Analysis of Box Girder Bridges
by Cylindrical Shell Theory
and Model Test

by

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CONTENT

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ABSTRACT

This thesis describes a general method for analyzing the stresses and deflections in box girder bridges by means of the degenerate cylindrical shell theory. The structure is taken as an degenerate form of cylindrical shell with edge beam and with another plate connected at bottom. Gibson's cylindrical shell equation is used to determine the bending moments and membrane stresses in each plate element. This method is ideally suitable for analyzing structures with simply supported ends. The applied loads are resolved into harmonic components of the Fourier Series. Other boundary conditions are suggested using the same method for further development. The analysis allows the complete interaction of one plate on another to be examined, and thus stresses throughout the complete structure due to uniformly distributed load, line load and partial surface load for any one plate may be analyzed.

General computer programs based on the developed method are described. The input requires only the geometry and material properties of the structure, magnitude and locations of the applied loading. Final results for all internal forces and displacements at selected points are calculated and printed.

Tests were conducted on three box girder bridge models simply supported at both ends, made from perspex consisting of three cells in which both strains and deflections were measured under a variety of loading conditions. All models were tested in the elastic range.

Experimental results were compared with results derived from the degenerate cylindrical shell theory and the elementary theory.

CHAPTER 1

INTRODUCTION

1.1 AN OUTLINE OF THE PROBLEM

The box configuration has played an important role in structural applications for many years. Box-shaped cross-sections are closely related to other thin-walled closed cross-sections, the most readily recognigable ones being the box section girder bridge, and wing and fuselage of an aircraft. Box girder bridges are an established feature of the highway system. They form a very beautiful and aesthetically pleasing bridge for an elevated highways with its broad unbroken soffit when viewed from beneath. The very large torsional rigidity of the closed cellular section provides structural efficiency, as any non-symmetric loads are supported by all the deck section. This structural efficiency means that such bridges may be lighter than other bridges and they are particularly adaptable to prefabrication and to the standardization of details. Relatively large segments can be prefabricated and transported to the site and erected in a short time.

Concrete box girder bridges have found wide usage both as simple span and continuous structures, primarily in the span ranges, between 60 and 100 feet. Transverse diaphrams are used at both the end and interior support points and in some cases additional interior diaphrams are utilized between supports. For continuous bridges interior supports are often provided by single columns giving a graceful appearance to the structures.

For larger span, up to 160 feet, cast-in-place, simply supported post-tensioned prestressed concrete box girders have been used extensively.

Another type of cellular bridge which has received considerable attention is the composite steel-concrete box girder bridge. This bridge consists of a concrete deck acting integrally with cellular steel boxes. The individual steel boxes are spaced uniformly over the width of the bridge.

1.2 REVIEW OF PREVIOUS WORK

The problems of folded plate structures and box sections are closely related and thus a review of the research on folded plates is essential to understand the behaviour of closed cellular structures. Folded plate analysis are usually based on either the "ordinary theory" or the "elasticity theory". The ordinary theory assumes that the membrane stresses in each plate can be calculated by the elementary beam theory, and that slab bending is defined by means of transverse one-way slab action only[23]. In contrast the elasticity methods utilize the plane stress elasticity theory and classical two-way thin plate bending theory to determine simultaneously the membrane stresses and slab moments in each plate.

Gibson and Gardner's work published in 1965[6] [7] [8] indicated that the degenerate cylindrical shell theory can be used successfully to solve the problem of folded plate structures.

Gibson[6] hypothesis that if the radius of a cylindrical shell tends to infinity, then the shell will behave as a flat plate and called it the "degenerate shell theory". This method was only used for plates simply supported at the transverse boundaries.

A number of papers have been written on the analysis of continuous folded plates. Yitzhaki[24] presented an approximate solution based on the ordinary theory for certain special cases of longitudinal boundary conditions. Lee[11] used a finite-difference technique to analyze structures with arbitrary end conditions in which an extrapolation procedure was utilized to improve the accuracy of the solution.

The application of a finite element analysis to an open section folded plate was presented by Rockey and Evans[15] in which they used a rectangular element processing 8 degrees of freedom for in plane stresses and 12 degrees of freedom for plate bending to develop an element stiffness matrix for the folded plate problem.

Box girder bridges have been analysed in 4 major ways - a simple beam in bending and torsion, considering the bridge as an equivalent grillage of longitudinal and transverse girder of appropriate stiffness, a boxed cellular structure and the finite element theory.

Scordelis[16] treated the box girder bridge as a series of rectangular plates interconnected along the longitudinal joints to form a cellular structure simply supported at the end diaphrams, classical thin plate theory and plane stresses elasticity theory were used to determine the bending moments and membrane stresses in each plate element. The solution was based on a direct-stiffness method in which a harmonic analysis using Fourier series was utilized to analyze structures with arbitrary loading and boundary conditions.

Three approach were used by Scordelis and Lo[10][17] in the investigation of continuous box girder bridges. The first approach, designated the "folded plate method"[19], used a combination of a displacement (stiffness) method and a force (flexibility) method in the solution. The second approach called the "finite" segment method" was based on the ordinary theory for folded plates. In this analysis the basic structural element used is formed by dividing each plate element to a finite number of segments longitudinally, compatibility and equilibrium conditions are then satisfied at selected points along the four edges of each segment in the structure. The third approach specified as the "finite element method", element stiffness matrices were developed based on 24 degrees of freedom for each element. This method can be used for arbitrary loadings and boundary conditions. It can also treat the cases of varying dimensional and material properties throughout the structure, as well as the cases of cutouts in the plates.

Abel-Samad, Wright and Robinson[1] developed two methods for analysis for closed section girders following the basic concepts of thin-walled beam behavior. A thin-walled beam is characterized by three dimensions of different orders of magnitude. That is, the length is large compared to the width of the cross-section which in turn is large compared to the thickness of the elements of the cross-section. Therefore, in formulating the behavior of an individual plate element, it is permissible to neglect longitudinal moments, and transverse axial deformations, and to assume that longitudinal deformations are linearly distributed across the width. The first method developed, called the plate-element method, was similar to Scordelis's folded plate method. The second method named the generalized coordinate method. It employs a formulation of equilibrium equations for the cross-section, and initial parameter solution method like that of Vlasov[22]. This approach permits consideration of flexible interior diaphrams and arbitrary support conditions.

Mattock and Johnston[12] presented an analytical and experimental study of composite steel-concrete box girder bridges. Formulas were given for determining wheel load distribution for this type of bridges based on these studies.

1.3 SCOPE OF THE THESIS

This thesis is an extension of the degenerate cylindrical shell theory to analyse the problems of box girder bridges.

The problem is considered in four stages using both analytical and experimental technique to complement one another.

The stages are as follows:

Stage 1

A IBM 360-65 computer program was written for the analysis of box girder bridges, treating it as a degenerate cylindrical shell. The program provides a rapid solution for box girder structures simply supported with rigid diaphrams at the two ends.

Stage 2

Three perspex box girder models, simply supported at two ends consisted of three bays were tested in the elastic range under conditions of uniformly distributed load, longitudinal line load and partial surface load. Strains were measured both in longitudinal and transverse directions of the outer surface along the midspan cross section. Deflections were measured in vertical direction of the midspan.

Stage 3

Results were compared between the "degenerate cylindrical shell theory" "Elementary beam theory" and the experimental investigation.

Stage 4

Two methods have been suggested for the end conditions other than the simply supported case.

(1) Using basic function instead of harmonic function, using modified input informations the same computer program can be used to solve the problems.

(2) A force method can be used to solve the continuous box girder sections which is similar in concept to that used to analyse a continuous beam.

CHAPTER 2

THEORETICAL INVESTIGATION

2.1 INTRODUCTION

This theoretical investigation is concerned with the elastic analysis of box girders under uniformly distributed loads, line loads and partial surface loads.

(1) Degenerate Cylindrical Shell Theory

Both cylindrical shells and box girders are highly indeterminate structures. Gibson and Gardner's work showed that the problem of fold plate structures can be solved by using the degenerate form of cylindrical shells. A flat surface may be generated from a cylindrical surface by causing the half angle of the latter to decrease to a sufficiently small value. Thus, if the half angle of the cylindrical surface is taken as 1° and R is its radius as shown in Fig. 1.

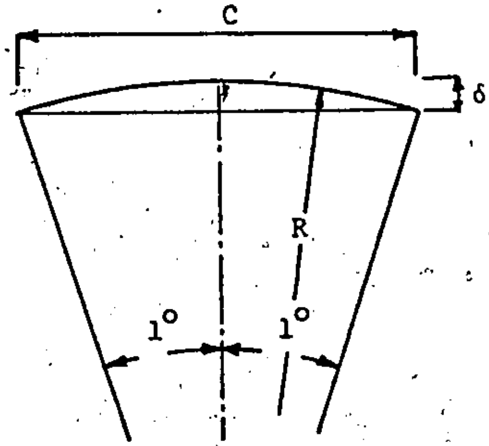


Fig. 1

Then the ratio of the rise δ to the chord width C which is a measure of the flatness of the surface is

$$\delta/C = R(1 - \cos 1^{\circ})/2R \sin 1^{\circ} = 1/230 \qquad (2.1)$$

and this may be considered sufficiently flat for all practical purpose.

Gardner's work[8] indicated that a close agreement between the classical plate theory and degenerate cylindrical shell theory gives the confirmation of the applicability of the method for the design of flate plates.

Theoretical testing were carried out for the cylindrical shell theory to determine how much reduction in the half angle was necessary to achieve maximum accuracy. It was found that reducing the half angle below 1° made no significant difference in the final answers.

As the chord width C is given by

$$C = 2R \sin 1^{\circ} \qquad (2.2)$$

then the radius required to generate a given chord width will be

$$R = C/2 \sin 1^{\circ} \qquad (2.3)$$

Another justification of using the degenerated cylindrical surface to dealing with the plate problem is, if to consider Gibson's cylindrical shell partial differential equation[5]

$$R^8\left[\left(\frac{\partial^2}{R^2\partial\phi^2} + \frac{\partial^2}{\partial x^2}\right)^4\right] w + AR^6/I \;\frac{\partial^4 w}{\partial x^4}$$

$$= \frac{R^6}{EI}\left[R^2\left(\frac{\partial^2}{R^2\partial\phi^2} + \frac{\partial^2}{\partial x^2}\right)^2 Z + R\,\frac{\partial^2}{R^2\partial\phi^2}\left(\frac{\partial Y}{R\partial\phi} - \frac{\partial X}{\partial x}\right)\right.$$

$$\left. + 2R\,\frac{\partial^3 Y}{R\partial\phi\partial x^2}\right] \qquad (2.4)$$

dividing each side by R^8

If $R \to \infty$, then $R\partial\phi \to y$ and let $Y \to 0$, $X \to 0$

The equation reduces to

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{Z}{EI} \qquad (2.5)$$

and the plate equation is obtained.

Thus a general computer program for determining the stresses and deflections in multi-cylindrical shell could equally well be used for determining stresses in multi-folded plate structures provided that the radii and half angles were suitably modified. In the cylindrical shell problem the shell is defined in terms of its geometrical dimensions, length radius, half angle and thickness. Since those dimensions are input data, no changes are necessary in the logic of the computer program.

The Author's work extended Gibson and Gardner's work by adding one more shell surface to the cylindrical shell with edge beams,and using the degenerate method to form the box girder, see Fig. 2

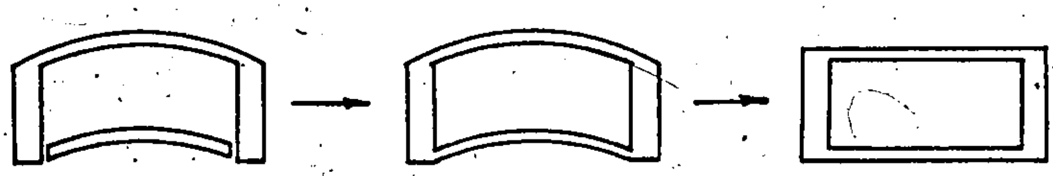


Fig. 2

(2) Elementary Beam Theory

In this elementary beam theory, it is considered that the whole structure acts as a beam simply supported with diaphrams at both ends. For the symmetrical loading condition, forces and displacement are simply determined as a general beam. For a unsymmetrical loading conditions, twisting effect was added into the analysis. This elementary theory provides the designer very quickly a good ideal of the suitably of his chosen dimensions.

2.2 DEGENERATE CYLINDRICAL SHELL THEORY

Basic assumptions used are as follows:

(1) The structure is monolithic and the joints are rigid.

(2) Each plate of the box girder is rectangular of uniform thickness and is made of an isotropic and homogenous elastic material.

(3) The thickness is small compared with the overall dimensions of the boundary surfaces.

(4) The relation between forces and deformations is linear so that superposition is valid.

(5) The structure is simply supported at its extreme ends. Transverse diaphrams at the end and interior supports are infinitely rigid in their own plane, but perfectly flexible normal to their own plane.

(6) The flanges are treated as a degenerated form of cylindrical shell and the webs are treated as edge beam.

(7) In all plates and beams, plane sections remain plane after deformation

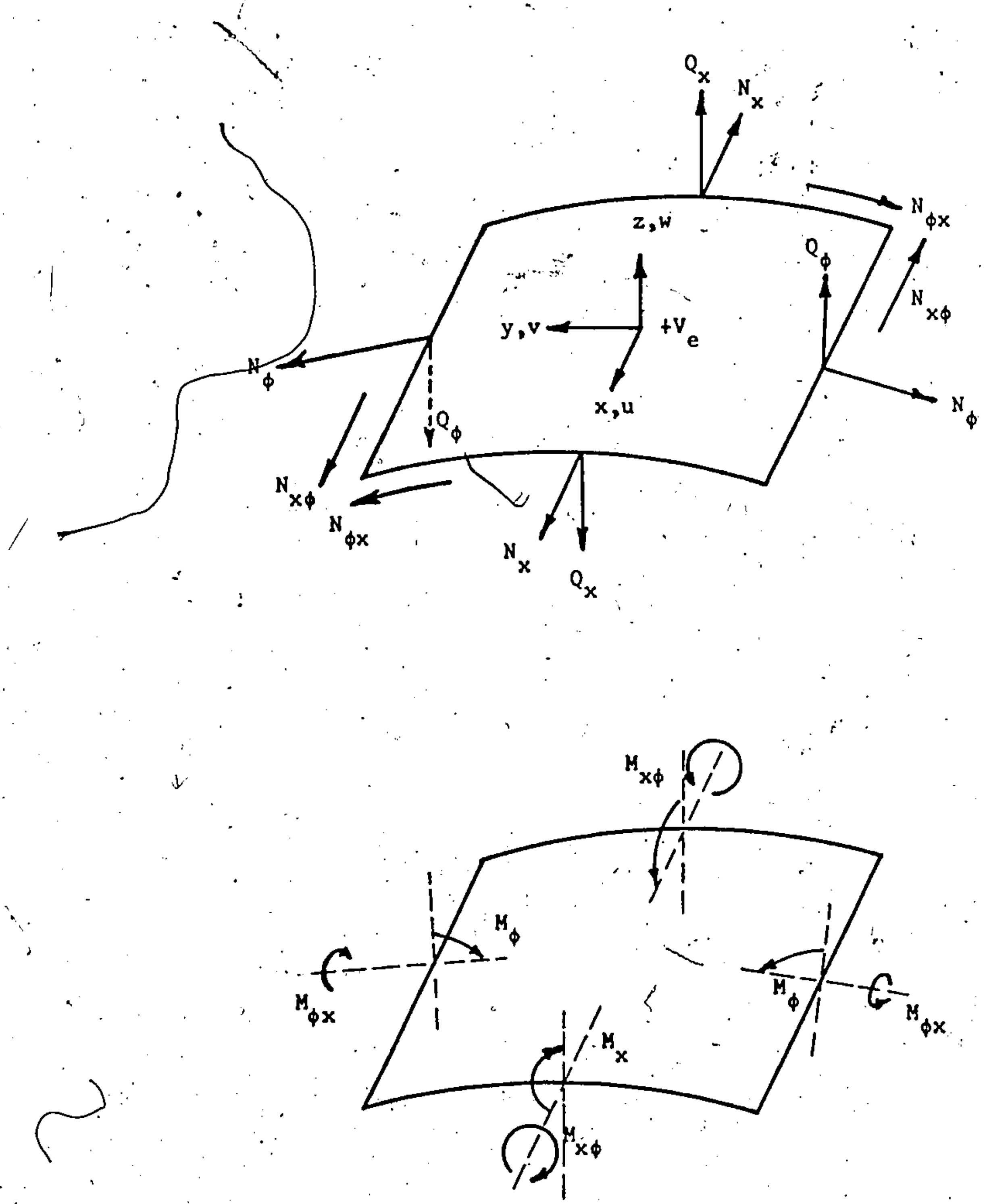


Fig. 3 General Notation

It is assumed that all forces act at the centroids of the shell sections all moments are about the centroids. The origin of the axes is the middle surface at its mid-point, both transversely and longitudinally.

Equilibrium Equations

Consider the modified stress diagram shown in Fig. 4 of the element of the shell of longitudinal length dx and transverse length $Rd\phi$. Now the only forces affecting the equilibrium of the shell element in the direction of x are shown in Fig. 4. Examining the equilibrium in this direction gives

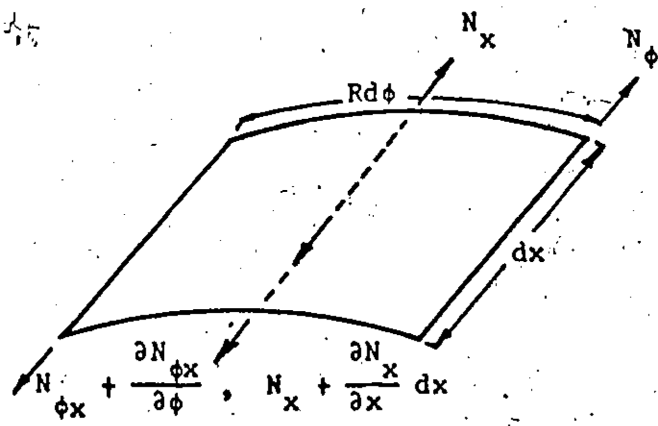


Fig. 4. Longitudinal Shell Stress

$$\frac{\partial Nx}{\partial x} dxRd\phi + \frac{\partial N\phi x}{R\partial \phi} Rd\phi dx + X \cdot Rd\phi dx = 0 \tag{2.6}$$

which reduces to the stress equilibrium equation

$$\frac{\partial Nx}{\partial x} + \frac{\partial N\phi x}{R\partial \phi} + X = 0 \tag{2.7}$$

Similarly, by examination of the equilibrium of the shell element in the other two directions, $y = R\phi$ and z. The following equilibrium equations hold

$$\frac{\partial N x\phi}{\partial \phi} + \frac{\partial N\phi}{R\partial \phi} + Y = 0 \tag{2.8}$$

$$\frac{\partial Qx}{\partial x} + \frac{\partial Q\phi}{R\partial \phi} + \frac{N\phi}{R} - Z = 0 \tag{2.9}$$

Examination of the moment equilibrium of the shell likewise yields the three moment equilibrium equations.

$$\frac{\partial Mx}{\partial x} + \frac{\partial M\phi x}{R\partial \phi} - Q_x = 0 \tag{2.10}$$

$$\frac{\partial M\phi}{R\partial \phi} + \frac{\partial Mx\phi}{\partial x} - Q_\phi = 0 \tag{2.11}$$

$$Nx\phi = N\phi x \tag{2.12}$$

Thus six equations involving ten unknows, and in order that the problem may be solved, it is necessary to determine other equations. These equations are obtained from the stress strain and strain displacement equations.

Strain Displacement Relationships

$$S_x = \frac{\partial u}{\partial x} - Z\frac{\partial^2 w}{\partial x^2} \tag{2.13}$$

$$2\,S_{x\phi} = \frac{\partial u}{\partial \phi} / (R + Z) + \frac{R+Z}{R} \cdot \frac{\partial v}{\partial x}$$

$$- \left(\frac{Z}{R-Z} + \frac{Z}{R}\right)\frac{\partial^2 w}{\partial x \partial \phi} \tag{2.14}$$

$$S_\phi = \frac{\partial v}{R\partial\phi} - \frac{Z}{R(R+Z)} \cdot \frac{\partial^2 w}{\partial\phi^2} + \frac{w}{R+Z} \qquad (2.15)$$

where S_x and S_ϕ are the pure strains in the direction of x and $y = R\phi$, respectively and $S_{x\phi}$ is the shear strain about the z axis as shown in Fig. 5.

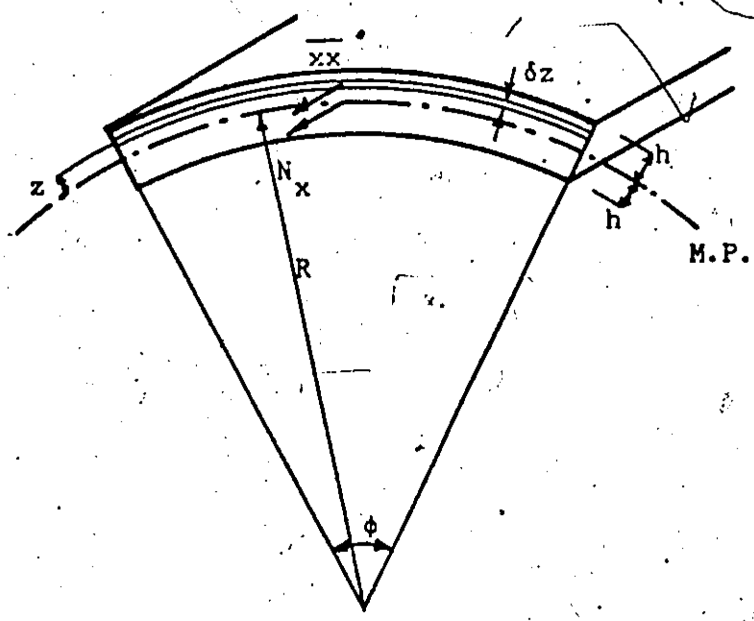


Fig. 5 Longitudinal Stress on Shell Element

Stress Strain Relationships

Now if $\bar{x}\bar{x}$ and $\bar{\phi}\bar{\phi}$ are the normal stresses in the shell per unit area and $x\phi$ is the shear stress then, the stress strain relationships may be expressed in following form.

$$\bar{x}\bar{x} = \frac{E}{G}(S_x + \eta S_\phi) \qquad (2.16)$$

$$\bar{x}\bar{\phi} = \frac{E}{1+\eta} S_{x\phi} = \bar{\phi}\bar{x} \qquad (2.17)$$

$$\overline{\phi\phi} = \frac{E}{G}\,(S_\phi + \eta S_x) \qquad (2.18)$$

E is Young's modulus, η is poissions ratio, $G = 1 - \eta^2$

Force and Moment Displacement Equations

By integrating the stresses over a section of width $R\phi = y = 1$, or $dx = 1$ and depth 2h, as shown in Fig. 5. N_ϕ , N_x , $N_{x\phi}$ can be derived in terms of u, v and w. Again by consideration of the moments of the force produced by the stresses acting on elements of the shell about the neutral surfaces, the moments M_x , M_ϕ , $M_{x\phi}$ may also be stated in terms of the displacements u, v, and w with the aid of the stress equilibrium equation. The following equations are obtained.

$$M_x = \frac{EI}{G}\left(\frac{\partial^2 w}{\partial x^2} + \eta\,\frac{\partial^2 w}{R^2\partial\phi^2}\right)$$

$$M_\phi = \frac{EI}{G}\left(\frac{\partial^2 w}{R^2\partial^2\phi} + \eta\,\frac{\partial^2 w}{\partial x^2}\right)$$

$$Q_\phi = \frac{EI}{G}\left(\frac{\partial^3 w}{R\partial\phi^2\partial x^2} + \eta\,\frac{\partial^3 w}{R^3\partial\phi^3}\right)$$

$$Q_x = \frac{EI}{G}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{R\partial x\partial\phi^2}\right)$$

$$M_{x\phi} = M_{\phi x} = \frac{EI}{1+\eta}\left(\frac{\partial^2 w}{R\partial x\partial\phi}\right)$$

$$N_\phi/R = -\frac{EI}{G}\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{R\partial x^2\partial\phi^2} + \frac{\partial^4 w}{R^4\partial\phi^4}\right) + Z$$

$$\frac{\partial N x\phi}{\partial \phi} = \frac{EI}{G} \cdot R \cdot \left(\frac{\partial^5 w}{R \partial x^4 \partial \phi} + 2 \frac{\partial^5 w}{R^3 \partial x^2 \partial \phi^3} + \frac{\partial^5 w}{R^5 \partial \phi^5}\right) - \frac{\partial Z}{\partial \phi} - Y$$

$$\frac{\partial^2 (Nx)}{\partial x^2} = - \frac{EI}{G} \left(\frac{\partial^6 w}{R \partial x^4 \partial \phi^2} + 2 \frac{\partial^6 w}{R^3 \partial x^2 \, \partial \phi^3} + \frac{\partial^6 w}{R^5 \partial \phi^6}\right)$$

$$+ \frac{\partial Z}{R \partial \phi^2} + \frac{\partial Y}{R \partial \phi} - \frac{\partial X}{\partial x}$$

$$N_x = \frac{EA}{G} \left[\frac{\partial u}{\partial x} + \eta \left(\frac{\partial v}{\partial \phi} + w\right)/R\right]$$

$$N_\phi = \frac{EA}{G} \left[\left(\frac{\partial v}{\partial \phi} + w\right)/R + \eta \frac{\partial u}{\partial x}\right]$$

$$N_{x\phi} = N_{\phi x} = \frac{EA}{2(1 + \eta)} \left(\frac{\partial u}{R \partial \phi} + \frac{\partial v}{\partial x}\right) \qquad (2.19)$$

where $I = \frac{2}{3} h^3$ is the moment of inertia of unit width of the shell

$A = 2h$ the area of the unit width.

$G = 1 - \eta^2$

Compatibility Equation

By successive partial differentiational elimination a partial differential equation in one of the variables 'w' can be obtained.

$$R^8\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{R^2\partial\phi^2}\right)^4 w + R^6\,(1 - \eta^2)\,\frac{A}{I}\,\frac{\partial^4 w}{\partial x^4}$$

$$- \frac{R^6(1 - \eta^2)}{EI}\left\{ R^6\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{R^2\partial\phi^2}\right)^2 Z + \frac{\partial}{\partial\phi}\left[\left[(2 + \eta)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{R^2\partial\phi^2}\right] Y\right]\right.$$

$$\left. + R\,\frac{\partial}{\partial x}\left[\left(\eta\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{R^2\partial\phi^2}\right)\right] X \right\} \qquad (2.20)$$

The solution of this compatibility equation consists of the complementary function and a particular integral relevant to the type of external loading utilized.

The Complementary Function

Equating the right-hand side of equation (2.20) to zero, the solution of the resulting homogeneous equation is

$$w = A_M e^{M\phi} \cdot \cos nkx = A_M e^{M\phi} \cdot \cos kx$$

on utilizing only the first term, i.e., $n = 1$ on substituting this solution for w in the equation, the following indicial equation results

$$(M^2 - k^2R^2)^4 - R^6(1 - \eta^2)\,\frac{A}{I} \cdot k^4 = 0 \qquad (2.21)$$

The solution of this indicial equation is then

$$M = \pm\,(m_1 \pm in_1)\,\frac{D'}{4\sqrt{2}} \quad \text{and} \quad \pm\,(m_2 \pm in_s)\,\frac{D'}{4\sqrt{2}}$$

where

$$m_1 = \sqrt{\frac{\sqrt{(1+\gamma)^2+1}+(1+\gamma)}{2}}, \qquad m_2 = \sqrt{\frac{\sqrt{(1-\gamma)^2+1}-(1-\gamma)}{2}}$$

$$n_1 = \sqrt{\frac{\sqrt{(1+\gamma)^2}-1-(1+\gamma)}{2}} \qquad n_2 = \sqrt{\frac{\sqrt{(1-\gamma)^2}+1+(1-\gamma)}{2}}$$

where,

$$\frac{D'}{\sqrt[4]{2}} = \sqrt[8]{3(1-\eta^2)}\ \sqrt[4]{\frac{R}{2h}}\sqrt{\frac{R\pi}{L}} \quad \text{and} \quad \gamma = \frac{\frac{R\pi}{L}\cdot\sqrt{\frac{2h}{R}}}{\sqrt[4]{3(1-\eta^2)}}.$$

There are eight complex values for M and thus the solution for 'w' may be written

$$w = (A_1 e^{M_1\phi} + \ldots\ldots + A_8 e^{M_8\phi})\cos kx \qquad (2.22)$$

where A_1 to A_8 are arbitrary complex constants. Now these arbitrary complex constants must be so chosen that the complete solution for 'w' is real, on doing this it is found that 'w' eventually takes the following form

$$w = 2\begin{bmatrix} +(a\cos\beta_1\phi - b\sin\beta_1\phi)e^{-\alpha_1\phi} + (c\cos\beta_1'\phi - d\sin\beta_1'\phi)e^{-\alpha_1'\phi} \\ +(e\cos\beta_1\phi - f\sin\beta_1\phi)e^{-\alpha_1\phi} + (g\cos\beta_1'\phi - h\sin\beta_1'\phi)e^{-\alpha_1'\phi}\end{bmatrix}$$

(2.23)

Where a to h are now real arbitrary constants.

Substitution of this value of 'w' in the equations (2.19) then determines the values of all required functions in terms of a to h. These arbitrary constants are determined by the boundary conditions, existing at the two edges of the shell.

The values of the complementary functions for all required quantities are readily tabulated as following depending whether the function is obtained by an even or odd number of differentiations of w.

TABLE 1

Powers of Indicial Roots

α_1 β_1	Pm_1 Pn_1	α'_1 β'_1	Pm_2 Pn_2
α_2 β_2	$P^2(1+\gamma)$ P^2	α'_2 β'_2	$P^2(\gamma-1)$ P^2

where $P = D'/\sqrt{2}$.

TABLE 2

Shell Coefficients of the Complementary Function (Even)

$$F = \bar{R}\,\phi \begin{bmatrix} +\{(aB_1 - bB_2)\cos\beta_1\phi - (aB_2 + bB_1)\sin\beta_1\phi\}e^{+\alpha_1\phi} \\ +\{(cB_3 - dB_4)\cos\beta'_1\phi - (cB_4 + dB_3)\sin\beta'_1\phi\}e^{+\alpha'_1\phi} \\ +\{(eB_1 - fB_2)\cos\beta_1\phi + (eB_2 + fB_1)\sin\beta_1\phi\}e^{-\alpha_1\phi} \\ +\{(gB_3 - hB_4)\cos\beta'_1\phi + (gB_4 + hB_3)\sin\beta'_1\phi\}e^{-\alpha'_1\phi} \end{bmatrix} \qquad (A)$$

F	R	B_1	B_2	B_3	B_4
$M\phi$	$2EI/R^2G$ $\times \cos kx$	$\alpha_2 - \eta k^2R^2$	β_2	$\alpha'_2 - \eta k^2R^2$	β'_2
Mx	$-2EIk^2/G$ $\times \cos kx$	$1 - \eta\alpha_2/k^2R^2$	$-\eta\beta_2/k^2R^2$	$1 - \eta\alpha'_2/k^2R^2$	$-\eta\beta'_2/k^2R^2$
Qx	$-2EIk^3/G\gamma$ $\times \sin kx$	$+1$	$+1$	-1	$+1$
$N\phi$	$-2EIR\,.\,2k^4/G\gamma^2$ $\times \cos kx$	0	$+1$	0	-1
Nx	$+2EIR\,.\,2k^4/G\gamma^2$ $\times \cos kx$	-1	$+(1+\gamma)$	$+1$	$+(1-\gamma)$
u	$+IRk^3/GA\gamma^2$ $\times \sin kx$	-1	$1+\gamma(1+\eta)$	$+1$	$(1-\gamma(1+\eta))$
w	$2\cos kx$	$+1$	0	$+1$	0

where $G = 1 - \eta^2$.

TABLE 3
Shell Coefficients for Complementary Functions (Odd)

$$F = \overline{R}\begin{bmatrix} +\{(aB_1 - bB_2)\cos\beta_1\phi - (aB_2 + bB_1)\sin\beta_1\phi\}e^{+\alpha_1\phi} \\ +\{(cB_3 - dB_4)\cos\beta_1'\phi - (cB_4 + dB_3)\sin\beta_1'\phi\}e^{+\alpha_1'\phi} \\ -\{(eB_1 - fB_2)\cos\beta_1\phi + (eB_2 + fB_1)\sin\beta_1'\phi\}e^{-\alpha_1\phi} \\ -\{(gB_3 - hB_4)\cos\beta_1'\phi + (gB_4 + hB_3)\sin\beta_1'\phi\}e^{-\alpha_1'\phi} \end{bmatrix} \quad \text{-(B)}$$

F	$\overline{R}$	B_1	B_2	B_3	B_4
$Q\phi$	$2EIk^3/G(\sqrt{\gamma})^3 \times \cos kx$	$m_1 - n_1$	$m_1 + n_1$	$-(m_2 + n_2)$	$m_2 - n_2$
$Q'\phi$	$2EIk^3/G(\sqrt{\gamma})^3 \times \cos kx$	$m_1[1 - \gamma(1 - \eta)] - n_1$	$m_1 + n_1[1 - \gamma(1 - \eta)]$	$-m_2[1 + \gamma(1 - \eta)] - n_2$	$m_2 - n_2[1 + \gamma(1 - \eta)]$
$Nx\phi$	$4EIRk^4/G(\sqrt{\gamma})^5 \times \sin kx$	$-n_1$	$+m_1$	$+n_2$	$-m_2$
v	$-2IR \cdot 2k^3/GA(\sqrt{\gamma})^7 \times \cos kx$	$m_1 + n_1[1 - \gamma(1 + \eta)]$	$n_1 - m_1[1 - \gamma(1 + \eta)]$	$-m_2 + n_2[1 + \gamma(1 + \eta)]$	$-n_2 - m_2[1 + \gamma(1 + \eta)]$
θ	$+1 \cdot \cos kx$	$-\frac{2\alpha_1}{R} + \frac{(\bar{R} \cdot B_1)_v}{R}$	$-\frac{2\beta_1}{R} + \frac{(\bar{R} \cdot B_2)_v}{R}$	$-\frac{2\alpha_1'}{R} + \frac{(\bar{R} \cdot B_3)_v}{R}$	$-\frac{2\beta_1'}{R} + \frac{(\bar{R} \cdot B_4)_v}{R}$
$Mx\phi$	$-2EIk/(1 + \eta)R \times \sin kx$	α_1	β_1	α_1'	β_1'

Particular Integrals

A uniformly distributed load q can be treated as a Fourier Series. The particular integral, part of the solution, depends upon the type of load on the shell. Two types of load are considered, the uniformly distributed load and the partial uniformly distributed load.

Uniformly Distributed Load

$$p' = \frac{4q}{\pi} \cos kx = p \cos kx \qquad (2.24)$$

where $p = q \cdot 4/\pi$ (i.e., only the first term of the Fourier half range series having been utilized).

(1) $X = 0$, (2) $Y = p \sin \phi \cdot \cos kx$

(3) $Z = -p \cos \phi \cdot \cos kx$.

Substituting these values in equation (2.20), we find that the solution of the resulting equation is

$$w = Cp \cdot \cos\phi \, . \, \cos kx$$

where

$$C = \left(\frac{-R^4(1 - \eta^2)}{EI}\right)\left(\frac{(Rk)^4 + (4 + \eta)(Rk)^2 + 2}{(1 + k^2R^2)^4 + R^6(1 - \eta^2)Ak^4/I}\right)$$

and on substituting this value of w in equation (2.19), the particular integrals are found to be:

(a) $M_x = \frac{-EI}{G}(k^2 + \eta/R^2)C \cdot p \cdot \cos\phi \cdot \cos kx$,

(b) $M_\phi = -\frac{EI}{G}(\eta k^2 + 1/R^2)C \cdot p \cdot \cos\phi \cdot \cos kx$,

(c) $M_{x\phi} = \frac{EI}{1 - \eta}(+ k/R)C \, . \, p \, . \, \sin\phi \, . \, \sin kx$,

(d) $Q_x = \frac{EI}{G}(k^3 + k/R^2)C \, . \, p \, . \, \cos\phi \, . \, \sin kx$,

(e) $Q_\phi = \frac{EI}{G}(k^2/R + 1/R^3)C \, . \, p \, . \, \sin\phi \, . \, \cos kx$,

(f) $N_\phi = - R(H_3 + 1)p \, . \, \cos\phi \, . \, \cos kx$,

(g) $N_{x\phi} = - \frac{1}{k}(H_3 + 2)p \, . \, \sin\phi \, . \, \sin kx$,

(h) $N_x = - \frac{1}{k^2R}(H_3 + 2)p \, . \, \cos\phi \, . \, \cos kx$,

(i) $u = - \frac{1}{kEA}\left[\frac{-1}{k^2R}(H_3 + 2) + \eta R(H_3 + 1)\right] p \, . \, \cos\phi \, . \, \sin kx$,

(j) $v = \frac{1}{k^2EA}\left[\eta(H_3 + 3) + (H_3 + 2)\cdot(2 + 1/k^2R^2)\right]p \, . \, \sin\phi \, . \, \cos$

$= \left[v_p\right] p \, . \, \sin\phi \, . \, \cos kx$,

(k) $w = C \, . \, p \, . \, \cos\phi \, . \, \cos kx$,

(l) $\theta = \frac{1}{R}\left[C + \left[v_p\right]\right] p \cdot \sin\phi \cdot \cos kx$, where $\left[v_p\right]$ is given by (j)

where $H_3 = \frac{EI}{1 - \eta^2}(k^4 + 2k^2/R^2 + 1/R^4)C.$

Partial Uniformly Distributed Load

A partial load is considered to be uniformly distributed over the shade rectangular area Fig. 6 with side u and C_2. This type of load can be considered as a partial load in the longitudinal direction acting over the full width of an equivalent plate C_2.

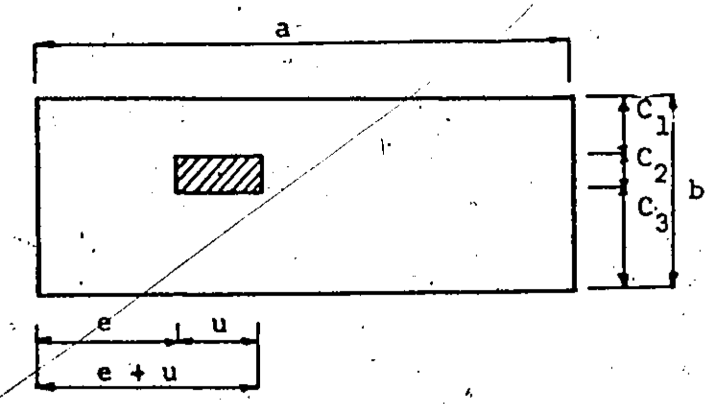


Fig. 6 Partial Uniformly Distributed Load

(i) In transverse direction

The partially loaded plate can be treated as three connected plates, as shown in Fig. 7.

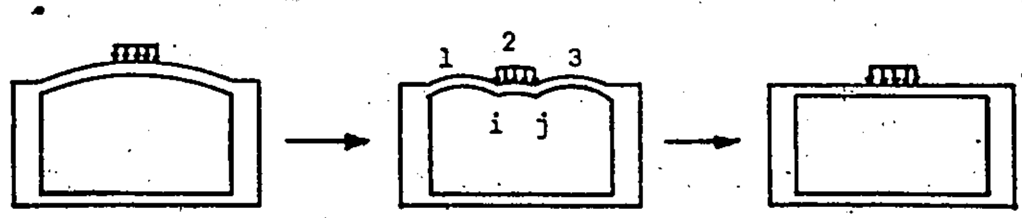


Fig. 7 Partial Surface Load in Transverse Direction

As each plate contains eight unknowns, there are total 24 unknowns. However, each junction line (i, j) provides 8 equations, 4 displacement equilibrium equations and 4 force equilibrium equations, giving a total of 16 equations. Another 8 equations will be found from the junction between plate 1, plate 3 and edge beams. If the unknowns of plate 2 are considered as constants. The other 16 unknowns relating to plate 1 and plate 3 can be found in terms of the 8 unknowns of plate 2; i.e., solving for the stress resultants can be reduced to finding 8 unknowns. This three plate system has been arranged, as an alternative to the single plate, in the computer program.

(ii) In longitudinal direction

By developing the load in the series

$$\frac{2}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi x}{a}\int_{e}^{e-u} q\sin\frac{m\pi\xi}{a}\,d\xi$$

$$=\sum_{m=1}^{\infty}\left(\frac{2q}{a}\int_{e}^{e-u}\sin\frac{m\pi\xi}{a}\,d\xi\right)\sin\frac{m\pi x}{a}$$

$$=\sum_{m=1,3,5\ldots}(-1)^{\frac{(m-1)}{2}}\frac{2q}{m\pi}\left(-\cos\frac{m\pi(e+u)}{a}+\cos\frac{m\pi e}{a}\right)\sin\frac{m\pi x}{a} \tag{2.26}$$

It can be seen that the partial load in longitudinal direction can be expressed by a sine series, since q, a, e and u are given.

Previously it was assumed

$$p = \frac{4q}{m\pi} \cos \frac{m\pi x}{a}$$

thus the same computer program can be used to solve the problems under this kind of loading by substituting

$$(-1)^{\frac{(m - 1)}{2}} \left(\frac{2q}{m\pi} \left(- \cos \frac{m\pi(e + u)}{a} + \cos \frac{m\pi(e)}{a} \right) \right) \text{ instead of } \frac{4q}{m\pi},$$

similar to the case of partial load acting on the edge beam.

<u>General shell-edge conditions</u>

The conditions existing at the shell edge are the longitudinal deflection, the lateral deflection, the vertical deflection, and finally the edge rotation.

By considering the equation convention used, shown in Fig. 8, the above displacements at $- \phi_k$ and $- \phi_k$ are easily shown to be (due to the orthogonal transformation of the axes from $- \phi_k$ to $- \phi_k$)

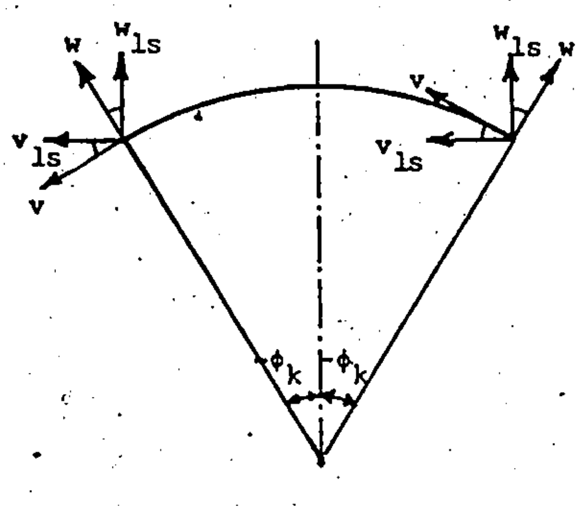


Fig. 8 Shell Displacements

(1) At $+\phi_k$, u is outwards from the plane of the paper and therefore + ive by the original convention.

At $-\phi_k$, u is outwards from the plane of the paper and therefore + ive by convention.

(2) At $+\phi_k$, $v_{1s} = v \cos \phi_k + w \sin \phi_k$

At $-\phi_k$, $v_{1s} = v \cos \phi_k - w \sin \phi_k$

(3) At $+\phi_k$, $v_{1s} = v \cos \phi_k - w \sin \phi_k$

At $-\phi_k$, $v_{1s} = v \cos \phi_k + w \sin \phi_k$

(4) At $\pm\phi_k$, θ is positive if the transverse tangent rotates outwards.

(2.27)

Fig. 9 illustrates the forces and moments existing at the shell edges $+\phi_k$ and $-\phi_k$ due to orthogonal transformation of the edge forces and moments from $+\phi_k$ to $-\phi_k$.

