \vee, \sum_{(\lambda_i)_I}) \in \mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$ and $C \in \mathbf{Stbl}(M)$,

$$\varepsilon_{(M, \vee, \sum_{(\lambda_i)_I})}(C) = \bigvee_{d \in C} \left(\sum_{(d(m))_{\text{supp}(d)}} m \right).$$

- (c) **The multiplication**(μ): for $(S, \vee) \in \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Set})$ and $C \in \mathbf{Stbl}(\mathbf{Stbl}(S))$,

$$\mu_{(S, \vee)}(C) = \widetilde{\bigoplus_{D \in C}} \left(\widetilde{\bigoplus_{(D(C))_{\text{supp}(D)}}} C \right).$$

(Note that the every $C \in \text{supp}(D)$ is an element of $\mathbf{Stbl}(S)$. Thus, C is a \vee -stable superconvex set.)

Proposition 4.3.8. *The category $\mathbf{Stbl}\text{-Alg}$ is equivalent to the category $\mathbf{Mod}(\mathbf{MC}, \mathbf{Set})$. Thus, \mathbf{Stbl} constructs the free infinite mixed choice models over arbitrary semilattices.*

Theorem 4.3.9. *We can factorize the geometrically superconvex powerset functor, \mathcal{G}_{scvx} , through the non-empty powerset functor \mathcal{P}^* by using the \vee -stable superconvex functor \mathbf{Stbl} . In other words, $\mathcal{G}_{scvx} \cong \mathbf{Stbl} \circ \mathcal{P}^*$.*

4.4 \vee -Stability and Posetal Models of Mixed Choice

In this section we return our focus to finite theories of nondeterministic and probabilistic choice. However, we now want to describe our construction of free finite mixed choice models over finite semilattices over posetal models, instead of set models. In other words, we want to define the posetal version of the \vee -stable convex functor. However, as mentioned in Chapter 3, there are three different types of nondeterministic theories to consider when considering ordered theories. We begin this chapter by recalling the basic algebra constructions used to capture probabilistic and nondeterministic choice over **Poset**.

4.4.1 Posetal Models of Probabilities and Nondeterminism

Posetal Models of Probabilities

We begin by presenting the posetal probabilistic algebra. As previously mentioned, the probabilistic algebra functor over partially ordered sets is a particular case of the construct described in depth in Claire Jones' thesis [20].

Definition 4.4.1 (Probabilistic Algebra). The **probabilistic algebra** over **Poset**, $\mathcal{V} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{P}_{fin}, \mathbf{Poset})$ is defined as follows:

- (a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{V}((X, \sqsubseteq)) = (\mathcal{D}_{fin}(X), \preceq),$$

where \preceq is the *distributions order* over (X, \sqsubseteq) . Recall that for $d, d' \in \mathcal{D}_{fin}(X)$, $d \preceq d'$ if and only if for all $Y \subseteq X$, $d(\uparrow_{\sqsubseteq} Y) \leq d'(\uparrow_{\sqsubseteq} Y)$, where $d(Y) = \sum_{y \in Y} d(y)$.

- (b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$, $d \in \mathcal{V}(X)$ and $x' \in X'$,

$$\mathcal{V}f(d) = d_f = \sum d(x)\delta_{f(x)}.$$

Posetal Models of Nondeterminism

Next we present the posetal definition of the three powerdomains for nondeterminism. As previously stated, each of the following algebras gives rise to models of nondeterminism with associative, commutative and idempotent nondeterministic choice operators. However, in the cases of the Hoare and Smyth algebras two extra axioms concerning the ordering are assumed, $A \sqsubseteq A \boxplus B$ and $A \boxplus B \sqsubseteq A$, respectively.

Definition 4.4.2 (Convex Algebra Functor). The **convex algebra functor**, $\mathcal{P} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{P}((X, \sqsubseteq)) = ((\mathcal{P}(X), \sqcup), \sqsubseteq_{EM}),$$

- (i) $\mathcal{P}(X)$ is the set of all order-convex closures under \sqsubseteq of finitely generated subsets of (X, \sqsubseteq) (i.e $\mathcal{P}(X) = \{\overline{Y} \mid Y \in \mathcal{P}_{fin}^*(X)\}$, where \overline{Y} is the smallest order-convex set containing Y).
- (ii) \sqcup is union closed under order-convexity (i.e., $\overline{Y_1} \sqcup \overline{Y_2} = \overline{Y_1 \cup Y_2}$).
- (iii) \sqsubseteq_{EM} is the *Egli-Milner order* over (X, \sqsubseteq) . For $\overline{Y_1}, \overline{Y_2} \in \mathcal{P}(X)$, $\overline{Y_1} \sqsubseteq_{EM} \overline{Y_2}$ if and only if $\downarrow Y_1 \subseteq \downarrow Y_2$ and $\uparrow Y_2 \subseteq \uparrow Y_1$.

(b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\overline{Y} \in \mathcal{P}(X)$,

$$\mathcal{P}f(\overline{Y}) = \overline{f[Y]}.$$

Definition 4.4.3 (Lower (Hoare) Nondeterministic Algebra Functor). The **lower ND algebra functor**, $\mathcal{H} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{ND}^*, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{H}((X, \sqsubseteq)) = ((\mathcal{H}(X), \cup), \sqsubseteq).$$

- (i) $\mathcal{H}(X)$ is the set of all lower sets under \sqsubseteq of finitely generated subsets of (X, \sqsubseteq) (i.e $\mathcal{H}(X) = \{\downarrow_{\sqsubseteq} Y \mid Y \in \mathcal{P}_{fin}^*(X)\}$, where $\downarrow_{\sqsubseteq} Y$ is the set of all elements of X which are smaller than some element in Y).

(b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\downarrow_{\sqsubseteq} Y \in \mathcal{H}(X)$,

$$\mathcal{H}f(\downarrow_{\sqsubseteq} Y) = \downarrow_{\sqsubseteq'} f[Y].$$

Definition 4.4.4 (Upper (Smyth) Nondeterministic Algebra Functor). The **upper ND algebra functor**, $\mathcal{S} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{S}((X, \sqsubseteq)) = ((\mathcal{S}(X), \cup), \supseteq).$$

(i) $\mathcal{S}(X)$ is the set of all upper sets under \sqsubseteq of finitely generated subsets of (X, \sqsubseteq) (i.e. $\mathcal{S}(X) = \{\uparrow_{\sqsubseteq} Y \mid Y \in \mathcal{P}_{fin}^*(X)\}$, where $\uparrow_{\sqsubseteq} Y$ is the set of all elements of X which are bigger than some element in Y).

(b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\uparrow_{\sqsubseteq} Y \in \mathcal{S}(X)$,

$$\mathcal{S}f(\uparrow_{\sqsubseteq} Y) = \uparrow_{\sqsubseteq'} f[Y].$$

4.4.2 Convex Algebra Functors

As stated previously, there is a algebra functor for mixed choice associated to each different type of nondeterminism present for posetal models. We recall their definitions below.

The biconvex algebra functor captures the mixed choice theory associated to the nondeterministic operations described by the convex algebra functor.

Definition 4.4.5 (Biconvex Algebra Functor). The **biconvex algebra functor**, $\mathcal{PV} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{PV}((X, \sqsubseteq)) = ((\mathcal{PV}(X), \overline{\boxplus}, \overline{\boxplus}_\lambda), \sqsubseteq_{EM \preceq})$$

(i) $\mathcal{PV}(X)$ is the set of order-convex closures under \preceq of finitely generated convex subsets over (X, \sqsubseteq) (i.e., $\mathcal{PV}(X) = \{\overline{C} \mid C \in \mathcal{G}_{conv}(X)\}$, where \overline{C} is the smallest order-convex, convex subset of $\mathcal{V}(X)$ containing C).

- (ii) $\overline{\boxplus}$ and $\overline{\oplus_\lambda}$ are the order-convex closures of \boxplus and \oplus_λ . For $\overline{C_1}, \overline{C_2} \in \mathcal{PV}(X)$, we have that

$$\begin{aligned}\overline{C_1} \overline{\boxplus} \overline{C_2} &= \overline{(C_1 \boxplus C_2)} \\ \overline{C_1} \overline{\oplus_\lambda} \overline{C_2} &= \overline{(C_1 \oplus_\lambda C_2)}\end{aligned}$$

- (iii) $\sqsubseteq_{EM\preceq}$ is the *Egli-Milner order* over $\mathcal{V}((X, \sqsubseteq))$.

- (b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\overline{C} \in \mathcal{PV}(X)$,

$$\mathcal{PV}f(\overline{C}) = \overline{f[C]_{cvx}}.$$

The convex lower ND algebra functor captures the mixed choice theory associated to the ND behavior described by the Hoare algebra.

Definition 4.4.6 (Convex Lower Nondeterministic Algebra Functor). The **convex lower ND algebra functor**, $\mathcal{HV} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is defined as follows:

- (a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{HV}((X, \sqsubseteq)) = ((\mathcal{HV}(X), \boxplus^\dagger, \oplus_\lambda^\dagger), \sqsubseteq).$$

- (i) $\mathcal{HV}(X)$ is the set of lower sets under \preceq of finitely generated convex subsets over (X, \sqsubseteq) (i.e., $\mathcal{HV}(X) = \{\downarrow_{\preceq} C \mid C \in \mathcal{G}_{cvx}(X)\}$, where $\downarrow_{\preceq} C$ is the set of all distribution in $\mathcal{V}(X)$ which are smaller than some distribution in C).
- (ii) \boxplus^\dagger and \oplus_λ^\dagger are the appropriate closures of \boxplus and \oplus_λ . For $\downarrow_{\preceq} C_1, \downarrow_{\preceq} C_2 \in \mathcal{HV}(X)$, we have that

$$\begin{aligned}\downarrow_{\preceq} C_1 \boxplus^\dagger \downarrow_{\preceq} C_2 &= \downarrow_{\preceq} (C_1 \boxplus C_2) \\ \downarrow_{\preceq} C_1 \oplus_\lambda^\dagger \downarrow_{\preceq} C_2 &= \downarrow_{\preceq} (C_1 \oplus_\lambda C_2)\end{aligned}$$

- (b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\downarrow_{\preceq} C \in \mathcal{HV}(X)$,

$$\mathcal{HV}f(\downarrow_{\preceq} C) = \downarrow_{\preceq'} (f[C]_{cvx}).$$

The convex upper ND algebra functor captures the mixed choice theory associated to the ND behavior described by the Smyth algebra.

Definition 4.4.7 (Convex Upper Nondeterministic Algebra Functor). The **convex upper ND algebra functor**, $\mathcal{SV} : \mathbf{Poset} \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $(X, \sqsubseteq) \in \mathbf{Poset}$,

$$\mathcal{SV}((X, \sqsubseteq)) = ((\mathcal{SV}(X), \boxplus^\dagger, \oplus_\lambda^\dagger), \supseteq).$$

- (i) $\mathcal{SV}(X)$ is the set of upper sets under \preceq of finitely generated convex subsets over (X, \sqsubseteq) (i.e., $\mathcal{SV}(X) = \{\uparrow_{\preceq} C \mid C \in \mathcal{G}_{conv}(X)\}$, where $\uparrow_{\preceq} C$ is the set of all distributions in $\mathcal{V}(X)$ which are larger than some distribution in C).
- (ii) \boxplus^\dagger and \oplus_λ^\dagger are the appropriate closures of \boxplus and \oplus_λ . For $\uparrow_{\preceq} C_1, \uparrow_{\preceq} C_2 \in \mathcal{SV}(X)$, we have that

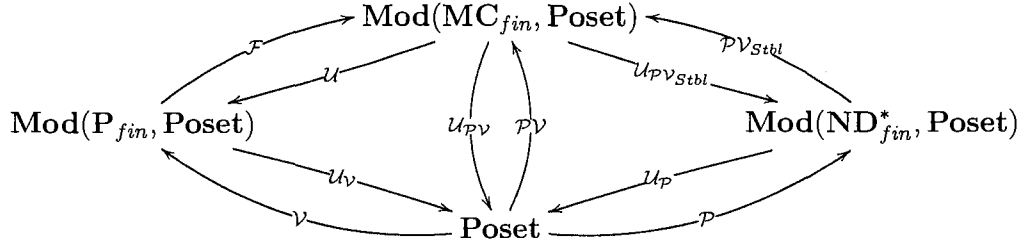
$$\begin{aligned} \uparrow_{\preceq} C_1 \boxplus^\dagger \uparrow_{\preceq} C_2 &= \uparrow_{\preceq} (C_1 \boxplus C_2) \\ \uparrow_{\preceq} C_1 \oplus_\lambda^\dagger \uparrow_{\preceq} C_2 &= \uparrow_{\preceq} (C_1 \oplus_\lambda C_2) \end{aligned}$$