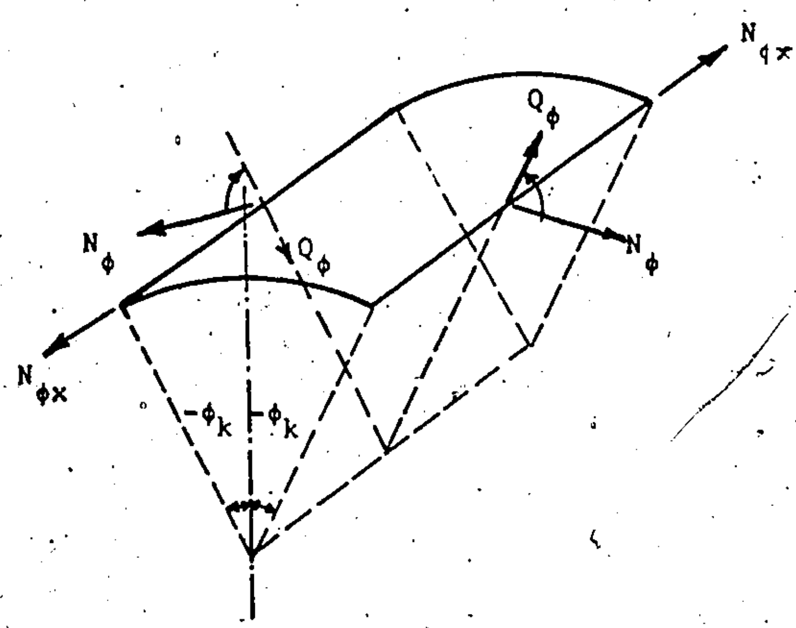


Fig. 9 Shell Moments and Forces

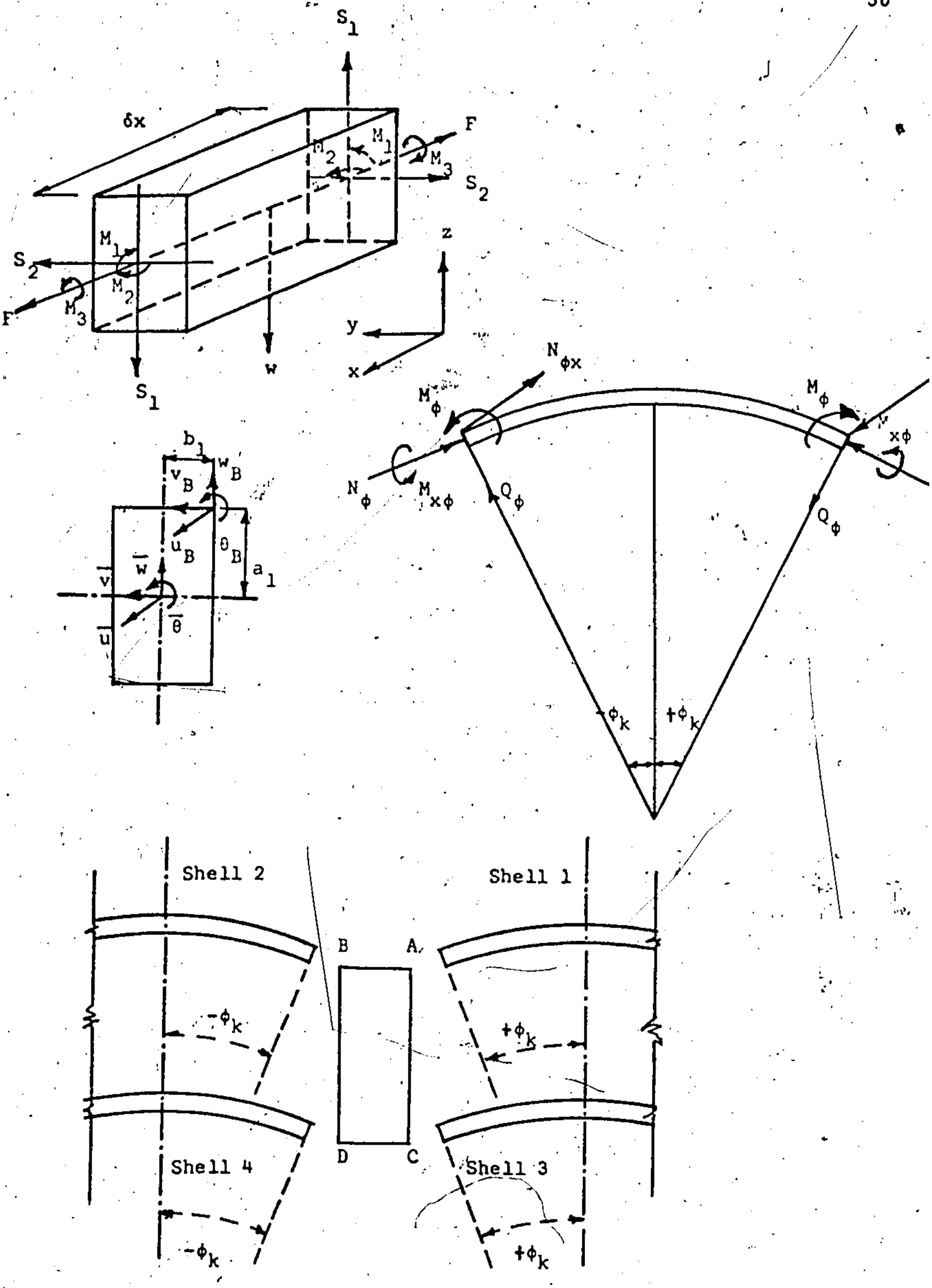


Fig. 10 Edge Beam Forces, Moments and Displacements

General edge beam conditions

The sets of shell forces and moments operating on the edge beam at four corners be as shown in Fig. 10. By considering the equilibrium of the edge beam element δx with shell 1(A), the following equilibrium equations can be easily established.

$$\partial F/\partial x - N_{x\phi} = 0$$

$$\partial S_1/\partial x - (Q_\phi \cos\phi_k + N_\phi \sin\phi_k) + w = 0$$

$$\partial S_2/\partial x + (Q_\phi \sin\phi_k - N_\phi \cos\phi_k) = 0$$

$$\partial M_1/\partial x - M_{x\phi} \cos\phi_k + N_{x\phi}\, a_1 - S_1 = 0$$

$$\partial M_2/\partial x - M_{x\phi} \sin\phi_k - N_{x\phi}\, b_1 + S_2 = 0$$

$$\partial M_3/\partial x + M_\phi + (Q_\phi \cos\phi_k + N_\phi \sin\phi_k)\, b_1 + (Q_\phi \sin\phi_k - N_\phi \cos\phi_k)\, a_1 = 0 \qquad (2.28)$$

The following relationship between the displacements of the edge of the edge beam u_B, v_B, w_B and θ_B and the displacement of the centroid of the edge beam can be found.

$$u_B = \bar{u} + a_1 \left(\frac{\partial \bar{w}}{\partial x}\right) - b_1 \left(\frac{\partial \bar{v}}{\partial x}\right)$$

$$v_B = \bar{v} + a_1\, \bar{\theta}$$

$$w_B = \bar{w} + b_1\, \bar{\theta}$$

$$\theta_B = \bar{\theta} \qquad (2.29)$$

By reducing these equations the following relationships are eventually established.

$$M_1 = -\frac{1}{k^2}(Q'_\phi \cos\phi_k + N_\phi \sin\phi_k - a_1 k N_{\phi x} - w)$$

$$M_2 = -\frac{1}{k^2}(Q'_\phi \sin\phi_k - N_\phi \cos\phi_k + b_1 k N_{\phi x})$$

$$\partial M_3/\partial x = -(M_\phi + a_1(Q_\phi \sin\phi_k - N_\phi \cos\phi_k)$$

$$+ b_1(Q_\phi \cos\phi_k - N_\phi \sin\phi_k))$$

$$F = -N_{x\phi}/k \tag{2.30}$$

$$u_B = +\frac{1}{k}\left(\frac{F}{A_1 E} - a_1 \frac{M_1}{EI_1} + b_1 \frac{M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} + \frac{a_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2}\left(\frac{M_1}{EI_1} + \frac{b_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2}\left(\frac{\partial M_3}{\partial x} \cdot \frac{1}{EI_3}\right) \tag{2.31}$$

Similarly, at corner B

$$M_1 = -\frac{1}{k^2}(-Q'_\phi \cos\phi_k + N_\phi \sin\phi_k + N_{\phi x} a_1 k - w)$$

$$M_2 = -\frac{1}{k^2}(Q'_\phi \sin\phi_k + N_{x\phi} b_1 k + N_\phi \cos\phi_k)$$

$$\partial M_3/\partial x = (M_\phi + a_1 (- Q_\phi \sin \phi_k - N_\phi \cos \phi_k)$$

$$+ b_1 (- Q_\phi \cos \phi_k + N_\phi \sin \phi_k)$$

$$F = - N_{\phi x}/k \tag{2.32}$$

$$u_B = \frac{1}{k} (\frac{F}{A_1 E} + a_1 \frac{M_1}{EI_1} + b_1 \frac{M_2}{EI_2})$$

$$v_B = - \frac{1}{k^2} (\frac{M_2}{EI_2} - \frac{a_1}{EI_3} \frac{\partial M_3}{\partial x})$$

$$w_B = - \frac{1}{k^2} (\frac{M_1}{EI_1} + \frac{b_1}{EI_3} \frac{\partial M_3}{\partial x})$$

$$\theta_B = - \frac{1}{k^2} (\frac{1}{EI_3} \frac{\partial M_3}{\partial x}) \tag{2.33}$$

At corner C

$$M_1 = - \frac{1}{k^2} (Q_\phi' \cos \phi_k + N_\phi \sin \phi_k + a_1 k N_{x\phi} - w)$$

$$M_2 = - \frac{1}{k^2} (Q_\phi' \sin \phi_k - N_\phi \cos \phi_k + b_1 k N_{x\phi})$$

$$\partial M_3/\partial x = -\left(M_\phi + b_1 (Q_\phi \cos \phi_k + N_\phi \sin \phi_k)\right.$$

$$\left.- a_1 (Q_\phi \sin \phi_k + N_\phi \cos \phi_k)\right)$$

$$F = - N_{\phi x}/k \tag{2.34}$$

$$u_B = \frac{1}{k} \left(\frac{F}{A_1 E} + a_1 \frac{M_1}{EI_1} + b_1 \frac{M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2} \left(\frac{M_2}{EI_2} - \frac{a_1}{EI_3} \frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2} \left(\frac{M_1}{EI_1} + \frac{b_1}{EI_3} \frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2} \left(\frac{\partial M_3}{\partial x} \cdot \frac{1}{EI_3}\right) \tag{2.35}$$

At corner D

$$M_1 = -\frac{1}{k^2} (- Q_\phi' \cos \phi_k + N_\phi \sin \phi_k - a_1 k N_{\phi x} - w)$$

$$M_2 = -\frac{1}{k^2} (+ Q_\phi' \sin \phi_k + N_\phi \cos \phi_k + b_1 k N_{x\phi})$$

$$\partial M_3/\partial x = - (- M_\phi + b_1(Q_\phi \cos \phi_k - N_\phi \sin \phi_k) - a_1(Q_\phi \sin \phi_k + N_\phi \cos \phi_k))$$

$$F = N_{x\phi}/k \tag{2.36}$$

$$u_B = \frac{1}{k} \left(\frac{F}{A_1 E} + a_1 \frac{M_1}{EI_1} - b_1 \frac{M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2} \left(+ \frac{M_2}{EI_2} - \frac{a_1}{EI_3} \frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2} \left(+ \frac{M_1}{EI_1} - \frac{b_1}{EI_3} \frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2} \left(\frac{1}{EI_3} \frac{\partial M_3}{\partial x}\right) \tag{2.37}$$

General junction beam conditions

At a junction beam all sets of shell forces and moments will operate on the beam. So, by combining the four previous cases, the junction-beam equations become

$$M_1 = -\frac{1}{k^2}\Big[(Q_\phi' \cos\phi_k + N_\phi \sin\phi_k - a_1 k N_{\phi x})_A$$

$$+(-Q_\phi' \cos\phi_k + N_\phi \sin\phi_k + a_1 k N_{\phi x})_B$$

$$+(Q_\phi' \cos\phi_k + N_\phi \sin\phi_k + a_1 k N_{\phi x})_C$$

$$+(-Q_\phi' \cos\phi_k + N_\phi \sin{}_{\phi k} - a_1 k N_{\phi x})_D - w\Big]$$

$$M_2 = -\frac{1}{k^2}\Big[(Q_\phi' \sin\phi_k - N_\phi \cos\phi_k + b_1 k N_{\phi x})_A$$

$$+ (Q_\phi' \sin\phi_k + N_\phi \cos\phi_k + b_1 k N_{\phi x})_B$$

$$+ (Q_\phi' \sin\phi_k - N_\phi \cos\phi_k + b_1 k N_{x\phi})_C$$

$$+ (Q_\phi' \sin\phi_k + N_\phi \cos\phi_k + b_1 k N_{x\phi})_D\Big]$$

$$\frac{\partial M_3}{\partial x} = -(M_\phi + a_1(Q_\phi \sin\phi_k - N_\phi \cos\phi_k) + b_1(Q_\phi \cos\phi_k + N_\phi \sin\phi_k))_A$$

$$-(M_\phi - a_1(Q_\phi \sin\phi_k - N_\phi \cos\phi_k) - b_1(Q_\phi \cos\phi_k - N_\phi \sin\phi_k))_B$$

$$-(M_\phi - a_1(Q_\phi \sin\phi_k - N_\phi \cos\phi_k) + b_1(Q_\phi \cos\phi_k + N_\phi \sin\phi_k))_C$$

$$+(M_\phi - a_1(Q_\phi \sin\phi_k - N_\phi \cos\phi_k) - b_1(Q_\phi \cos\phi_k - N_\phi \sin\phi_k))_D$$

$$F = -\frac{1}{k}\left[(N_{x\phi})_A - (N_{x\phi})_B + (N_{x\phi})_C - (N_{x\phi})_D\right] \qquad (2.38)$$

and at corner A

$$u_B = +\frac{1}{k}\left(\frac{F}{A_1E} - \frac{a_1M_1}{EI_1} + \frac{b_1M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} + \frac{a_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} + \frac{b_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2}\left(\frac{\partial M_3}{\partial x}\cdot\frac{1}{EI_3}\right) \qquad (2.39)$$

At corner B

$$u_B = +\frac{1}{k}\left(\frac{F}{A_1E} - \frac{a_1M_1}{EI_1} - \frac{b_1M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} + \frac{a_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2}\left(\frac{M_1}{EI_1} - \frac{b_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2}\left(\frac{1}{EI_3}\cdot\frac{\partial M_3}{\partial x}\right) \tag{2.40}$$

At corner C

$$u_B = \frac{1}{k_1}\left(\frac{F}{A_1E} + \frac{a_1M_1}{EI_1} + \frac{b_1M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} - \frac{a_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2}\left(\frac{M_1}{EI_1} + \frac{b_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2}\left(\frac{1}{EI_3}\cdot\frac{\partial M_3}{\partial x}\right) \tag{2.41}$$

at corner D

$$u_B = \frac{1}{k_1}\left(\frac{F}{A_1E} + \frac{a_1M_1}{EI_1} - \frac{b_1M_2}{EI_2}\right)$$

$$v_B = -\frac{1}{k^2}\left(\frac{M_2}{EI_2} - \frac{a_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$w_B = -\frac{1}{k^2}\left(\frac{M_1}{EI_1} - \frac{b_1}{EI_3}\frac{\partial M_3}{\partial x}\right)$$

$$\theta_B = -\frac{1}{k^2}\left(\frac{1}{EI_3}\cdot\frac{\partial M_3}{\partial x}\right) \tag{2.42}$$

The extreme outside bay of the section are similar to the general junction beam condition but with only two sets of shell forces acting on the edge beam.

Properties of the edge beams

The 2nd moment of area about Y axis (I_1), the 2nd moment of area about Z axis (I_2) and the area of edge beam (A_1) are easily determined.

$$I_1 = \frac{4}{3} b_1 a_1^3 \qquad I_2 = \frac{4}{3} a_1 b_1^3$$

$$A_1 = 4a_1 b_1$$

Timoshenko, in his book Theory of Elasticity, shows that the torsional moment M_3, in a beam of rectangular cross-section, is related to the twist per unit length in the following manner

$$M_3 = \frac{1}{3} Gk_1(2b_1)^3 (2a_1) \frac{\partial\bar{\theta}}{\partial x} \tag{2.43}$$

where $k_1 = (1 - \frac{192}{\pi^5} \sum \frac{1}{n^5} \tanh \frac{n\pi a_1}{2b_1})$ and the torsional rigidity

$$G = \frac{E}{2(1 + n)}$$

Therefore $\dfrac{\partial M_3}{\partial x} = \dfrac{1}{3} Gk_1(2b_1)^3 (2a_1) \dfrac{\partial^2\bar{\theta}}{\partial x^2}$

and this is the moment-twist equation that we have used

$$\frac{\partial M_3}{\partial x} = EI_3 \frac{\partial^2\bar{\theta}}{\partial x^2}$$

Thus,

$$I_3 = k_1(2b_1)^3 (2a_1)/6(1 + n)$$

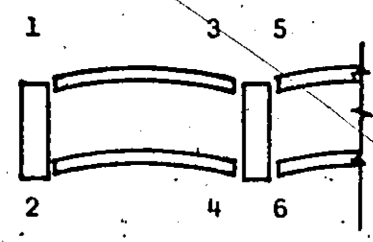


Fig. 11

<u>Boundary conditions</u>

$(\ )^J_B$ displacements of junction beam

$(\ )_s$ displacements of shell.

at joint 1 and 2

(1) $(u)^J_B = (u)_s$

(2) $(v)^J_B = (v)_s$

(3) $(w)^J_B = (w)_s$

(4) $(\theta)^J_B = (\theta)_s$

For each joint these give four equations in a_1 to h_1, a_2 to h_2, a_3 to h_3 and a_4 to h_4

at joint 3, 4, 5 and 6

(1) $(u)^J_B = (u)_s$

(2) $(v)^J_B = (v)_s$

(3) $(w)^J_B = (w)_s$

(4) $(\theta)^J_B = (\theta)_s$

For each joint these give four equations in a_1 to h_1, a_2 to h_2, a_3 to h_3 and a_4 to h_4 .

The stresses in a box girder can be determined by considering the box as an assemblage of flat plates. Each plate can be analysed by the degenerate shell theory using a cylindrical shell program with a modified radius and half angle as input data as described in the introduction.

Single box section girder

The above description concerned about the general theory of cylindrical shell with edge beam. A computer program was written to treat the single box section as four degenerate cylindrical shells without edge beam.

(1) under uniformly distributed load, use four bays

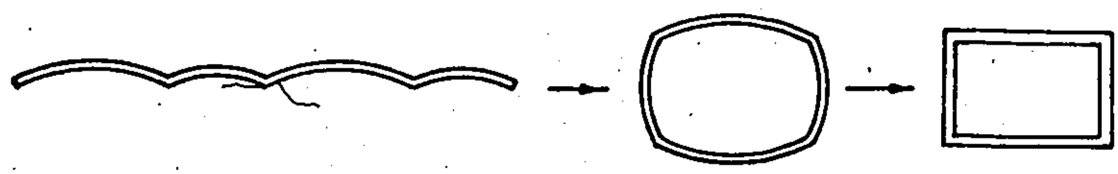


Fig. 12 Single Box Girder under U.D.L.

(2) under partial or line load use six bays.

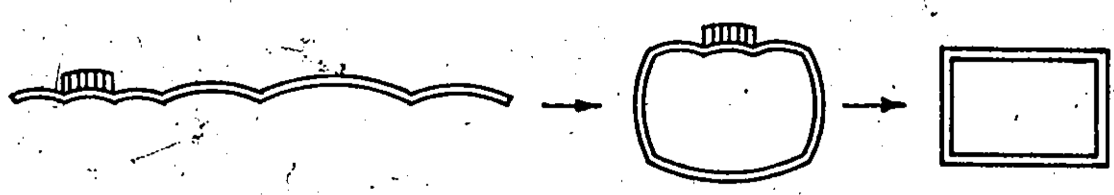


Fig. 13 Single Box Girder under Partial Surface Load

by this way both flange and web are treated as plate. Logically, the results should be more accurate than using cylindrical shell with edge beam theory which consider the flange as a plate and the web as a beam. Those two sets of results were compared and found the difference is not large.

Computer analysis of the problem

It is worthy of comment that without the use of a computer the theoretical calculation in this thesis would have proved extremely tedious.

A general computer program had been written to perform the analysis of the degenerate shell method described previously. The program was written in FORTRAN IV language for the IBM-360/65 computer

Detailed descriptions of the input, output, sign conventions of this program are given in Appendix A. The flow chart of the program is shown in Fig. 13a.

A brief description of the program is given below.

(a) Input data

(1) Dimension of the structure in terms of span, number of bays.

(2) Dimension and material properties of each flange and web in terms of radius,half angle, slope, --- etc.

(3) Magnitudes and locations of externally applied loads.

(b) Output

(1) The complete input data is printed as a check.

(2) For each flange and web, all internal forces and displacements are printed at the selected point of midspan.

(c) Limitation

The program has no limitation except the size of the memory of the computer. The present program was written to accept

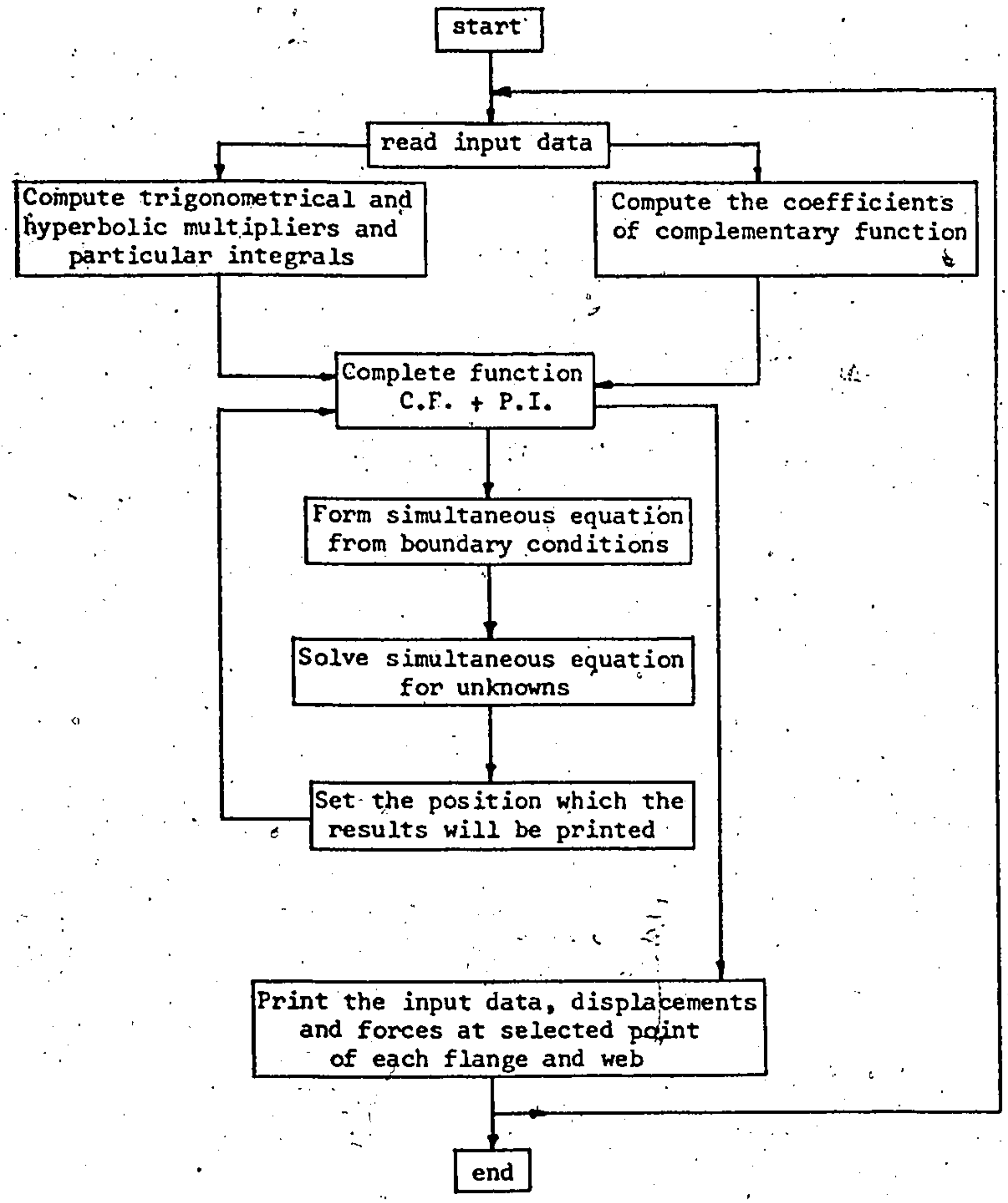


Fig. 13a Flow Chart of the Program

up to 20 bays which would be adequate in practice.

2.3 ELEMENTARY BEAM THEORY

In this elementary theory it is considered that the whole structure acts as a beam simply supported at both ends. Under symetrical loading conditions, the longitudinal moments are simply determined as for a simply supported beam and then the longitudinal forces are determined from a knowledge of the 2nd moment of area of the cross-section. Under the unsymmetrical loading condition the torsional effect will be added into the system. The advantage of this method of calculating stresses is that it involves at the most one page of calculation and serves as a ready means of checking the suitability of dimensions.

(1) <u>Symmetrical loading conditions</u>

Assume the deflections are uniform across the transverse direction.

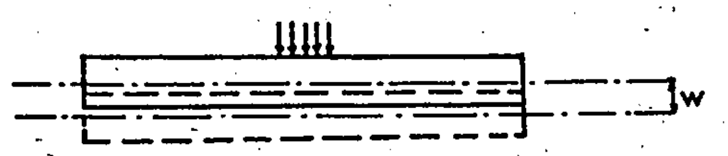


Fig. 14 Symmetrical Loading Conditions

then apply the ordinary beam theory to the system.

The following assumptions are implied in the elementary beam theory:

(a) The deformations of the cross-section in its plane are negligible.

(b) M_y and $M_{x\phi}'$ may be ignored.

(c) The strain γ_{xy} caused by the shear force $N_{x\phi}$ and poisson's ratio effects are neglected.

If L is the span of the girder between supports, the maximum bending moment M at the center of the structure will be given by

$$M = wL^2/8$$

From which stresses may be determined in the usual way and so that the force in the top plate (N_x) will be

$$(N_x)_{top} = \frac{M}{I} \cdot y \cdot 2A$$

The force in the bottom plate

$$(N_x)_{bottom} = \frac{M}{I} \cdot y' \cdot 2A$$

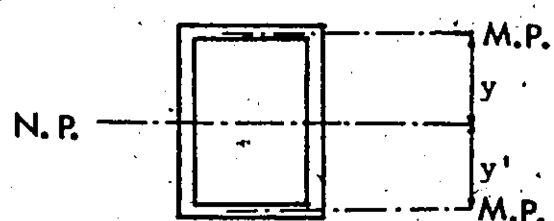


Fig. 15

I: the 2nd moment of area of the cross-section.

y: the distance between N.A. and the central plane of the top flange

y': the distance between N.A. and the central plane of the bottom flange

2A: the thickness of the plate.

Stress of top plate $(f)_{top} = (N_x)_{top}/2A$

Stress of bottom plate $(f)_{bottom} = (N_x)_{bottom}/2A$

Vertical shear stress in the web

V: shearing force

Q: moment of area respect to N.A

b: width of beam

I: moment of inertia of the section

for Box section b is not uniform, so

$$\tau = \frac{VQ}{bI} = \frac{V}{I}\left(\frac{h^2}{4} - y^2\right) + \frac{Vb}{2It}\left(\frac{d^2}{4} - \frac{h^2}{4}\right)$$

The maximum value of τ occurs at N.A. where $y = 0$ and is given by

$$\tau_{max} = \frac{Vh^2}{4I} + \frac{Vb}{2It}\left(\frac{d^2}{4} - \frac{h^2}{4}\right)$$

$$(N_{x\phi})_{max} = \tau_{max} \cdot t$$

at the top or bottom of the web where $y = \frac{1}{2} h$

$$\tau = \frac{Vb}{2It}\left(\frac{d^2}{4} - \frac{h^2}{4}\right)$$

$$N_{x\phi} = \tau \cdot t = \frac{Vb}{2I}\left(\frac{d^2}{4} - \frac{h^2}{4}\right)$$

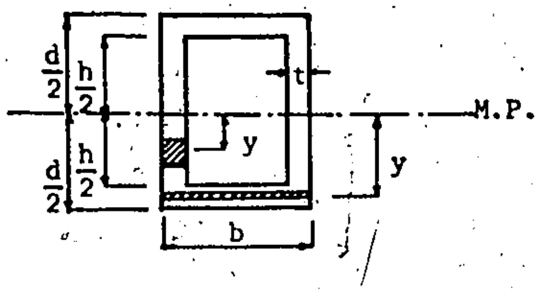


Fig. 16

The stresses of transverse direction cannot be found by beam theory. It can be solved by considering it as a one-way slab. Since those stresses are not the most important one and the elementary beam theory here only used to estimate the dimensions of the structure so it has not been calculated here.

(2) Unsymmetrical loading conditions

In those case the system will be considered as a combination of the symmetrical loading and the torsional effect. The unsymmetrical load can be replaced by a load acting on the centroid of the cross-section plus a torque which equal to load times eccentric arm. Because the bridge has a finite width the deflection is increased by the effect of torsion on one side of the section and decreased on the other side. The magnitude of the bending stress is proportional to the deflections, thus the problem reduces to finding the deflections. It is assumed the deflection curve due to torque is an inclined straight line as shown in Fig. 17.

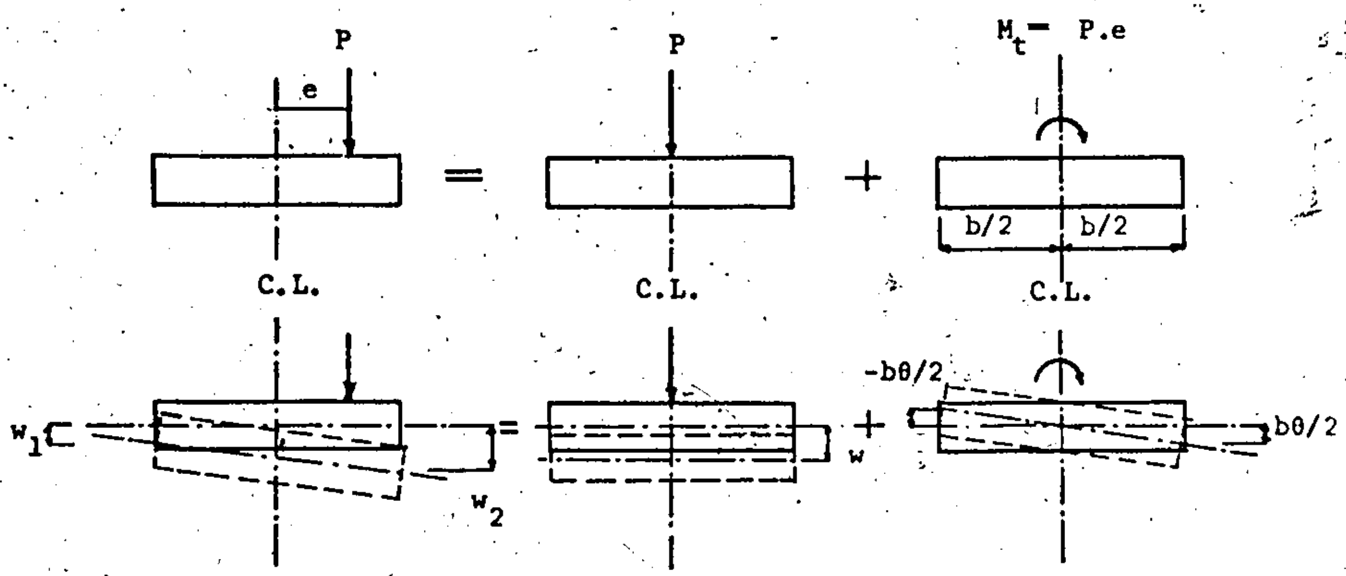


Fig. 17 Unsymmetrical Loading Condition

$$w_1 = w - b\theta/2$$

$$w_2 = w + b\theta/2$$

w can be obtained from elementary beam theory as mentioned in last article.

$$\theta = \frac{M_t}{J}$$

$$M_t = P \cdot e \qquad \text{---- twisting moment}$$

For a box cross-section using the thin tube method.

$$J = \frac{4A^2G}{\int \frac{ds}{t}}$$

A is the mean of the areas enclosed by the outer and the inner boundaries of the cross-section. s is the length of the center line of the section. t is the thickness.

(a) Maximum bending stress

It is assumed that the load condition is a uniformly distributed load on one half of the bridge and that bridge is torsionally fixed at the supports. The distribution of torque can be found.

$$\theta_{max} = \frac{wb^2L^2}{64\ GJ}$$

Hence,

$$\frac{\sigma_2}{\sigma} = \frac{w + \frac{b}{2}\theta}{w} = \frac{\frac{5}{384} w \cdot \frac{b}{2} \cdot \frac{L^4}{EI} + \frac{1}{128} \frac{wb^2L^2}{GJ}}{\frac{5}{384} w \frac{b}{2} \frac{L^4}{EI}}$$

(b) The maximum shearing stress occurs at the center of the long side of the cross-section

$$\tau_{max} = \frac{M_t}{\alpha bc^2}$$

b is the length of the long side and c is the length of the short side of the cross-section. G is modulus of rigidity. α and β are constants depending on the value of b/c, which is given in Timoshenko's book "The theory of elasticity"(20).

CHAPTER 3

EXPERIMENTAL INVESTIGATION

3.1 INTRODUCTION

Experimental investigations on full sized structures would incur very high costs for which reason most tests are conducted on models.

Model tests are an accepted technique in the analysis of redundant structures. They can be divided into the two broad categories as direct and indirect tests. In direct model tests the model is made such that all its properties are identical to those of the proposed design but on a reduced scale. If the scale loads are applied then the stresses in the model can be scaled up to give the state of stress in the prototype. In the indirect model test a small structure is constructed and the results from the test are used to verify a proposed theory which is then used to design the full sized structure.

All the tests described below are indirect model tests with the model studies being used to check the accuracy with the degenerate cylindrical shell theory in predicting the behavior of box girder bridge. Perspex was chosen to be the model material as its low modulus of elasticity (0.42×10^6 psi), enables a high response (strain) to be obtained for a low load.

3.2 PROPERTIES OF PERSPEX**

Perspex is poly-methyl-methacrylate, a water-white solid formed by the polymerisation of methyl methacrylate which is a volatile, clear organic liquid of boiling point 100°C.

One of the outstanding properties of "perspex" actylic materials in clear form is their very high light transmission, combing high surface finish with complete absence of colour. The resistance of 'perspex' to outdoor exposure is also outstanding and in this respect it is markedly superior to other thermoplastics. 'perspex does not have a sharp melting point but softens gradually as the temperature is increased. At a temperature of 150° to 160° it is sufficiently rubber-like to be shaped easily and simply.

It is noted that the mechanical properties of perspex depend markedly on the temperature at which they are measured the rate at which the 'perspex' is stressed or strained, and to a lesser extent on the presence of absorbed water which tends to act as a plasticiser, but this last effect can usually be ignored.

Values for the perspex mechanical properties are list in table (4) mechanical properties of perspex at 20°C.

** Those informations came from "perspex properties" published by Imperial Chemical Industries Limited, Plastic Division, Welwyn, Garden City, England.

TABLE 4

Property	Units	Mean Value	Standard Deriviation	Remarks
Tensile strength	lb/sq.in	12,000	500	Straining rate 1% per sec.
Flexural strength	lb/sq.in	20,000	1,000	
Shear strength	lb/sq.in	1,100	500	
Young's modulus in flexure	lb/sq.in	4.2×10^5		
Poisson's ratio		0.35		

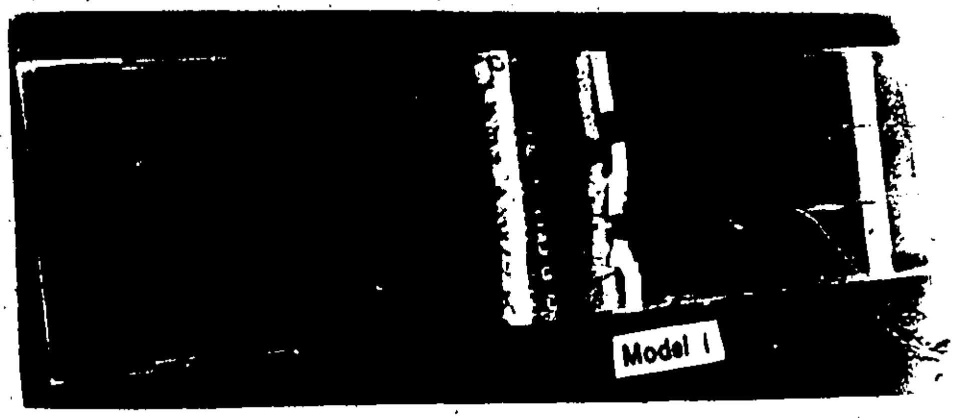
3.3 MODELS AND TESTING APPARATUS, DESCRIPTION OF MODELS

The models, constructed in perspex, were used to verify the theory using a homogenous elastic material. Models of three different dimensions were used which are given in table (5).

The thickness of flanges of Model 1 and 2 are small compared with the overall dimensions of the boundary surface. Those models will be used to prove the thin plate theory. The thickness of flanges of Model 3 was quite thick. This model will be used to check the experimental results from a thick plate system with the theoretical results from thin plate theory.

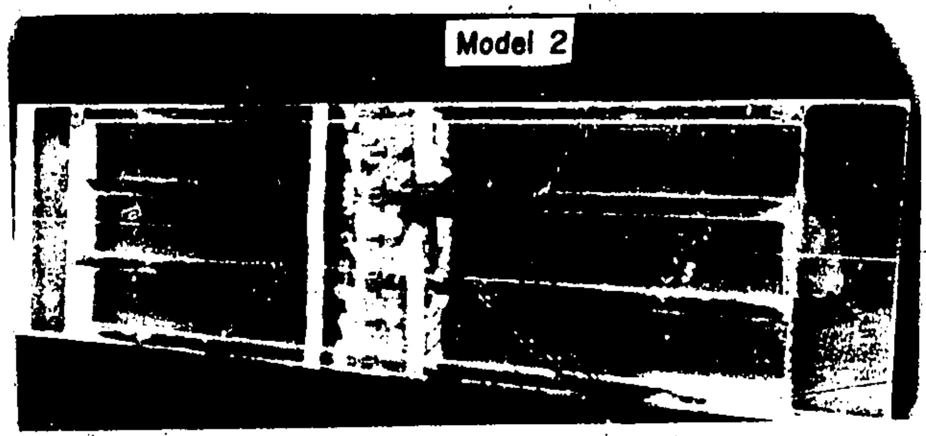
The models were simply supported at their longitudinal boundaries with solid diaphrams at the support points. Another diaphrams with thickness 3/16 inch was used at the midspan of the Model 3.

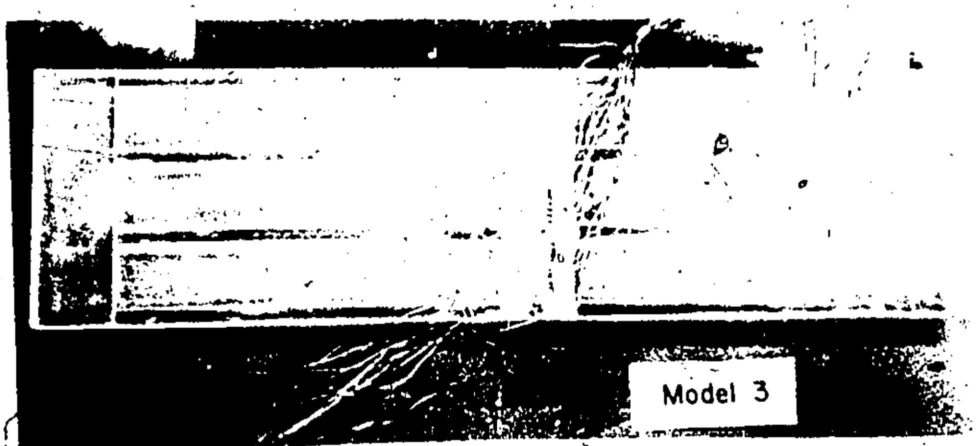
The models are made by simply glueing perspex plates together.



model 1

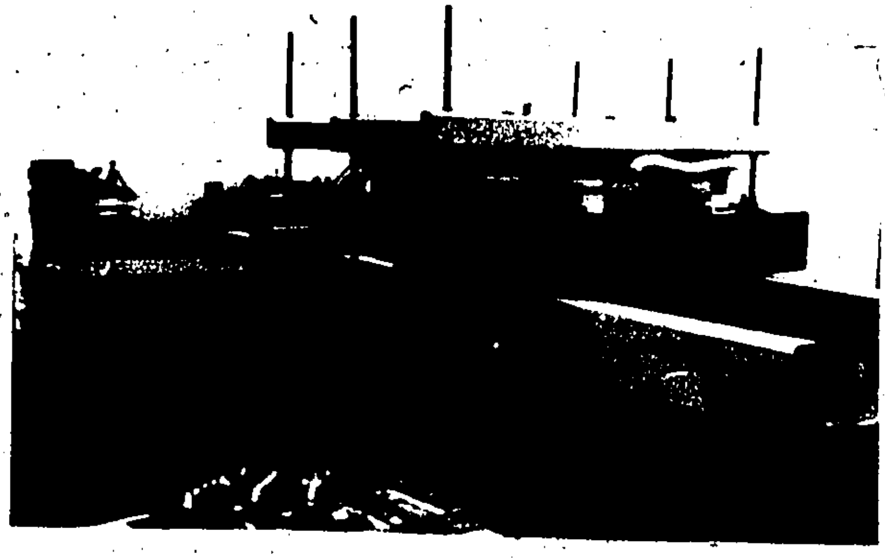
model 2

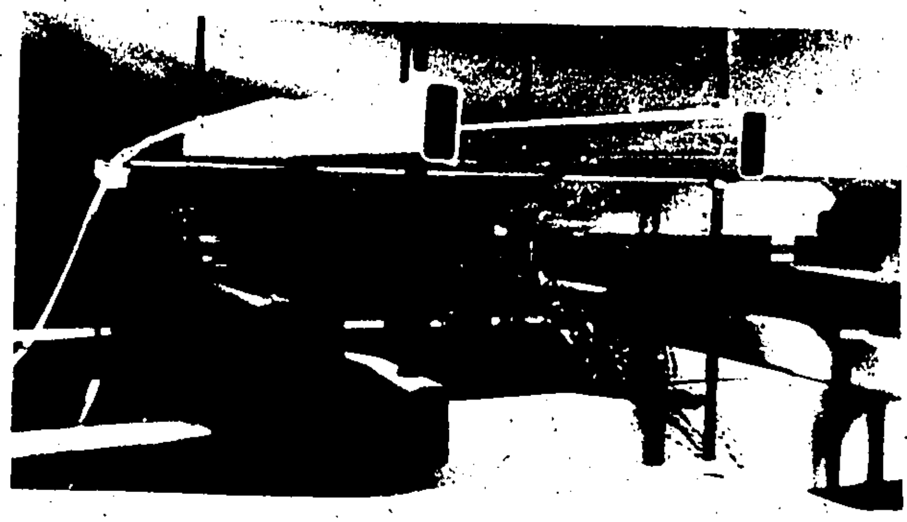




model 3

plate 1 showing the detail of models





air bags

plate 2 showing the u.d.l. system

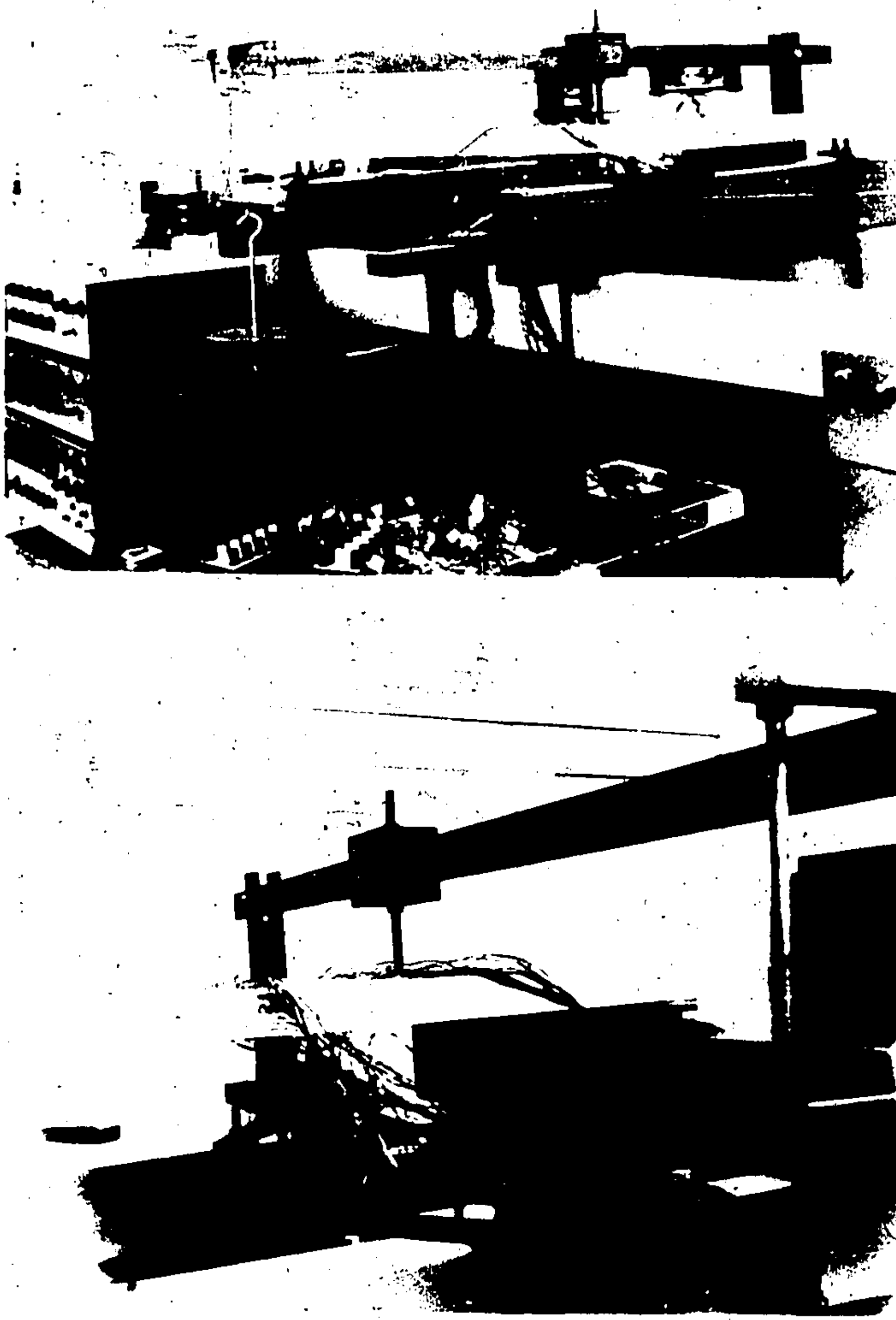


plate 3 showing the point loading system

TABLE 5

Dimensions of Models

Model No.	No. of Box	Thickness of flange	Width of flange	Thickness of web	Depth of web	Width of model	Span length of model
		a	b	c	d	w	
1	3	3/16"	11/3"	1/4"	2"	12"	36"
2	3	3/16"	3"	3/4"	2"	12"	36"
3	3	3/4"	3"	3/4"	2"	12"	36"

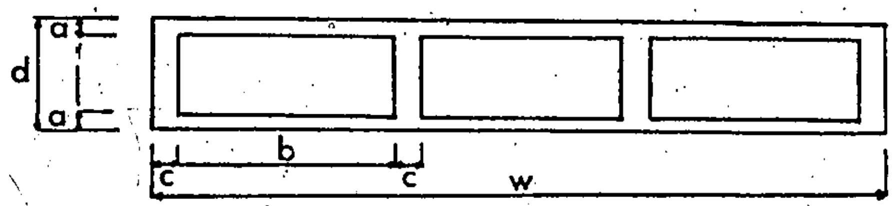


Fig. 20

Loading System

The types of loading which are used to study the behaviour of box girder bridges are uniformly distributed loading, longitudinal line loading and partial surface loading.

(1) Uniformly distributed loading and line loading system.

An air pressure bag loading system, similar to that described by Gardner[8] in his experimental investigation of folded plates was to give the uniformly distributed and line loads. The air bag was placed on

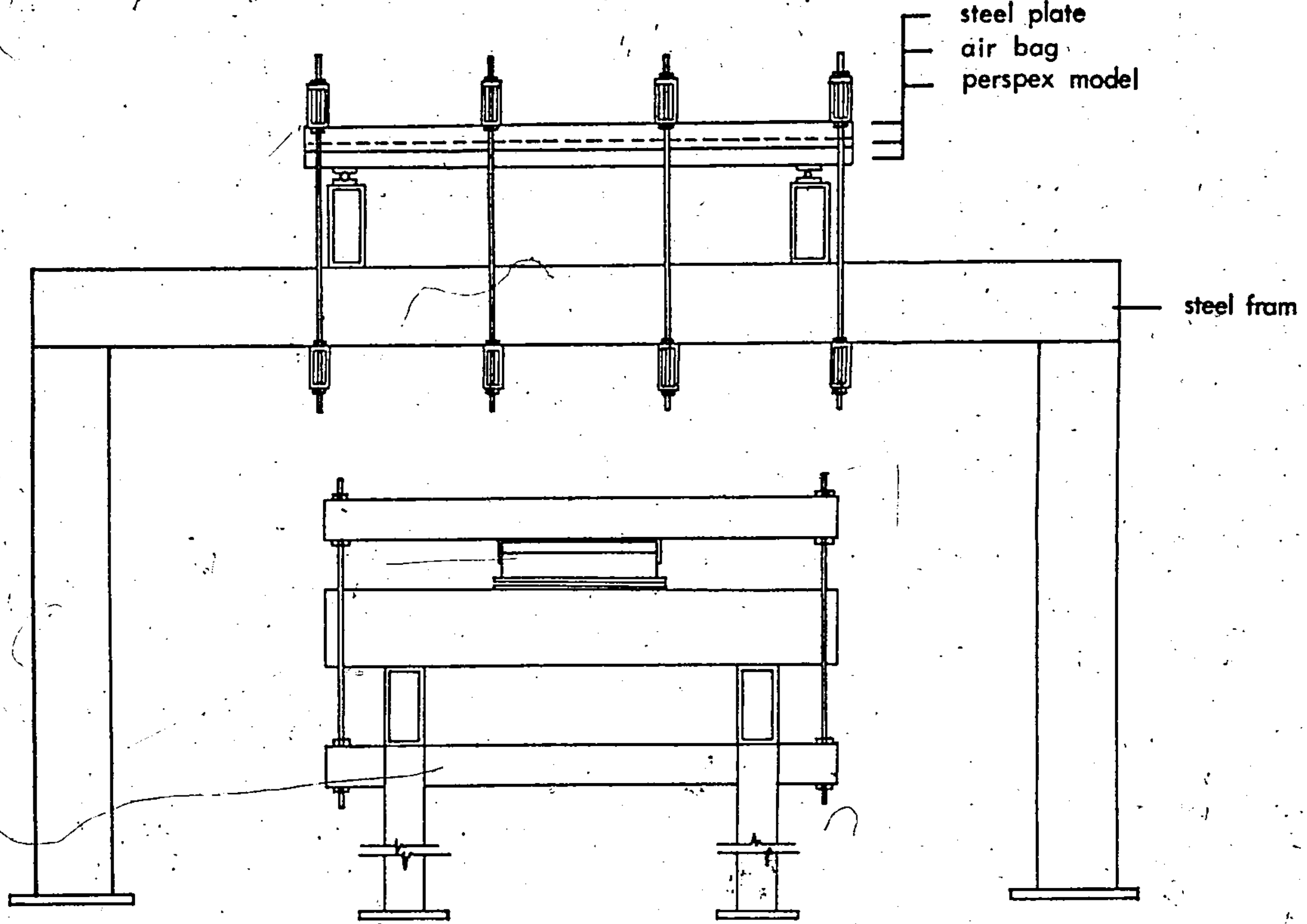


fig.(21) side view of testing apparatus

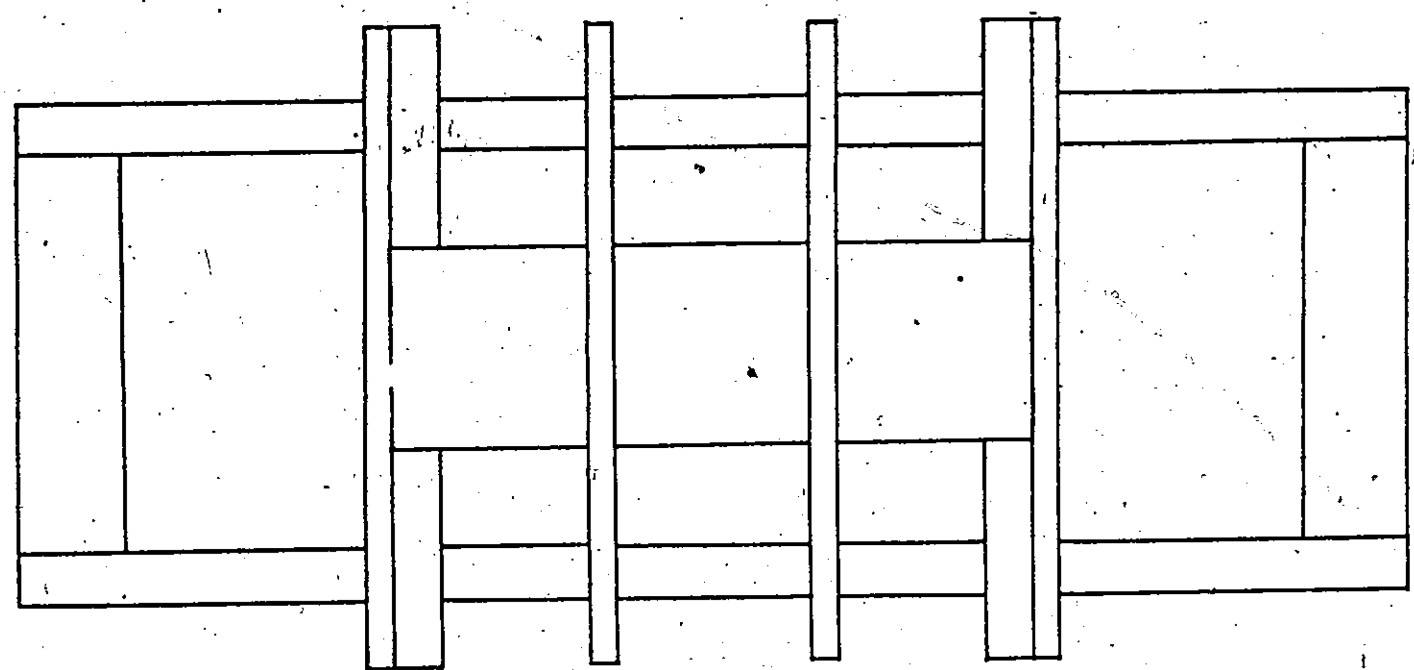


fig.(22) top view of testing apparatus

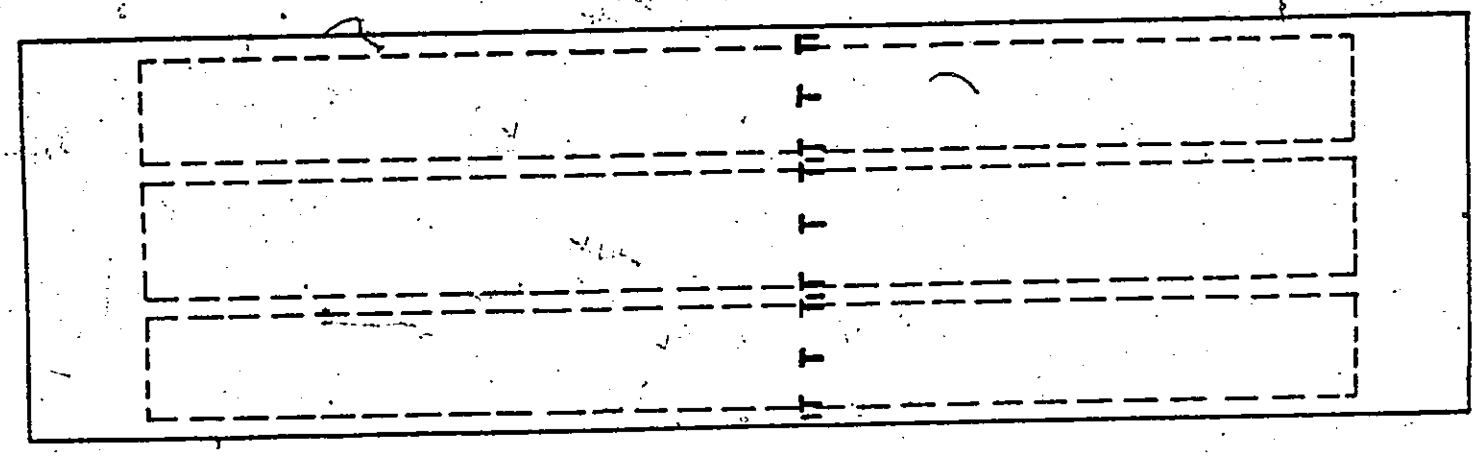


fig.(23) arrangement of strain gauges on models

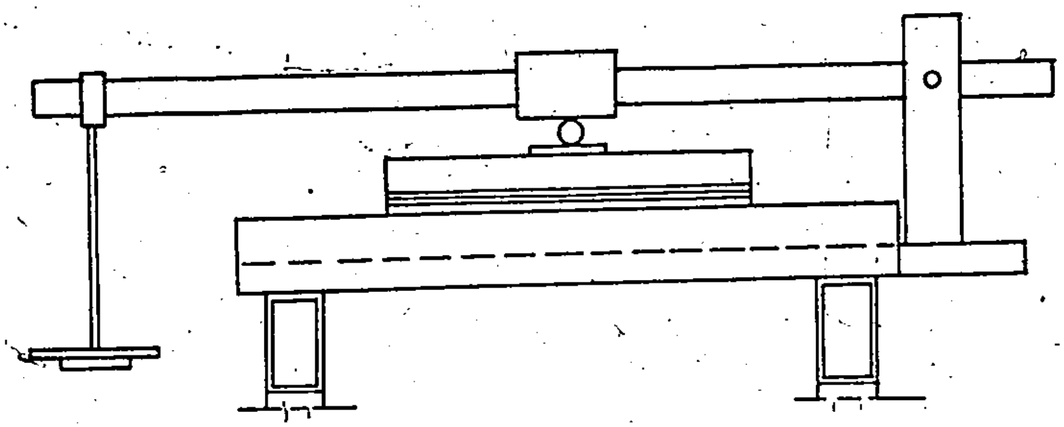


fig.(24) partial surface loading system

the top surface of the model and air bag inflated to provide the load. The top reaction for the air bag was provided by a steel plate firmly anchored to the test frames. The pressure in the air bags was measured by a manometer using water as the fluid medium. Details of the air bags and the uniformly distributed loading system are shown in Plate (2).

(2) Partial surface loading system.

A simple lever device was developed in applying the partial surface loading. The idea is shown in Fig. 25.

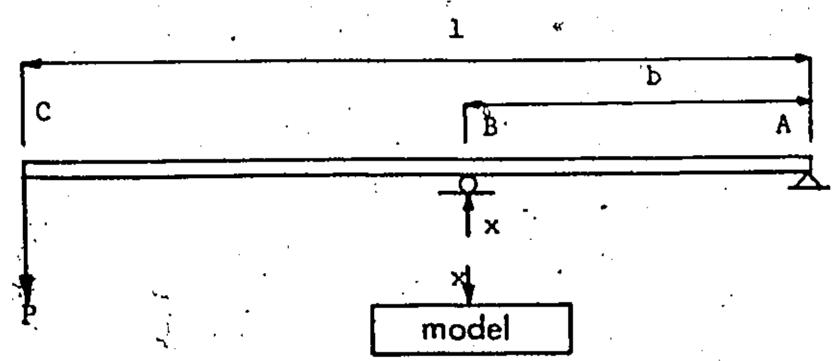


Fig. 25 The Idea of Partial Surface Loading System

An overhanding structure was made. A is a hinge support, B is an intermediate support, l and b are arm length of CA and AB respectively. If a load P is applied on the point C, the load acting on the model will be

$$X = P \times l/b$$

The partial load device is shown in detail in Plate (3).

3.4 TESTING OF THE MODEL

The strains both in the longitudinal and transverse directions on the top and bottom surface at midspan were measured by resistance strain gauges fixed on the models with Eastman 910 adhesive. Normal deflections of the models were measured by a series of dial gauges which were rigidly mounted on the testing rig.

The testing was carried out using uniformly distributed load line load and partial surface load. The uniformly distributed load and line load was carried out by covering the surface of the model with foam rubber pads and placing a series of air pressure bags, the pressure used in those two cases was 1 psi. The partial surface load was carried out as indicated in Fig. 24 and 100 psi load was used in this case.

The deflections were recorded for loads of .1, .2, .3, .4, .5, and 1.0 psi. This test showed the models behaved in a sensibly linear manner.

Before loading the model the dial gauges were set to zero, which remained unchanged for the whole test. The strain gauges were individually balanced before each loading. The model was then loaded and the strains and deflections measured. The load was removed and strain readings and deflections again taken. The readings due to the load were taken as the difference between readings at load and the readings at the subsequent unloaded condition. The strain gauges were then balanced in preparation for the next test. The tests on each type of loading were repeated until three sensibly consistent results were obtained.

3.5 ANALYSIS OF EXPERIMENTAL RESULTS

From the above tests, strains had been obtained for the outer surface of the model along the midspan cross-section both in longitudinal and transverse directions. The quantities ε_x, ε_y and w are ploted as strains with respect to their various positions as shown in the results.

Accuracy of Results

The minimum reading of an electric resistance strain gauge is approximately 2×10^{-6}. Considering perspex models with a modulus of elasticity of about 4.2×10^{5} psi, then the stress can be measured to an accuracy of 0.84 psi. In a box girder the forces and moments vary considerably with location and any error in placing the gauges leads to unknown inaccuracies.

Deflections were measured using 0.001 inch dial gauges. Errors can be present in the system due to the model not being perfectly monolithic.

In addition to the above errors for a supposedly ideal material, there are errors due to the use of perspex as testing material.

(1) Under a sustained load or rapidly repeated load, the creep of the material occurs. The effect of creep of perspex was minimised by balancing the gauges before each test and the time taken for a test was small, less than an hour.

(2) Temperature and humidity were relevant to the much great extent they influence strain gauge reading. This was

minimized by avoiding changes in the ambient conditions, averaging the testing results, and using a half bridge system. The half bridge system was a dummy gauge mounted on perspex to balance the circuit and it was assumed that all changes in temperature, humidity, etc., suffered by the active gauge, were suffered to some extent by the dummy gauge thus giving no net response.

CHAPTER 4

DISCUSSION

4.1 ANALYSIS OF THEORETICAL RESULTS

It is not possible to obtain the direct forces and moments in the top and bottom members of the box section from strain measurements on the external faces. Therefore the forces and moments obtained as results from the computer program were converted into surface strains using the following relationships.

$$\sigma_x = \frac{N_x}{h} + \frac{My}{I}$$

$$\sigma_y = \frac{N_y}{h} + \frac{My}{I}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \eta \frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \eta \frac{\sigma_x}{E}$$

$$I = \frac{h^3}{12(1 - \eta^2)}$$

The modulus of elasticity was taken to be 4.2×10^5 psi and Poisson's ratio to be 0.35.

The experimental and calculated magnitudes of the surface strains and deflections are plotted against position across the midspan section.

The agreement, or otherwise, between the measured and predicted values are discussed model by model.

4.2 DISCUSSION OF RESULTS

Model 1

The ratio of the plate thickness to plate width to plate length was 1 : 19.8 : 192 for the model.

(a) Uniformly distributed load

By observation of the graphs (Fig. A1 to Fig. C15) it can be seen that the deflections calculated using the first order solution of the degenerate shell theory overestimate the experimental results by some 15%. The deflections calculated from the simple beam theory overestimate the measured values by about 25%.

The measured magnitude of the surface strains in the longitudinal direction are overestimated by 15% and 25% from the degenerate shell theory and the simple beam theory respectively.

Similar, rapid variations, of the transverse strain is found from both the degenerate theory and the measured results. The simple beam method can only assume uniform strains in the transverse direction and is thus realistic.

(b) Line load

The degenerate shell theory and simple beam analysis overestimate the experimental values of deflections and surface strains for both the cases of the centre bay only loaded or an edge bay loaded with a U.D.L. by 20%.

For the case of the bridge loaded on one of the exterior bays the simple beam theory still predicts the measured deflections with reasonable accuracy.

(c) Partial surface load.

In general, the errors between prediction and experiment of the partial surface loading conditions are larger than the errors of the uniformly distributed loading condition and line loading condition. Since double sine series were developed in the degenerate shell theory of this case, the errors should be larger, logically. Thus higher order Fourier Series are needed to express the partial surface load for use on the degenerate shell theory.

Using only the first order of Fourier Series, it can be found that all theoretical results overestimate the experimental results, but results predicted by the degenerate shell theory still hold quite well. The distribution of strains in the transverse direction is quite complicated but the degenerate shell theory reasonably predicts the experimental results. The simple beam method can not be used to predict strains in the transverse direction.

Model 2

The dimensions of the flanges in this model are in the ratio of thickness, width, length of 1 : 16 : 192.

The discussion of the results for this model is basically the same as for Model 1. Both Model 1 and Model 2 were designed so that thickness of the flanges was such that thin plate theory could be used. The degenerate shell theory, based on thin shell theory, gives good agreement with the experimental results. Thus the change in web thickness between Model 1 and Model 2 does not affect the errors between the prediction and measured results.

Model 3

This model was designed not to conform to the assumptions of the thin shell theories with relative dimensions of flange thickness, width, length of 1 : 4 : 48. This model had a 0.375 inch thick diaphram at midspan. The degenerate shell theory only assume the diaphrams are uniformly rigid in their own plane, and not allow consideration of the flexibility of the diaphrams.

(a) Uniformly distributed load

Poor agreement is obtained between the results from degenerate theory and experiment, however reasonable agreement is obtained between the simple beam theory and experiment. The experimental strains in transverse direction are quite small, but the variation is not clear.

(b) Line load and partial surface load

The same discussion as (a) can be hold in those two cases.

Very poor agreement between the experimental and theoretical results are obtained in this model. This looks like quite natural, since the thickness of the flanges are not small comparing with other dimensions. This means that proposed degenerate cylindrical shell theory can not be used to dealing with the thick plate. Actually the problem of the thick plate is a three-dimensional problem of elasticity. From the graphs it can be seen the degenerate shell theory underestimates the experimental results. The experimental deflections are nearly double the results from the degenerate shell theory, the strains still larger. This is quite dangerous if used in design problem. Good agreement is obtained between the experimental results and the results from simple beam theory, this means that simple beam theory can again be used to estimate the initial dimensions of the section even in the problem of thick plate.

The strains in the transverse direction have much great errors than the strains in the longitudinal direction. This is probably due to the assumptions used in the theory not being exact. The theory assumes that the edge beam cross-section remains undistorted. In practice the beam yields and stress redistribution take place with the high theoretical stress peaks not being measured in practice. However, in all cases the theoretical results are conservative, up to 50% in some cases.

CHAPTER 5

CONCLUSIONS

From the detailed discussion outlined in Chapter 4 the following conclusions can be down.

(1) The simple beam theory can be used for the preliminary design of cellular section bridges irrespective of the dimensions of the plate elements.

(2) The results presented in this thesis show that, in general, the agreement between the experimental results and the degenerate cylindrical shell theory is quite good, provided the thin shell (plate) assumptions hold. For the uniformly distributed load case the errors between the theoretical results and experimental results of deflection and longitudinal strain is about 15%, which is usual for an analysis using the first term only of the Fourier series. In the transverse direction the agreement between the measured and predicted results is less good, partially due to distorsion of the edge beam.

For the line loading case the errors between the measured and predicted values of deflection and longitudinal strain is some 20%. The difference in the values of the strain in the transverse direction is great due to edge beam deformation. The use of only a first term Fourier series for the concentrated load case would be expected to overestimated the experimental results. Even in this case the theory gives reasonable error between predicted and measured results for the longitudinal strain and deflection. The lack agreement between the theoretical and experimental values of transverse strain is poor due to beam distorsion.

(3) The experimental results described in this thesis show that the thin plate analysis can be used provided that the plate thickness is less than 1/16 of the other two dimensions.

CHAPTER 6

ALTERNATIVE LONGITUDINAL BOUNDARY CONDITIONS

6.1 INTRODUCTION

The theory developed in Chapter 2 is limited in its application to box girder bridges simply supported. For this particular condition the stresses and displacements can be expressed in Fourier series. The method breaks down if we have to deal with other boundary conditions.

Other end conditions may be dealt by a method described by Morice (12). His method suitably modified, then it may be used to solve box girder structures with other end conditions using the same computer program by degenerate cylindrical shell theory presented previously.

A force method can be used to solve the continuous box girder sections which is similar in concept to that used to analyze a continuous beam.

6.2 BASIC FUNCTION METHOD

In Morice's method, basic function is used instead of trigonometric functions. Basic function, first introduced by Inglis(9) have the interesting property that they reproduce themselves on four successive differentiations.

Inglis has developed the mathematical relations for basic functions which are of the form.

$$F(x) = A\ (\cosh\ kx - \cos kx) - (\sinh\ kx - \sin kx)$$

The substitution into this expression of the boundary condition equations leads to a sequence of values of kL and A for various types of boundary conditions.

The expansion of any form of function w in basic functions is of the type

$$w = w_1 F_1(x) + w_2 F_2(x) + \ldots\ldots + w_n F_n(x) \qquad (6.1.1)$$

The determination of coefficient w_n is carried out in the same way as is done in Fourier analysis by multiplying each side of equation (6.1.1) by $F_n(x)$ and integrating from 0 to L. Then, due to orthogonality of the basic functions, we have

$$\int_0^L wF_n(x)dx = w_n \int_0^L F_n^2(x)dx$$

Hence we can write

$$w = \frac{\int_0^L wF_1 dx}{\int_0^L F_1^2 dx} F_1 + \ldots\ldots + \frac{\int_0^L wF_n dx}{\int_0^L F_n^2\, dx} F_n + \ldots\ldots$$

Let us consider three sets of boundary conditions

(1) Both Ends Fixed

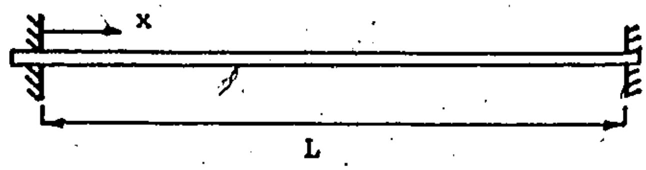


Fig. 26

$$\text{at } x = 0 \quad f(x) = 0 \quad A(\cosh kx - \cos kx) - (\sinh kx - \sin kx) = 0$$

$$\text{at } x = 0 \quad f'(x) = 0 \quad A(\sinh kx + \sin kx) - (\cosh kx - \cos kx) = 0$$

$$\text{at } x = 0 \quad f(x) = 0 \quad A(\cosh kx - \cos kx) - (\sinh kx - \sin kx) = 0$$

$$\text{at } x = 0 \quad f'(x) = 0 \quad A(\sinh kx + \sin kx) - (\cosh kx - \cos kx) = 0$$

Substitute those boundary conditions into basic function.

$$A \cdot 0 = 0$$

$$A \cdot 0 = 0$$

$$A(\cosh kL - \cos kL) - (\sinh kL - \sin kL) = 0$$

$$A(\sinh kL + \sin kL) - (\cosh kL - \cos kL) = 0$$

So

$$A = \frac{\sinh kL - \sin kL}{\cosh kL - \cos kL} = \frac{\cosh kL - \cos kL}{\sinh kL + \sin kL}$$

Hence $\cosh kL \cos kL = 1$

This equation can be solved by computer programme.

$k_1L = 3.927$ $\qquad$ $A_1 = 1.000$

$k_2L = 7.069$ $\qquad$ $A_2 = 1.000$

$k_3L = 10.210$ $\qquad$ $A_3 = 1.000$

In the case of a simply supported ends, the function used in sin kx where the boundary conditions lead to the requirement that

$$kL = \pi,\ 2\pi\ \ldots\ldots n\pi$$

A comparison of the solutions of the three cases given above and the simply supported case, we can find the applicable equivalent length to the simply supported case that means use the modified length instead of original length, the same computer programme can be used to get the answers. In general, we shall be concerned with loading which is uniformly distributed in the direction of coordinate x. Using the basic function analysis described above. We find that the coefficients of the first three terms of the series for a unit uniform loading are as follow:

TABLE 6

	gd_1	gd_2	gd_3
Encastered	0.8165	- 0.0347	0.3789
Cantilever	0.5750	0.4434	0.2753
Mopped Cantilever	0.8584	0.0825	0.3700

It can be seen from the above the behaviour of box girders can be theoretically described by a combination of basic function and

The first three values of kL and A are:

$k_1L = 4.730$ $A_1 = 1.018$

$k_2L = 7.854$ $A_2 = 0.999$

$k_3L = 10.996$ $A_3 = 1.000$

(2) The Cantilever

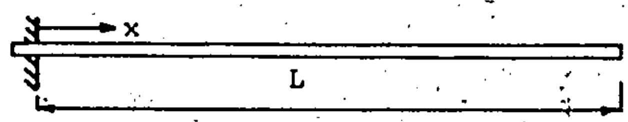


Fig. 27

Using the same technique as before. In this case the application of the boundary conditions the first three values of kL and A are:

$k_1L = 1.875$ $A_1 = 1.362$

$k_2L = 4.695$ $A_2 = 0.982$

$k_3L = 7.854$ $A_3 = 1.001$

(3) The Propped Cantilever

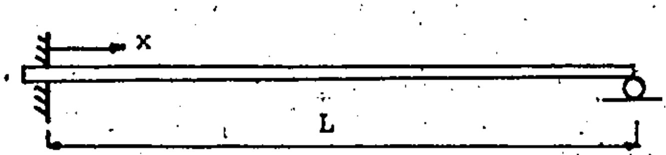


Fig. 28

In this case the boundary conditions give the following coefficient degenerate cylindrical shell theory.

Although basic equation is not strictly a solution to the Gibson equation (8), it is a solution to the Shorer equation. It was felt that little error would be introduced by using the program developed from Gibson's equation.

Since the difference between Gibson and Shorer equations is the appearance of second, fourth, and six order terms in the former. These terms represent small terms in the element and have a negligible effect in the engineering design. The first three orders of $f(\phi)$ were calculated from computer program using three equivalent length, and multiplied by the appropriate values of a $f(x)$ to give the absolute deflection w at various location in the plate. The other terms M_x, M_y, N_x, N_y...... etc., can also be calculated in the same manner.

Consider the case of a box section girder continuous over two equal spans

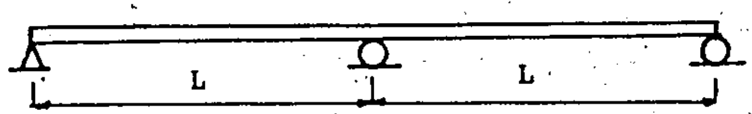


Fig. 29

It is obvious that because of symmetry, each of the girders will have boundary conditions corresponding to a propped cantilever. Consider one of girders, say B,C which is clamped at B and propped at C, it can be solved by the technique mentioned previously.

The disadvantage of this method is we have to know the mode shape first otherwise this method breaks down. For example, for the case of concentrated load or unsymmetrical case, it is very difficult to figure the mode shape. It needs advanced study.

6.3 FORCE METHOD

A force method of analyses can be used to solve the continuous box girder bridges. This method is ideally suited to continuous box girder bridges which have simple supports at the two ends since a harmonic analysis using Fourier series can be used. Other end conditions can be solved by the concept suggested previously.

The conception of the force method is similar to that used to analyze a continuous beam. A primary structure is selected in which the redundants X are taken as the reaction forces at the interior supports since a rigid transverse diaphrams is assured to exist at the interior support, the displacements and rotations in the plane of the diaphram of all points on this cross-section of the bridge should be zero under the influence of the external load P and redundant forces X. The redundant forces X are represented by a set of three joint forces at each joint, consisting of vertical, horizontal and rotational components in the plane of the transverse diaphrams. Thus the number of redundants equal to 3n, n is the number of joints.

The analysis is carried out in the following sequence of

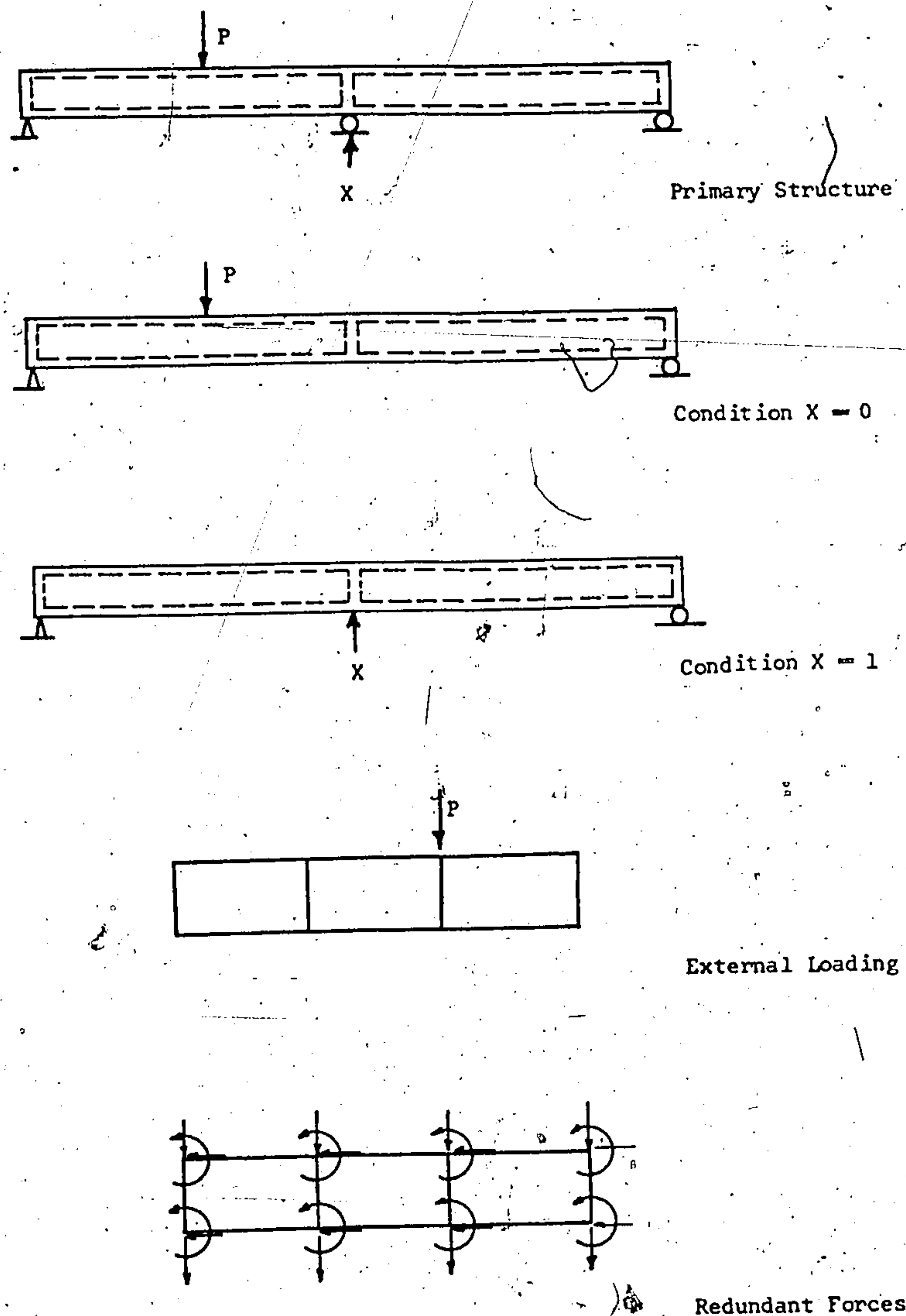


Fig. 30 Force Method

steps:

1. With the redundants set equal to zero, the structure is analyzed under the given external loading P. Because of the simple supports at the two ends, a harmonic analysis using the degenerate cylindrical shell theory may be used to determine displacements at any desired points on the bridge. We can represent the displacements in vector form.

$$\left[\delta_p\right] = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} \tag{6.3.1}$$

2. The structure is then analyzed for unit values of each of the redundant forces, X, and the corresponding flexibility matrix F is formed.

$$\left[\delta\right] = \left[F\right]\left[X\right] \tag{6.3.2}$$

3. The total displacement under the superposition of the external loading p and the redundants X must be zero at each of the points where the redundants exist.

$$\left[\delta\right] = \left[\delta\right]_p + \left[\delta\right]_x = 0 \tag{6.3.3}$$

4. Substituting (6.3.2) into (6.3.3) the redundant forces may be found.

$$\left[\delta\right]_p + \left[F\right]\left[X\right] = 0$$

$$\left[X\right] = \left[F\right]^{-1}\left[\delta\right]_p$$

5. The simply supported structure can be analyzed, subjected to the known loading and the known redundant forces, to determine the final stresses and displacements in the continuous box girder bridge.

APPENDIX A

RESULTS

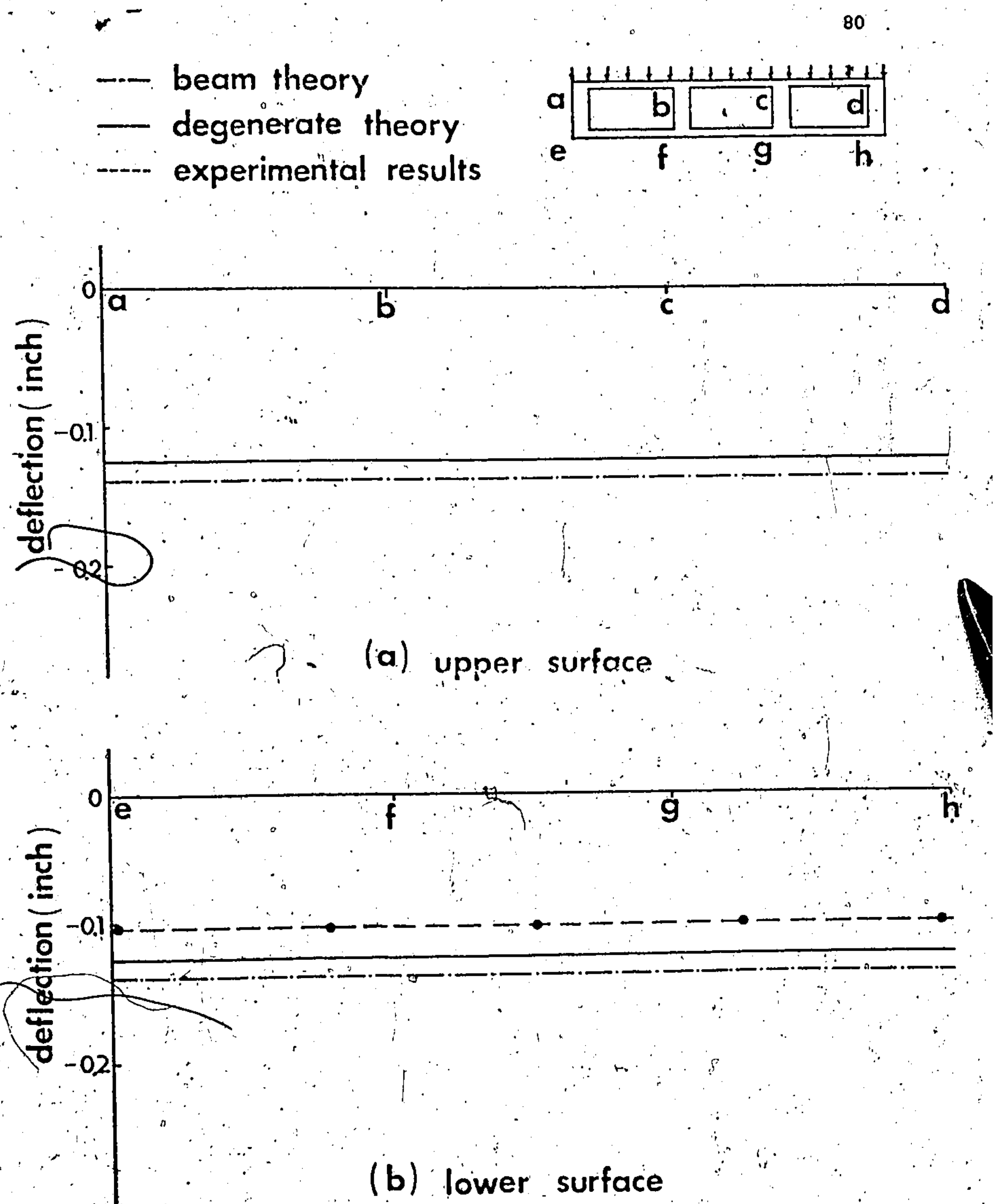


FIG. A1 Comparison of Calculated and Measured Deflections for Model 1 under a U.D.L. of 1 psi

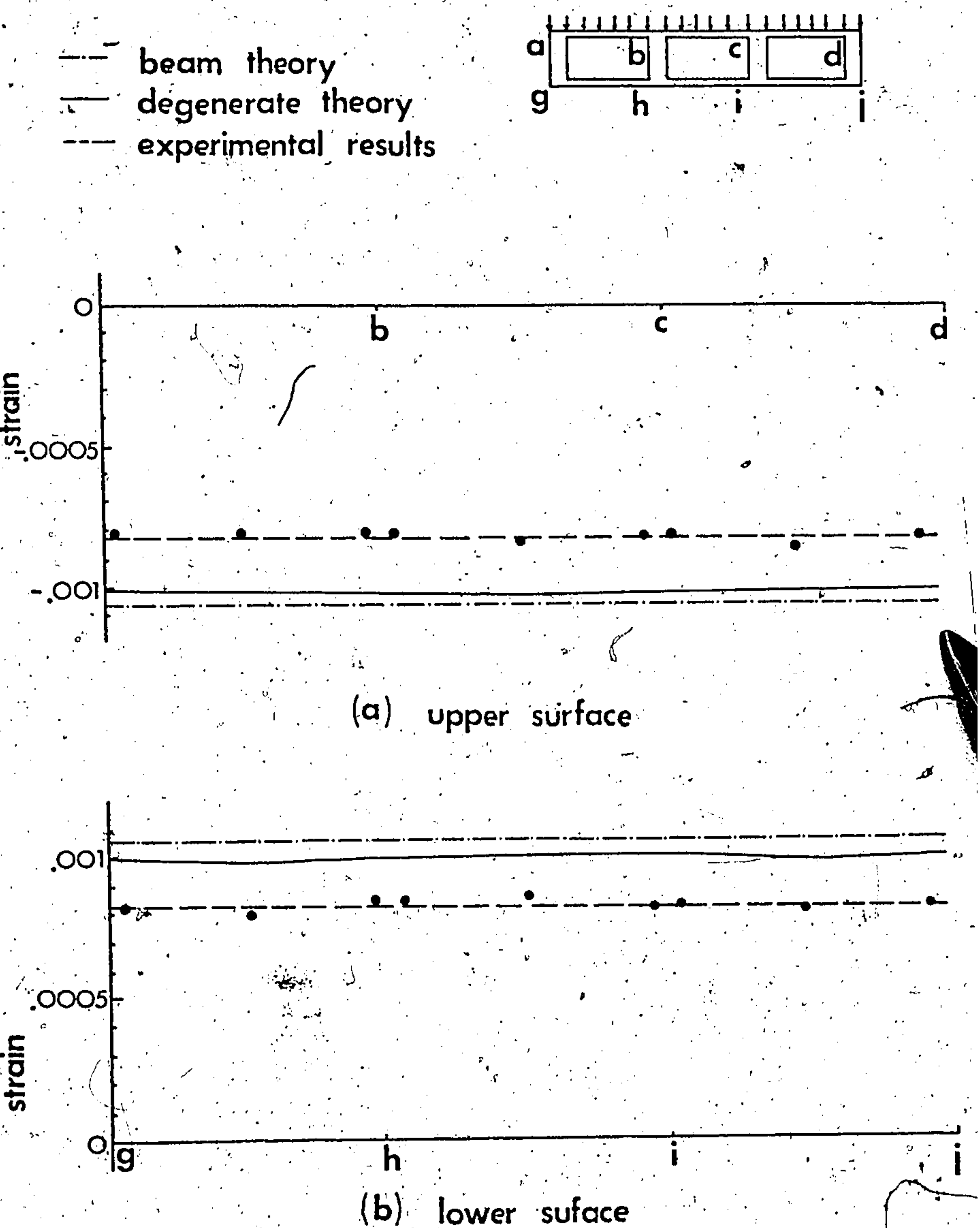
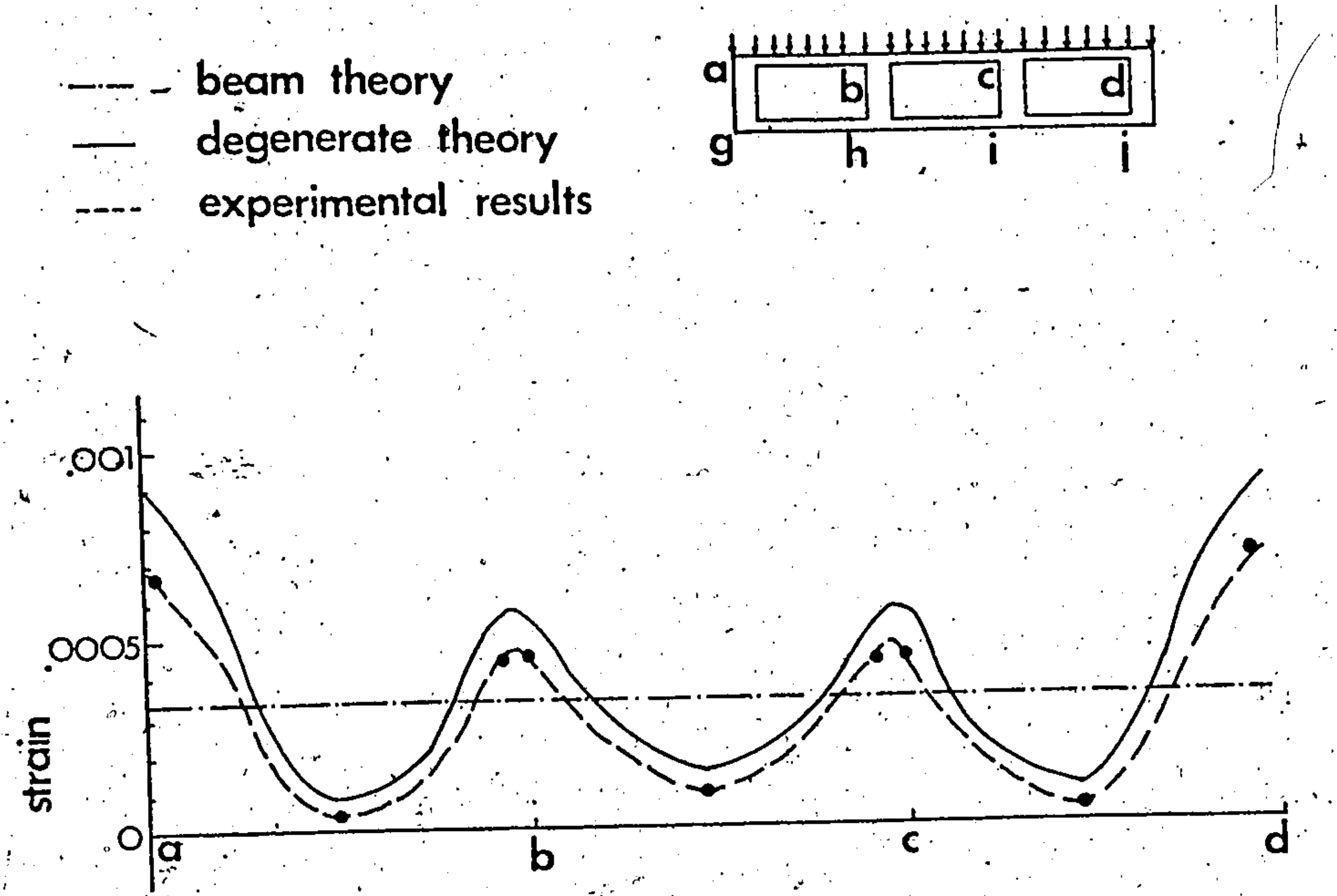
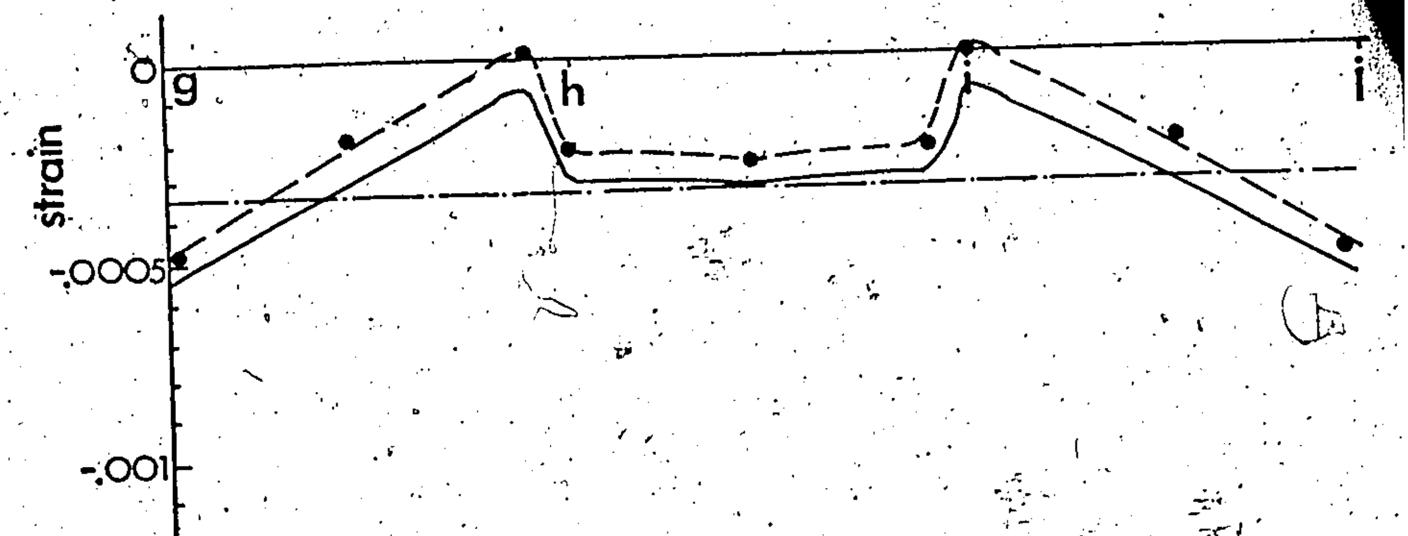