(b) **On morphisms:** For $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\uparrow_{\preceq} C \in \mathcal{SV}(X)$,

$$\mathcal{SV}f(\uparrow_{\preceq} C) = \uparrow_{\preceq'} (f[C]_{conv}).$$

4.4.3 \vee -Stable Convex Algebra Functors

As we have noted during our constructions over \mathbf{Set} , the above convex algebra constructions are obtained by following the left hand side of our recurring diagram, modified for the posetal case (the case involving the convex algebra functor is shown in Figure 15). Based on our work on finding set-theoretical models for mixed choice, we will construct the functors necessary to obtain posetal mixed choice models by following the right hand side of the diagram. For each of the possible convex algebra functors capturing mixed choice, there will be an appropriately ordered \vee -stable functor. We begin by defining the \vee -stable biconvex functor, which constructs mixed



Where each functor is part of an adjunction pair i.e. $\mathcal{P}\mathcal{V} \dashv \mathcal{U}_{\mathcal{P}\mathcal{V}}, \mathcal{V} \dashv \mathcal{U}_{\mathcal{V}}, \mathcal{P} \dashv \mathcal{U}_{\mathcal{P}}, \mathcal{P}\mathcal{V}_{Stbl} \dashv \mathcal{U}_{\mathcal{P}\mathcal{V}_{Stbl}}$ and $\mathcal{U} \dashv \mathcal{F}$, such that $\mathcal{F} \circ \mathcal{V} \cong \mathcal{P}\mathcal{V}$.

Figure 15: Mixed Choice Factored over **Poset**

choice models over arbitrary ordered semilattices, using the most familiar interpretation of posetal nondeterminism.

We've observed in the definition of the convex algebra functor and in the biconvex algebra functor that in order to capture this type of nondeterminism we consider particular sets of *order-convex* convex subsets of $\mathcal{V}((S, \sqsubseteq))$. Thus, we shall have to extend our previous concepts of \vee -stability in order to construct a family of convex subsets which are \vee -stable and order-convex under \preceq ; i.e. *order \vee -stable* convex sets.

Definition 4.4.8 (Order \vee -Stable). Given an ordered semilattice $((S, \sqsubseteq), \vee)$. A convex subset $C \in \mathcal{Cvx}(S)$ is **order \vee -stable** if C is \vee -stable (i.e., for every facial element $F \in \mathcal{F}^{elt}(S)$, $F \subseteq C \Rightarrow F^\downarrow \subseteq C$) and order-convex under \preceq (i.e., for any $d \in \mathcal{V}((S, \sqsubseteq))$, if there exists $d_1, d_2 \in C$ such that $d_1 \preceq d \preceq d_2$, then $d \in C$).

Both \vee -stability and order-convexity behave well with respect to arbitrary intersections, yielding the following result.

Proposition 4.4.9. *Suppose we are given an ordered semilattice $((S, \vee), \sqsubseteq)$. Then \vee -stability and order-convexity are preserved under arbitrary non-empty intersection.*

Thus, as previously done over **Set**, we can define a closure operation called the *order \vee -stable closure* over any convex subset of $\mathcal{V}((S, \sqsubseteq))$.

Definition 4.4.10 (Order \vee -Stable Closure). Suppose we are given an ordered semilattice $((S, \sqsubseteq), \vee)$ and a convex subset $C \in \mathcal{Cvx}(S)$. The **order \vee -stable closure** of

C **under** \preceq , denoted by $C^{\bar{\cdot}}$, is the smallest convex subset of $\mathcal{V}((S, \sqsubseteq))$ which is both \vee -stable and order-convex under \preceq .

In the case of **Set**, we have defined a family of partially \vee -stable-closed sets constructed from a particular convex set C , i.e. $C^{\dagger_{ptl}}$. We showed how this recursively defined family could be used to construct the \vee -stable closure of C . Now we shall define its counterpart for partial order \vee -stable closures.

Definition 4.4.11 (Partial Order \vee -Stable Closures). Given an ordered semilattice $((S, \vee), \sqsubseteq)$ and a convex set $C \in \text{Cvx}(S)$, the **set of all partial order \vee -stable closures of C** , $C^{\bar{\cdot}_{ptl}}$, is given by the following recursive definition.

- (a) $C \in C^{\bar{\cdot}_{ptl}}$,
- (b) If $Y \in C^{\bar{\cdot}_{ptl}}$ and there exists $F \in \mathcal{F}^{elt}(S)$ such that $F \subseteq Y$, then $\overline{(F^{\downarrow} \boxplus Y)} \in C^{\bar{\cdot}_{ptl}}$.

Proposition 4.4.12 (Properties of $C^{\bar{\cdot}_{ptl}}$).

- (a) For each $Y \in C^{\bar{\cdot}_{ptl}}$, $Y \in \mathcal{PV}((S, \sqsubseteq))$, thus there exists $A \in \mathcal{G}_{cvx}(S)$ such that $Y = \overline{A}$.
- (b) For each $Y \in C^{\bar{\cdot}_{ptl}}$, $C \subseteq Y$,
- (c) $C^{\bar{\cdot}_{ptl}}$ is closed under \boxplus in $\mathcal{PV}((S, \sqsubseteq))$ (i.e., If $Y, Y' \in C^{\bar{\cdot}_{ptl}}$, then $Y \boxplus Y' \in C^{\bar{\cdot}_{ptl}}$).

Each of the properties of Proposition 4.4.12 follows from the recursive definition of partial order \vee -stable closures. The proof is quite similar to the proof of Proposition 4.2.19 given when dealing with models over **Set**.

Theorem 4.4.13. If C is a convex set in $\text{Cvx}(S)$ then $C^{\bar{\cdot}} = \bigcup_{Y \in C^{\bar{\cdot}_{ptl}}} Y$.

Proof: Many of the necessary conditions we need to show are similar to their set theoretical counterparts given in the proof of Theorem 4.2.20. Their remains to show that $\bigcup_{Y \in C^{\bar{\cdot}_{ptl}}} Y$ is order convex. Let $d, d' \in \bigcup_{Y \in C^{\bar{\cdot}_{ptl}}} Y$, then there exists $Y_d, Y_{d'} \in C^{\bar{\cdot}_{ptl}}$

such that $d \in Y_d$ and $d' \in Y_{d'}$. This implies that $d, d' \in Y_d \overline{\boxplus} Y_{d'}$ which is order convex and a partial order \vee -stable closure of C (i.e., $Y_d \overline{\boxplus} Y_{d'} \in C^{\overline{\uparrow}ptt}$). Therefore, any d'' such that $d \preceq d'' \preceq d'$ must be in $Y_d \overline{\boxplus} Y_{d'}$ and hence in $\bigcup_{Y \in C^{\overline{\uparrow}ptt}} Y$. \square

Proposition 4.4.14 (Mixed Choice Model). *Suppose we are given an ordered semilattice $((S, \vee), \sqsubseteq)$. The set of all order \vee -stable closures on $\mathcal{G}_{cvx}(S)$, $\mathcal{PV}_{Stbl}(S)$, together with the operations \boxplus^* and \oplus_λ^* , the order \vee -stable closures of \boxplus and \oplus_λ from $\mathcal{G}_{cvx}(S)$, forms a mixed choice model $(\mathcal{PV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*)$ over **Poset** when equipped with the Egli-Milner ordering \preceq_{EM} .*

Before we can prove Proposition 4.4.14 and show that $((\mathcal{PV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*), \preceq_{EM})$ is in fact a mixed choice model over **Poset**, we prove some technical lemmas which we shall require.

Lemma 4.4.15. *Suppose we are given an ordered semilattice $((S, \vee), \sqsubseteq)$. Let C be an order \vee -stable convex set over $((S, \vee), \sqsubseteq)$. Then, for any $C' \in \mathcal{PV}(S)$ and $F \in \mathcal{F}^{elt}(S)$ such that $C' \oplus_\lambda F \subseteq C$ we have that $\overline{(C' \oplus_\lambda F^\downarrow)} \subseteq C$.*

Proof: Since C is \vee -stable we know that $C' \oplus_\lambda F \subseteq C$ implies $C' \oplus_\lambda F^\downarrow \subseteq C$, as shown in Lemma 4.2.22. Moreover, since C is also order-convex we must have that $\overline{(C' \oplus_\lambda F^\downarrow)} \subseteq C$. \square

Lemma 4.4.16. *Let $C_1, C_2 \in \mathcal{G}_{cvx}S$. Then*

$$(a) (C_1^{\overline{\uparrow}} \oplus_\lambda C_2^{\overline{\uparrow}})^{\overline{\uparrow}} = (C_1 \oplus_\lambda C_2)^{\overline{\uparrow}},$$

$$(b) (C_1^{\overline{\uparrow}} \boxplus C_2^{\overline{\uparrow}})^{\overline{\uparrow}} = (C_1 \boxplus C_2)^{\overline{\uparrow}}.$$

Proof: Let $C_1, C_2 \in \mathcal{G}_{cvx}(S)$.

$$(a) \text{ We show that } (C_1^{\overline{\uparrow}} \oplus_\lambda C_2^{\overline{\uparrow}})^{\overline{\uparrow}} \subseteq (C_1 \oplus_\lambda C_2)^{\overline{\uparrow}} \text{ and } (C_1^{\overline{\uparrow}} \oplus_\lambda C_2^{\overline{\uparrow}})^{\overline{\uparrow}} \supseteq (C_1 \oplus_\lambda C_2)^{\overline{\uparrow}}.$$

(\subseteq) It will be enough to show that $C_1^{\overline{\uparrow}} \oplus_\lambda C_2^{\overline{\uparrow}} \subseteq (C_1 \oplus_\lambda C_2)^{\overline{\uparrow}}$; hence we need to show that for every $Y_1 \in C_1^{\overline{\uparrow}ptt}$ and $Y_2 \in C_2^{\overline{\uparrow}ptt}$, $Y_1 \oplus_\lambda Y_2 \subseteq (C_1 \oplus_\lambda C_2)^{\overline{\uparrow}}$.

We proceed by simultaneous structural induction on Y_1 and Y_2 .

- (i) **Base Case:** Let $Y_1 = C_2$ and $Y_1 = C_2$, then $Y_1 \oplus_\lambda Y_2 = C_1 \oplus_\lambda C_2 \subseteq (C_1 \oplus_\lambda C_2)^\dagger$.
- (ii) **Induction Step:** Let $Y_1 = \overline{(Y'_1 \boxplus F_1^\downarrow)}$, and $Y_2 = \overline{(Y'_2 \boxplus F_2^\downarrow)}$, where $Y'_1 \in C_1^{\dagger_{\mu}}$, $Y'_2 \in C_2^{\dagger_{\mu}}$, and $F_1, F_2 \in \mathcal{F}^{elt}(S)$ such that $F_1 \subseteq Y'_1$ and $F_2 \subseteq Y'_2$. By I.H we assume that $Y'_1 \oplus_\lambda Y'_2 \subseteq (C_1 \oplus_\lambda C_2)^\dagger$.

$$\begin{aligned} Y_1 \oplus_\lambda Y_2 &= \overline{(Y'_1 \boxplus F_1^\downarrow)} \oplus_\lambda \overline{(Y'_2 \boxplus F_2^\downarrow)} \\ &\subseteq \overline{((Y'_1 \oplus_\lambda Y'_2) \boxplus (Y'_1 \oplus_\lambda F_2^\downarrow) \boxplus (F_1^\downarrow \oplus_\lambda Y'_2) \boxplus (F_1^\downarrow \oplus_\lambda F_2^\downarrow))} \\ &= \overline{(Y'_1 \oplus_\lambda Y'_2)} \boxplus \overline{(Y'_1 \oplus_\lambda F_2^\downarrow)} \boxplus \overline{(F_1^\downarrow \oplus_\lambda Y'_2)} \boxplus \overline{(F_1^\downarrow \oplus_\lambda F_2^\downarrow)} \end{aligned}$$

Since $(F_1 \oplus_\lambda F_2), (F_1 \oplus_\lambda Y'_2)$ and $(Y'_1 \oplus_\lambda F_2)$ are all subsets of $Y'_1 \oplus_\lambda Y'_2 \subseteq (C_1 \oplus_\lambda C_2)^\dagger$ and since $(C_1 \oplus_\lambda C_2)^\dagger$ is order \vee -stable, we have that $\overline{(F_1^\downarrow \oplus_\lambda F_2^\downarrow)}, \overline{(F_1^\downarrow \oplus_\lambda Y'_2)}$ and $\overline{(Y'_1 \oplus_\lambda (F_1^\downarrow))}$ are contained in $(C_1 \oplus_\lambda C_2)^\dagger$. Thus, $Y_1 \oplus_\lambda Y_2 \subseteq (C_1 \oplus_\lambda C_2)^\dagger$.

(\supseteq) Since $C_1 \subseteq C_1^\dagger$ and $C_2 \subseteq C_2^\dagger$, we have that $C_1 \oplus_\lambda C_2 \subseteq C_1^\dagger \oplus_\lambda C_2^\dagger$. Therefore, $(C_1 \oplus_\lambda C_2)^\dagger \subseteq (C_1^\dagger \oplus_\lambda C_2^\dagger)^\dagger$.

- (b) By similar arguments and observations as those made for part (a), we have that part (b) holds. □

Proof: [Proposition 4.4.14] Since the elements of $\mathcal{PV}_{Stbl}(S)$ are in particular order-convex sets under \preceq , then the Egli-Milner order \preceq_{EM} , is a well-defined order relation on $\mathcal{PV}_{Stbl}(S)$. Next, we show that the operations \boxplus^* and \oplus_λ^* are well-defined and respect the Egli-Milner ordering \preceq_{EM} and satisfy the required axioms. Suppose $\lambda \in [0, 1], A_1^\dagger = B_1^\dagger$ and $A_2^\dagger = B_2^\dagger$, where $A_1, A_2, B_1, B_2 \in \mathcal{G}_{cvx}(S)$.

- (a) \boxplus^* and \oplus_λ^* are well-defined.

(i) We show that $A_1^\dagger \oplus_\lambda^* A_2^\dagger = B_1^\dagger \oplus_\lambda^* B_2^\dagger$.

$$\begin{aligned} A_1^\dagger = B_1^\dagger, A_2^\dagger = B_2^\dagger &\Rightarrow A_1^\dagger \oplus_\lambda A_2^\dagger = B_1^\dagger \oplus_\lambda B_2^\dagger \\ &\Rightarrow (A_1^\dagger \oplus_\lambda A_2^\dagger)^\dagger = (B_1^\dagger \oplus_\lambda B_2^\dagger)^\dagger \\ &\Rightarrow A_1^\dagger \oplus_\lambda^* A_2^\dagger = B_1^\dagger \oplus_\lambda^* B_2^\dagger \end{aligned}$$

(ii) Similarly, we have that $A_1^\dagger \boxplus^* A_2^\dagger = B_1^\dagger \boxplus^* B_2^\dagger$.

(b) \boxplus^* and \oplus_λ^* respect the Egli-Milner ordering, \preceq_{EM} .

Suppose $A_1^\dagger \preceq_{EM} B_1^\dagger$ and $A_2^\dagger \preceq_{EM} B_2^\dagger$, where $A_i, B_i \in \mathcal{G}_{cvx}(S)$.

(i) We show that $A_1^\dagger \oplus_\lambda^* A_2^\dagger \preceq_{EM} B_1^\dagger \oplus_\lambda^* B_2^\dagger$.

$$\downarrow_{\preceq}(A_1 \oplus_\lambda A_2) \subseteq \downarrow_{\preceq}(B_1 \oplus_\lambda B_2) \text{ and } \uparrow_{\preceq}(B_1 \oplus_\lambda B_2) \subseteq \uparrow_{\preceq}(A_1 \oplus_\lambda A_2)$$

Let $d \in \downarrow_{\preceq}(A_1 \oplus_\lambda A_2)$, then there exists $d_{A_i} \in A_i$ such that $d \preceq d_{A_1} \oplus_\lambda d_{A_2}$. Moreover, we know that there exists $d_{B_i} \in B_i$ such that $d_{A_i} \preceq d_{B_i}$. Thus, $d \preceq d_{A_1} \oplus_\lambda d_{A_2} \preceq d_{B_1} \oplus_\lambda d_{B_2}$ which implies that $d \in \downarrow_{\preceq}(B_1 \oplus_\lambda B_2)$ and thus $\downarrow_{\preceq}(A_1 \oplus_\lambda A_2) \subseteq \downarrow_{\preceq}(B_1 \oplus_\lambda B_2)$. Similarly, we get that $\uparrow_{\preceq}(B_1 \oplus_\lambda B_2) \subseteq \uparrow_{\preceq}(A_1 \oplus_\lambda A_2)$.

(ii) By similar arguments as made above in (i), we have that $A_1^\dagger \boxplus^* A_2^\dagger \preceq_{EM} B_1^\dagger \boxplus^* B_2^\dagger$.

(c) \boxplus^* and \oplus_λ^* satisfy the mixed choice theory axioms.

The necessary mixed choice theory axioms are all satisfied, since \boxplus and \oplus_λ satisfy them and that \boxplus^* and \oplus_λ^* are the order \vee -stable closures of \boxplus and \oplus_λ .

□

We now have all the necessary information to define the \vee -stable biconvex functor, \mathcal{PV}_{Stbl} .

Definition 4.4.17 (\vee -Stable Biconvex Functor). The \vee -stable biconvex functor, $\mathcal{PV}_{Stbl} : \text{Mod}(\text{ND}_{fin}^*, \text{Poset}) \rightarrow \text{Mod}(\text{MC}_{fin}, \text{Poset})$ is defined as follows:

(a) **On objects:** Given $((S, \sqsubseteq), \vee) \in \text{Mod}(\text{ND}_{fin}^*, \text{Poset})$,

$$\mathcal{PV}_{Stbl}(((S, \sqsubseteq), \vee)) = ((\mathcal{PV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*), \preceq_{EM}).$$

(i) $\mathcal{PV}_{Stbl}(S)$ is the set of all order-convex \vee -stable closures of the finitely generated convex subsets over (S, \sqsubseteq) (i.e., $\mathcal{PV}_{Stbl}(S) = \{C^\dagger \mid C \in \mathcal{G}_{cvx}(S)\}$, where C^\dagger is the smallest order-convex, \vee -stable convex subset over $((S, \vee), \sqsubseteq)$ containing C).

(ii) \boxplus^* and \oplus_λ^* are the order-convex \vee -stable closures of \boxplus and \oplus_λ .

$$\begin{aligned} C_1^\dagger \boxplus^* C_2^\dagger &= (C_1 \boxplus C_2)^\dagger \\ C_1^\dagger \oplus_\lambda^* C_2^\dagger &= (C_1 \oplus_\lambda C_2)^\dagger \end{aligned}$$

(iii) \preceq_{EM} is the *Egli-Milner order* over $\mathcal{V}((S, \sqsubseteq))$

(b) **On morphisms:** Let $f : ((S, \sqsubseteq), \vee) \rightarrow ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $C^\dagger \in \mathcal{PV}_{Stbl}(S)$,

$$\mathcal{PV}_{Stbl}f(C^\dagger) = (f[C]_{cvx})^\dagger.$$

In the following theorem, we show that the \vee -stable biconvex functor defined above is indeed the left adjoint to the forgetful functor from the posetal models of mixed choice to the posetal models of nondeterministic choice.

Theorem 4.4.18. *The \vee -stable biconvex functor $\mathcal{PV}_{Stbl} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is left adjoint to the forgetful functor $\mathcal{U}_{\mathcal{PV}_{Stbl}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. Thus it forms a monad $\mathbb{P}\mathcal{V}_{Stbl} = (\mathcal{U}_{\mathcal{PV}_{Stbl}} \circ \mathcal{PV}_{Stbl}, \eta, \mu)$ over $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. We include the adjunction and monad structures:*

(a) **The unit**(η): for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $s \in S$,

$$\eta_{((S, \vee), \sqsubseteq)}(s) = \{\delta_s\}^\dagger.$$

(b) **The counit**(ε): for $((M, \vee, +_\lambda), \sqsubseteq) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ and $N^\dagger \in \mathcal{PV}_{Stbl}(M)$,

$$\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N^\dagger) = \bigvee_{d \in \text{gen}(N)} \left(\sum_{(d(m))_{\text{supp}(d)}} m \right).$$

(c) **The multiplication**(μ): for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $\mathcal{N}^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{PV}_{Stbl}(S))$,

$$\mu_{((S, \vee), \sqsubseteq)}(\mathcal{N}^\dagger) = \boxplus_{D \in \text{gen}(\mathcal{N})}^* \left(\bigoplus_{(D(C^\dagger))_{\text{supp}(D)}}^* C^\dagger \right).$$

Proof: We must show that the unit and counit maps are well defined and satisfy the necessary conditions.

(a) The unit is a well-defined ordered semilattice morphism.

(i) **Well-defined:** Consider $s, s' \in S$ such that $s = s'$, clearly $\eta_{((S, \vee), \sqsubseteq)}(s) = \eta_{((S, \vee), \sqsubseteq)}(s')$.

(ii) **Preserves the order:** Consider $s, s' \in S$ such that $s \sqsubseteq s'$, clearly $\downarrow_{\leq} \{\delta_s\} \subseteq \downarrow_{\leq} \{\delta_{s'}\}$ and $\uparrow_{\leq} \{\delta_{s'}\} \subseteq \uparrow_{\leq} \{\delta_s\}$. Therefore,

$$\eta_{((S, \vee), \sqsubseteq)}(s) \preceq_{EM} \eta_{((S, \vee), \sqsubseteq)}(s').$$

(iii) **Semilattice morphism:** Given an ordered semilattice $((S, \vee), \sqsubseteq)$ and $s, s' \in S$,

$$\begin{aligned} \eta_{((S, \vee), \sqsubseteq)}(s \vee s') &= \{\delta_{s \vee s'}\}^{\bar{\Gamma}} \\ &= (\{\delta_s\} \boxplus \{\delta_{s'}\})^{\bar{\Gamma}} \\ &= \{\delta_s\}^{\bar{\Gamma}} \boxplus^* \{\delta_{s'}\}^{\bar{\Gamma}} \\ &= \eta_{((S, \vee), \sqsubseteq)}(s) \boxplus^* \eta_{((S, \vee), \sqsubseteq)}(s') \end{aligned}$$

(b) The counit is a well-defined mixed choice morphism.

(i) **Well-defined:** Let $N_1^{\bar{\Gamma}}, N_2^{\bar{\Gamma}} \in \mathcal{PV}_{Stbl}(M)$, such that $N_1^{\bar{\Gamma}} = N_2^{\bar{\Gamma}}$. We must show that $\varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\Gamma}}) = \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_2^{\bar{\Gamma}})$.

For any $d \in \text{gen}(N_2)$, define $d^{\flat} = \sum_{(d(m))_{\text{supp}(d)}} m \in M$. Recall that $\sum_{(d(m))_{\text{supp}(d)}} m$ represents the convex combination in $(M, \vee, +\lambda)$ of the form $d(m_1)m_1 + d(m_2)m_2 + \dots + d(m_k)m_k$ where $\text{supp}(d) = \{m_1, m_2, \dots, m_k\}$. We show that $d^{\flat} \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\Gamma}})$ where \leq_{\vee} is the order generated by the semilattice structure in $((M, \vee, +\lambda), \sqsubseteq)$.

Given a $d \in \text{gen}(N_2)$, since $d \in N_2 \subseteq N_2^{\bar{\Gamma}} = N_1^{\bar{\Gamma}}$, there exists a partial completion $Y \in N_1^{\bar{\Gamma}^{\text{ptu}}}$ such that $d \in Y$. We show by structural induction on $Y \in N_1^{\bar{\Gamma}^{\text{ptu}}}$ that $(\forall d \in Y) d^{\flat} \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\Gamma}})$.