FIG. A2

Comparison of Calculated and Measured Longitudinal Strains for Model 1 under a U. D. L. of .1 psi



(a) upper surface



(b) lower surface

FIG. A3

Comparison of Calculated and Measured Transverse Strains for Model 1 under a U.D.L. of 1 psi

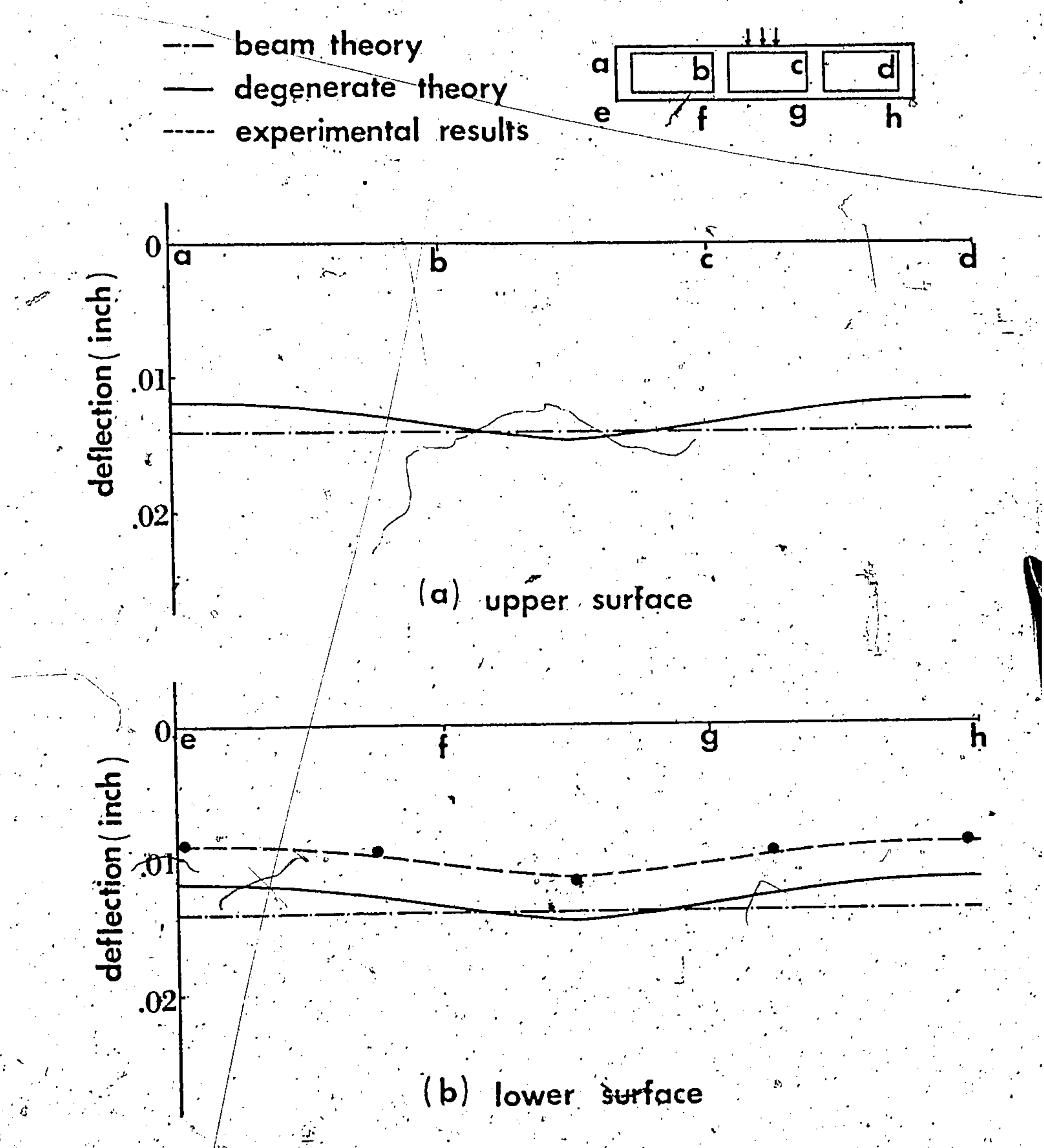


FIG. A4 Comparison of Calculated and Measured Deflections for Model 1 under a Line Load of 1 psi

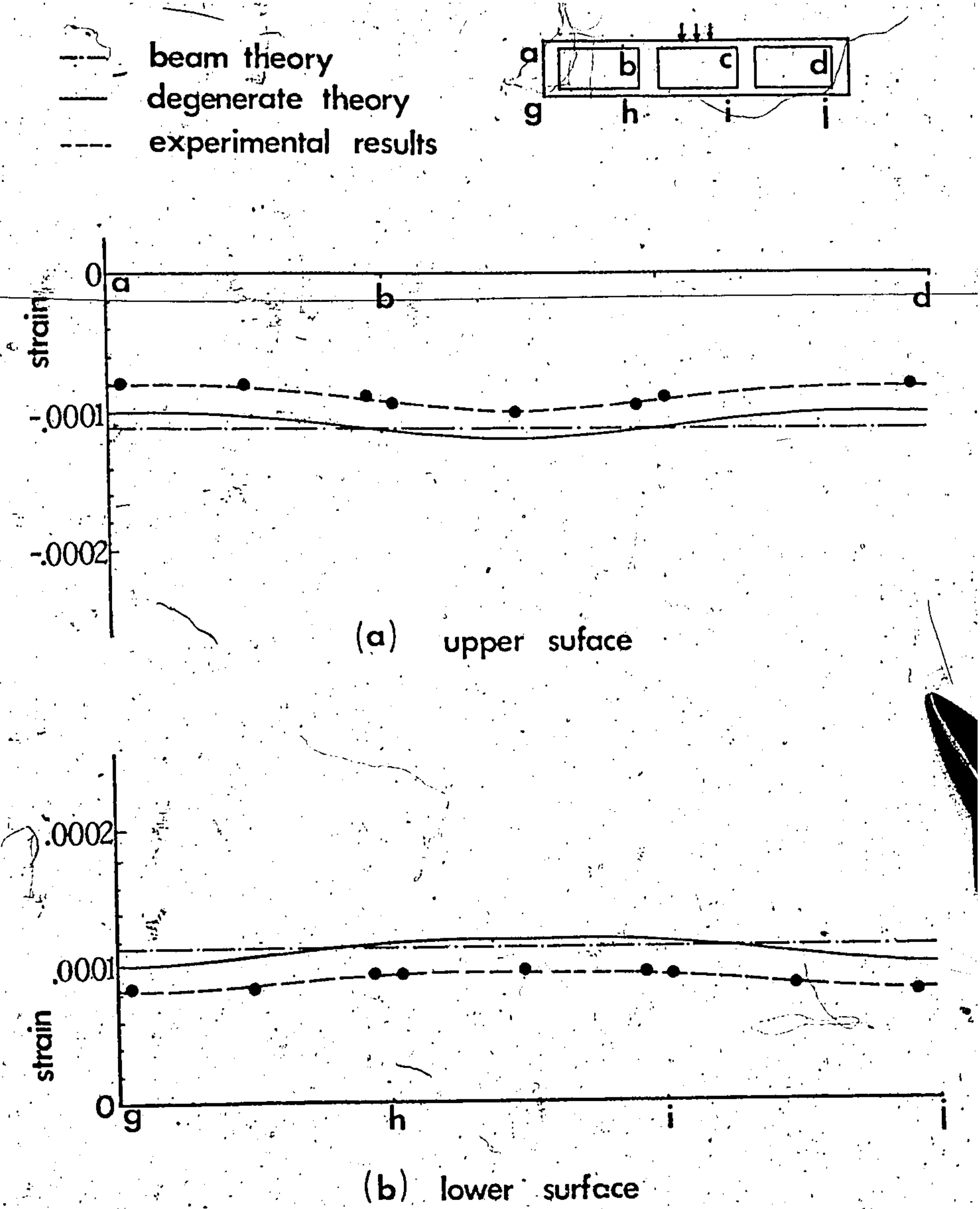


FIG. A5

Comparison of Calculated and Measured longitudinal Strains for Model 1 under a line load of 1 psi

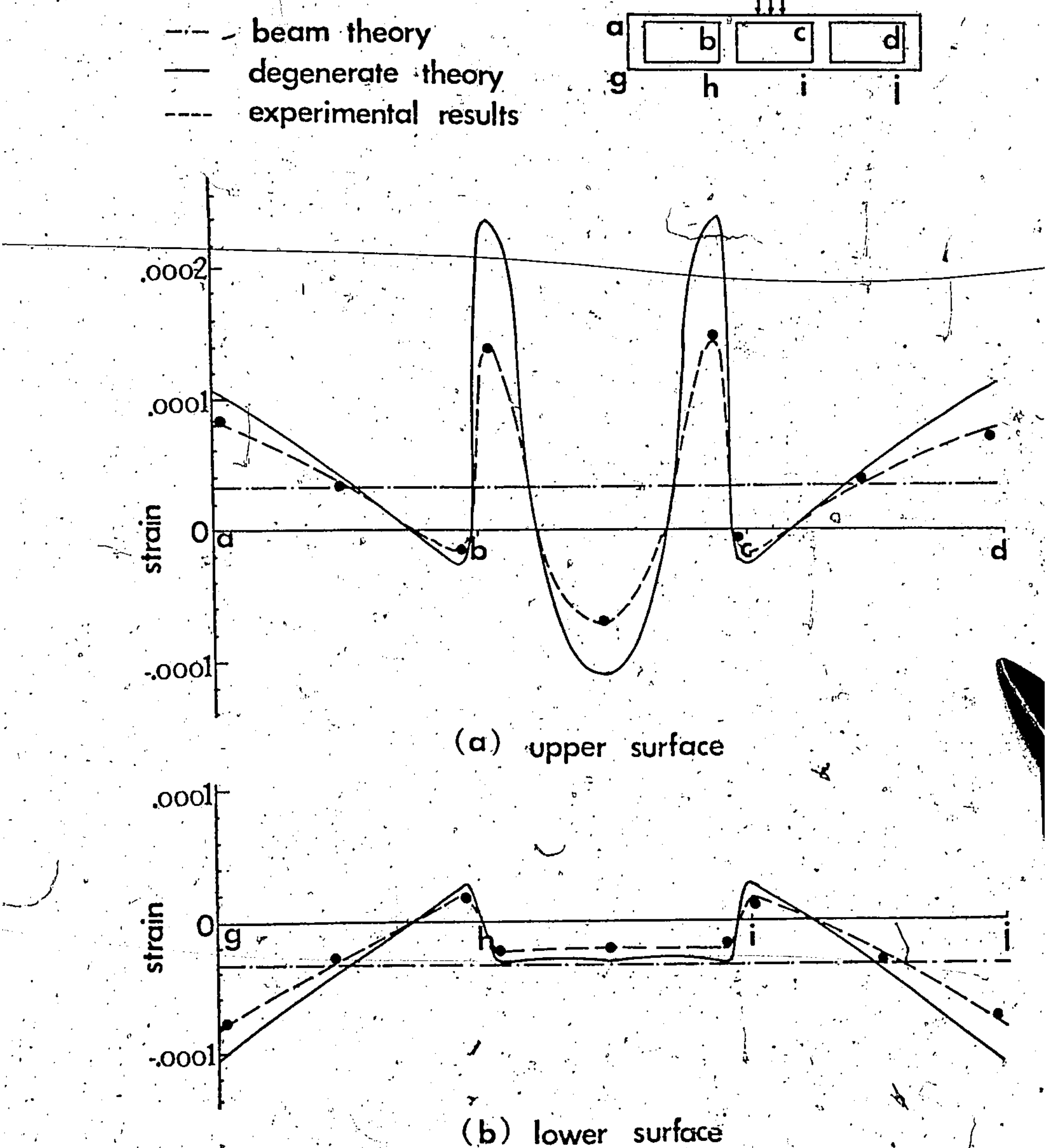


FIG. A6
Comparison of Calculated and Measured Transverse Strains for Model 1 under a line load of 1 psi

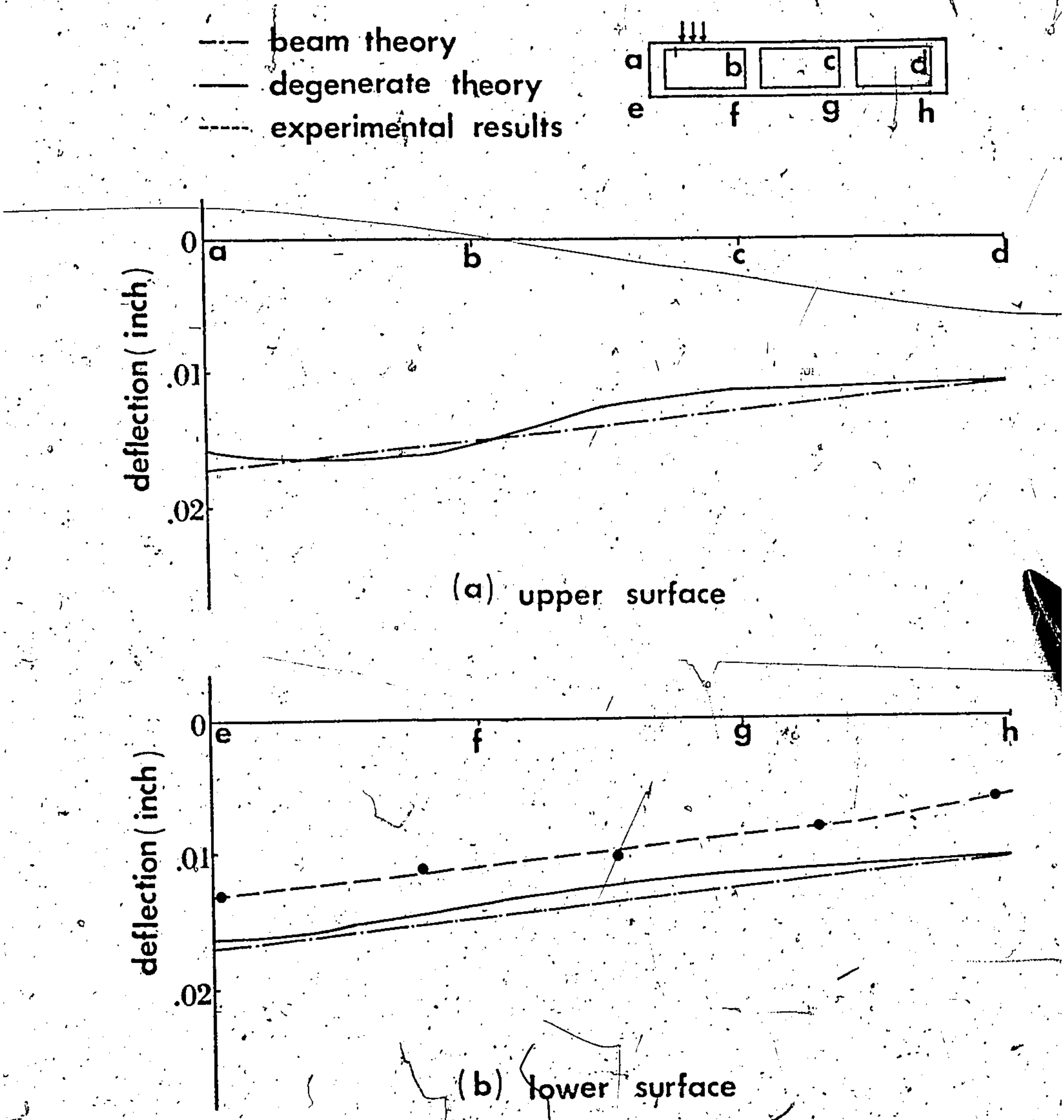


FIG.A7 Comparison of Calculated and Measured Deflections for Model 1 under a Line Load of 1 psi

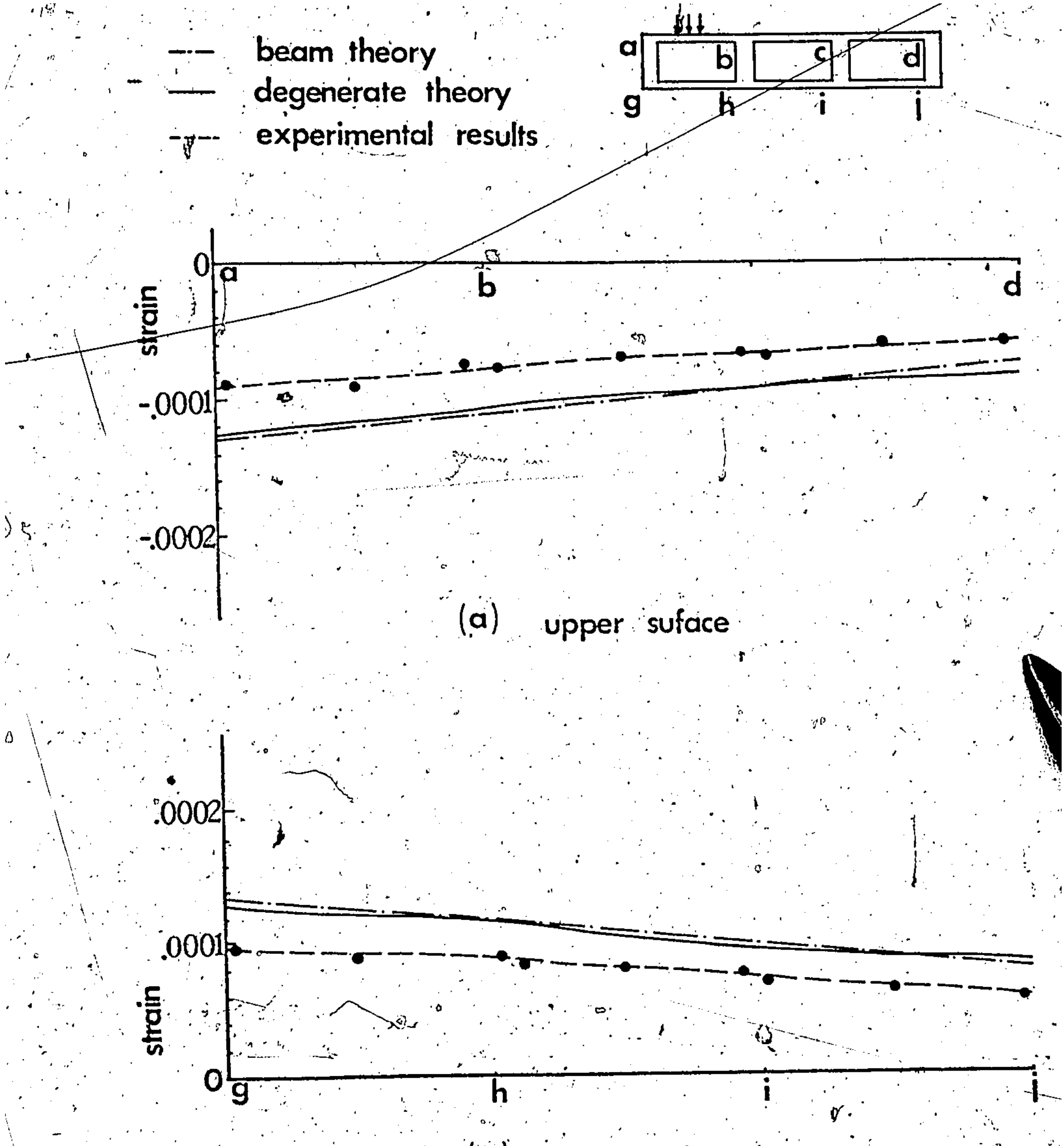


FIG. A 8

Comparison of Calculated and Measured longitudinal Strains for Model 1 under a line load of 1 psi

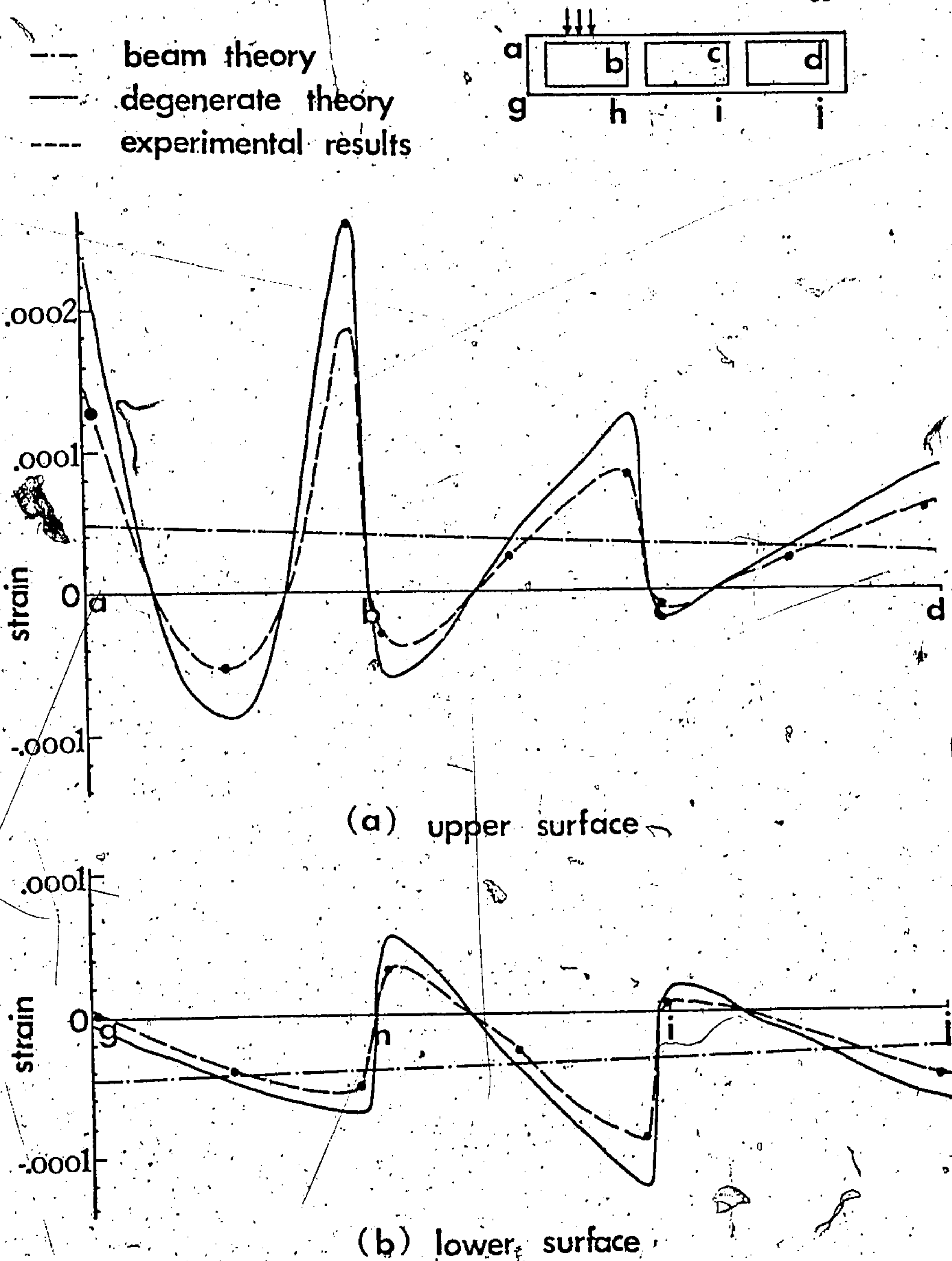


FIG. A9

Comparison of Calculated and Measured Transverse Strains for Model 1 under a line load of 1 psi

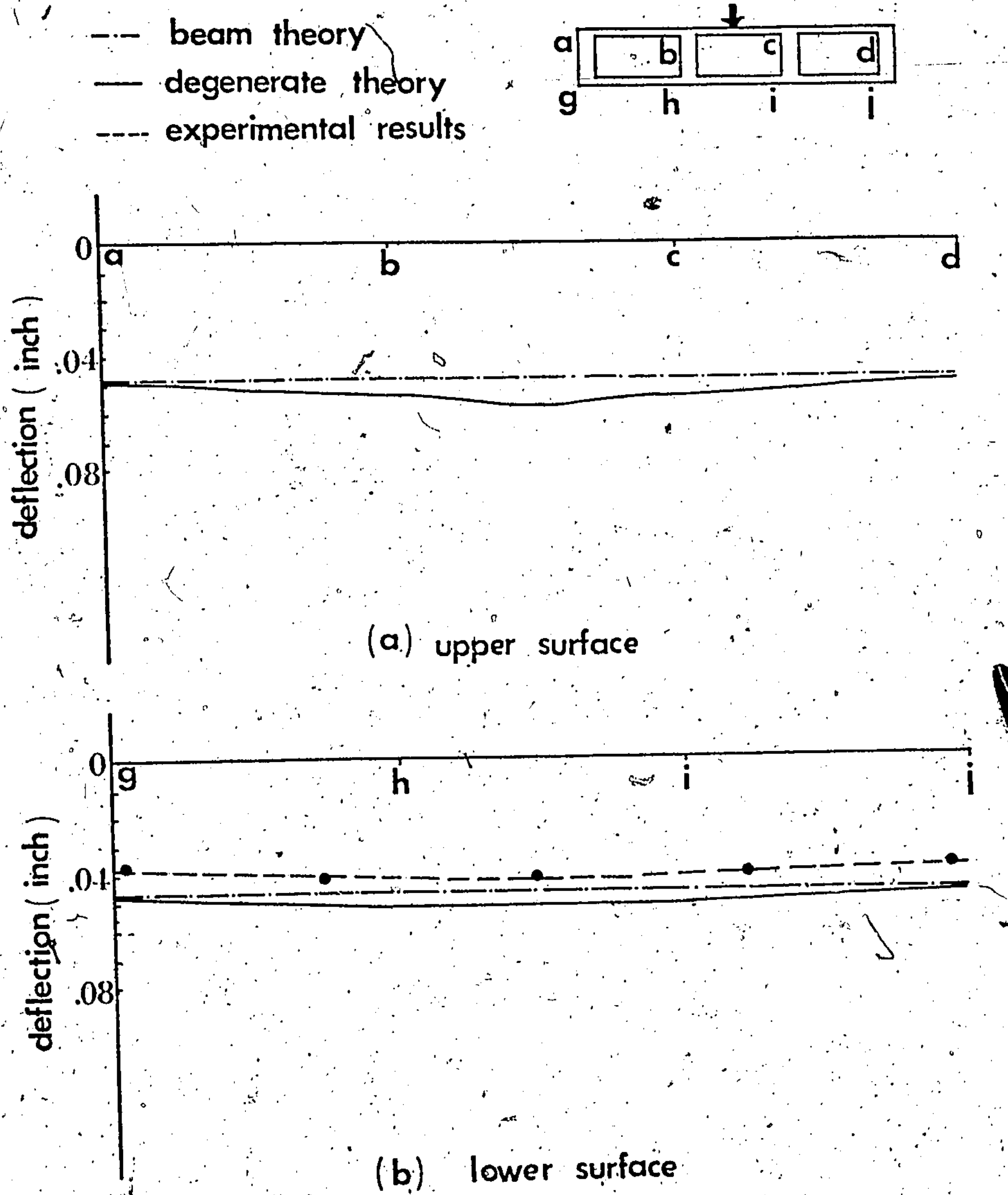


FIG. A10 Comparison of Calculated and Measured Deflections for Model 1 under a Partial Load of 100 psi

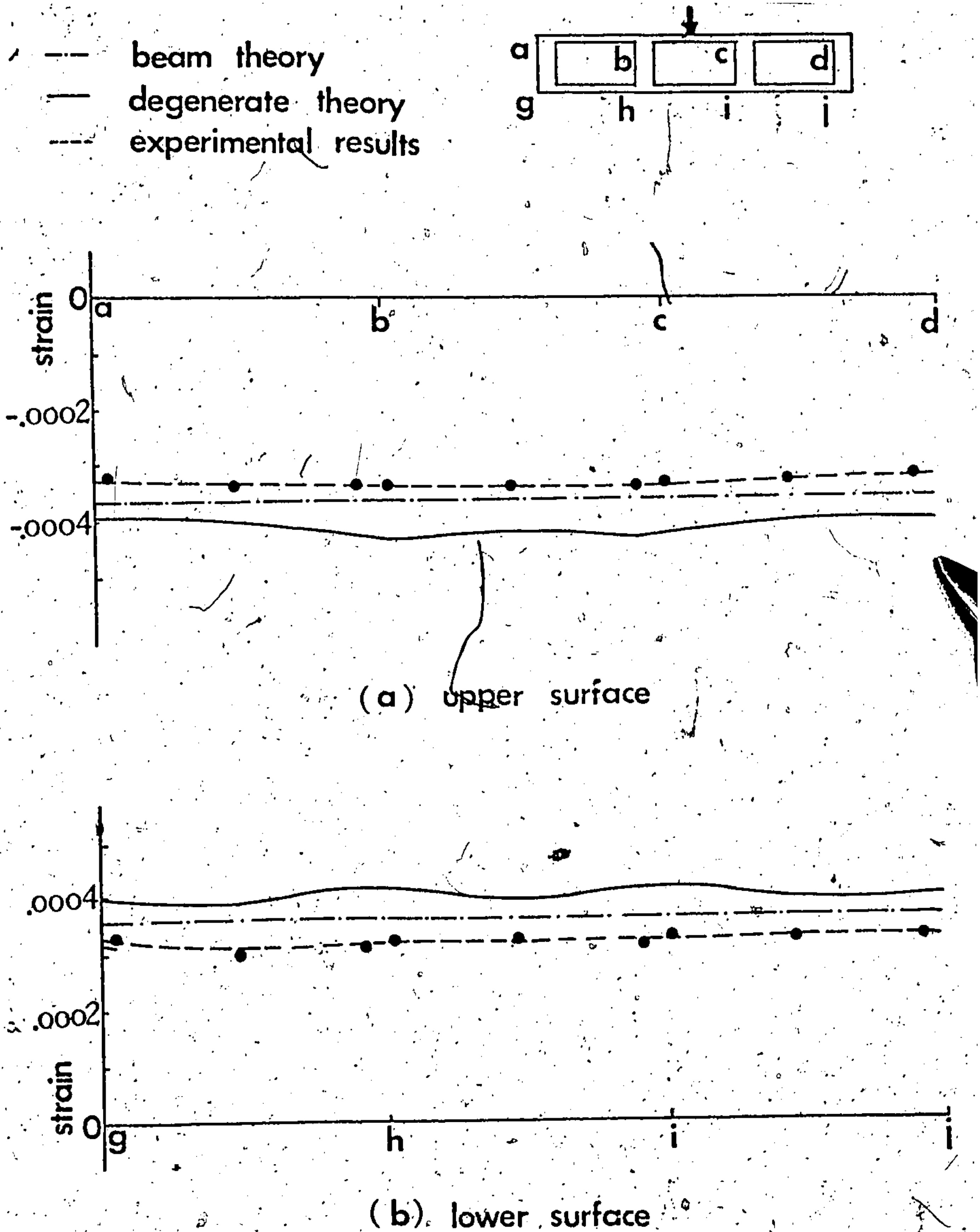


FIG. A11

Comparison of Calculated and Measured longitudinal Strains for Model 1 under a partial Load of 100 psi

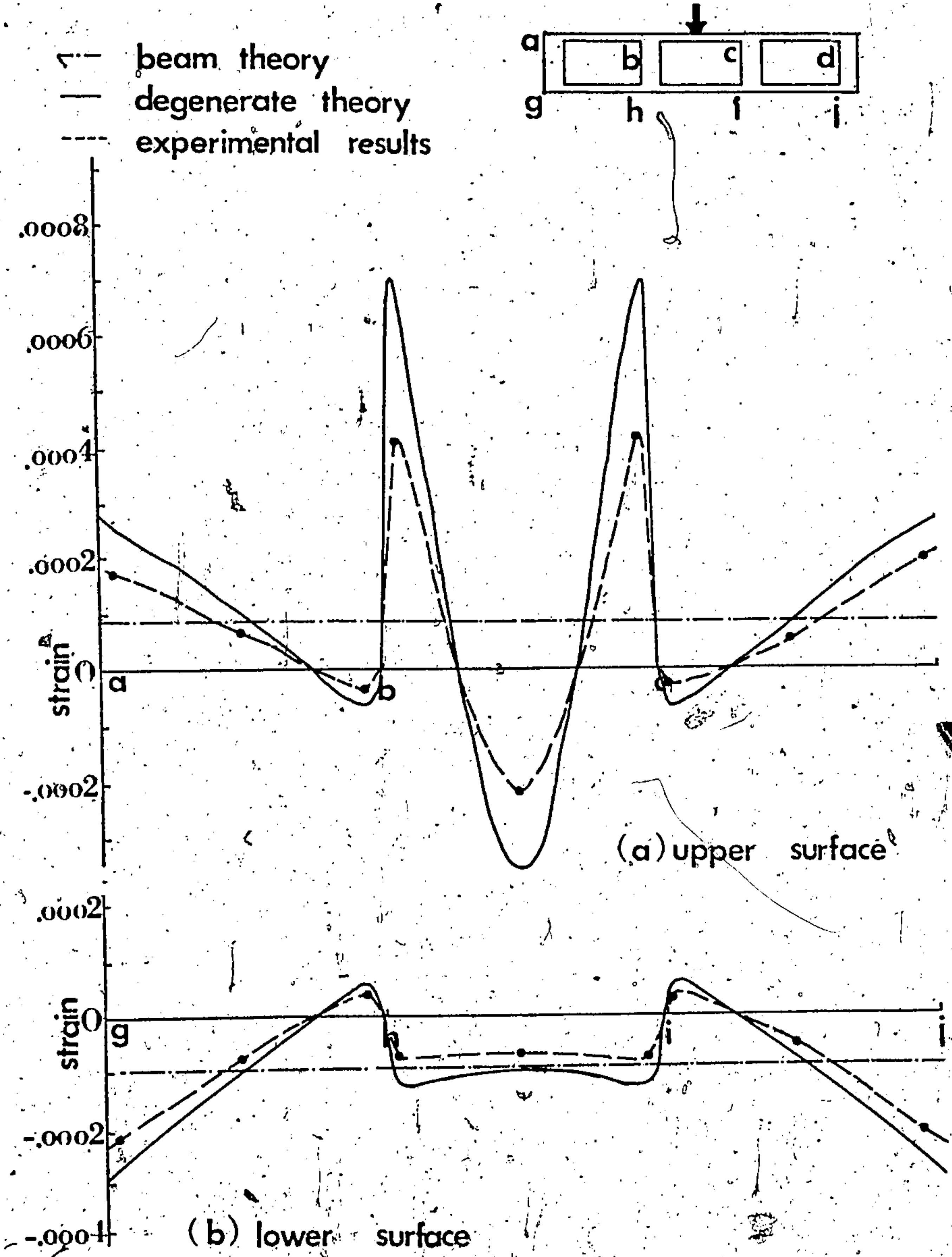


FIG. A 12

Comparison of Calculated and Measured Transverse Strains for Model 1 under a Partial Load of 100 psi

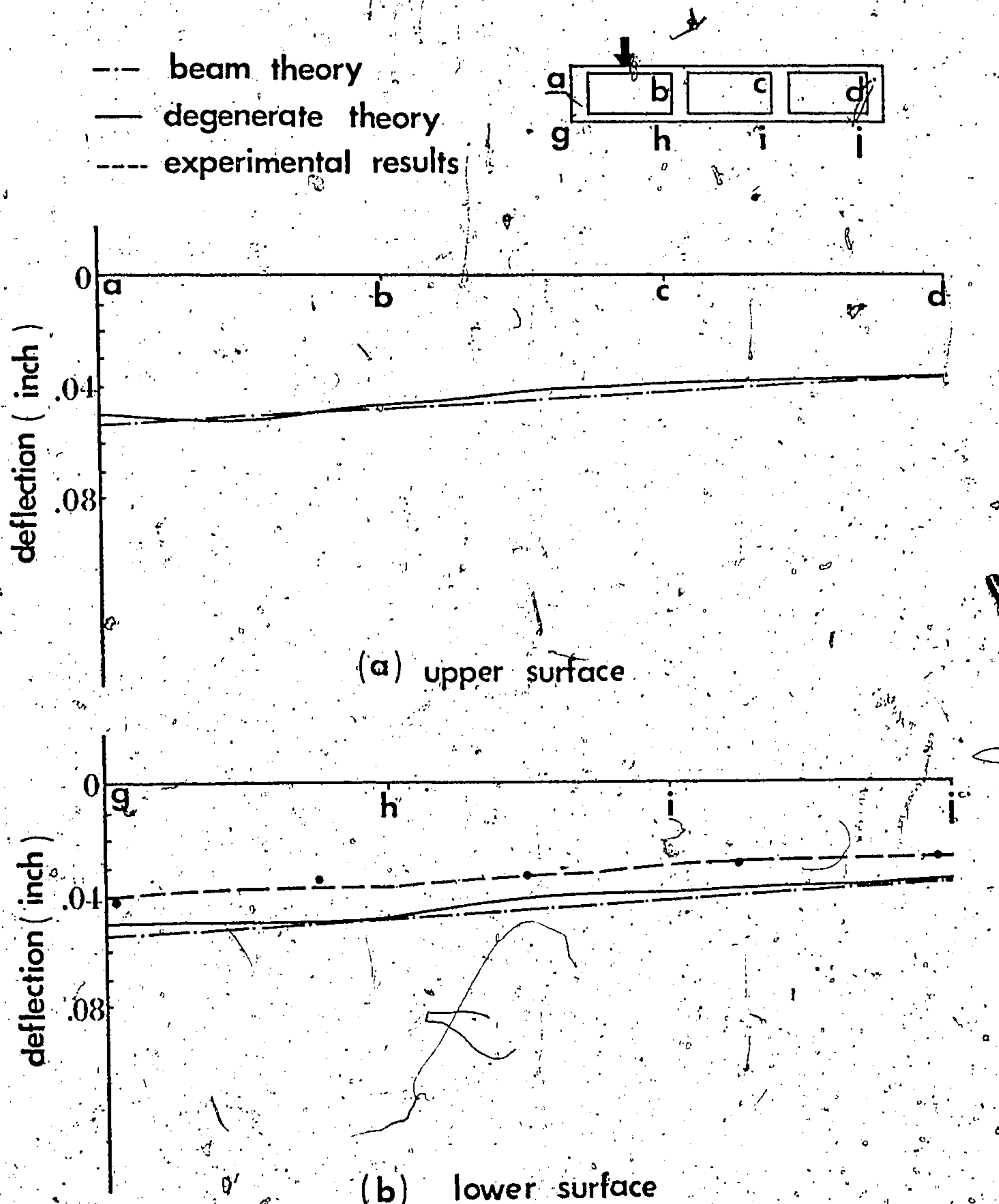


FIG. A13 Comparison of Calculated and Measured Deflections for Model 1 under a Partial Load of 100 psi

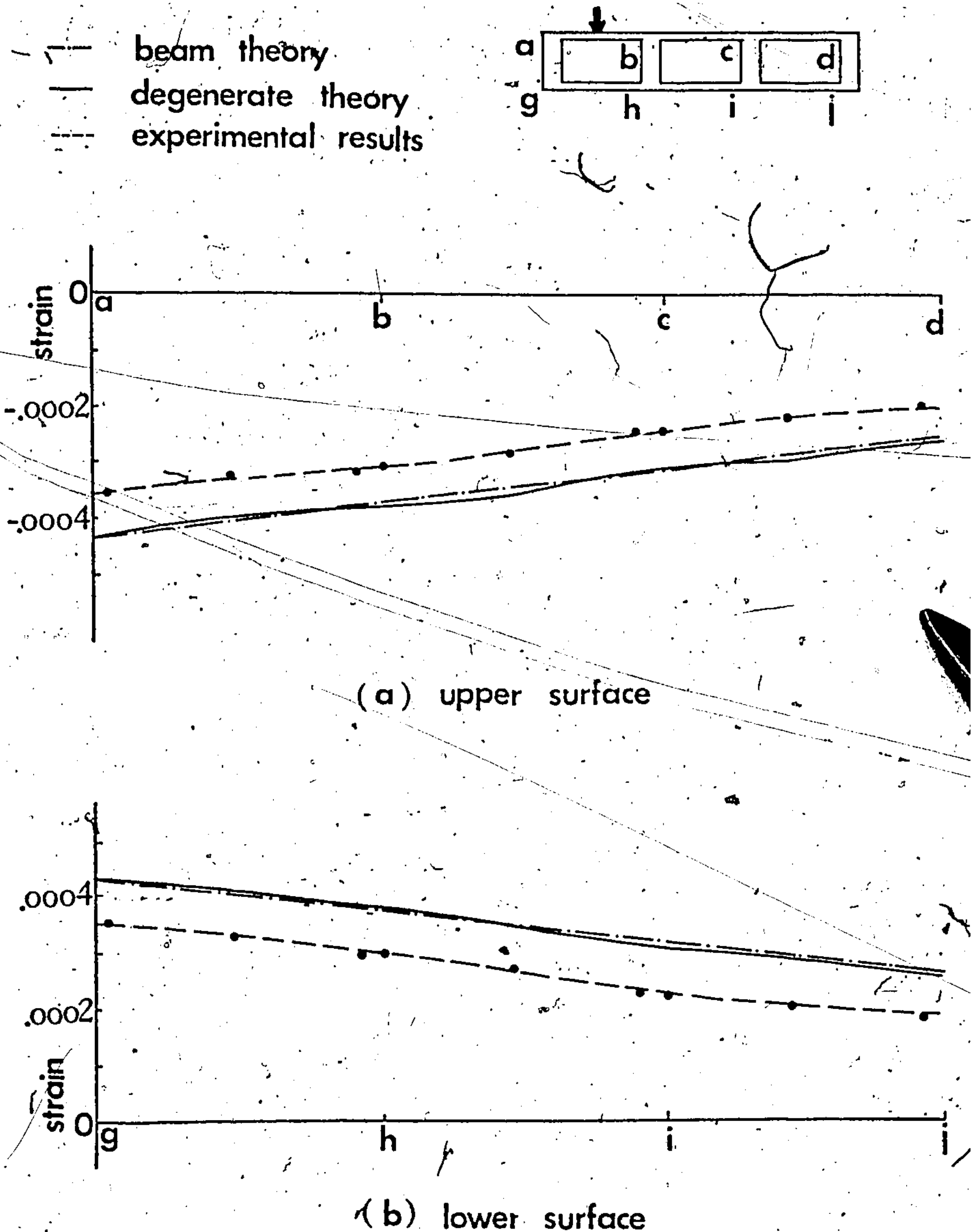


FIG.A14

Comparison of Calculated and Measured longitudinal Strains for Model 1 under a partial Load of 100 psi

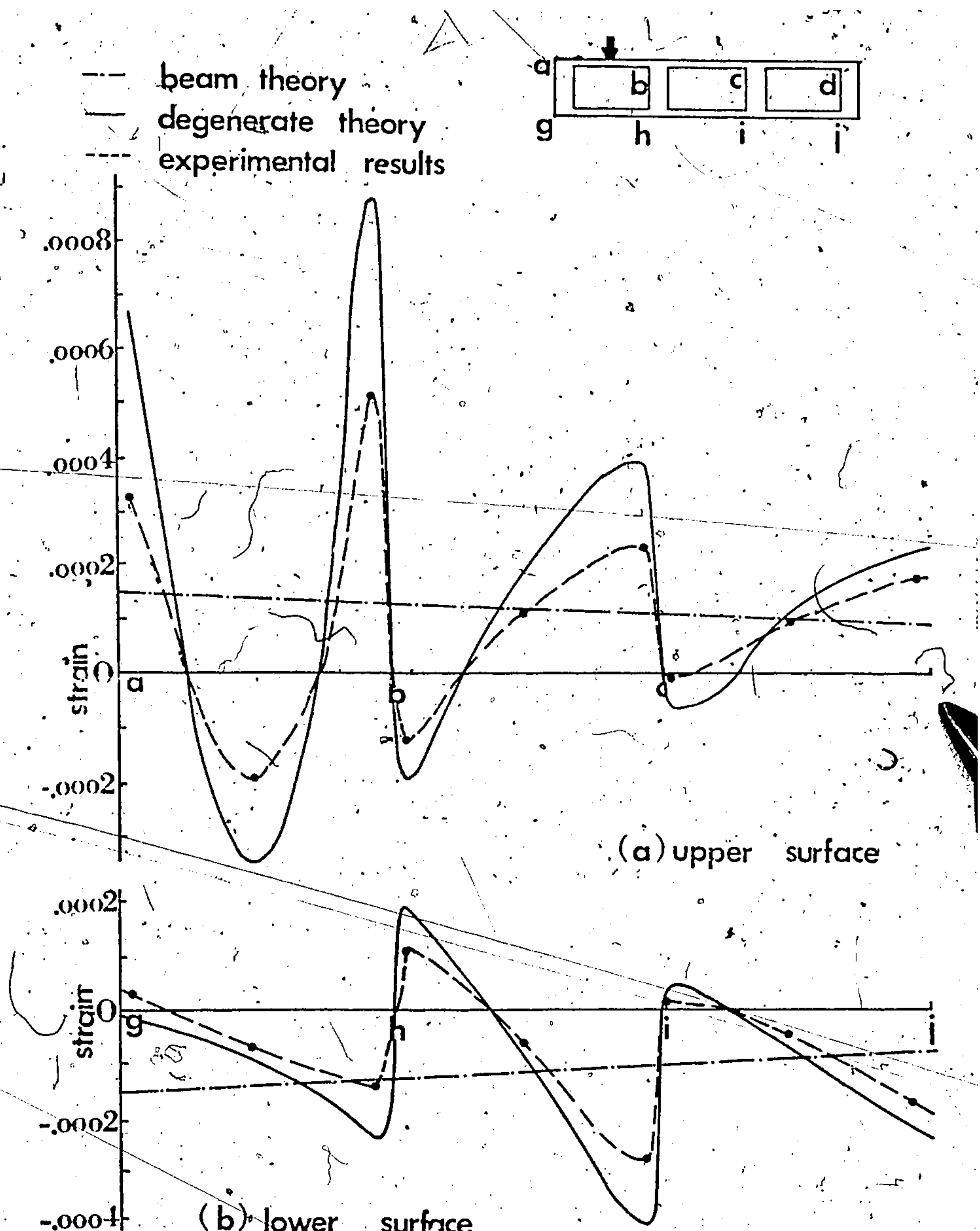


FIG. A15

Comparison of Calculated and Measured Transverse Strains for Model 1 under a Partial Load of 100 psi

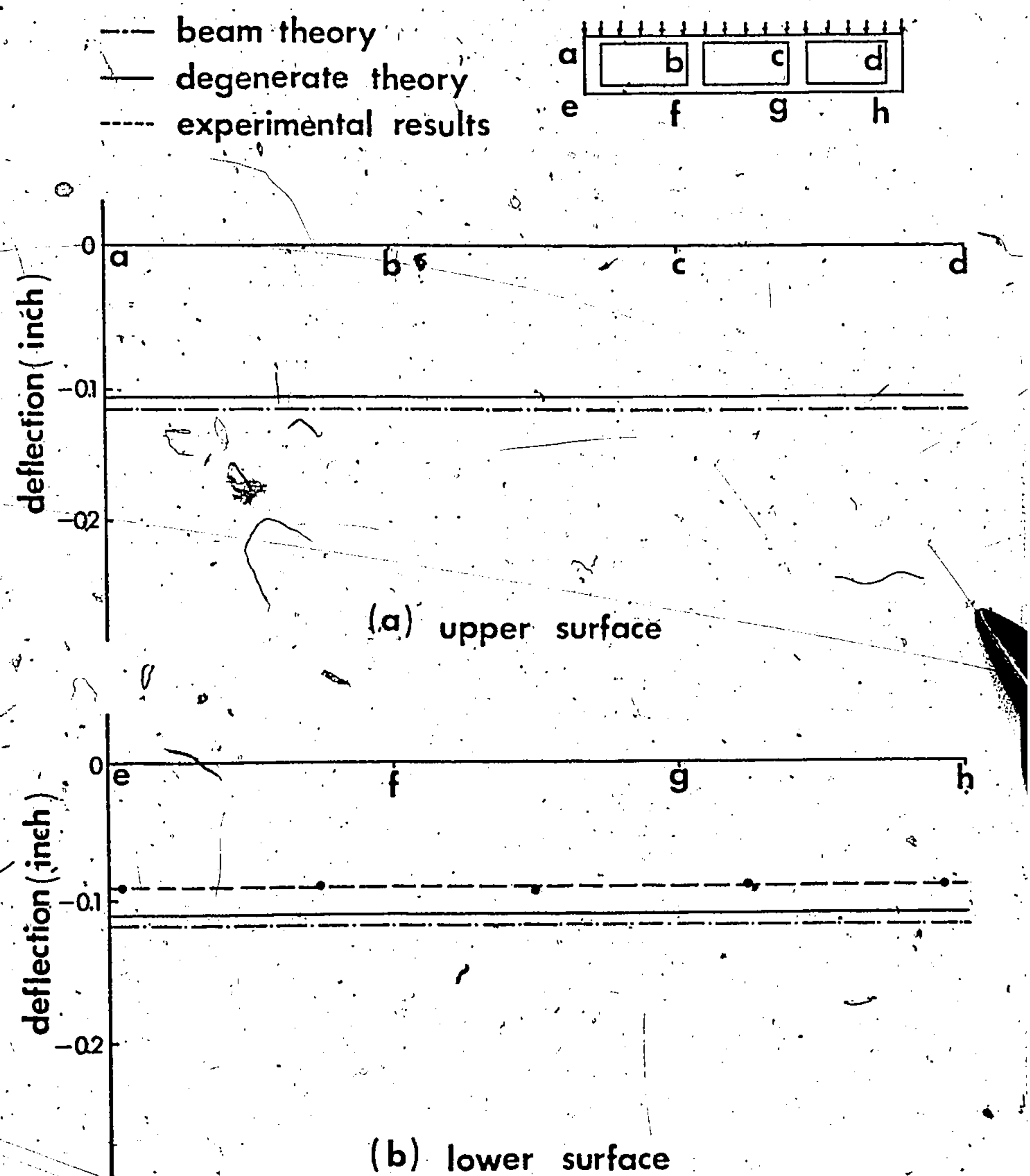


FIG. B1 Comparison of Calculated and Measured Deflections for Model 2 under a U.D.L. of 1 psi

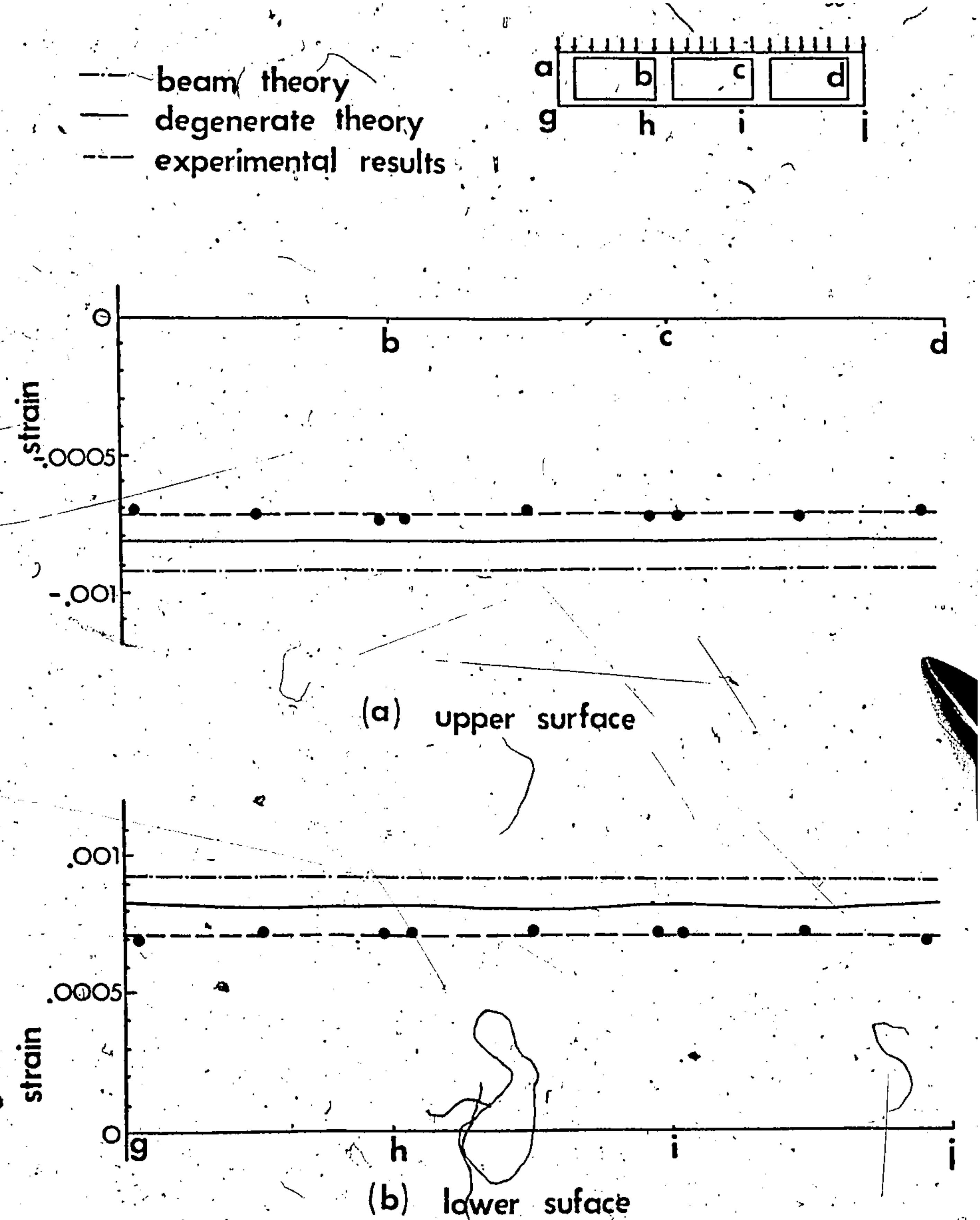


FIG. B2

Comparison of Calculated and Measured Longitudinal Strains for Model 2 under a U.D.L. of 1 psi

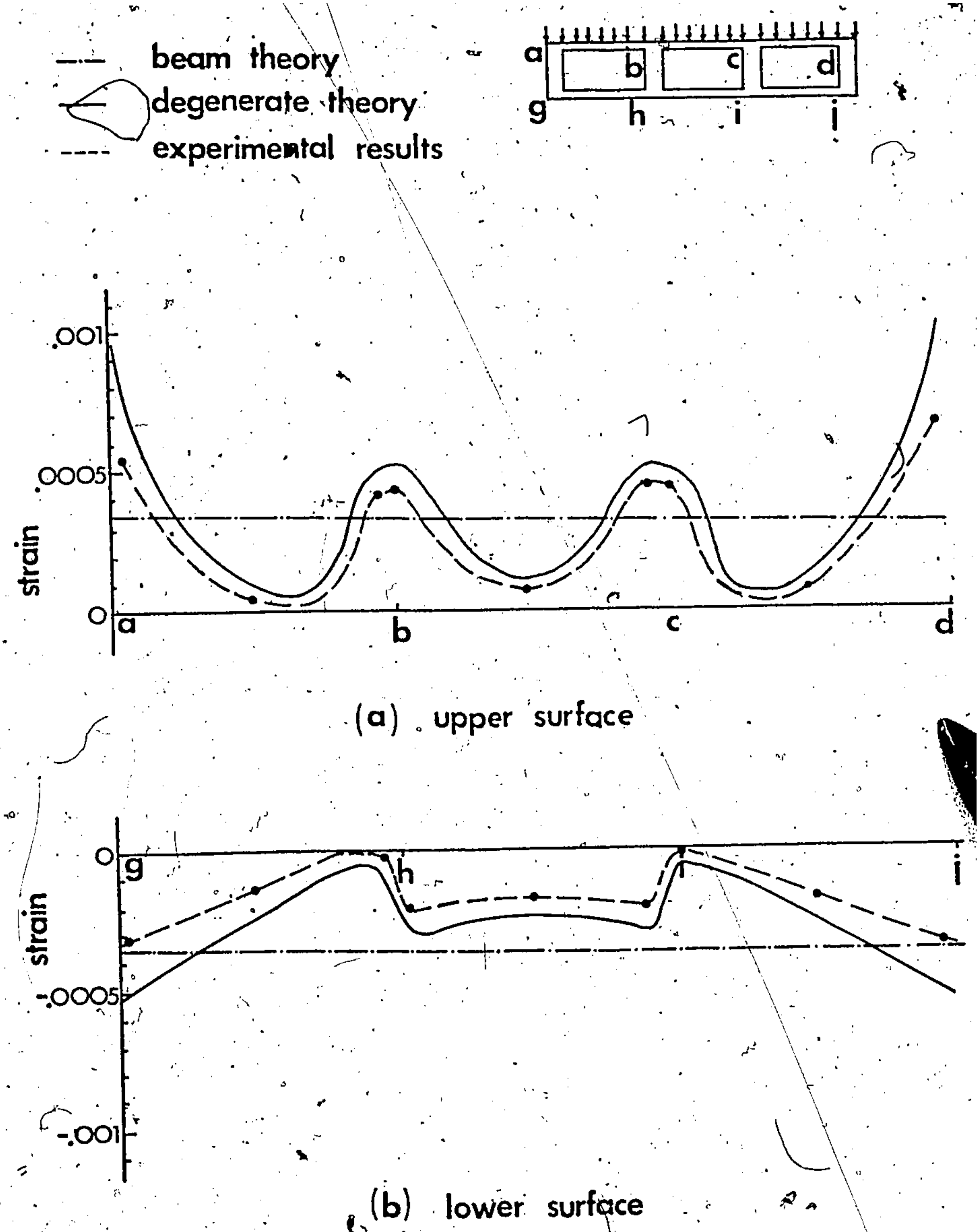


FIG. B3

Comparison of Calculated and Measured Transverse Strains for Model 2 under a U.D.L. of 1 psi

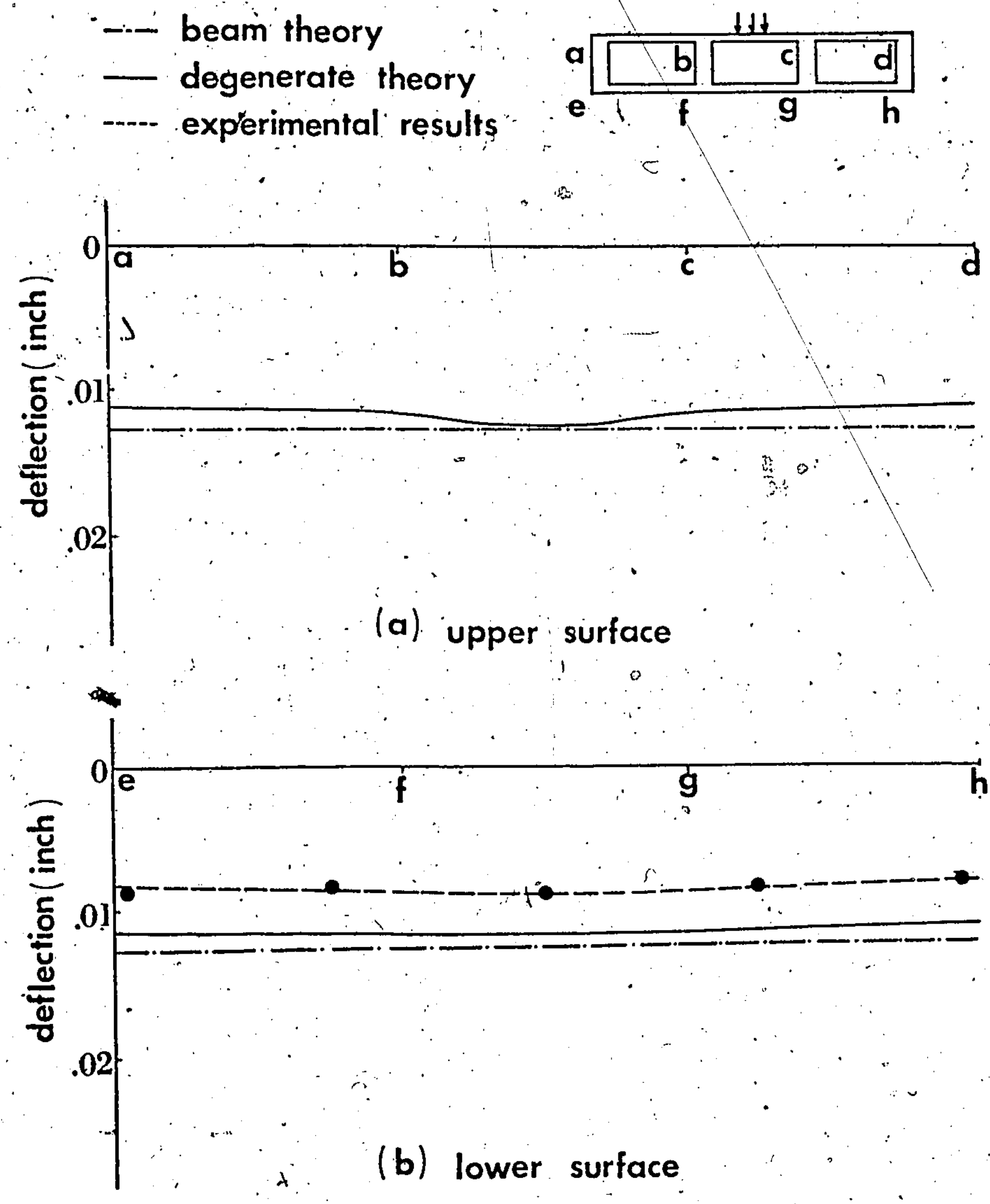


FIG.B4 Comparison of Calculated and Measured Deflections for Model 2 under a Line Load of 1 psi

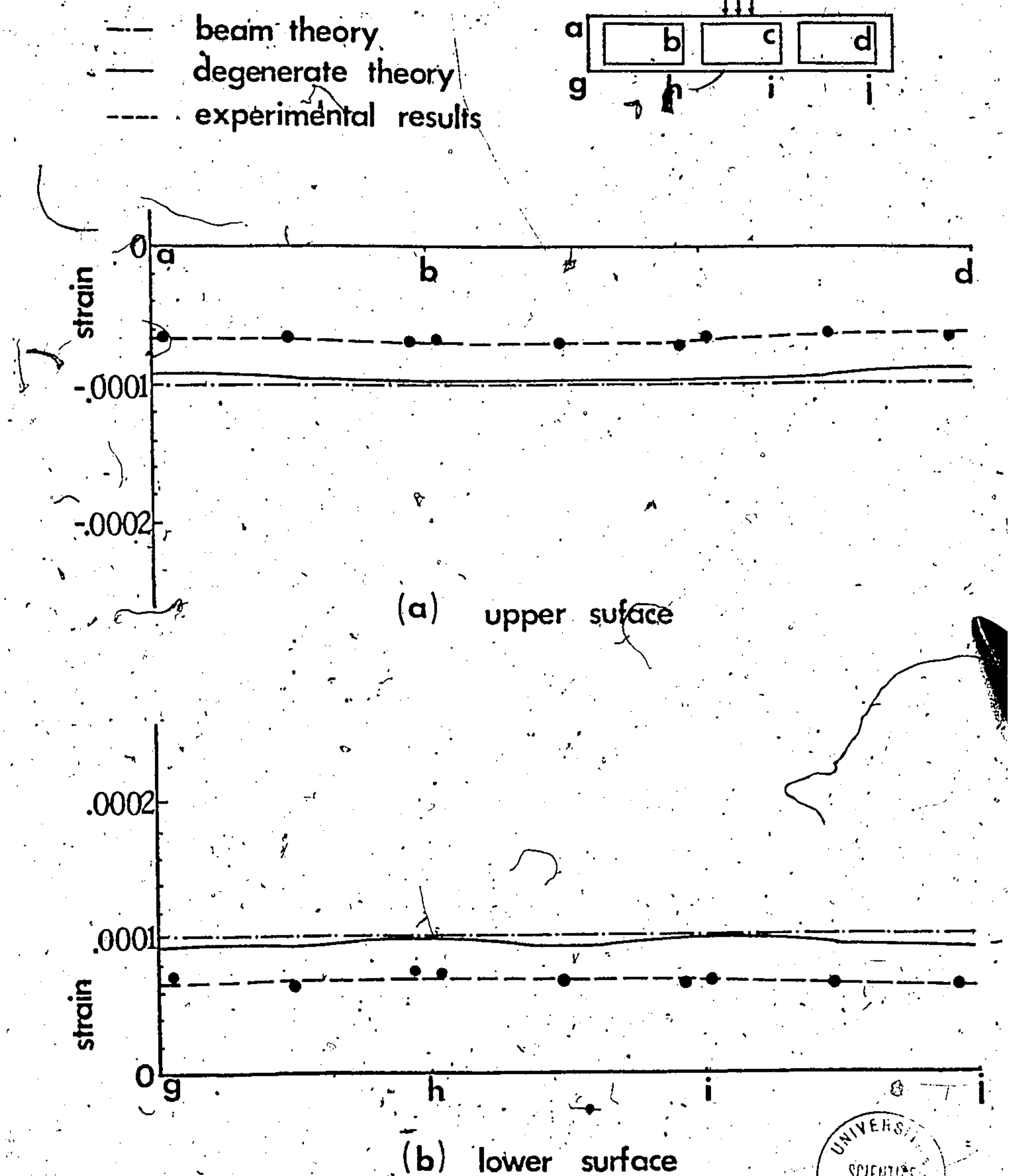


FIG. B5

Comparison of Calculated and Measured longitudinal Strains for Model 2 under a line load of 1 psi

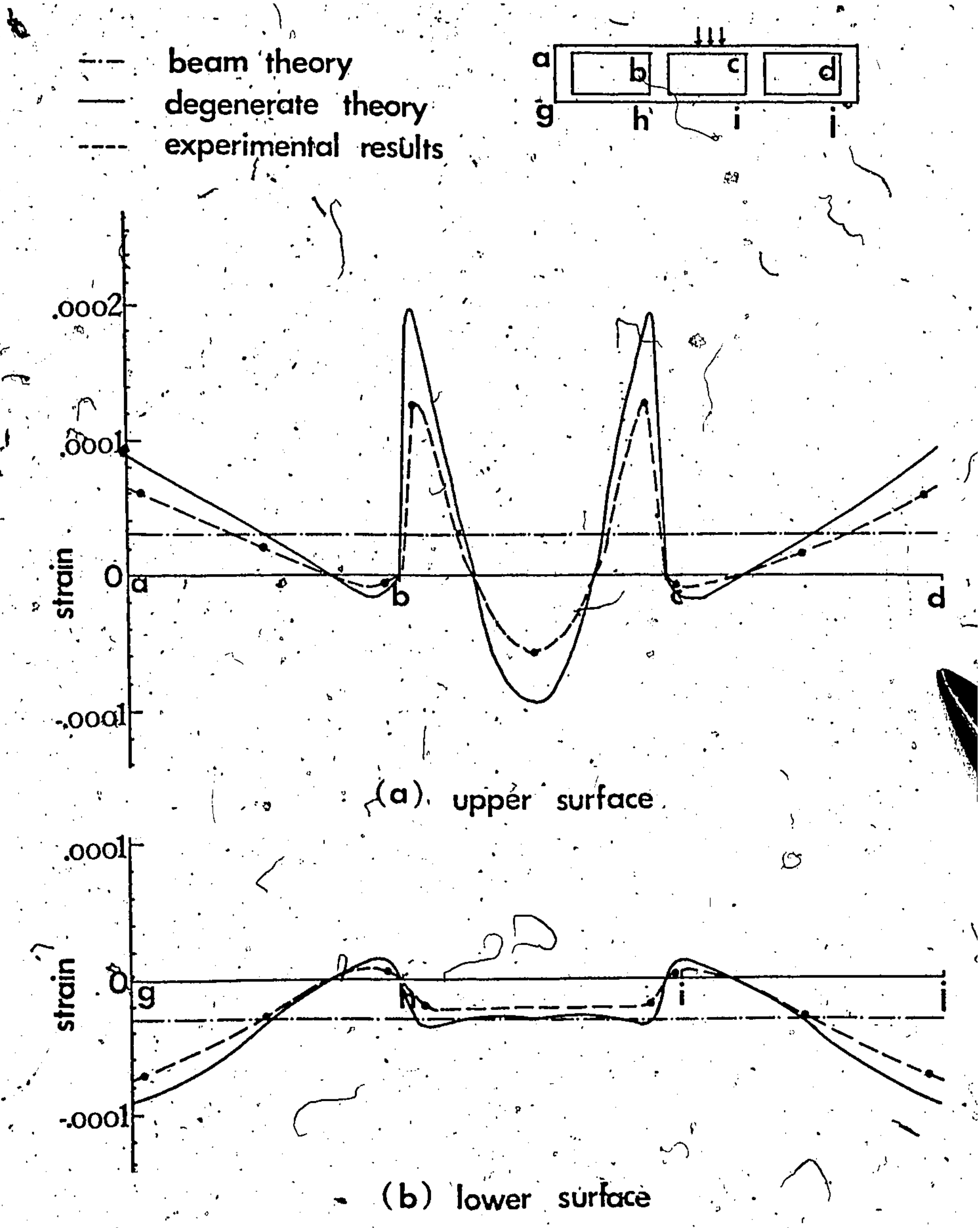


FIG. B6

Comparison of Calculated and Measured Transverse Strains for Model 2 under a line load of 1 psi

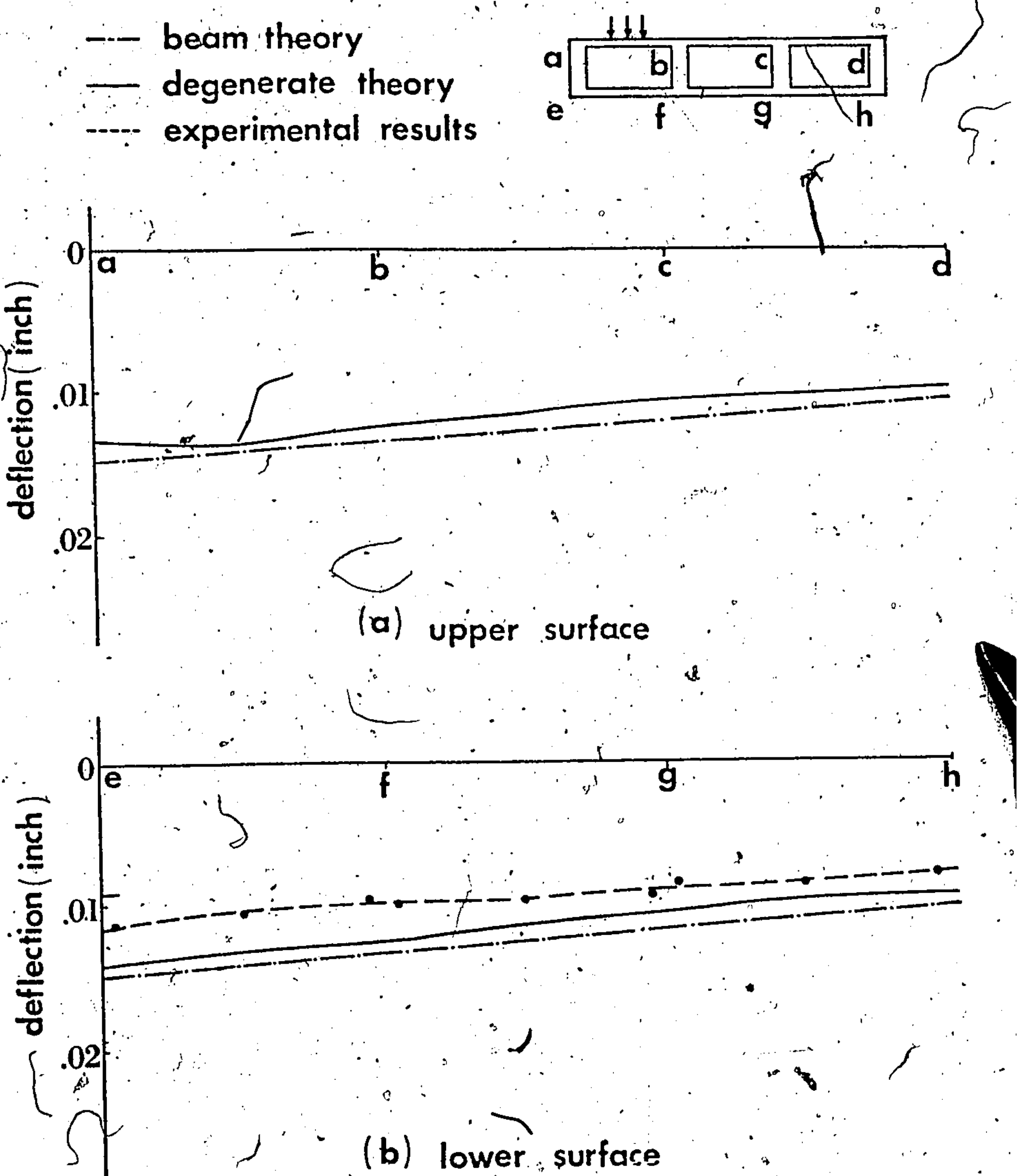


FIG. B7 Comparison of Calculated and Measured Deflections for Model 2 under a Line Load of 1 psi

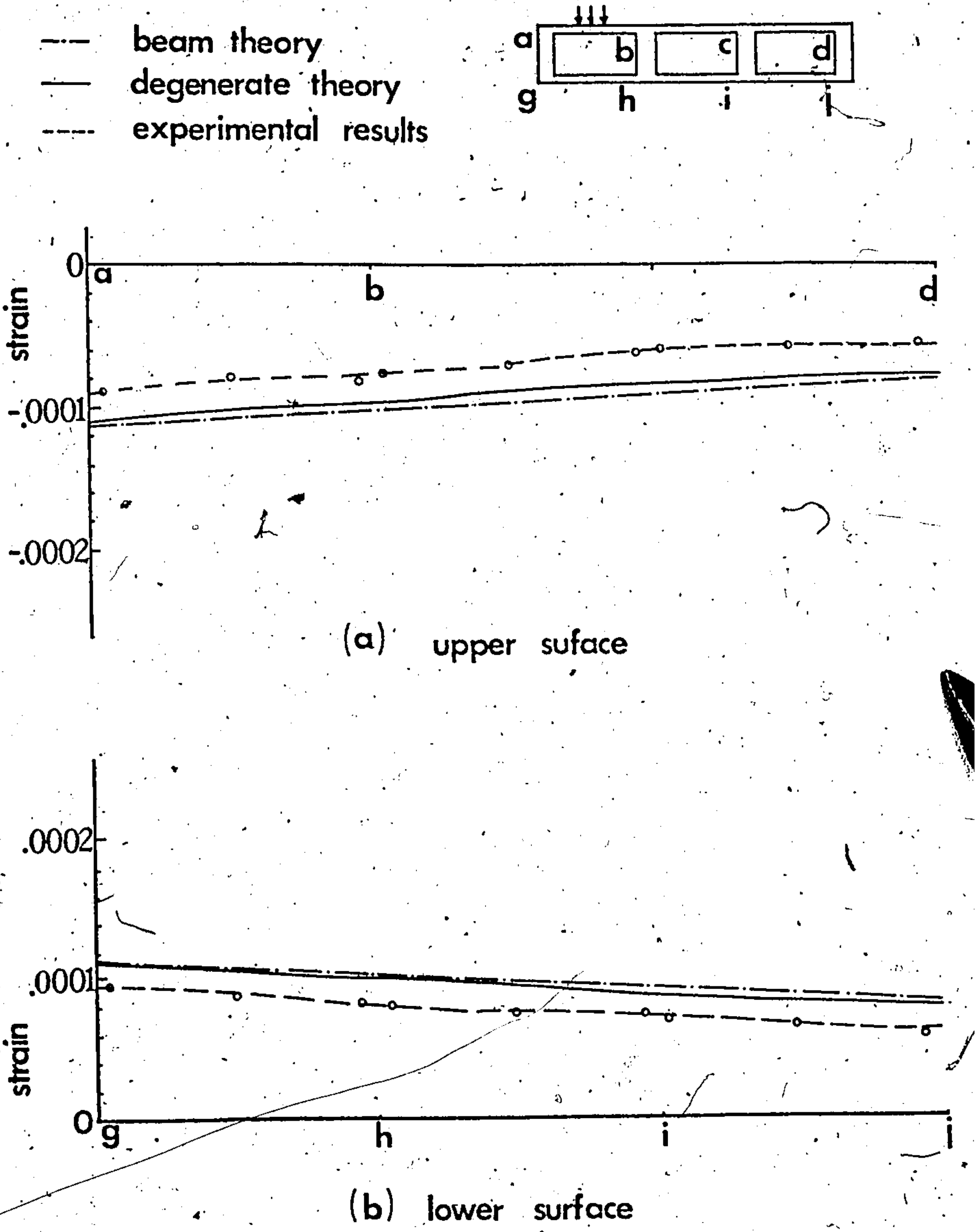


FIG. B8

Comparison of Calculated and Measured longitudinal Strains for Model 2 under a line load of 1 psi

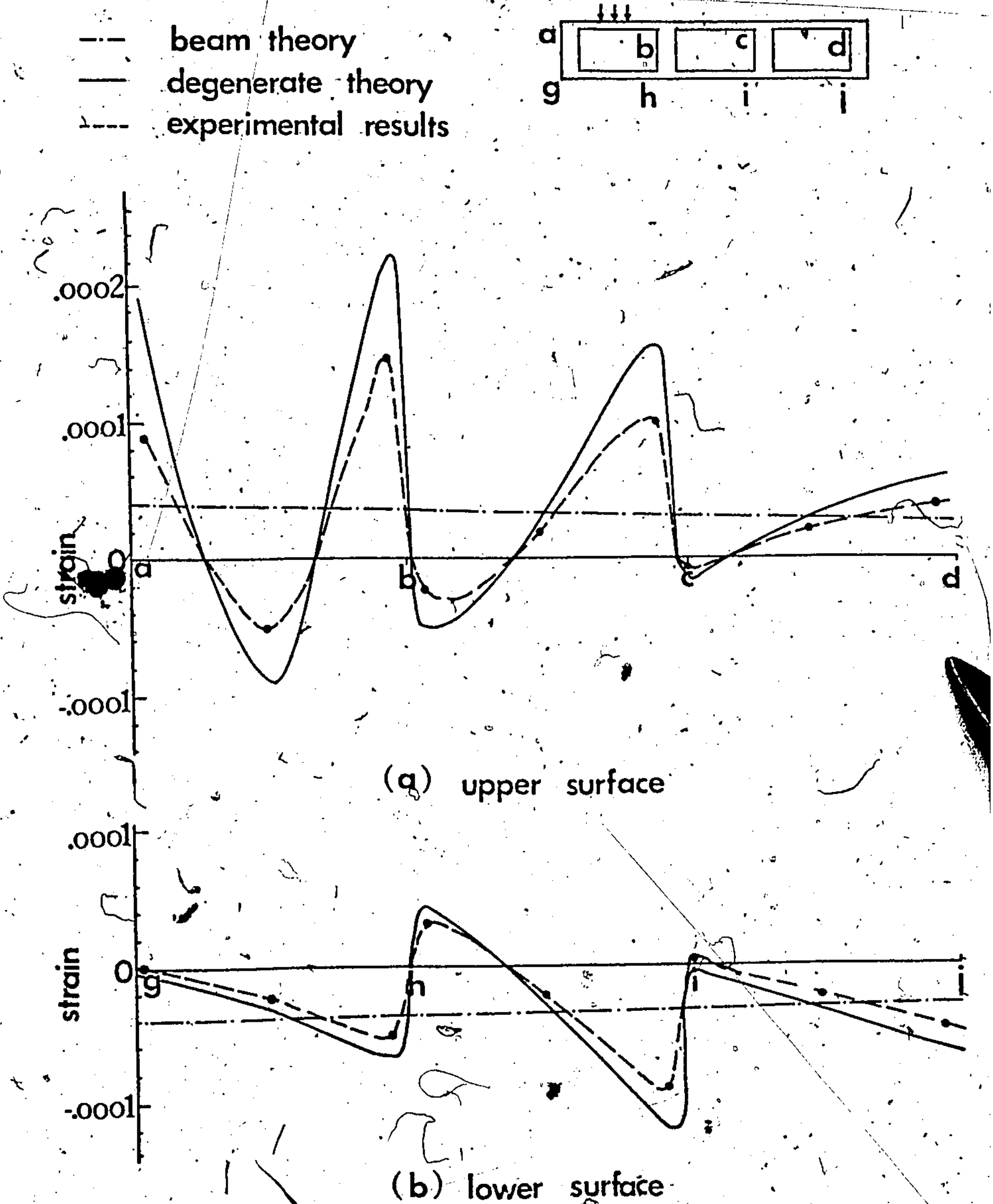


FIG. B9

Comparison of Calculated and Measured Transverse Strains for Model 2 under a line load of 1 psi

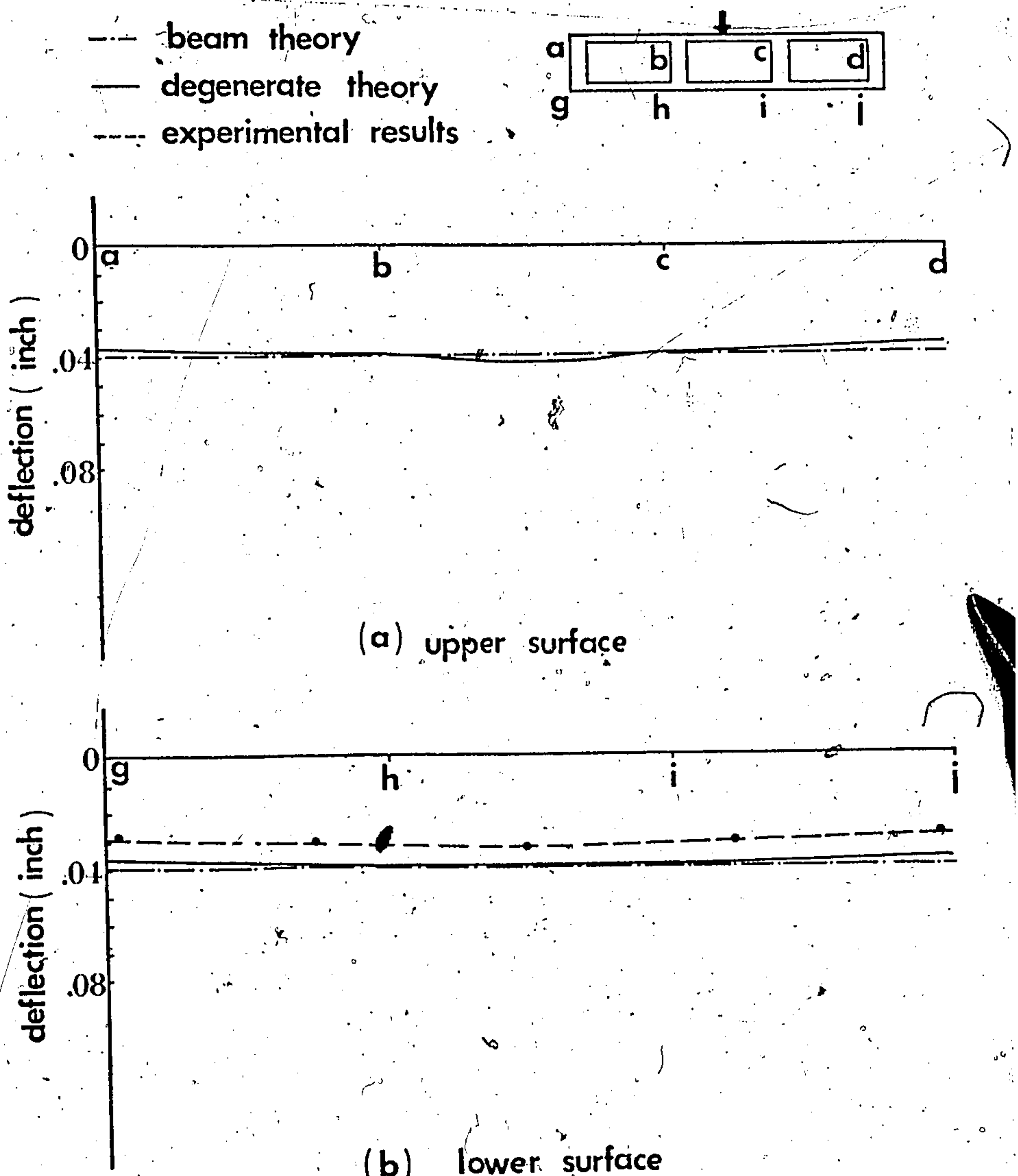


FIG. B10 Comparison of Calculated and Measured Deflections for Model 2 under a Partial Load of 100 psi

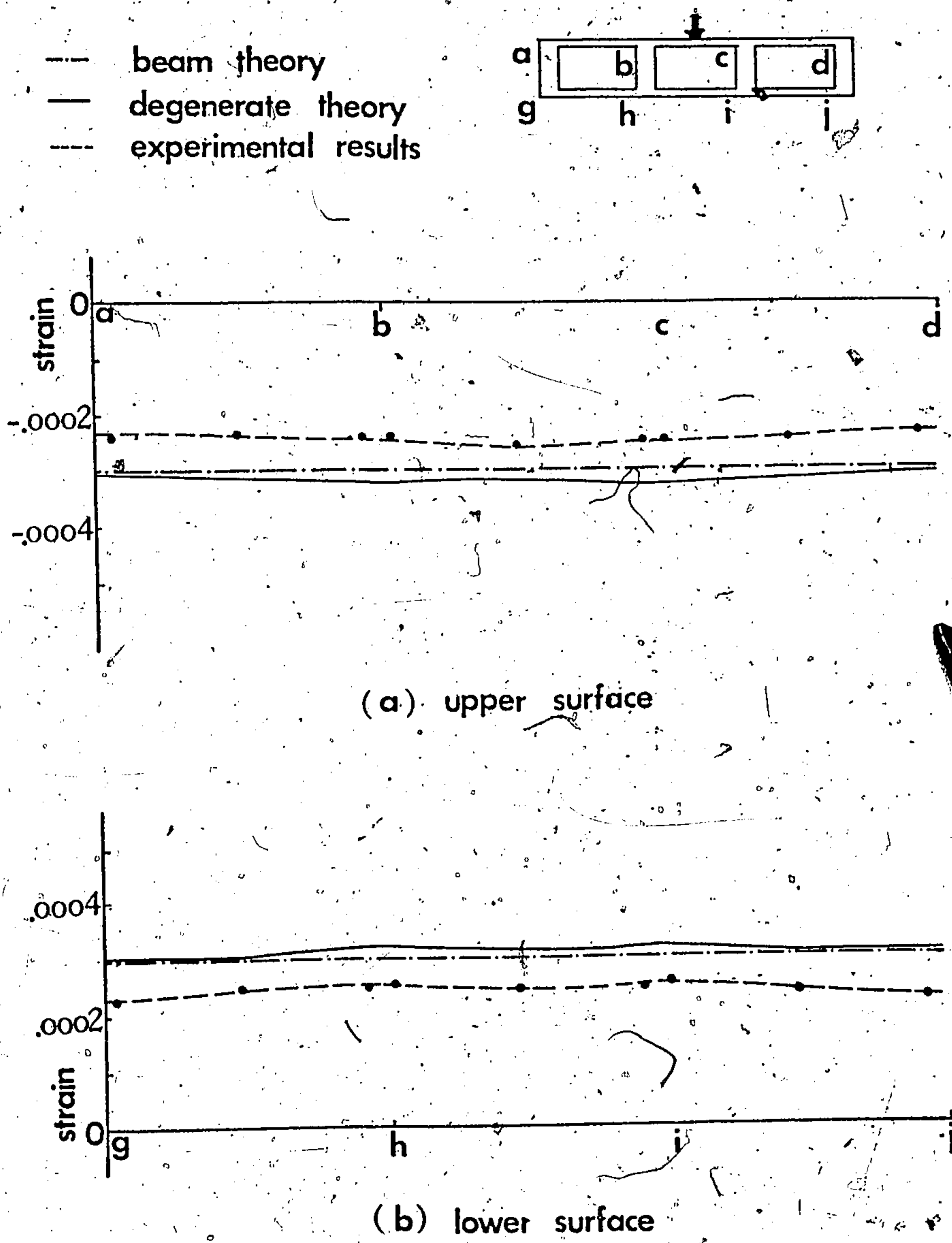


FIG. B11

Comparison of Calculated and Measured longitudinal Strains for Model 2 under a partial Load of 100 psi