- (1) **Base Case:** $Y = N_1$. If $d \in Y = N_1$, then $d = \bigoplus_{(\lambda_i)_I} d_i$ where $|I| < \infty$, and for each $i \in I$, $d_i \in \text{gen}(N_1)$. Since $d^b = \sum_{(\lambda_i)_I} d_i^b$ and for each $i \in I$, $d_i^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$, we have that $d^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$.
- (2) **Structural Case:** Let $Y = \overline{(Y' \boxplus F^{\downarrow})}$, where $F \in \mathcal{F}^{elt}(M)$, $Y' \in N_1^{\bar{\dagger} \text{pt}}$, $F \subseteq Y'$ and by induction hypothesis the property holds for any $d' \in Y'$ (i.e., for any $d' \in Y'$, $(d')^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$). Since $d \in Y$, there exists $d_1, d_2 \in Y' \boxplus F^{\downarrow}$ such that $d_i = d'_i \oplus_{\rho_i} f_i$ where $\rho_i \in [0, 1]$, $d'_i \in Y'$ and $f_i \in F^{\downarrow}$, such that $d_1 \preceq d \preceq d_2$.

Claim 1. Let $F = \bigoplus_{(\lambda_i)_I} \mathcal{V}(T_i)$, where $T_i \subseteq_{fin} M$. The distribution $f^* = \bigoplus_{(\lambda_i)_I} \delta_{(\vee T_i)} \in F^{\downarrow}$ has the following property:

$$(f^*)^b = \sum_{(\lambda_i)_I} (\bigvee T_i) \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}}).$$

Indeed, let $\vec{t} \in \prod_I T_i$ and consider the distributions of the form $f_{\vec{t}} = \bigoplus_{(\lambda_i)_I} \delta_{(\pi_i(\vec{t}))} \in F$. Since $F \subseteq Y'$, then by the induction hypothesis we have that for each $\vec{t} \in \prod_I T_i$, $(f_{\vec{t}})^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$. This implies that $\bigvee_{\vec{t} \in \prod_I T_i} (f_{\vec{t}})^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$. By the distributivity property of $+\lambda$ over \vee from $((M, \vee, +\lambda), \sqsubseteq)$, we have that

$$(f^*)^b = \bigvee_{\vec{t} \in \prod_I T_i} (f_{\vec{t}})^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}}).$$

Claim 2. $(\forall f \in F^{\downarrow}) f^b \leq_{\vee} (f^*)^b$.

Indeed, consider $f \in F^{\downarrow}$, then $f = \bigoplus_{(\lambda_i)_I} \delta_{m_i}$ where for each $i \in I$, $m_i \leq_{\vee} \bigvee T_i$. Therefore, we have that $f^b \leq_{\vee} (f^*)^b$.

Therefore,

$$d^b = (d')^b +_{\rho} f^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$$

Thus, $d^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$, $\forall d \in \text{gen}(N_2)$, which implies that

$$\varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_2^{\bar{\dagger}}) = \bigvee_{d \in \text{gen}(N_2)} d^b \leq_{\vee} \varepsilon_{((M, \vee, +\lambda), \sqsubseteq)}(N_1^{\bar{\dagger}})$$

Similarly, we can prove that $\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^\dagger) \leq_\vee \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^\dagger)$. Hence, we have that $\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}$ is well-defined.

- (ii) **Preserves the order relation:** Suppose $N_1^\dagger, N_2^\dagger \in \mathcal{PV}_{Stbl}(M)$ such that $N_1^\dagger \preceq_{EM} N_2^\dagger$. We must show that $\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^\dagger) \sqsubseteq \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^\dagger)$.

Claim 3. Suppose $d, d' \in \mathcal{V}(M)$ such that $d \preceq d'$, then $d^\flat \sqsubseteq (d')^\flat$.

Indeed, let S be the subsemilattice of (M, \vee) generated by the support of d , $supp(d)$. We construct a family of possibly empty subsets of $supp(d')$, indexed by elements of S , as follows: for every $k \in S$, we define $m' \in X_k$ iff $\vee(supp(d) \cup \downarrow_{\sqsubseteq} m') = k$. From this definition we can observe that:

- (1) $\sum_{(d(X_k))_S} k \sqsubseteq (d')^\flat$, and
- (2) $d^\flat \sqsubseteq \sum_{(d(X_k))_S} k$.

Thus, if $d_1 \in gen(N_1)$, there exists $d_2 \in N_2$ such that $d_1 \preceq d_2$. By our previous claim, we obtain $d_1^\flat \sqsubseteq d_2^\flat \sqsubseteq \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^\dagger)$. Thus $\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^\dagger) \sqsubseteq \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^\dagger)$.

- (iii) **Mixed choice morphism:** we must show that the counit is a mixed choice morphism. Given an ordered mixed choice model $((M, \vee, +_\lambda), \sqsubseteq)$ and $N_1^\dagger, N_2^\dagger \in \mathcal{PV}_{Stbl}(M)$,

- (1) A ND-choice morphism.

$$\begin{aligned}
& \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^\dagger \boxplus N_2^\dagger) \\
&= \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}((N_1 \boxplus N_2)^\dagger) \\
&= \bigvee_{d \in gen(N_1 \boxplus N_2)} d^\flat \\
&= \left(\bigvee_{d' \in gen(N_1)} (d')^\flat \right) \vee \left(\bigvee_{d'' \in gen(N_2)} (d'')^\flat \right) \\
&= \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^\dagger) \vee \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^\dagger)
\end{aligned}$$

(2) A **P**-choice morphism.

$$\begin{aligned}
& \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^{\bar{\top}} \oplus_\lambda^* N_2^{\bar{\top}}) \\
&= \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}((N_1 \oplus_\lambda N_2)^{\bar{\top}}) \\
&= \bigvee_{d \in \text{gen}(N_1 \oplus_\lambda N_2)} d^{\flat} \\
&= \bigvee_{d' \in \text{gen}(N_1)} \bigvee_{d'' \in \text{gen}(N_2)} ((d')^{\flat} +_\lambda (d'')^{\flat}) \\
&= \left(\bigvee_{d' \in \text{gen}(N_1)} (d')^{\flat} \right) +_\lambda \left(\bigvee_{d'' \in \text{gen}(N_2)} (d'')^{\flat} \right) \\
&= \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_1^{\bar{\top}}) +_\lambda \varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N_2^{\bar{\top}})
\end{aligned}$$

(c) The required equations are satisfied. Given an ordered semilattice $((S, \vee), \sqsubseteq)$ and $C^{\bar{\top}} \in \mathcal{PV}_{Stbl}(S)$,

$$\begin{aligned}
& \varepsilon_{(\mathcal{PV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*)} \circ \mathcal{PV}_{Stbl} \eta_{((S, \vee), \sqsubseteq)}(C^{\bar{\top}}) \\
&= \varepsilon_{(\mathcal{PV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*)}((\eta_{((S, \vee), \sqsubseteq)}[C]_{cvx})^{\bar{\top}}) \\
&= \bigboxplus_{d \in \text{gen}(\eta_{((S, \vee), \sqsubseteq)}[C]_{cvx})}^* d^{\flat}
\end{aligned}$$

However, we have that $\text{gen}(\eta_{((S, \vee), \sqsubseteq)}[C]_{cvx}) = \{d_{\eta_{((S, \vee), \sqsubseteq)}} \mid d \in \text{gen}(C)\}$, where

$$d_{\eta_{((S, \vee), \sqsubseteq)}} = \bigoplus_{(d(s))_{\text{supp}(d)}} \delta_{(\eta_{((S, \vee), \sqsubseteq)}(s))} = \bigoplus_{(d(s))_{\text{supp}(d)}} \delta_{\{\delta_s\}^{\bar{\top}}}.$$

$$d_{\eta_{((S, \vee), \sqsubseteq)}}^{\flat} = \bigoplus_{(d(s))_{\text{supp}(d)}}^* \{\delta_s\}^{\bar{\top}}$$

Hence,

$$\begin{aligned}
\bigboxplus_{d \in \text{gen}(\eta_{S, \vee}[C]_{cvx})}^* d^{\flat} &= \bigboxplus_{d \in \text{gen}(C)}^* d_{\eta_{((S, \vee), \sqsubseteq)}}^{\flat} \\
&= C^{\bar{\top}}
\end{aligned}$$

Given a mixed choice model $((M, \vee, +_\lambda), \sqsubseteq)$ and $m \in M$,

$$\begin{aligned}
& \mathcal{U}_{\mathcal{PV}_{Stbl} \mathcal{E}(\mathcal{PV}_{Stbl}(M), \boxplus^*, \oplus_\lambda^*)} \circ \eta_{(M, \vee)}(m) \\
&= \mathcal{U}_{\mathcal{PV}_{Stbl} \mathcal{E}(\mathcal{PV}_{Stbl}(M), \boxplus^*, \oplus_\lambda^*)}(\{\delta_m\}^\dagger) \\
&= m
\end{aligned}$$

□

Finally, we show that what we have obtained following the right-hand path is equivalent to the previous constructions following the left hand path.

Theorem 4.4.19. *The composition of the \vee -stable biconvex functor with the convex algebra functor, $\mathcal{PV}_{Stbl} \circ \mathcal{P}$, is isomorphic to the biconvex functor \mathcal{PV} .*

Proof: We define two inverse natural transformations:

- (a) $\phi : (\mathcal{PV}_{Stbl} \circ \mathcal{P}) \Rightarrow \mathcal{PV}$, where for an $(X, \sqsubseteq) \in \mathbf{Poset}$, the ordered mixed choice morphism $\phi_{(X, \sqsubseteq)} : (\mathcal{PV}_{Stbl} \circ \mathcal{P})(X, \sqsubseteq) \rightarrow \mathcal{PV}(X, \sqsubseteq)$ is defined as follows. If $C^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{P}(X))$, we define

$$\phi_{(X, \sqsubseteq)}(C^\dagger) = \overline{\left(\boxplus_{d \in \text{gen}(C)} \left(\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} (\mathcal{V}(\bar{Y})) \right) \right)}.$$

- (i) $\phi_{(X, \sqsubseteq)}$ is well-defined. Suppose $C_1^\dagger, C_2^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{P}(X))$ such that $C_1^\dagger = C_2^\dagger$. We must show that $\phi_{(X, \sqsubseteq)}(C_1^\dagger) = \phi_{(X, \sqsubseteq)}(C_2^\dagger)$. Let $d \in \text{gen}(C_2)$. This implies that $d \in C_1^\dagger$ and thus, that there exists a $Y \in C_1^{\dagger \text{ptl}}$ such that $d \in Y$. We shall show by structural induction on Y that for all $d \in Y, X_d = \overline{\left(\bigoplus_{((d(\bar{Y}))_{\text{supp}(d)})} (\mathcal{V}(\bar{Y})) \right)} \subset \phi_{(X, \sqsubseteq)}(C_1^\dagger)$.

Base Case: Let $Y = C_1$, then $d = \bigoplus_{(\rho_j)_J} d_j$ where $|J| < \infty$, and for each $j \in J, d_j \in \text{gen}(C_1)$. Thus

$$X_d = \overline{\bigoplus_{(\rho_j)_J} X_{d_j}} \subset \phi_{(X, \sqsubseteq)}(C_1^\dagger)$$

Induction Step: Let $Y = \overline{(Y' \boxplus F^\dagger)}$, where $F = \bigoplus_{(\lambda_i)_I} \mathcal{V}(T_i) \in \mathcal{F}^{elt}(\mathcal{P}(X)), Y' \in C_1^{\dagger \text{ptl}}, F \subseteq Y'$ and that by induction hypothesis

we have that for every $d' \in Y'$, $X_{d'} \subseteq \phi_{(X, \sqsubseteq)}(C_1^\dagger)$. Since, $d \in Y = \overline{(Y' \boxplus F^\perp)}$, then there exists $d_1, d_2 \in Y' \boxplus F^\perp$ such that $d_1 \preceq d \preceq d_2$, where for $i \in \{1, 2\}$ there exists $\rho_i \in [0, 1]$, $d'_i \in Y'$ and $f_i \in F^\perp$ such that $d_i = d'_i \oplus_{\rho_i} f_i$.

Claim 1. $\overline{(\bigoplus_{(\lambda_i)_I} (\mathcal{V}(\bigcup T_i)))} \subseteq \phi_{(X, \sqsubseteq)}(C_1^\dagger)$.