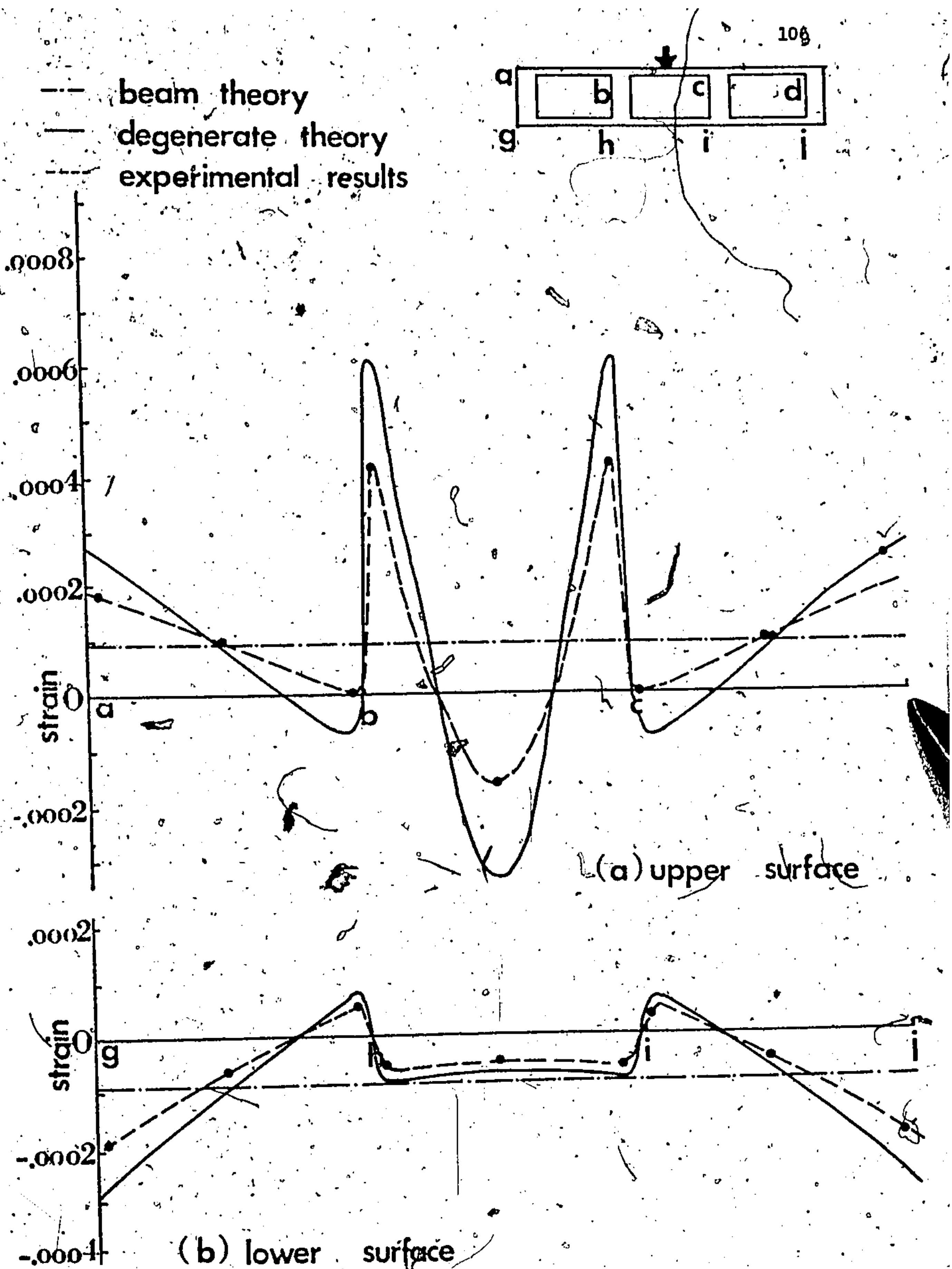


FIG. B12

Comparison of Calculated and Measured Transverse Strains for Model 2 under a Partial Load of 100 psi

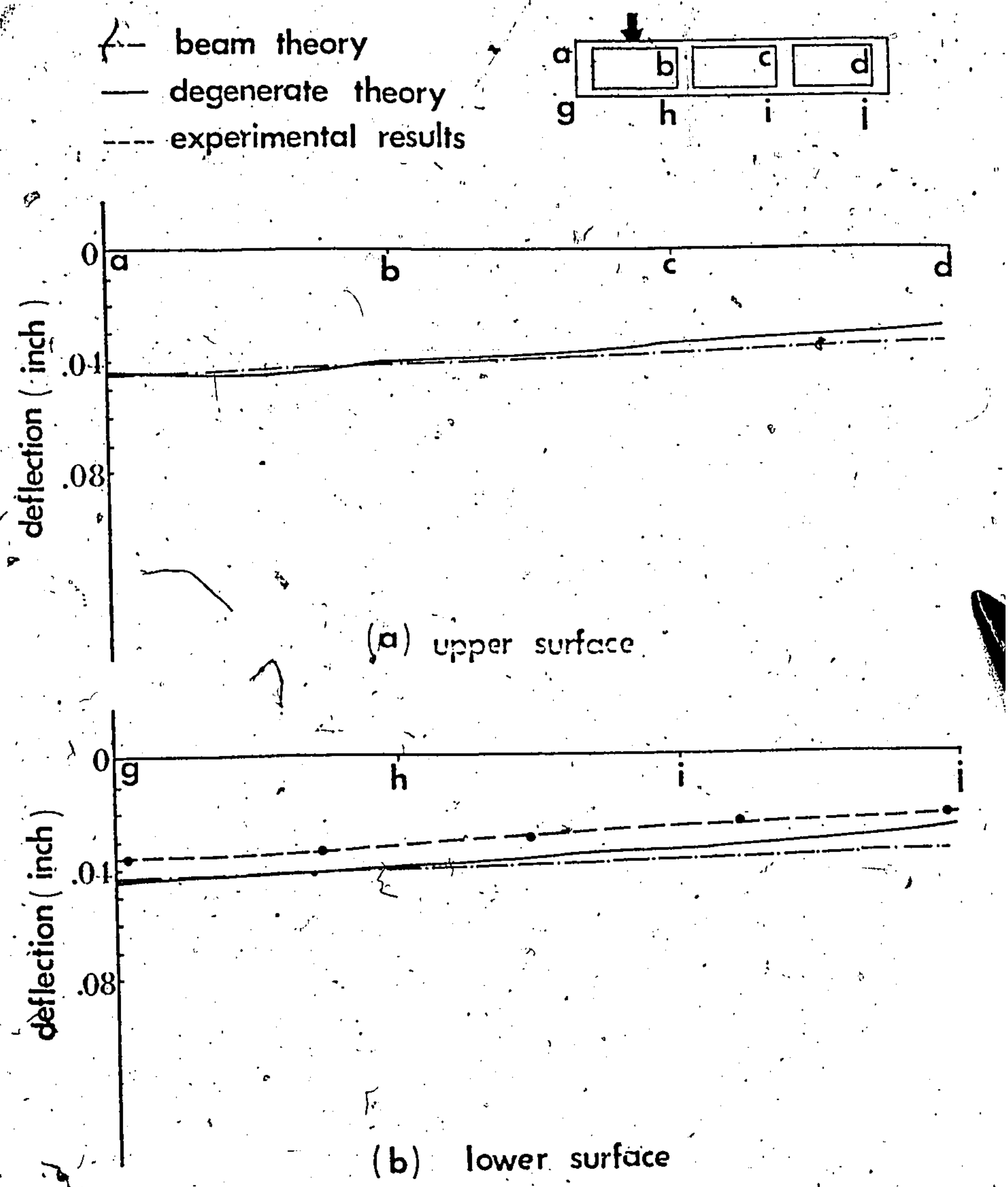


FIG. B13 Comparison of Calculated and Measured Deflections for Model 2 under a Partial Load of 100 psi

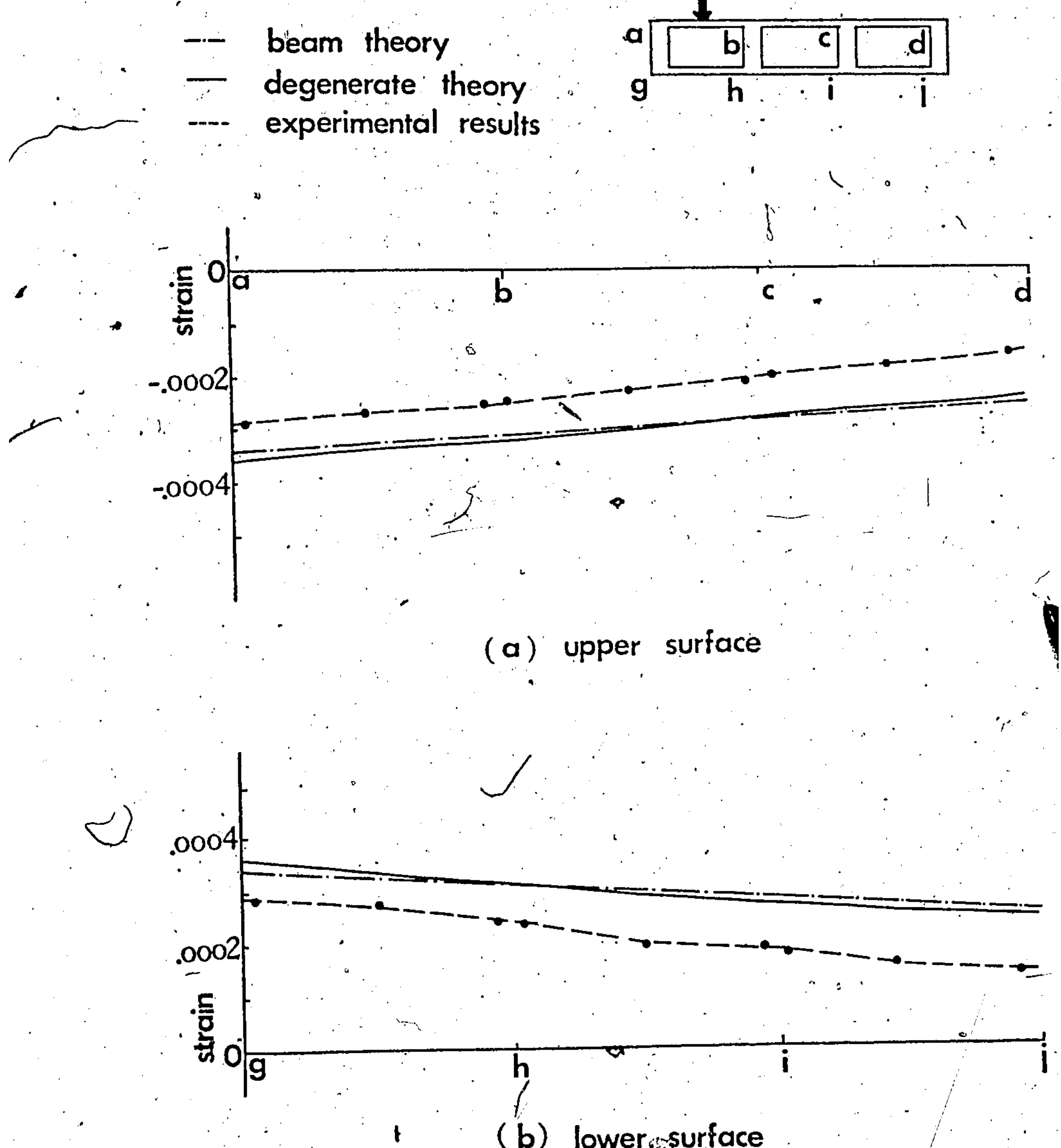


FIG. B14

Comparison of Calculated and Measured longitudinal Strains for Model 2 under a partial Load of 100 psi

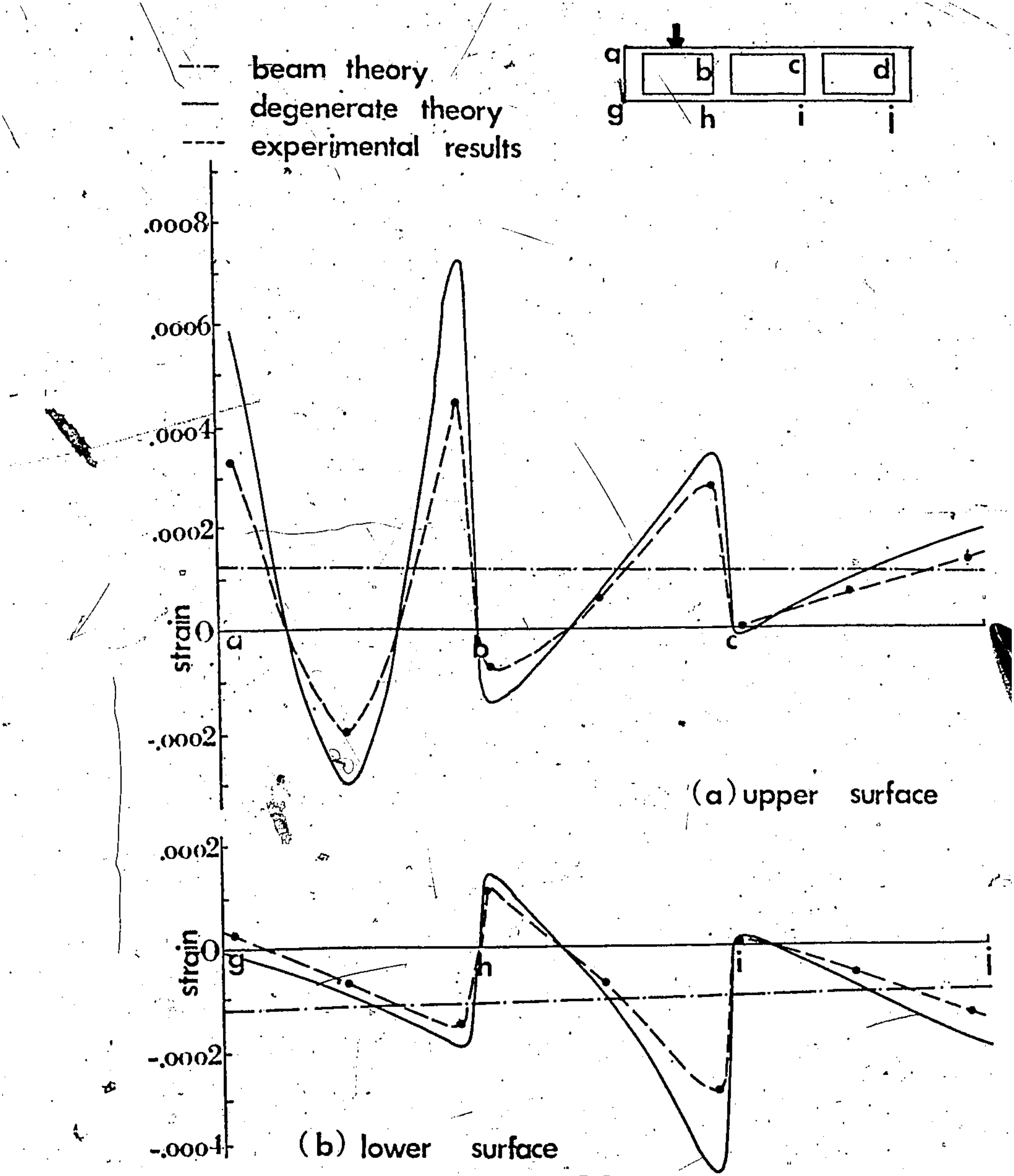


FIG. B 15

Comparison of Calculated and Measured Transverse Strains for Model 2 under a Partial Load of 100 psi

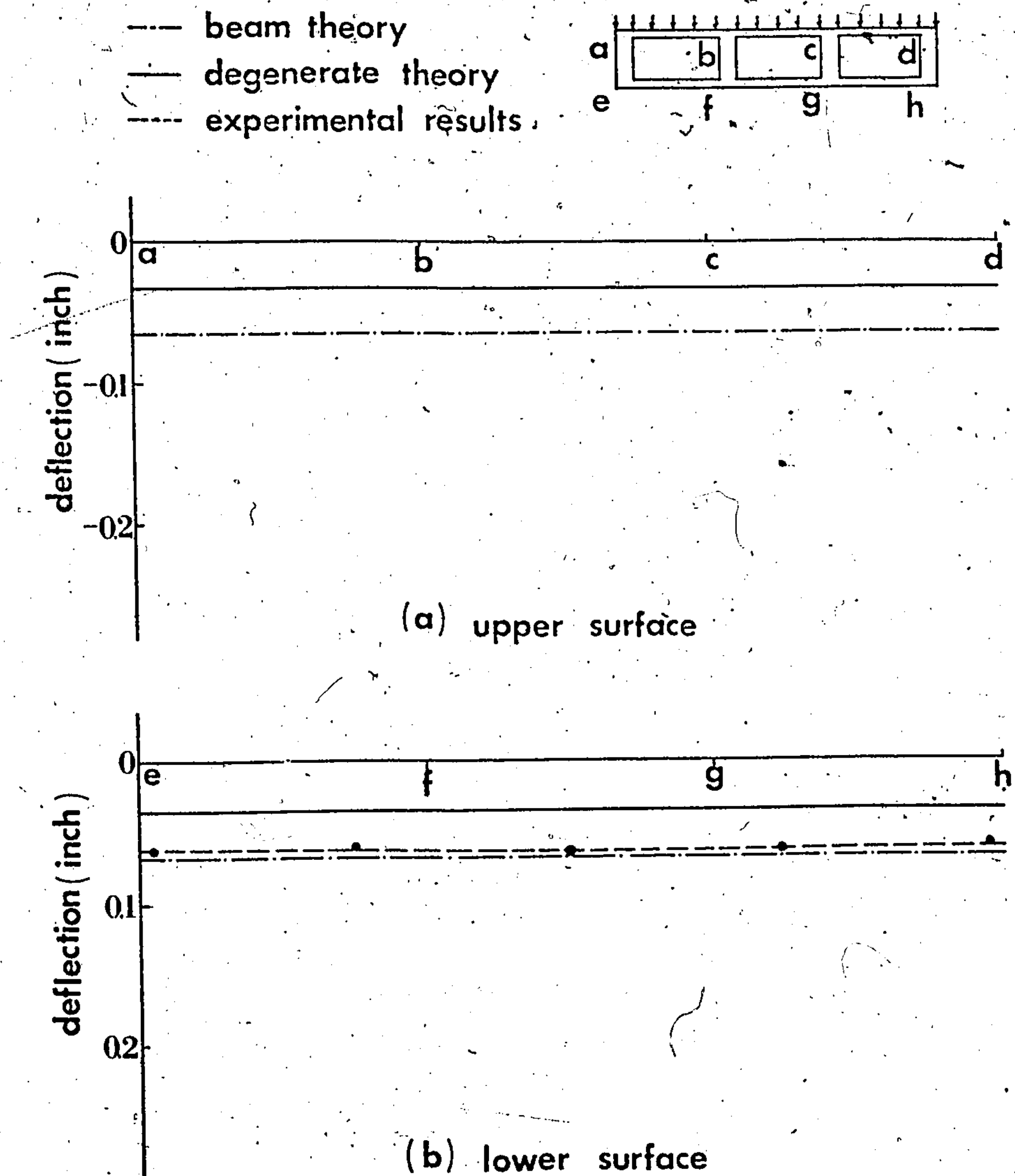


FIG.C1 Comparison of Calculated and Measured Deflections for Model 3 under a U.D.L. of 1 psi

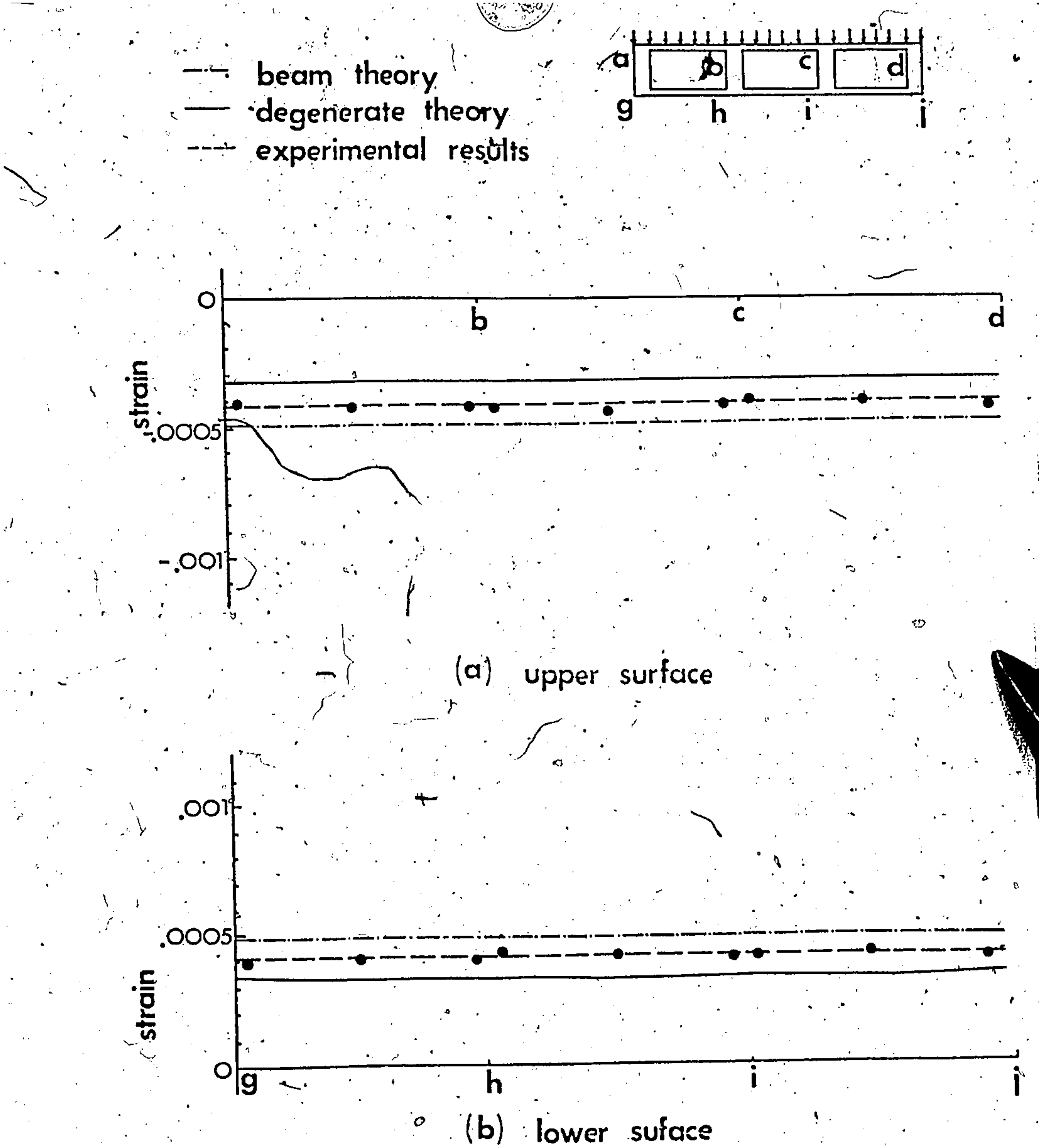


FIG.C 2

Comparison of Calculated and Measured Longitudinal Strains for Model 3 under a U. D. L. of 1 psi

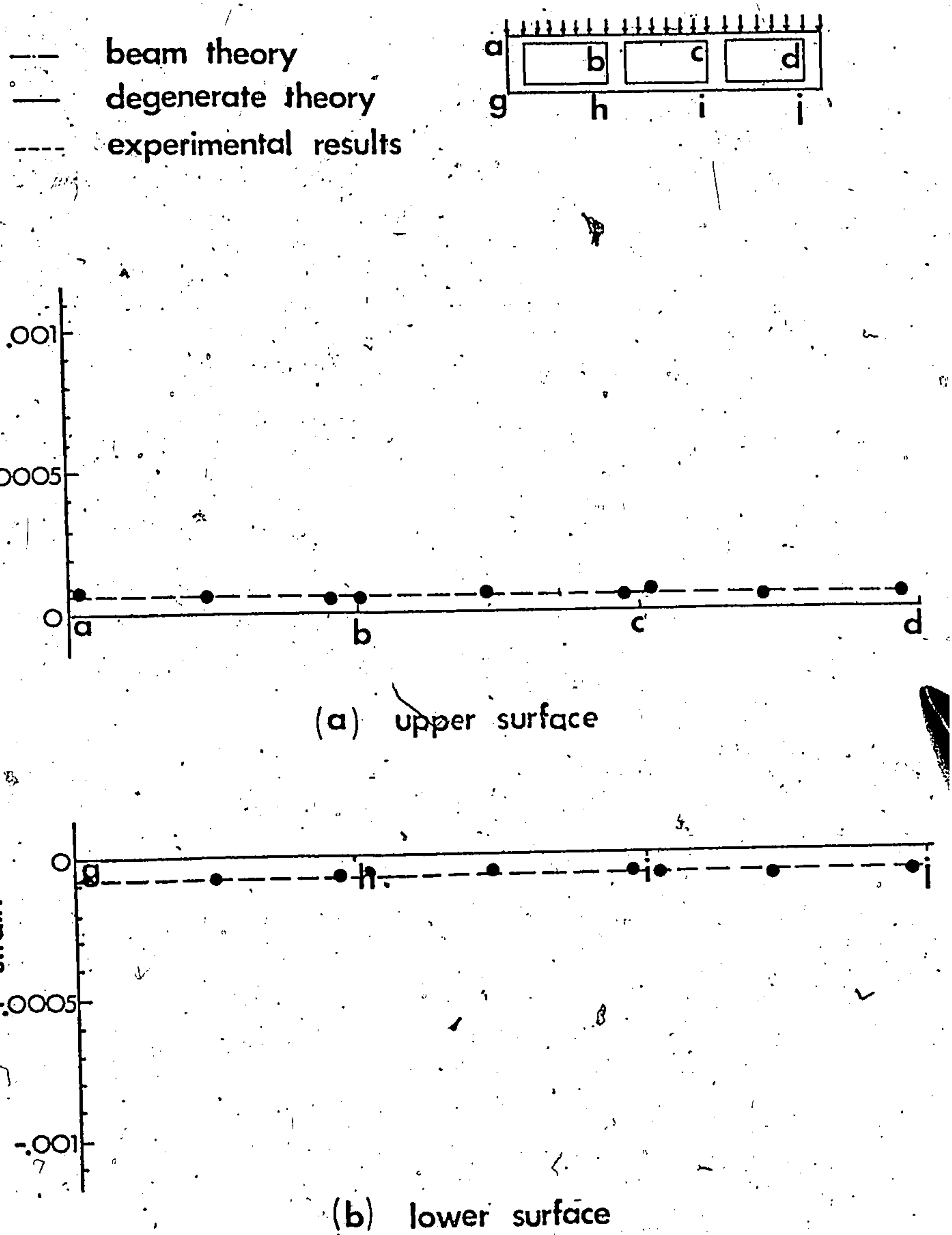


FIG. C3

Comparison of Calculated and Measured Transverse Strains for Model 3 under a U.D.L. of 1 psi

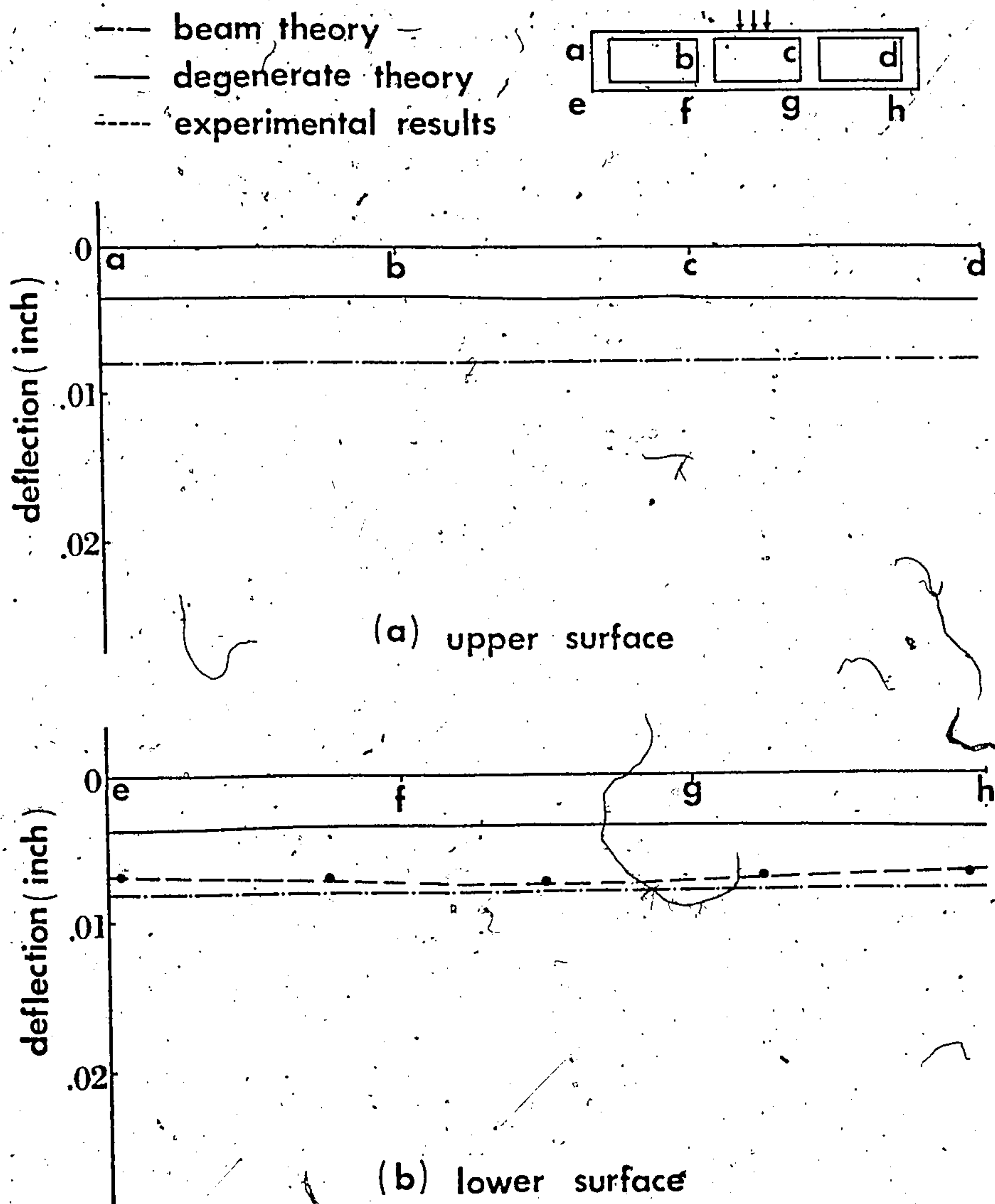


FIG.C4 Comparison of Calculated and Measured Deflections for Model 3 under a Line Load of 1 psi

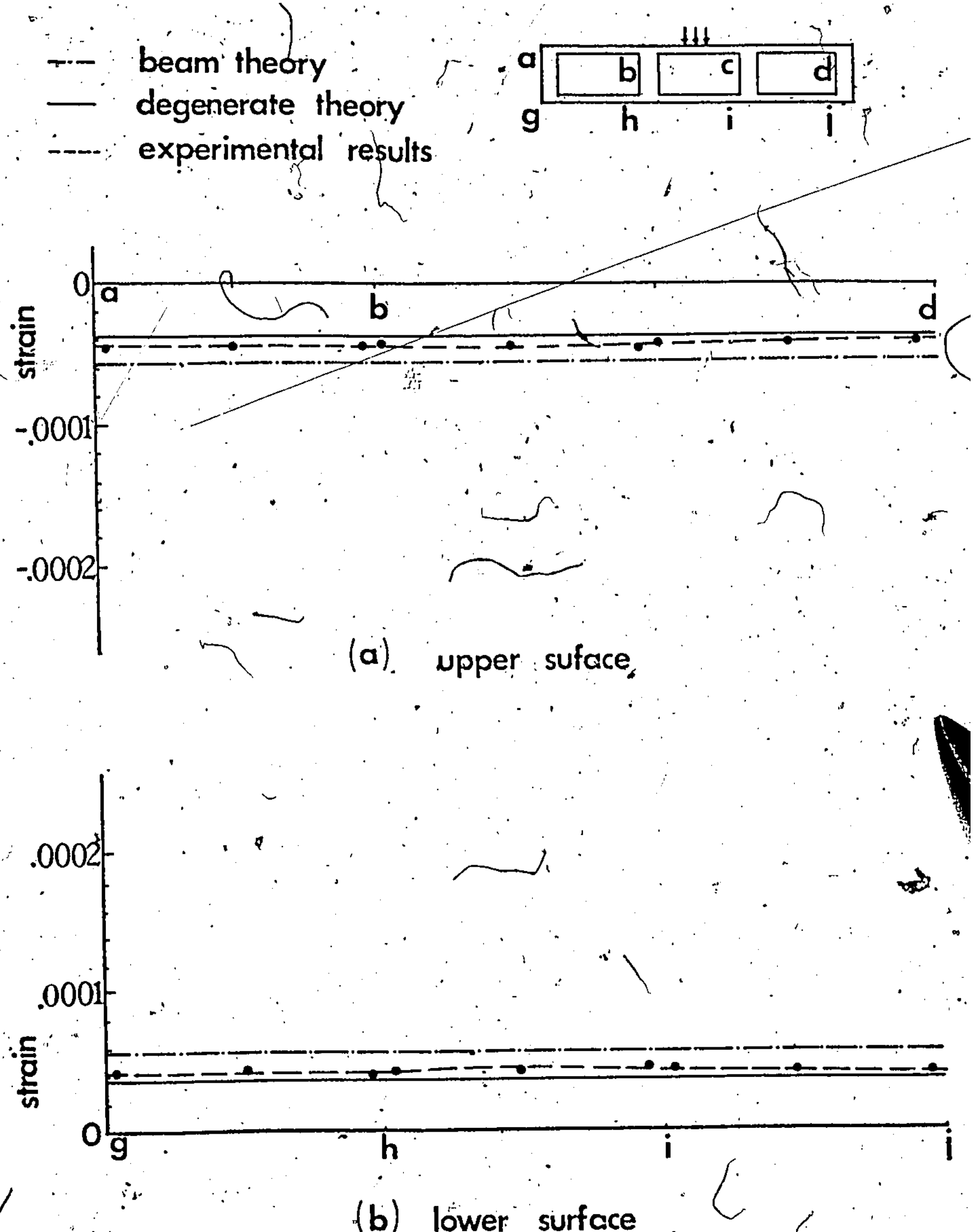


FIG. C5
Comparison of Calculated and Measured longitudinal Strains for Model 3 under a line load of 1 psi

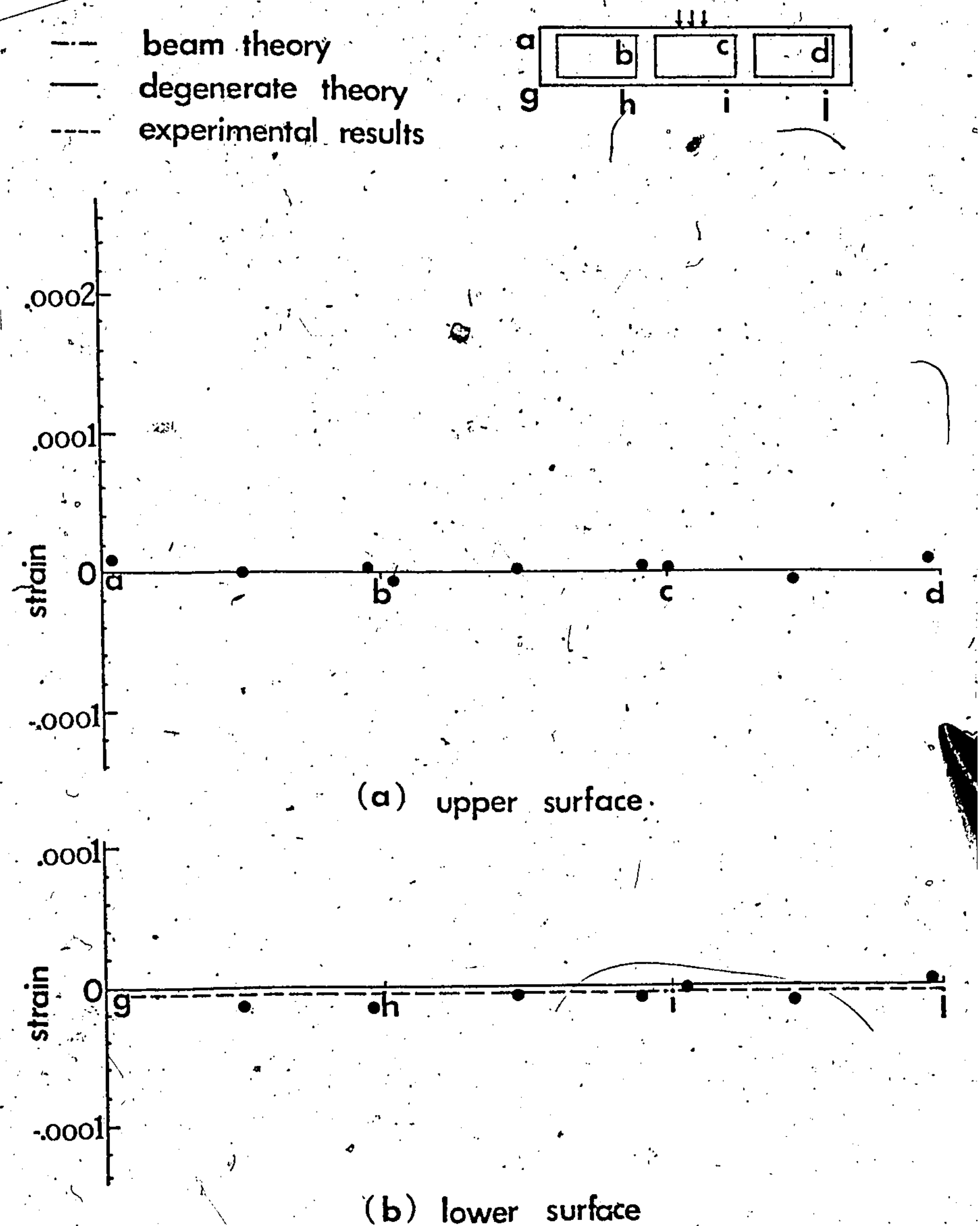


FIG.C6

Comparison of Calculated and Measured Transverse Strains for Model 3 under a line load of 1 psi

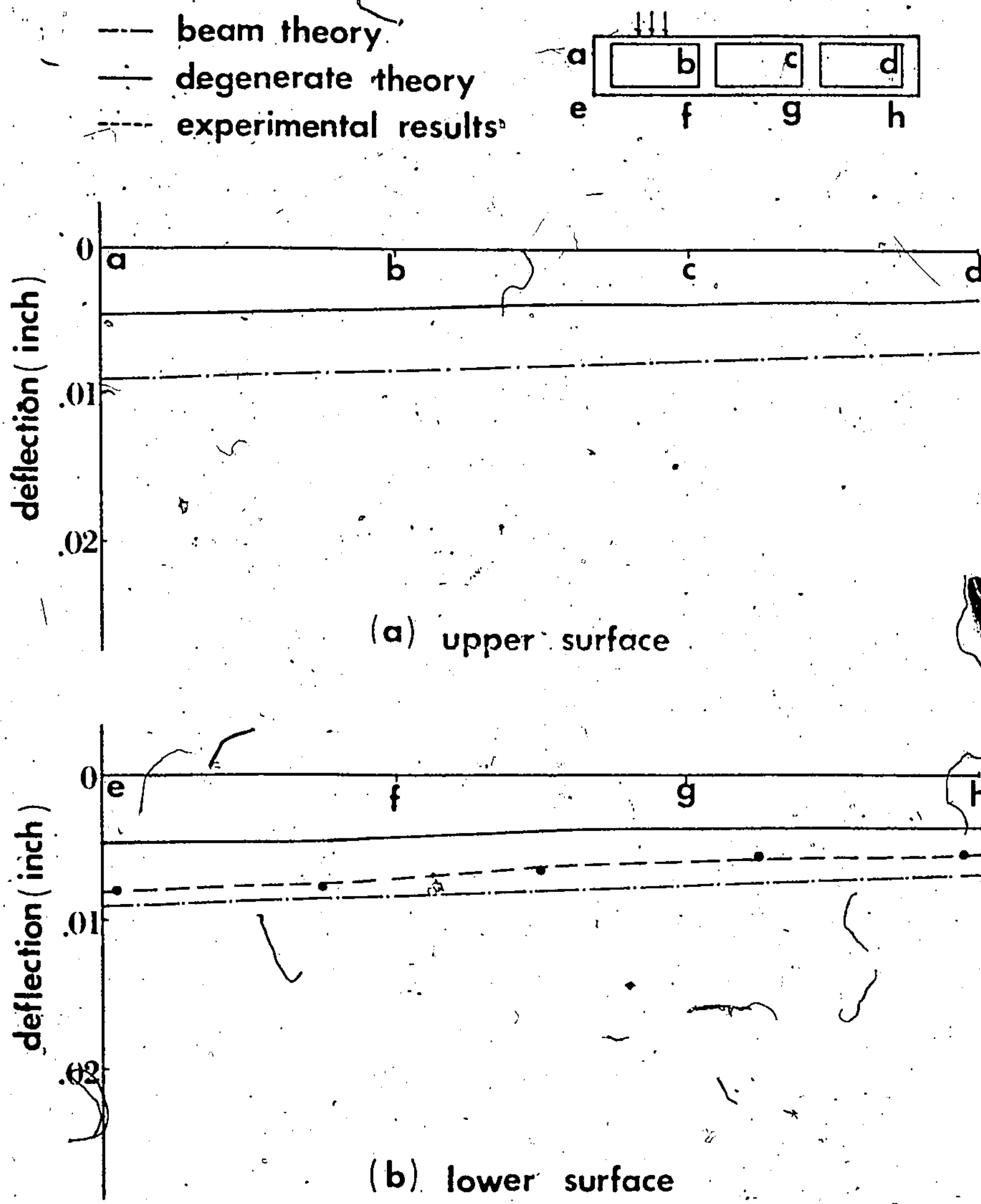


FIG.C7 Comparison of Calculated and Measured Deflections for Model 3 under a Line Load of 1 psi

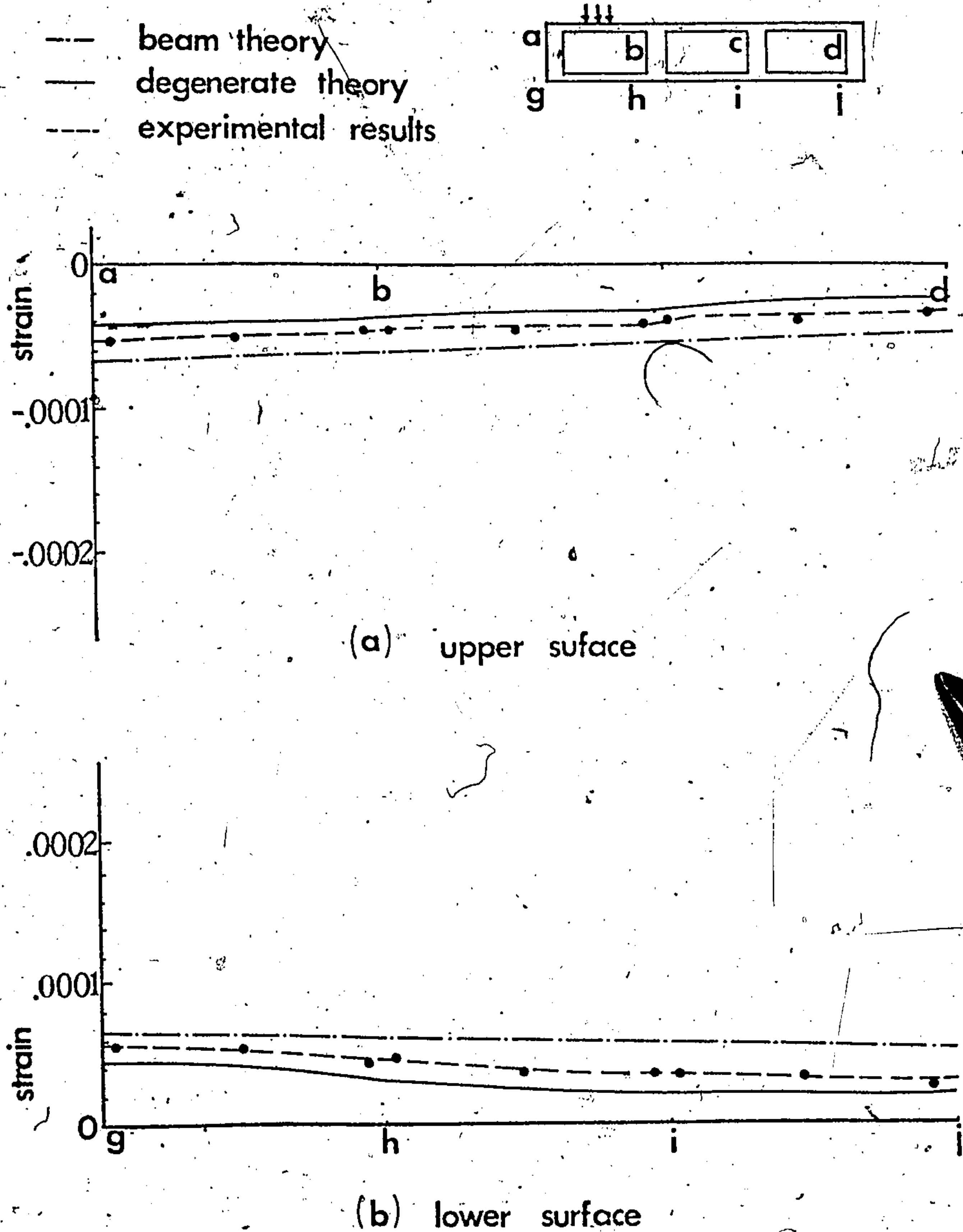


FIG.C8

Comparison of Calculated and Measured longitudinal Strains for Model 3 under a line load of 1 psi