For every $\vec{t} \in \prod_I T_i$, consider the distribution $f_{\vec{t}} = \bigoplus_{(\lambda_i)_I} \delta_{\pi_i(\vec{t})} \in F$. Since $F \subseteq Y'$, we have that for every $\vec{t} \in \prod_I T_i$ that $X_{f_{\vec{t}}} \subseteq \phi_{(X, \sqsubseteq)}(C_1^\dagger)$. Therefore, by the convexity of $\phi_{(X, \sqsubseteq)}(C_1^\dagger)$, we have that $\overline{\bigboxplus_{\vec{t} \in \prod_I T_i} X_{f_{\vec{t}}}} \subseteq \phi_{(X, \sqsubseteq)}(C_1^\dagger)$ and by the distributivity of $\overline{\bigoplus_\lambda}$ over $\overline{\bigboxplus}$ we have that

$$\begin{aligned} \overline{\bigboxplus_{\vec{t} \in \prod_I T_i} X_{f_{\vec{t}}}} &= \overline{\bigboxplus_{\vec{t} \in \prod_I T_i} (\bigoplus_{(\lambda_i)_I} \mathcal{V}(\pi_i(\vec{t})))} \\ &= \overline{(\bigoplus_{(\lambda_i)_I} (\bigboxplus_{\vec{Y} \in T_i} \mathcal{V}(\vec{Y})))} \\ &= \overline{(\bigoplus_{(\lambda_i)_I} \mathcal{V}(\bigcup T_i))} \end{aligned}$$

Finally, $f \in F^\perp$ implies that $X_f = \overline{(\bigoplus_{(\lambda_i)_I} \mathcal{V}(\vec{Y}_i))}$ where $\vec{Y}_i \subseteq \bigcup T_i$, therefore, $X_f \subseteq \overline{(\bigoplus_{(\lambda_i)_I} \mathcal{V}(\bigcup T_i))} \subseteq \phi_{(X, \sqsubseteq)}(C_1^\dagger)$. Therefore, since $d = d' \oplus_\rho f$ our result follows.

(ii) $\phi_{(X, \sqsubseteq)}$ is a natural transformation. We prove that the necessary equations hold. Let $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $C^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{P}(X))$.

$$\begin{aligned} \phi_{(X', \sqsubseteq')} \circ \mathcal{PV}_{Stbl}(\mathcal{P}f(C^\dagger)) &= \phi_{(X', \sqsubseteq')}(\overline{(\mathcal{P}f[C]_{cvx})^\dagger}) \\ &= \overline{\bigboxplus_{d' \in \text{gen}(\mathcal{P}f[C]_{cvx})} (\bigoplus_{(d'(\vec{Y}))_{\text{supp}(d')}} \mathcal{V}(\vec{Y}))} \\ &= \overline{\bigboxplus_{d \in \text{gen}(C)} (\bigoplus_{(d(\vec{Y}))_{\text{supp}(d)}} \mathcal{V}(\overline{f[\vec{Y}]}))} \\ &= \mathcal{P}f(\overline{\bigboxplus_{d \in \text{gen}(C)} (\bigoplus_{(d(\vec{Y}))_{\text{supp}(d)}} \mathcal{V}(Y))}) \\ &= \mathcal{P}f \circ \phi_{(X, \sqsubseteq)}(C^\dagger) \end{aligned}$$

(iii) $\phi_{(X, \sqsubseteq)}$ is a mixed choice morphism. Let $C_1^\dagger, C_1^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{P}(X))$. It

preserves the **ND** operations.

$$\begin{aligned}
& \phi_{(X, \sqsubseteq)}(C_1 \bar{\boxplus}^* C_2 \bar{\boxplus}) \\
&= \phi_{(X, \sqsubseteq)}((C_1 \boxplus C_2) \bar{\boxplus}) \\
&= \overline{\left(\bigsqcup_{d \in \text{gen}(C_1 \boxplus C_2)} \left(\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} \nu(\bar{Y}) \right) \right)} \\
&= \overline{\left(\bigsqcup_{d \in \text{gen}(C_1)} \left(\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} \nu(\bar{Y}) \right) \right)} \bar{\boxplus} \overline{\left(\bigsqcup_{d' \in \text{gen}(C_2)} \left(\bigoplus_{(d'(\bar{Y}))_{\text{supp}(d')}} \nu(\bar{Y}) \right) \right)} \\
&= \phi_{(X, \sqsubseteq)}(C_1 \bar{\boxplus}) \bar{\boxplus} \phi_{(X, \sqsubseteq)}(C_2 \bar{\boxplus})
\end{aligned}$$

It preserves the **P** operations.

$$\begin{aligned}
& \phi_{(X, \sqsubseteq)}(C_1 \bar{\boxplus}_\lambda^* C_2 \bar{\boxplus}) \\
&= \phi_{(X, \sqsubseteq)}((C_1 \boxplus_\lambda C_2) \bar{\boxplus}) \\
&= \overline{\left(\bigsqcup_{d \in \text{gen}(C_1 \boxplus_\lambda C_2)} \left(\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} \nu(\bar{Y}) \right) \right)} \\
&= \overline{\left(\bigsqcup_{d \in \text{gen}(C_1)} \left(\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} \nu(\bar{Y}) \right) \right)} \bar{\boxplus}_\lambda \overline{\left(\bigsqcup_{d' \in \text{gen}(C_2)} \left(\bigoplus_{(d'(\bar{Y}))_{\text{supp}(d')}} \nu(\bar{Y}) \right) \right)} \\
&= \phi_{(X, \sqsubseteq)}(C_1 \bar{\boxplus}) \bar{\boxplus}_\lambda \phi_{(X, \sqsubseteq)}(C_2 \bar{\boxplus})
\end{aligned}$$

- (b) $\psi : \mathcal{PV} \Rightarrow (\mathcal{PV}_{\text{Stbl}} \circ \mathcal{P})$, where for an $(X, \sqsubseteq) \in \mathbf{Poset}$, the ordered mixed choice morphism $\psi_{(X, \sqsubseteq)} : \mathcal{PV}((X, \sqsubseteq)) \rightarrow (\mathcal{PV}_{\text{Stbl}} \circ \mathcal{P})((X, \sqsubseteq))$ on $\bar{C} \in \mathcal{PV}(X)$ is given by

$$\psi_{(X, \sqsubseteq)}(\bar{C}) = (\eta_{(X, \sqsubseteq)}[C]_{\text{cvx}}) \bar{\boxplus}.$$

where $\eta : 1_{\mathbf{Poset}} \Rightarrow \mathcal{U}_{\mathcal{P}} \circ \mathcal{P}$ is the unit for the convex algebra monad.

- (i) ψ_X is well-defined. Consider $\bar{C}_1, \bar{C}_2 \in \mathcal{PV}(X)$, and $d \in \text{gen}(C_2)$, this implies that $d \in \bar{C}_1$. Thus, there exists $d_1, d_2 \in C_1$ such that $d_1 \preceq d \preceq d_2$. However, since $\mathcal{P}\eta_{(X, \sqsubseteq)}$ preserves order this implies that $\bigoplus_{((d_1(x)))_{\text{supp}(d_1)}} \delta_{\{x\}} \preceq \bigoplus_{((d(x)))_{\text{supp}(d)}} \delta_{\{x\}} \preceq \bigoplus_{((d_2(x)))_{\text{supp}(d_2)}} \delta_{\{x\}}$, which implies that $\bigoplus_{((d(x)))_{\text{supp}(d)}} \delta_{\{x\}} \in \psi_{(X, \sqsubseteq)}(\bar{C}_1)$.

(ii) ψ is a natural transformation. This follows from the naturality of η .

Let $f : (X, \sqsubseteq) \rightarrow (X', \sqsubseteq') \in \mathbf{Poset}$ and $\overline{C} \in \mathcal{PV}(X)$, then

$$\begin{aligned}
 \mathcal{PV}_{Stbl} \mathcal{P}f \circ \psi_{(X, \sqsubseteq)}(\overline{C}) &= \mathcal{PV}_{Stbl} \mathcal{P}f((\eta_{(X, \sqsubseteq)}[C]_{cvx})^{\dagger}) \\
 &= (\mathcal{P}f[\eta_X[C]_{cvx}]_{cvx})^{\dagger} \\
 &= ((\mathcal{P}f \circ \eta_{(X, \sqsubseteq)})[C]_{cvx})^{\dagger} \\
 &= ((\eta_{(X', \sqsubseteq')} \circ f)[C]_{cvx})^{\dagger} \\
 &= \psi_{(X', \sqsubseteq')}(\overline{(f[C]_{cvx})}) \\
 &= \psi_{(X', \sqsubseteq')} \circ \mathcal{P}f(\overline{C}).
 \end{aligned}$$

(iii) $\psi_{(X, \sqsubseteq)}$ is a mixed choice morphism. Let $\overline{C}_1, \overline{C}_2 \in \mathcal{PV}(X)$.

It preserves ND operations

$$\begin{aligned}
 \psi_{(X, \sqsubseteq)}(\overline{(C_1 \boxplus C_2)}) &= (\eta_{(X, \sqsubseteq)}[C_1 \boxplus C_2]_{cvx})^{\dagger} \\
 &= (\eta_{(X, \sqsubseteq)}[C_1]_{cvx} \boxplus \eta_{(X, \sqsubseteq)}[C_2]_{cvx})^{\dagger} \\
 &= (\eta_{(X, \sqsubseteq)}[C_1])^{\dagger} \boxplus^* (\eta_{(X, \sqsubseteq)}[C_2]_{cvx})^{\dagger} \\
 &= \psi_{(X, \sqsubseteq)}(\overline{C_1}) \boxplus^* \psi_{(X, \sqsubseteq)}(\overline{C_2})
 \end{aligned}$$

It preserves P operations.

$$\begin{aligned}
 \psi_{(X, \sqsubseteq)}(\overline{(C_1 \oplus_{\lambda} C_2)}) &= (\eta_{(X, \sqsubseteq)}[C_1 \oplus_{\lambda} C_2]_{cvx})^{\dagger} \\
 &= (\eta_{(X, \sqsubseteq)}[C_1]_{cvx} \oplus_{\lambda} \eta_{(X, \sqsubseteq)}[C_2]_{cvx})^{\dagger} \\
 &= (\eta_{(X, \sqsubseteq)}[C_1])^{\dagger} \oplus_{\lambda}^* (\eta_{(X, \sqsubseteq)}[C_2]_{cvx})^{\dagger} \\
 &= \psi_{(X, \sqsubseteq)}(\overline{C_1}) \oplus_{\lambda}^* \psi_{(X, \sqsubseteq)}(\overline{C_2})
 \end{aligned}$$

(c) These natural transformations are inverses.

(i) Let $\overline{C} \in \mathcal{PV}(X)$.

$$\begin{aligned}
 \phi_{(X, \sqsubseteq)} \circ \psi_{(X, \sqsubseteq)}(\overline{C}) &= \phi_{(X, \sqsubseteq)}((\eta_{(X, \sqsubseteq)}[C]_{cvx})^{\dagger}) \\
 &= \left(\bigsqcup_{d \in \text{gen}(\eta_{(X, \sqsubseteq)}[C]_{cvx})} \left(\bigoplus_{(d(\{x\}))_{\text{supp}(d)}} \mathcal{V}(\{x\}) \right) \right) \\
 &= \left(\bigsqcup_{d \in \text{gen}(C)} \left(\bigoplus_{(d(x))_{\text{supp}(d)}} \mathcal{V}(\{x\}) \right) \right) \\
 &= \overline{C}
 \end{aligned}$$

(ii) Let $C^\dagger \in \mathcal{PV}_{Stbl}(\mathcal{P}(X))$.

$$\begin{aligned}
\psi_{(X, \sqsubseteq)} \circ \phi_{(X, \sqsubseteq)}(C^\dagger) &= \psi_{(X, \sqsubseteq)}(\overline{\bigsqcup_{d \in \text{gen}(C)} (\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}} \mathcal{V}(\bar{Y}))})}) \\
&= \bigsqcup_{d \in \text{gen}(C)}^* (\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}}^* \psi_{(X, \sqsubseteq)}(\overline{\mathcal{V}(\bar{Y})})) \\
&= \bigsqcup_{d \in \text{gen}(C)}^* (\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}}^* (\eta_{(X, \sqsubseteq)}[\mathcal{V}(\bar{Y})]_{\text{cvx}})^\dagger) \\
&= \bigsqcup_{d \in \text{gen}(C)}^* (\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}}^* (\mathcal{V}(\{\{y\} \mid y \in \bar{Y}\}))^\dagger) \\
&= \bigsqcup_{d \in \text{gen}(C)}^* (\bigoplus_{(d(\bar{Y}))_{\text{supp}(d)}}^* \{\delta_{\bar{Y}}\}^\dagger) \\
&= C^\dagger
\end{aligned}$$

□

In what follows, we present the construction necessary to obtain the corresponding \vee -stable functors associated to the convex lower and convex upper ND algebra functors. The approach is very similar to the one we have just shown for the biconvex algebra functor. In each case we must refine our original \vee -stable convex sets used as models over **Set** in order that they are also either downclosed or upclosed under \preceq .