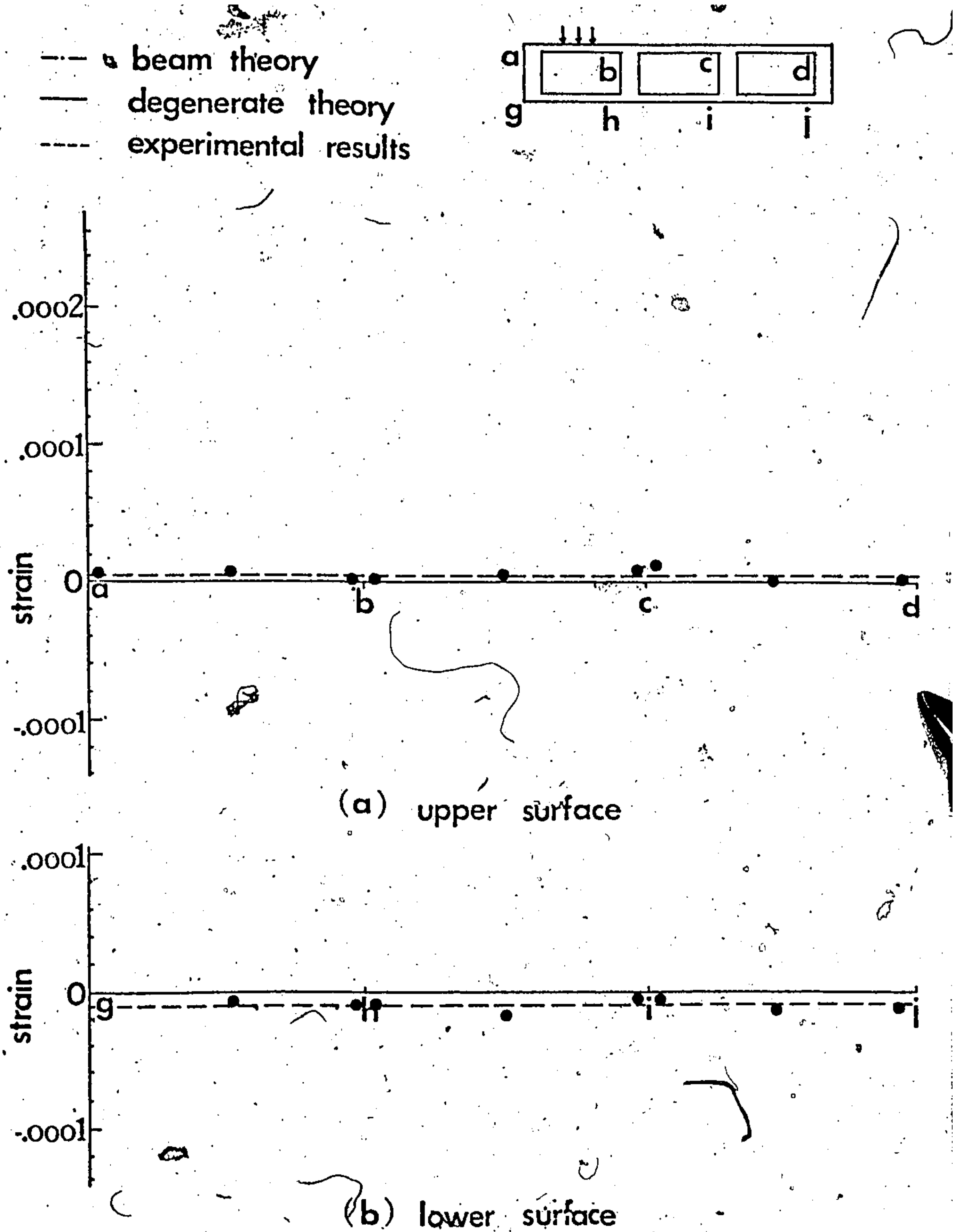


FIG. C9
Comparison of Calculated and Measured Transverse Strains for Model 3 under a line load of 1 psi

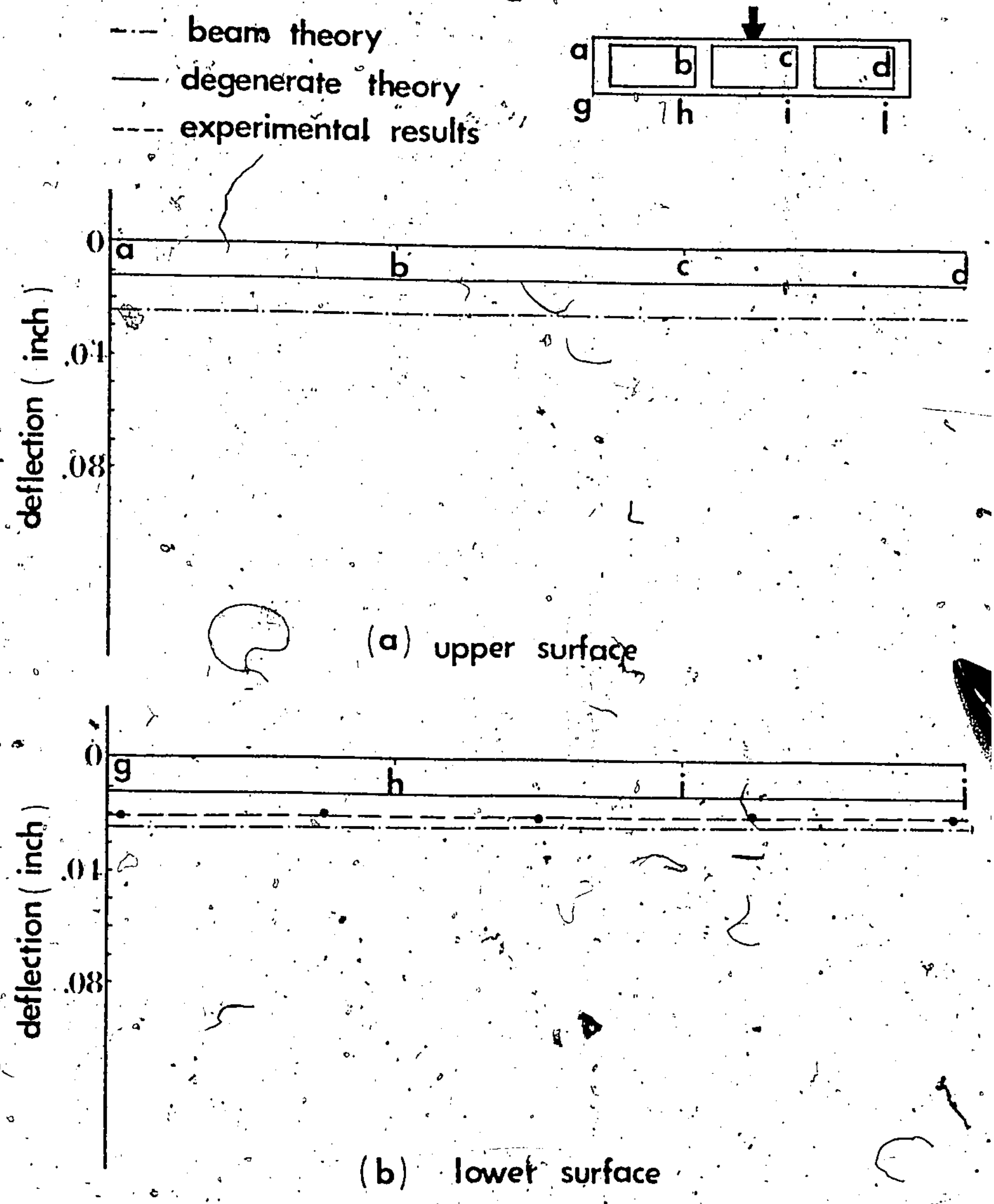


FIG.C10 Comparison of Calculated and Measured Deflections for Model 3 under a Partial Load of 100 psi.

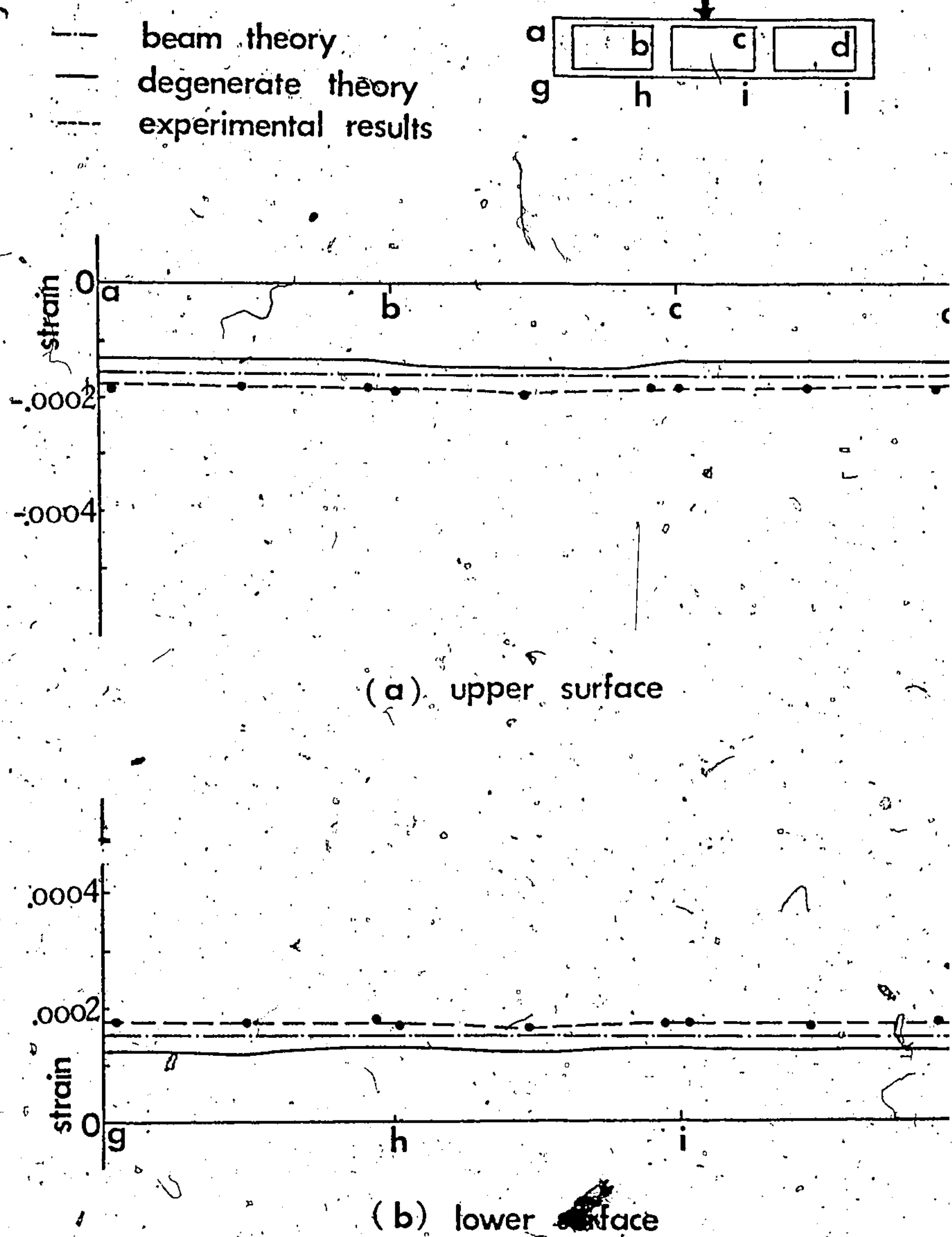


FIG. C11

Comparison of Calculated and Measured longitudinal Strains for Model 3 under a partial Load of 100 psi

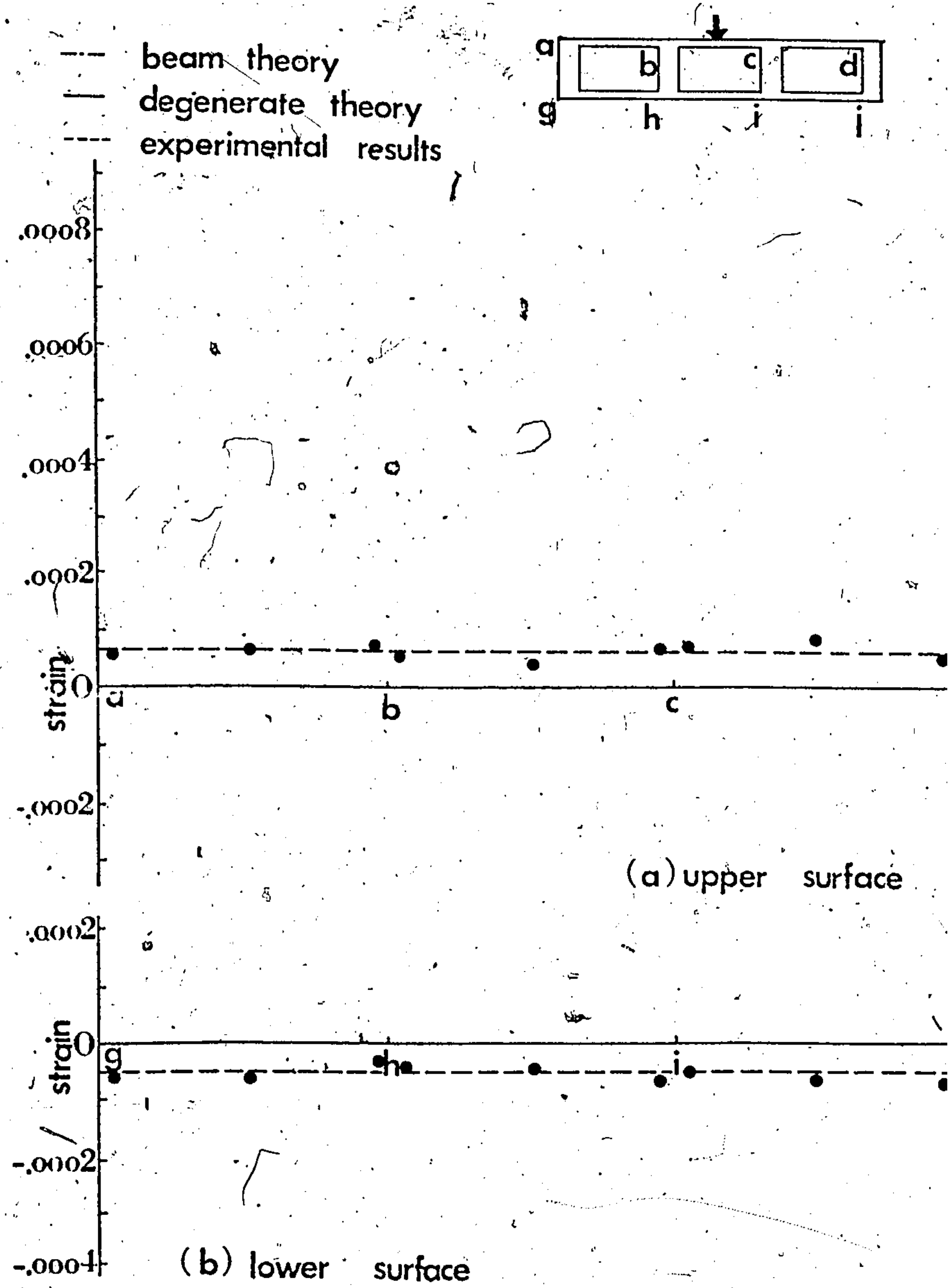


FIG.C12

Comparison of Calculated and Measured Transverse Strains for Model 3 under a Partial Load of 100 psi

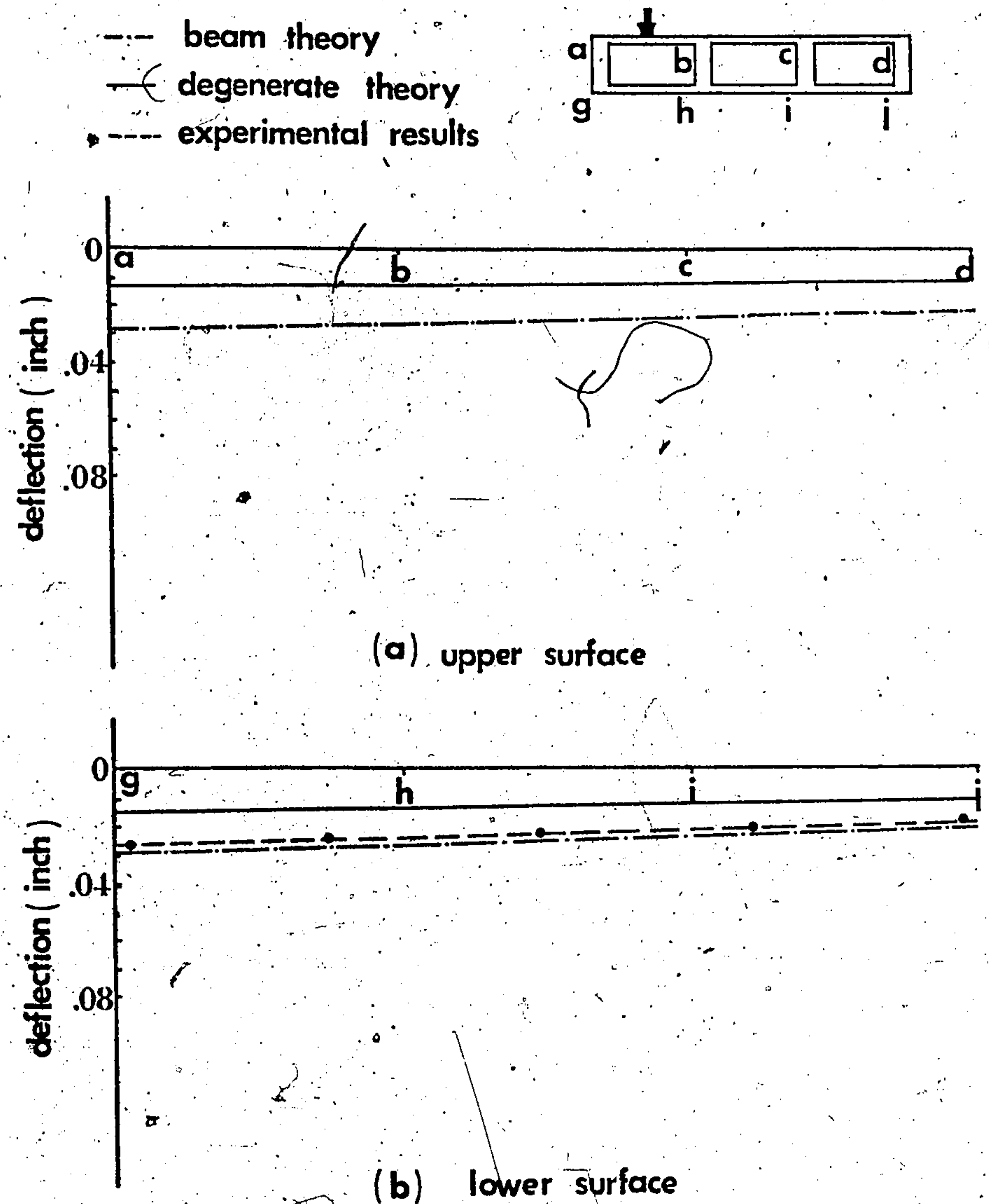


FIG. C13 Comparison of Calculated and Measured Deflections for Model 3 under a Partial Load of 100 ps

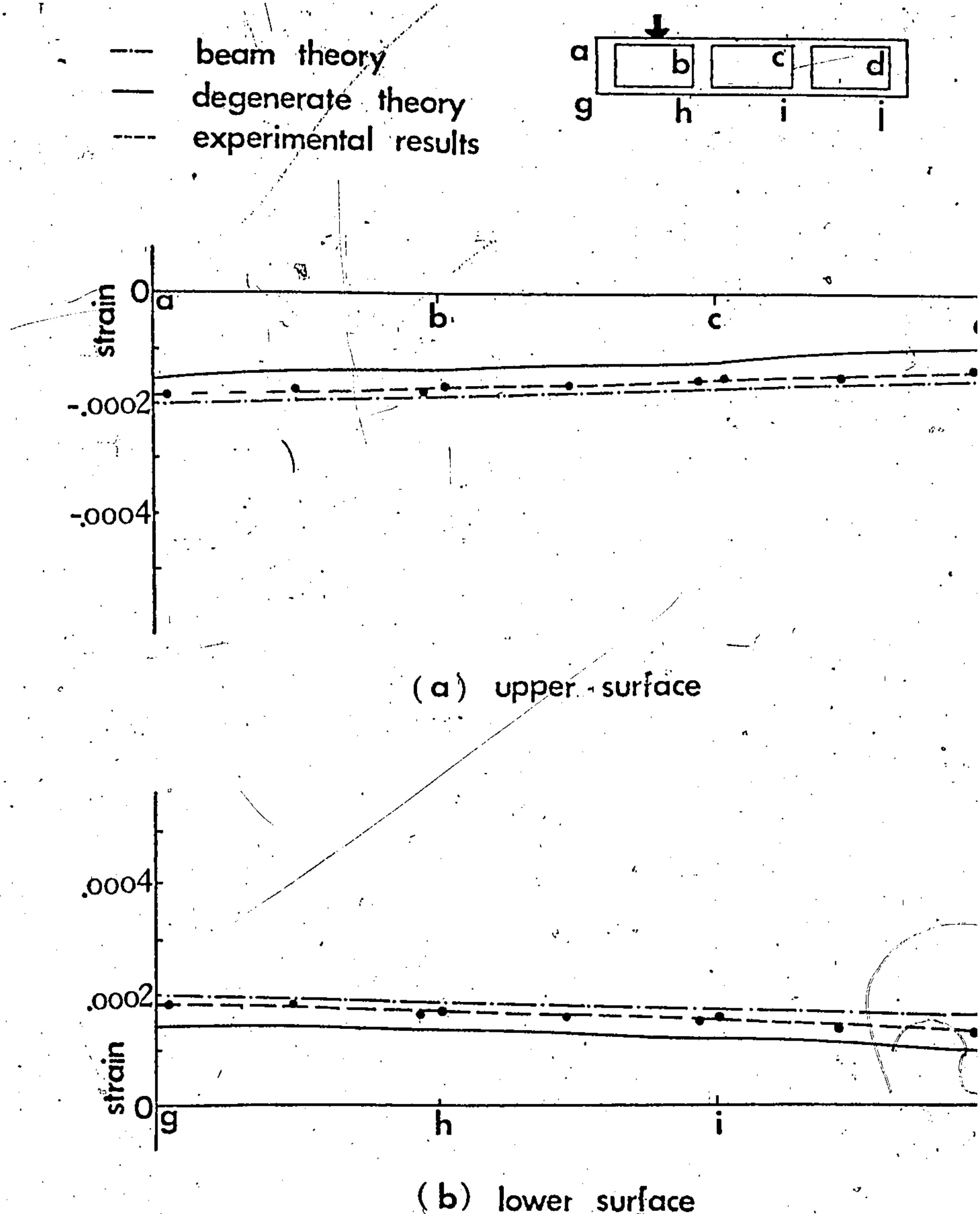


FIG.C14

Comparison of Calculated and Measured longitudinal Strains for Model 3 under a partial Load of 100 psi

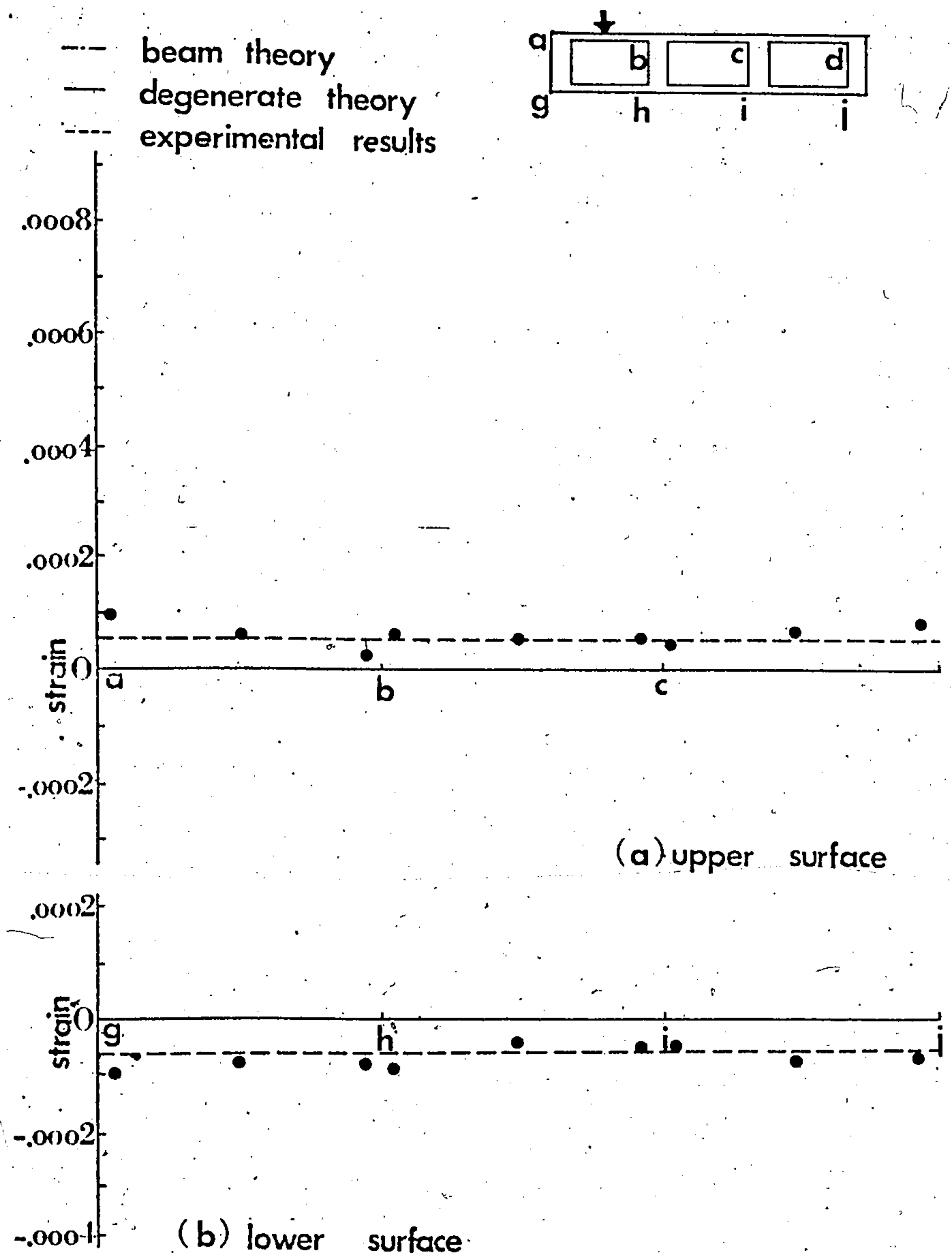


FIG.C15

Comparison of Calculated and Measured Transverse Strains for Model 3 under a Partial Load of 100 psi

APPENDIX B

(1) Purpose

The programme, provides solution for box girder structures simply supported at the two ends having rigid diaphrams ~~at two~~ supports. Uniformly distributed load, partial surface loads as well as line loads may be applied anywhere on the structures. The internal forces, moments and displacements in each plate at any point may be found.

(2) Restriction

Restriction as to the maximum number of bays are 20. The dimension statements will have to be modified if the number of bays are over 20.

(3) Description

The computer solution uses a degenerate method of cylindrical shell theory. The Gibson-equations are used to evaluate the program. A harmonic analysis is used for the loads. The program is written in FORTRAN IV language.

(4) Form of Input Data

1) First Card (I2)

Col. 1 to 2 NN number of bays

2) Next Card (2I5, 2F10.5)

Col. 1 to 5 KK Control Card

Col. 5 to 10 KKKK Control Card

Col. 11 to 20 E1 Lower integrating limit

Col. 20 to 30 E2 Upper integrating limit

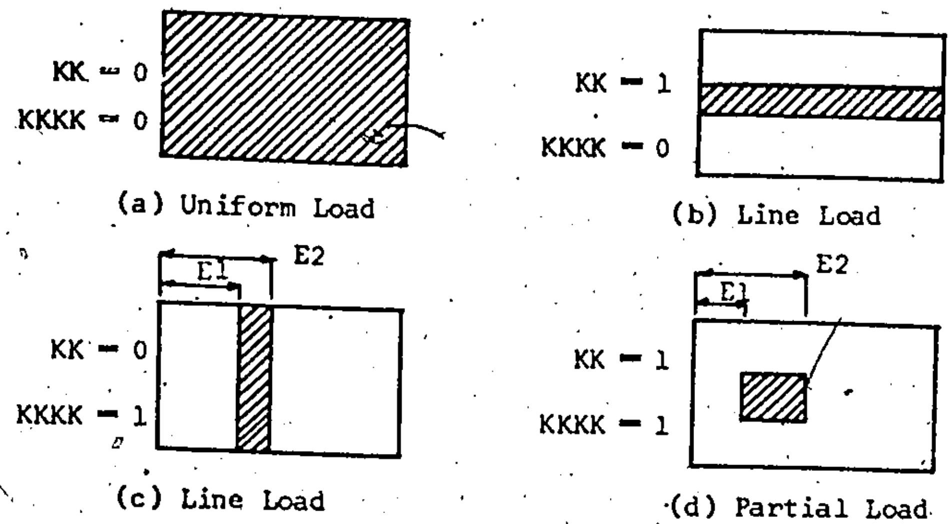


Fig. 31 Index of Various Loading Conditions

3) Next Cards (10F8.3) Data Cards

In case of (a) and (c) each plate needs one set of Cards

In case of (b) and (d) each plate needs three sets of Cards

Every set of Cards contains two cards

The first one of the set.

Col.	1 to 8	A(1)	Span length
Col.	9 to 16	A(2)	Width of the flange
Col.	17 to 24	A(3)	Thickness of the flange
Col.	25 to 32	A(4)	Half angle of the flange
Col.	33 to 40	A(5)	Mode of Fourier Series

Col.	41 to 48	A(6)	Slope of the flange
Col.	49 to 56	A(7)	Load per unit area
Col.	65 to 72	A(9)	Control label for loads acting on the edge beam.

A(9) equal to 0 for uniformly distributed load

A(9) equal to 1 for partial load acting on the left hand side edge beam.

A(9) equal to 2 for partial load acting on the right hand side edge beam

A(9) equal to 3 for partial load acting on both right and left hand side

The second one of the set

Col.	1 to 8	A(11)	Half depth of left hand side edge beam
Col.	9 to 16	A(12)	Half width of left hand side edge beam
Col.	17 to 24	A(13)	Lower integration limit for left hand side edge beam under partial load
Col.	25 to 32	A(14)	Upper integration limit for left hand side edge beam under partial load
Col.	33 to 40	A(15)	Load acting on the left hand side edge beam per unit length.

Col. 41 to 48	A(16)	Half depth of right hand side edge beam
Col. 49 to 56	A(17)	Half width of right hand side edge beam
Col. 57 to 64	A(18)	Lower integration limit for left hand side edge under partial load
Col. 65 to 72	A(19)	Upper integration limit for right hand side edge beam under partial load
Col. 73 to 80	A(20)	Load acting on the left hand side edge beam per unit length.

The input order of plates see Fig. (32)

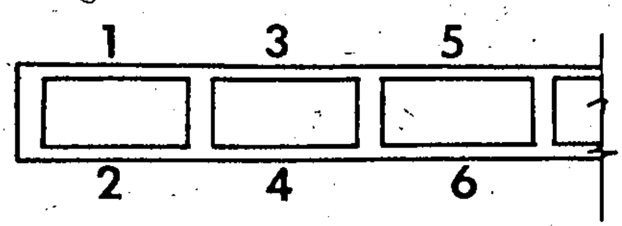


Fig. 32

4) All above control and data cards are repeated for next problem to be solved.

5) One blank card is added at the end of the data deck.

REMARK:

Since each plate consist of 8 unknowns, the number of unknowns equal to 16N, if the section consists of N bays, and 16N simultaneous equations will be solved. In order to get more accurate results, the double precision is used in matrix inversion and multiplication.

OUTPUT DESCRIPTION

a) Input Check Print Out

The complete input is printed and may be used to check up on possible errors in punching field specification and order of the cards.

b) Results

1) M_ϕ transverse moment per unit length

2) M_x longitudinal moment per unit length

3) Q_x normal shear on transverse section per unit length

4) N_ϕ transverse membrane force per unit length

5) N_x longitudinal membrane force per unit length

6) u x E longitudinal displacement

7) w x E normal displacement

8) Q_ϕ normal shear on longitudinal section per unit length

9) Q'_ϕ $Q_\phi + \frac{\partial M_{x\phi}}{\partial x}$

10) $N_{x\phi}$ membrane shear per unit length

11) v x E transverse displacement

12) θ rotation of the shell

13) $M_{x\phi}$ torsional moment per unit length

14) M_1 longitudinal moment of edge beam per unit length

15) M_2 transverse moment of edge beam per unit length

16) M_3 torsional moment of edge beam per unit length

17) θ rotation of the edge beam

DESCRIPTION OF SUBROUTINES

CISSY2 - Compute the coefficients of complementary functions and particular functions.

COEFF1 - Compute trigonometrical and hyperbolic multiples and forces.

STIF - Computation of rigidities of edge beam.

ABMOD - Compute the coefficient of torsional rigidity

ODD - Shell force resultants obtained by an odd number of the differentiations of the normal displacements as in table.

EVEN - Shell force resultants obtained by a even number of the differentiations of the normal displacements as in table.

CISSY - Plate displacements for uniform load use only.

CISSY1 - Plate displacements for partial and line loads use only.

CISSY5 - Edge beam forces and displacements if the plates joint the edge beam at the top and left hand side corner of the edge beam. Displacement equilibrium equations.

CISSY6 - Edge beam forces and displacements, if the plates joint the edge beam at the top and right hand side corner of the edge beam. Displacement equilibrium equations.

CISSY7 - Edge beam forces and displacements if the plate joint the edge beam at the bottom and left hand side corner of the edge beam. Displacement equilibrium equations.

CISSY8 - Edge beam forces and displacements if the plate joint the edge beam at the bottom and right hand side corner of the edge beam. Displacement equilibrium equations.

MULT - Matrix multiplication.

INVERT - Matrix inversion.

MAIN

```
C
C
C     ANALYSIS OF SIMPLY SUPPORTED BOX GIRDER BRIDGE BY DEGENERATE
C     CYLINDRICAL SHELL THEORY WITH EDGE BEAM---S.Y.CHANG   JULY,1969
C
      DOUBLE PRECISION B1,B2,B3,B4,B,BC,T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION U0(9),A11(400),A12(1200),B5(20,8,9),BC5(20,8,9)
      DIMENSION B1(48,48),B2(48,48),B3(48,9),B4(48,9),B(8,17),BC(8,17)
      DIMENSION KA1(20),KA2(20)
  400 KKK=0
      DO 830 I=1,48
      S(I,1)=0.
      DO 830 J=1,48
  830 T(I,J)=0.
      DO 6995 I=1,8
      B(I,17)=C.
 6995 BC(I,17)=0.
      READ(1,99) NN
      IF(NN.EQ.0) GO TO 3000
      WRITE(3,6000) NN
      MM=2*NN
      NNB=MM-1
      MT=MM+2
      DO 1000 MA=1,MM
      MMA=0
      READ(1,4001) KK,KKKK,E1,E2
      WRITE(3,4220) KK,KKKK,E1,E2
      KA1(MA)=KK
      IF(KK.EQ.1) GO TO 8001
  300 READ(1,100) (A(I),I=1,20)
      KAA=((A(5)-1.)/2.)
      KAA=(-1)**KAA
      CCC=A(4)*3.14159265/180.
      A(2)=A(2)/(2.*SIN(CCC))
      IF(A(9).EQ.0.) GO TO 4500
      IF(A(9).EQ.1.) GO TO 4501
```

MAIN

```
      AX1=A(5)*3.14159265*A(19)/A(1)
      AX2=A(5)*3.14159265*A(18)/A(1)
      A(20)=A(20)/2.*(COS(AX2)-COS(AX1))
      IF(A(9).EQ.3.) GO TO 4501
      GO TO 4500
 4501 AX1=A(5)*3.14159265*A(14)/A(1)
      AX2=A(5)*3.14159265*A(13)/A(1)
      A(15)=A(15)/2.*(COS(AX2)-COS(AX1))
 4500 IF((KK.EQ.0).AND.(KKKK.EQ.0)) GO TO 4000
      AX1=A(5)*3.14159265*E2/A(1)
      AX2=A(5)*3.14159265*E1/A(1)
      A(7)=A(7)/2.*(COS(AX2)-COS(AX1))
 4000 DO 1999 I=1,20
      MN=20*MA+I-20
 1999 A11(MN)=A(I)
      CALL CISSY2
      GO TO 9990
 8001 CONTINUE
      DO 9000 MMA=1,3
      READ(1,100) (A(I),I=1,20)
      CCC=A(4)*3.14159265/180.
      A(2)=A(2)/(2.*SIN(CCC))
      IF(A(9).EQ.0.) GO TO 4600
      IF(A(9).EQ.1.) GO TO 4601
      AX1=A(5)*3.14159265*A(19)/A(1)
      AX2=A(5)*3.14159265*A(18)/A(1)
      A(20)=A(20)/2.*(COS(AX2)-COS(AX1))
      GO TO 4600
 4601 AX1=A(5)*3.14159265*A(14)/A(1)
      AX2=A(5)*3.14159265*A(13)/A(1)
      A(15)=A(15)/2.*(COS(AX2)-COS(AX1))
 4600 CONTINUE
      IF((MA.NE.2).AND.(KK.EQ.1).AND.(KKKK.NE.1)) GO TO 4006
      AX1=A(5)*3.14159265*E2/A(1)
      AX2=A(5)*3.14159265*E1/A(1)
      A(7)=A(7)/2.*(COS(AX2)-COS(AX1))
 4006 CONTINUE
      DO 1998 I=1,20
      MN=20*MMA+I-20+MA*60-60
 1998 A12(MN)=A(I)
      CALL CISSY2
      IF(MMA.EQ.2) GO TO 8002
      IF(MMA.EQ.3) GO TO 8003
```

MAIN

```
      IF(MA.EQ.1) GO TO 1401
      CALL CISSY7
 1404 CONTINUE
      KKK=1
      A(4)=-A(4)
      CALL COEFF1
      CALL CISSY1
      DO 7001 I=1,4
      B(I,17)=B(I,17)+Z(I,9)
      DO 7001 J=1,8
 7001 B(I,J)=Z(I,J)
      CALL CISSY
      DO 7002 I=5,8
      IA=I-4
      B(I,17)=B(I,17)+E(IA,9)
      DO 7002 J=1,8
 7002 B(I,J)=E(IA,J)
      GO TO 9000
 8002 CALL COEFF1
      CALL CISSY1
      DO 7003 I=1,4
       B(I,17)=B(I,17)+Z(I,9)
      DO 7003 J=9,16
      JA=J-8
 7003 B(I,J)=Z(I,JA)
      CALL CISSY
      DO 7004 I=5,8
      IA=I-4
      B(I,17)=B(I,17)+E(IA,9)
      DO 7004 J=9,16
      JA=J-8
 7004 B(I,J)=E(IA,JA)
      A(4)=-A(4)
      CALL COEFF1
      CALL CISSY1
      DO 7007 I=1,4
      BC(I,17)=BC(I,17)+Z(I,9)
      DO 7007 J=9,16
      JA=J-8
 7007 BC(I,J)=Z(I,JA)
      CALL CISSY
      DO 7008 I=5,8
      IA=I-4
```

MAIN

```
      BC(I,17)=BC(I,17)+E(IA,9)
      DO 7008 J=9,16
      JA=J-8
 7008 BC(I,J)=E(IA,JA)
      GO TO 9000
 8003 CALL COEFF1
      CALL CISSY1
      DO 7005 I=1,4
      BC(I,17)=BC(I,17)+Z(I,9)
      DO 7005 J=1,8
 7005 BC(I,J)=Z(I,J)
      CALL CISSY
      DO 7006 I=5,8
      IA=I-4
      BC(I,17)=BC(I,17)+E(IA,9)
      DO 7006 J=1,8
 7006 BC(I,J)=E(IA,J)
      KKK=0
      IF(MA.EQ.NNB) GO TO 901
      A(4)=-A(4)
      CALL CISSY5
 1504 CONTINUE
 9000 CONTINUE
      DO 7200 I=1,8
      DO 7201 J=1,8
      B1(I,J)=B(I,J)
 7201 B2(I,J)=BC(I,J)
      DO 7202 IA=1,9
      JA=IA+8
      B3(I,IA)=B(I,JA)
 7202 B4(I,IA)=BC(I,JA)
 7200 CONTINUE
      CALL INVERT(B1,8)
      CALL INVERT(B2,8)
      CALL MULT(B1,B3,B3,8,8,9)
      CALL MULT(B2,B4,B4,8,8,9)
      DO 7300 I=1,8
      DO 7300 J=9,17
      IA=J-8
      B(I,J)=B3(I,IA)
 7300 BC(I,J)=B4(I,IA)
      DO 50 I=1,8
      DO 50 J=1,9
```

MAIN

```
      B5(MA,I,J)=B3(I,J)
   50 BC5(MA,I,J)=B4(I,J)
      KKK=2
      IF(MA.EQ.1) GO TO 9800
      GO TO 9801
 9800 CONTINUE
      CALL CISSY
      DO 8500 M=1,4
      DO 8500 I=1,9
      IA=I+8
      DO 8500 J=1,8
 8500 X(M,I)=T(M,J)*B(J,IA)+X(M,I)
      DO 8600 I=1,4
      S(I,1)=S(I,1)-X(I,9)
      DO 8600 J=1,8
 8600 T(I,J)=-X(I,J)
      CALL CISSY
      DO 8501 M=1,4
      DO 8501 I=1,9
      IA=I+8
      DO 8501 J=1,8
      JA=M+8
 8501 X(M,I)=T(JA,J)*B(J,IA)+X(M,I)
      DO 8601 I=1,4
      IA=8+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8601 J=1,8
 8601 T(IA,J)=-X(I,J)
      IF(NN.EQ.1) GO TO 9802
      GO TO 9803
 9801 CONTINUE
      NA=MA*2-1
      LA=NA+2
      LB=LA-3
      LC=LB-2
      CALL CISSY
      DO 8508 M=1,4
      DO 8508 I=1,9
      IA=8+I
      IAA=(LA-1)*4+M
      DO 8508 J=1,8
      JA=(MA-1)*8+J
 8508 X(M,I)=X(M,I)+T(IAA,JA)* B(J,IA)
```

MAIN

```
      DO 8608 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8608 J=1,8
      JA=(MA-1)*8+J
 8608 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8509 M=1,4
      DO 8509 I=1,9
      IA=8+I
      IAA=(LB-1)*4+M
      DO 8509 J=1,8
      JA=(MA-1)*8+J
 8509 X(M,I)=X(M,I)+T(IAA,JA)* B(J,IA)
      DO 8609 I=1,4
      IA=(LB-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8609 J=1,8
      JA=(MA-1)*8+J
 8609 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8510 M=1,4
      DO 8510 I=1,9
      IA=8+I
      IAA=(LC-1)*4+M
      DO 8510 J=1,8
      JA=(MA-1)*8+J
 8510 X(M,I)=X(M,I)+T(IAA,JA)* B(J,IA)
      DO 8610 I=1,4
      IA=(LC-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8610 J=1,8
      JA=(MA-1)*8+J
 8610 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8511 M=1,4
      DO 8511 I=1,9
      IA=8+I
      IAA=(NA-1)*4+M
      DO 8511 J=1,8
      JA=(MA-1)*8+J
 8511 X(M,I)=X(M,I)+T(IAA,JA)* B(J,IA)
      DO 8611 I=1,4
```

MAIN

```
      IA=(NA-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8611 J=1,8
      JA=(MA-1)*8+J
 8611 T(IA,JA)=-X(I,J)
      IF(MA.EQ.NNB) GO TO 9802
 9803 CONTINUE
      NA=MA*2
      LA=NA+2
      LB=LA+3
      LC=LB-2
      CALL CISSY
      DO 8504 M=1,4
      DO 8504 I=1,9
      IA=8+I
      IAA=(LA-1)*4+M
      DO 8504 J=1,8
      JA=(MA-1)*8+J
 8504 X(M,I)=X(M,I)+T(IAA,JA)*BC(J,IA)
      DO 8604 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8604 J=1,8
      JA=(MA-1)*8+J
 8604 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8505 M=1,4
      DO 8505 I=1,9
      IA=8+I
      IAA=(LB-1)*4+M
      DO 8505 J=1,8
      JA=(MA-1)*8+J
 8505 X(M,I)=X(M,I)+T(IAA,JA)*BC(J,IA)
      DO 8605 I=1,4
      IA=(LB-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8605 J=1,8
      JA=(MA-1)*8+J
 8605 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8506 M=1,4
      DO 8506 I=1,9
      IA=8+I
```

MAIN

```
      IAA=(LC-1)*4+M
      DO 8506 J=1,8
      JA=(MA-1)*8+J
 8506 X(M,I)=X(M,I)+T(IAA,JA)*BC(J,IA)
      DO 8606 I=1,4
      IA=(LC-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8606 J=1,8
      JA=(MA-1)*8+J
 8606 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8507 M=1,4
      DO 8507 I=1,9
      IA=8+I
      IAA=(NA-1)*4+M
      DO 8507 J=1,8
      JA=(MA-1)*8+J
 8507 X(M,I)=X(M,I)+T(IAA,JA)*BC(J,IA)
      DO 8607 I=1,4
      IA=(NA-1)*4+I
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8607 J=1,8
      JA=(MA-1)*8+J
 8607 T(IA,JA)=-X(I,J)
      GO TO 9850
 9802 CONTINUE
      CALL CISSY
      DO 8502 M=1,4
      DO 8502 I=1,9
      IA=I+8
      IAA=M+8*MM-4
      DO 8502 J=1,8
      JA=J+8*(MA-1)
 8502 X(M,I)=T(IAA,JA)*BC(J,IA)+X(M,I)
      DO 8602 I=1,4
      IA=I+8*MM-4
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8602 J=1,8
      JA=J+8*(MA-1)
 8602 T(IA,JA)=-X(I,J)
      CALL CISSY
      DO 8503 M=1,4
      DO 8503 I=1,9
```

MAIN

```
      IA=8+I
      IAA=M+8*MM-12
      DO 8503 J=1,8
      JA=J+8*(MA-1)
 8503 X(M,I)=T(IAA,JA)*BC(J,IA)+X(M,I)
      DO 8603 I=1,4
      IA=I+8*MM-12
      S(IA,1)=S(IA,1)-X(I,9)
      DO 8603 J=1,8
      JA=J+8*(MA-1)
 8603 T(IA,JA)=-X(I,J)
 9850 CONTINUE
      KKK=0
      GO TO 1000
 9990 CONTINUE
      IF(MA.EQ.1) GO TO 1001
      IF(MA.EQ.2) GO TO 1111
      IF(MA.EQ.NNB) GO TO 1003
      IF(MA.EQ.MM) GO TO 1113
      MA1=(MA+1)/2
      MA2=MA/2
      IF(MA1.EQ.MA2) GO  TO 1112
      GO TO 1002
 1401 CONTINUE
 1001 CONTINUE
      CALL COEFF1
      CALL STIF
      DO 2 I=1,9
      Z(1,I)=-(U(I)*D9+YA(I)*D10-A(11)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10-YA(I)*D9+A(12)*C9*W(I))/C9**2
      Z(3,I)=(XA(I)+A(11)*(V(I)*D10-YA(I)*D9)+A(12)*(V(I)*D9+YA(I)*D1
      Z(3,I)=-Z(3,I)
    2 Z(4,I)=-W(I)/C9
      Z(1,9)=Z(1,9)+2.0*A(15)/(3.1416*C9**2*A(5))
      DO 3 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3-A(11)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
    3 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      DO 9 I=1,4
      IA=8+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 9 J=1,8
```

MAIN

```
    9 T(IA,J)=-Z1(I,J)
      DO 601 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
  601 Z1(2,I)=-(Z(2,I)/A3+A(11)*Z(3,I)/A4)/C9**2
      CALL CISSY
      DO 8 I=1,4
      S(I,1)=S(I,1)+E(I,9)
      DO 8 J=1,8
    8 T(I,J)=E(I,J)
      IF(MMA.EQ.1) GO TO 1404
      IF(NN-1) 900,901,900
  900 A(4)=-A(4)
      CALL CISSY5
      GO TO 1000
 1111 CALL COEFF1
      CALL STIF
      DO 202 I=1,9
      Z(1,I)=-(U(I)*D9+YA(I)*D10+A(11)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10-YA(I)*D9+A(12)*C9*W(I))/C9**2
      Z(3,I)=(XA(I)-A(11)*(V(I)*D10-YA(I)*D9)+A(12)*(V(I)*D9+YA(I)*D10)
      Z(3,I)=-Z(3,I)
  202 Z(4,I)=-W(I)/C9
      Z(1,9)=Z(1,9)+2.0*A(15)/(3.1416*C9**2*A(5))
      DO 604 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3+A(11)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
      Z1(4,I)=-Z(3,I)/(A4*C9**2)
  604 CONTINUE
      DO 750 I=1,4
      S(I,1)=S(I,1)-Z1(I,9)
      DO 750 J=1,8
      JA=8+J
  750 T(I,JA)=-Z1(I,J)
      DO 203 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
  203 Z1(2,I)=-(Z(2,I)/A3-A(11)*Z(3,I)/A4)/C9**2
      CALL CISSY
      DO 751 I=1,4
      IA=8+I
      S(IA,1)=S(IA,1)+E(I,9)
      DO 751 J=1,8
      JA=8+J
```

MAIN

```
 751 T(IA,JA)=E(I,J)
     IF(NN-1) 910,911,910
 910 A(4)=-A(4)
     CALL CISSY6
     GO TO 1000
1002 CALL CISSY7
     A(4)=-A(4)
     CALL CISSY5
     GO TO 1000
1112 CALL CISSY8
     A(4)=-A(4)
     CALL CISSY6
     GO TO 1000
1403 CONTINUE
1003 CALL CISSY7
 901 A(4)=-A(4)
     CALL COEFF1
     CALL STIF
     D10=-D10
     DO 65 I=1,9
     Z(1,I)=-(-U(I)*D9+YA(I)*D10+A(16)*C9*W(I))/C9**2
     Z(2,I)=-(U(I)*D10+YA(I)*D9+A(17)*C9*W(I))/C9**2
     Z(3,I)=XA(I)+A(16)*(-V(I)*D10-YA(I)*D9)+A(17)*(-V(I)*D9+YA(I)*D10)
  65 Z(4,I)=W(I)/C9
     Z(1,9)=Z(1,9)+2.0*A(20)/(3.1416*C9**2*A(5))
     DO 617 I=1,9
     Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
     Z1(2,I)=-(Z(2,I)/A3-A(16)*Z(3,I)/A4)/C9**2
     Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
 617 Z1(4,I)=-Z(3,I)/(A4*C9**2)
     DO 760 I=1,4
     IA=I+8*MM-4
     S(IA,1)=S(IA,1)-Z1(I,9)
     DO 760 J=1,8
     JA=J+8*(MA-1)
 760 T(IA,JA)=-Z1(I,J)
     DO 66 I=1,9
     Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
  66 Z1(2,I)=-(Z(2,I)/A3+A(16)*Z(3,I)/A4)/C9**2
     D10=-D10
     CALL CISSY
     DO 761 I=1,4
     IA=I+8*MM-12
```

```
      S(IA,1)=S(IA,1)+E(I,9)
      DO 761 J=1,8
      JA=J+8*(MA-1)
  761 T(IA,JA)=F(I,J)
      IF(MMA.EQ.3) GO TO 1504
      GO TO 1000
 1113 CALL CISSY8
  911 A(4)=-A(4)
      CALL COEFF1
      CALL STIF
      D10=-D10
      DO 265 I=1,9
      Z(1,I)=-(-U(I)*D9+YA(I)*D10-A(16)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10+YA(I)*D9+A(17)*C9*W(I))/C9**2
      Z(3,I)=XA(I)+A(16)*(+V(I)*D10+YA(I)*D9)+A(17)*(-V(I)*D9+YA(I)*D1
  265 Z(4,I)=W(I)/C9
      Z(1,9)=Z(1,9)+2.0*A(20)/(3.1416*C9**2*A(5))
      DO 620 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3+A(16)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
  620 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      DO 37 I=1,4
      IA=I+8*MM-12
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 37 J=1,8
      JA=J+8*(MA-1)
   37 T(IA,JA)=-Z1(I,J)
      DO 266 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
  266 Z1(2,I)=-(Z(2,I)/A3-A(16)*Z(3,I)/A4)/C9**2
      D10=-D10
      CALL CISSY
      DO 36 I=1,4
      IA=I+8*MM-4
      S(IA,1)=S(IA,1)+E(I,9)
      DO 36 J=1,8
      JA=J+8*(MA-1)
   36 T(IA,JA)=E(I,J)
 1000 CONTINUE
      NA=8*MM
      CALL INVERT(T,NA)
      CALL MULT(T,S,S,NA,NA,1)
```

MAIN

```
      DO 800 NA=1,MM
      DO 800 I=1,8
      IA=I+8*NA-8
  800 X(NA,I)=S(IA,1)
      DO 200 MA=1,MM
      KK=KA1(MA)
      IF(KK.EQ.1) GO TO 6200
      DO 201 I=1,20
      MN=20*MA+I-20
      A(I)=A11(MN)
      WRITE(3,1800) A(I)
  201 CONTINUE
      DO 80 JA=1,8
   80 E3(JA,1)=-X(MA,JA)
      GO TO 5990
 6200  KKK=1
      DO 6199 MMA=1,3
      DO 6201 I=1,20
      MN=20*MMA+I-20+MA*60-60
      A(I)=A12(MN)
 6201 WRITE(3,1800) A(I)
      DO 5999 I=1,8
 5999 E3(I,1)=0.
      DO 5998 I=1,8
      IA=MA*8+I-8
 5998 B(I,1)=-S(IA,1)
      GO TO (5987,5988,5989),MMA
 5988 DO 5980 I=1,8
 5980 E3(I,1)=B(I,1)
      GO TO 5990
 5987 DO 5981 I=1,8
      DO 5981 J=1,8
 5981 E3(I,1)=E3(I,1)+B(J,1)*B5(MA,I,J)
      DO 5982 I=1,8
 5982 E3(I,1)=-(E3(I,1)+B5(MA,I,9))
      GO TO 5990
 5989 DO 5983 I=1,8
      DO 5983 J=1,8
 5983 E3(I,1)=E3(I,1)+B(J,1)*BC5(MA,I,J)
      DO 5984 I=1,8
 5984 E3(I,1)=-(E3(I,1)+BC5(MA,I,9))
      GO TO 5990
 5991 CONTINUE
```

MAIN

```
 6199 CONTINUE
      KKK=0
      GO TO 5912
 5990 CONTINUE
      CALL CISSY2
      A4=A(4)/4.
      DO 11 I=1,13
      DO 1 N=1,9
      A(4)=5-N
      A(4)=A(4)*A4
      D1=COS(C2*A(4))
      D2=SIN(C2*A(4))
      D3=EXP(C1*A(4))
      D4=EXP(-C1*A(4))
      D5=COS(C4*A(4))
      D6=SIN(C4*A(4))
      D7=EXP(C3*A(4))
      D8=EXP(-C3*A(4))
      D9=COS(A(4)+A6)
      D10=SIN(A(4)+A6)
      IF(I-7) 7100,7100,7101
 7100 CALL EVEN(U,I)
      GO TO 7102
 7101 CALL ODD(U,I)
 7102 U1=0.
      DO 12 J=1,8
      U1=U1+U(J)*E3(J,1)
   12 CONTINUE
      U0(N)=U1+U(9)
      GO TO(90,1,1,90,1,1,1,90,90,90,1,1,1),I
   90 IF(KKK-1) 85,84,85
   84 IF((N.EQ.1).AND.(MMA.EQ.1)) GO TO 91
      IF((N.EQ.9).AND.(MMA.EQ.3)) GO TO 92
      GO TO 1
   85 IF(N.EQ.1) GO TO 91
      IF(N.EQ.9) GO TO 92
      GO TO 1
   91 T(1,I)=U0(N)
      GO TO 1
   92 E(1,I)=U0(N)
    1 CONTINUE
      WRITE(3,101) (U0(N),N=1,9)
   11 CONTINUE
```

MAIN

```
      IF(KKK.EQ.1) GO TO 5991
 5912 CONTINUE
      LA=MA/2
      LB=(MA+1)/2
      IF(LA.EQ.LB) GO TO 96
      D10=-D10
      Z(1,2)=-(-E(1,9)*D9+E(1,4)*D10+A(16)*C9*E(1,10))/C9**2+A(20)/
     1(3.1416*C9**2*A(5))
      Z(2,2)=-(E(1,9)*D10+E(1,4)*D9+A(17)*C9*E(1,10))/C9**2
      Z(3,2)=E(1,1)+A(16)*(-E(1,8)*D10-E(1,4)*D9)+A(17)*(-E(1,8)*D9+
     1E(1,4)*D10)
      Z(4,2)=E(1,10)/C9
      Z(1,1)=-(T(1,9)*D9+T(1,4)*D10-A(11)*C9*T(1,10))/C9**2+A(15)/(
     13.1416*C9**2*A(5))
      Z(2,1)=-(T(1,9)*D10-T(1,4)*D9+A(12)*C9*T(1,10))/C9**2
      Z(3,1)=-(T(1,1)+A(11)*(T(1,8)*D10-T(1,4)*D9)+A(12)*(T(1,8)*D9+
     1T(1,4)*D10))
      Z(4,1)=-(T(1,10))/C9
      GO TO 200
   96 CONTINUE
      D10=-D10
      Z(1,4)=-(-E(1,9)*D9+E(1,4)*D10-A(16)*C9*E(1,10))/C9**2+A(20)/
     1(3.1416*C9**2*A(5))
      Z(2,4)=-(E(1,9)*D10+E(1,4)*D9+A(17)*C9*E(1,10))/C9**2
      Z(3,4)=E(1,1)+A(16)*(+E(1,8)*D10+E(1,4)*D9)+A(17)*(-E(1,8)*D9+
     1E(1,4)*D10)
      Z(4,4)=E(1,10)/C9
      Z(1,3)=-(T(1,9)*D9+T(1,4)*D10+A(11)*C9*T(1,10))/C9**2+A(15)/(
     13.1416*C9**2*A(5))
      Z(2,3)=-(T(1,9)*D10-T(1,4)*D9+A(12)*C9*T(1,10))/C9**2
      Z(3,3)=-(T(1,1)-A(11)*(T(1,8)*D10-T(1,4)*D9)+A(12)*(T(1,8)*D9+
     1T(1,4)*D10))
      Z(4,3)=-(T(1,10))/C9
      IF(MA-2) 93,93,94
   93 DO 2000 I=1,4
 2000 T(MA,I)=Z(I,1)+Z(I,3)
      T(MA,1)=T(MA,1)+2.0*A(15)/(3.1416*C9**2*A(5))
      DO 2001 I=1,4
      Z(I,5)=Z(I,2)
 2001 Z(I,6)=Z(I,4)
      IF(MM-2) 200,95,200
   94 DO 2002 I=1,4
 2002 T(MA,I)=Z(I,1)+Z(I,3)+Z(I,5)+Z(I,6)
```

MAIN

```
      DO 2003 I=1,4
      Z(I,5)=Z(I,2)
 2003 Z(I,6)=Z(I,4)
      IF(MM-MA) 200,95,200
   95 DO 2004 I=1,4
 2004 T(MT,I)=Z(I,5)+Z(I,6)
      T(MT,1)=T(MT,1)+2.*A(20)/(3.1416*C9**2*A(5))
  200 CONTINUE
      DO 2005 I=2,MT,2
 2005 WRITE(3,2100)(T(I,J),J=1,4)
   99 FORMAT(2I2)
  100 FORMAT(10F8.4)
  101 FORMAT(9(E12.4))
  102 FORMAT(F12.4)
  103 FORMAT(8(E12.4))
 1800 FORMAT(F12.5)
 2100 FORMAT(4E12.4)
 4001 FORMAT(2I5,2F10.5)
 4220 FORMAT('    KK=',I5,'    KKKK=',I5,'    E1=',F10.5,'    E2=',F10.5)
 6000 FORMAT('    NN=',I3/)
      GO TO 400
 3000 STOP
      END
```

CISSY

```
      SUBROUTINE CISSY
C
C     PLATE DISPLACEMENTS FOR UNIFORM LOAD USE ONLY
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      IF(KKK-2) 2,3,2
    3 DO 6 I=1,4
      DO 6 J=1,9
    6 X(I,J)=0.
      RETURN
    2 CONTINUE
      CALL EVEN(U,6)
      CALL EVEN(W,7)
      CALL ODD(V,11)
      CALL ODD(XA,12)
      IF((KKK.EQ.1).AND.(A(4).LT.0.)) GO TO 11
      IF((KKK.EQ.1).AND.(A(4).GT.0.)) GO TO 10
      DO 4 I=1,9
      E(1,I)=-Z1(1,I)+U(I)
      E(2,I)=-Z1(2,I)+V(I)*D9+W(I)*D10
      E(3,I)=-Z1(3,I)+W(I)*D9-V(I)*D10
    4 E(4,I)=-Z1(4,I)+XA(I)
      RETURN
   10 DO 20 I=1,9
      E(1,I)=-U(I)
      E(2,I)=-V(I)*D9-W(I)*D10
      E(3,I)=-W(I)*D9+V(I)*D10
   20 E(4,I)=XA(I)
      RETURN
   11 DO 21 I=1,9
      E(1,I)=U(I)
      E(2,I)=V(I)*D9+W(I)*D10
      E(3,I)=-V(I)*D10+W(I)*D9
   21 E(4,I)=-XA(I)
      RETURN
      END
```

CISSY1

```
      SUBROUTINE CISSY1
C
C
C     PLATE DISPLACEMENTS FOR PARTIAL AND LINE LOADS USE ONLY
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9),R(9),U0(9)
      IF(A(4)) 1,1,2
    1 DO 5 I=1,9
      Z(1,I)=-W(I)
      Z(2,I)=XA(I)
      Z(3,I)=-YA(I)*D9+U(I)*D10
    5 Z(4,I)=YA(I)*D10+U(I)*D9
      GO TO 3
    2 DO 6 I=1,9
      Z(1,I)=W(I)
      Z(2,I)=-XA(I)
      Z(3,I)=YA(I)*D9-U(I)*D10
    6 Z(4,I)=-U(I)*D9-YA(I)*D10
    3 CONTINUE
      RETURN
      END
```

CISSY2

```
      SUBROUTINE CISSY2
C
C
C     COMPUTE THE COEFFICIENTS OF COMPLEMENTARY FUNCTIONS AND
C     PARTICULAR FUNCTIONS
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      A6=3.1416*A(6)/180.
      A(6)=(A(3)**3)/12.
      A(4)=3.1416*A(4)/180.
      BA=3.-3.*A(10)**2
      BA=SQRT(BA)
      BA=SQRT(BA)
      B1=SQRT(A(3)/A(2))
      B1=3.1416*B1*A(2)/(A(1)*BA)
      BA=SQRT(BA)
      A1=SQRT((1.+B1)**2+1.)+1.+B1
      A1=SQRT(0.5*A1)
      A2=SQRT((1.+B1)**2+1.)-1.-B1
      A2=SQRT(0.5*A2)
      A3=SQRT((1.-B1)**2+1.)-1.+B1
      A3=SQRT(0.5*A3)
      A4=SQRT((1.-B1)**2+1.)+1.-B1
      A4=SQRT(0.5*A4)
      C1=SQRT(A(2)/A(3))
      C2=3.1416*C1*A(2)/A(1)
      BA=BA*SQRT(C2)
      C1=BA*A1
      C2=BA*A2
      C3=BA*A3
      C4=BA*A4
      C5=BA*BA*(1.+B1)
      C6=BA*BA
      C7=BA*BA*(-1.+B1)
      C8=BA*BA
```

CISSY2

```
      C9=3.1416*A(5)/A(1)
      G0=1.-A(10)**2
      G(1)=2.0*A(6)/(G0*A(2)**2)
      G(2)=-2.0*A(6)*C9**2/G0
      G(3)=G(2)*C9/B1
      G(4)=-4.0*C9**4*A(6)*A(2)/(G0*B1**2)
      G(5)=-G(4)/B1
      G(6)=G(5)/(A(3)*C9)
      G(7)=2.
      G(8)=2.0*A(6)*C9**3/(G0*((B1)**1.5))
      G(9)=G(8)
      G(10)=2.0*G(9)*A(2)*C9/B1
      G(11)=-4.*A(6)*A(2)*C9**3/(G0*A(3)*(B1**3.5))
      G(12)=1.0/A(2)
      G(13)=-2.0*A(6)*C9/((1.0+A(10))*A(2))
      H(1,1)=C5-A(10)*C9**2*A(2)**2
      H(1,2)=C6
      H(1,3)=C7-A(10)*C9**2*A(2)**2
      H(1,4)=C8
      H(2,1)=1.0-A(10)*C5/((C9*A(2))**2)
      H(2,2)=-A(10)*C6/((C9*A(2))**2)
      H(2,3)=1.0-A(10)*C7/((C9*A(2))**2)
      H(2,4)=-A(10)*C8/((C9*A(2))**2)
      H(3,1)=1.
      H(3,2)=1.
      H(3,3)=-1.
      H(3,4)=1.
      H(4,1)=0.
      H(4,2)=1.
      H(4,3)=0.
      H(4,4)=-1.
      H(5,1)=-1.
      H(5,2)=1.0+B1
      H(5,3)=1.
      H(5,4)=1.0-B1
      H(6,1)=-1.
      H(6,2)=1.0+B1+B1*A(10)
      H(6,3)=1.
      H(6,4)=1.0-B1-B1*A(10)
      H(7,1)=1.
      H(7,2)=0.
      H(7,3)=1.
      H(7,4)=0.
```

CISSY2

```
H(8,1)=A1-A2
H(8,2)=A1+A2
H(8,3)=-A3-A4
H(8,4)=A3-A4
H(9,1)=A1*(1.0-B1*(1.0-A(10)))-A2
H(9,2)=A1+A2*(1.0-B1*(1.0-A(10)))
H(9,3)=-A3*(1.0+B1*(1.0-A(10)))-A4
H(9,4)=A3-A4*(1.0+B1*(1.0-A(10)))
H(10,1)=-A2
H(10,2)=A1
H(10,3)=A4
H(10,4)=-A3
H(11,1)=A1+A2*(1.0-B1*(1.0+A(10)))
H(11,2)=A2-A1*(1.0-B1*(1.0+A(10)))
H(11,3)=-A3+A4*(1.0+B1*(1.0+A(10)))
H(11,4)=-A4-A3*(1.0+B1*(1.0+A(10)))
H(12,1)=-2.0*C1+G(11)*H(11,1)
H(12,2)=-2.0*C2+G(11)*H(11,2)
H(12,3)=-2.0*C3+G(11)*H(11,3)
H(12,4)=-2.0*C4+G(11)*H(11,4)
H(13,1)=C1
H(13,2)=C2
H(13,3)=C3
H(13,4)=C4
F0=C9*A(2)
F1=   A(6)*(((1.0+F0**2)**4)+(A(2)**6)*(G0*A(3)*(C9**4))/A(6))
F2=-(A(2)**4)*((F0**4)+(4.0+A(10))*F0**2+2.0)*G0
F1=4.0*F2*A(7)/(3.1416*F1*A(5))
A(7)=4.0*A(7)/(3.1416*A(5))
F(1)=-F1*A(6)*(A(10)*C9**2+1.0/(A(2)**2))/G0
F(2)=-F1*A(6)*(C9**2+A(10)/(A(2)**2))/G0
F(13)=C9*A(6)*F1/((1.0+A(10))*A(2))
F(3)=F1*A(6)*(C9**3+C9/(A(2)**2))/G0
F(8)=F1*A(6)*(C9**2/A(2)+1.0/(A(2)**3))/G0
F(9)=F1*A(6)*(C9**2/A(2)+1.0/(A(2)**3)+C9**2/((1.0+A(10))*A(2)))
F(9)=F(9)/G0
H3=A(6)*F1*((C9**4)+2.0*C9**2/(A(2)*A(2))+1.0/(A(2)**4))/G0
F(4)=-A(2)*(H3+A(7))
F(10)=-(H3+2.0*A(7))/C9
F(5)=-(H3+2.0*A(7))/(C9**2*A(2))
F(11)=(H3+2.0*A(7))*(2.0+1.0/(C9**2*A(2)**2))
F(11)=F(11)+A(10)*(H3+3.0*A(7))
F(11)=F(11)/(C9**2*A(3))
```

CISSY2

```
F(6)=(-(H3+2.0*A(7))/(C9**2*A(2))+A(10)*A(2)*(H3+A(7)))/(C9*A(3))
F(7)=F1
F(12)=(F1+F(11))/A(2)
F13=-4.*A(8)*GO/(3.1416*A(6)*((C9**4)+GO*A(3)/(A(6)*(A(2)**2))))
F3(7)=F13
F3(2)=-A(6)*C9*C9*F13/GO
F3(13)=0
F3(3)=A(6)*(C9**3)*F13/GO
F3(8)=0
F3(9)=0
F3(4)=-A(2)*(A(6)*(C9**4)*F13/GO+4.*A(8)/3.1416)
F3(10)=0
F3(5)=0
F3(6)=-F3(4)*A(10)/A(3)
F3(11)=0
F3(12)=0
F3(1)=-A(6)*A(10)*C9*C9*F13/GO
RETURN
END
```

CISSY5

```
      SUBROUTINE CISSY5
C
C
C     EDGE BEAM FORCES AND DISPLACEMENTS, IF THE PLATES JOINT THE
C     EDGE BEAM AT THE TOP AND LEFT HAND SIDE CORNER OF THE EDGE
C     BEAM.   DISPLACEMENT EQUILIBRIUM EQUATIONS
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      CALL COEFF1
      CALL STIF
      D10=-D10
      DO 5 I=1,9
      Z(1,I)=-(-U(I)*D9+YA(I)*D10+A(16)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10+YA(I)*D9+A(17)*C9*W(I))/C9**2
      Z(3,I)=XA(I)+A(16)*(-V(I)*D10-YA(I)*D9)+A(17)*(-V(I)*D9+YA(I)*D10)
    5 Z(4,I)=W(I)/C9
      Z(1,9)=Z(1,9)+1.0*A(20)/(3.1416*C9**2*A(5))
      DO 602 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3-A(16)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
  602 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      NA=MA*2
      LA=NA+2
      LB=LA+3
      LC=LB-2
      DO 700 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 700 J=1,8
      JA=(MA-1)*8+J
  700 T(IA,JA)=-Z1(I,J)
      DO 603 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2+A(17)*Z(2,I)/A3)/C9
  603 Z1(3,I)=-(Z(1,I)/A2+A(17)*Z(3,I)/A4)/C9**2
```

CISSY5

```
      DO 708 I=1,4
      IA=(LB-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 708 J=1,8
      JA=(MA-1)*8+J
  708 T(IA,JA)=-Z1(I,J)
      DO 6 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2+A(17)*Z(2,I)/A3)/C9
    6 Z1(2,I)=-(Z(2,I)/A3+A(16)*Z(3,I)/A4)/C9**2
      DO 716 I=1,4
      IA=(LC-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 716 J=1,8
      JA=(MA-1)*8+J
  716 T(IA,JA)=-Z1(I,J)
      DO 306 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
  306 Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
      D10=-D10
      CALL CISSY
      DO 724 I=1,4
      IA=(NA-1)*4+I
      S(IA,1)=S(IA,1)+E(I,9)
      DO 724 J=1,8
      JA=(MA-1)*8+J
  724 T(IA,JA)=E(I,J)
      RETURN
      END
```

CISSY6

```
      SUBROUTINE CISSY6
C
C
C     EDGE BEAM FORCES AND DISPLACEMENTS, IF THE PLATES JOINT THE
C     EDGE BEAM AT THE TOP AND RIGHT HAND SIDE CORNER OF THE EDGE
C     BEAM.   DISPLACEMENT EQUILIBRIUM EQUATIONS.
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION A11(400),A12(60)
      CALL COEFFI
      CALL STIF
      D10=-D10
      DO 205 I=1,9
      Z(1,I)=-(-U(I)*D9+YA(I)*D10-A(16)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10+YA(I)*D9+A(17)*C9*W(I))/C9**2
      Z(3,I)=XA(I)+A(16)*(+V(I)*D10+YA(I)*D9)+A(17)*(-V(I)*D9+YA(I)*D10)
  205 Z(4,I)=W(I)/C9
      Z(1,9)=Z(1,9)+1.0*A(20)/(3.1416*C9**2*A(5))
      DO 606 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3+A(16)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
  606 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      NA=MA*2
      LA=NA-2
      LB=LA+3
      LC=LB+2
      DO 701 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 701 J=1,8
      JA=(MA-1)*8+J
  701 T(IA,JA)=-Z1(I,J)
      DO 605 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(16)*Z(1,I)/A2+A(17)*Z(2,I)/A3)/C9
```

CISSY6

```
605 Z1(3,I)=-(Z(1,I)/A2+A(17)*Z(3,I)/A4)/C9**2
    DO 709 I=1,4
    IA=(LB-1)*4+I
    S(IA,1)=S(IA,1)-Z1(I,9)
    DO 709 J=1,8
    JA=(MA-1)*8+J
709 T(IA,JA)=-Z1(I,J)
    DO 206 I=1,9
    Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2+A(17)*Z(2,I)/A3)/C9
206 Z1(2,I)=-(Z(2,I)/A3-A(16)*Z(3,I)/A4)/C9**2
    DO 717 I=1,4
    IA=(LC-1)*4+I
    S(IA,1)=S(IA,1)-Z1(I,9)
    DO 717 J=1,8
    JA=(MA-1)*8+J
717 T(IA,JA)=-Z1(I,J)
    DO 506 I=1,9
    Z1(1,I)=(Z(4,I)/A1+A(16)*Z(1,I)/A2-A(17)*Z(2,I)/A3)/C9
506 Z1(3,I)=-(Z(1,I)/A2-A(17)*Z(3,I)/A4)/C9**2
    DIO=-DIO
    CALL CISSY
    DO 725 I=1,4
    IA=(NA-1)*4+I
    S(IA,1)=S(IA,1)+E(I,9)
    DO 725 J=1,8
    JA=(MA-1)*8+J
725 T(IA,JA)=E(I,J)
    RETURN
    END
```

CISSY7

```
      SUBROUTINE CISSY7
C
C
C     EDGE BEAM FORCES AND DISPLACEMENTS, IF THE PLATES JOINT THE
C     EDGE BEAM AT THE BOTTOM AND LEFT HAND SIDE CORNER OF THE EDGE
C     BEAM.   DISPLACEMENT EQUILIBRIUM EQUATIONS
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION A11(400),A12(60)
      CALL COEFFI
      CALL STIF
      DO 52 I=1,9
      Z(1,I)=-(U(I)*D9+YA(I)*D10-A(11)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10-YA(I)*D9+A(12)*C9*W(I))/C9**2
      Z(3,I)=(XA(I)+A(11)*(V(I)*D10-YA(I)*D9)+A(12)*(V(I)*D9+YA(I)*D10))
      Z(3,I)=-Z(3,I)
   52 Z(4,I)=-W(I)/C9
      Z(1,9)=Z(1,9)+1.0*A(15)/(3.1416*C9**2*A(5))
      DO 608 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3-A(11)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
  608 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      NA=MA*2-1
      LA=NA+2
      LB=LA-3
      LC=LB-2
      DO 702 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 702 J=1,8
      JA=(MA-1)*8+J
  702 T(IA,JA)=-Z1(I,J)
      DO 607 I=1,9
      Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2-A(12)*Z(2,I)/A3)/C9
```

CISSY7

```
607 Z1(3,I)=-(Z(1,I)/A2-A(12)*Z(3,I)/A4)/C9**2
    DO 710 I=1,4
    IA=(LB-1)*4+I
    S(IA,1)=S(IA,1)-Z1(I,9)
    DO 710 J=1,8
    JA=(MA-1)*8+J
710 T(IA,JA)=-Z1(I,J)
    DO 53 I=1,9
    Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2-A(12)*Z(2,I)/A3)/C9
 53 Z1(2,I)=-(Z(2,I)/A3+A(11)*Z(3,I)/A4)/C9**2
    DO 718 I=1,4
    IA=(LC-1)*4+I
    S(IA,1)=S(IA,1)-Z1(I,9)
    DO 718 J=1,8
    JA=(MA-1)*8+J
718 T(IA,JA)=-Z1(I,J)
    DO 353 I=1,9
    Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
353 Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
    CALL CISSY
    DO 726 I=1,4
    IA=(NA-1)*4+I
    S(IA,1)=S(IA,1)+E(I,9)
    DO 726 J=1,8
    JA=(MA-1)*8+J
726 T(IA,JA)=E(I,J)
    RETURN
    END
```

CISSY8

```
      SUBROUTINE CISSY8
C
C
C     EDGE BEAM FORCES AND DISPLACEMENTS, IF THE PLATES JOINT THE
C     EDGE BEAM AT THE BOTTOM AND RIGHT HAND SIDE CORNER OF THE EDGE
C     BEAM.   DISPLACEMENT EQUILIBRIUM EQUATIONS.
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION A11(400),A12(60)
      CALL COEFF1
      CALL STIF
      DO 262 I=1,9
      Z(1,I)=-(U(I)*D9+YA(I)*D10+A(11)*C9*W(I))/C9**2
      Z(2,I)=-(U(I)*D10-YA(I)*D9+A(12)*C9*W(I))/C9**2
      Z(3,I)=(XA(I)-A(11)*(V(I)*D10-YA(I)*D9)+A(12)*(V(I)*D9+YA(I)*D10))
      Z(3,I)=-Z(3,I)
  262 Z(4,I)=-W(I)/C9
      Z(1,9)=Z(1,9)+1.0*A(15)/(3.1416*C9**2*A(5))
      DO 619 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
      Z1(2,I)=-(Z(2,I)/A3+A(11)*Z(3,I)/A4)/C9**2
      Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
  619 Z1(4,I)=-Z(3,I)/(A4*C9**2)
      NA=MA*2-1
      LA=NA-2
      LB=LA-3
      LC=LB+2
      DO 707 I=1,4
      IA=(LA-1)*4+I
      S(IA,1)=S(IA,1)-Z1(I,9)
      DO 707 J=1,8
      JA=(MA-1)*8+J
  707 T(IA,JA)=-Z1(I,J)
      DO 618 I=1,9
      Z1(1,I)=(Z(4,I)/A1-A(11)*Z(1,I)/A2-A(12)*Z(2,I)/A3)/C9
```

CISSY8

```
618  Z1(3,I)=-(Z(1,I)/A2-A(12)*Z(3,I)/A4)/C9**2
     DO 715 I=1,4
     IA=(LB-1)*4+I
     S(IA,1)=S(IA,1)-Z1(I,9)
     DO 715 J=1,8
     JA=(MA-1)*8+J
715  T(IA,JA)=-Z1(I,J)
     DO 263 I=1,9
     Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2-A(12)*Z(2,I)/A3)/C9
263  Z1(2,I)=-(Z(2,I)/A3-A(11)*Z(3,I)/A4)/C9**2
     DO 723 I=1,4
     IA=(LC-1)*4+I
     S(IA,1)=S(IA,1)-Z1(I,9)
     DO 723 J=1,8
     JA=(MA-1)*8+J
723  T(IA,JA)=-Z1(I,J)
     DO 563 I=1,9
     Z1(1,I)=(Z(4,I)/A1+A(11)*Z(1,I)/A2+A(12)*Z(2,I)/A3)/C9
563  Z1(3,I)=-(Z(1,I)/A2+A(12)*Z(3,I)/A4)/C9**2
     CALL CISSY
     DO 731 I=1,4
     IA=(NA-1)*4+I
     S(IA,1)=S(IA,1)+E(I,9)
     DO 731 J=1,8
     JA=(MA-1)*8+J
731  T(IA,JA)=E(I,J)
     RETURN
     END
```

COEFF1

```
      SUBROUTINE COEFF1
C
C
C     COMPUTE TRIGONOMETRICAL AND HYPERBOLIC MULTIPLES AND FORCES
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13)
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      D1=COS(C2*A(4))
      D2=SIN(C2*A(4))
      D3=EXP(C1*A(4))
      D4=EXP(-C1*A(4))
      D5=COS(C4*A(4))
      D6=SIN(C4*A(4))
      D7=EXP(C3*A(4))
      D8=EXP(-C3*A(4))
      D10=SIN(A(4)+A6)
      D9=COS(A(4)+A6)
      CALL ODD(U,9)
      CALL ODD(V,8)
      CALL ODD(W,10)
      CALL EVEN(XA,1)
      CALL EVEN(YA,4)
      RETURN
      END
```

STIF

```
      SUBROUTINE STIF
C
C
C     COMPUTATION OF RIGIDITIES OF EDGE BEAM
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13)
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      IF(A(4)) 1,1,2
    2 A1=4.0*A(11)*A(12)
      A2=4.0*A(12)*A(11)**3/3.0
      A3=4.0*A(11)*A(12)**3/3.
      CALL ABMOD(A(11),A(12),AP6)
      A4=2.0*AP6*A3/(1.0+A(10))
      GO TO 3
    1 A1=4.0*A(16)*A(17)
      A2=4.0*A(17)*A(16)**3/3.
      A3=4.0*A(16)*A(17)**3/3.
      CALL ABMOD(A(16),A(17),AP6)
      A4=2.0*AP6*A3/(1.0+A(10))
    3 CONTINUE
      RETURN
      END
```

ABMOD

```
      SUBROUTINE ABMOD(A00,A01,AP6)
C
C
C     COMPUTE THE COEFFICIENT OF TORSIONAL RIGIDITY
C
C
      IF((A00-A01).LT.0.0) GO TO 500
      AP1=A00
      AP2=A01
      GO TO 501
  500 AP1=A01
      AP2=A00
  501 CONTINUE
      AP5=0
      DO 502 N=1,21,2
      AP3=N
      AP4=3.1416*AP3*AP1/(2.0*AP2)
      AP7=TANH(AP4)
      AP5=AP5+AP7/(AP3**5)
  502 CONTINUE
      AP6=1.-192.*AP2*AP5/(AP1*(3.1416**5))
      RETURN
      END
```

EVEN

```
      SUBROUTINE EVEN(R,I)
C
C
C     SHELL FORCE RESULTS OBTAINED BY A EVEN NUMBER OF
C     DIFFERENTIATIONS OF THE NORMAL DISPLACEMENTS AS IN TABLE
C
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION R(9)
      R(1)=G(I)*H(I,1)*D1*D3-G(I)*H(I,2)*D2*D3
      R(2)=-G(I)*H(I,2)*D1*D3-G(I)*H(I,1)*D2*D3
      R(3)=G(I)*H(I,3)*D5*D7-G(I)*H(I,4)*D6*D7
      R(4)=-G(I)*H(I,4)*D5*D7-G(I)*H(I,3)*D6*D7
      R(5)=G(I)*H(I,1)*D1*D4+G(I)*H(I,2)*D2*D4
      R(6)=-G(I)*H(I,2)*D1*D4+G(I)*H(I,1)*D2*D4
      R(7)=G(I)*H(I,3)*D5*D8+G(I)*H(I,4)*D6*D8
      R(8)=-G(I)*H(I,4)*D5*D8+G(I)*H(I,3)*D6*D8
      R(9)=F(I)*D9+F3(I)
      RETURN
      END
```

ODD

```
      SUBROUTINE ODD(R,I)
C
C
C     DIFFERENTIATIONS OF THE NORMAL DISPLACEMENTS AS IN TABLE
C
      DOUBLE PRECISION T,S
      COMMON T,S
      COMMON A,U,V,W,XA,YA,E,H,F,G,F3,Z,Z1,E3,X,C9
      COMMON C1,C2,C3,C4,A1,A2,A3,A4,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
     1ANN,A6,MA,KKK
      DIMENSION T(48,48),S(48,1)
      DIMENSION A(20),U(9),V(9),W(9),XA(9),YA(9),E(4,10),H(13,4),F(13),
     1G(13),F3(13),Z(4,9),Z1(4,9),E3(8,1),X(20,9)
      DIMENSION R(9)
      R(1)=G(I)*H(I,1)*D1*D3-G(I)*H(I,2)*D2*D3
      R(2)=-G(I)*H(I,2)*D1*D3-G(I)*H(I,1)*D2*D3
      R(3)=G(I)*H(I,3)*D5*D7-G(I)*H(I,4)*D6*D7
      R(4)=-G(I)*H(I,4)*D5*D7-G(I)*H(I,3)*D6*D7
      R(5)=-G(I)*H(I,1)*D1*D4-G(I)*H(I,2)*D2*D4
      R(6)=+G(I)*H(I,2)*D1*D4-G(I)*H(I,1)*D2*D4
      R(7)=-G(I)*H(I,3)*D5*D8-G(I)*H(I,4)*D6*D8
      R(8)=+G(I)*H(I,4)*D5*D8-G(I)*H(I,3)*D6*D8
     1      R(9)=F(I)*D10+F3(I)
      RETURN
      END
```

INVEPT

```
      SUBROUTINE INVERT(A,N)
C
C
C     MATRIX INVERSION
C
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(48,48),INDEX(48,2)
      DO 108 I=1,N
  108 INDEX(I,1)=0
      II=0
  109 AMAX=-1.
      DO 110 I=1,N
      IF(INDEX(I,1))110,111,110
  111 DO 112 J=1,N
      IF(INDEX(J,1))112,113,112
  113 TEMP=DABS(A(I,J))
      IF(TEMP-AMAX)112,112,114
  114 IROW=I
      ICOL=J
      AMAX=TEMP
  112 CONTINUE
  110  CONTINUE
      IF(AMAX)225,115,116
  116 INDEX(ICOL,1)=IROW
      IF(IROW-ICOL)119,118,119
  119 DO 120 J=1,N
      TEMP=A(IROW,J)
      A(IROW,J)=A(ICOL,J)
  120 A(ICOL,J)=TEMP
      II=II+1
      INDEX(II,2)=ICOL
  118 PIVOT=A(ICOL,ICOL)
      A(ICOL,ICOL)=1.
      PIVOT=1./PIVOT
      DO 121 J=1,N
  121 A(ICOL,J)=A(ICOL,J)*PIVOT
      DO 122 I=1,N
      IF(I-ICOL)123,122,123
  123 TEMP=A(I,ICOL)
      A(I,ICOL)=0.
      DO 124 J=1,N
  124 A(I,J)=A(I,J)-A(ICOL,J)*TEMP
```

INVERT

```
  122   CONTINUE
        GO TO 109
  125   ICOL=INDEX(II,2)
        IROW=INDEX(ICOL,1)
        DO 126 I=1,N
        TEMP=A(I,IROW)
        A(I,IROW)=A(I,ICOL)
  126   A(I,ICOL)=TEMP
        II=II-1
  225   IF(II)125,127,125
  115   WRITE(6,1001)
 1001   FORMAT(1H0,2X,11H ZERO PIVOT)
  127   CONTINUE
        RETURN
        END
```

MULT

```
      SUBROUTINE MULT(A,B,D,I,J,K)
C
C
C     MATRIX MULTIPLICATION
C
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(48,48),B(48,9),C(48,9),D(48,9)
      DO 11 L=1,I
      DO 11 N=1,K
   11 C(L,N)=0.
      DO 2 L=1,I
      DO 2 N=1,K
      DO 2 M=1,J
    2 C(L,N)=C(L,N)+A(L,M)*B(M,N)
      DO 3 N=1,K
      DO 3 L=1,I
    3 D(L,N)=C(L,N)
      RETURN
      END
```

APPENDIX C

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APPENDIX D

LIST OF SYMBOLS USED

a - h	Numerical constants
a_1	Half depth of edge beam
b_1	Half width of edge beam
c	Transverse span of shell
$gd_1 gd_2 gd_3$	Numerical constants
k	Trigonometric parameter
$m_1 m_2 n_1 n_2 p$	Numerical constants
u	Longitudinal displacement of shell
u_B	Longitudinal displacement of edge beam
v	Transverse displacement of shell
v_B	Transverse displacement of edge beam
v_{1s}	Displacement in y - direction
w	Normal displacement of shell
w_B	Normal displacement of edge
w_{1s}	Displacement in z - direction
x	Cartesian coordinate axis
y	Cartesian coordinate axis
z	Cartesian coordinate axis
A	Half thickness of shell
$A_1 A_2 A_3$	Numerical constants
$B_1 B_2 B_3 B_4$	Numerical constants
C, H_3	Numerical constants
E	Young's modulus
F	Force in the edge beam in x - direction

I_1	Moment of inertia about y axis
I_2	Moment of inertia about z axis
I_3	Torsional rigidity of edge beam
$k_1 k_2 k_3$	Numerical constants
L	Longitudinal span of the shell
M_1	Longitudinal moment of edge beam per unit length
M_2	Transverse moment of edge beam per unit length
M_3	Torsional moment of edge beam per unit length
M_x	Longitudinal moment of shell per unit length
M_ϕ	Transverse moment of shell per unit length
$M_{x\phi}$	Torsional moment of shell per unit length
N_x	Longitudinal membrane force per unit length
N_ϕ	Transverse membrane force per unit length
$N_{x\phi}$	Membrane shearing stress per unit length
Q_x	Normal shear on transverse section per unit length
Q_ϕ	Normal shear on longitudinal section per unit length
R	Radius of curvature of shell element
S_1	Force in the edge beam in z direction
S_2	Force in the edge beam in y direction
X	Load in x direction
Y	Load in y direction
Z	Load in z direction
θ	Rotation of shell
θ_B	Rotation of edge beam
ϕ_k	Half angle of shell in transverse direction

$\alpha_1 \alpha_2 \alpha_1' \alpha_2'$	Numerical constants
$\beta_1 \beta_2 \beta_1' \beta_2'$	Numerical constants
γ	Numerical constants
δ	Rise of shell over span c
η	Poisson ratio
$\{\delta\}_p$	Displacement matrix due to external load
$\{\delta\}_x$	Displacement matrix due to redundant force
$\{X\}$	Redundant forces matrix
$\{F\}$	Flexibility matrix