Remark 4.4.20. For the remaining two constructions, the lower \vee -stable functor and the upper \vee -stable functor, we omit the proofs which are routine modifications of the previous proofs presented when defining the \vee -stable biconvex functor.

Definition 4.4.21 (Lower \vee -Stable). Suppose we are given an ordered semilattice $((S, \sqsubseteq), \vee)$, a convex subset $C \in \text{Cvx}(S)$ is **lower \vee -stable** if C is \vee -stable (i.e., for every facial element $F \in \mathcal{F}^{elt}(S)$, $F \subseteq C \Rightarrow F^\perp \subseteq C$) and downclosed under \preceq (i.e., for any $d \in \mathcal{V}((S, \sqsubseteq))$, if there exists $d' \in C$ such that $d \preceq d'$, then $d \in C$).

Definition 4.4.22 (Lower \vee -Stable Closure). Suppose we are given an ordered semilattice $((S, \sqsubseteq), \vee)$ and a convex subset $C \in \text{Cvx}(S)$. The **lower \vee -stable closure of C under \preceq** , denoted by C^{dstbl} , is the smallest convex subset of $\mathcal{V}((S, \sqsubseteq))$ which is both \vee -stable and downclosed under \preceq .

Proposition 4.4.23 (Mixed Choice Model). *Suppose we are given an ordered semi-lattice $((S, \vee), \sqsubseteq)$. The set of all lower \vee -stable closures on $\mathcal{G}_{cvx}(S)$, $\mathcal{HV}_{Stbl}(S)$, together with the operations \boxplus^* and \oplus_λ^* , the lower \vee -stable closures of \boxplus and \oplus_λ from $\mathcal{G}_{cvx}(S)$, forms a mixed choice model $(\mathcal{HV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*)$ over **Poset** when equipped with the inclusion ordering.*

Definition 4.4.24 (\vee -Stable Lower Convex Functor). The \vee -stable lower convex functor, $\mathcal{HV}_{Stbl} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $((S, \sqsubseteq), \vee) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$,

$$\mathcal{HV}_{Stbl}(((S, \sqsubseteq), \vee)) = ((\mathcal{HV}_{Stbl}(S), \boxplus^*, \oplus_\lambda^*), \sqsubseteq).$$

- (i) $\mathcal{HV}_{Stbl}(S)$ is the set of all lower \vee -stable closures of the finitely generated convex subsets over (S, \sqsubseteq) (i.e., $\mathcal{HV}_{Stbl}(S) = \{C^{dstbl} \mid C \in \mathcal{G}_{cvx}(S)\}$, where C^{dstbl} is the smallest downclosed, \vee -stable convex subset over $((S, \vee), \sqsubseteq)$ containing C).
- (ii) \boxplus^* and \oplus_λ^* are the lower \vee -stable closures of \boxplus and \oplus_λ .

$$C_1^{dstbl} \boxplus^* C_2^{dstbl} = (C_1 \boxplus C_2)^{dstbl}$$

$$C_1^{dstbl} \oplus_\lambda^* C_2^{dstbl} = (C_1 \oplus_\lambda C_2)^{dstbl}$$

(b) **On morphisms:** Let $f : ((S, \sqsubseteq), \vee) \rightarrow ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $C^{dstbl} \in \mathcal{HV}_{Stbl}(S)$,

$$\mathcal{HV}_{Stbl}f(C^{dstbl}) = (f[C]_{cvx})^{dstbl}.$$

Theorem 4.4.25. *The \vee -stable lower convex functor $\mathcal{HV}_{Stbl} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is left adjoint to the forgetful functor $\mathcal{U}_{\mathcal{HV}_{Stbl}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. Thus it forms a monad $\mathbb{H}\mathcal{V}_{Stbl} = (\mathcal{U}_{\mathcal{HV}_{Stbl}} \circ \mathcal{HV}_{Stbl}, \eta, \mu)$ over $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. We include the adjunction and monad structures:*

(a) **The unit** (η) : for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $s \in S$,

$$\eta_{((S, \vee), \sqsubseteq)}(s) = \{\delta_s\}^{dstbl}.$$

(b) **The counit**(ε): for $((M, \vee, +_\lambda), \sqsubseteq) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ and $N^{dstbl} \in \mathcal{HV}_{Stbl}(M)$,

$$\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N^{dstbl}) = \bigvee_{d \in \text{gen}(N)} \left(\sum_{(d(m))_{\text{supp}(d)}} m \right).$$

(c) **The multiplication**(μ): for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $\mathcal{N}^{dstbl} \in \mathcal{HV}_{Stbl}(\mathcal{HV}_{Stbl}(S))$,

$$\mu_{((S, \vee), \sqsubseteq)}(\mathcal{N}^{dstbl}) = \boxplus_{D \in \text{gen}(\mathcal{N})}^* \left(\bigoplus_{(D(C^{dstbl}))_{\text{supp}(D)}}^* C^{dstbl} \right).$$

Theorem 4.4.26. *The composition of the \vee -stable lower convex functor with the Hoare algebra functor, $\mathcal{HV}_{Stbl} \circ \mathcal{H}$, is isomorphic to the lower convex functor \mathcal{HV} .*

Definition 4.4.27 (Upper \vee -Stable). Given an ordered semilattice $((S, \sqsubseteq), \vee)$. A convex subset $C \in \mathcal{Cvx}(S)$ is **upper \vee -stable** if C is \vee -stable (i.e., for every facial element $F \in \mathcal{F}^{elt}(S)$, $F \subseteq C \Rightarrow F^\downarrow \subseteq C$) and upclosed under \preceq (i.e., for any $d \in \mathcal{V}((S, \sqsubseteq))$, if there exists $d' \in C$ such that $d' \preceq d$, then $d \in C$).

Definition 4.4.28 (Upper \vee -Stable Closure). Given an ordered semilattice $((S, \sqsubseteq), \vee)$ and a convex subset $C \in \mathcal{Cvx}(S)$. The **upper \vee -stable closure of C under \preceq** , denoted by C^{rustbl} , is the smallest convex subset of $\mathcal{V}((S, \sqsubseteq))$ which is both \vee -stable and upclosed under \preceq .

Proposition 4.4.29 (Mixed Choice Model). *Suppose we are given an ordered semilattice $((S, \vee), \sqsubseteq)$. The set of all upper \vee -stable closures on $\mathcal{G}_{cvx}(S)$, $\mathcal{SV}_{Stbl}(S)$, together with the operations \boxplus^* and \bigoplus_λ^* , the upper \vee -stable closures of \boxplus and \bigoplus_λ from $\mathcal{G}_{cvx}(S)$, forms a mixed choice model $(\mathcal{SV}_{Stbl}(S), \boxplus^*, \bigoplus_\lambda^*)$ over \mathbf{Poset} when equipped with the reverse inclusion ordering.*

Definition 4.4.30 (\vee -Stable Upper Convex Functor). The **\vee -stable upper convex functor**, $\mathcal{SV}_{Stbl} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is defined as follows:

(a) **On objects:** Given $((S, \sqsubseteq), \vee) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$,

$$\mathcal{SV}_{Stbl}(((S, \sqsubseteq), \vee)) = ((\mathcal{SV}_{Stbl}(S), \boxplus^*, \bigoplus_\lambda^*), \supseteq).$$

- (i) $\mathcal{SV}_{Stbl}(S)$ is the set of all upper \vee -stable closures of the finitely generated convex subsets over (S, \sqsubseteq) (i.e., $\mathcal{SV}_{Stbl}(S) = \{C^{ustbl} \mid C \in \mathcal{G}_{cvx}(S)\}$, where C^{ustbl} is the smallest upclosed, \vee -stable convex subset over $((S, \vee), \sqsubseteq)$ containing C).
- (ii) \boxplus^* and \oplus_λ^* are the upper \vee -stable closures of \boxplus and \oplus_λ .

$$\begin{aligned} C_1^{ustbl} \boxplus^* C_2^{ustbl} &= (C_1 \boxplus C_2)^{ustbl} \\ C_1^{ustbl} \oplus_\lambda^* C_2^{ustbl} &= (C_1 \oplus_\lambda C_2)^{ustbl} \end{aligned}$$

- (b) **On morphisms:** Let $f : ((S, \sqsubseteq), \vee) \rightarrow ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $C^{ustbl} \in \mathcal{SV}_{Stbl}(S)$,

$$\mathcal{SV}_{Stbl}f(C^{ustbl}) = (f[C]_{cvx})^{ustbl}.$$

Theorem 4.4.31. *The \vee -stable upper convex functor $\mathcal{SV}_{Stbl} : \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ is left adjoint to the forgetful functor $\mathcal{U}_{\mathcal{SV}_{Stbl}} : \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset}) \rightarrow \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. Thus it forms a monad $\mathbb{S}\mathcal{V}_{Stbl} = (\mathcal{U}_{\mathcal{SV}_{Stbl}} \circ \mathcal{SV}_{Stbl}, \eta, \mu)$ over $\mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$. We include the adjunction and monad structures:*

- (a) **The unit**(η): for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $s \in S$,

$$\eta_{((S, \vee), \sqsubseteq)}(s) = \{\delta_s\}^{ustbl}.$$

- (b) **The counit**(ε): for $((M, \vee, +_\lambda), \sqsubseteq) \in \mathbf{Mod}(\mathbf{MC}_{fin}, \mathbf{Poset})$ and $N^{ustbl} \in \mathcal{SV}_{Stbl}(M)$,

$$\varepsilon_{((M, \vee, +_\lambda), \sqsubseteq)}(N^{ustbl}) = \bigvee_{d \in \mathit{gen}(N)} \left(\sum_{(d(m))_{\mathit{supp}(d)}} m \right).$$

- (c) **The multiplication**(μ): for $((S, \vee), \sqsubseteq) \in \mathbf{Mod}(\mathbf{ND}_{fin}^*, \mathbf{Poset})$ and $\mathcal{N}^{ustbl} \in \mathcal{SV}_{Stbl}(\mathcal{SV}_{Stbl}(S))$,

$$\mu_{((S, \vee), \sqsubseteq)}(\mathcal{N}^{ustbl}) = \boxplus_{D \in \mathit{gen}(\mathcal{N})}^* \left(\bigoplus_{(D(C^{ustbl}))_{\mathit{supp}(D)}}^* C^{ustbl} \right).$$

Theorem 4.4.32. *The composition of the \vee -stable upper convex functor with the Smith algebra functor, $\mathcal{SV}_{Stbl} \circ S$, is isomorphic to the upper convex functor $S\mathcal{V}$.*

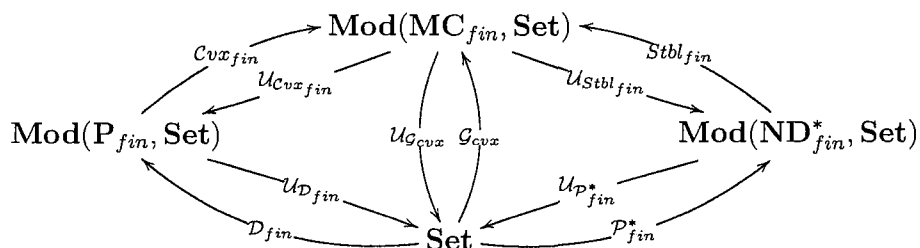
Chapter 5

Conclusion

We began our work by presenting an in-depth study of the theories of nondeterministic choice and probabilistic choice and we showed how to appropriately combine them in order to obtain a theory of mixed choice. We have considered many types of mixed choice depending upon the structure of each individual choice theory we were combining. We've discussed the theory of mixed choice which arises from the combination of the finite choice theories (\mathbf{P}_{fin} with \mathbf{ND}_{fin}^*) and those arising from the combination of the ordered theories (\mathbf{P}_{fin} with \mathbf{H}_{fin} or \mathbf{S}_{fin}). Moreover, we presented a new mixed choice theory based on combining infinitary variants of each individual choice theory (\mathbf{P} with \mathbf{ND}^*). Consequently, we presented a new type of convex structure, called superconvex sets, in order to describe the models of the infinite variant of the probabilistic choice theory.

For each mixed choice theory considered we first constructed their associated monad using the classical approach presented in the literature. In our framework this method amounts to ascending up the left-hand path in our reoccurring diagram

shown below.



As we have seen, this construction allows not only for the construction of free models for mixed choice over an arbitrary set, but also for the construction of free models for mixed choice over arbitrary models of probabilistic choice (i.e., over any convex set). Alternatively, we motivated and proved the existence of a second approach to construct free models of mixed choice which corresponds to ascending the right-hand side of the above diagram. By exploiting a result presented by Barr and Wells [2] on the existence of left adjoints, we determined that such an approach was possible. We've shown that this dual approach is equivalent to the method obtained by following the left-hand path, thus both methods construct the same free model for mixed choice theories over any appropriate category. However, as an important consequence of our approach, we were able to construct free models for mixed choice theories over arbitrary models for nondeterministic choice (i.e., over any semilattice).

After establishing the existence of the necessary functor, we presented the fundamental notions necessary to develop the concrete definitions of the left adjoint shown in the north-east corner of the above diagram, called the \vee -stable functor. We developed a new property on convex sets generated over a set equipped with a semilattice structure, called \vee -stability. This property was key in our definition, since it provided a convex set which preserved the inherent \vee -structure already present in the underlying set. Then we presented a closure operation induced by the \vee -stable property which enabled us to define the required \vee -stable functor. Finally, we showed how to modify the notion of \vee -stability, in order to capture each different type of mixed choice theory.

Finally, as an aside in the appendix, we provide an example of a mixed choice process calculus (called MCCS) equipped with an alternating operational semantics.

We end by sketching the necessary framework, in order to obtain a fully-abstract set-theoretical model for this calculus.

In the future, it would be interesting to apply the \vee -stable functor on existing non-standard models of nondeterminism, in particular those associated to process calculi which admit a non-standard definition of nondeterminism. This would provide a free mixed choice extension of such models, which could not previously be obtained by the available machinery.

One important aspect of combining mixed choice was how to properly capture the interaction between the nondeterministic and probabilistic choice operators, given by considering the distributive tensor between their respective Lawvere theories. We would be interested in generalizing our work to encompass not only the combining of nondeterminism and probability, but for any distributive combination of theories. In order to develop concrete definitions to the monads associated to their respective theories.

There is still a lot of directions we can consider for generalizing our construction. One such direction is to further develop our notion of \vee -stability over the category of domains \mathbf{Dom} , in order to extend the results already obtained for mixed choice models over ordered convex sets to ordered semilattices.

A second direction we can consider is to incorporate a higher degree of generalization in the probabilistic choice theory by considering continuous probabilistic combinations, instead of finite ones. In this case, we expect our discrete sums to be replaced by integrals over appropriate measures, as in the work of Mislove [32] and Danos, Desharnais and Panangaden [9].

One idea would be to consider the existing monad over \mathbf{Mes} associated to continuous probabilistic combinations as presented by Giry [11]. A major complication to this approach is that unlike our discrete distributions monad, \mathcal{D}_{fin} , the Giry monad does not admit a complete characterization for its algebras. There have been attempts to characterize the algebras for the Giry monad applied to subcategories of \mathbf{Mes} . In particular, Doberkat [10], fully characterizes the algebras of the Giry monad over Polish Spaces and continuous maps. However, this subcategory falls short in capturing the required generality of our setting. For these reasons, we believe more progress must

be done in studying the precise subcategory necessary to determine a characterization of the algebras of the Giry monad with the necessary level of generality.

Appendix A

A Mixed Choice Process Language

A.1 Mixed Choice CCS

In this section we sketch a development of a variant of Milner's CCS [28], called the *mixed choice calculus* (=MCCS). This calculus will admit an internal sum, an external sum and a probabilistic choice operator. We define a strictly alternating operational semantics for the calculus and show that the sub-calculus of probabilistic processes forms a model for the theory of mixed choice. Since this is a sketch, we omit the proofs.

Definition A.1.1 (Mixed Choice CCS). Let \mathcal{N} be a countable set of labels, define the set $Name = \mathcal{N} \cup \overline{\mathcal{N}} \cup \{\tau\}$, where \mathcal{N} represents input prefixes, $\overline{\mathcal{N}} = \{\bar{x} \mid x \in \mathcal{N}\}$ represents output prefixes and $\tau \notin \mathcal{N}$ represents an internal action prefix. The processes of the **mixed choice CCS**, (= MCCS) are defined by mutual recursion as follows: Let $\alpha \in Name$ and $a \in \mathcal{N}$,

$$\begin{aligned} N & ::= 0 \quad | \quad \alpha.P \quad | \quad N + N' \quad | \quad (\nu a)N \quad | \quad N|N' \\ P & ::= (1)N \quad | \quad P \oplus_\lambda P' \quad | \quad P \boxplus P' \quad | \quad (\nu a)P \quad | \quad P|P' \end{aligned}$$

Processes of the form P are **probabilistic state** processes and processes of the form N are **nondeterministic state** processes. We define $P\text{-Proc}$ as the subset of the language consisting of all probabilistic state processes, $ND\text{-Proc}$ as the subset of all nondeterministic state processes and $MC\text{-Proc}$ the set of all processes.

Definition A.1.2 (MCCS Alternating Operational Semantics). We define the strictly alternating operational semantics for MCCS as follows:

- (a) **For nondeterministic state processes** (as seen in Figure 16): in the alternating operational semantics a nondeterministic state process will evolve into a probabilistic state process. We use $Name$, as the set of transition labels for nondeterministic transitions.

$\frac{}{\alpha.P \xrightarrow{\alpha} P} \text{ (PRE)}$	$\frac{N \xrightarrow{\alpha} P \quad \alpha \neq a, \bar{a}}{(\nu a)N \xrightarrow{\alpha} (\nu a)P} \text{ (ND-RES)}$
$\frac{N_1 \xrightarrow{\alpha} P}{N_1 + N_2 \xrightarrow{\alpha} P} \text{ (EXTSUM1)}$	$\frac{N_2 \xrightarrow{\alpha} P}{N_1 + N_2 \xrightarrow{\alpha} P} \text{ (EXTSUM2)}$
$\frac{N_1 \xrightarrow{\alpha} P}{N_1 N_2 \xrightarrow{\alpha} P (1)N_2} \text{ (ND-PAR1)}$	$\frac{N_2 \xrightarrow{\alpha} P}{N_1 N_2 \xrightarrow{\alpha} (1)N_1 P} \text{ (ND-PAR2)}$
$\frac{N_1 \xrightarrow{\alpha} P_1 \quad N_2 \xrightarrow{\bar{\alpha}} P_2}{N_1 N_2 \xrightarrow{\tau} P_1 P_2} \text{ (COM)}$	

Figure 16: Operational Semantics for Nondeterministic State Processes

- (b) **For probabilistic state processes** (as seen in Figure 17): in the alternating operational semantics, a probabilistic state process will evolve into a nondeterministic state process according to a probabilistic distribution. Thus we choose $\kappa \notin Name$ to be the label for a probabilistic transition (which can be thought of as the toss of a biased coin) and $\mu, \mu_i, \mu' \in \mathcal{D}_{fin}(ND-Proc)$ probabilistic distributions over nondeterministic processes (i.e., $\mu, \mu_i, \mu' : ND-Proc \rightarrow [0, 1]$). In particular, a process P which can either evolve to process N_1 with probability $\frac{1}{3}$ or N_2 with probability $\frac{2}{3}$ will make the following transition $P \xrightarrow{\kappa} \mu$, where $\mu(N_1) = \frac{1}{3}, \mu(N_2) = \frac{2}{3}$ and for all other nondeterministic state processes $N \in ND-Proc, \mu(N) = 0$.

The semantic rules for ND state processes are defined as usual. As we can see,

$\frac{-}{(1)N \xrightarrow{\kappa} \delta_N} \text{ (ONE)}$	$\frac{P \xrightarrow{\kappa} \mu \quad P \xrightarrow{\kappa} \mu' \quad \lambda \in [0, 1]}{P \xrightarrow{\kappa} \mu \oplus_\lambda \mu'} \text{ (CONV)}$
$\frac{P \xrightarrow{\kappa} \mu}{(\nu a)P \xrightarrow{\kappa} (\nu a)\mu} \text{ (P-RES)}$	$\frac{P_1 \xrightarrow{\kappa} \mu_1 \quad P_2 \xrightarrow{\kappa} \mu_2}{P_1 \oplus_\lambda P_2 \xrightarrow{\kappa} \mu_1 \oplus_\lambda \mu_2} \text{ (CHOICE)}$
$\frac{P_1 \xrightarrow{\kappa} \mu_1 \quad P_2 \xrightarrow{\kappa} \mu_2}{P_1 \boxplus P_2 \xrightarrow{\kappa} \mu_1 \boxplus \mu_2} \text{ (INTSUM)}$	$\frac{P_1 \xrightarrow{\kappa} \mu_1 \quad P_2 \xrightarrow{\kappa} \mu_2}{P_1 P_2 \xrightarrow{\kappa} \mu_1 \mu_2} \text{ (P-PAR)}$
<p>(P-RES): $(\nu a)\mu = \sum \mu(N)\delta_{(\nu a)N}$,</p>	
<p>(CONV/CHOICE): $\mu_1 \oplus_\lambda \mu_2 = (\lambda)\mu_1 + (1 - \lambda)\mu_2$</p>	
<p>(INTSUM): $\mu_1 \boxplus \mu_2 = (\Lambda)\mu_1 + (1 - \Lambda)\mu_2$ such that Λ is a fresh probabilistic variable,</p>	
<p>(P-PAR): $\mu_1 \mu_2 = \sum \mu_1(N_1)\mu_2(N_2)\delta_{N_1 N_2}$.</p>	

Figure 17: Operational Semantics for Probabilistic State Processes

(EXTSUM1) and (EXTSUM2) impose an external behavior on the nondeterministic sum operator $+$, since the next state of such a process will be determined first by its environment, then by the scheduler. The only rules which have been modified from their usual formulations are (ND-PAR1) and (ND-PAR2). Since the semantics must remain alternating, the interleaving property of the parallel composition of ND state processes must be defined carefully. We can not make a transition into the parallel composition of mixed terms. For this reason we always lift the inactive component into its probabilistic image under guaranteed choice.

The semantic rules for probabilistic state processes merit some clarification. The rules (ONE) and (CHOICE) are chosen to be the obvious interpretation of a unary or binary probabilistic choice. The rule (INTSUM) is faithful to our intuitive definition of the internal nondeterministic sum. We have previously stated that such a process should have the capability to behave as any possible distribution between its terms. Thus, this rule implies that any internal nondeterministic sum process must always have an uncountable number of possible transitions. The rule (P-RES) simply states

that the restriction operator on the process $(\nu a)P$ is “pushed” onto the support of the distributions of P . Finally, one could interpret a rule (**P-PAR'**) which states that if P_1, P_2 have transitions into μ_1, μ_2 respectively, then $P_1|P_2$ will have a transition into a distribution μ whose support consists of processes of the form $N_1|N_2$ where N_i is in the support of μ_i . The associated probabilistic weight for $N_1|N_2$ is the product of the weights for N_1 and N_2 in their respective distributions, i.e. $\mu(N_1|N_2) = \mu_1(N_1)\mu_2(N_2)$. However, due to the possible presence of internal nondeterministic sums in both terms of the parallel composition, (**P-PAR'**) does not capture all possible transitions for P . For example, consider a process of the form $P = P_1|P_2$ where $P_i = (1)a_i \boxplus (1)b_i$. Given that $(1)a_i \boxplus (1)b_i \xrightarrow{\kappa} (\rho_i)\delta_{a_i} + (1-\rho_i)\delta_{b_i}$ for any $\rho_i \in [0, 1]$, we have that by (**INTSUM**) $P \xrightarrow{\kappa} \mu = (\rho_1)(\rho_2)\delta_{a_1|a_2} + (\rho_1)(1-\rho_2)\delta_{a_1|b_2} + (1-\rho_1)(\rho_2)\delta_{b_1|a_2} + (1-\rho_1)(1-\rho_2)\delta_{b_1|b_2}$. Through many runs of the process P , the scheduler could have chosen to place in parallel $(1)a_1$ and $(1)a_2$ a third of the time, $(1)a_1$ and $(1)b_2$ another third of the time and $(1)b_1$ and $(1)b_2$ for the remaining third. Thus P must be able to transition into the distribution $\frac{1}{3}\delta_{a_1|a_2} + \frac{1}{3}\delta_{a_1|b_2} + \frac{1}{3}\delta_{b_1|b_2}$. This is not a possible outcome using (**P-PAR'**), for any choice of ρ_1, ρ_2 in the distribution μ . However, it is a convex combination of two such simple transitions (i.e. $\frac{2}{3} \sum \delta_{a_1}(N_1)(\frac{1}{2}\delta_{a_2} + \frac{1}{2}\delta_{b_2})(N_2)\delta_{N_1|N_2} + \frac{1}{3} \sum \delta_{b_1}(N_1)\delta_{b_2}(N_2)\delta_{N_1|N_2}$). Thus, we have defined the rule (**P-PAR**) above as the convex closure of the transitions arising from (**P-PAR'**).

Next we define a congruence relation on **MCCS** by combining notions of bisimulation arising in nondeterministic calculi and probabilistic calculi. The following definition is an adaptation from the definition presented in Bandini and Segala [1].

Definition A.1.3 (Mixed Choice Bisimulation). An equivalence relation \mathcal{S} , is a **mixed choice bisimulation** relation if it satisfies the following conditions:

- (a) Whenever NSN' ,
 - (i) if $N \xrightarrow{\alpha} P$ then there exists a P' such that $N' \xrightarrow{\alpha} P'$ and PSP' .
 - (ii) if $N' \xrightarrow{\alpha} P'$ then there exists a P such that $N \xrightarrow{\alpha} P$ and $P'SP$.
- (b) Whenever PSP' ,

- (i) if $P \xrightarrow{\kappa} \mu$, then there exists a μ' such that $P' \xrightarrow{\kappa} \mu'$ and $(\forall N \in ND\text{-Proc})$
 $\sum_{\{N' | NSN'\}} \mu(N') = \sum_{\{N' | NSN'\}} \mu'(N')$.
- (ii) if $P' \xrightarrow{\kappa} \mu'$, then there exists a μ such that $P \xrightarrow{\kappa} \mu$ and $(\forall N \in ND\text{-Proc})$
 $\sum_{\{N' | NSN'\}} \mu(N') = \sum_{\{N' | NSN'\}} \mu'(N')$.

We say R and R' are **mixed choice bisimilar**, denoted $R \cong_{mc} R'$, if there exists a mixed bisimulation \mathcal{S} such that RSR' .

Proposition A.1.4. *The mixed choice bisimulation forms a congruence relation for MCCS. The probabilistic state processes of the MCCS syntax modulo the mixed choice bisimulation forms a model for the mixed choice theory.*

A.2 A Set-Theoretical Model for MCCS

Motivated by work of Fiore, Moggi and Sangiorgi [13] and Ian Stark [47], we give a fully-abstract set-theoretic model for MCCS. Even though the cited authors works involved defining fully-abstract presheaf models for the π -calculus, we use similar techniques in order to define our model.

We distinguish the following important sets for our model,

- (a) The **object of labels**, $L = \{a_1, a_2, \dots\}$, which will be used to construct the object of prefixes.
- (b) The **object of prefixes**, Pre , will model the set of prefixes available in our language. It is defined such that

$$Pre = \underbrace{L}_{\text{input}} + \underbrace{L}_{\text{output}} + \underbrace{1}_{\{\tau\}}$$

where we will denote input action labels as a_1, a_2, \dots , output action labels as $\bar{a}_1, \bar{a}_2, \dots$ and let τ label internal actions.

- (c) The *object of nondeterministic state processes*, Nd , and the *object of probabilistic state processes*, Pb , are sets representing the set of nondeterministic

and probabilistic state processes respectively. They will be defined such that (Nd, Pb) is the simultaneous least fixed point of the following equations:

$$\begin{aligned}\mathcal{F}_1 X &= \mathbb{P}_{fin}^*(Pre \times \mathcal{F}_2 X) \\ \mathcal{F}_2 Y &= \mathbb{G}_{cvx}(\mathcal{F}_1 Y)\end{aligned}$$

It is important to note that we use the monad \mathbb{P}_{fin}^* in the equation \mathcal{F}_1 in order to model nondeterministic state processes, since they admit an external non-deterministic sum. We have already seen that the probabilistic state processes form a model for mixed choice, i.e. contain a probabilistic choice and an internal nondeterministic sum. For this reason we use the monad \mathbb{G}_{cvx} in the equation for \mathcal{F}_2 .

We construct the following set of functions from the above objects to define the interpretation of the M CCS operations.

- (a) The **guaranteed choice** function $guar : Nd \rightarrow Pb$, such that

$$guar(X) = \text{conv}(\{\delta_X\}).$$

- (b) The n -ary **probabilistic choice** functions $pchoice : S \times (Pb)^n \rightarrow Pb$, where $S = \{(\lambda_i) \in ([0, 1])^n \mid \sum_{i=1}^n \lambda_i = 1\}$, such that

$$pchoice(\vec{\lambda}, \vec{Y}) = \text{conv}(\{(\lambda_1)\mu_1 + \dots + (\lambda_n)\mu_n \mid \mu_i \in \text{gen}(Y_i)\}).$$

- (c) The n -ary **internal nondeterministic sum** function $intsum : (Pb)^n \rightarrow Pb$, such that

$$intsum(\vec{Y}) = \text{conv}\left(\bigcup_{i=1}^n \text{gen}(Y_i)\right).$$

- (d) The **probabilistic restriction** function $pres : L \times Pb \rightarrow Pb$, such that

$$pres(a, Y) = \text{conv}(\{pres(a, \mu) \mid \mu \in \text{gen}(Y)\}),$$

where $pres(a, \mu) = \sum_{X \in Nd} \mu(X) \delta_{nres(a, X)}$.

- (e) The **probabilistic parallel composition** function $ppar : Pb \times Pb \rightarrow Pb$, such that

$$ppar(Y_1, Y_2) = \text{conv}(\{ppar(\mu_1, \mu_2) \mid \mu_i \in \text{gen}(Y_i)\}),$$

$$\text{where } ppar(\mu_1, \mu_2) = \sum_{\{(X_1, X_2) \mid X_i \in Nd\}} \mu_1(X_1) \mu_2(X_2) \delta_{npar(X_1, X_2)}.$$

- (f) The **prefixing** function $pre : Pre \times Pb \rightarrow Nd$, such that

$$pre(\alpha, Y) = \{(\alpha, Y)\}.$$

- (g) The **nondeterministic restriction** function $nres : L \times Nd \rightarrow Nd$, such that

$$nres(a, X) = \{(\alpha, pres(a, Y)) \mid \alpha \neq a, \bar{a} \text{ and } (\alpha, Y) \in X\}.$$

- (h) The n -ary **external nondeterministic sum** functions $endsum : (Nd)^n \rightarrow Nd$, such that

$$endsum(\vec{X}) = \bigcup_{i=1}^n X_i.$$

- (i) The **left merge** function $lmerge : Nd \times Nd \rightarrow Nd$, such that

$$lmerge(X_1, X_2) = \{(\alpha, ppar(Y, guar(X_2))) \mid (\alpha, Y) \in X_1\}.$$

- (j) The **right merge** function $rmerge : Nd \times Nd \rightarrow Nd$, such that

$$rmerge(X_1, X_2) = \{(\alpha, ppar(guar(X_1), Y)) \mid (\alpha, Y) \in X_2\}.$$

- (k) The **synchronization** function $synch : Nd \times Nd \rightarrow Nd$, such that

$$synch(X_1, X_2) = \{(\tau, ppar(Y_1, Y_2)) \mid (\alpha, Y_1) \in X_1 \text{ and } (\bar{\alpha}, Y_2) \in X_2\}.$$

- (l) The **nondeterministic parallel composition** function $npar : Nd \times Nd \rightarrow Nd$ such that

$$npar(X_1, X_2) = \text{endsum}(lmerge(X_1, X_2), rmerge(X_1, X_2), synch(X_1, X_2)).$$

Lemma A.2.1. *Given $P_i, P \in P\text{-Proc}$, the functions ppar and pres defined on discrete valuations, as described above, preserve convex combinations of discrete valuations.*

Given the above specifications we can now define an interpretation map from the mixed choice processes to the objects in our model.

Definition A.2.2. Define the interpretation map, $\llbracket - \rrbracket : MC\text{-Proc} \rightarrow Nd + Pb$, as shown in Figure 18.

On ND State Processes:	On Prob. State Processes:
$\llbracket 0 \rrbracket = \emptyset$	$\llbracket (1)N \rrbracket = \text{guar}(\llbracket N \rrbracket)$
$\llbracket \alpha.P \rrbracket = \text{pre}(\alpha, \llbracket P \rrbracket)$	$\llbracket P_1 \oplus_\lambda P_2 \rrbracket = \text{pchoice}((\lambda, 1 - \lambda), (\llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket))$
$\llbracket N_1 + N_2 \rrbracket = \text{endsum}(\llbracket N_1 \rrbracket, \llbracket N_2 \rrbracket)$	$\llbracket P_1 \boxplus P_2 \rrbracket = \text{intsum}(\llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket)$
$\llbracket (\nu a)N \rrbracket = \text{nres}(a, \llbracket N \rrbracket)$	$\llbracket (\nu a)P \rrbracket = \text{pres}(a, \llbracket P \rrbracket)$
$\llbracket N_1 N_2 \rrbracket = \text{npar}(\llbracket N_1 \rrbracket, \llbracket N_2 \rrbracket)$	$\llbracket P_1 P_2 \rrbracket = \text{ppar}(\llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket)$

Figure 18: Interpretation map for MCCS processes

The following results will show that our model is fully-abstract with regards to mixed choice bisimulation. First we state some lemmas that allows us to define a relation between the interpretation of a process and its possible transitions. These lemmas are key in the proof of the upcoming full-abstraction theorem.

Lemma A.2.3.

(a) *If $N \xrightarrow{\alpha} P$, then $(\alpha, \llbracket P \rrbracket) \in \llbracket N \rrbracket$.*

(b) *If $(\alpha, Y) \in \llbracket N \rrbracket$, then $\exists P \in P\text{-Proc}$ such that $N \xrightarrow{\alpha} P$ and $\llbracket P \rrbracket = Y$.*

Lemma A.2.4. *If for $1 \leq i \leq n$, $P \xrightarrow{\kappa} \mu_i$ then for any $\lambda_i \in [0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$, $P \xrightarrow{\kappa} \sum_{i=1}^n \lambda_i \mu_i$.*

Lemma A.2.5.

(a) *If $P \xrightarrow{\kappa} \mu$, then $\exists \eta^\mu \in \llbracket P \rrbracket$ such that $\eta^\mu(X) = \sum_{\{N' | \llbracket N' \rrbracket = X\}} \mu(N')$.*

- (b) If $\mu \in \llbracket P \rrbracket$, then $\exists \eta_\mu \in \text{conv}(ND\text{-Proc})$ such that $P \xrightarrow{\kappa} \eta_\mu$ and
- $$\mu(X) = \sum_{\{N' \mid \llbracket N' \rrbracket = X\}} \eta_\mu(N').$$

Our main result for this section is the following full-abstraction theorem. This theorem allows use to show how strongly our model reflects the operational semantics of our calculus. It states that for any two processes: their interpretations will coincide in our model if and only if they are mixed choice bisimilar.

Theorem A.2.6 (Full Abstraction).

- (a) $\llbracket N \rrbracket = \llbracket N' \rrbracket \iff N \cong_{mc} N'$
- (b) $\llbracket P \rrbracket = \llbracket P' \rrbracket \iff P \cong_{mc} P'$

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